ALGANT Master Program

Dipartimento di Matematica — Fakultät für Mathematik

An Introduction to Blow-Ups of Quasi-Smooth Closed Derived Subschemes

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Contents

Introduction

As commented by Toën in his review paper [38] on Derived Algebraic Geometry, "Derived Algebraic Geometry is an extension of algebraic geometry whose main purpose is to propose a setting to treat geometrically 'special' situations (typically 'bad' intersections, quotients by 'bad' actions, ...), as opposed to generic situations (transversal intersections, quotients by free and proper actions, ...)."
In such a formulation the influence of Grothendieck's method is apparent: rather than confronting problems directly, it is meaningful to create an ambient world in which statements can naturally find a solution. Nevertheless, an even more relevant standpoint is the attitude towards problem-solving, which in practice implies the existence of several notions of "Higher" or "Derived Algebraic Geometry". These tend to feature also different practical strategies and not always yield equivalent results. All of them are based on some notion of "Higher Algebra", which again depends on the context at stake (see for instance the section "Comparison with Spectral Higher Algebra"). Philosophically speaking, there is no ultimate formalism, but each approach has its own features, strengths and weaknesses. And even the limits of comparability among the several approaches are not to be understood as a bug, but as an expression of different descriptive potentials.

In this dissertation, we choose to follow the theory developed by Lurie, Toen and Vezzosi in several contributions - see for instance [26], [39], [40] and the voluminous Ph.D. Thesis by Lurie subdivided into several volumes DAG-n. We refer the reader interested in historical details to the aforementioned review [38].
We will adopt the language of ∞-categories, and aim at attempting a maximally model independent exposition. For such a reason, we extensively leverage on the construction of the "non-abelian localization" - or "animation" in the language of Scholze and Cesnavicius in [3] - to construct most of the objects of interest.

Our exposition aims at providing an (almost) self-contained introduction to the construction of blow-ups of quasi-smooth closed "derived" sub-schemes via "animation", as presented in [17]. We will extensively discuss most of the prerequisites needed, since the literature available is sometimes rather intricate and many statements are expressed in different formalisms.

Blow-ups are a recurring construction in algebraic geometry. Intuitively, one tries to resolve local "bad behaviours" (so singularities) of schemes at some closed "centre"; this is achieved by successive local deformations of it along the "normal" direction which induce isomorphisms on its complement. This can be made precise, for instance, via the celebrated Hironaka's Theorem on the Resolution of Singularities.
In classical algebraic geometry, the blow-up at a closed subscheme - often called the "centre" - is defined via a universal property. It can always be constructed as the projectivization of the Rees algebra associated to the ideal-sheaf which defines the closed immersion considered. The centre will then be deformed into the "exceptional divisor", i.e. the projectivized normal cone of its ideal-sheaf.
Such features are maintained in the ∞-world, where the universal property is furthermore strengthened. The construction has been discovered in several steps by Khan-Rydh [17], Hekking [13] and Hekking-Khan-Rydh [18]. Subsequently, it has been enhanced by the three authors to greater generality; an overview of the state of the art can be found, for instance, in Hekking's Ph.D. dissertation.
Our goal is to provide an extensive introduction to Higher Algebra and Derived Algebraic Geometry in order to present the first paper [17] in the series above, namely *'Virtual Cartier Divisors and Blow-Ups'* by A. Khan and D. Rydh. Then, we will also compute some examples of blow-ups of quasi-smooth closed sub-schemes.
As is often the case in the ∞-world, defining the objects of interest is highly non-trivial; in particular, here the problem lied in the lack of a theory of graded algebras, which could have allowed for a generalization of the classical construction similar in shape. The two authors paved the way towards it, which has been achieved by Hekking in [13]. Indeed, they circumvented the algebraic problems by providing a more geometric description of the blow-up in a "nice" setting - namely considering blow-ups of quasi-smooth closed immersions, so "derived" zero-loci of maps into affine spaces.
In such situations, it arises as the moduli stack solving the problem of classifying all squares which express some relative "homotopical" notion of effective Cartier divisors. The latter morally amounts to closed immersions of "derived" schemes which are locally cut-out by a single equation and lie over the canonical inclusion of the origin into some affine space.

The "niceness" lies in the fact that, although "derived" rings do not supply a Factor Theorem, such zero-loci can be characterized locally by a universal property similar in spirit to that of ring quotients. For a closed immersion, this turns out to be a very algebraic feature, which is controlled by both a compactness condition and the local freeness of the associated conormal sheaf, whence the terminology "quasi-smooth".
Remarkable is that such a geometric approach via a moduli problem can actually be extended, so as to construct blow-ups at arbitrary closed immersions. This is achieved in the subsequent paper [18].

In order to appreciate these subtleties, the exposition is organized as follows.
The first part is devoted to some foundational work. Initially, we review higher algebra as in SAG [26],25, but we choose to present the theory independently of the model of spectra; in particular, we slightly contaminate the exposition by providing the (almost complete) proofs of some statements in the introduction to the Ph.D. thesis of A. Khan [15], so as to obtain presheaves of animated algebras CAlg^{Δ} and modules Mod, MOD. The construction of the latter is formal and - although at some steps it still remains conjectural - it should be further generalizable. This also allows to construct symmetric monoidal structures on the ∞-categories of modules and rings in a compatible way. We do not know references in the literature for such constructions, but the ideas are all well-known to experts and many were suggested to the author by his advisor Prof. Marc Hoyois.
Then, we introduce several useful constructions, among which of paramount importance are the "(algebraic) cotangent complex" and "(homotopy) quotient rings".
In view of the discussion above, let us remark that our exposition is not equivalent to the one via spectra. However, animated $\mathbb{Q}$-algebras retrieve connective rational E_∞-rings, and in such a setting also the corresponding connective module categories agree (see section 3.3 or [26],25.1.2). Strikingly, animated modules over some animated ring $\underline{A}$ are not in general the module objects in the ∞-category $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ of $\underline{A}$-algebras (see [26],25.3.3).

Thereafter, again following Khan's thesis, we approach Derived Algebraic Geometry as follows: the right Kan extension of the previous constructions Mod and CAlg^{Δ} amount to the presheaves of quasi-coherent modules and algebras, while "derived" (Zariski-)stacks are presented as a generalization of the formalism of "functors of points". Then, we define several interesting classes of relative schemes and provide a comparison with the more "topological" perspective. So, we will finally be ready to introduce "quasi-smoothness", "virtual Cartier divisors" and to present the main section of the thesis on "derived" blow-ups.

At the end, the dissertation will be concluded with three Appendices on "Animation", "Symmetric Monoidal ∞-categories" and "Sheaves and ∞-topoi". Here we collect useful results on the topics from various sources and translate them into our language.

As for the pre-requisites, on the categorical level we assume familiarity with the formalism of ∞-categories, as exposed in Lurie's [24] or in the more gentle introduction [20] by Land. Nevertheless, we will recall the constructions needed whenever this allows relevant insights.
Moreover, due to time constraints, sometimes we had to reference statements without being able to provide appropriate proofs. In particular this concerns facts about (pre-)stable categories - for which we refer to Lurie's [23],1, [21], or to the Appendix C in [26] - and symmetric monoidal structures - see [23] for an exposition in the language of operads and [26] for the applications to the categories of quasi-coherent modules and algebras needed.
For what concerns the more geometric parts of the essay, we adopted as main references [7], lecture notes given by Prof. Guido Kings in Regensburg last Winter Semester and occasionally the Stacks Project [37]. We assume the content of a standard two-semesters-long course on the theory of schemes, in particular about the formalism of the "functor of points". Most of this will be generalized to the "derived" setting. In particular, whenever possible we will attempt a comparison between the ∞-worlds and the classical one, as well as reduce proofs to their classical analogues.
Finally, there are a few arguments with a more model-theoretic flavour or invoking spectral sequences; however, this will be reduced to a minimum, so they can be safely skipped and we will not recall the notions needed.

1 Preliminaries

1.1 Conventions

In a sense that will become more precise in what follows, studying DAG in the language of animated (commutative) rings - or equivalently simplicial (commutative) rings á la Lurie - can be thought of as an $(\infty, 1)$-categorification of classical AG.
We will, then, extensively employ the language of ∞-categories, attempting a model independent study, in the spirit of Gaitsgory and Rozenblyium (see [8]), and adopt the model of quasi-categories only when strictly necessary. Our main sources are the comprehensive HTT,[24] by Lurie as well as an introductory review of it [20] published by Land. Such a formalism will be assumed to be a prerequisite, so that well-known results will be reported freely. We will still elaborate on some more involved proofs whenever this might convey deeper intuition.
For the rest of our dissertation let us fix three Grothendieck universes $\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathcal{U}''$. Unless otherwise specified, all the choices of sets, the constructions of universals, etc. are to be understood as performed within $\mathcal{U}$, and hence involving (locally) "small" categories or sets. We will omit the word "small" in the notation, whenever this does not involve any risk of confusion. On the contrary, the terminology "large" and "very large" will refer to constructions in $\mathcal{U}'$ and $\mathcal{U}''$, respectively. Moreover, the work "class" will be used if we do not want to specify the size of our sets.
A list of some recurrent pieces of notation follows:

- Cat_X will denote the $(X+1,1)$-category of $(X,1)$-categories, for $0 \leq X \leq \infty$. In what follows, we will refer to $(\infty, 1)$-categories simply as ∞-categories. These can be grouped into the ∞-category Cat_∞, obtained (via one of the many equivalent constructions) as the homotopy-coherent nerve of the Kan-enrichment of $\mathrm{Cat} = \mathrm{Cat}_1$.

- We will denote the (homotopy-coherent) nerve by $\mathcal{N}$ and distinguish between the two only in the risk of confusion.

- Let $\subseteq_{f.f.}$ denote 'fully faithful embeddings'.

- Spc is our notation for the category of spaces. Such an ∞-category arises for instance as the localization of Kan - or equivalently ∞-groupoids Gpd_∞ - at the class he of (weak) homotopy equivalences.

- When dealing with purely categorical contents, $\mathcal{C} \in Cat$ will denote a 1-category, whereas $C \in \mathrm{Cat}_\infty$ will refer to an ∞-category. However, if clear from the context, we will prefer the notation $\mathcal{C}$ over C also for ∞-categories.

- When needed, we will distinguish between (co)limits in Cat and Cat_∞; to this extent, we write an apex 1 when referring to the former and no apex otherwise. Unless otherwise specified, the expression "homotopy (co)limits" will refer to (co)limits in Cat_∞, and the adjective will be most often omitted. Also, several classes of (co)limits will be considered. In general, by e.g. colim^{XXX} we will mean a colimit of shape from the family XXX. Moreover, let $\varinjlim$ denote directed (or filtered) colimits.

- In the more geometrical context, let Sch denote the ∞-category of "derived" schemes. On the other hand, classical schemes form the ordinary category $\mathrm{Sch}^{\mathrm{cl}}$, which will will be retrieved as the essential image of the functor $(-)^{\mathrm{cl}}$ extracting "classical underlying schemes". We will drop the adjective "derived" and mean all schemes to be such; on the contrary, the attribute "classical" will always highlight when we perform constructions in $\mathrm{Sch}^{\mathrm{cl}}$.

- Moreover, we will call "base-change" any fibred-product of schemes. It will always be specified if this occurs along any special class of morphisms.

1.2 Ajoint Functor Theorems

In the present section we will recall some useful adjoint functor theorems for presentable ∞-categories, almost verbatim generalizations of their 1-categorical analogous. Our main sources are Lurie in [24],4.3 and its review by Land, as in [20],5.2.
First we need to introduce come piece of terminology about continuity properties of ∞-categories.

Definition 1.2.0.1. *(colim-dense subcategory)* A full subcategory $C_0 \subseteq_{f.f.} C$ in Cat_∞ is called colim-**dense** in case it colim-generates C, i.e. for each object $x \in C$ there is a diagram $p : K \to C_0 \subseteq C$ s.t. $x \simeq \mathrm{colim}_K\, p$.

Definition 1.2.0.2. *(κ-filtered limit)* For a regular cardinal κ, we define a **κ-filtered colimit** (or κ-directed limit) $\underrightarrow{\mathrm{colim}}^\kappa$ a colimit indexed by a κ-filtered $I \in \mathrm{Cat}_\infty$, i.e. an ∞-category I s.t. each diagram $J \to I$ over a κ-small $J \in sSet$ admits a cone $J \star \Delta^0 \to I$.

Such definitions highlight the key properties of a relevant class of ∞-categories, namely those which are "accessible". As we will see, the terminology comes from the fact that they are (almost) a locally small free Ind_κ-completion of an essentially small ∞-category with κ-cpt objects; as such, the knowledge about their 'well-behaved' properties can be 'accessed' through their κ-cpt objects.

Definition 1.2.0.3. *(Accessible ∞-category)* Given a regular cardinal κ, we say that an ∞-category $C \in \mathrm{Cat}_\infty$ is **κ-accessible** in case

- it is locally small;
- it admits κ-filtered colimits;
- it has an essentially small (i.e. equivalent to a small ∞-category) full subcategory $C_0 \subseteq_{f.f.} C$ which consists of κ-compact objects and which generates C under κ-filtered colimits (so, it is $\underrightarrow{\mathrm{colim}}^\kappa$-dense in C).

We then say that $C \in \mathrm{Cat}_\infty$ is **accessible** if it is κ-accessible for some regular cardinal κ.

A relevant subclass of accessible categories is that of the presentable ones:

Definition 1.2.0.4. *(Presentable ∞-category)* An ∞-category $C \in \mathrm{Cat}_\infty$ is **presentable** if it is both accessible and cocomplete.

As we will remark, such a class enjoys extremely rich properties. In what follows we will present two results on the existence of adjoints for functors from e.g. presentable categories.
Although we will apply these theorems only to functors from presentable categories, we anyway present them in greater generality, in analogy with the 1-categorical formulations. Notice that, again on the lines of such a similarity, also in the ∞-world the construction of adjoints is almost tautological.

Theorem 1.2.0.5. *(I Adjoint Functor Theorem, [20],5.2.2) Let $C \in \mathrm{Cat}_\infty$ be locally small and cocomplete; assume that it admits a full subcategory $C_0 \subseteq_{f.f.} C$ which is essentially small and* colim*-dense.*
Then, for any locally small ∞-category $D \in \mathrm{Cat}_\infty$, any functor $F : C \to D$ has a right adjoint iff it is cocontinuous.

Theorem 1.2.0.6. *(II Adjoint Functor Theorem, [20],5.2.14) Let $C \in \mathrm{Cat}_\infty$ be (locally small) accessible and complete; consider any $D \in \mathrm{Cat}_\infty$ which is locally small, complete and such that each of its objects d is κ-compact, for some (wlog uniform over* $\mathrm{ob}D$*) regular cardinal $\kappa = \kappa(d)$.*
Then, $F : C \to D$ has a left adjoint iff it is continuous and accessible (i.e. F preserves eventually continuous directed colimits, which means that there is some cardinal κ such that F preserves ν-directed limits whenever $\nu > \kappa$.)

An almost direct consequence of the I Adjoint Functor Theorem (together with [24],5.3.4.13) is the following result:

Corollary 1.2.0.7. *([20],5.2.5) A locally small and cocomplete ∞-category C which is also* colim-*generated by an essentially small subcategory is also complete. In particular, presentable categories are bi-complete.*

The II Adjoint Functor Theorem (together with the same [24],5.3.4.13) then supplies also an almost converse implication to the previous corollary.

Corollary 1.2.0.8. *([20],5.2.19) A complete and accessible ∞-category is also cocomplete and, hence, presentable.*

1.3 Representability Theorems

We record here for future reference two highly non-trivial results on the (co)representability of functors from a presentable ∞-category into spaces. As shown in Lurie's [24],5.2.2, they can be used to prove the Adjoint Functor Theorems for presentable categories.

Theorem 1.3.0.1. *(Representability Criterion, [24],5.5.2.2) Let $C \in \mathrm{Cat}_\infty$ be a presentable ∞-category. A functor $C^{op} \to \mathrm{Spc}$ is representable by some $c \in C$ iff it preserves small limits, i.e. it takes small colimits in C to limits in* Spc.

Theorem 1.3.0.2. *(Corepresentability Criterion, [24],5.2.2.7) Let $C \in \mathrm{Cat}_\infty$ be a presentable ∞-category. A functor $C \to \mathrm{Spc}$ is co-representable by some $c \in C$ iff it is accessible and preserves small limits.*

2 Topics in Classical Algebraic Geometry

In view of our future discussion on blow-ups of quasi-smooth schemes, in this section we briefly review its classical counterpart. We will start by introducing the notion of "regularity" and sketch its connection with the Koszul complex. Then, we will move to classical blow-ups of closed sub-schemes, with a main focus on the case of the inclusion of the origin in the n-th affine space.
Our goal is to record the classical facts on which our "derived" work grounds. So, this section is meant both as some sort of recollection and as an occasion to stress on the viewpoint which we aim at generalizing. Therefore, we have no claim of completeness. Our main references are [37], [17],2 and [7], but they are quoted rather freely.

2.1 Regular Immersions

We start by recalling some classical facts about (Koszul) regular immersions. In order to avoid confusion, we prefer to distinguish between the customary terminology of the StacksProject [37] and the one introduced by Khan and Rydh in our main source [17] for regularity in the context of "derived" schemes. As we will observe soon, they will turn out to coincide for locally Noetherian schemes.

Definition 2.1.0.1. *(Koszul complex, [17],2.1.1)* Define the **Koszul complex** of a commutative ring $A \in$ CRing to be the chain complex given by multiplication by any n-tuple of elements $f := (f_1, \dots, f_n) \subseteq A$:

- $n = 1 : \mathrm{Kosz}_A(f) := (f : A \to A) \in \mathrm{Ch}(\mathrm{CRing})$ in homological degrees 1 and 0;
- n arbitrary: $\mathrm{Kosz}_A(f) := \otimes_{i=1}^n \mathrm{Kosz}_A(f_i) = \otimes_{i=1}^n (f_i : A \to A)$.

Remark. For $n = 1$, the only non-trivial homology of the Koszul complex for $f \in A$ are $\mathrm{H}_0(\mathrm{Kosz}_A(f)) = A/\mathrm{Im}(f)$ and $\mathrm{H}_1(\mathrm{Kosz}_A(f)) = \mathrm{Ann}_A(f) := \mathrm{Ker}(f : A \to A)$.
Hence, $f \in A$ is a non-zero divisor of A iff $\mathrm{Kosz}_A(f) \to \mathrm{H}_0(\mathrm{Kosz}_A(f)) \cong A/f$ is a quasi-isomorphism, i.e. $\mathrm{Kosz}_A(f)$ is acyclic in positive degrees. This motivates the following definition.

Definition 2.1.0.2. *(Regular sequence, [37],15.30.1)* A tuple $(f_1, \dots, f_n)$ of elements in a commutative ring $A \in$ CRing is:

- **Koszul regular** iff $\mathrm{Kosz}_A(f_1, \dots, f_n)$ is acyclic in positive degree;

- **regular** iff the module $A/(f_1, \dots, f_n) \neq 0$ is non-trivial and, for each $0 \leq m \leq n$ and $f_0 := 0$, f_i is regular in $A/(f_1, \dots, f_{m-1})$.

Remark. For $n > 1$, the second condition is stronger (see [37],15.30.2) and it does depend on the order of the sequence. However, for a Noetherian commutative ring A they coincide whenever the sequence $(f_1, \dots, f_n) \subseteq \mathrm{rad}(A)$ (e.g. always for A Noetherian and local). Moreover, in such a case also the second condition does not depend on the order of the sequence. For $n = 1$, the two conditions are the same.

Remark. (Free resolutions of quotients) For a sequence $(f_1, \dots, f_n) \subseteq A \in \mathrm{CRing}$, the degree 0 homology of the Koszul complex is isomorphic to the quotient of the ring by the ideal generated by the sequence: $\mathrm{H}_0(\mathrm{Kosz}_A(f_1, \dots, f_n) \cong A/(f_1, \dots, f_n)$. Then, $\mathrm{Kosz}_A(f_1, \dots, f_n) \to A/(f_1, \dots, f_n)$ supplies a free resolution of the quotient whenever $(f_1, \dots, f_n)$ is Koszul regular. In particular, this can be used to compute the values of some derived functors at the quotient.

Lemma 2.1.0.3. *(Properties of Koszul complexes, [17],2.1.2) Let $A \in \mathrm{CRing}$ be a commutative ring and consider a sequence $(f_1, \dots, f_n) \subseteq A$. Then,*

1. (Free resolutions of quotients): $\mathrm{Kosz}_A(f_1, \dots, f_n) \to A/(f_1, \dots, f_n)$ *supplies a free resolution of the quotient whenever the sequence $(f_1, \dots, f_n)$ is Koszul regular.*

2. (Stability by extension of scalars): *For any map $\phi : A \to B$ in* CRing*, there is a quasi-isomorphism of chain complexes $B \otimes_A \mathrm{Kosz}_A(f_1, \dots, f_n) \to \mathrm{Kosz}_B((\phi f_1, \dots, \phi f_n)$ in $\mathrm{Ch}(B)$.*

3. (Computing Koszul complexes): *$(f_1, \dots, f_n)$ is the datum of a map $(f_1, \dots, f_n) : \mathbb{Z}[t_1, \dots, t_n] \to A$ acting by $t_i \mapsto f_i$. Then, $H_*\big(\mathrm{Kosz}_A(f_1, \dots, f_n)\big) \cong \mathrm{Tor}_*^{\mathbb{Z}[t_i]_i}\big(A, \mathbb{Z}[t_i]_i/(t_i)_i\big)$ are (quasi-)isomorphic in the derived category $\mathcal{D}(A)$.*

Proof. (1) : For a sequence $f := (f_1, \dots, f_n) \subseteq A \in \mathrm{CRing}$, the degree 0 homology of the Koszul complex is isomorphic to the quotient of the ring by the ideal generated by the sequence. Then, consider the canonical map $\mathrm{Kosz}_A(f) \to \mathrm{H}_0\big(\mathrm{Kosz}_A(f)\big) \cong A/(f)$.

(3) : Let $t := (t_1, \dots, t_n)$ and $f := (f_1, \dots, f_n)$ denote the n-tuples at stake. The Tor-groups can be computed as $\mathrm{H}_*\big(A \otimes_{\mathbb{Z}[t]} \mathrm{Kosz}_{\mathbb{Z}[t]}(t)\big)$ via the free resolution $\mathrm{Kosz}_{\mathbb{Z}[t]}(t) \to \mathbb{Z}[t]/(t)$ induced by the Koszul regular sequence (t). Finally, the Koszul complex is stable under extension of scalars, so that the two chain complexes in the statement are quasi-isomorphic as claimed. ☐

Definition 2.1.0.4. *(Regular Immersion, [17],2.1.3)* A closed immersion $Z \twoheadrightarrow X$ of classical schemes in $\mathrm{Sch}^{\mathrm{cl}}$ is a **regular immersion** iff the corresponding $\mathcal{O}_X$-ideal-sheaf $\mathcal{I}$ is Zar-locally generated by a Koszul regular sequence.

Remark. For X locally Noetherian, the ideal sheaf is locally Koszul regular iff locally regular, so we recover the standard definition of a l.c.i. (or also "regular") immersion.

Remark. ([17],2.1.3) Let $Z \twoheadrightarrow X$ be a regular immersion cut-out by the $\mathcal{O}_X$-ideal-sheaf $\mathcal{I}$. Then, the conormal sheaf $\mathcal{N}_{Z/X} \cong \mathcal{I}/\mathcal{I}^2$ is locally free of finite rank (since it is locally a base-change of $\Omega^1_{\mathbb{Z}[t]} \cong \mathbb{Z}^n$ for some indeterminates $t = (t_1, \dots, t_n)$), and the relative cotangent sheaf can be canonically identified with the suspension of the latter: $\mathcal{L}_{Z/X} \cong \mathcal{N}_{Z/X}[1]$.

Proposition 2.1.0.5. *(Properties of regular immersions) "Being a regular immersion" is stable under composition, flat base-change and is local on the base.*

Proof. We regard such properties as "static shadows" of those of quasi-smooth closed immersions. In 4.5.2.2,ii we will observe that a closed immersion of classical schemes is regular iff it is quasi-smooth; then, the stated properties can be seen as follows. 4.5.2.2,i implies the first two: recall that taking the base-change of classical schemes in Sch coincides with that in $\mathrm{Sch}^{\mathrm{cl}}$ whenever it is performed along a flat morphism; as for the third one, argue in view of the characterization of quasi-smooth closed immersions 4.5.2.3 together with 3.6.2.6. ☐

A regular immersion of codimension 1 yields a prominent example of a closed subscheme, namely an effective Cartier divisor.

Definition 2.1.0.6. *(Effective Cartier Divisor)* Let $X \in \mathrm{Sch}^{\mathrm{cl}}$ be a classical scheme. Define an **effective Cartier divisor** on X to be equivalently:

- *([37],31.13.1):* a closed classical subscheme $D \twoheadrightarrow X$ whose $\mathcal{O}_X$-ideal sheaf is an invertible $\mathcal{O}_X$-module.
- *([17],2.1.3):* a classical scheme D equipped with a regular closed immersion $D \twoheadrightarrow X$ of Krull codimension 1.

In what follows, all divisors will be assumed to be effective, so we will omit the latter specification. Moreover, the following property characterizes Cartier divisors among all locally principal sub-schemes of X. We will prove only one directions, since the other one is not needed.

Lemma 2.1.0.7. *(Cartier divisors have dense complements, [7],IV-19) Let $Z \twoheadrightarrow X$ in $\mathrm{Sch}^{\mathrm{cl}}$ be a locally principal subscheme. Then, Z is a Cartier divisor iff its complement $X \setminus Z \subseteq X$ is schematically dense, i.e. its schematic closure is the whole X.*

Proof. Assume $Z \subseteq X$ to be a Cartier divisor, and let's show that its complement $X \setminus Z$ is schematically dense. The statement is local, so wlog $X = \mathrm{Spec}(R)$, and there exists some non-zero-divisor $f \in R$ for which $Z = \mathrm{Spec}(R/(f))$ and $X \setminus Z = \mathrm{Spec}(R[f^{-1}])$. Notice that any closed subscheme $Z' \subseteq X$ such that $X \setminus Z \subseteq Z' \setminus X$ corresponds to a factorization $R \twoheadrightarrow R/I(Z') \to R[f^{-1}]$ of the localization map at f. However, the latter is injective, because f is a non-zero-divisor; hence, for any such Z' it must hold $Z' \cong X$, i.e. the schematic closure of $X \setminus Z$ in X is the whole ambient scheme. □

2.2 Classical Blow-Ups: the Affine Case

Now, let us briefly review the classical construction of the blow-up of a scheme at a closed subscheme. Our main reference will be [7],IV, which will be quoted rather freely. In the rest of this dissertation, whatever appears here will be denoted with an apex $^{\mathrm{cl}}$.
The blow-up of a closed immersion $i : Z \twoheadrightarrow X$ of classical schemes is defined via a universal property as a classifying classical scheme for Cartier divisors obtained as a base-change along i. The bulk of the work is then to prove its existence.

Definition 2.2.0.1. *(Classical Blow-Up, [7],IV-16)* Let $Z \twoheadrightarrow X$ be a closed immersion of classical schemes in $\mathrm{Sch}^{\mathrm{cl}}$. Define the **blow-up** of X at Z, write $\pi_{Z/X} : \mathrm{Bl}_Z(X) \to X$, by the following universal property:

1. $E_Z X := \mathrm{Bl}_Z(X) \times_X Z \twoheadrightarrow \mathrm{Bl}_Z(X)$ is a Cartier divisor;
2. $\pi_{Z/X} : \mathrm{Bl}_Z(X) \to X$ is universal (read "terminal") for property (1): given any other Cartier divisor $Z \times_X S \to S$ in $\mathrm{Sch}^{\mathrm{cl}}_{/X}$, there exists a unique factorization of the structure map $S \to \mathrm{Bl}_Z(X) \to X$ via $\pi_{Z/X}$.

   In other words, if $\mathrm{CaDiv}_{/(Z,X)} \subseteq \mathrm{Mor}(\mathrm{Sch}^{\mathrm{cl}})_{/(Z,X)}$ denotes the subset of Cartier divisors $i^*(S) \twoheadrightarrow S$ over $i : Z \twoheadrightarrow X$ - so sitting in a cartesian square together with the structural morphisms - then the object-function of $(\pi_{Z/X})_* : \mathrm{Sch}^{\mathrm{cl}}_{/\mathrm{Bl}_Z(X)} \to \mathrm{Sch}^{\mathrm{cl}}_{/X}$ factors bijectively through $\mathrm{CaDiv}_{/(Z,X)}$.

The universal Cartier divisor $E_Z X := \mathrm{Bl}_Z(X) \times_X Z$ is called the **exceptional divisor** of X at Z.

Example 2.2.0.2. *(Degenerate cases)* The definition is not empty; for instance consider the following degenerate cases. For a classical scheme $X \in \mathrm{Sch}^{\mathrm{cl}}$:

- The blow-up of id_X is the empty-scheme;
- The blow-up of $\emptyset \twoheadrightarrow X$ is X itself;

- The blow-up of a Cartier divisor $i : Z \twoheadrightarrow X$ is isomorphic to the scheme itself via the structure map $\pi_{X/X} : \mathrm{Bl}_X(X) \overset{\simeq}{\rightarrow} X$.

Before delving into existence issues, the next Lemma records some features following directly from the universal property.

Lemma 2.2.0.3. *(Properties of the classical blow-up) Let $i : Z \twoheadrightarrow X$ be a closed immersion. Then, the following properties hold:*

1. *Assume the existence of the blow-up $(\mathrm{Bl}_Z(X), \pi_{Z/X})$. For any open $U \hookrightarrow X$, there is an isomorphism $\mathrm{Bl}_{U\cap Z}(U) \cong \pi^{-1}_{Z/X}(U) \subseteq \mathrm{Bl}_Z(X)$.*

2. *Blow-ups can be determined locally over the base. More precisely, for any affine Zariski cover $\{U_\alpha\}_\alpha$ for the base X, whenever existing the blow-ups $\{\mathrm{Bl}_{U_\alpha \cap Z}(U_\alpha)\}_\alpha$ can be glued to $\mathrm{Bl}_Z(X)$.*

3. *The structure map $\pi_{Z/X}$ induces an isomorphism $\mathrm{Bl}_Z(X) \setminus E_Z(X) \to X \setminus Z$ "away from Z".*

Proof. (1) : It follows from the locality on the base of both regular immersions and of the principality of the ideal cutting out the source.
(2) : The transition morphisms are induced by the classifying property of blow-ups and restrict to isomorphisms on the (open) intersections by (1); then, their glueing enjoys the universal property of blow-ups, as it follows from the locality on the base of both regular immersions and of the principality of the ideal cutting out the source.
(3) : $X \setminus Z \hookrightarrow X$ is an open immersion; hence, by (1) it holds:

$$\mathrm{Bl}_Z(X) \setminus E_Z(X) \cong \pi^{-1}_{Z/X}(X \setminus Z) \cong \mathrm{Bl}_{(X\setminus Z)\cap Z}(X \setminus Z) \cong \mathrm{Bl}_\emptyset(X \setminus Z) \cong X \setminus Z$$

□

The universal property of the blow-up of $Z \twoheadrightarrow X$ is "local on the base" over X, so even the simplest example is already rather enlightening.

Theorem 2.2.0.4. *(Blow-up of $\mathbb{A}^n$ at $\{0\}$) Let $i : \{0\} \twoheadrightarrow \mathbb{A}^n_R = \mathrm{Spec}(R[t_1, \dots, t_n])$ in $\mathrm{Sch}^{\mathrm{cl}}$ denote the closed embedding of the origin into the n-th affine space over any ring $R \in \mathrm{CRing}$. Then,*

1. *There exists the blow-up $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \cong \mathrm{Proj}_R(\mathrm{Sym}^*_R(t_1, \dots, t_n))$ of $\mathbb{A}^n_R$ at $\{0\}$; it is the glueing datum of the affine open charts $\mathrm{Spec}(A_k) \cong \mathbb{A}^n_R$ with $A_k := R[t_k\, , t_r/t_k : r \neq k]$ along the identifications $\{A_k[t_r^{-1}] \cong A_r[t_k^{-1}]\}_{r\neq k}$.*

   The structure map $\pi_{\{0\}/\mathbb{A}^n_R} : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \to \mathbb{A}^n_R$ is the composite $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \twoheadrightarrow \mathbb{A}^n_R \times_{\mathrm{Spec} R} \mathbb{P}^{n-1}_R \to \mathbb{A}^n_R$.

2. *$E_{\{0\}}(\mathbb{A}^n_R) := \pi^{-1}_{\{0\}/\mathbb{A}^n_R}(\{0\}) \twoheadrightarrow \mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R)$ is the universal Cartier divisor over $\{0\} \twoheadrightarrow \mathbb{A}^n_R$ in the sense of the definition above.*

   Moreover, there is a canonical isomorphism $E_{\{0\}}(\mathbb{A}^n_R) \cong \mathbb{P}^{n-1}_R \cong \mathbb{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbb{A}^n_R})$ with the projectivization of the conormal sheaf $\mathcal{N}_{\{0\}/\mathbb{A}^n_R} \cong \mathcal{I}/\mathcal{I}^2$ for the ideal-sheaf $\mathcal{I}$ defining the inclusion of the origin.

3. *The structure map $\pi_{\{0\}/\mathbb{A}^n_R} : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \to \mathbb{A}^n_R$ is a regular immersion, is proper, and induces an equivalence with the base $\mathbb{A}^n_R$ "away from the origin": $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \setminus E_{\{0\}}(\mathbb{A}^n_R) \overset{\simeq}{\rightarrow} \mathbb{A}^n_R \setminus \{0\}$.*

Our aim for this section is to collect the results from the classical affine setting on which to ground the more general proof later on. So, for convenience we will split the argument in several Lemmas and omit those proofs, such as e.g. glueing arguments, which are shadows of ∞-categorical phenomena and do not generalize. Let's start with a more explicit description of the affine charts above. It is the content of the following useful although rather technical lemma.

Lemma 2.2.0.5. *(A_k quotient of a regular sequence, [2],VII,1.8,ii) The rings $A_k = R[t_k\, , t_r/t_k : r \neq k]$ can be written as quotients by regular sequences.*

Proof. Let $t := (t_1, \dots, t_n)$, $y^k := (y_r :\neq k)$ denote tuples of indeterminates, and define a tuple of relations $\rho^k := (\rho_r := t_k y_r - t_r : r \neq k)$. Let's unwind the definition of the structural morphism $\gamma_k : \mathrm{Spec}(A_k) \to \mathbb{A}^n_R$ between $R[t]$-algebras:

$$\begin{aligned} \gamma_k : R[t_1, \dots, t_n] &\longrightarrow \frac{R[t_1, \dots, t_n][y_r]_{r\neq k}}{(\rho_r := t_k y_r - t_r : r \neq k)} = A_k \\ t_k &\longmapsto t_k \\ (\forall r \neq k) \qquad t_r &\longmapsto t_r = t_k y_r \end{aligned}$$

Then, observe that the sequence $\rho^k \subseteq R[t, y^k]$ is regular.
Indeed, it is an extract of the sequence $(t, \rho^k) \subseteq R[t, y^k]$, and the latter is regular, since it is obtained from (t, y^k) by multiplication with the invertible matrix M acting by the blocks: $M_{1,1} := (\mathrm{Id}_n)$, $M_{1,2} := (0_{n,n-1})$, $M_{2,1} := (-\mathrm{Id}_{n-1,n})$, $M_{2,2} := (t_k \mathrm{Id}_{n-1})$. □

We are now ready to provide the construction of the blow-up of the inclusion of the origin into the n-th affine space.

Construction 2.2.0.6. *(The Blow-Up* $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R)$*)* Consider the affine schemes $\mathrm{Spec}(A_k) \in \mathrm{Sch}^{\mathrm{cl}}_{/\mathbb{A}^n}$ above as lying under $\mathrm{Spec}(R(t_1, \dots, t_n))$ for the map induced by the inclusion. As sub-algebras of $R(t_1, \dots, t_n)$, the localizations $A_k[t_r^{-1}] \cong A_r[t_k^{-1}]$ are canonically isomorphic $R[t_1, \dots, t_n]$-algebras, so that they yield a glueing datum describing the patches of $\pi : Y := \cup \mathrm{Spec}(A_k) \to \mathbb{A}^n_R$; the structural morphism π is induced by the canonical inclusion $R[t_1, \dots, t_n] \subseteq R(t_1, \dots, t_n)$, which is clearly compatible with the glueing datum.

Claim 1. *The obtained glueing is* $Y \cong \mathrm{Proj}_R(\mathrm{Sym}^*_R(t_1, \dots, t_n))$.

Proof. Assume $n > 1$, otherwise it is trivial. In the proof, we will need the following tuples of indeterminates: $t := (t_1, \dots, t_n)$, $T/T_k := (T_r/T_k : r \neq k)$ and $y^k := (y_r : r \neq k)$. Let's describe the second term. Recall first that $\mathrm{Sym}^*_R(t) \cong R[t]/J$, where J is the ideal generated by all the relations $(t_r T_k - T_r t_k)_{r,k}$; here we let t_r, T_r denote the copy of the indeterminate t_r in degree $0, 1$, respectively.
Then, the affine chart $D_+(T_k)$ of the projectivization where T_k is invertible, is $\mathrm{Spec}(R[t][T/T_k]/I_k)$ with I_k generated by the relations $(t_r - T_r/T_k t_k)_{r\neq k}$. Consider the tuple of relations $\rho^k := (\rho_r := t_k y_r - t_r : r \neq k) \subseteq R[t][y^k]$; then the change of variables $y^k := T/T_k$ allows us to write $\mathrm{Spec}(R[t][T/T_k]/I_k) \cong \mathrm{Spec}(R[t][y^k]/(\rho^k)) \cong \mathrm{Spec}(A_k)$ as in 2.2.0.5. ■

So, we state the following Claim, which will be proved soon. We will start by verifying the first property in the definition of a blow-up.

Claim 2. *([7],IV-18) The pair* (Y, π) *exhibits the blow-up* $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R)$.

Claim 3. *([7],IV-17) The exceptional divisor* $E := E_{\{0\}}(\mathbb{A}^n_R)$ *exhibits a Cartier divisor* $E \twoheadrightarrow Y$. *In particular, this supplies an isomorphism* $E \cong \mathbb{P}^{n-1}_R$, *since the two schemes have isomorphic Zariski atlases.*

Proof. (Of Claim 2) Since "being a Cartier divisor" is Zar-local on the base, the Claim amounts to $E_k \twoheadrightarrow \mathrm{Spec}(A_k)$ being a Cartier divisor for each k. This is true, because the exceptional divisor $E := E_{\{0\}}(\mathbb{A}^n_R)$ can be described on the atlas for the blow-up as the quotient $E_k := E \times_{\{0\}} \mathrm{Spec}(A_k) = \mathrm{Spec}(A_k/(t_k))$. So, we are left to prove the following computation in CRing: for a tuple of indeterminates $t := (t_1, \dots, t_n)$,

$$A_k \otimes_{R[t]} R[t]/(t) \cong A_k/(t_k)$$

In order to see this, recall the action of the structure map γ_k from the previous Lemma 2.2.0.5 and observe that $(t_k, \rho_r) = (t_k, t_r)$ for each $r \neq k$; hence, one has the sought isomorphism: for a tuple of indeterminates $y^k := (y_r : r \neq k)$, and relations $\rho^k := (\rho_r := t_k y_r - t_r : r \neq k)$:

$$A_k \otimes_{R[t]} \frac{R[t]}{(t)} \cong \frac{R[t][y^k]}{(t) + (\rho^k)} \cong \frac{R[t_1, \dots, t_n][y^k]}{(t_k) + (\rho^k)} \cong \frac{A_k}{(t_k)}$$

■

Sketch. (Of Claim 1) We need to check the universality of $E \twoheadrightarrow Y$ among all Cartier divisors $(f : S \to \mathbb{A}^n_R,\ f^{-1}(V(t_1, \dots, t_n)) \twoheadrightarrow S)$ over $(\{0\}, \mathbb{A}^n_R)$.

Up to a glueing argument, we can assume wlog that $S := \mathrm{Spec}(B)$ be an affine scheme represented by some *local* $R[t_1, \ldots, t_n]$-algebra B (with structure map $f^\flat$ induced by f) and $f^{-1}(\{0\}) \cong \mathrm{Spec}(B/(x))$ for some non-zero divisor $x \in B$.
Moreover, we can assume $x = f^\flat(t_k)$ for some k. In order to see this, abuse notation and let $t := (t_1, \ldots, t_n)$ denote the n-tuple generating the maximal ideal $(t_1, \ldots, t_n)B$; let again $(t)B$ denote the extension of the maximal ideal $(t) \subseteq R[t]$ under $f^\flat$. by assumption, there is an equality $(t)B = (x)B$, which yields n-tuples $\alpha, \beta \in B^n$ such that $x = {}^t\alpha \cdot f^\flat(t)$ and $f^\flat(t) = \beta x$, so that $x = ({}^t\alpha \cdot \beta)x$. Now, being $x \neq 0$ and B local, an application of Nakayama's Lemma yields that the product ${}^t\alpha \cdot \beta \in B^\times = B \setminus (t)$ must be a unit, i.e. $\alpha_k\beta_k \notin (t)$ for some k, iff $\alpha_k, \beta_k \notin (t)$.
Hence, one can write $(x) = f^\flat(x_k)$ up to a unit. In particular, we can write $f^\flat(t) = \beta f^\flat(x_k)$. Then, consider the map $\phi^\flat : A_k \to B$ acting as $\phi^\flat(t/t_k) \mapsto \beta$. It gives rise to the sought commutative triangle $f^\flat = b^\flat \circ \pi^\flat$ with the structure map $\pi : Y \to \mathbb{A}^n_R$. Finally, by construction $\phi^\flat$ is also the unique map (up to rescaling by a unit of B) of $R[t_1, \ldots, t_n]$-algebras which can sit in such a triangle. ■

Remark. In 3.8.3.2 we will compute $\mathcal{N}_{\{0\}/\mathbb{A}^n_R} \cong R^n$; then, the isomorphism of 2.2.0.6,ii follows from the very definitions:

$$\mathbb{P}(\mathcal{N}_{\{0\}/\mathbb{A}^n_R}) \cong \mathbb{P}(R^n) = \mathrm{Proj}(\mathrm{Sym}^*_R((R^n)^\vee)) \cong \mathrm{Proj}(R[t_1, \ldots, t_n]) \cong \mathbb{P}^{n-1}_R \cong E_{\{0\}}(\mathbb{A}^n_R)$$

The next construction is not necessary for our future arguments; however, in view of [7],IV-21 fosters intuition on how blow-ups emerge.

Construction 2.2.0.7. *(*$\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ *as the closure of a graph, [7],IV-17)* Let $t := (t_1, \ldots, t_n)$ denote the n-tuple of indeterminates of $R[t_1, \ldots, t_n]$ and consider the map $\alpha_t : \mathbb{A}^n_R \setminus \{0\} \hookrightarrow \mathbb{P}^{n-1}_R$ induced by the surjection

$$\begin{aligned} \alpha_t^\flat : \mathcal{O}^{\oplus n}_{\mathbb{A}^n_R \setminus \{0\}} &\longrightarrow \mathcal{O}_{\mathbb{A}^n_R \setminus \{0\}} \\ a &\longmapsto {}^t a \cdot t \end{aligned}$$

More precisely, we are considering the first composite in the following factorization:

$$\alpha_t : \mathrm{Spec}(R[t]) \setminus \{0\} \longrightarrow \mathrm{Proj}(R[t]) \subseteq \coprod_k \mathrm{Spec}(R[t_r/t_k : r \neq k])$$

where $\{\mathrm{Spec}(R[t_r/t_k : r \neq k]\}_k$ forms an affine Zariski atlas of the scheme $\mathrm{Proj}(R[t]) \cong \mathbb{P}^{n-1}_R$.
Let $\mathrm{graph}(\alpha_t)_k \subseteq \mathbb{A}^n_R \times_{\mathrm{Spec}(R)} \mathbb{P}^{n-1}_R$ denote the graph of the restriction of α_t to the open affine chart $D(t_k) = \mathrm{Spec}(R[t][t_k^{-1}])$, and observe that it is isomorphic to $\mathrm{Spec}(A_k[t_k^{-1}])$; the latter is in turn a (schematically) dense open of the blow-up Z.
Hence, we obtain a description of the blow-up $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \cong \overline{\mathrm{graph}(\alpha_t)}$ as the schematic closure of the graph of the previous map into $\mathbb{A}^n_R \times_{\mathrm{Spec}(R)} \mathbb{P}^{n-1}_R$.
In particular, this supplies a factorization describing the structural map of the blow-up of $(\{0\}, \mathbb{A}^n_R)$:

$$\pi_{\{0\}/\mathbb{A}^n_R} : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \twoheadrightarrow \mathbb{A}^n_R \times_{\mathrm{Spec}(R)} \mathbb{P}^{n-1}_R \to \mathbb{A}^n_R$$

where the last composite is induced by the structural map $R[t] \subseteq R[t] \otimes_R R(t)$.
Moreover, as a by-product we can explicitly describe the canonical identification $\pi_{\{0\}/\mathbb{A}^n_R} : Y \setminus E \cong \mathbb{A}^n_R \setminus \{0\}$ of 2.2.0.3,iii. Indeed, the latter can be obtained as the glueing of the isomorphism of charts $D_{\mathbb{A}^n_R \setminus \{0\}}(t_k) = \mathrm{Spec}(R[t][t_k^{-1}]) \cong \mathrm{Spec}(A_k[t_k^{-1}]) = D_{Y \setminus E}(t_k)$.

Finally, the remaining properties of the structural morphism will be a consequence of the following construction.

Construction 2.2.0.8. *(*$\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ *as a closed regular subscheme of* $\mathbb{P}^{n-1}_{\mathbb{A}^n}$*)* As before, let $t := (t_1, \ldots, t_n)$, $y := (y_1, \ldots, {}_n)$ and $z = (z_1, \ldots, z_n)$ denote n-tuples of indeterminates. By 2.2.0.5, for each $1 \leq k \leq n$ there is a regular immersion into the $(n-1)$-dimensional projective space over $\mathbb{A}^n_R$:

$$\mathrm{Spec}(A_k) \cong \mathrm{Spec}\left(\frac{R[t][y_r : r \neq k]}{(\rho_r := t_k y_r - t_r : r \neq k)}\right) \twoheadrightarrow \mathrm{Spec}(R[t][y_r : r \neq k]) \subseteq \mathrm{Proj}(R[t][z]) \cong \mathbb{P}^{n-1}_{\mathbb{A}^n_R}$$

where we identified $\mathrm{Spec}(R[t][y_r : r \neq k]) \cong \mathrm{Spec}(R[t][z_r/z_k : r \neq k])$ with the k-th affine chart of the projective space.
The regular immersions obviously glue on the intersections among charts, since - for each $r \neq k$ - inverting t_r in $\mathrm{Spec}(A_k)$ amounts to inverting y_r. Thus, our construction glues to a closed immersion $\iota : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n_R) \twoheadrightarrow \mathbb{P}^{n-1}_{\mathbb{A}^n_R}$, which is furthermore regular, since the latter notion can be checked Zar-locally on the base.

Moreover, the structural map $\mathbb{P}^{n-1}_X \to X$ of the projective X-scheme is always proper and regular, so we are left to check that the embedding ι above is proper. This can be done Zar-locally on the base by 4.1.4.22, and there the three properties in the definition of properness are clearly satisfied by construction.

We close this brief review with a more general construction describing the blow-up of any closed immersion. This will be based on the Rees algebra: given a ring $R \in \mathrm{CRing}$ and an ideal $I \subseteq R$, the **Rees algebra** of I is the graded R-algebra defined as $\mathrm{Rees}(I) := \oplus_{n<\omega} I^n$, where we set $I^0 := R$.
This construction can then be globalized as follows: let $i : Z \twoheadrightarrow X$ be a closed immersion in $\mathrm{Sch}^{\mathrm{cl}}$, and let $\mathcal{I} := \mathrm{Ker}(i^\flat) \subseteq \mathcal{O}_X$ denote the defining ideal-sheaf; then, define the Rees algebra associated to $\mathcal{I}$ as the sheaf of quasi-coherent algebras $\mathcal{R}(\mathcal{I}) := \oplus_{n<\omega} \mathcal{I}^n \in \mathrm{QCohAlg}(X)$, where we again set $\mathcal{I}^0 := \mathcal{O}_X$.

Theorem 2.2.0.9. *(General construction of blow-ups, [7],IV-23) Let $i : Z \twoheadrightarrow X$ be a closed immersion in $\mathrm{Sch}^{\mathrm{cl}}$. Then, the blow-up of i is realized by the scheme $\mathrm{Bl}_Z(X) := \mathrm{Proj}_X(\mathcal{R}(\mathcal{I}))$.*

We will not prove the latter Theorem, because it will not be needed in our dissertation. Instead, let us comment on how to retrieve the blow-up of $\{0\} \twoheadrightarrow \mathbb{A}^n_R$ as a particular case. The same reasoning applies to an arbitrary regular immersion into a Noetherian scheme.

Example 2.2.0.10. *(The affine case)* Let $i : \{0\} \twoheadrightarrow \mathbb{A}^n$ denote the regular closed immersion of the origin in the n-th affine space, and let $\mathcal{I}$ denote the quasi-coherent ideal with global section $(t_1, \ldots, t_n)$. In 2.2.0.4 we showed the isomorphism $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \cong \mathrm{Proj}_R(\mathrm{Sym}^*_R(\mathcal{I}))$.
On the other hand, as stated in [7],IV-26 (and proved for instance in [28],Ch.1,Thm.1) there is an isomorphism $\mathrm{Sym}^*_R(\mathcal{I}) \cong \mathrm{Rees}(\mathcal{I})$, because the sequence $(t_1, \ldots, t_n) \subseteq R[t_1, \ldots, t_n]$ is regular. Hence, we retrieve the description of the Theorem above.

3 Higher Commutative Algebra

In this section we attempt an introduction to "higher algebra" in a model independent fashion and adopting the language of animated rings with possibly no reference to the parent notion of $\mathbb{E}_\infty$-rings. Our goal is to develop the algebraic foundations of "derived algebraic geometry".
All the material presented is well-known to experts, and can be found for instance in the following sources: [23] for what concerns higher algebra, [26],25 for an introduction to animated rings (called simplicial rings by Lurie), and [15],0.4 for the statements regarding the existence of presheaves of animated algebras and modules. The previous expositions have then been reviewed by Cesnavicius and Scholze in [3], where also the terminology *"anima,-ae"* (latin term for the English "soul") and *"animated widget"* are introduced.
In the algebraic and geometric context, we will stick to the latter convention, in that the author finds it to be more evocative in expressing an 'internalization'-like procedure of classical algebraic notions into the ∞-category *Ani* of "animae", namely spaces. However, for the sake of consistency with the generally accepted language of ∞-categories, we will continue to write *Spc* for the ∞-category of spaces in the most categorical passages.

Disclaimer. Let us mention that many proofs have been expanded in full details and some of them have been freely (so often without comments) more or less adapted to the chosen setting and goals. As a consequence, the exposition, terminology or notation in the references provided might be slightly different. The author takes up full responsibility for misunderstandings and wrong statements, and no claim of originality is made.

Aknowledgement. For what concerns many parts of the current exposition, the author is highly indebted to his advisor, Prof. Hoyois, for the many introductory talks, prompts and insights that he has given to him on the subject.

3.1 Recollection on Animation

Definition 3.1.0.1. Let $\mathcal{C} \in \mathrm{Cat}$ be a cocomplete category which is generated under 1-sifted colimits (in Cat) by the small full subcategory of its compact projective (or strongly finitely presented, denoted by *cpt+proj*) objects $\mathcal{C}^{\mathrm{sfp}}$.
Define the **animation** of $\mathcal{C}$ by $\mathrm{Ani}(\mathcal{C}) := \mathcal{P}_\Sigma(\mathcal{C}^{\mathrm{sfp}}) = \mathrm{Fun}^\times(\mathcal{N}(\mathcal{C}^{\mathrm{sfp}})^{op}, \mathrm{Spc})$.
As we will recall, the latter will turn out to describe $\mathrm{Ani}(\mathcal{C})$ as the free ∞-*sInd*-completion of $\mathcal{C}^{\mathrm{sfp}}$.

In other words, $\mathrm{Ani}(\mathcal{C})$ will be the smallest enlargement of $\mathcal{C}$ having all sifted colimits and satisfying the universal property of sInd-completions:

$$[\mathrm{UP}:\mathrm{Ani}]: \qquad \forall A \in \mathrm{Cat}_\infty \text{ w/ } \mathrm{colim}^{sift}, \quad \mathrm{Fun}_\Sigma\big(\mathrm{Ani}(\mathcal{C}), A\big) \xrightarrow{\simeq} \mathrm{Fun}\big(\mathcal{C}^{\mathrm{sfp}}, A\big)$$

where Fun_Σ denotes the full sub-∞-category spanned by those functors which preserve sifted colimits (see A.3.0.2).

We include here just a short summary of the $\mathcal{P}_\Sigma$-construction and refer the unexperienced reader to the related Appendix for further details.

Lemma 3.1.0.2. *(Properties of* Ani*) There exists an animation assignment* Ani *as in the previous definition. Moreover, for a cocomplete $\mathcal{C} \in \mathrm{Cat}$ generated under 1-sifted colimits by its compactly projective objects,* Ani *satisfies the following properties:*

1. *(A.2.0.2)* $\mathrm{Ani}(\mathcal{C})$ *is presentable, being it an accessible localization of the ∞-category of presheaves $\mathcal{P}(\mathcal{C}^{sfp})$. In particular,* $\mathrm{Ani}(\mathcal{C})$ *is bi-complete.*

2. *(A.7.0.2)* $\mathrm{Ani} := \mathrm{Ani}(\mathrm{Set}) \simeq \mathrm{Spc}$, *so that we can write* $\mathrm{Ani}(\mathcal{C}) = \mathrm{Fun}^\times\big((\mathcal{C}^{\mathrm{sfp}})^{op}, \mathrm{Ani}\big)$ *for each 1-category $\mathcal{C}$ as before. In other words, we regard animated widgets as generalized cartesian widget-objects in* Ani.

3. *(*A.5.0.7*) Post-composing with the truncation functor $\tau_{\leq n}$:* $\mathrm{Ani} \to \mathrm{Ani}$ *(see the Appendix) to an animated widget $(\mathcal{C}^{\mathrm{sfp}})^{op} \to$* Ani *induces a truncation functor $\overline{\tau}_n$:* $\mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{C})$, *which is left adjoint to the canonical fully faithful inclusion* $\mathrm{Ani}(\mathcal{C})_{\leq n} \subseteq_{f.f.} \mathrm{Ani}(\mathcal{C})$ *of $(\leq n)$-connected widgets. In particular, for $\mathcal{C} \in \mathrm{Cat}$ as before, there is a fully faithful inclusion $\mathcal{C} \subseteq_{f.f.}$* $\mathrm{Ani}(\mathcal{C})$ *which identifies $\mathcal{C}$ with the static (i.e. 0-truncated) widgets* $\mathrm{Ani}(\mathcal{C})_{\leq 0} \subseteq_{f.f.} \mathrm{Ani}(\mathcal{C})$.

Furthermore, by a closer inspection of Lurie's $\mathcal{P}_\Sigma$-construction, we can further specialize the previous description:

Lemma 3.1.0.3. *(A.3.0.2) In the previous setting, the universal property* $[\mathrm{UP}:\mathrm{Ani}]: \ \mathrm{Fun}_\Sigma((\mathcal{C}^{\mathrm{sfp}})^{op}, A) \simeq \mathrm{Fun}(\mathrm{Ani}(\mathcal{C}), A)$ *is realized by restriction along the factorization of the Yoneda embedding*

$$j : \mathcal{C}^{\mathrm{sfp}} \overset{j}{\hookrightarrow} \mathrm{Ani}(\mathcal{C}) = \mathrm{Fun}^\times((\mathcal{C}^{\mathrm{sfp}})^{op}, \mathrm{Spc}) \subseteq_{f.f.} \mathrm{Fun}((\mathcal{C}^{\mathrm{sfp}})^{op}, \mathrm{Spc})$$

The essential image of j consists of those finite-product-preserving functors $F : (\mathcal{C}^{\mathrm{sfp}})^{op} \to \mathrm{Spc}$ s.t. $F \simeq \mathrm{LKE}_j(F \circ j)$, i.e. which are the left Kan extensions along j of their restrictions to the cpt+proj's.
Moreover, assuming further that A has also finite coproducts, then any $F \in \mathrm{Fun}(\mathrm{Ani}(\mathcal{C}), A)$ preserves (finite) coproducts iff its restriction $F \circ j$ to cpt+proj's does so. Hence, in such a setting, F preserves all small colimits iff its restriction $F \circ j$ does so.

We are now ready to characterize animations as free sInd-completions: $\mathrm{Ani}(C)$ is the the datum of a presentable ∞-category together with a Yoneda embedding of $\mathcal{C}$ as cpt+proj colim^{sift}-generators, i.e. $\mathrm{Ani}(C) \simeq \mathrm{sInd}(\mathcal{C}^{\mathrm{sfp}})$.

Lemma 3.1.0.4. *(*$\mathrm{Ani}(\mathcal{C})$ *as* sInd*-completion, A.3.0.2 and A.4.0.3) Given a small finitely cocartesian $\mathcal{C} \in$* Cat, *its animation* $\mathrm{Ani}(\mathcal{C})$ *can be characterized up to equivalence by:*

- $\mathrm{Ani}(\mathcal{C}) \in \mathrm{Cat}_\infty$ *is presentable;*
- *the Yoneda embedding restricts to a fully faithful coproduct-preserving functor $j : \mathcal{C}^{\mathrm{sfp}} \subseteq \mathrm{Ani}(\mathcal{C})$;*

- *the essential image of the previous functor consists of those cpt+proj's in* $\mathrm{Ani}(\mathcal{C})$ *which generate the latter under sifted colimits.*

Moreover, our assignment Ani shows some kind of functoriality over a suitable sub-graph of Cat supported by a proper class:

Definition 3.1.0.5. Let $F : \mathcal{C} \to \mathcal{D}$ in Cat be a functor of cocomplete categories which preserves 1-sifted colimits. Then, it can be animated to a functor $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$ such that

1. $\mathrm{Ani}(F)$ preserves sifted colimits;
2. $\mathrm{Ani}(F)_{|\mathcal{C}^{\mathrm{sfp}}} : \mathcal{C}^{\mathrm{sfp}} \xrightarrow{F} \mathcal{D} \subseteq_{f.f.} \mathrm{Ani}(\mathcal{D})$ agrees with F on compact projective objects;
3. $\mathrm{Ani}(F)$ commutes with $\pi_0 := \tau_{\leq 0}$ in Cat_∞, i.e. $\pi_0 \circ \mathrm{Ani}(F) = F \circ \pi_0$.

Lemma 3.1.0.6. *(Composition of animated functors, A.6.0.6) Let* $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ *in* Cat *be cocomplete and projectively generated. Consider a pair of composable functors* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ *which preserve* 1*-sifted colimits. Then,*

- *There exists a natural transformation (i.e.* 2*-cell)* $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \to \mathrm{Ani}(G \circ F)$ *in* Cat_∞*;*
- *Assume further that*
  - *either* $F(\mathcal{C}^{\mathrm{sfp}}) \subseteq \mathrm{Ind}(\mathcal{D}^{\mathrm{sfp}})$ *in* $\mathcal{D}$*, i.e.* F *sends cpt+proj objects in* $\mathcal{C}$ *to directed colimits of cpt+proj's in* $\mathcal{D}$*;*
  - *or* $\mathrm{Ani}(G)\big(F(\mathcal{C}^{\mathrm{sfp}})\big) \subseteq \mathcal{E}$ *in* $\mathrm{Ani}(\mathcal{E})$*, i.e. the restriction of the composite of animated functors to cpt+proj's in* $\mathcal{C}$ *is again static;*

  Then, our comparison map is an isomorphism in Cat_∞*:* $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \xrightarrow{\simeq} \mathrm{Ani}(G \circ F)$*.*

Remark. As mentioned in the Appendix, we can regard the $\mathcal{P}_\Sigma$-construction as a non-abelian localization functor, and $\mathrm{Ani}(F)$ as a left-derived functor of F.

3.2 Animated Rings and Modules

Let us introduce, now, the ∞-categories of animated commutative rings, modules and algebras. Our main references for the current section are [26],25 and [3],5, of which we will merge notations. We will also drop the adjective "commutative", meaning all the rings considered to be so.
We will define the ∞-category $\mathrm{Ani}(\mathbb{Z}) = \mathrm{CAlg}^{\Delta}$ of animated commutative $\mathbb{Z}$-algebras and regard it as the 'ambient category' within which to define each of the ∞-categories $\mathrm{CAlg}^{\Delta}_{\underline{R}}$ spanned by commutative $\underline{R}$-algebras for any arbitrary animated ring $\underline{R}$.
However, differently from the classical case, in order to define the ∞-categories of animated modules over arbitrary animated rings, we will perform a construction similar to that of quasi-coherent modules (see 3.2.5.11).
This will lead us to consider an ∞-category of "animated modules with their animated ring of scalars" together with a canonical forgetful functor to the ∞-category of animated rings. The fibres of such a construction will turn out to be precisely the ∞-categories of modules over the corresponding animated rings; in particular, such ∞-categories will be defined simultaneously and our construction will be functorial via the extension-restriction of scalars adjunction.
We remark that one could define the ∞-categories of algebras over arbitrary animated rings in a similar, but significantly simplified fashion (see 3.2.3.1). This is the case because algebra structures - as opposed to module structures - can be expressed via slices, which makes it easier to functorially interchange them.
Our next step will be to endow the categories at stake with intrinsic as well as compatible monoidal structures, so as to be able to perform also in the derived setting many useful constructions of classical algebraic geometry. This will be carried on at the end of the dedicated Appendix B, where also the needed terminology is reviewed.
Finally, we will briefly compare our perspective with the better-studied - although not equivalent - notion of $\mathbb{E}_\infty$-rings.

Definition 3.2.0.1. For a ring $R \in \mathrm{CRing}$, let $\mathrm{Poly}_R \subseteq_{f.f.} R\text{-Alg} \in \mathrm{Cat}$ denote the full subcategory of polynomial rings over R in finitely many variables.
Define the category of **animated R-algebras** by $\mathrm{CAlg}_R^{\Delta} := \mathrm{Ani}(R\text{-Alg}) = \mathcal{P}_\Sigma(\mathrm{Poly}_R) \in \mathrm{Cat}_\infty$.

Notation. In what follows, we maintain the notation R also for the embedded copy of $R \in \mathrm{CRing}$ into the ∞-category $\mathrm{CAlg}_{R'}^{\Delta}$ of the animated R'-algebras at stake. On the other hand, by writing $\underline{S}$ we will refer to an arbitrary animated R'-algebra in such a category, so not necessarily living in the essential image of j.
As it will be made precise later on, such a slight abuse of notation will be proven to be consistent with the theory developed, as well as motivated by the fact that the embedded essential copy of R'-Alg into $\mathrm{Ani}(R'\text{-Alg})$ is 'static'.

3.2.1 Homotopy groups

As a starting point of our digression, recall the following property of animation. Being animated rings functors of ∞-categories into *Spc*, truncation of spaces imports such a notion into Ani(CRing), thus allowing us to define homotopy groups over the latter category. This will be achieved by means of a more general construction, as introduced in [24],5.5.8.26 and presented in the Appendix on 'Animation'. Let us restate the result in the language of animated algebras.

Construction 3.2.1.1. *(Truncation and Homotopy groups of animated rings)* Fix an integer n and let $\tau_{\leq n}^{\mathrm{Spc}}$ denote the n-truncation functor of spaces. It is left adjoint to the fully faithful inclusion of n-truncated spaces $\subseteq^{\mathrm{Spc}}$, i.e. it sits in a Bousfield localization adjunction

$$\tau_{\leq n}^{\mathrm{Spc}} : \mathrm{Spc} \rightleftarrows \tau_{\leq n}^{\mathrm{Spc}}\mathrm{Spc} : \subseteq^{\mathrm{Spc}}$$

Now, for any $R \in \mathrm{CRing}$, we say that $\underline{S} \in \mathrm{CAlg}_R^{\Delta}$ is ***n*-truncated** provided that the space $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(\underline{X}, \underline{S})$ is n-truncated for each $\underline{X} \in \mathrm{CAlg}_R^{\Delta}$, iff for each $\underline{X} \in \mathrm{Poly}_R$. Indeed, by A.2.0.4 each animated R-algebra is the geometric realization of a simplicial object $\underline{X} \simeq \mathrm{colim}^{sift}\, X_\bullet$ with each X_m an arbitrary coproduct of polynomial R-algebras, mapping spaces commute with colimits in the contravariant variable, and limits in Spc preserve truncation properties.
Define the ∞-category of ***n*-truncated animated R-algebras** to be the full subcategory $\tau_{\leq n}\mathrm{CAlg}_R^{\Delta} \subseteq_{f.f.} \mathrm{CAlg}_R^{\Delta}$ generated by the n-truncated objects.
Notice that the functor $\tau_{\leq n}^{\mathrm{Spc}}$ preserves finite products, so that it induces a functor $\tau_{\leq n} \simeq (\tau_{\leq n}^{\mathrm{Spc}})_* : \mathrm{CAlg}_R^{\Delta} \to \mathrm{CAlg}_R^{\Delta}$, which is again a (Bousfield) localization functor.
Its essential image consists of n-truncated objects and, as proven in [24],5.5.8.26, it is precisely $\tau_{\leq n}\mathrm{CAlg}_R^{\Delta}$. Hence, we can identify $\tau_{\leq n}$ with the n-truncation functor of CAlg_R^{Δ}, and we have the following Bousfield localization adjunction

$$\tau_{\leq n} : \mathrm{CAlg}_R^{\Delta} \rightleftarrows \tau_{\leq n}\mathrm{CAlg}_R^{\Delta} : \subseteq$$

By the construction, we observe that the latter adjunction is then induced via post-composition by the former one, so, since the right adjoint to $\tau_{\leq n}$ is essentially unique, one has that $\subseteq\ \simeq\ (\subseteq^{\mathrm{Spc}})_*$.
On this streamline, define the **homotopy groups** of an animated R-algebra $\underline{S}$ to be

$$\pi_*(\underline{S}) := \pi_*\big(\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[t], \underline{S})\big)$$

Observe that the truncation functors $\{\tau_{\leq n}\}_n$ are compatible with such a notion of homotopy groups. In other words, $\underline{S} \in \mathrm{CAlg}_R^{\Delta}$ is n-truncated iff $\pi_i(\underline{S}) \cong 0$ for each $i > n$. This follows from our previous observation on the verification of n-truncatedness and from the fact that Poly_R is generated by $R[t]$ under finite coproducts.

Remark. As an application of the Whitehead Theorem together with the Yoneda Lemma, observe that π_* - or better $\pi_*\big(\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[t], -)\big)$ - is a conservative functor, in that it detects isomorphisms of animated rings, i.e. any morphism of animated rings which induces isomorphisms in homotopy must have been already an equivalence itself in the ∞-category CAlg_R^{Δ}. The digression at the end of this subsection will explain in which sense this is not accidental.

Moreover, notice that we defined homotopy groups of an animated R-algebra $\underline{S}$ to be those of its 'underlying space' $\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[t], \underline{S}) \in \mathrm{Spc}$. Let us make this somewhat more precise.

Lemma 3.2.1.2. $R = \mathrm{Map}(-, R) = \mathrm{Map}_{\mathcal{N}(\mathrm{CRing})}(-, R) \in \mathrm{CAlg}_R^{\Delta}$ *is essentially the initial animated ring.*

Proof. R has the stated form by an application of A.3.0.6. The essential uniqueness follows from [20],4.1.3, which states that the subcategory of initial objects is, if inhabited, a contractible ∞-groupoid. $\square$

More generally, being $R\text{-Alg}^{\mathrm{sfp}} \cong \mathrm{Poly}_R$ generated by $R[t]$ under finite coproducts, an application of the Yoneda Lemma allows us to completely describe any $\underline{S} \in \mathrm{CAlg}_R^{\Delta}$ on points:

$$\mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[t_1, ..., t_n], \underline{S}) \simeq \underline{S}(R[t_1, ..., t_n]) \simeq \underline{S}(R[t])^n$$

Hence, intuitively, the information contained in $\underline{S}$ "can be reconstructed" from its evaluation at $R[t]$.
In view of this, the next definition generalizes both the fact that the underlying set of any $S \in R\text{-Alg}$ is $\mathrm{Hom}_{\mathrm{CRing}}(R[t], S) \simeq for(S)$ and the analogous construction for symmetric monoidal ∞-categories as in B.2.0.7.

Definition 3.2.1.3. *(Underlying space)* Given an animated R-algebra $\underline{S} \in \mathrm{CAlg}_R^{\Delta}$, we define its **underlying space** $for\underline{S}$ to be its evaluation at the embedded copy $R[t] \in \mathrm{CAlg}_R^{\Delta}$, namely

$$for\underline{S} := \mathrm{Map}_{\mathrm{CAlg}_R^{\Delta}}(R[t], \underline{S}) \simeq \underline{S}(R[t])$$

When it is clear from the context, we will just write S in place of $for\underline{S}$.
The set of connected components of $for\underline{S}$ retrieves the **static part** of the animated R-algebra $\underline{S}$:

$$S := \pi_0(\underline{S}) \cong \mathrm{Hom}_{ho\mathrm{CAlg}_R^{\Delta}}(R[t], \underline{S}) \simeq \pi_0(for\underline{S})$$

Remark. By abstract nonsense, i.e. since $\pi_0 = \tau_{\leq 0}$ has a right adjoint and by the fact that $\mathrm{Poly}_R \simeq R\text{-Alg}^{\mathrm{sfp}}$ is static, it is straightforward to infer that the all of $R\text{-Alg} \subseteq_{f.f.} \mathrm{Ani}(\mathrm{CAlg}_R^{\Delta})$ is static. Furthermore, notice that the static part of an animated ring coincides with the underlying space iff the animated ring itself is static. In particular, this motivates our choice of not changing the notation for the (static) essential image of CRing into Ani(CRing).

We close this subsection with a digression on a model theoretic approach to the definition of animated rings. In particular, this motivates the last bit of terminology.

Digression: A model theoretic approach to the theory of Animated Rings.

As in the general abstract context of the $\mathcal{P}_\Sigma$-construction, the introduction of homotopy groups of animated rings gives us a way to consistently define a useful class of weak equivalences on CAlg_R^{Δ}. These turn out to arise from the weak equivalences of a model structure on CRing with respect to which $\mathcal{P}_\Sigma$ is a localization functor. Let us expand on this. As observed in [22],4.1.2, an application of A.6.0.3 to the category $s(R\text{-Alg})_{\mathrm{proj}}$ of simplicial commutative R-algebras endowed with the projective model structure yields a canonical equivalence of ∞-categories $\mathcal{N}(s(R\text{-Alg})^{\circ}_{\mathrm{proj}}) \simeq \mathrm{CAlg}_R^{\Delta}$, where $s(R\text{-Alg})^{\circ}_{\mathrm{proj}}$ denotes the full subcategory of fibrant-cofibrant objects.

Moreover, an extremely useful feature of homotopy groups is "recalling" the algebraic information of animated algebras. This will be made precise in the next result. For the sake of simplicity, we will state and prove it for $R = \mathbb{Z}$. As we will see in the next subsection, this will not cause any loss of generality.

Lemma 3.2.1.4. *(Module structure on homotopy groups, [22],4.1.6) Let* $A \in \mathrm{Ani}(\mathrm{CRing})$ *be an animated ring. Then, its homotopy groups* $\pi_i\underline{A} \in \mathrm{Grp}$ *are all* $\pi_0\underline{A}$*-modules, i.e.* $\pi_*(\underline{A})$ *is a graded commutative ring.*

Proof. Let us start by providing some context. As we have just observed, any animated ring $\underline{A}$ can be seen as a homotopy equivalence class in $s\mathrm{CRing}^{\circ}_{\mathrm{proj}}$ represented by some fibrant-cofibrant simplicial ring $\mathcal{A}$.
Moreover, the underlying space functor $for : \mathrm{Ani}(\mathrm{CRing}) \to \mathrm{Ani} \simeq \mathrm{Spc}$, given by evaluation at the free object on the point (see A.7.0.2) lies under the forgetful functor $\theta : s\mathrm{CRing}_{\mathrm{proj}} \to \mathrm{sSet}_{\mathrm{Quillen}}$ and can be obtained by localizing the latter at weak homotopy equivalences.
Equivalently, we can obtain *for* by localizing the composite $|-| \circ \theta : s\mathrm{CRing}_{\mathrm{proj}} \to \mathrm{CGHaus}_{\mathrm{Quillen}}$ at weak homotopy equivalences; here $|-| \simeq \mathrm{colim}^{sift}_{\Delta^{op}} : \mathrm{sSet} \to \mathrm{CGHaus}$ denotes the geometric realization functor into

compactly generated Hausforff spaces. And the equivalence of the two models comes from the fact that the adjunction $|-|\colon \mathrm{sSet}_{\mathrm{Quillen}} \rightleftarrows \mathrm{CGHaus}_{\mathrm{Quillen}} : \mathrm{Sing}(-)$ is a Quillen equivalence.
Since all the functors involved preserve finite products by construction, $|-| : s\mathrm{CRing} \to \mathrm{CGHaus}$ actually Landds in the category CRing(CGHaus) of compactly generated topological Hausdorff rings. Thus, we can extract from each animated ring $\underline{A}$ an underlying topological ring A.

Observe that in both $s\mathrm{CRing}_{\mathrm{proj}}$ and $\mathrm{CRing}(\mathrm{CGHaus}_{\mathrm{proj}})$ there is a notion of graded commutative homotopy ring associated to the objects at stake and with the expected properties: $\pi_*\mathcal{A}$ (see [11],I.7 for the simplicial case) and the more classical $\pi_* A$ (see e.g. [22],4.1.6 for the topological case); and they are compatible:

- given the suitable notion of sphere object, one posits $\pi_*(-) = \mathrm{Hom}(\mathbb{S}^*, -)$;
- then, multiplication of elements with the same degree is induced under the Yoneda Lemma by the algebra structure of the target, and
- one takes care of elements of different degrees by pre-composing with the smash product of spheres;
- finally, the graded-commutativity is induced by the fact that the natural map $\mathbb{S}^{m+n} \to \mathbb{S}^{n+m}$ swapping the tuples of coordinates has degree equal to the sign of the associated permutation, namely $(-1)^{mn}$.

Hence, we are left to show that all these notions of homotopy groups agree, namely that $\pi_*\underline{A} \simeq \pi_*\mathcal{A} \simeq \pi_* A$. But this is now a direct consequence of our construction; being the topological one analogous, let us spell out e.g. the simplicial case: an object $\mathcal{A}$ in an "algebraic" simplicial model category $s\mathrm{CRing} \subseteq \mathrm{sSet}$ with an initial object $\Delta^0 \simeq *$ is isomorphic to the hom-set $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, \mathcal{A}) \cong \mathrm{Hom}_{s\mathrm{CRing}}(\mathbb{Z}[*], \mathcal{A})$, under the freeness adjunction $\mathbb{Z}[-] \vdash \subseteq$; then,

$$\pi_n\mathcal{A} \cong \mathrm{Hom}_{\mathrm{sSet}}(\mathbb{S}^n, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{sSet}}(\mathbb{S}^n, \mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, \mathcal{A})) \cong \mathrm{Hom}_{\mathrm{sSet}}(\mathbb{S}^n, \mathrm{Hom}_{s\mathrm{CRing}}(\mathbb{Z}[x], \mathcal{A})$$

Finally, localization at weak homotopy equivalences yields $\mathrm{Map}_{\mathrm{Spc}}\big(\mathbb{S}^n, \mathrm{Map}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[x], \underline{A})\big) \simeq \pi_n(\underline{A})$, as desired. ☐

3.2.2 Base-Change

Our next aim is to include in our theory all animated commutative algebras and modules over *any* animated ring. In order to achieve such a goal, we will import the classical constructions of restriction and extension of scalars.

Definition 3.2.2.1. *(Derived tensor product)* Let it be given a span of morphisms of animated R'-algebras $\underline{S} \leftarrow \underline{R} \to \underline{T}$ in $\mathrm{CAlg}^{\Delta}_{R'}$.
Define the **derived tensor product** of $\underline{S}$ and $\underline{T}$ over $\underline{R}$ as the colimit in $\mathrm{CAlg}^{\Delta}_{R'}$ of such a span:

$$\underline{S} \otimes^{L}_{\underline{R}} \underline{T} := \mathrm{colim}(\underline{S} \leftarrow \underline{R} \to \underline{T})$$

Remark. The tensor product of animated rings is indeed well-defined, since the category of animated widgets is always presentable, and hence in particular cocomplete.

We will now derive the classical base-change adjunction:

Lemma 3.2.2.2. *(Change of base ring, [26],25.1.4) Given a morphism $\phi : R \to R'$ in* CRing*, extension of scalars induces a map $f := (- \otimes_R R') : \mathrm{Poly}_R \to \mathrm{Poly}_{R'}$, which in turn yields an adjunction in Cat_∞ of the form:*

$$F \simeq \mathrm{LKE}_j(j \circ f)\colon \mathrm{CAlg}^{\Delta}_{R} \rightleftarrows \mathrm{CAlg}^{\Delta}_{R'} : (j \circ f)^* \simeq G$$

*F is called **extension of scalars**, whereas G is the **restriction of scalars**.*

Proof. Consider the extension of scalars $f : \mathrm{Poly}_R \to \mathrm{Poly}_{R'}$. Clearly, f and j preserve 1-sifted colimits and $\mathrm{CAlg}^{\Delta}_{R'}$ has sifted colimits, so we can apply Theorem 3.1.0.3,i) to $j \circ f$, so as to obtain an essentially unique F as in the following square.

$$
\begin{array}{ccc}
\mathrm{Fun}_\Sigma(\mathrm{Poly}_R, \mathrm{CAlg}^{\Delta}_{R'}) & \xrightarrow{\simeq} & \mathrm{Fun}(\mathrm{CAlg}^{\Delta}_R, \mathrm{CAlg}^{\Delta}_{R'}) \\
j_{R'} \circ f & \mapsto & F \ \mid \ F \circ j \simeq j \circ f
\end{array}
\qquad
\begin{tikzcd}
\mathrm{Poly}_R \arrow[r, "f"'] \arrow[d, hook, "j"] & \mathrm{Poly}_{R'} \arrow[d, hook, "j"'] \\
\mathrm{CAlg}^{\Delta}_R \arrow[r, "F"] & \mathrm{CAlg}^{\Delta}_{R'}
\end{tikzcd}
$$

Moreover, since f preserves finite coproducts, then by *ibid*,iii) also F does so, and hence preserves all coproducts.
Now, a functor into a cocomplete category preserves coproducts and sifted colimits iff it is cocontinuous. Hence, we can conclude by the I Adjoint Functor Theorem 1.2.0.5 that F admits a right adjoint $G : \mathrm{CAlg}^{\Delta}_{R'} \to \mathrm{CAlg}^{\Delta}_R$. As proved in 3.1.0.3,ii), $F \simeq \mathrm{LKE}_j(F \circ j)$, which is in turn equivalent to $\mathrm{LKE}_j(j \circ f)$. Hence, being the right adjoint G to F essentially unique, it must be equivalent to the restriction $(j \circ f)^*$. □

Remark. Consider the tensor product functor $\underline{S} \mapsto \underline{S} \otimes^L_R R'$ in CAlg^{Δ}_R and notice that it agrees with the restriction of F to $\mathrm{Poly}_R \simeq (R\text{-Alg})^{\mathrm{sfp}}$; hence, by [UP : Ani], they actually do define the same functor. In other words, we can interpret $(- \otimes^L_R R)$ as a derived functor with respect to $\mathcal{P}_\Sigma$, whence the adjective "derived". Moreover, since $F \circ j \simeq j \circ f$, one has that for any static animated R-algebra R' and any vector of indeterminates X, $R[X] \otimes^L_R R' \simeq R[X] \otimes_R R' \cong R'[X]$, as expected.

Warning. ([22],4.1.16) Observe that, being it a homotopy colimit (where 'homotopy' in our setting means '∞-categorical' or - equivalently - with respect to the aforementioned projective model structure on simplicial algebras), the derived tensor product of two static animated R-algebras needs not be again static.
Indeed, by the Digression above on a model theoretical approach and the Dold-Kan correspondence, animated rings can be regarded as fibrant-cofibrant chain-complexes in $\mathrm{Ch}_{\geq 0}(\mathrm{CRing})_{\mathrm{proj}}$, i.e. as equivalence classes of non-negatively graded chain complexes valued in CRing with respect to the projective model structure; hence, under such an identification, $\pi_*(S \otimes^L_R R') \cong Tor^R_*(S, R')$ by [22],4.1.14. Therefore, in particular $\pi_0(S \otimes^L_R R') \simeq S \otimes_R R'$ retrieves the ordinary tensor product as the underlying ring, and - for the symmetric monoidal structure on the ∞-category CAlg^{Δ}_R of 3.2.3.1 - we observe that the left adjoint functor $\pi_0 : \mathrm{CAlg}^{\Delta}_R \to R\text{-Alg}$ is a symmetric monoidal functor (in the sense of B.2.0.3).

Remark. $G \simeq (j \circ f)^*$ acts as the restriction along the base-change of static animated polynomial rings $f : R[t] \mapsto R'[t]$ (in turn induced by restriction along $\phi : R \to R'$), whence the analogy between G and the operation of restriction of scalars.

Moreover, from the very definition of homotopy groups, restriction of scalars induces a canonical equivalence

$$\pi_*(G(\underline{S})) \simeq \pi_*(\underline{S})$$

which, in particular, in the static case $\pi_*(G(R')) \simeq \pi_*(R')$ allows us to canonically regard $G(R') \in \left(\mathrm{CAlg}^{\Delta}_R\right)^{\mathrm{static}}$ as a static object and then to (again canonically) identify it with the embedded copy of $R' \in R\text{-Alg}$.
It becomes then reasonable to wonder whether we can actually relate animated algebras in $\mathrm{CAlg}^{\Delta}_{R'}$ as slices in CAlg^{Δ}_R whenever we can regard R' as an R-algebra. This is the content of the next result.

Lemma 3.2.2.3. *([26],25.1.4.2) Given $\phi : R \to R'$ in* CRing*, restriction of scalars induces a canonical equivalence*

$$\overline{G} : \mathrm{CAlg}^{\Delta}_{R'} \simeq (\mathrm{CAlg}^{\Delta}_{R'})_{R'/} \xrightarrow{G} (\mathrm{CAlg}^{\Delta}_R)_{G(R')/} \simeq (\mathrm{CAlg}^{\Delta}_R)_{R'/}$$

Proof. In order to prove our result, we will use the extension of scalars F to provide a left adjoint $\overline{F}$ to $\overline{G}$. Then, we will show that such an adjunction is actually an equivalence via the following technical result.

Claim 1. *([23],pp.688) For a functor $G : D \to C$ in Cat_∞,* tfae*:*

1. *G is an equivalence;*
2. *G is conservative and admits a left adjoint s.t. the unit of the adjunction is an equivalence.*
3. *G is conservative and admits a fully faithful left adjoint;*

Proof. We will work with the incarnation of quasi-categories.
(1) $\iff$ (2) : Let η and ϵ denote respectively the unit and counit of the adjunction $G^L \vdash G$. As in [20],5.1.14, our adjunction is determined by the fact that unit and counit sit in the triangle identities; then, in particular $\eta G \circ G^L\epsilon \simeq id_G$. Now, by the definition of unit, ηG is an equivalence, so that also $G\epsilon$ is such; by [20],2.2.2, the latter amounts to the fact that $G\epsilon$ corresponds to point-wise equivalences, iff ϵ does so (G is conservative), iff ϵ is itself an equivalence. Finally, recall that G^L and G are mutually inverse equivalences iff both the unit and counit are equivalences in the respective functor categories.
(2) $\iff$ (3) : As in the proof of [20],5.1.8, we see that G admits a fully faithful left adjoint G^L iff the unit of $G^L \vdash G$ is an equivalence. ■

Now, let's define the left adjoint to $\overline{G}$. Recall that, for a static ring map $R \to R'$, we canonically identified $R' \simeq G(R') \in \mathrm{CAlg}^{\Delta}_{R}$; then, the unit η of the adjunction $F \vdash G$ induces a map $\eta : F(R') \to R'$ in $\mathrm{CAlg}^{\Delta}_{R'}$.

Claim 2. *Unwinding the definition of $\overline{G}$, the latter admits a left adjoint $\overline{F}$ which acts on animated R-algebras as the extension of scalars along η, namely $\overline{F}(\underline{A}) := F(\underline{A}) \otimes^{L}_{F(R')} R'$ on $\mathrm{CAlg}^{\Delta}_{R}$.*

Proof. Let's recall the action of $\overline{G}$ on objects $\underline{B} \in \mathrm{CAlg}^{\Delta}_{R'}$ and let's parallely define the one of $\overline{F}$ on $[R' \to \underline{A}] \in (\mathrm{CAlg}^{\Delta}_{R})_{R'/}$:

- $\overline{G} : \mathrm{CAlg}^{\Delta}_{R'} \to (\mathrm{CAlg}^{\Delta}_{R})_{R'/}$ such that
$$\overline{G} : \underline{B} \overset{\subseteq}{\longmapsto} [R' \to \underline{B}] \overset{G}{\mapsto} [G(R') \to G(\underline{B})] \simeq [R' \to G(\underline{B})]$$
- $\overline{F} : (\mathrm{CAlg}^{\Delta}_{R})_{R'/} \to \mathrm{CAlg}^{\Delta}_{R'}$ such that
$$\overline{F} : [R' \to \underline{A}] \overset{F}{\mapsto} [F(R') \to F(\underline{A})] \overset{\subseteq}{\longmapsto} [F(\underline{A}) \leftarrow F(R') \to R'] \xmapsto{colim} F(\underline{A}) \otimes^{L}_{F(R')} R'$$

Since F, $\subseteq$, colim respect colim^{sift}, by $[UP : \mathrm{Ani}]$ the functor $\overline{F}_{|(\mathrm{Poly}'_R)_{R'/}}$ yields a well-defined animated functor (with the claimed action on objects), so that we are left to prove that our candidate is indeed the left adjoint to $\overline{G}$.
In order to show this, let us first exhibit a wannabe adjunction natural equivalence ϕ. First, notice that the latter can be written as a functor
$$\phi : (\mathrm{CAlg}^{\Delta}_{R})_{R'/} \times \mathrm{CAlg}^{\Delta}_{R'} \to \mathrm{Fun}\big(\Delta^1, (\mathrm{CAlg}^{\Delta}_{R})_{R'/} \times \mathrm{CAlg}^{\Delta}_{R'}\big)$$
Now, consider bi-functors $\mathrm{Map}(\star, -)$ as in [20],4.2.5, and let $\psi_{R'/} : F_{R'/} \vdash G_{R'/}$ denote the base-change adjunction equivalence in the under-slice. Then, the following composition
$$\phi := (\mathrm{Map}(\star, -) \circ \langle \mathrm{colim}^*, id\rangle) \circ (\mathrm{Map}(\star, -) \circ \langle \subseteq, id\rangle) \circ \psi_{R'/} \circ \mathrm{Map}(\star, -)$$
defines a functor of ∞-categories, and hence a natural transformation of the mapping spaces at stake.
Thus, by [20],2.2.2 it suffices to show that the natural transformation ϕ is a point-wise equivalence. To this end, consider the following chain of equivalences of mapping spaces for $[R' \to \underline{A}] \in (\mathrm{CAlg}^{\Delta}_{R})_{R'/}$ and $\underline{B} \in \mathrm{CAlg}^{\Delta}_{R'}$:
$$\begin{aligned}\mathrm{Map}(\overline{F}[R' \to \underline{A}], \underline{B}) = \mathrm{Map}(F(\underline{A}) \otimes^{L}_{F(R')} R', \underline{B}) &\simeq_{(1)} \mathrm{Map}([F(\underline{A}) \leftarrow F(R') \to R'], \underline{B}) \\ &\simeq_{(2)} \mathrm{Map}_{F(R')/}(F(\underline{A}), \underline{B}) \times \mathrm{Map}_{F(R')/}(R', \underline{B}) \\ &\simeq_{(3)} \mathrm{Map}_{R'/}(\underline{A}, G(\underline{B})) \simeq \mathrm{Map}([R' \to \underline{A}], [R' \to G(\underline{B})])\end{aligned}$$
where the numbered equivalences are so deduced. (1) : corresponds to the universal property of colim, as in [20],4.3.4. (2) : amounts to the following manipulation of slice categories, which we perform again in the incarnation of quasi-categories, even though it should follow from any reasonable model-independent definition of a slice-category.
Let's first fix some notation: call $\mathcal{D} := \mathrm{CAlg}^{\Delta}_{R'}$ and $\phi : \Lambda^2_1 \simeq \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \to \mathcal{D}$ the functor with diagram $[F(\underline{A}) \leftarrow F(R') \to R']$. Then, as in [20],4.3.1, the LHS sits in the cartesian square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}}(\phi, \underline{B}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Lambda^2_1 \star \Delta^0, \mathcal{D}) \arrow[d, "\subseteq^*"] \\
\Delta^0 \arrow[r, "(\phi, \underline{B})"] & \mathrm{Fun}(\Lambda^2_1, \mathcal{D}) \times \mathcal{D}
\end{tikzcd}$$

Since $(-\star\Delta^0)$ is a left adjoint (to the corresponding slice), the latter can be equivalently rewritten as follows:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}}(\phi, \underline{B}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^{\{0,1\}} \star \Delta^0, \mathcal{D}) \times_{\mathrm{Fun}(\Delta^{\{1\}} \star \Delta^0, \mathcal{D})} \mathrm{Fun}(\Delta^{\{1,2\}} \star \Delta^0, \mathcal{D}) \arrow[d, "\subseteq^*"] \\
\Delta^0 \arrow[r] & \mathrm{Fun}(\Delta^{\{0,1\}}, \mathcal{D}) \times_{\mathrm{Fun}(\Delta^{\{1\}}, \mathcal{D})} \mathrm{Fun}(\Delta^{\{1,2\}}, \mathcal{D}) \times \mathcal{D}
\end{tikzcd}$$

where the lower horizontal arrow is now $(\phi, \underline{B}) \simeq \big(F_{|\Delta^{\{0,1\}}} \times_{F_{|\Delta^{\{1\}}}} F_{|\Delta^{\{1,2\}}}, \underline{B}\big)$.

Moreover, $(- \times \mathcal{D})$ distributes with the fibred-product, so that we can regard the latter cartesian square as a 'cube' yielding the desired expression for $\mathrm{Map}_{\mathcal{D}}(\phi, \underline{B})$.

Finally, (3) comes from the adjunction property of slices, which is a reformulation of [20],5.1.16. ■

Now, let's show that $\overline{G}$ satisfies the assumptions of (the technical) Claim 1.

Claim 3. *$\overline{G}$ is conservative.*

Proof. $\pi_*(\overline{G}(\underline{B})) \simeq \pi_*(\underline{B})$, so $\overline{G}$ detects equivalences. ■

Claim 4. *The unit $u : id_{(\mathrm{CAlg}^{\Delta}_R)_{R'/}} \to \overline{G} \circ \overline{F}$ is an equivalence.*

Proof. By [20],2.2.2 it suffices to check that the unit corresponds to point-wise equivalences, so fix $\underline{A} \in \mathrm{CAlg}^{\Delta}_R$ and consider $u_{\underline{A}} : \underline{A} \to \overline{G} \circ \overline{F}(\underline{A})$ under $u_{R'}$. We will prove our statements after two reduction steps.

First, we claim that wlog $\underline{A} \in ((\mathrm{CAlg}^{\Delta}_R)_{R'/})^{\mathrm{sfp}}$ is cpt+proj.

This is based on the following important technical result: *given an adjunction $F \vdash G$ with the source of F being generated by cpt+proj's, F preserves cpt+proj's iff G commutes with sifted colimits.*

To see the latter, first recall that, as observed in [24],5.5.8.20, an object x of an ∞-category C with sifted colimits is cpt+proj iff $\mathrm{Map}_C(x, -)$ commutes with such colimits. Then, unwinding the definitions, for any sifted diagram χ one has:

- $\mathrm{Map}(x, \mathrm{colim}^{sift}\, G(\chi)) \simeq_{(1)} \mathrm{colim}^{sift}\, \mathrm{Map}(x, G(\chi)) \simeq \mathrm{colim}^{sift}\, \mathrm{Map}(Fx, \chi)$, where (1) comes from assuming x cpt+proj;
- $\mathrm{colim}^{sift}\, \mathrm{Map}(Fx, \chi) \simeq_{(2)} \mathrm{Map}(Fx, \mathrm{colim}^{sift}\, \chi) \simeq \mathrm{Map}(x, G(\mathrm{colim}^{sift}\, \chi))$ with (2) corresponding precisely to the fact that F preserves the cpt+proj object x.

Thus, the stated equivalence follows from an application of the Yoneda Lemma and the fact that $C^{\mathrm{sfp}} \subseteq_{f.f.} C$ is dense, so that checking the property of G against C^{sfp} amounts to checking it against the whole of C.

Now, notice that $F \simeq (- \otimes^L_R R')$ does indeed preserve cpt+proj objects: by [24],5.5.8.25,ii), $(\mathrm{CAlg}^{\Delta}_R)^{\mathrm{sfp}}$ consists of (retracts of) $j(\mathrm{Poly}_R) \subseteq \mathrm{EssIm}(j)$, and those are sent by F to (retracts of) $j(\mathrm{Poly}_{R'}) \simeq (\mathrm{CAlg}^{\Delta}_{R'})^{\mathrm{sfp}}$. Therefore, G commutes with sifted colimits, and the latter further implies that the functor "evaluation of the unit" $u_{(-)} : (\mathrm{CAlg}^{\Delta}_R)_{R'/} \to \mathrm{Fun}\big(\Delta^1, (\mathrm{CAlg}^{\Delta}_R)_{R'/}\big)$ preserves sifted colimits, as well.

Furthermore, a closer inspection of cpt+proj's in $(\mathrm{CAlg}^{\Delta}_R)_{R'/}$ allows us to consider wlog only those animated rings of the form $\underline{A} \simeq A_0 \otimes^L_R R'$ for a polynomial ring $A_0 \simeq R[X] \in \mathrm{CAlg}^{\Delta}_R$, with X being some finite tuple of indeterminates.

Indeed, as $\overline{G}$ is conservative and preserves sifted colimits (because G does so), we can apply [23],4.3.7.18 to infer that cpt+proj's in the target $\mathrm{CAlg}^{\Delta}_{R'}$ of $\overline{F}$ are retracts of $\mathrm{EssIm}(\overline{F}_{|(\mathrm{CAlg}^{\Delta}_R)^{\mathrm{sfp}}_{R'/}})$. The latter can now be rewritten as $\overline{F}\big(\big(j(\mathrm{Poly}_R)\big)_{R'/}\big)$, because, as already recalled, $(\mathrm{CAlg}^{\Delta}_R)^{\mathrm{sfp}} \simeq j(\mathrm{Poly}_R)$ and colimits in overcategories commute with the forgetful functor. Then, $\overline{F}(\underline{A})$ for $\underline{A}$ cpt+proj is obtained up to homotopy by applying F to an object of $\big(j(\mathrm{Poly}_R)\big)_{R'/}$; from the very definition of $\otimes^L$, the latter is wlog into $j(\mathrm{Poly}_R)\otimes^L_R R'$, as required.

Finally, as a last step we claim that $u_{\underline{A}}$ is an equivalence for those animated rings as before.

In order to show this last part, we observe that the map $u_{\underline{A}}$ induces the identity in homotopy:

$$\begin{aligned} R'[X] \simeq \pi_*(R[X] \otimes^L_R R') \simeq \pi_*(\underline{A}) \overset{\pi_*(u_{\underline{A}})}{\longrightarrow} \pi_*(\overline{G} \circ \overline{F})(\underline{A}) &\simeq \pi_*(\overline{F}(\underline{A})) \\ &\simeq \pi_*(F(\underline{A}) \otimes^L_{F(R')} R') \\ &\simeq \pi_*(F(A_0) \otimes^L_{F(R)} F(R') \otimes^L_{F(R')} R') \\ &\simeq \pi_*(F(A_0) \otimes^L_{F(R)} R') \\ &\simeq \pi_*(F(A_0)) \\ &\simeq \pi_*(R'[X]) = R'[X] \end{aligned}$$

Indeed, the latter turns out to be a map in $R'[X]$-Alg $\simeq \pi_0(R'[X])$-Alg, as observed in Lemma 3.2.1.4. ■

We are finally done by Claim 1.

□

As a corollary, we are now allowed to merge our notation and regard (compatibly) all categories CAlg^{Δ}_R as slice ∞-categories $(R \downarrow \mathrm{Ani}(\mathrm{CRing}))$.

Corollary 3.2.2.4. *([26],25.1.4.3) For any $R \in \mathrm{CRing}$, restriction of scalars induces an equivalence $\mathrm{CAlg}^{\Delta}_R \simeq \mathrm{Ani}(\mathrm{CRing})_{R/}$.*

3.2.3 The Presheaf of Animated Algebras

The previous Corollary hints us at the following generalization, so as to include in our theory arbitrary '$\underline{A}$-algebra structures'. It will be the content of the following definition-proposition.

Definition 3.2.3.1. For any $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$, define the ∞-category of **animated $\underline{A}$-algebras** by $\mathrm{CAlg}^{\Delta}_{\underline{A}} := \mathrm{Ani}(\mathrm{CRing})_{\underline{A}/}$. The present definition restricts to the previous one whenever $\underline{A}$ is representable.

More generally, there is a **presheaf of animated algebras**, say $\mathrm{CAlg}^{\Delta} : \mathrm{Ani}(\mathrm{CRing})^{op} \to \mathrm{Cat}_\infty$, with values in *presentable closed symmetric monoidal* ∞-categories and whose action on the 1-skeleton is:

- $\underline{A} \mapsto \mathrm{CAlg}^{\Delta}(\underline{A}) = \mathrm{CAlg}^{\Delta}_{\underline{A}} = \mathrm{CAlg}^{\Delta}_{\underline{A}/}$
- $(\phi : \underline{A} \to \underline{A'}) \mapsto (\phi^* : \mathrm{CAlg}^{\Delta}_{\underline{A'}} \to \mathrm{CAlg}^{\Delta}_{\underline{A}})$ defines restriction of scalars along maps of animated rings.

Moreover, for each $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$, $\mathrm{CAlg}^{\Delta}_{\underline{A}} \simeq \lim(\mathrm{CAlg}^{\Delta}_R \mid R_{/\underline{A}} \in \mathrm{CRing}^{op}_{\underline{A}/})$.

Before proving the existence of such a presheaf, let us state a useful Lemma, which will be applied very often in what follows.

Lemma 3.2.3.2. *Let $\mathcal{C} \in \mathrm{Cat}_\infty$ be a small ∞-category with finite products and consider its animation $\mathcal{P}_\Sigma(\mathcal{C})$. Let it be given any two presheaves $\mathcal{F}, \mathcal{G}$ in the latter, and consider sifted diagrams (wlog over the same indexing simplicial set) $p, q : K \to \mathcal{C}^{\mathrm{sfp}}$ whose colimits are $\mathcal{F}, \mathcal{G}$, respectively. Then, there is a canonical equivalence in Spc:*

$$\mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \simeq \lim_K \mathrm{Map}_{\mathcal{C}}(p, q)$$

In particular, any map $\phi : \mathcal{F} \to \mathcal{G}$ can be seen as a limit of some natural transformation $\psi : p \to q$; this is well-defined up to contractible homotopy by [24],5.1.2.2.

Proof. Commit a slight abuse and let $|-|:=\mathrm{colim}_K^{sift}$. The diagrams p,q as above exist by A.2.0.4. Then, there is the following chain of equivalences:

$$\mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(\mathcal{F},\mathcal{G})\simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(|p|,|q|)\simeq_{(a)}\lim_K\lim_K \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(p,q)\simeq_{(b)}\lim_K \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(p,q)$$

where: (a) : is because mapping spaces commute with limits in the first variable and then, $\mathrm{EssIm}(p)\subseteq_{f.f.}\mathcal{C}^{\mathrm{sfp}}$ implies that we can take out also the sifted-colimit in the covariant argument; (b) : K is sifted iff the diagonal $K\to K\times K$ is cofinal. □

Proof. (Of 3.2.3.1) Let us define the functor of ∞-categories $(\mathrm{CAlg}^{\Delta})^{op}:\mathrm{Ani(CRing)}\to\mathrm{Cat}_\infty$. We will anticipate part of the argument of 3.2.5.11, although in a much simpler setting, due to 3.2.2.3.

Claim 1. *The source functor* $ev_0:\mathrm{Fun}(\Delta^1,\mathrm{Ani(CRing)})\to\mathrm{Ani(CRing)}$ *is a bi-cartesian fibration.*

Proof. ev_0 is a cartesian fibration as a consequence of Joyal's Theorem. Alternatively, the following proof applies as well. We are left to show that it is also a cocartesian fibration.
From the very definition, a pair of morphisms $(\phi:\underline{A}\to\underline{B}\,,\,\phi':\underline{A}'\to\underline{B}')$ in Ani(CRing) is ev_0-cocartesian iff for each morphism of animated rings $\zeta:\underline{X}\to\underline{X}'$, the following square of mapping spaces is (homotopy) cartesian.

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Fun}(\Delta^1,\mathrm{Ani(CRing)})}([\underline{B}\to\underline{B}'],\zeta) \arrow[r, "(-)\circ(\phi,\phi')"] \arrow[d, "ev_0"'] & \mathrm{Map}_{\mathrm{Fun}(\Delta^1,\mathrm{Ani(CRing)})}([\underline{A}\to\underline{A}'],\zeta) \arrow[d, "ev_0"] \\
\mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B},\underline{X}) \arrow[r, "(-)\circ\phi"] & \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{A},\underline{X})
\end{tikzcd}$$

By [29],3.3.18, a commutative square of spaces is (homotopy) cartesian iff the fibres of the vertical maps are equivalent. So, taken any $\psi\in\mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B},\underline{X})$, let us compare the fibre over any $\psi\mapsto\psi\circ\phi$. By 3.2.2.3, we can write it as

$$(-)\circ(\phi')\quad:\quad \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{B}'}}(\underline{B}',\underline{X}')_\psi\to \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}'}}(\underline{A}',\underline{X}')_{\psi\circ\phi}$$

Then, the special choice $\phi'=\phi$ yields an ev_0-cocartesian lift of $\phi:\underline{A}\to\underline{B}$. ■

Hence, the source functor sits in a classifying cartesian square via the Straightening theorem, thus defining a presheaf of ∞-categories, say $(\mathrm{CAlg}^{\Delta})^{op}:\mathrm{Ani(CRing)}\to\mathrm{Cat}_\infty$:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1,\mathrm{Ani(CRing)}) \arrow[r] \arrow[d, "\mathrm{Cart}\ni"', "ev_0"] & \mathrm{Cat}_{\infty//*} \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Ani(CRing)} \arrow[r, "(\mathrm{CAlg}^{\Delta})^{op}"] & \mathrm{Cat}_\infty
\end{tikzcd}$$

Notice that the action of the restriction of $(\mathrm{CAlg}^{\Delta})^{op}_{|\mathrm{Poly}}$ is the declared one. Then, consider the left derived functor as in A.3.0.2 of such a restriction; in other words, form the left Kan extension along the Yoneda embedding $j:\mathrm{Poly}\hookrightarrow\mathrm{Ani(CRing)}$:

$$(\mathrm{CAlg}^{\Delta})^{op}_1\simeq \mathrm{LKE}_j\big((\mathrm{CAlg}^{\Delta})^{op}_{|\mathrm{Poly}}\big)$$

We are left to prove the equivalence $(\mathrm{CAlg}^{\Delta})^{op}\simeq\mathrm{LKE}_j(\mathrm{CAlg}^{\Delta}_{|\mathrm{Poly}})^{op}\simeq(\mathrm{CAlg}^{\Delta})^{op}_1$. Let us postpone it and discuss first how to conclude.
Provided that, the factorization $j:\mathrm{Poly}\hookrightarrow\mathrm{CRing}\hookrightarrow\mathrm{Ani(CRing)}$ induces an equivalence of left Kan extensions: $(\mathrm{CAlg}^{\Delta})^{op}\simeq\mathrm{LKE}_j\big((\mathrm{CAlg}^{\Delta})^{op}_{|\mathrm{CRing}}\big)$, where the latter is taken along $j:\mathrm{CRing}\hookrightarrow\mathrm{Ani(CRing)}$.
Then, by inspection, one would obtain the expected action on animated rings:

$$(\mathrm{CAlg}^{\Delta})^{op}(\underline{A})\simeq\mathrm{colim}\,\big((\mathrm{CAlg}^{\Delta})^{op}(R)\mid R\in\mathrm{CRing}_{/\underline{A}}\big)$$

Hence, passing to the *op*-functor exhibits $\mathrm{CAlg}^{\Delta}\simeq\mathrm{RKE}_j(\mathrm{CAlg}^{\Delta}_{|\mathrm{CRing}^{op}})$ as a right Kan extension, and

$$\mathrm{CAlg}^{\Delta}(\underline{A})\simeq\lim\big(\mathrm{CAlg}^{\Delta}_R\mid R\in(\mathrm{CRing}_{/\underline{A}})^{op}\big)$$

Notice that everything is well-defined, in that all (co)limits considered are indeed small, since they are computed on the ∞-category of Grothendieck elements and Ani(CRing) has small mapping spaces. Moreover, by 3.1.0.2 each CAlg_R^Δ is presentable for R static and small limits of presentable ∞-categories are again presentable by [24],5.5.3.18; so, CAlg^Δ would indeed take values in Pr^L, as stated.

CLAIM 2. CAlg_1^Δ *takes morphisms in* $\mathrm{Ani(CRing)}$ *to "restriction of scalars" functors. In other words, for any animated ring* $\underline{A} \in \mathrm{Ani(CRing)}$, $\mathrm{CAlg}_1^\Delta(\underline{A}) \simeq \mathrm{CAlg}_{\underline{A}}^\Delta$ *and, for any morphism of animated rings* $\phi : \underline{A} \to \underline{A}'$, $\mathrm{CAlg}_1^\Delta(\phi) \simeq \phi^*$ *acts as pre-composition by* ϕ.

Proof. By construction, the restriction $\left(\mathrm{CAlg}_1^\Delta\right)^{op}_{|\mathrm{Poly}} \simeq \left(\mathrm{CAlg}^\Delta\right)^{op}_{|\mathrm{Poly}}$ acts as stated; indeed, by 3.2.2.3 pre-composition with any map $\phi : R \to R'$ in CRing induces a "restriction of scalars" functor $\phi^* : \mathrm{CAlg}_R^\Delta \to \mathrm{CAlg}_{R'}^\Delta$ between the corresponding ∞-categories of animated algebras. Thus, we need to show that our definition generalizes restriction of scalars as in 3.2.2.2 to arbitrary animated rings. Let $\phi : \underline{A} \to \underline{A}'$ denote a morphism in Ani(CRing).
Being Ani(CRing) the sInd-completion of Poly, by A.2.0.4 there exist sifted-diagrams $p : K \to \mathrm{Poly}_{/\underline{A}}$ and $p' : K \to \mathrm{Poly}_{/\underline{A}'}$ whose sifted colimit (in Ani(CRing) or equivalently in $\mathcal{P}(\mathrm{Poly})$ by A.2.0.2) retrieves $\underline{A} \simeq \overline{p}(\infty)$ and $\underline{A}' \simeq \overline{p'}(\infty)$; here $\overline{p}, \overline{p'}$ denote some choice (in the presentable ∞-category Ani(CRing)) of colimiting cones for the diagrams above.
Now, we defined $\left(\mathrm{CAlg}_1^\Delta\right)^{op}$ as a left Kan extension along the Yoneda embedding $j : \mathrm{Poly} \hookrightarrow \mathrm{Ani(CRing)}$, so by construction it commutes with colimits of animated rings. Hence, we obtain

$$\left(\mathrm{CAlg}_1^\Delta\right)^{op}(\underline{A}) \simeq \mathrm{colim}\left(\mathrm{CAlg}_1^\Delta\right)^{op}(p)$$

and the same holds for p' and $\underline{A}'$. Moreover, according to 3.2.3.2, (up to equivalence) the map $\phi : \underline{A} \to \underline{A}'$ arises as the colimit $\phi \simeq \overline{\psi}(\infty)$ of some natural transformation $\psi : p \to p'$ in Ani(CRing).
Being a right Kan extension, CAlg_1^Δ commutes with all small limits in $\mathrm{Ani(CRing)}^{op}$, hence

$$\mathrm{CAlg}_1^\Delta(\phi) \simeq \mathrm{CAlg}_1^\Delta(\psi(\infty)) \simeq \lim_K \mathrm{CAlg}_1^\Delta(\psi) \simeq \lim \mathrm{Map}(\psi, -) \simeq \mathrm{Map}(\lim \psi, -) \simeq \mathrm{Map}(\phi, -)$$

■

CLAIM 3. *Define* $\mathrm{CALG}^\Delta := \int(\mathrm{CAlg}^\Delta)_1^{op}$. *Then, being it the straightening of* $(\mathrm{CAlg}^\Delta)^{op}$, *one has an equivalence* $\mathrm{Fun}(\Delta^1, \mathrm{Ani(CRing)}) \simeq \mathrm{CALG}^\Delta$. *In particular,* $\mathrm{CAlg}^\Delta \simeq \mathrm{CAlg}_1^\Delta$, *as desired.*

Proof. Both presheaves agree on their restriction to Poly, so the universal property of left Kan extensions yields a comparison map $\alpha : (\mathrm{CAlg}^\Delta)_1^{op} \to (\mathrm{CAlg}^\Delta)^{op}$ which restricts to an equivalence on static rings. We wish to prove that α is an equivalence, which is true, by [20],2.2.2 and since α is point-wise such. ■

Finally, each category of animated algebras is closed symmetric monoidal. The proof is analogous to B.4.0.2, so we need to show first, that each $\mathrm{CAlg}_{\underline{A}}^\Delta$ be the animation of its set of cpt+proj-generators. The proof of this is postponed to 3.2.3.4.
Provided that, the encoded tensor product of each $(\mathrm{CAlg}_{\underline{A}}^\Delta)^\otimes$ will give point-wise left-adjuncts to the action of $\mathrm{CAlg}^\Delta(-)$ on morphisms. Therefore, transition morphisms of CAlg^Δ will be linear with respect to the symmetric monoidal structures considered. □

Lemma 3.2.3.3. *(Animated Base-Change Adjunction) A morphism of animated rings* $\phi : \underline{A} \to \underline{A}'$ *induces the* symmetric monoidal ***animated base-change*** *adjunction:*

$$(- \otimes^L_{\underline{A}} \underline{A}') \colon \mathrm{CAlg}_{\underline{A}}^\Delta \rightleftarrows \mathrm{CAlg}_{\underline{A}'}^\Delta : \phi^*$$

As before, the left adjoint is called extension of scalars, *while we refer to the right adjoint, which is a transition map in* CAlg^Δ, *as* restriction of scalars.

Proof. Recall that, by definition, $\phi^* = \mathrm{CAlg}^\Delta(\phi)$ acts as pre-composition by ϕ. Being categories of animated algebras presentable, we want to show that ϕ^* commutes with limits and is accessible, so as to apply the II Adjoint Functor Theorem 1.2.0.6 and conclude the existence of a left-adjoint to it.
ϕ^* clearly commutes with all small limits, so let us consider any filtered diagram $F : I \to \mathrm{CAlg}_{\underline{A}'}^\Delta$. We wish $(\underline{A} \to \mathrm{colim}\, F) \simeq \mathrm{colim}(\underline{A} \to F)$, but the latter holds, because filtered diagrams are weakly contractible (A.1.0.4) and weakly-contractible colimits in an under-category are computed after forgetting the slice-structure. Indeed, the forgetful functor from an under-category is evaluation at the target, which is a left fibration and hence preserves weakly contractible colimits by [24],4.4.2.9.

Therefore, we conclude the existence of a left-adjoint to ϕ^*, say $F : \mathrm{CAlg}^{\Delta}_{\underline{A}} \to \mathrm{CAlg}^{\Delta}_{\underline{A}'}$.

Claim. *The left adjoint* $(-) \otimes^L_{\underline{A}} \underline{A}'$ *to* ϕ^* *is compatible with static restriction of scalar. Hence, the latter adjunction indeed enhances base-change over a static ring.*

Proof. Consider the following diagram, where the square of restrictions of scalars commutes by the construction.

$$\begin{tikzcd}
\mathrm{CAlg}^{\Delta}_{\underline{A}} \arrow[r, shift left, "(-\otimes^L_{\underline{A}}\underline{A}')"] \arrow[d, shift left, "\phi^*_{R,\underline{A}}"] & \mathrm{CAlg}^{\Delta}_{\underline{A}'} \arrow[l, shift left, "\phi^*_{\underline{A},\underline{A}'}"] \arrow[d, equal] \\
\mathrm{CAlg}^{\Delta}_{R} \arrow[r, shift left, "(-\otimes^L_{R}\underline{A}')"] \arrow[u, shift left, "(-\otimes^L_R\underline{A})"] & \mathrm{CAlg}^{\Delta}_{\underline{A}'} \arrow[l, shift left, "\phi^*_{R,\underline{A}'}"]
\end{tikzcd}$$

Since adjunctions compose,

$$(- \otimes^L_{\underline{A}} \underline{A}') \circ (- \otimes^L_R \underline{A}) \vdash \phi^*_{R,\underline{A}} \circ \phi^*_{\underline{A},\underline{A}'} \simeq \phi^*_{R,\underline{A}'}$$

But then, the left-adjoint to $\phi_{R,\underline{A}'}$ is essentially unique, so that it must hold also $(- \otimes^L_R \underline{A}') \simeq (- \otimes^L_{\underline{A}} \underline{A}') \circ (- \otimes^L_R \underline{A})$, which is precisely the commutativity of the square of extensions of scalars. ■

Finally, as observed at the end of 3.2.3.1, after the next Lemma we will have that the "base-change" adjunction amounts to the fact that the symmetric monoidal structure on $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ is closed. □

We conclude this subsection with a presentation of ∞-categories of animated $\underline{A}$-algebras by means of the $\mathcal{P}_\Sigma$-construction. This is analogous to 3.2.5.14

Lemma 3.2.3.4. *([23],7.2.2.15) Let* $\underline{A} \in \mathrm{Ani(CRing)}$*, and let* $\mathrm{Poly}_{\underline{A}} := \underline{A} \otimes^L_{\mathbb{Z}} \mathrm{Poly} \subseteq_{f.f.} \mathrm{CAlg}^{\Delta}_{\underline{A}}$ *denote the* ∞*-category spanned by a set of representatives of finitely generated polynomial* $\underline{A}$*-algebras. Then,* $\mathrm{CAlg}^{\Delta}_{\underline{A}} \simeq \mathcal{P}_\Sigma(\mathrm{Poly}_{\underline{A}})$*.*

Proof. By 3.2.3.3, we can consider the base-change adjunction $\underline{A} \otimes^L_{\mathbb{Z}} (-) \colon \mathrm{Ani(CRing)} \rightleftarrows \mathrm{CAlg}^{\Delta}_{\underline{A}} : \phi^*$.
Since the right-adjoint ϕ^* is conservative and preserves sifted colimits, and since Ani(CRing) is cpt+proj-generated, by [23],4.7.3.18 we conclude that extension of scalars $\underline{A} \otimes^L_{\mathbb{Z}} (-)$ preserves and detects cpt+proj's, i.e. $\mathrm{Poly}_{\underline{A}} \simeq (\mathrm{CAlg}^{\Delta}_{\underline{A}})^{\mathrm{sfp}}$, and that also $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ is cpt+proj-generated.
Thus, being $\mathrm{Poly}_{\underline{A}}$ essentially small by assumption, we can apply the $\mathcal{P}_\Sigma$-construction to it, and this yields the whole category $\mathrm{CAlg}^{\Delta}_{\underline{A}}$. □

3.2.4 Animated Modules

Our next goal is to adopt the same strategy to define animated modules over static rings.

Definition 3.2.4.1. *(Animated modules)* For a ring $R \in \mathrm{CRing}$, let FFree_R denote the 1-category of finite free R-modules. Being it $\mathrm{FFree}_R \simeq \mathrm{Mod}(R)^{\mathrm{sfp}}$, we can define the presentable ∞-category of **animated** R**-modules** by $\mathrm{Mod}_R := \mathrm{Ani}(\mathrm{Mod}(R)) = \mathrm{Fun}^\times(\mathrm{FFree}_R^{op}, \mathrm{Spc})$.

Moreover, it is straightforward to import all definitions and terminology provided insofar to the case of animated modules. These will be used freely.
In particular, as expected the previously defined base-change adjunction extends to animated modules.

Lemma 3.2.4.2. *(Base-Change Adjunction for Modules) By B.4.0.2,* Mod_R *can be endowed with a closed symmetric monoidal structure* $\mathrm{Mod}^{\otimes}_R$ *arising from Day convolution.*
Given a morphism $\phi : R \to R'$ *in* CRing*, the encoded tensor product* $\otimes_R$ *yields an adjunction in* Cat_∞ *of the form*

$$(-) \otimes_R R' \simeq \mathrm{LKE}_j(j \circ f) \colon \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_{R'} : [R', -]$$

where $f := (-) \otimes_R R' : \mathrm{FFree}_R \to \mathrm{FFree}_{R'}$ *is the restriction to the monoidal structure on* $\mathrm{Mod}(R)^{\otimes}$*.*
The functor $(-) \otimes_R R'$ *is called* ***extension of scalars****, whereas* $\phi^* := [R', -]$ *is the* ***restriction of scalars****.*
Again, restriction of scalars induces a canonical equivalence $\pi_*(\phi^* \underline{M}]) \simeq \pi_*(\underline{M})$*; so, in particular, in the static case we can identify* $\phi^*(R')^n \in (\mathrm{Mod}_R)^{\mathrm{static}} \simeq \mathrm{Mod}(R) \ni (R')^n$*.*
Moreover, notice that the choice of the 'fixed argument' is perfectly symmetric.

Remark. The obvious functor $\iota_{|\mathrm{Poly}_R} : \mathrm{Poly}_R \hookrightarrow \mathrm{Mod}(R) \subseteq_{f.f.} \mathrm{Mod}_R$ forgetting the ring structure is symmetric monoidal and extends to a symmetric monoidal forgetful functor $for : \mathrm{CAlg}_R^{\Delta} \to \mathrm{Mod}_R$. See 3.7.1.2.

For the sake of completeness, we record here a stability result for the ∞-category of animated modules. However, its proof will be a by-product of our subsequent constructions, so that it is postponed.

Lemma 3.2.4.3. *(*Mod *is pre-stable) For each* $R \in \mathrm{CRing}$*, the* ∞*-category* Mod_R *of animated* R*-modules is pre-stable.*

3.2.5 The Presheaf of Animated Modules

We will now define simultaneously ∞-categories of animated modules over any arbitrary animated ring, and we will group them into the ∞-category Ani(CRMod) of 'animated modules together with their (animated) ring of scalars'.
This turns out to be not only a formal exercise, since it implies strong compatibility properties of such categories.
In view of our subsequent presentation of quasi-coherent sheaves of modules, we group them into the following Theorem, of which we provide an almost complete proof.

Acknowledgement. The author is indebted to Prof. Marc Hoyois who suggested to look at [5],A for the symmetric monoidal enhancement of the Straightening Equivalence; moreover, he also taught him about Day convolution, as in [10] or [23],2.4.6, and its universal property [23],4.8.1.12.

Theorem 3.2.5.1. *There exists an* ∞*-category* MOD *over* Ani(CRing) *which enjoys the following properties:*

- MOD *can be endowed with a relative closed symmetric monoidal structure* $pr_1^{\boxtimes} : \int \mathrm{MOD}^{\boxtimes} \to \mathrm{Ani(CRing)}^{\amalg}$ *over the cocartesian* $\mathrm{Ani(CRing)}^{\amalg}$ *extending the external tensor product via Day convolution. Its underlying fibration is bi-cartesian and classifies the following point-wise adjoint functors, whose action on the objects of* Ani(CRing) *retrieves the corresponding* pr_1*-fibres of* MOD*.*

  - *The straightened version* $\mathrm{MOD}^{\boxtimes}$ *of* $pr_1^{\boxtimes}$ *lies over the "co-presheaf of animated modules"*

    $$\mathrm{MOD} : \mathrm{Ani(CRing)} \to \mathrm{Pr}^{L,\otimes,\mathrm{pre-Ex}}$$

    taking values in presentable pre-stable closed symmetric monoidal ∞*-categories. Its action on morphisms can be informally described by "extension of scalars".*
  - *its op-straightened version is the "presheaf of animated modules":*

    $$\mathrm{Mod} : \mathrm{Ani(CRing)}^{op} \to \mathrm{Pr}^{L,\mathrm{lax},\mathrm{Pre-Ex}}$$

    taking values into presentable pre-stable lax closed symmetric monoidal ∞*-categories. Its action on morphisms can be informally described by "restriction of scalars".*

- *The point-wise adjunction* $\mathrm{MOD} \vdash \mathrm{Mod}$ *generalizes the "base-change" adjunction of 3.2.4.2. Units and counits are monoidal natural transformations.*

  Moreover, restriction of scalars along any $\phi : \underline{A} \to \underline{A}'$ *in* Ani(CRing) *induces an equivalence in homotopy: informally, it allows to regard* $\underline{A}'$*-modules as* $\underline{A}$*-modules;*

- *Functors in* Cat_∞ *with source* MOD *can be defined wlog on the animation of a* 1*-category* $\mathcal{C}$ *as in 3.2.5.2.*

The proof of the Theorem is somewhat laborious and we divide it into multiple results of independent interest.

MOD **as Animation.**

Proposition 3.2.5.2. *(*MOD *as animation of* CRMod*, [26],25.2.1.2 revisited) Let* CRMod *denote the* 1*-category whose objects are pairs* $(A, M \in \mathrm{Mod}(A))$ *of a module together with the corresponding ring of scalars and morphisms consist of arrows* $(\phi, f) : (A, M) \to (B, N)$ *with* $\phi : A \to B$ *a ring morphism and* $f : M \to \phi^* N$ *A-linear; equivalently, define its arrows as* $f : B \otimes_A M \to N$ *under the base-change adjunction. Then, its cpt+proj's are of the form* $(A := \mathbb{Z}[X], A^n)$ *for any finite tuple of variables* X*. Define the category* $\mathrm{Ani}(\mathrm{CRMod}) := \mathrm{Fun}^{\times}((\mathrm{CRMod}^{\mathrm{sfp}})^{op}, \mathrm{Spc})$ *of animated "modules with rings of scalars".*

Proof. Let $\mathcal{C}$ denote the full subcategory of CRMod spanned by those objects of the form $(A := \mathbb{Z}[X], A^n)$ for any finite tuple of variables X, say of length $|X|$. Since sifted colimits commute with finite products (see A.1.0.6), clearly $\mathcal{C}$ generates CRMod under 1-sifted-colimits.

Notice that $\mathcal{C}$ is the closure under finite coproducts of the full subcategory of CRMod generated by $C := (\mathbb{Z}[t], 0)$ and $D := (\mathbb{Z}, \mathbb{Z})$. Indeed, it is easy to (explicitly) check that finite coproducts in CRMod are computed component-wise: $(A, M) \coprod (B, N) \cong (A \otimes B, M \oplus N)$.

Finally, we need to show that actually $\mathcal{C} \subseteq \mathrm{CRMod}^{\mathrm{sfp}}$, and hence they are the same. By the properties of cpt+proj's, it actually suffices to check that the two generators C, D are cpt+proj.
Let $p = (p', p'') : K \to \mathrm{CRMod}$ be a 1-sifted diagram, and recall that sifted colimits commute with finite products A.1.0.6:

$$\mathrm{Map}_{\mathrm{CRMod}}\big((\mathbb{Z}[t], 0), \mathrm{colim}^{sift}(p', p'')\big) \simeq \mathrm{Map}_{\mathrm{CRMod}}\big((\mathbb{Z}[t], 0), (\mathrm{colim}^{sift}\, p', \mathrm{colim}^{sift}\, p'')\big)$$

Now, there is only one $\mathbb{Z}[t]$-linear morphism from 0, so the latter amounts to

$$\mathrm{Map}_{\mathrm{CRing}}\big(\mathbb{Z}[t], \mathrm{colim}^{sift}\, p'\big) \simeq \mathrm{colim}^{sift}\, \mathrm{Map}_{\mathrm{CRing}}\big(\mathbb{Z}[t], p'\big) \simeq \mathrm{colim}^{sift}\, \mathrm{Map}_{\mathrm{CRMod}}\big((\mathbb{Z}[t], 0), (p', p'')\big)$$

which proves $(\mathbb{Z}[t], 0) \in \mathrm{CRMod}^{\mathrm{sfp}}$. On the other hand, consider $(\mathbb{Z}, \mathbb{Z})$: since there is only one map of commutative rings from $\mathbb{Z}$,

$$\mathrm{Map}_{\mathrm{CRMod}}\big((\mathbb{Z}, \mathbb{Z}), (\mathrm{colim}^{sift}\, p', \mathrm{colim}^{sift}\, p'')\big) \simeq \mathrm{Map}_{\mathrm{Ab}}\big(\mathbb{Z}, for(\mathrm{colim}^{sift}\, p'')\big) \simeq \mathrm{colim}^{sift}\, \mathrm{Map}_{\mathrm{Ab}}\big(\mathbb{Z}, p''\big)$$

and finally $\mathrm{Map}_{\mathrm{Ab}}\big(\mathbb{Z}, p''\big) \simeq \mathrm{Map}_{\mathrm{CRMod}}\big((\mathbb{Z}, \mathbb{Z}), (p', p'')\big)$ as before. □

Digression. By our construction, given any element of $\mathcal{C}$ its second component has always a module structure on the first one. Sifted colimits of diagrams in $\mathcal{C}$ *should* still enjoy such a property by abstract non-sense, since they are computed component-wise (see A.1.0.6). This will motivate the definition of $\mathrm{Mod}_{\underline{A}}$ over any animated ring $\underline{A}$.
In a very special case, we are able to provide a more concrete proof of the fact that such a property enjoys some colimit stability; perhaps such an attempt can be generalized to the setting of higher algebra via the object of endomorphisms of [23],4.7.3.
Consider a *filtered* diagram $p = (p', p'') : I \to \mathcal{C}$ consisting of *static pairs* (and hence also with a static colimit). Then, we can check that $\varinjlim p'' \in \mathrm{Mod}_{\mathrm{colim}\, p''}$ by means of the following observation: given a $\mathbb{Z}$-module M, denote by $\mathrm{End}_{\mathbb{Z}}(M)$ the (non-commutative) ring of its endomorphisms; it is easy to check that the (commutative) rings $R \in \mathrm{CRing}$ for which M can be endowed with an R-module structure are precisely the sub-objects in CRing of the centre of $\mathrm{End}_{\mathbb{Z}}(M)$.
Indeed, on the one hand, if M has an R-module structure $\lambda : R \times M \to M$, then, for each $r \in R$, the $\mathbb{Z}$-linear multiplication λ_r induces a $\mathbb{Z}$-endomorphism of M and the axioms of scalar multiplication yield $\lambda_R \in \mathrm{CRing}$. Conversely, for a commutative sub-ring $R \leq \mathrm{End}_{\mathbb{Z}}(M)$, $\lambda : (f, m) \mapsto f(m)$ determines a R-scalar structure on M. So, under the assumption of stasis, p induces a filtered diagram $q : I \to \mathrm{Fun}(\Delta^1, \mathrm{CRing})$ which amounts to a natural transformation $ev_1 \circ p \hookrightarrow \mathrm{End}_{\mathbb{Z}}(ev_2 \circ p)$. The latter is point-wise a monomorphism and filtered colimits commute with kernels, so that also the colimit of p will enjoy the desired property.

Let us define the ∞-category of animated modules over any arbitrary animated ring $\underline{A}$.
In order to do so, observe first that Ani(CRMod) comes equipped with canonical split projections to Ani(CRing), $\mathrm{Mod}_{\mathbb{Z}}$ respectively, together with a 'common' diagonal section, given as follows.

Lemma 3.2.5.3. Ani(CRMod) *is canonically equipped with projections* $pr_1 := \mathrm{Ani}(pr_1) : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Ani}(\mathrm{CRing})$*,* $pr_2 := \mathrm{Ani}(pr_2) : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Mod}_{\mathbb{Z}}$*;* pr_1 *is split, with section induced by the diagonal map* $\mathrm{diag} := \mathrm{Ani}(\mathrm{diag})$*.*
In particular, the objects of Ani(CRMod) *can be written as pairs* $(\underline{A}, \underline{M})$ *for some* $\underline{M} \in \mathrm{Mod}_{\mathbb{Z}}$*; moreover, one has that* $\underline{M} \in \mathrm{Mod}_R$ *for each* $R \in \mathrm{Poly}_{/\underline{A}}$ *in the slice over* $\underline{A}$*.*

Proof. With reference to the previous Proposition, the projection on the first component $pr_1 : \mathcal{C} \to \mathrm{Poly}$ in Cat_1 induces a functor $\mathcal{C} \to \mathrm{Poly} \hookrightarrow \mathrm{Ani}(\mathrm{CRing})$ under post-composition by the Yoneda embedding. Then, the animation of the latter yields the sought $\mathrm{Ani}(pr_1) : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Ani}(\mathrm{CRing})$.
Similarly, one defines the projection on the second component $\mathrm{Ani}(pr_2) : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Ani}(\mathrm{Mod}_{\mathbb{Z}})$.

Moreover, $\mathrm{Ani}(pr_1)$ comes equipped with a canonical section, as induced by the diagonal inclusion $\mathrm{Ani}(\mathrm{diag}) : \mathrm{Ani}(\mathrm{CRing}) \to \mathrm{Ani}(\mathrm{CRMod})$.
Indeed, post-composing $\mathrm{Ani}(pr_1)$ to the animation of the following inclusion gives the identity:

$$\mathrm{Poly} \xrightarrow{\mathrm{diag}} \mathcal{C} \hookrightarrow \mathrm{CRing} \times \mathrm{Mod}(\mathbb{Z}) \hookrightarrow \mathrm{Ani}(\mathrm{CRing}) \times \mathrm{Mod}_{\mathbb{Z}}$$

We remark that the comparison natural transformation in $\mathrm{Ani}(\mathrm{CRing}) \to \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{CRing}))$ between the composition and the identity functor can be defined by animating its action on Poly; then one can check point-wise that the latter transformation is an equivalence, since left-derived functors preserve sifted colimits.

Furthermore, post-composition of Ani(diag) by $\mathrm{Ani}(pr_i)$ yields $\mathrm{Ani}(\mathrm{CRing}) \ni \underline{A} \mapsto \underline{A} \in \mathrm{Mod}_{\mathbb{Z}}$, so that Ani(diag) does actually act as the diagonal map.

In particular, the objects of Ani(CRMod) can be written as pairs $(\underline{A}, \underline{M})$ for some $\underline{M} \in \mathrm{Mod}_{\underline{Z}}$. Such a construction can be performed with Poly_R, FFree_R in place of Poly, FFree, so we actually infer that $\underline{M} \in \mathrm{Mod}_R$ for any $R \in \mathrm{Poly}_{/\underline{A}}$ in the slice over $\underline{A}$. Indeed, since restriction of scalars induces equivalences in homotopy, we can identify the animated modules $\underline{M}_R \simeq \underline{M}_{\mathbb{Z}}$. $\square$

In particular, Ani(diag) actually acts as the diagonal, and thus allows us to consider the pairs $(\underline{A}, \underline{A}_{\mathbb{Z}})$ as objects of Ani(CRMod).

In view of 3.2.5.3, we would like to denote an arrow in Ani(CRMod) between $(\underline{A}, \underline{M}) \to (\underline{B}, \underline{N})$ as a pair (ψ, f_R), where - for each R in $\mathrm{Poly}_{/\underline{A}}$ - we consider some map $f : \underline{A}_R \to \underline{N}_R$ in Mod_R.
In other words, for each suitable R we are writing arrows in Ani(CRMod) under the canonical maps $(pr_1^R, pr_2^R) : \mathrm{Ani}(\mathrm{CRMod}_R) \to \mathrm{Ani}(\mathrm{CAlg}_R^{\Delta}) \times \mathrm{Mod}_R$.

Lemma 3.2.5.4. *For each* $R \in \mathrm{CRing}$*, the canonical functor* $(pr_1^R, pr_2^R) : \mathrm{Ani}(\mathrm{CRMod}_R) \to \mathrm{Ani}(\mathrm{CAlg}_R^{\Delta}) \times \mathrm{Mod}_R$ *is conservative.*

Proof. In order to see this, consider the forgetful functors $for_1^R, for_2^R : \mathrm{Ani}(\mathrm{CRMod}_R) \to \mathrm{Spc}$ induced by evaluation at $(R[t], 0)$ and (R, R), respectively. Since by A.2.0.2 colimits in $\mathrm{Ani}(\mathrm{CRMod}_R)$ are computed point-wise, we can study the action of for_1^R and for_2^R on objects at the level of the Yoneda embedding of Poly_R, where they yield the spaces lying under the first and the second component, respectively. Hence, by left-deriving there are equivalences $for \circ pr_i^R \simeq for_i^R$, where for takes the corresponding underlying spaces. So, we are left to prove that the product functor (for_1^R, for_2^R) is conservative. Let $\eta : F \to G$ be a natural transformation between "animated modules with animated rings of scalars" such that $(for_1^R, for_2^R)(\eta)$ is an equivalence in Spc. Since $\mathcal{C}_R := \mathrm{CRMod}_R^{\mathrm{sfp}}$ is the closure under coproducts of $(R[t], 0)$ and (R, R) and both F, G take coproducts in $\mathcal{C}_R$ to products, our assumption means that η is an equivalence point-wise, so we conclude by [20],2.2.2. $\square$

This justifies the notation above. We adopted a version relative to R, because we also would like to go the other way around and "reconstruct" the original map by such a presentation. Indeed, in a reasonable category of "modules with rings of scalars", there should be a way of forgetting scalar structure, and hence "identifying copies of the same" module via restriction of scalars. So, morally for the backward direction we should keep track of all the successive "forgetting steps".

The closed symmetric monoidal ∞-category $\mathrm{Ani}(\mathrm{CRMod})^{\boxtimes}$**.**

Moreover, B.4.0.2 and the subsequent remark allow for a more intrinsic description of Ani(CRMod).

Proposition 3.2.5.5. *(*Ani(CRMod) *closed symmetric monoidal)* Ani(CRMod) *can be endowed with a closed symmetric monoidal structure* $\mathrm{Ani}(\mathrm{CRMod})^{\boxtimes}$ *which extends via Day convolution the external product on* CRMod*.*

Informally, the encoded tensor product can be described as follows: for any $\mathcal{F}, \mathcal{G} \in \mathrm{Ani}(\mathrm{CRMod})$*, let* $p := (p'.p''), q := (q'.q'') : K \to \mathrm{CRMod}^{\mathrm{sfp}}$ *be the corresponding sifted resolutions (wlog from the same sifted indexing simplical set); their tensor product is the localization of their sifted realization*

$$\mathcal{F} \boxtimes \underline{G} := L \circ |(p', p'') \circledast (q', q'')| \simeq L \circ |(p', p'') \boxtimes (q', q'')|$$

where $\circledast$ *denotes the Day convolution with respect to* $\mathrm{Spc}^{\times}$ *and the external product* $(\mathrm{CRMod}^{\mathrm{sfp}})^{\boxtimes}$*, while* $L : \mathcal{P}(\mathrm{CRMod}^{\mathrm{sfp}}) \to \mathrm{Ani}(\mathrm{CRMod})$ *is the localization functor of A.2.0.2.*

In particular, the closure part yields a "base-change" adjunction which generalizes 3.2.4.2: for any morphism of animated rings $\phi : \underline{A} \to \underline{B}$*, there is an adjunction*

$$(-) \boxtimes \mathrm{diag}(\phi) \colon \mathrm{Ani}(\mathrm{CRMod}) \rightleftarrows \mathrm{Ani}(\mathrm{CRMod}) : \phi^*$$

Corollary 3.2.5.6. *(*(pr_1, pr_2) *is symmetric monoidal)* $\mathrm{Ani}(\mathrm{CRing}) \times \mathrm{Mod}_{\mathbb{Z}} \simeq \mathrm{Ani}(\mathrm{Poly} \times \mathrm{FFree})$ *admits a closed symmetric monoidal structure via Day convolution of the external product, as in B.4.0.2.*
Then, $(pr_1, pr_2) : \mathrm{Ani}(\mathrm{CRMod})^{\boxtimes} \to \big(\mathrm{Ani}(\mathrm{CRing}) \times \mathrm{Mod}_{\mathbb{Z}}\big)^{\boxtimes} \simeq \mathrm{Ani}(\mathrm{CRing})^{\otimes} \times \mathrm{Mod}_{\mathbb{Z}}^{\otimes}$ *is a symmetric monoidal functor.*

Proof. The animation of an additive category is an additive ∞-category, namely finite coproducts in Ani(CRMod) coincide with finite products. In view of A.1.0.6, this can be checked on $\mathrm{CRMod}^{\mathrm{sfp}}$. $\square$

We are now ready to define modules over arbitrary animated rings.

Definition 3.2.5.7. *(Animated Modules)* Let $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ be an animated ring. Define the ∞-category $\mathrm{Mod}_{\underline{A}}$ of **animated modules over** $\underline{A}$ to be the fibre of $pr_1 : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Ani}(\mathrm{CRing})$ over $\underline{A}$.

Observe that a specialization of the generalized "base-change" adjunction to $\phi = 1_{\mathrm{diag}(\underline{A})}$ supplies for a way of reproducing the form of morphisms in CRMod.

Corollary 3.2.5.8. *(Morphisms of* Ani(CRing)*) In particular, a morphism* $\mathcal{F} \to \mathcal{G}$ *in* Ani(CRMod) *amounts to a pair* $(\phi : \underline{A} \to \underline{B}, f)$ *of maps in* Ani(CRing) *and* $\mathrm{Mod}_{\underline{B}}$*, respectively.*

Proof. We will employ the notation of Appendix B. Let $\psi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Ani}(\mathrm{CRing}) \simeq \mathrm{Ani}(\mathrm{CRing})^{\boxtimes}_{\langle 1 \rangle}$ as in the statement. Let $p := (p', p''), q := (q', q'') : K \to \mathrm{CRMod}^{\mathrm{sfp}}$ be sifted realizations for $\mathcal{F} \simeq |p|$ and $\mathcal{G} \simeq |q|$, respectively; here $| - | := \mathrm{colim}_K^{sift}$. Then, consider the map

$$\psi \boxtimes 1_{\mathrm{diag}(|q'|)} : \mathcal{F} \boxtimes \mathrm{diag}(|q'|) \to \mathcal{G} \boxtimes \mathrm{diag}(|q'|) \simeq \mathcal{G}$$

As observed again in B.4.0.2, by construction we can compute $\boxtimes$-tensor-products via resolutions; hence, we obtain a pair: for $\underline{B} := |q'|$,

$$\psi \boxtimes 1_{\mathrm{diag}(\underline{B})} = \big(\phi : |p'| \to |q'| \, , \, f : |q' \otimes_{q'} p''| \to |q''|\big)$$

where $\phi : \underline{A} := |p'| \to |q'| =: \underline{B}$ and $f : \underline{M} := |q' \otimes_{q'} p''| \to |q''| =: \underline{N}$ for animated modules $\underline{M}, \underline{N} \in \mathrm{Mod}_{\underline{B}}$. $\square$

The co-presheaf of modules MOD**.**

Lemma 3.2.5.9. *(*pr_1 *is bi-cartesian)* $pr_1 : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Ani}(\mathrm{CRing})$ *is a bi-cartesian fibration, since each arrow* $\phi : \underline{A} \to \underline{B}$ *admits both a cartesian and a cocartesian lift in* Ani(CRMod)*.*
Moreover, the class of pr_1*-cocartesian edges is closed under* $\boxtimes$*-tensor product with objects in* Ani(CRMod)*.*

Proof. We will prove only the cocartesian part, since the other one is analogous. The point being that we can check such properties after (pr_1, pr_2).
We need to show that the, for each $\phi : \underline{A} \to \underline{B}$ in Ani(CRing), there exists a morphism $\Phi : \mathcal{F} \to \mathcal{G}$ in Ani(CRMod) such that - for each $\mathcal{X} \in$ Ani(CRMod) - the following square of mapping spaces is (homotopy) cartesian:

$$\begin{tikzcd} \mathrm{Map}(\mathcal{G},\mathcal{X}) \arrow[r, "\Phi^*"] \arrow[d, "pr_1"] & \mathrm{Map}(\mathcal{F},\mathcal{X}) \arrow[d, "pr_1"] \\ \mathrm{Map}(\underline{B}, pr_1(\mathcal{X})) \arrow[r, "\phi^*"] & \mathrm{Map}(\underline{A}, pr_1(\mathcal{X})) \end{tikzcd}$$

Set $\underline{R} := pr_1(\mathcal{X})$. By [29],3.3.18, a commutative square of spaces is (homotopy) cartesian iff the fibres of the vertical maps are equivalent. So, taken any $\psi \in \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B}, \underline{R})$, let us compare the fibres over $\psi \mapsto \psi \circ \phi$.
In view of 3.2.3.2, all mapping spaces are isomorphic to co-sifted limits of mapping spaces between diagrams with values in either $\mathrm{CRMod}^{\mathrm{sfp}}$ or Poly. Up to re-indexing, we can assume that the limits are all over the same index set; moreover, being the diagonal of a sifted set cofinal, we obtain wlog the co-sifted limit of a square of mapping spaces between diagrams as above. Now, limits commute with pull-backs, so we can assume that all the objects in our diagram are indeed either in $\mathrm{CRMod}^{\mathrm{sfp}}$ or Poly.
Then, in any given fibre we can check the property of being cartesian in the image of (pr_1, pr_2), so as mapping spaces of Ani(CRing) $\times \mathrm{Mod}_{\mathbb{Z}}$.
This forces $pr_2(\Phi) : \underline{B} \otimes_{\underline{A}} pr_2(\mathcal{G}) \to pr_2(\mathcal{F})$ to be an equivalence, and there are plenty of them. (See 3.2.5.17 for a more explicit proof of this last step; the proof applies to our case whenever it involves only modules on static rings.) As a by-product, we also proved the following interesting result.

Claim. *The conservative symmetric monoidal functor* (pr_1, pr_2) : Ani(CRMod) $\to$ Ani(CRing) $\times \mathrm{Mod}_{\mathbb{Z}}$ *preserves and detects* pr_1*-cocartesian and* pr_1*-cartesian edges over* Ani(CRing).

Finally, let's show that pr_1-cocartesian edges are closed under $\boxtimes$-tensor product with (the identity of) objects in Ani(CRMod).
Let $f : \mathcal{F} \to \mathcal{G}$ be pr_1-cocartesian in Ani(CRMod) and consider any other $\mathcal{X} \in$ Ani(CRMod).
Notice that, as a particular case of 3.2.5.6, there is an equivalence at the level of the encoded tensor products:

$$(pr_1, pr_2) \circ \big((-) \boxtimes \mathcal{X}\big) \simeq (pr_1, pr_2)(-) \boxtimes (pr_1, pr_2)(\mathcal{X})$$

Then, we conclude by the Claim above and the generalized "base-change" adjunction. □

Proposition 3.2.5.10. *(*Ani(CRMod) *relative symmetric monoidal* ∞*-category) Consider the closed symmetric monoidal structure map* $p^{\boxtimes} : \int \mathrm{Ani(CRMod)}^{\boxtimes} \to \Delta$. *Then, it factors through the symmetric monoidal* ∞*-category* $\mathrm{Ani(CRing)}^{\amalg} \to \Delta$ *by a cocartesian fibration* $pr_1^{\boxtimes}$ *which exhibits* $\mathrm{Ani(CRMod)}^{\boxtimes} \in \mathrm{SymMon}_{\mathrm{Ani(CRing)}^{\amalg}}$ *as a relative symmetric monoidal* ∞*-category and retrieves* pr_1 *as the underlying cocartesian fibration.*

Proof. In view of [5],A.6 and the Lemma above, we are left to exhibit an inner fibration $pr_1^{\boxtimes}$ lifting pr_1. Fibrewise, define $(pr_1^{\boxtimes})_{\langle n\rangle} \simeq pr_1^n$ by the Segal condition. One should prove that this extends to a factorization at the level of the classifying functors, but we omit the argument, due to time constraints. □

Therefore, by B.5.0.3, there exists a co-presheaf MOD : Ani(CRing) $\to$ SymMon taking values in presentable closed symmetric monoidal ∞-categories. Its action can be informally described as follows:

- Obj: $\underline{A} \mapsto \mathrm{MOD}(\underline{A}) = \mathrm{Mod}_{\underline{A}}^{\otimes}$.

  Here the symmetric monoidal structure is induced by $\mathrm{Ani(CRMod)}^{\boxtimes}$ and (assuming 3.2.5.14 for now) it retrieves the one of B.4.0.2; in particular, its encoded tensor product acts as follows: for each $\underline{M}, \underline{M}' \in \mathrm{Mod}_{\underline{A}}$, one has $\underline{M} \otimes_{\underline{A}} \underline{M}' = \mathrm{diag}(\underline{A})_*(\underline{M} \boxtimes \underline{M}')$.

- Mor: $(\phi : \underline{A} \to \underline{B}) \mapsto (\mathrm{MOD}(\phi) : (-) \boxtimes \mathrm{diag}(\phi) : \mathrm{Mod}(\underline{A})^{\otimes} \to \mathrm{Mod}_{\underline{B}}^{\otimes})$;

The presheaf of modules Mod**.**

Now, being $pr_1 \in \mathrm{CoCart}(\mathrm{Ani}(\mathrm{CRing}))$ a cocartesian fibration, the Straightening Theorem provides a canonical way of functorially defining all the "restriction of scalars" functors on the fibres of Ani(CRMod).

Lemma 3.2.5.11. *(The presheaf of animated modules) There exists a presheaf on* Ani(CRing)*, say* Mod*, with values in* presentable pre-stable closed symmetric monoidal ∞*-categories. Informally,* Mod *acts as follows:*

- Obj: $\mathrm{Mod} : \underline{A} \quad \mapsto \quad \mathrm{Mod}(\underline{A}) = \mathrm{Mod}_{\underline{A}} \simeq \lim(\mathrm{Mod}_R \mid R_{/\underline{A}} \in (\mathrm{CRing}_{/\underline{A}})^{op})$
- Mor: $\mathrm{Mod} : (\phi : \underline{A} \to \underline{A}') \quad \mapsto \quad \mathrm{Mod}(\phi) := (\phi^* : \mathrm{Mod}_{\underline{A}'} \to \mathrm{Mod}_{\underline{A}})$ *which generalizes restriction of scalars.*

Proof. By the Straightening Theorem [24],3.2, $pr_1 \in \mathrm{Cart}(\mathrm{Ani}(\mathrm{CRing}))$ is classified via a cartesian square by a functor $\mathrm{Ani}(\mathrm{CRing}) \to \mathrm{Cat}_\infty$; let's call it Mod^{op}:

$$\begin{tikzcd}
\mathrm{Ani}(\mathrm{CRMod}) \arrow[r] \arrow[d, "\mathrm{Cart}\ni pr_1"'] & \mathrm{Cat}_{\infty//*} \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Ani}(\mathrm{CRing}) \arrow[r, "\mathrm{Mod}^{op}"] & \mathrm{Cat}_\infty
\end{tikzcd}$$

Notice that the restriction of Mod to Poly^{op} acts informally as claimed.

Claim. $\mathrm{Mod} : \mathrm{Ani}(\mathrm{CRing})^{op} \to \mathrm{Cat}_\infty$ *takes colimits in* Ani(CRing) *to limits of* ∞*-categories.*

Proof. Let $p : K \to \mathrm{Ani}(\mathrm{CRing})^{op}$ be a diagram of animated rings, and let $\overline{p}$ denote its limit cocone.
We wish to show that $\mathrm{Mod}_{\overline{p}} = \mathrm{Mod}(\lim p) \simeq \lim \mathrm{Mod}(p)$. To this end, consider the following diagram of cartesian squares:

$$\begin{tikzcd}
\mathrm{Mod}_p \arrow[r] \arrow[d] & \mathrm{Mod}_{\overline{p}} \arrow[r] \arrow[d] & \mathrm{Ani}(\mathrm{CRMod}) \arrow[r] \arrow[d, "pr_1"] & \mathrm{Cat}_{\infty//*} \arrow[d, "\pi_{\mathrm{univ}}"] \\
K \arrow[r, hook] & \overline{K} \arrow[r, "\overline{p}"] & \mathrm{Ani}(\mathrm{CRing})^{op} \arrow[r, "\mathrm{Mod}"] & \mathrm{Cat}_\infty
\end{tikzcd}$$

By [24],3.3.4.2, being $\overline{p}$ a limit cocone of its restriction p, we have cartesian equivalences

$$\lim \mathrm{Mod}_p \simeq \mathrm{Mod}_p^\natural \subseteq \mathrm{Mod}_{\overline{p}(\infty)}^\natural$$

where we used the language of marked simplicial sets: for cartesian fibrations $f : X^\natural \to S$ and $g : Y^\natural \to S$, $X^\natural$ denotes the pair (X, E) with E the set of f-cartesian edges of X, while $h : X^\natural \to Y^\natural$ is a shorthand for a morphism of cartesian fibrations $h : f \to g$, i.e. a morphism $h : X \to Y$ which takes E_X to E_Y. Moreover, such a language endows sSet with a simplicial model structure, which we denote by sSet^+; therefore, by 'cartesian equivalence' we mean a morphism of marked simplicial sets in sSet^+ which becomes an isomorphism in homotopy if we invert all distinguished edges (in this case, those which are cartesian).
Now, by [24],2.4.2.3 the base-change of a cartesian fibration is again cartesian, so we deduce that on the fibres Mod_p and $\mathrm{Mod}_{p(\infty)}$ of the bi-cartesian fibration pr_1 one can mark as cartesian edges only the equivalences (see [20],3.1.6 and the proof of 3.2.5.9).
Hence, in our case, the previous cartesian equivalence is indeed an equivalence of ∞-categories, as claimed. ■

Then, by A.3.0.2, Mod^{op} is the left Kan extension of its restriction to Poly.

Claim. Mod *recovers the classical Grothendieck construction on the* 1*-skeleton.*

Remark. Assume that the action of Mod^{op} on morphisms generalizes "restriction of scalars" for animated modules over static rings. Then, according to [8],I.1.4, we need to show that Mod^{op} acts on the 1-skeleton as follows:

- Obj: triples $(\underline{A} \in \mathrm{Ani}(\mathrm{CRing}), \mathrm{Mod}^{op}(\underline{A}) \ni \underline{M})$;

- Mor: pairs $(\phi, f) = (\phi, (\phi^*)^{op}, f) : (\underline{A}, \mathrm{Mod}_{\underline{A}} \ni \underline{M}) \to (\underline{A}', \mathrm{Mod}_{\underline{A}'} \ni \underline{N})$ with $\phi : \underline{A} \to \underline{A}'$ in Ani(CRing) and $f : \underline{M} \to \phi^* \underline{N}$ in $\mathrm{Mod}_{\underline{A}}$.

Therefore, fibres in Ani(CRMod) would be described by the values of the classifying functor Mod^{op}, and thus recover $pr_1^{-1}(\underline{A}) \simeq \mathrm{Mod}(\underline{A}) = \mathrm{Mod}_{\underline{A}}$ over any animated ring $\underline{A}$; furthermore, the enhanced Straightening construction (see [8],I.1.6) would yield $pr_1^{-1}[\phi : \underline{A} \to \underline{B}] \simeq \cup_{\underline{M},\underline{N}} \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\phi_* \underline{M}, \underline{N})$ over morphisms.
In particular, we are finally allowed to regard morphisms in Ani(CRMod) as pairs $(\phi, f) : (\underline{A}, \underline{M}) \to (\underline{B}, \underline{N})$ s.t. $\phi : \underline{A} \to \underline{B}$ in Ani(CRing) and $f : \underline{M} \to \phi^* \underline{N}$ in $\mathrm{Mod}_{\underline{A}}$, thus extending the notation for morphisms in CRMod.

Proof. The factorization of the Yoneda embedding $j : \mathrm{Poly} \hookrightarrow \mathrm{CRing} \hookrightarrow \mathrm{Ani(CRing)}$ allows us to regard the latter as a left Kan extension of $\mathrm{Mod}^{op}_{|\mathrm{CRing}}$ along the second embedding, so that we can describe the action of its opposite Mod on the 1-skeleton of $\mathrm{Ani(CRing)}^{op}$ as follows:

- Obj: For $\underline{A} \in \mathrm{Ani(CRing)}$, one has $\mathrm{Mod}_{\underline{A}} = \mathrm{Mod}(\underline{A}) = \lim(\mathrm{Mod}_R \mid R \in \mathrm{CRing}^{op}_{\underline{A}/}) \in \mathrm{Pr}^L$.

- Mor: For $\phi : \underline{A} \to \underline{A}'$, one has $\mathrm{Mod}(\phi) : \mathrm{Mod}_{\underline{A}'} \to \mathrm{Mod}_{\underline{A}}$, where the latter will turn out to be the limit of a diagram of restrictions of scalars corresponding to the diagrams of representables over $\underline{A}$ and $\underline{A}'$.

Claim 1. *Let $\phi : \underline{A} \to \underline{A}'$ be a morphism in* Ani(CRing). *Then, we can enhance the base-change adjunction for animated modules over static rings to an adjunction* $(-) \otimes^L_{\underline{A}} \underline{A}' : \mathrm{Mod}_{\underline{A}} \rightleftarrows \mathrm{Mod}_{\underline{A}'} : \phi^*$.

Proof. Let $p : \mathrm{CRing}_{/\underline{A}} \to \mathrm{Ani(CRing)}_{/\underline{A}}$ be the diagram of representables over $\underline{A}$ and let p' denote the one over $\underline{A}'$. As already seen, if $\psi : p \to p'$ denotes the natural transformation of the two diagrams, the limit map $\overline{\psi}(\infty)$ is equivalent to ϕ (by 3.2.3.2) and induces $\phi^* = \mathrm{Mod}(\phi) \simeq \lim \mathrm{Mod}(\psi)$, because the right Kan extension Mod commutes with limits.
Being categories of animated modules presentable, we want to show that ϕ^* commutes with limits and is accessible, so as to apply the II Adjoint Functor Theorem 1.2.0.6 and conclude the existence of a left-adjoint to it.

Claim 1.1. $\phi^* : \mathrm{Mod}_{\underline{A}'} \to \mathrm{Mod}_{\underline{A}}$ *preserves small limits.*

Proof. Let us work in the incarnation of quasi-categories; however, let us remark that we need a model only to devise the existence of a path-object Δ^1, the fact that sSet is cartesian closed and the one that 'being invertible' is a property of natural transformations which can be checked point-wise.
Let $F : I \to \mathrm{Mod}_{\underline{A}'} \simeq \lim(\mathrm{Mod} \circ p')$ be a diagram; under the limit adjunction, the following 1-simplices correspond to each other:

$$F \in \mathrm{Map}_{\mathrm{Cat}_\infty}(I, \lim(\mathrm{Mod} \circ p')) \quad \simeq \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CRing}_{/\underline{A}'}, \mathrm{Cat}_\infty)}(\mathrm{const}_I, \mathrm{Mod} \circ p') \ni \hat{F}$$

Then, by the universal property of limits, post-composition by the natural transformation $\psi^* = \mathrm{Mod}(\psi) : \mathrm{Mod} \circ p' \to \mathrm{Mod} \circ p$ induces

$$G \in \mathrm{Map}_{\mathrm{Cat}_\infty}(I, \lim(\mathrm{Mod} \circ p)) \quad \simeq \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CRing}_{/\underline{A}}, \mathrm{Cat}_\infty)}(\mathrm{const}(I), \mathrm{Mod} \circ p) \ni \hat{G}$$

with $G \simeq (\lim \mathrm{Mod}(\psi)) \circ F \simeq \mathrm{Mod}(\phi) \circ F$.
Similarly, the well-defined limit functors $\lim'_I$ and $\lim_I$ yield:

$$\lim{}'_I \in \mathrm{Map}_{\mathrm{Cat}_\infty}\big(\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}'}), \lim(\mathrm{Mod} \circ p')\big) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CRing}_{/\underline{A}'}, \mathrm{Cat}_\infty)}\big(\mathrm{const}_{\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}'})}, \mathrm{Mod} \circ p'\big) \ni \widehat{\lim{}'_I}$$

where we can describe informally the action of the latter natural transformation at the object $R'_{/\underline{A}'}$ (see the proof of [20],5.1.24):

$$\widehat{\lim{}'_I} : \mathrm{const}\big(\mathrm{Fun}(I, \lim \mathrm{Mod} \circ p')\big) \longrightarrow \mathrm{Mod} \circ p' \qquad \mathrm{w/} \qquad \widehat{\lim{}'_I}(R') : F \to \lim_I F_{R'}$$

where $F_{R'}$ denotes the representative of F in $\mathrm{Fun}(I, \mathrm{Mod}_{R'})$. Then, an analogous reasoning holds for $\lim_I \in \mathrm{Map}_{\mathrm{Cat}_\infty}\big(\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}}), \lim(\mathrm{Mod} \circ p)\big)$. The following observation will allow us not to distinguish between $\lim_I$ and $\lim'_I$, which amounts to the claim.

Consider the following adjoint diagrams. It suffices to show the commutativity of the left-most square, which would imply in particular $\lim_I G \simeq \phi^* \lim'_I F$. This in turn corresponds to the fact that the right-most square, which a priori lives in $\mathrm{Cat}_{(\infty,2)}$, actually commutes up to homotopy, and hence is indeed in Cat_∞.

$$\begin{tikzcd}
\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}'}) \arrow[r, "\lim'_I"] \arrow[d, "(\phi^*)_*"'] & \lim(\mathrm{Mod} \circ p') \arrow[d, "\phi^*"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}}) \arrow[r, "\lim_I"] & \lim(\mathrm{Mod} \circ p)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{const}\big(\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}'})\big) \arrow[r, "\widehat{\lim'_I}"] \arrow[d, "(\phi^*)_*"'] & \mathrm{Mod} \circ p' \arrow[d, "\psi^*"] \arrow[dl, Rightarrow, "\hat{\alpha}"'] \\
\mathrm{const}\big(\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}})\big) \arrow[r, "\widehat{\lim_I}"] & \mathrm{Mod} \circ p
\end{tikzcd}$$

In other words, we need to show that the possibly non-invertible 2-cell $\hat{\alpha}$ of natural transformations is an equivalence. To this end, observe that $\hat{\alpha}$ can be regarded as a functor

$$\hat{\alpha} : \mathrm{CRing}_{/\underline{A}'} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Cat}_\infty)$$

and hence as a natural transformation $\Delta^1 \to \mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(\mathrm{CRing}_{/\underline{A}'}, \mathrm{Cat}_\infty)\big)$, so we are left to check pointwise that it is an equivalence. In other words, we want the canonical map

$$\hat{\alpha}(R'/\underline{A}') : \psi(R')^*(\lim_I F_{R'}) \to \lim_I (\psi(R')^* F_{R'})$$

to be invertible, but this holds, because restriction of scalars of static rings commutes with limits. ■

Remark. We actually proved a more general result. In order to ease future referencing let us state it again as a Lemma.

Lemma 3.2.5.12. *(Multi-dimensional limits) Consider a natural transformation $\psi : p \to p'$ of diagrams in $\mathrm{Fun}(I, \mathrm{Cat}_\infty)$ over any arbitrary indexing category I, and let $\phi := \overline{\psi}(\infty) : \overline{p}(\infty) \to \overline{p}'(\infty)$ denote the vertex of a limit cocone of ψ.*

Consider two diagrams of natural transformations $\mathcal{H} : \mathrm{Map}(K, p(-)) \to p$ and $\mathcal{H}' : \mathrm{Map}(K, p'(-)) \to p'$, together with commutative squares $\psi \circ \mathcal{H} \simeq \mathcal{H}' \circ \psi_*$.

Let H, H' denote the limit of the two functors over I. Then, also $\phi \circ H \simeq H' \circ \phi_*$.

Claim 1.2. *$\phi^* : \mathrm{Mod}_{\underline{A}'} \to \mathrm{Mod}_{\underline{A}}$ preserves filtered colimits.*

Proof. Analogous to the previous one: replace $\lim_I$, $\widehat{\lim_I}$ by $\underrightarrow{\mathrm{colim}}_I$, $\widehat{\underrightarrow{\mathrm{colim}}_I}$ (and the same with the prime). Observe that this is indeed the only change, because we never used the universal property of $\lim_I$ or $\underrightarrow{\mathrm{colim}}_I$ until the very last step and again ψ^* commutes object-wise with filtered colimits. ■

Therefore, we conclude the existence of a left-adjoint to ϕ^*, say $F : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}'}$. In analogy with the classical setting, call it $(-) \otimes^L_{\underline{A}} \underline{A}'$.

To conclude the proof of our claim, we are left to check that also the newly defined extension of scalars functor is compatible with the base-change adjunctions over static rings. In particular, this will imply that the "base-change" adjunction at stake amounts to the closure property of the symmetric monoidal structure induced by the fibre-wise restriction of $\mathrm{Ani}(\mathrm{CRMod})^{\boxtimes}$.

Claim 1.3. *The left adjoint $(-) \otimes^L_{\underline{A}} \underline{A}'$ to ϕ^* is compatible with static restriction of scalar.*

Proof. Consider the following diagram, where the square of restrictions of scalars commutes by the construction.

$$\begin{tikzcd}
\mathrm{Mod}_{\underline{A}} \arrow[r, shift left, "(-\otimes^L_{\underline{A}} \underline{A}')"] \arrow[d, shift left, "\phi^*_{R,\underline{A}}"] & \mathrm{Mod}_{\underline{A}'} \arrow[l, shift left, "\phi^*_{\underline{A},\underline{A}'}"] \arrow[d, equal] \\
\mathrm{Mod}_R \arrow[r, shift left, "(-\otimes^L_R \underline{A}')"] \arrow[u, shift left, "(-\otimes^L_R \underline{A})"] & \mathrm{Mod}_{\underline{A}'} \arrow[l, shift left, "\phi^*_{R,\underline{A}'}"]
\end{tikzcd}$$

Since adjunctions compose,

$$(- \otimes^L_{\underline{A}} \underline{A}') \circ (- \otimes^L_R \underline{A}) \vdash \phi^*_{R,\underline{A}} \circ \phi^*_{\underline{A},\underline{A}'} \simeq \phi^*_{R,\underline{A}'}$$

But then, the left-adjoint to $\phi_{R,\underline{A}'}$ is essentially unique, so that it must hold also $(- \otimes^L_R \underline{A}') \simeq (- \otimes^L_{\underline{A}} \underline{A}') \circ (- \otimes^L_R \underline{A})$, which is precisely the commutativity of the square of extensions of scalars. ■

Remark. In particular, the last Claim 1.3 will allow us to use such an adjunction in the proof of 3.2.5.14.

Finally, the construction of a symmetric monoidal structure on $\mathrm{Mod}_{\underline{A}}$ and the proof of $\mathrm{Mod}_{\underline{A}}$ pre-stable is postponed. $\square$

Corollary 3.2.5.13. *(Restriction of scalars is conservative) Restriction of scalars is conservative.*

Proof. By 3.2.4.2, $\mathrm{Mod}(\phi)$ is conservative for any map $\phi : R \to R'$ of static rings. Now, let $\phi : \underline{A} \to \underline{B}$ be any morphism in Ani(CRing); by 3.2.3.2 we can assume that $\phi\,\mathrm{colim}^{sift}(\psi : p \to q)$ for some natural transformation of sifted realizations $\underline{A} \simeq |p|$ and $\underline{B} \simeq |q|$. So, since $\mathrm{Mod}(-)$ preserves limits, and $\mathrm{Mod}(\psi)$ induces point-wise isomorphisms in homotopy, it follows that also $\mathrm{Mod}(\phi)$ does the same, hence it is conservative. $\square$

Let us present an a-posteriori definition of the ∞-category $\mathrm{Mod}_{\underline{A}}$, for $\underline{A} \in$ Ani(CRing), which leverages on the $\mathcal{P}_\Sigma$-construction and is rather enlightening. This is a translation in our language of the first part of (3) $\implies$ (1) in the proof of Lazard's Theorem in [23],7.2.2.15.

Lemma 3.2.5.14. *($\mathrm{Mod}_{\underline{A}}$, [23],7.2.2.15) Let $\underline{A} \in$ Ani(CRing), and let $\mathrm{FFree}_{\underline{A}} := \underline{A} \otimes^L_{\mathbb{Z}} \mathrm{FFree}_{\mathbb{Z}} \subseteq_{f.f.} \mathrm{Mod}_{\mathbb{Z}}$ denote the ∞-category spanned by a set of representatives of finitely generated free $\underline{A}$-modules. Then, $\mathrm{Mod}_{\underline{A}} \simeq \mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})$.*

Proof. By 3.2.5.11, we can consider the base-change adjunction $\underline{A} \otimes^L_{\mathbb{Z}} (-) \colon \mathrm{Mod}_{\mathbb{Z}} \rightleftarrows \mathrm{Mod}_{\underline{A}} : \phi^*$.
Since the right-adjoint ϕ^* is conservative and preserves sifted colimits, and since $\mathrm{Mod}_{\mathbb{Z}}$ is cpt+proj-generated, by [23],4.7.3.18 we conclude that extension of scalars $\underline{A} \otimes^L_{\mathbb{Z}} (-)$ preserves and detects cpt+proj's, i.e. $\mathrm{FFree}_{\underline{A}} \simeq \mathrm{Mod}^{\mathrm{sfp}}_{\underline{A}}$, and that also $\mathrm{Mod}_{\underline{A}}$ is cpt+proj-generated.
Thus, being $\mathrm{FFree}_{\underline{A}}$ essentially small by assumption, we can apply the $\mathcal{P}_\Sigma$-construction to it, and this yields the whole category $\mathrm{Mod}_{\underline{A}}$. $\square$

Once we have the base-change adjunction for animated modules over arbitrary animated rings, we can apply B.4.0.2 to infer that the presheaf Mod does indeed take values into $\mathrm{SymMon}_{\mathrm{lax}}$. The compatibility of the various symmetric monoidal structures is a direct consequence of the construction.

Proposition 3.2.5.15. *($\mathrm{Mod}_{\underline{A}}$ closed symmetric monoidal) Let $\underline{A} \in$ Ani(CRing). Then, the ∞-category $\mathrm{Mod}_{\underline{A}}$ is closed symmetric monoidal.*

Pre-stability.

Then, let us record a stability result of the categories of animated modules over arbitrary animated rings. We will employ the language of the next section, since this will not imply any circularity.

Proposition 3.2.5.16. *($\mathrm{Mod}_{\underline{A}}$ is pre-stable) Consider the fibre $\mathrm{Mod}_{\underline{A}}$ of MOD $\to$ Ani(CRing) over the animated ring $\underline{A}$. Then, $\mathrm{Mod}_{\underline{A}}$ is a connective pre-stable ∞-category.*

Proof. By the construction of MOD = Ani(CRMod) as in 3.2.5.2, we have an equivalence of ∞-categories MOD $\simeq \mathrm{SCRMod}^{\mathrm{cn}}$, since they both are the animation of the same ordinary category $\mathcal{C}$.
Hence, they have equivalent fibres over Ani(CRing) $\simeq \mathrm{CAlg}^{\Delta}$, so that $\mathrm{Mod}_{\underline{A}} \simeq \mathrm{Mod}^{\mathrm{cn}}_{\underline{A}^\circ}$, which is pre-stable with stabilization $\mathrm{Mod}_{\underline{A}^\circ}$. $\square$

A technical Lemma.

We close this section with a technical Lemma where we characterize pr_1-cocartesian morphisms of MOD over Ani(CRing).

Lemma 3.2.5.17. *(pr_1-Cocartesian morphisms in MOD) Consider the cocartesian fibration pr_1 : MOD $\to$ Ani(CRing) in Cat_∞. A morphism (ϕ, f) in MOD is cocartesian over the map of animated rings $\phi : \underline{A} \to \underline{B}$ iff, letting v denote the counit of the base-change adjunction induced by ϕ, for each $\underline{R} \in \mathrm{CAlg}^{\Delta}_{\underline{B}}$, the map $v \circ f \otimes^L \underline{R}$ is an equivalence in $\mathrm{Mod}_{\underline{R}}$.*

Proof. From the very definition, an arrow $(\phi, f) : (\underline{A}, \underline{M}) \to (\underline{B}, \underline{N})$ in MOD is pr_1-cocartesian iff, for each $(\underline{R}, \underline{Y}) \in$ MOD, the following square of mapping spaces is homotopy cocartesian:

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{MOD}}\big((\underline{B},\underline{N}),(\underline{R},\underline{Y})\big) \arrow[r, "(-)\circ(\phi,f)"] \arrow[d, "pr_1"] & \mathrm{Map}_{\mathrm{MOD}}\big((\underline{A},\underline{M}),(\underline{R},\underline{Y})\big) \arrow[d, "pr_1"] \\
\mathrm{Map}_{\mathrm{CAlg}^\Delta}\big(\underline{B},\underline{R}\big) \arrow[r, "(-)\circ\phi"] & \mathrm{Map}_{\mathrm{CAlg}^\Delta}\big(\underline{A},\underline{R}\big)
\end{tikzcd}$$

Being both functors of mapping spaces Kan fibrations, as in [29],3.3.18 the latter property is equivalent to a fibre-wise equivalence over ϕ^*. So, let $\psi \in \mathrm{Map}_{\mathrm{CAlg}^\Delta}(\underline{B}, \underline{R})$ and let us compare the fibres over $\psi \mapsto \psi \circ \phi$

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(\underline{N}, \psi^*\underline{Y}) \arrow[r, "\phi^*"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\phi^*\underline{N}, \phi^*\psi^*\underline{Y}) \arrow[r, "-\circ f"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{M}, (\psi\circ\phi)^*\underline{Y}) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathrm{Mod}_{\underline{R}}}\big(\underline{N}\otimes^L_{\underline{B}}\underline{R}, \underline{Y}\big) \arrow[r, "-\circ\left(v(\underline{N})\otimes^L_{\underline{B}}\underline{R}\right)"] & \mathrm{Map}_{\mathrm{Mod}_{\underline{R}}}\big(\phi^*\underline{N}\otimes^L_{\underline{A}}\underline{R}, \underline{Y}\big) \arrow[r, "-\circ\left(f\otimes^L_{\underline{A}}\underline{R}\right)"] & \mathrm{Map}_{\mathrm{Mod}_{\underline{R}}}\big(\underline{M}\otimes^L_{\underline{A}}\underline{R}, \underline{Y}\big)
\end{tikzcd}$$

The base-change adjunction for animated modules yields the lower arrows, which acts as precomposition with $\big(v(\underline{N}) \circ f\big) \otimes^L \underline{R}$, for $v(\underline{N}) : \phi^*\underline{N} \otimes_{\underline{A}} \underline{B} \to \underline{N}$ the counit of $(- \otimes^L_{\underline{A}} \underline{B}) \vdash \phi^*$.
Therefore, we conclude that an arrow $(\phi, f) : (\underline{A}, \underline{M}) \to (\underline{B}, \underline{N})$ in MOD is pr_1-cocartesian iff the $\underline{A}$-linear morphism $f : \underline{M} \to \phi^*\underline{N}$ becomes an equivalence after tensoring with each animated algebra $\underline{R} \in \mathrm{CAlg}^\Delta_{\underline{B}}$. $\square$

Remark. In order to grasp some more intuition, consider the fibre over a static ring $\underline{A} = A$, and let us move to the setting of simplicial rings and modules. There, we would need to consider the induced map under $(- \otimes^L_{\underline{A}} \underline{R}) \simeq \mathrm{Tor}^A_\bullet(-, \underline{R})$ in the derived category $\mathcal{D}(A)$. Then, the condition $(v \circ f) \otimes^L_{\underline{A}} \underline{R}$ being an equivalence would amount to $\mathrm{Tor}^A_\bullet(v \circ f, \underline{R})$ being a quasi-isomorphism in $\mathcal{D}(A)$. Considering the long exact sequence in homology, this is equivalent to $\mathrm{Tor}^A_\bullet(\mathrm{Coker}(v \circ f), \underline{R})$ being quasi-isomorphic to 0, i.e. $\mathrm{Coker}(v \circ f)$, $\underline{R}$ being Tor-independent over A.

3.3 Comparison with Spectral Higher Algebra

In this subsection we briefly compare our construction of higher algebra with widely studied one over $\mathbb{E}_\infty$-ring spectra.
We will neither attempt an introduction, nor give many definition regarding the latter approach. However, we will recollect here a couple of relevant comparison statements, as presented by Lurie in both *SAG*,[26] and *HA*,[23].

Definition 3.3.0.1. *(Ring and module spectra)*

- Let CAlg(Sp) denote the ∞-category of **ring spectra** (or **$\mathbb{E}_\infty$-rings**). Similarly to CRing not being abelian, also the latter does not come equipped with a canonical t-structure, but we can still consider its 'connective' and 'static' parts by importing the terminology of Sp. Namely, we call 'connective part' the full subcategory $\mathrm{CAlg(Sp)}^{\mathrm{cn}}$ spanned by those object with vanishing homotopy in negative degrees; on the other hand, we call 'heart' its static part $\mathrm{CAlg(Sp)}^\heartsuit = \mathrm{CAlg(Sp)}_0$.

- For $\underline{A} \in$ Ani(CRing), let $\underline{A}^\circ$ be its underlying $\mathbb{E}_\infty$-ring (see [26],25.1.2.1). Denote by $\mathrm{Mod}_{\underline{A}^\circ}$ the corresponding category of **$\underline{A}^\circ$-module spectra**. The latter is canonically endowed with a t-structure, and we write $\mathrm{Mod}^{\mathrm{cn}}_{\underline{A}^\circ}$ for its 'connective part' and $\mathrm{Mod}^\heartsuit_{\underline{A}^\circ}$ for its 'heart'. Moreover, let us observe that the equivalence $\mathrm{Mod(Sp)} \simeq \mathrm{Mod}_{\mathbb{Z}^\circ}$ holds.

- Now, there is a canonical inclusion $\mathrm{CAlg}^\Delta \subseteq_{f.f.} \mathrm{CAlg(Sp)}^{\mathrm{cn}} \subseteq_{f.f.} \mathrm{CAlg(Sp)}$, (see [26],25.1.2.2), which allows us to construct the following category of **'module spectra with scalar ring spectrum'**: $\mathrm{SCRMod} := \mathrm{CAlg}^\Delta \times_{\mathrm{CAlg(Sp)}} \mathrm{Mod(Sp)}$.

  Objects of SCRMod have the form $(\underline{A}, \underline{M}) \in \mathrm{CAlg(Sp)}^{\mathrm{cn}} \times \mathrm{Mod}_{\underline{A}^\circ}$ and we call them connective whenever $\underline{M}$ is connective; this defines $\mathrm{SCRMod}^{\mathrm{cn}} := \mathrm{CAlg}^\Delta \times_{\mathrm{CAlg(Sp)}} \mathrm{Mod(Sp)}^{\mathrm{cn}}$.

Whenever we are considering algebras over $\mathbb{Q}$, we can also prove the essential surjectivity of the canonical inclusion $\mathrm{CAlg}^{\Delta} \subseteq_{f.f.} \mathrm{CAlg}(\mathrm{Sp})^{\mathrm{cn}}$, so that we have the following comparison maps.

Lemma 3.3.0.2. *(Comparisons) The following comparison equivalences hold:*

- (Ring spectra, [26],25.1.2): *In the rational setting, i.e. for* $\underline{A} \in \mathrm{CAlg}^{\Delta}_{\mathbb{Q}}$, $\mathrm{CAlg}(\mathrm{Sp})^{\mathrm{cn}}_{\underline{A}^{\circ}} \simeq \mathrm{CAlg}^{\Delta}_{\underline{A}}$ *and* $\mathrm{CAlg}(\mathrm{Sp})^{\heartsuit} \simeq \mathrm{CRing}$.
- (Module spectra, [26],25.2.1.2): $\mathrm{Mod}^{\mathrm{cn}}_{\underline{A}^{\circ}} \simeq \mathrm{Mod}_{\underline{A}}$ *and* $\mathrm{Mod}(\mathrm{Sp})^{\heartsuit} \simeq \mathrm{Ab}$.
- (Module spectra with scalar ring spectra, [26],25.2.1.2): $\mathrm{SCRMod}^{\mathrm{cn}} \simeq \mathrm{MOD} \simeq \mathrm{Ani}(\mathrm{CRMod})$ *and* $\mathrm{SCRMod}^{\heartsuit} \simeq \mathrm{CRMod}$.

Remark. The expository choice is aiming at clarity and compactness of statements, but does not reflect the logical order of comparison. Indeed, one first notices that the two ∞-categories at stake have the same static part; this yields a canonical inclusion of the 'animated' object into the 'spectral' one, and they are left to check its essential surjectivity. We stress on the fact that the last step is precisely when the assumption of rationality comes into play.

Moreover, let us briefly sketch the idea behind the proof. The last point follows from the fact that both ∞-categories arise as the animation of $\mathcal{C}$ (see [26],25.2.1.2 and 3.2.5.2), and this will a posteriori induce also the identifications of module categories, which can be regarded as fibres of MOD over CAlg^{Δ} (see 3.2.5.1). In particular, the various fibres of MOD over CAlg^{Δ} are equivalent to the corresponding pre-stable categories of connective module spectra. So, their stabilization will recover the whole corresponding stable categories of module spectra.

Finally, we observe that the symmetric monoidal structure of any ∞-category of animated modules $\mathrm{Mod}_{\underline{A}}$ is precisely the one on the corresponding $\mathrm{Mod}^{\mathrm{cn}}_{\underline{A}^{\circ}}$, which is in turn preserved by the stabilization (see [26],C.1).

3.4 Flatness

In this section, we review the prominent notion of flatness for animated modules, and hence algebras. As in the classical setting, restriction along flat morphisms will define flat morphisms of (derived) affine schemes. The author is indebted to Prof. Marc Hoyois for having offered him introductory lectures to the subject; more details can be found in Lurie's 'Higher Algebra' section [23],7.2.2.

Notation. For the sake of clarity, we will denote $\otimes^{L}$ by $\otimes$; this highlights the fact that the former endows the ∞-categories at stake with a symmetric monoidal structure, so our notation is not misleading.

We will have to work in the stabilization $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}}$ of our ∞-categories of animated modules. Therefore, we will start by stating (without proof) a useful although technical Lemma in which pre-stable ∞-categories - such as $\mathrm{Mod}_{\underline{A}}$ - are defined, and their relation with their stabilization - here $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}}$ - is provided. In order to avoid confusion with the standard literature, let us recall from the previous section that, whenever $\underline{A} \in \mathrm{CAlg}^{\Delta}_{\mathbb{Q}}$, in our notation one can identify $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}} \simeq \mathrm{Mod}_{\underline{A}^{\circ}}$ and $\mathrm{Mod}_{\underline{A}} \simeq \mathrm{Mod}^{\mathrm{cn}}_{\underline{A}^{\circ}}$.

Lemma 3.4.0.1. *(Pre-stable ∞-category) Let C be a pre-stable ∞-category, namely a pointed ∞-category which embeds fully faithfully in its stabilization C^{Ex} and whose essential image is closed under finite colimits and extensions (see [26],C.1.2.3). Then, a cartesian square in C of a surjection (on π_0) is also cocartesian when regarded in C^{Ex} ([26],C.1.1.2.c). On the other hand, any cocartesian square in C is also cartesian both in C and in C^{Ex} ([26],C.1.2.6).*

Proposition 3.4.0.2. *(Flatness) Let* $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$. *An animated* $\underline{A}$*-module* $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ *is **flat** whenever one of the following equivalent conditions hold:*

1. (Homological flatness): *the functor of animated* $\underline{A}$*-modules* $\underline{M} \otimes_{\underline{A}} (-) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}}$ *is (left-)exact;*
2. (Preserving static objects): $\underline{M} \otimes_{\underline{A}} (-)$ *preserves static objects;*

3. (Preserving n-truncated objects): $\underline{M} \otimes_{\underline{A}} (-)$ *preserves n-truncated objects, hence it commutes with the n-th truncation functor;*

4. (Fibre-wise flatness, [23],7.2.2.10): *The following conditions on the homotopy groups of $\underline{A}$ and $\underline{M}$ hold:*
   - *$\pi_0\underline{M}$ is an (ordinary) flat $\pi_0\underline{A}$-module;*
   - *for each $i \in \mathbb{Z}$, the counit of 3.2.4.2 induces an isomorphism $\pi_i\underline{A} \otimes_{\pi_0\underline{A}} \pi_0\underline{M} \to \pi_i\underline{M}$ of (ordinary) $\pi_0\underline{A}$-modules.*

Proof. (2) $\iff$ (3) : One implication is clear, so let us show that for $\underline{M} \otimes_{\underline{A}} (-)$ preserving static objects implies preserving the n-truncated ones. Let us argue by induction on $n \geq 0$. The induction starts by hypotheses, so let us assume that $\underline{M} \otimes_{\underline{A}} (-)$ preserves $(n-1)$-truncated objects for $n \geq 1$.
$\mathrm{Mod}_{\underline{A}}$ is presentable, so we can consider the Postnikov tower of any n-truncated $\underline{N} \in \mathrm{Mod}_{\underline{A}}$. Furthermore, they converge, meaning that their transition morphisms are again suitable restrictions of truncations. (See A.5.0.5.)
The fibre of the n-th transition morphism sits in the following cartesian square:

$$\begin{tikzcd} \underline{N} \arrow[r] & \tau_{\leq n-1}\underline{N} \\ \pi_n\underline{N}[n] \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Since $\mathrm{Mod}_{\underline{A}} \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$ embeds fully faithfully in its stabilization, by the technical Lemma 3.4.0.1 our square is also cocartesian in the bigger category. Now, $\underline{M} \otimes_{\underline{A}} (-)$ preserves colimits, hence in particular push-outs and suspension, so that we obtain the following cocartesian square:

$$\begin{tikzcd} \underline{M} \otimes_{\underline{A}} \underline{N} \arrow[r] & \underline{M} \otimes_{\underline{A}} \tau_{\leq n-1}\underline{N} \\ \big(\underline{M} \otimes_{\underline{A}} \pi_n\underline{N}\big)[n] \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

By the induction premise and our assumption (2), $\underline{M} \otimes_{\underline{A}} \tau_{\leq n-1}\underline{N}$ and $\underline{M} \otimes_{\underline{A}} \pi_n\underline{N}$ must be still $(n-1)$- and 0-truncated respectively. In particular, the n-th suspension of the latter must be n-truncated.
Furthermore, the new square (in $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$) is also cartesian by the technical Lemma 3.4.0.1; therefore, the induced long exact sequence in homotopy shows that $\underline{M} \otimes_{\underline{A}} \underline{N}$ is n-truncated, as required.
Finally, we need to show that $\underline{M} \otimes_{\underline{A}} (-)$ commutes with the n-th truncation functor $\tau_{\leq n}$. To this end, notice that - for any n-truncated animated $\underline{A}$-module $\underline{N}$, so on the essential image of $\tau_{\leq n}$ - the properties of the t-structure on $\mathrm{Mod}_{\underline{A}}$ yield a cofibre-sequence

$$\tau_{\geq n}(\underline{M} \otimes_{\underline{A}} (\underline{N})) \to \underline{M} \otimes_{\underline{A}} \tau_{\leq n}(\underline{N}) \to \tau_{\leq n}(\underline{M} \otimes_{\underline{A}} (\underline{N}))$$

Then, by assumption and the previous part the first term vanishes, so that we obtain the sought equivalence between the last two by inspecting the associated long exact sequence in homotopy. ■

(3) $\implies$ (4) : Consider the bicartesian (see the technical Lemma 3.4.0.1) square in $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$ associated to the fibre of the n-th transition map in the Postnikov tower of the copy of $\underline{A} \in \mathrm{Mod}_{\underline{A}}$:

$$\begin{tikzcd} \tau_{\leq n}\underline{A} \arrow[r] & \tau_{\leq n-1}\underline{A} \\ \pi_n\underline{A}[n] \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

As before, tensoring by $\underline{M}$ yields the cocartesian square with fiber the n-suspension of the static - by (3) - $\underline{A}$-module $\underline{M} \otimes_{\underline{A}} \pi_n\underline{A}$:

$$\begin{tikzcd}
\underline{M} \otimes_{\underline{A}} \tau_{\leq n}\underline{A} \arrow[r] & \underline{M} \otimes_{\underline{A}} \tau_{\leq n-1}\underline{A} \\
\big(\underline{M} \otimes_{\underline{A}} \pi_n\underline{A}\big)[n] \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Being it static and since π_0 symmetric monoidal, $\underline{M} \otimes_{\underline{A}} \pi_n\underline{A} \simeq \pi_0\big(\underline{M} \otimes_{\underline{A}} \pi_n\underline{A}\big) \simeq \pi_0\underline{M} \otimes_{\pi_0\underline{A}} \pi_n\underline{A}$.
But now, by (3) the functor $\underline{M} \otimes_{\underline{A}} (-)$ commutes with the n-th truncation functor, so the latter cocartesian square becomes

$$\begin{tikzcd}
\tau_{\leq n}\underline{M} \arrow[r] & \tau_{\leq n-1}\underline{M} \\
\big(\pi_0\underline{M} \otimes_{\pi_0\underline{A}} \pi_n\underline{A}\big)[n] \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Again by the technical Lemma 3.4.0.1, our cocartesian square in $\mathrm{Mod}_{\underline{A}}$ is also cartesian. So, it exhibits $\big(\pi_0\underline{M} \otimes_{\pi_0\underline{A}} \pi_n\underline{A}\big)[n]$ as the fibre of the n-th transition morphism in the Postnikov tower of $\underline{M}$. But the latter fibre is $\pi_n\underline{M}[n]$ and therefore the canonical comparison map in (4) must be an equivalence, as needed. ■

(4) $\implies$ (2) : We can regard any static $\underline{A}$-module N as lying under some animated module, i.e. $N \cong \pi_0\underline{N}$.

Claim. *wlog* $N = \pi_0\underline{A}$.

Proof. $\underline{M} \otimes_{\underline{A}} N \simeq \underline{M} \otimes_{\underline{A}} \pi_0\underline{N} \simeq \underline{M} \otimes_{\underline{A}} (\pi_0\underline{A} \otimes_{\pi_0\underline{A}} \pi_0\underline{N})$. Now, assume that $\underline{M} \otimes_{\underline{A}} (-)$ preserves the static part of the animated base-ring, i.e. that

$$\underline{M} \otimes_{\underline{A}} \pi_0\underline{A} \simeq \pi_0(\underline{M} \otimes_{\underline{A}} \pi_0\underline{A}) \cong \pi_0\underline{M} \otimes_{\pi_0\underline{A}} \pi_0\underline{A} \cong \pi_0\underline{M}$$

Then, we can conclude that

$$\underline{M} \otimes_{\underline{A}} N \simeq (\pi_0\underline{M} \otimes_{\pi_0\underline{A}} \pi_0\underline{A}) \otimes_{\pi_0\underline{A}} \pi_0\underline{N} \cong \pi_0\underline{M} \otimes_{\pi_0\underline{A}} \pi_0\underline{N} \cong \pi_0(\underline{M} \otimes_{\underline{A}} \underline{N})$$

which is static, as desired. ■

Finally, let's prove the statement for $\underline{N} = \underline{A}$ the base ring, namely, that $\underline{M} \otimes_{\underline{A}} \pi_0\underline{A} \simeq \pi_0(\underline{M} \otimes_{\underline{A}} \pi_0\underline{A}) \simeq \pi_0\underline{M}$. Consider the Tor-spectral sequence of [22],4.1.14: $E_2^{p,q} : \mathrm{Tor}_p^{\pi_*\underline{A}}(\pi_*\underline{M}, \pi_*(\pi_0\underline{A}))_q \implies \pi_{p+q}(\underline{M} \otimes_{\underline{A}} \pi_0\underline{A})$. Being $\pi_0\underline{A}$ static, it degenerates at the second page and gives the sought isomorphisms in homotopy. □

Remark. This means that, in each degree i, we are requiring the canonical maps $\pi_0\underline{R} \to \pi_0\underline{M}$ and $\pi_i\underline{R} \to \pi_i\underline{M}$ to sit in a cocartesian square, i.e. to have equivalent cofibers. This generalizes the classical intuition of flatness as being a condition on (relative) affine schemes which implies fibres to 'vary continuously' over the base.

Remark. Clearly, for a (connective) animated ring $\underline{A} \in \mathrm{Ani(CRing)}$, all flat $\underline{A}$-modules in $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$ are again connective, and hence all already contained in $\mathrm{Mod}_{\underline{A}}$.
Moreover, if $\underline{R} \simeq \pi_0\underline{R}$ is static, then we recover the classical condition: $\underline{M} \in \mathrm{Mod}_R$ is flat iff $\underline{M} \simeq \pi_0\underline{M}$ is static and $\pi_0\underline{M}$ is R-flat.

An important special case of the notion of flatness is the following.

Definition 3.4.0.3. *(Faithful flatness)* A map of animated rings $f : \underline{A} \to \underline{B}$ in Ani(CRing) is said to be **faithfully flat** if the extension of scalars functor $\underline{B} \otimes_{\underline{A}}^L (-) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}_{\underline{B}}$ is both exact and conservative.

Remark. Since $\pi_0 : \mathrm{Mod}_{\underline{B}} \to \mathrm{Mod}(\pi_0\underline{B})$ is conservative on flat modules, a faithfully flat map $\phi : \underline{A} \to \underline{B}$ induces a faithfully flat map of static rings $\pi_0\phi : \pi_0\underline{A} \to \pi_0\underline{B}$ on connected components.

Let us record for future reference an immediate although fundamental property of flatness: it is preserved under extension of scalars.

Lemma 3.4.0.4. *(Flatness is stable under extension of scalars) Let $\phi : \underline{A} \to \underline{B}$ be any morphism in* Ani(CRing) *and consider a flat animated module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$. Then, also its extension of scalars $\underline{B} \otimes_{\underline{A}} \underline{M} \in \mathrm{Mod}_{\underline{B}}$ is flat.*

Proof. Consider the functor $(\underline{M} \otimes_{\underline{A}} \underline{B}) \otimes_{\underline{B}} (-)$. For each $\underline{N} \in \mathrm{Mod}_{\underline{B}}$, it holds $\underline{M} \otimes_{\underline{A}} \underline{B} \otimes_{\underline{B}} \underline{N} \simeq \underline{M} \otimes_{\underline{A}} \phi^* \mathcal{N}$; but now the right-adjoint ϕ^* (see 3.2.5.11) preserves truncation properties and $\underline{M}$ is $\underline{A}$-flat, so $\underline{M} \otimes_{\underline{A}} \underline{B} \otimes_{\underline{B}} \underline{N}$ is static whenever $\mathcal{N}$ is such. We conclude by 3.4.0.2. $\square$

As in the classical setting, we can define an analogous homological notion of projectiveness. Indeed, we call projective $\underline{A}$-modules, the objects of $\mathrm{Proj}(\mathrm{Mod}_{\underline{A}})$. We observe that this is in agreement with [23],7.2.2.4, because in our notation $\mathrm{Mod}_{\underline{A}} \simeq \mathrm{Mod}^{\mathrm{cn}}_{\underline{A}^\circ}$ consists only of connective $\underline{A}$-modules.

Lemma 3.4.0.5. *([23],7.2.2.7, [23],7.2.2.14) Let* $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ *and let* $\mathrm{Flat}_{\underline{A}} \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}$ *denote the full subcategory spanned by flat* $\underline{A}$*-modules. Then,*

- $\mathrm{Flat}_{\underline{A}}$ *is closed under finite coproducts, retracts and filtered colimits.*
- *Projective* $\underline{A}$*-modules are precisely the retracts of free* $\underline{A}$*-modules.*
- $\mathrm{Free}_{\underline{A}} \subseteq_{f.f.} \mathrm{Proj}(\mathrm{Mod}_{\underline{A}}) \subseteq_{f.f.} \mathrm{Flat}_{\underline{A}}$.

Proof. (1): The tensoring functor $(-) \otimes^L_{\underline{A}} \underline{M}$ preserves colimits in each variable.

$(2), (3)$: [23],7.2.2.6 enhances the classical homological descriptions of projectives; this allows us to prove both (2) and (3) in the classical way. $\square$

The following result is an example of the enhancement, operated by flatness, of static algebraic properties to higher homotopical degrees.

Proposition 3.4.0.6. *(Flatness enhances projectiveness, [23],7.2.2.18) Let* $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ *be an animated ring. Then, a flat* $\underline{A}$*-module* $\underline{M} \in \mathrm{Flat}_{\underline{A}}$ *is projective (resp. free) iff* $\pi_0\underline{M}$ *is a projective (resp. free)* $\pi_0\underline{A}$*-module.*

Proof. (2) : Recall that the left-adjoint π_0 preserves retracts and direct sums, so if an $\underline{A}$-module $\underline{M}$ is projective (resp. free), also its static part is such. Let's prove the converse implication.

Claim. *For* $\underline{M} \in \mathrm{Flat}_{\underline{A}}$*, if* $\pi_0\underline{M}$ *is* $\pi_0\underline{A}$*-free, then also* $\underline{M}$ *is* $\underline{A}$*-free.*

Proof. Let X be a basis for the free static module module $\pi_0\underline{M}$, namely $\phi : \oplus_{x \in X}(\pi_0\underline{A})x \cong \pi_0\underline{M}$. Then, multiplication by $x \in \pi_0\underline{M}$ induces maps

$$\{\phi_x : \pi_0\underline{A} \to \pi_0\underline{M} : x \in X\} \in \mathrm{Hom}_{\mathrm{Mod}(\pi_0\underline{A})}(\pi_0\underline{A}, \pi_0\underline{M})$$

and hence maps $\{\Phi_x : \underline{A} \to \underline{M} : x \in X\}$, since $\pi_0 : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}(\pi_0\underline{A})$ is surjective on connected components. These in turn canonically assemble into $\Phi : \oplus_X \underline{A}x \to \underline{M}$. The latter map of flat $\underline{A}$-modules induces an equivalence after the conservative functor π_0, so that it must be an equivalence itself, as desired.

Claim. *For* $\underline{M} \in \mathrm{Flat}_{\underline{A}}$*, if* $\pi_0\underline{M}$ *is* $\pi_0\underline{A}$*-projective, then also* $\underline{M}$ *is* $\underline{A}$*-projective.*

Proof. Being $\pi_0\underline{M}$ a $\pi_0\underline{A}$-projective module, there exists some free module $F_0 \in \mathrm{Mod}(\pi_0\underline{A})$ s.t. $F_0 \cong \pi_0\underline{M} \oplus G_0$. Up to replacing F_0 by $F_0^{(\omega)}$, by Eilenberg's trick we can assume also G_0 to be free.
As before, we can find a lift for the projection $p : F_0 \twoheadrightarrow \pi_0\underline{M}$, namely there exists some free $\underline{A}$-module $\underline{F}$ with $\pi_0\underline{F} \cong F_0$ together with a map $g : \underline{F} \to \underline{M}$ s.t. $\pi_0(g) = p$.
We wish to show that $\underline{G} := \mathrm{Fib}(g)$ *is a free* $\underline{A}$*-module.* Being its static part $\pi_0\underline{G} \cong \ker(p : F_0 \twoheadrightarrow \pi_0\underline{M}) \simeq G_0$ a $\pi_0\underline{A}$-free module, by the first part it suffices to prove that $\underline{G}$ be $\underline{A}$-flat via 3.4.0.2,iv. In other words, we are left to prove that for each $n > 0$ there is an isomorphism $\pi_n\underline{G} \cong \pi_n\underline{A} \otimes_{\pi_0\underline{A}} \pi_0\underline{G}$.
To this end, fix $n > 0$ and consider the following commutative diagram of static $\pi_0\underline{A}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \pi_n\underline{A} \otimes_{\pi_0\underline{A}} \pi_0\underline{G} \arrow[r] \arrow[d, "\phi'"] & \pi_n\underline{A} \otimes_{\pi_0\underline{A}} \pi_0\underline{F} \arrow[r, "``p"'"'] \arrow[d, "\phi", "\cong"'] & \pi_n\underline{A} \otimes_{\pi_0\underline{A}} \pi_0\underline{M} \arrow[r] \arrow[d, "\phi''", "\cong"'] & 0 \\
0 \arrow[r] & \pi_n\underline{G} \arrow[r] & \pi_n\underline{F} \arrow[r, "\pi_n(g)"] & \underline{M} \arrow[r] & 0
\end{tikzcd}$$

By the flatness of $\underline{F}$ and $\underline{M}$, the two vertical arrows ϕ and ϕ'' are isomorphisms, so, provided the exactness of the two rows, the Snake Lemma would imply that also ϕ is an isomorphism, as desired. Then, let's take care of the exactness part.
First of all observe that homotopy groups commute with fibres, so both the bottom sequence and the top one before tensoring by $\pi_n\underline{A}$ are exact both on the left and in the middle.
The exactness on the right of both sequences is a consequence of the flatness of $\underline{F}$ and $\underline{M}$: being ϕ, ϕ'' isomorphisms, the two arrows "p" $:= \pi_n\underline{A} \otimes_{\pi_0\underline{A}} \pi_0(g)$ and $\pi_n(g)$ are isomorphic; then, being $\pi_n\underline{A} \otimes_{\pi_0\underline{A}} (-)$ right-exact, $p : \pi_0\underline{F} \twoheadrightarrow \pi_0\underline{M}$ induces the needed surjections.
Moreover, again by the flatness of $\pi_0\underline{M}$ and F_0, an inspection of the Tor long exact sequence yields the exactness on the left of both rows.

Therefore, we established that $\underline{G} = \mathrm{Fib}(g)$ is a free $\underline{A}$-module. Consider again the fibre sequence in $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$

$$\underline{G} = \mathrm{Fib}(g) \overset{\gamma}{\hookrightarrow} \underline{F} \overset{g}{\twoheadrightarrow} \underline{M}$$

We wish to show that the sequence splits, namely that $\underline{G}$ be a direct summand of $\underline{F}$: being $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$ stable, this will allow us to identify $\underline{M} \simeq \mathrm{Cofib}(\gamma)$ with the other summand.
Our last claim amounts to proving that $\underline{G}$ is a direct summand of $\underline{F}$ in the homotopy category $ho\mathrm{Mod}_{\underline{A}}$, i.e. that γ admits a retraction, say $\nu : \underline{F} \to \underline{G}$ s.t. $\nu \circ \gamma$ is homotopic to the identity $1_{\underline{G}}$.
Let's construct such a retraction ν by the usual lifting procedure. $\pi_0\underline{M}$ is projective, so $\pi_0\underline{G} \cong G_0 \subseteq F_0 \cong \pi_0\underline{F}$ splits, say via some retraction $\nu_0 : \pi_0\underline{F} \twoheadrightarrow \pi_0\underline{G}$. Being $\underline{F}$ free, ν_0 lifts to some map $\nu : \underline{F} \to \underline{G}$. Then, the composite $\nu \circ \gamma : \underline{G} \to \underline{G}$ induces the identity on connected components, i.e. $\pi_0(\nu \circ \gamma) \cong 1_{\pi_0\underline{G}}$. Finally, being π_0 conservative on flat modules, $\nu \circ \gamma$ must have been already an equivalence. So, it admits a quasi-inverse and hence there is a homotopy $\nu \circ \gamma \sim 1_{\underline{G}}$. This shows that ν is a retraction of γ in the homotopy category, as needed. □

We close this section with Lurie's enhancement of the celebrated Theorem of Lazard. We will only sketch the proof, so as to give an idea of how the reviewed tools come into play, but we will omit all the technicalities relative to spectral sequences.

Theorem 3.4.0.7. *(Lazard's Theorem, [26],7.2.2.15) Let $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ and $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ be both* connective. TFAE:

1. *$\underline{M}$ is flat.*
2. *$\underline{M} \simeq \varinjlim \left(\underline{A}^{n_i} \mid n_i < \infty, i \in I\right)$ admits a presentation as a filtered colimit of finitely generated free $\underline{A}$-modules.*
3. *$\underline{M} \simeq \varinjlim \left(\underline{P}_i \mid \underline{P}_i \in \mathrm{Proj}(\mathrm{Mod}_{\underline{A}}), i \in I\right)$ admits a presentation as a filtered colimit of projective $\underline{A}$-modules.*

Proof. (Sketch) (2) $\implies$ (3): Clear. (3) $\implies$ (1): This is the previous Lemma. Let's sketch (1) $\implies$ (3). Our aim is to reduce the problem to the classical Lazard's Theorem.
Let $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$. By 3.2.5.14, we have $\mathrm{Mod}_{\underline{A}} \simeq \mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})$.
Thus, by [24],5.1.5.5 such an equivalences is the left Kan extension along the Yoneda embedding of its restriction to $\mathrm{FFree}_{\underline{A}}$. Unwinding the definitions, this means that each $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ admits a presentation as the colimit of a diagram $p : \mathrm{FFree}_{\underline{A}}/\underline{M} \simeq \mathrm{FFree}_{\underline{A}} \times_{\mathrm{Mod}_{\underline{A}}} (\mathrm{Mod}_{\underline{A}})_{/\underline{M}} \to \mathrm{Mod}_{\underline{A}}$.
We are left to show that $\mathrm{FFree}_{\underline{A}}/\underline{M}$ is actually filtered whenever $\underline{M}$ is flat over $\underline{A}$. In other words, we wish any of its finite sub-diagrams to have a colimit cone extension. This statement is implied by the following two properties of $\mathrm{FFree}_{\underline{A}}/\underline{M}$:

- *(Extending finite diagrams)* For each $\{X_i\}_{i=1}^n \subseteq \mathrm{FFree}_{\underline{A}}/\underline{M}$, there exists a module $\mathrm{FFree}_{\underline{A}}/\underline{M}$ together with morphisms $(X_i \to X)_i$.

  This is clear, since $\mathrm{FFree}_{\underline{A}}$ is stable under finite coproducts.

- *(Refinement to a colimit cone)* Given any modules $X,\ Y \in \mathrm{FFree}_{\underline{A}}/\underline{M}$, integer $n \geq 0$, and morphism $\mathbb{S}^n \to \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}/\underline{M}}(X,Y) \in \mathcal{H} = h o\mathrm{Spc}$, there exists a map $Y \to Z$ in $\mathrm{FFree}_{\underline{A}}/\underline{M}$ s.t. $\mathbb{S}^n \to \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}/\underline{M}}(X,Z)$ is null-homotopic.

  The proof of the latter statement is both technical and involved, and thus omitted. In particular, we observe that this is precisely where ne applies the classical version of Lazard's Theorem.

Then, given any finite sub-diagram $q : J \to \mathrm{FFree}_{\underline{A}}/\underline{M}$, by the first property we can extend it to a cone and by the second one we can choose an initial extension: let $X := \coprod_J q(j)$ extend $q_{|J}$; we claim that $X \simeq \operatorname{colim} q_{|J}$. Let us argue by contradiction: assume that there exists some $Y \in \mathrm{FFree}_{\underline{A}}/\underline{M}$ for which some homotopy group, say the m-th, of $\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}/\underline{M}}(X,Y)$ is non-trivial. By the second property, there exists a map $Y \to Z$ which kills that homotopy group, so s.t. $\pi_m\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}/\underline{M}}(X,Z) \simeq 0$. But now, we assumed $X \simeq \coprod_J p(j)$, so that we conclude that each $\pi_m\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}/\underline{M}}(p(j),Z) \simeq 0$, so that also $\pi_m\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}/\underline{M}}(p(j),Y) \simeq 0$, which yields a contradiction. $\square$

3.4.1 Example: Localization of Animated Rings and Modules

In this subsection we will discuss the construction of localization of animated rings and modules, as presented in [22],4.1.18 and [13],2.7. Lurie's approach generalizes the equivalence between the two most useful definitions of classical localization: by means of a universal property and via a co-base-change along the canonical map $\mathbb{Z}[x] \to \mathbb{Z}[x,x^{-1}]$.
We will prove the equivalence by an adaptation of the argument given by J. Hekking in [13],2.7.2.
For the sake of clarity, we work only in the category Ani(CRing), however let us observe that it is straightforward to reformulate all the statements and proofs *mutatis mutandis* in any other slice $\mathrm{CAlg}^{\Delta}_{\underline{R}}$ of $\underline{R}$-algebras.

Let us begin by introducing the notion of the space of units of an animated ring.

Definition 3.4.1.1. *(Space of units, [13],2.7)* For any animated ring $\underline{A} \in \mathrm{Ani(CRing)}$ define its **space of units** by $\underline{A}^\times := \mathrm{Map}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[t^{\pm1}],\underline{A})$.

The next lemma shows that the notation is not misleading, in that $\underline{A}^\times$ turns out to be the full subspace of $\mathit{for}\underline{A}$ spanned by $\pi_0\underline{A}^\times \cong (\pi_0\underline{A})^\times \leq \pi_0\underline{A}$ as abelian groups. The proof is an adaptation of the first paragraph of [13],2.7, of [22],4.1.19 and of the last part of [22],4.1.18. Moreover, the author is indebted to Prof. Marc Hoyois, who suggested the second part of the argument.

Lemma 3.4.1.2. *([13],2.7) The canonical comparison morphism $\underline{A}^\times \to \mathit{for}\underline{A}$ induced by pre-composition with the localization map $\mathbb{Z} \to \mathbb{Z}[t^{\pm1}]$ is a monomorphism in* Spc.

Proof. Recall that $\pi_0\underline{A} \in \mathrm{CRing}$ by 3.2.1.4. We will show that $\pi_0(\underline{A}^\times) \cong (\pi_0\underline{A})^\times$ and $\pi_n(\underline{A}^\times) \cong \pi_n\underline{A}$ for each $n > 0$. Then, by looking at the long exact sequence in homotopy, this will allow us to conclude that the fibres of the canonical map $\underline{A}^\times \to \mathit{for}\underline{A}$ are either empty or contractible, i.e. that such a map is a monomorphism. We will start with a computation.

Claim 1. *([22],4.1.19) As static animated rings, $\mathbb{Z}[x^{\pm1}] \simeq \mathbb{Z}[x,y] \otimes^L_{\mathbb{Z}[t]} \mathbb{Z}$*

Proof. More precisely, we need to show that the following commutative square in Ani(CRing) is cocartesian.

$$\begin{tikzcd} \mathbb{Z}[t] \arrow[r, "t\mapsto 1"'] \arrow[d, "t\mapsto xy"'] & \mathbb{Z} \arrow[d] \\ \mathbb{Z}[x,y] \arrow[r, "y\mapsto x^{-1}"] & \mathbb{Z}[x,x^{-1}] \end{tikzcd}$$

The static part of the square is a push-out, i.e. $\mathbb{Z}[x,y] \otimes_{\mathbb{Z}[t]} \mathbb{Z} \cong \mathbb{Z}[x^{\pm1}]$ in CRing, so we are left to show that the derived tensor product is indeed static.
Since the co-angle consists of static animated rings, for each $n \geq 0$ the following isomorphism of abelian groups holds:

$$\pi_n(\mathbb{Z}[x,y] \otimes^L_{\mathbb{Z}[t]} \mathbb{Z}) \cong \mathrm{Tor}_n^{\mathbb{Z}[t]}(\mathbb{Z},\mathbb{Z}[x,y])$$

and we are done if we observe that the right-hand side vanishes whenever $n > 0$.
Indeed, $xy \in \mathbb{Z}[x, y]$ is not a zero-divisor, so consider the long exact sequence induced by applying $\mathrm{Tor}_\bullet^{\mathbb{Z}[t]}(\mathbb{Z}, -)$ to the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}[t] \longrightarrow \mathbb{Z}[x, y] \longrightarrow \mathbb{Z}[x, y]/\mathbb{Z}[t] \cong \mathbb{Z}[x] \oplus \mathbb{Z}[y] \longrightarrow 0$$

Then, the result follows by recalling that $\mathrm{Tor}_n^{\mathbb{Z}[t]}(\mathbb{Z}, \mathbb{Z}[t]^{(k)}) \simeq 0$ whenever $n > 0$. ■

Now, consider the fibre-square obtained by applying $\mathrm{Map}_{\mathbb{Z}/}(-, \underline{A})$ to the cocartesian square above and recalling that $m : \underline{A}(\mathbb{Z}[x, y]) \simeq \underline{A} \times \underline{A} \to \underline{A} \simeq \underline{A}(\mathbb{Z}[t])$ and $e : \underline{A}(\mathbb{Z}) \simeq * \to \underline{A}$ define the multiplication and the unit of $\underline{A}$, respectively (see 3.2.1.4 for more details on the topological algebra structure on $\underline{A}$). It exhibits $\underline{A}^\times \simeq \mathrm{Fib}_e(m : \underline{A} \times \underline{A} \to \underline{A})$:

$$\begin{tikzcd}
\underline{A}^\times \arrow[r] \arrow[d] & \star \arrow[d, "e"] \\
\underline{A} \times \underline{A} \arrow[r, "m"] & \underline{A}
\end{tikzcd}$$

So, inspection of the induced long exact sequence in homotopy pointed by e:

$$\pi_n(\underline{A}^\times, e) \longrightarrow \pi_n(\underline{A} \times \underline{A}, (e, e)) \cong \pi_n(\underline{A}, e) \times \pi_n(\underline{A}, e) \xrightarrow{m} \pi_n(\underline{A}, e)$$

induces isomorphisms $\pi_0(\underline{A}^\times, e) \simeq \pi_0(\underline{A}, e)^\times$ and $\pi_n(\underline{A}^\times, e) \simeq \pi_n(\underline{A}, e)$, since $\pi_n(\underline{A}, e)^\times \cong \ker(\pi_n(m))$.
Finally, observe that the isomorphisms of homotopy groups above do not depend on the chosen base-point. Indeed, this is always the case when we are dealing with topological groups and the space $\underline{A}^\times$ inherits a group-structure from $\underline{A}$, namely the one induced under $\mathrm{Map}(-, \underline{A})$ by the co-multiplication $\mathbb{Z} \to \mathbb{Z}[x, x^{-1}]$ of the square above.

□

We are now ready to characterize localizations of animated rings by using $\mathbb{Z}[x] \to \mathbb{Z}[x^{\pm 1}]$ as a prototype. The proof is an adaptation of [22],4.1.18.

Proposition 3.4.1.3. *(Universal property of localizations, [22],4.1.18) Consider a morphism $f : \underline{A} \to \underline{B}$ in* $\mathrm{Ani(CRing)}$ *together with an element $a \in \pi_0\underline{A} \in \mathrm{CRing}$ s.t. $f(a) \in (\pi_0\underline{B})^\times$ is invertible. Then,* TFAE:

1. ([UP : loc]) *For each $\underline{R} \in \mathrm{Ani(CRing)}$, there is an equivalence of spaces $f^* : \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B}, \underline{R}) \xrightarrow{\simeq} \mathrm{Map}'(\underline{A}, \underline{R})$, where $\mathrm{Map}'(\underline{A}, \underline{R}) \subseteq_{f.f.} \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{A}, \underline{R})$ is the full subspace spanned by those maps $h : \underline{A} \to \underline{R}$ s.t. $h(a) \in (\pi_0\underline{R})^\times$;*

2. *For each $n \geq 0$, there is an isomorphism $\pi_n\underline{A} \otimes_{\pi_0\underline{A}} (\pi_0\underline{A})[a^{-1}] \xrightarrow{\cong} \pi_n\underline{B}$ in* Ab.

   Equivalently, $\pi_0\underline{B} \cong (\pi_0\underline{A})[a^{-1}]$ *and $\underline{B}$ is flat on $\underline{A}$ (see 3.4.0.2).*

In particular for each pair $(\underline{A}, a \in \pi_0\underline{A})$ as before, there exists some $\underline{B} \in$ $\mathrm{Ani(CRing)}$ *together with a map $f : \underline{A} \to \underline{B}$ satisfying both the conditions above. Call such a candidate $\underline{B}$ the* ***localization of $\underline{A}$ at*** *a and write $\underline{B} = \underline{A}[a^{-1}]$.*
This is well-defined up to contractible ambiguity, since the universal property [UP : loc] *implies that any two maps satisfying* (1) $\iff$ (2) *must be equivalent.*

Proof. Let us start by proving the existence of localizations.

Claim ($\exists$). *Given any animated ring $\underline{A} \in$* $\mathrm{Ani(CRing)}$ *together with a connected component $a \in \pi_0\underline{A}$, there exists some animated ring $\underline{B}$ together with some map $f : \underline{A} \to \underline{B}$ in* $\mathrm{Ani(CRing)}$ *which satisfies both* (1) *and* (2)*. Denote them by $\underline{A}[a^{-1}]$ and $\phi : \underline{A} \to \underline{A}[a^{-1}]$.*
Proof. Evaluation at the variable x induces an identification of static rings in CRing:

$$ev_x : \mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[x], \pi_0\underline{A}) \xrightarrow{\cong} \pi_0\underline{A} = \pi_0\mathrm{Map}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[x], \underline{A})$$
$$(\hat{a} : \mathbb{Z}[x] \to \pi_0\underline{A}) \longmapsto [a : \mathbb{Z}[x] \to \underline{A}]$$

so that we can think the connected component $a \in \pi_0\underline{A}$ as obtained by an assignment $\hat{a} : x \mapsto a$.

Define $\underline{B} := \underline{A} \otimes^L_{\mathbb{Z}[x]} \mathbb{Z}[x^{\pm 1}]$ and $f : \underline{A} \to \underline{B}$ by forming the following co-cartesian diagram in Ani(CRing):

$$\begin{tikzcd}
\mathbb{Z}[x] \arrow[r, "a"] \arrow[d] & \underline{A} \arrow[d, "f"] \\
\mathbb{Z}[x, x^{-1}] \arrow[r] & \underline{B}
\end{tikzcd}$$

(1) : Then, let us verify that f satisfies the first property, i.e. that, for each $\underline{R} \in$ Ani(CRing), pre-composition by f induces an equivalence $\mathrm{Map}(\underline{B}, \underline{R}) \to \mathrm{Map}'(\underline{A}, \underline{R})$ in Spc.
To this end, consider the following diagram, in which the first square - hence the outer rectangle - is cartesian by construction:

$$\begin{tikzcd}
\mathrm{Map}(\underline{B}, \underline{R}) \arrow[r] \arrow[d, "f^*"] & \mathrm{Map}(\mathbb{Z}[x^{\pm 1}], \underline{R}) \arrow[r, equal] \arrow[d] & \underline{R}^{\times} \arrow[d, hook] \\
\mathrm{Map}(\underline{A}, \underline{R}) \arrow[r, "a^*"] & \mathrm{Map}(\mathbb{Z}[x], \underline{R}) \arrow[r, equal] & for\underline{R}
\end{tikzcd}$$

Moreover, the rightmost vertical map is a monomorphism by the previous Lemma 3.4.1.2, so that f^* turns out to be a monomorphism as well.
Let $\mathrm{Map}'(\underline{A}, \underline{R}) \subseteq_{f.f.} \mathrm{Map}(\underline{A}, \underline{R})$ denote the essential image of f^*, and observe that the canonical map $\mathrm{Map}(\underline{B}, \underline{R}) \to \mathrm{Map}'(\underline{A}, \underline{R})$ is an equivalence, so that we have the following cartesian square.

$$\begin{tikzcd}
\mathrm{Map}'(\underline{A}, \underline{R}) \arrow[r] \arrow[d, hook] & \underline{R}^{\times} \arrow[d, hook] \\
\mathrm{Map}(\underline{A}, \underline{R}) \arrow[r, "a^*"] & for\underline{R}
\end{tikzcd}$$

Now, by the construction $\mathrm{Map}'(\underline{A}, \underline{R}) \simeq \mathrm{Map}(\underline{B}, \underline{R})$ is spanned by some arrows $h : \underline{A} \to \underline{R}$ s.t. $h(a) = a^*(h) \in \underline{R}^{\times}$ is invertible. So, we are left to show that each such map $h : \underline{A} \to \underline{R}$ s.t. $h(a) \in \underline{R}^*$ is a point in $\mathrm{Map}'(\underline{A}, \underline{R})$.
But this is implied by the universal property of pull-backs: any such map h amounts to a pair of arrows $(h, h(a)) : \Delta^0 \to \mathrm{Map}(\underline{A}, \underline{R}) \times \underline{R}^{\times}$ extending the angle, so that h factors through $\mathrm{Map}'(\underline{A}, \underline{R})$.

(2) : Let us now check also the second property, namely that f is flat and that $\pi_0 \underline{B} \simeq (\pi_0 \underline{A})[a^{-1}]$.
For what concerns flatness, recall that, by 4.1.4.7, such a property is stable under extension of scalars, so it is implied by the flatness of $\mathbb{Z}[x] \to \mathbb{Z}[x^{\pm 1}]$.
As for the static part, instead, we argue as follows: π_0 is a left-adjoint (see A.5.0.7), hence it commutes with the push-out of animated rings (given by $\otimes^L$): $\pi_0 \underline{B} \simeq \pi_0 \underline{A} \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}] = (\pi_0 \underline{A})[a^{-1}]$, since by construction $\hat{a} : \mathbb{Z}[x] \to \pi_0 \underline{A}$ acts as $x \mapsto a$ by evaluation at x. ■

(1) $\implies$ (2) : Let it be given a map $f : \underline{A} \to \underline{B}$ and an element $a \in \pi_0 \underline{A}$ s.t. $f(a) \in \underline{B}^{\times}$.
Let $\phi : \underline{A} \to \underline{A}[a^{-1}]$ be as in the Claim. Then, let us prove that $(\underline{A}[a^{-1}], \phi) \simeq (\underline{B}, f)$, so that $(\underline{B}, f)$ is forced to satisfy also (2).
This is the usual verification that universal property define objects up to contractible ambiguity.
Notice first that $(\underline{B}, f)$ extends the angle $\mathbb{Z}[x^{\pm 1}] \leftarrow \mathbb{Z}[x] \to \underline{A}$ with push-out $(\underline{A}[a^{-1}], \phi)$, so there is a canonical comparison map $\hat{f} : \underline{A}[a^{-1}] \to \underline{B}$. Hence, pre-composition by f, ϕ, $\hat{f}$ induces a commutative triangle of representable functors:

$$\begin{tikzcd}
\mathrm{Map}(\underline{B}, -) \arrow[rr, "\hat{f}^*"] \arrow[dr, "f^*"'] & & \mathrm{Map}(\underline{A}[a^{-1}], -) \arrow[dl, hook, "\phi^*"] \\
& \mathrm{Map}(\underline{A}, -) &
\end{tikzcd}$$

Now, as we showed in the Claim, the point-wise essential image of ϕ^* is $\mathrm{Map}'(\underline{A}, \underline{R})$, so let $\mathrm{Map}'(\underline{A}, -) :$ Ani(CRing) $\to$ Spc denote $\mathrm{EssIm}(\phi^*) \in \mathcal{P}(\mathrm{Ani(CRing)}^{op})$ and choose a factorization for ϕ^* with respect to the factorization system (EffEpi, Mono) on the (large) topos $\mathcal{P}(\mathrm{Ani(CRing)}^{op})$ (see [24],5.2.8.16):

$$\phi^* : \mathrm{Map}(\underline{A}[a^{-1}]) \xrightarrow{\alpha} \mathrm{Map}'(\underline{A}, -) \overset{\beta}{\hookrightarrow} \mathrm{Map}(\underline{A}, -)$$

Then, we are left to show that $\alpha \circ f^* : \mathrm{Map}(\underline{B}, -) \to \mathrm{Map}'(\underline{A}, -)$ is an equivalence. By [20],2.2.2 it suffices to prove this point-wise for each $\underline{R} \in \mathrm{Ani(CRing)}$, which holds by assumption.

(2) $\implies$ (1) : Let $f : \underline{A} \to \underline{B}$ be a morphism in Ani(CRing) and pick up any $a \in \pi_0 \underline{A}$ such that $f(a) \in \underline{B}^\times$. For a given $\underline{R} \in \mathrm{Ani(CRing)}$, define $\mathrm{Map}'(\underline{A}, \underline{R}) := \mathrm{Map}(\underline{A}, \underline{R}) \times_{for\underline{R}} \underline{R}^\times$ to be the full subspace of $\mathrm{Map}(\underline{A}, \underline{R})$ spanned by those maps $h : \underline{A} \to \underline{R}$ such that $h(a) = a^*(h) \in \underline{R}^\times$.
Similarly, form also the pull-back $\mathrm{Map}'(\underline{B}, \underline{R}) := \mathrm{Map}(\underline{B}, \underline{R}) \times_{for\underline{R}} \underline{R}'$, namely as the full subspace of $\mathrm{Map}(\underline{B}, \underline{R})$ spanned by those maps $h : \underline{B} \to \underline{R}$ such that $h(fa) \in \underline{R}^\times$.
Consider the following diagram; by the universal property of pull-backs, the restriction of f^* to $\mathrm{Map}'(\underline{B}, \underline{R})$ gives a comparison map $\mathrm{Map}'(\underline{B}, \underline{R}) \to \mathrm{Map}'(\underline{A}, \underline{R})$:

$$\begin{tikzcd}
\mathrm{Map}'(\underline{B}, \underline{R}) \arrow[r, dotted] \arrow[d, hook] \arrow[rr, bend left] & \mathrm{Map}'(\underline{A}, \underline{R}) \arrow[r] \arrow[d, hook] & \underline{R}^\times \arrow[d, hook] \\
\mathrm{Map}(\underline{B}, \underline{R}) \arrow[r, "f^*"] & \mathrm{Map}(\underline{A}, \underline{R}) \arrow[r, "a^*"] & for\underline{R}
\end{tikzcd}$$

Let us record a useful observation: the pair $(\underline{B}, 1_{\underline{B}})$ satisfies condition (1), since by (2) it holds that $\underline{B}[f(a)^{-1}] \simeq \underline{B}$.
This implies in particular that the inclusion $\mathrm{Map}'(\underline{B}, \underline{R}) \subseteq_{f.f.} \mathrm{Map}(\underline{B}, \underline{R})$ is actually an equivalence.
Moreover, the previous observation amounts to the fact that the following diagram commutes and consists of cocartesian squares:

$$\begin{tikzcd}
\mathbb{Z}[x] \arrow[r, "a"] \arrow[d] & \underline{A} \arrow[r, "f"] \arrow[d, "f"] & \underline{B} \arrow[d, equal] \\
\mathbb{Z}[x^{\pm 1}] \arrow[r] & \underline{B} \arrow[r, equal] & \underline{B}
\end{tikzcd}$$

as a consequence, f^{op} is a monomorphism in $\mathrm{Ani(CRing)}^{op}$, so that f^* is a monomorphism as well. This, means that there is an equivalence $\mathrm{Map}(\underline{B}, \underline{R}) \simeq \mathrm{EssIm}(f^*)$, and the latter can be identified with $\mathrm{Map}'(\underline{B}, \underline{R})$. Therefore, we are left to show that the restriction of f^* induces an equivalence of spaces $\mathrm{Map}'(\underline{B}, \underline{R}) \simeq \mathrm{Map}'(\underline{A}, \underline{R})$; being the latter a monomorphism, it suffices to prove that it is surjective on π_0 (so that its fibres are all contractible).
To this end, consider again the previous rectangle of cocartesian squares. A point in $\mathrm{Map}'(\underline{A}, \underline{R})$ is equivalent to a map $h : \underline{A} \to \underline{R}$ s.t. $h(a) = a^*(h) \in \underline{R}^\times$, namely to a map h extending the co-angle $\mathbb{Z}[x^{\pm 1}] \leftarrow \mathbb{Z}[x] \to \underline{A}$. By the universal property of push-outs, h factors through $f : \underline{B} \to \underline{A}$, and hence through $\underline{B}[f(a)^{-1}]$ by considering the rightmost square. Thus, the factorization

$$h : \underline{A} \xrightarrow{f} \underline{B} \xrightarrow{\exists g} \underline{R}$$

yields some point $g \in \mathrm{Map}'(\underline{B}, \underline{R})$, i.e. such that $g(fa) \in \underline{R}^\times$, lying over $h \in \mathrm{Map}'(\underline{A}, \underline{R})$. This proves the surjectivity on π_0 and hence the statement. □

Let us include also the proof provided by Lurie in [22],4.1.18. Many ideas are the same, although in a different order. However, we choose to present both proofs because here we make a heavy use of the spectral sequence in [23],7.2.2.13, which somehow hides the universal property of localizations behind an extensive use of flatness. Namely, the latter property will allow us to reduce questions about the existence of certain factorizations to the static case, where they can be addressed by means of the universal property of static localizations.
In the latter sense, the following proof is an instance of the meta-principle asserting that flatness "animates" properties of static objects to higher homotopical degree.

Proof. *([22],4.1.18)* (2) $\implies$ (1) : Notice first that f^{op} in Ani(CRing)^{op} is a monomorphism.
Indeed, consider the cokernel pair (g, g') of f in Ani(CRing), namely the following cocartesian square of animated rings:

$$\begin{tikzcd} \underline{A} \arrow[r, "f"] \arrow[d, "f"] & \underline{B} \arrow[d, "g'"] \\ \underline{B} \arrow[r, "g"] & \underline{B}' \end{tikzcd}$$

For the flat map $f : \underline{A} \to \underline{B}$, [22],4.1.14 gives a spectral sequence

$$E_2^{p,q} : \mathrm{Tor}_p^{\pi_*\underline{A}}(\pi_*\underline{B}, \pi_*\underline{B})_q \implies \pi_{p+q}\underline{B}'$$

which degenerates at the second page and amounts to the following isomorphisms of abelian groups as in [23],7.2.2.13 (in the second page, only the vertical axis is non-trivial):

$$\mathrm{Tor}_0^{\pi_0\underline{A}}(\pi_0\underline{B}, \pi_n\underline{B}) \xrightarrow{\cong} \pi_n(\underline{B}')$$

In other words, for each $n \geq 0$, we have the sought isomorphisms in Ab:

$$\pi_n\underline{B}' \cong \pi_n\underline{B} \otimes_{\pi_0\underline{A}} \pi_0\underline{B} \cong_{(2)} \pi_n\underline{A} \otimes_{\pi_0\underline{A}} (\pi_0\underline{A})[a^{-1}] \otimes_{\pi_0\underline{A}} (\pi_0\underline{A})[a^{-1}] \cong \pi_n\underline{A} \otimes_{\pi_0\underline{A}} (\pi_0\underline{A})[a^{-1}] \cong_{(2)} \pi_n\underline{B}$$

Thus, both $g, g' : \underline{B} \to \underline{B}'$ are equivalences, i.e. the cofibres of f are either empty or contractible.
Therefore, pre-composition with f induces a monomorphism (i.e. a fully faithful functor) of mapping spaces $f^* : \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B}, \underline{R}) \hookrightarrow \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{A}, \underline{R})$ for each $\underline{R} \in$ Ani(CRing). Let $\mathrm{Map}'(\underline{A}, \underline{R})$ denote its essential image.
Since $f(a) \in (\pi_0\underline{B})^\times$ is invertible by assumption, $\mathrm{Map}'(\underline{A}, \underline{R})$ must be spanned by some maps $h : \underline{A} \to \underline{R}$ which carry a to an invertible $h(a) \in (\pi_0\underline{R})^\times$. We are left to show that our space contains all such maps, i.e. that each map $h : \underline{A} \to \underline{R}$ s.t. $h(a) \in (\pi_0\underline{R})^\times$ factors through f.
To this end, form the aside push-out of f and g and let's check that the induced map f' is an equivalence by showing that it induces isomorphisms in homotopy. Being f flat by (2), the spectral sequence from [22],4.1.14 again degenerates at the second page, so that one obtains the following chain of isomorphisms in Ab:

$$\begin{tikzcd} \underline{A} \arrow[r, "g"] \arrow[d, "f"] & \underline{R} \arrow[d, "f'"] \\ \underline{B} \arrow[r] & \underline{R}' \end{tikzcd}$$

$$\pi_n\underline{R}' \cong \mathrm{Tor}_0^{\pi_0\underline{A}}(\pi_0\underline{B}, \pi_n\underline{R}) \cong \pi_n\underline{R} \otimes_{\pi_0\underline{A}} \pi_0\underline{B} \cong_{(2)} \pi_n\underline{R} \otimes_{\pi_0\underline{A}} (\pi_0\underline{A})[a^{-1}] \cong \pi_n\underline{R}$$

where the last isomorphisms comes from the fact that $h(a)$ is already invertible in the graded commutative ring $\pi_*\underline{R}$.

Claim. *Given any animated ring $\underline{A} \in$ Ani(CRing) together with a connected component $a \in \pi_0\underline{A}$, there exists some animated ring $\underline{B}$ together with some map $f : \underline{A} \to \underline{B}$ in* Ani(CRing) *which satisfies both* (1) *and* (2).

Proof. Evaluation at the variable x induces an identification of static rings in CRing:

$$ev_x : \mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[x], \pi_0\underline{A}) \xrightarrow{\cong} \pi_0\underline{A} = \pi_0\mathrm{Map}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[x], \underline{A})$$
$$(\hat{a} : \mathbb{Z}[x] \to \pi_0\underline{A}) \longmapsto [a : \mathbb{Z}[x] \to \underline{A}]$$

so that we can think the connected component $a \in \pi_0\underline{A}$ as obtained by an assignment $\hat{a} : x \mapsto a$.
Define $\underline{B} := \underline{A} \otimes^L_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}]$ and $f : \underline{A} \to \underline{B}$ by forming the following co-cartesian diagram in Ani(CRing):

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[r, "a"] \arrow[d] & \underline{A} \arrow[d, "f"] \\ \mathbb{Z}[x, x^{-1}] \arrow[r] & \underline{B} \end{tikzcd}$$

Then, let us verify that f satisfies the second property. To this end, let us distinguish two cases:

- $n = 0$: π_0 is a left-adjoint, hence it commutes with the push-out of animated rings (given by $\otimes^L$): $\pi_0\underline{B} \simeq \pi_0\underline{A} \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}] = (\pi_0\underline{A})[a^{-1}]$, since by construction $\hat{a} : \mathbb{Z}[x] \to \pi_0\underline{A}$ acts as $x \mapsto a$ by evaluation at x.

- $n > 0$: $\mathbb{Z}[x, x^{-1}]$ is $\mathbb{Z}[x]$-flat, so again the spectral sequence of [22],4.1.14 degenerates at the second page, where only the vertical axis is non-trivial. Hence, we obtain isomorphisms in Ab for each $n \geq 0$:

$$\pi_n \underline{B} \cong \mathrm{Tor}_0^{\mathbb{Z}[x]}(\pi_n \underline{A}, \mathbb{Z}[x, x^{-1}]) \cong \pi_n \underline{A} \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}]$$

  Now, by 3.2.1.4, we have an isomorphism $\pi_n \underline{A} \cong \pi_n \underline{A} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{A}$, so that we can continue the chain by

$$\pi_n \underline{A} \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}] \cong (\pi_n \underline{A} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{A}) \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}] \cong \pi_n \underline{A} \otimes_{\pi_0 \underline{A}} (\pi_0 \underline{A})[a^{-1}]$$

Finally, recall that we already proved (2) $\implies$ (1). ■

(2) $\implies$ (2) : Let $f : \underline{A} \to \underline{B}$ satisfy (1); by the Claim there is some map $f' : \underline{A} \to \underline{B}$ which satisfies both (1) and (2). In particular, both f and f' satisfy the universal property of localizations, so that they must be equivalent in Ani(CRing), which implies that f must satisfy (2) as well. □

As a corollary, we observe that localizations are well-behaved under change of base-ring.

Corollary 3.4.1.4. *(Base-change of localizations, [22],4.1.20) With reference to the previous notation, for the following commutative square in* Ani(CRing)

$$\begin{tikzcd} \underline{A} \arrow[r, "f"] \arrow[d, "g"] & \underline{A}[a^{-1}] \arrow[d] \\ \underline{A}' \arrow[r, "f'"] & \underline{B}' \end{tikzcd}$$

TFAE:

1. *f' exhibits $\underline{B}' \simeq \underline{A}'[g(a)^{-1}]$ as the localization of $\underline{A}'$ at $g(a)$;*
2. *the square above is co-cartesian, i.e. $\underline{B}' \simeq \underline{A}[a^{-1}] \otimes^L_{\underline{A}} \underline{A}'$.*

Proof. Consider the following commutative diagram. The depicted equivalences are given by the previous two characterizations of localizations. Then, the equivalence of the two statements is established as follows: (1) holds iff the dotted arrow exists and is an equivalence; and (2) amounts to the top horizontal arrow being an equivalence.

$$\begin{tikzcd} \mathrm{Map}(\underline{A}[a^{-1}], \underline{R}) \times_{\mathrm{Map}(\underline{A}, \underline{R})} \mathrm{Map}(\underline{A}', \underline{R}) \arrow[rr, "(2)"'] \arrow[d, "\simeq"', "f^*"] & & \mathrm{Map}(\underline{B}', \underline{R}) \arrow[d, "(f')^*"] \arrow[dl, dotted, "(1)"] \\ \mathrm{Map}'(\underline{A}, \underline{R}) \times_{\mathrm{Map}(\underline{A}, \underline{R})} \mathrm{Map}(\underline{A}', \underline{R}) \arrow[r, "\simeq"] & \mathrm{Map}'(\underline{A}', \underline{R}) \arrow[r, hook, "f.f."] & \mathrm{Map}(\underline{A}', \underline{R}) \end{tikzcd}$$

□

Moreover, let us record a generalization of the previous definition to the case of modules.

Definition 3.4.1.5. *(Localization of modules)* For an animated ring $\underline{A} \in$ Ani(CRing), let $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ be an $\underline{A}$-module. Define the **localization of $\underline{M}$ at** $a \in \pi_0 \underline{A}$ by $\underline{M}[x^{-1}] := \underline{M} \otimes^L_{\underline{A}} \underline{A}[x^{-1}]$.

Remark. By the previous Corollary, localization of modules is compatible with the extension of scalars adjunction of 3.2.5.11.

3.5 (Homotopy) Quotient Rings

In this section, we will develop the algebraic formalism supplying for a local description of "quasi-smooth closed immersions". We aim at generalizing the notion of a "regular sequence" in a "homotopy-coherent way".
We refer to the subsection on "Regular Immersions" for a comparison with the classical terminology. As it is to be expected, the homological viewpoint admits more fruitful generalizations to the ∞-world, and in particular when dealing with DAG. One reason can be the fact that, in general, the Factor Theorem for rings

does not generalize to the ∞-world, so that ring quotients are not well-behaved, e.g. closed immersions need *not* be monomorphisms.
Therefore, we will enhance the notion of Koszul regularity. More pictorially, as commented by Khan and Rydh in [17],2.2, the basic idea here is to regard the Koszul complex $\mathrm{Kosz}_{\underline{A}}(f_1,\dots,f_n)$ as the "ring of functions" on a "quasi-smooth" (derived) subscheme of $\mathrm{Spec}(\underline{A})$. Let us define more precisely our "rings of functions" and explore the connection with Koszul complexes.
On this streamline, we will see in the section "Quasi-Smooth Closed Immersions" that, in some sense, the notion of "quotient rings" works as a "system of coordinates" for a scheme. Noteworthy is how the good-behaviour of the latter will be controlled by the cotangent complex / sheaf, so an algebraic object.

For future reference, let us give a name to maps arising by the following useful observation.

Definition 3.5.0.1. *(Coordinate maps)* Let $\underline{A} \in \mathrm{Ani(CRing)}$ be an animated ring, and consider a sequence $(f_1,\dots,f_n) \subseteq \pi_0\underline{A}$ of connected components. Under the identification

$$\begin{array}{ccccccc} ev_{(f_1,\dots,f_n)} : \mathrm{Hom}_{ho\mathrm{Ani(CRing)}}(\mathbb{Z}[t_i]_{i=1}^n, \underline{A}) & \xrightarrow{\cong} & \pi_0\mathrm{Map}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[t_i]_{i=1}^n, \underline{A}) = (\pi_0\underline{A})^n & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[t_i]_{i=1}^n, \pi_0\underline{A}) \\ [t_i \mapsto f_i] & \longmapsto & (f_1,\dots,f_n) & \longmapsto & (t_i \mapsto f_i) \end{array}$$

the sequence $(f_1,\dots,f_n)$ corresponds to an essentially unique choice of a map $\mathbb{Z}[t_1,\dots,t_n] \to \underline{A}$.
Call **coordinate maps** any such lift $(t_i \mapsto f_i) : \mathbb{Z}[t_1,\dots,t_n] \to \underline{A}$ which recovers the sequence $(f_1,\dots,f_n)$ at the level of connected components. Write simply the tuple $(f_1,\dots,f_n)$ for short.

Remark. In particular, the very definition of an animated ring $\underline{A} \in \mathrm{Ani(CRing)} \simeq \mathrm{Fun}^\times(\mathrm{Poly},\mathrm{Spc})$ implies that $\mathrm{Map}(\mathbb{Z}[t_1,\dots,t_n],\underline{A}) \simeq for\underline{A}^n$: at the level of objects, then, evaluation at the n-tuple of indeterminates characterizes maps $\mathbb{Z}[t_1,\dots,t_n] \to \underline{A}$ by n-tuples of coordinate maps (whence the terminology).

Definition 3.5.0.2. *(Quotient ring, [17],2.3.1)* Let $\underline{A} \in \mathrm{Ani(CRing)}$ be an animated ring, and consider a sequence $(f_1,\dots,f_n) \subseteq \pi_0\underline{A}$ of connected components. Define the **quotient of $\underline{A}$ by** $(f_1,\dots,f_n)$ as the base-change:

$$\underline{A} \,/\!\!/\, (f_1,\dots,f_n) := \underline{A} \otimes^L_{\mathbb{Z}[t_1,\dots,t_n]} \mathbb{Z}[t_1,\dots,t_n]/(t_1,\dots,t_n)$$

of a choice of coordinate maps $t_1,\dots,t_n \mapsto f_1,\dots,f_n$ (see 3.5.0.1) along the canonical quotient $\mathbb{Z}[t_1,\dots,t_n] \to \mathbb{Z}[t_1,\dots,t_n]/(t_1,\dots,t_n) \simeq \mathbb{Z}$:

$$\begin{tikzcd} \mathbb{Z}[t_1,\dots,t_n] \arrow[r] \arrow[d, "t_1,\dots,t_n \mapsto f_1,\dots,f_n"'] & \mathbb{Z}[t_1,\dots,t_n]/(t_1,\dots,t_n) \arrow[d] \\ \underline{A} \arrow[r] & \underline{A} \,/\!\!/\, (f_1,\dots,f_n) \end{tikzcd}$$

Remark. More geometrically, $\mathrm{Spec}(\underline{A} \,/\!\!/\, (f_1,\dots,f_n)) \simeq \mathrm{Spec}(\underline{A}) \times_{\mathbb{A}^n} \{0\} \in \mathrm{Sch}$ (see 4.3.2.6), which will correspond to a "homotopy coherent" choice of a "local frame" of coordinate functions $(f_1,\dots,f_n)$ together with an origin $0 \in \mathrm{Spec}(\underline{A})$.

Lemma 3.5.0.3. *(Properties of quotient rings, [17],2.3.1-2-3) Let $\underline{A} \in \mathrm{Ani(CRing)}$ be an animated ring, and consider a coordinate maps $(f_1,\dots,f_n)$ (see 3.5.0.1). Then, the quotient $\underline{A} \,/\!\!/\, (f_1,\dots,f_n)$ satisfies the following properties:*

1. $\pi_0(\underline{A} \,/\!\!/\, (f_1,\dots,f_n)) \simeq \pi_0(\underline{A})/(f_1,\dots,f_n)$;

2. *($[\underline{A}/*]$ quotient $*$-torsor in $\mathrm{Mod}_{\underline{A}}$): Forgetting the $\underline{A}$-algebra structure via 3.7.1.2 yields the underlying $\underline{A}$-module:*

$$\underline{A} \,/\!\!/\, (f_1,\dots,f_n) \simeq \otimes^L \mathrm{Cofib}(f_i : \underline{A} \to \underline{A}) \in \mathrm{Mod}_{\underline{A}}$$

3. *(Compatibility with ordinary regular quotients): If $\underline{A} := R \in \mathrm{CRing}$ is a static ring, then the canonical map $R \,/\!\!/\, (f_1,\dots,f_n) \to R/(f_1,\dots,f_n)$ retrieves the classical quotient iff $(f_1,\dots,f_n) \subseteq R$ is regular.*

Proof. (1) : It is clear, because the left-adjoint 0-truncation functor π_0 (see A.5.0.7) preserves push-outs of animated rings.

(2) : One could prove the statement directly in the model of simplicial rings and modules: take a sifted resolution of $\underline{A}$ by $\mathbb{Z}[t_1, \ldots, t_n]$-polynomial algebras, forget the algebra structure (*for* which commutes with sifted colimits by the adjunction 3.7.1.2, because $\mathrm{Mod}_{\mathbb{Z}[t_i]_{i=1}^n}$ is cpt+proj-generated and Sym^* preserves cpt+proj's) and reducing - via the Dold-Kan correspondence [23],1.2.4.1 - to the properties of Koszul complexes, as in 2.1.0.3,iii. Notice that the connected components of the coordinate maps $[t_i \mapsto f_i]$ arise functorially from the $\mathbb{Z}[t_1, \ldots, t_n]$-algebra structure of the diagram resolving $\underline{A}$.

However, it is instructive to argue by induction on the length of the sequence $(f_1, \ldots, f_n)$. The induction starts establishing an equivalences of $\underline{A}$-modules between $\mathit{for}(\underline{A} /\!\!/ (f))$ and the quotient of $\underline{A}$ by the group-action induced by the multiplication map $f : \underline{A} \to \underline{A}$.

START. $\underline{A} /\!\!/ (f) \simeq \mathrm{Cofib}(f : \underline{A} \to \underline{A}) \in \mathrm{Mod}_{\underline{A}}$.

Proof. Define the multiplication map $f := (\cdot f) : \underline{A} \to \underline{A}$ by the base-change $\underline{A} \to \underline{A} \otimes^L_{\mathbb{Z}[t]} \mathbb{Z}[t] \simeq \underline{A}$ of multiplication by t. Since $\otimes^L_{\mathbb{Z}[t]}$ is co-continuous separately in each variable and $\mathrm{Cofib}(t : \mathbb{Z}[t] \to \mathbb{Z}[t]) \simeq \mathbb{Z}[t]/(t)$ in $\mathrm{Mod}_{\mathbb{Z}[t]}$, we conclude that also $\mathrm{Cofib}(f : \underline{A} \to \underline{A}) \simeq \underline{A} \otimes^L_{\mathbb{Z}[t]} \mathrm{Cofib}(t : \mathbb{Z}[t] \to \mathbb{Z}[t]) \simeq \underline{A} /\!\!/ (f)$ in $\mathrm{Mod}_{\underline{A}}$. ■

INDUCTION STEP. *Let $(f_1, \ldots, f_n) \subseteq \underline{A}$ be a n-sequence of coordinate maps, and suppose that the $\underline{A}$-module lying under the $(n-1)$-quotient is $\underline{A} /\!\!/ (f_1, \ldots, f_{n-1}) \simeq \otimes^L_{i \leq n-1} \mathrm{Cofib}(f_i : \underline{A} \to \underline{A})$. Then, also $\underline{A} /\!\!/ (f_1, \ldots, f_n) \simeq \otimes^L_i \mathrm{Cofib}(f_i : \underline{A} \to \underline{A})$ in $\mathrm{Mod}_{\underline{A}}$.*

Proof. The following pasting of cocartesian diagrams in Ani(CRing) proves that "passing to quotients" is an associative operation:

$$\begin{tikzcd}
 & \mathbb{Z}[t_n] \arrow[r] \arrow[d, "\simeq"] & \frac{\mathbb{Z}[t_n]}{(t_n)} \arrow[d, "\simeq"] \\
 & \left(\frac{\mathbb{Z}[t_1,\ldots,t_{n-1}]}{(t_1,\ldots,t_{n-1})}\right)[t_n] \arrow[r] \arrow[d, "\simeq"] & \left(\frac{\mathbb{Z}[t_1,\ldots,t_{n-1}]}{(t_1,\ldots,t_{n-1})}\right)\frac{\mathbb{Z}[t_n]}{[t_n]} \arrow[d, "\simeq"] \\
\mathbb{Z}[t_n][t_1, \ldots, t_n] \arrow[r] \arrow[d, "(t_i \mapsto f_i)"] & \frac{\mathbb{Z}[t_n][t_1,\ldots,t_{n-1}]}{(t_1,\ldots,t_{n-1})} \arrow[r] \arrow[d, "(t_n \mapsto f_n)"] & \frac{\mathbb{Z}[t_1,\ldots,t_n]}{(t_1,\ldots,t_n)} \arrow[d] \\
\underline{A} \arrow[r] & \underline{A} /\!\!/ (f_1, \ldots, f_{n-1}) \arrow[r] & \underline{A} /\!\!/ (f_1, \ldots, f_n)
\end{tikzcd}$$

where the bottom left square is cocartesian, because we can compute the push-out of the coangle by pre-composition as in the following pasting of cocartesian squares in Ani(CRing):

$$\begin{tikzcd}
\mathbb{Z}[t_1, \ldots, t_{n-1}] \arrow[r] \arrow[d] \arrow[dd, bend right, "(t_i \mapsto f_i)"'] & \frac{\mathbb{Z}[t_1,\ldots,t_{n-1}]}{(t_1,\ldots,t_{n-1})} \arrow[d] \\
\mathbb{Z}[t_1, \ldots, t_n] \arrow[r] \arrow[d] & \frac{\mathbb{Z}[t_1,\ldots,t_n]}{(t_1,\ldots,t_{n-1})} \arrow[d] \\
\underline{A} \arrow[r] & \mathrm{PO}
\end{tikzcd}$$

Finally, we conclude by inspection of the bottom row in the big diagram: $\underline{A} /\!\!/ (f_1, \ldots, f_n) \simeq \big(\underline{A} /\!\!/ (f_1, \ldots, f_{n-1})\big) /\!\!/ (f_n)$, so by forgetting the $\underline{A}$-algebra structure and by applying both the induction premise and start, one obtains that:

$$
\begin{aligned}
\mathit{for}\big(\underline{A} \,/\!\!/\, (f_1,\dots,f_n)\big) &\simeq \big(\mathrm{Cofib}(f_n : \underline{A} \,/\!\!/\, (f_1,\dots,f_{n-1}) \longrightarrow \underline{A} \,/\!\!/\, (f_1,\dots,f_{n-1}))\big) \\
&\simeq \mathit{for}\big(\underline{A} \,/\!\!/\, (f_1,\dots,f_{n-1})\big) \otimes^L_{\underline{A}} \mathrm{Cofib}(f_n : \underline{A} \to \underline{A}) \\
&\simeq \otimes^L_i \mathrm{Cofib}(f_i : \underline{A} \to \underline{A})
\end{aligned}
$$

since we recall once more that $\otimes^L_{\underline{A}}$ is co-continuous separately in each variable. ■

(3) : For a static ring $R \in \mathrm{CRing}$ we have a chain of equivalences of graded rings:

$$
\begin{aligned}
\pi_*\big(R \,/\!\!/\, (f_1,\dots,f_n)\big) &= \pi_*\big(R \otimes^L_{\mathbb{Z}[t_1,\dots,t_n]} \mathbb{Z}[t_1,\dots,t_n]/(t_1,\dots,t_n)\big) \\
&\cong_{(i)} \mathrm{Tor}_*^{\mathbb{Z}[t_1,\dots,t_n]}\big(R, \mathbb{Z}[t_1,\dots,t_n]/(t_1,\dots,t_n)\big) \\
&\simeq_{(ii)} \mathrm{H}_*\big(\mathrm{Kosz}_R(f_1,\dots,f_n)\big) \\
&\cong_{(iii)} R/(f_1,\dots,f_n)
\end{aligned}
$$

where the stated equivalences are obtained as follows: (i) : since all rings are static; (ii) : 2.1.0.3,iii, (iii) : 2.1.0.3,i. □

Example 3.5.0.4. *(Non-regular quotients need not be static)* For a static ring $R \in \mathrm{CRing}$, $R \,/\!\!/\, (0) \in \mathrm{Ani(CRing)}$ has underlying chain complex given by the square-zero extension $R \oplus R[1]$ (with zero differential). In particular, even such a simple example gives a non-static animated ring, since $\pi_0(R/\!\!/(0)) \cong R \cong \pi_1(R/\!\!/(0))$. Indeed, $\pi_*(R \,/\!\!/\, (0)) \cong \mathrm{H}_*\mathrm{Kosz}_R(0) \cong \mathrm{H}_*(0 : R \to R)$.
In more geometric terms: being surjective on connected components, the map $\mathrm{Spec}(R \,/\!\!/\, (0)) \twoheadrightarrow \mathrm{Spec}(R)$ is a closed immersion, but it is *not* a monomorphism: it is the identity at the level of underlying classical schemes, but carries homotopical information, namely the identification with the origin expressing the "triviality" of the coordinate function 0 (see 3.5.0.5).

The next Lemma gives some more mathematical content to the notion of "triviality" of coordinate maps.

Lemma 3.5.0.5. *(Universal property of quotients, [17],2.3.5) Let $\underline{A} \in \mathrm{Ani(CRing)}$ be an animated ring and consider a sequence $f_1,\dots,f_n \in \underline{A}$. Then, for each $(\phi : \underline{A} \to \underline{B}) \in \mathrm{CAlg}^{\Delta}_{\underline{A}}$ there is an equivalence of spaces:*

$$
\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}\big(\underline{A} \,/\!\!/\, (f_1,\dots,f_n), \underline{B}\big) \simeq \prod_{i=1}^{n} \mathrm{Map}_{\mathit{for}\underline{B}}(f_i', 0)
$$

where $f_i' := \phi(f_i)$. In other words, for any $\underline{A}$-algebra $\phi : \underline{A} \to \underline{B}$, there is a natural equivalence between the space of $\underline{A}$-algebra morphisms $\underline{A} \,/\!\!/\, (f_1,\dots,f_n) \to \underline{B}$ and that one of paths $\{f_i' := \phi(f_i) \simeq 0\}_{i=1}^n$ in the underlying space $\mathit{for}\underline{B}$.

Proof. Let $\underline{B} \in \mathrm{CAlg}^{\Delta}_{\underline{A}}$ be an animated $\underline{A}$-algebra; then, the very definition yields an equivalence in Spc:

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}\big(\underline{A} \,/\!\!/\, (f_1,\dots,f_n), \underline{B}\big) &\simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}(\underline{A}, \underline{B}) \times_{\mathrm{Map}(\mathbb{Z}[t_1,\dots,t_n], B)} \mathrm{Map}_{\mathrm{Ani(CRing)}}\big(\mathbb{Z}, \underline{B}\big) \\
&\simeq_{(i)} \mathrm{Path}_{\mathrm{Map}(\mathbb{Z}[t_1,\dots,t_n], \underline{B})}\big((f_1',\dots,f_n'), (0,\dots,0)\big) \\
&\simeq_{(ii)} \mathrm{Path}_{\mathit{for}\underline{B}^n}\big((f_1',\dots,f_n'), (0,\dots,0)\big) \\
&\simeq_{(iii)} \prod_{i=1}^{n} \mathrm{Path}_{\mathit{for}\underline{B}}(f_i', 0)
\end{aligned}
$$

where the stated equivalences can be deduced as follows:

- (i) : for an $\underline{A}$-algebra $\underline{B}$, both mapping spaces $\mathrm{Map}(\underline{A}, \underline{B})$ and $\mathrm{Map}(\mathbb{Z}, \underline{B})$ are contractible and spanned by the structure maps, so that the angle corresponds to an identifications of the choices of coordinate maps:

$$
(f_1',\dots,f_n') : * \simeq \underline{B}(\underline{A}) \to \mathrm{Map}(\mathbb{Z}[t_1,\dots,t_n], \underline{B}) \leftarrow \underline{B}(\mathbb{Z}) \simeq * : (0,\dots,0)
$$

- (ii) : From the very definitions: $\mathit{for}\underline{B}^n \simeq \mathrm{Map}(\mathbb{Z}[x], \underline{B})^n \simeq \mathrm{Map}(\mathbb{Z}[t_1,\dots,t_n], \underline{B})$;

- (iii) : *for* is a right-adjoint (see 3.7.1.2), and pull-backs commute with products. □

For future reference, let us also include a useful computation. Morally, one should be able to generalize it, so as to show that quotients behave similarly to ordinary successive quotients by principal ideals. However, the author has no full proof of it, as of yet.

Lemma 3.5.0.6. *(Quotient by principal ideals, [17],3.1.5) Consider an animated ring* $\underline{A} \in \mathrm{Ani(CRing)}$ *and let* $f \in \underline{A}$ *be a coordinate map (see 3.5.0.1). Then, for each* $g \in \underline{A}$*, there is a canonical equivalence*

$$\mathrm{Map}_{for\underline{A}/\!\!/(f)}(g, 0) \simeq \mathrm{Fib}_g(f : \underline{A} \to \underline{A})$$

where $\mathrm{Fib}_g(f : \underline{A} \to \underline{A})$ *is the space of pairs* $(a, \alpha : fa \simeq g)$ *in* $\underline{A}$*.*

Proof. By 3.5.0.3,ii, at the level of underlying $\underline{A}$-modules $\underline{A} /\!\!/ (f) \simeq \mathrm{Cofib}(f : \underline{A} \to \underline{A})$ in $\mathrm{Mod}_{\underline{A}}$.
As in 3.4.0.1, cofibre sequences in a pre-stable category (such as $\mathrm{Mod}_{\underline{A}}$) are exact, so that we obtain equivalences - at the level of underlying spaces - between the horizontal fibres in the following cartesian square:

$$\begin{tikzcd} for\underline{A} \arrow[r, "f"] \arrow[d] & for\underline{A} \arrow[d] \\ \{0\} \arrow[r] & for(\underline{A} /\!\!/ (f)) \end{tikzcd}$$

In particular, for any $g \in \underline{A}$ whose image in the quotient vanishes (i.e. is homotopy to 0), this gives the sought equivalence $\mathrm{Fib}_g(f : for\underline{A} \to for\underline{A}) \simeq \mathrm{Path}_{\underline{A}/\!\!/(f)}(g, 0) \simeq \mathrm{Map}_{for(\underline{A}/\!\!/(f))}(g, 0)$. □

Remark. Consider $\underline{A}$-algebra map $\phi : \underline{A} \to \underline{A} /\!\!/ (f)$ and let $g := \phi(f)$; the universal property of quotients 3.5.0.5 induces an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}(\underline{A} /\!\!/ (f)\,, \underline{A} /\!\!/ (f)) \simeq \mathrm{Map}_{for\underline{A}/\!\!/(f)}(g, 0) \simeq \mathrm{Fib}_g(f : \underline{A} \to \underline{A})$$

which identifies the space of endomorphisms of $\underline{A} /\!\!/ (f)$ with the space of pairs $(a, \alpha : fa \simeq g)$ in $\underline{A}$.

3.6 Locally free modules

In this section we expand on the notion of "finitely generated projective" animated modules, for which we privilege the terminology "being locally free of finite rank". Translated into the language of DAG, this will allow us to develop the theory of the Picard group of a (derived) scheme.
We will begin by a quick review of (almost) perfect animated modules as presented in [23],7.4.2 and then move to the bulk of the work, for which our main reference is [26],2.9.
For the sake of readability, we will have to freely adopt the language of stable ∞-categories. We refer the unexperienced reader to e.g. [21].

3.6.1 (Almost) Perfect Modules

In this subsection we introduce the notion of (almost) perfect animated modules. This is meant as a translation into our language of the classical theory of perfect modules over a static ring. The latter has been widely studied, since it forms the subcategory of compact objects in the derived category of the given ring. Such a perspective also motivates the choices we make in our exposition.
Intuition can be developed thanks to Lurie's observation right below [23],7.4.2.7: for $\underline{A} \in \mathrm{Ani(CRing)}$ an $\underline{A}$-module is perfect if it can be built from finitely many copies of $\underline{A}$ by means of shifts, extensions and retractions.
However, to quote again Lurie, such a notion is too rigid for many practical applications, so that we will consider a weakening of it, thus allowing infinitely many copies of $\underline{A}$ as "elementary bricks". This will give rise to the class of almost perfect $\underline{A}$-modules.

Definition 3.6.1.1. *(Perfect module, [23],7.4.2.1-2)* For an animated ring $\underline{A} \in \mathrm{Ani(CRing)}$, an $\underline{A}$-module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ is called **perfect** iff it is a compact object in $\mathrm{Mod}_{\underline{A}}$. Let $\mathrm{Perf}(\underline{A}) := (\mathrm{Mod}_{\underline{A}})^{\mathrm{fp}}$ denote the full subcategory spanned by perfect $\underline{A}$-modules.

Lemma 3.6.1.2. *(*$\mathrm{Mod}_{\underline{A}}$ *is compactly generated, [23],7.2.4.2) For an animated ring* $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$*, the category* $\mathrm{Mod}_{\underline{A}}$ *is compactly-generated, namely* $\mathrm{Mod}_{\underline{A}} \simeq \mathrm{Ind}(\mathrm{Perf}(\underline{A}))$*.*

Proof. Unwinding the definitions, we need to show that $\mathrm{Ind}(\mathrm{Perf}(\underline{A})) \simeq \mathrm{Mod}_{\underline{A}}$. To this end, we will apply the Ind-versions of A.3.0.2 and A.4.0.3, namely [24],5.3.5.10 and [24],5.3.5.11. Namely, the inclusion of the cpt-objects $\mathrm{Perf}(\underline{A}) \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}$ admits a left-derived functor $F : \mathrm{Ind}(\mathrm{Perf}(\underline{A})) \to \mathrm{Mod}_{\underline{A}}$, since the target category has filtered colimits ($\mathrm{Mod}_{\underline{A}}$ is presentable). And the latter is an equivalence iff its restriction to the Yoneda embedding is fully faithful and has compact and lim-dense essential image in the target $\mathrm{Mod}_{\underline{A}}$. The first two conditions are satisfied by assumption, so we are left to prove that F is essentially surjective.
Recall that $\mathrm{Mod}_{\underline{A}} \simeq \mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})$, so by A.3.0.1 we can express each $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ as the geometric realization of a simplicial $\underline{A}$-module which is degree-wise a coproduct of copies of cpt+proj's in $\mathrm{Mod}_{\underline{A}}$, i.e. a coproduct of copies of $\underline{A}$.
Thus, since $\underline{A} \in \mathrm{FFree}_{\underline{A}} \simeq \mathrm{Mod}_{\underline{A}}^{\mathrm{sfp}} \subseteq_{f.f.} \mathrm{Perf}(\underline{A})$, we are done if we show that the essential image of F is closed under sifted colimits in $\mathrm{Mod}_{\underline{A}}$. But this holds (as in A.3.0.2) since the inclusion $\mathrm{Perf}(\underline{A}) \subseteq \mathrm{Mod}_{\underline{A}}$ is right-exact, i.e. it preserves all finite colimits in $\mathrm{Perf}(\underline{A})$. □

Proposition 3.6.1.3. *(Properties of* $\mathrm{Perf}(\underline{A})$*, [23],7.4.2.5) Let* $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ *be an animated ring. Then, the following properties hold*

1. $\mathrm{Perf}(\underline{A})$ *is the smallest pre-stable (see 3.4.0.1) full subcategory of* $\mathrm{Mod}_{\underline{A}}$ *which contains* $\underline{A}$ *and is closed under retracts.*
2. *For any* $\underline{M} \in \mathrm{Perf}(\underline{A})$*,* $\pi_k\underline{M} \in \mathrm{Mod}(\pi_0\underline{A})$ *is finitely presented provided that* $\pi_m\underline{M} \cong 0$ *for each* $m < k$*.*

Proof. (1) : $\mathrm{Mod}_{\underline{A}}^{\mathrm{fp}}$ is idempotent complete by a similar argument to A.4.0.5,ii and clearly $\underline{A} \in \mathrm{Mod}_{\underline{A}}^{\mathrm{fp}}$.
Moreover, notice that also the (pre)stability properties of $\mathrm{Perf}(\underline{A})$ and $\mathrm{Mod}_{\underline{A}}$ agree: by 3.6.1.2, the Yoneda embedding $\mathrm{Perf}(\underline{A}) \subseteq_{f.f.} \mathrm{Ind}(\mathrm{Perf}_{\underline{A}}) \simeq \mathrm{Mod}_{\underline{A}}$ allows us to compute limits and colimits into $\mathrm{Mod}_{\underline{A}}$ and, by inspecting the corresponding long exact homotopy sequences, $\mathrm{Mod}_{\underline{A}}^{\mathrm{fp}}$ admits the (co)fibre sequences which exist in $\mathrm{Mod}_{\underline{A}}$.
Finally, let us observe that $\mathrm{FFree}_{\underline{A}}$ constitutes a set of compact generators of $\mathrm{Mod}_{\underline{A}}$ (see the Examples in Appendix A: the free-adjunction carries a set FinSet of generators for $\mathrm{Set}^{\mathrm{fp}}$ to a set of generators $\mathrm{FFree}_{\underline{A}}$ for $\mathrm{Mod}_{\underline{A}}^{\mathrm{fp}}$), so we conclude again by a similar argument to A.4.0.5,ii and bt the closure properties of stable ∞-categories. ■

(2) : Notice first that we can assume wlog $k = 0$. Indeed, by assumption there exists some $\underline{N} \in \mathrm{Mod}_{\underline{A}}$ s.t. $\underline{M} \simeq \Sigma^k\underline{N}$ is a k-suspension of $\underline{M}$; then, the adjunction $\Sigma \vdash \Omega$ yields $\pi_k(\underline{M}) \simeq \pi_0(\Omega^k\underline{M}) \simeq \pi_0(\underline{N})$.
Now, observe that in the truncation adjunction (see A.5.0.7):

$$\pi_0 = \tau_{\leq 0} : \mathrm{Mod}_{\underline{A}} \rightleftarrows \mathrm{Mod}(\pi_0\underline{A}) : \subseteq$$

$\mathrm{Mod}_{\underline{A}}$ is compactly generated (by 3.6.1.2) and the right-adjoint preserves filtered colimits, so that the left one must preserve compact objects (and the two conditions on the adjuncts are actually equivalent, same proof as in the first part of 3.2.2.3, Claim 4).
Thus, since $\mathrm{Mod}(\pi_0\underline{A})^{\mathrm{fp}}$ is the full subcategory of static $\pi_0\underline{A}$-modules spanned by those that are finitely presented, we conclude that $\pi_0(\underline{M}) \in \mathrm{Mod}(\pi_0\underline{A})^{\mathrm{fp}}$ must be finitely presented, as wished. ■ □

As already mentioned, being perfect is a very strong condition: over a static ring $R \in \mathrm{CRing}$, a module $\underline{M} \in \mathrm{Mod}_R$ is perfect iff its Postnikov tower exhibits a bounded resolution of $\pi_0\underline{M}$ by finitely generated free modules; hence, it forces $\pi_0\underline{M}$ to be strongly finitely presented of finite homological dimension, and in general - i.e. unless the base ring R be regular - the latter notion is much stronger than being just finitely generated. For such a reason, we will now present a weakening of the latter, thus allowing perfect modules to have also unbounded resolutions by finitely many copies of $\underline{R}$. To this end, let us introduce the notion of an almost compact object in a compactly-generated ∞-category.

Definition 3.6.1.4. *(Almost compact, [23],7.2.4.8)* Let $C \in \mathrm{Cat}_\infty$ be a compactly-generated ∞-category. We say that an object $x \in C$ is **almost compact** iff, for each $n \geq 0$, its n-truncation $\tau_{\leq n}x$ is compact in $(\mathrm{Mod}_{\underline{A}})_{\leq n}$.

Remark. Unless C consists only of truncated objects, "almost compactness" is a priori weaker than "compactness", because the commutativity with filtered limits is tested only against diagrams of (arbitrarily) truncated objects. In particular, all "compact" objects are also "almost compact".

Definition 3.6.1.5. *(Almost perfect, [23],7.2.4.10)* Let $\underline{A} \in \mathrm{Ani(CRing)}$. An $\underline{A}$-module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ is said **almost perfect** iff it is an almost compact $\underline{A}$-module.
Let $\mathrm{APerf}(\underline{A}) \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}$ denote the full subcategory spanned by almost perfect $\underline{A}$-modules.

Proposition 3.6.1.6. *(Properties of* APerf*, [23],7.2.4.11) Let $\underline{A} \in \mathrm{Ani(CRing)}$ be an animated ring. Then,*

1. $\mathrm{APerf}(\underline{A}) \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}$ *is a pre-stable full subcategory closed under retracts and the formation (in* $\mathrm{Mod}_{\underline{A}}$*) of the geometric realizations of its simplicial objects in* $s\mathrm{APerf}(\underline{A})$.
2. $\mathrm{Perf}(\underline{A}) \subseteq_{f.f.} \mathrm{APerf}(\underline{A})$.
3. *Every almost perfect $\underline{A}$-module can be obtained as the geometric realization of a degree-wise finite free simplicial $\underline{A}$-module $\underline{X}_\bullet \in s\mathrm{Mod}_{\underline{A}}$, i.e. such that $\underline{X}_n \simeq \underline{A}^{(n)}$ for each $n \geq 0$.*

Proof. (1) : We will only prove that $\mathrm{APerf}(\underline{A})$ is closed under geometric realizations in $\mathrm{Mod}_{\underline{A}}$ of its simplicial objects.
Being $\mathrm{Mod}_{\underline{A}}$ cpt-generated (by 3.6.1.2), the left-adjunct $\tau_{\leq n}$ preserves compact objects, because the inclusion $(\mathrm{Mod}_{\underline{A}})_{\leq n} \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}$ preserves filtered colimits. So, we are left to check that $\mathrm{Perf}(\underline{A})$ is closed under geometric realizations (in $\mathrm{Mod}_{\underline{A}}$) of its simplicial objects; but this holds, since the right-exactness of the inclusion $\mathrm{Perf}(\underline{A}) \subseteq_{f.f.} \mathrm{Mod}_{\underline{A}}$ implies the cocontinuity of its left-derived functor $\mathrm{Ind}(\mathrm{Perf}(\underline{A})) \hookrightarrow \mathrm{Mod}_{\underline{A}}$.

(3) : It is a non-trivial consequence of the Dold-Kan correspondence [23],1.2.4.1; the proof is omitted. □

Similarly to A.4.0.5, the previous Proposition allows us to characterize $\mathrm{APerf}(\underline{A})$ as follows.

Proposition 3.6.1.7. *(Universal property* $\mathrm{APerf}(\underline{A})$*, [23],7.2.4.12) Let $\underline{A} \in \mathrm{Ani(CRing)}$ be an animated ring. For each ∞-category $\mathcal{D} \in \mathrm{Cat}_\infty$ admitting geometric realizations of its simplicial objects, the inclusion* $\mathrm{FFree}_{\underline{A}} \subseteq_{f.f.} \mathrm{APerf}(\underline{A})$ *induces an equivalence*

$$\mathrm{Fun}_\Sigma(\mathrm{APerf}(\underline{A}), \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{FFree}_{\underline{A}}, \mathcal{D})$$

where $\mathrm{Fun}_\Sigma(\mathrm{APerf}(\underline{A}), \mathcal{D}) \subseteq_{f.f.} \mathrm{Fun}(\mathrm{APerf}(\underline{A}), \mathcal{D})$ *denotes the full subcategory spanned by those functors which preserve geometric realizations of simplicial almost perfect modules.*
In other words, $\mathrm{APerf}(\underline{A})$ *is the* free completion of $\mathrm{FFree}_{\underline{A}}$ under geometric realizations *(in* $\mathrm{Mod}_{\underline{A}}$*) of simplicial objects in* $\mathrm{FFree}_{\underline{A}}$.

Proof. The proof is an application of A.3.0.1. Let $\mathcal{C} \subseteq_{f.f.} \mathcal{P}(\mathrm{FFree}_{\underline{A}})$ be the smallest full subcategory containing the essential image of the Yoneda embedding $j : \mathrm{FFree}_{\underline{A}} \hookrightarrow \mathcal{P}(\mathrm{FFree}_{\underline{A}})$ and being closed under geometric realizations (in $\mathcal{P}(\mathrm{FFree}_{\underline{A}})$, and hence in $\mathrm{Mod}_{\underline{A}}$ by A.2.0.2) of $s\mathrm{FFree}_{\underline{A}}$. Call again $j : \mathrm{FFree}_{\underline{A}} \hookrightarrow \mathcal{C}$ the (factorization of the) Yoneda embedding. Then, for each $\mathcal{D} \in \mathrm{Cat}_\infty$ admitting geometric realizations, restriction along j induces an equivalence (by A.3.0.1)

$$\mathrm{Fun}_\Sigma(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{FFree}_{\underline{A}}, \mathcal{D})$$

In particular, the inclusion $\mathrm{FFree}_{\underline{A}} \subseteq_{f.f.} \mathrm{APerf}(\underline{A})$ extends to a fully faithful (by [24],5.3.5.11,i) functor $F : \mathcal{C} \hookrightarrow \mathrm{APerf}(\underline{A})$, and we are left to prove that the latter is an equivalence.
By an application of [24],5.3.5.11,ii, this amounts to proving the essential surjectivity of F, which is the content of 3.6.1.6,iii. □

Warning. The reader should beware that (almost) perfect modules are complicated. In particular, as commented by Lurie right below [23],7.2.4.12, in general the t-structure on $\mathrm{Mod}_{\underline{A}}$ does not descend to one on $\mathrm{APerf}(\underline{A})$, and we cannot expect to recover the nerve of the 1-category of finitely presented $\pi_0\underline{A}$-modules as the heart of $\mathrm{APerf}(\underline{A})$ via $\pi_0 : \mathrm{APerf}(\underline{A}) \to \mathrm{Mod}(\pi_0\underline{A})$. This happens, however when the base animated ring $\underline{A}$ is left-coherent, see [23],7.2.4.19.

An interesting question to be addressed is the interplay between (almost) perfectness and flatness. Indeed, flatness lifts properties of static modules to higher homotopical degrees, so it will allow us to bridge the gap between compactness and its weakened version.

Proposition 3.6.1.8. *(*$\mathrm{Proj} = \mathrm{Flat} \cap \mathrm{APerf}$*, [23],7.2.4.20) For an animated ring* $\underline{A} \in \mathrm{Ani(CRing)}$ *and an* $\underline{A}$*-module* $\underline{M}$*,* TFAE:

- $\underline{M}$ *is finitely generated and projective, i.e. it is a retract of a finite free* $\underline{A}$*-module.*
- $\underline{M}$ *is flat and almost perfect.*

Proof. (1) $\implies$ (2) : This is clear, because $\mathrm{Perf}(\underline{A})$ is closed under retracts. (2) $\implies$ (1) : Let $\underline{M} \in \mathrm{APerf}(\underline{A}) \cap \mathrm{Flat}(\underline{A})$. Then, for each $n \geq 0$ it holds $\tau_{\leq n}\underline{M} \in \mathrm{Perf}(\underline{A})_{\leq n}$; choose some n and notice that a truncated version of 3.6.1.3,ii implies that $\pi_0\underline{M} = \pi_0(\tau_{\leq n}\underline{M})$ is a finitely presented flat $\pi_0\underline{A}$-module, hence projective. Then, by 3.4.0.6 also $\underline{M}$ is $\underline{A}$-projective.
In order to prove the finiteness of $\underline{M}$, we will employ the usual lifting argument of 3.4.0.6. Indeed, by the above there exists some finite free $\pi_0\underline{A}$-module F_0 together with a splitting projection $p : F_0 \twoheadrightarrow \pi_0\underline{M}$; this is in turn induced by some map $g : \underline{F} \twoheadrightarrow \underline{M}$ in $\mathrm{Mod}_{\underline{A}}$ with $\underline{F}$ finite $\underline{A}$-free and $\pi_0(g) = p$. Then, being $\underline{M}$ projective the latter must split as well, so that $\underline{M}$ is indeed a retract of a finite free $\underline{A}$-module. □

Remark. In particular, a flat and almost perfect module is a fortiori perfect.

In order to turn to a more geometric approach, we wish to study also stability results for almost perfect modules.

Proposition 3.6.1.9. *(*APerf *is stable under base-change, [26],2.7.3.1) Almost perfectness is stable under base-change. In other words, for any map* $f : \underline{A} \to \underline{B}$ *in* $\mathrm{Ani(CRing)}$ *and almost perfect module* $\underline{M} \in \mathrm{APerf}(\underline{A})$*, also* $\underline{B} \otimes^L_{\underline{A}} \underline{M}$ *is almost* $\underline{B}$*-perfect.*
Moreover, base-change along faithfully flat maps detects almost perfectness.

Proof. We will prove only the first part, which is a straightforward consequence of 3.6.1.6,iii: choose a degree-wise finite free simplicial $\underline{A}$-module $\underline{X}_\bullet \in s\mathrm{Mod}_{\underline{A}}$ s.t. $\underline{M} \simeq |\underline{X}_\bullet|$ and tensor by $\underline{B}$:

$$\underline{B} \otimes^L_{\underline{A}} \underline{M} \simeq \underline{B} \otimes^L_{\underline{A}} |\underline{X}_\bullet| \simeq |\underline{B} \otimes^L_{\underline{A}} \underline{X}_\bullet|$$

which remains degree-wise finite $\underline{B}$-free, since $\otimes^L_{\underline{A}}$ preserves colimits separately in each variable. □

Let us briefly recall the definition of the **fpqc site** on $\mathrm{Sch}^{\mathrm{Aff}}$ (see 4.1.2.4): it is the Grothendieck topology (see C.3.0.2) "generated" by *finite* families of jointly surjective (so, such that the map from the coproduct is an effective epimorphism) flat morphisms of affine schemes; here to "generate" means taking the closure under base-change along open immersions of affine schemes.

Translated into the language of algebra, covering families for the spectrum of some $\underline{A} \in \mathrm{Ani(CRing)}$ for the fpqc site on $\mathrm{Sch}^{\mathrm{Aff}}$ correspond to finite families of flat $\underline{A}$-algebra maps $\{\phi_i : \underline{A} \to \underline{A}_i\}_{i=1}^n$ such that the canonical map $\phi : \underline{A} \to \prod_{i=1}^n \underline{A}_i$ is faithfully flat (see the characterization in 4.1.4.5).

Let us observe that an affine Zariski cover (see 4.1.3.2) is also a fpqc-covering. So, locality of almost perfectness with respect to affine Zariski cover will be a consequence of fpqc-locality.

Proposition 3.6.1.10. *(*APerf *is local) Almost perfectness is local for the fpqc sites on* $\mathrm{Sch}^{\mathrm{Aff}}$*, and hence also with respect to affine Zariski covers (see 4.1.3.2).*
More explicitly, let it be given the datum of finitely many flat maps of animated rings $\phi_i : \underline{A} \to \underline{A}_i$ *such that the canonical map* $\phi : \underline{A} \twoheadrightarrow \prod_i \underline{A}_i$ *is faithfully flat.*
For any $\underline{M} \in \mathrm{Mod}_{\underline{A}}$*, if* $\underline{M} \otimes^L_{\underline{A}} \underline{A}_i \in \mathrm{APerf}(\underline{A}_i)$ *for each* $i = 1, \ldots, n$*, then* $\underline{M} \in \mathrm{APerf}(\underline{A})$*.*

Proof. By the converse of 3.6.1.9, it suffices to show that $\underline{M} \otimes^L_{\underline{A}} (\prod_i \underline{A}_i)$ is almost $(\prod_i \underline{A}_i)$-perfect. This will be achieved in two steps.

Claim. *([26],2.7.0.8) Call* $\underline{M} := \prod_{i=1}^n \underline{M}_i$, $\underline{A} := \prod_{i=1}^n \underline{A}_i$. *Then,* $\underline{M}_i \in \mathrm{APerf}(\underline{A}_i)$ *for each* i *iff* $\underline{M} \in \mathrm{APerf}(\underline{A})$.

Proof. Unwinding the definition, our assumption is that, for each $n \geq 0$, $\tau_{\leq n}\underline{M}_i \in (\mathrm{Mod}_{\underline{A}_i})^{\mathrm{fp}}_{\leq n}$ is compact, and we need to show that the same holds for the truncations of $\underline{M} \in \mathrm{Mod}_{\underline{A}}$. Hence, we need to show that $\underline{M}$ is $\underline{A}$-compact iff each $\underline{M}_i$ is $\underline{A}_i$-compact.
This is implied by the following observations. First of all notice that $\mathrm{Mod}_{\underline{A}} \simeq \prod_i \mathrm{Mod}_{\underline{A}_i}$, because the presheaf of modules $\mathrm{Mod} : \mathrm{Ani(CRing)} \to \mathrm{Cat}_\infty$ is a right Kan extension (see 3.2.5.11); hence, in particular mapping space functors are component-wise: $\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}} \simeq \prod_i \mathrm{Map}_{\mathrm{Mod}_{\underline{A}_i}}$ in $\mathrm{Fun}(\mathrm{Tw}(\mathrm{Mod}_{\underline{A}}), \mathrm{Spc})$. Moreover, also filtered diagrams in $\mathrm{Mod}_{\underline{A}}$ are obtained as finite products of diagrams in the various $\mathrm{Mod}_{\underline{A}_i}$'s, because for each filtered indexing ∞-category I, $\mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}}) \simeq \prod_i \mathrm{Fun}(I, \mathrm{Mod}_{\underline{A}_i})$
Finally, in $\mathrm{Mod}_{\underline{A}}$ the commutativity with filtered diagrams of mapping spaces from compact objects can be checked component-wise: as already observed, filtered diagrams in $\mathrm{Mod}_{\underline{A}}$ are finite products of filtered diagrams in the various $\mathrm{Mod}_{\underline{A}_i}$, filtered colimits commute with finite products and mapping spaces are continuous in the covariant argument. ■

Claim. $\underline{M} \otimes^L_{\underline{A}} (-) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}}$ *commutes with finite products.*

Proof. We will prove that finite products in $\mathrm{Mod}_{\underline{A}}$ coincide with finite coproducts: this will allow us to conclude by the fact that $\underline{M} \otimes^L_{\underline{A}} (-)$ preserves colimits separately in each variable.
Fix some $m \geq 0$, and consider the m-ary product and coproduct functors. Being they left-derived functors, it suffices to check their equivalence on $\mathrm{FFree}_{\underline{A}}$.
Let $p : K \to \mathrm{Poly}_{/\underline{A}}$ be a sifted simplicial diagram of finite polynomial algebras over $\underline{A}$, whose geometric realization gives $|p| \simeq \underline{A}$. Observe that the functor $\mathit{for} : \mathrm{CAlg}^{\Delta}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}}$ forgetting the $\underline{A}$-algebra structure preserves sifted colimits: $\mathrm{Mod}_{\underline{A}}$ is cpt+proj-generated and its left-adjoint $L\mathrm{Sym}^*_{\underline{A}} : \mathrm{Mod}_{\underline{A}} \to \mathrm{CAlg}^{\Delta}_{\underline{A}}$ (see 3.7.1.2) preserves cpt+proj's (it is an explicit computation, see the proof of 3.2.2.3). Thus, $\mathit{for} \circ p : K \to (\mathrm{Mod}_{\mathbb{Z}})_{/\underline{A}}$ is a sifted simplicial diagram in $\mathrm{Mod}_{\mathbb{Z}}$ consisting of static $\mathbb{Z}$-modules and with geometric realization $|\mathit{for} \circ p| \simeq \mathit{for}\underline{A}$.
Since finite products commute with sifted colimits (a), we have the following chain of equivalences:

$$(\mathit{for}\underline{A})^{\times m} \simeq |\mathit{for} \circ p|^{\times m} \simeq_{(a)} |(\mathit{for} \circ p)^{\times m}| \simeq_{(b)} |\coprod_m (\mathit{for} \circ p)| \simeq \coprod_m |\mathit{for} \circ p| \simeq \coprod_m \mathit{for}\underline{A}$$

where (b) follows from the fact that (co)limits in functor categories are computed object-wise and that finite coproduct and finite product in the abelian 1-category $\mathrm{Mod}(\mathbb{Z})$ are isomorphic. ■ □

3.6.2 Locally Free Modules of Finite Rank

In this subsection we specialize the previous digression to finitely generated projective modules, which we call "locally free of finite rank", thus turning to a more geometric approach.

Definition 3.6.2.1. *(Locally free of finite rank, [26],2.9.1.1)* For $\underline{A} \in \mathrm{Ani(CRing)}$ an animated ring and an animated $\underline{A}$-module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$, TFAE:

1. $\underline{M}$ is **locally free of finite rank** (fg-loc.free);
2. $\underline{M}$ is finitely generated and projective, i.e. it is a direct summand of a finite $\underline{A}$-free module;
3. $\underline{M}$ is fg-loc.free iff it is $\underline{A}$-flat and almost $\underline{A}$-perfect (see 3.6.1.8);
4. $\underline{M}$ is a dualizable object for the symmetric monoidal structure $\mathrm{Mod}^{\otimes}_{\underline{A}}$ (see [26],2.9.1.5).

Being locally free of finite rank is stable under base-change and is local, so that it will serve well for geometric purposes.

Proposition 3.6.2.2. *(fg-loc.free is local and stable under base-change, [26],2.9.1.4) The property of "being fg-loc.free" is stable under base-change and* Zar*-local (i.e. it can be checked on an affine Zariski cover as in 4.1.3.2).*

Proof. It suffices to show that both flatness and the property of being almost perfect enjoys the stated stability. As for the stability under base-change, these amount to propositions 3.4.0.4 for flatness and 3.6.1.9 for almost perfectness.
For what concerns the Zariski stability, instead, we postpone the proof to the more geometric sections of the essay: see 4.1.4.6 and 3.6.1.10. □

In what follows, we will consider mostly locally free modules which have uniform rank on some affine Zariski cover of the base ring (see 4.1.3.2).

Definition 3.6.2.3. *(n-loc.free,* [26],2.9.2.1) Let $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ be an animated ring. We say that an $\underline{A}$-module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ is **locally free of rank** n (n-loc.free) (≥ 0) iff

- $\underline{M}$ is locally free of finite rank;
- $\underline{M}$ has a well-defined rank, uniformly on points, i.e. for each field $K \in \mathrm{CAlg}^{\Delta}_{\underline{A}}$, $\pi_0(K \otimes^L_{\underline{A}} \underline{M}) \simeq K^n$ in Vect_K.

Remark. ([26],2.9.2.2) In the second condition, we can wlog restrict to those geometric points at algebraically closed fields.
Indeed, if $\underline{M}$ has rank n at K, then it has such rank also at any algebraic field extension $K[x]/(\lambda(x))$ of K: being π_0 symmetric monoidal and K-flat,

$$\pi_0\left(\frac{K[x]}{\lambda(x)} \otimes^L_{\underline{A}} \underline{M}\right) \cong \pi_0\left(\frac{K[x]}{\lambda(x)} \otimes^L_K (K \otimes^L_{\underline{A}} \underline{M})\right) \cong \frac{K[x]}{\lambda(x)} \otimes_K K^n \cong \left(\frac{K[x]}{\lambda(x)}\right)^n$$

In other words, the rank of $\underline{M}$ at K is preserved along chains of algebraic extensions of K, so that it must coincide with the one at any choice of an algebraic closure of K.

Moreover, the next result ensures that if $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ admits rank uniformly at geometric points, then its rank is stable on some affine Zariski cover (see 4.1.3.2) of $\underline{A}$ by distinguished opens. In more geometric terms, this allows for the existence of trivializing atlases for vector bundles (see 4.3.2.7).

Proposition 3.6.2.4. *(The rank is uniform on charts, [26],2.9.2.3) Consider an animated ring $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ and let $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ be a locally free $\underline{A}$-module of finite rank.*
Then, there exists a partition of units $\{x_i\}_{i=1}^m$ of the connected components $\pi_0\underline{A}$ s.t. the localization $\underline{M}[x_i^{-1}] := \underline{A}[x_i^{-1}] \otimes^L_{\underline{A}} \underline{M}$ is $\underline{A}[x_i^{-1}]$-free of rank n_i.
If further $\underline{M}$ is locally $\underline{A}$-free of rank n, then $\underline{M}$ has uniform rank wlog $n_i = n$ for each i.

Proof. Consider the set $\mathcal{E}$ of all connected components of $\underline{A}$ such that the locally free $\underline{A}$-module of finite rank $\underline{M}$ is free of finite rank on the distinguished open on which x is invertible:

$$\mathcal{E} := \{x \in \pi_0\underline{A} \mid \exists n_x \geq 0 : \underline{M}[x^{-1}] \simeq \underline{A}[x^{-1}]^{n_x}\} \subseteq \pi_0\underline{A}$$

The statement amounts to proving that $\mathcal{E}$ generates the unit ideal. Let's argue by contradiction, so assume that there exists some maximal ideal $\mathfrak{m} \subseteq \underline{A}$ containing $\mathcal{E}$ and let $k := \pi_0\underline{A}/\mathfrak{m} \cong (\pi_0\underline{M})_{\mathfrak{m}}/\mathfrak{m}$ be its residue field.
Then, being modules over a field free, there exists some $n' \geq 0$ s.t. $\pi_0(k \otimes^L_{\underline{A}} \underline{M}) \cong k^{n'}$ is a finite-dimensional k-vector space *(with $n' = n$ in the case $\underline{M}$ is n-loc.free)*.
Hence, choose elements $\underline{y} = \{y_i\}_{i=1}^{n'} \subseteq \pi_0\underline{M}$ whose images form a basis of $k \otimes_{\pi_0\underline{A}} \pi_0\underline{M} \cong \pi_0(k \otimes^L_{\underline{A}} \underline{M}) \cong k^{n'}$ (recall that π_0 is symmetric monoidal and that $\underline{M}$ is $\underline{A}$-flat).
It follows by Nakayama's Lemma that the images of $\underline{y} = \{y_i\}_{i=1}^{n'}$ lift to generators of the local ring $(\pi_0\underline{M})_{\mathfrak{m}}$. Indeed, by definition $\pi_0\underline{M}$ is finitely generated over $\pi_0\underline{A}$, and by construction one has both that the maximal ideal $\mathfrak{m} \cong \mathrm{rad}\big((\pi_0\underline{A})_{\mathfrak{m}}\big)$ coincides with the Jacobson radical of the local ring and that $\{y_i\}_i$ generates the quotient $\pi_0(k \otimes^L_{\underline{A}} \underline{M}) \cong \pi_0(M)/\mathfrak{m}(\pi_0(\underline{M}))$. Thus we conclude by a version ([37],10.20.1,viii) of Nakayama's Lemma.
Now, being $\pi_0\underline{A}^{n'}$ a compact $\pi_0\underline{A}$-module,

$$(\pi_0\underline{A})_{\mathfrak{m}} \otimes_{\pi_0\underline{A}} \underline{y} \;:\; \oplus_i(\pi_0\underline{A})y_i \twoheadrightarrow (\pi_0\underline{M})_{\mathfrak{m}} \cong \underrightarrow{\mathrm{colim}}\left((\pi_0\underline{M})[z^{-1}] \mid z \in \pi_0\underline{A} \setminus \mathfrak{m}\right)$$

factors through some $\pi_0\underline{M}[x^{-1}]$ and can be rewritten as $\hat{\underline{y}} : \oplus_i(\pi_0\underline{A}[x^{-1}])\hat{y}_i \twoheadrightarrow (\pi_0\underline{M})[x^{-1}]$ via the extension of scalars along $\pi_0\underline{A} \to \pi_0\underline{A}[x^{-1}]$; finally, the base-change $\hat{\underline{y}} \cong \pi_0\underline{A}[x^{-1}] \otimes_{\pi_0\underline{A}} \underline{y}$ of the surjective map $\underline{y}$ is again such.
As in 3.4.1.3, $\pi_0(\underline{A}[x^{-1}]) \cong (\pi_0\underline{A})[x^{-1}]$, so that also $\pi_0(\underline{M}[x^{-1}]) \cong (\pi_0\underline{M})[x^{-1}]$. Hence, the surjection $\hat{\underline{y}}$ comes from a map of $\underline{A}$-modules $g : \underline{A}[x^{-1}]^{n'} \to \underline{M}[x^{-1}]$ which retrieves the surjection $\hat{\underline{y}} = \pi_0(g)$ on connected components. Then, since passing to localizations preserves the projectivity of $\underline{M}$, g admits a homotopy inverse, say $\psi : \underline{M}[x^{-1}] \to \underline{A}[x^{-1}]^{n'}$.
Consider the $n' \times n'$ matrix X over $\pi_0\underline{A}[x^{-1}]$ determined by the composite $\psi \circ g : \underline{A}[x^{-1}]^{n'} \to \underline{A}[x^{-1}]^{n'}$.
Its determinant $\det(X) \in \pi_0\underline{A}[x^{-1}]$ must have the form $\det(X) = x' \cdot x^{-a}$ for some $a \geq 0$. Consider its numerator $x' := x^a \det X \in \pi_0\underline{A}$ and observe that $x' \notin \mathfrak{m}$, because $\hat{\underline{y}} = \pi_0(g)$ induces an isomorphism of vector spaces $y = k \otimes_{\pi_0\underline{A}} \hat{\underline{y}} : k^{n'} \cong \pi_0(k \otimes^L_{\underline{A}} \underline{M})$, so that $k \otimes_{\pi_0\underline{A}} \pi_0(\psi \circ g) = id_{k^{n'}}$. Thus, $\det X \in (\pi_0\underline{A}[x^{-1}, (x')^{-1}])^\times$ is invertible in the localization of $\pi_0\underline{A}[x^{-1}]$ at x'.
Furthermore, localizing g at x' induces a map $\underline{A}[x^{-1}, (x')^{-1}]^{n'} \to \underline{M}[x^{-1}, (x')^{-1}]$ which acts on connected components as the localizations of $\hat{\underline{y}}$ at x'. But the latter is an isomorphism: being localizations right-exact and $\hat{\underline{y}}$ surjective, it stays surjective; moreover, it is injective, since it is the first composite of an automorphism of $\pi_0\underline{A}[x^{-1}, (x')^{-1}]$ corresponding to the invertible matrix X.
Hence, being π_0 conservative on flat modules, also the map $\underline{A}[x^{-1}, (x')^{-1}]^{n'} \to \underline{M}[x^{-1}, (x')^{-1}]$ induced by g turns out to be an equivalence. This means that $xx' \in \pi_0\underline{A}$ is actually an element of $\mathcal{E} \subseteq \mathfrak{m}$, which contradicts $xx' \in \mathfrak{m}$. □

Proposition 3.6.2.5. *(n-loc.free is local, [26],2.9.2.4) Being "locally free of rank n" is stable under base-change and* Zar*-local (i.e. it can be checked on an affine Zariski cover as in 4.1.3.2).*

Proof. By 3.6.2.2, being "finitely generated locally free" is stable under base-change and Zar-local. Moreover, the rank of a fg-loc.free $\underline{A}$-module is clearly stable under base-change. Hence, we are left to prove the Zar-locality part; as before, we will actually prove the stronger flat-locality, which translates into algebraic terms as follows.

Claim. *For $\underline{A} \in \mathrm{Ani(CRing)}$, consider $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ fg-loc.free and any faithfully flat map $\underline{A} \to \prod_{\alpha=1}^n \underline{A}_\alpha$. If each base-change $\underline{M} \otimes^L_{\underline{A}} \underline{A}_\alpha$ is n-loc.free on $\underline{A}_\alpha$, then also $\underline{M}$ is n-loc.free on $\underline{A}$.*

Proof. We need to prove condition 3.6.2.3,ii, namely that base-change along each field $k \in \mathrm{CAlg}^\Delta_{\underline{A}}$ yields an n-dimensional k-vector space $\pi_0(k \otimes^L_{\underline{A}} \underline{M}) \cong k^n$.
The faithful flatness of $\underline{A} \to \prod_\alpha \underline{A}_\alpha$ implies that there exists some $\alpha \in I$ for which $\underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M} \neq 0$ is non-vanishing.
Indeed, also the induced static map $\pi_0\underline{A} \to \prod_\alpha \pi_0\underline{A}_i$ in $\pi_0\underline{A}$-Alg is faithfully flat, so that

$$0 \neq \pi_0(\prod_{\alpha=1}^n \underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M}) \cong \prod_{\alpha=1}^n \pi_0(\underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M})$$

where finite products in $\mathrm{Mod}_{\underline{A}}$ commute with $\otimes^L_{\underline{A}}$ as in the proof of 3.6.1.9. Hence, there exists one such index α for which $\pi_0(\underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M}) \neq 0$, which implies that already $\underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M} \neq 0$, as desired.

Then, consider any maximal ideal $\mathfrak{m}$ of $\pi_0(k \otimes^L_{\underline{A}} \underline{A}_\alpha)$, and let K denote its residue field. Being the latter a field extension of k and since ideals of $\pi_0(k \otimes^L_{\underline{A}} \underline{A}_\alpha)$ are in particular also $\underline{A}$-modules, our claim is reduced to checking the isomorphism $\pi_0(K \otimes^L_{\underline{A}} \underline{M}) \cong K^n$ in Vect_K (see the Remark right below 3.6.2.3). But this is a consequence of the following manipulation: being $\underline{M}$ flat over $\underline{A}$, wlog $K \in \mathrm{Mod}_{\underline{A}_\alpha}$, and π_0 symmetric monoidal, one has that

$$\pi_0\big(K \otimes^L_{\underline{A}} \underline{M}\big) \cong \pi_0\big(K \otimes^L_{\underline{A}_\alpha} (\underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M})\big) \cong K^n$$

where the latter isomorphism comes from the assumption that $\underline{A}_\alpha \otimes^L_{\underline{A}} \underline{M}$ is n-loc.free over $\underline{A}_\alpha$. □

We close this subsection with an easy but useful 2-out-of-3 property for exact sequences of finitely generated locally free modules.

Lemma 3.6.2.6. *(2-out-of-3 for fg- and n-loc.free) Consider a cofibre sequence $\underline{M}' \to \underline{M} \to \underline{M}''$ in $\mathrm{Mod}_{\underline{A}}$. Then, all modules are fg-loc.free iff any two of them are such. Moreover, the ordinary rule to add ranks along exact sequences holds true.*

Proof. We will use the third characterization of fg-loc.free. As for the compactness part, testing the commutativity against filtered diagrams induces a fibre sequence of spaces, and we can conclude by inspection of the induced long exact sequence in homotopy. So, we are left to prove that also flatness enjoys the 2-out-of-3 property. But this is clear: we can work in the stabilization $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$, so our cofibre sequence is exact and we can again conclude by inspection of the long exact sequence in homotopy.
Finally, if any two modules are n-, m-loc.free, then we want to extend the ordinary rule to add ranks along exact sequences. To this end, as in the definition 3.6.2.3 consider the base-change of the sequence (which is still exact in $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$) by any field $K \in \mathrm{CAlg}_{\underline{A}}^{\Delta}$ and apply π_0. Then, we obtain a short exact sequence in Vect_K to which we can apply the rank rule, thus proving that also the third fg-loc.free $\underline{A}$-module must have the right rank, uniformly defined on points. $\square$

3.7 Universal Tensor Algebras

Construction 3.7.0.1. *(Construction: Derived Symmetric Powers, [26],25.2.2.1)*
For $A \in \mathrm{CRing}$ and $M \in \mathrm{Mod}(A)$, define the (static part of the) derived symmetric powers of M over A by

$$\mathrm{CSym}_A^n(M) := \pi_0(T_A^n M)/\Sigma_n \in \mathrm{Mod}(A)$$

where $T_A^n(M)$ denotes the n-th graded part of the (derived) tensor A-algebra functor evaluated at M, namely $T_A^n(M) := M^{\otimes_A^L n}$, whose static part is then quotiented out by the action $\Sigma_n \curvearrowright T^n M$ of the symmetric group Σ_n on $M^{\otimes_A n}$ which permutes the various copies of M in the product.

Remark. For a free A-module $M = \oplus_{i=1}^m A x_i$, one has that $\mathrm{CSym}_A^n A^m \simeq \oplus_{|\underline{\alpha}|=n} A(\underline{x}^{\otimes \underline{\alpha}}) \in \mathrm{FFree}_A$ (where $\underline{\alpha}$ denotes a multi-index of weight $|\alpha| = n$); the latter is a free A-module whose rank equals the number of symmetric monomials of degree n in m indeterminates, namely $\binom{n+m-1}{n}$.

Let $\mathcal{C} \subseteq_{f.f.} \mathrm{CRMod}$ denote the full subcategory of cpt+proj's, i.e. generated by those pairs of the form $(A := \mathbb{Z}[X], A^n)$ with X any finite tuple of indeterminates of length $|X| = m$. By the Remark, $\mathrm{CSym}_{(-)}^n$ induces the following map:

$$f: \quad \mathcal{C} \longrightarrow \mathcal{C} \quad \subseteq_{f.f.} \quad \mathrm{Ani}(\mathrm{CRMod}) = \mathrm{MOD}$$
$$(A, M) \mapsto (A, \mathrm{CSym}_A^n M)$$

$$\begin{tikzcd}
\mathrm{MOD} \arrow[rr, "F"] \arrow[dr, "pr_1"'] & & \mathrm{MOD} \arrow[dl, "pr_1"] \\
& \mathrm{Ani}(\mathrm{CRing}) &
\end{tikzcd}$$

which admits an essentially unique colim^{sift}-preserving extension $F : \mathrm{Ani}(\mathrm{CRMod}) \to \mathrm{Ani}(\mathrm{CRMod})$ sitting in the aside commutative triangle.
Observe that the stated triangle is given by the fact that f is a functor over $\mathrm{Ani}(\mathrm{CRing})$ and that $pr_1 = \mathrm{Ani}(pr_1)$ (see 3.2.5.3) preserves sifted colimits.
Hence, one can describe the action of F on objects by the assignment $F(\underline{A}, \underline{M}) = (\underline{A}, \mathrm{LSym}_{\underline{A}}^n \underline{M})$, for some animated $\underline{A}$-module $\mathrm{LSym}_{\underline{A}}^n \underline{M}$.
Moreover, notice that F extends to MOD the definition of $\mathrm{CSym}_{(-)}^n$ which we introduced only over CRMod, since also the latter functor preserves sifted colimits (both $\otimes$ and the quotient by Σ_n preserve both filtered colimits and reflexive coequalizers, the former separately in each variable, and hence as a multi-functor).
The latter induces a well-defined notion of n**-th derived symmetric tensor algebra** over the whole of $\mathrm{Mod}_{\underline{A}}$, whose functoriality corresponds precisely to F.

The techniques presented in the previous paragraph 3.7.0.1 yield also the other Derived Powers functors. We report the full (analogous) construction, so as to ease referencing.

Construction 3.7.0.2. *(Construction: Derived Exterior Powers, [26],25.2.2.2)*
For $A \in \mathrm{CRing}$ and $M \in \mathrm{Mod}(A)$, define the (static part of the) derived exterior powers of M over A by

$$C\Lambda^n_A(M) := \pi_0(T^n_A M)/\mathrm{Alt}_n$$

where again $T^n_A(M) = M^{\otimes^L_A n}$ denotes the n-th graded part of the (derived) tensor A-algebra functor evaluated at M, and where we quotient its static part by the submodule Alt_n generated by those elementary n-tensors whose components are not all distinct.

Remark. For a free A-module $M = \oplus_{i=1}^m Ax_i$, one has that $C\Lambda^n_A A^m \simeq \oplus_{|\underline{\alpha}|=n\,,\,\forall i:1\leq\alpha_i<\alpha_{i+1}\leq m} A(\underline{x}^{\wedge\underline{\alpha}}) \in \mathrm{FFree}_A$ (where $\underline{\alpha}$ denotes a multi-index of weight $|\alpha| = n$) is a free A-module of rank $\binom{m}{n}$.

Let $\mathcal{C} \subseteq_{f.f.}$ CRMod denote the full subcategory of cpt+proj's, i.e. generated by those pairs of the form $(A := \mathbb{Z}[X], A^n)$ with X any finite tuple of indeterminates of length $|X| = m$. By the Remark, $C\Lambda^n_{(-)}$ induces the following map:

$$f: \quad \mathcal{C} \longrightarrow \mathcal{C} \quad \subseteq_{f.f.} \quad \mathrm{Ani(CRMod)} = \mathrm{MOD}$$
$$(A, M) \mapsto (A, C\Lambda^n_A M)$$

which admits an essentially unique colim^{sift}-preserving extension $F : \mathrm{Ani(CRMod)} \to \mathrm{Ani(CRMod)}$ sitting in the aside commutative triangle.

$$\begin{tikzcd} \mathrm{MOD} \arrow[rr, "F"] \arrow[dr, "pr_1"'] & & \mathrm{MOD} \arrow[dl, "pr_1"] \\ & \mathrm{Ani(CRing)} & \end{tikzcd}$$

Hence, one can describe the action of F on objects by the assignment $F(\underline{A}, \underline{M}) = (\underline{A}, L\Lambda^n_{\underline{A}}\underline{M})$, for some animated $\underline{A}$-module $L\Lambda^n_{\underline{A}}\underline{M}$.
Moreover, notice that F extends to MOD the definition of $C\Lambda^n_{(-)}$ which we introduced only over CRMod, since also the latter functor preserves sifted colimits (both $\otimes$ and the quotient by Alt_n preserve both filtered colimits and reflexive coequalizers, the former separately in each variable, and hence as a multi-functor).
The latter induces a well-defined notion of **n-th derived exterior tensor algebra** over the whole of $\mathrm{Mod}_{\underline{A}}$, whose functoriality corresponds precisely to F.

Construction 3.7.0.3. *(Construction: Derived Divided Powers, [26],25.2.2.3)*
For $A \in$ CRing and $M \in \mathrm{Mod}(A)$, define the (static part of the) derived divided powers of M over A, say $C\Gamma^n_A(M)$, via the universal property in [19],3.1.

Remark. For $A \in$ Poly and $M \in \mathrm{FFree}_A$ it turns out that $C\Gamma^n_A(M)$ can actually be described as the 'collection' of invariants in $T^n_A(M)$ for the action $\Sigma_n \curvearrowright M$, as in 3.7.0.1.
More explicitly, for $A \in$ Poly and $M \in \mathrm{FFree}_A$, $C\Gamma^n_A(M)$ is determined as (a graded A-algebra) by the following properties: writing st_{Σ_n} for the stabilizer of the aforementioned action $\Sigma_n \curvearrowright M^{\otimes n}$,

$$\begin{cases} C\Gamma^n_A(At) := st_{\Sigma_n}(\pi_0(T^n_A At)) = A(\frac{t^n}{n!}) \\ C\Gamma_A(M \oplus N) \cong C\Gamma_A(M) \otimes_A C\Gamma_A(N) & [[19], 3.19] \end{cases}$$

Then, for a free A-module $M = \oplus_{i=1}^m Ax_i$, one has that $C\Gamma^n_A A^m \in \mathrm{FFree}_A$ is a free A-module of rank $\binom{n+m-1}{n}$.

Let $\mathcal{C} \subseteq_{f.f.}$ CRMod denote the full subcategory of cpt+proj's, i.e. generated by those pairs of the form $(A := \mathbb{Z}[X], A^n)$ with X any finite tuple of indeterminates of length $|X| = m$. By the Remark, $C\Gamma^n_{(-)}$ induces the following map:

$$f: \quad \mathcal{C} \longrightarrow \mathcal{C} \quad \subseteq_{f.f.} \quad \mathrm{Ani(CRMod)} = \mathrm{MOD}$$
$$(A, M) \mapsto (A, C\Gamma^n_A M)$$

which admits an essentially unique colim^{sift}-preserving extension $F : \mathrm{Ani(CRMod)} \to \mathrm{Ani(CRMod)}$ sitting in the aside commutative triangle.

$$\begin{tikzcd} \mathrm{MOD} \arrow[rr, "F"] \arrow[dr, "pr_1"'] & & \mathrm{MOD} \arrow[dl, "pr_1"] \\ & \mathrm{Ani(CRing)} & \end{tikzcd}$$

Hence, one can describe the action of F on objects by the assignment $F(\underline{A}, \underline{M}) = (\underline{A}, L\Gamma^n_{\underline{A}}\underline{M})$, for some animated $\underline{A}$-module $L\Gamma^n_{\underline{A}}\underline{M}$.
Moreover, notice that F extends to MOD the definition of $C\Gamma^n_{(-)}$ which we introduced only over CRMod, since also the latter functor preserves sifted colimits (by [19],3.14-16, it preserves filtered colimits and reflexive coequalizers).
The latter induces a well-defined notion of **n-th derived divided power algebra** over the whole of $\mathrm{Mod}_{\underline{A}}$, whose functoriality corresponds precisely to F.

Remark. One might wonder the origin of the degree-wise equality of dimensions of Derived Powers and Symmetric Powers. It turns out to be indeed very deep, in that it stems from some 'duality' in the aforementioned constructions. More precisely, for $A \in \mathrm{CRing}$ and $M \in \mathrm{Mod}(A)$, the natural evaluation-pairing on tensors $T^n_A(M) \times T^n_A(M^\vee) \to A$ induces a canonical natural isomorphism

$$C\Gamma^n_A(M^\vee) \cong \mathrm{CSym}^n_A(M)^\vee$$

where $(-)^\vee = \mathrm{Hom}_A(-, A)$ denotes the dualization functor of A-modules. For more details, we refer to the first section of the Appendix in [35].

On the other hand, let us remark that the construction of Derived External Powers is instead self-dual, so that dualization does not yield any other interesting universal tensor algebra. Indeed, with notation as before, the previous pairing induces a canonical natural iso $C\Lambda^n_A(M^\vee) \cong \big(C\Lambda^n_A(M)\big)^\vee$.

Remark. Furthermore, let us specify the values of our newly introduced functors in degree 0 and 1. As expected, for $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ and $\underline{M} \in \mathrm{Mod}_{\underline{A}}$,

- $n = 0$: $\mathrm{LSym}^0_{\underline{A}}(\underline{M}) \simeq L\Lambda^0_{\underline{A}}(\underline{M}) \simeq L\Gamma^0_{\underline{A}}(\underline{M}) \simeq \underline{A}$;
- $n = 1$: $\mathrm{LSym}^1_{\underline{A}}(\underline{M}) \simeq L\Lambda^1_{\underline{A}}(\underline{M}) \simeq L\Gamma^1_{\underline{A}}(\underline{M}) \simeq \underline{M}$

3.7.1 Construction: Derived Symmetric Algebra

In this subsection we will assemble the n-th degree derived symmetric powers into the derived symmetric algebra. We will drop the L in the notation of the latter and refer to it simply by $\mathrm{Sym}^*_{\underline{A}}(\underline{M})$, for $\underline{A}$ an animated ring and $\underline{M}$ any $\underline{A}$-module. Noteworthy, by construction Sym^* will be a left-adjoint to the functor "forgetting the algebra structure"; in other words, each $\mathrm{Sym}^*_{\underline{A}}(\underline{M})$ will be described as the "free" $\underline{A}$-algebra generated by $\underline{M}$. This subsection follows Lurie's [26],25.2.2.6.

Construction 3.7.1.1. The canonical (non-full) inclusion $\iota : \mathrm{CAlg} \hookrightarrow \mathrm{Mod}$ (obtained by left Kan extending the one at the level of Poly) induces a map $\mathrm{CAlg}^\Delta_{\underline{A}/} \simeq \mathrm{CAlg}^\Delta_{\underline{A}} \hookrightarrow \mathrm{Mod}_{\underline{A}}$ which (informally) acts on objects as $[\phi : \underline{A} \to \underline{B}] \mapsto (\underline{A}, \phi^*\underline{B})$.
Then, we obtain a functor between their un-straightenings $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{CRing})) \simeq \int \mathrm{CAlg}^\Delta$ and $\mathrm{MOD} \simeq \int \mathrm{Mod}$, so that the previous maps assemble into the following one:

$$U : \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{CRing})) \longrightarrow \mathrm{MOD}$$
$$[\phi : \underline{A} \to \underline{B}] \mapsto (\underline{A}, \phi^*\underline{B})$$

Observe that it preserves both limits and sifted colimits (so, in particular the filtered ones).

Therefore, we can apply the II Adjoint Functor Theorem 1.2.0.6 to obtain an adjunction:

$$\Phi\colon \mathrm{MOD} \overset{\perp}{\rightleftarrows} \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{CRing})) : U$$

With a slight abuse of notation which will readily become clear, we can informally describe its action on objects as follows.

Lemma 3.7.1.2. *Unwinding the definition, the adjunction $\Phi \vdash U$ lies over* $\mathrm{Ani}(\mathrm{CRing})$*. Then, we can (informally) describe the action of Φ on objects as:*

$$\Phi : (\underline{A}, \underline{M}) \mapsto [\underline{A} \to \mathrm{Sym}^*_{\underline{A}}(\underline{M})]$$

for some $\underline{A}$-algebra $\mathrm{Sym}^*_{\underline{A}}(\underline{M}) \in \mathrm{CAlg}^\Delta_{\underline{A}}$.

Proof. Recall first the following result on ∞-colimits: *the forgetful functor of an over-category commutes with colimits* ([24],1.2.13.8), i.e. the post-composition of a diagram p with the forgetful functor $for : \mathcal{C}/F \to \mathcal{C}$ admits a colimit $\overline{for(p)}$, iff p already admits a colimit $\overline{p} \in \mathcal{C}/F$ and - in such case - they coincide: $for(\overline{p}) \simeq \overline{for(p)}$.

Then, we claim that the left adjoint Φ is actually defined fibre-wise over Ani(CRing), so that it sits in the aside triangle as well. Here, we change our notation, namely let $ev_1 := pr_1 : \mathrm{MOD} \to \mathrm{Ani(CRing)}$; this stresses on the adjunction relation between the functors (cfr. [20],5.1).

$$\begin{tikzcd} \mathrm{MOD} \arrow[rr, shift left, "\Phi"] \arrow[dr, "ev_1"'] & & \mathrm{Fun}(\Delta^1, \mathrm{Ani(CRing)}) \arrow[ll, shift left, "U"] \arrow[dl, "ev_0"] \\ & \mathrm{Ani(CRing)} & \end{tikzcd}$$

In order to show this, since Φ is defined by means of the II Adjoint Functor Theorem, we shall closely inspect the proof of 1.2.0.6.
Let us introduce some shorthand: call $\mathcal{C} := \mathrm{Fun}(\Delta^1, \mathrm{Ani(CRing)})$, $\mathcal{D} := \mathrm{MOD}$, so that $U : \mathcal{C} \to \mathcal{D}$.
It suffices to show that, for each object $d \in \mathcal{D}$, the over-category $(\mathcal{C}_{d/})/\mathrm{Ani(CRing)}$ admits a weakly initial set $\mathcal{S}$, i.e. a small subset $\mathcal{S}$ of objects in the over-category such that - for any other object x of our over-category - there exists an arrow with source in $\mathcal{S}$ and target x.
$\mathcal{C}$ is presentable, so let κ be a regular cardinal for which there is a $\varinjlim^{\kappa}$-dense subcategory $\mathcal{C}^{\kappa} \subseteq_{f.f.} \mathcal{C}$.
Closely following the proof of 1.2.0.6 in [20],5.2.14, define $\mathcal{S} := \{(x, \alpha : d \to Ux) \mid x \in \mathcal{C}^{\kappa}\} \subseteq ob(\mathcal{C}_{d/})$; take any $(z, \beta : d \to Uz) \in \mathcal{C}_{d/}$ and let $\underline{A} := ev_0(z) \in \mathrm{Ani(CRing)}$. We claim that $\mathcal{S}/\underline{A}$ is the required weakly initial set of $(\mathcal{C}_{d/})/\underline{A}$.
In order to prove it, first notice that, since $\mathcal{C}$ is a sInd-completion, we can regard z as the sifted colimit of some (sifted) diagram $p : K \to \mathcal{C}^{\kappa}$. The canonical morphisms of colimits give a map $ev_0(p(k)) \to \underline{A}$, so that we may actually assume that also $Im(p) \subseteq \mathcal{C}/\underline{A}$.
By the previously stated [24],1.2.13.8, the over-slice projection $(\mathcal{C}_{d/})/\underline{A} \to \mathcal{C}_{d/}$ commutes with colimits, so that also $[z \to \underline{A}] \simeq \mathrm{colim}^{sift}[p(-) \to \underline{A}]$.
In turn, U preserves sifted colimits, so $U(z) \simeq \mathrm{colim}^{sift}\, U(p)$. Then, again by [24],1.2.13.8, since forgetful functors of over-slices - such as $\mathcal{D}/\underline{A} \to \mathcal{D}$ - reflect colimits, we can regard the latter colimit as living over $\underline{A}$.
Finally, strongly-compact objects remain such in over-slice categories. Indeed, again for [24],1.2.3.18 over-slice forgetful functors commute with colimits, so that, for any sifted diagram q, the mapping space equivalence

$$\mathrm{Map}_{\mathcal{D}}(d, \mathrm{colim}^{sift}\, q(-)) \simeq \mathrm{colim}^{sift}\, \mathrm{Map}_{\mathcal{D}}(d, q(-))$$

restricts to the following one over $\underline{A}$: fibres embed faithfully, so

$$\mathrm{Map}_{\mathcal{D}/\underline{A}}(d, \mathrm{colim}^{sift}(q(-)) \simeq \mathrm{colim}^{sift}\, \mathrm{Map}_{\mathcal{D}/\underline{A}}(d, q(-))$$

Hence, we can conclude (for $q := p$) that there is some $k \in K$ s.t. the map $\beta/\underline{A} : d/\underline{A} \to Uz/\underline{A}$ is represented in the colim_K^{sift} of mapping spaces by some $\delta/\underline{A} \in \mathrm{Map}_{\mathcal{D}/\underline{A}}(d, p(k))$. In other words, $\beta/\underline{A}$ factors through $\delta/\underline{A}$; then, we are done by the fact that $Im(p)/\underline{A} \subseteq \mathcal{S}/\underline{A}$. $\square$

We are finally ready to define the symmetric algebra associated to a pair $(\underline{A}, \underline{M}) \in \mathrm{MOD}$. In what follows we will write $\mathrm{Sym}^* \vdash for$ for the adjunction of the Lemma above. This will be applied quite often in the more geometric sections in order to define vector bundles on schemes (such as e.g. the affine space).

Construction 3.7.1.3. *(Symmetric Algebra)* Consider the following functor respecting fibres over Ani(CRing) and call it CSym^*, in agreement with the previous abuse of notation:

$$\mathrm{CSym}^*_{(-)}(-) : \mathrm{CRMod}^{\mathrm{sfp}} \xrightarrow{\Phi} \mathrm{Fun}(\Delta^1, \mathrm{Ani(CRing)}) \xrightarrow{U} \mathrm{MOD}$$

Remark. On the ordinary category $\mathrm{CRMod}^{\mathrm{sfp}}$, we recover the adjunction defining the universal property of symmetric algebras. More explicitly, for any $\big(A := \mathbb{Z}[X], M := \oplus_{i=1}^{m} \mathbb{Z}[X](y_i)\big) \in \mathrm{CRMod}^{\mathrm{sfp}}$,

$$\Phi(A, M) \cong (\mathbb{Z}[X] \xrightarrow{str} \mathbb{Z}[X, \underline{y}]) \quad \Longrightarrow \quad (U \circ \Phi)(A, M) \cong \big(\mathbb{Z}[X], \oplus_{n \geq 0} \mathrm{CSym}^n_{\mathbb{Z}[X]}(M)\big) = \big(A, \mathrm{Sym}^*_A(M)\big)$$

Moreover, by the construction CSym^* preserves sifted colimits in $\mathrm{CRMod}^{\mathrm{sfp}}$, so that it admits a left derived functor $\mathrm{LSym}^* : \mathrm{MOD} \to \mathrm{MOD}$.
Over each fibre $\mathrm{Mod}_{\underline{A}}$, the second projection of the latter coincides with the previously defined $\oplus_{n \geq 0} \mathrm{LSym}^n_{\underline{A}}$, since they agree on fibres of CRMod.
Thus, with reference to the construction of Derived Symmetric Powers, one has the expected canonical equivalence:

$$\forall (\underline{A}, \underline{M}) \in \mathrm{MOD} : \quad \mathrm{Sym}^*_{\underline{A}}(\underline{M}) \simeq \oplus_{n \geq 0} \mathrm{LSym}^n_{\underline{A}}(\underline{M})$$

3.7.2 Properties of Universal Tensor Algebra functors

In this subsection we will investigate some properties of the previously introduced Universal Tensor Algebra functors; we will short the latter as UTA-f.
Recall that by 3.2.5.1 each morphism $\phi : \underline{A} \to \underline{B}$ in Ani(CRing) induces a base-change adjunction which extends 3.2.4.2, namely $(-) \otimes^L_{\underline{A}} \underline{B} \colon \mathrm{Mod}\underline{A} \rightleftarrows \mathrm{Mod}_{\underline{B}} : \phi^*$.

Lemma 3.7.2.1. *(Stability under base-change, [26],25.2.3.1) Let $F_n : \mathrm{MOD} \to \mathrm{MOD}$ be a UTA-f. as before, so acting on objects by $(\underline{A}, \underline{M}) \mapsto (\underline{A}, LF_n(\underline{M}))$. Then, denoting by $pr_1 : \mathrm{MOD} \to \mathrm{Ani(CRing)}$ the canonical projection, F preserves pr_1-cocartesian morphisms.*
In other words, for each morphism $\phi : \underline{A} \to \underline{B}$ and each module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$,

- $\underline{B} \otimes^L_{\underline{A}} \mathrm{LSym}^n_{\underline{A}}(\underline{M}) \xrightarrow{\simeq} \mathrm{LSym}^n_{\underline{B}}(\underline{B} \otimes_{\underline{A}} \underline{M})$
- $\underline{B} \otimes^L_{\underline{A}} L\Lambda^n_{\underline{A}}(\underline{M}) \xrightarrow{\simeq} L\Lambda^n_{\underline{B}}(\underline{B} \otimes_{\underline{A}} \underline{M})$
- $\underline{B} \otimes^L_{\underline{A}} L\Gamma^n_{\underline{A}}(\underline{M}) \xrightarrow{\simeq} L\Gamma^n_{\underline{B}}(\underline{B} \otimes_{\underline{A}} \underline{M})$

Proof. Notice that, by 3.2.5.17, the stated equivalences correspond to the fact that F preserves pr_1-cocartesian morphisms. Then, we need to prove that the given canonical morphisms are indeed equivalences.

We will deal only with the case of symmetric powers, the other proofs are analogous. In particular, let us show that for any morphism of animated rings $\phi : \underline{A} \to \underline{B}$ and any $\underline{A}$-module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$, the canonical map $\alpha_{\underline{B},\underline{M}} : \underline{B} \otimes^L_{\underline{A}} \mathrm{LSym}^n_{\underline{A}}(\underline{M}) \to \mathrm{LSym}^N_{\underline{A}}(\underline{M})$ is an equivalence. In order to achieve this, we will undertake some reduction steps, so as to reduce the statement to the classical setting.

- WLOG $\underline{M} \simeq \underline{A} \otimes^L_{\mathbb{Z}} M_0$ *with* $M_0 \simeq \mathbb{Z}^{(n)}$*:*

  Fix a morphism $\phi : \underline{A} \to \underline{B}$. Since both $\mathrm{LSym}^n_{\underline{A}}(-)$ and $\underline{B} \otimes^L_{\underline{A}} (-)$ commute with sifted colimits, also the functor $\alpha_{\underline{B},(-)} : \mathrm{Mod}_{\underline{A}} \to \mathrm{Fun}(\Delta^1, \mathrm{Mod}_{\underline{B}})$, which acts on objects as $\underline{M} \mapsto \alpha_{\underline{B},\underline{M}}$, does (see [24],5.1.2.3). Hence, wlog $\underline{M} \simeq \underline{A}^{(n)}$.

  In particular, the monoidal structure on MOD allows us to write (up to homotopy) $\underline{M} \simeq \underline{A} \otimes^L_{\mathbb{Z}} M_0$ with $M_0 \simeq \mathbb{Z}^{(n)}$.

- WLOG $\underline{A} \simeq \mathbb{Z}$*, so we can reduce to* Poly*:*

  Consider the following commutative triangle:

$$\begin{tikzcd}
 & \underline{B} \otimes^L_{\underline{A}} \underline{A} \otimes^L_{\mathbb{Z}} \mathrm{LSym}^n_{\mathbb{Z}}(M_0) \arrow[dl, "\underline{B} \otimes^L_{\underline{A}} \alpha_{(\underline{A}, M_0)}"'] \arrow[dr, "\alpha_{\underline{B} \otimes^L_{\underline{A}} \underline{A}, M_0}"] & \\
\underline{B} \otimes^L_{\underline{A}} \mathrm{LSym}^n_{\underline{A}}(\underline{A} \otimes^L_{\mathbb{Z}} M_0) \arrow[rr, "\alpha_{\underline{B},\underline{M}}"'] & & \mathrm{LSym}^n_{\underline{B}}(\underline{B} \otimes^L_{\underline{A}} \underline{A} \otimes^L_{\mathbb{Z}} M_0)
\end{tikzcd}$$

  If we assume the canonical map $\alpha_{\underline{C},M_0}$ to be an equivalence for each morphism $\psi : \mathbb{Z} \to \underline{C}$ in Ani(CRing), then, in particular both $\alpha_{\underline{A},M_0}$ and $\alpha_{\underline{B} \otimes^L_{\underline{A}} \underline{A}, M_0}$ will be equivalences, and hence, by the 2-out-of-3 property, also $\alpha_{\underline{B},\underline{M}}$ will be so.

- WLOG $\underline{B} \simeq \mathbb{Z}[X]$ *for some finite tuple of indeterminates X, so we can reduce to the ordinary setting:*

  In view of the previous reduction steps, let us now fix some $(\underline{A}, \underline{M}) \simeq (\mathbb{Z}, \mathbb{Z}^{(n)}) \in \mathrm{Poly}$ and let us consider $\alpha_{\underline{B}} = \alpha_{(\underline{B},M)}$ as a functor of $\underline{B}$, namely $\alpha_{\underline{B}} : \mathrm{Ani(CRing)} \to \mathrm{Fun}(\Delta^1, \mathrm{MOD})$.

  Since it commutes with sifted colimits (it is the same argument as before), wlog $\underline{B} \simeq \mathbb{Z}[X]$ for some finite tuple of indeterminates X.

Therefore, we are left to prove that the following morphism (of static modules) is an iso:

$$\mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathrm{CSym}^n_{\mathbb{Z}}(\mathbb{Z}^{(n)}) \longrightarrow \mathrm{CSym}^n_{\mathbb{Z}[X]}(\mathbb{Z}^{(n)})$$

The latter claim follows from the explicit construction of symmetric powers: on the free $\mathbb{Z}$-module $\mathbb{Z}^m \cong \oplus_{y \in \underline{y}} \mathbb{Z}(y)$ with $\underline{y}$ a tuple of m free generators, our map becomes the canonical base change isomorphism:

$$\mathbb{Z}[X] \otimes_{\mathbb{Z}} (-) : \quad \mathbb{Z}[X] \otimes_{\mathbb{Z}} \left(\oplus_{|\underline{\alpha}|=n} \mathbb{Z}(\underline{y}^{\otimes \underline{\alpha}}) \right) \xrightarrow{\simeq} \oplus_{|\underline{\alpha}|=n} \mathbb{Z}[X](\underline{y}^{\otimes \underline{\alpha}})$$

□

Moreover, let us record here an important Corollary, which will be used later on in order to define vector bundles in DAG.

Corollary 3.7.2.2. *(Local freeness, [26],25.2.3.2) Let $(\underline{A}, \underline{M}) \in \mathrm{MOD}$ be animated modules with rings of scalars s.t. $\underline{M}$ is (locally) free of rank r over $\underline{A}$. Then, the following UTA-f's $\mathrm{MOD} \to \mathrm{MOD}$ preserve (local) freeness (see 3.6.2.3):*

- $\mathrm{LSym}^n_{\underline{A}}(\underline{M})$ *is (locally) free of rank* $\binom{n+r-1}{n}$;
- $L\Lambda^n_{\underline{A}}(\underline{M})$ *is (locally) free of rank* $\binom{r}{n}$;
- $L\Gamma^n_{\underline{A}}(\underline{M})$ *is (locally) free of rank* $\binom{n+r-1}{n}$.

Proof. Claim. *wlog $\underline{M} \simeq \underline{A} \otimes_{\mathbb{Z}} \mathbb{Z}^r$ free.*

Proof. Let $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ be a locally free module of rank r. By 3.6.2.4, there exists a partition of unity $\{x_i\}_{i=1}^m$ of $\underline{A}$ such that each $\underline{M}[x_i^{-1}]$ is $\underline{A}[x_i^{-1}]$-free of rank r.
Then, assume that, for each i, the given UTA-f for $\underline{A}[x_i^{-1}]$, $\underline{M}[x_i^{-1}]$ is a free $\underline{A}[x_i^{-1}]$-free module of the right rank. By 3.7.2.1, they can be written as localizations at $\underline{A}[x_i^{-1}]$ of the original UTA-f for $(\underline{A}, \underline{M})$. Hence, we conclude by 3.6.2.5. ■

As in the previous Proposition, *wlog* $(\underline{A}, \underline{M}) \simeq (\mathbb{Z}, \mathbb{Z}^r)$. Hence, the result follows from an explicit computation, as performed in the definition of the UTA-f at stake. □

We will finally provide a couple of results comparing ordinary and derived Universal Tensor Algebra functors of static modules.

Corollary 3.7.2.3. *(UTA-f. preserve flatness, [26],25.2.3.3) Let $\underline{A} \in \mathrm{Ani(CRing)}$ and $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ be flat. Then, also $L\mathrm{Sym}^n_{\underline{A}}(\underline{M})$, $L\Lambda^n_{\underline{A}}(\underline{M})$ and $L\Gamma^n_{\underline{A}}(\underline{M})$ are flat $\underline{A}$-modules.*

Proof. By the ∞-Lazard's Theorem 3.4.0.7, our flat animated $\underline{A}$-module admits a presentation $\underline{M} \simeq \underrightarrow{\mathrm{colim}}(\underline{A}^{n_i} \mid i \in I)$ as a filtered colimit of finitely generated free $\underline{A}$-modules over an arbitrary directed set I. Our UTA-f. commute with sifted colimits by the construction via animation, so that wlog $\underline{M} \simeq \underline{A}^n$ is finitely generated free. Then, we conclude by the previous result. □

In particular, over a static ring $A \in \mathrm{CRing}$ and for any flat (hence discrete) A-module $\underline{M} = M$, we recover the static versions of our UTA-f.

Proposition 3.7.2.4. *(Compatibility of UTA-f, [26],25.2.3.4) Let $A \in \mathrm{CRing}$ be a static ring and consider $M \in \mathrm{Mod}_A$ flat (hence static). Then, there are canonical isomorphisms of static A-modules which recover the classical UTA-f:*

- $\alpha : L\mathrm{Sym}^n_A(M) \simeq C\mathrm{Sym}^n_A(M)$;
- $\beta : L\Lambda^n_A(M) \simeq \Lambda^n_A(M)$;
- $\gamma : L\Gamma^n_A(M) \simeq \Gamma^n_A(M)$

Proof. Let us prove only the first one, the rest is analogous. We will employ the usual machinery. Consider the restrictions of the functors $L\mathrm{Sym}^n$ and $C\mathrm{Sym}^n$ to the ordinary category CRMod:

$$\mathrm{Sym}^n : \mathrm{CRMod} \longrightarrow \mathrm{MOD} \qquad\qquad \mathrm{Sym}^n_s : \mathrm{CRMod} \longrightarrow \mathrm{MOD}$$
$$(A, M) \mapsto (A, L\mathrm{Sym}^n_A(M)) \qquad\qquad (A, M) \mapsto (A, C\mathrm{Sym}^n_A(M))$$

Let $\mathcal{C} \subseteq_{f.f.} \mathrm{CRMod}^{\mathrm{sfp}}$ be as in 3.2.5.2, and observe that, by the construction, $\mathrm{Sym}^n_{|\mathcal{C}} \simeq \mathrm{Sym}^n_{s\,|\mathcal{C}}$ and $\mathrm{Sym}^n \simeq \mathrm{LKE}_j(\mathrm{Sym}^n_{|\mathcal{C}})$. Hence, by the universal property of left Kan extensions, there must be an essentially unique comparison map $\alpha : \mathrm{Sym}^n \to \mathrm{Sym}^n_s$ which restricts to an isomorphism on $\mathcal{C}$.
Observe that, on the static fibre $\mathrm{Mod}(A)$ of CRMod over A, the latter corresponds to a natural transformation $\alpha_A : L\mathrm{Sym}^n_A(-) \to C\mathrm{Sym}^n_A(-)$.

Claim. *$\alpha_{A,M}$ is an equivalence whenever M is flat.*

Proof. By Lazard's Theorem, the static module M admits a presentation as a filtered colimit of finitely generated free A-modules. Now, for a fixed A, the functor corresponding to α_M commutes with directed colimits. Hence, we can assume wlog $M \simeq A^n$ finitely generated free.
In such a case, write $M \simeq A \otimes_{\mathbb{Z}} \mathbb{Z}^n$ and observe that we can consider $\alpha_{\mathbb{Z}^n}$, since by 3.7.2.1 α commutes with base-change. But then, $\alpha_{\mathbb{Z}^n}$ is an equivalence by the construction: see the Remark at 3.7.0.1. □

Warning. As commented by Lurie at [26],25.2.3.5, for $A \in \mathrm{CRing}$ and a static $M \in \mathrm{Mod}(A)$ which is however *not* flat, the UTA-f considered might be non-static, in the same way as the "derived" tensor product $\otimes^L$ of static modules is in general non-static. In particular, $L\Lambda^n_A(M)$, $L\Gamma^n_A(M)$ need not coincide with the usual functors from commutative algebra, even though their underlying connected components do recover the usual construction.
On the other hand, on the case of Derived Symmetric Powers we can say more: in [26],25.2.6, Lurie compares our construction $L\mathrm{Sym}$ with the simplicial one over $\underline{A}$, seen as a connected $\mathbb{E}_\infty$-ring; the latter recovers the classical commutative algebra and the two coincide whenever we are working with a rational animated ring, namely a $\mathbb{Q}$-algebra $\underline{A}$.
Notice that this ought to be expected, since the rationality condition turns out to guarantee the equivalence of the two formalisms. In other words, we are implying that the construction of Symmetric Powers coincides with the one in the setting of $\mathbb{E}_\infty$-rings, whenever we are working rationally over $\underline{A} \in \mathrm{CAlg}^{\Delta}_{\mathbb{Q}}$, so that it can be entirely retrieved from the (spectral) symmetric monoidal structure on $\mathrm{Mod}_{\underline{A}}$.

3.8 The (Relative) Algebraic Cotangent Complex and Derivations

In this section we present the derived version of the ubiquitous construction of the module $\Omega_{\pi_0\underline{B}/\pi_0\underline{A}}$ of differentials for a map of classical $\pi_0\underline{A}$-algebras $\pi_0\underline{A} \to \pi_0\underline{B}$. This will lead to the notion of the relative (algebraic) cotangent complex between the corresponding animated algebras $L_{\underline{B}/\underline{A}}$, as introduced by Lurie in [26],25.3.1.
The topics discussed in this section admit a natural translation into the language of DAG with respect to which they correspond to the conormal sheaf. This will be discussed in section 4.4 and will also be regarded as motivating the algebraic counterpart.

3.8.1 The Algebraic Cotangent Complex

In view of the Lurie's introduction in [23],7.4.1, sometimes intuition will be conveyed in the language of ∞-group actions and the corresponding principal ∞-bundles. From such a perspective, a square-free extension of $\underline{A} \in \mathrm{Ani(CRing)}$ by $\underline{M} \in \mathrm{Mod}_{\underline{A}}$ should be seen as an ∞-action (see [30],3.1) of $M \in \mathrm{Grp}_\infty(\mathrm{Spc})$ on $A \in \mathrm{Spc}$ at the level of underlying spaces.
Our focus being the geometric version of such a construction, we will limit such comments and simply freely adopt the needed terminology; we refer the unfamiliar reader to the relative section of Appendix C or to the extensive exposition [30], where the theory of ∞-bundles is revisited in the modern language of ∞-categories and ∞-topoi.

Remark. Let us briefly comment on the legitimacy of our choice. From the very definition, animated widgets from $\mathcal{C} \in \mathrm{Cat}_1$ are product preserving functors $(\mathcal{C}^{\mathrm{sfp}})^{op} \to \mathrm{Spc}$; when $\mathcal{C}$ is an 'algebraic category' we proved at the end of Appendix A that a system of cpt+proj generators is given by taking free $\mathcal{C}$-widgets on FinSet, so that animated $\mathcal{C}$-widgets turn out to satisfy the Segal condition (recall that they take the point in Set to the one in Spc), and hence to be ∞-group objects in the ∞-topos Spc in the sense of C.1.0.5. Therefore, the framework provided by [30] applies.

Construction 3.8.1.1. *(Trivial square-zero extension)* With notation as in 3.2.5.2, consider the full subcategory $\mathcal{C} \subseteq_{f.f.} \mathrm{MOD}$ spanned by $\{(A, M) \mid A \in \mathrm{Poly},\ M \cong A^{(n)}\, \exists n \in \mathbb{N}\}$. Recall that for each $A \in \mathrm{CRing}$ and $M \in \mathrm{Mod}(A)$ we can endow $A \oplus M$ with an A-algebra structure via $(a, m) \cdot (a', m') := (aa', am' + a'm)$. Hence, we can define a functor

$$F : \mathcal{C} \to \mathrm{CRing} \subseteq \mathrm{Ani}(\mathrm{CRing})$$
$$(A, M) \mapsto A \oplus M$$

Let $\oplus : \mathrm{MOD} \to \mathrm{Ani}(\mathrm{CRing})$ denote the left-derived functor of F (see A.3.0.2). Notice that it preserves sifted colimits by the construction.

Let us record some useful properties of the functor $\oplus$.

Lemma 3.8.1.2. *(Properties of $\oplus$) Let $\oplus : \mathrm{MOD} \to \mathrm{Ani}(\mathrm{CRing})$ be the trivial square-zero extension functor. Then, for any $(\underline{A}, \underline{M}) \in \mathrm{MOD}$*

1. *([26],25.3.1.2) $\oplus$ preserve stasis, i.e. each $(A, M) \in \mathrm{CRMod}$ is taken to $A \oplus M \in \mathrm{CRing}$;*
2. *([26],25.3.1.3) $\underline{M} \simeq 0$ in $\mathrm{Mod}_{\underline{A}}$ yields a canonical equivalence $\underline{A} \oplus \underline{M} \simeq \underline{A}$;*
3. *([26],25.3.1.3) We can canonically regard $\underline{A} \oplus \underline{M} \in \mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{A}}$.*

Proof. (1) : By the construction. (2) : Let $\mathcal{E} \subseteq_{f.f.} \mathrm{MOD}$ be the full subcategory spanned by $\{(\underline{B}, \underline{N}) \in \mathrm{MOD} \mid \underline{N} \simeq 0\}$. By A.4.0.5 and the construction 3.2.5.2, the small set $\mathcal{S} := \{(A, 0) \mid A \in \mathrm{Poly}\}$ forms a set of cpt+proj generators of $\mathcal{E}$ and the restriction $\oplus_{|\mathcal{E}}$ agrees with the first projection $pr_1 : (A, M) \mapsto A$ over CRMod, and hence over $\mathcal{S}$. Thus, since $\oplus_{|\mathcal{E}}$ preserves sifted colimits, we obtain an equivalence $\oplus_{|\mathcal{E}} \simeq pr_1$ over $\mathcal{E}$. Informally, on objects this means that, for each $\underline{M} \simeq 0 \in \mathrm{Mod}_{\underline{A}}$, there is a canonical equivalence $\underline{A} \oplus \underline{M} \simeq \underline{A}$ as left Kan extensions over Poly.
(3) : Given any $\underline{M} \in \mathrm{Mod}_{\underline{A}}$, there is a canonical sequence of arrows $0 \to \underline{M} \to 0$ in the pointed category $\mathrm{Mod}_{\underline{A}}$. Then, in view of part (2), an application of $\underline{A} \oplus (-)$ yields arrows $\underline{A} \to \underline{A} \oplus \underline{M} \to \underline{A}$, thus exhibiting $\underline{A} \oplus \underline{M} \in \mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{A}}$ in a canonical way. ☐

Lemma 3.8.1.3. *The canonical map $\underline{A} \oplus \underline{M} \to \underline{A}$ can be regarded as a projection on the first component and it sits in a fibre sequence in $\mathrm{Mod}_{\underline{A}}$:*

$$\underline{M} \hookrightarrow \underline{A} \oplus \underline{M} \twoheadrightarrow \underline{A}$$

with the last map being an effective epimorphism. Hence, the trivial square-zero extension $\underline{A} \oplus \underline{M} \to \underline{A}$ is the trivial ∞-bundle on $\underline{A}$ with fibre $\underline{M}$ (cfr. Appendix C).

Proof. Recall first that there is a canonical forgetful functor $for : \mathrm{Ani}(\mathrm{CRing}) \to \mathrm{Mod}_{\mathbb{Z}}$ which acts fibre-wise as $\mathrm{CAlg}^{\Delta}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}}$ and that preserves sifted colimits (being it itself a left-derived functor).
Notice that $for \circ \oplus : \mathrm{MOD} \to \mathrm{Mod}_{\mathbb{Z}}$ is equivalent to the binary product $\times : \mathrm{MOD} \subseteq \mathrm{Mod}_{\mathbb{Z}} \times \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}}$, since they both are left-derived functors of the static binary product $\times_{|\mathcal{C}} : \mathcal{C} \subseteq \mathrm{Mod}(\mathbb{Z}) \times \mathrm{Mod}(\mathbb{Z}) \to \mathrm{Mod}(\mathbb{Z})$. Then, in $\mathrm{Mod}_{\underline{A}}$ the canonical map $\underline{A} \oplus \underline{M} \to \underline{A}$ corresponds to $pr_1 : \underline{A} \times \underline{M} \to \underline{A}$, as one can see from the following left square describing the functoriality of $\oplus$. So, as for additive 1-categories, we will stick to the direct sum notation $\oplus$ for the functor $for \circ \oplus$.

$$\begin{tikzcd}
\mathrm{MOD} : \arrow[d, "\oplus"'] & (\underline{A}, \underline{M}) \arrow[r] \arrow[d, "\oplus"'] & (\underline{A}, 0) \arrow[d, "pr_1"] \\
\mathrm{Ani}(\mathrm{CRing}) : & \underline{A} \oplus \underline{M} \arrow[r] & \underline{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Mod}_{\underline{A}} : \arrow[d, "\underline{A}\oplus(-)"'] & \underline{M} \arrow[r, two heads] \arrow[d, "\underline{A}\oplus(-)"'] & 0 \arrow[d] \\
\mathrm{Mod}_{\underline{A}} : & \underline{A} \oplus \underline{M} \arrow[r, two heads] & \underline{A}
\end{tikzcd}$$

Finally, by a similar reasoning we deduce that $\oplus_{|\mathrm{Mod}_{\underline{A}}}$ is also equivalent to the binary coproduct in $\mathrm{Mod}_{\underline{A}}$, so that we obtain the stated fibre-sequence.
Moreover, being $\underline{M} \twoheadrightarrow 0$ surjective on π_0, also $\underline{A} \times \underline{M} \to \underline{A}$ is clearly such, since $\oplus \simeq for \circ \oplus \simeq \times$ is exact. $\square$

Remark. The surjectivity on π_0 implies that the fibre sequence above in the pre-stable $\mathrm{Mod}_{\underline{A}}$ gives a fibre-cofibre sequence in the stabilization $\mathrm{Mod}_{\underline{A}}^{\mathrm{Ex}}$ (see 3.4.0.1)

Similarly to the classical setting, we will now define square-zero extensions of $\underline{A}$ by $\underline{M}$ (or $\underline{A}$-derivations on $\underline{M}$) as twists of the trivial square-zero extension $\underline{A} \oplus \underline{M}$, namely sections of the trivial $\underline{M}$-bundle $\underline{A} \oplus \underline{M} \to \underline{A}$.

Definition 3.8.1.4. *(Space of derivations, [26],25.3.1.4)* The **space of derivations** associated to the pair $(\underline{A}, \underline{M}) \in \mathrm{MOD}$ is $\mathrm{Der}(\underline{A}, \underline{M}) := \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, \underline{A} \oplus \underline{M}) \in \mathrm{Spc}$. Call derivations its points.

Construction 3.8.1.5. *(Functoriality of Derivations)* We will show that the construction $[(\underline{A}, \underline{M}) \mapsto \mathrm{Der}(\underline{A}, \underline{M})]$ defines a functor $\mathrm{Der} : \mathrm{MOD} \to \mathrm{Spc}$.
Fibre-wise over $\underline{A} \in \mathrm{Ani(CRing)}$ it can be described as $\mathrm{Der}(\underline{A}, -) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Spc}$, namely as a choice of composition for $\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, -) \circ (\underline{A} \oplus (-)_{|\mathrm{Mod}_{\underline{A}}})$ (up to contractible ambiguity).

More generally, let F denote a choice of the composition

$$F \simeq \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\Delta})} \circ (\mathrm{id}_{(pr_1)^{op}}, \oplus \to pr_1) : \mathrm{MOD} \to \mathrm{Tw}(\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\Delta})) \to \mathrm{Spc}$$

Under the Straightening Theorem [24],3.2, the co-presheaf F is classified by a right-fibration $q : \int F \to \mathrm{MOD}$; the 1-truncation of $\int F$ retrieves the classical Grothendieck constructions, so that the objects of $\int F$ are triples of the form

$$[(\underline{A}, \underline{M}), \sigma \in \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathrm{CAlg}^{\Delta})}(1_{\underline{A}}, \underline{A} \oplus \underline{M} \to \underline{A})]$$

Let $\int \mathrm{Der} \subseteq_{f.f.} \int F$ denote the full sub-category spanned by those triples as before with squares $\sigma \in \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, \underline{A} \oplus \underline{M})$, so such that the lower horizontal map is the identity.
Such an assignment defines a sub-co-presheaf $\mathrm{Der} \leq F$ which acts on objects as $(\underline{A}, \underline{M}) \mapsto \mathrm{Der}(\underline{A}, \underline{M})$.

Our next goal is to classify such fibre bundles, namely to prove that Der is fibre-wise representable by some moduli stack, which we will call the (absolute) algebraic cotangent complex. Moreover, we will be able to describe the corresponding universal arrow as follows.

Definition 3.8.1.6. *(Universal derivation)* For any pair $(\underline{A}, \underline{M_0}) \in \mathrm{MOD}$, each derivation $\eta \in \mathrm{Der}(\underline{A}, \underline{M_0})$ induces a map

$$ev_\eta := \eta^* \circ (\underline{A} \oplus -) : \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{M_0}, \underline{M}) \to \mathrm{Der}(\underline{A}, \underline{M})$$

We call **universal** a derivation $d \in \mathrm{Der}(\underline{A}, \underline{M_0})$ for some $\underline{M_0} \in \mathrm{Mod}_{\underline{A}}$ s.t. ev_d is an equivalence for each $\underline{M} \in \mathrm{Mod}_{\underline{A}}$.

Proposition 3.8.1.7. *(Algebraic Cotangent Complex, [26],25.3.1.5) For each animated ring $\underline{A} \in \mathrm{Ani(CRing)}$, the functor $\mathrm{Der}(\underline{A}, -) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Spc}$ is co-represented by $L_{\underline{A}} \in \mathrm{Mod}_{\underline{A}}$ and admits a universal derivation $d \in \mathrm{Der}(\underline{A}, L_{\underline{A}})$). We call $L_{\underline{A}}$ the **(algebraic) cotangent complex** associated to $\underline{A}$.*

Proof. The proof is now by abstract non-sense and amounts to an application of 1.3.0.2. Let us show that the functor $\mathrm{Der}(\underline{A}, -) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Spc}$ preserves limits and is accessible.
Notice first that the corepresentable functor $\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, -)$ enjoys such properties, so we are left to consider the restriction of $\oplus$ to $\mathrm{Mod}_{\underline{A}}$. Now, the latter commutes with sifted colimits by construction, so it is in particular accessible.
As for limits, instead, by Yoneda Lemma we need to test the commutativity of $\underline{A} \oplus (-)_{|\mathrm{Mod}_{\underline{A}}}$, $\lim_K$ with any K-indexed diagram $p : K \to \mathrm{Mod}_{\underline{A}}$. Since $\mathrm{CAlg}^{\Delta}_{\underline{A}} \simeq \mathcal{P}_\Sigma(\mathrm{Poly}_{\underline{A}})$, by 3.2.3.4, it suffices to test the commutativity with limits against finitely generated polynomial $\underline{A}$-algebras.
The following chain of equivalences holds:

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}\big(\underline{A}[t_i]_{i=1}^n, \underline{A}\oplus \lim_K p\big) &\simeq_{(a)} \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}\big(\underline{A}^n, \mathit{for}\circ(\underline{A}\oplus \lim_K p)\big)\\
&\simeq_{(b)} \lim_K \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}\big(\underline{A}^n, \mathit{for}\circ(\underline{A}\oplus p)\big)\\
&\simeq \lim_K \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}(\mathrm{Sym}^*(\underline{A}^n), \underline{A}\oplus p)\\
&\simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}(\underline{A}[t_i]_{i=1}^n, \lim_K \underline{A}\oplus p)
\end{aligned}
$$

where

- (a) : there is a fibre-wise adjunction $\Phi = L\mathrm{Sym}^* \colon \mathrm{Mod}_{\mathbb{Z}} \rightleftarrows \mathrm{Ani}(\mathrm{CRing}) : \mathit{for}$ (see 3.7.1.2); moreover, by 3.7.2.1, finitely generated polynomial $\underline{A}$-algebras can be written as
$$L\mathrm{Sym}^*_{\underline{A}}(\underline{A}^n) \simeq L\mathrm{Sym}^*_{\underline{A}}(\underline{A}\otimes^L_{\mathbb{Z}} \mathbb{Z}^n) \simeq \underline{A}\otimes^L_{\mathbb{Z}} \mathbb{Z}[t_i]_{i=1}^n \simeq \underline{A}[t_i]_{i=1}^n$$
- (b) : mapping spaces commute with limits in the second variable; moreover, by 3.8.1.3, the composition $\mathit{for}\circ\oplus$ amounts to the binary product in $\mathrm{Mod}_{\underline{A}}$, hence it commutes with small limits.

Thus, an application of 1.3.0.2 implies that $\mathrm{Der}(\underline{A}, -)$ is corepresentable by some $L_{\underline{A}} \in \mathrm{Mod}_{\underline{A}}$.
Unwinding the definitions, under the Straightening Theorem [24],3.2 this amounts to the following commutative cube:

$$
\begin{tikzcd}
\int \mathrm{Map}(L_{\underline{A}}, -) \arrow[rr] \arrow[dr, "\mathrm{St}(\delta)", "\simeq"'] \arrow[dd] & & \mathrm{Spc}_{//*} \arrow[dr, equals] \arrow[dd] & \\
& \int \mathrm{Der}(\underline{A}, -) \arrow[rr] \arrow[dd] & & \mathrm{Spc}_{//*} \arrow[dd] \\
\mathrm{Mod}_{\underline{A}} \arrow[rr, "\mathrm{Map}(L_{\underline{A}},-)"] \arrow[dr, equals] & & \mathrm{Spc} \arrow[dl, Rightarrow, "\delta"', "\simeq"] \arrow[dr, equals] & \\
& \mathrm{Mod}_{\underline{A}} \arrow[rr, "\mathrm{Der}(\underline{A},-)"] & & \mathrm{Spc}
\end{tikzcd}
$$

At the level of objects, the identification $\mathrm{St}(\delta) : \int \mathrm{Map}(L_{\underline{A}}, -) \simeq \int \mathrm{Der}(\underline{A}, -)$ corresponds to a bijection of triples.

$$\mathrm{St}(\delta)_0 : [(\underline{A}, \underline{M}), f \in \mathrm{Map}_{\underline{A}}(L_{\underline{A}}, \underline{M})] \leftrightarrow [(\underline{A}, \underline{M}), \eta \in \mathrm{Der}(\underline{A}, \underline{M})]$$

Claim. *The identification* $[(\underline{A}, L_{\underline{A}}), 1_{L_{\underline{A}}} \in \mathrm{Map}_{\underline{A}}(L_{\underline{A}}, L_{\underline{A}})] \leftrightarrow [(\underline{A}, L_{\underline{A}}), d \in \mathrm{Der}(\underline{A}, L_{\underline{A}})]$ *gives the universal derivation* d*.*

Proof. Under the Yoneda Lemma (see [20],4.2.10, [20],4.2.11) the following equivalence in $\mathrm{Fun}(\mathrm{Mod}_{\underline{A}}, \mathrm{Spc})$ is induced by evaluation at $1_{L_{\underline{A}}}$:

$$\mathrm{Map}_{\underline{A}}(L_{\underline{A}}, -) \simeq \mathrm{Map}_{\mathrm{Spc}}\big(\mathrm{Map}_{\underline{A}}(L_{\underline{A}}, L_{\underline{A}}), \mathrm{Map}_{\underline{A}}(L_{\underline{A}}, -)\big)$$

Then, by the functoriality of δ we obtain a commutative square of equivalences where the left vertical arrow acts as conjugation by δ:

$$
\begin{tikzcd}
\mathrm{Map}_{\mathrm{Spc}}\big(\mathrm{Map}_{\underline{A}}(L_{\underline{A}}, L_{\underline{A}}), \mathrm{Map}_{\underline{A}}(L_{\underline{A}}, -)\big) \arrow[r, "ev_{\mathrm{id}}"] \arrow[d, "(\delta_*)(\delta^{-1}_{L_{\underline{A}}})^*"] & \mathrm{Map}_{\underline{A}}(L_{\underline{A}}, -) \arrow[d, "\delta"] \\
\mathrm{Map}_{\mathrm{Spc}}\big(\mathrm{Der}(\underline{A}, L_{\underline{A}}), \mathrm{Der}(\underline{A}, -)\big) \arrow[r, "ev_d"] & \mathrm{Der}(\underline{A}, -)
\end{tikzcd}
$$

Let us spell out the action on objects. Informally, this means that we have a family of commutative squares indexed by $f : L_{\underline{A}} \to \underline{M}$ in $\mathrm{Mod}_{\underline{A}}$ and which is compatible up to higher coherent homotopy:

$$
\begin{tikzcd}
\mathrm{Map}_{\underline{A}}(L_{\underline{A}}, L_{\underline{A}}) \arrow[r, "f\circ(-)"] \arrow[d, "\delta"] & \mathrm{Map}_{\underline{A}}(L_{\underline{A}}, \underline{M}) \arrow[d, "\delta"] \\
\mathrm{Der}(\underline{A}, L_{\underline{A}}) \arrow[r, "(\underline{A}\oplus -)\circ(f\circ -)"] & \mathrm{Der}(\underline{A}, \underline{M})
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1_{L_{\underline{A}}} \arrow[r] \arrow[d] & f \arrow[d] \\
d \arrow[r] & \delta(f)
\end{tikzcd}
$$

which exhibits $\delta(f) \simeq (\underline{A} \oplus f) \circ d \simeq d^*(\underline{A} \oplus -)(f) = ev_d(f)$ as in the definition of a universal arrow. $\square$

3.8.2 The Relative Algebraic Cotangent Complex

In this section we introduce a relative version of the algebraic cotangent complex. The first step is to observe that our moduli space does indeed define a functor. In order to show this, we will use that the algebraic cotangent complex of any static polynomial algebra is the corresponding usual module of derivations (see 3.8.3.1). For expository reasons, we postpone the proof to a section on examples; the reader should remark that this causes no cyclic argument.

Proposition 3.8.2.1. *(The cotangent complex functor, [26],25.3.1.8) The construction* $[\underline{A} \mapsto (\underline{A}, L_{\underline{A}})]$ *extends to a functor* $\mathrm{Ani(CRing)} \to \mathrm{MOD}$ *of ∞-categories. In particular, any map $f : \underline{A} \to \underline{B}$ of animated rings induces a morphism $\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}} \to L_{\underline{B}}$ in* $\mathrm{Mod}_{\underline{B}}$.

Proof. Consider the functor $L : \mathrm{Poly} \to \mathrm{CRMod} \subseteq \mathrm{MOD}$ acting as

- Obj: $A := \mathbb{Z}[t_i]_{i=1}^n \mapsto (A, \Omega_A)$;
- Mor: $(f : A \to B) \mapsto (B, B \otimes_A \Omega_A)$

Let $\mathcal{L} : \mathrm{Ani(CRing)} \to \mathrm{MOD}$ denote its left-derived functor; its action on objects can be described as follows. Let $\underline{A} \in \mathrm{Ani(CRing)}$ be the sifted colimit of some diagram $p : K \to \mathrm{Poly}$. Then, under the base-change adjunction for animated modules, $L(p) \simeq (for)^*(\Omega_p) : K \to \mathrm{FFree}_{\mathbb{Z}}$ where $for : \mathrm{Mod}(p) \to \mathrm{Mod}(\mathbb{Z})$ level-wise forgets the scalar structure (see 3.8.3.1 for the computation); equivalently, we can express the functoriality of $\Omega_{(-)}$ in $\mathrm{Mod}_{\underline{A}}$ by extending scalars along the canonical natural transformation $\psi : p \to \underline{A}$.
So, since $pr_1 : \mathrm{MOD} \to \mathrm{Ani(CRing)}$ preserves colimits, we can regard our diagram as living over $pr_1(\mathcal{L}(\underline{A})) \simeq \underline{A}$.
Now, the second projection preserves colimits in each fibre of $pr_1 : \mathrm{MOD} \to \mathrm{Ani(CRing)}$, so our construction yields a chain of equivalences:

$$\mathcal{L}(\underline{A}) \simeq \mathrm{colim}_K^{sift}(p, \underline{A} \otimes^L_p \Omega_p) \simeq (\underline{A}, |\underline{A} \otimes^L_p \Omega_p|)$$

where we used that sifted colimits in over-slices can be computed in the above category. Then, we are left to show the following claim.

Claim. *([26],25.3.1.8) If $\underline{A} \simeq |p|$ is the geometric realization of a simplicial diagram $: K \to \mathrm{Poly}$, then $L_{\underline{A}} \simeq |\underline{A} \otimes^L_p L_p| \simeq |\underline{A} \otimes^L_p \Omega_p|$.*

Proof. By assumption, $\mathrm{Map}_{\underline{A}}(L_{\underline{A}}, -) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, \underline{A} \oplus -) \simeq \mathrm{colim}_K^{sift} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(p, p \oplus -)$, since the diagonal of a sifted simplicial set is cofinal and $\oplus$ commutes with sifted colimits separately in each variable. Now, the functoriality of both $\Omega_{(-)}$ and the co-representability equivalence yields a canonical morphism under extension of scalars along the colimit map of diagrams $\psi : p \to \underline{A}$:

$$\mathrm{colim}_K^{sift} \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{A} \otimes^L_p \Omega_p, -) \to \mathrm{colim}_K^{sift} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(p, p \oplus -)$$

Finally, the latter turns out to be an equivalence, because it is such level-wise on K. Therefore, we have the sought equivalence of mapping spaces and we conclude by the Yoneda Lemma:

$$\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(|\underline{A} \otimes^L_p \Omega_p|, -) \simeq \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(L_{\underline{A}}, -)$$ ■

For what concerns the action of $\mathcal{L}$ on morphisms, take any $f : \underline{A} \to \underline{B}$ in $\mathrm{Ani(CRing)}$ and let's argue as in 3.2.5.11 (with the same notation). Namely, let $\psi : p \to q$ be a natural transformation of sifted diagrams wlog $K \to \mathrm{Poly}$ whose geometric realization is $f : \underline{A} \to \underline{B}$. Then, $\mathcal{L}(f) \simeq |L(\psi)|$ with $L(\psi) : L(p) \to L(q)$ in $\mathrm{Mod}(p)$. By the animated extension of scalars, post-composing with the canonical colimit map yields a natural transformation $\underline{B} \otimes^L_{\underline{A}} L(p) \to |L(q)|$ in $\mathrm{Mod}_{\underline{B}}$; since $\otimes^L$ commutes with sifted colimits separately in each variable, the latter canonically corresponds to a map $\underline{B} \otimes^L_{\underline{A}} |L(p)| \to |L(q)|$. This is the sought canonical map $\mathcal{L}(f) : \underline{B} \otimes^L_{\underline{A}} \mathcal{L}(\underline{A}) \to \mathcal{L}(\underline{B})$. $\square$

In particular, our functorial construction supplies for a fibre-wise left-adjoint to the trivial square-zero extension functor.

Proposition 3.8.2.2. *(Square-zero extension is a right-adjoint, [26],25.3.2.3) For* $\underline{B} \in \mathrm{Ani(CRing)}$ *let* $\underline{B} \otimes^L_{(-)} L_{(-)} : \mathrm{Ani(CRing)} \to \mathrm{Mod}_{\underline{B}}$ *denote the functor extending the construction* $[\underline{A} \mapsto \underline{B} \otimes^L_{\underline{A}} L_{\underline{A}}]$ *of 3.8.2.1. Then, there is an adjunction:*

$$\underline{B} \otimes^L_{(-)} L_{(-)} \colon \mathrm{CAlg}^{\Delta}_{/\underline{B}} \rightleftarrows \mathrm{Mod}_{\underline{B}} : \underline{B} \oplus (-)_{|\mathrm{Mod}_{\underline{B}}}$$

In particular, $\underline{B} \otimes^L_{(-)} L_{(-)}$ *is co-continuous.*

Proof. We will apply the dual of [20],5.1.10 so as to promote to an adjunction the following assignments:

- Obj: $(\phi : \underline{A} \to \underline{B}) \mapsto \underline{B} \otimes^L_{\underline{A}} L_{\underline{A}}$
- Mor: $1_\phi \mapsto \big(u_{\underline{A}} : \underline{A} \to \underline{B} \oplus (\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}})\big)$ over $\underline{B}$ corresponding to the $\underline{A}$-algebra structure.

In order to achieve this, we need to show that, for each $\underline{N} \in \mathrm{Mod}_{\underline{B}}$, the point-wise triangle identity is an equivalence of spaces. Let us first spell the details of the wannabe unit assignment.

Claim 1. *Given* $\phi : \underline{A} \to \underline{B}$ *in* $\mathrm{Ani(CRing)}$*, the functoriality on the first variable of* $\oplus : \mathrm{MOD} \to \mathrm{Ani(CRing)}$ *yields natural equivalences:*

$$\underline{B} \otimes^L_{\underline{A}} (\underline{A} \oplus (-)_{|\mathrm{Mod}_{\underline{A}}}) \simeq \underline{B} \oplus \big(\underline{B} \otimes^L_{\underline{A}} (-)_{|\mathrm{Mod}_{\underline{A}}}\big)$$
$$\phi^*(\underline{B} \oplus (-)_{|\mathrm{Mod}_{\underline{B}}}) \simeq \underline{A} \oplus \phi^*(-)_{|\mathrm{Mod}_{\underline{B}}}$$

Proof. As for the first one, in the following 3D diagram the top face commutes, since all the others do.

$$\begin{tikzcd}
\mathrm{Mod}_{\underline{A}} \arrow[rr, "\oplus_{|\mathrm{Mod}_{\underline{A}}}"] \arrow[dr, hook] \arrow[dd, "\underline{B}\otimes_{\underline{A}}(-)"] & & \mathrm{CAlg}^{\Delta}_{\underline{A}} \arrow[dr, "\mathrm{CALg}^{\Delta}"] \arrow[dd, "\underline{B}\otimes_{\underline{A}}(-)"] & \\
& \mathrm{MOD} \arrow[rr, dotted, "\oplus"] & & \mathrm{Ani(CRing)} \\
\mathrm{Mod}_{\underline{B}} \arrow[rr, "\oplus_{|\mathrm{Mod}_{\underline{B}}}"] \arrow[ur, hook] & & \mathrm{CAlg}^{\Delta}_{\underline{B}} \arrow[ur] &
\end{tikzcd}$$

The second equivalence is analogous. ■

Hence, the assignment on morphisms is the canonical map obtained by post-composing the unit $\eta_{\underline{A},\underline{B}}$ of the adjunction $\underline{B} \otimes^L_{\underline{A}} (-) \vdash \phi^*$ to the universal derivation $d : \underline{A} \to \underline{A} \oplus L_{\underline{A}}$:

$$u_{\underline{A}} : \underline{A} \xrightarrow{d} \underline{A} \oplus L_{\underline{A}} \xrightarrow{\eta_{\underline{A},\underline{B}}} \underline{B} \otimes^L_{\underline{A}} (\underline{A} \oplus L_{\underline{A}})$$

Now, consider the following diagram, where the wannabe triangular equivalence sits in the top row.

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}}, \underline{N}) \arrow[r, "\underline{B}\oplus(-)"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{B} \oplus (\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}}), \underline{B} \oplus \underline{N}) \arrow[r, "(-)\circ u_{\underline{A}}"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{A}, \underline{B} \oplus \underline{N}) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(L_{\underline{A}}, \phi^*\underline{N}) \arrow[r, "\underline{A}\oplus(-)"] & \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A} \oplus L_{\underline{A}}, \underline{A} \oplus \phi^*\underline{N}) \arrow[r, "(-)\circ d"] & \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, \underline{A} \oplus \phi^*\underline{N})
\end{tikzcd}$$

The lower composite is an equivalence, since it retrieves the functor of spaces evaluating at the universal derivation $d : \underline{A} \to \underline{A} \oplus L_{\underline{A}}$ of 3.8.1.7.
As for the vertical maps, thanks to Claim 1 the left and middle vertical arrows are equivalences induced by the extension of scalars for animated modules (see 3.2.5.11). Moreover, the right map is an equivalence by the following claim.

Claim 2. *There is an equivalence of spaces* $\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{A}, \underline{B} \oplus \underline{N}) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, \underline{A} \oplus \phi^*\underline{N})$

Proof. Notice that ϕ sits in the aside cartesian square in $\mathrm{Mod}_{\underline{A}}$ (see 3.8.1.3); the latter was already cartesian before applying $for : \mathrm{CAlg}^{\Delta}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}}$, since forgetting the $\underline{A}$-algebra structure reflects limits (see the proof of 3.8.1.7).
So, the universal property of pull-backs yields the sought equivalence of spaces:

$$
\begin{tikzcd}
\underline{A} \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, equal, bend right] & & \\
& \underline{A} \oplus \phi^* \underline{N} \arrow[r, "\phi \oplus id"] \arrow[d, "pr_1"'] & \phi^*(\underline{B} \oplus \underline{N}) \arrow[d, "pr_1"] \\
& \underline{A} \arrow[r, "\phi"] & \phi^*(\underline{B})
\end{tikzcd}
$$

$$\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{A} \overset{\phi}{\to} \underline{B}, \underline{B} \oplus \underline{N} \overset{pr_1}{\to} \underline{B}) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A} = \underline{A}, \underline{A} \oplus \phi^* \underline{N} \overset{pr_1}{\to} \underline{A})$$

■

Finally, by the construction of our equivalence and the description of the assignment in Claim 1, also the rightmost square commutes, as wished. □

Definition 3.8.2.3. *(The relative algebraic cotangent complex, [26],25.3.2.1)* Let $f : \underline{A} \to \underline{B}$ be any morphism in Ani(CRing). Define the **relative algebraic cotangent complex** of $\underline{B}$ over $\underline{A}$ as $L_{\underline{B}/\underline{A}} := \mathrm{cofibre}\big(\underline{B} \otimes^L L_{\underline{A}} \to L_{\underline{B}}\big)$, i.e. as the cofibre in MOD of the map $\mathcal{L}(f)$ from the previous proposition.

As in the absolute case, also the relative cotangent complex co-represents a space generalizing relative derivations. The author learnt the proof in a seminar talk given by L. Pol, see [34],2.10.

Proposition 3.8.2.4. *(Universal Property of $L_{\underline{B}/\underline{A}}$, [26],25.3.2.4) Given any morphism $\phi : \underline{A} \to \underline{B}$ in* Ani(CRing) *and pair $(\underline{B}, \underline{N}) \in$ MOD, there is a natural equivalence of spaces:*

$$\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(L_{\underline{B}/\underline{A}}, \underline{N}) \simeq \mathrm{Map}_{\mathrm{Calg}^{\Delta}_{\underline{A}//\underline{B}}}(\underline{B}, \underline{B} \oplus \underline{N}) =: \mathrm{Der}_{\underline{A}}(\underline{B}, \underline{N})$$

Proof. Consider the fibre-sequence of mapping spaces induced by applying $\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(-, \underline{N})$ to the cofibre-sequence $\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}} \to L_{\underline{B}} \to L_{\underline{B}/\underline{A}}$ in $\mathrm{Mod}_{\underline{B}}$, namely

$$\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(L_{\underline{B}/\underline{A}}, \underline{N}) \to \mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(L_{\underline{B}}, \underline{N}) \to \mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}}, \underline{N}) \simeq \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(L_{\underline{A}}, \phi^* \underline{N})$$

where the last equivalence follows by the base-change adjunction for animated modules 3.2.5.11. Then, by the universal property of the algebraic cotangent complex, the last arrow can be equivalently rewritten as

$$\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{B}, \underline{B} \oplus \underline{N}) \to \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A}, \underline{A} \oplus \phi^* \underline{N})$$

and we are left to show the following claim.

Claim. *The last arrow acts as $(-) \circ \phi : \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{B}, \underline{B} \oplus \underline{N}) \to \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{A}, \underline{B} \oplus \underline{N})$.*

Proof. Similarly to Claim 2 in the proof of 3.8.2.2, there is an equivalence of spaces:

$$\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{A} \overset{\phi}{\to} \underline{B}, \underline{B} \oplus \underline{N} \overset{pr_1}{\to} \underline{B}) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{A}}}(\underline{A} = \underline{A}, \underline{A} \oplus \phi^* \underline{N} \overset{pr_1}{\to} \underline{A})$$

so that the target of our map has the desired form, and we conclude by the Yoneda Lemma together with the commutativity condition of structure maps over $\underline{B}$. ■

Then, the fibre of pre-composition by ϕ gives the desired natural equivalence.

$$\mathrm{Fib}(- \circ \phi) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/\underline{B}}}(\underline{A}/\underline{B}, \underline{A}/(\underline{B} \oplus \underline{N})) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}}}(\underline{B}, \underline{B} \oplus \underline{N})$$

□

We close this section with a list of useful properties of the relative algebraic cotangent complex.

Proposition 3.8.2.5. *(Properties of $L_{\underline{B}/\underline{A}}$) The functor relative algebraic cotangent complex L satisfies the following properties:*

1. ([26],25.3.2.4) *For any extension of scalars $\underline{B} \simeq \underline{A} \otimes^L_{\underline{A}'} \underline{B}'$ in* Ani(CRing) *there is a canonical equivalence $L_{\underline{B}/\underline{A}} \simeq \underline{B} \otimes^L_{\underline{B}'} L_{\underline{B}'/\underline{A}'}$.*

2. ([26],25.3.2.5) *Any composition of maps of animated rings* $\underline{A} \to \underline{B} \to \underline{C}$ *induces a canonical cofibre sequence in the stabilization* $\mathrm{Mod}_{\underline{C}}^{\mathrm{Ex}}$*:*

$$\underline{C} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}} \longrightarrow L_{\underline{C}/\underline{A}} \longrightarrow L_{\underline{C}/\underline{B}}$$

Moreover, the canonical map $\underline{C} \otimes^L_{\underline{A}} L_{\underline{B}/\underline{A}} \to L_{\underline{C}/\underline{A}}$ *is an equivalence whenever its source and target are equivalent* $\underline{C}$*-modules.*

3. *Any map* $\phi : \underline{A} \to \underline{B}$ *in* $\mathrm{Ani(CRing)}$ *induces a canonical map* $\epsilon_\phi : \underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi) \to L_{\underline{B}/\underline{A}}$ *in* $\mathrm{Mod}_{\underline{B}}^{\mathrm{Ex}}$*, namely the* ***Hurewicz map*** *associated to* ϕ*.*

4. ([26],25.3.6.6) *A morphism* $\phi : \underline{A} \to \underline{B}$ *in* $\mathrm{Ani(CRing)}$ *is an equivalence iff* $\pi_0(\phi)$ *is an iso and* $L_{\underline{B}/\underline{A}} \simeq 0$*.*

Proof. (1) : Consider a co-cartesian square in Ani(CRing) yielding the extension of scalars $\underline{B} \simeq \underline{A} \otimes^L_{\underline{A}'} \underline{B}'$ as in the statement.

$$\begin{tikzcd} \underline{A}' \arrow[r, "g"] \arrow[d, "\alpha"] & \underline{B}' \arrow[d, "\beta"] \\ \underline{A} \arrow[r, "f"] & \underline{B} \end{tikzcd}$$

By the universal property of the relative cotangent complex 3.8.2.4, for each $\underline{N} \in \mathrm{Mod}_{\underline{B}}$ one has the following chain of equivalences, so we conclude by the Yoneda Lemma:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}\big(\underline{B} \otimes^L_{\underline{B}'} L_{\underline{B}'/\underline{A}'}, \underline{N}\big) &\simeq_{(a)} \mathrm{Map}_{\mathrm{Mod}_{\underline{B}'}}\big(L_{\underline{B}'/\underline{A}'}, \beta^* \underline{N}\big) \\
&\simeq_{(b)} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}'//\underline{B}'}}\big(\underline{B}', \underline{B}' \oplus \beta^* \underline{N}\big) \\
&= \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}'//\underline{B}'}}\big(g^* \underline{B}', g^*(\underline{B}' \oplus \beta^* \underline{N})\big) \\
&\simeq_{(c)} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}'//\underline{B}'}}\big(g^* \underline{B}', \underline{A}' \oplus (\beta \circ g)^* \underline{N}\big) \\
&\xrightarrow{\underline{A} \otimes^L_{\underline{A}'}(-)}_{(d)} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}}}\big(\underline{B}, \underline{A} \oplus (\underline{A} \otimes^L_{\underline{A}'} (f \circ \alpha)^* \underline{N})\big) \\
&\xrightarrow{v_{\underline{N}}}_{(e)} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}}}\big(\underline{B}, \underline{A} \oplus f^* \underline{N}\big) \\
&\simeq_{(f)} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}}}\big(f^* \underline{B}, f^*(\underline{B} \oplus \underline{N})\big) \\
&= \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}}}\big(\underline{B}, \underline{B} \oplus \underline{N}\big) \\
&\simeq_{(g)} \mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(L_{\underline{B}/\underline{A}}, \underline{N})
\end{aligned}$$

where we used:

- (a) : It is extension of scalars for animated modules along $\beta : \underline{B}' \to \underline{B}$, see 3.2.5.11;
- (b) : It is the universal property for the relative algebraic cotangent complex, see 3.8.2.4; the next line is then a more explicit rewriting via restriction of scalars along $g : \underline{A}' \to \underline{B}'$, which is implicit whenever we regard a $\underline{B}'$-algebra as a $\underline{A}'$-algebra via 3.2.2.3;
- (c) : By Claim 1 in 3.8.2.2, $\oplus$ commutes with restriction of scalars along g;
- (d) : Apply the functor $\underline{A} \otimes^L_{\underline{A}'} (-)$, namely extension of scalars along $\alpha : \underline{A}' \to \underline{A}$ of 3.2.3.3; moreover, use the fact that the square is (commutative and) co-cartesian;
- (e) : Post-compose with the counit $v_{\underline{N}}$ of the adjunction $\underline{A} \otimes^L_{\underline{A}'} (-) \colon \mathrm{CAlg}^{\Delta}_{\underline{A}'//\underline{B}'} \rightleftarrows \mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}} : \alpha^*$;
- (f) : It is again the commutativity of f^* and $\oplus$; as in (b), in the standard notation f^* is omitted;

- (g) : It is again the universal property of the relative algebraic cotangent complex, see 3.8.2.4.

Finally, observe that the composition of (d) and (e) is an equivalence by the triangle identity. ■

(2) : By the functoriality of $L_{(-)}$ as in 3.8.2.1, we have the following diagram of morphisms of cofibre sequences in $\mathrm{Mod}_{\underline{C}}$, which induces the dotted arrows at the level of cofibres:

$$\begin{tikzcd}
E_1 : \arrow[d, "\phi_1"'] & \underline{C} \otimes^L_{\underline{A}} L_{\underline{A}} \arrow[r] \arrow[d, equal] & \underline{C} \otimes^L_{\underline{B}} L_B \arrow[r] \arrow[d] & \underline{C} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}} \arrow[d, dotted] \\
E_2 : \arrow[d, "\phi_2"'] & \underline{C} \otimes^L_{\underline{A}} L_{\underline{A}} \arrow[r] \arrow[d] & L_{\underline{C}} \arrow[r] \arrow[d, equal] & L_{\underline{C}/\underline{A}} \arrow[d, dotted] \\
E_3 : & \underline{C} \otimes^L_{\underline{B}} L_{\underline{B}} \arrow[r] & L_{\underline{C}} \arrow[r] & L_{\underline{C}/\underline{B}}
\end{tikzcd}$$

We are left to show that the sequence $\mathcal{E}$ in $\mathrm{Mod}_{\underline{C}}$ formed by the induced dotted arrows is cofibre.
To this end, we will apply Yoneda Lemma and check the universal property of push-outs against any $\underline{M} \in \mathrm{Mod}_{\underline{C}}$.
In what follows we will let E_i denote both the sequence itself and the composite of the arrows forming it. The interpretation considered will be clear from the context.
Now, observe that the commutativity of the diagram and the universal property of cofibres (for the last sequence E_3) yield the following chain of equivalences:

$$\begin{aligned}
\mathrm{Map}_{\underline{C}}([* \leftarrow \underline{C} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}} \to L_{\underline{C}/\underline{A}}], \underline{M}) &\xrightarrow{E_1^*} \mathrm{Map}_{\underline{C}}([* \leftarrow E_2], \underline{M}) \\
&\simeq \mathrm{Map}_{\underline{C}}([* \leftarrow \underline{C} \otimes^L_{\underline{A}} L_{\underline{A}} \to E_3], \underline{M}) \\
&\simeq \mathrm{Map}_{\underline{C}}([* \leftarrow E_2 \to L_{\underline{C}/\underline{B}}], \underline{M}) \\
&\simeq E_2^*\big(\mathrm{Map}_{\underline{C}}([* \leftarrow L_{\underline{C}/\underline{A}} \to L_{\underline{C}/\underline{B}}], \underline{M})\big) \\
&\simeq E_1^*\big(\mathrm{Map}_{\underline{C}}([* \leftarrow \mathcal{E}], \underline{M})\big)
\end{aligned}$$

Putting everything together, we obtain the equivalence of mapping spaces:

$$E_1^*\big(\mathrm{Map}_{\underline{C}}([* \leftarrow \underline{C} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}} \to L_{\underline{C}/\underline{A}}], \underline{M})\big) \simeq E_1^*\big(\mathrm{Map}_{\underline{C}}([* \leftarrow \mathcal{E}], \underline{M})\big)$$

which allows us to conclude by the Yoneda Lemma that $[* \leftarrow \mathcal{E}]$ is a cocartesian square extending the coangle $[* \leftarrow \underline{C} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}} \to L_{\underline{C}/\underline{A}}]$, as wished.
Finally, the last part of the statement follows by inspection of our construction together with the proof of 3.8.2.1: it holds for the static parts by the construction of the I fundamental sequence for the module of differentials, and can be extended to the derived setting, since cofibre sequences commute with sifted resolutions. ■

(3) : Omissis.
(4) : One direction is clear; for the other one, assume both that ϕ induces an isomorphism $\pi_0(\phi) : \pi_0(\underline{A}) \to \pi_0(\underline{B})$ and that $L_{\underline{B}/\underline{A}}$. We will need a technical Lemma 3.8.2.6 on the connectivity of the Hurewicz map, which we postpone.
Assume by contradiction that ϕ be *not* an equivalence and consider the cofibre sequence of ϕ in $\mathrm{Mod}_{\underline{A}}$ (by 3.4.0.1 it can be computed in the pre-stable category, since $\pi_0(\phi)$ and all of its de-suspensions are isomorphisms). By the conservativity of π_*, our assumption amounts to the existence of a smallest $n \geq 0$ for which $\pi_n \mathrm{Cofib}(\phi) \neq 0$. Then, consider the Hurewicz map $\epsilon_\phi : \underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi) \to L_{\underline{B}/\underline{A}}$.
By the following technical Lemma 3.8.2.6, the n-connectivity of $\mathrm{Cofib}(\phi)$ implies that $\mathrm{Fib}(\epsilon_\phi)$ is at least $(n+1)$-connected. Hence, ϵ_ϕ induces an isomorphism $\pi_n(\underline{B} \otimes^L \mathrm{Cofib}(\phi)) \to \pi_n(L_{\underline{B}/\underline{A}})$. But now, the latter vanishes by assumption, so that we obtain $\pi_n(\underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi)) \cong 0$ which yields the sought contradiction. □

Remark. Since $\underline{B}' \otimes^L_{\underline{B}'} (-)$ preserves colimits (hence in particular cofibre sequences), the first property basically makes sense of the statement that the algebraic cotangent complex preserves cocartesian diagrams in Ani(CRing). In other words, we proved that for any extension of scalars $\underline{B} \simeq \underline{A} \otimes^L_{\underline{A}'} \underline{B}'$ in Ani(CRing), the following square is cocartesian:

$$\begin{tikzcd}
\underline{B} \otimes^L_{\underline{A}'} L_{\underline{A}'} \arrow[r] \arrow[d] & \underline{B} \otimes^L_{\underline{A}} L_{\underline{A}} \arrow[d] \\
\underline{B} \otimes^L_{\underline{B}'} L_{\underline{B}'} \arrow[r] & L_{\underline{B}}
\end{tikzcd}$$

We close this subsection with a technical Lemma on some useful connectivity properties of the Hurewicz map. The proof is omitted.

Lemma 3.8.2.6. *(Connectivity of the Hurewicz map, [26],25.3.6.1) Let $\phi : \underline{A} \to \underline{B}$ in* Ani(CRing) *be a morphism of animated rings. Then, the Hurewicz map $\epsilon_\phi : \underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi) \to L_{\underline{B}/\underline{A}}$ of 3.8.2.5,iii satisfies the following connectivity properties:*

- $\pi_0(\epsilon_\phi)$ *is surjective;*
- *If* $\mathrm{Fib}(\phi)$ *is connective, then* $\mathrm{Fib}(\epsilon_\phi)$ *is* 2*-connective;*
- *If* $\mathrm{Fib}(\phi)$ *is m-connective for some $m > 0$, then* $\mathrm{Fib}(\epsilon_\phi)$ *is $(m + 3)$-connective.*

3.8.3 Examples

In this subsection we include some computations and remarks that will be used when dealing with the geometric version of the cotangent complex.

Let us start by showing that our construction does indeed generalize the classical module of differentials.

Lemma 3.8.3.1. *([26],25.3.1.7) Let $A := \mathbb{Z}[x_s|s \in S] \in$* CRing *be a static polynomial algebra (on an arbitrary set of generators S). Recall that static A-derivations are represented by $(\Omega_A = \oplus_S[dx_s]A \in \mathrm{Mod}(A)\,, d : A \to \Omega_A)$, where the universal derivation acts as $d : A \ni f \mapsto (f, \sum_S \partial_s f dx_s) \in A \oplus \Omega_A$.*
Then, (Ω_A, d) co-represent also $\mathrm{Der}(A, -) : \mathrm{Mod}_A \to \mathrm{Spc}$. In particular, the algebraic cotangent complex L_A of any static polynomial algebra A is itself static.

Proof. We have to show that $d^*(A \oplus -) : \mathrm{Map}_{\mathrm{Mod}_A}(\Omega_A, \underline{M}) \to \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/A}}(A, A \oplus \underline{M})$ is an equivalence of spaces for each $\underline{M} \in \mathrm{Mod}_A$.
Notice that wlog S is a finite set, so both A and Ω_A are cpt+proj. Indeed, $\Omega_A = \oplus_S[dx_s]A \cong \underrightarrow{\mathrm{colim}}\big(\oplus_F [dx_s]A \mid F \subseteq S \text{ finite}\big)$ and $A \cong \underrightarrow{\mathrm{colim}}\big(\mathbb{Z}[t]^{\otimes |F|} \mid F \subseteq S \text{ finite}\big)$ can be written as filtered 1-colimits of finite free modules and finitely generated polynomial algebras, respectively. Now, by [6],7.3, filtered 1-colimits are also ∞-colimits, so we obtain presentations of Ω_A and A by cpt+proj objects of the corresponding ∞-categories. Hence, we can assume that the left component of our mapping spaces is cpt+proj. Now, since $\oplus$ commutes with sifted colimits separately in each variable, wlog S is finite.
Thus, for $p : K \to \mathrm{Mod}_A$ a sifted diagram with colimit $\underline{M}$, we can carry the K-indexed colimit out of both sides, so that also $\underline{M}$ can be assumed to live in FFree_A.
Therefore, we reduced our problem to the static case, where the statement holds true by the classical theory. □

Example 3.8.3.2. Let $t := (t_1, \dots, t_n)$ denote a tuple of n variables and consider the quotient projection $\mathbb{Z}[t] \to \mathbb{Z}[t]/(t) \simeq \mathbb{Z}$. Then, $L_{\mathbb{Z}/\mathbb{Z}[t]} \simeq \mathbb{Z}^n[1]$.
In the language of DAG, this will correspond to the closed point inclusion $0 \twoheadrightarrow \mathbb{A}^n$ into the n-th affine space.

Proof. As in 3.8.2.2, $\mathbb{Z} \otimes^L_{(-)} L_{(-)} \simeq L_{(-)}$ is a left-adjunct, so it takes the inizial object $\mathbb{Z} \in \mathrm{Ani(CRing)}$ to the initial object $0 \in \mathrm{Mod}_{\mathbb{Z}}$. Hence, there is a cofibre sequence

$$\begin{tikzcd}
\mathbb{Z} \otimes^L_{\mathbb{Z}[\underline{t}]} L_{\mathbb{Z}[\underline{t}]} \arrow[r] \arrow[d, "\simeq"] & L_{\mathbb{Z}} \arrow[r] \arrow[d, "\simeq"] & L_{\mathbb{Z}/\mathbb{Z}[\underline{t}]} \arrow[d, equal] \\
\Omega_{\mathbb{Z}[\underline{t}]} \arrow[r] & 0 \arrow[r] & L_{\mathbb{Z}/\mathbb{Z}[\underline{t}]}
\end{tikzcd}$$

which can be written as in the bottom row by 3.8.3.1, thus exhibiting the relative cotangent complex as a suspension in $\mathrm{Mod}_{\mathbb{Z}}$: $L_{\mathbb{Z}/\mathbb{Z}[\underline{t}]} \simeq \Sigma\Omega_{\mathbb{Z}[\underline{t}]} \simeq \mathbb{Z}^{|\underline{t}|}[1]$. $\square$

The following result allows us to determine the algebraic cotangent complex of the symmetric $\underline{A}$-algebra of an $\underline{A}$-module over the base animated ring. The author learned the proof in a seminar talk by Luca Pol, see [34],2.12.

Lemma 3.8.3.3. *([26],25.3.2.2) Let $(\underline{A}, \underline{M}) \in \mathrm{MOD}$ and set $\underline{B} := L\mathrm{Sym}^*_{\underline{A}}(\underline{M})$. Then, by the universal property of the Derived Symmetric Algebra 3.7.1.2 we deduce that $L_{\underline{B}/\underline{A}} \simeq \underline{B} \otimes^L_{\underline{A}} \underline{M} \simeq L\mathrm{Sym}^*_{\underline{A}}(\underline{M}) \otimes^L_{\underline{A}} \underline{M}$.*

Proof. For each $\underline{N} \in \mathrm{Mod}_{\underline{B}}$ with $\underline{B} = L\mathrm{Sym}^*_{\underline{A}}(\underline{M})$, there is a chain of natural equivalences of mapping spaces, so we conclude by the Yoneda Lemma:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(L_{\underline{B}/\underline{A}}, \underline{N}) &\simeq_{(a)} \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}//\underline{B}}}(\underline{B}, \underline{B} \oplus \underline{N}) \\
&\simeq_{(b)} \mathrm{Fib}\big(\mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}(\underline{B}, \underline{B} \oplus \underline{N}) \to \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{\underline{A}}}(\underline{B}, \underline{B})\big) \\
&\simeq_{(c)} \mathrm{Fib}\big(\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{M}, \phi^*(\underline{B} \oplus \underline{N}) \to \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{M}, \phi^*\underline{B})\big) \\
&\simeq_{(d)} \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{M}, \phi^*\underline{N}) \\
&\simeq_{(e)} \mathrm{Map}_{\mathrm{Mod}_{\underline{B}}}(\underline{B} \otimes^L_{\underline{A}} \underline{M}, \underline{N})
\end{aligned}$$

Here the equivalences are obtained as follows:

- (a) : It is the universal property of the relative cotangent complex 3.8.2.4;
- (b) : By 3.2.2.3, we can identify $\mathrm{CAlg}^{\Delta}_{\underline{A}} \simeq \mathrm{CAlg}^{\Delta}_{\underline{A}/}$; then, the equivalence with the fibre follows from the definition of maps over $\underline{B}$;
- (c) : It follows from the universal property of the Derived Symmetric $\underline{A}$-Algebra $\underline{B} = L\mathrm{Sym}^*_{\underline{A}}(\underline{M})$, see 3.7.1.2;
- (d) : Since mapping space and *for* : $\mathrm{CAlg}^{\Delta}_{\underline{A}} \to \mathrm{Mod}_{\underline{A}}$ commute with limits in the second variable (see 3.8.1.7 for the latter), this is an application of the Yoneda Lemma to the fibre sequence of 3.8.1.3;
- (e) : It is the extension of scalars for animated modules, see 3.2.5.11.

$\square$

4 A primer on Derived Algebraic Geometry

4.1 Derived Schemes

In this section we will introduce the definition of a derived scheme, following the doctoral thesis of A. Khan [15] and referring to the Appendices for details about the results involved as well as for motivation.
We will generalize the classical formalism of 'functors of points' by introducing the notion of stacks as sheaves for the Zariski site, which will correspond then to 'local functors of points', and considering the full subcategory spanned by those stacks which further admit a finite affine Zariski covering, corresponding in turn to classical schemes. These will be, indeed, our candidates for derived schemes.

A number of interesting local properties of relative schemes will be introduced, such as flatness, compactness conditions, open and closed immersions, being qcqs (i.e. quasi-compact and quasi-separated) on the base, etc.
Thereafter, as in classical algebraic geometry, we will relate the formalism of 'functors of points' to the topological description of schemes. This will be achieved thanks to the tools developed in Appendix C, primarily Čech descent.
Finally, we will devote the remaining of this section to introducing the machinery that will be employed in the construction of blow-ups of quasi-smooth derived schemes. Such constructions will be of independent interest and will also serve as excellent examples in order to achieve a more thorough understanding of the theory itself.
In what follows by *scheme* or *stack* we will always mean their higher version and explicitly write *classical* whenever we want to consider the ordinary concepts.

4.1.1 Pre-Stacks

Warning. Choose a chain of three universes $\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathcal{U}''$ and adopt the terminology "small", "large" and "very large" to distinguish smallness properties along the hierarchy. Observe that, in general, taking presheaves and animation do not preserve smallness; in particular, if we assume CRing to be small, then Ani(CRing) will be a large ∞-category and taking presheaves - so functors into the large ∞-category of large spaces $\widehat{\mathrm{Spc}}$ - will yield the very large $\mathcal{P}(\mathrm{Ani}(\mathrm{CRing})^{op}) = \mathrm{Fun}(\mathrm{Ani}(\mathrm{CRing}), \widehat{\mathrm{Spc}})$; then the localization to sheaves on such a large site (which we will define later on) gives again a very large $\mathrm{Sh}(\mathrm{Ani}(\mathrm{CRing}), \mathrm{Spc})$.
When dealing with stacks, we will remark the implications of having only large ∞-topoi at our disposal. Moreover, for the sake of readability, we drop the hat in the notation: Spc is automatically the ∞-category of large spaces whenever we deal with stacks, and the ordinary one if we consider schemes; otherwise, it will be clear from the context when we can restrict to its full sub-category of small spaces (e.g. when considering mapping spaces in the locally small - although large - ∞-category Ani(CRing)).

Definition 4.1.1.1. *(Prestack, [15],4.2.1)* A **prestack** is a presheaf $\mathrm{Ani}(\mathrm{CRing}) \to \mathrm{Spc}$. Let PreStack $:=$ $\mathcal{P}(\mathrm{Ani}(\mathrm{CRing})^{op})$ denote the ∞-category of prestacks.
For any animated ring $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$, write $\mathrm{Spec}(\underline{A}) := \mathrm{Map}_{\mathrm{Ani}(\mathrm{CRing})}(\underline{A}, -)$ for the co-represented prestack, and consider the full subcategory of **affine schemes**, $\mathrm{Sch}^{\mathrm{Aff}} \subseteq_{f.f.} \mathrm{PreStack}$, spanned by the image of the Yoneda embedding.
Moreover, given any animated ring $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$, we call **$\underline{A}$-points** of a prestack $S \in \mathrm{PreStack}$ the morphisms $s : \mathrm{Spec}(\underline{A}) \to S$, i.e. the points of the space $S(\underline{A}) \in \mathrm{Spc}$ under the Yoneda Lemma.

Similarly to the case of animated rings, we consider the classical prestack underlying a higher one.

Definition 4.1.1.2. *(Classical underlying stack, [15],4.2.3)* Define $\mathrm{PreStack}^{\mathrm{cl}} = \mathcal{P}(\mathrm{CRing}^{op})$ and, given any $S \in \mathrm{PreStack}$, call its **underlying classical prestack** the presheaf $S^{\mathrm{cl}} := j^*(S) \in \mathrm{PreStack}^{\mathrm{cl}}$ obtained as a restriction along the Yoneda embedding $j : \mathrm{CRing} \hookrightarrow \mathrm{Ani}(\mathrm{CRing})$.
Furthermore, the 0-truncation adjunction $\pi_0 \vdash j$ of A.5.0.7 induces the following adjunction of presheaves by passing to the underlying classical prestack:

$$\pi_0^* \colon \mathrm{PreStack}^{\mathrm{cl}} \rightleftarrows \mathrm{PreStack} : j^* =: (-)^{\mathrm{cl}}$$

Remark. For $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$, it holds $\mathrm{Spec}(\underline{A})^{\mathrm{cl}} \simeq \mathrm{Spec}(\pi_0(\underline{A}))$ and they span the full sub-category of *classical affine schemes* in $\mathrm{Sch}^{\mathrm{Aff}}$. Moreover, $\mathrm{Spec}(\underline{A})^{\mathrm{cl}}$ is actually an 'ordinary' affine scheme, namely it factors through Set.
Indeed, for each static ring $B \in \mathrm{CRing}$, the 0-truncation adjunction $\pi_0 \vdash j$ yields:

$$(\mathrm{Spec}(\underline{A}))^{\mathrm{cl}}(B) \simeq (\mathrm{Spec}(\underline{A}) \circ j)(B) \simeq \mathrm{Map}_{\mathrm{Ani}(\mathrm{CRing})}(\underline{A}, j(B)) \simeq \mathrm{Hom}_{\mathrm{CRing}}(\pi_0\underline{A}, B) = \mathrm{Spec}(\pi_0\underline{A})(B) \in \mathrm{Set}$$

Definition 4.1.1.3. *(Quasi-Coherent Modules, [15],4.3.1-4)* For an affine scheme $S := \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, define $\mathrm{QCoh}(S) := \mathrm{Mod}_{\underline{A}}$; in particular, write $\mathcal{O}_{\mathrm{Spec}(\underline{A})} := \underline{A} \in \mathrm{Mod}_{\underline{A}}$ for the quasi-coherent copy of the base-ring.

Define the presheaf of **quasi-coherent modules** to be the one of presentable pre-stable symmetric monoidal ∞-categories

$$\mathrm{QCoh} : \mathrm{PreStack} \to \mathrm{Pr}^L_\otimes \quad \text{s.t.} \quad \mathrm{QCoh} := \mathrm{Ran}_j(\mathrm{Mod})$$

which is the (large) right Kan extension along the Yoneda embedding $\mathrm{Ani}(\mathrm{CRing})^{op} \hookrightarrow \mathrm{PreStack}$ of the (large) presheaf $\mathrm{Mod} : \mathrm{Ani}(\mathrm{CRing})^{op} \to \mathrm{SymMon}^L$ occurring in the definition of MOD (as in 3.2.5.11).

Informally, we can write its action on objects by a large inverse limit over all generalized points of S:

$$\mathrm{QCoh}(S) := \lim \left(\mathrm{QCoh}(\mathrm{Spec}(\underline{B})) \mid \mathrm{Spec}(\underline{B}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/S}\right)$$

At each step, the monoidal **unit** is given by the quasi-coherent module corresponding to the base-ring, namely $\mathcal{O}_S \in \mathrm{QCoh}(S)$, represented by $\left(\mathcal{O}_{S,s} := \mathcal{O}_{\mathrm{Spec}(\underline{B})} \mid (s : \mathrm{Spec}(\underline{B}) \to S) \in \mathrm{Sch}^{\mathrm{Aff}}_{/S}\right)$.

Moreover, as in the classical setting, a morphism $f : S \to T$ in PreStack induces a transition functor f^* for the presheaf QCoh which can be described similarly to 3.2.5.11: $f : S \to T$ is the large colimit of a natural transformation $\psi : \mathrm{Spec}(q) \to \mathrm{Spec}(p)$ as induced by the ∞-Density Theorem [24],5.1.5.3 and which we informally denote as follows:

$$f \simeq \mathrm{colim}\left(\psi_{\underline{B},\underline{A}} : \mathrm{Spec}\underline{B}/S \to \mathrm{Spec}\underline{A}/T \mid \mathrm{Spec}(\underline{B}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/S}\,,\, \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/T}\right)$$

Let us describe the object-wise action of the functor f, with reference to the notation of the proof of 3.2.5.11. Let $P \in \mathrm{Mod}_p = \mathrm{QCoh}(\mathrm{Spec}(p))$ be a large diagram of affine representatives of a quasi-coherent module $\underline{N} \simeq \lim P$ over T. Then, ψ yields by extension of scalars a new large diagram of quasi-coherent modules $q \otimes^L_p P$ in the fibre $\mathrm{Mod}_q = \mathrm{QCoh}(\mathrm{Spec}(q))$ over the algebras CAlg^{Δ}_q.
Hence, as in 3.2.5.11, each f^* is a symmetric monoidal and (large) colimit-preserving functor. In particular, whenever we can take small diagrams over S and T, then each f^* admits a right-adjoint f_*.
As in the classical case, we refer to f^* as the **inverse image** functor and call f_* the **direct image** functor. Accordingly to the previous derivation, f_* acts on objects by restriction of scalars, namely (in the obvious notation) f_* takes $Q \in \mathrm{Mod}_q$ to $\psi^*(Q) \in \psi^*(\mathrm{Mod}_q) \simeq \mathrm{Mod}_p$.

Remark. Let $S \in \mathrm{PreStack}$ be such that $\mathrm{QCoh}(S)$ is defined over a small diagram. Then, in view of the adjunction in 3.2.5.1, $\mathrm{QCoh}(S)$ can be endowed with a closed symmetric monoidal structure via Day convolution by positing $\mathrm{QCoh}(S)^{\circledast} \simeq \mathrm{LKE}_j(\mathrm{MOD}^{\boxtimes})$ as in 3.2.5.1.

Warning. We already observed that diagrams over a stack are potentially large and that the presheaf ∞-category $\mathcal{P}(\mathrm{Ani}(\mathrm{CRing})^{op})$ is a large ∞-topos, namely it is not accessible in the sense of essentially small ∞-categories. In particular, this does not really allow straightforward applications of Adjoint Functor Theorems.
However, this is not a major issue: whenever the prestacks S, T arise as the colimit of a small diagram of representables (read classical schemes), then both $\mathrm{QCoh}(S)$, $\mathrm{QCoh}(T)$ are still presentable: they are small limits of presentable categories, and $\mathrm{Pr}^R \subseteq \mathrm{Cat}_\infty$ is closed under small limits in the ambient ∞-category by [24],5.5.3.18. Moreover, we will define schemes so as to admit only essentially small affine over-slices, so over them we will be able to work with presentable ∞-categories and carry on the usual business.

An analogous digression with the presheaf of animated algebras 3.2.3.1 in place of 3.2.5.11 describes quasi-coherent algebras QCohAlg as a right Kan extension of CAlg^{Δ} along the Yoneda embedding.

Definition 4.1.1.4. *(Quasi-Coherent Algebras, [15],4.7.1)* For an affine scheme $S := \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, define $\mathcal{O}_{\mathrm{Spec}(\underline{A})}\text{-Alg} := \mathrm{QCohAlg}(S) := \mathrm{CAlg}^{\Delta}_{\underline{A}}$; in particular, write $\mathcal{O}_{\mathrm{Spec}(\underline{A})}(\underline{A}) := \underline{A} \in \mathrm{CAlg}^{\Delta}_{\underline{A}}$ for the quasi-coherent copy of the base-ring (see 4.2.2.2).

Define the presheaf of **quasi-coherent algebras** to be the one of presentable symmetric monoidal ∞-categories

$$\mathrm{QCohAlg} : \mathrm{PreStack} \to \mathrm{SymMon}^L \quad \text{s.t.} \quad \mathrm{QCohAlg} := \mathrm{RKE}_j(\mathrm{CAlg}^{\Delta})$$

which is the (large) right Kan extension along the Yoneda embedding $\mathrm{Ani}(\mathrm{CRing})^{op} \hookrightarrow \mathrm{PreStack}$ of the (large) presheaf $\mathrm{CAlg}^{\Delta} : \mathrm{Ani}(\mathrm{CRing})^{op} \to \mathrm{SymMon}^L_{\mathrm{lax}}$ as in 3.2.3.1.

Informally, we can write its action on objects by a large inverse limit over all generalized points of $S \in \mathrm{PreStack}$:

$$\mathcal{O}_s\text{-Alg} := \mathrm{QCohAlg}(S) = \lim\left(\mathcal{O}_{\mathrm{Spec}(\underline{A})}\text{-Alg} \mid \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/S}\right)$$

At each prestack S, the monoidal structure is induced by that of the corresponding ∞-category $\mathrm{QCoh}(S)$ of quasi-coherent modules on S. Similarly, the functoriality on arrows is expressed by direct and inverse image functors with respect to the extension of scalars adjunction for algebras.

4.1.2 Stacks

Definition 4.1.2.1. *(Flatness, f.p. open immersions, [15],4.4.2)* Let $f : T := \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A}) =: S$ be a morphism $\mathrm{Sch}^{\mathrm{Aff}}$. We say that:

- f is **flat** iff the inverse image $f^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(T)$ is an exact functor.

  Equivalently, iff the induced morphism of animated rings is flat (see 3.4.0.2), iff the morphism of underlying classical schemes $f_{\mathrm{cl}} : \mathrm{Spec}(\pi_0\underline{B}) \to \mathrm{Spec}(\pi_0\underline{A})$ is flat (i.e. $\pi_0\underline{B}$ is a flat $\pi_0\underline{A}$-module) and the canonical base-change comparison morphism of $\pi_0\underline{B}$-algebras (i.e. the counit of 3.2.3.3) $\pi_i\underline{A}\otimes^L_{\pi_0\underline{A}}\pi_0\underline{B} \to \pi_i(\underline{B})$ is invertible for each i.

- f is **of finite presentation** (f.p.) iff $\underline{B} \in \mathrm{CRing}_{\underline{A}/}$ is a compact (i.e. finitely presented) $\underline{A}$-algebra.

- f is an **open immersion** iff f is flat and the morphism of underlying classical schemes f^{cl} is an open immersion, i.e. for each $g : \mathrm{Spec}(R) \to \mathrm{Spec}(\pi_0\underline{A}) = S^{\mathrm{cl}}$, the base-change $g^*(f^{\mathrm{cl}})$ has source $\mathrm{Spec}(\pi_0\underline{B} \otimes_R \pi_0\underline{A}) \cong \mathrm{Spec}(R)_{\underline{a}}$ for some ideal $\underline{a} \subseteq \pi_0\underline{A}$, where the latter classical scheme is defined on C-valued points by $\mathrm{Spec}(R)_{\underline{a}}(C) := \{\phi : R \to C \,|\, C = \phi(\underline{a})C\}$.

  Equivalently, iff it is a flat f.p. monomorphism (i.e. (-1)-truncated, so with an invertible diagonal $\Delta : T \to T \times_S T$).

Before proving the claimed equivalences of definitions, let us motivate the stated generalizations.

Motivation. The definitions given generalize the following:

- *Flat* (equiv. definition): classical flatness is [37],II.29.25.2; the higher homotopical condition clearly has no counterpart in classical algebraic geometry. At the level of derived categories, it could be stated as a Tor-independence property: $\mathrm{Tor}_i^{\pi_0\underline{A}}(\pi_i\underline{A}, \pi_0\underline{B}) \cong \pi_i\underline{B}$ for each $i \in \mathbb{N}$ (see [37],15.61); however, in the classical formalism, this is an 'external' condition on the objects involved, more than a property of their structure. On the other hand, in DAG this means that the maps $\pi_0\underline{A} \to \pi_0\underline{B}$ and $\pi_i\underline{A} \to \pi_i\underline{B}$ sit in the obvious cocartesian square, i.e. that their cofibers are equivalent. See 3.4.0.2 for more details.

- *f.p.:* [37],II.29.21.2.

- *Open immersion:* The definition is self-explanatory, whereas the equivalent condition is definitely non-trivial, but the goal is providing a definition of open immersions which can be formulated in the language of commutative higher algebra and which does not require any specification of a topology on affine spectra.

  The equivalence in the classical setting can be found in [37],II.41.14.1: as a consequence ([37],II.29.23.2) of Chevalley's Theorem ([37],I.10.29.10) and Going Down ([37],I.10.41.1) for flat maps ([37],I.10.39.19), flat f.p. morphisms of affine schemes are open ([37],I.10.41.8). Then, the part relative to being an 'immersion' is self-explanatory: the condition on the diagonal generalizes the similar characterization of monomorphism in terms of kernel pair which holds in any category with fibred products.

Noteworthy is however to observe the different role played by hypotheses of different 'nature'. For future reference, we will enclose this in a vague meta-definition.

Definition 4.1.2.2. *(Meta-definition: classes of properties)* We group generalizations of properties of Classical Algebraic Geometry into three classes:

- *algebraic assumptions:* (e.g. compactness, such as f.p.) they pertain to the representing animated objects;

- *homotopical assumptions:* (e.g. flatness) they relate different homotopy-degrees in the Postnikov tower and allow us to consistently lift 'topological' properties to the ∞-world. In the example of flatness, the latter is the combination of a 'homotopical' property, i.e. higher coherence, together with a 'topological' property, i.e. the fact that fibres behave nicely at the level of underlying classical schemes; the lifting part will be made meaningful in the next proof.

- *topological assumptions:* (e.g. properties of immersions) they are relative to the underlying classical scheme, so the object of study of Classical Algebraic Geometry, and can be lifted to the ∞-world by means of good higher consistence (i.e. flatness).

Proof. Flatness:
For a morphism of affine pre-stacks $f : T = \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A}) = S$ in $\mathrm{Sch}^{\mathrm{Aff}}$, the inverse image f^* is exact whenever the extension of scalars $\mathcal{O}_T \otimes_{\mathcal{O}_S} (-)$ is such. Hence, we conclude by the digression in 3.4.0.2, where it is proven that a morphism $f : \underline{A} \to \underline{B}$ of animated algebras is flat iff the induced extension of scalars $(-) \otimes^L_{\underline{A}} \underline{B}$ is exact, iff the last equivalent condition holds.

Open immersions:
Claim 1. *A flat morphism* $f : T = \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A}) =: S$ *in* $\mathrm{Sch}^{\mathrm{Aff}}$ *is a monomorphism iff* f_{cl} *is such.*

Proof. f is a monomorphism iff the diagonal map $\Delta : T \to T \times_S T$ is an equivalence, iff the canonical multiplication $m : \underline{B} \otimes_{\underline{A}} \underline{B} \to \underline{B}$ is such. By flatness all tensor products in homotopy are static, so we can consider the following commutative square for each i:

$$\begin{tikzcd}
(\pi_i \underline{A} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{B}) \otimes_{\pi_i \underline{A}} (\pi_i \underline{A} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{B}) \arrow[r, "\pi_i(m)"] \arrow[d, "\simeq"'] & \pi_i \underline{A} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{B} \arrow[d, "\simeq"'] \\
\pi_i \underline{A} \otimes_{\pi_0 \underline{A}} \left(\pi_0 \underline{B} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{B}\right) \arrow[r, "\pi_i(\underline{A}) \otimes \pi_0(m)"] & \pi_i \underline{A} \otimes_{\pi_0 \underline{A}} \pi_0 \underline{B}
\end{tikzcd}$$

Hence, by Whitehead Theorem, m is an equivalence iff $\pi_0(m)$ is such. ■

Then, one can show that, for flat monomorphisms, being étale lifts to the derived setting. See for instance [23],7.5.0.6.

Finally, as already observed (see [37],II.41.14.1), an open immersion of classical affine schemes is a flat f.p. monomorphism and we proved that, for a flat f, f_{cl} enjoys such properties iff f does. □

We are now ready to define the Zariski site on the ∞-category of affine schemes. This generalizes the classical small Zariski site, as presented in [27],III.3 (see also [37],II.34.3).

Definition 4.1.2.3. *(Zariski Site, [15],4.4.3)* Define the (small) **Zariski site** on $\mathrm{Sch}^{\mathrm{Aff}}$ to be the Grothendieck site (see C.3.0.2) associated to the following pre-topology (or basis):
For a small set $I \in \mathrm{Set}$, we say that $\{j_\alpha : U_\alpha \to S \text{ in } \mathrm{Sch}^{\mathrm{Aff}} \mid \alpha \in I\} \in Zar$ iff, for each $\alpha \in I$, the map j_α is an open immersion and the inverse images $j^*_\alpha : \mathrm{QCoh}(S) \to \mathrm{QCoh}(U_\alpha)$ are jointly conservative, i.e. a morphism ϕ in $\mathrm{QCoh}(S)$ is an isomorphism iff $j^*_\alpha(\phi)$ iso in each $\mathrm{QCoh}(U_\alpha)$.
We write *'Zar-covering'* for a covering of the Zariski site.

Remark. In other words, we are requiring our covering sieves to detect whether two affine schemes coincide by checking their affine open sub-schemes.

Remark. The reader should beware that we stated only "generating" families for the corresponding covering sieves, which are obtained by closing under pre-composition in the site $\mathrm{Sch}^{\mathrm{Aff}}$.

Remark. The family of functors $j^*_\alpha : \mathrm{QCoh}(S) \to \mathrm{QCoh}(U_\alpha)$ is jointly conservative iff the canonical map $j := \coprod_\alpha j_\alpha$ induces a conservative functor $j^* : \mathrm{QCoh}(S) \to \prod_\alpha \mathrm{QCoh}(U_\alpha) \simeq \mathrm{QCoh}(\coprod_\alpha U_\alpha)$.
Indeed, $\coprod U_\alpha$ corresponds to a product in Ani(CRing) and hence is preserved by QCoh^{op}, which is a right Kan extension over Ani(CRing), being it the composite of two right Kan extensions. Moreover, the canonical product projections form a jointly conservative family.

Sometimes we do not need the full strength of all the requirements in the definition of Zariski covers. For instance, let us give a name to a finer site (so one with "more covers") which will often occur when studying descent.

Definition 4.1.2.4. *(fpqc-site)* Define the (small) **fpqc site** on $\mathrm{Sch}^{\mathrm{Aff}}$ to be the Grothendieck site (see C.3.0.2) associated to the following pre-topology (or basis):
For a *finite* set $I \in \mathrm{Set}$, we say that $\{f_\alpha : S_\alpha \to S \in \mathrm{Sch}^{\mathrm{Aff}} \mid \alpha \in I\}$ generates a fpqc-covering iff each f_α is flat (and of finite presentation), and the inverse images $f_\alpha^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(S_\alpha)$ are jointly conservative.

As already anticipated, we define 'stacks' as local (in the sense of Bousfield localisations) prestacks with respect to the Zariski site, thus generalizing 'local functors of points'.

Definition 4.1.2.5. *(Stacks, [15],4.4.4)* Define a **stack** to be a prestack satisfying descent with respect to the Zariski site, i.e. to be a sheaf in $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}}, \mathrm{Zar})$. Call $\mathrm{Stack} := \mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}}, \mathrm{Zar}) \subseteq_{f.f.} \mathrm{PreStack}$ the full subcategory spanned by stacks. By extension, for any site τ on $\mathrm{Sch}^{\mathrm{Aff}}$, we will call τ-stacks the sheaves in $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}}, \tau)$.

Digression. The very large ∞-topos Stack "behaves like" an ∞-topos (see Appendix C on ∞-topoi and sheaves).
As already observed in 4.1.1, $\mathrm{Sch}^{\mathrm{Aff}}$ gives rise to a large site, and sheaves on large sites are in general very large ∞-topoi and need not be small. In particular, what fails is precisely being accessible (hence presentable) with respect to the smallest universe $\mathcal{U}$.
A consequence of such an observation is that, for example, the Representability Theorem or the Adjoint Functor Theorems do not hold any more, unless allowing large diagrams.
However, in most cases we will be dealing with schemes (which will defined soon) on the Zariski site, and such a pathology can be solved as in the classical case (see [37],34.3.5): we restrict our constructions to involve only a small set of schemes and a small set of (small) families of their covers; then one can define a small site in which to operate: this will not alter anything in the constructions and will have the advantage of allowing us to work into a topos. As expected, however, one should be very careful when comparing constructions among the different sites arising in such a way.
In particular, such an observation allows us to still construct sheafification functors as in C.5.1.2, thus enforcing in such very large sheaves ∞-categories all Giraud's axiom (apart from accessibility). Moreover, all the $\mathcal{U}$-small constructions which could be performed in a $\mathcal{U}$-small ∞-topos remain $\mathcal{U}$-small in Stack.
We observe, that such a feature really depends on the possibility of defining Zar by means of covering sieves consisting of compact morphisms. For instance, the same observation holds for the étale site, but should not be taken for granted, since e.g. it does not hold for the *fpqc*-site.
Nevertheless, in order to focus on the more algebraic/geometric aspects of the subject, we choose to avoid focussing too much on set-theoretical issues.

Lemma 4.1.2.6. (Zar is sub-canonical) *The site* Zar *is sub-canonical, i.e. all representable presheaves* $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$ *are indeed* Zar*-sheaves.*

Proof. We need to show that, for each $S := \mathrm{Spec}(\underline{B}) \in \mathrm{Sch}^{\mathrm{Aff}}$ and each diagram $\mathrm{Spec}(p) : I \to \mathrm{Sch}^{\mathrm{Aff}}$ with limit $\lim \mathrm{Spec}(p) = S$, it holds $\mathrm{Spec}(\underline{A})(S) \simeq \lim \mathrm{Spec}(\underline{A})(\mathrm{Spec}(p))$.
Under Yoneda Lemma, the diagram $\mathrm{Spec}(p)$ into $\mathrm{Sch}^{\mathrm{Aff}}$ amounts to a diagram p into $\mathrm{Ani}(\mathrm{CRing})$, so that our condition becomes $\mathrm{Map}(\underline{A}, \underline{B}) \simeq \lim \mathrm{Map}(\underline{A}, \underline{B}_\alpha)$; the latter clearly holds, since Map preserves limits in the second variable. $\square$

4.1.3 Schemes

The next definition generalizes open immersions in the formalism of functors of points.

Definition 4.1.3.1. *(Open immersion,[15],4.5.2)* Let $j : U \to S$ be a morphism in Stack. Then,

- For $S \in \mathrm{Sch}^{\mathrm{Aff}}$: j is an **open immersion** (write $\hookrightarrow$) into an affine scheme iff
  - j is a monomorphism;
  - there exists a family of open immersions of affine schemes $(j_\alpha : U_\alpha \hookrightarrow S)_\alpha \subseteq \mathrm{Sch}^{\mathrm{Aff}}$ which factors through U by an effective epimorphism $\coprod_\alpha U_\alpha \twoheadrightarrow U$ (see C.1.0.7).
- For an arbitrary $S \in \mathrm{Stack}$: j is an open immersion of stacks iff, for each $s : \mathrm{Spec}(\underline{A}) \to S$ $\underline{A}$-point of S, the base-change $s^*(j) : U \times_S \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{A})$ is an open immersion into an affine scheme.

Given an open immersion $j : U \hookrightarrow S$ in Stack, we call U an **open sub-stack** of S.

Remark. Intuitively, an open immersion of arbitrary stacks is defined to be 'locally on the base' an open immersion of stacks into an affine scheme. Moreover, as we will observe in 4.1.4.10, they are precisely monomorphisms being also open maps of stacks.

We postpone statements and proofs of the properties of open immersions of stacks to the section consisting of examples of relative schemes. Nevertheless, we will refer to them in showing many properties of schemes.

In the classical setting, schemes are defined to be local functors of points which admit a small cover by affine open sub-functors. Let us extend such a definition to the setting of stacks.

Definition 4.1.3.2. *(Zariski cover, [15],4.5.3)* A **Zariski cover** of a stack $S \in \mathrm{Stack}$ is a family of open immersions of stacks $\mathcal{U} := (j_\alpha : U_\alpha \hookrightarrow S)_\alpha$ such that the canonical map $j : U := \coprod_\alpha U_\alpha \twoheadrightarrow S$ is an effective epimorphism in the very large ∞-topos Stack.
Furthermore, the family $\mathcal{U}$ is called an **affine Zariski cover** iff each $U_\alpha \in \mathrm{Sch}^{\mathrm{Aff}}$.

Remark. By C.1.0.8 and 4.1.4.8, (affine) Zariski covers are transitive: let $(U_\alpha \hookrightarrow S)_\alpha$ be a (affine) Zariski cover of $S \in \mathrm{Stack}$ and consider (affine) Zariski covers $(V_\beta^\alpha \hookrightarrow U_\alpha)_\beta$ of each U_α; then $(V_\beta^\alpha \hookrightarrow S)_{\alpha,\beta}$ is again a (affine) Zariski cover of S.

The next lemma relates affine Zariski covers of affine schemes and Zar-coverings.

Lemma 4.1.3.3. *Consider an affine $S = \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$. Then, a small affine Zariski cover for S is a Zar-covering.*

Proof. Let $(j_\alpha : U_\alpha := \mathrm{Spec}(\underline{B}_\alpha) \hookrightarrow S)_\alpha$ be open immersions in $\mathrm{Sch}^{\mathrm{Aff}}$ s.t. the canonical map $j : \mathcal{U} := \coprod U_\alpha \simeq \mathrm{Spec}(\prod \underline{B}_\alpha) \twoheadrightarrow S$ is an effective epimorphism in Stack. We want to show that $j^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathcal{U})$ is conservative. We recall that, from the very definition, an open immersion of affine schemes is flat; hence, we are done by the characterization of faithful-flatness 4.1.4.5. ☐

We observe, however, that the terminology is purposely confusing: Zariski covers are coverings for the following site on Stack. Recall the construction of finitary sites as in C.5.2.2. Let $S := \mathrm{EffEpi}(\mathrm{Stack})$ be the class of effective epimorphisms in Stack, and consider the sieves which are generated by Zariski covers. EffEpi(Stack) satisfies the assumptions of C.5.2.2 by C.1.0.8.
EffEpi(Stack) is closed under large (so not very large) coproducts, so we can lift the assumptions of finiteness in the generation of sieves, and our assignment specifies a Grothendieck site on Stack.
As already observed, Zar is sub-canonical, so that $\mathrm{Sch}^{\mathrm{Aff}} \subseteq \mathrm{Stack}$ and we wonder whether Stack admits an analogous site with coverings generated by (small) *affine* Zariski covers.
This is no longer true, because an arbitrary stack needs not admit an affine Zariski cover. Then, we will define $\mathrm{Sch} \subseteq_{f.f.} \mathrm{Stack}$ to be the maximal subcategory admitting such a site.

Definition 4.1.3.4. *(Schemes, [15],4.5.4)* Define a **scheme** to be a stack which admits a small affine Zariski cover. Let $\mathrm{Sch} \subseteq_{f.f.} \mathrm{Stack}$ be the full subcategory spanned by schemes.

Derived schemes enjoy stability properties analogous to those of the classical ones.

Lemma 4.1.3.5. *(Open sub-prestacks are schemes) Let $U \hookrightarrow X$ be an open immersion in* PreStack *with $X \in \mathrm{Sch}$. Then, also $U \in \mathrm{Sch}$.*

Proof. Claim. $U \in \mathrm{Stack}$.

Proof. Let $S \in \mathrm{Sch}^{\mathrm{Aff}}$ and consider any of its Zar-covering $\mathcal{J} \in \mathrm{Zar}(S)$. According to C.5.3.3, we need to check the sheaf condition relative to $\mathcal{J}$, namely that $\lim U_{|\mathcal{J}} \simeq U(S) \simeq \mathrm{Map}(S, U)$.
Since $X \in \mathrm{Stack}$, it holds in particular $\lim X_{|\mathcal{J}} \simeq X(S) \simeq \mathrm{Map}(S, X)$. Then, the open immersion of prestacks $j : U \looparrowright X$ induces the following square in PreStack:

$$\begin{tikzcd}
\lim U_{|\mathcal{J}} \arrow[r, hook] \arrow[d, hook, "j_{|\mathcal{J}}"] & \mathrm{Map}(S, U) \arrow[d, hook, "j_*"] \\
\lim X_{|\mathcal{J}} \arrow[r, "\simeq"] & \mathrm{Map}(S, X)
\end{tikzcd}$$

We wish to show that such a square in Spc is cartesian. By [29],3.3.18, it amounts to check that the fibres of the horizontal maps are equivalent.
To this end, take any point $\phi : \Delta^0 \to \mathrm{Map}(S, X)$ and let $\phi_{\mathcal{J}} : \Delta^0 \to \lim X_{|\mathcal{J}}$ denote its copy in the limit. Consider the following cub induced by taking the fibre over ϕ.

$$\begin{tikzcd}
(\lim j_{|\mathcal{J}})^{-1}(\phi_{\mathcal{J}}) \arrow[rr] \arrow[dr, dotted] \arrow[dd, hook] & & \lim U_{|\mathcal{J}} \arrow[dr] \arrow[dd, hook, "j_{|\mathcal{J}}" near start] & \\
& j_*^{-1}(\phi) \arrow[rr] \arrow[dd, hook] & & \mathrm{Map}(S, U) \arrow[dd, hook, "j_*"] \\
\Delta^0 \arrow[rr, "\phi_{\mathcal{J}}" near start] \arrow[dr, equal] & & \lim X_{|\mathcal{J}} \arrow[dr, "\simeq"] & \\
& \Delta^0 \arrow[rr, "\phi"] & & \mathrm{Map}(S, X)
\end{tikzcd}$$

We will now freely apply results from the appendix on truncatedness. The leftmost vertical maps are monomorphisms of spaces, so that they have (-1)-truncated fibres. Hence, by [24],5.5.6.14, also the comparison map between fibres must be (-1)-truncated.
We wish to show that it is actually (-2)-truncated, i.e. an equivalence. In other words, we want it to have contractible fibres. Since we already know them to be either empty or contractible, let's prove that the fibre over any lift $\psi : S \to U$ of $\phi : S \to X$ is not empty, i.e. that there exists $\psi_{\mathcal{J}} \in \lim U_{|\mathcal{J}}$ which corresponds to ψ and lifts $\phi_{\mathcal{J}}$.
But this can be checked at the level of the underlying classical prestacks (so after applying j^*) and turns out to be true, because $j^{\mathrm{cl}} : U^{\mathrm{cl}} \hookrightarrow X^{\mathrm{cl}}$ makes U^{cl} into an open (hence local) subfunctor of the classical scheme X^{cl}. ■

Claim. *The stack U admits an affine Zariski cover.*

Proof. Let $(j_\alpha^X : X_\alpha \looparrowright X)_\alpha$ be an affine Zariski cover of X and consider the following cartesian squares consisting of open immersions (by 4.1.4.8) and defining a Zariski cover of U (by C.1.0.8):

$$\begin{tikzcd}
U_\alpha := X_\alpha \times_X U \arrow[r] \arrow[d, "j_\alpha^U"] & X_\alpha \arrow[d, "j_\alpha^X"] \\
U \arrow[r, "j"] & X
\end{tikzcd}$$

In particular, the fact that $(j_\alpha^X)^*(j) : U_\alpha \looparrowright X_\alpha$ is an open immersion into the affine scheme X_α is witnessed by a family of open immersions $(V_\beta^\alpha \looparrowright X_\alpha)_\beta$ of affine schemes s.t. $\coprod_\beta V_\beta^\alpha \twoheadrightarrow U_\alpha$. We claim that $(V_\beta^\alpha \to U_\alpha)_\beta$ is an affine Zariski cover of the latter, i.e. that each $V_\beta^\alpha \to U_\alpha$ is an open immersion.
In other words, we need to show that, for each $\underline{A}$-point of U_α, the base-change $V_\beta^\alpha \times_{U_\alpha} \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{A})$ is an open immersion. Our schemes sit in the following diagram with cartesian squares:

$$\begin{tikzcd}
V^\alpha_\beta \times_{U_\alpha} \mathrm{Spec}(\underline{A}) \arrow[r] \arrow[d] & V^\alpha_\beta \arrow[d] \arrow[dr, hook, "op"] & \\
\mathrm{Spec}(\underline{A}) \arrow[r, "\forall"] & U_\alpha := X_\alpha \times_X U \arrow[r, hook, "op"'] \arrow[d, hook, "op"', "j^U_\alpha"] & X_\alpha \arrow[d, hook, "op"', "j^X_\alpha"] \\
& U \arrow[r, hook, "j", "op"'] & X
\end{tikzcd}$$

The post-composition of the upper-left square by the monomorphism $U_\alpha \hookrightarrow X_\alpha$ is still cartesian and exhibits $V^\alpha_\beta \times_{U_\alpha} \mathrm{Spec}(\underline{A}) \simeq V^\alpha_\beta \times_{X_\alpha} \mathrm{Spec}(\underline{A}) \hookrightarrow \mathrm{Spec}(\underline{A})$, as wished. ■ □

Proposition 4.1.3.6. *(Stability under coproducts and base-change, [15],4.5.4)* Sch *is closed under coproducts and base-change in* Stack. *In particular coproducts in* Sch *are disjoint and universal.*

Proof. The very large ∞-topos Stack is in particular small bi-complete; let's show that affine Zariski covers are 'stable' under such operations.

Base-change: The proof is analogous to the classical setting. First, observe that the fibre-product in Stack of the angle $\mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{B}) \leftarrow \mathrm{Spec}(\underline{C})$ is the affine scheme $\mathrm{Spec}(\underline{A} \otimes^L_{\underline{B}} \underline{C})$. Indeed, by an adaptation of [24],5.1.3.2 the co-Yoneda embedding Spec sends colimits in Ani(CRing) to limits in $\mathrm{Sch}^{\mathrm{Aff}}$, and the latter is a subcategory of Stack because Zar is sub-canonical.

Then, let $X \to Z \leftarrow Y$ be an angle in Sch and form its fibre-product in Stack; we are left to show that the latter admits an affine Zariski cover.

Let $(j^Z_\alpha : Z_\alpha \hookrightarrow Z)_\alpha$ be an affine Zariski cover for $Z \in \mathrm{Sch}$, so s.t. $\coprod Z_\alpha \twoheadrightarrow Z$ is an effective epimorphism. The base-changes $j^X_\alpha : X_\alpha := X \times_Z Z_\alpha \to X$ and $j^Y_\alpha : Y_\alpha := Y \times_Z Z_\alpha \to Y$ of j^Z_α are again open immersions by 4.1.4.8, and thus, by 4.1.3.5, they both live in Sch and form Zariski covers of X and Y, respectively.

Choose affine Zariski covers $(j^X_{\alpha,\beta} : X^\alpha_\beta \hookrightarrow X_\alpha)_\beta$ and $(j^Y_{\alpha,\beta} : Y^\alpha_\beta \hookrightarrow Y_\alpha)_\beta$. Then, again by applying 4.1.4.8 three times, the affine schemes

$$X^\alpha_\beta \times_{Z_\alpha} Y^\alpha_\beta \simeq (X^\alpha_\beta \times_Z Y) \times_{Z_\alpha} (Z \times_Z Y^\alpha_\beta)$$

form a family of open immersions $(X^\alpha_\beta \times_{Z_\alpha} Y^\alpha_\beta \hookrightarrow X \times_Z Y)_{\alpha,\beta}$ whose source is affine by the first part of the argument. We are left to check the covering property. By the construction, we have the following cube where $\twoheadrightarrow$ denotes effective epimorphisms.

$$\begin{tikzcd}
\coprod_{\beta,\alpha}(X^\alpha_\beta \times_{Z_\alpha} Y^\alpha_\beta) \arrow[rr] \arrow[dd] \arrow[dr] & & \coprod_{\alpha,\beta} Y^\alpha_\beta \arrow[dd, two heads] \arrow[dr] & \\
& \coprod_{\alpha,\beta} X^\alpha_\beta \arrow[rr] \arrow[dd, two heads] & & \coprod Z_\alpha \arrow[dd, two heads] \\
X \times_Z Y \arrow[rr] \arrow[dr] & & Y \arrow[dr] & \\
& X \arrow[rr] & & Z
\end{tikzcd}$$

Since coproducts in Stack are universal, also the top face is cartesian, so that the vertical map at the back-right corner is an effective epimorphism. Let us briefly prove this latter fact, which holds more generally in a semi-topos: call f such a map, and let f_T denote the other vertical maps, for $T = X, Y, Z$; we need to show that $\mathrm{colim}\, \check{C}(f)_{|\Delta^{op}} \simeq \mathrm{cod}(f)$. We claim that the canonical comparison map $\check{C}(f) \to \check{C}(f_X) \times_{\check{C}(f_Z)} \check{f}_Y$ is an equivalence: the check nerve of a map u is level-wise a fibre-product of copies of the source of u over its target, so that the canonical map is a level-wise equivalence and we conclude by [20],2.2.2. Then, being the f_T's all effective epimorphisms, we conclude by the fact that colimits in a semi-topos are universal.

Coproducts: Let $\{X_i\}_I$ be a family of schemes and choose an affine Zariski cover for each of them, say $\mathcal{X}_i := (X^i_\alpha \hookrightarrow X_i \mid \alpha \in A_i)$. Then, $\cup_I \mathcal{X}_i$ forms an affine Zariski cover for $\coprod_I X_i$. Indeed, effective epimorphisms are stable under coproduct by [24],6.2.3.11. □

Similarly to the case of classical prestacks, we can retrieve the ordinary notion of scheme as an example of our theory.

Definition 4.1.3.7. *(Classical schemes, [15],4.5.6)* Call **classical scheme** a classical stack on the site generated by Zar-coverings such that it admits a small affine Zariski cover.
The ordinary full subcategory $\mathrm{Sch}^{\mathrm{cl}} \subseteq_{f.f.} \mathrm{Sh}(\mathrm{CRing}^{op}, \mathrm{Zar}, \mathrm{Set})$ spanned by classical schemes retrieves the ordinary notion of schemes. Here Set denotes consistently the large 1-category of large sets.
Moreover, for $X \in \mathrm{Sch} \subseteq_{f.f.} \mathrm{Stack}$, the underlying classical prestack $X^{\mathrm{cl}} \in \mathrm{PreStack}^{\mathrm{cl}}$ takes values in Set and is a classical scheme. Call it the **underlying classical scheme** of X.

Proof. Let $X \in \mathrm{Sch}$ be a Zariski-stack with an affine Zariski cover $(j_\alpha : U_\alpha \looparrowright X)_\alpha$. We already know that the underlying classical stack $X^{\mathrm{cl}} \in \mathrm{Sch}^{\mathrm{cl}} \subseteq_{f.f.} \mathrm{PreStack}^{\mathrm{cl}}$ factors through Set. Hence, wish to show that the family $(j_\alpha)_\alpha$ reduces to a Zar-covering by ordinary open affine schemes.
The condition $j_\alpha : U_\alpha \looparrowright X$ open immersion reduces to the fact that the base-change along each $\underline{A}$-point of X is an open immersion into an affine scheme. At the level of classical schemes, for each $x : \mathrm{Spec}(\underline{A}) \to X$ this corresponds to the rightmost cartesian square.

$$\begin{tikzcd}
U_\alpha \times_X \mathrm{Spec}(\underline{A}) \arrow[r] \arrow[d, hook, "x^*(j_\alpha)"', "op"] & U_\alpha \arrow[d, hook, "j_\alpha"', "op"] \\
\mathrm{Spec}(\underline{A}) \arrow[r, "x"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
U_\alpha^{\mathrm{cl}} \times_{X^{\mathrm{cl}}} \mathrm{Spec}(\pi_0\underline{A}) \arrow[r] \arrow[d, hook] & U_\alpha^{\mathrm{cl}} \arrow[d, hook, "j_\alpha^{\mathrm{cl}}"] \\
\mathrm{Spec}(\pi_0\underline{A}) \arrow[r, "x^{\mathrm{cl}}"] & X^{\mathrm{cl}}
\end{tikzcd}$$

Notice that the rightmost square has the stated form, since pre-composition with the Yoneda embedding $j : \mathrm{CRing} \hookrightarrow \mathrm{Ani}(\mathrm{CRing})$ is a right-adjoint, so that it preserves limits, thus in particular monomorphisms and fibre products.
Moreover, the fact that $x^*(j_\alpha)$ is an open immersion is witnessed by a family of open immersions of affine schemes $(j^\alpha_\beta : V^\alpha_\beta \looparrowright \mathrm{Spec}(\underline{A}))_\beta$ with $p_\alpha : \coprod_\beta V^\alpha_\beta \twoheadrightarrow U_\alpha \times_X \mathrm{Spec}(\underline{A})$ being an effective epimorphism.
At the level of classical schemes, each family $\{(j^\alpha_\beta)^{\mathrm{cl}}\}_\beta$ consists again of open immersions and the restriction along the Yoneda embedding of the corresponding effective epimorphism amounts to an ordinary glueing of schemes.
Indeed, restriction along the Yoneda embedding admits also a right-adjoint via a right Kan extension, thus it is a bi-continuous functor; in particular, it preserves effective epimorphisms, so that, by C.5.3.7,

$$U_\alpha^{\mathrm{cl}} \times_{X^{\mathrm{cl}}} \mathrm{Spec}(\pi_0\underline{A}) \cong \mathrm{colim}\, \check{C}(p_\alpha)_{|\Delta^{op}} \cong \mathrm{colim}\, \check{C}(p_\alpha)_{|\Delta^{\leq 2}}$$

exhibits $\{V^\alpha_\beta\}_\beta$ as an open cover of the classical scheme $U_\alpha^{\mathrm{cl}} \times_{X^{\mathrm{cl}}} \mathrm{Spec}(\pi_0\underline{A})$.
In other words, each $\{(j^\beta_\alpha)^{\mathrm{cl}}\}_\beta$ witnesses the fact that $(x^*(j_\alpha))^{\mathrm{cl}}$ is an open immersion into a classical affine scheme. Thus, also j_α^{cl} is an open immersion of classical schemes in the formalism of functors of points, as desired. ☐

We close this section by introducing an important class of schemes, consisting of those which are 'quasi-compact and quasi-separated'. Indeed, these can be described by finitely many affine charts, so that they enhance algebraic compactness to the scheme-theoretic one.

Definition 4.1.3.8. *(qcqs, [15],4.5.5)* Consider two schemes X, $Y \in \mathrm{Sch}$ and a morphism $f : X \to Y$ between them.

- $X \in \mathrm{Sch}$ is **quasi-compact** (qc) iff each Zariski cover $(j_\alpha : U_\alpha \looparrowright X \,|\, \alpha \in I)$ admits a finite Zariski sub-cover $(j_\alpha : U_\alpha \looparrowright X \,|\, \alpha \in F \subseteq A \text{ finite})$ (see C.5.2.1).
- $f : X \to Y$ in Sch is quasi-compact iff the base-change $X \times_Y \mathrm{Spec}(\underline{A})$ along any $\underline{A}$-point of Y is qc, i.e. iff it has qc fibres.
- $Y \in \mathrm{Sch}$ is **quasi-separated** (qs) iff the intersection $U \times_Y V$ of any two affine open sub-schemes $U, V \looparrowright Y$ is quasi-compact.

Remark. As already observed, Sch $\subseteq_{f.f.}$ Stack is the maximal subcategory which admits a (non-finitary) site as in C.5.2.2 with respect to EffEpi(Stack) and those sieves generated by affine Zariski covers. Now, we address the issue of defining a *finitary* site on Sch with respect to EffEpi(Stack) and with sieves generated by finite affine Zariski covers. Again, Sch does not admit such a site, and the maximal subcategory on which to restrict our investigation is precisely the full subcategory $\mathrm{Sch}^{\mathrm{qc}} \subseteq_{f.f.}$ Sch of quasi-compact schemes.

4.1.4 Examples of Relative Schemes

All the examples of relative schemes we will consider will be of course defined compatibly with the Zariski site, and a general strategy to prove statements about the properties of such maps will be a reduction to their restrictions to affine charts.
A very favourable situation occurs when a property **P** of a morphism of stacks $f : X \to Y$ can be checked "locally on the base", meaning that, for each affine chart y of the base, it suffices to prove **P** on the restriction of the fibre f_y to an affine cover of X_y. Clearly such a feature needs not be true in full generality, but, if we restrict to morphisms of schemes, then luckily it will be enjoyed by quite a big class of interesting examples. Therefore, let us state it more precisely.

Definition 4.1.4.1. *(*Zar*-locality on the base)* Let $f : X \to Y$ be a morphism in Stack and consider an affine Zariski cover $\mathcal{Y}$ of the base Y such that, for each affine chart $y \in \mathcal{Y}$, the fibre X_y over the latter admits an affine Zariski cover $\mathcal{X}_y := \{j^y_\alpha : X^y_\alpha \hookrightarrow X_y\}_\alpha$. Then, we say that a property **P** of f is Zar**-local on the base** iff P holds whenever each restriction $f \circ j^y_\alpha$ satisfies **P**.

Remark. Let τ denote any "small" site on $\mathrm{Sch}^{\mathrm{Aff}}$. What we actually need is that, for a "small" amount of affine charts $y \in \mathcal{Y}$ "τ-covering" the base, our notion of "small τ-cover" may provide a suitable cofinal class $\mathcal{X} \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/X_y}$ on which to check algebraic properties of the quasi-coherent algebra $\mathcal{O}_{X_y} \in \mathrm{QCohCAlg}(X_y)$ (see the section on "Animated Schemes").
The notion of "smallness" then depends on the feature at stake: for compactness properties we will need to work with finitely many fibres X_y and to consider a limit over finite classes $\mathcal{U}_y$; when dealing with (co)limits, then we need small indexing diagrams - i.e. a small amount of fibres and small cofinal classes - so that we can relax the size constraint on affine Zariski covers. In other cases, one might even extend the definition to morphisms of stacks and not really care about size issues.

Affine morphisms.

Definition 4.1.4.2. *(Affine maps, [15],4.6.1)* A morphism of schemes $f : X \to Y$ in Sch is **affine** iff it has affine fibres, i.e. for each $x \in X(\underline{A})$, the fibre $Y \times_X \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$ is affine.

A morphism of stacks $f : S \to T$ in Stack is **affine** iff its schematic fibres are affine maps of schemes: for each map $X \to T$ with $X \in$ Sch, the base-change $X_S \to X$ is affine in Sch. So, iff each $\underline{A}$-valued point is an affine scheme.

Remark. Let $f : X \to Y$ be a map in Sch with $Y \in \mathrm{Sch}^{\mathrm{Aff}}$ affine. Then, f is affine iff also $X \in \mathrm{Sch}^{\mathrm{Aff}}$.
Indeed, if f is affine, base-changing along any isomorphic copy of Y forces X to be affine.

Proposition 4.1.4.3. *(Stability properties of affine morphisms) Affine morphisms are stable under base-change and composition.*

Proof. Composition: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be affine morphisms in Sch. For any $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, consider the following composition of cartesian squares:

$$\begin{tikzcd}
X \times_Z \mathrm{Spec}(\underline{A}) \arrow[r] \arrow[d] & Y \times_Z \mathrm{Spec}(\underline{A}) \arrow[r] \arrow[d] & \mathrm{Spec}(\underline{A}) \arrow[d, "\forall"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

and notice that $Y \times_Z \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$ because g is an affine morphism; hence, since f affine also $X \times_Z \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, as needed.

Base-change: Let $f : X \to Y$ in Sch be affine, and consider the base-change $X \times_Y Y'$ by any map $g : Y' \to Y$ in Sch. For any $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, consider the following composition of cartesian squares:

$$\begin{tikzcd}
X' \times_Y \mathrm{Spec}(\underline{A}) \arrow[r] \arrow[d] & \mathrm{Spec}(\underline{A}) \arrow[d, "\forall"] \\
X' = X \times_Y Y' \arrow[r, "g^*(f)"] \arrow[d] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

$X' \times_Y \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$ because f is affine, thus we conclude that also $g^*(f)$ is such. □

Proposition 4.1.4.4. *(Cancellation property of affine morphisms, [13],2.4.3) Let $S \xrightarrow{f} T \xrightarrow{g} Z$ be composable morphisms in* Stack*. If g is affine, then f is affine iff $g \circ f$ is such.*

Proof. One direction follows from the stability under composition, as in 4.1.4.3. Conversely, assume that both g and $g \circ f$ are affine. Assume for the moment the following Claim.
Claim. *Reduce wlog to the composition $pr \circ i = 1_S : S \hookrightarrow T \xrightarrow{affine} S$.*
Then, we are left to consider the following map in Sch obtained by base-changing along an arbitrary $\underline{A}$-valued point $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/S}$:

$$1_{\mathrm{Spec}(\underline{A})} : \mathrm{Spec}(\underline{A}) \overset{i_{\mathrm{Spec}(\underline{A})}}{\hookrightarrow} \mathrm{Spec}(\underline{A})_T \underset{\text{affine}}{\xrightarrow{pr_{\mathrm{Spec}(\underline{A})}}} \mathrm{Spec}(\underline{A})$$

Now, being $pr_{\mathrm{Spec}(\underline{A})} : \mathrm{Spec}(\underline{A})_T \to \mathrm{Spec}(\underline{A})$ an affine morphism over an affine scheme, also the base-change $\mathrm{Spec}(\underline{A})_T$ is forced to be affine; hence, the map $i_{\mathrm{Spec}(\underline{A})} : \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{A})_T$ in $\mathrm{Sch}^{\mathrm{Aff}}$ must be affine, as desired.

Proof. (Of the Claim) The following extension of a cartesian square yields a factorization:

$$f : S \overset{i}{\hookrightarrow} S \times_Z T \xrightarrow{pr_2} T$$

By 4.1.4.3, affine morphisms stable under base-change, so both pullback projections pr_1, pr_2 are affine. Then, being they also stable under composition (see *ibid.*), it suffices to show that i is affine. In particular, we can replace g by the composite $g \circ pr_2 := S \times_Z T \to T$. ■

$$\begin{tikzcd}
S \arrow[dr, dotted, hook, "i"] \arrow[drr, bend left, "f"] \arrow[ddr, equal, bend right] & & \\
& S \times_Z T \arrow[r, "pr_2"] \arrow[d, "pr_1"] & T \arrow[d, "g", "\mathrm{Aff}"'] \\
& S \arrow[r, "g \circ f", "\mathrm{Aff}"'] & Z
\end{tikzcd}$$

□

Flat morphisms.

We recall the definition of a flat map of prestacks: $f : T \to S$ in PreStack is flat iff $f^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(T)$ is an exact functor. In particular, whenever $f : S := \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A}) =: T$ is a map in $\mathrm{Sch}^{\mathrm{Aff}}$ such a condition amounts to the flatness of the corresponding morphism $f^{\#} : \underline{A} \to \underline{B}$ in Ani(CRing).
Let us record some stability properties of flat morphisms. First of all, notice that they are clearly stable under composition.
Let us start with a characterization of faithful flatness, which will be used often to prove the locality of some given properties, such as flatness itself.

Lemma 4.1.4.5. *(Characterization of faithful flatness) For a map $\phi : \underline{A} \to \underline{B}$ in* Ani(CRing) tfae:

1. *ϕ is faithfully flat, i.e. the induced extension of scalars functor $\underline{B} \otimes^L_{\underline{A}} (-) : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}_{\underline{B}}$ is both exact and conservative (see 3.4.0.3).*

2. *ϕ is flat and $\phi^* : \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ is an effective epimorphism in the (very large) ∞-topos* Stack.

Proof. (1) $\implies$ (2) : We need to show that a faithfully flat map ϕ induces a surjection $\pi_0\phi^* : \mathrm{Spec}(\underline{B})^{\mathrm{cl}} = \mathrm{Spec}(\pi_0\underline{B}) \to \mathrm{Spec}(\pi_0\underline{A}) = \mathrm{Spec}(\underline{A})^{\mathrm{cl}}$ on connected components; since also $\pi_0\phi$ in CRing is faithfully flat, this comes from the classical theory.

(2) $\implies$ (1) : Let it be given a map $f : \underline{M} \to \underline{M}'$ in $\mathrm{Mod}_{\underline{A}}$ which is sent to an equivalence $\phi^*(f) : \underline{M} \otimes^L_{\underline{A}} \underline{B} \to \underline{M}' \otimes^L_{\underline{A}} \underline{B}$ in $\mathrm{Mod}_{\underline{B}}$. ϕ is flat, so by 3.4.0.2 in homotopy we obtain morphisms in $\mathrm{Mod}(\pi_0\underline{B})$:

$$\pi_i(\phi^*(f)) : \pi_i(\underline{M}) \otimes_{\pi_0(\underline{A})} \pi_0(\underline{B}) \xrightarrow{\cong} \pi_i(\underline{M}') \otimes_{\pi_0(\underline{A})} \pi_0(\underline{B})$$

But now the effective epimorphism $\mathrm{Spec}(\underline{B}) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ induces a surjection of classical schemes $\mathrm{Spec}(\pi_0\underline{B}) \cong \mathrm{Spec}(\underline{B})^{\mathrm{cl}} \twoheadrightarrow \mathrm{Spec}(\underline{A})^{\mathrm{cl}} \cong \mathrm{Spec}(\pi_0\underline{B})$, i.e. $\pi_0(\phi) : \pi_0\underline{A} \to \pi_0\underline{B}$ in CRing is faithfully flat. Thus, each iso $\pi_i(\phi^*(f))$ comes from an isomorphism $\pi_i(f)$, and we infer the desired conservativity of ϕ^* by Whitehead Theorem. $\square$

We are now ready to present the anticipated local properties of flatness.

Proposition 4.1.4.6. *(Flatness is local, [26],2.7.) Let $f : S \to T$ be a morphism in* Sch *and assume S to admit a* finite *affine Zariski cover, say $\{j_\alpha : U_\alpha := \mathrm{Spec}(\underline{R}_\alpha) \hookrightarrow S\}_\alpha$. Then f is flat iff each composite $f \circ j_\alpha$ is such.*

Proof. Choose a small set $\mathcal{U}'_T := \{U \hookrightarrow T \,|\, U \in \mathrm{Sch}^{\mathrm{Aff}}_{/T}\}$ of affine charts of T containing the Čheck nerve of an affine Zariski cover for T, and let $\mathcal{U}_T := \langle \mathcal{U}'_T \rangle \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/T}$ denote the full subcategory of spanned by such a set. Observe that the inclusion $\mathcal{U}_T \subseteq \mathrm{Sch}^{\mathrm{Aff}}_{/T}$ is cofinal.
Then, let $\mathcal{U}'_S := f^*(\mathcal{U}_T) \subseteq_{f.f.} \mathrm{Sch}_{/S}$ denote the base-change of $\mathcal{U}_T$ along f; notice that $\mathcal{U}'_S$ needs not live into $\mathrm{Sch}^{\mathrm{Aff}}_{/S}$, unless f is affine. Hence, define $\mathcal{U}_S \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_S$ to be spanned by open affine refinements of $\mathcal{U}'_S$: for each $X \in \mathcal{U}'_S$, choose an affine Zariski cover $\mathcal{Z}(X) := \{j_U : U \hookrightarrow X\}_U$ and let $\mathcal{U}_S \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/S}$ be the union $\mathcal{U}_S := \cup(\mathcal{Z}(X) \,|\, X \in \mathcal{U}'_S)$; observe that for a scheme T such a colimit is small, because both $\mathcal{U}_T$ (hence $\mathcal{U}'_S$) and each $\mathcal{Z}(X)$ are small.
Moreover, since both base-change and restriction to affine Zariski covers preserve limits of quasi-coherent modules, remark that also $\mathcal{U}_S \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/S}$ is cofinal. In particular, the construction is independent of the choices made.
In what follows, we adopt a more evocative terminology and refer to $\mathcal{U}_{(\star)}$ as the set of "affine open patches of $(\star)$". So, by construction we can regard the given map as being determined over affine open patches:

$$f^* : \mathrm{QCoh}(S) \simeq \lim\big(\mathrm{QCoh}(U) \,|\, U \in \mathcal{U}_S\big) \longrightarrow \lim\big(\mathrm{QCoh}(V) \,|\, V \in \mathcal{U}_T\big) \simeq \mathrm{QCoh}(T)$$

is the limit point $f^* = \overline{\psi}(\infty)$ of a natural transformation $\psi : p_S \to p_T$ between the restrictions $p_S := (\mathrm{QCoh}(-)|\mathcal{U}_S)$ and $p_T := (\mathrm{QCoh}(-)|\mathcal{U}_T)$.
Consider any (co)limit extension $F : \overline{K} \to \mathrm{QCoh}(S)$ of a finite K-indexed diagram and call $H := F(\infty)$ its (co)limit point. Consider $f^* \circ F : \overline{K} \to \mathrm{QCoh}(T)$. The exactness of f^* amounts to proving that $f^*(H)$ is still the (co)limit point of the restriction $(f^* \circ F)_{|K}$.
In what follows, will now refer to a straightforward adaptation of the proof of Claim 1.2 in 3.2.5.11 (it suffices to change the notation, as prompted by the bold Remark right below it).
Observe that F is the limiting diagram of a cocone of natural transformations on $\mathcal{U}_S$, say $\mathcal{F} : \mathrm{const}_{\mathcal{U}_S}(K) \to p_S$, which represents F on the affine patches of S. Similarly, by taking limits over K point-wise on $\mathcal{U}_S$ one obtains a diagram of natural transformations $\mathcal{H} : \mathrm{Map}(K, p_S(-)) \to p_S$ representing the (co)limit point H on the affine patches of S.
Let $\mathcal{H}' : \mathrm{Map}(K, p_T(-)) \to p_T$ denote the (co)limiting points over K of $\psi \circ \mathcal{F}$ point-wise on $\mathcal{U}_T$, and call H' the limiting point of $\mathcal{H}'$ over $\mathcal{U}_T$.
By the bold Remark in the proof of 3.2.5.11, we obtain the required commutativity $f^* \circ H \simeq H'$ provided that this holds point-wise on $\mathcal{U}_S$, i.e. that $\psi \circ \mathcal{H} \simeq \mathcal{H}'$.

In other words, we proved that it suffices to show the statement for a map $f : \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ with $\mathrm{Spec}(\underline{A})$ an open patch of T. The given affine Zariski open cover induces one of $\mathrm{Spec}(\underline{B})$, so we are left to consider the following claim.

Claim. *Let $f : \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{B})$ be a map in $\mathrm{Sch}^{\mathrm{Aff}}$, and let it be given any affine Zariski open cover $\{j_\alpha : \mathrm{Spec}(\underline{B}_\alpha) \hookrightarrow \mathrm{Spec}(\underline{B})\}_\alpha$ in $\mathrm{Sch}^{\mathrm{Aff}}$. Then, f is flat iff, for each α, $f \circ j_\alpha$ is such.*

Proof. We will actually prove a stronger result, namely that flatness being local for the fpqc-site, i.e. we need not require the maps j_α to be open immersions, but we assume precisely j_α to be flat for each α and $j : \coprod \mathrm{Spec}(\underline{B}_\alpha) \twoheadrightarrow \mathrm{Spec}(\underline{B})$ to be an effective epimorphism. Algebraically, this means that each $j^\#_\alpha : \underline{B} \to \underline{B}_\alpha$ is flat and that $j^\# : \underline{B} \to \prod_\alpha \underline{B}_\alpha$ is faithfully flat (see the characterization in 4.1.4.5 above). On the other hand, our statement becomes that $f^\# : \underline{A} \to \underline{B}$ be flat iff $j^\#_\alpha \circ f^\# : \underline{A} \to \underline{B}_\alpha$ be such.
One direction is clear. For the converse claim, observe that it suffices to show that the composite $j^\# \circ f^\# : \underline{A} \to \underline{B} \to \prod \underline{B}_\alpha$ is flat whenever $j^\#_\alpha \circ f^\#$ is such for each α. Indeed, being j^* conservative, the exactness of $(j \circ f)^*$ would imply also that of f^*.
By 3.4.0.2, this means proving that $\pi_0(j^\# \circ f^\#) : \pi_0\underline{A} \to \prod \pi_0(\underline{B}_\alpha)$ be flat and that $\prod \pi_n(\underline{B}_\alpha) \simeq \pi_n\underline{A} \otimes^L_{\pi_0\underline{A}} \prod \pi_0\underline{B}_\alpha$.
The second claim is the assumption that $j^\#_\alpha \circ f^\#$ is flat for each α. As for the first claim, instead it comes from the classical theory: consider static algebras $\{B_i\}_{i=1}^n$ over $A \in \mathrm{CRing}$; we will show that $(\prod_i B_i) \otimes_A (-)$ is exact iff each $B_i \otimes_A (-)$ is such.
To this end, recall first that, in the abelian category $\mathrm{Mod}(A)$, finite products are isomorphic to finite coproducts and both functors are identified with the direct sum $\oplus$; in particular the functor $\oplus$ is exact and commutes with the tensor product $\otimes$, so that $(\oplus_i B_i) \otimes_A (-) \cong \oplus \circ \{B_i \otimes_A (-)\}_i$, and the latter composite turns out to be exact precisely when each B_i is A-flat. ■ □

Remark. We actually proved that the flatness of a map $f : S \to T$ in *Stack* can be checked locally on the base, namely on a system of maps in $\mathrm{Sch}^{\mathrm{Aff}}$ whose targets range over *all* the affine open patches of the target in some (possibly large) class $\mathcal{U}_T$ being cofinal to $\mathrm{Sch}^{\mathrm{Aff}}_{/T}$.
The compactness properties of the target T affect our understanding of the term "locally": such a verification process can be simplified by reduction to those patches of a "nice" affine Zariski "cover" of the target (possibly large unless T is a scheme). In practice, this means being able to choose "nice" versions of $\mathcal{U}_T$.
Then, the given "local" assumption on the source S shapes the verification itself, which is the content of the last Claim.
In particular, for a map of schemes with a quasi-compact source, we obtain the fpqc-locality (and hence the Zar-locality) of flatness.

Proposition 4.1.4.7. *(Flatness is stable under base-change) Let $f : X \to Z$ be a morphism in* Stack*, and consider any other map of stacks $g : Y \to Z$. If f is flat, then also the base-change $g^*(f) : X \times_Z Y \to Y$ is flat.*

Proof. By the previous result 4.1.4.6, the question is now local on Y and $X \times_Z Y$. By 4.1.3.6, one can determine an affine Zariski cover of the fibre-product by taking all possible fibre-products of suitable choices of affine Zariski covers of X, Y, Z, locally on Z. So the statement reduces to check that the extension of scalars of flat maps in $\mathrm{Ani}(\mathrm{CRing})$ is flat, which is the content of 3.4.0.4. □

Open immersions.

Proposition 4.1.4.8. *(Properties of open immersions) Open immersions of stacks (or schemes) are stable under composition and base-change.*

Proof. Composition. Let $j : X \hookrightarrow Y$, $j' : Y \hookrightarrow Z$ be open immersions of stacks. Pick up an arbitrary point $s \in Z(\underline{A})$, and let it be given a family of open immersions witnessing j' open immersion, namely $(j'_\alpha : U'_\alpha \hookrightarrow \mathrm{Spec}(\underline{A}))_\alpha$ with $U'_\alpha := \mathrm{Spec}(\underline{B}_\alpha)$ affine schemes s.t. $\coprod U'_\alpha \twoheadrightarrow Y \times_Z \mathrm{Spec}(\underline{A})$ is an effective epimorphism.
Then, being also j an open immersion, pick up families of open immersions as in the definition for each of the points $j'_\alpha \in Y(\underline{B}_\alpha)$, namely $(j^\alpha_\beta : U^\alpha_\beta \hookrightarrow U'_\alpha)_\alpha$ with $U^\alpha_\beta := \mathrm{Spec}(\underline{C}^\alpha_\beta)$ affine schemes s.t. $\coprod_\beta U^\alpha_\beta \twoheadrightarrow U'_\alpha$ is an effective epimorphism.
Then, by C.1.0.8 also $\coprod_{\alpha,\beta} U^\alpha_\beta \twoheadrightarrow Y \times_Z \mathrm{Spec}(\underline{A})$ is an effective epimorphism.

Hence, $(j'_\alpha \circ j^\beta_\alpha : U^\alpha_\beta \hookrightarrow U'_\alpha \hookrightarrow \mathrm{Spec}(\underline{A}))_{\alpha,\beta}$ is the family of open immersions of affine schemes witnessing $j' \circ j$ open immersion.

Base-change. Let $j : U \hookrightarrow X$ be an open immersion of stacks and consider any $g : Y \to X$ in Sch. We need to show that, for each $\underline{A}$-point $y : \mathrm{Spec}(\underline{A}) \to Y$, the base-change $g^*(j) \times_Y \mathrm{Spec}(\underline{A}) : U \times_X \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{A})$. But this is clear, because the fibre-product is along $[g \circ y : \mathrm{Spec}(\underline{A}) \to X] \in X(\underline{A})$, so that $g^*(j) \times_Y y \simeq j \times_X (g \circ y)$ is an open immersion by the assumption on j. □

Now, as the topological intuition would suggest, we prove that open immersions of stacks are open monomorphisms. Furthermore, this turns out to characterize open immersions whenever the source admits an open cover.

Definition 4.1.4.9. *(Open morphism)* Let $f : S \to T$ in Stack be a morphism of stacks. Then,

- For $T \in \mathrm{Sch}^{\mathrm{Aff}}$: f is **open** into an affine scheme iff there exists a family of open immersions of affine schemes $(j_\alpha : U_\alpha \hookrightarrow T)_\alpha \subseteq \mathrm{Sch}^{\mathrm{Aff}}$ which factors through S by an effective epimorphism $\coprod_\alpha U_\alpha \twoheadrightarrow T$ (see C.1.0.7).
- For an arbitrary $T \in$ Stack: f is open iff the base-change of f along any affine point of T is open into an affine scheme.

Lemma 4.1.4.10. *(Characterization of open maps) Let $f : S \to T$ in* Stack *be an open map of stacks; then, post-composition by f preserves open immersions.*

Proof. Let $j : U \hookrightarrow S$ be an open immersion in Stack, and let $s : \mathrm{Spec}(\underline{R}) \to T$ be any $\underline{R}$-valued point of the target. We need to show that $s^*(f \circ j) : U \times_T \mathrm{Spec}(\underline{R}) \to U \times_S \mathrm{Spec}(\underline{R}) \to \mathrm{Spec}(\underline{R})$ is again an open immersion.
f is open, so there exists a family of affine open charts $\{V_j \hookrightarrow \mathrm{Spec}(\underline{R})\}_{j \in J}$ which covers $S \times_T \mathrm{Spec}(\underline{R})$. Moreover, by 4.1.4.8 open immersions are stable under base-change, so also the first composite in $s^*(f \circ j)$ must be an open immersion; hence, for each j there exists a family of affine open immersions $\{V^j_\alpha \hookrightarrow V_j\}_{\alpha \in I_j}$ which induces an effective epimorphism $\coprod_{I_j} V^j_\alpha \twoheadrightarrow U \times_T \mathrm{Spec}(\underline{R})$.
Then, again by 4.1.4.8, we obtain a family of affine open immersions $\{V^j_\alpha \hookrightarrow V_j \hookrightarrow \mathrm{Spec}(\underline{R})\}_{j,\alpha}$ indexed by $\cup_J I_j$. Finally, it covers $U \times_T \mathrm{Spec}(\underline{R})$ because each I_j-indexed subfamily does and effective epimorphisms are stable under coproducts. □

Remark. If S admits an open cover (e.g. is a scheme), then the condition above is clearly both necessary and sufficient.

Closed immersions.

Definition 4.1.4.11. *(Closed immersion, [15],4.6.2)* Let $i : Z \to X$ be a morphism in Sch. Then,

- For $Z := \mathrm{Spec}(\underline{B})$, $X := \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$: i is a **closed immersion** (write $\twoheadrightarrow$) of affine schemes iff its transpose $\underline{A} \to \underline{B}$ induces a surjection of connected components $\pi_0(\underline{A}) \to \pi_0(\underline{B})$.
- For arbitrary schemes Z, $X \in$ Sch: i is a closed immersion iff
  - i is affine;
  - i is a closed immersion on affine fibres: for each $x \in X(\underline{A})$, the base-change $Z \times_X \mathrm{Spec}(\underline{A}) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ is a closed immersion of affine schemes.

  Equivalently, iff $i^{cl} : Z^{\mathrm{cl}} \twoheadrightarrow X^{\mathrm{cl}}$ is a closed immersion of ordinary schemes.

In such a case, we say that Z is a **closed subscheme** of X, as witnessed by the closed immersion $i : Z \twoheadrightarrow X$.

Remark. In particular, a subscheme Z of X is closed iff $Z^{\mathrm{cl}} \subseteq X^{\mathrm{cl}}$ is closed at the level of underlying classical schemes.

Proposition 4.1.4.12. *(Properties of closed immersions) Closed immersions are stable under composition and base-change.*

Proof. Composition: Let $X \overset{i}{\twoheadrightarrow} Y \overset{j}{\twoheadrightarrow} Z$ be closed immersions in Sch, i.e. affine morphisms of schemes which are closed immersions at the level of affine fibres. Composition of affine morphisms is again affine, so we are left to check the second condition. For any $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, consider the following composition of cartesian squares:

$$\begin{tikzcd} X \times_Z \mathrm{Spec}(\underline{A}) \arrow[r, "i'"] \arrow[d] & Y \times_Z \mathrm{Spec}(\underline{A}) \arrow[r, "j'"] \arrow[d] & \mathrm{Spec}(\underline{A}) \arrow[d, "\forall"] \\ X \arrow[r, "i"] & Y \arrow[r, "j"] & Z \end{tikzcd}$$

and notice that both $Y \times_Z \mathrm{Spec}(\underline{A})$, $X \times_Z \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$, because j, i are affine morphisms, so that their composition clearly is such. Hence, the condition of being closed immersions on fibres induces surjections of classical schemes proving that $j \circ i$ is a closed immersion at the level of fibres:

$$\pi_0(\underline{A}) \overset{j'}{\twoheadrightarrow} \pi_0\mathcal{O}(Y \times_Z \mathrm{Spec}(\underline{A})) \overset{i'}{\twoheadrightarrow} \pi_0\mathcal{O}(X \times_Z \mathrm{Spec}(\underline{A}))$$

Base-change: Let $g^*(i)$ be the base-change in Sch of a closed immersion $i : X \twoheadrightarrow Y$ along any morphism $g : Y' \to Y$. $g^*(i)$ is again affine, so let's check that it is also a closed immersion at the level of fibres. Consider the following composition of cartesian squares.

$$\begin{tikzcd} X' \times_Y \mathrm{Spec}(\underline{A}) \arrow[r, "i'"] \arrow[d] & \mathrm{Spec}(\underline{A}) \arrow[d, "\forall"] \\ X' = X \times_Y Y' \arrow[r, "g^*(i)"] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

The rightmost vertical composite is some map $\mathrm{Spec}(\underline{A}) \to Y$, so, being i a closed immersion on affine fibres, we obtain the needed epimorphism of underlying classical schemes:

$$(g^*(i)')^{\mathrm{cl}} : \pi_0(\underline{A}) \overset{(i')^{\mathrm{cl}}}{\twoheadrightarrow} \pi_0\mathcal{O}(X \times_Y \mathrm{Spec}(\underline{A})) \cong \pi_0\mathcal{O}(X' \times_Y \mathrm{Spec}(\underline{A}))$$

□

Proposition 4.1.4.13. *(Complementary open immersion, [15],4.6.3) Let $i : Z \twoheadrightarrow X$ be a closed immersion of schemes and consider the sub-prestack of X obtained by base-change along the classical sub-prestack which is informally defined on $\underline{A}$-points as follows: $U(\underline{A})_0 := \{x \in X(\underline{A})_0 \mid \mathrm{Spec}(\underline{A}) \times_X Z = \emptyset\}$, where $\emptyset \in \mathrm{Stack}$ is the initial stack.*
*Then, U is a scheme and $j : U \hookrightarrow X$ is called the **complementary open immersion** to i.*

Proof. Claim. *There exists a sub-prestack U of X, which has the stated $\underline{A}$-points for $\underline{A} \in \mathrm{Ani(CRing)}$. Call it a complement of Z in X.*

Proof. Let us first make sense of the definition of the prestack U. Let $(-) \times_X Z = i^* : \mathrm{Sch}^{\mathrm{Aff}}_{/X} \to \mathrm{Sch}^{\mathrm{Aff}}_{/Z}$ denote the functor of schemes "base-change by i", as induced by restricting the base-change functor of the large topos Stack to affine schemes; notice that this is well-defined, because i is an affine morphism. Let $\mathcal{I} := (i^*)^{-1}(\emptyset) \subseteq \mathrm{Sch}^{\mathrm{Aff}}_{/X}$ denote the pre-image under i^* of the full subcategory $\emptyset \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/Z}$, as in the following cartesian square in Cat_∞.

$$\begin{tikzcd} \mathcal{I} := (i^*)^{-1}(\emptyset) \arrow[r] \arrow[d, hook] & \emptyset \arrow[d, hook] \\ \mathrm{Sch}^{\mathrm{Aff}}_{/X} \arrow[r, "i^*"] & \mathrm{Sch}^{\mathrm{Aff}}_{/Z} \end{tikzcd}$$

Since taking pre-images preserves faithfully full monomorphisms, $\mathcal{I} \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/X}$ is a full subcategory of affine schemes over X. Moreover, the latter comes equipped with a canonical forgetful functor $ev_0 : \mathcal{I} \to \mathrm{Sch}^{\mathrm{Aff}}$. We claim it to be a right-fibration.
In order to see this, first notice that $ev_0 : \mathrm{Sch}^{\mathrm{Aff}}_{/X} \to \mathrm{Sch}^{\mathrm{Aff}}$ is a right-fibration, so that by [20],3.1.22 all morphisms in $\mathrm{Sch}^{\mathrm{Aff}}_{/X}$ (thus a fortiori all those in $\mathcal{I}$) are ev_0-cartesian. Hence, it suffices to show that each morphism in $\mathrm{Sch}^{\mathrm{Aff}}$ with target in $\mathcal{I}$ admits a lift in $\mathcal{I}$. But this is clear, since for each $\mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ s.t. $Z \times_X \mathrm{Spec}(\underline{A}) \simeq \emptyset$, one has that also $Z \times_X \mathrm{Spec}(\underline{B}) \simeq (Z \times_X \mathrm{Spec}(\underline{A})) \times_{\mathrm{Spec}(\underline{A})} \mathrm{Spec}(\underline{B}) \simeq \emptyset$; hence, $\mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$, having both source and target in $\mathcal{I} \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/X}$, must itself live in $\mathcal{I}$, as wished.
Therefore, the right-fibration $ev_0 : \mathcal{I} \to \mathrm{Sch}^{\mathrm{Aff}}$ represents a prestack $U : \mathrm{Sch}^{\mathrm{Aff}} \simeq \mathrm{Ani}(\mathrm{CRing})^{op} \to \mathrm{Spc}$ under the Straightening Theorem [24],3.2:

$$\begin{tikzcd}
\mathcal{I} \arrow[r] \arrow[d, "ev_0", "\mathrm{RFib}\ni"'] & \mathrm{Spc}_* \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Sch}^{\mathrm{Aff}} \arrow[r, "U"] & \mathrm{Spc}
\end{tikzcd}$$

The action of U on objects retrieves the classical Grothendieck construction, so that, for each $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/X}$, one has that $U(\underline{A}) := (i^*|\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{Spec}(\underline{A})//X})^{-1}(\emptyset) \in \mathrm{Spc}$ yields the corresponding fibre of ev_0.
Moreover, under the aforementioned Straightening equivalence of ∞-categories $\mathrm{RFib}(\mathrm{Sch}^{\mathrm{Aff}}) \simeq \mathcal{P}(\mathrm{Sch}^{\mathrm{Aff}})$, the fully faithful inclusion $\mathcal{I} \subseteq \mathrm{Sch}^{\mathrm{Aff}}_{/X}$ allows us to regard $j : U \hookrightarrow X$ as a subobject in PreStack. ■

Claim. *$U \times_X Z \simeq \emptyset$.*

Proof. Form the fibre-product $U \times_X Z$ and consider the following diagram in PreStack induced by any of its $\underline{A}$-points:

$$\begin{tikzcd}
 & & \mathrm{Spec}(\underline{A}) \arrow[dll, equals, bend right] \arrow[d, dotted, "\exists!"] \arrow[dddr, bend left, "ev_2\circ\alpha"] & \\
\mathrm{Spec}(\underline{A}) \arrow[d, "\alpha"] & & \mathrm{Spec}(\underline{A}) \times_X Z \simeq \emptyset \arrow[ll] \arrow[d] & \\
U \times_X Z \arrow[rr, "ev_2"] \arrow[dr] \arrow[drrr, dotted] & & Z \arrow[dr, hook, "i"] & \\
 & U \arrow[rr, hook, "j"] & & X
\end{tikzcd}$$

The existence of a map $\mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(0) \simeq \emptyset$ implies that $\underline{A}$ is a 0-algebra, and hence is trivial. In other words, $\mathrm{Sch}^{\mathrm{Aff}}_{/U\times_X Z} \simeq * \simeq \mathrm{Sch}^{\mathrm{Aff}}_{/\emptyset}$ and we conclude by the Yoneda Lemma in PreStack. ■

Claim. *Any complement U of Z in X is a stack.*

Proof. Follow the proof of 4.1.3.5 until the last step to reduce the statement to the case of underlying classical schemes. There, $U^{\mathrm{cl}} \hookrightarrow X^{\mathrm{cl}}$ is an open complement of the closed classical subscheme $Z^{\mathrm{cl}} \twoheadrightarrow X^{\mathrm{cl}}$, so we can apply 4.1.3.5. ■
Remark. We do not really need this last claim, we make such an observation just because we defined closed immersions only for stacks.

Claim. *$j : U \hookrightarrow X$ is an open immersion of stacks.*

Proof. Notice first that wlog $X \in \mathrm{Sch}^{\mathrm{Aff}}$.
Indeed, we need to show that the base-change along each $\underline{A}$-point of X yields an open immersion $U \times_X \mathrm{Spec}(\underline{A}) \hookrightarrow \mathrm{Spec}(\underline{A})$. On the other hand, from the very definition we know that $Z \times_X \mathrm{Spec}(\underline{A}) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ is a closed immersion of affine schemes. Now, the intersection of the two sub-stacks is empty:

$$(U \times_X \mathrm{Spec}(\underline{A})) \times_{\mathrm{Spec}(\underline{A})} (Z \times_X \mathrm{Spec}(\underline{A})) \simeq U \times_X Z \times_X \mathrm{Spec}(\underline{A}) \simeq \emptyset \times_X \mathrm{Spec}(\underline{A}) \simeq \emptyset$$

because by C.1.0.8 we can cover the last fibre-product by the empty set. Thus, we retrieve our setting over the arbitrarily chosen affine scheme $\mathrm{Spec}(\underline{A})$.

Therefore, we need to find open immersions $U_\alpha \looparrowright X$ in $\mathrm{Sch}^{\mathrm{Aff}}$ s.t. $\coprod U_\alpha \twoheadrightarrow U$ is an effective epimorphism. Observe that, for each $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}_{/U}$, $\mathrm{Spec}(\underline{A}) \times_X Z \simeq \emptyset$. Moreover, recall that by the ∞-Density Theorem [24],5.1.5.3, we can write $\mathrm{colim}_{\mathrm{Sch}^{\mathrm{Aff}}_{/U}} \mathit{for} \simeq U$, and (by the sheaf condition on covers) the colimit can be computed by restricting along the cofinal inclusion $\mathcal{U} \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/U}$ of the wlog small family of open affine schemes in $\mathcal{U}$; hence, by [24],6.2.3.13, we conclude that $\coprod \mathcal{U} \twoheadrightarrow \mathrm{colim}_{\mathcal{U}} \mathit{for} \simeq U$ is an effective epimorphism.

□

Morphism of Finite Presentation.

Let us start by introducing some topological compactness property, namely the relative notion of qcqs schemes. In view of the section on "Animated Schemes", such topological statements and proofs are entirely classical, and hence omitted.

Definition 4.1.4.14. *(Quasi-compact, separated and quasi-separated)* A morphism of schemes $f : X \to Y$ is **quasi-compact** (or *qc* for short) iff there is an affine Zariski cover $\mathcal{Y}$ of the base Y with quasi-compact pre-image $f^{-1}(y) \in \mathrm{Sch}^{\mathrm{qc}}$, iff every $\underline{A}$-valued point $y \in Y(\underline{A})$ has quasi-compact pre-image $f^{-1}(y)$.

A morphism of schemes $f : X \to Y$ is **(quasi-)separated** (or *qs* for short) iff its diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion (resp. quasi-compact).

Example 4.1.4.15. qc: Closed immersions are always quasi-compact, but an open immersion needs not be. **Separated:** (Maps of) Affine schemes are always separated.

Proposition 4.1.4.16. *(Properties of qcqs and separated) "Being qc" and "Being separated (resp. qs)" are stable under base-change, composition, and* Zar*-local on the base.*
For a separated morphism $f : X \to Y$ in Sch *with Y separated, then the intersection of any two affine subschemes of X is again affine.*

We now enlarge the picture by adding some algebraic compactness, namely finite presentation. Since in DAG we work with homotopy colimits, compactness expresses now in a broad ranges of facets: see the section on "Almost Perfect Modules" for the algebraic context. However, we will not really need such flexibility at the level of morphisms of schemes, so we state only the direct analogous of the classical notion.

Definition 4.1.4.17. *(Finite presentation)* A morphism of schemes $f : X \to Y$ in Sch is **locally of finite presentation** (or *lfp* for short) iff, Zar-locally on the base Y (see 4.1.4.1), it is of the form $\mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ in $\mathrm{Sch}^{\mathrm{Aff}}$ for $\underline{B}$ a compact $\underline{A}$-algebra. More precisely:

- for $f : X := \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A}) =: Y$ in $\mathrm{Sch}^{\mathrm{Aff}}$: f corresponds to a finitely presented $\underline{A}$-algebra $f^\flat : \underline{A} \to \underline{B}$, i.e. $\underline{B} \in (\mathrm{CAlg}^{\Delta}_{\underline{A}})^{\mathrm{fp}}$ is compact.
- for an arbitrary $f : X \to Y$ in Sch: for any chart $y : \mathrm{Spec}(\underline{A}) \to Y$ of an affine Zariski cover $\mathcal{Y}$ of the base and affine Zariski covers $\mathcal{X}_y := \{X^y_\alpha \looparrowright X_y\}_\alpha$ of the fibres $X_y := X \times_X \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{A})$, the restriction $f_{|X^y_\alpha} : X^y_\alpha \to \mathrm{Spec}(\underline{A})$ in $\mathrm{Sch}^{\mathrm{Aff}}$ is lfp.

The morphism $f : X \to Y$ in Sch is furthermore **of finite presentation** (or *fp* for short) iff it is quasi-compact and quasi-separated and locally of finite presentation.

Warning. In view of the introductory paragraph, for a morphism of schemes $f : X \to Y$ being lfp is *not* equivalent to being flat with f^{cl} lfp. In particular, in general it holds only that f lfp implies f^{cl} lfp.

Example 4.1.4.18. Any open immersion is lfp and fp iff it is quasi-compact.

Proposition 4.1.4.19. *(Properties of lfp morphisms) "Being lfp" is stable under composition, base-change, and is fpqc- and* Zar*-local on the base.*

Proof. Zariski locality follows from the very definition of lfp, so that the two stability claims become straightforward by the stability properties of compact algebras. Finally, let's consider fpqc-locality. We can assume wlog that $f : \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ be affine and - for $\phi := f^\flat : \underline{A} \to \underline{B}$ in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ - we need to prove that wlog, for any commutative square whose horizontal maps are faithfully flat and the a_i's and b_i's are flat:

$$\begin{tikzcd} \underline{A} \arrow[r, two heads, "\prod a_i"] \arrow[d, "\phi"] & \prod_i \underline{A}_i \arrow[d, "\prod \phi_i"] \\ \underline{B} \arrow[r, two heads, "\prod b_i"] & \prod_i \underline{B}_i \end{tikzcd}$$

B is a compact $\underline{A}$-algebra whenever each $\underline{B}_i$ is a compact $\underline{A}_i$-algebra.
Observe that, with a similar argument to the proof of 3.6.1.10,Claim1 (replace the presheaf Mod with CAlg^{Δ} of 3.2.3.1), our assumption implies in particular that $\prod \underline{B}_i$ is $\prod \underline{A}_i$-compact, and hence - being $\prod a_i$ faithfully flat - also $\underline{A}$-compact. So, we are left to prove the following statement.

Claim 1. *Consider the following triangle of animated rings, where the horizontal map ψ is faithfully flat. Then, $\underline{B}$ is $\underline{A}$-compact whenever $\underline{C}$ is such.*

$$\begin{tikzcd} & \underline{A} \arrow[dl] \arrow[dr] & \\ \underline{B} \arrow[rr, two heads, "\psi"] & & \underline{C} \end{tikzcd}$$

Proof. $\psi : \underline{B} \to \underline{C}$ is faithfully flat, i.e. the induced extension of scalars functor $\underline{C} \otimes^L_{\underline{B}} (-)$ is both exact (between the stabilized categories of modules) and conservative. So, one is left to test the compactness condition in Ani(CRing) against any filtered diagram $p : I \to \mathrm{Mod}_{\underline{B}}$. Due to time constraints we omit the rest of the argument. ☐

Proposition 4.1.4.20. *(Cancellation property of lfp morphisms) Let $g \circ f : X \to Y \to Z$ be a choice of a composition in* Sch*, and assume g to be locally of finite presentation. Then, f is locally of finite presentation iff $g \circ f$ is locally of finite presentation.*

Proof. One direction follows from the stability under composition of lfp morphisms. Conversely, assume that both $g \circ f$ and g are lfp and let's profe that also f is such. By the locality on the base, we are left to show wlog the algebraic counterpart: let $\psi \circ \phi : \underline{A} \to \underline{B} \to \underline{C}$ be a composition in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$, and assume that both $\underline{B}$ and $\underline{C}$ are $\underline{A}$-compact; then, $\underline{C}$ is also $\underline{B}$-compact.
In order to see this, let's test the compatibility of $\mathrm{Map}_{\underline{B}/}(\psi^*\underline{C}, -)$ with filtered limits against any filtered diagram $p : I \to \mathrm{CAlg}^{\Delta}_{\underline{B}}$:

$$\begin{aligned} \mathrm{Map}_{\underline{B}/}\big(\psi^*\underline{C}, \lim p\big) &\simeq \mathrm{Map}_{\underline{A}/}\big((\psi \circ \phi)^*\underline{C}, \phi^*(\lim p)\big) \times_{\mathrm{Map}(\underline{A},\underline{B})} \{\phi\} \\ &\simeq_{(i)} \left(\lim \mathrm{Map}_{\underline{A}/}\big((\psi \circ \phi)^*\underline{C}, \phi^* p\big)\right) \times_{\mathrm{Map}(\underline{A},\underline{B})} \{\phi\} \\ &\simeq_{(ii)} \lim \left(\mathrm{Map}_{\underline{B}/}\big((\psi)^*\underline{C}, p\big)\right) \end{aligned}$$

where (i) follows by the $\underline{A}$-compactness of $\underline{C}$ together with the fact that restriction of scalars along ϕ preserves sifted colimits, since $\mathrm{Mod}_{\underline{B}}$ is cpt+proj-generated and its left-adjoint $\underline{B} \otimes^L_{\underline{A}} (-)$ preserves cpt+proj's; and (ii) is due to the universality of colimits in the ∞-topos Spc. ☐

Proper Morphisms. We now introduce what should be regarded as a "compact" relative scheme. This is a topological notion, hence intrinsically classical.

Definition 4.1.4.21. A morphism of schemes $f : X \to Y$ in Sch is **proper** iff the underlying morphism $f^{\mathrm{cl}} : X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$ of classical schemes is such.
Equivalently, iff it is separated, of finite type (i.e. $\mathcal{O}_X$ is a finitely generated $\mathcal{O}_Y$ algebra) and universally closed (i.e. any base-change is a closed map).

Remark. The equivalence of the two definitions above is evident: closedness - so also separatedness - is defined at the level of the underlying classical schemes, whereas being of finite type amounts to the existence of a surjection (on π_0) $\mathcal{O}_Y^n \twoheadrightarrow \mathcal{O}_X$ in $\mathrm{QCohAlg}(\mathcal{O}_Y)$, which supplies for a system of local generators.

Let us now record some properties of proper morphisms. They can be all immediately reduced to the classical case, so the proofs are omitted; see e.g. [37],29.41 for a more complete discussion.

Proposition 4.1.4.22. *(Properties of proper morphisms) "Being a proper morphism of stacks" is local on the base and stable under base-change and composition.*

4.2 Schemes as Animated Ringed Spaces

In this section we generalize the classical approach to algebraic geometry. Namely, after having enhanced the ordinary category $\mathcal{GS}$ of geometric space to the ∞-category $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ of animated ringed spaces (an example of 'structured ringed spaces', see [22]), we will equivalently characterize schemes as a special class of hypercomplete objects of the latter.
Among schemes, we will characterize those which are affine by means of a universal property; as a by-product, this will allow us to introduce the structure sheaf of a scheme and to regard it as a forgetful functor.
Thereafter, a brief digression on the truncation of schemes and on the relationship between Sch and $\mathrm{Sch}^{\mathrm{cl}}$ will follow.
Finally, we conclude our discussion by a result comparing our current point of view with the approach via stacks.

4.2.1 Animated Ringed Spaces and Schemes

In this subsection we introduce the notion of animated ringed spaces. For expository reasons, the more technical results will be discussed in the Appendix on ∞-Sheaves.

Definition 4.2.1.1. *(Animated Ringed Spaces, [26],1.1.2.5)* With notation as in Construction C.5.4.2, define the ∞-category of **animated ringed spaces** by $\mathrm{Top}_{\mathrm{Ani(CRing)}}$. An element of the latter has the form $(X, \mathcal{O}_X)$ and we refer to $\mathcal{O}_X$ as the **structure sheaf** of the space X.

Remark. The ∞-category $\mathrm{Top}_{\mathcal{N}\mathrm{CRing}}$ retrieves the nerve of the ordinary category of ringed spaces, consisting of topological spaces equipped with a sheaf of static rings.

Construction 4.2.1.2. *(The underlying ringed space, [26],1.1.2.6)* Let $X \in \mathrm{Top}$ be a topological space and consider a MOD-valued sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{MOD}}(X)$, where we recall that MOD is the ∞-category of "animated modules with animated ring of scalars" as in 3.2.5.1.
For each $n \geq 0$, the ordinary presheaf $[U \mapsto \pi_n \mathcal{F}(U)] \in \mathrm{Psh}(X, \mathrm{Ab})$ can be sheafified to

$$\pi_n \mathcal{F} := L(U \mapsto \pi_n \mathcal{F}(U)) \in \mathrm{Sh}_{\mathrm{Ab}}(X)$$

In particular, for $(X, \mathcal{O}_X) \in \mathrm{Top}_{\mathrm{Ani(CRing)}}$, this yields sheaves

- $n = 0 : \pi_0 \mathcal{O}_X \in \mathrm{Sh}_{\mathrm{CRing}}(X)$;
- $n > 0 : \pi_n \mathcal{O}_X \in \mathrm{Sh}_{\mathrm{Mod}(\pi_0 \mathcal{O}_X)}(X)$

So, we define the **underlying ringed space** functor by taking the 0-truncation:

$$\begin{aligned} \pi_0 : \mathrm{Top}_{\mathrm{Ani(CRing)}} &\longrightarrow \mathrm{Top}_{\mathrm{CRing}} \\ (X, \mathcal{O}_X) &\longmapsto (X, \pi_0 \mathcal{O}_X) \end{aligned}$$

Warning. ([26],1.1.2.7) For each $n \geq 0$ and $U \in \mathrm{Open}(X)$, the canonical map $\pi_n(\mathcal{O}_X(U)) \to (\pi_n \mathcal{O}_X)(U)$ needs not be an isomorphism, because we had to sheafify the n-th homotopy structure sheaf. However, for any animated ringed space lying over an *affine* scheme, taking sections of homotopy structure sheaves is the same as considering the homotopy groups of their evaluations. This motivates the third requirement in the definition of a scheme.

Definition 4.2.1.3. *(Animated scheme, [26],1.1.2.8)* A (connective) **animated scheme** is an animated ringed space $(X, \mathcal{O}_X)$ such that:

1. the underlying ringed spaces $(X, \pi_0\mathcal{O}_X) \in \mathrm{Sch}^{\mathrm{cl}}$ is a classical scheme;
2. for each $n \geq 0$, $\pi_n\mathcal{O}_X \in \mathrm{QCoh}(\mathrm{Spec}(\mathcal{O}_X(X))^{\mathrm{cl}})$;
3. For each open subset $U \subseteq X$ with affine underlying classical geometric space $(U, \pi_0(\mathcal{O}_X)_{|U}) \in \mathrm{Sch}^{\mathrm{Aff,cl}}$, the canonical map $\pi_n(\mathcal{O}_X(U)) \to (\pi_n\mathcal{O}_X)(U)$ is an isomorphism.

4.2.2 The Spectrum of an Animated Ring

In the current subsection, we define affine animated schemes as "homotopy-coherent data" lying over an affine classical scheme. Our next concern will then be enhancing the usual glueing procedure, so as to recover classical algebraic geometry as the static part of our construction.
We incidentally observe that such an approach will amount to defining the free quasi-coherent module of rank one $\mathcal{O}_{\mathrm{Spec}(\underline{A})}$ on an affine animated scheme $\mathrm{Spec}(\underline{A})$. Then, the aforementioned "glueing procedure" will allow us to generalize our construction to an arbitrary base. This will be made more precise in the next section on Locally Free Sheaves.

Definition 4.2.2.1. *(Affine, [26],1.1.4.1)* For an animated ring $\underline{A} \in \mathrm{Ani(CRing)}$, consider the topological space $|\mathrm{Spec}\underline{A}| := \mathrm{Spec}(\pi_0\underline{A}) \in \mathrm{Top}$ lying under the spectrum of $\underline{A}$, namely the spectrum of the corresponding underlying classical ring. We say that an open subset $U \in \mathrm{Open}(|\mathrm{Spec}(\underline{A})|)$ is **affine** iff $U \subseteq \mathrm{Spec}(\underline{A})^{\mathrm{cl}}$ is an open affine chart.

Proposition 4.2.2.2. *(Spectrum of an animated ring, [26],1.1.4.3) For an animated ring $\underline{A} \in \mathrm{Ani(CRing)}$, there exists a sheaf $\mathcal{O}_{\underline{A}} \in \mathrm{Sh}_{\mathrm{Ani(CRing)}}(|\mathrm{Spec}(\underline{A})|)$ together with a map $\phi : \underline{A} \to \mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|)$ such that the following properties hold:*

1. *For each $x \in \pi_0\underline{A}$, let $D(x) := \{\mathfrak{p} \in |\mathrm{Spec}(\underline{A})| \,|\, x \notin \mathfrak{p}\}$ denote the affine chart of $\mathrm{Spec}(\underline{A})^{\mathrm{cl}}$ where x is invertible; then, the composite*
$$\underline{A} \xrightarrow{\phi} \mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|) \longrightarrow \mathcal{O}_{\underline{A}}(D(x))$$
exhibits $\mathcal{O}_{\underline{A}}(D(x))$ as the localization $\underline{A}[x^{-1}]$ in $\mathrm{Ani(CRing)}$ (see 3.4.1.3).
2. *At the level of connected components, the composite*
$$\pi_0\underline{A} \xrightarrow{\pi_0\phi} \pi_0\mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|) \longrightarrow (\pi_0\mathcal{O}_{\underline{A}})(|\mathrm{Spec}(\underline{A})|)$$
induces an isomorphism which retrieves $\pi_0\mathcal{O}_{\underline{A}} \cong \mathcal{O}_{\pi_0\underline{A}}$ in $\mathrm{Sh}_{\mathrm{CRing}}(\mathrm{Spec}\underline{A}^{\mathrm{cl}})$.
3. *$\mathrm{Spec}(\underline{A}) := (|\mathrm{Spec}(\underline{A})|, \mathcal{O}_{\underline{A}}) \in \mathrm{Top}_{\mathrm{Ani(CRing)}}$ is an animated scheme. Define the **spectrum** of the animated ring $\underline{A}$ to be $\mathrm{Spec}(\underline{A})$.*

Remark. In particular, being localizations defined via a universal property, by (1) the map ϕ turns out to be an equivalence with quasi-inverse $\mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|) \simeq \mathcal{O}_{\underline{A}}(D(1)) \simeq \underline{A}[1^{-1}] \simeq \underline{A}$.

Remark. *(Classical case, [26],1.1.4.7)* With reference to C.5.4.6, let $X := |\mathrm{Spec}R|$ for some static ring $R \in \mathrm{CRing}$, $\mathcal{C} := \mathrm{CRing}$, and $\mathcal{U}_e := \{D(r) \,|\, r \in R\}$. Then, there exists an essentially unique classical affine scheme $(X = |\mathrm{Spec}R|, \mathcal{O}_X) \in \mathrm{Sch}^{\mathrm{Aff,cl}}$, together with a map $\phi : R \to \mathcal{O}_{\mathrm{Spec}(R)}|\mathrm{Spec}(R)|$ as in the statement. Indeed, by C.5.3.7 descent for sheaves with values in ordinary categories retrieves the classical construction.

Proof. *(of 4.2.2.2)* The proof of 4.2.2.2 relies on a couple of technical results from the Appendix on ∞-Sheaves, namely C.5.4.3 and C.5.4.6. We will also import our notation from such a digression.

$(\exists)$: Let $X := |\mathrm{Spec}(\underline{A})| \in \mathrm{Top}$ and consider the quasi-compact basis $\mathcal{U}_e := \{D(x) \mid x \in \pi_0\underline{A}\}$ consisting of all the open affine subsets $D(x)$ of X where x is invertible. By C.5.4.6 it suffices to define the wannabe structure sheaf $\mathcal{O} := \mathcal{O}_{\underline{A}} : \mathrm{Open}(X)^{op} \to \mathrm{Ani(CRing)}$ on its restriction $\mathcal{O}_e$ to $\mathcal{U}_e$.

Claim. *There exists an essentially unique sheaf* $\mathcal{O}_e \in \mathrm{Sh}(\mathcal{U}_e, \mathrm{CAlg}^{\Delta}_{\underline{A}})$ *with values* $\mathcal{O}_e(D(x)) \simeq \underline{A}[x^{-1}] \in \mathrm{CAlg}^{\Delta}_{\underline{A}}$. *In particular,* $\pi_0\mathcal{O}_e \simeq \mathcal{O}_{|\mathrm{Spec}(\underline{A})^{\mathrm{cl}}}$ *in* $\mathrm{Sh}(X, \mathrm{CRing})$.[1]

Proof. Let $\overline{\mathcal{O}} := \mathcal{O}_{\mathrm{Spec}(\pi_0\underline{A})}$ denote the structure sheaf of the underlying classical scheme $\mathrm{Spec}(\underline{A})^{\mathrm{cl}}$, which comes equipped with a natural family of localization maps $\phi_0 : \{\phi_0(x) : \pi_0\underline{A} \to \pi_0\underline{A}[x^{-1}] = \overline{\mathcal{O}}(D(x))\}_{D(x)\in\mathcal{U}_e}$. Let $\overline{\mathcal{O}}_e := \overline{\mathcal{O}}_{|\mathcal{U}_e} : \mathcal{U}_e^{op} \to \pi_0\underline{A}\text{-Alg}$ denote its restriction to the quasi-compact basis $\mathcal{U}_e$. By [23],7.5.0.6, there exists an essentially unique presheaf $\mathcal{O}_e : \mathcal{U}_e^{op} \to \mathrm{CAlg}^{\Delta}_{\underline{A}}$ lifting $\overline{\mathcal{O}}_e$ in such a way that, for each $D(x) \in \mathcal{U}_e$, the value of $\mathcal{O}_e(D(x)) \in \mathrm{CAlg}^{\Delta}_{\underline{A}}$ is a (essentially unique) choice of a localization of $\underline{A}$ at x via $\phi(x) : \underline{A} \to \underline{A}[x^{-1}]$ (see 3.4.1.3).

Let us check that $\mathcal{O}_e \in \mathcal{P}(\mathcal{U}_e, \mathrm{CAlg}^{\Delta}_{\underline{A}})$ does indeed define a sheaf. In other words, we have to show the following property: for each finite family $I \subseteq \mathcal{U}_e$ of distinguished opens of X, let $\gamma : P_f(I) \to \mathcal{U}'(I)^{op}$ be the right-cofinal map of C.5.4.5,ii; then, the canonical map $\theta : \mathcal{O}_e(\cup I) \to \lim \mathcal{O} \circ \gamma$ is an equivalence.

We will momentarily drop the subscript e in the notation. Observe first that the map $\mathcal{O}(\cup I) \to \prod_I \mathcal{O}(i)$ in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ is faithfully flat, because it is flat (by the properties of localizations) and the underlying map on $\overline{\mathcal{O}}_e \cong \pi_0\mathcal{O}$ induced by the cover I of $\cup I$ is also faithfully flat. Hence, it suffices to prove that θ is equivalence after tensoring with each $\mathcal{O}(i)$, i.e. that each of the following maps is an equivalence:

$$\theta_i : \mathcal{O}(i) \to \mathcal{O}(i) \otimes^L_{\mathcal{O}(\cup I)} \lim \mathcal{O} \circ \gamma$$

Now, since $\mathcal{O}(i) \otimes^L_{\mathcal{O}(\cup I)} (-) : \mathrm{Mod}^{\mathrm{Ex}}_{\mathcal{O}(\cup I)} \to \mathrm{Mod}^{\mathrm{Ex}}_{\mathcal{O}(\cup I)}$ is an exact functor of stable ∞-categories (see the proof of 3.6.1.9, Claim 2), it preserves finite limits, so that we are left to consider the maps:

$$\theta_i : \mathcal{O}(i) \to \lim\big(\mathcal{O}(i) \otimes^L_{\mathcal{O}(\cup I)} \mathcal{O} \circ \gamma\big) \simeq \lim\big(\mathcal{O} \circ [S \mapsto \gamma(\{i\} \cup S)]\big)$$

where the latter equality is obtained by 3.4.1.4. [2]

Now, the construction $[S \mapsto \mathcal{O} \circ \gamma(\{i\} \cup S)]$ is the right Kan extension of its restriction to the over-poset $P_f(I)_{/i}$ of those finite subsets of I containing i (recall that the order is given by the reverse inclusion). But now, $i = \max\big(P_f(I)_{/i}\big) \simeq \big(P_f(I)_{/i}\big)^{\mathrm{term}}$, so that the canonical map θ_i is an equivalence as desired. ■

Define $\mathcal{O} := \mathrm{Ran}(\mathcal{O}_e) \in \mathcal{P}(X, \mathrm{Ani}(\mathrm{CRing}))$ to be a choice of a right Kan extension of $\mathcal{O}_e$. By the construction, we conclude that $\mathcal{O} \in \mathrm{Sh}(X, \mathrm{Ani}(\mathrm{CRing}))$ is furthermore a sheaf, as in C.5.4.3. We are now left to check the stated properties.

(1) : This follows by the construction: $\mathcal{O}(D(x)) = \mathcal{O}_e(D(x)) \simeq \underline{A}[x^{-1}]$.

(2) : By construction, the construction $[U \mapsto \pi_0\mathcal{O}(U)] \in \mathrm{Psh}(|\mathrm{Spec}(\underline{A})|, \mathrm{CRing})$ agrees with the classical structure sheaf $\overline{\mathcal{O}} = \pi_0\mathcal{O}$ on the quasi-compact basis $\mathcal{U}_e$, so that they must be isomorphic in $\mathrm{Sh}(X, \mathrm{CRing})$ by C.5.4.6.

(3) : Let us check that $(|\mathrm{Spec}(\underline{A})|, \mathcal{O}) \in \mathrm{Top}_{\mathrm{Ani(CRing)}}$ is indeed an animated scheme.
The underlying geometric space is clearly an affine scheme by construction, so the first requirement is met.
Then, let's check that each homotopy group of the structure sheaf is a quasi-coherent module on the underlying affine scheme. For any fixed $n \geq 0$, the homotopy group $\pi_n\underline{A} \in \mathrm{Mod}(\pi_0\underline{A})$ is the global section of a (wlog classical) quasi-coherent module $\widetilde{\pi_n\underline{A}} \in \mathrm{QCoh}(\mathrm{Spec}(\underline{A}))$ given by $\widetilde{\pi_n\underline{A}}(D(x)) = \pi_0\underline{A}[x^{-1}] \otimes_{\pi_0\underline{A}} \pi_n\underline{A} \cong \pi_n\underline{A}[x^{-1}] = \pi_n\mathcal{O}(D(x))$ (since $\underline{A}[x^{-1}]$ is $\underline{A}$-flat) on distinguished opens. By C.5.4.6, we can lift the equality on $\mathcal{U}_e$ to the whole of $|\mathrm{Spec}(\underline{A})|$, so that $\pi_n\mathcal{O} \cong \widetilde{\pi_n\underline{A}} \in \mathrm{Sh}(X, \mathrm{CRing})$ as quasi-coherent sheaves.

Remark. In particular, this implies the flatness of $\mathcal{O}$ on each open affine chart $\mathrm{Spec}(\pi_0\underline{B}) \hookrightarrow X$.
Indeed, being $\pi_0\underline{B}$ a flat $\pi_0\underline{A}$-algebra by assumption, $\pi_n\mathcal{O}(\mathrm{Spec}(\pi_0\underline{B})) \simeq \widetilde{\pi_n\underline{A}}(\mathrm{Spec}(\pi_0\underline{B})) \cong \pi_0\underline{B} \otimes_{\pi_0\underline{A}} \pi_n\underline{A}$; moreover, $\pi_0\mathcal{O}(\mathrm{Spec}(\pi_0\underline{B})) \cong \widetilde{\pi_0\underline{A}}(\mathrm{Spec}(\pi_0\underline{B})) \cong \pi_0\underline{B}$.

[1] Lurie works in greater generality: consider the quasi-compact basis $\mathcal{U}_e$ consisting of all the open affine charts of X; then, there exists an essentially unique sheaf $\mathcal{O}_e \in \mathrm{Sh}(\mathcal{U}_e, \mathrm{CAlg}^{\Delta}_{\underline{A}})$ taking values into flat $\underline{A}$-algebras via $\phi_U : \underline{A} \to \mathcal{O}_e(U)$ and retrieving $\pi_0\mathcal{O}_e(U) \cong \overline{\mathcal{O}}_e(U)$ on connected components. This is obtained via an analogous construction, the only difference being that he constructs more generally $\mathcal{O}_e$ as an étale-lift (see [23],7.5.0.6) of $\overline{\mathcal{O}}_e : \mathcal{U}_e^{op} \to \pi_0\underline{A}\text{-Alg}^{\text{ét}}$.

[2] The argument works also in greater generality (as proven by Lurie in [26],1.4.4.3): if we assume the basis $\mathcal{U}_e$ consisting of those open affine charts of X, then we can lift via [23],7.5.0.6 the classical fibre-product identity $\overline{\mathcal{O}}(i) \otimes_{\overline{\mathcal{O}}(V)} \overline{\mathcal{O}}(\cap S) \cong \overline{\mathcal{O}}\big(i \cap (\cap S)\big) \cong \overline{\mathcal{O}}\big(\cap(S \cup \{i\})\big)$. (In order to avoid confusion: for a set T of spaces, we write $\cap T := \cap_{t\in T} t$ as a $\#T$-ary function.)

Finally, we need to show that the canonical map $\pi_n(\mathcal{O}(U)) \to (\pi_n\mathcal{O})(U)$ is an isomorphism over each open affine patch $(U = \mathrm{Spec}(\pi_0\underline{B}), \pi_0(\mathcal{O})_{|U})$ of the classical geometric space lying under $(X, \mathcal{O})$.
First, let us observe that wlog $n = 0$: being $\pi_n(\mathcal{O}(U))$ a flat $\pi_0\underline{A}$-algebra, we can write $\pi_n(\mathcal{O}(U)) \cong \pi_0(\mathcal{O}(U)) \otimes_{\pi_0\underline{A}} \pi_n\underline{A}$; on the other hand, $(\pi_n\mathcal{O})(U) \cong \widetilde{\pi_n\underline{A}}(\mathrm{Spec}(\underline{B})) \cong \pi_0\underline{B} \otimes_{\underline{A}} \pi_n\underline{A}$, so we are left to show that $\pi_0(\mathcal{O}(U)) \cong \pi_0\underline{B} = (\pi_0\mathcal{O})(U)$.
The latter isomorphism is a consequence of our construction: with a slight abuse of notation, let $I := \{D(i)\}_{i\in I} \subseteq \mathcal{U}_e$ be a finite cover of $U = \cup I \in \mathrm{Sch}^{\mathrm{Aff,cl}}$ by distinguished opens of X; then, as before the map $\mathcal{O}(U) \to \prod_I \underline{A}[i^{-1}]$ is faithfully flat and, after having tensored by $\mathcal{O}(D(i)) \cong \underline{A}[i^{-1}]$, one is left to check that $\pi_0(\underline{A}[i^{-1}]) \cong (\pi_0\underline{A})[i^{-1}]$; finally, the latter holds true by virtue of 3.4.1.3. $\square$

Warning. ([26],1.1.4.9) We can regard a ring $R \in \mathrm{CRing}$ as a static object in $\mathrm{Ani(CRing)}_{\leq 0}$. This yields a possible conflict in terminology, since we adopt the same notation for both:

- $\mathrm{Spec}(R) \cong (|\mathrm{Spec}(R)|, \overline{\mathcal{O}}) \in \mathrm{Sh}(|\mathrm{Spec}(R)|, \mathrm{CRing})$, obtained by embedding into ∞-sheaves of rings the structure sheaf of the classical scheme;

- $\mathrm{Spec}(R) \cong (|\mathrm{Spec}(R)|, \mathcal{O}) \in \mathrm{Sh}(|\mathrm{Spec}(R)|, \mathrm{Ani(CRing)})$ as in 4.2.2.2.

The two objects are indeed different. However, as observed by Lurie they are interchangeable data:

- $\overline{\mathcal{O}} \cong \pi_0\mathcal{O}$;

- $\mathcal{O} \simeq L(\overline{\mathcal{O}})$, where $L : \mathcal{P}(|\mathrm{Spec}(R)|, \mathrm{Ani(CRing)}) \to \mathrm{Sh}(|\mathrm{Spec}(R)|, \mathrm{Ani(CRing)})$ is the sheafification functor and we let again $\overline{\mathcal{O}}$ denote its copy in the image of the fully faithful embedding induced by the 0-truncation of coefficients.

4.2.3 The ∞-Category of Animated Schemes and the Universal Property of Spec

Our next aim is to define an ∞-category Sch' of all the animated schemes. Similarly to the classical case, we will set up the theory within an ambient ∞-category enhancing the ordinary one of locally ringed spaces; this will give substance to the aforementioned idea of a "glueing procedure".
In the next subsection, we will finally observe that our two approaches to the theory of schemes in the end produce an equivalence of ∞-categories $\mathrm{Sch} \simeq \mathrm{Sch}'$.

Definition 4.2.3.1. *(Animated locally ringed spaces, [26],1.1.5.3-4)* An animated ringed space $(X, \mathcal{O}_X) \in \mathrm{Top}_{\mathrm{Ani(CRing)}}$ is an **animated locally ringed space** iff its underlying ringed space $(X, \pi_0\mathcal{O}_X) \in \mathcal{GS}$ is a geometric space (or locally ringed space). In particular, observe that animated schemes are in $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$.
Define the **∞-categories** of animated locally ringed spaces and **animated schemes** by the chain of "full subcategories" $\mathrm{Sch}' \subseteq_{f.f.} \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}} \subseteq_{f.f.} \mathrm{Top}_{\mathrm{Ani(CRing)}}$ sitting in the following diagram of cartesian squares:

$$\begin{tikzcd}
\mathrm{Sch}' \arrow[r] \arrow[d, "\pi_0"] & \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}} \arrow[r] \arrow[d, "\pi_0"] & \mathrm{Top}_{\mathrm{Ani(CRing)}} \arrow[d, "\pi_0"] \\
\mathrm{Sch}^{\mathrm{cl}} \arrow[r, hook] & \mathcal{GS} \arrow[r, hook] & \mathrm{Top}_{\mathrm{CRing}}
\end{tikzcd}$$

Remark. In particular, $\mathrm{Map}_{\mathrm{Sch}'}\big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\big) \simeq \mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}}\big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\big)$, so that a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in Sch' is a pair (f, α) in $\mathrm{Top}_{\mathrm{Ani(CRing)}}$ such that $\alpha : \mathcal{O}_Y \to f_*\mathcal{O}_X$ in $\mathrm{Sh}(Y, \mathrm{Ani(CRing)})$ is a local map at each stalk $(\pi_0\alpha)_f(x)$ for $x \in X$.

The next result characterizes affine schemes by means of a universal property: as it will be manifest from the proof, this enhances the classical adjunction $\mathcal{O}^{op} \vdash \mathrm{Spec}$ induced by $\mathrm{Hom}_{\mathrm{Sch}^{\mathrm{cl}}}\big((X, \mathcal{O}_X), \mathrm{Spec}(R)\big) \simeq \mathrm{Hom}_{\mathrm{CRing}}\big(R, \mathcal{O}_X(X)\big)$ for each ring $R \in \mathrm{CRing}$ and geometric space $(X, \mathcal{O}_X) \in \mathcal{GS}$.
We start with a technical Lemma, which amounts to the truncation adjunction $\pi_0 \vdash \subseteq$ for $\mathrm{Sh}(|\mathrm{Spec}(\underline{A})|, \mathrm{Ani(CRing)})$.

Lemma 4.2.3.2. *(Technical lemma, [26],1.1.8.1) Let* $(X, \pi_0\mathcal{O}_X) \in \mathrm{Top}_{\mathrm{Ani(CRing)}}$ *be a* static *animated ringed space. Then, for any* $(Y, \mathcal{O}_Y) \in \mathrm{Top}_{\mathrm{Ani(CRing)}}$*:*

$$\mathrm{Map}_{\mathrm{Top}_{\mathrm{Ani(CRing)}}}\big((X, \pi_0\mathcal{O}_X), (Y, \mathcal{O}_Y)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{CRing}}}\big((X, \pi_0\mathcal{O}_X), (Y, \pi_0\mathcal{O}_Y)\big)$$

In other words, taking the 0*-truncation induces a fully faithful functor of slices under* $(X, \pi_0\mathcal{O}_X)$*.*

Proof. We wish to show that, for any given continuous map $f : X \to Y$ in Top, the following functor of mapping spaces is an equivalence:

$$\mathrm{Map}_{\mathrm{Sh}(Y,\mathrm{Ani(CRing)})}\big(\mathcal{O}_Y, f_*\pi_0\mathcal{O}_X\big) \xrightarrow{\pi_0} \mathrm{Hom}_{\mathrm{Sh}(Y,\mathrm{CRing})}\big(\pi_0\mathcal{O}_Y, f_*\pi_0\mathcal{O}_X\big)$$

Provided the commutativity of pushing-forward along f and taking the underlying ringed space, this is a consequence of the fact that we defined the 0-truncation adjunction $\pi_0 \vdash \subseteq$ for homotopy structure sheaves as the composite of the sheafification localization $L \vdash \subseteq$ after the 0-truncation adjunction $\pi_0^{\mathrm{Psh}} \vdash \subseteq^{\mathrm{Psh}}$ for presheaves of A.5.0.7.
Hence, we are left to show that $f_*(\pi_0\mathcal{O}_X) \simeq \pi_0(f_*\mathcal{O}_X)$ as static sheaves of rings.
In order to see this, it suffices to check the isomorphism as static presheaves, i.e. on each open patch. To this end, observe that the presheaf of static rings $[U \mapsto \pi_0\mathcal{O}_X(U)]$ is *already* a sheaf, so that $\pi_0\mathcal{O}_X = \pi_0^{\mathrm{Psh}}\mathcal{O}_X$ and it satisfies $\pi_0(\mathcal{O}_X(U)) \xrightarrow{\cong} (\pi_0\mathcal{O}_X)(U)$ for each open subset $U \in \mathrm{Open}(X)$. Thus, for each $V \in \mathrm{Open}(Y)$, we have the desired isomorphism:

$$\pi_0(f_*\mathcal{O}_X(V)) = \pi_0(\mathcal{O}_X(f^{-1}V)) \cong (\pi_0\mathcal{O}_X)(f^{-1}V) = (f_*(\pi_0\mathcal{O}_X))(V) \qquad \square$$

In particular, the previous technical Lemma 4.2.3.2 allows us to retrieve classical schemes and (locally) ringed spaces as 0-truncations of our new gadgets.

Corollary 4.2.3.3. *Suitable restrictions of the* 0*-truncation functor* $\pi_0 : \mathrm{Top}_{\mathrm{Ani(CRing)}} \to \mathrm{Top}_{(\mathrm{CRing})}$ *induce the following equivalences of* ∞*-categories:*

- ([26],1.1.8.2) $(\mathrm{Top}_{\mathrm{Ani(CRing)}})_0 \simeq \mathrm{Top}_{\mathrm{CRing}}$*;*
- ([26],1.1.8.3) $(\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}})_0 \simeq \mathcal{GS}$*;*
- ([26],1.1.8.4) $\mathrm{Sch}'_0 \simeq \mathrm{Sch}^{\mathrm{cl}}$.

([26],1.1.8.5) Moreover, a homotopy inverse to the latter equivalence supplies a fully faithful embedding $\mathrm{Sch}^{\mathrm{cl}} \hookrightarrow \mathrm{Sch}'$ *with essential image spanned by static animated schemes.*

Proof. The first equivalence follows from the essential surjectivity of π_0. The second one is then automatic. Let's prove the third statement.
By the second claim, we can regard each classical scheme $(X, \mathcal{O}) \in \mathrm{Sch}^{\mathrm{cl}}$ as a static animated locally ringed space $(X, \pi_0\mathcal{O}_X) \in (\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}})_0$. Finally, the static $(X, \mathcal{O}_X)$ is clearly an animated scheme (iff a static animated scheme). $\square$

After such a brief technical digression, we are finally ready to introduce the aforementioned adjunction.

Proposition 4.2.3.4. *(Universal property of* Spec*, [26],1.1.5.5) For each* $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ *and* $\mathrm{Spec}(\underline{A}) = (|\mathrm{Spec}(\underline{A})|, \mathcal{O}_{\underline{A}})$ *with* $\underline{A} \in \mathrm{Ani(CRing)}$*, the equivalence* $\alpha : \underline{A} \to \mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|)$ *of 4.2.2.2 induces an equivalence:*

$$ev_{|\mathrm{Spec}(\underline{A})|} \circ \mathrm{pr}_2 : \mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}}\big((X, \mathcal{O}_X), \mathrm{Spec}(\underline{A})\big) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Ani(CRing)}}\big(\underline{A}, \mathcal{O}_X(X)\big)$$

Proof. We wish to show that the above functor of mapping spaces has contractible fibres; this will be achieved in several steps. Fix some map $\phi : \underline{A} \to \mathcal{O}_X(X)$ and let $Z := \mathrm{Map}\big((X, \mathcal{O}_X), \mathrm{Spec}(\underline{A})\big) \times_{\mathrm{Map}(\underline{A}, \mathcal{O}_X(X))} \{\phi\}$ denote the fibre over ϕ.
Remark. The same construction as in the classical adjunction implies that - point-wise - the essential image of $\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}} \subseteq_{f.f.} \mathrm{Map}_{\mathrm{Top}_{\mathrm{Ani(CRing)}}}$ is spanned by those maps which are completely determined by the arrow of structure sheaves. This is the content of the first two Claims.

Claim 1. *The second projection* $ev_2 : (f, \Phi) \to \Phi$ *is surjective on connected components. Let it be given a map of structure sheaves* $\Phi : \mathcal{O}_{\underline{A}} \to \mathcal{O}_X$ *and call* $\phi := \Phi(|\mathrm{Spec}(\underline{A})|)$*. Then, we can make it into a pair* (f, Φ) *over* ϕ *as follows: for each* $x \in X$*, let* $k(x) := (\pi_0\mathcal{O}_X)_x/\mathfrak{m}_x$ *denote its residue field and consider the stalk map* $\pi_0\phi_x : \pi_0\underline{A} \to \pi_0\mathcal{O}_X(X) \to k(x)$ *in* CRing*; let* $\mathfrak{p}_x := \ker(\pi_0\phi_x) \in \mathrm{Spec}(\pi_0\underline{A})$ *and define the following continuous map:*

$$\begin{aligned} f : X &\longrightarrow |\mathrm{Spec}(\underline{A})| \\ x &\longmapsto \mathfrak{p}_x \end{aligned}$$

Proof. The continuity of f can be checked on the basis consisting of the distinguished opens of $|\mathrm{Spec}(\underline{A})|$: for each $a \in \pi_0\underline{A}$, we want $U := X \setminus f^{-1}(V(a)) = \{x \in X \mid a \notin \mathfrak{p}_x\}$ to be an open subspace of X.
To this end, for each $x \in U$ we wish to have an open $x \in V \subseteq U$. Pick up any $x \in U$; by construction, $\pi_0\phi_x(a) \in (\pi_0\mathcal{O}_X)^\times$ is invertible, so let $s \in (\pi_0\mathcal{O}_X)^\times$ be a multiplicative inverse to a. Let $\overline{s} \in \pi_0(\mathcal{O}_X(V))$ be a lift of s on some open neighbourhood $V \subseteq X$ of x; up to shrinking V, wlog $\overline{s} \in \pi_0(\mathcal{O}_X(V))^\times$ is invertible and equal to $\phi_V(a)^{-1}$ for $\phi_V : \pi_0\underline{A} \to \pi_0\mathcal{O}_X(X) \to \pi_0\mathcal{O}_X(V)$. But such an invertibility means precisely that V consists of points v for which $a \notin \mathfrak{p}_v$, so that $V \subseteq U$, as needed. ■

Claim 2. *Identify the fibre* Z *over* ϕ *with the fibre over* ϕ *of the following map, with* f *as in Claim 1:*

$$\mathrm{Map}_{\mathrm{Sh}(|\mathrm{Spec}(\underline{A})|,\mathrm{Ani}(\mathrm{CRing}))}\big(\mathcal{O}_{\underline{A}}, f_*\mathcal{O}_X\big) \longrightarrow \mathrm{Map}_{\mathrm{Ani}(\mathrm{CRing})}\big(\underline{A}, \mathcal{O}_X(X)\big)$$

Proof. We need to show that the second projection is a monomorphism over ϕ, so as to conclude by the fact that (EffEpi, Mono) gives a factorization system in the ∞-topos Spc (see [24],5.2.8.16).
To this end, construct f in Top over ϕ as in Claim 1; we will prove that, for any given morphism $(g, \Gamma) : (X, \mathcal{O}_X) \to \mathrm{Spec}(\underline{A})$ in $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ over ϕ, it holds $g = f$.
Pick up any $x \in X$, and let $\gamma := \Gamma(|\mathrm{Spec}(\underline{A})|)$. Observe that the following map factors through $(\pi_0\underline{A})_{g(x)}$:

$$\pi_0\phi_x : \pi_0\underline{A} \xrightarrow{\pi_0\phi} \pi_0\mathcal{O}_X(X) \longrightarrow (\pi_0\mathcal{O}_X)_x \longrightarrow k(x)$$

Indeed, by the universal property of localizations, this amounts to the fact that each $a \in g(x)$ is carried to a unit $\pi_0\phi_x(a) \in (\pi_0(\mathcal{O}_X)_x)^\times = (\pi_0(\mathcal{O}_X)_x) \setminus \mathfrak{m}_x$. Now, $(g, \Gamma) \in Z$ implies $\gamma = \Gamma(|\mathrm{Spec}(\underline{A})|) = \phi$ and, being γ a local map, it sends $\gamma : g(x)(\pi_0\underline{A})_{g(x)} \mapsto \mathfrak{m}_x$. This means that $g(x)(\pi_0\underline{A})_{g(x)} \subseteq \gamma^{-1}(\mathfrak{m}_x)$; but now the right hand-side must be an ideal and the left hand-side is the maximal one, so that the equality holds.
In particular, our argument amounts to the desired equality $g(x) = \ker(\pi_0\gamma_x) = \ker(\pi_0\phi_x) = f(x)$ for each $x \in X$. ■

Claim 3. *By C.5.4.6, we can compute mapping spaces in* $\mathrm{Sh}(|\mathrm{Spec}(\underline{A})|, \mathrm{Ani(CRing)})$ *by restricting the sheaves to the quasi-compact basis* $\mathcal{U}_e$ *consisting of all the quasi-distinguished sets of* $\mathrm{Spec}(\underline{A})$*. Then, our fibre is a static Hom-space:* $Z \simeq \mathrm{Hom}_{\mathrm{Psh}(\mathcal{U}_e,\pi_0\underline{A}\text{-Alg})}\big(\pi_0(\mathcal{O}_{\underline{A}})_{|\mathcal{U}_e}, \pi_0(f_*\mathcal{O}_X)_{|\mathcal{U}_e^{op}}\big)$*.*

Proof. We wish to show that $Z \simeq \pi_0 Z$ in Spc. In view of the discussion above, we can write

$$\begin{aligned} Z &\simeq \mathrm{Map}_{\mathrm{Sh}(\mathcal{U}_e,\mathrm{Ani(CRing)})}\big((\mathcal{O}_{\underline{A}})_{|\mathcal{U}_e}, (f_*\mathcal{O}_X)_{|\mathcal{U}_e}\big) \times_{\mathrm{Map}(\underline{A},\mathcal{O}_X(X))} \{\phi\} \\ &\simeq \mathrm{Map}_{\mathrm{Sh}(\mathcal{U}_e,\mathrm{CAlg}^{\Delta}_{\underline{A}})}\big((\mathcal{O}_{\underline{A}})_{|\mathcal{U}_e}, (f_*\mathcal{O}_X)_{|\mathcal{U}_e}\big) \end{aligned}$$

where the latter equivalence follows from the fact that $(\mathcal{O}_{\underline{A}})_{|\mathcal{U}_e}$ takes values into localizations of $\underline{A}$ for the $\underline{A}$-algebra structure induced by the equivalence $\alpha : \underline{A} \to \mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|)$ (see 4.2.2.2) and that ϕ specifies an $\underline{A}$-algebra structure on the global section $\mathcal{O}_X(X)$.
Now, we invoke the previous technical Lemma 4.2.3.2. As a consequence, it suffices to show that we can replace both $\underline{A}$ and $(\mathcal{O}_{\underline{A}})_{|\mathcal{U}_e}$ by their static parts.
Finally, this is implied by the usual lifting-result [23],7.5.0.6. ■

Claim 4. *The Hom-set* $\mathrm{Hom}_{\mathrm{Psh}(\mathcal{U}_e,\pi_0\underline{A}\text{-Alg})}\big(\pi_0(\mathcal{O}_{\underline{A}})_{|\mathcal{U}_e}, \pi_0(f_*\mathcal{O}_X)_{|\mathcal{U}_e^{op}}\big)$ *is contractible whenever the following technical condition holds:* $\pi_0\phi_{D(a)}(a) \in (\pi_0(f_*\mathcal{O}_X)(D(a)))^\times$ *is a unit for each* $a \in \pi_0\underline{A}$*.*

Proof. The technical condition shows that the image $\pi_0\phi(\pi_0\underline{A})$ is a field of global sections into $\pi_0\mathcal{O}_X(X)$; hence, every two arrows in the Hom-set are homotopic via a path through the constant map at the global section 1. ■

Claim 5. *Technical classical condition:* $\pi_0\phi_{D(a)}(a) \in (\pi_0(f_*\mathcal{O}_X)(D(a)))^\times$ *is a unit for each* $a \in \pi_0\underline{A}$*.*

Proof. Multiplication by the elements of $\pi_0\underline{A}$ via the given map $\phi : \pi_0\underline{A} \to \pi_0\mathcal{O}_X(X)$ yields a scalar structure (i.e. a static sub-algebra sheaf) $\pi_0\underline{A} \leq \mathrm{End}_{\mathrm{Sh}(\mathcal{U}_e, \pi_0\underline{A}\text{-Alg})}(\pi_0\mathcal{O}_X)$.
For a fixed $a \in \pi_0\underline{A}$, let $U := f^{-1}D(a)$ and consider the corresponding $\pi_0\underline{A}$-scalar structure on the sections at U. We wish to show that the multiplication map $(\cdot a)_{|U} = \big(\cdot\, \pi_0\phi_{D(a)}(a)\big)$ induces an automorphism of $\pi_0\mathcal{O}_X(U) = \pi_0(f_*\mathcal{O}_X)(D(a))$, i.e. that $\pi_0\phi_{D(a)}(a) \in \pi_0(f_*\mathcal{O}_X)(D(a))^\times$ is a unit.
In order to see this, observe that, for each $x \in U$, $\pi_0\phi_x(a) \in (\pi_0\mathcal{O}_X)_x^\times$ is a unit by construction (see Claim 1); furthermore, there exists an open $x \in V_x \subseteq U$ on which the latter lifts to an invertible section $\pi_0\phi_{V_x}(a) \in (\pi_0\mathcal{O}_X(V_x))^\times$. In other words, $(\cdot a)_{|V_x}$ is an isomorphism of $\pi_0(\mathcal{O}_X)_{|V_x}$. Being our choice of x arbitrary, this proves that the multiplication map by a has contractible fibre $\mathrm{Fib}(\cdot a)_{|U} \simeq *$ on U, i.e. the glueing on U of the isomorphisms on the various patches V_x's is an isomorphism of $\pi_0\mathcal{O}_X(U)$, as desired. ☐

Construction 4.2.3.5. *(Global sections and* Spec *adjunction, [26],1.1.5.6-7)* The global section functor determines a forgetful functor:

$$\mathcal{O}^{op} : \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}} \longrightarrow \mathrm{Ani(CRing)}^{op}$$
$$(X, \mathcal{O}_X) \overset{\Gamma(X;-)}{\longmapsto} \mathcal{O}_X(X)$$

The universal property of Spec 4.2.3.4 amounts to an adjunction $\mathcal{O}^{op} \vdash \mathrm{Spec}$, where the right-adjoint acts on objects by:

$$\mathrm{Spec} : \mathrm{Ani(CRing)}^{op} \longrightarrow \mathrm{Sch}' \subseteq_{f.f.} \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$$
$$\underline{A} \longmapsto \mathrm{Spec}(\underline{A})$$

Moreover, Spec is fully faithful, so that $\mathcal{O} \vdash \mathrm{Spec}$ exhibits $\mathrm{Ani(CRing)}^{op}$ as a right Bousfield localization of $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ and restricts to an equivalence $\mathrm{Ani(CRing)}^{op} \simeq \mathrm{Sch}^{\mathrm{Aff}}$.

Proof. we prove that the counit $v_{\underline{A}} : \mathcal{O}(\mathrm{Spec}(\underline{A})) \to \underline{A}$ can be obtained as the quasi-inverse to the equivalence $\alpha : \underline{A} \to \mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|)$. By [20],5.1.10, we will promote to an adjunction the following assignments:

- Obj: $\underline{A} \mapsto \mathrm{Spec}(\underline{A})$
- Mor: $1_{\underline{A}} \mapsto (v_{\underline{A}} : \mathcal{O}(\mathrm{Spec}(\underline{A})) \to \underline{A})$

In order to achieve it, we need to prove that, for each $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$, the point-wise triangle identity is an equivalence of spaces:

$$\mathrm{Map}\big((X, \mathcal{O}_X), \mathrm{Spec}(\underline{A})\big) \xrightarrow{\mathcal{O}^{op}} \mathrm{Map}_{\mathrm{Ani(CRing)}^{op}}\big(\mathcal{O}_X(X), \mathcal{O}(\mathrm{Spec}(\underline{A}))\big) \xrightarrow{op\,\circ\,(v_{\underline{A}})_*} \mathrm{Map}_{\mathrm{Ani(CRing)}}\big(\underline{A}, \mathcal{O}_X(X)\big)$$

and we conclude by the Proposition if we notice that - up to working in the opposite category - one has:

$$\mathcal{O}\big(\mathrm{Map}\big((X, \mathcal{O}_X), \mathrm{Spec}(\underline{A})\big)\big) \simeq ev_{|\mathrm{Spec}(\underline{A})|} \circ pr_2\big(\mathrm{Map}\big(\mathcal{O}_{\underline{A}}(|\mathrm{Spec}(\underline{A})|), f_*(\mathcal{O}_X)(|\mathrm{Spec}(\underline{A})|)\big)\big)$$

Finally, the fully faithfulness of Spec is a consequence of the construction: by inspection of the proof of [20],5.1.10, for each $\underline{A}$, $\underline{B} \in \mathrm{Ani(CRing)}$, the action of Spec on the corresponding mapping spaces is defined so as to obtain the commutativity of the triangle below:

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}}\big(\mathrm{Spec}(\underline{B}), \mathrm{Spec}(\underline{A})\big) \arrow[rr, "\simeq", "\mathrm{UP}"'] & & \mathrm{Map}_{\mathrm{Ani(CRing)}}\big(\underline{A}, \mathcal{O}_{\underline{B}}|\mathrm{Spec}(\underline{B})|\big) \\
& \mathrm{Map}_{\mathrm{Ani(CRing)}}\big(\underline{A}, \underline{B}\big) \arrow[ul, dotted] \arrow[ur, "\simeq", "\beta_*"'] &
\end{tikzcd}$$

where $\beta : \underline{B} \to \mathcal{O}(|\mathrm{Spec}(\underline{B})|)$ denotes the equivalence expressing the algebra structure of 4.2.2.2. ☐

The $\mathcal{O}^{op} \vdash \mathrm{Spec}$ adjunction also allows us to characterize those schemes which are affine. This will be the goal of the next four results.

Proposition 4.2.3.6. *(Local characterization of affine schemes, [26],1.1.6.1) Let $f : (X, \mathcal{O}_X) \to \mathrm{Spec}(\underline{A})$ be a map in $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ such that:*

(a) Its transpose $\alpha : \underline{A} \to \mathcal{O}_X(X)$ under 4.2.3.5 is an equivalence.

(b) The underlying locally ringed space $(X, \pi_0\mathcal{O}_X) \in \mathcal{GS}$ is affine;

(c) The homotopy sheaves $\pi_n\mathcal{O}_X \in \mathrm{QCoh}(X)$ are quasi-coherent for each $n \geq 0$.

Then, TFAE:

1. *f is an equivalence;*

2. *$(X, \mathcal{O}_X) \in \mathrm{Sch}'$ is an animated scheme*

Proof. (1) $\implies$ (2) : It is clear, because $\mathrm{Spec}(\underline{A}) \in \mathrm{Sch}'$ and the full subcategories are closed under equivalences.

(2) $\implies$ (1) : Consider an animated scheme $(X, \mathcal{O}_X) \in \mathrm{Sch}'$ lying - by (b) - over an affine classical scheme $\mathrm{Spec}(R) := (X, \pi_0\mathcal{O}_X) \in \mathrm{Sch}^{\mathrm{Aff,cl}}$ and - by (c) - with quasi-coherent homotopy modules $\{\widetilde{M_n}\}_n \subseteq \mathrm{QCoh}(\mathrm{Spec}(R))$ for some $\{M_n\}_n \subseteq \mathrm{Mod}(R)$ with $M_0 = R$.[3]

Let's show that the continuous map $f : X \to |\mathrm{Spec}(\underline{A})|$ is a homeomorphism.

By the compatibility of homotopy structure sheaves of schemes with truncation, we have an isomorphism $\pi_n(\mathcal{O}_X(U)) \to (\pi_n\mathcal{O}_X)(U) = \widetilde{M_n}(U)$ over each distinguished open $U \subseteq X$, and in particular on the whole space.

Now, by (a), the transpose α induces isomorphisms $\pi_n(\alpha) : \pi_n\underline{A} \xrightarrow{\cong} \pi_n(\mathcal{O}_X(X)) \cong M_n$ for each $n \geq 0$. This holds in particular for $n = 0$, which yields $\alpha : \pi_0\underline{A} \cong \pi_0\mathcal{O}_X(X) \cong R$, i.e. f induces an isomorphism $\pi_0(f) : (X, \pi_0\mathcal{O}_X) \cong \mathrm{Spec}(R) \to \mathrm{Spec}(\pi_0\underline{A})$ and in particular a homeomorphism $f : X \to |\mathrm{Spec}(\underline{A})|$.

Finally, under the identification $\mathrm{Sh}(|\mathrm{Spec}(\underline{A})|) \simeq \mathrm{Sh}(X)$, we are left to check that f induces an equivalence $\mathcal{O}_{\underline{A}} \to \mathcal{O}_X$ in $\mathrm{Sh}(X, \mathrm{Ani}(\mathrm{CRing}))$. Equivalently, by C.5.4.6 it suffices to check it over the quasi-compact basis $\mathcal{U}_e$ of all distinguished opens of X. Hence, being π_* conservative, we need to check that, for each $n \geq 0$ and $x \in \pi_0\underline{A}$, the maps $\pi_n(f)(D(x)) : \pi_n\mathcal{O}_{\underline{A}}(D(x)) \to \pi_n\mathcal{O}_X(D(x))$ are isomorphisms. Fix $n \geq 0$; the statement now follows by considering the following commutative triangle:

$$\begin{tikzcd}
 & (\pi_0\mathcal{O}_X)(D(x)) \otimes_R M_n \arrow[dl, "\simeq"', "(i)"] \arrow[dr, "\simeq", "(ii)"'] & \\
\pi_n\mathcal{O}_{\underline{A}}(D(x)) \arrow[rr, "\pi_n(f)(D(x))"] & & \pi_n\mathcal{O}_X(D(x))
\end{tikzcd}$$

where the slanting maps are equivalences by the following arguments:

- (i) : by (a), for $m = 0, n$ there is an isomorphism $\pi_m(\alpha) : \pi_m\underline{A} \cong M_m$, which in turn induces an isomorphism $(\pi_0\mathcal{O}_X)(D(x)) \otimes_R M_n \cong \pi_0\underline{A}[x^{-1}] \otimes_{\pi_0\underline{A}} \pi_n\underline{A} \cong \pi_n\underline{A}[x^{-1}] \cong \pi_n\mathcal{O}_{\underline{A}}(D(x))$;

- (ii) : $(\pi_0\mathcal{O}_X)(D(x)) \otimes_R M_n \cong \widetilde{M_n}(D(x)) \cong (\pi_n\mathcal{O}_X)(D(x))$ by (c) and the second property of the definition of animated schemes.

□

Corollary 4.2.3.7. *(Global characterization of affine schemes, [26],1.1.6.2) For an animated locally ringed spaces $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ lying over a classical scheme $(X, \pi_0\mathcal{O}_X) \in \mathrm{Sch}^{\mathrm{cl}}$ and quasi-coherent homotopy structure sheaves $\{\pi_n\mathcal{O}_X\}_n \subseteq \mathrm{QCoh}(\pi_0\mathcal{O}_X)$.* TFAE:

1. *Each $x \in X$ admits an open neighbourhood $x \in U \subseteq X$ over which the restriction $(U, (\mathcal{O}_X)_{|U}) \in \mathrm{Sch}'$ is affine.*

2. *$(X, \mathcal{O}_X)$*

[3] So, in our setting the assumption $(X, \mathcal{O}_X) \in \mathrm{Sch}'$ amounts to further assuming the compatibility with truncation for homotopy structure sheaves, i.e. property (3) in the definition of an animated scheme.

Proof. (1) $\implies$ (2) : Notice first that $(X, \mathcal{O}_X)$ is very close to a scheme, and what is missing is only the compatibility of the homotopy structure sheaves with truncation. As already observed at the end of the proof of 4.2.2.2, the isomorphisms at stake can be checked on the quasi-compact basis $\mathcal{U}_e$ consisting of all the distinguished opens of X, and there is holds by (1), since we can cover each (affine) open $U \subseteq X$ by distinguished opens.

(2) $\implies$ (1) : Consider $(X, \mathcal{O}_X) \in \mathrm{Sch}'$ and pick up any $x \in X$. Since $(X, \pi_0 X) \in \mathrm{Sch}^{\mathrm{cl}}$, there exists some open neighbourhood $x \in U \subseteq X$ over which $(U, (\pi_0\mathcal{O}_X)_{|U}) \in \mathrm{Sch}^{\mathrm{Aff,cl}}$. Set $\underline{A} \simeq \mathcal{O}_X(U)$, and consider its transpose $f : (U, (\mathcal{O}_X)_{|U}) \to \mathrm{Spec}(\underline{A})$ under 4.2.3.5. By assumption, $(X, \mathcal{O}_X)$ - and hence its restriction to U - is an animated scheme, so that f turns out to be an equivalence by the previous Proposition 4.2.3.6. ☐

As a consequence, the following two corollaries formalise two key intuitive facts:

- For an animated scheme, "being affine" can be checked on its static part;
- For an animated locally ringed space, "being a scheme" can be tested locally on X.

Moreover, the latter translates in more topological terms the definition of a scheme as a stack admitting an affine Zariski cover.

Corollary 4.2.3.8. *(Topological characterization of schemes, [26],1.1.6.3-4) An animated scheme* $(X, \mathcal{O}_X) \in \mathrm{Sch}'$ *is affine iff its static part* $(X, \pi_0\mathcal{O}_X) \in \mathrm{Sch}^{\mathrm{Aff,cl}}$ *is such.*
Moreover, an animated locally ringed space $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ *is an animated scheme iff each* $x \in X$ *admits an open neighbourhood* $x \in U \subseteq X$ *over which the restriction* $(U, (\mathcal{O}_X)_{|U})$ *is an animated scheme.*

Proof. The first claim follows from the proof of 4.2.3.7,(2) $\implies$ (1). The second statement is a consequence of 4.2.3.7,(1) $\implies$ (2) together with the fact that the assumption of *loc.cit.* are local on X. ☐

4.2.4 Functor of Points and the Comparison Theorem

In the current subsection, we finally prove the equivalence between the ∞-category Sch' of animated schemes and the one Sch of stacks admitting an affine Zariski cover. Morally, this amounts to achieving a better description of the co-Yoneda embedding $\mathrm{Ani(CRing)} \hookrightarrow \mathrm{PreStack}$.

Warning. We summarize the discussion in 4.1.1 and 4.1.2. Let $\widehat{\mathrm{Spc}}$ denote the ∞-category of *large* spaces. As when dealing with Stack, we will have to consider very large ∞-topoi of sheaves on large sites; those can be regarded in turn as full subcategories of presheaves into $\widehat{\mathrm{Spc}}$. Such ∞-categories are large-presentable, but not small-accessible; however, for the rest they "behave like ∞-topoi", in that all other Giraud's Axioms can be enforced. As in the previous section, we drop the hat in the notation, so Spc will automatically denote the large ∞-topos of large spaces, and it will be clear from the context when we can restrict to the ∞-topos of spaces (e.g. when considering mapping spaces in the locally small - although large - ∞-category Ani(CRing)).

Definition 4.2.4.1. *(Animated functor of points, [26],1.6.1.1)* For each animated locally ringed space $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$, define the prestack (i.e. presheaf, see 4.1.1.1) $h_X \in \mathrm{PreStack} = \mathcal{P}(\mathrm{Sch}^{\mathrm{Aff}}) = \mathrm{Fun}(\mathrm{Sch}^{\mathrm{Aff}}, \mathrm{Spc})$ by the Yoneda embedding of $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$:

$$h_X : \underline{R} \mapsto \mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}}\big(\mathrm{Spec}(\underline{R}), (X, \mathcal{O}_X)\big)$$

Remark. The fully faithful embedding $\mathrm{Sch}^{\mathrm{Aff}} \hookrightarrow \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ allows us to consider the restriction $h_{|\mathrm{Sch}^{\mathrm{Aff}}}$ as an extension of the Yoneda Lemma of $\mathrm{Sch}^{\mathrm{Aff}}$. As observed in 4.1.2.6, the Zariski site on $\mathrm{Sch}^{\mathrm{Aff}}$ is sub-canonical, meaning that the representable presheaves of the form $h_{\mathrm{Spec}(\underline{A})}$ are already sheaves. As a consequence, also h_X is a sheaf for any $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$. This is the content of Theorem [26],1.6.2.1, which we will not prove.

The next result establishes that functors of points arise naturally; indeed, as we will see, they will assemble into the Yoneda embedding of the large ∞-topos $\mathrm{Stack} := \mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$ into the very large ∞-topos $\mathrm{Sh}(X, \mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})^{op})$ of families of open sub-stacks of X. Let us first record a computation, which will appear in the proof.

Lemma 4.2.4.2. *(Fibre-products of animated locally ringed spaces) Consider an angle of animated locally ringed spaces in* $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$*:*

$$(X,\mathcal{O}_X) \xrightarrow{(a,\alpha)} (Y,\mathcal{O}_Y) \xleftarrow{(b,\beta)} (Z,\mathcal{O}_Z)$$

and form the pull-back in Top *under the first projection:*

$$\begin{tikzcd} X\times_Z Y \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, dotted, "r"] & Y \arrow[d, "b"] \\ X \arrow[r, "a"] & Z \end{tikzcd}$$

Then, the fibre-product in $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ *of the angle can be described by:*

$$h_X \times_{h_Z} h_Y \simeq h_{(-)}\Big((X,\mathcal{O}_X)\times_{(Z,\mathcal{O}_Z)}(Y,\mathcal{O}_Y)\Big) \simeq h_{(-)}\Big(X\times_Z Y\ ,\ p_*\mathcal{O}_X \coprod_{r_*\mathcal{O}_Z} q_*\mathcal{O}_Y\Big)$$

Proof. The first equivalence follows from the very definition of $h_{(-)}$ as the composition of the Yoneda embedding of $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ after the fully faithful inclusion $\mathrm{Sch}^{\mathrm{Aff}} \hookrightarrow \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$. Indeed, mapping spaces commute with limits in the second component.
Hence, we need to characterize fibre-products of animated locally ringed spaces. It suffices to check that, for each morphism from $(S,\mathcal{O}_S)$ into the pull-back, the corresponding continuous map $f: S\to X\times_Z Y$ in Top induces:

$$\mathrm{Map}'\Big(p_*\mathcal{O}_X \coprod_{r_*\mathcal{O}_Z} q_*\mathcal{O}_Y, f_*\mathrm{Spec}(\underline{A})\Big) \simeq \mathrm{Map}'\Big(p_*\mathcal{O}_X \leftarrow r_*\mathcal{O}_Z \rightarrow q_*\mathcal{O}_Y, f_*\mathrm{Spec}(\underline{A})\Big)$$

where Map' denotes mapping sub-spaces of $\mathrm{Sh}(X\times_Z Y, \mathrm{Ani(CRing)})$ consisting of maps whose static part is local at each stalk. The locality property can be rephrased by requiring that $\pi_0\mathrm{Map}'$ yields a Hom-set of structure sheaves coming from Hom-sets of $\mathcal{GS}$.
More generally, recall that $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ is defined as the fibre-product of the angle

$$\mathcal{GS} \overset{f.f.}{\hookrightarrow} \mathrm{Top}_{\mathrm{CRing}} \overset{\pi_0}{\leftarrow} \mathrm{Top}_{\mathrm{Ani(CRing)}}$$

Thus, we are left to show that the needed isomorphism holds in each ∞-category of the angle, which is now clear, since it follows from either the definition or the classical setting. ☐

Proposition 4.2.4.3. *([26],1.6.3.1) For any animated locally ringed space* $(X,\mathcal{O}_X)\in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$*, each restriction* $(U,(\mathcal{O}_X)_{|U})\in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ *to the open* $U\in \mathrm{Open}(X)$ *represents a stack* $h_U \in \mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$ *(i.e. a Zariski-sheaf, see 4.1.2.5).*
They are grouped into the construction $[U\mapsto h_U]$*, which determines a sheaf* $h_{(-)}\in \mathrm{Sh}(X, \mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})^{op})$*.*

Proof. The first statement follows from Theorem [26],1.6.2.1.
Now, with reference to C.5.4.3, consider any family $I\subseteq \mathrm{Open}(X)$ and let $\mathcal{U}'(I) := \cup_I \mathrm{Open}(i)$. The sheaf condition C.5.3.2 of $h_{(-)}$ amounts to proving that, for any arbitrary choice of I as above, the canonical map $\mathrm{colim}\, h_{|\mathcal{U}'(I)} \to h_{\cup I}$ is an equivalence in the very large ∞-topos $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$. In regard to this, let us avoid confusion by recalling that we would like to have $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})^{op}$-valued sheaves. And given that $\mathcal{U}'(I)$ needs not be small in the large site $\mathrm{Sch}^{\mathrm{Aff}}$, we are in need to deal with a very large ∞-topos (see the introductory Warning).
Now, let us simplify the condition above. In the previous Remark we observed that, being the Zariski site subcanonical (see 4.1.2.6), the Yoneda embedding $j: \mathrm{Ani(CRing)} \hookrightarrow \mathcal{P}(\mathrm{Sch}^{\mathrm{Aff}}, \mathrm{Spc})$ factors through $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$. Then, up to allowing large diagrams, $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$ is still colim-generated by the co-representables, as in the ∞-Density Theorem [24],5.1.5.3. In particular, being colimit in $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$ universal, it suffices to prove the following Claim.

CLAIM. *For each* $\phi: j(\underline{R}) \to h_{\cup I}$*, the canonical map* $\mathrm{colim}\,\big(j(\underline{R})\times_{h_{\cup I}} h_{|\mathcal{U}'(I)}\big) \to j(\underline{R})$ *is an equivalence in* $\mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})$.

Proof. Let us fix some notation: set $V := \cup I$ and let $f : |\mathrm{Spec}(\underline{R})| \to V$ in Top denote the first projection of the given $\phi \in \mathrm{Map}(\mathrm{Spec}(\underline{R}), (V, (\mathcal{O}_X)_{|V}))$.
We will start by inspecting the fibre-products. By 4.2.4.2, for each $U \in \mathcal{U}'(I)$ we can describe the fibre-product $j(\underline{R}) \times_{h_V} h_U \simeq j_U(\underline{R}) \subseteq j(\underline{R})$ in $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ by the sheaf of direct summands

$$j_U(\underline{R}) : \underline{A} \mapsto j_U(\underline{R})(\underline{A}) \leq_\oplus \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{R}, \underline{A})$$

where the latter subspace $j_U(\underline{R})(\underline{A})$ is spanned by all those maps $\gamma : \underline{R} \to \underline{A}$ in Ani(CRing) which fit in the extension of the following cartesian square:

$$\begin{tikzcd}
{|\mathrm{Spec}(\underline{A})|} \arrow[dr, dotted] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right] & & \\
& f^{-1}(U) \arrow[r, hook, "op"'] \arrow[d] & {|\mathrm{Spec}(\underline{R})|} \arrow[d, "f"] \\
& U \arrow[r, hook, "op"'] & V
\end{tikzcd}$$

Now, let us rephrase the statement of the Claim. Consider the sieve $C^{(0)}_{/\underline{R}}$ spanned by the class of sheaf inclusions:

$$\{j_U(\underline{R}) \hookrightarrow j(\underline{R}) \mid U \in \mathcal{U}'(I)\}$$

and observe that the construction $[U \mapsto j_U(\underline{R})]$ (so a choice of the composition $j(\underline{R}) \times_{h_V} (-) \circ h_{(-)}$) determines a left-cofinal inclusion $\mathcal{U}'(I) \hookrightarrow C^{(0)}_{/\underline{R}}$.
Hence, the statement amounts to the equivalence of the canonical map

$$\operatorname{colim} \mathrm{id}_{C^{(0)}_{/\underline{R}}} \simeq \operatorname{colim} \big(j_U(\underline{R}) \mid U \in \mathcal{U}'(I)\big) \longrightarrow j(\underline{R})$$

namely that $j(\underline{R}) \in \big(C^{(0)}_{/\underline{R}}\big)^{\mathrm{term}}$, i.e. that $C^{(0)}_{/\underline{R}}$ is the sieve over $j(\underline{R})$ generated by the identity $1_{j(\underline{R})}$. In yet another reformulation, this amounts to the fact that the Čheck nerve $C^{(0)}_{/\underline{R}} \simeq \check{C}(1_{j(\underline{R})})_{|\Delta^{op}}$ Zar-covers $j(\underline{R})$ (so its augmentation is an effective epimorphism).
To see this, since I covers $V = \cup I$, choose some partition of unity $\{x_k\}_{k=1}^m \subseteq \pi_0 \underline{R}$ in such a way that each inclusion of distinguished opens $D(x_k) = |\mathrm{Spec}(\underline{R}[x_k^{-1}])| \hookrightarrow |\mathrm{Spec}(\underline{R})|$ factors through $f^{-1}(k)$ for some $k \in I$. But then the maps $\{j(\underline{R}[x_k^{-1}]) \hookrightarrow k(\underline{R})\}_{k=1}^m \subseteq C^{(0)}_{/\underline{R}} \in \mathrm{Zar}(\mathrm{Spec}(\underline{R}))$ generate a Zariski cover of $j(\underline{R})$ and sit in the cartesian squares above, so that our sieve is indeed covering as wished. $\square$

As a Corollary, we finally obtain the Comparison Theorem between our two approaches do DAG.

Corollary 4.2.4.4. *(Comparison Theorem, [26],1.6.3.3) The restriction to scheme induces a fully faithful embedding*

$$h_{(-)} : \mathrm{Sch}' \hookrightarrow \mathrm{Fun}(\mathrm{Ani(CRing)}, \mathrm{Spc}) \simeq \mathcal{P}(\mathrm{Sch}^{\mathrm{Aff}}) = \mathrm{PreStack}$$

whose essential image is the full subcategory of Sh(Ani(CRing)) *spanned by those local functors of points admitting an affine Zariski cover (see 4.1.3.2), namely those sheaves h_X with $(X, \mathcal{O}_X) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}$ such that there exists a family $\{U_i\}_I \subseteq \mathrm{Open}(X)$ which induces an effective epimorphism $\coprod h_{U_i} \twoheadrightarrow h_X$.*
In particular, $h_{(-)} : \mathrm{Sch}' \hookrightarrow \mathrm{Sch} \subseteq_{f.f.} \mathrm{PreStack}$ establishes an equivalence of ∞-categories.

Proof. $h_{(-)}$ is fully faithful: Fix any two animated schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y) \in \mathrm{Sch}'$. In order to prove that $h_{(-)}$ induces an equivalence on the mapping space between the two, it suffice to show that it does whenever we replace $(X, \mathcal{O}_X)$ by its restriction $(U, (\mathcal{O}_X)_{|U})$ to any open $U \in \mathrm{Open}(X)$. Then, we wish that the following full subcategory of $\mathrm{Open}(X)$ be the whole one:

$$\mathcal{U} := \langle \big\{U \in \mathrm{Open}(X) \mid \theta_U : \mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ani(CRing)}}}\big((U, (\mathcal{O}_X)_{|U}), (Y, \mathcal{O}_Y)\big) \to \mathrm{Map}_{\mathrm{PreStack}}\big(h_U, h_Y\big)\big\}\rangle$$

Claim. *$\mathcal{U}$ is stable under glueing: for each $V \in \mathrm{Open}(X)$, let*

$$\mathrm{Cov}(V) := \{I \subseteq \mathcal{U} \mid V = \cup I\,,\, \forall S \in P_f(I),\, \cap S \subseteq \mathcal{U}\}$$

Then, $\mathrm{Cov}(V) \neq \emptyset$ *implies* $V \in \mathcal{U}$.

Proof. Fix any $V \in \mathrm{Open}(X)$, and let $I \subseteq \mathcal{U}$ be a covering family for V (i.e. $\cup I = V$) such that $I = \{U_i\}_I \in \mathrm{Cov}(V)$. By 4.2.4.3, the functor $h_{(-)}$ - hence the functor associated to the natural transformation $\theta_{(-)}$ - is a sheaf on $\mathrm{Open}(X)$; now by [26],1.6.2.1, h_Y is a hypercomplete sheaf, so that the sheaf condition of $\theta_{(-)}$ at the covering sieve generated by I can be checked on the Čheck nerve of $\coprod_I U_i \twoheadrightarrow \cup I = V$. In other words, since we consider $\mathrm{Sh}(\mathrm{Ani}(\mathrm{CRing}))^{op}$ as coefficients, one has the equivalence $\theta_V \simeq \mathrm{colim}\,\theta_{|\check{C}(I \to V)}$. But now, being colimits in Set universal, the simplicial object lying under the Čheck nerve acts as:

$$\check{C}(I \to V)_{|\Delta^{op}} : [n] \mapsto \big(\coprod_I U_i\big)^{\times_V n} \cong \{\cap S \mid S \in P_f(I)\,,\, \#S = n\}$$

and we conclude by the facts that, by assumption, $\mathcal{U}$ contains each $\cap S$. ■

Then, we will attempt three successive generalizations:

1. $\mathcal{U}$ contains the open affine charts of X, over which $h_{(-)}$ restricts to the Yoneda embedding.
2. Let $U \in \mathrm{Open}(X)$ be such that there exists some open affine chart $V \in \mathrm{Open}(X)$ for which $U \subsetneq V \subsetneq X$. Then, we observe that $U \in \mathcal{U}$.

   Indeed, being $(X, \mathcal{O}_X)$ an animated scheme, the open subset U is the topological space lying under some classical scheme, and hence admits a cover $I = \{U_i\}_I$ by affine open charts. Now, being $U_i \hookrightarrow V$ also open affine charts of the affine animated scheme V, they can be covered by distinguished opens of V, i.e. spectra of localizations of $\underline{A}$. In particular, this implies that all the finite intersections of I are again distinguished opens and hence open affine charts of U. By (1), $I \subseteq \mathcal{U}$; moreover, by construction $\cup I = U$ and, for each finite subset $S \in P_f(I)$, $\cap S \in \mathcal{U}$. Hence, $I \in \mathrm{Cov}(U)$ and we conclude by the Claim that also $U \in \mathcal{U}$, as desired.
3. We finally show that any arbitrary open subspace $U \in \mathrm{Open}(X)$ belongs to $\mathcal{U}$.

   As before, there exists some cover $I := \{U_i\}_I \subseteq \mathrm{Open}(X)$ consisting of open affine charts. Since $\cup I = U$ and $I \subseteq \mathcal{U}$ by (1), an application of the Claim would imply that $U \in \mathcal{U}$ whenever $\mathcal{U}$ contains all finite intersections $\cap S$ with $S \in P_f(I)$. But the latter condition follows from (2): for a given $S \in P_f(I)$ and any affine open $s \in S$, $\cap S \subseteq s$, and $s \in \mathcal{U}$ by (1).

Essential image of $h_{(-)}$*:* Any sheaf as in the statement belongs to the essential image of $h_{(-)}$, because the latter functor is a sheaf extending the Yoneda embedding $h_{(-)} : \mathrm{Sch}^{\mathrm{Aff}} \hookrightarrow \mathrm{PreStack}$.
The converse inclusion is a consequence of 4.2.4.3: being $h_{(-)}$ a $\mathrm{Sh}(\mathrm{Ani}(\mathrm{CRing}))^{op}$-sheaf on X with hypercomplete values, h_X is the colimit of the values of $h_{(-)}$ at the Čheck nerve of any open cover $I := \{U_i\}_I \subseteq \mathrm{Open}(X)$ of X; then, observe that we can choose an I consisting of affine open charts of X: being $(X, \mathcal{O}_X) \in \mathrm{Sch}'$ an animated scheme, its underlying ringed space is a classical scheme.
Hence, taking the Čheck nerve $\coprod_I U_i \twoheadrightarrow \cup_I U_i = X$ of a small affine Zariski cover gives the sought effective epimorphism $\coprod h_{U_i} \twoheadrightarrow h_X$ (here we use that the Yoneda embedding $h_{|\mathrm{Sch}^{\mathrm{Aff}}} : \mathrm{Sch}^{\mathrm{Aff}} \hookrightarrow \mathrm{PreStack}$ commutes with coproducts of affine schemes, see [26],1.6.2.4). □

4.3 Vector Bundles

In this section we will regard quasi-coherent sheaves from an equivariant point of view. This will provide interesting geometric insights into the algebraic constructions we introduced so far, such as symmetric algebras and cotangent complexes.
We will start by defining the relative spectrum of a quasi-coherent algebra, and then use this to obtain vector bundles of schemes from locally free sheaves.

4.3.1 The Relative Spectrum

In this subsection we will extend to quasi-coherent algebras the equivalence $\mathrm{Ani(CRing)}^{op} \simeq \mathrm{Sch}^{\mathrm{Aff}}$ induced by the adjunction $\mathcal{O}^{op} \vdash \mathrm{Spec}$ of 4.2.3.5. However, the reader should beware that this will not provide an alternative proof of such an equivalence, since the proof of the enhancement will be a reduction to the "algebraic" case.

Proposition 4.3.1.1. *(*Spec *of quasi-coherent algebra, [13],2.4.3) Let* $\mathrm{Aff}_{/T} \subseteq_{f.f.} \mathrm{Stack}_{/T}$ *denote the full sub-slice spanned by those stacks which are affine over* T*. There is an equivalence in* Cat_∞*:*

$$\mathcal{O} : (\mathrm{Aff}_{/T})^{op} \longrightarrow \mathrm{QCohAlg}(T)$$

Define the ***relative spectrum*** $\mathrm{Spec}_T : \mathrm{QCohAlg}(T)^{op} \to \mathrm{Aff}_{/T}$ *to be the quasi-inverse to* $\mathcal{O}^{op}$*.*

Proof. First recall that, for each affine map of stacks $S \to T$ and each $\underline{A}$-valued $\mathrm{Spec}(\underline{A}) \to T$, the base-change $S_{\mathrm{Spec}(\underline{A})} := S \times_T \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{A})$ is an affine map in Sch from the very definition, so that $S_{\mathrm{Spec}(\underline{A})} \in \mathrm{Sch}^{\mathrm{Aff}}$ is forced to be affine.
Then, the restriction of the bi-functor $\mathcal{O} \circ \big((-) \times_T (-)\big)$ to $\mathrm{Sch}^{\mathrm{Aff}}_{/T}$ yields a (possibly large) diagram of functors:

$$\mathcal{O}\big((-) \times_T \mathrm{Spec}(\underline{A})\big) : \big(\mathrm{Aff}_{/T}\big)^{op} \longrightarrow \mathrm{QCohAlg}(\mathrm{Spec}(\underline{A})) = \mathrm{CAlg}^{\Delta}_{\underline{A}}$$

$$\textsc{Obj}: (S \xrightarrow{\text{affine}} T) \longmapsto \mathcal{O}\big(S \times_T \mathrm{Spec}(\underline{A})\big)$$

whose (large) limit gives a functor:

$$\mathcal{O} : \big(\mathrm{Aff}_{/T}\big)^{op} \longrightarrow \mathrm{QCohAlg}(T)$$

$$\textsc{Obj}: (S \xrightarrow{\text{affine}} T) \longmapsto \mathcal{O}(S) := \mathrm{colim}\, \mathcal{O}\big(S \times_T \mathrm{Spec}(-)\big)_{|\left(\mathrm{Sch}^{\mathrm{Aff}}_{/T}\right)^{op}}$$

In other words, we set

$$\mathcal{O}(-) \simeq \mathrm{Ran}\Big(\mathcal{O} \circ \big((-) \times_T (\star)\big)_{|\left(\mathrm{Sch}^{\mathrm{Aff}}_{/T}\right)^{op}}\Big)$$

as the (large) right Kan extension along the Yoneda embedding of the slice consisting of those affine schemes over T into the second copy of Stack.
Let's show that $\mathcal{O}$ is an equivalence of ∞-categories.
The statement is local on T: by construction, it suffices to show that each $\mathcal{O}((-) \times_T \mathrm{Spec}(\underline{A}))$ is such. Hence, wlog $T = \mathrm{Spec}(\underline{A}) \in \mathrm{Sch}^{\mathrm{Aff}}$; but now $\mathrm{Aff}_{/\mathrm{Spec}(\underline{A})} \simeq \mathrm{Sch}^{\mathrm{Aff}}_{/\mathrm{Spec}(\underline{A})}$ has a terminal object, so that $\mathcal{O}(-) \simeq \mathcal{O}((-) \times_{\mathrm{Spec}(\underline{A})} \mathrm{Spec}(\underline{A}))$ and the claim is trivial. □

Remark. The properties of affine morphisms enhance the identification in 3.2.2.3 of the operations "taking under-slices" and "restricting the algebra structure": by the cancellation property 4.1.4.4, composable arrows of stacks $S \to T \to Z$ induces equivalences in Cat_∞:

$$\big(\mathrm{Aff}_{/Z}\big)_{/(T/Z)} \simeq \mathrm{Aff}_{/T} \qquad \Longleftrightarrow \qquad \mathrm{QCohAlg}(Z)_{(Z/T)/} \simeq \mathrm{QCohAlg}(T)$$

In particular, having $\mathcal{O}$ and QCohAlg compatible behaviours under restriction of scalars, the presheaf $\mathrm{QCohAlg}(-)$ induces a presheaf $\mathrm{Aff}(-)$ grouping all affine morphisms of stacks, and the equivalence of their values becomes tautologically functorial.

Remark. From the very definition of affine morphisms of stack, over any scheme $X \in \mathrm{Sch}$ the anti-equivalence restricts to one between the ∞-categories of schemes which are affine over X, on the one hand, and that of quasi-coherent $\mathcal{O}_X$-algebras on the other: $\mathcal{O}^{op} : \mathrm{Aff}_{/X} \rightleftarrows \mathrm{QCohAlg}(X)^{op} : \mathrm{Spec}_X$.
Moreover, as already observed, restricting further the equivalence to $\mathrm{Sch}^{\mathrm{Aff}}$ yields wlog the one $\mathrm{Spec} : \mathrm{Ani(CRing)}^{op} \simeq \mathrm{Sch}^{\mathrm{Aff}} : \mathcal{O}^{op}$ of 4.2.3.5. In particular, this supplies a formula to compute the action of Spec_S on objects: with notation as in 4.3.1.2, for any $\mathcal{A} \in \mathcal{O}_S\text{-Alg}$,

$$\mathrm{Spec}_T(\mathcal{A}) : \big(\mathrm{Aff}_{/T}\big)^{op} \longrightarrow \mathrm{Spc}$$

$$\textsc{Obj}: (p : S \xrightarrow{\text{affine}} T) \longmapsto (p^*\mathcal{A})^\vee = \mathrm{Map}_{\mathcal{O}_S\text{-Alg}}(p^*\mathcal{A}, \mathcal{O}_S)$$

where the notation $(-)^\vee$ means dualizing with respect to the monoidal structure of $\mathcal{O}_X$-Alg, in the sense of the following definition.

Definition 4.3.1.2. *(Dualization in* QCoh*)* For a prestack $S \in \text{PreStack}$, the presheaf of quasi-coherent modules on S represented by the structure sheaf $\mathcal{O}_S$ defines a **dualization functor** for $\text{QCoh}(S)$ by

$$(-)^\vee : \text{QCoh}(S) \longrightarrow \text{Spc}$$
$$\mathcal{M} \longmapsto \mathcal{M}^\vee := \text{Map}_{\text{QCoh}(S)}\big(\mathcal{M}, \mathcal{O}_S\big)$$

A similar discussion can be carried on with $\mathcal{O}_S$-algebras and $\text{Map}_{\mathcal{O}_S\text{-Alg}}$.

4.3.2 Vector Bundles over Stacks

In the current subsection, we will define vector bundles over stacks. This is a direct generalization of the classical case whenever the base stack admits a small cover by open affine charts. Let us start with a description of their fibres: it is well-posed in view of 3.6.2.5.

Definition 4.3.2.1. *(Locally free quasi-coherent module of rank n, [26],2.9.3.1)* Let $S \in \text{Stack}$ be a stack. A quasi-coherent module $\mathcal{F} \in \text{QCoh}(S)$ is said to be **locally free of finite rank** (fg-loc.free) iff the base-change $f^*\mathcal{F} \in \text{QCoh}(\text{Spec}(\underline{A})) \simeq \text{Mod}_{\underline{A}}$ along any $\underline{A}$-valued point $f : \text{Spec}\underline{A} \to S$ is a finitely generated locally free $\underline{A}$-module.
If furthermore $\mathcal{F}$ has uniform rank, say equal to n, on all $\underline{A}$-valued points (see 3.6.2.4), then we say that the quasi-coherent module $\mathcal{F}$ is **locally free of rank** n (n-loc.free).
Let $\text{QCoh}^n(S)$, $\text{QCoh}^{\text{fg}}(S) \subseteq_{f.f.} \text{QCoh}(S)^\simeq$ denote the spaces of n-loc.free (respectively fg-loc.free) quasi-coherent modules on the stack S.

Let us record some properties of locally free quasi-coherent modules of finite rank; they are all globalizations of the corresponding features for locally free modules of finite rank.

Proposition 4.3.2.2. *(Properties of locally free quasi-coherent modules of finite rank) Let $S \in \text{Stack}$ be a stack. Then, the following properties of locally free $\mathcal{O}_S$-modules of finite rank hold:*

- *"Being locally free of finite rank (resp. of rank n)" is fpqc-local (hence a fortiori* Zar*-local), stable under base-change, and satisfies the 2-out-of-3 property on exact sequences (see 3.6.2.6)*
- *$\mathcal{F} \in \text{QCoh}(S)$ is fg-loc.free iff $\mathcal{F}$ is a dualizable object for the closed symmetric monoidal structure $\text{QCoh}(S)^\otimes$ (see 4.3.1.2 and [26],2.9.1.5).*

Moreover, locally free quasi-coherent modules generate free quasi-coherent algebras via a globalization of the Derived Symmetric Algebra we defined in 3.7.1.

Definition 4.3.2.3. *(Quasi-coherent* Sym*, [15],4.7.4)* For a stack $S \in \text{Stack}$, consider a quasi-coherent module $\mathcal{F} \in \text{QCoh}(S)$, and define the **quasi-coherent symmetric algebra** $\text{Sym}^*_{\mathcal{O}_S}$ by:

- $S = \text{Spec}(\underline{A}) \in \text{Sch}^{\text{Aff}} : \text{Sym}^*_{\mathcal{O}_{\text{Spec}(\underline{A})}}(\mathcal{F}) := \text{Sym}^*_{\underline{A}}(\mathcal{F}_{|\underline{A}}) \in \mathcal{O}_{\underline{A}}\text{-Alg} = \text{CAlg}^{\Delta}_{\underline{A}}$;
- $S \in \text{Stack}$ arbitrary: define $\text{Sym}^*_{\mathcal{O}_S}(\mathcal{F}) := \lim\big(\text{Sym}^*_{\mathcal{O}_{S,s}}(\mathcal{F}_{|s}) \mid s \in \text{Sch}^{\text{Aff}}_{/S}\big) \in \mathcal{O}_S\text{-Alg}$ as the (large) limit over all the $\underline{A}$-valued points $s : \text{Spec}(\underline{A}) \to S$.

Remark. $\text{Sym}^*_{\mathcal{O}_S}$ is well-defined over the diagram of $\mathcal{F}$, because the derived symmetric algebra Sym^* commutes with base-change: see 3.7.2.1. Moreover, by 4.1.1 the limit defining $\text{Sym}^*_{\mathcal{O}_S}(\mathcal{F})$ can be assumed to be small whenever $S \in \text{Sch}$.

We are now ready to introduce vector bundles over stacks.

Definition 4.3.2.4. *(Vector bundle, [15],4.7.6)* Let $S \in \text{Stack}$ be a stack, and consider a locally free quasi-coherent module $\mathcal{F} \in \text{QCoh}(S)$ of rank n. Define the **vector bundle** associated to $\mathcal{F}$ as the affine S-stack:

$$\mathbb{V}_S(\mathcal{F}) := \text{Spec}_S\big(\text{Sym}^*_{\mathcal{O}_S}(\mathcal{F}^\vee)\big) \in \text{Aff}_{/S}$$

where $(-)^\vee$ denotes the dualization functor in $\text{QCoh}^\otimes$ of 4.3.1.2.

Remark. Let $\mathcal{F} \in \mathrm{QCoh}(S)$ and $E := \mathbb{V}_S(\mathcal{F})$ be the associated vector bundle. As in the affine case, under the globalized Spec-adjunction 4.3.1.1 it holds that global sections $\sigma : S \to E$ (of the structural morphism) are closed immersions corresponding to maps $\sigma^\flat : \mathcal{F}^\vee \to \mathcal{O}_S$, so global sections $\sigma^\flat \in \Gamma(S, \mathcal{F})$ before the dualization. A special instance of global section is the "trivial embedding" of the base, which is defined as follows.

Construction 4.3.2.5. *(Zero-section)* Let $\mathcal{F} \in \mathrm{QCoh}(S)$ be a quasi-coherent sheaf of rank n on the stack $S \in \mathrm{Stack}$, and let $E := \mathbb{V}_S(\mathcal{F})$ denote its associated vector bundle. On the open affine charts of a trivializing atlas for $\mathcal{F}$, one can write $E \simeq \mathrm{Spec}_{\mathrm{Spec}(\underline{A})}(\mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee))$ for some $(\underline{A}, \underline{M}) \in \mathrm{MOD}$ with $\underline{M}$ locally $\underline{A}$-free of rank n.
Define the **zero-section** of $\mathcal{F}$ as the canonical map $0 : S \to E$ associated to the "quotient" map $0^\flat$: $\mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) \to \mathrm{Sym}^0_{\underline{A}}(\underline{M}^\vee) \simeq \underline{A}$ under the globalized Spec-adjunction 4.3.1.1; here $0^\flat$ is induced in $\mathrm{Mod}_{\underline{A}}$ by $1_{\underline{A}}$ and $L\mathrm{Sym}^k_{\underline{A}}(\underline{M}^\vee) \to 0$ for $k > 0$.
In particular, observe that the zero-section $0 : X \to E$ is always a closed immersion locally of finite presentation. As in 4.4.1.3, its conormal sheaf recovers the dual of the quasi-coherent sheaf: $\mathcal{N}_0 \simeq \mathcal{L}^\vee$. Hence, by 4.5.2.3 it will follow that zero-sections are furthermore quasi-smooth closed immersions (see 4.5.2.1).

As a particular case, we can define the affine space $\mathbb{A}^n_S$ over any arbitrary base-scheme S. For $S = \mathrm{Spec}(\mathbb{Z})$, this retrieves the usual notion $\mathbb{A}^n \simeq \mathrm{Spec}(\mathbb{Z}[t_1, \dots, t_n])$.

Definition 4.3.2.6. *(Affine space)* For an arbitrary stack $S \in \mathrm{Stack}$, define the **n-th affine space on** S as the following base-change:

$$\mathbb{A}^n_S := \mathrm{Spec}_S\big(\mathrm{Sym}^*_{\mathcal{O}_S}((\mathcal{O}^n_S)^\vee)\big) \simeq \mathrm{Spec}_S(\mathcal{O}_S[t_i]^n_{i=1}) \simeq \mathbb{A}^n \times_{\mathrm{Spec}(\mathbb{Z})} S$$

Remark. The equivalence $\mathrm{Sym}^*_{\mathcal{O}_S}\big((\mathcal{O}^n_S)^\vee\big) \simeq \mathcal{O}_S[t_i]^n_{i=1}$ can be checked as follows. The right-hand side is the quasi-coherent algebra defined by the base-change $\mathcal{O}_S[t_i]^n_{i=1} \simeq \mathcal{O}_S \otimes^L_{\mathbb{Z}} \mathbb{Z}[t_i]^n_{i=1}$, so the statement is Zar-local; then, the equivalence $\mathrm{Sym}^*_{\underline{A}}((\underline{A}^n)^\vee) \simeq \underline{A}[t_{i=1}]^n$ for any $\underline{A} \in \mathrm{Ani(CRing)}$ is obtained by tensoring $\mathrm{Sym}^*_{\mathbb{Z}}\big((Z^n)^\vee\big) \simeq \mathbb{Z}[t_1, \dots, t_n]$ along $\mathbb{Z} \to \underline{A}$.

Remark. (Classical affine spaces, [15],4.7.7) For $S \in \mathrm{Stack}$, $n \in \mathbb{N}$, the structure map $\mathbb{A}^n_S \to S$ is induced by the quasi-coherent algebra structure $\mathcal{O}_S \to \mathcal{O}_S[t_i]^n_{i=1}$ and is flat (check it on open affine charts). In particular, one has $(\mathbb{A}^n_S)^{\mathrm{cl}} \simeq \mathbb{A}^n_S \otimes^L_S S^{\mathrm{cl}} \simeq \mathbb{A}^n_{S^{\mathrm{cl}}}$; so, our construction retrieves the classical one as a particular case.

Remark. (Zero-section of an affine space) For an affine space the zero-section is particularly simple. Indeed, being the symmetric algebra stable under base-change by 3.7.2.1, on a trivializing atlas for $\mathcal{O}_S$ - so with $S = \mathrm{Spec}(\underline{A})$ for some $\underline{A} \in \mathrm{Ani(CRing)}$ - we have

$$0 : \underline{A}[t_1, \dots, t_n] \to \underline{A} \simeq \underline{A}[t_1, \dots, t_n] /\!\!/ (t_1, \dots, t_n) := \underline{A}[t_1, \dots, t_n] \otimes^L_{\mathbb{Z}[t_1, \dots, t_n]} \mathbb{Z}[t_1, \dots, t_n]/(t_1, \dots, t_n)$$

along a choice of a lift $\mathbb{Z}[t_1, \dots, t_n] \to \underline{A}[t_1, \dots, t_n]$ for the connected component $[t_i \mapsto t_i] : \mathbb{Z}[t_1, \dots, t_n] \to \pi_0\underline{A}[t_1, \dots, t_n]$ in $ho\mathrm{Ani(CRing)}$. The equivalence $\underline{A} \simeq \underline{A}[t_1, \dots, t_n] /\!\!/ (t_1, \dots, t_n)$ can be inferred by a diagram chasing argument, which we report with only one indeterminate x:

$$\begin{aligned}
\mathrm{Map}_{\mathbb{Z}[x]/}(\underline{A} /\!\!/ (x)\,,\, \underline{B}) &\simeq \mathrm{Map}_{\mathbb{Z}[x]/}(\underline{A}[x] \leftarrow \mathbb{Z}[x] \to \mathbb{Z}[x]/(x)\,,\, \underline{B}) \\
&\simeq \mathrm{Map}_{\mathbb{Z}/}(\underline{A}[x] \leftarrow \underline{A} \leftarrow \mathbb{Z} \to \mathbb{Z}[x] \to \mathbb{Z}[x]/(x)\,,\, \underline{B}) \\
&\simeq \mathrm{Map}_{\mathbb{Z}/}(\mathbb{Z}[x]/(x) \leftarrow \mathbb{Z} \to \underline{A}\,,\, \underline{B}) \\
&\simeq \mathrm{Map}(\underline{A}\,,\, \underline{B})
\end{aligned}$$

Animated rings such as $\underline{A}[t_1, \dots, t_n] /\!\!/ (t_1, \dots, t_n)$ will be extensively studied in the section "Quotients of Rings".
In particular, we proved directly that the zero-section of an affine space is always a quasi-smooth closed immersion (see 4.5.2.1). As observed in 4.4.1.3, this holds for any vector bundle, but will only follow from the more abstract characterization 4.5.2.3.

In order to foster the intuition, let us spell out the action of the zero-section $0 : S \to E = \mathbb{V}_S(\mathcal{F})$ of an arbitrary locally free quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(S)$ of rank n on some stack $S \in \mathrm{Stack}$. Let's do it locally on a trivializing atlas for $\mathcal{F}$, so that wlog $S = \mathrm{Spec}(\underline{A})$, $\mathcal{F} = \underline{M} \in \mathrm{QCoh}(\mathrm{Spec}(\underline{A})) = \mathrm{Mod}_{\underline{A}}$ and $0^\flat : \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) \to \underline{A}$. Since both $(-)^\vee$ and (the left adjoint) $\mathrm{Sym}^*_{\underline{A}}$ preserve finite coproduct, for some $\underline{X} \in \mathrm{Mod}_{\underline{A}}$ such that $\underline{M} \oplus \underline{X} \simeq \underline{A}^n$ it holds

$$0^\flat : \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) \longrightarrow \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) \otimes^L_{\underline{A}} \mathrm{Sym}^*_{\underline{A}}(\underline{X}^\vee) \simeq \underline{A}[t_1, \dots, t_n] \longrightarrow \underline{A}[t_1, \dots, t_n] /\!\!/ (t_1, \dots, t_n) \simeq \underline{A}$$

Then, we know that the zero-section is quasi-smooth, so - up to further shrinking our charts - it is wlog of the form

$$0^\flat : \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) \to \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) /\!\!/ (f_1, \dots, f_r) \simeq \underline{A}$$

for some $(f_1, \dots, f_r) \subseteq \pi_0 \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee)$ not-necessarily-regular-generators exhibiting the latter as a static $\pi_0\underline{A}$-algebra of finite presentation.

The next result allows for a more intuitive approach: a vector bundle amounts to a "glueing" of affine spaces over a trivializing atlas consisting of open affine charts. Both the statement and the proof are a translation of 3.6.2.4 into more geometric terms; we will also refer to the algebraic counterpart for some more technical steps.

Proposition 4.3.2.7. *(Trivializing atlas) Let $S \in \mathrm{Stack}$ admitting a small cover by open affine schemes. Then, a vector bundle $E := \mathbb{V}_S(\mathcal{F}) \to S$ on $S \in \mathrm{Stack}$ admits a trivializing atlas, namely there exists a (small) cover $\mathcal{E}$ of S by affine open charts on which E restricts to affine spaces.*

Proof. By the construction of QCoh and the fact that $\mathrm{Sym}^* \circ (-)^\vee$ preserves (small) limits, it suffices to prove the claim for $S = \mathrm{Spec}(\underline{A})$, $\mathcal{F} := \underline{M} \in \mathrm{QCoh}(\mathrm{Spec}(\underline{A})) = \mathrm{Mod}_{\underline{A}}$ for some animated ring $\underline{A} \in \mathrm{Ani(CRing)}$ and some locally free $\underline{A}$-module $\underline{M}$ of ranks n. Then, we need to show that $\mathrm{Spec}(\underline{A})$ admits an affine Zariski cover on which $\underline{M}$ restricts to a free module of rank n. So, the proof continues as in 3.6.2.4, which we translate into more geometric terms.

Define $\mathcal{E}$ to be the set of distinguished opens $D(x)$ of $\mathrm{Spec}(\underline{A})^{\mathrm{cl}}$ over which $\widetilde{\underline{M}} \in \mathrm{QCoh}(\mathrm{Spec}(\underline{A}))$ with global section $\underline{M}$ corresponds to a free module of finite rank n_x. We argue by contradiction that $\mathcal{E}$ is not a trivializing atlas for $\widetilde{\underline{M}}$, i.e. that it does not cover $\underline{A}$; namely, assume the existence of some closed point $\epsilon_{\mathfrak{m}} : V(\mathfrak{m}) = \mathrm{Spec}(k = k(\mathfrak{m})) \twoheadrightarrow \mathrm{Spec}(\underline{A})^{\mathrm{cl}} \to \mathrm{Spec}(\underline{A})$ not belonging to any of the opens in $\mathcal{E}$.

We want to show that $\widetilde{\underline{M}}$ be free in an open neighbourhood of such a point $\mathfrak{m}$.
By Nakayama's Lemma we can afford a local description: a choice of a k-basis $\{y_i\}_i$ of the inverse image of $\widetilde{\underline{M}}$ at the closed point $\{\mathfrak{m}\}$ yields global generators of $\widetilde{M}$ on $\mathcal{D}^{\mathrm{cl}} := \mathrm{Spec}(\pi_0(\underline{A})_{\mathfrak{m}}) = \cap\{\mathfrak{m} \in D(z)\}$ (as in the proof of 3.4.0.6, we can argue at the level of connected components). In other words, we obtain a surjection $\underline{\hat{y}} : \mathcal{O}^{n'}_{\mathcal{D}^{\mathrm{cl}}} \twoheadrightarrow \underline{M}_{|\mathcal{D}^{\mathrm{cl}}}$. *(Here we can assume $n' = n$ iff $\underline{M}$ is n-loc.free.)*
Then, we claim that we can find an open neighbourhood $U \ni \mathfrak{m}$ which belongs to $\mathcal{E}$, thus reaching the sought contradiction.
Choose any open neighbourhood $D(x)$ of $\mathfrak{m}$ and consider a lift of $\underline{\hat{y}}$ to a surjection $g : \mathcal{O}^{n'}_{D(x)} \twoheadrightarrow \widetilde{\underline{M}}_{|D(x)}$. By the projectivity of $\Gamma(D(x), \widetilde{M})$, this admits a section ψ, i.e. $g \circ \psi$ is homotopic to the identity. We are left to show that also $\psi \circ g$ is homotopic to the identity of the trivial $D(x)$-vector bundle of rank n'.
To this end, we consider the matrix X associated to the linear map $\Gamma(D(x), \pi_0(\psi \circ g))$ in Vect_k. The vanishing locus $V(\det X)$ cannot contain $\mathfrak{m}$, because the first composite $\pi_0(g)$ of the pull-back of $\pi_0(\psi \circ g)$ to the specialization $\{\mathfrak{m}\}$ is an isomorphism. In other words, $\mathfrak{m} \in \mathcal{D} \subseteq D(\det X)$. If we let x' denote the numerator of $\det(X)$, one has $D(\det X) = D(xx')$. Moreover, we can regard X as inducing an automorphism of the quasi-coherent sheaf $\mathcal{O}_{D(x)}$ which is then invertible on $D(xx')$, thus implying the triviality of $\widetilde{M}_{|D(xx')}$. It follows that $xx' \in \mathcal{E}$, which is the desired contradiction. □

Another prominent class of example consists of line bundles, namely locally free quasi-coherent sheaves of rank 1, which constitute the class of invertible objects for the symmetric monoidal structure on the given ∞-category of quasi-coherent modules.

Definition 4.3.2.8. *(Line bundles, [26],2.9.4.1)* A quasi-coherent module $\mathcal{F} \in \mathrm{QCoh}(S)$ on a stack $S \in \mathrm{Stack}$ is a **line bundle** iff it is locally free of rank 1.

We refer to the full subspace $\underline{\mathrm{Pic}}(S) \subseteq_{f.f.} \mathrm{QCoh}(S)^{\simeq}$ of line bundles on S as the **Picard group** of S.

Remark. Whenever $S = X \in \mathrm{Sch}$ is a scheme, its connected components $\mathrm{Pic}(X) := \pi_0\underline{\mathrm{Pic}}(X)$ retrieve the classical Picard group of the underlying classical scheme X^{cl}.

Proposition 4.3.2.9. *(Line bundles are invertible, [26],2.9.4.2) For a scheme $X \in \mathrm{Sch}$, a quasi-coherent module $\mathcal{F} \in \mathrm{QCoh}(X)$ is a line bundle iff it is invertible for $\mathrm{QCoh}(X)^{\otimes}$.*

Finally, observe that the data of Picard groups over schemes assembles into a fpqc-sheaf (on a very large site).

Construction 4.3.2.10. *($\underline{\mathrm{Pic}}$ fpqc-stack)* The construction $[X \mapsto \underline{\mathrm{Pic}}(X)]$ assembles into a fpqc-stack on $\mathrm{Sch}^{\mathrm{Aff}}$.

Proof. Define $\int \underline{\mathrm{Pic}}' \subseteq_{f.f.} \int \mathrm{QCoh}$ as the full subcategory spanned by locally free quasi-coherent modules of rank 1. By the construction, the projection on the first component $(pr_1 : \int \mathrm{QCoh} \to \mathrm{Sch}^{\mathrm{Aff}}) \in \mathrm{CoCart}(\mathrm{Sch}^{\mathrm{Aff}})$ restricts to a cocartesian fibration $pr_1 : \int \underline{\mathrm{Pic}}' \to \mathrm{Sch}^{\mathrm{Aff}}$; to the latter remains associated a sub-prestack $\underline{\mathrm{Pic}}' \leq \mathrm{QCoh}$.
Then, by 3.6.2.5 the condition of "being locally free of rank 1" is fpqc-local, so that the fpqc-sheaf $\mathrm{QCoh}(-) \in \mathrm{Sh}_{\mathrm{fpqc}}(\mathrm{Sch}^{\mathrm{Aff}}, \mathrm{Cat}_\infty)$ restricts to the fpqc-stack $\underline{\mathrm{Pic}}'$. Finally, $\underline{\mathrm{Pic}} := \underline{\mathrm{Pic}}'(-)^{\simeq} \leq \mathrm{QCoh}(-)^{\simeq}$ gives the sought fpqc-sheaf. $\square$

Finally, we close this section with a brief digression on the projectivization of locally free modules of finite rank.

Definition 4.3.2.11. *(Projectivization, [15],6.1.2)* Let $S \in \mathrm{Sch}$ be a scheme, $\mathcal{F} \in \mathrm{QCoh}^{\mathrm{fg}}(S)$ a locally free quasi-coherent sheaf of finite rank, and consider any S-scheme $Y \in \mathrm{Sch}_{/S}$. Let $\mathrm{Proj}_S(\mathcal{F})(Y)$ denote the space of all surjections $(f : Y \to S, f^*\mathcal{F}^\vee \twoheadrightarrow \mathcal{L})$ onto some line bundle $\mathcal{L} \in \mathrm{QCoh}^1(Y)$. Such a construction defines a sheaf on $\mathrm{Sch}_{/S}$, which admits a classifying S-scheme $\mathbb{P}_S(\mathcal{F})$, call it the **projectivization of $\mathcal{F}$ on S.**

Remark. By 3.6.2.1 $\mathcal{F}$ is projective at each affine point $\mathrm{Spec}(\underline{A}) \to S$, so $\mathrm{Proj}_S(\mathcal{F})(\mathrm{Spec}(\underline{A}))$ is the space of all direct summands of rank 1 in the dual of $\mathcal{F}_{|\mathrm{Spec}(\underline{A})} \in \mathrm{QCoh}^{\mathrm{fg}}(\underline{A}) \subseteq \mathrm{APerf}(\underline{A}) \cap \mathrm{Flat}_{\underline{A}}$; morally, it is then the space of all the equations cutting-out hyperplanes of $f^*\mathcal{F}$, as it should be.

Remark. Projectivization can be equivalently described as follows: by post-composition with the zero-section of $\mathcal{L}$, the data above correspond to a monomorphisms in degree 1 of globalized symmetric algebras $\mathrm{Sym}^*_S(f^*\mathcal{F}^\vee) \hookrightarrow \mathrm{Sym}^*_Y(\mathcal{L})$. So, $\mathbb{P}_S(\mathcal{F})(f : Y \to S)$ is also the space of the "lines" $\mathbb{V}_Y(\mathcal{L}) \subseteq f^*\mathbb{V}_S(\mathcal{F})$ of the total S-space for F.

Remark. A derived analogue to the theory of graded algebras has been laid down in [13], where the statements in the previous definition-proposition are proven.

Remark. (Classical projectivization, [15],6.1.2) Let $S \in \mathrm{Sch}$, $\mathcal{F} \in \mathrm{QCoh}^n(S)$ be as in the definition above. The structural morphism $\mathbb{P}_S(\mathcal{F}) \to S$ can be proven to be flat, so one has $\mathbb{P}_S(\mathcal{F})^{\mathrm{cl}} \simeq \mathbb{P}_S(\mathcal{F}) \times_S S^{\mathrm{cl}} \simeq \mathbb{P}_{S^{\mathrm{cl}}}(i^*(\mathcal{F}))$, where $i : S^{\mathrm{cl}} \to S$ is the canonical map from the underlying classical scheme. Hence, our construction agrees with the classical one whenever both S and $\mathcal{F}$ are such.

4.4 The Conormal Sheaf

In this subsection we translate the construction of the (relative) algebraic cotangent complex in the language of DAG. To this end, we will follow the handouts of a seminar talk given by L. Pol, see [34],3. Moreover, we will freely use the notation and results of section 3.8 on the "Algebraic Cotangent Complex", to which we refer the unexperienced reader.

Definition 4.4.0.1. *(Space of derivations of a scheme)* Let $f : Y \to X$ be a morphism in Sch and $y : \mathrm{Spec}(\underline{A}) \to Y$ an $\underline{A}$-point of Y. Given an animated module $\underline{M} \in \mathrm{Mod}_{\underline{A}}$, consider the following commutative square of spaces, with $ev := ev_1 : \underline{A} \oplus \underline{M} \to \underline{A}$ a retraction of the trivial square-zero extension d^{triv} of 3.8.1.1.

$$\begin{tikzcd}
Y(\underline{A}\oplus\underline{M}) \arrow[r, "ev_*"] \arrow[d] & Y(\underline{A}) \arrow[d, "f(\underline{A})"] \arrow[dl, Rightarrow, "\eta"'] \\
X(\underline{A}\oplus\underline{M}) \arrow[r, "ev_*"] & X(\underline{A})
\end{tikzcd}$$

Define the **space of derivations** at y of Y into $\underline{M}$ as the fibre "at y" of the canonical comparison map: $Y(\underline{A}\oplus\underline{M}) \longrightarrow Y(\underline{A})\times_{X(\underline{A})} X(\underline{A}\oplus\underline{M})$.

$$\mathrm{Der}_y(Y/X,\underline{M}) := \mathrm{Fib}_y\big(Y(\underline{A}\oplus\underline{M}) \to Y(\underline{A})\times_{X(\underline{A})} X(\underline{A}\oplus\underline{M})\big)$$

Remark. By "fibre at y" in the definition of the space $\mathrm{Der}_y(Y/X,\underline{M})$ we actually mean the fibre at the following point: the commutativity of the square gives a homotopy $\eta : ev_*(d_*^{\mathrm{triv}}(f(y))) \simeq f(y)$ in $X(\underline{A})$ between $f(y)$ and the following composite:

$$X(\underline{A}\oplus\underline{M}) \ \ni\ d_*^{\mathrm{triv}}(f(y)) : \mathrm{Spec}(\underline{A}\oplus\underline{M}) \xrightarrow{d^{\mathrm{triv}}} \mathrm{Spec}(\underline{A}) \xrightarrow{y} Y \xrightarrow{f} X$$

Then, we take $y\times_\eta d_*^{\mathrm{triv}}(f(y)) \in Y(\underline{A})\times_{X(\underline{A})} X(\underline{A}\oplus\underline{M})$ as a base-point.

Definition 4.4.0.2. *(f admits a cotangent complex)* Let $f : Y\to X$ be a morphism in Sch.

- *(Local version, [34],3.2)* Given any $\underline{A}$-valued point $y : \mathrm{Spec}(\underline{A}) \to Y$ for which $\mathrm{Der}_y(Y/X,-)$ is co-representable, we define the **cotangent complex of** f **at** y to be the co-representing $\underline{A}$-module L_y, i.e. such that there is a natural equivalence $\mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(L_y,\underline{M}) \simeq \mathrm{Der}_y(Y/X,\underline{M})$ for each $\underline{M}\in\mathrm{Mod}_{\underline{A}}$.

  Whenever $\mathrm{Der}_y(Y/X,-)$ is co-representable (i.e. exists such a L_y) we say that f *admits a cotangent complex at* y and write equivalently $y^*L_{Y/X} = y^*L_f := L_y$.

- *(Global version, [34],3.4)* A quasi-coherent module $\mathcal{L}\in\mathrm{QCoh}(Y)$ is a **cotangent complex of** f iff, for each $\underline{R}$-point $y\in Y(\underline{R})$, $y^*\mathcal{L}\simeq y^*L_f$ is a cotangent complex of f at y.

  Whenever there exists such an $\mathcal{L}$, we say that f *admits a cotangent complex* and write equivalently $\mathcal{L}_f \simeq \mathcal{L}_{Y/X} := \mathcal{L}$.

Remark. As observed in 4.4.1.1, the notation is consistent.

Let us now record some properties of the cotangent complex of a morphism of schemes. In proving them, we will reduce the statements to the open affine patches and then refer to the following subsection on Examples for such special cases.

Proposition 4.4.0.3. *(Properties of $\mathcal{L}_{Y/X}$) The following properties hold:*

1. ([34],3.5) *Let $Z \xrightarrow{p} Y \xrightarrow{q} X$ be composable morphisms in* Sch*, and assume that there exists $\mathcal{L}_q$. Then,*
   - *there exists $\mathcal{L}_p$ iff $\mathcal{L}_{q\circ p}$ exists;*
   - *in such a case, there is an exact sequence in the stabilization* $\mathrm{QCoh}(Z)^{\mathrm{Ex}}$*: $p^*\mathcal{L}_q \to \mathcal{L}_{q\circ p} \to \mathcal{L}_p$.*
2. ([34],3.6) *Any open immersion $j : U \hookrightarrow X$ in* Sch *admits a trivial cotangent complex $\mathcal{L}_{U/X}\simeq 0$.*
3. (3.8.2.5,iv) *A morphism of schemes $f : Y\to X$ in* Sch *admitting a cotangent complex $\mathcal{L}_{Y/X}$ is an equivalence iff it induces an isomorphism $f^{\mathrm{cl}} : X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$ at the level of underlying classical schemes and $\mathcal{L}_{X/Y}\simeq 0$ vanishes.*

Proof. (1) : We will reduce the question Zar-locally on X. There it will be answered in the Example 4.4.1.1. In order to see this, let $\mathcal{U}\subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/X}$ denote the full subcategory spanned by open affine patches of X, and recall that $\mathrm{QCoh}_{/\mathcal{O}_X}$ is determined on open affine patches (it is a consequence of the sheaf-condition of X). In other words, $\mathcal{U}\subseteq \mathrm{Sch}^{\mathrm{Aff}}_{/X}$ is cofinal for QCoh and we can write:

$$\mathrm{QCoh}(X) \simeq \lim\big(\mathrm{QCoh}(\mathrm{Spec}\underline{A}) \simeq \mathrm{Mod}_{\underline{A}} \mid j : \mathrm{Spec}(\underline{A}) \hookrightarrow X\,,\ \underline{A}\in\mathrm{Ani}(\mathrm{CRing})\big)$$

Now, by construction base-change preserves limits of quasi-coherent modules, so $q^*\mathcal{U}$ (resp. $(q\circ p)^*\mathcal{U}$) consist of a cofinal full subcategory of the open affine patches of Y (resp. Z).
So, suppose to have cofibre sequences Zar-locally on X.

In other words, we assume that for each $(j := j_{\underline{A}} : \mathrm{Spec}(\underline{A}) \hookrightarrow X) \in \mathcal{U}$ - and with notation as in the aside rectangle of cartesian squares - there is a cofibre sequence:

$$\begin{tikzcd} (q\circ p)^*U \arrow[r, "p''"] \arrow[d, hook, "j''"] & q^*U \arrow[r, "q'"] \arrow[d, hook, "j'"] & U \arrow[d, hook, "j"] \\ Z \arrow[r, "p"] & Y \arrow[r, "q"] & X \end{tikzcd}$$

$$(p'')^*\mathcal{L}_{q'} \longrightarrow \mathcal{L}_{q'\circ p''} \longrightarrow \mathcal{L}_{p''}$$

The existence of two cotangent complexes as in the statement means that two out of three Zar-locally defined quasi-coherent modules assemble into global ones.
Furthermore, observe that we can assume wlog to be in the case in which the first two terms of the cofibre sequences of quasi-coherent modules assemble into global quasi-coherent modules on Z.
Indeed, if we know the existence of both $\mathcal{L}_q$ and $\mathcal{L}_p$, for each $U \in \mathcal{U}$ we can de-suspend the sequences in $\mathrm{QCoh}((q\circ p)^*U)^{\mathrm{Ex}}$, thus obtaining a fibre - equivalently cofibre - sequence whose first two terms assemble into global quasi-coherent modules on Z.
Thus, we can assume wlog that $\mathcal{L}_q$ and $\mathcal{L}_{q\circ p}$ exist and we need to show that also $(\mathcal{L}_{p''} \,|\, j'' \in (q\circ p)^*\mathcal{U})$ assembles into $\mathcal{L}_p \in \mathrm{QCoh}(Z)$, i.e. that it defines an object in the previous limit.
Equivalently we want that, for each morphism $(g : U/X \to V/X) \in \mathcal{U}$, the functoriality of $\mathcal{L}_{(-)}$ over the open affine patches in $(q\circ p)^*\mathcal{U}$ (induced by the universal property of cofibre sequences) corresponds to equivalences $g^*\mathcal{L}_{(q\circ p)^*V/q^*V} \simeq \mathcal{L}_{(q\circ p)^*U/q^*U}$.
But this holds true for the first two quasi-coherent modules $p^*\mathcal{L}_q$, $\mathcal{L}_{q\circ p}$ in the (co)fibre sequence, so it must hold true also for the last one by inspecting the induced long exact sequences in homotopy. ■

(2) : It suffices to show that $u^*\mathcal{L}_{U/X} \simeq 0$ for each $\underline{R}$-point $u \in U(\underline{R})$, i.e. that, for each $\underline{M} \in \mathrm{Mod}_{\underline{R}}$, the canonical map $U(\underline{R}\oplus\underline{M}) \to U(\underline{R}) \times_{X(\underline{R})} X(\underline{R}\oplus\underline{M})$ has contractible fibre at $(u, d_*^{\mathrm{triv}}(ju))$.
The open immersion j is in particular a monomorphism, so by diagram-chasing the canonical comparison map is a monomorphism, too. In other words, its fibres are either empty or contractible. However, $d_*^{\mathrm{triv}}(ju) \in U(\underline{R}\oplus\underline{M})$ lies over $(u, d_*^{\mathrm{triv}}(ju))$, so the fibre over the latter must be contractible, as needed. □

Theorem 4.4.0.4. *(Existence of $\mathcal{L}$, [34],3.7) Any morphism $f : Y \to X$ in* Sch *admits a cotangent complex $\mathcal{L}_{Y/X} \in \mathrm{QCoh}(Y)$.*

Proof. Let us start with a reduction step to the absolute case, i.e. to the relative case over $\mathbb{Z}$.

Claim. *wlog $X = \mathrm{Spec}(\mathbb{Z})$*

Proof. Assume that $Y/\mathbb{Z}$ and $X/\mathbb{Z}$ admit cotangent complexes over $\mathrm{Spec}(\mathbb{Z})$, say $\mathcal{L}_Y$ and $\mathcal{L}_X$. Then, an application of 4.4.0.3,i to the sequence $Y \to X \to \mathrm{Spec}(\mathbb{Z})$ yields the sought $\mathcal{L}_{Y/X} := \mathrm{Cofib}(f^*\mathcal{L}_X \to \mathcal{L}_Y)$. ■

We need to show that there exists $\mathcal{L}_Y := \mathcal{L}_{Y/\mathrm{Spec}(\mathbb{Z})}$. Recall that - as a consequence of the sheaf-condition and the existence of an affine Zariski cover - quasi-coherent modules on a scheme are determined by their values on open affine charts, i.e.

$$\mathcal{U} := \langle \{U := \mathrm{Spec}(\underline{A}) \hookrightarrow Y \mid \underline{A} \in \mathrm{Ani(CRing)}\}\rangle \subseteq_{f.f.} \mathrm{Sch}^{\mathrm{Aff}}_{/Y}$$

is cofinal, so that we can write

$$\mathrm{QCoh}(Y) \simeq \lim \mathrm{QCoh}_{|\mathcal{U}} \simeq \lim \big(\mathrm{QCoh}(\mathrm{Spec}\underline{A}) \simeq \mathrm{Mod}_{\underline{A}} \mid j : \mathrm{Spec}(\underline{A}) \hookrightarrow Y \,,\, \underline{A} \in \mathrm{Ani(CRing)}\big)$$

Therefore, it suffices to construct a compatible system of quasi-coherent modules

$$\big(j^*\mathcal{L}_Y \mid j : U := \mathrm{Spec}(\underline{A}) \hookrightarrow Y \,,\, \underline{A} \in \mathrm{Ani(CRing)}\big) = \big(j^*\mathcal{L}_Y \mid j \in \mathcal{U}\big)$$

To this end, notice that there exists $\mathcal{L}_U \simeq \mathcal{L}_{\mathrm{Spec}(\underline{A})} = L_{\underline{A}}$ (by 4.4.1.1), and that the open immersion j admits a well-defined trivial cotangent complex, namely $\mathcal{L}_{U/Y} \simeq 0$ (by 4.4.0.3,ii). Thus, whenever it exists, $\mathcal{L}_Y$ must sit in the cofiber sequence $j^*\mathcal{L}_Y \to \mathcal{L}_U \to \mathcal{L}_{U/Y}$ of 4.4.0.3,i. In other words, $j^*\mathcal{L}_Y \simeq L_{\underline{A}}$ whenever $\mathcal{L}_Y$ exists, so we have a Zar-local candidate for the restriction of the latter quasi-coherent module to the open patch $U = \mathrm{Spec}(\underline{A})$.
Therefore, set $(j^*\mathcal{L}_Y := L_{\underline{A}})_{j\in\mathcal{U}}$ and let us show that it defines an object in the previous limit.

Observe first that the construction $[\underline{A} \mapsto j^*_{\underline{A}}\mathcal{L}_Y]$ is functorial on $\mathcal{U}$: the Spec-construction is fully faithful, so (in the incarnation of quasi-categories) all (≥ 1)-simplices are in Ani(CRing) and $L_{(-)}$ is functorial on Ani(CRing).
Thus, it suffices to show that $j^*_{(-)}\mathcal{L}_Y$ respects the relation defining the limit as a subobject of the product of categories of quasi-coherent modules. Indeed, recall that limits are constructed via products and pull-backs.

In other words, we want that, for each morphism $g : U := \mathrm{Spec}(\underline{B})/Y \to V := \mathrm{Spec}(\underline{A})/Y$ of open affine patches, the canonical comparison map $\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}} \to L_{\underline{B}}$ induces an equivalence $g^*\mathcal{L}_V \simeq \mathcal{L}_U$ in $\mathrm{QCoh}(U) \simeq \mathrm{Mod}_{\underline{B}}$.
To this end, notice that g induces a cofibre sequence by 4.4.0.3,i:

$$g^*\mathcal{L}_{V/Y} \to \mathcal{L}_{U/Y} \to \mathcal{L}_{U/V}$$

Now, the first two terms vanish, because j_U and j_V are open immersions; so also $\mathcal{L}_{U/V} \simeq 0$. But the question is now algebraic: by the Example 4.4.1.1 and 3.8.2.5,ii, there is a cofiber sequence

$$\begin{tikzcd}
g^*\mathcal{L}_V \arrow[r] \arrow[d, equal] & \mathcal{L}_U \arrow[r] \arrow[d, equal] & \mathcal{L}_{U/V} \simeq 0 \arrow[d, "\simeq"] \\
\underline{B} \otimes^L_{\underline{A}} L_{\underline{A}} \arrow[r] & L_{\underline{B}} \arrow[r] & L_{\underline{B}/\underline{A}}
\end{tikzcd}$$

which forces an equivalence $g^*\mathcal{L}_V \simeq \mathcal{L}_U$ between the first two terms, as wished. □

In view of the exact sequence of cotangent sheaves 4.4.0.3,i and of the shift occurring in the cotangent complex relative to the inclusion of the origin in the affine space 3.8.3.2, let us give a name to the de-suspension of the cotangent complex of a map of schemes. Needless to observe that such an object plays a prominent role in the modern approach to algebraic geometry.

Definition 4.4.0.5. *(Conormal sheaf)* Define the **conormal sheaf** of a map of schemes $f : Y \to X$ in Sch as the de-suspension $\mathcal{N}_{Y/X} := \mathcal{L}_{Y/X}[-1]$

4.4.1 Examples

Example 4.4.1.1. *([34],3.1-3)* Let $f : Y := \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A}) =: X$ be in $\mathrm{Sch}^{\mathrm{Aff}}$. Then, we retrieve $\mathrm{Der}_{1_{\underline{B}}}(Y/X, \underline{M}) \simeq \mathrm{Der}_{\underline{A}}(\underline{B}, \underline{M})$. In particular, f admits a cotangent complex at $1_{\underline{B}}$, namely $L_{Y/X} := L_{\underline{B}/\underline{A}}$. Moreover, by base-change at any $\underline{R}$-point $y : \mathrm{Spec}(\underline{R}) \to Y$ - i.e. $y \in \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B}, \underline{R})$ - f admits a cotangent complex $L_y = y^*L_{Y/X} \simeq \underline{R} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}}$.

Proof. Let $\phi : \underline{A} \to \underline{B}$ in Ani(CRing) correspond to $f : Y \to X$ in $\mathrm{Sch}^{\mathrm{Aff}}$, and recall first that the base-point in our notation is $1_{\underline{B}} \times_{(\phi^*, ev_*)} (d^{\mathrm{triv}} \circ \phi)$. We have the following chain of equivalences in Spc:

$$\begin{aligned}
\mathrm{Der}_{1_{\underline{B}}}(Y/X, \underline{M}) &= \mathrm{Fib}_{1_{\underline{B}}}\big(\mathrm{Map}_{\mathrm{CAlg}^\Delta}(\underline{B}, \underline{B} \oplus \underline{M}) \to \mathrm{Map}_{\mathrm{CAlg}^\Delta}(\underline{B}, \underline{B}) \times_{\mathrm{Map}_{\mathrm{CAlg}^\Delta}(\underline{A}, \underline{B})} \mathrm{Map}_{\mathrm{CAlg}^\Delta}(\underline{A}, \underline{B} \oplus \underline{M})\big) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}^\Delta_{\underline{A}//\underline{B}}}(\underline{B}, \underline{B} \oplus \underline{M}) \\
&= \mathrm{Der}_{\underline{A}}(\underline{B}, \underline{M})
\end{aligned}$$

Informally, under the Yoneda Lemma, the canonical comparison map allows us to regard the maps in the source $\mathrm{Map}_{\mathrm{CAlg}^\Delta}(\underline{B}, \underline{B} \oplus \underline{M})$ as pairs consisting of:

- maps $\underline{B} \to \underline{B}$ of animated rings coming from the $\underline{A}$-algebra structure $\phi : \underline{A} \to \underline{B}$ of $\underline{B}$, and
- maps $\underline{A} \to \underline{B} \oplus \underline{M}$ obtained by regarding square-zero extensions of $\underline{B}$ by $\underline{M}$ as maps of $\underline{A}$-algebras.

Then, taking the fibre at $1_{\underline{B}} \times_\phi (d^{\mathrm{triv}} \circ \phi)$ amounts to considering those maps in the source which are $\underline{A}$-algebra square-zero extensions of $\underline{B}$ by $\underline{M}$.
Moreover, by 3.8.2.4 the functor $\mathrm{Der}_{\underline{A}}(\underline{B}, \underline{M})$ is co-represented by $L_{\underline{B}/\underline{A}}$, so f does indeed admit $L_{\underline{B}/\underline{A}}$ as a cotangent complex at $1_{\underline{B}}$.

Finally, let us show that, for any $y \in \mathrm{Map}_{\mathrm{Ani(CRing)}}(\underline{B}, \underline{R})$, we obtain L_y as the base-change $\underline{R} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}}$. In order to see this, observe that there is a cofibre sequence of spaces induced by the composable arrows of animated rings $\underline{A} \xrightarrow{\phi} \underline{B} \xrightarrow{y} \underline{R}$:

$$\mathrm{Der}_{1_{\underline{R}}}(\mathrm{Spec}(\underline{R})/\mathrm{Spec}(\underline{A}), \underline{M}) \xrightarrow{f^*} \mathrm{Der}_{1_{\underline{R}}}(\mathrm{Spec}(\underline{R})/\mathrm{Spec}(\underline{B}), \underline{M}) \xrightarrow{y^*} \mathrm{Der}_y(\mathrm{Spec}(\underline{B})/\mathrm{Spec}(\underline{A}).\underline{M})$$

as given by the following composition of morphisms of fibre angles (one can show that this is again a fibre sequence as in the proof of 3.8.2.5,ii):

$$[1_{\mathrm{pt}}, 1_{\underline{R}}, 1_{\underline{R}} \times_{\phi^*} \phi^*] \circ [1_{\mathrm{pt}}, y^*, y^* \times_{\phi^*} \phi^*]$$

Now, notice that 3.8.2.5,ii yields a morphism of fibre sequences in Spc for each $\underline{M} \in \mathrm{Mod}_{\underline{R}}$, where the solid vertical arrows are equivalences by the previous part:

$$\begin{tikzcd}
\mathrm{Der}_{1_{\underline{R}}}(\mathrm{Spec}(\underline{R})/\mathrm{Spec}(\underline{A}), \underline{M}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Der}_{1_{\underline{R}}}(\mathrm{Spec}(\underline{R})/\mathrm{Spec}(\underline{B}), \underline{M}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Der}_y(\mathrm{Spec}(\underline{B})/\mathrm{Spec}(\underline{A}).\underline{M}) \arrow[d, dotted] \\
\mathrm{Map}_{\mathrm{Mod}_{\underline{R}}}(L_{\underline{R}/\underline{A}}, \underline{M}) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}_{\underline{R}}}(L_{\underline{R}/\underline{B}}, \underline{M}) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}_{\underline{R}}}(\underline{R} \otimes^L_{\underline{B}} L_{\underline{B}/\underline{A}}, \underline{M})
\end{tikzcd}$$

Hence, also the induced dotted vertical arrow is an equivalence, as inferred by considering the associated long exact sequences in homotopy. □

Remark. In particular, by the functoriality of QCoh, the notation $y^* L_{Y/X}$ (resp. $\mathcal{L}_{Y/X}$) for the local (resp. global) cotangent complex of f is consistent.

Example 4.4.1.2. In particular, the closed immersion of the origin of the n-th affine space, $\{0\} \twoheadrightarrow \mathbb{A}^n$, admits a cotangent complex and it holds $\mathcal{L}_{\{0\}/\mathbb{A}^n} \simeq \mathbb{Z}^n[1] \in \mathrm{QCoh}(\mathrm{Spec}(\mathbb{Z}))$. In particular, $\mathcal{N}_{\{0\}/\mathbb{A}^n} \simeq \mathbb{Z}^n$.

Proof. Let $\underline{t}$ be a tuple of n variables. The closed immersion can be rewritten as

$$\{0\} \simeq \mathrm{Spec}(\mathbb{Z}[\underline{t}]/(\underline{t})) \twoheadrightarrow \mathrm{Spec}(\mathbb{Z}[\underline{t}]) \simeq \mathbb{A}^n$$

so, by the previous example, we know that - at the (only) point $r : \mathrm{Spec}(\underline{R}) \to \mathrm{Spec}(\mathbb{Z}) = \{0\}$ - it holds $r^*(\mathcal{L}_{\{0\}/\mathbb{A}^n}) \simeq r^* L_{\mathbb{Z}/\mathbb{Z}[\underline{t}]} \simeq \underline{R} \otimes^L_{\mathbb{Z}} \mathbb{Z}^n[1] = \underline{R}^n[1]$. □

Let us now include a couple of slight variations of the previous example. We refer to 4.3.2.5 and 4.5.2.1 for the terminology needed.

Example 4.4.1.3. *(Conormal sheaf of zero-sections)* Let $\mathcal{L} \in \mathrm{QCoh}(S)$ be a locally free quasi-coherent module of rank n on a stack $S \in \mathrm{Stack}$. Let $p : L := \mathbb{V}_S(\mathcal{L}) \to X$ denote the associated vector bundle and consider its zero-section $0 : X \to L$. Then, $\mathcal{L}_0 \simeq \mathcal{L}^\vee[-1] \simeq (\mathcal{L}[1])^\vee$, so that $\mathcal{N}_0 \simeq \mathcal{L}^\vee$.

Proof. Locally on a trivializing atlas for $\mathcal{L}$, our setting amounts to a factorization of the identity:

$$1_{\underline{A}} : \underline{A} \xrightarrow{p^\flat} \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee) \xrightarrow{0^\flat} \underline{A}$$

for $(\underline{A}, \underline{M}) \in \mathrm{MOD}$ such that $\mathcal{L}_{|\mathrm{Spec}(\underline{A})} = \underline{M} \in \mathrm{Mod}_{\underline{A}}$. By 3.8.2.5,ii, the latter induces in turn a cofibre sequence of relative cotangent complexes:

$$\underline{A} \otimes^L_{\underline{B}} L_{p^\flat} \to L_{1_{\underline{A}}} \to L_{0^\flat}$$

with $\underline{B} := \mathrm{Sym}^*_{\underline{A}}(\underline{M}^\vee)$. Then, being $L_{1_{\underline{A}}} \simeq 0$ trivial, the exactness in $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}}$ allows us to compute $L_{0^\flat}[-1] \simeq \underline{A} \otimes^L_{\underline{B}} L_{p^\flat} \simeq \underline{M}^\vee$ by 3.8.3.3. □

Example 4.4.1.4. *(Conormal sheaf of* qSmCl*)* Let $i : Z := \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \twoheadrightarrow \mathrm{Spec}(\underline{A}) =: X$ be a quasi-smooth closed immersion of affine schemes, and recall that $\underline{A} /\!\!/ (f_1, \ldots, f_n) \simeq \underline{A} \otimes^L_{\mathbb{Z}[t_1,\ldots,t_n]} \mathbb{Z}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)$. Then, by 3.8.3.2 and 4.4.1.1, the base-change along $\mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \to \{0\}$ yields $\mathcal{N}_{Z/X} \simeq \big(\underline{A} /\!\!/ (f_1, \ldots, f_n)\big)^n$.

4.5 Smoothness and Quasi-Smoothness

In this subsection we introduce classes of relative schemes which correspond to a choice of a "system of local coordinates" on the base scheme. The mathematical nature of such frames will be determined by the properties of the class at stake.

4.5.1 Smooth and Étale Morphisms.

Let us start with a brief review of smooth and étale morphisms of schemes. Our goal is to focus on quasi-smoothness, so we take the latter notions as motivational. As an outcome, we will not provide any proof.

Definition 4.5.1.1. *(Smooth and Étale morphisms, [15],5.3.2)* A morphism of schemes $f : Y \to X$ in Sch with quasi-compact and quasi-separated source $Y \in \mathrm{Sch}^{\mathrm{qcqs}}$ is:

1. **smooth** iff, Zar-locally on the base X, it is locally of finite presentation and the cotangent complex $\mathcal{L}_{Y/X}$ is locally free of finite rank.

   Equivalently, iff f is flat and the underlying morphism of classical schemes $f^{\mathrm{cl}} : Y^{\mathrm{cl}} \to X^{\mathrm{cl}}$ in $\mathrm{Sch}^{\mathrm{cl}}$ is smooth.

2. **étale** iff, Zar-locally on the base X, it is locally of finite presentation and the cotangent complex $\mathcal{L}_{Y/X}$ vanishes.

   Equivalently, iff f is flat and the underlying morphism of classical schemes $f^{\mathrm{cl}} : Y^{\mathrm{cl}} \to X^{\mathrm{cl}}$ in $\mathrm{Sch}^{\mathrm{cl}}$ is étale.

For any scheme $S \in \mathrm{Sch}$, let Sm_S, $\mathrm{\acute{E}t}_S \subseteq_{f.f.} \mathrm{Sch}_S^{\mathrm{qcqs,lfp}}$ denote the full subcategories of S-smooth and S-étale schemes.

Example 4.5.1.2. From the very definition, open immersions are étale and étale maps are in particular smooth.
Moreover, any vector bundle is smooth on the base, so in particular affine and projective spaces are such.

4.5.2 Quasi-Smooth Closed Immersions.

In this subsection we will introduce the notion of "quasi-smooth closed immersions" as zero-loci of coordinate maps (see 3.5.0.1). This will be the most important class of relative schemes for the rest of our dissertation. Morally, we will see here in which sense the notion of "(homotopy) quotient rings" works as a "homotopy-coherent system of coordinates" for a scheme. Noteworthy is how the good-behaviour of the latter is controlled by the cotangent complex / sheaf, so an algebraic object.

Definition 4.5.2.1. *(Quasi-Smooth closed immersion, [17],2.3.6)* A closed immersion of schemes $i : Z \twoheadrightarrow X$ in Sch is **quasi-smooth** iff, Zar-locally on the base X, there exists a map $f : X \to \mathbb{A}^n$ sitting in the following cartesian square in Sch:

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\ \{0\} \arrow[r, "0"] & \mathbb{A}^n \end{tikzcd}$$

where $0 : \{0\} \twoheadrightarrow \mathbb{A}^n$ denotes the (closed) inclusion of the origin. In other words, we require i to be locally of the form $\mathrm{Spec}(\underline{A} /\!\!/ (f_1, \dots, f_n)) \to \mathrm{Spec}(\underline{A})$ for some coordinate maps $(f_1, \dots, f_n) \subseteq \underline{A}$.
Sometimes we will denote by qSmCl the property "being quasi-smooth" of a closed immersion of schemes.

Lemma 4.5.2.2. *(Properties of quasi-smooth closed immersions, [17],2.3.6) The following properties hold:*

1. *"Being a quasi-smooth closed immersion" is stable under base-change and* Zar*-local on the base.*

2. *A closed immersion $i^{\mathrm{cl}} : Z^{\mathrm{cl}} \twoheadrightarrow X^{\mathrm{cl}}$ of classical schemes in* $\mathrm{Sch}^{\mathrm{cl}}$ *is quasi-smooth iff it is regular (see 2.1.0.4).*

Proof. (2) : It follows from 3.5.0.3,iii and 2.1.0.3,i-iii.
(1) : Locality on the base is by definition. As for stability under base-change, the question is now local on the base. Then, let it be given a quasi-smooth closed immersion $i : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \dots, f_n)) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ in $\mathrm{Sch}^{\mathrm{Aff}}$; for any map $\phi : \mathrm{Spec}(\underline{A}) \to \mathrm{Spec}(\underline{B})$ in $\mathrm{Sch}^{\mathrm{Aff}}$, the base-change along ϕ retrieves a quasi-smooth closed immersion $\phi^*(i) : \mathrm{Spec}(\underline{B} /\!\!/ (\phi^\flat(f_1), \dots, \phi^\flat(f_n))) \to \mathrm{Spec}(\underline{B})$, since $\otimes^L$ is associative:

$$\underline{B} \otimes^L_{\underline{A}} \left(\underline{A} /\!\!/ (f_1, \dots, f_n) \right) \simeq \underline{B} \otimes^L_{\mathbb{Z}[t_1, \dots, t_n]} \mathbb{Z}[t_1, \dots, t_n]/(t_1, \dots, t_n) \simeq \underline{B} /\!\!/ (\phi(f_1), \dots, \phi(f_n)) \qquad \square$$

Remark. Any (non-empty) static scheme admits non-static quasi-smooth sub-schemes: on the affine chart $\mathrm{Spec}(R)$ corresponding to some static ring $R \in \mathrm{CRing}$, consider the quasi-smooth closed immersion $\mathrm{Spec}(R /\!\!/ (0)) \to \mathrm{Spec}(R)$ of Example 3.5.0.4.
Morally, this is a *non-static* generalization of Koszul regular immersions: for a sequence of coordinate maps, not being "topologically" Koszul regular (on the underlying schemes) is not a bug, but a feature: it means that they carry also higher "homotopical" information.

The next result characterizes quasi-smoothness algebraically in terms of the conormal sheaf (see 4.4.0.5), thus opening for a comparison with an enhancement of the more familiar notion of "smoothness".

Proposition 4.5.2.3. *(Characterization of quasi-smooth closed immersions, [17],2.3.8) A closed immersion of schemes $i : Z \twoheadrightarrow X$ in* Sch *is quasi-smooth iff it is locally of finite presentation (see 4.1.4.17) and the conormal sheaf $\mathcal{N}_{Z/X} = \mathcal{L}_{Y/X}[-1] \in \mathrm{QCoh}(Z)^{\mathrm{Ex}}$ is locally free of finite rank (see 4.3.2.1).*
In particular, "being a quasi-smooth closed immersion" is fpqc-local.

Proof. First observe that - by the properties 4.5.2.2, 4.1.4.19, 4.3.2.1 of the objects at stake - the statement is Zar-local and stable under base-change, so that we can reduce it wlog to $i : Z := \mathrm{Spec}(\underline{B}) \twoheadrightarrow \mathrm{Spec}(\underline{A}) =: X$.

($\Longrightarrow$) : Up to base-change along $\underline{A}$, wlog $i : \{0\} \twoheadrightarrow \mathbb{A}^n$ is the inclusion of the origin. Then, by 3.8.3.2 $\mathcal{N}_{\{0\}/\mathbb{A}^n} \simeq \mathcal{L}_{\{0\}/\mathbb{A}^n}[-1] \simeq \widetilde{\mathbb{Z}}^n$ is a locally free quasi-coherent module of finite rank. Moreover, i is locally finitely presented: $\mathbb{Z}$ is a compact $\mathbb{Z}[t_1, \dots, t_n]$-algebra, since extension of scalars induces an equivalence $\mathrm{Map}_{\mathbb{Z}[t_1,\dots,t_n]/}(\mathbb{Z}, -) \simeq \mathrm{Map}(\mathbb{Z}, -) \simeq \mathrm{Map}_{\mathbb{Z}[t_1,\dots,t_n]/}(\mathbb{Z}[t_1, \dots, t_n], -)$ of mapping spaces of (the given slice of) animated rings and $\mathrm{Poly}_{\mathbb{Z}[t_1,\dots,t_n]} \simeq (\mathrm{CAlg}^{\Delta}_{\mathbb{Z}[t_1,\dots,t_n]})^{\mathrm{sfp}} \subseteq (\mathrm{CAlg}^{\Delta}_{\mathbb{Z}[t_1,\dots,t_n]})^{\mathrm{fp}}$.

($\Longleftarrow$) : As in the introduction, let's work on a trivializing atlas for $\mathcal{L}_i$; so, assume $i : \mathrm{Spec}(\underline{B}) \to \mathrm{Spec}(\underline{A})$ for some surjective (on π_0) map $\phi := i^\flat : \underline{A} \to \underline{B}$ in $(\mathrm{CAlg}^{\Delta}_{\underline{A}})^{\mathrm{fp}}$ exhibiting $\underline{B}$ as a compact $\underline{A}$-algebra and with the shifted cotangent complex $L_{\underline{B}/\underline{A}}[-1] \in \mathrm{Mod}^{\mathrm{Ex}}_{\underline{B}}$ being free of finite rank, say n. (Recall that the source of a closed immersion into an affine scheme is affine by 4.2.3.8, so that we obtain such a $\phi = i^\flat$ by either the Yoneda Lemma or, equivalently, by the Spec-adjunction 4.2.3.5).
We need to exhibit a sequence $(f_1, \dots, f_n) \subseteq \underline{A}$ for which $\underline{B} \simeq \underline{A} /\!\!/ (f_1, \dots, f_n)$. To this end, we will first exhibit a morphism $\underline{A} /\!\!/ (f_1, \dots, f_n) \to \underline{B}$ in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ and then prove that it is an equivalence via property 3.8.2.5,iv of the conormal sheaf.
Recall that, by the universal property of quotients 3.5.0.5, the datum of a morphism $\underline{A} /\!\!/ (f_1, \dots, f_n) \to \underline{B}$ in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ corresponds to a sequence $(f_1, \dots, f_n) \subseteq \underline{A}$ together with homotopies $\{f'_i := \phi(f_i) \simeq 0\}_i$ in $for\underline{B}$.
We will construct such a sequence of pairs by lifting a basis of connected components in the free $\pi_0(\underline{B})$-module $\pi_1(L_{\underline{B}/\underline{A}}) \cong \pi_0(L_{\underline{B}/\underline{A}}[-1])$ lying under the shifted relative cotangent complex.

Claim 1. *Take the fibre $\underline{F} := \mathrm{Fib}(\phi : \underline{A} \to \underline{B})$ in $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}}$; the Hurewicz map of 3.8.2.5,iii induces an identification $\pi_1(L_{\underline{B}/\underline{A}}) \cong \pi_0(\underline{F} \otimes^L_{\underline{A}} \underline{B})$, so that the latter is a locally free $\pi_0\underline{B}$-module of rank n.*

Proof. Let's construct an isomorphism $\pi_0(\underline{F} \otimes^L_{\underline{A}} \underline{B}) \cong \pi_0(L_{\underline{B}/\underline{A}}[-1]) \cong \pi_1(L_{\underline{B}/\underline{A}})$. To this end, consider the fibre sequence in $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{B}}$ associated to $\underline{B} \otimes^L_{\underline{A}} \phi$, and observe that its tail can be computed by tensoring with $\underline{B} \otimes^L_{\underline{A}} (-)$ the cofibre sequence in $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}}$ for $\phi : \underline{A} \to \underline{B}$:

$$\underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi)[-1] \simeq \underline{B} \otimes^L_{\underline{A}} \underline{F} \longrightarrow \underline{B} \xrightarrow{\underline{B} \otimes^L \phi} \underline{B} \otimes^L_{\underline{A}} \underline{B} \longrightarrow \underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi)$$

Indeed, being $\pi_0(\phi)$ surjective, by 3.4.0.1 we can compute the terms (in the degrees above) of the sequence for ϕ in the pre-stable category $\mathrm{Mod}_{\underline{A}}$. Then, tensoring along ϕ is exact in the pre-stable category $\mathrm{Mod}_{\underline{A}}$ (even though it is exact in its stabilization iff ϕ is flat), so we obtain the exact sequence above.
Now, consider the Hurewicz map $\epsilon_\phi : \underline{B} \otimes^L_{\underline{A}} \mathrm{Cofib}(\phi) \to L_{\underline{B}/\underline{A}}$ of 3.8.2.5,iii; again by the surjectivity of ϕ on connected components, by 3.8.2.6 we conclude that the connectivity of $\underline{F} = \mathrm{Fib}(\phi)$ implies the 2-connectivity of $\mathrm{Fib}(\epsilon_\phi)$, i.e. the desired map $\pi_1(\epsilon_\phi) \cong \pi_0(\epsilon_\phi[-1]) : \pi_0(\underline{B} \otimes^L_{\underline{A}} \underline{F}) \to \pi_0(L_{\underline{B}/\underline{A}}[-1])$ is an isomorphism. ■

Claim 2. *A basis $\{df_1, \ldots, df_n\}$ for the n-loc.free $\pi_0(\underline{F} \otimes^L_{\underline{A}} \underline{B})$ lifts to elements $\{\tilde{f}_1, \ldots, \tilde{f}_n\} \subseteq \pi_0(F)$ which wlog $\pi_0\underline{A}$-generate the latter by Nakayama's Lemma.*

Proof. Let $I := \ker(\pi_0(\phi) : \pi_0\underline{A} \twoheadrightarrow \pi_0\underline{B})$ denote the ideal of the structure map, which is surjective at the level of connected components by assumption. Being π_0 symmetric monoidal, we can write:

$$\pi_0(\underline{F} \otimes^L_{\underline{A}} \underline{B}) \cong \pi_0\underline{F} \otimes_{\pi_0\underline{A}} \pi_0\underline{B} \cong \pi_0(\underline{F})/I\pi_0(\underline{F})$$

Choose lifts $\{\tilde{f}_i\}_i \subseteq \pi_0(\underline{F})$ of the basis $\{df_i\}_i$ of the quotient. We claim that $\{\tilde{f}_i\}_i$ $\pi_0\underline{A}$-generate $\pi_0\underline{F}$.
In order to see this, we need a few remarks. First, by the following Remark, being ϕ surjective on π_0, we can compute its fibre $\underline{F}$ in $\mathrm{Mod}_{\underline{A}}$, and the latter operation commutes with $\pi_0 := \mathrm{Map}_{\mathrm{Spc}}(\mathbb{S}^0, \mathrm{Map}_{\mathrm{Mod}_{\underline{A}}}(\underline{A}, -)$ (see 3.2.1.4). Then, $\pi_0\underline{F} \cong \pi_0(\mathrm{Fib}(\phi)) \cong \mathrm{Fib}(\pi_0\phi)$, and the latter is in turn isomorphic to $\ker(\pi_0\phi) \cong I \in \mathrm{Mod}(\pi_0\underline{A})$, which is finitely generated by assumption.
Now, observe that for this Claim we can assume wlog $\pi_0\underline{A}$ to be local, since the surjectivity of $\oplus_i[\tilde{f}_i]\pi_0(\underline{A}) \to \pi_0(\underline{F})$ is a local property.
Therefore, $\{\tilde{f}_1, \ldots, \tilde{f}_n\}$ generate $\pi_0(\underline{F})$ by Nakayama's Lemma as in [37],10.20.1.vii. ■

Remark. Let $\phi : R \to S$ be in CRing. *Then, ϕ is surjective iff the fibre $\mathrm{Fib}(\phi) \in \mathrm{Mod}^{\mathrm{Ex}}_R$ can actually be computed in Mod_R and it is static and can be identified with $\ker(\phi) \in \mathrm{Mod}(R)$.*

Proof. It suffices to prove that ϕ is surjective iff $\mathrm{Fib}(\phi) \simeq \ker(\phi)$ is static. The rest is a consequence of 3.4.0.1.
The claim can be seen by inspecting the induced long exact sequence in homotopy:

$$\pi_k(\mathrm{Fib}(\phi)) \to \pi_k(R) \to \pi_k(S) \to \pi_{k-1}(\mathrm{Fib}(\phi))$$

Indeed, the satiticity of R, S imply that $\pi_k(\mathrm{Fib}(\phi)) \cong 0$ whenever $k \neq 0, -1$; finally, $\pi_0(\mathrm{Fib}(\phi)) \simeq \ker(\phi)$ and $\pi_{-1}(\mathrm{Fib}(\phi)) \cong \mathrm{coker}(\phi)$ vanishes iff ϕ is surjective. ■

As in 3.5.0.1, lift the latter generators to a sequence of coordinate maps $(f_1, \ldots, f_n) \subseteq \underline{F}(\underline{A}^n) \hookrightarrow \underline{A}(\mathbb{Z}[t_1, \ldots, t_n]) \simeq for\underline{A}^n$, together with a choice of paths $\{f_i' \simeq 0 \in \mathrm{Path}_{for\underline{B}}(f_i', 0)\}_{i=1}^n$ witnessing the commutativity of the fibre square in $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{A}}$:

$$\begin{tikzcd} \underline{F} \arrow[r, hook] \arrow[d] & \underline{A} \arrow[d, "\phi"] \\ 0 \arrow[r, hook] & \underline{B} \end{tikzcd} \qquad \begin{tikzcd} f_i \arrow[r] \arrow[d] & f_i \arrow[d] \\ 0 \arrow[r] & f_i' := \phi(f_i) \end{tikzcd}$$

As already observed, by the universal property of quotients in 3.5.0.5, such a datum amounts to an essentially unique morphism $\Phi : \underline{A} /\!\!/ (f_1, \ldots, f_n) \to \underline{B}$ in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$.

Claim 4. *The morphism $\Phi : \underline{A} /\!\!/ (f_1, \ldots, f_n) \to \underline{B}$ in $\mathrm{CAlg}^{\Delta}_{\underline{A}}$ is an equivalence.*

Proof. By property 3.8.2.5,iv, Φ is an isomorphism iff it induces an isomorphism $\pi_0(\Phi)$ on connected components and its relative cotangent complex $L_\Phi \simeq 0$ vanishes.
The first condition follows by the construction: as in the Remark and in the proof of Claim 2, $\pi_0(\underline{F}) \cong \ker(\pi_0\phi)$ and $\pi_0(\phi)$ is surjective, so that 3.5.0.3,i at (3) implies:

$$\pi_0(\underline{B}) \cong \pi_0(\underline{A})/\pi_0(\underline{F}) \cong_{(1)} \pi_0(\underline{A})/(\tilde{f}_1, \ldots, \tilde{f}_n) \cong_{(2)} \pi_0(\underline{A})/(f_1, \ldots, f_n) \cong_{(3)} \pi_0(\underline{A} /\!\!/ (f_1, \ldots, f_n))$$

and - reading from right to left - we retrieve the action of the map Φ as described in the proof of 3.5.0.5 (apart from (1) and (2) which are the algebraic identifications of CLaim 2).
Let's turn to the relative cotangent complex L_Φ. Consider the exact sequence in $\mathrm{Mod}^{\mathrm{Ex}}_{\underline{B}}$ as in 3.8.2.5,ii:

$$L_{(\underline{A}/\!\!/(f_1,\ldots,f_n))/\underline{A}} \otimes^L_{\underline{A}/\!\!/(f_1,\ldots,f_n)} \underline{B} \quad\longrightarrow\quad L_{\underline{B}/\underline{A}} \quad\longrightarrow\quad L_{\underline{B}/\underline{A}/\!\!/(f_1,\ldots,f_n)} = L_\Phi$$

The vanishing of L_Φ amounts to the first map being an equivalence. In order to see the latter, notice first that $L_{\underline{B}/\underline{A}}[-1] \simeq \underline{B}^n$ by assumption, so that $L_{\underline{B}/\underline{A}} \simeq \underline{B}^n[1]$ by suspending in $\mathrm{Mod}_{\underline{B}}^{\mathrm{Ex}}$. On the other hand, by 3.8.2.5,i, the very definition $\underline{A} /\!\!/ (f_1, \ldots, f_n) \simeq \underline{A} \otimes^L_{\mathbb{Z}[t_1,\ldots,t_n]} \mathbb{Z}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)$ allows us to rewrite

$$L_{(\underline{A}/\!\!/(f_1,\ldots,f_n))/\underline{A}} \simeq L_{\{0\}/\mathbb{Z}[t_1,\ldots,t_n]} \otimes^L_{\mathbb{Z}[t_1,\ldots,t_n]} \underline{A} /\!\!/ (f_1, \ldots, f_n) \simeq \big(\underline{A} /\!\!/ (f_1, \ldots, f_n)\big)^n[1]$$

Finally, extending scalars along Φ computes also the first term as $\underline{B}^n[1]$. Thus, by the last part of 3.8.2.5,ii the first map of the exact sequence is an equivalence, as wished. ■

Thus, on a trivializing atlas for the conormal sheaf $\mathcal{N}_{Z/X}$, the closed immersion $i : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ is quasi-smooth.

Finally, for what concerns the last statement on the fpqc-locality, this is now automatic: the algebraic characterization of quasi-smoothness is fpqc-local by 3.6.2.2 and 4.1.4.19. □

Remark. The proof is constructive: let it be given a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$, so a closed immersion of finite presentation admitting a conormal sheaf $\mathcal{N}_i$ which is locally free of finite rank; then, we can choose a trivializing neighbourhood $\mathcal{Z} = \{Z_i \hookrightarrow Z\}_I$ of Z on which i restricts to maps $\mathrm{Spec}(\underline{A}_i /\!\!/ (f^i_1, \ldots, f^i_{n_i})) \twoheadrightarrow \mathrm{Spec}(\underline{A}_i)$ for an affine Zariski cover $\{\mathrm{Spec}(A_i)\}_I$ of X; for each i, this can be obtained by lifting a basis $\{df^i_j\}_{j=1}^{n_i}$ for the free module $\mathcal{N}_i$ to a sequence $\{f^i_j\}_{j=1}^{n_i}$ of coordinate maps in $\underline{A}$ together with homotopies $\{f^i_j \simeq 0\}_{i=1}^{n_i}$.

Example 4.5.2.4. *(Closed immersions in* Sm_S*)* Let $S \in \mathrm{Sch}$ be a scheme. Any closed immersion of *quasi-smooth* S-schemes is itself such.
In particular, closed immersions which admit smooth retractions (e.g. zero-sections of vector bundles) are quasi-smooth.

Let's prove it. Let $i : Z \twoheadrightarrow X$ be a closed immersion of smooth S-schemes. Notice first that i is locally finitely presented, because the class of lfp morphisms enjoys the right-cancellation property 4.1.4.20. As for the condition on the cotangent complexes, consider the shifted exact sequence in $\mathrm{QCoh}(Z)^{\mathrm{Ex}}$ of 4.4.0.3,i: $i^*\mathcal{N}_{X/S} \to \mathcal{N}_{Z/S} \to \mathcal{N}_{Z/X}$; by assumption the first two quasi-coherent modules are locally free of finite rank, so we conclude by 4.3.2.2.

We close our digression on quasi-smooth morphisms by introducing a notion of "(co)dimension" for such coordinate "frames".

Definition 4.5.2.5. *(Virtual codimension, [17],2.3.11)* Define Zar-locally on the source the **virtual codimension** of a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$ to be the rank of the (fg-loc.free) conormal sheaf $\mathcal{N}_{Z/X} \in \mathrm{QCoh}(\mathcal{O}_Z)^{\mathrm{Ex}}$. Denote it by either $\mathrm{codim.vir}(X, Z)$ or $\mathrm{codim.vir}(i)$.

Remark. In other words, in view of the proof of 4.5.2.3 we define $\mathrm{codim.vir}(i)$ on a trivializing neighbourhood for $\mathcal{N}_{Z/X}$ by positing $\mathrm{codim.vir}(\mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)), \mathrm{Spec}(\underline{A})) := n$. The adjective "virtual" is probably due to the fact that our "coordiante frames" carry homotopical as well as topological information.

Proposition 4.5.2.6. *(Properties of* $\mathrm{codim.vir}$*) The virtual codimension of a quasi-smooth closed immersion* $i : Z \twoheadrightarrow X$ *in* Sch *enjoys the following properties.*

1. $\mathrm{codim.vir}(i)$ *is stable under base-change and fpqc-local;*

2. *Define the **topological codimension*** $\mathrm{codim.top}(X, Z)$ *of a closed immersion* $i : Z \twoheadrightarrow X$ *to be the (Krull) codimension of the underlying classical closed immersion* $i^{\mathrm{cl}} : Z^{\mathrm{cl}} \twoheadrightarrow X^{\mathrm{cl}}$*, i.e. morally the minimum among the Krull codimensions of the closed irreducible components of* $(Z^{\mathrm{cl}}, X^{\mathrm{cl}})$*.*

   Suppose that the classical scheme X^{cl} *lying under the target is locally Noetherian. Then, the two notions of dimension are comparable and*

   (a) $\mathrm{codim.vir}(X, Z) \geq \mathrm{codim.top}(X, Z)$*;*
   (b) the equality at a point $x \in X$ *holds whenever* $Z \times_X X^{\mathrm{cl}} \in \mathrm{Sch}$ *is classical* Zar*-locally around* x*; in particular, the virtual and topological codimension of a regular closed immersion coincide;*

(c) if further X^{cl} is Cohen-Macauley (see [37],28.8: X^{cl} is locally given by Cohen-Macauley Noetherian rings as in [37],10.104) at $x \in X^{\mathrm{cl}}$, then the equality at x holds iff $Z \times_X X^{\mathrm{cl}}$ is classical around x.

Proof. (1) : Is clear from the definition and 4.5.2.3, 3.6.2.5, 3.6.1.9.

$(2), a$: The question is local on the base, so let's assume wlog $i : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{A})$ for a Noetherian ring $\pi_0\underline{A} \in \mathrm{CRing}$. We need to show that the Krull codimension of the canonical quotient map $\pi_0(\underline{A}) \twoheadrightarrow \pi_0(\underline{A})/(f_1, \ldots, f_n)$ is at most n. Let us recall more precisely the definition of Krull codimension for i^{cl}. First recall that the Krull dimension of a Noetherian static ring $R \in \mathrm{CRing}$ is is the maximum among the dimensions of its closed irreducible components (see [9],11.7).

Recall also that - being the static ring $\pi_0\underline{A}$ Noetherian - the ideal $(f_1, \ldots, f_n)$ admits a minimal finite primary decomposition, with associated primes $\mathrm{Ass}(f_1, \ldots, f_n)$ (combine [9],8.21-25-27); in other words, if we let $\mathrm{Min}(f_1, \ldots, f_n)$ denote the set of all the isolated prime ideals $\mathfrak{p} \in \mathrm{Ass}(f_1, \ldots, f_n)$ of $(f_1, \ldots, f_n)$ (by [9],8.30, read "minimal" primes over the ideal),

$$(f_1, \ldots, f_n) = \cap \mathrm{Ass}(f_1, \ldots, f_n) = \cap \mathrm{Min}(f_1, \ldots, f_n)$$

This gives a decomposition into closed irreducible components of

$$Z^{\mathrm{cl}} = \mathrm{Spec}(\pi_0(\underline{A})/(f_1, \ldots, f_n)) = \cup \{V(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Min}(\mathfrak{p})\}$$

Then, by definition $\mathrm{codim.top}(\mathrm{Z}^{\mathrm{cl}}, X^{\mathrm{cl}}) = \mathrm{codim}(\mathfrak{q})$ is the codimension (or height) of the minimal prime $\mathfrak{q} \in \mathrm{Min}(f_1, \ldots, f_n)$ which realizes the maximum in:

$$\dim(\pi_0\underline{A}/(f_1, \ldots, f_n)) = \max\{\dim(\pi_0\underline{A}/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Min}(f_1, \ldots, f_n)\}$$

Intuitively, we are forcing the equality in [9],11.5,b: $\dim(\pi_0\underline{A}) = \dim(\pi_0\underline{A}/\mathfrak{q}) + \mathrm{codim}(\mathfrak{q})$.

Finally, the Corollary [9],11.17 to Krull's Hauptidealsatz implies that the minimal prime $\mathfrak{q}$ over $(f_1, \ldots, f_n)$ has codimension at most n, so that $\mathrm{codim.top}(Z^{\mathrm{cl}}, X^{\mathrm{cl}}) \leq n$, as wished. ■

$(2).b$: As before, wlog $i : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{A})$; unpacking the assumption, there exists some open neighbourhood $x \in \mathrm{Spec}(\underline{A}[g^{-1}]) \subseteq \mathrm{Spec}(\underline{A})$ such that, for $\underline{B} := \underline{A}[g^{-1}]$,

$$\pi_0\underline{B} /\!\!/ (f_1, \ldots, f_n) \simeq \underline{B} /\!\!/ (f_1, \ldots, f_n) \otimes^L_{\underline{B}} \pi_0\underline{B} \simeq \pi_0(\underline{B})/(f_1, \ldots, f_n)$$

where the first equality follows from 3.5.0.3,i. By 3.5.0.3,iii this amounts to $(f_1, \ldots, f_n) \subseteq \pi_0\underline{B}$ being a regular sequence.

The claim is to show that the inequality obtained via Krull Hauptidealsatz is indeed an equality $n = \mathrm{codim}(\mathfrak{q})$, for $\mathfrak{q} \subseteq \pi_0\underline{B}$ some minimal prime ideal over $(f_1, \ldots, f_n)$ realizing the Krull codimension of i^{cl} (as in the proof of $(2), a$). To this end, notice that we can recursively write $\pi_0\underline{B}/(f_1, \ldots, f_n)$ by successive quotients by a single regular equation f_i each; this yields an n-long increasing sequence of prime ideals with $\mathfrak{p}_n = \mathfrak{q}$ and such that, for each $1 \leq i \leq n$, $\mathfrak{p}_i \in \mathrm{Min}(f_1, \ldots, f_i)$ is a minimal prime ideal over the first i equations. Hence, by [9],11.19, each $\mathfrak{p}_i$ has codimension i and the statement follows.

In particular, for a regular closed immersion $i : Z^{\mathrm{cl}} \twoheadrightarrow X^{\mathrm{cl}}$ in $\mathrm{Sch}^{\mathrm{cl}}$, $\mathrm{codim.vir}(i) = \mathrm{codim.top}(i)$: on a trivializing atlas for $\mathcal{N}_i$, we can write $i : \mathrm{Spec}(R /\!\!/ (f_1, \ldots, f_n)) \simeq \mathrm{Spec}(R/(f_1, \ldots, f_n)) \to \mathrm{Spec}(R)$ for some static ring $R \in \mathrm{CRing}$, and the sufficient condition is satisfied.

$(2).c$: Again, wlog $i : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{A})$ for a *local* Noetherian Cohen-Macauley ring $\pi_0\underline{A}$, i.e. such that the inequality of [9],11.5,b is always an equality. Then, we conclude by [37],10.104.2: a sequence in $\pi_0\underline{A}$ is regular iff it induces the "right" codimension, which in turn equates the virtual one. □

Remark. We require X^{cl} to be locally Noetherian, because otherwise the Krull dimension itself is not "well-behaved" and the bound need not hold. For instance, in such cases Krull's Hauptidealsatz [9],11.15 fails because of pathological examples of a non-locally Noetherian static ring $R \in \mathrm{CRing}$ over which $\mathrm{codim.top}(\mathbb{A}^n_R, \{0\}) > n + \dim(R)$; incidentally, this is definitely not the notion of topological (co)dimension we wish for, so we will not try to include such instances in the theory.

4.6 Cartier Divisors

In this section we generalize the various notions of divisors, as well as the relative comparisons.

4.6.1 Virtual Cartier Divisors

Definition 4.6.1.1. *(Virtual Cartier divisor, [17],3.1.1-2)* Let $X \in \mathrm{Sch}$ be a scheme. Define a **virtual (effective) Cartier divisor** on X to be a pair (D, i_D) with $i_D : D \twoheadrightarrow X$ a quasi-smooth closed immersion in Sch exhibiting D as a closed subscheme of X of virtual codimension 1.
Let $\mathrm{VDiv}(X) \subseteq_{f.f.} (\mathrm{Sch}^{\mathrm{cl}}_{/X})^{\simeq}$ denote the sub-space spanned by virtual Cartier divisors on X.

Remark. Unwinding the definition, we require the quasi-smooth closed immersion $i_D : D \twoheadrightarrow X$ to be of the form $\mathrm{Spec}(\underline{A} /\!\!/ (f)) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ on a trivializing atlas for $\mathcal{N}_{D/X}$.

Example 4.6.1.2. (Classical Cartier divisors, [17],3.1.3) (Effective) Cartier divisors on a classical scheme are an instance of virtual (effective) Cartier divisors: $\mathrm{CaDiv}(X^{\mathrm{cl}}) \subseteq \mathrm{VDiv}(X^{\mathrm{cl}})$. This is a consequence of 2.1.0.6 together with 4.5.2.6,2,ii.

The next result assembles the data of virtual Cartier divisors on schemes into an fpqc-stack.

Construction 4.6.1.3. *(*VDiv*, [17],3.1.2)* The construction $[X \mapsto \mathrm{VDiv}(X)]$ determines a sub-fpqc-stack of $[X \mapsto (\mathrm{Sch}_{/X})^{\simeq}]$.

Proof. Let $(\mathrm{Sch}_{/(-)})^{op} : \mathrm{Sch} \to \mathrm{Spc}$ be the presheaf represented by the target map $(ev_1 : \mathrm{Fun}(\Delta^1, \mathrm{Sch}) \to \mathrm{Sch}) \in \mathrm{CoCart}(\mathrm{Sch})$ under the Straightening Theorem [24],3.2. Post-compose it with the core-groupoid functor $(-)^{\simeq}$, so as to obtain the presheaf $\mathcal{F} := (\mathrm{Sch}_{/(-)})^{\simeq}$.
Unwinding the definitions, we can informally describe $\mathrm{Map}_{\int \mathcal{F}}(D'/X, D/Y)$ as the spaces of squares over "admissible maps" in $\mathrm{Map}'(X,Y) \subseteq_{f.f.} \mathrm{Map}_{\mathrm{Sch}}(X,Y)$, where the latter embedding is the essential image of $\{1_X\} \subseteq \mathrm{Map}_{\mathrm{Sch}}(X,Y)$ whenever $X \simeq Y$ and an equivalence otherwise.
Consider the full subcategory $\int \mathrm{VDiv} \subseteq_{f.f.} \int(\mathrm{Sch}_{(-)})^{\simeq}$ spanned by the 1-skeletal data:

- Obj: $\mathrm{VDiv}(X)$ for $X \in \mathrm{Sch}$;
- Mor: For any pair of virtual Cartier divisors $i_{D'} : D' \twoheadrightarrow X$, $i_D : D \twoheadrightarrow Y$, define $\mathrm{Map}_{\mathrm{VDiv}}(i_{D'}, i_D) \subseteq_{f.f.} \mathrm{Map}_{\int \mathcal{F}}(i_{D'}, i_D)$ to be the subspace spanned by cocartesian squares over "admissible maps".

Denote again $ev_1' : \int \mathrm{VDiv} \to \mathrm{Sch}$ for the restricted target projection. Observe that $ev_1' \in \mathrm{LFib}(\mathrm{Sch})$ is a left-fibration, since by construction it is cocartesian and its fibres are ∞-groupoids (see [20],3.1.23). Hence, again by the Straightening Theorem [24],3.2 it classifies a presheaf $\mathrm{VDiv} : \mathrm{Sch} \to \mathrm{Spc}$, which arises together with a monomorphism $\mathrm{VDiv} \hookrightarrow \mathcal{F}$ (its fibres are -1-truncated, as it can be seen at the level of unstraightened categories). In other words, we obtain a presheaf $\mathrm{VDiv} \leq \mathcal{F} = (\mathrm{Sch}_{(-)})^{op}$ in PreStack.
We are left to check the sheaf condition on fpqc-covers of $\mathrm{Sch}^{\mathrm{Aff}}$. This is a straightforward consequence of our fpqc-local definition of the notions "quasi-smooth closed immersion" and "virtual codimension", hence of "virtual Cartier divisor". $\square$

4.6.2 Generalized Cartier Divisors

Definition 4.6.2.1. (Generalized effective Cartier divisor, [17],3.2.1) Let $X \in \mathrm{Sch}$ be a scheme. A **generalized** (effective) **Cartier divisor** over X is an invertible $\mathcal{O}_X$-twist, namely a pair $(\mathcal{L}, s : \mathcal{L} \to \mathcal{O}_X)$ in $\mathrm{QCoh}(X)$ with $\mathcal{L}$ locally free of rank 1.
Define $\mathrm{GDiv}(X) := \left(\underline{\mathrm{Pic}}'(X)_{/\mathcal{O}_X}\right)^{\simeq} \subseteq_{f.f.} \left(\mathrm{QCoh}(X)_{/\mathcal{O}_X}\right)^{\simeq}$ to be the full subspace of generalized Cartier divisors on X.

Remark. We write $\underline{\mathrm{Pic}}'$ in place of $\underline{\mathrm{Pic}}$ (as in 4.3.2.8), because this time we consider the core groupoid of the slice category over the structure sheaf: otherwise all comparison maps s would be invertible...

Construction 4.6.2.2. (GDiv is a fpqc-stack, [17],3.2.4) The construction $[X \to \mathrm{GDiv}(X)]$ assembles into a fpqc-stack $\mathrm{Sch}^{\mathrm{Aff}} \to \mathrm{Spc}$.

Proof. Observe that the slice of two sheaves is still a sheaf: one can check the sheaf condition for e.g. coverings by [24],5.2.2.2. Then, apply the usual argument with the Straightening Theorem [24],3.2, so as to define the slice fpqc-stack GDiv: the fpqc-stacks $\underline{\mathrm{Pic}}'$ and $\mathcal{O}(-)$ (see 4.3.1.1) induce a fpqc-stack $\underline{\mathrm{Pic}}'(-)_{/\mathcal{O}_{(-)}}$, which stays a fpqc-stack after passing to core groupoids. □

We will now prove the main result of this sections, thus showing the equivalences of the notions of virtual and generalized Cartier divisors. Let us start with the useful construction of a comparison map GDiv → VDiv.

Construction 4.6.2.3. *(Zero-loci as* VDiv*, [17],3.2.3)* Let $(\mathcal{L}, s) \in \mathrm{GDiv}(X)$ be a generalized Cartier divisor on X. It gives rise to a virtual Cartier divisor as follows.
Let $\sigma : X \to L$ be the global section of the vector bundle $L := \mathbb{V}_X(L)$ on X which corresponds to $s := \sigma^\flat : \mathcal{L} \to \mathcal{O}_X$ (see 4.3.2.4, and 4.3.1.1). Let $0 : X \twoheadrightarrow L$ denote the zero-section of $\mathcal{L}$. Take the zero-locus D of σ, i.e. form the following cartesian square:

$$\begin{tikzcd} D \arrow[r, "i_D"] \arrow[d, "\sigma_{|D}"] & X \arrow[d, "\sigma"] \\ X \arrow[r, "0"] & L \end{tikzcd}$$

Such data exhibits $(D, i_D : D \twoheadrightarrow X) \in \mathrm{VDiv}(X)$ as a virtual Cartier divisor on X with conormal sheaf $\mathcal{N}_{D/X} \simeq \mathcal{L}_{|D}$.

Proof. The zero-section $0 : X \to L$ is quasi-smooth by 4.5.2.4 (or more directly by 4.3.2.5) with conormal sheaf $\mathcal{N}_0 \simeq \mathcal{L}^\vee$. Then, by 4.5.2.2,i also i_D is a quasi-smooth closed immersion, and we obtain $\mathcal{N}_{i_D} \simeq (\sigma_{|D})^* \mathcal{N}_0 = \mathcal{L}^\vee_{|D}$. □

We close this subsection by stating (without proof) a theorem comparing our notions of divisors.

Theorem 4.6.2.4. *(Comparison of* VDiv*,* GDiv*, [17],3.2.6) There are canonical isomorphisms of derived stacks:*

$$\mathrm{VDiv} \xrightarrow{\simeq} \mathrm{GDiv} \xrightarrow{\simeq} [\mathbb{A}^1/\mathbb{G}_m]$$

where $[\mathbb{A}^1/\mathbb{G}_m]$ is the quotient stack (see 4.6.3.1) of the affine line $\mathbb{A}^1$ by the canonical "scaling" $\mathbb{G}_m$ action.

4.6.3 Quotient Stacks

In the (very large) ∞-topos Stack it makes sense to consider group stacks, namely objects of Grp(Stack) as in B.1.0.9 or, equivalently, contractible groupoids as in C.1.0.5.

Definition 4.6.3.1. *(*Quotient stack, [16],4.25) For a group stack $G \in \mathrm{Grp(Stack)}$, define the G-action groupoid of a stack $U \in \mathrm{Stack}$ as in C.2.0.1, and refer to $(U /\!\!/ G)_\bullet$ as a *"stack S with a G-action"*. Define the **quotient stack** $[U/G]$ of U by G as the geometric realization $[U/G] := |(U /\!\!/ G)_\bullet|$ in Stack. By construction, the canonical map $S \to [U/G]$ exhibits a *G-torsor*, as in C.2.0.2.

Remark. ([16],4.26) As it is often the case with sheaves, the sheafification functor L needs not be right-exact, and in general colimits of sheaves are computed as the sheafification of colimits in the presheaf category: $\mathrm{colim}^{\mathrm{Stack}} \simeq L \circ \mathrm{colim}^{\mathrm{PreStack}}$. Similarly, also here the canonical map $\mathrm{colim}^{\mathrm{PreStack}}(U /\!\!/ G)_\bullet \to [U/G]$ comparing the geometric realizations in PreStack and Stack respectively (see C.5.1.2) is in general *not* an equivalence, but exhibits the latter as the sheafification of the former.

Let us now include a useful computation of the points of a quotient stack, namely we will translate the classification result for G-torsors into a more geometric language (see C.2.0.5 for the topos-theoretic proof).

Theorem 4.6.3.2. *(Functor of points of a quotient stack, [16],4.28) Let $(G /\!\!/ U)_\bullet$ be a stack with a G-action; its quotient stack $[U/G]$ can be described as follows: its functor of points is the space $[U/G](-) \simeq \mathrm{Map}_{\mathrm{Stack}^G}(-, (G /\!\!/ U)_\bullet)$ spanned by*

- *coangles $(\pi, f) : T \twoheadleftarrow Y \to U$ with π being a G-torsor and f a G-equivariant map;*

- *where an equivalence of two coangles $(f,\pi) \to (f',\pi')$ is a G-equivariant equivalence $\phi : Y \to Y'$ with the datum of commutative squares expressing the compatibility with π, π' and f, f', respectively.*

In particular, for $U = \mathrm{Spec}(\mathbb{Z}) \simeq *$, the quotient $[*/G]$ recovers the ***classifying stack*** *$\mathbb{B}G$.*

Example 4.6.3.3. *(Classification of QCoh^n via $\mathbb{B}\mathrm{GL}_n$, [16],4.32)* Define the **n-th general linear group** scheme by:

$$\mathrm{GL}_n := \mathrm{Map}_{\mathrm{Stack}}(\mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}], -) \in \mathrm{Sch}^{\mathrm{Aff}}$$

More generally, for an arbitrary stack $S \in \mathrm{Stack}$ define the stack $\mathrm{GL}_n(S) := \mathrm{GL}_n \times_{\mathrm{Spec}(\mathbb{Z})} S \in \mathrm{Stack}_{/S}$ by base-change along the canonical map to the point.
Let now $X \in \mathrm{Sch}$ be a scheme. To a locally free sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ of rank n (see 3.6.2.3), it remains associated a GL_n-torsor

$$\pi : \mathrm{Iso}_X(\mathcal{O}_X^n, \mathcal{F}) \to X$$

whose total space represents the core-groupoid of the internal mapping space between $\mathcal{O}_X^n$ and $\mathcal{F}$ in the closed symmetric monoidal ∞-category $\mathrm{QCoh}(X)^{\otimes}$; recall that the action on objects is by restriction along the given relative scheme:

$$\begin{aligned} \mathrm{Map}_{\mathcal{O}_X}\left(\mathcal{O}_X^n, \mathcal{F}\right)^{\simeq}(-) : \left(\mathrm{Sch}_{/X}\right)^{op} &\longrightarrow \mathrm{Spc} \\ (t : T \to X) &\longmapsto \mathrm{Map}_{\mathcal{O}_T}\left(\mathcal{O}_T^n, t^*\mathcal{F}\right)\right)^{\simeq} \end{aligned}$$

Then, functoriality in the second variable determines an equivalence

$$\mathrm{Map}_{\mathcal{O}_X}\left(\mathcal{O}_X^n, -\right)^{\simeq} : \mathrm{QCoh}^n(X) \xrightarrow{\simeq} \mathrm{GL}_n\mathrm{Bund}(X)$$

between n-loc.free quasi-coherent modules over X (see 4.3.2.1) and GL_n-torsors over X. In particular, locally free quasi-coherent modules on X of rank n are classified by maps $X \to \mathbb{B}\mathrm{GL}_n$ in Sch.

4.7 Blow-up of Quasi-Smooth Closed Derived Sub-Schemes

This section is the goal of the project. We have finally developed all the machinery necessary to present the construction of blow-ups of quasi-smooth closed derived sub-schemes. It is due to Khan and Rydh and appeared in the paper [17], which will be our main reference for the topic.
We will start by introducing a relative notion of virtual Cartier divisors on some X-scheme S over some base quasi-smooth closed immersion $i : Z \twoheadrightarrow X$. By varying $S_{/X}$, such objects will then be grouped into a stack of X-schemes, namely $\mathrm{Bl}_Z(X)(-)$. This will turn out to be representable by a classifying moduli-X-scheme, namely the blow-up $\mathrm{Bl}_Z(X)$ of the pair (X, Z); in other words, the latter arises as the X-scheme classifying all the relative virtual Cartier divisors lying over (X, Z).
Finally, in the main result of the dissertation, it will be proven that the X-scheme $\mathrm{Bl}_Z(X)$ enjoys the "derived" version of the same properties as the classical blow-up. Moreover, the construction above is compatible with the classical one, which is retrieved whenever the pair (X, Z) is classical.
Such a geometric construction of blow-ups has then been generalized in subsequent papers by Hekking [13], Hekking-Khan-Rydh [18], in order to encompass all pairs (X, Z) of relative closed schemes and not only quasi-smooth closed immersions. However, due to time constraints, this will not be discussed in our dissertation.

4.7.1 The Blow-Up Stack

Let us start by introducing a relative notion of virtual Cartier divisors. We will provide three equivalent definitions, thus allowing more flexibility in manipulations.

Definition 4.7.1.1. *(Relative Virtual Divisor, [17],4.1.1)* Consider the following square Q in Sch, where $f : S \to X$ is any map in Sch and $i : Z \twoheadrightarrow X$ is a quasi-smooth closed immersion. Then, Q exhibits a **Virtual Cartier Divisor on S lying over** (X, Z) (or "relative divisor" for short) iff it satisfies:

$$\begin{tikzcd} D \arrow[r, "i_D", "\mathrm{VDiv}"'] \arrow[d, "g"'] & S \arrow[d, "\forall f"] \\ Z \arrow[r, "i", "\mathrm{qSmCl}"'] & X \end{tikzcd}$$

(a) $(i_D : D \twoheadrightarrow S) \in \mathrm{VDiv}(S)$ is a virtual Cartier divisor on S;

(b) The underlying classical square Q^{cl} in $\mathrm{Sch}^{\mathrm{cl}}$ is cartesian;

(c) The comparison map $g^*\mathcal{N}_{Z/X} \to \mathcal{N}_{D/S}$ is surjective (on π_0).

Let $\mathrm{VDiv}(S)_{/(X,Z)} \subseteq_{f.f.} \mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^1 \times \Delta^1, \mathrm{Sch})$ denote the subspace of those squares in Sch which satisfy the previous properties.

(Local formulation, [17],4.1.3,ii): Equivalently, we can consider the following set of axioms: let i, f be as before and denote $S_Z := S \times_X Z$; then a virtual Cartier divisor $D \in \mathrm{VDiv}(S)$ on S lying over (X, Z) is a closed relative scheme $(h : D \twoheadrightarrow S_Z) \in \mathrm{Sch}_{/S_Z}$ such that:

$$\begin{tikzcd} D \arrow[rr, "i_D", "\mathrm{VDiv}"'] \arrow[dr, "h"] \arrow[dd, "g"'] & & S \arrow[dd, "\forall f"] \\ & S_Z \arrow[ur, "pr_1"'] \arrow[dl, "pr_2"] & \\ Z \arrow[rr, "i", "\mathrm{qSmCl}"'] & & X \end{tikzcd}$$

(A) The composite $(i_D : D \twoheadrightarrow S_Z \twoheadrightarrow S) \in \mathrm{VDiv}(S)$ is a virtual Cartier divisor on S;

(B) $h : D \to S_Z$ induces an isomorphism $D^{\mathrm{cl}} \cong S_Z^{\mathrm{cl}} \in \mathrm{Sch}^{\mathrm{cl}}$ at the level of classical underlying schemes;

(C) $h^*\mathcal{N}_{S_Z/S} \to \mathcal{N}_{D/S}$ is surjective on π_0 (i.e. $\mathcal{L}_{D/S_Z}$ is 2-connective).

(Connectivity of the square, [17],4.1.3,iii): Equivalently, with notation as in the local formulation above, a virtual Cartier divisor $D \in \mathrm{VDiv}(S)$ on S lying over (X, Z) is a closed relative scheme $(h : D \twoheadrightarrow S_Z) \in \mathrm{Sch}_{/S_Z}$ such that the composite $(i_D = pr_1 \circ h : D \to S) \in \mathrm{VDiv}(S)$ is a virtual Cartier divisor on S and the induced map $h^\flat : \mathcal{O}_{S_Z} \to h_*\mathcal{O}_D$ is 1-connected.

Let us mention that the definition above encompasses all effective Cartier divisors arising as the base-change of a regular immersion of classical schemes along any other map in $\mathrm{Sch}^{\mathrm{cl}}$: this will be the content of 4.6.1.2. More examples will be provided in the homonymous subsection, whereas we start by proving the equivalences of the various formulations.
Notice that in practice we could carry on all the future arguments without the third one, which is perhaps the reason why it was presented as a Remark by Khan and Rydh. However, we opt for including it as part of the definition; this is motivated by Definition [18],5.10 in their subsequent work, in that it better highlights the "technicality" of axioms (B) and (C).

Proof. (Of the equivalence) Let's start with the local rephrasing. The pairs of axioms (a), (A) and (b), (B) are equivalent: the only difference being that the second formulation takes as part of the data a choice of a comparison map $h : D \to S_Z$.
As for the third requirement, let's construct the map of conormal sheaves $g^*\mathcal{N}_i \to \mathcal{N}_{i_D}$ in (c): it is the de-suspension of the lower-left corner in the diagram of exact sequences induced by the two composites $D \to X$ via 4.4.0.3,i:

$$\begin{tikzcd} & i_D^*\mathcal{L}_f \arrow[d] & \\ g^*\mathcal{L}_i \arrow[r] \arrow[dr, dotted] & \mathcal{L}_{D/X} \arrow[r] \arrow[d] & \mathcal{L}_g \\ & \mathcal{L}_{i_D} & \end{tikzcd}$$

The map for (C) is defined in a similar way. Moreover, for $pr_1 := S_Z \to S$ and $pr_2 := S_Z \to Z$ the pull-back projections, the composite $g = h \circ pr_2$ yields a factorization $(g^*\mathcal{L}_i \to \mathcal{L}_{i_D}) \simeq (h^*pr_2^*\mathcal{L}_i \to h^*\mathcal{L}_{pr_1} \to \mathcal{L}_{i_D})$. So, with the obvious change of notation, it suffices to record the following observation.

Claim 1. *The comparison map $g^*\mathcal{L}_i \to \mathcal{L}_{i_D}$ is an isomorphism whenever Q is cartesian.*

Proof. The local definition of the cotangent sheaf (so, 3.8.2.5,i) implies that such a map is an equivalence, since it is such on an affine Zariski cover. ■

Then, let us show that the local formulation is equivalent to the connectivity requirements on the square Q. We will prove the following statement.

Claim 2. *The closed immersion $h : D \twoheadrightarrow S_Z$ corresponds (under the globalized* Spec*-adjunction 4.3.1.1) to a surjection $h^\flat : \mathcal{O}_{S_Z} \to h_*\mathcal{O}_D$ such that:*

- *(B) iff $\pi_0(h^\flat)$ is an isomorphism;*
- *(B) implies the equivalence: (C) iff $\pi_1(h^\flat)$ is surjective.*

Hence, in particular the local formulation above posits connectivity requirements for $h^\flat$:

$$(B) + (C) \iff \pi_0(h^\flat) \text{ iso} \quad \& \quad \pi_1(h^\flat) \text{ epi}$$

Proof. We will refer to the notation adopted in the picture above. By construction, (B) iff $\pi_0(h^\flat)$ is an isomorphism. Moreover, notice that TFAE:

1. $(C) : \pi_1(h^*\mathcal{L}_{pr_1}) \to \pi_1(\mathcal{L}_{i_D})$ is surjective;
2. $\mathcal{L}_h$ is 2-connective;
3. $\pi_0\big(\mathrm{Fib}(h^*(h^\flat))\big) \cong 0$;
4. $\pi_1\big(h^*(h^\flat)\big)$ surjective.

Let's show the equivalence of such statements. (1) $\iff$ (2) : From the exact sequence of 4.4.0.3,i it follows that (1) iff $\pi_1(\mathcal{L}_h) \cong 0$; moreover, $\pi_0(\mathcal{L}_h) \cong 0$ always holds for closed immersions such as h.
(2) $\iff$ (3) : is a consequence of the connectivity properties of the Hurewicz map $\epsilon_{h^\flat}$ (see 3.8.2.6): as we already observed at the beginning of the proof of 4.5.2.3, there is an isomorphism $\pi_1(\mathcal{L}_h) = \pi_1(L_{h^\flat}) \cong \pi_0(\mathrm{Fib}(h^*(h^\flat)))$.
(3) $\iff$ (4) : this is the long exact sequence in homotopy induced by the fibre sequence of $h^*(h^\flat)$.
Now, we want to use the equivalence above to prove that - whenever also (B) holds - the surjectivity of $\pi_1(h^*(h^\flat)) \cong \pi_1(\mathcal{O}_D \otimes^L_{\mathcal{O}_{S_Z}} h^\flat)$ is equivalent to that of $\pi_1(h^\flat)$. Let us reformulate the statement; consider the following cocartesian square exhibiting the "cofibre pair" of $h^\flat$, where we drop h_* because $|D| = |S_Z|$ by (B):

$$\begin{tikzcd}
\mathcal{O}_{S_Z} \arrow[r, "h^\flat"] \arrow[d, "h^\flat"] & \mathcal{O}_D \arrow[d, "h^*(h^\flat)"] \\
\mathcal{O}_D \arrow[r, "h^*(h^\flat)"] & \mathcal{O}_D \otimes^L_{\mathcal{O}_{S_Z}} \mathcal{O}_D
\end{tikzcd}$$

Our claim is that the upper horizontal arrow induces a surjection in π_1 iff the lower horizontal one does. One direction is clear, so let's prove the converse. This is in turn equivalent to showing that e.g. the lower horizontal arrow induces an isomorphism on π_1. Indeed, after applying π_1 the square above becomes the cokernel pair of $\pi_1(h^\flat)$, and a morphism of static modules is surjective iff its cokernel pair is trivial.
So, we are left to show that $\pi_1(h^*(h^\flat))$ is injective; which is equivalent to the vanishing $\pi_0(\mathrm{Cofib}(h^*(h^\flat))) \cong 0$, as one can see by inspection of the long exact sequence in homotopy.
Now, since π_0 is both symmetric monoidal and a left-adjoint, the latter can be written as $\mathrm{coker}(R/I \otimes_R \pi_0(h^\flat))$ for some static ring $R := \pi_0(\mathcal{O}_{S_Z})$ and ideal I cutting out the closed subscheme $\pi_0\mathcal{O}_D$. But then, $\pi_0(h^\flat)$ is surjective, since h is a closed immersion, so we are done. □

Remark. (Obstruction to being cartesian, [17],4.1.3,i) If a virtual Cartier divisor $D \in \mathrm{VDiv}(S)_{/(X,Z)}$ is exhibited by a cartesian square Q as before, then (see Claim 1 in the proof above) the comparison map in Property (c) is an equivalence $g^*\mathcal{N}_{Z/X} \simeq \mathcal{N}_{D/S}$.

On the other hand, in order for Q to be cartesian, D must be the pull-back of a virtual Cartier divisor $(X, Z) \in \mathrm{VDiv}(X)$, because the virtual codimension is stable under base-change as in 4.5.2.6,i.

As our notation suggests, we regard $\mathrm{VDiv}(S)_{/(X,Z)}$ as the following subspace of squares (i.e. morphisms of relative schemes) over (X, Z):

$$\mathrm{VDiv}(S)_{/(X,Z)} \subseteq_{f.f.} \left((\mathrm{Sch}_{/S})_{/(X,Z)}\right)^{\simeq}$$

The latter is in general not a slice of $\mathrm{VDiv}(S)$; however, since h is the obstruction to Q being cartesian, its connectivity properties intuitively measure "how far" this is from being true.

Our next goal is promoting such a construction to a functor $\mathrm{Bl}_Z(X) : (\mathrm{Sch}_{/X})^{op} \to \mathrm{Spc}$. This will be possible by the following Lemma.

Lemma 4.7.1.2. *("Being a relative divisor" is local on the base) For a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$, the conditions (A), (B), (C) are stable under base-change in* Sch *and* Zar*-local on the base X.*

Proof. It will be achieved in two steps; in the proof we will also spell out more precisely what we exactly mean by locality on the base for relative Cartier divisors. By 4.5.2.2,i, there exists an affine Zariski cover $\mathcal{X} := \{j^X_\alpha : X_\alpha \hookrightarrow X\}_\alpha$ such that i is a quasi-smooth closed immersion iff its restrictions $i_\alpha := (j^X_\alpha)^*(i) : Z_\alpha \twoheadrightarrow X_\alpha$ are such, with $Z_\alpha := Z \times_X X_\alpha$. For any map $f : S \to X$ in Sch and any square Q exhibiting a relative virtual Cartier divisor $D \in \mathrm{VDiv}(S)_{/(X,Z)}$, form the following cube by base-change along j^X_α, so with all faces apart from the front and back one being cartesian:

$$\begin{tikzcd}
D_\alpha \arrow[rr, "i_{D_\alpha}"] \arrow[dr] \arrow[dd] & & S_\alpha \arrow[dr] \arrow[dd] & \\
& D \arrow[rr, "i_D"] \arrow[dd] & & S \arrow[dd, "f"] \\
Z_\alpha \arrow[rr, "i_\alpha"] \arrow[dr, "j^Z_\alpha"'] & & X_\alpha \arrow[dr, "j^X_\alpha"'] & \\
& Z \arrow[rr, "i"] & & X
\end{tikzcd}$$

Set $f_\alpha := (j^X_\alpha)^*(f) : S_\alpha \to X_\alpha$, and let Q_α denote the back face, which exhibits the "restriction" to (Z_α, X_α) of the front-face.
Indeed, notice that all maps of the form $T_\alpha \hookrightarrow T$ (for the various values of T at stake) assemble into affine Zariski covers by 4.1.4.8 and C.1.0.8,i. The reader should beware, however, that only the X_α and Z_α need be affine open charts (the latter because i is a closed immersion), unless f is an affine morphism of schemes.

Now, let's check the properties in the second/third formulation.
(A) is local over X by construction, 4.5.2.2,i and 4.5.2.6,i. As for (B) and (C), let $h : D \twoheadrightarrow Z \times_X S$ and $h_\alpha : D_\alpha \twoheadrightarrow Z_\alpha \times X_\alpha S_\alpha$ denote the comparison maps.
By construction, one can verify by the universal property of pull-backs that $h_\alpha \simeq (j^X_\alpha)^*(h)$. Then, since pulling-back along a cover induces a family of jointly conservative base-change functors, the connectivity properties of h (or better of $h^\flat$) are also local over X.

Finally, let's show that we can indeed assume the back face to be consisting of affine schemes.
Indeed, one could repeat a similar procedure with affine Zariski covers $\{j^{S_\alpha}_\beta : S^\alpha_\beta \hookrightarrow S_\alpha\}_\beta$ exhibiting the locality on the base S_α of each statement "i_{D_α} is a quasi-smooth closed immersion". This yields a cube with a trivial bottom face witnessing $i_\alpha = i_\alpha$; paste it to the back face Q_α and call Q^α_β the new back face. As before, the quasi-smooth closed immersions $i_{D^\alpha_\beta} : D^\alpha_\beta \twoheadrightarrow S^\alpha_\beta$ obtained as the base-change $(j^{S_\alpha}_\beta)^*(i_{D_\alpha})$ are clearly virtual Cartier divisors. Moreover, a similar verification yields $h^\alpha_\beta \simeq (j^{S_\alpha}_\beta)^*(h)$ for the comparison maps, so we can conclude as above by $\{j^{S_\alpha}_\beta\}_\beta$ forming a cover of S_α. $\square$

Proposition 4.7.1.3. *(The Blow-Up Stack, [17],4.1.4) Fix a closed immersion $i := (X, Z) : Z \twoheadrightarrow X$ in* Sch*. Define* $\mathrm{Bl}_Z(X)(S \to X) \subseteq_{f.f.} (\mathrm{Sch}_{/S_Z})^{\simeq} \in \mathrm{Spc}$ *to be the subspace of those virtual Cartier divisors on S lying* *over (X,Z). Then, the construction $[(f:S\to X)\mapsto \mathrm{Bl}_Z(X)(f:S\to X)]$ defines the* ***Blow-Up stack****, equipped with the structural morphism $\pi_{Z/X}$:*

$$[\mathrm{Bl}_Z(X):(\mathrm{Sch}^{\mathrm{Aff}}_{/X})^{op}\to \mathrm{Spc}]\in \mathrm{Stack}_{/X} \qquad w/ \qquad \pi_{Z/X}:\mathrm{Bl}_Z(X)\to X$$

Proof. We will adopt the second formulation to construct the presheaf $\mathrm{Bl}_Z(X)$ as a subobject of the following presheaf of spaces, obtained by a choice of the composition $(-)^{\simeq}\circ \mathrm{Sch}_{/(-)}\circ(Z\times_X(-))$:

$$\begin{aligned}\mathcal{F}:\left(\mathrm{Sch}^{op}_{/X}\right)&\longrightarrow \mathrm{Spc}\\ (f:S\to X)&\longmapsto (\mathrm{Sch}_{/S_Z})^{\simeq}\end{aligned}$$

The presheaf $\mathcal{F}$ is obtained by the usual construction via the Straightening Theorem [24],3.2 as in 4.3.2.8. Then, the stability under base-change of the definition above will allow us to regard $\mathrm{Bl}_Z(X)$ as the stated sub-presheaf. Thereafter, the sheaf condition on Zar-covers of S over X will be an automatic consequence of the local nature of our axioms in the second formulation (see 4.7.1.2). In particular, the restriction $\mathrm{Bl}_Z(X):(\mathrm{Sch}^{\mathrm{Aff}}_{/X})^{op}\to\mathrm{Spc}$ will define a stack over X, with structural morphism $\pi_{Z/X}:\mathrm{Bl}_Z(X)\to X$; this classifies then relative Cartier divisors over (X,Z) under the Yoneda Lemma. Let us briefly mention that we do indeed obtain a stack by virtue of the following useful observation.

Remark. ([15],4.7.2) There are canonical equivalences of ∞-categories $\mathrm{Sh}(\mathrm{Sch}_{/X})\simeq \mathrm{Sh}(\mathrm{Sch})_{/X}\simeq \mathrm{Sh}(\mathrm{Sch}^{\mathrm{Aff}})_{/X}$.

Sketch. We just sketch the argument. As for the first equivalence, it is a manipulation with Yoneda Lemma and holds furthermore at the level of presheaves; then, it is preserved by the sheafification, since both the forgetful functor of over-slices commutes with limits and C.5.3.3. For what concerns the second one, instead, the restriction functor induces an equivalence fibre-wise, since schemes admit a small affine Zariski cover: Sch is in the colimit closure of $\mathrm{Sch}^{\mathrm{Aff}}$, so that sheaf condition C.5.3.3 on Sch can be checked on the restriction to $\mathrm{Sch}^{\mathrm{Aff}}$. ■

So, let's construct $\mathrm{Bl}_Z(X)$. Recall that, up to equivalence, the straightened category of $\mathcal{F}$ has objects of the form $(f:S\to X,(h:D\to S_Z)\in\mathcal{F}(f),\mathcal{F}(f))$; then, define $\int\mathrm{Bl}_Z(X)\subseteq_{f.f.}\int\mathcal{F}$ to be the full subcategory spanned by those $h:D\to S_Z$ satisfying the local version of the conditions in 4.7.1.1. This yields the following pasting of cartesian squares, where we let $q:\int\mathrm{Bl}_Z(X)\to(\mathrm{Sch}_{/X})^{op}$ denote the restriction of the straightening projection $\mathrm{St}(\mathcal{F})$.

$$\begin{tikzcd}
\int \mathrm{Bl}_Z(X) \arrow[r, hook] \arrow[d, "q"] & \int \mathcal{F} \arrow[r] \arrow[d, "\mathrm{St}(\mathcal{F})", "\mathrm{LFib}\ni"'] & \mathrm{Spc}_{//*} \arrow[d] \\
(\mathrm{Sch}_{/X})^{op} \arrow[r, equal] & (\mathrm{Sch}_{/X})^{op} \arrow[r, "\mathcal{F}"] & \mathrm{Spc}
\end{tikzcd}$$

By the Straightening Theorem [24],3.2, we are left to prove that also q is a left-fibration. Indeed, then $\mathrm{Bl}_Z(X):=\mathrm{UnSt}(q):(\mathrm{Sch}_{/X})^{op}\to\mathrm{Spc}$ yields the sought presheaf of schemes over X, together with a map $\mathrm{Bl}_Z(X)\to\mathcal{F}$ in $\mathcal{P}(\mathrm{Sch}_{/X})$, which is readily seen to be a monomorphism (check it on fibres, at the level of straightened categories).
Moreover, observe that it suffices to show that q is a cartesian fibration (i.e. cocartesian on $(\mathrm{Sch}_{/X})^{op}$). Indeed, provided that, we can conclude by an application of [20],3.1.22: we constructed $\int\mathrm{Bl}_Z(X)\subseteq_{f.f.}\int\mathcal{F}$ as a fully faithful embedding - so an edge in the source is q-cocartesian iff its copy in the target is $\mathrm{St}(\mathcal{F})$-cocartesian - and a left fibration is a cocartesian one with all edges being cocartesian.
Then, let's prove that each commutative triangle $g:(f':S'/X)\to(f:S/X)$ in $\mathrm{Sch}_{/X}$ with $\mathrm{Bl}_Z(X)(f)\neq\emptyset$ admits a lift in $\int\mathrm{Bl}_Z(X)$; this is then automatically (q-) $\mathrm{St}(\mathcal{F})$-cocartesian.
To this end, notice that the stability under base-change of 4.7.1.1 yields a base-change functor, whose action on objects can be described as follows:

$$\begin{aligned}g^*:\mathrm{Bl}_Z(X)(f)&\longrightarrow \mathrm{Bl}_Z(X)(f')\\ (i_D:D\to S)&\longmapsto (i_{D'}:=g^*(i_D):D':=S'\times_S D\to S')\end{aligned}$$

Since g^* also gives an arrow connecting $g^*(i_D)\to i_D$, we are done. □

Remark. ([17],4.1.6) Defining quasi-smoothness fpqc-locally on the base, we can extend our discussion to define a fpqc-stack $\mathrm{Bl}_Z(X)$ over X classifying virtual Cartier divisors lying over (X, Z). Also the properties of the latter stack generalize *mutatis mutandis*.

Before stating the main Theorem, let us record one more stability property of relative virtual Cartier divisors.

Lemma 4.7.1.4. *(Change of base-pair) Let $i : Z \twoheadrightarrow X$ and $i' : X \twoheadrightarrow Y$ be composable quasi-smooth closed immersions in* Sch. *Then, relative virtual Cartier divisors over (X, Z) are also such over (Y, Z), i.e. over the composition $i' \circ i : Z \twoheadrightarrow Y$.*

Proof. With notation as in the second formulation of 4.7.1.1, let Q be a square exhibiting a virtual Cartier divisor $i_D : D \twoheadrightarrow S$ over $i : Z \twoheadrightarrow X$, and let Q' denote the square obtained by pasting the morphism of arrows $i \to i' \circ i$ to the bottom.
Then, Q' clearly satisfies properties (a) + (b), since pull-backs are preserved by post-composition with monomorphisms and closed immersions are injective at the level of the underlying classical schemes. As for property (c), consider the exact sequence of 4.4.0.3,i associated to the composition $i' \circ i$ and take the base-change along $g : D \to Z$:

$$(i \circ g)^* \mathcal{N}_{i'} \to g^* \mathcal{N}_{i' \circ i} \to g^* \mathcal{N}_i$$

So, we conclude by considering the following composition: $g^* \mathcal{N}_{i' \circ i} \twoheadrightarrow g^* \mathcal{N}_i \twoheadrightarrow \mathcal{N}_{i_D}$, where the second composite is surjective by assumption, while the first one is such by the exactness of the sequence. □

We are now ready to state the main Theorem about Blow-Ups of quasi-smooth schemes. The proof is very articulated and will be the content of the next subsection.

Theorem 4.7.1.5. *(Properties of the Blow-Up stack, [17],4.1.5) For a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$ in* Sch*, the following statements hold true:*

1. qSmCl-Functoriality:
   - (a) *The stack $\mathrm{Bl}_Z(X) \in \mathrm{Stack}_{/X}$ is representable by a scheme.*
   - (b) *The construction $[(Z \twoheadrightarrow X) \mapsto (\pi_{Z/X} : \mathrm{Bl}_Z(X) \to X)]$ on* qSmCl *commutes with base-change in* Sch*, i.e. $\mathrm{Bl}_Z(X) \times_X X' \simeq \mathrm{Bl}_{Z \times_X X'}(X')$.*
   - (c) *The construction $[i \mapsto \pi_i]$ is covariantly functorial in X along* qSmCl*, i.e. for each $X \twoheadrightarrow Y$ in* qSmCl *there exists a canonical quasi-smooth closed immersion $\mathrm{Bl}_Z(X) \twoheadrightarrow \mathrm{Bl}_Z(Y)$ in $\mathrm{qSmCl}_{/Y}$.*
2. Exceptional Divisor:
   - (a) *There is a canonical closed immersion $\mathbb{P}(\mathcal{N}_{Z/X}) \twoheadrightarrow \mathrm{Bl}_Z(X)$ exhibiting the projectivized normal bundle as the universal virtual Cartier divisor lying over (X, Z).*
   - (b) *The structure map $\pi_{Z/X} : \mathrm{Bl}_Z(X) \to X$ is quasi-smooth and proper and induces an equivalence with the base X away from Z: $\mathrm{Bl}_Z(X) \setminus \mathbb{P}_Z(\mathcal{N}_{Z/X}) \xrightarrow{\simeq} X \setminus Z$.*
3. Comparison with classical blow-ups:
   - (a) *For any classical schemes Z^{cl}, $X^{\mathrm{cl}} \in \mathrm{Sch}^{\mathrm{cl}}$, the blow-up $\mathrm{Bl}_{Z^{\mathrm{cl}}}(X^{\mathrm{cl}}) \in \mathrm{Sch}^{\mathrm{cl}}$ is a classical scheme and coincides with the classical construction: $\mathrm{Bl}_{Z^{\mathrm{cl}}}(X^{\mathrm{cl}}) \cong \mathrm{Bl}^{\mathrm{cl}}_{Z^{\mathrm{cl}}}(X^{\mathrm{cl}})$.*
   - (b) *In general, for $\mathcal{I} := \mathrm{Fib}(i^\flat : \mathcal{O}_X \to i_* \mathcal{O}_Z)$ the quasi-coherent ideal-sheaf corresponding to the closed immersion $i : Z \twoheadrightarrow X$, the classical scheme lying under the Blow-Up is $\big(\mathrm{Bl}_Z(X)\big)^{\mathrm{cl}} \cong \mathbb{P}^{\mathrm{cl}}_{X^{\mathrm{cl}}}(\pi_0 \mathcal{I})$, where $\mathbb{P}^{\mathrm{cl}}(-) = C\mathrm{Proj}(C\mathrm{Sym}^*(-))$ denotes the projectivized classical symmetric algebra.*
4. Degenerate cases:
   - (a) *The Blow-Up of X at a virtual Cartier divisor $(i : Z \twoheadrightarrow X) \in \mathrm{VDiv}(X)$ is equivalent to the scheme itself via the structure map $\pi_{Z/X} : \mathrm{Bl}_Z(X) \xrightarrow{\simeq} X$.*
   - (b) *The Blow up of a scheme X at itself (i.e. at $i = 1_X$) is the empty scheme $\mathrm{Bl}_Z(X) \simeq \emptyset$.*

4.7.2 Proof of the Main Theorem

As usual, we will prove the main Theorem in two steps: the affine case will be dealt with directly, whereas we will reduce the general quasi-smooth setting to the latter one.
Nevertheless, let us point out one major difference occurring when dealing with the "derived" affine special case as opposed to the "classical" one, which is due to the peculiarities of the ∞-world. Namely, in the classical setting we exhibit a candidate for the blow-up of $\mathbb{A}^n$ at $\{0\}$ via a description on charts. Here, however, a scheme cannot be constructed by hands: the glueing datum is a diagram with an infinite chain of "higher coherences". Therefore, the existence of the blow-up stack $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ is proven by abstract nonsense as in 4.7.1.3, and only thereafter we investigate its properties; in particular, we will show that the affine Zariski atlas of the classical blow-up $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$ does provide one for the blow-up stack at the inclusion of the origin in the n-th affine space, so that the latter stack turns out to be a classical scheme isomorphic to $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$.

The Affine Case.

There are many analogies with the standard argument in the classical case, for which we refer for instance to [7],IV-17,18. In particular, the ordinary construction hints at an example of a family of classical virtual Cartier divisors on $(\mathbb{A}^n, \{0\})$: the bulk of the work will be then to show that such a family does indeed provide a Zariski atlas for $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$.
Notation. For convenience, we will follow the authors conventions: fix $n \geq 0$ and consider the quasi-smooth closed immersion of the origin $i : \{0\} = \mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)) \twoheadrightarrow \mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n]) = \mathbb{A}^n$; let $Y := \mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ denote its blow-up stack.

Example 4.7.2.1. *(Charts of* $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$*, [17],4.2.1)* For each $1 \leq k \leq n$, define the static rings

$$A_k := \mathbb{Z}[t_1/t_k, \ldots, t_n/t_k, t_k] \subseteq \mathbb{Z}(t_1, \ldots, t_n) \in \mathrm{CRing}$$

and observe that the following squares of classical schemes define points of $\mathrm{VDiv}(\mathrm{Spec}(A_k))_{/(\mathbb{A}^n, \{0\})}$.

$$\begin{tikzcd}
\mathrm{Spec}(A_k/(t_k)) \arrow[r, "\mathrm{VDiv}"'] \arrow[d] & \mathrm{Spec}(A_k) \arrow[d, "\gamma_k"] \\
\mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)) \arrow[r, "i", "\mathrm{qSmCl}"'] & \mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n])
\end{tikzcd}$$

By the construction, for each k the virtual Cartier divisor $\mathrm{Spec}(A_k/(t_k)) \twoheadrightarrow \mathrm{Spec}(A_k)$ is classified by a canonical morphism $j_k : \mathrm{Spec}(A_k) \to Y$.

Proof. Let's check that $\mathrm{Spec}(A_k/(t_k)) \twoheadrightarrow \mathrm{Spec}(A_k)$ is a virtual Cartier divisor over i, namely that it satisfies the axioms in the first version of 4.7.1.1:

- (a) : t_k is a regular element of A_k, so by 3.5.0.3,iii the quotients $A_k /\!\!/ (t_k) \simeq A_k/(t_k)$ are canonically equivalent and the upper row is a quasi-smooth closed immersion.
- (b) : The square is cartesian (in $\mathrm{Sch}^{\mathrm{cl}}$) iff computing the static tensor product gives:

  $$A_k \otimes_{\mathbb{Z}[t_1,\ldots,t_n]} \mathbb{Z}[t_1, \ldots, t_n]/(t_1, \ldots, t_n) \simeq A_k/(t_k)$$

  In order to see this, first first recall the action of the map γ_k between subalgebras of $\mathbb{Z}(t_1, \ldots, t_n)$ as in 2.2.0.5:

  $$\begin{aligned}
  \gamma_k : \mathbb{Z}[t_1, \ldots, t_n] &\longrightarrow \frac{\mathbb{Z}[t_1, \ldots, t_n][y_r]_{r \neq k}}{(\rho_r := t_k y_r - t_r : r \neq k)} = A_k \\
  t_k &\longmapsto t_k \\
  (\forall r \neq k) \qquad t_r &\longmapsto t_r = t_k y_r
  \end{aligned}$$

Then, since $(t_k, \rho_r) = (t_k, t_r)$ for each $r \neq k$, it holds:

$$A_k \otimes_{\mathbb{Z}[t_1,\dots,t_n]} \frac{\mathbb{Z}[t_1,\dots,t_n]}{(t_1,\dots,t_n)} \cong \frac{\mathbb{Z}[t_1,\dots,t_n][y_r]_{r\neq k}}{(t_1,\dots,t_n)+(\rho_r : r\neq k)} \cong \frac{\mathbb{Z}[t_1,\dots,t_n][y_r]_{r\neq k}}{(t_k)+(\rho_r := t_k y_r - t_r : r\neq k)} \cong \frac{A_k}{(t_k)}$$

- (c) : This is 4.8.1.1.

□

Remark. Recall that in 2.2.0.5 we proved that *the sequence* $(\rho_r := t_r - t_k y_r)_{r\neq k} \subseteq \mathbb{Z}[t_r, y_r]_{r\neq k}$ *is regular.* Therefore, 3.5.0.3,iii yields an identification as static $\mathbb{Z}[t_1,\dots,t_n]$-algebras:

$$A_k \cong \frac{\mathbb{Z}[t_1,\dots,t_n][y_r]_{r\neq k}}{(\rho_r := t_k y_r - t_r : r\neq k)} \simeq Z[t_1,\dots,t_n, y_r : r\neq k] \mathbin{/\!\!/} (\rho_r : r \neq k)$$

In particular, notice that the squares above *cannot* be cartesian in Sch, because codim.vir$(\mathbb{A}^n, \{0\}) = n$; the fibre-product in Sch of the angle would then be represented by $A_k \mathbin{/\!\!/} (t_k, t_r = 0 : r \neq k)$, namely it keeps trace of the identifications induced by the relations $\{\rho_r\}_{r\neq k}$ above as higher homotopical data.

Proposition 4.7.2.2. *(Zariski atlas for* $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$*, [17],4.2.2)*

1. *For each* $1 \leq k \leq n$*, the classifying morphism* $j_k : \mathrm{Spec}(A_k) \to Y$ *is a monomorphism (i.e. with* (-1)*-truncated fibres).*
2. *The family* $\{j_k : \mathrm{Spec}(A_k) \hookrightarrow Y\}_{k=1}^n$ *defines an affine cover of the stack* Y*, i.e. the induced morphism* $j : \coprod_k \mathrm{Spec}(A_k) \to Y$ *is an effective epimorphism in* Stack.
3. *The affine cover above is Zariski-open, so the blow-up stack* Y *is a classical scheme isomorphic to the classical blow-up* $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$*.*

Being the proof significantly long, we follow the authors' choice and split it into three parts.

Proof. (Of 4.7.2.2,i, [17],4.2.3) We need to show that the fibres of $\phi_k : \mathrm{Spec}(A_k) \to Y$ over $\mathbb{A}^n$ are (-1)-truncated. Observe first that it suffices to check it over the $\mathrm{Sch}^{\mathrm{Aff}}_{/\mathbb{A}^n}$-points of Y.
Indeed, "being m-truncated" in Sch is a property in Stack which can be checked in PreStack, since the sheafification functor is left-exact. Now, as in A.5.0.7, a map $f : S \to T$ in PreStack is m-truncated iff $\mathrm{Map}(R, f) \in \mathrm{Spc}_{\leq m}$ for each $S \in$ PreStack. Then, the Density Theorem [24],5.1.5.3 allows us to write $R \simeq \mathrm{colim}_{\mathrm{Sch}^{\mathrm{Aff}}_{/R}} for$, so that $\mathrm{Map}(R, f) \simeq \lim_{\mathrm{Sch}^{\mathrm{Aff}}_{/R}} \mathrm{Map}(for, f)$; finally, since taking the limit preserves truncation properties, it suffices to check that $\mathrm{Map}(\mathrm{Spec}(\underline{A}), f)$ be m-truncated for each $\underline{A} \in \mathrm{Ani(CRing)}$.
Thus, we need to prove that, for each affine scheme $S := \mathrm{Spec}(\underline{R}) \in \mathrm{Sch}^{\mathrm{Aff}}$ and each coordinate map $t_i \mapsto f_i : S \to \mathbb{A}^n$, the following functor of spaces is a monomorphism:

$$\theta : \mathrm{Map}_{/\mathbb{A}^n}(S, \mathrm{Spec}(A_k)) \to \mathrm{Map}_{/\mathbb{A}^n}(S, Y) \simeq Y(S \to \mathbb{A}^n)$$

To this end, let us start by obtaining a better homotopical description of both the source and the target.

Claim 1. *Morally, the datum of a map* $S \to \mathrm{Spec}(A_k)$ *over* $\mathbb{A}^n$ *is determined by coordinate maps* $t_k \to f_k$ *and* $t_r/t_k \mapsto f_r/f_k$ *for* $r \neq k$*. More precisely,*

$$\mathrm{Map}_{/\mathbb{A}^n}(S, \mathrm{Spec}(A_k)) \simeq \mathrm{Map}_{Z[t_1,\dots,t_n]/}(A_k, \underline{R}) \simeq \prod_{r\neq k} \mathrm{Fib}_{f_r}(f_k : \underline{R} \to \underline{R})$$

Proof. As already observed in the Remark above, A_k is a static $\mathbb{Z}[t_1,\dots,t_n]$-algebra generated by the indeterminates $\{y_r\}_{r\neq k}$ with regular relations $(\rho_r = t_k y_r - t_r)_{r\neq k}$. Recall that by construction any map $\psi : A_k \to \underline{R}$ factors through the structure map $(t_i \mapsto f_i) : \mathbb{Z}[t_1,\dots,t_n] \to \underline{R}$; then, let $\rho'_r := f_k \psi(y_r) - f_r$ denote the image under ψ of the relation ρ_r. One has the following chain of equivalences in Spc:

$$\begin{aligned}\mathrm{Map}_{\mathbb{Z}[t_1,\dots,t_n]/}(A_k, \underline{R}) &\simeq \mathrm{Map}_{\mathbb{Z}[t_1,\dots,t_n]/}\big(\mathbb{Z}[t_1,\dots,t_n, y_r : r\neq k] \mathbin{/\!\!/} (\rho_r : r \neq k), \underline{R}\big) \\ &\simeq_{(i)} \prod_{r\neq k} \mathrm{Path}_{for\underline{R}}(\rho'_r, 0) \\ &\simeq_{(ii)} \prod_{r\neq k} \mathrm{Fib}_{f_r}(f_k : \underline{R} \to \underline{R})\end{aligned}$$

where (i) is implied by the universal property of quotients 3.5.0.5; while (ii) can be proved as follows. Recall first that the additive group structure on the topological algebra $for\underline{B}$ (see 3.2.1.4) is compatible with paths; hence, for any given map $\psi : A_k \to \underline{R}$ and $\rho'_r := \psi(\rho_r) = f_k\psi(y_r) - f_r$, there is an equivalence $\mathrm{Path}_{for\underline{R}}(\rho'_r, 0) \simeq \mathrm{Path}_{for\underline{R}}(f_k\psi(y_r), f_r)$. In oder words, the relation ρ'_r imposes on $\psi(y_r)$ precisely the condition of being an element $(\psi(y_r) \in for\underline{R}\ ,\ \alpha_r : f_k\psi(y_r) \simeq f_r)$ of the fibre $\mathrm{Fib}_{f_r}(f_k : \underline{R} \to \underline{R})$ of the multiplication map $f_k : \underline{R} \to \underline{R}$ over f_r, as wished.

Hence, maps $S/\mathbb{A}^n \to \mathrm{Spec}(A_k)/\mathbb{A}^n$ can be identified with tuples $a := (a_r, \alpha_r)_{r\neq k}$ of coordinate maps $(a_r)_{r\neq k} \subseteq \underline{R}$ together with paths $(\alpha_r : f_k a_r \simeq f_r \in \mathrm{Path}_{\underline{R}}(f_k a_r, f_r))_{r\neq k}$. ■

Claim 2. *The objects of the target $Y(S \to \mathbb{A}^n)$ can be canonically regarded as pairs consisting of an $\underline{R}$-algebra $\psi : \underline{R} \to \underline{R}'$ endowed with a collection of paths $(\psi(f_r) \simeq 0)_{r\neq k}$ into $for\underline{R}'$.*

Proof. The construction of the blow-up stack yields the following canonical embedding:

$$Y(t_i \mapsto f_i : S \to \mathbb{A}^n) \simeq \mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \leq \left(\mathrm{Sch}^{\mathrm{Aff}}_{/S\times_{\mathbb{A}^n}\{0\}}\right)^{\simeq} \simeq \left(\mathrm{CAlg}^{\Delta}_{\underline{R}/\!\!/(f_1,\ldots,f_n)}\right)^{\simeq}$$

Hence, the objects of $Y(t_i \mapsto f_i : S \to \mathbb{A}^n)$ can be canonically identified with pairs consisting of an R-algebra $\psi : \underline{R} \to \underline{R}'$ together with an $\underline{R}$-algebra morphism $\underline{R} /\!\!/ (f_1, \ldots, f_n) \to \underline{R}'$. Now, the universal property of quotients 3.5.0.5 supplies an identification of each $\underline{R} /\!\!/ (f_1, \ldots, f_n) \to \underline{R}'$ with an object of $\prod_{r\neq k} \mathrm{Map}_{for\underline{R}'}(\psi(f_r), 0)$, whence the stated characterization. ■

Claim 3. *One can describe the action of the functor θ on objects by:*

$$\theta :\ a := (a_r\,,\, \alpha_r : f_k a_r \simeq f_r)_{r\neq k} \longmapsto \theta(a) := (\underline{R} /\!\!/ (f_k)\,,\, \theta(a)_r : f_r \simeq 0)_{r=1}^n$$

where the paths $\theta(a)_r \in \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r, 0)$ are defined as follows:

- $r = k : \theta(a)_k : f_k \simeq 0$ *is the "tautological" path given by the identification between f_k and 0 in $\underline{R} /\!\!/ f_k$;*
- $r \neq k : \theta(a)_r := (\theta(a)_k \cdot a_r) \circ \alpha_r^{-1} : f_r \simeq f_k a_r \simeq 0 \cdot a_r \simeq 0.$

Proof. A virtual Cartier divisor on $S = \mathrm{Spec}(\underline{R})$ must have the form $\mathrm{Spec}(\underline{R} /\!\!/ (f_k)) \in \mathrm{Sch}_{/S\times_{\mathbb{A}^n}\{0\}}$, whence the choice of $\underline{R}/\!\!/(f_k)$ for $\underline{R}'$ - with notation as in Claim 2. Then, we can write $\theta(a) = (\underline{R}/\!\!/(f_k)\,,\, \theta(a)_r : f_r \simeq 0)$, and we are left to specify the paths.

It suffices to exhibit candidates for paths in $\underline{R} /\!\!/ (f_k)$ between the f_r's in 0, since such a datum completely determines (up to homotopy equivalence in the mapping space) an algebra map $\underline{R} /\!\!/ (f_1, \ldots, f_n) \to \underline{R} /\!\!/ (f_k)$.

Now, notice that the algebra structure $\underline{R} \to \underline{R} /\!\!/ (f_k)$ entails the choice of a homotopy $f_k \simeq 0$, call it $\theta(a)_k : f_k \simeq 0$.

Furthermore, pre-composition by the classifying map $\mathcal{O}_Y \to A_k$ preserves the paths induced by $A_k \to \underline{R}$ which correspond then to some homotopies of $\underline{R} /\!\!/ (f_k)$; in particular we still have the α_r's at our disposal and we can perform the compositions above, so as to obtain $\theta(a)_r : f_r \simeq 0$ for each $r \neq k$. ■

We are finally ready to prove our statement, namely that θ is fully faithful. This amounts to the following Claim.

Claim 4. *For each pair of points $a, a' \in \mathrm{Map}_{/\mathbb{A}^n}\big(S, \mathrm{Spec}(A_k)\big)$, the space of identifications of their images under θ can be written as follows:*

$$\mathrm{Map}_{Y(f:S\to\mathbb{A}^n)}\big(\theta(a), \theta(a')\big) \simeq \mathrm{Path}_{Y(f:S\to\mathbb{A}^n)}\big(\theta(a), \theta(a')\big) \simeq \prod_{r\neq k} \mathrm{Map}_{\underline{R}/\!\!/(f_k)}\big(\theta(a)_r\,,\, \theta(a')_r\big)$$

Proof. There is a chain of equivalences in Spc, which can be deduced as argued in the following bullet-list:

$$\begin{aligned}
\mathrm{Map}_{Y(f:S/\mathbb{A}^n)}\big(\theta(a), \theta(a')\big) &\simeq_{(a)} \mathrm{Map}_{\underline{R}/\!\!/(f_1,\ldots,f_n)/}\big(\underline{R} /\!\!/ (f_k), \underline{R} /\!\!/ (f_k)\big) \\
&\simeq_{(b)} \mathrm{Fib}_{\theta(a')}\Big(\theta(a)^* : \mathrm{Map}_{\underline{R}/}\big(\underline{R} /\!\!/ (f_k), \underline{R} /\!\!/ (f_k)\big) \to \mathrm{Map}_{\underline{R}/}\big(\underline{R} /\!\!/ (f_1, \ldots, f_n), \underline{R} /\!\!/ (f_k)\big)\Big) \\
&\simeq_{(c)} \mathrm{Fib}_{(\theta(a'))_r}\Big(\mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_k, 0) \xrightarrow{((\alpha_r^{-1})^*)_r} \prod_{r=1}^{n} \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r, 0)\Big) \\
&\simeq_{(d)} \prod_{r\neq k} \mathrm{Path}_{\mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r,0)}\big(\theta(a)_r, \theta(a')_r\big)
\end{aligned}$$

- (a) : This is Claim 2.
- (b) : Under the identification $\mathrm{Map}_{\underline{R}/\!\!/(f_1,\dots,f_n)/}(\underline{R}/\!\!/(f_1,\dots,f_n),\underline{R}/\!\!/(f_k)) \simeq \{\theta(a')\}$, consider the following cartesian diagram: computing the corresponding tensor product in $\mathrm{CAlg}^{\Delta}_{\underline{R}}$ corresponds to extending the scalars in the right-upper corner along the structure map $\theta(a)$.

$$\begin{tikzcd}
\mathrm{Fib}_{\theta(a')}(\theta(a)^*) \arrow[r] \arrow[d] & \mathrm{Map}_{\underline{R}/}(\underline{R} /\!\!/ (f_k), \underline{R} /\!\!/ (f_k)) \arrow[d, "\theta(a)^*"] \\
\mathrm{Map}_{\underline{R}/\!\!/(f_1,\dots,f_n)/}(\underline{R} /\!\!/ (f_1,\dots,f_n), \underline{R} /\!\!/ (f_k)) \simeq * \arrow[r, "\theta(a')"] & \mathrm{Map}_{\underline{R}/}(\underline{R} /\!\!/ (f_1,\dots,f_n), \underline{R} /\!\!/ (f_k))
\end{tikzcd}$$

- (c) : This is precisely the universal property of quotient rings 3.5.0.5, together with the description above of the action of θ on objects.
- (d) : For each r, the fibre can be computed by the following cartesian square, where we evidence the commutativity 2-simplex (so a suitable homotopy in the lower-right mapping space):

$$\begin{tikzcd}
\mathrm{Fib} \arrow[r] \arrow[d] & \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_k, 0) \arrow[d, "(\alpha_r^{-1})^*"] \arrow[dl, Rightarrow, "h"] \\
\{\theta(a')_r\} \arrow[r] & \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r, 0)
\end{tikzcd}$$

Being paths invertible, we can identify $\mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_k,0) \simeq \mathrm{Path}_{\underline{R}/\!\!/(f_K)}(0,0) \times \{\theta(a)_k\}$ under the post-composition of loops at 0, so that the action of the functor $(\alpha_r^{-1})^*$ can be described on components by

$$(\mathrm{id},(\alpha_r^{-1})^*) : \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(0,0) \times \{\theta(a)_k\} \longrightarrow \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(0,0) \times \{\theta(a)_r\} \simeq \mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r,0)$$

In particular, the essential image of $(\alpha_r^{-1})^*$ is equivalent (via homotopies in $\mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r,0)$, so morphisms of paths) to the point $\{\theta(a)_r\}$. Hence, there is an identification of Fib with the mapping space $\mathrm{Path}_{\mathrm{Path}_{\underline{R}/\!\!/(f_k)}(f_r,0)}\big(\theta(a)_r, \theta(a')_r\big)$ of the two choices of a pointing. Finally, observe that, for $r = k$, the functor α_r is the identity, so the fibre is contractible. ■

Therefore, the functor $\theta : \mathrm{Path}_{\mathrm{Map}_{/\mathbb{A}^n}(S,\mathrm{Spec}(A_k))}(a,a') \to \mathrm{Map}_{Y(S)}(\theta(a),\theta(a'))$ can be identified by Claim 3 with the following map, which is induced by the canonical one between the fibres over f_r and 0 of $f_k : \underline{R} \to \underline{R}$ and $\{0\} \to for\underline{R} /\!\!/ (f_k)$:

$$\prod_{r\neq k} \mathrm{Map}_{\mathrm{Fib}_{f_r}(f_k:\underline{R}\to\underline{R})}\big((a_r,\alpha_r)\,,\,(a'_r,\alpha'_r)\big) \longrightarrow \prod_{r\neq k} \mathrm{Map}_{for\underline{R}/\!\!/(f_k)}\big(\theta(a)_r\,,\,\theta(a')_r\big)$$

and the latter is an equivalence component-wise, as we computed in 3.5.0.6. □

Proof. (Of 4.7.2.2,ii, [17],4.2.4) We need to show that the canonical map $\coprod_k \mathrm{Spec}(A_k) \twoheadrightarrow Y$ is an effective epimorphism in Stack. We already know that $\coprod_{S\in\mathrm{Sch}^{\mathrm{Aff}}_{/Y}} S \to \mathrm{colim}_{\mathrm{Sch}^{\mathrm{Aff}}_{/Y}} for \simeq Y$ is an effective epimorphism by C.1.0.9 and the Density Theorem [24],5.1.5.3; then, by C.1.0.8 it suffices to show that such a map factors as $\coprod_{\mathrm{Sch}^{\mathrm{Aff}}_{/Y}} S \to \coprod_k \mathrm{Spec}(A_k) \to Y$. To this end, we will prove the following Claim.

Claim. *For each map $f : S \to \mathbb{A}^n$ and each relative virtual Cartier divisor $(D \twoheadrightarrow S) \in \mathrm{VDiv}(S)_{/(\mathbb{A}^n,\{0\})}$, there exists a map $S \to \mathrm{Spec}(A_k)$ fitting in the following diagram, where the top square is cartesian:*

$$\begin{tikzcd}
D \arrow[r, "i_D", "\mathrm{VDiv}"'] \arrow[d] \arrow[dd, bend right=60, "g"'] & S \arrow[d, "\exists"] \arrow[dd, bend left=60, "\forall f"] \\
\mathrm{Spec}(A_k/(t_k)) \arrow[r, "\mathrm{VDiv}"'] \arrow[d] & \mathrm{Spec}(A_k) \arrow[d] \\
\{0\} \arrow[r, "i", "\mathrm{qSmCl}"'] & \mathbb{A}^n
\end{tikzcd}$$

Proof. Let it be given f and i_D as in the statement, and write $S = \mathrm{Spec}(\underline{R})$ for some $\underline{R} \in \mathrm{Ani(CRing)}$; let $f : \mathrm{Spec}(\underline{R}) \to \mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n]) = \mathbb{A}^n$ be given by the coordinate maps $(f_1, \ldots, f_n)$. Moreover, let $\mathcal{O}_D \in \mathrm{CAlg}^{\Delta}_{\underline{R}}$ be the animated ring representing the closed subscheme D, which is affine by virtue of 4.2.3.8. Then, by 4.5.2.3 the virtual Cartier divisor i_D admits a locally free conormal sheaf $\mathcal{N}_{i_D} \in \mathrm{Mod}_{\mathcal{O}_D}$ of rank 1. Moreover, observe that we can choose its generator among those of $\mathcal{N}_{\{0\}/\mathbb{A}^n}$, say df_k. Indeed, axiom (c) of Definition 4.7.1.1 yields a surjection on connected components:

$$g^*\mathcal{N}_{\{0\}/\mathbb{A}^n} \simeq \mathcal{O}_D \otimes^L_{\mathbb{Z}} \left(\oplus_{k=1}^n [df_k]\mathbb{Z} \right) \simeq \oplus_{k=1}^n [df_k]\mathcal{O}_D \twoheadrightarrow \mathcal{N}_{i_D}$$

Then, argue as in the proof of 4.5.2.3: it is constructive and allows us to express $D \simeq \mathrm{Spec}(\underline{B} \mathbin{/\!\!/} (f_k))$ on the trivializing open neighbourhoods $\mathrm{Spec}(B) \looparrowright D$ for $\mathcal{N}_{i_D}$. Actually, an adaptation of the last Claim in the aforementioned proof shows that $D \twoheadrightarrow S$ is also globally cut-out by the equation f_k, since the induced morphism $D \to S \times_{\mathbb{A}^n} \{0\} \simeq \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_k))$ is an equivalence.

Let us postpone the details of the adaptation, and focus on the latter composition. The canonical map $h : D \to S \times_{\mathbb{A}^n} \{0\}$ provides a map $h^\flat : \underline{R}/\!\!/(f_1, \ldots, f_n) \to \mathcal{O}_D \simeq \underline{R}/\!\!/(f_k)$ in $\mathrm{CAlg}^{\Delta}_{\underline{R}}$, namely a point in the essential image of the fully faithful functor θ as in 4.7.2.2,1.(Claim 3). By 4.7.2.2,1, the latter corresponds to a map $\mathrm{Spec}(\underline{R}) \to \mathrm{Spec}(A_k)$. Then, the global presentation $D \simeq \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_k)) \simeq \mathrm{Spec}(A_k/(t_k)) \times_{\mathrm{Spec}(A_k)} \mathrm{Spec}(\underline{R})$ means precisely that the upper square in the diagram above is cartesian.

Finally, we show that the map $D \to \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_k))$ is an equivalence, as claimed. We will apply 4.4.0.3,iii. Notice that we can work locally on the base S and on a trivializing neighbourhood for the conormal sheaf $\mathcal{N}_{i_D}$; in other words, in view of the second paragraph we can suppose our map to be of the form $\mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_k)) \to \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{R} \mathbin{/\!\!/} (f_k))$. The given composition clearly induces the identity at the level of the underlying classical schemes: the first composite $\pi_0(h^\flat)$ is an isomorphism by 4.7.1.1,B with the second composite being the inverse on connected components (by the universal property of classical pull-backs). Thus, we are left to prove the vanishing of the cotangent sheaf $L_{\underline{R}/\!\!/(f_k)/\underline{R}/\!\!/(f_k)}$.
To this end, observe that the identity at the level of the underlying classical schemes means that the factorization above is induced by taking the base-change of the composition

$$\mathrm{id} : \mathbb{Z}[x] \xrightarrow{x \mapsto t_k} \mathbb{Z}[t_1, \ldots, t_n] \xrightarrow{\text{quot}} \mathbb{Z}[t_1, \ldots, t_n]/(t_r : r \neq k) \simeq \mathbb{Z}[x]$$

along the coordinate maps $(f_k) : \mathbb{Z}[x] \to \underline{R}$ and $f := (f_1, \ldots, f_n) : \mathbb{Z}[t_1, \ldots, t_n] \to \underline{R}$. Therefore, the cotangent complex vanishes: $L_{\underline{R}/\!\!/(f_k)/\underline{R}/\!\!/(f_k)} \simeq L_{\mathrm{id}} \otimes^L_{\mathbb{Z}[t_k]} \underline{R} \simeq 0$, since it is stable under base-change by 3.8.2.5,i.

□

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Proof. (Of 4.7.2.2,3) Let's show first that the classifying maps $\{j_k : \mathrm{Spec}(A_k) \hookrightarrow Y\}_k$ are indeed open immersions.

In view of 4.6.2.3, we will use the language of generalized divisors. Let $\mathcal{I} \in \mathrm{QCoh}(Y)$ be the locally free ideal-sheaf of rank 1 cutting-out the exceptional divisor E. In other words, $\mathcal{I}$ is the image of the coordinate map $\pi := (p_1, \ldots, p_n) : \mathbb{Z}[t_1, \ldots, t_n] \to \mathcal{O}_Y$ inducing the structure map of the blow-up stack Y.
Indeed, the structure map of the $\mathbb{A}^n$-scheme E is given by some coordinate maps $e := (e_1, \ldots, e_n) : \mathbb{Z}[t_1, \ldots, t_n] \to \mathcal{O}_E$ which lift the $\mathbb{Z}[t_1, \ldots, t_n]$-generators of $\mathcal{O}_E$; now, by assumption (b), $\pi_0\mathcal{O}_E$ sits in the following co-cartesian square of the underlying static quasi-coherent sheaves:

$$\begin{tikzcd}
\mathbb{Z}[t_1, \ldots, t_n] \arrow[r, "t"] \arrow[d] & \pi_0\mathcal{O}_Y \arrow[d] \\
0 \arrow[r] & \pi_0\mathcal{O}_E
\end{tikzcd}$$

which exhibits $\pi_0(\mathcal{O}_Y)/\pi_0\mathcal{I} \cong \pi_0(\mathcal{O}_E) \cong \mathrm{Coker}(t) \simeq \pi_0(\mathcal{O}_y)/\mathrm{Im}(t)$, so that the local generator of $\mathcal{I}$ is given by the isomorphism $\pi_0\mathcal{I} \cong \mathrm{Im}(t)$ and we obtain the claimed universal factorization $\mathbb{Z}[t_1, \ldots, t_n] \to \mathcal{I} \to \mathcal{O}_Y$.

Consider the maps $\mathbb{Z}[t_k] \to \mathbb{Z}[t_1, \ldots, t_n] \to \mathcal{I}$, where the first map is the canonical inclusion $t_k \mapsto t_k$, and let $V_k := D_Y(p_k)$ denote its non-vanishing locus in Y. We claim that $V_k \simeq \mathrm{Spec}(A_k)$. By 4.7.2.2,ii, there is a map $V_k \to \mathrm{Spec}(A_k)$, so - in view of [20],2.2.2 and of the classifying property of Y - it suffices to show that the two Y-stacks have the same functor of points as sub-stacks of $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)(-)$. Let's unwind the definition of $V_k(S) \subseteq \mathrm{VDiv}(S)_{/(\mathbb{A}^n,\{0\})}$ at any affine scheme $S \in \mathrm{Sch}_Y$. Any map $\phi : S \to V_k$ corresponds to some relative virtual Cartier divisor $i_D : D \twoheadrightarrow S$ and can be described by an n-tuple of coordinate maps $\phi := (\phi_1, \ldots, \phi_n) : S \to \mathbb{A}^n$ which sit in a diagram:

$$\begin{tikzcd}
\mathbb{Z}[t_k] \arrow[r, hook, "t_k"] \arrow[dr, "\phi_k"'] & \mathbb{Z}[t_1, \ldots, t_n] \arrow[r] \arrow[d, "\phi"] & \mathcal{I} \arrow[dl, dotted, "\simeq"] \\
& \mathcal{I}_D &
\end{tikzcd}$$

with $\mathcal{I} \to \mathcal{I}_D$ an equivalence; here we denoted by $\mathcal{I}_D$ the ideal-sheaf defining $i_D : D \twoheadrightarrow S$. Arguing as at the very beginning, the identification of the two ideal-sheaves corresponds to an isomorphism $D \to V_S(\phi_k)$ expressing the relative virtual Cartier divisor D as the vanishing locus on S of the coordinate map ϕ_k.
So, the condition $(i_D : D \twoheadrightarrow S) \leftrightarrow (\phi : S \to V_k) \in V_k(S)$ is equivalent to i_D sitting in the following cartesian square:

$$\begin{tikzcd}
D = V_S(\phi_k) \arrow[r, "i_D"] \arrow[d] & S \arrow[d, "\phi"] \\
\mathrm{Spec}(A_k/(t_k)) \arrow[r] & \mathrm{Spec}(A_k)
\end{tikzcd}$$

where we again let $\phi : S \to \mathrm{Spec}(A_k)$ denote the factorization of the coordinate maps $\phi : S \to \mathbb{A}^n$; this is due to the fact that the condition $D \simeq V_S(\phi_k)$ means precisely that ϕ_k lifts a local generator $d\phi_k$ of the fibre of $\mathcal{O}_S \to \mathcal{O}_D$, and hence of the conormal sheaf $\mathcal{N}_{i_D}$, so we conclude by the above 4.7.2.2,ii. ■

Therefore, by 4.7.2.2,1-2 we obtain an affine Zariski cover $\{j_k : \mathrm{Spec}(A_k) \hookrightarrow Y\}_k$ for $Y \in \mathrm{Stack}$, so that the latter must indeed be a scheme. Furthermore, it is classical, since the cover consists of classical schemes. Indeed, consider the fibre sequence induced by the effective epimorphism $\coprod \phi_k$:

$$F \hookrightarrow \coprod \mathrm{Spec}(A_k) \twoheadrightarrow Y$$

The fibre of an effective epimorphism is -1-connected by definition, and it is also 0-truncated, since it is the subobject of a classical scheme. Thus, also F is classical, which forces Y to be such.
Moreover, $\{\mathrm{Spec}(A_k)\}_k$ is the standard cover for the classical blow-up $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$ of the n-th affine space at the origin; hence, the two schemes are isomorphic in $\mathrm{Sch}^{\mathrm{cl}}$. Indeed, $Y \cong \cup\mathrm{Spec}(A_k) \cong \mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$ in $\mathrm{Sch}^{\mathrm{cl}}$; in our language, this is an instance of a more general topos-theoretic fact: by C.1.0.8,iii, base-change along an effective epimorphism is conservative. □

Remark. (Universal property of $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$*, [17],4.2.6)* Noteworthy is that, provided it be classical, then by construction Y satisfies the same universal property as $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$, see 2.2.0.1: given any map $f : S^{\mathrm{cl}} \to \mathbb{A}^n$ in $\mathrm{Sch}^{\mathrm{cl}}$ such that the schematic fibre $f^{-1}(\{0\}) \in \mathrm{Sch}^{\mathrm{cl}} \in \mathrm{CaDiv}(S^{\mathrm{cl}})$ exhibits a classical Cartier divisor on S^{cl}, then the space $\mathrm{VDiv}(f)_{/(\mathbb{A}^n,\{0\})} = Y(f) \simeq \mathrm{Map}_{/\mathbb{A}^n}(S, Y)$ is contractible; in particular, there exists a unique map $S^{\mathrm{cl}} \to Y$ in $\mathrm{Sch}^{\mathrm{cl}}_{/\mathbb{A}^n}$, namely the classifying morphism for $f^{-1}(\{0\})$.
Indeed, Y classifies relative virtual Cartier divisors over $(\mathbb{A}^n, \{0\})$, and these include classical relative Cartier divisors by 4.6.1.2. Moreover, being the blow-up scheme Y classical implies that, for any $f : S^{\mathrm{cl}} \to \mathbb{A}^n$ in $\mathrm{Sch}^{\mathrm{cl}}$ as before, the space $\mathrm{VDiv}(f)_{/(X,Z)} \simeq \mathrm{Hom}_{/\mathbb{A}^n}(S^{\mathrm{cl}}, Y)$ is indeed static (i.e. a set); hence, we conclude by condition (B) in Definition 4.7.1.1.

Proof of the Main Theorem.

The proof is again very articulated, so we will split it accordingly to the claims stated in 4.7.1.5. However, the order will not always be maintained: the statements are about the representing schemes, whereas e.g. in the first paragraph we will provide proofs for the blow-up stacks. Let us start by a short summary of the strategy which will be adopted in the proof.

As a prerequisite, we will show first that the construction of the blow-up stack on qSmCl commutes with base-change in Sch. Then, we will address the question of whether the blow-up X-stack over (X, Z) - which was constructed in 4.7.1.3 - can be represented by a scheme. Since the statement Zar-local on the base (by 4.7.1.2) and since quasi-smooth closed immersions are defined to be zero-loci of coordinate maps, the preliminary step allows us to reduce it to the affine case $(X, Z) = (\mathbb{A}^n, \{0\})$: there it has already been answered affirmatively by 4.7.2.2,3.
Thereafter, we will exhibit the "universal" relative virtual Cartier divisor $\mathbb{P}(\mathcal{N}_{Z/X}) \twoheadrightarrow \mathrm{Bl}_Z(X)$ over (X, Z): being the statement local and stable under base-change, it can be reduced to relative divisors over the inclusion of the origin in the n-th affine space; there the closed immersion of the classical exceptional divisor into the blow-up supplies a candidate and the universality part will amount to the classical universal property of $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$.
Then, we will move on to the properties in 4.7.1.5,2.b of the universal structure map $\pi_{Z/X}$, and this will imply also 4.7.1.5,1.c.
Once the machinery is set up, it is meaningful to compare it with the classical construction and prove statements 4.7.1.5,3.a-b; finally, the degenerate cases in 4.7.1.5,4.a-b are almost trivial consequences of the definition 4.7.1.1 of relative virtual Cartier divisors.

1.(a)-(b). **Functoriality properties of the blow-up construction.**

Proof. (Of 4.7.1.5,1.b) The construction on qSmCl expressed in the statement is functorial in the following sense, which amounts to the claim.
Let $(i : Z \twoheadrightarrow X) \in$ qSmCl be a quasi-smooth closed immersion of schemes, $p : X' \to X$ any morphism in Sch, and $(i' : Z' \twoheadrightarrow X') \in$ qSmCl be the base-change of i along p (recall 4.5.2.2,i). Observe that base-change along p induces a natural equivalence of functors $i^*(-) \to (i')^*(-)$, and hence one of prestacks $\big(\mathrm{Sch}_{/(-)\times_X Z}\big)^{\simeq} \to \big(\mathrm{Sch}_{/(-)\times_{X'} Z'}\big)^{\simeq}$ with quasi-inverse post-composition by the morphism of relative schemes $(p, p^*(i))$. Since p^* preserves relative virtual Cartier divisors, the composite of their restrictions gives a canonical natural equivalence $\mathrm{Bl}_{Z'}(X')(-) \to \mathrm{Bl}_Z(X)(-) \to \mathrm{Bl}_Z(X)(-) \times_X X'$ of sub-stacks of $\big(\mathrm{Sch}_{/(-)\times_{X'} Z'}\big)^{\simeq}$. In other words, we proved the following statement: *([17],4.3.2) The canonical morphism* $\mathrm{Bl}_{Z'}(X') \to \mathrm{Bl}_Z(X) \times_X X'$ *in* Stack *is an equivalence.* □

Proof. (Of 4.7.1.5,1.a) We will reduce the problem to the affine case via two reduction steps.

Claim 1. *The statement is* Zar*-local on the base X (see 4.1.4.1). Hence, we can assume wlog i to be of the form* $\mathrm{Spec}(\underline{A}) \times_{\mathbb{A}^n} (\{0\} \twoheadrightarrow \mathbb{A}^n)$ *for some animated ring* $\underline{A} \in \mathrm{Ani(CRing)}$.

Proof. This can be argued as follows. By 4.5.2.2,i, being a quasi-smooth closed immersion is local on the base, so let it be given affine Zariski covers $\mathcal{X} := \{j^X_\alpha : X_\alpha \hookrightarrow X\}_\alpha$ of X and $\mathcal{Z} := \{j^Z_\alpha : Z_\alpha \hookrightarrow Z\}_\alpha$ of Z on which the restriction of i can be written as $i_\alpha := i_{|Z_\alpha} := (j^X_\alpha)^*(i) : Z_\alpha \twoheadrightarrow X_\alpha$. Assume that, for each α, there exists some $Y_\alpha \in \mathrm{Sch}_{/X_\alpha}$ representing the blow-up stack at i_α, i.e. such that $\mathrm{Bl}_{Z_\alpha}(X_\alpha)(-) \simeq \mathrm{Map}_{/X_\alpha}(-, Y_\alpha)$ in $\mathrm{Sch}_{/X_\alpha}$.
As already noticed, "being a relative Cartier divisor over (X, Z)" is Zar-local on the base X in view of the second formulation of 4.7.1.1, i.e. informally - and with abuse of notation in the choice of the index α - there is an isomorphism $\mathrm{colim}_\alpha \mathrm{Bl}_{Z_\alpha}(X_\alpha) \simeq \mathrm{Bl}_Z(X)$ of stacks over X; more precisely, the diagram of blow-up stacks over X is induced by the functoriality of 4.7.1.5,1.b with respect to the intersections among affine charts of X or of Z - so on the product of the Čech nerves of the covers $\check{C}(\mathcal{X}) \times \check{C}(\mathcal{Y})$. Then, we are left to show that, for $Y := \mathrm{colim}\, Y_\alpha := \mathrm{colim}\, \check{C}(\coprod Y_\alpha \times \coprod Y_\alpha \to \coprod Y_\alpha)$, the canonical map

$$\mathrm{colim}_\alpha \big(\mathrm{Map}_{/X_\alpha}(-, Y_\alpha)\big)(-) \xrightarrow{\simeq} \mathrm{colim}_\alpha \big(\mathrm{Map}_{/X}(-, Y_\alpha)\big) \longrightarrow \mathrm{Map}_{/X}(-, Y)$$

is an equivalence of stacks over X. Notice first that the first map is an equivalence, because we are post-composing with monomorphisms $X_\alpha \hookrightarrow X$.
Then, consider the composite. By [20],2.2.2, the equivalence can be tested point-wise by evaluation at any $f : S \to X$: for each α, set $S_\alpha := (j^X_\alpha)^*(S)$ and $f_\alpha := (j^X_\alpha)^*(f) : S_\alpha \to X_\alpha$; then, our map of spaces becomes the following canonical one, which is then clearly an equivalence:

$$\mathrm{colim}_\alpha \big(\mathrm{Map}_{/X_\alpha}(-, Y_\alpha)\big)(S) \simeq \mathrm{colim}_\alpha \big(\mathrm{Map}_{/X_\alpha}(S_\alpha, Y_\alpha)\big) \to \mathrm{Map}_{/X}(S, Y) \qquad \blacksquare$$

Claim 2. *We can assume wlog $i : \{0\} \twoheadrightarrow \mathbb{A}^n$ to be the inclusion of the origin in the n-th affine space.*

Proof. Let $0 : \{0\} \twoheadrightarrow \mathbb{A}^n$ denote the inclusion of the origin. In view of Claim 1, wlog $i : \mathrm{Spec}(\underline{A}) \times_{\mathbb{A}^n} 0$. Now, by 4.7.1.5,1.b one has $\mathrm{Bl}(i) \simeq \mathrm{Bl}(0) \times_{\mathbb{A}^n} \mathrm{Spec}(\underline{A})$, and the claim follows by the fact that mapping spaces commute with limits (hence fibre-products) in the covariant component. ■

Finally, by the affine case 4.7.2.2, we know that the blow-up stack $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ is represented by the classical scheme $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$. □

Remark. In particular, 4.7.1.5,1.b becomes a precise mathematical statement in terms of the schemes representing the corresponding blow-up stacks; this motivates the order of the statements in the main theorem. Moreover, as we have just shown their compatibility with base-change is equivalent to that at the stack level.

2.(a)-(b), 1.(c). **The universal relative Cartier divisor.**

Proof. (Of 4.7.1.5,2.a) Let $\pi_{Z/X} : \mathrm{Bl}_Z(X) \to X$ denote the blow-up X-scheme of 4.7.1.5,1.a. Then, by the usual "Yoneda-like argument", one has the following universal data:

- the identity morphism $\mathrm{Bl}_Z(X) \to \mathrm{Bl}_Z(X)$ classifies the "universal virtual Cartier divisor" lying over (X, Z); call it $D^{\mathrm{univ}}_{Z/X}$;
- it comes equipped with a canonical map $\pi_{\mathrm{univ}} : D^{\mathrm{univ}}_{Z/X} \to Z$ together with a canonical line bundle $\mathcal{L}^{\mathrm{univ}}_{Z/X} := \mathcal{N}_{D^{\mathrm{univ}}_{Z/X}/\mathrm{Bl}_Z(X)}$;
- condition (c) of 4.7.1.1 is then expressed by a canonical surjection $\pi^*_{\mathrm{univ}}\mathcal{N}_{Z/X} \twoheadrightarrow \mathcal{L}^{\mathrm{univ}}_{Z/X}$.

This is classified by a canonical morphism $D^{\mathrm{univ}}_{Z/X} \to \mathbb{P}_Z(\mathcal{N}_{Z/X})$ in $\mathrm{Sch}_{/Z}$. We will prove in three steps that it is invertible.

Claim 1. *The statement "$D^{\mathrm{univ}}_{Z/X} \to \mathbb{P}_Z(\mathcal{N}_{Z/X})$ in $\mathrm{Sch}_{/Z}$ is invertible" can be checked* Zar*-locally over X and is stable under base-change.*

Proof. Recall that, as in 4.7.1.5,1.a, the blow-up stack (and the construction supplying for its representing object) is Zar-local over X in the following sense. Being a quasi-smooth closed immersion is local on the base, so let it be given affine Zariski covers $\mathcal{X} := \{j^X_\alpha : X_\alpha \hookrightarrow X\}_\alpha$ of X and $\mathcal{Z} := \{j^Z_\alpha : Z_\alpha \hookrightarrow Z\}_\alpha$ of Z on which the restriction of i can be written as $i_\alpha := i_{|Z_\alpha} := (j^X_\alpha)^*(i) : Z_\alpha \twoheadrightarrow X_\alpha$. Then, as above, there is an isomorphism $\mathrm{colim}_\alpha \mathrm{Bl}_{Z_\alpha}(X_\alpha) \simeq \mathrm{Bl}_Z(X)$ both as X-stacks and representing X-schemes. In other words, the "classifying process" is Zar-local on the base Z.
Then, also the universal virtual Cartier divisor $D^{\mathrm{univ}}_{Z/X}$ over (X, Z) together with the relative data $\pi_{\mathrm{univ}}, \mathcal{L}^{\mathrm{univ}}_{Z/X}$ are local on the base Z, since they are classified by the identity of $\mathrm{Bl}_Z(X)$, which is in turn determined Zar-locally on Z by the identities.
The locality of the conormal sheaf $\mathcal{N}_{Z/X}$ follows from the very definition 4.4.0.5, so in addition we obtain also that of the surjection $\pi^*_{\mathrm{univ}}(\mathcal{N}_{Z/X}) \to \mathcal{L}^{\mathrm{univ}}_{Z/X}$. Finally, the construction of the projective Z-scheme $\mathbb{P}_Z(-)$ is local on the base Z (see 4.3.2.11), hence so is the classifying morphism $D^{\mathrm{univ}}_{Z/X} \to \mathbb{P}_Z(\mathcal{N}_{Z/X})$.
In view of 4.7.1.5,1.b, a similar argument shows the stability under base-change. ■

Now, locality allows us to reduce the statement to a base-change of $\{0\} \twoheadrightarrow \mathbb{A}^n$, which can then be neglected by virtue of the aforementioned stability properties. So, we are left to prove it in the affine case.

Claim 2. *$D^{\mathrm{univ}}_{Z/X} \to \mathbb{P}_Z(\mathcal{N}_{Z/X})$ in $\mathrm{Sch}_{/Z}$ is invertible for $(X, Z) = (\mathbb{A}^n, \{0\})$.*

Proof. As proven in 4.7.2.2,3, recall that $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \cong \mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n)$ is a classical scheme, and its fibre-product in $\mathrm{Sch}^{\mathrm{cl}}$ along the inclusion of the origin $\{0\} \twoheadrightarrow \mathbb{A}^n$ is isomorphic to the exceptional divisor. Now, we observed right below the Construction 2.2.0.6 that the latter is in turn isomorphic to the (classical) projectivization of the normal bundle: $E_{\{0\}}(\mathbb{A}^n) = \mathbb{P}^{\mathrm{cl}}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbb{A}^n}) \simeq \mathbb{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbb{A}^n})$, which coincides with the "derived" construction in Sch whenever it involves only classical objects (as remarked in our brief digression 4.3.2.11). Then, an application of the universal property of $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ - see right-below the proof of 4.7.2.2,3 - with

$S^{\mathrm{cl}} = \mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$, $f = \pi_{\{0\}/\mathbb{A}^n}$ and classifying map $\mathrm{id}_{\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)}$ implies that $\mathbb{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbb{A}^n}) \twoheadrightarrow \mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ is the unique virtual Cartier divisor on $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ over $(\mathbb{A}^n, \{0\})$, and it is classified by the identity. This forces the desired equality $D^{\mathrm{univ}}_{\{0\}/\mathbb{A}^n} = \mathbb{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbb{A}^n})$. ■ □

Construction 4.7.2.3. *(Universal property of Blow-ups)* To sum up, we proved that there is a canonical closed immersion $\mathbb{P}_Z(\mathcal{N}_{Z/X}) \twoheadrightarrow \mathrm{Bl}_Z(X)$ which exhibits the projectivized normal bundle as the universal virtual Cartier divisor lying over (X, Z), so as the exceptional divisor of the blow-up.
More precisely, let us spell out how the classification occurs. Let it be given a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$ and let $Y := \mathrm{Bl}_Z(X)$ and $E := \mathbb{P}_Z(\mathcal{N}_{Z/X})$ denote the Blow-up stack over i and the exceptional divisor, respectively. Then, we proved that - for any X-scheme $f : S \to X$ - the universal relative virtual Cartier divisor $E(f) \in \mathrm{VDiv}(Y(f))_{/(X,Z)}$ induces the following correspondence in Spc:

$$\mathrm{Map}_{/X}\big(S\,,\, Y\big) \longrightarrow \mathrm{Bl}_Z(X)(f : S \to X) \leq \big(\mathrm{Sch}_{/S_Z}\big)^{\simeq}$$
$$(\phi : S \to Y) \longmapsto (D := \phi^*(E(f)), h : D \twoheadrightarrow S_Z)$$

$$\begin{tikzcd}
D \arrow[r, dotted] \arrow[d, dotted, "h"'] \arrow[dr, dotted, "i_D"] & E(f) \arrow[r, "i_{\mathrm{univ}}"] \arrow[ddl, bend left] & Y(f) \arrow[dd, bend left, "\pi_{Z/X}"] \\
S_Z \arrow[r, "pr_2"] \arrow[d] & S \arrow[ur, dotted, "\phi"'] \arrow[d, "f"] & \\
Z \arrow[r, "i"] & X &
\end{tikzcd}$$

In other words, relative effective Cartier divisors over (X, Z) arise as base-changes of the universal one along classifying morphisms into the blow-up.

Proof. (Of the Construction) The construction above clearly gives a well-defined fully faithful functor, so we are left to show the essential surjectivity of the assignment. In other words, we wish that, for any given $\phi : S \to Y(f)$, the classified relative virtual Cartier divisor $i_D : D \twoheadrightarrow S$ over (X, Z) sits in a cartesian square exhibiting the pull-back of the angle $(\phi, i_{\mathrm{univ}})$. But this is a direct consequence of the construction of the stack $\mathrm{Bl}_Z(X)(-)$: being $\phi = \mathrm{id}_{Y(f)} \circ \phi$, we can regard $\mathrm{Bl}_Z(X)(\phi) \simeq \phi^*\mathrm{Bl}_Z(X)(\pi_{Z/X})$, as desired. □

Proof. (Of 4.7.1.5,2.b) By 4.5.2.2,i, 4.1.4.22, and C.1.0.8,i, the claim is local on the base X and stable under base-change. Hence, we can reduce it to $(X, Z) = (\mathbb{A}^n, \{0\})$. The properties needed can be proven in $\mathrm{Sch}^{\mathrm{cl}}$, so we refer to both 2.2.0.7 and 2.2.0.8; nevertheless, let us summarize our setting in the following Claim.

Claim. *Let $i : \{0\} \twoheadrightarrow \mathbb{A}^n$ be the inclusion of the origin. Then, the structure map $\pi_{\{0\}/\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)} : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \to \mathbb{A}^n$ is quasi-smooth and proper and induces an equivalence with the base $\mathbb{A}^n$ away from $\{0\}$: $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \setminus \mathbb{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbb{A}^n}) \xrightarrow{\simeq} \mathbb{A}^n \setminus \{0\}$.* □

Proof. (Proof of 4.7.1.5,1.c) Let $i : Z \twoheadrightarrow X$ and $i' : X \twoheadrightarrow Y$ be two composable quasi-smooth closed immersions in Sch. Recall that, by 4.7.1.4, a relative virtual Cartier divisor over $i : (X, Z)$ can be regarded as lying over $i' \circ i : (Y, Z)$. In particular this applies to the universal one on $\mathrm{Bl}_Z(X)$, which can be exhibited as a relative virtual Cartier divisor over (Y, Z) by the following square:

$$\begin{tikzcd}
\mathbb{P}_Z(\mathcal{N}_{Z/X}) \arrow[r] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d, "i' \circ \pi_{Z/X}"] \\
Z \arrow[r] & X
\end{tikzcd}$$

The latter is then classified by a canonical map $\mathrm{Bl}_Z(X) \twoheadrightarrow \mathrm{Bl}_Z(Y)$ over Y, which is a closed immersion, since the structure maps of blow-ups are such. So, we are left to show its quasi-smoothness. In view of 4.5.2.2,i, it suffices to argue locally over a cover of the base Y, say $\{Z, Y \setminus Z\}$.

- Over $Y \setminus Z$: by 4.7.1.5,2.b, the map becomes $i'_{|X\setminus Z} : X \setminus Z \simeq \mathrm{Bl}_Z(X) \to \mathrm{Bl}_Z(Y) \simeq Y \setminus Z$, which is the restriction of the quasi-smooth closed immersion i';

- Over Z: by the universal property of blow-ups 4.7.2.3, the classifying map restricts to a closed immersion between the projectivized conormal sheaves $\mathbb{P}_Z(\mathcal{N}_{Z/X}) \twoheadrightarrow \mathbb{P}_Z(\mathcal{N}_{Y/X})$; hence, it is quasi-smooth by 4.5.2.4, because both the source and the target are such over Z. □

3.(a)-(b). **Comparison with Classical Blow-Ups.**

Proof. (Of 4.7.1.5,3.a) Let $i : Z \twoheadrightarrow X$ be a quasi-smooth closed immersion in $\mathrm{Sch}^{\mathrm{cl}}$, i.e. a regular immersion of classical schemes. Both the constructions $[(X, Z) \mapsto \mathrm{Bl}_Z(X)]$ and $[(X, Z) \mapsto \mathrm{Bl}^{\mathrm{cl}}_Z(X)]$ are Zar-local over the base X in the same sense. Hence, we can reduce wlog to $X = \mathrm{Spec}(R)$, $Z = \mathrm{Spec}(R /\!\!/ (f_1, \ldots, f_n)) \cong \mathrm{Spec}(R/(f_1, \ldots, f_n))$ for some static ring $R \in \mathrm{CRing}$ and some regular sequence $(f_1, \ldots, f_n) \subseteq R$ (see 3.5.0.3,iii) such that Z ranges in a trivializing atlas for the conormal sheaf $\mathcal{N}_{Z/X}$. Adopt the following shorthand for tuples: $f := (f_1, \ldots, f_n)$ and $y^k := (y_r : r \neq k)$. The comparison will be established in two Lemmas: we first prove that $\mathrm{Bl}_Z(X) \in \mathrm{Sch}^{\mathrm{cl}}$ is classical, and then that it agrees with the classical construction $\mathrm{Bl}^{\mathrm{cl}}_Z(X)$.

Claim 1. *Let $\rho'^k : \mathbb{A}^n \to R[y]$ denote the n-tuple of coordinate functions $\rho_r := f_k y_r - f_r$ for $r \neq k$. Then, $\{\mathrm{Spec}(R[y^k]/(\rho'^k))\}_{k=1}^n$ forms a* classical *affine Zariski atlas for the blow-up scheme* $\mathrm{Bl}_Z(X)$.

Proof. 4.1.3.3 yields the claim with each $R[y^k]/(\rho'^k)$ replaced by $R[y^k] /\!\!/ (\rho'^k)$. Finally, observe that they are isomorphic: the sequence f is regular by assumption, so that also ρ'^k is such by an analogous argument to 2.2.0.5. Hence, the affine Zariski cover - and hence the blow-up of (Z, X) - is classical. ■

Moreover, such a cover $\{\mathrm{Spec}(R[y^k]/(\rho'^k))\}_k$ is also an affine Zariski cover of the classical blow-up. Indeed, the latter can be written as $\mathrm{Bl}^{\mathrm{cl}}_Z(X) \cong \mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n) \times_{\mathbb{A}^n} X$ by 2.2.0.3,i, and we showed in 4.7.2.2,iii that there is an isomorphism $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n) \cong \mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$.

□

Proof. (Of 4.7.1.5,3.b) From the very definition, quasi-smooth closed immersions arise as base-changes of the form $(i : Z \twoheadrightarrow X) \simeq X \times (\{0\} \twoheadrightarrow \mathbb{A}^n)$ along some coordinate maps on X (see 3.5.0.1).
Now, by 4.7.1.5,1.b the construction of the blow-up scheme is stable under base-change, so that $\mathrm{Bl}_Z(X) \simeq \mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \times_{\mathbb{A}^n} X$ in Sch. At the level of the underlying classical schemes, we obtain a base-change in $\mathrm{Sch}^{\mathrm{cl}}$: $\left(\mathrm{Bl}_Z(X)\right)^{\mathrm{cl}} \cong \mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n) \times^{\mathrm{cl}}_{\mathbb{A}^n} X^{\mathrm{cl}}$, because in 4.7.2.2,iii we already proved that $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n) \cong \mathrm{Bl}_{\{0\}}(\mathbb{A}^n) \simeq \left(\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)\right)^{\mathrm{cl}}$.
Moreover, the Claim is stable under base-change in $\mathrm{Sch}^{\mathrm{cl}}$, since the construction $[(i : Z \twoheadrightarrow X) \mapsto \pi_0 \mathrm{Fib}(i^\flat : \mathcal{O}_X \to i_* \mathcal{O}_Z)]$, the classical $C\mathrm{Sym}^*(-)$ and the classical $C\mathrm{Proj}$ are such.
Therefore, it can be reduced to the case of $(\mathbb{A}^n, \{0\})$, namely we have to show that $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}}(\mathbb{A}^n) \cong \mathrm{Proj}_{\mathbb{A}^n}(C\mathrm{Sym}^*(I))$, for $I = (t_1, \ldots, t_n)\mathbb{Z}[t_1, \ldots, t_n]$ the ideal describing the inclusion of the origin. This holds true, as observed in 2.2.0.10. □

Remark. ([17],4.3.9) As observed by Khan and Rydh, the blow-up $\mathrm{Bl}_Z(X)$ of a quasi-smooth closed immersion $i : D \twoheadrightarrow X$ cut-out by $\mathcal{I}$ is equivalent to the projectivized "derived" symmetric algebra $C\mathrm{Proj}(\mathrm{Sym}^*(\mathcal{I}))$ iff the virtual codimension $\mathrm{codim.vir}(i) \leq 2$.

4.(a)-(b). **The degenerate cases.**

Proof. (Of 4.7.1.5,4.a) We will show that, whenever $\mathrm{codim.vir}(i : Z \twoheadrightarrow X) = 1$, for any $f : S \to X$ a virtual Cartier divisor $i_D : D \twoheadrightarrow S$ lies over (X, Z) iff it is exhibited by a cartesian square. Then, the universal property of blow-ups 4.7.2.3 implies that $(X, Z) \simeq (\mathrm{Bl}_Z(X), E_Z(X))$, as wished.
In order to see the first claim, argue as follows. One direction is clear. Conversely, with notation as in 4.7.1.1 consider a relative virtual Cartier divisor $i_D : D \twoheadrightarrow S$ over (X, Z) and let $h : D \twoheadrightarrow S_Z$ denote the comparison morphism. Condition (C) yields a canonical surjection $h^*(\mathcal{N}_{S_Z/S}) \twoheadrightarrow \mathcal{N}_{i_D}$ in $\mathrm{QCoh}(D)$, which becomes then an isomorphism, as it can be checked on a common trivializing atlas, since both the quasi-coherent modules are locally free of rank 1. Hence, also their de-suspensions must be equivalent.

Then, the exact sequence of 4.4.0.3,i forces the vanishing of the cotangent complex $\mathcal{L}_h$. Moreover, by (B) h induces an isomorphism on π_0. Thus, we conclude by 4.4.0.3,iii that h is an isomorphism, i.e. that the square is cartesian. $\square$

Proof. *(Of 4.7.1.5,4.b)* Let $i := \mathrm{id}_X$ be the identity of X, and let S be any X-scheme. In order for a virtual Cartier divisor $i_D : D \twoheadrightarrow S$ to lie over (X, X), it must hold both that $i_D^{\mathrm{cl}} = \mathrm{id}_S$ and that $0 \simeq h^*\mathcal{N}_{\mathrm{id}_X} \to \mathcal{N}_{i_D}$ be surjective. The second condition forces $\mathcal{N}_{i_D} \simeq 0$, so that i_D must be an equivalence by 4.4.0.3,iii. But this would contradict the requirement $\mathrm{codim.vir}(i_D) = 1$, so we conclude that there cannot be any relative virtual Cartier divisor over (X, X), i.e. that $\mathrm{Bl}_X(X) = \emptyset$, as claimed. $\square$

4.8 Examples

4.8.1 Special Classes of Relative Virtual Cartier Divisors

Let us start by recording a couple of examples of relative virtual Cartier divisors.

Example 4.8.1.1. *(Relative* CaDiv *are* VDiv*, [17],4.1.2)* Consider a square Q^{cl} in $\mathrm{Sch}^{\mathrm{cl}}$ as above, so such that the angle $(i^{\mathrm{cl}}, f^{\mathrm{cl}}) : Z^{\mathrm{cl}} \to X^{\mathrm{cl}} \leftarrow S^{\mathrm{cl}}$ lives in $\mathrm{Sch}^{\mathrm{cl}}$ and i^{cl} is a regular closed immersion (see 4.5.2.2,ii). If the classical schematic fibre $(f^{\mathrm{cl}})^{-1}(Z^{\mathrm{cl}}) \in \mathrm{CaDiv}(S^{\mathrm{cl}})$ is a classical Cartier divisor on S, then $(f^{\mathrm{cl}})^{-1}(Z^{\mathrm{cl}}) \in \mathrm{VDiv}(S^{\mathrm{cl}})_{/(Z^{\mathrm{cl}}, X^{\mathrm{cl}})}$ also defines a virtual Cartier divisor on i^{cl}.

As a consequence, the we can produce many examples of classical Cartier divisors over (almost) arbitrary quasi-smooth closed immersions of schemes: consider a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$ in Sch and suppose that, Zar-locally on the base X, $i_{\underline{A}} : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{A})$ exhibits Z as being cut-out by a sequence $(f_1, \ldots, f_n) \subseteq \underline{A}$ such that wlog f_n is a non-zero-divisor in $\pi_0\underline{A}/(f_1, \ldots, f_{n-1})$; then, there exists some $f : S \to X$ in Sch giving rise to some classical Cartier divisor $(i_D : D \twoheadrightarrow S) \in \mathrm{CaDiv}(S)_{/(X,Z)}$.

Proof. By 4.6.1.2 and our assumption, we obtain conditions $(a) + (b)$. Condition (c), then, is equivalent to the surjectivity of $\pi_1(h^\flat : \mathcal{O}_{S_Z} \to \mathcal{O}_D)$ (defined as in the proof above, with $D = (f^{\mathrm{cl}})^{-1}(Z^{\mathrm{cl}})$), which automatically follows from D being classical.

Now, let it be given a quasi-smooth closed immersion $i : Z \twoheadrightarrow X$ in Sch, and let $i^{\mathrm{cl}} : Z^{\mathrm{cl}} \to X^{\mathrm{cl}}$ denote the induced map of underlying classical schemes. We will exhibit a map $g : S \to X^{\mathrm{cl}}$ in $\mathrm{Sch}^{\mathrm{cl}}$ and a *classical* Cartier divisor $i_D : D \twoheadrightarrow S$ sitting in a diagram as follows, where $g^{-1}(Z^{\mathrm{cl}})$ denote the classical schematic fibre:

$$\begin{tikzcd} D := g^{-1}(Z^{\mathrm{cl}}) \arrow[r, "i_D", "\mathrm{reg}"'] \arrow[d] & S \arrow[d, "g"] \\ Z^{\mathrm{cl}} \arrow[r, "i^{\mathrm{cl}}"] \arrow[d] & X^{\mathrm{cl}} \arrow[d] \\ Z \arrow[r, "i", "\mathrm{qSmCl}"'] & X \end{tikzcd}$$

We argue Zar-locally on the base X, so let $\underline{A} \in \mathrm{Ani}(\mathrm{CRing})$ be such that wlog $X = \mathrm{Spec}(\underline{A})$; then, we can write the quasi-smooth closed immersion as $i : \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \ldots, f_n)) \twoheadrightarrow \mathrm{Spec}(\underline{A})$ for some sequence $(f_1, \ldots, f_n) \subseteq \underline{A}$, so that $i^{\mathrm{cl}} : \mathrm{Spec}(\pi_0\underline{A}/(x_1, \ldots, x_m)) \to \mathrm{Spec}(\pi_0\underline{A})$ for some sub-sequence of $(f_1, \ldots, f_n)$ without repetitions. We need to exhibit some $\pi_0\underline{A}$-algebra R such that

$$i_D : g^{-1}(\mathrm{Spec}(\pi_0\underline{A}/(x_1, \ldots, x_m))) = \mathrm{Spec}(R/(x_1, \ldots, x_m)) \to \mathrm{Spec}(R)$$

is a closed immersion cut-out by a single non-zero divisor. If, up to permutations, $x_m = f_n$ is (Koszul) regular in $\pi_0\underline{A}/(x_1, \ldots, x_{m-1})$, then $R := \pi_0\underline{A}/(x_1, \ldots, x_{m-1})$ does the job. Finally, the glueing datum corresponding to the affine patches $\mathrm{Spec}(\pi_0\underline{A})$'s of X^{cl} allows to glue also our local candidates $\mathrm{Spec}(R)$'s to the desired scheme S. Then, $i_D \in \mathrm{VDiv}(S)_{/(Z^{\mathrm{cl}}, X^{\mathrm{cl}})}$ is also a relative virtual Cartier divisor over (X, Z): again conditions $(a) + (b)$ are clear and the surjectivity of $\pi_1(h^\flat)$ can be checked as above. $\square$

The next example is not necessary to the theory, but fosters intuition, since it shows that the "obvious" method of exhibiting relative virtual Cartier divisors does indeed work. Morally, given any quasi-smooth closed immersion of (virtual) codimension n, a composite of quasi-smooth closed immersions of codimension 1 and $n-1$ exhibits a relative virtual Cartier divisor over the former.

Example 4.8.1.2. *(Successive quotients exhibit relative virtual Cartier divisors)* Let $i : Z \twoheadrightarrow X$ be a quasi-smooth closed immersion in Sch of non-vanishing virtual codimension. Then, there exists a virtual Cartier divisor over (X, Z) on some X-scheme S/X in $\mathrm{Sch}_{/X}$.

Proof. Define S, $f : S \to X$ and $i_D : D := Z \twoheadrightarrow S$ by the following cube, where all vertical faces are cartesian.

$$\begin{tikzcd}
Z \arrow[rr, "i_D"] \arrow[dr] \arrow[drrr, dotted, "i"] \arrow[dd] & & S \arrow[dr, "f"] \arrow[dd] & \\
& \mathrm{PB} \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
\{0\} \arrow[rr] \arrow[dr] & & \mathbb{A}^1 \arrow[dr, "(t_i|i<n)"] & \\
& \mathbb{A}^{n-1} \arrow[rr, "(t_n)"] & & \mathbb{A}^n
\end{tikzcd}$$

We claim that the induced commutative square exhibits i_D as a relative Cartier divisor on S over i:

$$\begin{tikzcd}
Z \arrow[r, "i_D"] \arrow[d, equal] & S \arrow[d, "f"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

It suffices to check it Zar-locally on the base X, where one has:

$$\begin{tikzcd}
\mathrm{Spec}(\underline{A} /\!\!/ (f_1, \dots, f_n)) \arrow[r, "i_D"] \arrow[d, equal] & \mathrm{Spec}(\underline{A} /\!\!/ (f_1, \dots, f_{n-1})) \arrow[d, "f"] \\
\mathrm{Spec}(\underline{A} /\!\!/ (f_1, \dots, f_n)) \arrow[r, "i"] & \mathrm{Spec}(\underline{A})
\end{tikzcd}$$

Indeed, Zar-locally one has wlog $S = \mathrm{Spec}(\underline{R})$ for some $\underline{R} \in \mathrm{Ani(CRing)}$. Now, being the vertical right face cartesian, we can write

$$\begin{aligned}
\underline{R} \simeq \underline{A} \otimes^L_{\mathbb{Z}[t_1,\dots,t_n]} \frac{\mathbb{Z}[t_1,\dots,t_n]}{(t_1,\dots,t_{n-1})} &\simeq \underline{A} \otimes^L_{\mathbb{Z}[t_1,\dots,t_n]} \left(\mathbb{Z}[t_1,\dots,t_n] \otimes^L_{\mathbb{Z}[t_1,\dots,t_{n-1}]} \frac{\mathbb{Z}[t_1,\dots,t_{n-1}]}{(t_1,\dots,t_{n-1})} \right) \\
&\simeq \underline{A} \otimes^L_{\mathbb{Z}[t_1,\dots,t_{n-1}]} \frac{\mathbb{Z}[t_1,\dots,t_{n-1}]}{(t_1,\dots,t_{n-1})} \\
&\simeq \underline{A} /\!\!/ (f_1, \dots, f_{n-1})
\end{aligned}$$

On the other hand, as in the inductive proof of 3.5.0.3,ii, the front and left vertical faces of the cube express the identification $\underline{A} /\!\!/ (f_1, \dots, f_n) \simeq \big(\underline{A} /\!\!/ (f_1, \dots, f_{n-1})\big) /\!\!/ (f_n)$ which induces i_D.
Then, let's check the axioms of the first definition for the affine square above:

- (a) : The map i_D is a quasi-smooth closed immersion of codimension 1 - i.e. a virtual Cartier divisor on S - because is the base-change of the (quasi-smooth closed) inclusion of the origin $\{0\} \twoheadrightarrow \mathbb{A}^1$ (see 4.5.2.2,i and 4.5.2.6,i).
- (b) : At the level of classical schemes, the commutative square above is clearly cocartesian, since $\pi_0\underline{A}/(f_1, \dots, f_n) \cong \pi_0\underline{A}/(f_1, \dots, f_n) \otimes_{\pi_0\underline{A}} \pi_0\underline{A}/(f_1, \dots, f_{n-1})$.
- (c) : We need to check the surjectivity on π_0 of the canonical map $(1_Z)^*\mathcal{N}_i \to \mathcal{N}_{D/X} \to \mathcal{N}_{i_D}$; by 4.4.1.4 and the construction, this amounts to the following projection, so we are done:

$$\big(\underline{A} /\!\!/ (f_1, \dots, f_n)\big)^n \to \underline{A} /\!\!/ (f_1, \dots, f_n)$$

□

4.8.2 Computation: Affine Charts of Blow-Ups

In this subsection we record a computation of the affine charts of the blow-up of a quasi-smooth locally closed relative scheme.

Proposition 4.8.2.1. *(Blow-up of affine* qSmCl*, [17],4.3.7) Let* $i : Z = \mathrm{Spec}(\underline{R} /\!\!/ (f_1, \ldots, f_n)) \to \mathrm{Spec}(\underline{R}) = X$ *in* $\mathrm{Sch}^{\mathrm{Aff}}$ *be a quasi-smooth closed immersion cut-out by coordinate maps* $(f_1, \ldots, f_n) : \mathbb{Z}[t_1, \ldots, t_n] \to \underline{R}$ *(see 3.5.0.1). Then, the following affine schemes determine an affine Zariski atlas for* $\mathrm{Bl}_Z(X)$*:*

$$\mathrm{Spec}(\underline{R}) \times_{\mathbb{A}^n} \mathrm{Spec}(A_k) \simeq \mathrm{Spec}\big(\underline{R}[y_r : r \neq k] /\!\!/ (\rho'_r := f_k y_r - f_r : r \neq k)\big)$$

Proof. $\mathrm{Bl}_Z(R)$ is stable under base-change in Sch by 4.7.1.5,1.b; hence, the computation $\underline{R} /\!\!/ (f) \simeq R \otimes^L_{\mathbb{Z}[t]} \mathbb{Z}[t]/(t)$ implies the equivalence $\mathrm{Bl}_Z(X) \simeq \mathrm{Spec}(R) \times_{\mathbb{A}^n} \mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ in Sch. Then, intersecting the affine Zariski atlas $\{\mathrm{Spec}(A_k)\}_k$ for the blow-up of the inclusion of the origin $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ as in 4.7.2.2,iii yields an affine Zariski atlas for the blow-up, namely

$$\mathrm{Spec}(R) \times_{\mathbb{A}^n} \mathrm{Spec}(A_k) \simeq \mathrm{Spec}\big(R \otimes^L_{\mathbb{Z}[t_1,\ldots,t_n]} A_k\big) \quad \text{for } 1 \leq k \leq n$$

Thus, we are left to compute the tensor product. Let $t := (t_1, \ldots, t_n)$ and $y^k := (y_r : r \neq k)$ be n-tuples of indeterminates and let $f := (f_1, \ldots, f_n)$ denote the sequence of coordinate maps. Consistently, define also the n-tuple $\rho^k := (\rho_r := t_k y_r - t_r : r \neq k)$. With this notation, we can rewrite the description of the structural map $\gamma_k : \mathrm{Spec}(A_k) \to \mathbb{A}^n$ we provided in 2.2.0.5 as a regular immersion cut-out by the equations in ρ:

$$\begin{aligned} \gamma_k^\flat : \mathbb{Z}[t] &\longrightarrow \frac{\mathbb{Z}[t][y^k]}{(\rho^k)} = A_k \\ t_k &\longmapsto t_k \\ (\forall r \neq k) \qquad t_r &\longmapsto t_r = t_k y_r \end{aligned}$$

The expressions above assemble into coordinate maps $\rho^k : \mathbb{Z}[t] \to \mathbb{Z}[t][y^k]$, which together with f induce the following tensor product:

$$\underline{R} \otimes^L_{\mathbb{Z}[t]} A_k \simeq \underline{R} \otimes^L_{\mathbb{Z}[t]} \big(\mathbb{Z}[t][y^k] \otimes^L_{\mathbb{Z}[y^k]} \mathbb{Z}[y^k]/(y^k)\big) \simeq \underline{R}[y^k] \otimes^L_{\mathbb{Z}[y^k]} \mathbb{Z}[y^k](y^k)$$

Now, by the associativity of the monoidal structure $\otimes^L$ we are left to consider the latter algebra, which can be regarded as the claimed quotient ring. Indeed, it is given by the coordinate maps:

$$\rho'^k := \big(f \otimes^L_{\mathbb{Z}} Z[y^k]\big) \circ \rho^k : \mathbb{Z}[y^k] \longrightarrow \underline{R}[y^k]$$

which are morally obtained by replacing t with f in ρ^k, i.e. $\rho'_r = f_k y_r - f_r$ for $r \neq k$. $\square$

Example 4.8.2.2. Let us specialize the Proposition above into a couple of more explicit computations.

- Let $\underline{R} \in \mathrm{Ani(CRing)}$ be any animated ring, $X := \mathrm{Spec}(\underline{R})$, and let Z be cut-out by coordinate maps $(f_1, \ldots, f_n) = (0, \ldots, 0) : \mathbb{Z}[t_1, \ldots, t_n] \to \underline{R}$. Then, the blow-up $\mathrm{Bl}_Z(X)$ of $i : Z \twoheadrightarrow X$ admits the affine Zariski atlas:

$$\mathrm{Spec}(\underline{R}) \times_{\mathbb{A}^n} \mathrm{Spec}(A_k) \simeq \mathrm{Spec}(\underline{R}[y_r : r \neq k] /\!\!/ (\rho'_r = 0 : r \neq k))$$

- Let $R \in \mathrm{CRing}$ be a static ring, $X := \mathrm{Spec}(R)$, and let Z be cut-out by a regular sequence of coordinate maps $(f_1, \ldots, f_n) : \mathbb{Z}[t_1, \ldots, t_n] \to R$. Then, as observed in 4.7.1.5,3.a, the blow-up $\mathrm{Bl}_Z(X)$ of $i : Z \twoheadrightarrow X$ admits the affine Zariski atlas:

$$\mathrm{Spec}(R) \times_{\mathbb{A}^n} \mathrm{Spec}(A_k) \simeq \mathrm{Spec}\left(\frac{R[y_r : r \neq k]}{(\rho'_r = f_k y_r - f_r : r \neq k)}\right)$$

  In other words, by 2.2.0.10 we retrieve the classical description of blow-ups of regular immersions as the (classical) projectivization of the (classical) symmetric algebra generated by the defining ideal of regular equations: $\mathrm{Bl}_Z(X) \cong \mathrm{Bl}^{\mathrm{cl}}_Z(X) \cong C\mathrm{Proj}(C\mathrm{Sym}^*(f_1, \ldots, f_n))$.

A Animation

In this section we will present the construction $\mathcal{P}_\Sigma$, also known as *animation*. The main source is the work of Lurie in [24],5.5.8.
Unless otherwise specified, we will always be concerned with small ∞-categories which admit finite coproducts. An initial detour on sifted simplicial sets and colimits will provide us with the technical tools needed to develop the $\mathcal{P}_\Sigma$-construction.
This will be the content of the second subsection, whereas the third one will be mostly concerned with showing how to regard $\mathcal{P}_\Sigma(C)$ as the free sInd-completion of the full subcategory of compact and projective objects of C. In other words, $\mathcal{P}_\Sigma(C)$ will be obtained from C by freely adjoining sifted colimits, and hence, will enjoy the universal property of free sInd-completions. As we will remark also later on, it is indeed the latter fact which allows us to set up the theory of DAG, since it enables us to define functors of the ∞-categories involved by declaring their actions on the ordinary category of compact and projective objects of a suitable ∞-category C.
In the fourth subsection, we will briefly comment on how to regard $\mathcal{P}_\Sigma$ as a localization functor for a given 'non-abelian' model structure on $C \in \mathrm{Cat}_\infty$.
Finally, a synthetic presentation of [23],4.7.3 will close the section, aiming at determining how to characterize compact and projective objects of algebraic categories.

A.1 Detour on Sifted Colimits

We introduce the notion of 'sifted' simplicial sets. They are meant to be particularly nice indexing categories for 'sifted' diagrams, over which to form 'sifted' colimits. In the current subsection, we will stress on how they generalize the 1-categorical notion of 'siftedness', as introduced by Adamek and Rosicky in [1], in that they can again be described in terms of filtered colimits and reflexive coequalizers (namely, 'geometric realizations' in the ∞-categorical language).
We will then provide some examples of sifted colimits, which, as we will see, are manifestly ubiquitous when dealing with 'algebraic theories'.
Due to the nature of the objects involved, in the current subsections we will of course be working with simplicial sets and our results will be in the incarnation of quasi-categories.
We present, however, only an informal adaptation of the arguments proving the results stated, while we always reference to Lurie's or Land's formal proofs. Our expository choice is motivated by the following two reasons: first of all, the arguments are sometimes technical and fairly involved and anyway far from the scope of the present dissertation. On the other hand, we believe that manipulations of (co)limits become easier and more enlightening if one acquires enough intuition about the combinatorics of the objects involved.

Definition A.1.0.1. ([24],5.5.8.1) A simplicial set $K \in \mathrm{sSet}$ is called **sifted** if

- K is inhabited, i.e. is non-empty;
- the diagonal functor $\mathrm{diag} := \mathrm{const}_{\partial\Delta^1} : K \to K \times K$ is cofinal.

Consistently with the terminology of algebraic topology, we require a sifted simplicial set to be inhabited in order to avoid triviality; compare also with A.1.0.4.

Remark. As expected, filtered colimits in Cat_∞ are sifted: apply [24],5.3.1.20 to the simplicial set C with the auxiliary $K = \partial\Delta^1$.
We will now exhibit an interesting example of a sifted simplicial set. As a preparation, we state the following Criterion for a functor to be cofinal, due to Joyal.

Lemma A.1.0.2. *(Joyal Criterion of Cofinality, [24],4.1.3.1) Let $f : C \to D$ be a map in* sSet*, with $D \in \mathrm{Cat}_\infty$.* TFAE*:*

- *f is cofinal, i.e. for each diagram $p : D \to E$ of ∞-categories, f induces an equivalence* $\mathrm{colim}(p) \simeq \mathrm{colim}(p \circ f)$.

- *For every object $d \in D$, $C \times_D D_{d/}$ is weakly contractible.*

Remark. (Motivational, in Cat_∞*)* In [24],4.1.3, Lurie informally motivates the statement of Joyal's Theorem as follows. Given any diagram $p : D \to E$ of ∞-categories, we want an equivalence $\mathrm{colim}(p) \simeq \mathrm{colim}(p \circ f)$. Provided that both sides are well-defined, there is always a canonical morphism $\phi : \mathrm{colim}(p \circ f) \to \mathrm{colim}(p)$, so that we have the required cofinality iff there is a quasi-inverse ψ to ϕ. This amounts to defining a 'compatible' family of maps $\big(\psi_d : p(d) \to \mathrm{colim}(p \circ f) \mid d \in D\big)$.
The only reasonable candidate should be the one induced by a family of 'compatible' factorizations $p(d) \to p \circ f(c) \to \mathrm{colim}(p \circ f)$, where the first map arises from 'compatible' $d \to p \circ f(c)$, for some $c \in C$.
Now, observe that such an object c need not exist, and even in such a case it need not be unique. However, provided that the latter two conditions are satisfied, we are done.
The collection of such candidates for c is parametrized by the slice $C_{d/} \simeq C \times_D D_{d/} \in \mathrm{Cat}_\infty$. Hence, from this perspective, Joyal's statement amounts to the latter slice being weakly contractible, namely to the existence of a universal such object $c \in C$ as needed.

Lemma A.1.0.3. *(Natural numbers are co-sifted, [24],5.5.8.4)* $\mathcal{N}(\Delta)^{op} \in \mathrm{sSet}$ *is sifted. In particular, geometric realizations are sifted diagrams.*

Proof. (Sketch) Clearly $\mathcal{N}(\Delta)^{op}$ is inhabited, so we are left to show that the diagonal is cofinal. To this end, we will apply A.1.0.2 to diag, and hence show that, for each $d := ([m], [n]) \in \Delta \times \Delta$, relabelling $D_{d/} := \mathcal{N}(X := \Delta_{/[m]} \times_\Delta \Delta_{/[n]})$ (recall that we are dealing with *op*-categories) the slice $\Delta \times_{\Delta \times \Delta} D_{d/} \simeq D_{d/}$ is w. contractible.
Notice that it suffices to check it on the full subcategory $X^0 \subseteq_{f.f.} X$, spanned by the saturation of

$$\{\text{piece-wise 'continuous' paths in the rectangle } [m] \times [n]\}$$

Indeed, being X^0 a presentable and cocomplete category, the inclusion $X^0 \subseteq_{f.f.} X$ admits a left-adjoint, so that such an inclusion induces a w.h.equivalence under the nerve.
Finally, in order to prove that $\mathcal{N}(X^0)$ is w. contractible, we remark that $\mathcal{N}(X^0)$ is equivalent to the simplicial set $Sub(\Delta^m \times \Delta^n)$ of baricentric subdivisions of $\Delta^m \times \Delta^n$ (i.e. finite chains of sub-simplices), so that in homotopy $\pi_*(|\mathcal{N}(X^0)|) \simeq \pi_*(|Sub(\Delta^m \times \Delta^n)|) \simeq \pi_*(|\Delta^m \times \Delta^n|) \simeq *$. □

Remark. ([24],5.5.8.5) As another example of sifted colimits, observe that the geometric realization of simplicial objects $| - |$ is a colimit over a sifted diagram and should be thought of as the ∞-categorical analogue of the formation of reflexive coequalizers:

$$|X| := \mathrm{colim}_{\Delta^n/X} |\Delta^n| = \mathrm{colim}_{\Delta^n/X} \; |\Delta^n \underset{s_j}{\overset{d_i}{\rightleftarrows}} \Delta^{n+1}|_{i,j}$$

generalizes the '1-truncated version' $\mathrm{colim}\big(\, 0 \rightleftarrows 1 \,\big)^{op} \simeq \mathrm{colim}(id_{\Delta^{op}_{\leq 1}})$, which is the underlying diagram of a reflexive coequalizer.

Let us now present another application of Joyal's Criterion, which shows that sifted simplicial sets are very 'rigid'. Hence, as we will see afterwards, they have good commutativity properties with respect to other limits.

Lemma A.1.0.4. *(Sifted simplicial sets are w. contractible, [24],5.5.8.7) Any sifted simplicial set K is weakly contractible, i.e. the canonical map $K \to \Delta^0$ is a weak homotopy equivalence (see [25],3.2.6.1).*

Proof. Given any $x \in K$, by the Whitehead Theorem it suffices to show that K has w. contractible geometric realization: $\pi_*(|K|, x) \simeq *$.
Let us recall that cofinal morphisms (such as the diagonal diag) are w. h. equivalences ([24],4.1.1.3). Indeed, informally, given a diagram q, a cofinal map f for q amounts to fibre-wise equivalences of the cocartesian fibrations corresponding (under the Straightening-Unstraightening equivalence of [24],3.2) to $\mathrm{colim}(q)$ and $\mathrm{colim}(q \circ f)$; this, as shown in [20],3.1.27, means that f must be a w. h. equivalence itself.
From the latter we observe that the diagonal map induces:

$$\mathrm{diag} : \pi_*(|K|, x) \xrightarrow{\simeq} \pi_*(|K \times K|, \mathrm{diag}(x)) \simeq \pi_*(|K|, x) \times \pi_*(|K|, x)$$

which forces $\pi_*(|K|, x) \simeq *$, since - being K inhabited - $\pi_*(|K|, x) \neq \emptyset$. □

We now investigate the compatibility properties of sifted colimits and finite products. As in the 1-categorical case, we will obtain the commutativity of the two functors; this will be achieved in two lemmas.

Lemma A.1.0.5. *([24],5.5.8.6) Let $K \in \mathrm{sSet}$ be a sifted simplicial set, and consider ∞-categories C, D, E with K-indexed (sifted) colimits.*
If a functor $f : C \times D \to E$ of ∞-categories preserves K-indexed colimits separately in each variable, then it preserves sifted colimits of K-indexed diagram in the product category.

Proof. (Sketch) Consider K-indexed diagrams $p : K \to C$ and $q : K \to D$ and rename $\delta := \mathrm{diag} : K \to K \times K$. Then, we claim that we can decompose

$$\mathrm{colim}\left(f \circ (p \times q)\right) \simeq_{(1)} f \circ \left(\mathrm{colim}(p \times q)\right) \simeq_{(2)} \mathrm{colim}\left(K \xrightarrow{\delta} K \times K \xrightarrow{p \times q} C \times D \xrightarrow{f} E\right)$$

where the previous equivalences are due to the following facts:

1. *f admits colimits over $K \times K$, and these are computed by evaluating f at the colimit of $p \times q$.*

   Indeed, the assumption on f rewrites as the fact that, for any colimit cone $\overline{K} \simeq K \cup \{\infty\}$ extending K, $\overline{f \circ (p \times q)_{|\subseteq_i(K)}} : \overline{K} \to C \times D \to E$ is a colimit diagram. Now a technical Lemma (see [24],5.5.2.3) assures that we can compute the colimit over the product diagram by a 'Cantor diagonal argument' applied to the product of colimit cones, in other words that $\overline{K} \times \overline{K} \simeq \overline{K \times K}$.

   In practical terms, this means that the value of the colimit $\overline{f \circ p \times q}$ exists and it is computed by $f \circ \overline{p \times q}$.

2. *We can actually factor each colimit over $K \times K$ through δ*, i.e. δ is cofinal: this holds true by assumption. $\square$

Remark. Binary products in an ∞-topos (e.g. Spc, see Appendix C for a more general discussion) preserve small colimits separately in each variable.
Indeed, colimits in a topos $\mathcal{X}$ are universal, i.e. for each morphism $f : X \to Y$ in $\mathcal{X}$ there is a well-defined pull-back functor of slice topoi $f^* : \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$ which preserves all small colimits. Now, observe that a binary product in $\mathcal{X}$ corresponds to a pull-back square over the terminal object $*$ of $\mathcal{X}$, so that $X \times (-) \simeq (X \to *)^*(-)$ and similarly for the other variable.

Lemma A.1.0.6. *(Sifted colimits preserve finite products, [24],5.5.8.11) Let $K \in \mathrm{sSet}$ be sifted, and $X \in \mathrm{Cat}_\infty$ with finite products and sifted colimits (e.g. $X = \mathrm{Spc}$ or any ∞-topos). Assume further that each n-fold product preserves K-indexed colimits separately in each variable. Then, $\mathrm{colim}_K : \mathrm{Fun}(K, X) \to X$ preserves finite products.*

Proof. (Sketch) First recall that, as proved in the previous Lemma, colim_K commutes with binary products whenever $K \in \mathrm{sSet}$ is sifted. Now, notice that we can regard a finite product as the terminal object in the diagram of its binary sub-products. Therefore, in view of Lemma A.1.0.4, we are left to show the following:

Claim. *([24],4.4.4.9) colim_K preserves terminal objects whenever K is w. contractible.*

Proof. Our argument will amount to the definition of the copowering over $\mathcal{H} := Ho(\mathrm{Spc})$ of an ∞-category (in its quasi-categorical incarnation). The following digression summarizes [24],4.4.4.
First, let us fix some notation (for any arbitrary $K \in \mathrm{sSet}$). Fix a diagram $p : K \to C$ into an ∞-category C. In view of [24],4.4.4.5, we say that the left fibration $C_{p/} \to C$ is corepresentable if the under-category $C_{p/}$ has an initial object, or equivalently iff p has a colimit in C. In particular, let $c : \Delta^0 \to C$ be the inclusion of an object $c \in C$; in view of the Yoneda Lemma as formulated by Heberstreit in e.g. [20],4.2.12, write $\mathrm{Map}_C(c, -)$ for the functor corepresented by c in the homotopy category of Spc, namely $\mathcal{H}$. Now, let $p := \mathrm{const}_c : K \to C$ be the K-indexed constant diagram at $c \in C$.
For each $c' \in C$, the homotopy-coherent version of Straightening-Unstraightening yields a functorial identification $C_{p/} \times_C \{c'\} \simeq (C_{c/} \times_C \{c'\})^K$ of fibers of left fibrations (and hence an identification of these left fibrations over C in $\mathrm{LFib}(C)$), so that p is corepresented in $\mathcal{H}$ by the product $\mathrm{Map}_C(c, -)^{[K]}$, where $[K]$ denotes the homotopy class of K in $\mathcal{H}$.
In other words, following the formulation of [24],4.4.4.9, the objects of the fibre $C_{p/} \times_C \{c'\}$ are classified (up to equivalence) by maps $\psi : [K] \to \mathrm{Map}_C(c, c')$ in the homotopy category $\mathcal{H}$. Observe that, for each $z \in C$,

the latter induces a map $\mathrm{Map}_C(c', z) \to \mathrm{Map}_C(c, z)^{[K]}$. Now, one has that such a map ψ classifies a colimit for p iff, for each $z \in C$, the previous map actually corresponds to an iso $\mathrm{Map}_C(c', z) \simeq \mathrm{Map}_C(c, z)^{[K]}$ in $\mathcal{H}$. The latter statement is an immediate consequence of the fact that, in such a case, c' would be initial in $C_{p/}$. In order to grasp some intuition, this corresponds precisely to the classical definition of a colimit as a universal cone extension.

Remark. When such a colimit exists, then we denote it by $c \otimes K$, being it the K-copower of $c \in C$ over $\mathcal{H}$.

Finally, let us come back to our setting. For K w. contractible, $[K] = *$ and the fully faithfulness of the Yoneda functor ensures the existence of a ψ classifying a colimit for p. $\square$

A.2 The $\mathcal{P}_\Sigma$-Construction

We now introduce the $\mathcal{P}_\Sigma$-construction together with its first properties. For reasons which will be clear later on, the latter is also known as *'non-abelian localization'*; a more modern and evocative terminology has been however introduced by Cesnavicius and Scholze in [3],5, who refer to a special instance of the $\mathcal{P}_\Sigma$-construction as *'animation'*, a term which comes from 'anima,-ae', the latin word for 'soul'. The motivation behind such a terminology will become clear in the next sections.

Definition A.2.0.1. *($\mathcal{P}_\Sigma$-construction, [24],5.5.8.8)* Let $C \in \mathrm{Cat}_\infty$ admit finite coproducts and let $\mathcal{P}(C) := \mathrm{Psh}(C) = \mathrm{Fun}(C^{op}, \mathrm{Spc})$ denote the ∞-category of presheaves on C. Define $\mathcal{P}_\Sigma(C) \subseteq_{f.f.} \mathcal{P}(C)$ to be the full sub-∞-category spanned by finite-product-preserving functors from C^{op} to spaces: $\mathcal{P}_\Sigma(C) := \mathrm{Fun}^\times(C^{op}, \mathrm{Spc}) \subseteq_{f.f.} \mathcal{P}(C)$.

Before proving the properties of the $\mathcal{P}_\Sigma$-construction, we delve into a brief digression on strongly reflective localizations.

Digression: On strongly reflective localizations. We briefly review [24],5.5.4 - complementing with the relative *nLab* page [33] - aiming at introducing the language of S-local objects and morphisms, so as to describe Bousfield left-localizations in terms of those. The $\mathcal{P}_\Sigma$-construction will turn out to be a very well-behaved instance of such a class of localizations.
First, let us consider an ∞-category C and a class of morphisms $S \subseteq \mathrm{Mor}C$. We say that:

- $c \in C$ is a **S-local object** if $j(c)$ sends S to weak homotopy equivalences in Spc;
- $f \in \mathrm{Mor}C$ is a **S-local equivalence** if, for each S-local object $c \in C$, $j(c)(f)$ is a weak homotopy equivalence in Spc.

Intuitively, we are requiring a relative version of being a weak categorical equivalence (as in [20],2.2.28), since the latter property is to be checked on the (generally smaller) class S.

In addition, we say that a class S of morphisms in a cocomplete category $C \in \mathrm{Cat}_\infty$ is **strongly saturated** if it is stable under pull-backs along morphisms of C, it has the 2-out-of-3 property and the full subcategory of $\mathrm{Fun}(\Delta^1, C)$ generated by S is cocomplete.

One can show that, given a class S of morphisms of C, there is a well-defined notion of 'strong saturated closure' of S, which in particular must contain all weak categorical equivalences of C.
Moreover, as stated in [33],2.4, given a colimit preserving functor $F : C \to D$ of cocomplete ∞-categories and a strongly saturated class T of morphisms of the target D, then the pre-image $F^{-1}(T)$ is a strongly saturated class of the source C. In particular, this holds for the pre-image along F of the weak equivalences of its target D. Hence, for $F = \cap j(c) : C \to \mathrm{Spc}$ over S-local objects, one observes that the class of S-local equivalences in a category must be strongly saturated.

Let $\mathcal{W}_X$ denote the class of weak categorical equivalences of an ∞-category X. By [33],3.2 (or [24],5.5.4.2), a Bousfield left-localization $L\colon C \rightleftarrows D :\supseteq$ can be characterized in terms of $S := L^{-1}(\mathcal{W}_D)$-local objects and morphisms of C. More explicitly, denoting by $Loc := \subseteq \circ L$ the localization functor, $\{S\text{-local objects}\} \simeq Loc(C)$ and $S = \{S\text{-local equivalences}\}$.

For a presentable category X (such as e.g. $\mathcal{P}(C)$), a variant of the II Adjoint Functor Theorem allows us to prove that the previous characterization actually exhausts all the "reasonable" Bousfield left-localizations of X. Indeed, by [33] (or [24],5.5.4.15), the reflective subcategories $X^0 \subseteq_{f.f.} X$ are precisely those spanned by a *small set* $S^0 \subseteq Mor(X)$. Furthermore, as expected, for L the left-adjoint to $\subseteq$, $\{S^0\text{-local equivalences}\} = \overline{S^0} = L^{-1}(\mathcal{W}_{X^0})$.

We now claim that reflective subcategories X^0 of such a form are furthermore the **strongly reflective subcategories** of X, meaning that they are presentable, reflective and stable under equivalences in X. So, intuitively, they correspond to localizations at those morphisms which are relative weak categorical equivalences with respect to some small set.

Criterion. *(For strongly reflective localizations)* The full subcategories X^0 of a presentable $X \in \mathrm{Pr}^L$ which are spanned by the S^0-local objects of X for some *small set* $S^0 \subseteq Mor(X)$ are strongly reflective, i.e. are reflective, presentable and stable under equivalences in X.

Sketch. In view of the previous digression, we already have a Bousfield localization $L\colon X \rightleftarrows X^0 :\supseteq$ which exhibits X^0 as a reflective subcategory of X. Moreover, being X^0 spanned by the S^0-local objects, it is clearly stable under equivalences in X. Hence, will only need to prove that X^0 is presentable.
By [24],5.2.7.5, being X cocomplete, also $X^0 \simeq LX$ is so: for any diagram $q : K \to X^0$, we know that $p : K \to X^0 \subseteq X$ admits a colimit cone $\overline{p}$, which is preserved by the left-adjoint L, so that $L \circ \overline{p}$ is a colimit diagram of $L \circ p \simeq L\circ \subseteq \circ q \simeq q$ (where the equivalence is given by the counit).
Finally, the heart of our statement lies in the proof that LX is accessible whenever X is such; however, the latter is definitely technical and involved, so we defer it to Luries's exposition in [24],5.5.4.2/iii). ■

We close this digression by recording (without proof) three stability properties of strongly reflective localizations, which will be needed later on:

Properties. *(Of strongly reflective localizations)* Let $C \in \mathrm{Pr}^L$ be a presentable ∞-category. Then,

(a) *([24],5.5.4.17)* Pulling back the right adjoint of $F\colon C \rightleftarrows D :G$ in Pr^L along a strongly reflective embedding $C^0 \subseteq_{f.f.} C$ exhibits $D^0 := G^{-1}(C^0) \subseteq_{f.f.} D$ as a strongly reflective subcategory:

$$\begin{tikzcd} C^0 \arrow[r, hook, "f.f."] & C \arrow[d, shift left, "F"] \\ D^0 := G^{-1}(C^0) \arrow[u, "G_{|D^0}"] \arrow[r, hook, "f.f."] & D \arrow[u, shift left, "G"] \end{tikzcd}$$

(b) *([24],5.5.4.18)* Let $\{C_\alpha\}_A$ be a *small* family of strongly reflective subcategories of C induced by the *small* subsets $\{S_\alpha\}_A$ of C; then, also their intersection $\cap_A C_\alpha \subseteq_{f.f.} C$ is strongly reflective, as induced by localizing at $S := \cup_A S_\alpha$-local maps.

(c) *([24],5.5.4.19)* For a small simplicial set $K \in \mathrm{sSet}$, the full subcategory D spanned by

$$D := \langle \{\overline{p} : \overline{K} \to C \mid \overline{p} \simeq \lim p_{|K}\} \rangle \subseteq_{f.f.} \mathrm{Fun}(\overline{K}, C)$$

exhibits a strongly reflective subcategory $\lim_K(D) \subseteq_{f.f.} C$.

We are finally ready to state and prove the main proposition of this section, which investigates the properties of the $\mathcal{P}_\Sigma$-construction.

Proposition A.2.0.2. *(Properties of $\mathcal{P}_\Sigma$, [24],5.5.8.10) For a small ∞-category $C \in \mathrm{Cat}_\infty$ with finite coproducts, the following properties hold true:*

1. *$\mathcal{P}_\Sigma(C) \in \mathrm{Cat}_\infty$ is a Bousfield accessible localization of $\mathcal{P}(C)$ via $L\colon \mathcal{P}(C) \rightleftarrows \mathcal{P}_\Sigma(C) :\supseteq$.*

2. *The Yoneda embedding $j : C \hookrightarrow \mathcal{P}(C)$ factors through $j_\Sigma := L \circ j : C \hookrightarrow \mathcal{P}_\Sigma(C)$ and the latter preserves finite coproducts.*

3. *Given an adjunction $F\colon \mathcal{P}(C) \rightleftarrows D : G$ with $D \in \mathrm{Cat}_\infty$ a presentable category, G factors through $\mathcal{P}_\Sigma(C)$ iff $f := F \circ j : C \to D$ preserves finite coproducts.*

4. *$\mathcal{P}_\Sigma(C) \subseteq_{f.f.} \mathcal{P}(C)$ is stable under sifted colimits; in particular, pictorially $\mathcal{P}_\Sigma(C) \supseteq \mathrm{sInd}((C)$.*

5. *In the Bousfield localization $L\colon \mathcal{P}(C) \rightleftarrows \mathcal{P}_\Sigma(C) : \supseteq$, the unit $L : \mathcal{P}(C) \to \mathcal{P}_\Sigma(C) \subseteq_{f.f.} \mathcal{P}(C)$ preserves sifted colimits of presheaves.*

6. *$\mathcal{P}_\Sigma(C) \in \mathrm{Cat}_\infty$ is compactly generated.*

Proof. (1) : Rephrasing in the terminology of [24],5.5.4, we are asserting that $\mathcal{P}_\Sigma(C)$ is a strongly reflective subcategory of the presentable category $\mathcal{P}(C)$. We refer the unexperienced reader to the digression above for an introduction to left-localizations.
In view of property (b) as above, we will construct countably many strongly reflective subcategories $\mathcal{P}_n(C) \subseteq_{f.f.} \mathcal{P}(C)$ such that $\mathcal{P}_\Sigma(C) \simeq \cap_{n \leq \omega} \mathcal{P}_n(C)$. Moreover, if we let S_n denote the small sets of local equivalences inducing the localization $\mathcal{P}_n(C)$, then $\mathcal{P}_\Sigma(C)$ will be the localization at $S := \cup_n S_n$-local maps.
To this end, for each $n < \omega$ consider the full subcategory $\mathcal{P}_n(C) \subseteq_{f.f.} \mathcal{P}(C)$ spanned by those functors which preserve n-ary products in C^{op}, and observe that it is the left-localization at S_n-local maps, where:

$$S_n := \left\{ j(\prod_{i=1}^{n} x_i) \to \coprod_{i=1}^{m} j(x_i) \;\middle|\; (x_i)_{i=1}^n \in C^n \right\}$$

In order to see this, we need to check that, for each $\mathcal{F} \in \mathcal{P}_n(C)$ and each map in S_n, pre-composition with the latter induces an equivalence of mapping spaces. But this is clear: under the Yoneda Lemma, the previous map amounts to the canonical one between the corresponding evaluations and $\mathcal{F}$ preserves n-fold products by assumption, so the following is a homotopy equivalence:

$$\mathrm{Map}_{\mathcal{P}(C)}\big(\coprod j(x_i), \mathcal{F}\big) \simeq \prod \mathcal{F}(x_i) \to \mathcal{F}(\prod x_i) \simeq \mathrm{Map}_{\mathcal{P}(C)}\big(j(\prod x_i), \mathcal{F}\big)$$

Finally, we remark that, being C small by assumption, also S_n is a small set for each n, and hence it exhibits $\mathcal{P}_n(C)$ as a strongly reflective subcategory of $\mathcal{P}(C)$, as needed.

(2) : Being $\mathcal{P}_\Sigma(C)$ a full subcategory of $\mathcal{P}(C)$, the Yoneda embedding $j_C : C \to \mathcal{P}(C)$ factors through $\mathcal{P}_\Sigma(C)$ iff, for each $x \in C$, $j_C(x)$ preserves finite products in C^{op}. By [20],5.1.24, an n-coproduct is left adjoint to the n-constant functor, so that we obtain the required property:

$$\mathrm{Map}_C(\coprod_{i=1}^{n} z_i, x) \simeq \mathrm{Map}_{\mathrm{Fun}(\coprod *, C)}(z_i, \mathrm{const}_x) \simeq \prod_{i=1}^{n} \mathrm{Map}_C(z_i, x)$$

where the last equivalence follows from the dual to [20],4.3.24.

Now, we are left to check the 'dual property', namely that j carries finite coproducts in C to finite coproducts in $\mathcal{P}_\Sigma(C)$, i.e. that there is an equivalence of pre-sheaves $\mathrm{Map}_C(-, \coprod_F x_i) \simeq \coprod_F \mathrm{Map}_C(-, x_i) \in \mathcal{P}_\Sigma(C)$ over any finite F.
As an application of Yoneda Lemma (see [24],5.5.2.1 or - from a more ∞-categorical perspective - apply the homotopy-coherent version of the Yoneda Lemma to the natural transformation induced by evaluation at an object and conclude by [20]2.2.2), we can regard any representable functor $ev_y := \mathrm{Map}_C(-, y)$ as the composition of the opposite Yoneda embedding $j^{op} : C^{op} \to \mathcal{P}_\Sigma(C)^{op}$ followed by the functor represented by ev_y, namely $\mathrm{Map}_{\mathcal{P}_\Sigma(C)}(-, ev_y)$, which corresponds to 'evaluation at y' under the Yoneda Lemma.
Hence, it suffices to check that for any representable functor $e : \mathcal{P}_\Sigma(C)^{op} \to Spc$, the composition

$$C^{op} \xrightarrow{j^{op}} \mathcal{P}_\Sigma(C)^{op} \xrightarrow{e} \mathrm{Spc}$$

preserves all finite products in C^{op}. Finally, as observed in the first paragraph of [24],5.5.2, even more holds, namely that the composition preserves all small limits. Indeed, j^{op} does so by [24],5.1.3.2, while the evaluation e preserves lim, since in presheaves categories those are computed object-wise (see either [24],5.1.2.2 or the proof of [20],5.1.27).

(3) : Write $j_C(x) := \mathrm{Map}_C(-, x)$. The Yoneda Lemma (see [20],4.2.10) yields, for each $d \in D$, an equivalence of functors (in $\star$):

$$G(d)(\star) \simeq \mathrm{Map}_{\mathcal{P}(C)}(\mathrm{Map}_C(-,\star), G(d)) \simeq \mathrm{Map}_D(F \circ \mathrm{Map}_C(-,\star), d) \simeq j_D(d) \circ (F \circ j_C)^{op}(\star)$$

Call $f := F \circ j_C$ and assume D to be small. Then, (by [24],5.1.3.2) j_D preserves those limits which exist in the *small* ∞-category D^{op}, so, if f^{op} preserves $\prod^{<\omega}$, it follows that also $G(d) \simeq j_D(d) \circ f^{op}$ does so, and hence it belongs to the essential image of the f.f. embedding $\mathcal{P}_\Sigma(C) \subseteq_{f.f.} \mathcal{P}(C)$.

Conversely, let us assume that $G(d) \in \mathcal{P}_\Sigma(C)$ is $\prod^{<\omega}$-preserving. Then, we have a chain of equivalences:

$$\mathrm{Map}_D(f(\coprod_{i=1}^n x_i), d) \simeq G(d)(\prod_{i=1}^n x_i) \simeq \prod_{i=1}^n G(d)(x_i) \simeq \prod_{i=1}^n \mathrm{Map}_D(f(x_i), d) \simeq \mathrm{Map}_D(\coprod_{i=1}^n f(x_i), d)$$

where the latter equivalence is the dual of [24],5.1.3.2.

Therefore, we can conclude, by the fully-faithfulness of the Yoneda embedding, that f preserves finite coproducts.

Finally, the following claim shows that our reasoning generalizes also to a presentable category.

Claim. *wlog D is small.*

Proof. As part of the datum for a presentable category D, we are given a small full subcategory $D^0 \subseteq_{f.f.} D$ spanned by small objects and which generates the latter under filtered colimits $\varinjlim$. Since $\varinjlim$ commutes with both $\prod^{<\omega}$ (filtered diagrams are in particular sifted, so see A.1.0.6) and representable functors in the image of j_D (see [24],5.3.5.14), our reasoning generalizes to presentable categories. ■

(4) : *Spc* admits both $\prod^{<\omega}$ and colim^{sift}, so that also $\mathcal{P}(C)$ has them; moreover, by Lemma A.1.0.6, $\prod^{<\omega}$ and colim^{sift} commute. Thus, $\mathrm{colim}^{sift} : \mathrm{Fun}(K, \mathcal{P}_\Sigma(C)) \to \mathcal{P}(C)$ lands into the full subcategory $\mathcal{P}_\Sigma(C) \subseteq_{f.f.} \mathcal{P}(C)$. Indeed, by the proof of [20],5.1.27 (or, more directly, by [24],5.1.2.2), colimits of pre-sheaves are computed object-wise, so that, for a diagram $p : K \to \mathcal{P}_\Sigma(C)$ of pre-sheaves on C $p : K \to \mathcal{P}_\Sigma(C)$, one can compute its colimit at $\prod_{i=1}^n x_i \in C^{op}$ by:

$$\left(\mathrm{colim}_K^{sift}\, p(k)\right)(\prod_{i=1}^n x_i) \simeq \mathrm{colim}_K^{sift} \left(p(k)(\prod_{i=1}^n x_i)\right) \simeq \mathrm{colim}_K^{sift} \left(\prod_{i=1}^n p(k)(x_i)\right)$$

where the latter equivalence follows from the fact that p takes values into finite-product-preserving pre-sheaves on C. Then, as recalled $\prod^{<\omega}$ and colim^{sift} commute, so that we can rewrite the latter as follows:

$$\mathrm{colim}_K^{sift} \left(\prod_{i=1}^n p(k)(x_i)\right) \simeq \prod_{i=1}^n \mathrm{colim}_K^{sift} \left(p(k)(x_i)\right) \simeq \prod_{i=1}^n \left(\mathrm{colim}_K^{sift}\, p(k)\right)(x_i) \quad ■$$

(5) : Let $\overline{p} : \overline{K} \to \mathcal{P}(C)$ be a sifted colimit cone extending the (sifted) diagram $p := \overline{p}_{|K}$. Notice that $L(p) \in \mathcal{P}_\Sigma(C)$ has a colimit cone in the cocomplete ∞-category $\mathcal{P}(C)$. Now, since left adjoints preserve arbitrary colimits (whenever they exist in both the source and the target category), $L(\overline{p})$ must be again a sifted colimit cone for $L(p)$. Furthermore, since $L(p)$ takes values in $\mathcal{P}_\Sigma(C)$ and the latter category is closed under colim^{sift} by point (4), also $L(\overline{p})$ must take values into $\mathcal{P}_\Sigma(C)$, and hence $L(\overline{p}(\infty)) = L(\overline{p})(\infty) \in \mathcal{P}_\Sigma(C)$, as stated. ■

(6) : First recall that, by [24],5.3.5.12, the ∞-category of pre-sheaves $\mathcal{P}(C)$ is presentable, and hence compactly generated, i.e. there is some regular cardinal κ for which the full subcategory $\mathcal{P}(C)^{<\kappa} \subseteq_{f.f.} \mathcal{P}(C)$ spanned by the $< \kappa$-compact objects $\varinjlim^{<\kappa}$-generates $\mathcal{P}(C)$.

Let $L : \mathcal{P}(C) \to \mathcal{P}_\Sigma(C)$ be the Bousfield localization functor of point (1). Notice that, being it colimit-preserving and essentially surjective, $L(\mathcal{P}(C)^{<\kappa})$ $\varinjlim$-generates $\mathcal{P}_\Sigma(C)$. Then, to prove our statement it suffices to show that the former category consists of κ-compact objects.

To this end, observe that, for each $e \in \mathcal{P}(C)^{<\kappa}$ the unit $\eta : id_{\mathcal{P}(C)} \to L$ induces the triangle identity (point-wise described in [20],5.1.11):

$$\mathrm{Fun}(\mathcal{P}_\Sigma(C), \mathrm{Spc}) \ni f := \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(Le, -) \xrightarrow{[e \xrightarrow{\eta} Le]^* \circ \subseteq} \mathrm{Map}_{\mathcal{P}(C)}(e, -) =: f' \in \mathrm{Fun}(\mathcal{P}(C), \mathrm{Spc})$$

which is point-wise an equivalence, and hence, if we restrict the target to its essential image $\mathcal{P}_\Sigma(C)$, also an actual equivalence (see [20],2.2.2):

$$\eta^* \circ \subseteq : f \xrightarrow{\simeq} f'_{|\mathcal{P}_\Sigma(C)} \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(E, \subseteq (-))_{|\mathcal{P}_\Sigma(C)}$$

Now, f' is corepresented by the κ-compact object $e \in \mathcal{P}(C)$, so that, in particular, it commutes with filtered colimits in $\mathcal{P}_\Sigma(C)$ (recall that they exist there by point (4)). Thus, also f must be $\varinjlim^{<\kappa}$-cocontinuous, i.e. the representing object $Le \in \mathcal{P}_\Sigma(C)$ must be κ-compact, as needed. ■

□

We want now to provide an 'informal' converse to Property (4), namely to show that $\mathcal{P}_\Sigma(C) \subseteq \mathrm{sInd}(C)$, by proving that each pre-sheaf in $\mathcal{P}_\Sigma(C)$ can be regarded as the (sifted) geometric realization of a simplicial object in $\mathrm{Ind}(C)$. We added the adjective 'informal', because in-so-far free sInd-completions have not yet been defined. This will be indeed the content of the next subsection, in which we will spell out the universal property enjoyed by such classes.
We will first state a preliminary technical Lemma, which will not be proved.

Lemma A.2.0.3. *(Tree realization of pre-sheaves, [24],5.5.8.13) Let $C \in \mathrm{Cat}_\infty$ be small and consider a pre-sheaf $X \in \mathcal{P}(C)$. Then, there exists a simplicial object in $\mathcal{P}(C)$, say $Y_\bullet : \mathcal{N}(\Delta)^{op} \to \mathcal{P}(C)$ s.t.*

1. $|Y_\bullet| \simeq \mathrm{colim}^{sift}\, Y_\bullet \simeq X$.
2. *For each $n \geq 0$, there exists some small family $\{z_a\}_{A_n} \subseteq C$ s.t. $\mathcal{P}(C) \ni Y_n \simeq \coprod_{a \in A_n} j(z_a)$.*

We are now ready for the claimed result, namely that $\mathcal{P}_\Sigma(C)$ is the smallest full subcategory of $\mathcal{P}(C)$ which contains the essential image of the Yoneda embedding (A.2.0.2,ii) and is closed under filtered colimits and geometric realizations (A.2.0.2,iv).

Lemma A.2.0.4. *([24],5.5.8.14) Let $C \in \mathrm{Cat}_\infty$ admit finite coproducts. Then, for each pre-sheaf $X \in \mathcal{P}(C)$, $X \in \mathcal{P}_\Sigma(C)$ iff there exists some simplicial object $U_\bullet : \mathcal{N}(\Delta)^{op} \to Ind(C)$ whose geometric realization is $|U_\bullet| \simeq \mathrm{colim}^{sift}\, U_\bullet \simeq X$.*

Proof. ($\Longleftarrow$) First recall that both filtered colimits and geometric realizations are special sifted colimits. Moreover, by A.2.0.2,i) $j(C) \subseteq \mathcal{P}_\Sigma(C)$ and by A.2.0.2,iv) $\mathcal{P}_\Sigma(C)$ is stable under sifted colimits. Hence, $\mathrm{Ind}(C) \subseteq \mathrm{sInd}(C) \subseteq \mathcal{P}_\Sigma(C)$ and, in particular, for any simplicial object as before, $X \simeq \mathrm{colim}^{sift}\, U_\bullet \in \mathcal{P}_\Sigma(C)$.
($\Longrightarrow$) Let $Y_\bullet$ be a 'tree' for $X \simeq \mathrm{colim}^{sift}\, Y_\bullet$ as in the previous technical Lemma. Let $L : \mathcal{P}(C) \to \mathcal{P}_\Sigma(C)$ be the Bousfield localization functor of A.2.0.2,i). The counit of the adjunction is an equivalence (see the proof of [20],5.1.8) and the unit preserves sifted colimits by A.2.0.2,v), so that $X \simeq LX \simeq \mathrm{colim}^{sift}\, LY_\bullet$. Define $U_\bullet := L \circ Y_\bullet$; the last technical claim finishes the proof.

Claim. $LY_n \in \mathrm{Ind}(C)$ for each $n \in \mathbb{N}$.

Proof. By assumption, $Y_n \simeq \coprod_{a \in A_n} j(z_a)$ for $A := A_n$ small indexing sets. wlog $\#A < \infty$.
Indeed, consider the filtration $\mathcal{F} := \{F \subseteq A | \#F < \infty\}$ of the finite parts of A. Then, $Y_n \simeq \coprod_A j(z_a) \simeq \varinjlim \big(\coprod_F j(z_a) \mid F \in \mathcal{F} \big)$ and, since the left-adjoint L preserves (filtered) colimits, it suffices to show that $L\big(\coprod_F j(z_a) \big) \in \mathrm{Ind}(C)$ for an arbitrary $F \in \mathcal{F}$.

Now, as proved in A.2.0.2,ii), the Yoneda embedding j factors through $\mathcal{P}_\Sigma(C)$ and carries finite coproducts in C to finite coproducts in $\mathcal{P}_\Sigma(C)$. Thus, we are left to consider $L \circ j(\coprod_F z_a) \simeq L\big(\coprod_F j(z_a) \big)$. However, $L \circ j \simeq (L \circ \subseteq) \circ j_\Sigma \xrightarrow{\epsilon j_\Sigma} j_\Sigma$ is an equivalence, because the counit ϵ of the Bousfield localization is such.
Therefore, we reduced the claim to the statement $Y_n \simeq \varinjlim_{\mathcal{F}} j(\coprod_F z_a) \in Ind(C)$, which is true by assumption. □

A.3 The Universal Property of $\mathcal{P}_\Sigma$

In the current subsection we will show that the $\mathcal{P}_\Sigma$-construction enjoys the universal property of free sInd-completions, so it 'freely adjoins' sifted colimits. We will maintain the same notation and abbreviations of the previous sections.
We first need a technical result which allows us to use left Kan extensions so as to characterize free completions of $C \in \mathrm{Cat}_\infty$ obtained by freely adjoining a given class of universal objects.

Lemma A.3.0.1. *([24],5.3.5.8 as in [24],5.3.5.9) Let $C \in \mathrm{Cat}_\infty$ be small and let Σ denote some class of colimits existing on C (i.e. some set of diagrams into C which admit a colimit extension). Consider the* smallest *full subcategory $C_\Sigma \subseteq_{f.f.} \mathcal{P}(C)$ which contains the essential image of the Yoneda embedding $j(C)$ and is stable under colimits of type Σ, i.e. admits a well-defined functor colim^Σ.*
Let $D \in \mathrm{Cat}_\infty$ have colimits of type Σ. Then,

1. *For each functor $f : j(C) \to D$, there exists an extension $F : C_\Sigma \to D$ s.t. $F \simeq Lan_j(f)$ is a left Kan extension of its restriction $f \simeq F \circ j \simeq F_{|C}$.*
2. *Conversely, a functor $F : C_\Sigma \to D$ is the Left Kan extension of its restriction to C, namely $F \simeq Lan_j(F_{|C})$, iff $F_{|C}$ preserves colimits of type Σ.*

Proof. Recall that, in our setting, a left Kan extension $Lan_j(f) : C_\Sigma \to D$ of a functor $f : j(C) \to D$ can be thought of as the transpose of f under the adjunction $Lan_j := j_! \vdash j^*$, as induced by the restriction $j^* : \mathrm{Fun}(C_\Sigma, D) \to \mathrm{Fun}(j(C), D)$ along the Yoneda embedding. Such a model independent perspective makes sense, because in what follows we will be able to assume that wlog D is cocomplete; hence, we can apply [24],4.3.2.13 to show that each functor f admits a left Kan extension along j and then conclude, by [24],4.3.2.17, that such extensions can be grouped into the stated adjunction.
Let's begin with a reduction step:

Lemma. *([24],5.3.5.7) Given any $D \in \mathrm{Cat}_\infty$ (possibly not small) there exists a f.f. embedding $i : D \subseteq_{f.f.} D'$ into some $D' \in \mathrm{Cat}_\infty$ with all small colimits and s.t. i preserves colimits in D and detects those in D'.*

Proof. Define $D' := \mathrm{Fun}(D, \hat{\mathrm{Spc}})^{op}$, $i := \big(j : D \hookrightarrow \mathcal{P}(D)\big)^{op}$. Then, D' is co-complete by [24],5.1.2.2 (since Spc is bi-complete) and i has the stated co-continuity, since j commutes with all existing limits (by [24],5.1.3.2). ■

(1) : Fix an arbitrary functor $f : j(C) \to D$ and assume that wlog D embeds into some D' as in the reduction Lemma. For the rest of our reasoning, notice that we can identify $D \simeq i(D)$ into D' (see the first part of the proof of [24],5.3.6.2).
Then, an application of the theory of Kan extensions (see [24],4.3.2.13) to our setting yields, as in [24],5.1.5.5,ii), a left Kan extension $\overline{F} : \mathcal{P}(C) \to D$ of $f = i \circ f$, which is furthermore colimit-preserving (see *ibid*,i).
Now, being $D \subseteq_{f.f.} D'$ colim^Σ-stable (by assumption and because i preserves colimits), and being (again by assumption) C_Σ colim^Σ-generated by $j(C)$, we can conclude that the restriction $F := \overline{F}_{|C_\Sigma}$ factors through D. Therefore, we obtain a functor $F : C_\Sigma \to D$ as needed, namely s.t. it preserves colim^Σ and s.t. $F \simeq Lan_j(f)$.

(2) : ($\Longrightarrow$) Recall that Kan extensions are essentially unique, so, if F preserves colim^Σ, then each left Kan extension of f must be colim^Σ-cocontinuous.

($\Longleftarrow$) Now, let it be given some $F : C_\Sigma \to D$ as in (1) and consider any $F' : C_\Sigma \to D$ which restricts to $F_{|C}$ over C (i.e. $F'_{|C} \simeq F_{|C}$) and hence in particular s.t. $F_{|C}$ preserves colim^Σ; we claim that also $F' \simeq Lan_j(F'_{|C})$, so that (equivalently) $F' \simeq F$.
In order to see this, recall the $[UP : Lan]$: being $F \simeq Lan_j(F_{|C})$ and since the restrictions $F_{|C} \simeq F'_{|C}$ coincide, there must be a natural transformation $\alpha : F \to F'$ extending the identity transformation over C (more formally, such that $\alpha_{|C}$ is an equivalence).
Let $\mathcal{E} \subseteq_{f.f.} C_\Sigma$ be spanned by the 'equivalence-locus' of α over C_Σ, i.e. by $\{x \in C_\Sigma \mid \alpha_x : F(x) \xrightarrow{\sim} F'(x) \in \mathrm{Mor}(D)\}$.
Then, notice that $j(C) \subseteq \mathcal{E}$ and that the latter is stable under colimits of type Σ, since both F and F' are colim^Σ-cocontinuous. Thus, by the minimality of C_Σ, the previous embedding must be an equivalence $\mathcal{E} \simeq C_\Sigma$, which means (by [20],2.2.2) that $\alpha : F \to F'$ is an equivalence in $\mathrm{Fun}(C_\Sigma, D)$, as required. □

Proposition A.3.0.2. *(Left Derived Functors, [24],5.5.8.15) Let $C \in \mathrm{Cat}_\infty$ admit small coproducts and consider $D \in \mathrm{Cat}_\infty$ with both filtered colimits and geometric realizations; define*

$$\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D) \subseteq_{f.f.} \mathrm{Fun}(\mathcal{P}_\Sigma(C), D)$$

to be the full subcategory generated by those functors which preserve both filtered colimits and geometric realizations. Then,

1. $[UP : \mathrm{sInd}] : j_\Sigma : C \to \mathcal{P}_\Sigma(C)$ *induces an equivalence*

$$\theta : \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D) \xrightarrow{\simeq} \mathrm{Fun}(C, D)$$

Given any $f \in Fun(C, D)$, we say that the corresponding colim^{sift}*-preserving extension $F \in Fun_\Sigma(\mathcal{P}_\Sigma(C), D)$ determined under θ is the* ***left derived functor*** *of f.*

2. *Each $g \in \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D)$ preserves sifted colimits.*

3. *Assume further that D has also finite coproducts; then, $g \in \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D)$ is cocontinuous iff $g \circ j_\Sigma$ preserves finite coproducts.*

Proof. (1) : Let Σ denote the smallest class of diagrams in $\mathcal{P}(C)$ which contains both filtered diagrams and simplicial objects. As already observed, by A.2.0.4, $\mathcal{P}_\Sigma(C)$ is the smallest full subcategory of $\mathcal{P}(C)$ which contains the essential image of the Yoneda embedding and is closed under filtered colimits and geometric realizations, i.e. under colimits of type Σ.
So, we can apply A.3.0.1 in order to obtain an equivalence $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D) \simeq \mathrm{Lan}_j(j(C), D)$, where the latter denotes the full subcategory of $\mathrm{Fun}(j(C), D)$ spanned by left Kan extensions of functors in there; indeed, the two full subcategories are, up to homotopy, spanned by the same objects.

Now, by [24],4.3.2.15, $\mathrm{Lan}_j(j(C), D) \simeq \mathrm{Fun}(C, D)$, as desired. Indeed, in our notation, we can rewrite the latter statement as follows.
Let $\mathcal{K} := Lan_j(j(C), D) \subseteq_{f.f.} \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D)$ be spanned by those functors $G : \mathcal{P}_\Sigma(C) \to D$ s.t. $G \simeq Lan_j(G_{|C})$. Moreover, let $\mathcal{K}' \subseteq_{f.f.} \mathrm{Fun}_\Sigma(C, D)$ denote the full subcategory spanned by those functors $g : C \to D$ s.t. for each $F \in \mathcal{P}_\Sigma(C)$ the induced $C/F \to D$ admits a colimiting cone in D of type Σ. Notice that, under the assumptions on D, the latter condition is always satisfied by functors in $\mathrm{Fun}(C, D)$, i.e. $\mathcal{K}' \simeq \mathrm{Fun}(C, D)$. Now, the inference of [24],4.3.2.15 is that the canonical map $\mathrm{Lan}_j(j(C), D) = \mathcal{K} \to \mathcal{K}' \simeq \mathrm{Fun}(C, D)$ is a trivial fibration, and hence in particular an equivalence. ■

(2) : By the reduction step of A.3.0.1 (namely [24],5.3.5.7) a functor $g \in \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D)$ preserves sifted colimits iff its composite $e \circ g$ with any representable functor $e : D \to \mathrm{Spc}^{op}$ does so.
So, we can assume wlog $D \simeq \mathrm{Spc}^{op}$, which has sifted colimits. We will not need any other property of Spc, so we will keep writing D for the target and just assume that the latter admits sifted colimits.
Define the full subcategory $\mathrm{Fun}'_\Sigma(\mathcal{P}_\Sigma(C), D) \subseteq_{f.f.} \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D)$ spanned by those functors which preserve sifted colimits, and let's leverage on A.2.0.4 to prove that such an embedding is actually an equivalence.
To this end, observe that we can repeat our reasoning as in (1) to show that there is an equivalence $\theta' : \mathrm{Fun}'_\Sigma(\mathcal{P}_\Sigma(C), D) \to \mathrm{Fun}(C, D) \simeq_\theta \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), D)$. ■

(3) : Now suppose that D admits also finite coproducts.
($\Longrightarrow$) By A.2.0.2,ii), j_Σ preserves $\coprod^{<\omega}$, so we are done.

($\Longleftarrow$) As in the reduction step of A.3.0.1, wlog D is presentable. Again similarly to *loc.cit*, apply [24],5.1.5.5 to obtain a cocontinuous $G : \mathcal{P}(C) \to D$ s.t. $G \simeq Lan_j(G_{|j(C)} = g_{|j(C)})$. Then, by the I Adj. Functor Theorem 1.2.0.5, G admits a right adjoint $G^R : D \to \mathcal{P}(C)$. We would like the latter to factor through $\mathcal{P}_\Sigma(C)$, which (by A.2.0.2,iii) would be equivalent to $G \circ j$ being finite-coproducts-preserving.

Claim. wlog G extends g, i.e. $G_{|\mathcal{P}_\Sigma(C)} \simeq g$. Hence, in particular $G \circ j \simeq g \circ j$ preserves finite coproducts and so $G^R : D \to \mathcal{P}_\Sigma(C) \subseteq_{f.f.} \mathcal{P}(C)$, as desired.

Proof. Call $G_0 := G_{|\mathcal{P}_\Sigma(C)}$ the restriction of G. Notice that we obtain again a left Kan extension $G_0 \simeq Lan_j(g_{|j(C)})$, this time on $\mathcal{P}_\Sigma(C)$. Then, since both G_0 and g agree on $j(C)$, the $[UP : Lan]$ yields a natural transformation $\alpha : G_0 \to g$ which extends (up to homotopy) the identity on $j(C)$. As always, we let $\mathcal{E}$ denote the full subcategory of $\mathcal{P}_\Sigma(C)$ spanned by the 'equivalence-locus' of α in $\mathcal{P}_\Sigma(C)$. Again, we can conclude that the embedding was actually an equivalence $\mathcal{E} \simeq \mathcal{P}_\Sigma(C)$, since the latter contains $j(C)$ and is closed under both filtered limits and geometric realizations (this because both G_0 and g are so). In other words, $\alpha : G_0 \simeq g$ (by [20],2.2.2).

Thanks to such a factorization, we can then conclude the existence of a colimit-preserving functor $G' : \mathcal{P}_\Sigma(C) \to D$ s.t. $G \simeq G' \circ L$.

$$\begin{tikzcd}
C \arrow[r, hook, "j"] \arrow[dr, hook, "j_\Sigma"'] & \mathcal{P}(C) \arrow[rr, shift left, "G"] \arrow[d, shift left, "L"] & & D \arrow[ll, shift left, "G^R"] \arrow[dll, shift left, "G'^R"] \\
& \mathcal{P}_\Sigma(C) \arrow[u, hook, shift left] \arrow[urr, shift left, "G'"] & &
\end{tikzcd}$$

Indeed, the factorization through $\mathcal{P}_\Sigma(C)$ yields a functor of presentable categories $G'^R : D \to \mathcal{P}_\Sigma(C)$ s.t. $G^R \simeq \subseteq \circ G'^R$. The latter clearly preserves all small limits: having complete source and target, the right-adjoint $\subseteq$ not only preserves limits, but commutes with them; moreover, G'^R is accessible, because $G'^R \simeq L \circ G^R$ and both functors are so. Thus, G'^R admits a left-adjoint G'.

But now, since adjuncts are essentially unique, we recover the adjunction $G \vdash G^R$ as the composite of the induced $G' \vdash G'^R$ after $L \vdash \subseteq$, so that $G \simeq G' \circ L$. Moreover, being it a left adjoint, G' clearly preserves all small colimits.

Then, also $g \simeq G_{|\mathcal{P}_\Sigma(C)} \simeq G'$ must preserve all small colimits, as needed. ■ □

Remark. Let us briefly summarize the philosophy on which the proof of (2) grounds.
In each case we considered functors preserving colimits of a given type S, say $S = \Sigma, \Sigma'$, whose source was the corresponding free S-colimit completion of the original source C.
Now, [24],5.3.5.9 means that the 'intrinsic universality' of the colimit construction (we refer to both the proof and the role of [24],4.3.2.13 in the lemma) allows us to characterize such functors as left Kan extensions of functors $C \to D$, which preserve the right S-cocontinuity.
But, as we have seen in [24],4.3.2.15, if our target category has enough colimits, then left Kan extensions have automatically the needed cocontinuity and $\mathrm{Fun}(C, D)$ suffices to fully determine all the S-cocontinuous extensions at stake. Finally, we observe that, for similar statements (read: when we can adapt the first part of the proof of [24],5.3.6.2), the cocompleteness of the target D is an empty condition, because it can always be achieved up to a 'cunning' replacement, such as in [24],5.3.5.7.

As a Corollary, we will finally enhance to the ∞-world the decomposition of sifted colimits by means of filtered ones and geometric realizations.

Corollary A.3.0.3. *([24],5.5.8.17) Let $C \in \mathrm{Cat}_\infty$ have all small colimits (equivalently, all sifted colimits and finite coproducts) and let $f \in \mathrm{Fun}(C, D)$ for some $D \in \mathrm{Cat}_\infty$. Then, f preserves sifted colimits iff f preserves filtered colimits and geometric realizations.*

Proof. ($\Longrightarrow$) As we already noticed, both filtered colimits and geometric realizations are particular types of sifted colimits.
($\Longleftarrow$) Let $J \in \mathrm{sSet}$ be a (small) sifted simplicial set, and let $p : J \to C$ be a sifted diagram. Given a colimit cone $\overline{p} : \overline{J} \to C$ of the latter, we wish to show that $f \circ \overline{p} : \overline{J} \to D$ is again a colimit cone for $f \circ p$.
Let $\mathcal{J} \subseteq_{f.f.} \mathcal{P}(J)$ be the smallest full sub-∞-category which contains the essential image of the Yoneda embedding $j(J)$ and which is closed under finite coproducts.
By the usual A.3.0.1, there exists come $q : \mathcal{J} \to C$ which preserves finite coproducts and s.t. $p \simeq q \circ j$.
Then, by the first part of the above Proposition, we can factor $q \simeq \big(\mathcal{J} \xhookrightarrow{j_\Sigma} \mathcal{P}_\Sigma(\mathcal{J}) \xrightarrow{q'} C\big)$; furthermore, by *ibid.*iii), q' preserves all small colimits.
Remark. We needed to pass to $\mathcal{J}$ because, in order to apply the previous Proposition, we need the source of q to be an ∞-category with finite coproducts.
Now, we have that the composition $f \circ q' : \mathcal{P}_\Sigma(\mathcal{J}) \to D$ must live in $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{J}), D)$, since both functors preserve filtered colimits and geometric realizations. But then, by *ibid.*ii), $f \circ q'$ must preserve also sifted colimits.
Now, consider the (sifted) colimit cone $\overline{g} := \overline{j_\Sigma \circ j} : \overline{J} \to \mathcal{P}_\Sigma(\mathcal{J})$ of $g := j_\Sigma \circ j$.
Since q' preserves all small colimits, in particular $q' \circ \overline{g}$ is a colimit cone of $q' \circ j_\Sigma \circ j \simeq q \circ j \simeq p$, i.e. by $[UP : \mathrm{colim}]$ $q' \circ \overline{g} \simeq \overline{p}$. Hence, $f \circ \overline{p} \simeq f \circ (q' \circ \overline{g}) \simeq (f \circ q') \circ \overline{g}$.
Now, g is sifted, so $(f \circ q') \circ \overline{g}$ is a colimit cone of $f \circ q' \circ g \simeq f \circ p$, as desired. □

Remark. Notice that we use the characterization of sifted colimits in terms of filtered ones and geometric realizations of the previous Proposition precisely when we infer that $f \circ q'$ must preserve sifted colimits. The rest of the proof is mostly a formal 'divide et impera', in that it is basically a decomposition of the original diagram p into two 'nicer' pieces, $q' \circ g$, which we can handle.

Another application of Proposition A.3.0.2, is to obtain some kind of 'functoriality' of the $\mathcal{P}_\Sigma$-construction. In what follows, we will argue it up to 1-dimensional simplices (so spines) in Cat_∞.

Definition A.3.0.4. *(Total derived functors, [3],5.1.4)* Let $F : C \to D$ be a sifted-colimits-preserving functor of complete ∞-categories. Call **total left derived functor** of F the left derived functor

$$\mathcal{P}_\Sigma(F) : \mathcal{P}_\Sigma(C) \to \mathcal{P}_\Sigma(D)$$

obtained by applying A.3.0.2 to $F : C \to D \subseteq_{f.f.} \mathcal{P}_\Sigma(D)$. It is the essentially unique functor between the given non-abelian localizations which satisfies the following properties:

1. $\mathcal{P}_\Sigma(F)$ preserves sifted colimits;

2. Its restriction $\mathcal{P}_\Sigma(F)_{|C} : C \to D \subseteq_{f.f.} \mathcal{P}_\Sigma(D)$ agrees with F;

Despite the name, our notion is compatible with composition along 1-dimensional simplices of Cat_∞.

Proposition A.3.0.5. *(Composition of total derived functors) Let C, D, E be cocomplete ∞-categories, and consider composable sifted-colimits-preserving functors F, G between them. Then, one has an equivalence $\mathcal{P}_\Sigma(G) \circ \mathcal{P}_\Sigma(F) \simeq \mathcal{P}_\Sigma(G \circ F)$.*

Proof. In view of A.3.0.2 and [24],5.1.2.3 (or the proof of [20],5.1.27), it suffices to investigate the composition on the original categories C, D, E.
There, $\mathcal{P}_\Sigma(G) \circ \mathcal{P}_\Sigma(F)_{|C} \simeq \mathcal{P}_\Sigma(G)_{|F(C)} \simeq G_{|F(C)} \simeq G \circ F_{|C} \simeq \mathcal{P}_\Sigma(G \circ F)_{|C}$, which can be extended to an equivalence of total left derived functors. □

We close this section with a Lemma which states that non-abelian localization preserves initial or terminal objects.

Lemma A.3.0.6. *Let $C \in \mathrm{Cat}_\infty$ be a small ∞-category which admits finite coproducts. Assume that C has an initial object x. Then, $\mathcal{P}_\Sigma(C)^{\mathrm{init}}$ is a contractible ∞-groupoid essentially equivalent to the category with one object $j_\Sigma(x)$, where j_Σ is the factorization of the Yoneda embedding.*

Proof. As already observed, $C \subseteq_{f.f.} \mathcal{P}_\Sigma(C)$ via the Yoneda embedding j_Σ. Then, the copy of any initial object $x \in C$ in $\mathcal{P}_\Sigma(C)$ is equivalent to $\mathrm{Map}_C(-,x) : (C)^{op} \to \mathrm{Spc}$, and the latter is initial in $\mathcal{P}_\Sigma(C)$ by an application of Yoneda Lemma: for each $F \in \mathcal{P}_\Sigma(C)$,

$$\mathrm{Map}_{\mathcal{P}_\Sigma(C)}(\mathrm{Map}_C(-,x), F) \simeq \mathrm{Map}_{\mathcal{P}(C)}(\mathrm{Map}_C(-,x), F) \simeq F(x) \in \mathrm{Spc}^{\mathrm{term}}$$

where the latter equivalence is due to the fact that F preserves finite products (hence in particular terminal objects) and $x \in \left((C)^{op}\right)^{\mathrm{term}}$. But then $\mathrm{Spc}^{\mathrm{term}} \simeq \Delta^0$, so that each mapping space from $j_\Sigma(x)$ is indeed contractible.
Finally, as observed in [20],4.1.3, if inhabited, the subcategory of initial objects is a contractible ∞-groupoid. □

Remark. A dual statement holds for 'terminal' in place of 'initial' objects.

A.4 Compact and Projective Objects Determine Free sInd-Completions

In the current subsection our aim is to gain a better understanding of which ∞-categories can actually occur in the 'essential image' of the $\mathcal{P}_\Sigma$-construction.
In other words, given an arbitrary ∞-category D, we want to understand whether there exists some full subcategory $C \subseteq_{f.f.} D$ s.t. $\mathcal{P}_\Sigma(C) \simeq D$.
The idea is as follows: the $\mathcal{P}_\Sigma$-construction is a free *sInd*-completion, so, colim^{sift}-generating D from C, corresponds to the fact that the left derived functor $F : \mathcal{P}_\Sigma(C) \to D$ of the embedding $f : C \subseteq_{f.f.} D$ is an equivalence. In analogy with the study of presentable (and accessible) categories, we would like then to identify a class of 'elementary bricks' of D, that means, intuitively, a class of objects of D which 'detects' the fully faithfulness and the essential surjectivity of F, and we would like C to consist of such 'bricks'.

We will soon identify such 'nice' objects with the cpt+proj's of D, and we will call D^{sfp} the full subcategory spanned by those.
Then, we will show that, in order for $\mathcal{P}_\Sigma(C)$ to be a full subcategory of D (i.e. for F to be f.f.), C must be contained into the aforementioned full subcategory D^{sfp} of cpt+proj objects of D. Otherwise, the free $sInd$-completion generated by C turns out to be too big. Therefore, by putting constraint on the size of C, we can really morally regard D^{sfp} as consisting of 'elementary bricks' for D.
Furthermore, at the same time we will prove that, accordingly to the intuitive meaning of a free $sInd$-completion, D can be regarded as $\mathcal{P}_\Sigma(C)$ if and only if the copy of C in D is colim^{sift}-dense.
From such a perspective, we may then wonder whether it must necessarily hold $C \simeq D^{sfp}$. In order to address this question, we will call 'set of cpt-proj generators' of D any colim^{sift}-dense sub-class of 'elementary bricks' of D; this will turn out to actually describe the family of all the possible C's for which $D \simeq \mathcal{P}_\Sigma(C)$, and will finally allow us to provide a 'minimal model' for such a family.

The notion of being 'compact' is already well-known and relates to being an 'elementary brick' for a free Ind-completion. Let us define then the property of being projective.

Definition A.4.0.1. *(Projective object, [24],5.5.8.18)* Let $C \in \mathrm{Cat}_\infty$ have geometric realizations of its simplicial objects.
We say that an object $x \in C$ is **projective** iff $\mathrm{Map}_C(x,-) : C \to \mathrm{Spc}$ commutes with geometric realizations. Call $\mathrm{Proj}(C) \subseteq_{f.f.} C$ the full subcategory of C generated by its projective objects.

The following Lemma spells out two straightforward properties of projective objects.

Lemma A.4.0.2. *(Properties,[24],5.5.8.19-20) Let $C \in \mathrm{Cat}_\infty$ have finite coproducts and geometric realizations of its simplicial objects.*

1. *The subcategory of its projective objects* $\mathrm{Proj}(C)$ *has finite coproducts.*
2. *Assume further that C admits all small colimits. Then, $x \in C$ is cpt+proj iff* $\mathrm{Map}_C(x,-)$ *preserves sifted colimits.*

Proof. (1) : Recall that $\mathrm{Map}_C(\coprod_{i=1}^n x_i,-) \simeq \prod_{i=1}^n \mathrm{Map}_C(x_i,-)$ and that geometric realizations, being special sifted colimits, do commute with finite products.
(2) : ($\Longleftarrow$) is clear; ($\Longrightarrow$) Recall that cpt+proj means that $\mathrm{Map}_C(x,-)$ commutes with both filtered colimits and geometric realizations, respectively. In other words, with notation as in A.3.0.2, it corresponds to the fact that $\mathrm{Map}_C(x,-) \in \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(C), \mathrm{sSet})$. Hence, by *ibid.*ii), it also preserves sifted colimits, as desired. ☐

Warning. ([24],5.5.8.21) Let $\mathcal{A} \in \mathrm{Cat}$ be abelian, and recall that $P \in \mathcal{A}$ is called projective iff $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is (right) exact. The latter actually amounts to $\mathrm{Hom}_{\mathcal{A}}(P,-)$ preserving geometric realizations of simplicial objects in $\mathcal{A}$.
However, notwithstanding the similarity in terminology, the two notions are nevertheless distinct, since the canonical static-embedding $\mathrm{Set} \subseteq_{f.f.} \mathrm{Spc}$ (i.e. the right adjoint to the 0-truncation π_0) needs not preserve geometric realizations. Hence, if we let $\mathcal{A}_\infty$ denote the ∞-category underlying the model category $Ch_{\geq 0}(\mathcal{A})_{\mathrm{proj}}$, then $\mathrm{Hom}_{\mathcal{A}}(P,-) \simeq \pi_0\big(\mathrm{Map}_{\mathcal{A}_\infty}(\underline{P},-)\big)$ might preserve 'more' geometric realizations.

With reference to the introduction of the current subsection, we will now state and prove the Proposition implying that those categories C for which $D \simeq \mathcal{P}_\Sigma(C)$ must consist of 'elementary bricks' of D.

Proposition A.4.0.3. *([24],5.5.8.22) Let $C \in \mathrm{Cat}_\infty$ have finite coproducts and consider any $D \in \mathrm{Cat}_\infty$ which admits filtered colimits and geometric realizations.*
As in A.3.0.2, let $j : C \hookrightarrow \mathcal{P}(C)$ denote the Yoneda embedding and, for any functor $f : C \to D$, consider a left derived extension $F : \mathcal{P}_\Sigma(C) \to D$, so s.t. $f = F \circ j$ and F commutes with sifted colimits.
Consider the following three assumptions:

1. *f is fully faithful;*

2. The essential image of f consists of compact and projective objects;

3. D is generated by the essential image of f under filtered colimits and geometric realizations.

Then,

- $(1) + (2)$ *are equivalent to F being fully faithful.*
- (3) *is equivalent to F being essentially surjective; in particular,* $(1) + (2) + (3)$ *amount to F being an equivalence.*

Proof. Assume first $F : \mathcal{P}_\Sigma(C) \to D$ to be fully faithful.
(1) : being j f.f. clearly also $f = F \circ j$ is such.
(2) : Let $d \in \mathrm{EssIm}(f) \subseteq_{f.f.} D$, i.e. there is some $c \in C$ for which $d \simeq f(c)$. Then, from the very definition, $d \in D$ is cpt+proj iff $\mathrm{Map}_D(d, -) \simeq \mathrm{Map}_D(f(c), -) \simeq \mathrm{Map}_D(F \circ j_\Sigma(c), -)$ preserves filtered colimits and geometric realizations. Being F fully faithful, the latter is equivalent to the fact that $\mathrm{Map}_{\mathcal{P}_\Sigma(C)}(j_\Sigma(c), -)$ preserves such colimits.
Now, as in the proof of A.2.0.2,ii), observe that our functor can be identified with the composite $ev_c \circ \subseteq : \mathcal{P}_\Sigma(C) \subseteq \mathcal{P}(C) \to \mathrm{Spc}$, so that it must preserve both filtered colimits and geometric realizations, since each of the composite does so (see [24],5.5.8.14 and [24],5.1.2.2, respectively).

Conversely, assume conditions (1) and (2) on f, and let's show that F is f.f.

Claim 1. *F induces an equivalence of mapping spaces $C^{op} \times \mathcal{P}_\Sigma(C) \to \mathrm{Spc}$, i.e. if we denote by $\mathcal{P}'_\Sigma(C) \subseteq_{f.f.} \mathcal{P}_\Sigma(C)$ the full subcategory spanned by*

$$\{m \in \mathcal{P}_\Sigma(C) \mid \forall x \in C,\ F : \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(j_\Sigma(x), m) \xrightarrow{\simeq} \mathrm{Map}_D(f(x), F(m))\}$$

Then, $\mathcal{P}'_\Sigma(C) \simeq \mathcal{P}_\Sigma(C)$.

Proof. Notice that the representable presheaves of $\mathcal{P}_\Sigma(C)$ belong to $\mathcal{P}'_\Sigma(C)$, because both j and f are fully faithful (the latter by (1)).
Moreover, since by (2) $\mathrm{EssIm}(f)$ consists of cpt+proj objects of D, $\mathcal{P}'_\Sigma(C)$ is closed under filtered colimits and geometric realizations.
To see this, let $p : K \to \mathcal{P}'_\Sigma(C)$ be a sifted simplicial diagram, and let $\overline{p} : \overline{K} \to \mathcal{P}_\Sigma(C)$ be a colimit cone of p in $\mathcal{P}_\Sigma(C)$ (which exists by A.2.0.2,iv). Then,

$$\mathrm{Map}_D(f(x), F \circ \overline{p}) \simeq_{(a)} \mathrm{colim}_K \mathrm{Map}_D(f(x), F \circ p) \xleftarrow{F} \mathrm{colim}_K \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(j(x), p) \simeq_{(b)} \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(j(x), \overline{p})$$

where (a) holds because $f(x)$ is cpt+proj and F preserves sifted colimits, while (b) holds since, by the Yoneda Lemma, we can regard $\mathrm{Map}_{\mathcal{P}_\Sigma(C)}(j(x), -)$ as the left derived extension of $\mathrm{Map}_C(x, -)$.
Now, by the definition of $\mathcal{P}'_\Sigma(C)$, F induces a point-wise equivalence, so we actually obtain the desired one of K-colimits, i.e. $\overline{p}(\infty) \in \mathcal{P}'_\Sigma(C)$.
Thus, we have proved that $\big(j_\Sigma(C) \subseteq_{f.f.} \big)\mathcal{P}'_\Sigma(C) \subseteq_{f.f.} \mathcal{P}_\Sigma(C)$ is closed under sifted colimits, which in turn implies $\mathcal{P}'_\Sigma(C) \simeq \mathcal{P}_\Sigma(C)$ by the minimality of $\mathcal{P}_\Sigma(C)$. ■

Claim 2. *Let $\mathcal{P}''_\Sigma(C) \subseteq_{f.f.} \mathcal{P}_\Sigma(C)$ be the full subcategory spanned by*

$$\{m \in \mathcal{P}_\Sigma(C) \mid \forall n \in \mathcal{P}_\Sigma(C),\ \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(m, n) \xrightarrow{\simeq} \mathrm{Map}_D(F(m), F(n))\}$$

Then, $\mathcal{P}''_\Sigma(C) \simeq \mathcal{P}_\Sigma(C)$.

Proof. The previous claim means precisely that $\mathcal{P}''_\Sigma(C)$ contains the essential image of j_Σ as a full subcategory. Then, in view of the minimality of $\mathcal{P}_\Sigma(C)$, we are left to prove that $\mathcal{P}''_\Sigma(C)$ is closed under sifted colimits.
To this end, let again $p : K \to \mathcal{P}''_\Sigma(C)$ be a sifted diagram, and choose one of its colimit cones $\overline{p}$ in $\mathcal{P}_\Sigma(C)$. Then, we have

$$\mathrm{Map}_{\mathcal{P}_\Sigma(C)}(\overline{p}, n) \simeq_{(a)} \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(p, n) \xrightarrow{F} \mathrm{Map}_D(F(p), F(n)) \simeq_{(b)} \mathrm{Map}_D(F(\overline{p}), F(n))$$

where (a) is the definition of colimit and (b) holds because F preserves sifted colimits. Finally, as before, F induces a point-wise equivalence of mapping spaces, and hence one of K-colimits.
Therefore, once more it holds that $\mathcal{P}''_\Sigma(C) \simeq \mathcal{P}_\Sigma(C)$. ■

Thus, F is fully faithful.

Moreover, by the construction of $\mathcal{P}_\Sigma(C)$ as a free sifted-colimit completion of C, as well as by the one of left derived extensions, condition (3) is equivalent to F being essentially surjective. In other words, whenever F is fully faithful, (3) holds iff F is an equivalence. $\square$

The previous Proposition implies, in particular, that an ∞-category D occurs as a free $sInd$-completion of its cpt+proj objects D^{sfp} whenever the latter full subcategory is colim^{sift}-dense in it. Such an equivalence is witnessed by the left derived functor of the fully faithful embedding $D^{sfp} \subseteq_{f.f.} D$.
Now, we want to characterize those subcategories $C \subseteq_{f.f.} D$ for which $D \simeq sInd(C)$.
Consider the last part of the previous proof, namely that a functor $F : \mathcal{P}_\Sigma(C) \to D$ which preserves sifted colimits is essentially surjective iff the essential image of its restriction to C generates D under sifted colimits. This means, in our setting, that $D \simeq \mathcal{P}_\Sigma(C) \simeq \mathrm{sInd}(C)$ only if C is colim^{sift}-dense in D.
To this end, let us characterize those families of 'elementary bricks' which induce such dense C's.

Definition A.4.0.4. *(Set of cpt+proj generators, [24],5.5.8.23)* Let $D \in \mathrm{Cat}_\infty$ be cocomplete and consider a class S of objects of D. We say that S is a **set of cpt+proj generators** for D iff it satisfies the following properties:

1. $S \subseteq D^{\mathrm{sfp}}$ consists of cpt+proj objects of D;
2. the full subcategory of D spanned by S is essentially small;
3. S is dense in D, i.e. it generates D under small colimits.

Furthermore, we say that an ∞-category D is **projectively generated** (write 'cpt+proj-generated') iff there exists a set S of cpt+proj objects of D.

We are finally able to give our minimal model inducing categories which occur as free $sInd$-completions.

Proposition A.4.0.5. *([24],5.5.8.25) Let $D \in \mathrm{Cat}_\infty$ be cocomplete and let S be a set of cpt+proj generators of D. Then, the following two statements hold.*

- *Let S^0 be the closure of S under finite coproducts and let D^0 be the full subcategory of D spanned by S^0. Denote by $C \subseteq D^0$ a minimal model for D^0, i.e. an equivalent skeletal subcategory. Then, the left derived functor $F : \mathcal{P}_\Sigma(C) \to D$ of the embedding $f : C \subseteq D$ is an equivalence.*

  In particular, we conclude that a cocomplete ∞-category D admitting a set of cpt+proj generators is cpt+proj-generated and presentable.

- *We can characterize D^{sfp} as the closure of D^0 under retracts: for an object $x \in D$,* TFAE*:*
  1. *x cpt+proj.*
  2. *$e : D \to \widehat{\mathrm{Spc}}$ corepresented by x preserves sifted colimits.*
  3. *There exists some $x' \in D^0$ s.t. x is a retract of x'.*

Proof. As for the first statement, in order to show that F is an equivalence, we are left to prove the three conditions in the previous Proposition.
(1) : From the very definition, $F \circ j_\Sigma : C \to \mathcal{P}_\Sigma(C) \to D$ is the composite of $F \circ j_\Sigma = \subseteq_{f.f.}: D^0 \to \mathcal{P}_\Sigma(D^0) \to D$ after $C \simeq D^0$, so it is clearly fully faithful.
(2) : By assumption, we have an equivalence $C \simeq D^0$, so we are left to check the claim for $F \circ j_\Sigma \simeq \subseteq : D^0 \hookrightarrow \mathcal{P}_\Sigma(D^0) \to D$. To this end, recall that both the full subcategories of D respectively spanned by the compact and the projective objects are stable under finite coproducts, so that we have $\mathrm{EssIm}(F \circ j_\Sigma) \simeq D^0 \subseteq_{f.f.} D^{\mathrm{sfp}}$, as desired.
(3) : Arbitrary colimits are generated by finite coproducts and sifted colimits, so D^0 is colim^{sift}-dense in D. The former follows by [20],4.3.29 and a straightforward remark: an ∞-category E has all small colimits whenever it has small coproducts and coequalizers; now, small coproducts are obtained as filtered colimits of

the finite ones, while coequalizers of simplicially indexed diagrams are just truncations of geometric realizations (more generally, an application of Joyal's Criterion of Cofinality A.1.0.2 as in [24],6.5.3.7 allows us to neglect degeneracies when computing colimits of simplicial diagrams).

Now let's turn to the second statement. (2) $\iff$ (1) comes from the properties of cpt+proj ojects.
(1) $\implies$ (3) : Let $x \in D$ be cpt+proj. By the previous statement, wlog we can work in $\mathcal{P}_\Sigma(C)(\simeq D)$.
By A.2.0.4, there exists a simplicial object of $Ind(C)$, say $X_\bullet$, whose geometric realization is $|X_\bullet| \simeq x$ and s.t. each X_n is an arbitraty (small) coproduct of elements of C.
Now, being x projective, $1_x \in \mathrm{Map}_C(x,x) \simeq |\mathrm{Map}_C(x, X_\bullet)|$ corresponds to some class $[f : x \to X_\bullet]$ in the geometric realizations. Any representative f is a natural transformation $f : x \simeq \Delta^0 \to X_\bullet$, which in turn is described by some assignment $f : x \to X_0$ of constant simplicial sets.
Moreover, by the compactness of x, we can choose a factorization of the latter through some finite subcoproduct $c \in C$ (which is then cpt+proj) of X_0:

$$f = \left(x \xrightarrow{f_0} j(c) \xrightarrow{str} X_0\right)$$

Finally, our construction implies that x is a retract of such a $j(c)$ in D, as witnessed by the action on 0-simplices of a representative of the following composition:

$$\left[x \xrightarrow{f_0} j(c) \xrightarrow{\text{str}} X_0 \xrightarrow{\text{can}} x\right] \simeq 1_x$$

□

A.5 Truncation

In the current subsection, we briefly review the notion of truncation. Being proofs exquisitely technical and not particularly enlightening, we will be expository and provide almost none. The interested reader can refer to sections [24],5.5.6 and [24],5.5.8 in Lurie's *HTT*.

Definition A.5.0.1. *(Truncated spaces, [24],2.3.4.15)* For any space $X \in \mathrm{Spc}$ and any integer $n \geq -2$, we recursively define the property of being an n**-truncated space** as follows.

- $n = -2$: X is (-2)-truncated iff it is contractible;
- $n \geq -1$: X is n-truncated iff, for each $i > n$ and for each base-point $x \in X$, its homotopy groups are trivial, i.e. $\pi_i(X, x) \simeq *$.

We say that $f : X \to Y$ in Spc is an n**-truncated morphism** iff the homotopy fibers of f over any point of Y are n-truncated.

Observe first that the two definitions are clearly compatible: by inspecting the homotopy fibre sequence, the identity 1_X is n-truncated iff X is n-truncated. Now, the next step is to import the notion of truncation in any arbitrary ∞-category.

Definition A.5.0.2. Let $C \in \mathrm{Cat}_\infty$. For any integer $n \geq -1$, we say that

- $c \in C$ is an n**-truncated object** iff, for each other object $x \in C$, the space $\mathrm{Map}_C(x, c) \in \mathrm{Spc}$ is n-truncated. We then accordingly extend our definition so as to incorporate (-2)-truncated objects: $c \in C$ is (-2)-truncated iff it is terminal, i.e, for every other object $x \in C$, $\mathrm{Map}_C(x, c) \simeq *$ is a contractible space.
- $f : c \to c'$ in C is an n**-truncated morphism** iff, for every object $x \in C$, post-composition with f induces an n-truncated map of spaces, namely $f_* : \mathrm{Map}_C(x, c) \to \mathrm{Map}_C(x, c')$.

Notation. Let $C_{\leq n} \subseteq_{f.f.} C$ denote the full subcategory spanned by the n-truncated objects. Notice that, in particular, $C^{\mathrm{term}} = C_{\leq -2}$.

Furthermore, we call **static** the objects of $C_{\leq 0}$ and, by extension, we refer to the latter as forming the **static part** of C.

Such a terminology is motivated by the equivalence $C_{\leq 0} \simeq \mathcal{N}(\pi_0(C))$. In order to see this, recall that, by [24],2.3.4.18, a 0-truncated ∞-category such as $C_{\leq 0}$ is equivalent to a 1-category, and hence [24],2.3.4.5 yields an equivalence $C_{\leq 0} \simeq \mathcal{N}(\mathrm{ho}C_{\leq 0})$; finally, the images under ho and π_0 of an ∞-category coincide.

Moreover, a morphism $f : c \to c'$ in $C \in \mathrm{Cat}_\infty$ which is (-1)-truncated (so whose fibres are either empty or weakly contractible) is said to be a **monomorphism** of C; this is inspired by the latter condition being equivalent to fact that the target map induces a fully faithful functor of slices $(ev_1)_* : C_{/f} \to C_{/c'}$, which in turn is (up to homotopy) the categorical definition of being mono.

On the other hand, we observe that a morphism is (-2)-truncated (i.e. it has weakly contractible fibres) iff it is an equivalence.

In order to foster intuition, let us state without proof a couple of useful facts about truncation. They are all somehow consequences of the homotopy fibre sequence for mapping spaces (see [24],5.5.5.12 or [20],3.1.19).

Proposition A.5.0.3. *(Properties of truncation) Let C be an ∞-category. The following properties hold true for every $n \geq -2$:*

- ([24],5.6.6.5) *$C_{\leq n}$ is stable under all limits which exist in C.*
- ([24],5.5.6.14) *Let $f : x \to y$ be a morphism in C and assume y to be n-truncated; then, x is n-truncated iff f is n-truncated.*
- (Recursive character, [24],5.5.6.15) *Assume further that C has finite limits and that $n \geq -1$. Then, a morphism $f : x \to y$ is n-truncated iff the diagonal morphism $\delta : x \to x \times_y x$ is $(n-1)$-truncated.*

(Left exact functors preserve truncation, [24],5.5.6.15) *Consider now a left exact functor $F : C \to C'$ of ∞-categories which admit finite limits. Then, F preserves the property of objects and morphisms of being n-truncated.*

A fundamental feature of such a construction is that, whenever we can neglect set-theoretical obstructions to the formation of (co)limits (e.g. in the presentable setting), n-truncated objects span the essential image of a (accessible Bousfield) localization functor; such a left adjoint to the corresponding inclusion will then be called the n-th truncation functor.

Proposition A.5.0.4. *(Truncation functors, [24],5.5.6.18) Let $C \in \mathrm{Cat}_\infty$ be presentable and let $n \geq -2$ be an integer. Then, the inclusion $C_{\leq n} \subseteq_{f.f.} C$ gives rise to a (accessible Bousfield) localization functor $\tau_{\leq n}$, which sits in the adjunction*

$$\tau_{\leq n} : C \rightleftarrows C_{\leq n} : \subseteq$$

Moreover, also $C_{\leq n} \simeq \mathrm{EssIm}(\tau_{\leq n})$ is presentable.

In 'nice' settings, this allows us to recover it as the limit of a countable continuous filtration of successive truncations, known as Postnikov tower.

Proposition A.5.0.5. *(Postnikov towers, [24],5.5.6.26) Let Δ^∞ denote the extension of Δ by a maximum $[\infty]$ and consider a presentable ∞-category $C \in \mathrm{Pr}^L$.*
Call a tower in C any extended simplicial object $X_\bullet^\infty : \mathcal{N}(\Delta^\infty)^{op} \to C$. Define a ***Postnikov tower*** *of an object $x \in C$ to be a tower:*

$$X_\bullet^\infty \quad : \quad X_\infty^\infty = x \to \cdots \to \tau_{\leq n}(x) \to \tau_{\leq n-1}(x) \to \cdots \to \tau_{\leq 0}(x) = X_0^\infty$$

s.t. at each step $u(x) : x \to \tau_{\leq n} x$ is the unit of the truncation adjunction $\tau_{\leq n} \vdash \subseteq$.
Then, the extended simplicial object $X_\bullet^\infty$ is a limit cocone in C exhibiting $x \simeq \lim \tau_{\leq n}(x)$ iff all transition maps $\tau_{\leq n}(x) \to \tau_{\leq n-1}(x)$ are the units of the adjunction $\tau_{\leq n-1} \vdash \subseteq$ restricted to $C_{\leq n}$.

Such a feature encodes in some sense the 'homotopy theoretic' character of higher algebraic and geometric objects, as we remark while dealing with 'flatness'.

The next result investigates the problem of 'transporting' Postnikov towers. We will apply it to deduce that flat base-change commutes with truncation.

Proposition A.5.0.6. *(Postnikov towers are preserved, [24],5.5.6.28) Let $C,\ D \in \mathrm{Pr}^L$ be presentable ∞-categories and consider a left-exact and cocontinuous functor $F : C \to D$ between them. Then, F commutes with truncation functors, namely there is an equivalence $F \circ \tau^C_{\leq n} \simeq \tau^D_{\leq n} \circ F$ in $\mathrm{Fun}^L(C, D)$.*
In particular, F preserves Postnikov towers (even though it needs not preserve their convergence).

Let us conclude the current subsection with a useful application of the previously stated results, namely with a description of truncated objects in non-abelian localizations.

Lemma A.5.0.7. *([24],5.5.8.26) Let $C \in \mathrm{Cat}_\infty$ have finite coproducts. Post-composition with the n-th truncation functor $\tau_n : \mathrm{Spc} \to \mathrm{Spc}$ imports the notions of truncated objects in $\mathcal{P}_\Sigma(C)$ via the Bousfield localization*

$$\overline{\tau}_n := (\tau_{\leq n})_* \colon \mathcal{P}_\Sigma(C) \rightleftarrows \mathcal{P}_\Sigma(C)_{\leq n} : (\subseteq_{\leq n})_* =: \iota_n$$

The essential image of $\overline{\tau}_n$ is the presentable sub-∞-category of $\mathcal{P}_\Sigma(C)$ spanned by those finite-product-preserving presheaves on C which take n-truncated values, and the latter can be identified with the n-truncated objects of $\mathcal{P}_\Sigma(C)$, i.e. $\overline{\tau}_n$ is essentially surjective.

Proof. Observe first that $\tau_{\leq n} : \mathrm{Spc} \to \mathrm{Spc}_{\leq n}$ preserves finite products, since mapping spaces - and hence homotopy groups - commute with finite products in the second argument.
Post-composition with spaces-truncation to the objects of $\mathcal{P}_\Sigma(C) = \mathrm{Fun}^\times(C^{op}, \mathrm{Spc})$, defines a functor

$$\overline{\tau}_n := (\tau_{\leq n})_* : \mathcal{P}_\Sigma(C) \to \mathrm{Fun}^\times(C^{op}, \mathrm{Spc}_{\leq n}) \subseteq_{f.f.} \mathcal{P}_\Sigma(C)$$

Then, the Bousfield localization of spaces $\tau_{\leq n} \vdash \subseteq_n$ induces an adjunction (see [20],5.1.16)

$$\overline{\tau}_n = (\tau_{\leq n})_* \colon \mathcal{P}_\Sigma(C) \rightleftarrows \mathrm{Fun}^\times(C^{op}, \mathrm{Spc}_{\leq n}) : (\subseteq_n)_* =: \iota_n$$

Since post-composition with the inclusion of truncated objects is still fully faithful, we again obtain a Bousfield localization, with presentable essential image (see [24],5.5.4.15,ii).

We are now left to characterize the essential image of $\overline{\tau}_n$. Our claim is that $EssIm(\overline{\tau}_n) \simeq \mathcal{P}_\Sigma(C)_{\leq n}$ holds, thus giving substance to the intuition that '$\tau_{\leq n}$ imports the notion of truncated objects in $\mathcal{P}_\Sigma(C)$'.
Notice first that a functor $F : C^{op} \to \mathrm{Spc}$ takes n-truncated values iff $\mathrm{Map}_{\mathcal{P}(C)}(j(x), F) \in \mathrm{Spc}_{\leq n}$ for each presheaf on C represented by $x \in C$, so our claim amounts to the following equivalence:

$$\forall X \in \mathcal{P}_\Sigma(C),\ \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(X, F) \in \mathrm{Spc}_{\leq n} \qquad \Longleftrightarrow \qquad \forall x \in C,\ \mathrm{Map}_{\mathcal{P}_\Sigma(C)}(j(x), F) \in \mathrm{Spc}_{\leq n}$$

One direction is clear, so let's prove ($\Longleftarrow$). To this end, let $F : C^{op} \to \mathrm{Spc}_{\leq n}$ be a presheaf taking n-truncated values, and consider the full subcategory of $\mathcal{P}_\Sigma(C)$ spanned by those $X \in \mathcal{P}_\Sigma(C)$ s.t. $\mathrm{Map}_{\mathcal{P}_\Sigma(C)}(X, F) \in \mathrm{Spc}_{\leq n}$.
Notice that, by assumption, the latter contains $j(C)$; moreover, it is clearly closed under all small colimits in $\mathcal{P}_\Sigma(C)$ (whenever it is well-defined, the functor $\lim_K$ commutes with finite limits). Hence, as desired, it must be the all of $\mathcal{P}_\Sigma(C)$, by the minimality property of the latter. $\square$

A.6 The Model Theoretical Non-Abelian Localization

In this subsection, we will briefly comment on the fact that, for all the algebraic 1-categories of concern in constructing the foundations of DAG, the $\mathcal{P}_\Sigma$-construction can actually be regarded from a model categorical perspective.
This provides a recipe to simplify computations of colimits in the corresponding animated ∞-categories and motivates the name 'derived functor' to describe the functoriality of $\mathcal{P}_\Sigma$.
For the sake of consistency with the current literature inspired by [3], whenever we localize 1-categories of cpt+proj objects, we will refer to the $\mathcal{P}_\Sigma$-construction as 'animation'.

Consider a small 1-category $\mathcal{C} \in \mathrm{Cat}$ with finite products. Then, the opposite of its nerve $\mathcal{N}(\mathcal{C})^{op} \in \mathrm{Cat}_\infty$ is again small and has finite coproducts, so that we can apply the $\mathcal{P}_\Sigma$-construction to it, defined by $\mathcal{P}_\Sigma(\mathcal{N}(\mathcal{C})^{op}) = \mathrm{Fun}^\times(\mathcal{N}(\mathcal{C}), \mathrm{Spc})$.
Now, by [24],4.2.4.4, the latter is the ∞-category underlying the simplicial model category $\mathrm{sSet}^{\mathcal{C}}_{\mathrm{proj}}$ endowed with the projective model structure (so equivalences and fibrations are defined point-wise according to $\mathrm{sSet}_{\mathrm{Quillen}}$).

On the other hand, the category of finite-product-preserving presheaves is the underlying ∞-category of a model category $\mathcal{A}$ as in the following Lemma due to Quillen.

Lemma A.6.0.1. *([24],5.5.9.1) Let $\mathcal{C} \in \mathrm{Cat}_1$ be small and assume that it admits finite products. Let $\mathcal{A} := \mathrm{Fun}^{\times}(\mathcal{C}, \mathrm{sSet}) \in \mathrm{Cat}_1$ denote the category of finite-product-preserving functors $\mathcal{C} \to \mathrm{sSet}$. Then, $\mathcal{A}$ can be endowed with a simplicial model structure as follows:*

- *the equivalences $\mathcal{W}_{\mathcal{A}}$ are point-wise weak homotopy equivalences in $\mathrm{sSet}_{\mathrm{Quillen}}$;*
- *the fibrations $\mathrm{Fib}_{\mathcal{A}}$ are point-wise (Kan) fibrations in $\mathrm{sSet}_{\mathrm{Quillen}}$.*

Moreover, the embedding $\iota : \mathcal{A} \subseteq_{f.f.} \mathrm{sSet}^{\mathcal{C}}_{\mathrm{Quillen}}$ preserves both (trivial) fibrations and weak equivalences, so that it is a Quillen functor determining a Quillen adjunction $F \colon \mathrm{sSet}^{\mathcal{C}} \rightleftarrows \mathcal{A} : \iota$.

Now, each functor $f : \mathcal{N}(\mathcal{C}) \to \mathrm{Spc}$ can be represented under the coherent nerve by some $F : \mathcal{C} \to \mathrm{Kan}$ s.t. $f \simeq \gamma \circ \mathcal{N}(F)$, where $\gamma : \mathrm{Kan} \to \mathrm{Kan}[\mathrm{h.eq}^{-1}]$ is the localization at the (weak) homotopy equivalences of (the full subcategory of) fibrant objects in $\mathrm{sSet}_{\mathrm{Quillen}}$.
By our construction, $f \in \mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{C})^{op})$ iff its lift F is *weakly* finite-product-preserving, i.e. it preserves finite products up to homotopy equivalences in Kan.
The goal of Lurie in the very technical section [24],5.5.9 is to prove a refinement of [24],4.2.2.4 which allows the lift F to be chosen in such a way that it preserves finite products (so, up to isomorphism and not homotopy equivalence). This becomes more precise in the following Lemma of Bergner.

Proposition A.6.0.2. *([24],5.5.9.2) Let $\mathcal{C}$ be a small category with finite products and consider the Quillen adjunction $F \colon \mathrm{sSet}^{\mathcal{C}} \rightleftarrows \mathcal{A} : \iota$ of the previous Lemma.*
Then, the total right derived functor $\mathcal{R}\iota : \mathrm{ho}(\mathcal{A}) \longrightarrow \mathrm{ho}(\mathrm{sSet}^{\mathcal{C}})$ is a fully faithful embedding, and its essential image consists of the equivalence classes of those functors $f : \mathcal{C} \to \mathrm{sSet}$ which preserve finite products up to weak homotopy equivalence (and hence $[f]$ preserves finite products up to iso in the homotopy category).

By the universal property of localizations, this finally allows us to conclude that the ∞-category underlying $\mathcal{A}$ is precisely $\mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{C})^{op})$, so that we obtain a Quillen model for the latter.

Corollary A.6.0.3. *([24],5.5.9.3) Let $\mathcal{C} \in \mathrm{Cat}_1$ be a small category with finite products, and let $\mathcal{A}$ be as in the previous Lemma. Then, one has an equivalence of ∞-categories $\mathcal{N}(\mathcal{A}^{\circ}) \xrightarrow{\simeq} \mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{C})^{op})$, where $\mathcal{A}^{\circ} \subseteq_{f.f.} \mathcal{A}$ is the full subcategory spanned by fibrant-cofibrant objects.*

But then the Quillen adjunction $F \colon \mathrm{sSet}^{\mathcal{C}} \rightleftarrows \mathcal{A} : \iota$ promotes to one of total derived functors

$$\mathcal{L}F \colon \mathrm{ho}(\mathrm{sSet}^{\mathcal{C}}) \rightleftarrows \mathrm{ho}(\mathcal{A}) : \mathcal{R}\iota$$

which 'describes', under the coherent nerve, the Bousfield localization

$$L \colon \mathcal{P}(\mathcal{N}(\mathcal{C})^{op}) \rightleftarrows \mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{C})^{op}) : \subseteq$$

The latter observation is used in [24],5.5.9.14 to provide strategies for computing homotopy colimits of simplicial objects of $\mathcal{A}$, and hence geometric realizations of simplicial objects in the underlying ∞-category $\mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{C})^{op})$.

Proposition A.6.0.4. *([24],5.5.9.14) Let $\mathcal{C} \in \mathrm{Cat}_1$ admit finite products. Consider the following model categories:*

- $\underline{\mathcal{A}} := \mathrm{Fun}^{\times}(\mathcal{C}, \mathrm{sSet}) \subseteq_{f.f.} \mathrm{sSet}^{\mathcal{C}}_{\mathrm{proj}}$ *as in Quillen's Lemma;*
- $\mathcal{A} := \mathrm{Fun}^{\times}(\mathcal{C}, \mathrm{Set}) \subseteq_{f.f.} \mathrm{Set}^{\mathcal{C}}_{\mathrm{proj}}$.

Given a simplicial object of $\underline{\mathcal{A}}$, say $\underline{F} : \Delta^{op} \to \underline{\mathcal{A}}$, let $F : \Delta^{op} \times \Delta^{op} \to \mathcal{A}$ denote its adjoint bi-simplicial set of $\mathcal{A}$.
Then, the geometric realization of $\underline{F}$ can be computed as the simplicial set of $\mathcal{A}$ obtained by restricting F along the diagonal:

$$|\underline{F}| := \big(\mathrm{diag}^*(F) : \Delta^{op} \longrightarrow \Delta^{op} \times \Delta^{op} \xrightarrow{F} \mathcal{A}\big) \in \underline{\mathcal{A}}$$

In other words, $\mathrm{hocolim}(\underline{F}) \simeq |\underline{F}| \in \mathrm{ho}(\underline{\mathcal{A}})$.

Finally, let us carry on the discussion about composition of total left derived functors of animated categories. For this last part we will be concerned only with localizations of 1-categories consisting of cpt+proj objects.

In the spirit of DAG, we want to investigate istances of a presentable projectively generated 'algebraic' 1-category by studying the properties of 'richer' ∞-category which contain its embedded copy, namely the animation of the full subcategory spanned by the set of its projective generators. In particular, given a functor $F : \mathcal{C} \to \mathcal{D}$ between any two such categories, we would like for it to determine a functor of ∞-categories $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$. In view of Proposition A.3.0.2 and as discussed by Cesnavicius and Scholze in [3],5.1, this can be defined as follows.

Definition A.6.0.5. *(Animated functors, [3],5.1.4)* Let $F : \mathcal{C} \to \mathcal{D}$ be a 1-sifted-colimits-preserving functor of complete 1-categories. Define the **animated functor** of F, to be the left derived functor

$$\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) = \mathcal{P}_\Sigma((\mathcal{C}^{\mathrm{sfp}})^{op}) \to \mathcal{P}_\Sigma((\mathcal{D}^{\mathrm{sfp}})^{op}) = \mathrm{Ani}(\mathcal{D})$$

obtained by applying A.3.0.2 to $F : \mathcal{C}^{\mathrm{sfp}} \to \mathcal{D}^{sfp} \subseteq_{f.f.} \mathrm{Ani}(\mathcal{D})$. It is the essentially unique functor between the given non-abelian localizations which satisfies the following properties:

1. $\mathrm{Ani}(F)$ preserves sifted colimits;
2. Its restriction $\mathrm{Ani}(F)_{|\mathcal{C}^{\mathrm{sfp}}} : \mathcal{C}^{\mathrm{sfp}} \to \mathcal{D} \subseteq_{f.f.} \mathrm{Ani}(\mathcal{D})$ agrees with F;
3. $\pi_0 \circ \mathrm{Ani}(F) = F \circ \pi_0$.

Remark. The previous definition is indeed a particular case of $\mathcal{P}_\Sigma$ 'functoriality', but our animated functors now enjoy one more property: the 'nice' features of 'algebraic' categories allow us to apply a weaker version of the transport of Postnikov towers (see A.5.0.6).

Proposition A.6.0.6. *(Composition of total derived functors, [24],5.1.5) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be cocomplete and projectively generated 1-categories, and consider composable 1-sifted-colimits-preserving functors F, G between them. Then,*

- *There exists a natural transformation $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \to \mathrm{Ani}(G \circ F)$;*
- *Assume further that*
  - *either $F(\mathcal{C}^{\mathrm{sfp}}) \subseteq \mathrm{Ind}(\mathcal{D}^{\mathrm{sfp}})$ in $\mathcal{D}$*
  - *or $\mathrm{Ani}(G)\big(F(\mathcal{C}^{\mathrm{sfp}})\big) \subseteq \mathcal{E}$ in $\mathrm{Ani}(\mathcal{E})$ is static.*

  Then, the previous natural transformation is an equivalence $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \simeq \mathrm{Ani}(G \circ F)$.

Proof. Both animated functors preserve sifted colimits, so, by A.3.0.2 and [24],5.1.2.3 (or see the proof of [20],5.1.27), it suffices to define a natural transformation at the level of the full subcategories spanned by the sets of projective generators. Hence, let's compare their restriction to $\mathcal{C}^{\mathrm{sfp}}$; the unit of $\pi_0 \vdash \subseteq$ induces:

$$\big(\mathrm{Ani}(G)\circ\mathrm{Ani}(F)\big)_{|\mathcal{C}^{\mathrm{sfp}}} \simeq \mathrm{Ani}(G)_{|F(\mathcal{C}^{\mathrm{sfp}})} \longrightarrow \pi_0\circ\mathrm{Ani}(G)_{|F(\mathcal{C}^{\mathrm{sfp}})} \simeq_{(a)} (G\circ\pi_0\circ F)_{|\mathcal{C}^{\mathrm{sfp}}} \simeq_{(b)} G\circ F_{|\mathcal{C}^{\mathrm{sfp}}} \simeq \mathrm{Ani}(G\circ F)_{|\mathcal{C}^{\mathrm{sfp}}}$$

where (a) is implied by property (3) and (b) by the fact that F is already 0-truncated.
Now, such a natural transformation turns out to be a point-wise equivalence whenever $\mathrm{Ani}(G)(d) \simeq G(d)$; the two stated conditions imply this. Indeed, the second one clearly does. As for the first one, consider the full subcategory of $\mathrm{Ani}(\mathcal{C})$ which is spanned by $\{d \in \mathcal{D} \mid \mathrm{Ani}(G)(d) \xrightarrow{\simeq} G(d)\}$; the latter contains $\mathcal{D}^{\mathrm{sfp}}$ and is closed under filtered colimits, so that, by assumption, it must contain $F(\mathcal{C}^{\mathrm{sfp}})$, as needed. $\square$

As a last remark, we observe that such a 'pathology' with respect to the ∞-categorical case is analogous to the behaviour of total left derived functors in a model categorical perspective and should not be surprising. Indeed, in the latter case, we cannot expect our total derived functors, which are defined at the level of homotopy category and so on fibrant-cofibrant representatives, to be always compatible with composition of functors, which is instead defined on all objects of our model categories.

Here, again, our animated functors $\mathrm{Ani}(F)$, $\mathrm{Ani}(G)$ on the free sifted-colimit-completions of the respective 1-categories are (essentially) determined as follows. We first consider the action of the composable 1-sifted-colimit-preserving functors F and G between the 1-categories involved, and then we restrict them to the respective full subcategory of cpt+proj objects.
As shown, we can always relate $\mathrm{Ani}(G) \circ \mathrm{Ani}(F)$ and $\mathrm{Ani}(G \circ F)$ by the unit of the 0-truncation localization. Then, the obstruction to compatibility with composition lies in the following fact: in general, it might not be the case that F carries the cpt+proj's of the source to cpt+proj's of the target, as we would need in order to apply property (2). The two technical conditions, then, correspond to two situations in which we are able to govern such an obstruction.

A.7 Examples of Animation

In this last subsection, we will discuss some examples of animation which we will need in order to develop the theory of DAG.

Ex.1. $\mathrm{Ani}(\mathrm{Set}) \simeq \mathrm{Spc} := \mathrm{Ani}$.

Lemma A.7.0.1. *The compact and projective objects in* Set *are given by* $\mathrm{Set}^{\mathrm{sfp}} \cong \mathrm{FinSet}$.

Proof. Claim. *A set X is compact iff it is finite.*

Proof. We can equivalently characterize the property of being compact as follows: for every countable filtered system $Y_\bullet := (Y_n \ , \ i_n : Y_n \to Y_{n+1} \mid n < \omega)$ of sets with colimit $Y \in \mathrm{Set}$, and every function $f : X \to Y$, there exists some $n < \omega$ s.t. one has the canonical factorization $f : X \to Y_n \to Y$.
Then, if $\#X \geq \omega$ (i.e. X contains a copy of ω), then we can define a function $f : n \mapsto f(n) \in Y_n \setminus i_{n-1}(Y_{n-1})$ witnessing the non-compactness of X. ■

Let us now show that a finite set is also projective, i.e. $\mathrm{Hom}_{\mathrm{Set}}(X, -)$ commutes with reflexive coequalizers. We actually prove a stronger statement.
Claim. *Corepresentable presheaves on* Set always *commute with reflexive coequalizers.*

Proof. Consider a reflexive coequalizer in Set, with $f \circ s = id_B = g \circ s$:

$$(f, s, g) : A \underset{g}{\overset{f}{\rightleftarrows}} B \xrightarrow{q} \mathrm{Coeq}(f, g)$$

We need to show that $\mathrm{Coeq}(f_*, g_*) \cong Im(q_*)$.
By the $[UP : \mathrm{quot}]$ the map q_* factors uniquely through $\mathrm{Coeq}(f_*, g_*)$: the relations generating the equivalence on $\mathrm{Hom}(X, B)$ which yields $\mathrm{Coeq}(f, g)$ are $\{f_*(\phi) \sim g_*(\phi) \mid \phi \in \mathrm{Hom}(X, A)\}$; since they remain related after post-composition with q_*, the latter factors uniquely through $\mathrm{Coeq}(f_*, g_*)$ and we obtain a function $\mathrm{Coeq}(f_*, g_*) \to \mathrm{Im}(q_*)$, which is surjective by abstract non-sense. Let's prove its injectivity.
Let $\psi, \psi' : X \to B$ be functions s.t. $q_*(\psi) = q_*(\psi')$; by the reflexive property of our coequalizer, this is equivalent to $q \circ f(s \circ \psi(x)) = q \circ g(s \circ \psi'(x))$ for each $x \in X$, i.e. $f(s \circ \psi(x)) = g(s \circ \psi'(x))$ in $\mathrm{Coeq}(f, g)$ for each $x \in X$. This means that, for each $x \in X$, there exists some $a_x \in A$ s.t. $f(a_x) = f \circ (s \circ \psi(x))$ and $g(a_x) = g \circ (s \circ \psi'(x))$. But then we are done.
Indeed, we would like to write the previous condition as a generator of the relation on $\mathrm{Coeq}(f_*, g_*)$, so as to identify ψ and ψ' in the latter quotient. In other words, we want a function $\eta : X \to A$ s.t. both $f \circ (s \circ \psi)(x) = f \circ \eta(x)$ and $g \circ (s \circ \psi')(x) = g \circ \eta(x)$ hold for each $x \in X$. We achieve our goal by defining $\eta : x \mapsto a_x$. ■

□

Proposition A.7.0.2. *(Anima)* $\mathrm{Ani} := \mathrm{Ani}(\mathrm{Set}) = \mathrm{Fun}^{\times}\big(\mathrm{FinSet}^{op}, \mathrm{Spc}\big) \simeq \mathrm{Spc}$.

Proof. FinSet^{op} is generated by its initial object $*$ under finite products, so any animated set F is completely determined by its value on $*$. The latter can be sent to any space $F(*) \in \mathrm{Spc}$.
In other words, the functor evaluation at a point $ev_* : \mathrm{Fun}^{\times}(\mathrm{FinSet}^{op}, \mathrm{Spc}) \to \mathrm{Spc}$ is essentially surjective. In order for it to be an equivalence, we are left to show that it is also fully faithful, i.e. that

$$ev_* : \mathrm{Map}_{\mathrm{Ani}}(F, G) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Spc}}(F(*), G(*))$$

is an equivalence of spaces for every F, $G \in \mathrm{Ani}$. This follows from abstract nonsense: a natural transformation of symmetric monoidal functors is in turn symmetric monoidal.
We give anyway a more explicit argument in the incarnation of quasi-categories. Let us work with the minimal model $\Delta \simeq \mathrm{FinSet}$.
We recall that $\mathrm{Map}_{\mathcal{P}(\Delta)}(F,G) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\times}(\Delta^{op}, \mathrm{Spc})) \times_{\mathcal{P}(\Delta)\times\mathcal{P}(\Delta)} \Delta^0$ on the maps (ev_0, ev_1) and (F,G) respectively. $\mathrm{Map}_{\mathrm{Spc}}(\Delta^1, \mathrm{Spc})$ can be written in a similar way, so we are left to produce an equivalence of simplicial sets $ev_* : \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\times}(\Delta^{op}, \mathrm{Spc})) \to \mathrm{Fun}(\Delta^1, \mathrm{Spc})$.
The following assignment on n-simplices is a well defined functor of simplicial sets (again by the closed monoidal structure of sSet):

$$ev_* : \mathrm{Hom}(\Delta^{op}, \mathrm{Hom}(\Delta^n \times \Delta^1, \mathrm{Spc})) \longrightarrow \mathrm{Hom}(\Delta^n \times \Delta^1, \mathrm{Spc})$$
$$(\eta : \Delta^n \times (\Delta^1 \times \Delta^{op}) \to \mathrm{Spc}) \mapsto (\eta(*) : \Delta^n \times \Delta^1 \to \mathrm{Spc})$$

Moreover, the latter is clearly an equivalence, since it has an obvious point-wise mutual inverse, which is induced by the cartesian monoidal structure of Δ^{op}, namely $\eta_{(-)} : (\phi : F(*) \to G(*)) \mapsto (\eta_\phi : \Delta^{op} \to \mathrm{Fun}(\Delta^1, \mathrm{Spc}))$ given by $\eta_\phi([n]) := \phi^{\times n} : F(*)^n \to G(*)^n$. ☐

Ex.2. $\mathrm{Ani}(\mathcal{C}) \simeq \mathcal{P}_\Sigma\big(\mathrm{Retracts}(\mathrm{Free}(\mathrm{FinSet}))\big)$, *for* $\mathcal{C}$ *an 'algebraic' category such as* Grp, CRing, Ring, Mod(R)*, et similia.*

In what follows, we will present proofs for $\mathcal{C} = \mathrm{CRing}$, but the arguments are analogous for all the other examples stated, and indeed the results can be generalized to Lawvere theories. Therefore, let us carry on denoting our category with $\mathcal{C}$ and calling it 'algebraic'.

Lemma A.7.0.3. *Let* $\mathcal{C} \in \mathrm{Cat}$ *be an 'algebraic' category. Then the forgetful functor* $for : \mathcal{C} \to \mathrm{Set}$ *preserves sifted colimits.*

Proof. In the 1-categorical setting, our statement is equivalent to *for* preserving filtered colimits and reflexive coequalizers. It suffices to notice that both such colimits can be characterized as suitable quotients on the sets underlying the diagrams involved. This corresponds to the fact that *for* is representable by a cpt+proj object of the 'algebraic' category at stake. ☐

We now need a technical lemma from Lurie's *Higher Algebra*, which is an enhancement of A.4.0.3,ii).

Proposition A.7.0.4. *([23],4.7.3.18) Consider an adjunction* $F\colon C \rightleftarrows D : G$ *in* Cat_∞ *and assume that the following properties hold:*

1. $D \in \mathrm{Cat}_\infty$ *admits filtered colimits and geometric realizations, and* G *peserves them;*
2. $C \in \mathrm{Cat}_\infty$ *is projectively generated;*
3. G *is conservative.*

Then, the following three statements hold true:

1. D *is projectively generated (and has sifted colimits);*
2. *We can characterize cpt+proj's in* D *as follows:* $d \in D^{\mathrm{sfp}}$ *iff there exists some* $c \in C^{\mathrm{sfp}}$ *s.t.* d *is a retract of* $F(x)$*;*
3. G *preserves sifted colimits.*

Proof. Let D^0 denote the essential image of the restriction $F_{|C^{\mathrm{sfp}}}$.

Wlog $D^0 \subseteq D^{\mathrm{sfp}}$. Indeed, C is projectively generated, so, by spelling out the adjunction equivalence of mapping spaces, one can easily observe that in $F \vdash G$, the right adjoint G preserves filtered colimits and geometric realizations iff the essential image D^0 of the left adjoint consists of cpt+proj's. Then, one concludes by assumption (1).

Wlog D^0 is essentially small: up to taking a minimal model for D^0, the latter must be small, because by assumption (2) (and A.4.0.5) we can wlog assume C^{sfp} to be spanned by a small set of projective generators for C.

Moreover, observe that D^0 has finite coproducts and that they are preserved by $D^0 \subseteq_{f.f.} D$; in fact, C^{sfp} is stable under finite coproducts in C (again by A.4.0.5), which are in turn preserved by the left adjoint F.

Then, we can consider its sInd-completion $\mathcal{P}_\Sigma(D^0)$.

Claim. *The left derived functor $F : \mathcal{P}_\Sigma(D^0) \to D$ of the inclusion $D^0 \subseteq_{f.f.} D$ is an equivalence in* Cat_∞. *Hence, also $D \simeq \mathcal{P}_\Sigma(D^0)$ is projectively generated and admits sifted colimits.*

Proof. By the Criterion A.4.0.3 F is an equivalence iff D^0 is colim^{sift}-dense in D.
In order to see this, we will invoke a technical Lemma from Lurie's *'Higher Algebra'*, namely [23],4.3.7.14.
By assumptions (1) and (3), the hypotheses of the latter are satisfied, so that we can regard each object $d \in D$ as the geometric realization of a 'nice' simplicial object in D, say $d_\bullet$ which belongs point-wise to the essential image of F.
Now, by assumption (2), C is generated by C^{sfp} under sifted colimits, so that we can actually view d_n as belonging to the closure under sifted colimits of $\mathrm{EssIm}(F_{|C^{\text{sfp}}})$. Hence, as needed, D is generated by D^0 under sifted colimits. ■

We have then proved (1). Now, let's turn to the second statement. Being D^0 spanned by a small set of projective generators for D, we can conclude by A.4.0.3,ii) that $d \in D^{\text{sfp}}$ iff there is some $d' \in D^0$ s.t. d is a retract of d', iff there is some $c \in C^{\text{sfp}}$ s.t. d is a retract of $F(c) = d'$.
Finally, The last assertion follows from an application of A.3.0.2,ii): D^0 has finite coproducts and C has sifted colimits, so each functor $D \simeq \mathcal{P}_\Sigma(D^0) \to C$, such as G, which preserves filtered colimits and geometric realizations further preserves sifted colimits. □

Therefore, we can characterize the cpt+proj's of $\mathcal{C}$ via the adjunction $\mathrm{Free} \vdash for$, and obtain the following.

Lemma A.7.0.5. *(Cpt+Proj in 'algebraic' categories, [3],5.1.3) Under the adjunction* $\mathrm{Free} \vdash for$, $\mathcal{C}^{sfp} \subseteq_{f.f.} \mathcal{C}$ *is spanned by retracts of 'finite free' objects.*

Proof. Let us verify that $Free\colon \mathcal{C} \rightleftarrows \mathrm{Set} : for$ satisfies the conditions of the previous Lemma: (1) : $\mathcal{C}$ is cocomplete and for preserves 1-sifted colimits of $\mathcal{C}$; (2) : Set is projectively generated by FinSet; (3) : for is conservative. Thence, $\mathcal{C}^{\text{sfp}} \subseteq_{f.f.} \mathcal{C}$ is spanned by the retracts of Free(FinSet). □

Hence, up to closure under retracts we can provide the following list:

- $\mathrm{Grp}^{\text{sfp}} \cong$ f.g. free groups
- $\mathrm{Ab}^{\text{sfp}} \cong$ f.g. free abelian groups $=: \mathrm{FFree}_{\mathbb{Z}}$
- $\mathrm{Mod}(R)^{\text{sfp}} \cong \mathrm{FFree}_R$
- $\mathrm{CRing}^{\text{sfp}} \cong \mathrm{Poly}$
- $R{-}\mathrm{Alg}^{\text{sfp}} \cong \mathrm{Poly}_R$

These then yield the corresponding animated ∞-categories by the usual rule $\mathrm{Ani}(\mathcal{C}) = \mathrm{Fun}^\times\big((\mathcal{C}^{\text{sfp}})^{op}, \mathrm{Spc}\big)$.

B Symmetric Monoidal ∞-categories

In this section we briefly present the generalization of monoidal structures to the ∞-world.
Let us begin with a motivational review of the classical setting.

Definition B.0.0.1. *(Monoidal category: diagrammatic 'definition')* A **monoidal category** is the datum of a category $\mathcal{C} \in \mathrm{Cat}$ sitting in a diagram $\mathcal{C} \times \mathcal{C} \xrightarrow{m} \mathcal{C} \xleftarrow{e} *$, together with some coherence conditions.
The latter comes equipped with (natural isomorphisms) an associator α witnessing the associativity of the multiplication m, and left and right unitors λ, ρ expressing the fact that e is a generalized unit for m.
The monoidal structure is then called **symmetric** in the case m comes also with a natural isomorphism σ, called braiding, witnessing its commutativity.
Coherence conditions then consist in the data of two diagrams (invertible 2-cells), namely the so-called triangle and pentagon diagrams, which are technical coherence conditions making all diagrams involving only the monoidal data m, α, λ, ρ to be unambiguous, i.e. commutative.

The first part of such a 'definition' is somehow intuitive, in that it mimics the classical set-theoretical axioms of a monoid.
However, the process of vertical 2-categorification (via the introduction of an associator and of unitors together with the relative coherence conditions) implies a further level of subtlety, which requires and is expressed by coherence diagrams. All such information could be codified within a single 2-category by means of the globular formalism of orientals; see *nLab* for a more detailed insight.
Moreover, we deemed such new categorical conditions as 'technical', since they are a by-product of the chosen formalism, which keeps track, via 'weak commutativity' by means of invertible 2-paths, of 'how things can be equal' and not just of the identity relations between them, the latter corresponding to strict commutativity in Cat_2.
Moreover, notice that the reduction of coherence conditions to the two stated diagrams is a non-trivial 'technical' theorem proved by Kelly, which shortens the original list of coherence diagrams introduced by Mac Lane.
Surprisingly, as remarked in the dedicated *nLab* page, it seems not to exist an elementary 'magic wand' motivating the technology employed. Most naively, one could notice that a strict monoidal category is a monoid object internal to Cat equipped with the cartesian product; however, such an approach would already presume an understanding of $(\mathrm{Cat}, \times)$ as a monoidal category, and hence be circular. As remarked there, one could also attempt more sophisticated motivational approaches by regarding monoidal categories as algebras over (higher) monads. This might shed some light on such technicalities and yield generalisations to the ∞-world as done in [23] by Lurie. However, this basically moves the need to prove a Coherence Theorem for monoidal categories to that of obtaining a similar result for algebras over a monad, so that it ends up being not exquisitely 'motivational' and to mainly postpone technicalities to the homotopy-theoretical setting.

In this section, however, we will turn our attention to a functorial description of commutative monoids in an arbitrary monoidal category $\mathcal{C}$.
We will first consider, as a prototypical cocartesian monoidal structure, the category $(\mathrm{FinSet}_*, \coprod)$ of pointed finite sets, and we will equivalently express it with the formalism of correspondences; this will allow us to regard cartesian commutative monoids in any arbitrary symmetric monoidal category as generalized objects. An enhancement of such an approach will then yield the notion of a symmetric monoidal ∞-category.
Thereafter, in the next section, we will use our construction to induce non-necessarily cartesian commutative (∞-)monoids in any arbitrary symmetric monoidal (∞-)category $\mathcal{C} \in \mathrm{Cat}$ (resp. $C \in \mathrm{Cat}_\infty$).

B.1 Cartesian Commutative Monoids

Definition B.1.0.1. *(Pointed Finite Sets)* Consider the external base-point functor $(-)_+ : \mathrm{FinSet} \to \mathrm{FinSet}_*$, which acts as $I \mapsto I_+ := I \coprod *$ and extends morphisms in such a way that external base-points are respected.
We can identify the category of pointed finite sets FinSet_* with the following equivalent category Fin_*:

- Obj: $I \in \mathrm{FinSet}$
- Mor: Partially defined maps $f : I_+ \to J_+$ given by spans $I \hookleftarrow K \xrightarrow{f} J$, i.e. we are specifying the action of f on $K \subseteq I$ and sending $I \setminus K$ to the external basepoint of J.

- Comp: Fibred product of spans.

Here, the equivalence is given by $\mathrm{FinSet}_* \to \mathrm{Fin}_*$ acting as $[f : (I, i) \to (J, j)] \mapsto \widetilde{f} := \big(K := f^{-1}(J) \subseteq I, f_{|K}\big)$, with the obvious quasi-inverse $\eta : \widetilde{f} = (i, f) \mapsto f$ extending $f : K \to J$ to the whole of I.

In particular, for future reference, let us remark that Fin_* has families of distinguished maps, which will be called **Segal maps**: for each $(I_+, i \in I) \in \mathrm{FinSet}_*$ consider $\rho_i := \eta(id_{\{i\}}) : I_+ \to \{i\}_+$ which acts as $(i \neq)j \mapsto *$, thus extending the identity $id_{\{i\}} : i \mapsto i$.

In a sense that will be made precise later on, we want to view cartesian commutative monoids in a symmetric monoidal category $\mathcal{C}$ as monoidal functors from $(\mathrm{FinSet}_*, \coprod)$ to $(\mathcal{C}, \times)$.
In other words, we will regard commutative monoids in $\mathcal{C}$ as generalized objects of $\mathcal{C}$ whose internal monoid structure is (contravariantly) induced by the cocartesian symmetric monoidal structure on Fin_*, in turn induced by $(\mathrm{FinSet}_*, \coprod)$.

Definition B.1.0.2. *(Span exponentiation)* Given a symmetric cartesian monoidal category $(\mathcal{C}, \times)$, define a functor $\exp_{\mathcal{C}} : \mathrm{FinSet}_* \to \mathrm{Cat}$ as follows:

- Obj: $I_+ \mapsto \mathcal{C}^I$
- Mor: $[\ I \xleftarrow{\iota} K \xrightarrow{f} J\ ] \mapsto [\ \mathcal{C}^I \xrightarrow{\iota^*} \mathcal{C}^K \xrightarrow{\mu(f)} \mathcal{C}^J\ ]$

where μ acts by point-wise multiplication on the fiber of $f : K \to J$, i.e.

$$\mu(f) : [\nu : K \to \mathcal{C}] \mapsto \big[\widetilde{\nu} : j \mapsto \prod_{k \in f^{-1}(j)} \nu_{|f^{-1}(j)}(k)\big]$$

Remark. (The fibres of) f determines which objects of $\mathcal{C}^I$ are to be multiplied and μ "executes" the multiplications according to the cartesian symmetric monoidal structure on $\mathcal{C}$. So, the functoriality of $\exp_{\mathcal{C}}(-)$ depends on left legs of our spans, while the multiplication μ of $\mathcal{C}$ is defined on right legs.

Remark. In particular, μ carries Segal maps to projections: for $(I_+, i \in I)$, $\mu(\rho_i) : \mathcal{C}^I \xrightarrow{ev_i} \mathcal{C}$, since by definition $\rho_i^{-1}\{i\} = \{i\}$.

Notice that, for any cartesian (symmetric) monoidal category $(\mathcal{C}, \times)$, the functor $\exp_{\mathcal{C}} : (\mathrm{FinSet}_*, \coprod) \to (\mathrm{Cat}, \times)$ is symmetric monoidal, since it satisfies the following condition:

Definition B.1.0.3. *(Segal condition)* Let $(\mathcal{C}, \times) \in \mathrm{Cat}$ be a cartesian symmetric monoidal category. $F \in \mathrm{Fun}(\mathrm{FinSet}_*, \mathcal{C})$ satisfies the **Segal condition** iff, for each $I_+ \in \mathrm{FinSet}_*$, the Segal maps $\{\rho_i \mid i \in I\}$ induce an isomorphism $\prod \mu(\rho_i) : F(I) \xrightarrow{\simeq} \prod_I F(\{i\})$. In other words, iff $F : (\mathrm{FinSet}_*, \coprod) \to (\mathcal{C}, \times)$ is symmetric monoidal.

This is not an isolated case. Indeed, the next Lemma will characterize commutative monoid objects in a cartesian monoidal category as the class of objects which satisfy the Segal condition. In particular, we will be able to regard any cartesian symmetric monoidal category as a commutative monoid object in $(\mathrm{Cat}, \times)$.

As it will be apparent in the proof, the Segal condition allows us to characterize commutative monoids by means of an heuristic called the **'microcosm' principle**: our cartesian commutative monoid object (the 'microcosm') inherits its algebraic structure by the ambient one (here the cartesian symmetric monoidal structure of the ambient category, the 'macrocosm'). We refer to the homonymous *nLab* page for some more context.

Lemma B.1.0.4. *(Microcosm Principle) Let $\mathcal{C}$ be a cartesian symmetric monoidal category, and denote by $\mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \mathcal{C}) \subseteq_{f.f.} \mathrm{Fun}(\mathrm{FinSet}_*, \mathcal{C})$ the full subcategory of those functors which satisfy the Segal condition. Then, $\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \mathcal{C})$ is an equivalence of categories.*

Proof. ($\supseteq$): Take $F \in \mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \mathcal{C})$; the classical data are recovered by applying F to the following maps of FinSet_*: $\{0,1\}_+ \cong (* \coprod *)_+ \to *_+$ gives the multiplication, while $\emptyset_+ \to *_+$ the unit of the monoid $F(*)$. Commutativity corresponds to the functoriality of F with respect to the action $\mathbb{Z}/2 \curvearrowright \{0,1\}_+$, whereas associativity is again functoriality with respect to the ways of mapping $\{0,1,2\}_+ \to \{0,1\}_+$. Hence, we obtain a functor $ev_* : \mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \mathcal{C}) \to \mathrm{CMon}(\mathcal{C})$.

($\subseteq$): A quasi-inverse to the previous functor is simply given by sending any $M \in \mathrm{CMon}(\mathcal{C})$ to $\exp_M \in \mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \mathcal{C})$. Indeed, $ev_* \circ \exp_{(-)}(M) = M$ and $\exp_{(-)} \circ ev_*(F) = \exp_{F(*)} \cong F$ by the fact that a monoidal functor on $(\mathrm{FinSet}_*, \coprod)$ is determined by its value at the point $*$. □

Moreover, the latter characterization can be simplified by adopting a 2-categorical reformulation. Our generalization to the ∞-world will then be in the same spirit.

Lemma B.1.0.5. *(2-categorical Definition) Let* $\mathrm{Span} := \mathrm{Span}(\mathrm{FinSet})$ *denote the* 2*-category of* ***spans of finite sets*** *(also known as* ***correspondences****), with objects those of* FinSet*, morphisms spans of finite sets connecting the given source and target, and (weak) composition via weak pull-back of spans. In other words, we are basically considering a generalization of Fin_* where we do not require left maps to be monomorphisms. Then, with a slight abuse of notation, there is an equivalence of* 2*-categories* $\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}, \mathcal{C})$*, where the latter is the full subcategory of* $\mathrm{Fun}(\mathrm{Span}, \mathcal{C})$ *spanned by those pseudo-functors which take finite coproducts in* Span *to products in* $\mathcal{C}$*.*

Proof. First of all, notice that composition of morphisms in Fin_* can be seen as the rectification of composition of the corresponding morphisms in the 2-category Span, so that $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{Span}, \mathcal{C})$ in Cat_2. In order to see this, notice that, given a span $(f,g) : I \leftarrow K \to J$ in Span, the epi-mono factorization of Set associates to it a set of morphisms in Fin_*, namely

$$S(f,g) := \{(\mathrm{im}f, g \circ s) : I \hookleftarrow \mathrm{Im}f \to J \mid s : \mathrm{Im}f \hookrightarrow K \text{ section to } K \twoheadrightarrow \mathrm{Im}f\}$$

Given two composable spans (f_1, g_1) and (f_2, g_2), their composition in Span induces compositions of spans from $S(f_1, g_1)$ and $S(f_2, g_2)$ in Fin_*, which are clearly all homotopic in Span. Hence, the 2-categories of pseudo-functors $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ and $\mathrm{Fun}(\mathrm{Span}, \mathcal{C})$ are equivalent, as claimed.

Moreover, the equivalence factors as $\mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}^{op}, \mathcal{C})$ in Cat_2. Indeed, by an explicit check of the universal property, coproducts in Span (i.e. products in Span^{op}) are of the form $\big(X_i = X_i \hookrightarrow \coprod_I X_i \mid i \in I\big)$ and thus live in $\mathrm{Fin}_* \simeq \mathrm{FinSet}_*$; finally, coproducts in pointed finite sets boil down to those in Set. □

The *reasons* why we chose FinSet_* are on the one hand historical: as these notions were first introduced, 1-categories were more familiar than 2-categories. On the other hand, in our construction, considering *generalized objects from* FinSet_* makes everything less artificial: given a monoidal category $(\mathcal{C}, m, e)$, we can view the *biased* formation of the multiplication as post-composition of the given m to the formation of the binary-product category $\mathcal{C} \times \mathcal{C}$, while the unit is post-composition of e to the formation of the null-ary product category $*$; remark that the 'formation of the n-fold product' can be viewed as the evaluation at n of the prototypical Segal functor, namely the co-simplicial object $\delta(-) \in \mathrm{Fun}\big(\mathrm{FinSet}, \mathrm{Fun}(\mathrm{Cat}, \mathrm{Cat})\big)$, corresponding (point-wise) to the canonical co-monoidal structure $\delta^n : \mathcal{C} \to \mathcal{C}^{\times n}$.
Now, our coherence conditions allow us to pass unambiguously from the biased definition (providing only axioms for arities 0 and 2, namely those corresponding to δ^0 and δ^2) to the *unbiased* one (which instead provides axioms for all arities). The latter viewpoint amounts to specifying the multiplication as induced by the symmetric monoidal structure on FinSet (with skeleton Δ) via

$$\mathrm{FinSet} \simeq \Delta \xrightarrow{\mathrm{Im}(\delta)} \mathrm{Im}(\delta) \simeq \mathcal{C}^{\times(-)} \xrightarrow{m} \mathcal{C}$$

The latter fact is hidden in the proof of Lemma,B.1.0.4, and it is the reason why the given functors form a categorical equivalence, so that it assigns a very deep algebraic role to the 2-category of correspondences Span.

We are now ready to provide the anticipated generalization to the ∞-world. As in the classical case, we leverage on a "universal" cartesian commutative monoid $\mathrm{Cat}_{\infty}^{\times}$ to define arbitrary commutative monoids internally as commutative monoid objects. At the moment, such a perspective is mostly motivational, so we refer to [36],5.24 for a universal property defining a cartesian structure on a category with finite products.

Definition B.1.0.6. *(Commutative ∞-monoid - straightened, [36],4.12)* Let $C \in \mathrm{Cat}_\infty$ admit finite products; define the ∞-category of **commutative ∞-monoids** in C as

$$\mathrm{CMon}(C) := \mathrm{Fun}^\times(\mathrm{Span}, C) \simeq \mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, C)$$

where $\mathrm{Fun}^\times(\mathrm{Span}, C)$ is the full sub-∞-category of $\mathrm{Fun}(\mathrm{Span}, C)$ spanned by those functors which take finite coproducts to products, and it is equivalent to the full sub-∞-category of $\mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, C) \subseteq_{f.f.} \mathrm{Fun}(\mathrm{FinSet}_*, C)$ spanned by those functors satisfying the Segal condition (see B.1.0.5).

Remark. As we will observe in B.2.0.7, let it be given a commutative monoid $M \in \mathrm{CMon}(C)$ in an ∞-category C with finite products; then, the restriction along *any* (this is the "commutativity") of the two canonical inclusions $*_+ \to [1]_+$ induces an "encoded tensor product" $\otimes : M(*_+) \times M(*_+) \to M(*_+)$.

Definition B.1.0.7. *(Commutative ∞-groups - straightened, [36],4.12)* For a commutative monoid $M \in \mathrm{CMon}(C)$ in an ∞-category C with finite products, define the **shear map** $s : M \times M \to M \times M$ by acting as the projection pr_1 on the first coordinate and as the encoded tensor product $\otimes$ on the second one. We call M a **commutative group** in C iff the shear map s is an equivalence.
Define $\mathrm{CGrp}(C) \subseteq_{f.f.} \mathrm{CMon}(C)$ as the full subcategory spanned by commutative group objects in C.

Definition B.1.0.8. *(Symmetric monoidal ∞-category)* Define a **symmetric monoidal ∞-category** to be a commutative ∞-monoid in $\mathrm{Cat}_\infty^\times$. Moreover, we refer to arrows in $\mathrm{CMon}(\mathrm{Cat}_\infty^\times)$ as **symmetric monoidal functors**.

While developing the theory of derived algebraic geometry, we will have to be dealing with non-commutative group stacks, so let us include also a definition of a not necessarily commutative group object in a category with finite products. If the ambient category is a topos, notice that this retrieves a contractible groupoid object as in C.1.0.5. After this, we will restrict the focus of our presentation to commutative gadgets.

Definition B.1.0.9. *(Arbitrary monoids and group - straightened, [8],I.1.3.1.1-2-3)* Let $C \in \mathrm{Cat}_\infty$ be an ∞-category with finite products. We define an arbitrary **monoid** in C to be a functor $M : \Delta^{op} \to C$ satisfying the following axiom:

Arbitrary Segal condition, [aSeg]: For any $i, n \in \mathbb{N}$, consider the edges $\epsilon_i^n : [1] \to [n]$ acting as $0, 1 \mapsto i, i+1$. Then, $M : \Delta^{op} \to C$ satisfies the arbitrary Segal condition iff $M([0]) = *$ and the canonical maps $\prod \epsilon_i^n : M([n]) \to \prod M([1])$ are equivalences.

Let $\mathrm{Mon}(C) := \mathrm{Fun}^{\mathrm{aSeg}}(\Delta^{op}, C) \subseteq_{f.f.} \mathrm{Fun}(\Delta^{op}, C)$ denote the full subcategory of arbitrary monoids in C.

A monoid $M \in \mathrm{Mon}(C)$ comes equipped with canonical maps:

- the edge $[1] \to [2]$ acting as $0, 1 \mapsto 0, 2$ induces the *encoded tensor product* $\otimes : M([1]) \times M([1]) \simeq M([2]) \to M([1])$;
- the canonical edge $[1] \to [0]$ defines a point $1_M : * \simeq M([0]) \to M([1])$, namely the *unit* of the monoidal structure on M;
- the *shear map* $s := (pr_1, \otimes) : M([1]) \times M([1]) \to M([1]) \times M([1])$ is induced by the projection on the first component and the multiplication on the second one.

Define an arbitrary **group** in C as a monoid $G \in \mathrm{Mon}(C)$ for which the shear map is an equivalence. Let $\mathrm{Grp}(C) \subseteq_{f.f.} \mathrm{Mon}(C)$ denote the full subcategory of groups in C.

Let us close this subsection with a general observation. The reader should beware that for many applications (e.g. exhibiting symmetric monoidal structures on ∞-categories) such a definition is often not workable

enough, because it is hard in practice to construct functors into ∞-categories. In the next subsection we will provide a more manageable definition in terms of cocartesian fibrations.
Nevertheless, the perspective of Segal functors is very useful to exhibit "algebraic" objects in topoi; an example will be the animated rings and modules which we will extensively study in the section on "Higher Algebra" or group stacks, which will be introduced in "Quotient Stack" in order to classify vector bundles on schemes, as more generally discussed in "Principal ∞-bundles".

B.2 Commutative Monoids in Symmetric Monoidal ∞-categories

With the help of the Straightening Theorem [24],3.2, we can rephrase the previous functorial approach by means of cocartesian fibrations. We choose to work with Segal functors from FinSet_*, since the formalism is easier to be handled. So, let us start by introducing a bit of terminology.

Notation. It will be sometimes useful to identify $\mathrm{FinSet}_* \cong \mathrm{Fin}_*$ with its skeleton Δ, where we regard $[n] \in \Delta$ to be pointed by 0; under this identification, write $\langle n \rangle$ for the image of the latter, $\langle 1 \rangle$ for any $(*_+ =)* \cong \{i\}_+$ and $*$ stands here for $[0]$.
Moreover, let us give a name to two special families of arrows in Δ which will play a major role in constructing "algebraic" structures. Recall that arrows in Δ come from Fin_*, so are partially defined maps of finite sets; when dealing with them, we will stick to the following convention: i denotes an element of the source, whereas j one of the target:

- $f : \langle n \rangle \to \langle m \rangle$ is called **inert** iff $\# f^{-1}(j) = 1$ for each $j \neq 0$, i.e. f is an injective partially defined map; a prominent example are Segal maps $\rho_i^{\langle n \rangle} : \langle n \rangle \to \langle 1 \rangle$, as given by $\rho_i^{\langle n \rangle}(i) = 1$;
- $f : \langle n \rangle \to \langle m \rangle$ is called **active** iff $f^{-1}(0) = 0$, i.e. f is a (globally defined) map of finite sets; a prominent example is the canonical map $m : \langle n \rangle \to *$.

Let us start by rephrasing the needed terminology in terms of cocartesian fibrations.

Definition B.2.0.1. *(Segal condition - unstraightened)* Let $\mathrm{St}(F) : \int F \to \mathrm{FinSet}_*$ be a cocartesian fibration associated to the functor $F : \mathrm{FinSet}_* \simeq \Delta \to \mathrm{Cat}_\infty$ under the Straightening equivalence [24],3.2. Then, $\mathrm{St}(F)$ satisfies the **Segal condition** iff, for each $\langle n \rangle \in \Delta$, the canonical map of fibres

$$\prod_i \hat{\rho}_i : \big(\int F\big)_{\langle n \rangle} \longrightarrow \prod_i \big(\int F\big)_{\langle 1 \rangle}$$

induced by the Segal maps $\{\rho_i\}_i$ is an equivalence.

Definition B.2.0.2. Let $p : \mathcal{C} \to \Delta$ be a cocartesian fibration between ∞-categories. We say that an edge f in $\mathcal{C}$ is **inert** iff it is a p-cocartesian lift of an inert map $p(f)$ in Δ.

The following gives the unstraightened analogous to the straightened definition of "symmetric monoidal ∞-categories". Soon we will observe that - in a cartesian symmetric monoidal ∞-category - "straightened commutative monoids" coincide with "unstraightened commutative algebras"; then, we will retrieve "symmetric monoidal ∞-categories" as "unstraightened commutative algebras" in $\mathrm{Cat}_\infty^\times$, so the two approaches will be perfectly equivalent.

Definition B.2.0.3. *(Symmetric monoidal ∞-category - unstraightened, [36]5.3)* Consider the cartesian symmetric monoidal ∞-category $\mathrm{Cat}_\infty^\times$ of [36],5.24. Define the ∞-category of **symmetric monoidal ∞-categories** in $\mathrm{Cat}_\infty^\times$ as the full subcategory $\mathrm{SymMon} \subseteq_{f.f.} \mathrm{CoCart}(\Delta)$ spanned by those cocartesian fibrations which satisfy the unstraightened Segal condition B.2.0.1. In particular, the arrows of SymMon are morphisms of cocartesian fibrations between symmetric monoidal ∞-categories; call them **symmetric monoidal functors**.
Define $\mathrm{SymMon}_{\mathrm{lax}} \subseteq_{f.f.} (\mathrm{Cat}_\infty)_{/\Delta}$ as the full subcategory with the following 1-skeleton:

- Obj: symmetric monoidal ∞-categories;

- Mor: functors between symmetric monoidal ∞-categories which preserve inert edges (and not necessariliy all the cocartesian ones); call them **lax** symmetric monoidal functors.

Definition B.2.0.4. *(Commutative algebra in a symmetric monoidal ∞-category)* For a symmetric monoidal ∞-category $A^{\otimes} \in \mathrm{CMon}(\mathrm{Cat}_{\infty}^{\times})$, consider the pull-back in Cat_{∞} of the universal cocartesian fibration π_{univ} along $A^{\otimes}$ (see the Straightening Theorem [24],3.2):

$$\begin{tikzcd}
\int A^{\otimes} \arrow[r] \arrow[d, "\in \mathrm{CoCart}"] & \mathrm{Cat}_{\infty//*} \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{FinSet}_* \arrow[r, "A^{\otimes}"] \arrow[u, dotted, bend left, "M"] & \mathrm{Cat}_{\infty}
\end{tikzcd}$$

Define the ∞-category of **commutative algebras** in the symmetric monoidal ∞-category $A^{\otimes}$ to be the full subcategory $\mathrm{CAlg}(A^{\otimes}) \subseteq_{f.f.} \mathrm{Fun}(\mathrm{FinSet}_*, \int A^{\otimes})$ consisting of those sections of $\mathrm{St}(A^{\otimes})$ which preserve inert maps, i.e. spanned by lax symmetric monoidal functors $M : \mathrm{FinSet}_* \to \int A^{\otimes}$ sitting in the commutative triangle:

$$\begin{tikzcd}
\mathrm{FinSet}_* \arrow[rr, "M"] \arrow[dr, equal] & & \int A^{\otimes} \arrow[dl, "\mathrm{St}(A^{\otimes})"] \\
& \mathrm{FinSet}_* &
\end{tikzcd}$$

Remark. Under the Straightening Theorem we recover the ordinary intuition: morally, a "straightened lax symmetric monoidal functor" turns out to be the associate non-invertible natural transformation of symmetric monoidal ∞-categories: $\mathrm{UnSt}(M) : \mathrm{const}_{\mathrm{FinSet}_*} \Longrightarrow A^{\otimes}$.

As in the classical setting the functoriality of our algebraic structure is due to the "left morphisms" (i.e. spans whose right arrow is an isomorphism), here it is supplied by the inert maps in Δ. However, our requirement is actually redundant, in that sending the (inert) Segal maps to cocartesian edges suffices.

Lemma B.2.0.5. *(Functoriality can be checked on Segal maps) Equivalently, a commutative algebra M in the symmetric monoidal ∞-category $A^{\otimes} \in \mathrm{CMon}(\mathrm{Cat}_{\infty}^{\times})$ is a section M of $\mathrm{St}(A^{\otimes}) : \int A^{\otimes} \to \mathrm{FinSet}_*$ which takes Segal maps to cocartesian edges.*

Proof. One implication is clear, so let us show only that it suffices to impose our condition on Segal maps; we will work in FinSet_*, where inert maps correspond to injective "left morphisms". Assume that M sends Segal maps to cocartesian edges of $\int A^{\otimes}$, i.e. that the following edge be cocartesian:

$$\rho_i : [I \hookleftarrow \{i\} = \{i\}] \mapsto \left(\rho_i : M(I) \xrightarrow{\mathrm{res}_i} M(\{i\})\right)$$

Observe that the inclusions of sets $\{i\} \subseteq J \subseteq I$ exhibit the following composition $\rho_i^I = \rho_i^J \circ (I \to J) : I \to J \to \{i\}$ in FinSet_*:

$$\begin{tikzcd}
& & \{i\} \arrow[dl, hook] \arrow[dr, equal] & & \\
& J \arrow[dl, hook] \arrow[dr, equal] & & \{i\} \arrow[dl, hook] \arrow[dr, equal] & \\
I \arrow[rr, dotted] & & J \arrow[rr, dotted] & & \{i\}
\end{tikzcd}$$

Now, apply M; by assumption, it sends maps lying over Segal maps to cocartesian morphisms, so that, by [20],3.1.7, also $M(I \to J)$ must be cocartesian, as needed. $\square$

Moreover, observe that, for any symmetric monoidal ∞-category $A^{\otimes}$, inspection of the Straightening construction yields an inclusion $\mathrm{CAlg}(A^{\otimes}) \subseteq \mathrm{CMon}(A^{\otimes})$, so that commutative algebras are in particular commutative monoids, as it should be.

Lemma B.2.0.6. *(Commutative algebras satisfy the Segal condition) Let* $M \in \mathrm{CAlg}(A^{\otimes})$ *be a commutative algebra in a symmetric monoidal* ∞*-category. Then,* M *satisfies the Segal condition, i.e. for each* $I \in \mathrm{FinSet}_*$ *the canonical map* $M(I) \to \prod_I M(\{i\})$ *induced by the Segal maps is an equivalence. Hence, there is an inclusion* $\mathrm{CAlg}(\mathrm{A}^{\otimes}) \subseteq \mathrm{CMon}(A^{\otimes}) \simeq \mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*, \int A^{\otimes})$.

Proof. Recall that there is an identification describing the fibre $(\int A^{\otimes})_{\langle n\rangle} \simeq A^{\otimes}(\langle n\rangle)$, and that a symmetric monoidal ∞-category $A^{\otimes}$ satisfies the Segal condition, i.e. for each $I \in \mathrm{FinSet}$ the canonical map $A^{\otimes}(I) \to A^{\otimes}(*)^I$ is an equivalence.
Then, consider a commutative algebra $M \in \mathrm{CMon}(A^{\otimes})$. Being M a section of the "first projection" $\mathrm{St}(A^{\otimes})$, (up to equivalence) we can describe the action of M on objects as in the classical Grothendieck construction:

- $I = *(= *_+)$: $M(*) := (*, A^{\otimes}(*) \ni x)$;
- $I \in \mathrm{FinSet}_*$ arbitrary: $M(I) = (I, A^{\otimes}(I) \simeq A^{\otimes}(*)^I \ni (x_i)_I)$ for some tuple $(x_i)_I \in A^{\otimes}(*)^I$.

Then, observe that the Segal maps $(\rho_i : I \to \{i\})_I$ induce a canonical comparison arrow $M(I) \to \prod_I M(\{i\})$ given by the projections $(I, A^{\otimes}(*)^I, (x_i)_I) \to (\{i\}, A^{\otimes}(*), x_i)$. Finally, the functoriality of M carries the canonical isomorphism $\{i\} \cong *$ to an equivalence $M(\{i\}) \simeq M(*)$, so that in particular $x_i = x$ for each i. Thus, the previous comparison arrow is an equivalence and we retrieve the Segal condition for M. ☐

The latter observation allows the following generalization of Lemma B.1.0.4.

Definition B.2.0.7. *(Underlying algebra)* Given a symmetric monoidal ∞-category $A^{\otimes} : \mathrm{FinSet}_* \to \mathrm{Cat}_\infty$ (resp. a commutative algebra M), define its **underlying category** (resp. **underlying algebra**) to be $A^{\otimes}(*)$ (resp. $M(*)$).
The latter comes equipped with an unbiased **encoded tensor product** $\otimes$ (or just *tensor product*) given by the Segal condition and a representative of the canonical active map m as in the following triangle (similarly for M):

$$\begin{tikzcd}
A^{\otimes}(I) \arrow[r, "A^{\otimes}(m)"] \arrow[d, "\simeq"] & A^{\otimes}(*) \\
\prod_{i\in I} A^{\otimes}(\{i\}) \arrow[r, equal] & A^{\otimes}(*)^{\times I} \arrow[u, dotted, "\otimes"']
\end{tikzcd}$$

The more familiar biased definition is retrieved for $I = [1]_+$, which yields the familiar bi-functor $\otimes : A^{\otimes}(*) \times A^{\otimes}(*) \to A^{\otimes}(*)$ (similarly for M).

Remark. In particular, in agreement with the "microcosm principle", $A^{\otimes}$ acts at each algebra $M \in \mathrm{CAlg}(A^{\otimes}(*))$ as $\exp_M$, so that the environment-tensor induces the second operation of M via the construction of the previous section.
In other words, informally on the 1-skeleton one has that: for each $I_+ \to J_+$ in FinSet_+, $A^{\otimes}$ acts on $(f, g) : I_+ \to J_+$ at M by $A^{\otimes}(f, g) := m_g \circ res_f(-)$; i.e. it sends the span $I \xleftarrow{f} K \xrightarrow{g} J$ to the 1-morphism:

$$\begin{array}{ccccc}
A^{\otimes}(I) & \xrightarrow{\mathrm{res}_f} & A^{\otimes}(K) & \xrightarrow{m_g} & A^{\otimes}(J) \\
M(I) \simeq (M(\{i\}) \mid i \in I) \mapsto & & (M(\{i\}) | i \in K) \simeq M(K) \mapsto & & (M^{\otimes g^{-1}(j)} := \otimes_{k \in g^{-1}(j)} M(\{k\}) \mid j \in J) \simeq M(J)
\end{array}$$

where m_g acts as $\otimes$-multiplication on the fibres of g.

Remark. As expected, in general the inclusion is faithful but not full - and hence in particular not an equivalence. However, for a *cartesian* symmetric monoidal ∞-category (see [36],5.24) the notion of an algebra object coincides with that of a monoid object (see e.g. [36],5.27). Intuitively, this means that the cartesian monoidal structure of an "algebraic" object should be thought as an implicit datum.

Remark. Informally, by the Segal condition a commutative monoid $M \in \mathrm{CMon}(A^{\otimes})$ can be "represented" by the distinguished object $x \in A^{\otimes}$ of its underlying commutative monoid $M(*) = (*, A^{\otimes}(*) \ni x)$. In the applications we will abuse notation and mention only such an object in place of the commutative monoid at stake.

Remark. Notice that in our construction the multiplication is fibre-wise induced by the (symmetric) cartesian structure of Cat_∞ and is, therefore, intrinsically symmetric.
One could then define arbitrary monoidal ∞-categories to be "finitely additive" simplicial objects in Cat_∞ (see B.1.0.9); our symmetric monoidal ∞-categories would then turn out to be a special instance of the arbitrary ones, and one could express them in the more general setting via pulling back along a suitable comparison functor $\Delta^{op} \to \mathrm{FinSet}_*$ (see [8],3.3).

Arrows in Symmetric Monoidal ∞-Categories. In view of our future applications, we need a description of morphisms in unstraightened symmetric monoidal ∞-categories in terms of Δ. To this end, let us record here a couple of combinatorial results. We will start by introducing a piece of notation.

Notation. Let $A^\otimes : \Delta \to \mathrm{Cat}_\infty$ be a symmetric monoidal ∞-category, and let $\mathrm{St}(A^\otimes) : \underline{A}^\otimes := \int A^\otimes \to \Delta$ denote its unstraightened version. By the Segal condition we have canonical equivalences of fibres:

$$\underline{A}^\otimes_{\langle n\rangle} \simeq A^\otimes(\langle n\rangle) \simeq A^\otimes(\langle 1\rangle)^n \simeq \prod_{\langle n\rangle^\circ} \underline{A}^\otimes_{\langle 1\rangle}$$

where $(-)^\circ : \mathrm{FinSet}_* \simeq \Delta \to \mathrm{FinSet}$ forgets the pointing, i.e. it drops the 0.
So, we can identify an object x in the left-hand side with an n-tuple $(x_i)_{i=1}^n$; we will denote the latter by $\oplus_{i\in\langle n\rangle^\circ} x_i$.

The next Proposition explains in which sense the ambient symmetric monoidal ∞-category Cat_∞ acts at a commutative monoid $A^\otimes$ by $\exp_{A^\otimes}$.

Proposition B.2.0.8. *(Lifts in $\underline{A}^\otimes$, [36],5.14) Let $\underline{A}^\otimes \to \mathrm{FinSet}_*$ be an (unstraightened) symmetric monoidal ∞-category. Consider two objects $x := \oplus_{i\in\langle n\rangle^\circ} x_i \in \underline{A}^\otimes_{\langle n\rangle}$ and $y := \oplus_{j\in\langle m\rangle^\circ} y_j \in \underline{A}^\otimes_{\langle m\rangle}$ in $\underline{A}^\otimes$. Then, the fibre of the action*

$$\mathrm{St}(A^\otimes) : \mathrm{Map}_{\underline{A}^\otimes}(x, y) \to \mathrm{Map}_\Delta(\langle n\rangle, \langle m\rangle)$$

over any map $f : \langle n\rangle \to \langle m\rangle$ in Δ is the space

$$\mathrm{Fib}_f(\mathrm{St}(A^\otimes)) = \prod_{j\in\langle m\rangle^\circ} \mathrm{Map}_A\big(\otimes_{f(i)=j}\, x_i\,,\, y_j\big)$$

where $A = A^\otimes(\langle 1\rangle) \simeq A^\otimes(*)$ is the underlying symmetric monoidal ∞-category.

Proof. Inspection of the Straightening equivalence yields a lift $A^\otimes(f) : \underline{A}^\otimes_{\langle n\rangle} \to \underline{A}^\otimes_{\langle m\rangle}$ over $f : \langle n\rangle \to \langle m\rangle$.
Then, observe that the 1-truncation of the section $A^\otimes$ to $\mathrm{St}(\mathrm{Cat}_\infty^\times)$ can be described by means of our exp-construction, so the action of $A^\otimes(f)$ at x includes also an arrow in $\underline{A}^\otimes$ connecting x and its image $A^\otimes(x)$:

$$\exp_x(f) = \otimes_f \circ \mathrm{res}_f : x = \oplus_{i\in\langle n\rangle^\circ} x_i \longrightarrow \oplus_{j\in\langle m\rangle^\circ}\big(\otimes_{f(i)=j}\, x_i\big) = A^\otimes(f)(x)$$

Now, observe that such an equivalence is a $\mathrm{St}(A^\otimes)$-cocartesian edge over the identity $1_{\langle m\rangle}$. Then, by Lemma [20],3.19 the cocartesian fibration $\mathrm{St}(A^\otimes)$ induces a fibre sequence in Spc pointed by $f : \langle n\rangle \to \langle m\rangle$:

$$\mathrm{Map}_{\underline{A}^\otimes_{\langle m\rangle}}(A^\otimes(f)(x), y) \to \mathrm{Map}_{\underline{A}^\otimes}(x, y) \to \mathrm{Map}_\Delta(\langle n\rangle, \langle m\rangle)$$

Indeed, recall that $\mathrm{Map}_{\underline{A}^\otimes}(x, y) \simeq \cup_{g:\langle n\rangle\to\langle m\rangle} \mathrm{Map}_{\underline{A}^\otimes_{\langle m\rangle}}(A^\otimes(g)(x), y)$ by the construction of the universal fibration $\pi_{\mathrm{univ}} : \mathrm{Cat}_{\infty//*} \to \mathrm{Cat}_\infty$ and the fact that $\underline{A}^\otimes$ is a pull-back of π_{univ} along $A^\otimes$. ☐

Lemma B.2.0.9. *(Characterization of cocartesian lifts, [36],5.15) Let $\underline{A}^\otimes$ be an (unstraightened) symmetric monoidal ∞-category and consider a map $\phi : \oplus x_i \to \oplus y_j$ in $\underline{A}^\otimes$ living over $f := \mathrm{St}(A^\otimes)(\phi)$ in FinSet_*. By the Segal condition, the latter can be identified with (the product of) a family of maps $\{\phi_j : \oplus_{f(i)=j} x_i \to y_j\}_j$. Then, ϕ is a cocartesian lift of f iff ϕ_j is an equivalence for each j.*

Proof. By the above Proposition B.2.0.8, the space of lifts over f is a product of mapping spaces of the form $\mathrm{Map}_A(\otimes_{f(i)=j} x_i, y_j)$ over the identity $1_{\langle 1\rangle}$. So, from the very definition of $\mathrm{St}(A^\otimes)$-cocartesian edges, a lift $\phi = (\phi_j)_j$ over f is cocartesian iff each ϕ_j is such, iff each ϕ_j is an equivalence (see [20],3.1.6). ☐

Lax Symmetric Monoidal Functors. Transporting the symmetric monoidal structure preserves algebra objects.

Lemma B.2.0.10. *([36],5.19) A (unstraightened) lax symmetric monoidal functor $F : \underline{A}^{\otimes} \to \underline{B}^{\otimes}$ induces - under the post-composition* $\mathrm{Fun}(\mathrm{FinSet}_*, F)$ *- a functor* $\mathrm{CAlg}(A^{\otimes}) \to \mathrm{CAlg}(B^{\otimes})$.

Proof. The categories of commutative algebras are full subcategories of $\mathrm{Fun}^{\mathrm{Seg}}(\mathrm{FinSet}_*), \underline{A}^{\otimes})$ (resp. with $\underline{B}^{\otimes}$), so it suffices to show that $F \circ M : \mathrm{FinSet}_* \to \underline{B}^{\otimes}$ is a commutative algebra (i.e. a section of $\mathrm{St}(B^{\otimes})$ which preserves inert maps) whenever $M \in \mathrm{CAlg}(A^{\otimes})$. The section part is a consequence of Striaghtening, while the compatibility with inert maps amounts precisely to the fact that F is lax symmetric monoidal. $\square$

B.3 Localization of Symmetric Monoidal ∞-categories

Our goal for this section is to provide a verifiable condition for a (Bousfield) left-localization in order for it to be promoted to a symmetric monoidal adjunction. This will be our main ingredient to show that the $\mathcal{P}_{\Sigma}$-construction transports symmetric monoidal structures.

Definition B.3.0.1. *(Algebraic localization, [23],2.2.1.6 as in [36],8.1-2)* Let $A^{\otimes}$ be a symmetric monoidal ∞-category and let $L : C \to A[S^{-1}] \subseteq A$ be a left-localization of the underlying ∞-category $A = A^{\otimes}(*)$ with respect to some set $S \subseteq \mathrm{Mor}(A)$ of S-local weak equivalences. Then, the localization L is **compatible with the symmetric monoidal structure** (or **algebraic localization** for short) iff S is tensor-closed, i.e. iff one of the following two equivalent conditions holds:

1. For each local equivalence $(f : x \to y) \in S$ and each object $z \in A$, then also $(f \otimes 1_z : x \otimes z \to y \otimes z) \in S$ is a weak equivalence.

2. For every finite collection $\{f_i : x_i \to y_i\}_i \subseteq S$ of weak equivalences, their tensor product $(\otimes f_i : \otimes x_i \to \otimes y_i) \in S$ is again a weak equivalence.

We are now ready to state the main result of the Appendix.

Lemma B.3.0.2. *(Algebraic localizations are symmetric monoidal, [36],8.3) Let $A^{\otimes}$ be a symmetric monoidal ∞-category and set $A := A^{\otimes}(*)$ for the underlying ∞-category. Let $L : A \to A[S^{-1}] \subseteq A$ be an algebraic (Bousfield) left-localization functor of A with respect to S-local maps, for $S \subseteq \mathrm{Mor}(A)$ a small $\otimes$-tensor-closed set.*
Define $L\underline{A}^{\otimes} \subseteq_{f.f.} \underline{A}^{\otimes}$ to be the (unstraightened) full subcategory spanned by the class

$$\{\oplus_{i \in \langle n \rangle^{\circ}} Lx_i \mid \langle n \rangle \in \Delta \,,\, x_i \in A = A^{\otimes}(\langle 1 \rangle)\}$$

Then, the following statements hold true:

1. *The restriction $L\underline{A}^{\otimes} \to \mathrm{FinSet}_*$ of $\mathrm{St}(A^{\otimes})$ exhibits $L\underline{A}^{\otimes}$ as a symmetric monoidal ∞-category with underlying ∞-category $L(A)$.*

2. *$L \vdash \subseteq$ extends to a left-localization $L^{\otimes} : L\underline{A}^{\otimes} \rightleftarrows \underline{A}^{\otimes} : \subseteq$, i.e. such that $L^{\otimes}_{|A} = L$. Moreover, the unit $u^{\otimes}$ of $L^{\otimes} \vdash \subseteq$ can be chosen so that $\mathrm{St}(A^{\otimes})(u^{\otimes}) = \mathrm{id}_{\mathrm{FinSet}_*}$ retrieves the identity of FinSet_*.*

3. *$L^{\otimes}$ is a symmetric monoidal functor, and the inclusion $L\underline{A}^{\otimes} \subseteq \underline{A}^{\otimes}$ is a lax symmetric monoidal functor.*

Before proving the Proposition we need a technical Lemma; it is a particular case of Lurie's more general formulation.

Lemma B.3.0.3. *(Left localizations are morphisms in* CoCart*, [23],2.2.1.11) Let $p : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and let $L : \mathcal{E} \to L\mathcal{E} \subseteq \mathcal{E}$ be a left-localization whose unit induces $p \simeq p \circ L$. Then,*

1. *$L : \mathcal{E} \to L(\mathcal{E})$ preserves p-cocartesian edges;*

2. *The restriction $p : L(\mathcal{E}) \to \mathcal{C}$ is again a cocartesian fibration.*

Proof. (1) : Given any p-cocartesian edge $f : x \to y$ in $\mathcal{C}$, we want to show that also $p(f)$ is p-cocartesian. In other words, from the very definition (see [24],2.4.4.3) we want the following square of Kan complexes to be homotopy cartesian for each $z' \in L\mathcal{E}$:

$$\begin{tikzcd} \mathcal{E}_{Lf_/} \times_{\mathcal{E}} \{z'\} \arrow[r, "L(f)^*"] \arrow[d, "p"] & \mathcal{E}_{Lx_/} \times_{\mathcal{E}} \{z'\} \arrow[d, "p"] \\ \mathcal{C}_{p(Lf)_/} \times_{\mathcal{C}} \{p(z')\} \arrow[r, "pL(f)^*"] & \mathcal{C}_{p(Lx)_/} \times_{\mathcal{C}} \{p(z')\} \end{tikzcd}$$

In order to prove it, observe first that by assumption f is p-cocartesian, so that the corresponding square (drop L in the previous one) is homotopy cartesian in Kan. Now, pre-composition with the unit $u : \mathrm{id}_{\mathcal{E}} \to L$ of the localization $L \vdash \subseteq$ gives a natural transformation comparing the two squares, so we are left to show that the latter be an equivalence at each vertex (see [20],2.2.2).
To this end, notice that we can assume wlog $z' \in L(\mathcal{E})$ has the form $z' \simeq L(z)$ with $z = Le \in L(\mathcal{E})$, since $L^2 = L$. Thus, u induces equivalences of the upper vertices:

$$\mathcal{E}_{f_/} \times_{\mathcal{E}} \{Le\} \to \mathcal{E}_{Lf_/} \times_{\mathcal{E}} \{Le\} \qquad ; \qquad \mathcal{E}_{x_/} \times_{\mathcal{E}} \{Le\} \to \mathcal{E}_{Lx_/} \times_{\mathcal{E}} \{Le\}$$

As for the comparison maps of the lower vertices:

$$\mathcal{C}_{p(f)_/} \times_{\mathcal{C}} \{p(e)\} \to \mathcal{C}_{p(f)_/} \times_{\mathcal{C}} \{p(e)\} \qquad ; \qquad \mathcal{C}_{p(x)_/} \times_{\mathcal{C}} \{p(e)\} \to \mathcal{C}_{p(Lx)_/} \times_{\mathcal{C}} \{p(e)\}$$

the assumption $p \circ L \simeq p$ allows us to consider fibres over $p(Lz) = p(z) = p(Le) = p(e)$, so we can infer that they are also equivalences by remarking that $p(u) : p \to p \circ L = p$ amounts to the identity transformation of p, so that the maps between the first factors is always equivalent to the identity of the under-slices.

(2) : We need to show that, for any arbitrary choice of $x \in L\mathcal{E}$ and $f : p(x) \to c$ in $\mathcal{C}$, there exists a p-cocartesian lift $\bar{f} : x \to \bar{c}$ in $L\mathcal{E}$ of f. By assumption, f admits some p-cocartesian lift $f' : x \to e$ in $\mathcal{E}$ with $x \in L\mathcal{E}$; then, set $\bar{f} := L(f') : x \to L\bar{c}$ in $L\mathcal{E}$: it is p-cocartesian by (1) and, by assumption, $p \circ L \simeq p$, so that also $p(\bar{f}) = p \circ L(f') \simeq p(f') = f$ gives the sought edge up to the equivalence $p \circ u$. But now, p is also an isofibration, so - by Joyal's Lifting Horn's Theorem [20],2.1.10 - we can lift the equivalence $p \circ u : \bar{f} \to f$ to one in $L\mathcal{E}$, say $\tilde{f} \to f$ for some $\tilde{f}$ such that $p(\tilde{f}) = f$. Finally, recall that, from the very definition, p-cocartesian edges are stable under equivalence, so that $\tilde{f}$ is also p-cocartesian, as wished. □

We are now ready to prove the Proposition.

Proof. (of the Prop.B.3.0.2) (2) : Let's construct the localization functor $L^{\otimes} : \underline{A}^{\otimes} \to L\underline{A}^{\otimes}$. By the theory of Bousfield left-localizations, it suffices to specify a family of local equivalences $\mathcal{W} \subseteq \mathrm{Mod}(L\underline{A}^{\otimes})$ for which $L\underline{A}^{\otimes}$ consists of $\mathcal{W}$-local objects.
For $u : \mathrm{id}_A \to L$ the unit of the localization $L \vdash \subseteq$, define:

$$\mathcal{W} := \{\otimes_i u(x_i) : \otimes_{i \in \langle n \rangle^{\circ}} x_i \to \oplus_{i \in \langle n \rangle^{\circ}} L(x_i) \mid \langle n \rangle \in \Delta, x_i \in A = \underline{A}^{\otimes}_{\langle 1 \rangle}\}$$

We need to check that each object $\oplus_{j \in \langle m \rangle^{\circ}} Ly_j \in L\underline{A}^{\otimes} \subseteq_{f.f.} \underline{A}^{\otimes}$ is $\mathcal{W}$-local, namely that pre-composition with any map of $\mathcal{W}$ induces an equivalence of mapping spaces of $\underline{A}^{\otimes}$:

$$\big(\oplus_i u(x_i)\big)^* : \mathrm{Map}_{\underline{A}^{\otimes}}\big(\oplus_{i \in \langle n \rangle^{\circ}} L(x_i)\,,\, \oplus_{j \in \langle m \rangle^{\circ}} Ly_j\big) \longrightarrow \mathrm{Map}_{\underline{A}^{\otimes}}\big(\otimes_{i \in \langle n \rangle^{\circ}} x_i\,,\, \oplus_{j \in \langle m \rangle^{\circ}} Ly_j\big)$$

We can canonically decompose the arrow above as a disjoint union of its fibres over each of the maps $f : \langle n \rangle \to \langle m \rangle$ in Δ; now, by our previous computation B.2.0.8, the fibre over a fixed f in Δ is the product indexed by $j \in \langle m \rangle^{\circ}$ of the mapping spaces $\mathrm{Map}_A(\otimes_{f(i)=j} z_i, Ly_j)$ with $z_i = x_i$ or Lx_i.
Then, observing that pre-composition with the units $\oplus_{f(i)=j} u(x_i)$ lives in the fibre $\underline{A}^{\otimes}_{\langle n \rangle}$ over $\langle n \rangle$ (and hence respects the decomposition), it suffices to show that, for each $j \in \langle m \rangle^{\circ}$, pre-composition with the units $\oplus_{f(i)=j} u(x_i)$ induces an equivalence $\mathrm{Map}_A(\otimes_{f(i)=j} x_i, Ly_j) \to \mathrm{Map}_A(\otimes_{f(i)=j} Lx_i, Ly_j)$.
But now, the latter claim follows from the assumption that L is an algebraic left-localization of A with respect to the S-local maps (which comprise $S = \{u(x) : x \to Lx \mid x \in A\}$) with S $\otimes$-tensor-closed.

Thus, the fully faithful embedding $L\underline{A}^{\otimes} \subseteq_{f.f.} \underline{A}^{\otimes}$ admits a left adjoint $L^{\otimes}$ and the adjunction $L^{\otimes} \vdash \subseteq$ exhibits $L\underline{A}^{\otimes}$ as a left-localization of $\underline{A}^{\otimes}$ with respect to the $\mathcal{W}$-local-equivalences. Moreover, pre-composition with

the restriction to A preserves the adjunction, so that $L^{\otimes}_{|A}$ is still a left-adjoint to the restriction of the embedding, and hence it must coincide with L, as required.
Finally, our construction implies that the unit $u^{\otimes}$ takes as values edges which are the tensor product of values of u. Thus, since by assumption $\mathrm{St}(A^{\otimes}) \circ u : \mathrm{St}(A^{\otimes}) \simeq \mathrm{St}(A^{\otimes})L$ is the identity on $\langle 1 \rangle$, the composition $\mathrm{St}(A^{\otimes}) \circ u^{\otimes}$ takes values into the identities of (the objects of) Δ.

$(1), (3), i$: They are a consequence of our technical Lemma B.3.0.3:

- the map $\mathrm{St}(A^{\otimes}) \circ L^{\otimes}$ is a cocartesian fibration satisfying the Segal condition, because $L^{\otimes}$ preserves $\mathrm{St}(A^{\otimes})$-cocartesian edges, and a fortiori also the equivalences induced by the Segal maps (which in turn live over the canonical product equivalences, so we can conclude by [20],3.1.6);
- moreover, $L^{\otimes}$ is a symmetric monoidal functor, since it takes $\mathcal{W}$-local maps to equivalences, thus exhibiting $L^{\otimes}(\otimes x_i) \simeq \otimes L x_i$ as in the decomposition above (remark that the canonical active map m expresses any $\langle n \rangle$ as the pre-image of a point in $\langle 1 \rangle$).

$(3), ii$: We are left to prove that the inclusion $L\underline{A}^{\otimes} \subseteq_{f.f.} \underline{A}^{\otimes}$ is lax symmetric monoidal, i.e. that it preserves inert edges. Recall that an inert edge $\phi : \oplus_{i \in \langle n \rangle^{\circ}} x_i \to \oplus_{j \in \langle m \rangle^{\circ}} y_j$ in the unstraightened symmetric monoidal ∞-category $\underline{A}^{\otimes}$ is a $\mathrm{St}(A^{\otimes})$-cocartesian edge living over an inert map $f : \langle n \rangle \to \langle m \rangle$ in Δ. By our previous computations B.2.0.8 and B.2.0.9, recall that the $\mathrm{St}(A^{\otimes})$-cocartesian edge ϕ corresponds to a family $(\phi_j : \otimes_{f(i)=j} x_i \to y_j)_{j \in \langle m \rangle^{\circ}}$ consisting of equivalences in A, so over $\langle 1 \rangle$. Now, being it inert, $\# f^{-1}(j) = 1$ for each j, thus the tensor products are unary. The same reasoning holds in $L\underline{A}^{\otimes} \to \mathrm{FinSet}_*$, so the inclusion clearly respects inert edges. $\square$

B.4 Closed Symmetric Monoidal ∞-category

An ordinary symmetric monoidal category $(\mathcal{C}, \otimes)$ is called closed iff, for each $x \in \mathcal{C}$, (left or equivalently right) tensoring with x admits a right adjoint, say $x \otimes (-) \vdash [x, -]$. Such a notion immediately generalizes to symmetric monoidal ∞-categories as follows.

Definition B.4.0.1. *(Closed symmetric monoidal ∞-category, [23],4.1.1.15)* A symmetric monoidal ∞-category $C^{\otimes}$ is **closed** iff, for each $x \in C := C^{\otimes}(*)$, (left or equivalently right) tensoring (see B.2.0.7) with x admits a right adjoint, say $x \otimes (-) \vdash [x, -]$, i.e. there is a point-wise equivalence $\mathrm{Map}_C(x \otimes (-), \star) \simeq \mathrm{Map}_C(-, [x, \star])$.

By [20],5.1.10, adjunctions can be defined (and checked) point-wise via the triangle identities. So, the requirement amounts to the fact that, for each pair of objects $x,\ y \in C := C^{\otimes}(*)$, there exists a third object $y^x \in C$ together with an arrow $y^x \otimes x \to y$ inducing point-wise homotopy equivalences $\mathrm{Map}_C(\star, y^x) \simeq \mathrm{Map}_C(\star \otimes x, y)$.
Then, the construction $[y \mapsto y^x]$ can be promoted to an endofunctor of the ∞-category C supplying the right-adjoint to $(-) \otimes x$.

As an example, we will endow animated modules with a closed symmetric monoidal structure. The same reasoning holds for any ∞-category - such as Ani(CRing) - obtained via the $\mathcal{P}_{\Sigma}$-construction.

Construction B.4.0.2. *($\mathrm{Mod}_{\underline{A}}$ is a closed symmetric monoidal ∞-category)* For any animated ring $\underline{A} \in$ Ani(CRing), recall that $\mathrm{Mod}_{\underline{A}} \simeq \mathcal{P}_{\Sigma}(\mathrm{FFree}_{\underline{A}})$ (see 3.2.5.14) is a Bousfield left-localization of the category $\mathcal{P}(\mathrm{FFree}_{\underline{A}})$ of presheaves over $\mathrm{FFree}_{\underline{A}}$ with respect to the S-local equivalences, for a small set S as in the proof of A.2.0.2,i:

$$S = \left\{ j(\prod_{i=1}^{m} \underline{A}^{n_i}) \to \coprod_{i=1}^{m} j(\underline{A}^{n_i}) \;\middle|\; m < \omega \,,\, (\underline{A}^{n_i})_{i=1}^{m} \right\}$$

Now, consider the following symmetric monoidal ∞-categories:

- $\mathrm{FFree}_{\underline{A}}$ admits finite coproducts, so it can be endowed with a cocartesian symmetric monoidal structure $\mathrm{FFree}_{\underline{A}}^{\amalg}$ (via a construction dual to the cartesian one, see [36],5.24-29-30).

- $\mathrm{FFree}_{\underline{A}}$ admits a "non-trivial" symmetric monoidal structure defined as follows.
  (a) For $\underline{A} := R \in \mathrm{CRing}$ static, the classical tensor product of modules $\otimes_R$ induces a symmetric monoidal structure $\mathrm{FFree}_R^{\otimes}$;
  (a) For an arbitrary $\underline{A} \in \mathrm{Ani(CRing)}$, applying fibre-wise "extension of scalars" induces by the Segal condition a map $p_{\underline{A}}^{\otimes} : \mathrm{FFree}_{\underline{A}}^{\otimes} := \underline{A} \otimes_{\mathbb{Z}} \mathrm{FFree}_{\mathbb{Z}}^{\otimes} \to \Delta$. The "extension of scalars" functor takes $p_{\mathbb{Z}}^{\otimes}$-cocartesian edges to $p_{\underline{A}}^{\otimes}$-cocartesian ones, since it preserves finite products and by the characterization B.2.0.9. Hence, it defines a symmetric monoidal functor $\mathrm{FFree}_{\mathbb{Z}}^{\otimes} \to \mathrm{FFree}_{\underline{A}}^{\otimes}$.
- On the other hand, Spc admits a cartesian symmetric monoidal structure whose encoded tensor product $\times$ preserves small colimits separately in each variable (binary products are pull-backs on the terminal object, so we conclude by the universality of colimits in the ∞-topos Spc).

Then, by [23],4.8.1.12 each symmetric monoidal structure on $\mathrm{FFree}_{\underline{A}}$ induces one on $\mathcal{P}(\mathrm{FFree}_{\underline{A}})$, namely $\mathcal{P}(\mathrm{FFree}_{\underline{A}})^{\otimes}$, which is characterized (up to symmetric monoidal equivalence) by the following universal property:

- The Yoneda embedding $j : \mathrm{FFree}_{\underline{A}} \hookrightarrow \mathcal{P}(\mathrm{FFree}_{\underline{A}})$ can be extended to a symmetric monoidal functor;
- The tensor product $\otimes : \mathcal{P}(\mathrm{FFree}_{\underline{A}}) \times \mathcal{P}(\mathrm{FFree}_{\underline{A}}) \to \mathcal{P}(\mathrm{FFree}_{\underline{A}})$ preserves small colimits separately in each variable.

As observed in the subsequence [23],4.8.1.13, a candidate exhibiting such a structure is given by Day convolution (see [23],2.2.6.17 or [10]); in other words, the encoded tensor product $\circledast$ is the left Kan extension of $\times_{\mathrm{Spc}} \circ \times_{\mathcal{P}(\mathrm{FFree}_{\underline{A}})}$ along either the cocartesian tensor product $\amalg : \mathrm{FFree}_{\underline{A}} \times \mathrm{FFree}_{\underline{A}} \to \mathrm{FFree}_{\underline{A}}$ or the module-theoretic tensor product $\otimes_{\underline{A}} : \mathrm{FFree}_{\underline{A}} \times \mathrm{FFree}_{\underline{A}} \to \mathrm{FFree}_{\underline{A}}$.
Informally, this means that, for each pair of presheaves $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{P}(\mathrm{FFree}_{\underline{A}})$, in the co-cartesian case one has the following point-wise formula:

$$\mathcal{F}_0 \circledast \mathcal{F}_1(\underline{A}^n) \simeq \varinjlim \left(\mathcal{F}_0(\underline{A}^{n_0}) \times \mathcal{F}_1(\underline{A}^{n_1}) \mid (\underline{A}^{n_0}, \underline{A}^{n_1}, u : \underline{A}^{n_0+n_1} \to \underline{A}^n) \text{ in } \mathrm{FFree}_{\underline{A}} \right)$$

or along maps $u : \underline{A}^{n_0} \otimes_{\underline{A}} \underline{A}^{n_1} \simeq \underline{A}^{n_0 n_1} \to \underline{A}^n$ in the case of $\mathrm{FFree}_{\underline{A}}^{\otimes}$.
Hence, in order to endow $\mathrm{Mod}_{\underline{A}}$ with a symmetric monoidal structure, we are left to check one more Claim:

Claim 1. *The localization functor $L : \mathcal{P}(\mathrm{FFree}_{\underline{A}}) \to \mathcal{P}_{\Sigma}(\mathrm{FFree}_{\underline{A}})$ is algebraic; hence, we can apply B.3.0.2.*

Proof. It suffices to prove that the class S of morphisms inducing the localization is $\circledast$-tensor-closed; this follows almost tautologically by abstract non-sense and does not depend on the chosen symmetric monoidal structure on $\mathrm{FFree}_{\underline{A}}$. So, for each $M, N \in \mathrm{Mod}_{\underline{A}}$ and any map in S we wish the following to be an equivalence of mapping spaces:

$$\mathrm{Map}_{\mathcal{P}(\underline{A})}\left(\coprod_i j(\underline{A}^{n_i}) \circledast \underline{M}, \underline{N}\right) \longrightarrow \mathrm{Map}_{\mathcal{P}(\underline{A})}\left(j(\prod_i \underline{A}^{n_i}) \circledast \underline{M}, \underline{N}\right)$$

Since $\circledast$ preserves colimits (being it a left Kan extension), the following manipulation of the target allows us to forget about products and coproducts in the local map and concludes then the verification:

$$\mathrm{Map}\left(j(\prod_i \underline{A}^{n_i}) \circledast \underline{M}, \underline{N}\right) \simeq_{(i)} \mathrm{Map}\left(\prod_i j(\underline{A}^{n_i}) \circledast \underline{M}, \underline{N}\right) \simeq_{(ii)} \mathrm{Map}\left(\coprod_i j(\underline{A}^{n_i}) \circledast \underline{M}, \underline{N}\right)$$

where (i) holds because mapping spaces commute with limits in the covariant argument, while (ii) is obtained as follows: by A.2.0.2,ii, our mapping spaces are wlog in $\mathcal{P}_{\Sigma}(\mathrm{FFree}_{\underline{A}})$, and there finite products coincide with finite coproducts, since the set S above is taken by the localization L to equivalences of $\mathcal{P}_{\Sigma}(\mathrm{FFree}_{\underline{A}})$. ■

Thus, we have promoted $\mathrm{Mod}_{\underline{A}}$ to a symmetric monoidal ∞-category $\mathrm{Mod}_{\underline{A}}^{\otimes}$. Let's make a consistency remark about the computation of tensor products.

Claim 2. *The encoded tensor product $\otimes_{\underline{A}} := L \circ \circledast$ can be computed via sifted resolutions, where $\circledast$ is extended along either $\amalg$ or $(\otimes_{\underline{A}})_{|\mathrm{FFree}_{\underline{A}}}$.*

In other words, let it be given any $\underline{M}$, $\underline{N} \in \mathrm{Mod}_{\underline{A}}$, *together with sifted simplicial diagrams* $p, q : K \to \mathrm{FFree}_{\underline{A}}$ *whose geometric realizations exhibit* $|p| \simeq \underline{M}$ *and* $|q| \simeq \underline{N}$. *Then,* $\underline{M} \otimes_{\underline{A}} \underline{N} \simeq |p \otimes_{\underline{A}} q|$.

Proof. The statement is tautological: L preserves colimits, so we are left to show that the Day convolution can be computed via geometric resolutions, and this is clear: the Yoneda embedding is symmetric monoidal and the Day convolution commutes with colimits.
Nevertheless, in order to foster the intuition let us get our hands dirty with a more explicit (although rather informal) computation. There is a canonical map between the two modules in the statement, so (by virtue of [20],2.2.2) let's unwind the definition of the encoded tensor product $\otimes_{\underline{A}} := L \circ \circledast$ on objects: given any $\underline{M} \simeq |p|$, $\underline{N} \simeq |q| \in \mathrm{Mod}_{\underline{A}}$ as in the statement, $\underline{M} \otimes_A \underline{N} = L(\underline{M} \circledast \underline{N})$ preserves finite products in $\mathrm{FFree}_{\underline{A}}^{op}$, so it suffices to compute only the underlying space of the Day convolution on $\underline{A} \in \mathrm{FFree}_{\underline{A}}$: being the Yoneda embedding symmetric monoidal, we will drop it in the notation; there is a chain of equivalences of colimits indexed over $(\underline{A}^{n_0}, \underline{A}^{n_1}, u : \underline{A}^{n_0+n_1} \to \underline{A})$ in $\mathrm{FFree}_{\underline{A}}$

$$\underline{M} \circledast \underline{N}(\underline{A}) \simeq \underset{(n_0,n_1,u)}{\underrightarrow{\mathrm{colim}}} |p(\underline{A})|^{n_0} \oplus |q(\underline{A})|^{n_1} \simeq \underset{(n_0,n_1,u)}{\underrightarrow{\mathrm{colim}}} |p(\underline{A})^{n_0} \oplus q(\underline{A})^{n_1}| \simeq |\underset{(n_0,n_1,u)}{\underrightarrow{\mathrm{colim}}} p(\underline{A})^{n_0} \oplus q(\underline{A})^{n_1}|$$

and the last term is equivalent to the sought geometric realization $|p(\underline{A}) \circledast q(\underline{A})| \simeq \mathrm{colim}_K^{sift}\big(p(k) \otimes_{\underline{A}} q(k)\big)$, as desired. ■

Finally, let us show the closure part. Again, this follows from the presentability of non-abelian localizations of small categories and does not depend on the chosen symmetric monoidal structure to be animated.

Claim 3. $\mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})^{\otimes}$ *is a closed symmetric monoidal ∞-category. In other words, for each* $\underline{M} \in \mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})$, *the induced functor* $\underline{M} \otimes_{\underline{A}} (-)$ *"tensoring with* $\underline{M}$*" admits a right adjoint* $[\underline{M}, -]$.

Proof. It follows from abstract non-sense: by A.2.0.2 the ∞-category $\mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})$ is presentable, so we can apply the I Adjoint Functor Theorem 1.2.0.5 to the colimit-preserving functor $\underline{M} \otimes_{\underline{A}} (-) : \mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}}) \to \mathcal{P}_\Sigma(\mathrm{FFree}_{\underline{A}})$. ■

Remark. The same construction allows, for instance, the definition of $\mathrm{MOD}^{\boxtimes} := \mathrm{Ani}(\mathrm{CRMod})^{\boxtimes}$. This time we considered the external product $\boxtimes$ on $\mathrm{CRMod}^{\mathrm{sfp}}$, whose encoded tensor product acts informally as follows: given any $(A, A^n), (B, B^m) \in \mathrm{CRMod}^{\mathrm{sfp}}$, define $(A, A^n) \boxtimes (B, B^m) := (\iota_1)_* A^n \otimes_{A \otimes B} (\iota_2)_* B^m$ where ι_i is the canonical map into $A \otimes B$ and $(\iota_i)_*$ tensors by $A \otimes B$.

B.5 A Symmetric Monoidal enhancement of the Straightening Theorem

In this section we briefly present an enhancement of the Straightening Theorem which allows to construct functors with values in the ∞-category SymMon of symmetric monoidal ∞-categories. Our reference is the Appendix of [5],A.

We start with the definition of a relative symmetric monoidal ∞-category. Throughout this section, we will adopt the following notation.
Notation. Given a symmetric monoidal ∞-category $p_A^{\otimes} := \mathrm{St}(A^{\otimes}) : \underline{A} := \int A^{\otimes} \to \mathrm{FinSet}_* \simeq \Delta$ denote the corresponding underlying symmetric monoidal ∞-category with any of the following symbols: $A := A^{\otimes}(*) \simeq A^{\otimes}(\langle 1 \rangle) \simeq \underline{A}^{\otimes}_{\langle 1 \rangle}$; let $p_A : A \to \mathrm{FinSet}_*$ denote the canonical map. $p_A^{\otimes}$ induces an encoded tensor product, which will be denoted by either $\otimes$ or $\otimes_A$ in case of need.

Definition B.5.0.1. *(Relative symmetric monoidal ∞-category - unstraightened, [5],A.2)* Let $\pi_B^{\otimes} : \underline{B}^{\otimes} \to \Delta$ be a symmetric monoidal ∞-category. Define the ∞-category of **relative symmetric monoidal ∞-categories over** $B^{\otimes}$ as the full subcategory $\mathrm{SymMon}_{\underline{B}^{\otimes}} \subseteq_{f.f.} \mathrm{CoCart}(\underline{B}^{\otimes})$ spanned by those cocartesian fibrations $p^{\otimes} : \underline{A} \to \underline{B}^{\otimes}$ which satisfy the following axiom:
Relative Segal Condition, [rSeg]**:** A cocartesian fibration $p^{\otimes} : \underline{A}^{\otimes} \to \underline{B}^{\otimes}$ in SymMon satisfies the relative Segal condition iff for each $b = \oplus_{j \in \langle m \rangle^{\circ}} b_j \in \underline{B}^{\otimes}_{\langle m \rangle} \simeq B^m$, the (inert) Segal maps $\epsilon_j : b \to b_j$ induce an equivalence $\underline{A}_b^{\otimes} \to \prod_{j=1}^{n} \underline{A}_{b_j}^{\otimes}$.

Remark. ([5],A.3) As commented by the authors, clearly the composite $(p^{\otimes} \circ \pi_B^{\otimes} : \underline{A}^{\otimes} \to \Delta) \in \mathrm{CoCart}(\Delta)$ exhibits $\underline{A}^{\otimes}$ as a symmetric monoidal ∞-category.
Moreover, Drew and Gallauer proved in [5],A.4 that - with reference to the notation above - $p^{\otimes}$ is a morphism of cocartesian fibrations over Δ, so that $\mathrm{SymMon}_{\underline{B}^{\otimes}}$ can indeed be seen as a (not full) subcategory of the slice $\mathrm{SymMon}_{/\underline{B}^{\otimes}}$, whence our notation.

Consistently with the rest of the Appendix, let us propose also an equivalent straightened formulation.

Definition B.5.0.2. *(Relative commutative monoid - straightened, [5],A.10)* Let $B^{\otimes} \in \mathrm{Fun}^{\mathrm{Seg}}(\Delta, \mathrm{Cat}_{\infty})$ be a symmetric monoidal ∞-category. Define the ∞-category of **relative commutative monoids over** $B^{\otimes}$ as the full subcategory $\mathrm{CMon}_{B^{\otimes}}(\mathrm{Cat}_{\infty}) \subseteq_{f.f.} \mathrm{Fun}(\underline{B}^{\otimes}, \mathrm{Cat}_{\infty})$ spanned by those functors $M : \underline{B}^{\otimes} \to \mathrm{Cat}_{\infty}$ which satisfy the straightened relative Segal condition: with notation as above, functoriality along the Segal maps induces canonical equivalences $M(b) \to \prod_{j=1}^{m} M(b_j)$ for each $b = \oplus_{j \in \langle m \rangle^{\circ}} b_j \in \underline{B}^{\otimes}$.

Remark. ([5],A.11) Under the Straightening Theorem, there is a canonical equivalence $\mathrm{SymMon}_{\underline{B}^{\otimes}} \simeq \mathrm{CMon}_{B^{\otimes}}(\mathrm{Cat}_{\infty})$.

Lemma B.5.0.3. *(SymMon Straightening Equivalence, [5],A.12) Let $\underline{B}^{\amalg} \in \mathrm{SymMon}$ be a* cocartesian *symmetric monoidal ∞-category. Then, the Straightening Theorem induces an equivalence:*

$$\mathrm{SymMon}_{\underline{B}^{\amalg}} \simeq \mathrm{Fun}(B, \mathrm{CAlg}(\mathrm{Cat}_{\infty}))$$

Construction B.5.0.4. We report the informal description [5],A.13 of the symmetric monoidal Straightening equivalence in the Lemma B.5.0.3 above.
Let $B \in \mathrm{Cat}_{\infty}$ admit finite products, so that $\mathcal{B} := B^{op}$ has finite coproducts; endow it with the cocartesian symmetric monoidal structure $\mathcal{B}^{\amalg}$. Let's spell out both directions of the equivalence.

- *Suppose to have a functor $F : B^{op} \to \mathrm{CAlg}(\mathrm{Cat}_{\infty}) \simeq \mathrm{SymMon}$.* Its action on the 1-skeleton of B can be informally described as follows:

  – Obj: $b \mapsto F(b)^{\otimes} \in \mathrm{SymMon}$;

  – Mor: $(f : b' \to b) \mapsto (f^* : F(b) \to F(b'))$ in SymMon (so f^* is a symmetric monoidal functor).

  Under the equivalence of B.5.0.3, we obtain a relative symmetric monoidal ∞-category over $\underline{\mathcal{B}}^{\amalg}$, namely $(p^{\otimes} : \underline{A}^{\boxtimes} \to \underline{\mathcal{B}}^{\amalg}) \in \mathrm{SymMon}_{\underline{\mathcal{B}}^{\amalg}}$. Moreover, its underlying cocartesian fibration $p : A \to B$ is also induced by an application of the Straightening Equivalence to the post-composition of the given functor F with the forgetful functor into Cat_{∞}. On the 1-skeleton one retrieves the classical Grothendieck construction, so we can describe informally the underlying category A as follows:

  – Obj: triples $(b, F(b) \ni M)$ for $b \in B$;

  – Mor: triple of morphisms: $(f, F(f), \phi) : (b', F(b') \ni M') \to (b, F(b) \ni M)$ for some morphisms $f : b' \to b$ in B and $f^* M \to M'$ in $F(b')$;

  – External Product: $(b, M) \otimes (b', M') := M \boxtimes M' := (pr_1)^* M \otimes_{b \times b'} (pr_2)^* M'$ in $F(b \times b')^{\otimes}$ for pr_i the canonical projection from $b \times b'$.

  Then, by the relative Segal conditions, objects of $\underline{A}^{\boxtimes}$ are obtained as products of objects of A, while morphisms in $\underline{A}^{\boxtimes}$ over maps $f : d' \to d$ admit a description similar to B.2.0.8 for the corresponding tensor products.

  Moreover, in [5],A.6,ii it is observed that p-cocartesian edges are closed under tensor product along the identities of A; hence, the external product is compatible with the action of F: each $f : b' \to b$ in B induces a canonical equivalence $(f \amalg 1_{b'})^* (M \boxtimes M') \to f^* M \boxtimes M'$ in $F(b')$.

- *Conversely, let it be given a $\underline{\mathcal{B}}^{\amalg}$-relative symmetric monoidal ∞-category $p^{\otimes} : \underline{A}^{\boxtimes} \to \underline{\mathcal{B}}^{\amalg}$.* Then, the cocartesian fibration at the level of the underlying categories (see [5],A.6) defines a functor $F : B^{op} \to \mathrm{Cat}_{\infty}$ under the Straightening equivalence.

Informally, it acts on objects by taking the fibre $F : b \mapsto A_b$ into the underlying category A. Under the equivalence in B.5.0.3, the image of F factors through SymMon, so each fibre is endowed with a symmetric monoidal structure $A_b^{\otimes}$ such that the transition maps in the diagram of F are symmetric monoidal functors. Moreover, by an application of the previous discussion to the construction $[F : b \mapsto A_b^{\otimes}]$, such symmetric monoidal structures turn out to be induced by that of $\underline{A}^{\boxtimes}$.

In other words, the encoded tensor product of $\underline{A}_b^{\otimes}$ can be described as follows: for any $M, M' \in \underline{A}_b^{\otimes}$, $M \otimes_b M' := \Delta_*(M \boxtimes M')$ for $\Delta : b \to b \times b$ the diagonal map in B.

C The ∞-Topos of Sheaves

The notion of a topos plays a prominent role in modern Mathematics. In this section we will review the generalization of it to the ∞-world, according to Lurie's [24],6.
The author aims at motivating the several compatible definitions of ∞-sheaves which lay the foundations of Derived Algebraic Geometry. For such a reason, after a brief digression on ∞-topoi, we will immediately present Grothendieck sites on ∞-categories and D-valued ∞-sheaves on C, for any 'nice' ∞-categories C and D.
However, if in the classical case this would provide a complete description of topoi, in the ∞-world this is no longer true.
Indeed, every ordinary topos is equivalent to a Grothendieck topos, namely to a category of sheaves on some Grothendieck site. More explicitly, every ordinary topos is equivalent to a left exact localization of some presheaf category $\mathrm{Set}^{\mathcal{C}^{op}}$ and, for each ordinary category $\mathcal{C} \in \mathrm{Cat}_1$, such localizations are in bijection with Grothendieck topologies on $\mathcal{C}$.
Inspired by such a feature, we will define an ∞-topos to be an accessible exact Bousfield localization of a presheaf ∞-category over some small $C \in \mathrm{Cat}_\infty$.
However, what will no longer hold is the bijection between such localizations and Grothendieck topologies on C. More explicitly, we will observe that Grothendieck sites on an ∞-category C arise as 'topological' localizations of $\mathcal{P}(C)$. This will lead us to define ∞-sheaves as 'topologically'-local presheaves.
Then, as in the classical case, the notion of an ∞-sheaf will admit a number of more manageable compatible reformulations, according to the level of generality needed. This will be the content of our last section, which will also be concerned with providing a proof of such compatibilities, whenever they are simultaneously well-defined.

C.1 ∞-Topoi

In the current subsection we will generalize to the ∞-world the notion of an ordinary topos, both extrinsically and intrinsically.
In order to do so, we will state an enhanced ∞-version of Giraud's Theorem and motivate Čech nerves of morphisms. In what follows, the latter will generalize the classical notion of a sheaf on a topological space to the ∞-world. We defer to section [24],6.1 for more details.

Definition C.1.0.1. *(Extrinsic, [24],6.1.0.4)* Call **∞-topos** an accessible (left) exact Bousfield localization of the presheaf category over any small ∞-category.

In other words, to quote Lurie's comment, we are extrinsically characterizing ∞-topoi among all ∞-categories to be those which constitute the smallest class in Cat_∞ containing Spc and being stable under certain operations, such as left exact localizations and the formation of functor categories.

On the other hand, an enhancement of the well-known classification theorem for Grothendieck topoi provides a more intrinsic perspective.
The latter result is generally known as Giraud's Theorem and we defer to [27],A.1.1 for a detailed discussion.
In what follows, however, we will adopt a more elegant formulation for presentable categories, as stated by Lurie in [24],6.1.0.1.

Theorem C.1.0.2. *(Intrinsic, [24],6.1.0.6) For an ∞-category $\mathcal{X}$,* TFAE*:*

- *$\mathcal{X}$ is an ∞-topos;*
- *$\mathcal{X}$ satisfies the following version of* Giraud's Axioms*:*
  1. *$\mathcal{X}$ is presentable;*
  2. *colimits in $\mathcal{X}$ are universal, i.e. base-change along any morphism in $\mathcal{X}$ preserves colimits or, equivalently (by 1.2.0.5), any base-change functor admits a right-adjoint;*
  3. *coproducts in $\mathcal{X}$ are disjoint, i.e. given any product of two objects in $\mathcal{X}$, the base-change of a canonical inclusion along the other one is an initial object of $\mathcal{X}$;*
  4. *groupoid objects in $\mathcal{X}$ are effective.*

The first three axioms are an evident generalization of Giraud's, whereas the fourth one imports a notion of 'effective equivalence' in the ∞-world. A few comments on this are noteworthy, since it will lead us to introduce Čech nerves.

In the classical context, given $\mathcal{C} \in \mathrm{Cat}_1$, we define an equivalence relation $\mathcal{R}$ on $x \in \mathcal{C}$ to be a subobject of the self-cartesian product $\mathcal{R} \leq x \times x$ inducing a set-theoretical equivalence relation of generalized points.
Whenever $\mathcal{C}$ is finitely bi-complete, we notice that, for any $f : x \to y$, the kernel pair $x \times_y x$ is always an equivalence on the source x. Therefore, it makes sense to wonder whether they actually exhaust the whole class of equivalences. This is generally not the case, so we call *effective* those equivalences arising in such a way, namely as kernel pairs of epimorphisms. Consistently, by extension also these special epimorphisms are said to be effective.

Again quoting Lurie: being it intimately connected with the notion of epimorphism, a straightforward generalization of the notion of equivalence is however out of sight: the 'correct' corresponding notion of spaces is that of a 'surjection on path components', but homotopy kernel pairs need no longer be subobjects of homotopy self-products.
This pathological behaviour occurs because we are willing to represent kernel pairs 'internal to' spaces by the limit of a 1-truncated co-simplicial object of Spc.
In order to solve such an issue, we are forced to generalize kernel pairs with a suitable simplicial object of Spc and let an equivalence be the canonical morphism out of its geometric realization.
More explicitly, given a surjection $f : X \to Y$ in Spc, a 'kernel pair' for f should be a space consisting of pairs of connected components of X which are (universally) coequalized by f. The homotopy-coherence condition will be achieved by considering the geometric realization of a 'nice' simplicial object $U_\bullet$ in C starting with $U_0 \simeq X$, which will be called a *groupoid object* of C. On the other hand, the coequalizing property will mean that $U_\bullet$ will admit a particular 'coherent' augmentation $U_\bullet^+$ by f - called the *Čech nerve* of f - which will (geometrically) realize $U_\bullet$ as a simplicial resolution of Y; in other words, we will have the following exact diagram

$$U_{\bullet+1} \rightrightarrows U_0 \simeq X \xrightarrow{f} Y$$

with $|U_\bullet| \simeq U_\bullet^+(-1) \simeq Y$.
As in the classical case, such an f will be called *effective epimorphism*.

Before stating our definitions, however, we need to introduce some preliminary notation. First of all, for any $C \in \mathrm{Cat}_\infty$, write $sC := \mathrm{Fun}(\mathcal{N}(Delta)^{op}, C)$.
It will be convenient to perform a square-zero extension of our simplicial objects; to this end, define $\Delta_+ := \{[-1]\} \cup \Delta$ by 'adjoining' the iso-class of the linearly ordered set with no elements $[-1] := \emptyset$ together with the obvious canonical morphisms.
Then, let us call **augmented simplicial object** of an ∞-category C a functor $U_\bullet^+ : \mathcal{N}(\Delta_+)^{op} \to C$ and let its restriction $U_\bullet : \mathcal{N}(\Delta)^{op} \to C$ denote its underlying simplicial object. We will freely adopt all the terminology and conventions relative to simplicial sets, such as geometric realizations $|-|$; we will furthermore realize n-truncations by restricting along $\mathcal{N}(\Delta_+^{\leq n}) \subseteq \mathcal{N}(\Delta_+)$.

Definition C.1.0.3. *(∞-Groupoid Object, [24],6.1.2.6)* An **∞-groupoid object** in an ∞-category $C \in \mathrm{Cat}_\infty$ is a simplicial object $U_\bullet : \mathcal{N}(\Delta)^{op} \to C$ of C s.t. for every $n \in \mathbb{N}$ and every partition $[n] = S \cup S'$ with $S \cap S' = \{s\}$, then the following square is a pull-back in C:

$$\begin{tikzcd} U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\ U(S') \arrow[r] & U(\{s\}) \end{tikzcd}$$

Let $\mathrm{Gpd}(C) \subseteq_{f.f.} sC$ denote the full subcategory spanned by groupoid objects in C.

In other words, if $U_\bullet$ is 'right exact', i.e. $U_\bullet$ takes to a cartesian square any bi-cartesian square (so any short exact sequence) in Δ of the form:

$$\begin{tikzcd} {[0]} \arrow[r, "\{s\}"'] \arrow[d, "\{s\}"] & {[m]} \arrow[d, "S"'] \\ {[n-m]} \arrow[r, "S'"] & {[n]} \end{tikzcd}$$

Remark. Notice that the previous statement actually produces the correct 'definition' of a groupoid object, in that it corresponds to a Kan object in C, namely with the analogous of the right lifting property with respect to every horn inclusion.
This can be seen as follows. First, notice that a partition as in the statement amounts to one of the spine of the corresponding n-simplex, and that the right lifting property with respect to spine inclusions implies the one for all the horns of the right dimension. This allows us to proceed inductively on the length $n+1$ of the n-spine.
The inductive step for $n \geq 1$ goes as follows: A copy of a partition of the n-spine in $U_\bullet$ amounts to the angle $U([n-m]) \to U_0 \leftarrow U([m])$, while the fact that the square be cartesian is precisely the existence and uniqueness of the filler. Finally, the base of the induction with $n = 0$ is clear.
Moreover, let us remark that, as for the usual simplicial sets, the fact that every arrow is an equivalence amounts to being able to fill *all* 2-horns, so our terminology is not misleading.

Its coherent augmentation will then be achieved by the following extension.

Definition C.1.0.4. *(Čech Nerve, [24],6.1.2.11)* Consider an augmented simplicial object $U_\bullet^+ : \mathcal{N}(\Delta_+)^{op} \to C$ of an ∞-category C, and let $f : U_0 \to U_{-1}$ denote its 0-th degeneracy. We call $U_\bullet^+$ the **Čech nerve** of f whenever the following equivalent conditions hold:

- $U_\bullet^+$ is the right Kan extension of ${U_\bullet^+}_{|\mathcal{N}(\Delta_+^{\leq 0})^{op}}$ along the inclusion $\mathcal{N}(\Delta_+^{\leq 0}) \subseteq \mathcal{N}(\Delta_+)$.
- The underlying simplicial object of C, $U_\bullet$, is a groupoid object of C and the diagram ${U_\bullet^+}_{|\mathcal{N}(\Delta_+^{\leq 1})^{op}}$ is a pull-back square in C:

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d, "f"'] \\ U_0 \arrow[r, "f"] & U_{-1} \end{tikzcd}$$

Notice that unwinding the second definition each n-simplex of $U_\bullet^+$ will recover the familiar n-fold homotopy self-fibred product of U_0 over U_{-1} along f itself: $U_n \simeq U_0 \times_{U_{-1}} \cdots \times_{U_{-1}} U_0$.
Indeed, adding the final additional cartesian square to $U_\bullet$ fixes all the higher ones: given any partition of $[n]$ we can always post-compose cartesian squares of lower dimensions in each suitable direction and recover the whole stair; the leaves of our graphs of squares will then be zig-zag's consisting of copies of the angle $U_0 \to U_{-1} \leftarrow U_0$, thus yielding the stated expression for U_n.

We are finally ready to give content to the previously sketched intuition for an effective relation.

Definition C.1.0.5. *(Effective Groupoid, [24],6.1.2.14)* A simplicial object $U_\bullet$ in an ∞-category $C \in \mathrm{Cat}_\infty$ is said to be an **effective groupoid** if it can be extended to a colimit diagram $U_\bullet^+ : \mathcal{N}(\Delta_+)^{op} \to C$ with $U_\bullet^+$ being a Čech nerve.

Remark. As observed by Lurie at the end of section [24],1.2, already in Spc our requirement that every groupoid be effective is not that trivial. Indeed, any groupoid object $U_\bullet \in s\mathrm{Spc}$ with contractible $U_0 \simeq *$ can be regarded as a space U_1 equipped with a coherently associative multiplication operation (incarnated by the composition of morphisms). The augmenting square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & U_{-1} \end{tikzcd}$$

corresponds to U_1 being a loop space. Hence, requiring every groupoid object to be effective means asking that any associative multiplication on a space can be realized by a loop space.

Now, observe that augmenting a groupoid object and requiring everything to be compatibly made of pullbacks (so, the property of effectiveness) still needs not imply that the Čech nerve $U_\bullet^+$ is the geometric realization of its underlying simplicial object $|U_\bullet|$. In other words, our map $f : X = U_0^+ \to U_{-1}^+ = Y$ needs not be terminal, i.e. (-2)-truncated, in $\mathcal{X}_{/Y}$. However, the following results proves that there must be some 'compatible' monomorphism $f' : |U_\bullet| \hookrightarrow Y$, i.e. a (-1)-truncated object of $\mathcal{X}_{/Y}$.

Proposition C.1.0.6. *([24],6.2.3.4) Let $\mathcal{X} \in \mathrm{Cat}_\infty$ be a (semi)topos (so, we do not require coproducts to be disjoint), and consider a morphism $f : U_0 \to x$ in $\mathcal{X}$. Let $U_\bullet := \check{C}(f)_{|\Delta^{op}}$ denote the simplicial object underlying the Čech nerve of f. Let $v \simeq |U_\bullet| \in \mathcal{X}$ denote a realization of $U_\bullet$.*

$$\begin{tikzcd} U_0 \arrow[rr] \arrow[dr, "f"'] & & v \arrow[dl, "f'"] \\ & x & \end{tikzcd}$$

Then, the above triangle identifies f' with a (-1)-truncation of f in $\mathcal{X}_{/x}$.

We can then formulate the following criterion describing when an effective groupoid realizes its added object. We will define the class of effective epimorphisms to be the one consisting of those morphisms $f : X \to Y$ whose Čech nerve $\check{C}(f)$ realizes the target Y.

Corollary C.1.0.7. *(Effective Epimorphism, [24],6.2.3.5) Let $\mathcal{X} \in \mathrm{Cat}_\infty$ be a (semi)topos. For a morphism $f : U_0 \to x$ in $\mathcal{X}$,* TFAE:

1. *As an object over x, the (-1)-truncation of $f \in \mathcal{X}_{/x}$ is terminal, namely $\tau_{\leq -1}(f) \in (\mathcal{X}_{/x})^{term}$;*
2. *The Čech nerve $\check{C}(f)$ is a simplicial resolution of x, i.e. $|\check{C}(f)_{|\Delta^{op}}| \simeq x \simeq \check{C}(f)(-1)$.*

*In such a case, we call f an **effective epimorphism**.*

For future reference, let us record (without proof) some properties of effective epimorphisms.

Proposition C.1.0.8. *(Properties of effective epimorphisms) Let $\mathcal{X}$ be a (semi)topos and let* $\mathrm{EffEpi}(\mathcal{X})$ *denote the class of effective epimorphisms of $\mathcal{X}$. Then,*

- $\mathrm{EffEpi}(\mathcal{X})$ *is closed in $\mathcal{X}$ under small coproducts ([24],6.2.3.11), composition ([24],6.2.3.12), base-change ([24],6.2.3.15).*
- *Given two morphisms f, g and any choice of a composition $g \circ f$ in $\mathcal{X}$, then $g \circ f \in \mathrm{EffEpi}(\mathcal{X})$ implies $g \in \mathrm{EffEpi}(\mathcal{X})$ ([24],6.2.3.12).*
- *Base-change along any $g \in \mathrm{EffEpi}(\mathcal{X})$ detects effective epimorphisms of $\mathcal{X}$ ([24],6.2.3.15) and is conservative ([24],6.2.3.16).*

Moreover, effective epimorphisms arise 'naturally' as those canonical maps expressing simplicial colimits as quotients of a free coproduct by relations.

Proposition C.1.0.9. *(Colimits as quotients, [24],6.2.3.13) Let $\mathcal{X}$ be a (semi)topos and consider a simplicial diagram $p : K \to \mathcal{X}$ with colimit cone $\overline{p} : \overline{K} \to \mathcal{X}$. Let ∞ denote the terminal point of $\overline{K}$. Then, the canonical map (which is well-defined up to contractible ambiguity as in [20],4.3.21) $\coprod_{K_0} p(k) \twoheadrightarrow p(\infty)$ is an effective epimorphism.*

After a brief digression on group actions, the rest of this section will be devoted to proving that, as in the classical case, we can characterize sheaves on a site by the fact that they preserve the effectiveness of epimorphisms and finite products.

C.2 Group Actions and Principal ∞-Bundles

In this subsection, we briefly present an application of the notion of effective groupoids and of Giraud's Axioms, namely the theory of group actions and principal ∞-bundles. Two excellent expositions on the topic are [30] and [16],4, for a more intuitive overview.
Before getting started, let us anticipate a couple of advantages of adopting the ∞-categorical formalism: our notion of G-action will be automatically principal, so that our G-torsors (equivalently G-principal bundles) will be automatically locally trivial. This can be intuitively seen as follows: in the classical case, such facts are shadows of "higher homotopical coherence conditions", namely the effectiveness of groupoid objects in the first case and commutativity up to homotopy in the second one (which automatically encodes gluing conditions).
In what follows we will consider a topos $\mathcal{X}$ and work with its group objects $G \in \mathrm{Grp}(\mathcal{X})$, i.e. contractible groupoid objects in $\mathcal{X}$; this agrees with Definition B.1.0.9.

Definition C.2.0.1. *(Group action, [30],3.1)* Let $\mathcal{X}$ be a topos and consider a group object $G \in \mathrm{Grp}(\mathcal{X})$. Define a ***G*-action on** $P \in \mathcal{X}$ to be a groupoid object:

$$(P /\!\!/ G)_\bullet : \big(\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} P \times G \underset{d_1}{\overset{\rho = d_0}{\rightrightarrows}} P \big)$$

such that the following conditions are satisfied:

1. $d_1 : P \times G \to P$ is the first projection;
2. Base-change along the canonical map $P \to *$ induces a morphism $(P /\!\!/ G)_\bullet \to G_\bullet$ of groupoid objects consisting of cartesian squares:

$$\begin{tikzcd} P \times G^n \arrow[r, "pr_1"] \arrow[d, "pr_{\neq 1}"] & P \arrow[d, "!_P"] \\ G^n \arrow[r, "!_G"] & * \end{tikzcd}$$

([16],4.14) Equivalently, a G-action on P is a groupoid object $U_\bullet$ of $\mathcal{X}$ together with an equivalence $U_0 \simeq P$ and a morphism of groupoid objects $p : U_\bullet \to G_\bullet$ induced by $[0] \to [n]$ sending $0 \mapsto 0$ such that the squares $p_n \to p_0$ are always cartesian. In particular, this forces $U_\bullet \simeq (P /\!\!/ G)_\bullet$ as induced by some action $\rho : P \times G \to P$.

We will sometimes denote an action by $B(P;G)_\bullet := (P /\!\!/ G)_\bullet$ or simply $G \curvearrowright P$. Let $P /\!\!/ G := [P/G] := |P \times G_\bullet| \in \mathcal{X}$ denote the **quotient of the action** $G \curvearrowright P$.
Moreover, adopt the following notation for the full sub-slice ∞-category spanned by G-actions:

$$\mathcal{X}^G \subseteq_{f.f.} \mathrm{Gpd}(\mathcal{X})_{/(*/\!\!/G)}$$

Morphisms in $\mathcal{X}^G$ are called ***G*-equivariant**, in that they sit in a commutative triangle over $G_\bullet$.

Remark. $G_\bullet$ is an effective groupoid object by Giraud's Axioms, i.e. $G_\bullet \simeq \check{C}(!_G)_{|\Delta^{op}}$ and the latter resolution is colimiting. Write $\mathbb{B}G := [*/G] \simeq |G_\bullet|$ for the realization. Such a space will play a prominent role in the theory.

The equivalence in the statement supplies precise mathematical content to the following observation:

Remark. ([30],3.2) Informally, the action "itself" is given by the map $\rho := d_0 : P \times G \to P$, while requiring it to fit into such a simplicial diagram enforces the usual axioms of a group action "up to coherent homotopy". Indeed, we can inductively describe the degeneracy maps in $(P /\!\!/ G)_\bullet$; let's spell out their action in degree 2. For $X := [P/G]$, we have equivalences:

$$P \times G \times G \simeq (P \times_X P) \times_X P \simeq (P \times G) \times_X P$$

Then, the first two degeneracies are the base-change along $P \to X$ of those in degree 1, i.e. along the identification above $d_0 = \rho \times_X P \simeq \rho \times G$ and $d_1 = pr_1 \times G$. The last one is determined by the equivariance of $p : (P /\!\!/ G)_\bullet \to G_\bullet$, namely $d_2 : P \times m$ as induced by the multiplication $m : G \times G \to G$ (i.e. the self-action of G).

Hence, the simplicial identities in degree 2 establish the compatibility between the action $G \curvearrowright P$ and the multiplication $G \curvearrowright G$; in degree 3 they encode compatibility with the associativity-homotopy involved with acting with three group factors, and so on and so forth for higher compatibility.

Proof. (Of the equivalence) One direction is clear. Conversely, let it be given a groupoid object as in the definition. Recall that groupoid objects in the topos $\mathcal{X}$ are effective by Giraud's Axioms, so - for $X := [P/G] \simeq |U_\bullet|$ - it holds $U_\bullet \simeq \check{C}(P \to X)$. Now, the squares $p_n \to p_0$ are induced by the n-fold composition of the 0-th degeneracy maps, which forces $P^{\times_X n} \simeq P \times G^n$; in particular the canonical pull-back projections $(d_1, d_0) : P \times_X P \simeq P \times G \rightrightarrows P$ can be identified with (ρ, pr_1). □

Remark. (Principality condition, [30],3.7) G-actions are automatically principal, in that the effectiveness of groupoid objects in a topos retrieves - in coordinates - the equivalence $(\rho, pr_1) : P \times G \rightrightarrows P \times_X P$.

Remark. (Trivial action, [30],3.10) Let $G \in \mathrm{Grp}(\mathcal{X})$ be a group object in a topos $\mathcal{X}$. Any object $P \in \mathcal{X}$ can be endowed with a trivial action $P \times G_\bullet$ (so with $\rho = pr_1$). These correspond to base-changes $g^*(* \to \mathbb{B}G)$ along trivial maps $g : P \to * \to \mathbb{B}G$:

$$\begin{tikzcd} P \times G \arrow[r] \arrow[d, "pr_1"] & G \arrow[r] \arrow[d, "!_G"] & * \arrow[d, "\mathrm{pt}"] \\ P \arrow[r, "!_P"] & * \arrow[r] & \mathbb{B}G \end{tikzcd}$$

Definition C.2.0.2. *(G-torsors and GBund, [30],3.4)* Let $G \in \mathrm{Grp}(\mathcal{X})$ be a group object in a topos $\mathcal{X}$. A **G-torsor** (or G-principal bundle) on an object $X \in \mathcal{X}$ is a pair $(G \curvearrowright P, f : P \twoheadrightarrow X)$ such that: $f \in \mathrm{EffEpi}(\mathcal{X})$ is an effective epimorphism inducing an equivalence $(P /\!\!/ G)_\bullet \simeq \check{C}(f)_{|\Delta^{op}}$, i.e. f is the canonical map $f : (P /\!\!/ G)_0 \to [P/G] \simeq X$.

A morphism of G-torsors over X is an equivariant map in $\mathcal{X}^G_{/X}$. Such data can be grouped into the following ∞-category of G-torsors:

$$G\mathrm{Bund}(X) := \mathcal{X}^G \times_{\mathcal{X}} \{X\} \simeq \mathrm{Fib}_X\big(\mathcal{X}^G \subseteq_{f.f.} \mathrm{Gpd}(\mathcal{X})_{/(*/\!\!/G)} \overset{for}{\to} \mathrm{Gpd}(\mathcal{X}) \overset{\mathrm{colim}}{\to} \mathcal{X}\big)$$

Remark. In [16],4.19, A. Khan posits as definition an almost verbatim generalization of the classical approach: we morally defined a G-torsor over X as a G-action $(P /\!\!/ G)_\bullet$ such that $[P/G] \simeq X$ and endowed with a choice of both an identification $\alpha : [P/G] \simeq X$ and of an equivariant lift of $f := \alpha \circ (P \to [P/G])$ (a canonical one would be the map $f_\bullet := f \times G_\bullet$). The choice of an identification of the quotients is part of the datum of taking fibres in Cat_∞ over X.

Remark. We actually defined a presheaf of ∞-categories $G\mathrm{Bund} : \mathcal{X}^{op} \to \mathrm{Cat}_\infty$ which acts by pull-back on morphisms $f : X \to Y$. Indeed, as in [30],3.8, one can prove that $\mathcal{X}^G$ is stable under base-change:

any base-change $f^*[P_1/G]$ of the quotient of an action $(P_1 /\!\!/ G)_\bullet$ is still the quotient $[P_2/G]$ of an action $(P_2 /\!\!/ G)_\bullet \simeq f^*(P_2 /\!\!/ G)_\bullet$. This holds true because taking pull-backs preserves effective epimorphisms (see C.1.0.8,i) and Čheck nerves (recursively given by taking pull-backs) together with their colimits (by Giraud's Axioms, colimits in a topos are universal).

Our next aim is to exhibit a moduli stack classifying G-torsors for a given group object $G \in \mathrm{Grp}(\mathcal{X})$, i.e. representing a factorisation of the Cat_∞-presheaf $G\mathrm{Bund}' : \mathcal{X}^{op} \to \mathrm{Spc}$.

Proposition C.2.0.3. *(Universal G-principal bundle, [30],3.13) For each G-torsors $(P \to X) \in G\mathrm{Bund}(X)$, the canonical map $!_P : P \to *$ induces a cartesian square. The G-torsor $* \to \mathbb{B}G$ is called the **universal G-principal bundle**.*

$$\begin{tikzcd} P \arrow[r, "!_P"] \arrow[d] & * \arrow[d, "\mathrm{pt}"] \\ X \simeq [P/G] \arrow[r] & \mathbb{B}G \end{tikzcd}$$

Proof. First observe that - according to our definition - G-torsors are automatically locally trivial: since they are automatically locally principal, there exists always some $(U \twoheadrightarrow X) \in \mathrm{EffEpi}(\mathcal{X})$ for which $U \times G \simeq P \times_X P$ over U (e.g. take $U = P$). Then, consider a trivialization of $P \to X$:

$$\begin{tikzcd} U \times G \arrow[r, two heads] \arrow[d, "pr_1"] & P \arrow[r, "!_P"] \arrow[d] & * \arrow[d, "\mathrm{pt}"] \\ U \arrow[r, two heads] & X \arrow[r] & \mathbb{B}G \end{tikzcd}$$

We need to show that the right square in the above diagram is cartesian. Notice first that the left square is cartesian by definition of local trivialization, so - by C.1.0.8,i - also $U \times G \to P$ is an effective epimorphism; this is equivalent to $P \simeq |(U \times G)^{\times_P n}|$ by the effectiveness of groupoid objects.

Claim. $(U \times G)^{\times_P n} \simeq (U^{\times_P n}) \times G$

Proof. Let's argue by induction on n. For $n = 1$ it is clear, so assume $n > 1$: this follows by post-composing with the (cartesian) left-square and by applying the inductive premise:

$$(U \times G)^{\times_P (n+1)} \simeq U \times_X (U^{\times_X n}) \times G \simeq U^{\times_X (n+1)} \times G$$

■

Hence, we conclude by the following chain of equivalences:

$$P \simeq |(U \times G)^{\times_P n}| \simeq |(U^{\times_X n}) \times G| \simeq |\mathrm{pt}^*(U^{\times_X n})| \simeq \mathrm{pt}^*|U^{\times_X n}| \simeq \mathrm{pt}^* X$$

which is implied by (in this order) the Claim, the construction of trivial actions (this is a commutative extension of the original diagram, because also $U \to \mathbb{B}G$ is trivial), the universality of colimits, and $(U \twoheadrightarrow X) \in \mathrm{EffEpi}(\mathcal{X})$. □

Proposition C.2.0.4. *(GBund presheaf, [30],3.16) The construction $[X \mapsto G\mathrm{Bund}(X)]$ assembles into a presheaf $\mathcal{X} \to \mathrm{Spc}$.*

Proof. The definition is clearly functorial $\mathcal{X} \to \mathrm{Cat}_\infty$, so we are left to check that GBund takes values in $\infty-\mathrm{Gpd}$, and hence in Spc by localizing at weak equivalences. Consider $G\mathrm{Bund}(X)$ for some $X \in \mathcal{X}$, and let's prove that all its 1-simplices are equivalences. To this end, let $f : P_0 \to P_1$ be an equivariant map of G-torsors over X, i.e. a morphism of Čech nerves $f_\bullet : (P_1 /\!\!/ G)_\bullet \to (P_2 /\!\!/ G)_\bullet$ which retrieves $f_0 = f$ and induces an equivalence of geometric realizations $|f_\bullet| : [P_1/G] \simeq [P_2/G]$.
Our setting can be represented as in the diagram below, where the two slanted squares are cartesian by C.2.0.3; therefore, $f \simeq \mathrm{pt}^*(1_X)$ is forced to be an equivalence.

$$
\begin{tikzcd}
P_1 \arrow[rr, bend left, "f"] \arrow[dr] \arrow[dd] & & P_2 \arrow[dl] \arrow[dd] \\
& * \arrow[dd, two heads, "\mathrm{pt}"] & \\
X \arrow[rr, equal, bend left] \arrow[dr] & & X \arrow[dl] \\
& \mathbb{B}G &
\end{tikzcd}
$$

□

Finally, let us record a useful computation of the generalized points (so defined vie the Yoneda Lemma and the Density Theorem) of a G-torsor. This will specialize in the classical classification result for G-torsors and exhibit $\mathbb{B}G$ as the moduli stack representing $G\mathrm{Bund}(-)$.
We will mostly apply it when dealing with quotient stacks, so as to classify particularly interesting classes of vector bundles on a scheme.

Theorem C.2.0.5. *(Functor of points of a G-torsor, [16],4.28) Let $(G /\!\!/ P)_\bullet \in \mathcal{X}^G$ be an action groupoid, and let $p : (G /\!\!/ P)_\bullet \twoheadrightarrow [P/G]$ be the canonical G-torsor exhibiting its geometric realization. Define the* ***functor of points*** *of $[P/G]$ as $[P/G](-) := \mathrm{Map}_{\mathcal{X}}(-, [P/G])$. Then, base-change along p induces an equivalence*

$$[P/G](-) \simeq \mathrm{Map}_{\mathcal{X}^G}(-, (G /\!\!/ P)_\bullet)$$

The action on objects of $[P/G]$ can then be described as follows: for each $T \in \mathcal{X}$, $[P/G](T)$ is the space spanned by co-angles of the form:

$$
\begin{tikzcd}
(G /\!\!/ Y)_\bullet \arrow[r, "f_\bullet"] \arrow[d, two heads, "\pi"] & (G /\!\!/ P)_\bullet \\
T \simeq [Y/G] &
\end{tikzcd}
$$

for a G-equivariant map $f : Y \to P$ and a G-torsor π over the given T.

In particular, for $P = *$, we recover the classification of G-torsors: $\mathbb{B}G(-) = [*/G](-) \simeq G\mathrm{Bund}(-)$. In other words, $\mathbb{B}G$ is the moduli stack for $G\mathrm{Bund}$.

Proof. First recall that all groupoids in the topos $\mathcal{X}$ are effective by Giraud's Axioms. Hence, post-composition with p yields a map:

$$p_* \circ ev_0 : \mathrm{Map}_{\mathcal{X}^G}(-, (G /\!\!/ P)_\bullet) \to \mathrm{Map}_{\mathcal{X}}(-, [P/G])$$

Let's prove that it is an equivalence. By [20],2.2.2, it suffices to show the equivalence point-wise on $T \in \mathcal{X}$, and then fibre-wise on any given $\psi : T \to [P/G]$. In other words, we need to prove the contractility of the subspace of all the G-equivariant maps $f_\bullet : (G /\!\!/ Y)_\bullet \to (G /\!\!/ P)_\bullet$ over $\psi : T \simeq [Y/G] \to [P/G]$ for $(G /\!\!/ Y)_\bullet \to T$ any G-torsor over T.
To this end, it suffices to show that the essential image of $\{\psi\}$ under the functor "base-change along p" is equivalent to such a subspace. In other words, for any choice of a G-equivariant map f and of a G-torsor π over T sitting in the following left square, we wish that such a square be cartesian.

$$
\begin{tikzcd}
(G /\!\!/ Y)_\bullet \arrow[r, "\forall f"] \arrow[d, two heads, "\forall \pi"] & (G /\!\!/ P)_\bullet \arrow[r, "!_*"] \arrow[d, two heads, "p"] & * \arrow[d, two heads, "\mathrm{pt}"] \\
T \simeq [Y/G] \arrow[r, "\psi"] & {[P/G]} \arrow[r] & \mathbb{B}G
\end{tikzcd}
$$

Post-compose by the right square, which is cartesian by C.2.0.3; by the pasting law of pull-backs, the left square is then cartesian iff the total rectangle is such, and this holds true (again by C.2.0.3). □

C.3 Grothendieck sites

In this subsection we briefly review the Grothendieck sites. A more detailed exposition can be found in [24],6.2.1-2.
We will start by defining the notions of a sieve and of a Grothendieck site, which stem back to the pioneering work [39] by Toën and Vezzosi.

Definition C.3.0.1. *(Sieve, [24],6.2.2.1)* Let $C \in \mathrm{Cat}_\infty$. A **sieve** on C is a full subcategory $C^{(0)} \subseteq_{f.f.} C$ s.t. for every morphism $f : x \to c$ in C, $x \in C^{(0)}$ (i.e. f morphism in $C^{(0)}$) whenever $c \in C^{(0)}$.
Given any $c \in C$, a **sieve on** c is a sieve on the over-slice ∞-categories $C_{/c}$.
Moreover, given any morphism $f : x \to c$ in C and any sieve $C^{(0)}_{/c}$, we let $f^* C^{(0)}_{/c}$ denote the unique sieve on x s.t. post-composition by f induces the equivalence $C_{/f} \simeq C^{(0)}_{/c}$ as sieves of $C_{/c}$.

Notation. In what follows, we will introduce a little abuse of notation which will, however, make the reasoning with Grothendieck topology 'cleaner'. Namely, for any $c \in C$ and classes R, $S \subseteq C_{/c}$ of morphisms, let us denote by $S^*(R)$ the class $\{f^*(R) | f \in S\}$. Then, we can say, for instance, that a sieve $C^{(0)}_{/c}$ on c is spanned by a class of the form $(C_{/c})^*(R)$ for some $R \subseteq C_{/c}$.
With one additional abuse of notation, we will still use $S^*(R)$ together with symbols (informally) pertaining to the type 'object' of S or with functors (etc.) defined object-wise on S, whenever we want to describe a property which is enjoyed by *each* element of S.

Definition C.3.0.2. *(Grothendieck Topology, [24],6.2.2.1)* A **Grothendieck topology** Cov on an ∞-category $C \in \mathrm{Cat}_\infty$ is a specification for each $c \in C$ of a collection $\mathrm{Cov}(c)$ of *covering sieves* on C which satisfy the following properties:

1. (Maximum): For each $c \in C$, $C_{/c} \in \mathrm{Cov}(c)$
2. (Stability): For every covering sieve $C^{(0)}_{/c} \in \mathrm{Cov}(c)$, $(C_{/c})^*(C^{(0)}_{/c}) \subseteq \mathrm{Cov}(c)$.
3. (Transitivity): For each $c \in C$ and every two sieves $C^{(0)}_{/c}$, $C^{(1)}_{/c}$ with $C^{(0)}_{/c} \in \mathrm{Cov}(c)$ covering, then also $C^{(1)}_{/c}$ is covering whenever $(C^{(0)}_{/c})^*(C^{(1)}_{/c}) \subseteq \mathrm{Cov}(\mathrm{cod}(C^{(0)}_{/c}))$.

Remark. As in the classical case, the last two properties allow us to check the covering property on 'fibres' over maps of a sieve.

An interesting feature of our generalization is that we can extend the following property in a non-trivial way: a Grothendieck topology on the nerve of a 1-category $C \simeq \mathcal{N}(\mathcal{E})$ amounts to one on $\mathcal{E} \in \mathrm{Cat}_1$.

Proposition C.3.0.3. *(Topology on the homotopy category, [24],6.2.2.3) Let $C \in \mathrm{Cat}_\infty$ be an ∞-category. Then, a Grothendieck topology on C corresponds precisely to an ordinary Grothendieck site on the homotopy category $ho(C) \in \mathrm{Cat}_1$. In other words, for each $c \in C$, there is a bijection of specifications* $\mathrm{Cov}(c) \leftrightarrow \mathrm{ho}(\mathrm{Cov}(c)) \leftrightarrow \mathrm{Cov}_{ho}(c)$.

Proof. There is a canonical functor $\eta : ho(C_{/c}) \to ho(C)_{/c}$ of ordinary categories which is the identity on objects and acts on morphisms as follows. Let σ denote an arbitrary 2-simplex in C, namely a choice h of a composition of the two composable arrows f and g:

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "h"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

An arbitrary morphism $[\sigma]_\simeq \in \mathrm{Mor}(ho(C_{/c}))$ corresponds to the homotopy class of the arbitrary 2-simplex $\sigma \in C_2$ with $\sigma(\{2\}) = c$, namely to a choice of a composition of two composable arrows f, g over c.

On the other hand, an arbitrary morphism $\bar{\sigma} \in \mathrm{Mor}(ho(C)_{/c})$ is a 2-simplex in the nerve $\mathcal{N}(ho(C))$ with $\bar{\sigma}(\{2\}) = c$ and edges homotopy classes of the corresponding morphisms in C. In other words, $\bar{\sigma}$ is a composition of f, g without specifying the choice of the 2-cell realizing it.
Then, the canonical functor η simply acts on morphisms by forgetting the choice of a composition, thus inducing a surjection on Hom-sets.

But in view of our operative definition of sieves on $c \in C$ as classes $(C_{/c})^*(R)$ for some $R \subseteq C_{/c}$, for each $c \in C$ we obtain a bijective correspondence between sieves on c:

$$\{\text{Sieves on } ho(C)_{/c}\} \longrightarrow \{\text{Sieves on } ho(C_{/c})\}$$
$$(ho(C)_{/c})^*(R) \mapsto \eta^{-1}\big((ho(C)_{/c})^*(R)\big)$$

with inverse the action of η. Such a bijection is compatible with the axioms of a Grothendieck topology, so that it restricts to a bijection of covering sieves $ho(\mathrm{Cov}(c)) \cong \mathrm{Cov}_{ho}(c)$. □

Moreover, we can characterize covering sieves on $C \in \mathrm{Cat}_\infty$ by means of a bijection with presheaves in $\mathcal{P}(C)$.

Lemma C.3.0.4. *([24],6.2.2.4) Let $C \in \mathrm{Cat}_\infty$ be a small ∞-category. Then, the following functions induce a bijection between sieves on C and (-1)-truncated presheaves on C.*

$$C^{(0)}(-) : \mathcal{P}(C) \longrightarrow \{\textit{Sieves on } C\} \qquad \delta_{(-)} : \{\textit{Sieves on } C\} \longrightarrow \mathcal{P}(C)$$
$$U \mapsto C^{(0)}(U) := \langle\{x \in C \mid U(x) \neq \emptyset\}\rangle \qquad C^{(0)} \mapsto \big(\delta_{C^{(0)}} : C \to \Delta^1 \subseteq_{f.f.} \mathrm{Spc}^{op}\big)$$

where $\langle(-)\rangle$ denotes the full subcategory of the appropriate ∞-category spanned by the given set. Moreover, $\delta_{C^{(0)}}$ is the characteristic functor of the latter category, namely the unique functor $C \to \Delta^1$ s.t. $C^{(0)} \simeq \delta^{-1}_{C^{(0)}}(0)$. Finally, we regard $\Delta^1 \simeq \langle\{\emptyset, \Delta^0\}\rangle \simeq \mathrm{Spc}_{\leq -1} \subseteq_{f.f.} \mathrm{Spc}$.

Proof. The construction is self-explanatory. The factorization of $\delta_{C^{(0)}}$ through $\mathcal{P}(C)_{\leq -1}$ is A.5.0.7. □

We will be primarily interested in a relative version of such a construction, which recovers the classical definition of sieves as standard monomorphisms into representables.
Let us first introduce a bit of terminology. Let $\mathcal{X}$ be an ∞-topos, and recall that monomorphisms in $\mathcal{X}$ are (-1)-truncated morphisms; call $\mathrm{Mono}(\mathcal{X}) \subseteq_{f.f.} \mathrm{Fun}(\Delta^1, \mathcal{X})$ spanned by monomorphisms. Let $\mathrm{Sub}(c) := \mathrm{Mono}(\mathcal{X}_{/c})$ denote the poset of subobjects of c in $\mathcal{X}$. Notice that, by [24],6.2.1.4, they form a small set.

Lemma C.3.0.5. *(Sieves as subobjects, [24],6.2.2.5) Let $C \in \mathrm{Cat}_\infty$ be a small ∞-category and fix $c \in C$. Let $j : C \hookrightarrow \mathcal{P}(C)$ denote the Yoneda embedding and define, for any subobject $i : U \hookrightarrow j(c)$, the full subcategory:*

$$C_{/c}(U) \simeq \langle\big\{(f : x \to c) \in C_{/c} \mid \exists \sigma \in \mathcal{P}(C)_2 \ : \ \sigma_{\{0,2\}} = j(f) \ \wedge \ \sigma_{\{1,2\}} = i\big\}\rangle \subseteq_{f.f.} C_{/c}$$

i.e. $f \in C_{/c}$ s.t. $j(f) \in i_* i^*(\mathcal{P}(C)_{/c})$.
Then, $C_{/c}(U)$ is a sieve on $c \in C$, and our construction identifies equivalent subobjects of c, so that it defines a bijection:

$$C_{/c}(-) : \mathrm{Sub}(j(c)) \longrightarrow \{\textit{Sieves on } c\}$$
$$(i : U \hookrightarrow j(C)) \mapsto C_{/c}(U)$$

C.4 Topological Localization

As we already observed in the introduction to this section, sheaves ∞-categories on Grothendieck sites do not correspond to all exact left localizations of presheaf categories, but only to a special subclass of these, namely the so-called 'topological' localizations. In the current subsection we will briefly review this notion, as presented by Lurie in section [24],6.2.1.

First of all, we establish a criterion for a localization functor of an ∞-category with finite limits for being left-exact. The proof presented by Lurie is elementary, in that it relies on the theory of Bousfield localizations and is not a peculiarity of toposes.

Lemma C.4.0.1. *(Criterion for left-exact localizations, [24],6.2.1.1) Let $L : X \to Y$ be a (Bousfield) left localization functor of ∞-categories and assume that X admits all finite limits.* TFAE:

- *L is left-exact, i.e. L preserves finite limits of X.*
- *For any base-change f' of a morphism f in X, then $L(f')$ is an equivalence in Y whenever $L(f)$ is such, i.e. we are performing a localization L with respect to the strong saturation of a class of morphisms which is stable under pull-backs.*

Definition C.4.0.2. *(Topological Localization, [24],6.2.1.5)* Let $\mathcal{X} \in Pr^L$ be a presentable ∞-category, and consider a strongly saturated class of morphisms $\bar{S} \subseteq \mathrm{Mor}(\mathcal{X})$. We say that $\bar{S}$ is **topological** in case:

1. $\bar{S}$ is a strong saturation of some subclass S of $\mathrm{Mono}(\mathcal{X})$.
2. $\bar{S}$ is stable under pull-backs, i.e. for any base-change f' of $f \in \mathrm{Mor}(\mathcal{X})$, $f \in \bar{S}$ implies $f' \in \bar{S}$.

Consider now a left localization of a presentable ∞-category $\mathcal{X} \in Pr^L$, say $L : \mathcal{X} \to \mathcal{Y}$. L is called a **topological localization** whenever $\mathcal{Y} \simeq \mathcal{X}[\bar{S}^{-1}]$ for a topological class $\bar{S} \subseteq \mathrm{Mor}(\mathcal{X})$.

Proposition C.4.0.3. *(Properties: topological classes in ∞-topoi) Let $\mathcal{X} \in Pr^L$ be a presentable ∞-category with universal colimits, so e.g. an ∞-topos. Consider a strongly saturated class of morphisms $\bar{S} \subseteq \mathrm{Mor}(\mathcal{X})$.*

1. *([24],6.2.1.2) Assume further that the formation of pull-backs in $\mathcal{X}$ commutes with filtered colimits. If $\bar{S}$ is strongly generated by some* small *set $S \subseteq \mathrm{Mor}(\mathcal{X})$, then also the stabilization of $\bar{S}$ under pull-backs (so the smallest sup-class of S which is both strongly generated and stable under pull-backs) is generated by a small set.*
2. *([24],6.2.1.6) If $\bar{S}$ is topological, then it is generated by a* small *set of* $\mathrm{Mono}(\mathcal{X})$.

Proof. (Sketch) Let $U \subseteq ob(\mathcal{X})$ be a small colimit-dense set of objects in $\mathcal{X}$.
(1) : wlog $\mathcal{U} := \langle U \rangle \subseteq_{f.f.} \mathcal{X}$ has finite limits. Let S' be the closure of S in $\mathcal{U}$ under (co)base-change and formation of (co)diagonals. Morphisms of S' have source and target in U, so the latter is small. One should then check that $\bar{S'}$ is also closed under pull-backs; this is highly non-trivial and technical, so we defer to Lurie's work.
(2) : For each $x \in \mathcal{X}$, define $\mathrm{Sub}'(x) := \frac{\mathrm{Sub}(x) \cap \bar{S}}{\simeq}$. For each $u \in U$ and $\widetilde{u} \in \mathrm{Sub}'(u)$ choose a representative monomorphism $f_{\widetilde{u}} \in \bar{S}$. Let S_0 be the collection of all such representatives, and notice that it is small because, by [24],6.2.1.4 subobjects form small sets. Then, one can check that $\bar{S_0} = \bar{S}$. ☐

As a corollary, we obtain the main result of this subsection.

Proposition C.4.0.4. *(Topological localizations are accessible and left-exact, [24],6.2.1.7) Topological localizations of a presentable ∞-category $\mathcal{X} \in Pr^L$ with universal colimits, e.g. $\mathcal{X}$ an ∞-topos, are both accessible and left exact.*
In particular, a topological localization of the ∞-topos $\mathcal{P}(C)$ of presheaves on any small $C \in \mathrm{Cat}_\infty$ is again an ∞-topos.

Proof. By the theory of Bousfield left localizations, any localization of $\mathcal{X}$ corresponds to the strong saturation of a class S of local morphisms.
Since $\mathcal{X}$ is presentable and colimits are universal, by C.4.0.3, wlog S is a small set. Then, our localization is accessible by [24],5.5.4.2.
Moreover, by assumption $\bar{S}$ is stable under pull-backs, which means that the corresponding localization functor must be left-exact (see C.4.0.1). ☐

C.5 The ∞-Topos of Sheaves

In the current section we will study several definitions of ∞-categories of sheaves over small Grothendieck sites. They will achieve different levels of generality and any two of them will be proven to be equivalent whenever they both make sense.
Furthermore, we will sketch the proofs of the most relevant constructions, such as sheafification, and investigate in which sense sheaves are determined by the underlying Grothendieck site in homotopy. These perspectives appear and provide motivation to our approach to derived schemes.

C.5.1 Sheaves Characterize Sites

Definition C.5.1.1. (Sheaves as local objects, [24],6.2.2.6) Let (C, Cov) be a Grothendieck site on a small ∞-category $C \in \mathrm{Cat}_\infty$. Let S denote the set $\cup_{c \in C} \mathrm{Cov}(c) \subseteq \cup_{c \in C} \mathrm{Sub}(j(c))$ of all covering sieves on C.
Then, a presheaf $F \in \mathcal{P}(C)$ is a **sheaf** iff F is S-local. Define then the full subcategory $\mathrm{Sh}(C) \subseteq_{f.f.} \mathcal{P}(C)$ spanned by S-local presheaves.

Theorem C.5.1.2. (Sheafification is a topological localization, [24],6.2.2.7) *Let (C, Cov) be a Grothendieck site on a small ∞-category $C \in \mathrm{Cat}_\infty$. Then, $\mathcal{P}(C) \to \mathrm{Sh}(C)$ exhibits the latter as a topological localization. In particular, $\mathrm{Sh}(C)$ is an ∞-topos and one obtains a map of sets:*

$$\{(C, \mathrm{Cov}) \text{ site}\} \longrightarrow \{\mathcal{P}(C) \to [S^{-1}]\mathcal{P}(C) \text{ topological loc.}\}$$

Proof. (Sketch) Let $L : \mathcal{P}(C) \to \mathrm{Sh}(C)$ be the localization functor with respect to $S \simeq \cup_{c \in C} \mathrm{Cov}(c) \subseteq \cup_{c \in C} \mathrm{Sub}(j(c))$. S is generated by monomorphisms, so we need to show its closure under pull-backs or, equivalently by C.4.0.1, the left-exactness of L.

Let κ be a regular cardinal, s.t. for each $c \in C$ and $C^{(0)}_{/c} \in \mathrm{Cov}(c)$, the construction $\mathcal{F} \mapsto \lim \mathcal{F}_{|C^{(0)}_{/c}}$ determines a functor $\mathcal{P}(C) \to \mathrm{Spc}$ which commutes with κ-filtered colimits. By [24],5.3.3.3 it suffices to take $\kappa > \# ob(C) \# \mathrm{Mor}(C)$.
We will construct L by transfinite induction as the limit of a continuous chain of functors $(T_\alpha : \mathcal{P}(C) \to \mathcal{P}(\mathcal{C}) \mid \alpha < \kappa)$ in such a way that each step T_α will be left-exact.

The inductive step for successor ordinals will consist in an application of the **sheafification functor** $(-)^\dagger : \mathcal{P}(C) \to \mathcal{P}(C)$: let's construct it.
([24],6.2.2.8) On the ordinary site $(ho(C), \mathrm{Cov}_{ho})$ as in C.3.0.3, define $\mathrm{Cov}(ho(C)) \in \mathrm{Cat}_1$ to be the ordinary category with objects those pairs $(c, ho(C)^{(0)}_{/c} \in \mathrm{Cov}_{ho}(c))$ and morphisms those maps $f : c \to c'$ s.t. base-changing along them gives $f^*(ho(C)^{(0)}_{/c'}) = ho(C)^{(0)}_{/c}$.
This induces the ∞-category $\mathrm{Cov}(C) := C \times_{\mathcal{N}(hoC)} \mathcal{N}(\mathrm{Cov}(hoC))$ together with the forgetful functor $\rho : \mathrm{Cov}(C) \to C$. Notice that the fibre of ρ over $c \in C$ is $\mathcal{N}(\mathrm{Cov}_{ho}(c)) \cong \mathrm{Cov}(c)$ as posets.
([24],6.2.2.9) Then, consider the full subcategory $C^+ \subseteq_{f.f.} \mathrm{Fun}(\Delta^1, C) \times_{\mathrm{Fun}(\{1\}, C)} \mathrm{Cov}(C)$ spanned by $\{(x \to c \in C^{(0)}_{/c}) \mid C^{(0)}_{/c} \in \mathrm{Cov}(c)\}$.
The latter comes equipped with the following forgetful functors:

- $e : C^+ \to C$ given by $e = ev_0 \circ ev_1$;
- $\pi : C^+ \to \mathrm{Cov}(C)$ given by $\pi = ev_1 \circ ev_1 \times ev_2$.

Consider the restriction of presheaf categories induced by the functors e, π and ρ together with the corresponding Kan extensions $\pi^* \vdash \pi_*$ and $\rho_! \vdash \rho^*$.
Then, define the sheafification functor by

$$(-)^\dagger : \mathcal{P}(C) \xrightarrow{e^*} \mathcal{P}(C^+) \xrightarrow{\pi_*} \mathcal{P}(\mathrm{Cov}(C)) \xrightarrow{\rho_!} \mathcal{P}(C)$$

By closely inspecting the fibres of the functors involved as in [24],6.2.2.10-11, we can pictorially describe the action of $(-)^\dagger$ on a presheaf $\mathcal{F} \in \mathcal{P}(C)$ as follows:

$$\mathcal{F}^\dagger : c \mapsto \underset{C^{(0)}_{/c} \in \mathrm{Cov}(c)}{\underrightarrow{\mathrm{colim}}} \ \lim_{c' \in C^{(0)}_{/c}} \mathcal{F}(c')$$

Furthermore, notice that $(-)^\dagger \simeq \rho_! \circ \pi_* \circ e^*$ is left exact, since all functors involved are such.

We will now define our continuous chain of functors as follows. Let $[\kappa]$ be the initial segment of ordinals $\alpha \in \mathrm{Ord}$ which are less than κ. For each $\mathcal{F} \in \mathcal{P}(C)$, define inductively a functor $T\mathcal{F} : \mathcal{N}[\kappa] \to \mathrm{Fun}(\mathcal{P}, \mathcal{P}(C))$ on the spine $[\kappa]$ as follows:

- $T\mathcal{F}(0) := id_{\mathcal{P}(C)}$
- β^+ successor ordinal: $T\mathcal{F}(\beta^+) := \big(T\mathcal{F}(\beta)\big)^\dagger$

- α limit ordinal: $T\mathcal{F}(\alpha) := \underrightarrow{\mathrm{colim}}\big(T\mathcal{F}(\beta)\,,\, u_{\beta,\gamma} : T\mathcal{F}(\beta) \to T\mathcal{F}(\gamma) \mid \beta < \gamma < \alpha\big)$ with transition morphisms induced under continuity by $u_\beta := u_0(T\mathcal{F}(\beta))$ with $u_0 : id_{\mathcal{P}(C)} \to (-)^\dagger$ being the natural transformation of [24],6.2.2.13.

By the properties of the sheafification functor together with [24],5.3.3.3, $T\mathcal{F}(\alpha)$ is clearly left-exact at each step, and hence also at $T\mathcal{F}(\kappa)$, which remains defined by the regularity of κ.
Moreover, notice that for each $\alpha \leq \kappa$ we obtain a canonical comparison map $\mathcal{F} \to T\mathcal{F}(\alpha)$.

The last (technical) step will be to show that the construction $\mathcal{F} \mapsto T\mathcal{F}(\kappa)$ acts as the localization functor $L : \mathcal{F} \mapsto L\mathcal{F}$, i.e. that the canonical map $\mathcal{F} \to T\mathcal{F}(\kappa)$ is S-local and that $T\mathcal{F}(\kappa) \in \mathcal{P}(C)$ actually defines a sheaf in $\mathrm{Sh}(C, \mathrm{Cov})$. For these we defer to the proof by Lurie at [24],pp.582.
This will mean, indeed, that the canonical map $L(\lim \mathcal{F}_i) \to \lim L(\mathcal{F}_i)$ will be point-wise an equivalence, and hence an equivalence of functors. □

Now, we wonder whether also a converse result holds, namely whether the map of sets given by $(C, \mathrm{Cov}) \mapsto \big(\mathcal{P}(C) \to \mathrm{Sh}(C, \mathrm{Cov})\big)$ is bijective.
In other words, whether a topological localization of a category of presheaves $\mathcal{P}(C)$ on a small $C \in \mathrm{Cat}_\infty$ needs necessarily be a category of sheaves on some site on C.
Clearly, in order to classify sites by presheaves, we would like equivalent topological localizations to yield the same site. This will be ensured by the next Lemma.

Lemma C.5.1.3. *([24],6.2.2.16) Let (C, Cov) be a site on a small $C \in \mathrm{Cat}_\infty$, and let $L : \mathcal{P}(C) \to \mathrm{Sh}(C)$ denote the corresponding topological localization with respect to some S as in the definition of the sheaves ∞-topos.*
By the bijection of C.3.0.5, any sieve $C^{(0)}_{/c}$ on c corresponds to a unique subobject $i : U \hookrightarrow j(c) \in \mathrm{Sub}(j(c))$.
Then, $C^{(0)}_{/c}$ is covering iff $L(i)$ is an equivalence, i.e. $i \in \bar{S}$ belongs to the strong saturation of S.
In particular, equivalent localizations with respect to sets S, T (so for equivalent strong saturations $\bar{S}$, $\bar{T}$) yield the same Grothendieck site (C, Cov).

Proof. $(\implies)$: if $C^{(0)}_{/c} \in \mathrm{Cov}(c)$ is covering, then $i \in S$, and L takes S to equivalences of $\mathcal{P}(C)$.
$(\impliedby)$: Assume now that $L(i)$ is an equivalence, i.e. $i \in \bar{S}$ lives in the strong saturation. Then, by A.5.0.7, $\pi_0(Li) \simeq \tau_{\leq 0}(Li) \simeq L(\tau_{\leq 0} i)$ must be an equivalence. In the ordinary setting, the latter can be identified with a standard monomorphism $\eta : ho\mathcal{F} \subseteq_{f.f.} \mathrm{Hom}_{hoC}(-, c)$ in $\mathrm{Psh}(hoC, \mathrm{Set}) := \mathrm{Fun}(hoC, \mathrm{Set})$, where

$$ho\mathcal{F}(x) := \{(f : x \to c) \in hoC_{/c} \mid f \in hoC^{(0)}_{/c} \cong ho(C^{(0)}_{/c})\}$$

Being $L(ho\mathcal{F}) \in \mathrm{Sh}(hoC, \mathrm{Set})$ a sheaf, if η becomes an isomorphism after sheafification, then the identity $1_c \in L(ho\mathcal{F})(c)$ 'locally', by which we mean that there is an arbitrary family $\{f_i : c_i \to c \mid i \in I\}$ generating a covering sieve $hoC^{(1)}_{/c} \in ho\mathrm{Cov}(c)$ s.t. for each $i \in I$ it holds $f_i = f_i^*(1_c) \in ho\mathcal{F}(c_i)$. By C.3.0.3, $hoC^{(1)}_{/c} = ho(C^{(1)}_{/c})$, so this lifts to a similar condition on our sheaf $L\mathcal{F} \in \mathrm{Sh}(C)$. In particular, then, $f_i \in \mathcal{F}(c_i)$ implies $f_i \in C^{(0)}_{/c}$, and hence $C^{(1)}_{/c} \subseteq C^{(0)}_{/c}$. Thus, $C^{(1)}_{/c} \in \mathrm{Cov}(c)$, entails $C^{(0)}_{/c}$ covering, as required. □

Then, in view of the previous results, we can finally complete the classification of Grothendieck sites.

Proposition C.5.1.4. *(Sheaves classify sites, [24],6.2.2.17) Let $C \in \mathrm{Cat}_\infty$ be a small ∞-category. Then, Grothendieck sites over C correspond bijectively up to isomorphism to topological localizations of $\mathcal{P}(C)$:*

$$\{(C, \mathrm{Cov})\ \textit{site}\} \quad \overset{1:1}{\leftrightarrow} \quad \{\mathcal{P}(C) \to [S^{-1}]\mathcal{P}(C)\ \textit{topological loc.}\}_{/\cong}$$

Proof. (Sketch) By the previous results, we already have an injective map from left to right. We need to show that it is also surjective, namely that every topological localization $L : \mathcal{P}(C) \to [\bar{S}^{-1}]\mathcal{P}(C)$ comes from a Grothendieck site on C. Let $\bar{S} \subseteq \mathrm{Mor}(\mathcal{P}(C))$ be the strongly saturated class with respect to which we localize. Consider the small set $S := \{i : U \overset{j}{\hookrightarrow} (c) \in \mathrm{Sub}(j(c)) \mid c \in C \ \wedge\ i \in \bar{S}\}$ of $\bar{S}$-local subobjects of representables in $\mathcal{P}(C)$.

We will construct a site (C, Cov) s.t. $[\bar{S}^{-1}]\mathcal{P}(C)$ is equivalent to $\mathrm{Sh}(C, \mathrm{Cov})$.
It can be proved that the given $\bar{S}$ is actually the strong saturation of the constructed $S \subseteq \mathrm{Mono}\mathcal{P}(C)$. Hence, with reference to C.3.0.5, define for each $c \in C$ the specification $\mathrm{Cov}(c) := C^{(0)}_{/c}(S)$. Finally, one can show that (C, Cov) is actually a Grothendieck site: this will automatically yield $\mathrm{Sh}(C, \mathrm{Cov}) \simeq [S^{-1}]\mathcal{P}(C)$ by our construction. $\square$

C.5.2 Example: Finitary Sites

We provide now an interesting class of examples in view of our applications to Derived Algebraic Geometry. We defer to Lurie's Appendix [26],A.3.1 to *SAG* for more details.

Definition C.5.2.1. *([26],A.3.1.1)* Let (C, Cov) be a small site on $C \in \mathrm{Cat}_\infty$ which admits pull-backs. We say that a covering sieve $C^{(0)}_{/c} \in \mathrm{Cov}(c)$ is a **finite cover** whenever it is generated by a finite family of morphisms. A finite cover is then called a **covering morphism** in the case it admits a single generator.
We say that the site (C, Cov) is **finitary** iff every sieve in Cov refines (i.e. contains) a finite cover.

Remark. For any site (C, Cov) on $C \in \mathrm{Cat}_\infty$ with pull-backs, consider the finitary site Cov' consisting of those covering sieves of Cov which refine a finite cover. Then, (C, Cov') is the *finest* finitary topology on C which is *coarser* than the original one. Call Cov' the finitary sub-topology induced by Cov.

Proposition C.5.2.2. *(Abundance of finitary sites, [26],A.3.2.1) Let $C \in \mathrm{Cat}_\infty$ admit pull-backs and universal finite coproducts, and consider a class $S \subseteq \mathrm{Mor}(C)$ which enjoys the following stability properties (e.g. S topological):*

1. *S is stable under pull-backs;*
2. *S is closed under equivalence;*
3. *S is stable under finite coproducts and composition.*

Then, we can endow C with a finitary topology (C, Cov) defined as follows:

$$C^{(0)}_{/c} \in \mathrm{Cov}(c) \quad \iff \quad \exists \big\{ c_i \to c \mid i \in I \textit{ finite} \big\} \subseteq C^{(0)}_{/c} \quad \textit{s.t.} \quad \big(\coprod c_i \to c \big) \in S$$

We call such a site the ***site of covering morphisms*** *on C with respect to S.*

C.5.3 Čech Descent

In this subsection, we will generalize the usual definition of a sheaf on an ordinary site $(\mathcal{C}, \mathrm{Cov})$ with values in an arbitrary (enough complete) category $\mathcal{D}$ as a contravariant functor $\mathcal{C}^{op} \to \mathcal{D}$ which preserves finite products and 'effective equalizers'.

Definition C.5.3.1. *(D-valued C-sheaves)* Let (C, Cov) be a site on a small ∞-category $C \in \mathrm{Cat}_\infty$ and let $D \in \mathrm{Cat}_\infty$ have (enough) limits.
([26],A.3.2) Then, a functor $\mathcal{F} : C^{op} \to D$ is a **C-sheaf with values in** D iff for each $c \in C$ and covering sieve $C^{(0)}_{/c} \in \mathrm{Cov}(c)$, $\mathcal{F}(c) \simeq \lim \mathcal{F}_{|(C^{(0)}_{/c})^{op}}$.
([22],1.1.9) More explicitly, we are requiring the following extensions to be colimit diagrams in D^{op} with equivalent vertex at ∞:

$$\overline{C^{(0)}_{/c}} \subseteq_{f.f.} \overline{C_{/c}} \xrightarrow{for} C \xrightarrow{\mathcal{F}^{op}} D^{op}$$

In other words, a presheaf $\mathcal{F} : C^{op} \to D$ is a sheaf iff it is a left Kan extension along the Yoneda embedding j of the inclusion of each covering sieve.
Let $\mathrm{Sh}_D(C) \subseteq_{f.f.} \mathrm{Fun}(C^{op}, D)$ denote the full subcategory spanned by C-sheaves with values in D.

Remark. As for the equivalence of the two formulations, by the ∞-Density Theorem [24],5.1.5.3:

$$\begin{tikzcd}
\operatorname{colim}\big(C^{(0)}_{/c} \subseteq_{f.f.} C_{/c} \xrightarrow{for} C \overset{j}{\hookrightarrow} \mathcal{P}(C) \xrightarrow{\mathcal{F}^{op}} D^{op}\big) \simeq \operatorname{colim} \mathcal{F}^{op}_{|C^{(0)}_{/c}} \arrow[d] \\
\operatorname{colim}\big(C_{/c} \xrightarrow{\quad for \quad} C \overset{j}{\hookrightarrow} \mathcal{P}(C) \xrightarrow{\mathcal{F}^{op}} D^{op}\big) \simeq \mathcal{F}^{op}(c)
\end{tikzcd}$$

So, the second formulation amounts to requiring the canonical comparison map to be an isomorphism.

Let us show that we reduce to the previous definition, in the case of $D = \mathrm{Spc}$ and that $\mathrm{Sh}(C) \simeq \mathrm{Sh}_{\mathrm{Spc}}(C)$.

Lemma C.5.3.2. *(Sheaves are exact on covers) Let (C, Cov) be a small site. A presheaf $\mathcal{F} \in \mathcal{P}(C)$ is a sheaf iff for each $c \in C$ and each $C^{(0)}_{/c} \in \mathrm{Cov}(c)$ the canonical comparison map $\mathcal{F}(c) \to \lim \mathcal{F}_{|(C^{(0)}_{/c})^{op}}$ is an isomorphism.*

Proof. Let $L : \mathcal{P}(C) \to \mathrm{Sh}(C)$ be the topological localization with respect to $S = \cup_{c \in C} \mathrm{Cov}(c)$. $\mathcal{F} \in \mathcal{P}(C)$ is a sheaf iff it is S-local, namely iff for every subobject $i : U \hookrightarrow j(c) \in \mathrm{Sub}(j(c))$ it holds $\mathcal{F}(c) \simeq \mathrm{Map}_{\mathcal{P}(C)}(j(c), \mathcal{F}) \simeq \mathrm{Map}_{\mathcal{P}(C)}(U, \mathcal{F}) \simeq \mathcal{F}(U)$ in Spc.
Now, by the Density Theorem [24],5.1.5.3, we can write $U \simeq \operatorname{colim}_{j(x)/U} j(x)$ as a colimit of representables over the ∞-category of Grothendieck elements. Here, with reference to C.3.0.5, the latter is precisely $C^{(0)}_{/c}(U) \in \mathrm{Cov}(c)$. Therefore, we can test the sheaf condition by

$$\mathcal{F}(c) \longrightarrow \mathcal{F}(U) \simeq \lim_{(C^{(0)}_{/c}(U))^{op}} \mathrm{Map}(j(x), \mathcal{F}) \simeq \lim \mathcal{F}_{|(C^{(0)}_{/c}(U))^{op}}$$

and we conclude by applying again the above C.3.0.5: each sieve in $\mathrm{Cov}(c)$ corresponds to a subobject of the representable $j(c)$, so in our argument we are actually considering all covering sieves on c. □

Moreover, as proved by Lurie in [22],1.1.12, whenever the source is 'enough' cocomplete, so e.g. on an ∞-topos $\mathcal{X}$, then D-sheaves on $\mathcal{X}$ are precisely those functors $\mathcal{X}^{op} \to \mathcal{D}$ which preserve arbitrary limits, i.e. $\mathrm{Sh}_D(\mathcal{X}) \simeq \mathrm{Fun}^R(\mathcal{X}^{op}, D)$. More precisely, we have the following compatibility result.

Proposition C.5.3.3. *(Sheaves preserve limits, [22],1.1.12) Let (C, Cov) be a small site on $C \in \mathrm{Cat}_\infty$ and consider a topological localization $L : \mathcal{P}(C) \to \mathrm{Sh}_{\mathrm{Spc}}(C) = \mathrm{Sh}(C)$.*
Let $D \in \mathrm{Cat}_\infty$ admit arbitrary limits. Then, the canonical comparison map is an equivalence:

$$u_* := (L \circ j)_* : \mathrm{Sh}_D\big(\mathrm{Sh}(C)\big) \xrightarrow{\simeq} \mathrm{Sh}_D(C)$$

Proof. First recall that, by [24],5.1.5.6, right Kan extension along the Yoneda embedding j yields an equivalence $(j^{op})^* : \mathrm{Fun}^R(\mathcal{P}(C)^{op}, D) \simeq \mathrm{Fun}(C^{op}, D)$, where $\mathrm{Fun}^R(\mathcal{P}(C)^{op}, D) \subseteq_{f.f.} \mathrm{Fun}(\mathcal{P}(C)^{op}, D)$ is the full subcategory spanned by limit-preserving functors, namely by right-derived functors for the presheaf construction $\mathcal{P}$.

Now, [24],5.5.4.20 together with a formal manipulation, we can embed $\mathrm{Sh}_D(\mathrm{Sh}(C)) \overset{f.f.}{\hookrightarrow} \mathrm{Fun}^R(\mathcal{P}(C)^{op}, D)$ with essential image spanned by those functors $\mathcal{F} : \mathcal{P}(C)^{op} \to D$ s.t.

- $\mathcal{F}$ preserves all limits;
- for each cover $C^{(0)}_{/c} \in \mathrm{Cov}(c)$, if $i : U \hookrightarrow j(c) \in \mathrm{Sub}_{\mathcal{P}(C)}(j(c))$ is the corresponding subobject, then $\mathcal{F}(i) \in \mathrm{Mor}D$ is an equivalence.

By C.5.3.2 and the previous remark, this amounts to $\mathcal{F} \in \mathrm{Sh}_D(C)$.
Hence, we obtain an embedding of $\mathrm{Sh}_D(\mathrm{Sh}(C))$ into D-valued presheaves on C:

$$\mathrm{Sh}_D(\mathrm{Sh}(C)) \overset{f.f.}{\hookrightarrow} \mathrm{Fun}^R(\mathcal{P}(C)^{op}, D) \simeq \mathrm{Fun}(C^{op}, D)$$

with essential image $\mathrm{Sh}_D(C)$, as required. □

Lemma C.5.3.4. *Let C, D be ∞-categories. In any statement on D-valued presheaves on a site (C, Cov) involving (co)limits which exist in D, these (co)limits can be considered wlog in $D = \mathrm{Spc}$.*
In particular, given a presheaf $\mathcal{F} : C^{op} \to D$, $\mathcal{F} \in \mathrm{Sh}_D(C)$ iff $\mathrm{Map}_D(d, -) \circ \mathcal{F} \in \mathrm{Sh}(C)$ for each $d \in D$.

Proof. This is the first reduction step in [26],A.3.3.1. By [24],5.1.3.2, corepresentables (resp. representables) in $\mathcal{P}(D)$ commute with those limits (resp. colimits) which exist in D, so wlog they can be computed in Spc. ☐

Finally, the next result states the announced generalization. The proof is technical and involved, so we present the main ideas and defer to Lurie's [26],A.3.3.1 for more details.

Theorem C.5.3.5. *(Sheaves as preserving finite products and effective epimorphisms, [26],A.3.3.1) Let $C \in \mathrm{Cat}_\infty$ be a small ∞-category with pull-backs, and assume further that finite coproducts in C are universal and disjoint.*
Consider a class $S \subseteq \mathrm{Mor}(C)$ as in C.5.2.2, so stable under pull-backs, finite coproducts and composition, and closed under equivalences.
Moreover, let $D \in \mathrm{Cat}_\infty$ be an arbitrary ∞-category with finite products and admitting limits over Čech nerves.
Then, a functor $\mathcal{F} : C^{op} \to D$ is a D-valued C-sheaf on the site (C, Cov) of covering morphisms of C (see C.5.2.2) iff the following properties hold:

1. *$\mathcal{F}$ preserves finite products;*
2. *For each morphism $f : U_0 \to c \in S$, the augmented simplicial object*
$$\Delta_+ \overset{\check{C}(f)^{op}}{\longrightarrow} C^{op} \overset{\mathcal{F}}{\to} D$$
is a limit diagram, i.e. $\mathcal{F}(c) \simeq \lim \mathcal{F}_{|\check{C}(f)^{op}}$.

Proof. As observed in C.5.3.4, wlog $D = \mathrm{Spc}$.
$(\implies)$: Let $\mathcal{F} : C^{op} \to \mathrm{Spc}$ be a sheaf on C, and let us verify that it enjoys the stated properties.
(1) : For any finite coproduct $c := \coprod_{i=1}^n c_i \in C$ we will show by induction on n that the canonical map $\mathcal{F}(c) \to \prod_{i=1}^n \mathcal{F}(c_i)$ is an equivalence.

- $n = 0$. $c \in C^{\mathrm{init}}$ and the empty sieve $:= C^{\mathrm{init}} \in \mathrm{Cov}(c)$ is a covering sieve by the maximality axiom. But, then, $\mathcal{F}$ sheaf implies (by [24],6.2.2.18) that $\mathcal{F}(c) \in \mathrm{Spc}^{\mathrm{term}}$, which equivalent to the empty product.
- $n = 1$. There is nothing to prove.
- $n = 2$. Consider the finite cover $C^{(0)}_{/c} := \langle c_1 \to c \leftarrow c_2 \rangle$ generated by the canonical inclusions. Since $\mathcal{F}$ is a sheaf, $\mathcal{F} \simeq \lim \mathcal{F}_{|(C^{(0)}_{/c})^{op}}$, so we are left to prove that the latter is equivalent to $\mathcal{F}(c_1) \times \mathcal{F}(c_2)$.

  (Sketch of the proof.) Let $p : \Lambda^2_0 \to C^{(0)}_{/c}$ be the angle generating $C^{(0)}_{/c}$. *Assume* p to be cofinal, i.e. that the exactness of $\mathcal{F}$ on a finite cover generated by an angle can indeed be checked on such an angle. (This is a pretty involved, although instructive, application of Joyal's Theorem together with the following observation.)

  Let $\overline{p}$ denote a limiting cone for p, i.e. a pull-back, as represented by
$$\begin{tikzcd} c_1 \times_c c_2 \arrow[r] \arrow[d] & c_1 \arrow[d] \\ c_2 \arrow[r] & c_1 \coprod c_2 \simeq c \end{tikzcd}$$
  Since coproducts in C are disjoint, $c_1 \times_c c_2$ must be initial, so that $\mathcal{F}(c_1 \times_c c_2) \in \mathrm{Spc}^{\mathrm{term}}$. Therefore, it holds $\mathcal{F}_{|(\Lambda^2_0)^{op}} \simeq \mathrm{Ran}\mathcal{F}_{|\{1,2\}^{op}}$, and (by [24],4.3.2.7) the restriction along the inclusion $\{1, 2\} \subseteq \Lambda^2_0$ is cofinal: $\lim \mathcal{F}_{|(\Lambda^2_0)^{op}} \simeq \lim \mathcal{F}_{|\{1,2\}^{op}} \simeq \mathcal{F}(c_1) \times \mathcal{F}(c_2)$.

- $n > 2$. Set $d := \coprod_{i=1}^{n-1}$, so that $c \simeq d \coprod c_n$ and apply the induction assumption.

(2) : Let $f : U_0 =: X \hookrightarrow j(c)$ be a morphism in S and let $C^{(0)}_{/c} \in \mathrm{Cov}(c)$ be the covering sieve generated by f. Notice that, by the construction of Čech nerves, this is precisely the sieve corresponding to $U_\bullet \simeq \check{C}(f)_{|\Delta^{op}}$ as in C.3.0.5.
By the stability property of sieves under pull-backs, (with a slight abuse of notation) $U_\bullet : \Delta^{op} \xrightarrow{U_\bullet} C^{(0)}_{/X} \xrightarrow{f_*} C^{(0)}_{/c}$ factors through the corresponding sieve.
Since $\mathcal{F}$ is a sheaf, $\mathcal{F}(c) \simeq \lim \mathcal{F}_{|(C^{(0)}_{/c})^{op}}$, so that we are left to show that $U_\bullet : \Delta^{op} \to C^{(0)}_{/c}$ is coinitial, so as to obtain that $\lim \mathcal{F}_{|U^{op}_\bullet(\Delta)} \simeq \mathcal{F} \circ \check{C}(f)^{op}$ exhibits the latter as a limit diagram.
In order to see it, we will apply Joyal's Criterion A.1.0.2: we need to prove that for each $g : c' \to c \in C^{(0)}_{/c}$, the category $\mathcal{E} := \Delta^{op} \times_{C^{(0)}_{/c}} (C^{(0)}_{/c})_{g/}$ is weakly contractible.
Under the Straightening Theorem [24],3.2, the left fibration $ev_1 : \mathcal{E} \to \Delta^{op} \in \mathrm{LFib}(\Delta^{op})$ is classified by a functor $E : \Delta^{op} \to \mathrm{Spc}$.
By [24],3.3.4.5, it suffices to prove that its limit $\hat{E} : \Delta_+ \to \mathrm{Spc}$ classifies a weakly contractible category $\hat{\mathcal{E}}$.
Let us unwind the straightening construction: as a simplicial object of Spc, E_n is the fibre of $\mathcal{E}$ over $[n] \in \Delta^{op}$, namely

$$\mathcal{E}_{[n]} \simeq [n] \times_{C^{(0)}_{/c}} (C^{(0)}_{/c})_{g/} \simeq \mathrm{Map}_{C_{/c}}(c', U_n) \simeq \mathrm{Map}_{C_{/c}}(c', U_0 \times_c \cdots \times_c U_0) \simeq \mathrm{Map}_{C_{/c}}(c', U_0) \times_{\Delta^0} \cdots \times_{\Delta^0} \mathrm{Map}_{C_{/c}}(c', U_0)$$

In other words, E is the simplicial object lying under the čech nerve of $q : \mathrm{Map}_{C_{/c}}(c', U_0) \to \Delta^0$.
By the construction in C.3.0.5, since $g \in C^{(0)}_{/c}$, it follows that $E(0) = \mathrm{Map}_{C_{/c}}(c', U_0) \neq \emptyset$ is not the empty space; then, the (-1)-truncation of $E(0)$ is a point, which is terminal in Spc. So, in the ∞-topos Spc, q is an effective epimorphism of spaces by C.1.0.7, and hence it exhibits E as a simplicial realization of Δ^0, i.e. $|E| \simeq \check{C}(q)(-1) \simeq \Delta^0$ is the required weak equivalence.

$(\Longleftarrow)$: Let $\mathcal{F} : C^{op} \to D$ satisfy properties (1) and (2). We need to prove that $\mathcal{F}$ is exact on covering sieves of (C, Cov). For $c \in C$, consider a covering sieve $C^{(0)}_{/c} \in \mathrm{Cov}(c)$.

- $C^{(0)}_{/c}$ is a covering morphism: let $(f : c' \to c) \in C_{/c}$ generate $C^{(0)}_{/c}$ and consider its Čech nerve $\check{C}(f)$. As we have already proven, $\check{C}(f)_{|\Delta^{op}}$ is coinitial in $C^{(0)}_{/c}$, so (2) corresponds to the exactness of $\mathcal{F}$ on $C^{(0)}_{/c}$.

- $C^{(0)}_{/c}$ is a finite cover: *(Sketch of the proof)* let $\{f_i : c_i \to c \mid 1 \leq i \leq n\}$ be generating $C^{(0)}_{/c}$, with $f := \coprod f_i \in S$.

  Let $C^{(1)}_{/c}$ denote the sieve generated by f. Since we have an inclusion $\iota : C^{(0)}_{/c} \subseteq_{f.f.} C^{(1)}_{/c}$ (by the universal property of coproducts), also the latter is a covering sieve. Then, $\mathcal{F}$ is exact on $C^{(1)}_{/c}$ by the previous case.

  So, we are left to show that the restriction along ι preserves the limit. In order to see this, one can check that $\mathcal{F}_{|(C^{(1)}_{/c})^{op}} \simeq \mathrm{Ran}_\iota\big(\mathcal{F}_{|(C^{(0)}_{/c})^{op}}\big)$ and then conclude by the properties of right Kan extensions (see [24],4.3.2.7).

  The proof is omitted because technical; it is noteworthy to remark, however, that here it is exactly where we need to use all the assumptions on S together with (1).

- $C^{(0)}_{/c}$ is an arbitrary covering sieve: by assumption, it must refine some finite covering $C^{(1)}_{/c} := \langle\{f_i : c_i \to c \mid 1 \leq i \leq n\}\rangle \in \mathrm{Cov}(c)$ with $\coprod f_i \in S$.

  By the previous argument, $\mathcal{F}$ must preserve the exactness of the finite cover $C^{(1)}_{/c}$, so we are left to show that $\mathcal{F}(c) \simeq \lim \mathcal{F}_{|(C^{(1)}_{/c})^{op}}$.

For this it would suffice to check that $\mathcal{F}_{|(C^{(0)}_{/c})^{op}} \simeq \mathrm{Ran}_{\subseteq}\big(\mathcal{F}_{|(C^{(1)}_{/c})^{op}}\big)$.

Unwinding the definitions, this amounts to the fact that, for each $(g : x \to c) \in C^{(0)}_{/c}$, we can write $\mathcal{F}(x)$ as the limit of the under-slice $\mathcal{F}(x)/\mathcal{F}_{|(C^{(0)}_{/c})^{op}}$, which is in turn equivalent to $\mathcal{F}_{|g^*(C^{(1)}_{/c})^{op}}$.

But now, since $g^*(C^{(0)}_{/c}) \simeq \langle g^*(f_i) : c_i \times_c x \to x \mid i\rangle$, if also $\coprod g^*(f_i) \in S$, then we would be done by applying the previous point to the finite covering $g^*(C^{(0)}_{/c})$.

Finally, $\coprod g^*(f_i) \in S$ follows from the properties of S: write $\coprod g^*(f_i)$ as the following composite

$$\coprod(c_i \times_c x) \xrightarrow{\simeq} (\coprod c_i) \times_c x \longrightarrow u$$

where the first arrow is an equivalence, because finite coproducts in C are universal and $g^*(\coprod f_i)$ in S, which is closed under pull-backs; being the latter closed also under equivalence and composition, the whole composite must live in S.

□

Finally, we present a well-known characterization of descent for $\mathcal{C}$-valued sheaves with $\mathcal{C} \in \mathrm{Cat}_1$ an ordinary category.
We start with a Lemma reducing all simplicial limits of ordinary categories to "glueing problems".

Lemma C.5.3.6. *(Simplicial limits in ordinary categories) Let $p : \mathcal{N}(\Delta)^{op} \to \mathcal{D}$ be a simplicial diagram with values in an ordinary category $\mathcal{D}$. Then, $\lim p \simeq \lim p_{|\mathcal{N}(\Delta^{\leq 2})^{op}}$.*

Proof. Omitted, see [14],A.1. □

The previous useful lemma yields a straightforward characterization of sheaves on a 'nice' site (C, Cov) with values in an ordinary category $\mathcal{D}$ with enough limits.
In particular, we will apply it to sheaves on a 'geometric site' (e.g. the Zariski site) with values in an 'algebraic category' (namely some Lawvere theory as CRing of commutative rings or $\mathrm{Mod}(R)$ of R-modules).

Proposition C.5.3.7. *(Descent for sheaves with values in ordinary categories) Let $\mathcal{D} \in \mathrm{Cat}_1$ be an ordinary category with (enough) limits, and consider an ∞-category $C \in \mathrm{Cat}_\infty$ as in C.5.3.5, so with pull-backs and whose finite coproducts are universal and disjoint.*
For a 'nice' class of morphisms $S \subseteq \mathrm{Mor}(C)$ as in C.5.2.2 (so stable under pull-backs, finite coproducts and composition, and closed under equivalences), let (C, Cov) be the induced finitary site on C.
Let $\mathrm{Sh}_{\mathcal{D}}(C)$ denote the localization of the ∞-category of $\mathcal{D}$-valued presheaves $\mathcal{P}(C, \mathcal{D})$ on (C, Cov) at the class $S \subseteq \mathrm{Mor}(C)$.
Then, for a presheaf $\mathcal{F} \in \mathcal{P}(C, \mathcal{D})$, TFAE:

- *$\mathcal{F}$ is a sheaf in $\mathrm{Sh}_{\mathcal{D}}(C)$;*
- *$\mathcal{F}$ preserves 2-truncated simplicial limits;*
- *$\mathcal{F}$ preserves finite products and, for each morphism $(f : U_0 \to c) \in S$, the 2-truncation of the augmented co-simplicial object $\mathcal{F} \circ \check{C}(f)^{op}$ is a limit diagram, i.e. the following diagram is 'exact'*

$$\mathcal{F}(c) \quad \to \quad \Big(\mathcal{F}(U_0) \quad \rightrightarrows \quad \mathcal{F}(U_1) \times_{\mathcal{F}(U_0)} \mathcal{F}(U_1) \quad \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \quad \mathcal{F}(U_1) \times_{\mathcal{F}(U_0)} \mathcal{F}(U_1) \times_{\mathcal{F}(U_0)} \mathcal{F}(U_1)\Big)$$

Proof. The equivalence of the first two statements is C.5.3.3 together with the previous Lemma.
Then, in our setting, C.5.3.5 reduces the sheaf condition $\mathcal{F} \in \mathcal{P}(C, \mathcal{D})$ to preserving finite products and being exact on Čech nerves. Hence, we are left to show that the stated 2-truncation amounts to the exactness on Čech nerves. This is a straightforward consequence of the machinery developed: as proven in [24],6.5.3.7, we can neglect degeneracies while computing simplicial colimits, and we can 2-truncate our co-simplicial objects by the previous lemma. □

C.5.4 Example: Sheaves on Topological Spaces

A special case of C.5.3.2 retrieves the classical notion of sheaves on topological spaces.

Definition C.5.4.1. *([26],1.1.2.1)* Let $X \in \mathrm{Top}$ be a topological space and let $(\mathrm{Open}(X), \mathcal{U}')$ denote the Grothendieck site on the poset of open subsets of X.
Then, a presheaf $\mathcal{F} : \mathrm{Open}(X)^{op} \to C$ with values in a complete ∞-category $C \in \mathrm{Cat}_\infty$ is a C**-sheaf on** X iff, for each $V \in \mathrm{Open}(X)$ and each open covering $\mathcal{V}$ of V, there is a natural equivalence $\mathcal{F}(V) \simeq \lim \mathcal{F}_{|\mathcal{U}'(V)}$.
We will write the following shorthand: $\mathrm{Sh}_C(X) := \mathrm{Sh}_C(\mathrm{Open}(X), \mathcal{U}'(X))$

An application of the relative nerve construction [24],3.2.5.2, namely a particular case of the Straightening Theorem [24],3.2 for fibrant-objects, will allow us to consider an ∞-functor $\mathrm{Sh}_C(-) : \mathrm{Top} \to \mathrm{sSet}$ introducing compatibility of sheaf categories with respect to restrictions along continuous maps.

Construction C.5.4.2. *(Push-forward of sheaves, [26],1.1.2.2)* Let $C \in \mathrm{Cat}_\infty$ and consider any continuous map $\pi : X \to Y$ in Top. Define the **push-forward functor** along π to be the functor of ∞-categories of sheaves induced by the restriction along π:

$$\begin{aligned} \pi_* : \mathrm{Sh}_C(X) &\longrightarrow \mathrm{Sh}_C(Y) \\ \mathcal{F} &\longmapsto \mathcal{F}(\pi^{-1}(-)) \end{aligned}$$

By rectifying $\mathrm{Cat}_\infty \simeq \mathrm{sSet}_{\mathrm{Joyal}}[\mathrm{Joy}^{-1}]$, the construction $[\pi \mapsto \pi_*]$ induces a functor of ordinary categories:

$$\begin{aligned} \mathrm{Sh}_C(-)^{op} : \mathrm{Top} &\longrightarrow \mathrm{sSet} \\ X &\longmapsto Sh_C(X)^{op} \\ \pi &\longmapsto \pi_* \end{aligned}$$

Let Top_C denote nerve of Top relative to $\mathrm{Sh}_C(-)^{op}$ (see [24],3.2.5.2).
More explicitly, as observed in [26],1.2.1.3 the 1-truncation of the simplicial set Top_C can be described as follows:

- Obj: pairs $(X, \mathcal{F})$ with $\mathcal{F} \in \mathrm{Sh}_C(X)$;
- Mor: maps of pairs $(\pi, \alpha) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ with $\pi : X \to Y$ in Top and $\alpha : \mathcal{G} \to \pi_*\mathcal{F}$ in $\mathrm{Sh}_C(Y)$.

By the refinement of the Straightening Theorem for fibrant objects (as presented in [24],3.2.5.21) the cocartesian fibration given by the first projection $\mathrm{Top}_C \to \mathrm{Top}$ is associated to the nerve $\mathrm{Sh}_C(-)^{op} : \mathrm{Top} \to \mathrm{Cat}_\infty$ of the previous functor.

Furthermore, as in the classical case the datum of a sheaf on a topological space X, i.e. a sheaf on $\mathrm{Open}(X)$, is equivalent to that of a sheaf on some basis $\mathcal{U}_e$ for the topology of X. The next result attempts an enhancement of such a feature for a basis $\mathcal{U}_e$ of quasi-compact open subsets of X.

Proposition C.5.4.3. *(Characterization of sheaves over a basis, [26],1.1.4.4) Let $X \in \mathrm{Top}$ be a (locally compact) topological space and consider a complete ∞-category $C \in \mathrm{Cat}_\infty$. Suppose to be given a subset $\mathcal{U}_e \subseteq \mathrm{Open}(X)$ such that:*

1. *$\mathcal{U}_e$ consists of a pre-basis for the topology on X, i.e. if we write $\mathrm{Open}(X) = \cup\{\tau(x) | x \in X\}$ as the union of all filters of open neighbourhoods of the points of X, then $\mathcal{U}_e \cap \tau(x) \subseteq \tau(x)$ is always a cofinal sub-filter with respect to the order induced by reverse inclusion;*
2. *$\mathcal{U}_e$ is stable under finite intersections*
3. *$\mathcal{U}_e$ consists of quasi-compact sets.*

Then, the presheaf $\mathcal{F} : \mathrm{Open}(X)^{op} \to C$ is a sheaf in $\mathrm{Sh}_C(X)$ iff

- *[$\mathcal{U}_e$ is $\mathcal{F}$-dense into $\mathrm{Open}(X)$] : $\mathcal{F} \simeq \mathrm{Ran}(\mathcal{F}_{|\mathcal{U}_e^{op}})$;*

- [*Sheaf condition over $\mathcal{U}_e$*] : *For each* finite *subset $I \subseteq \mathcal{U}_e$, there is an equivalence*

$$\mathcal{F}(\cup I) \xrightarrow{\simeq} \lim\big(\mathcal{F}(\cap S)\,|\,\emptyset \neq S \subseteq I\big)$$

Remark. More explicitly, by Open(X) we denote the open cover site on Open(X) (see [27],pp.113): it is the Grothendieck site τ generated by a basis K consisting of finite families of open coverings. Here, generation must be understood as "generation of sieves", so by closing under pre-composition by continuous inclusions in Open(X).
The following Construction also fixes the notation that will appear in the proof of the Proposition.

Construction C.5.4.4. *(Coverings in the open cover topology)* Fix some arbitrary $U \in \mathrm{Open}(X)$; an element of $K(U)$ consists of a *finite* subset $I \subseteq \mathrm{Open}(X)$ such that $\cup I = U$. Then, I generates a covering $\mathcal{U}'(I) := \cup_I \mathrm{Open}(i) \subseteq \mathrm{Open}(\cup I)$ in $\tau(U)$.
Remark. Notice that $\mathcal{U}'(I)$ forms a basis of the topology on $U = \cup I$.

Before proving the Proposition, let us record a couple of properties of such a construction, which will be useful later on.

Lemma C.5.4.5. *(Computations: limits over the covering $\mathcal{U}'$, [26],1.1.4.4) Let $X \in$ Top be a (locally quasi-compact) topological space and assume to be given a basis $\mathcal{U}_e$ of X consisting of quasi-compact sets.*
Moreover, Let $\mathcal{F} \in \mathcal{P}(\mathrm{Open}(X), C)$ be a C-presheaf on X. Then, the following properties hold.

1. *For $I \subseteq \mathcal{U}_e$, the inclusion $\mathcal{U}'(I) \cap \mathcal{U}_e \hookrightarrow \mathcal{U}'(I)$ in* sSet *is right-cofinal; in particular, the sheaf condition of $\mathcal{F}$ with respect to the covering $\mathcal{U}'(I)$ can be tested on $\mathcal{U}'(I) \cap \mathcal{U}_e$.*

   ([26],1.1.4.6) Therefore, the sheaf condition of $\mathcal{F}$ can be tested with respect to the topology induced on $\mathcal{U}_e$ by intersecting coverings of the open cover site with $\mathcal{U}_e$, i.e. $F \in \mathrm{Sh}_C(X)$ iff $\mathcal{F}_{|\mathcal{U}_e^{op}} \in \mathrm{Sh}_C(\mathcal{U}_e)$. We refer to this fact as "$\mathcal{F}$ satisfies the sheaf condition over $\mathcal{U}_e$".

   we will refer to this fact by writing that "$\mathcal{F}$ satisfies the sheaf condition over $\mathcal{U}_e$".

2. *Let $I \subseteq \mathrm{Open}(X)$ and let $P_f(I)$ denote the poset of finite subsets of I. Then the following map in* sSet *is right-cofinal:*

$$\gamma : P_f(I) \longrightarrow \mathcal{U}'(I)^{op}$$
$$S \longmapsto \cap S$$

   In particular, the second condition in C.5.4.3 amounts to the "sheaf condition over $\mathcal{U}_e$".

3. *Suppose that $\mathcal{F}$ satisfies the sheaf condition over $\mathcal{U}_e$; choose an open covering $I \subseteq \mathrm{Open}(X)$ of X and consider $\mathcal{U}' := \mathcal{U}'(I)$. Then, $\mathcal{F}_{|\mathcal{U}_e^{op}} \simeq \mathrm{Ran}(\mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}')^{op}})$ is a right Kan extension along the inclusion $\mathcal{U}_e \cap \mathcal{U}' \subseteq \mathcal{U}_e$.*

Proof. (1) : This is a straightforward consequence of Joyal's Criterion A.1.0.2: the map is cofinal iff the nerve $\mathcal{N}\big(\mathcal{U}_e \cap \mathcal{U}'(I)_{V/}\big)$ is weakly contractible for each $V \in \mathcal{U}'(I)$, and this clearly holds, since the under-slice is inhabited (here we use $I \subseteq \mathcal{U}_e$) and stable under finite intersections; hence such a simplicial set is filtered with respect to the order induced by the reverse inclusion, and we conclude by A.1.0.4. ■

(2) : We will again apply Joyal's Criterion A.1.0.2: we need to prove that, for each $V \in \mathcal{U}'(I)$, the nerve $\mathcal{N}\big(P_f(I)_{V/}\big)$ is weakly contractible. Now, this holds true: the category is inhabited, since $V \in \mathcal{U}'(I) = \cup_I \mathrm{Open}(i)$ and $I \subseteq P_f(I)$; moreover, it is filtered, since it is stable under intersections by the construction. Hence, we conclude by A.1.0.4. ■

(3) : It suffices to show that the comparison map obtained via the universal property of Kan extensions is an equivalence point-wise in C, i.e. that, for each $V \in \mathcal{U}_e$, the canonical map $\theta : \mathcal{F}(V) \to \lim \mathcal{F}_{|\mathcal{U}_e \cap \mathcal{U}' \cap \mathrm{Open}(V)}$ is an equivalence.
Since $\mathcal{U}_e$ is a quasi-compact basis, V is itself quasi-compact, so - by the fact that I covers X - there exists some finite subset $J \subseteq \mathcal{U}' \cap \mathcal{U}_e$ exhibiting an open covering of V. Set $\mathcal{U}'' := \mathcal{U}'(J) \cap \mathcal{U}_e$, and observe that we have an equivalence $\mathcal{F}(V) \simeq \lim \mathcal{F}_{|\mathcal{U}''}$ by the assumption that $\mathcal{F}$ satisfies the sheaf condition over $\mathcal{U}_e$.
So, we are left to show that the right slanting arrow in the following commutative triangle is an equivalence:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[rr, "\theta"] \arrow[dr, "\simeq"'] & & \lim \mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}' \cap \mathrm{Open}(V))^{op}} \arrow[dl] \\
& \lim \mathcal{F}_{|(\mathcal{U}'')^{op}} &
\end{tikzcd}$$

Since $\mathcal{U}'' \subseteq \mathcal{U}'$ and fully faithful restrictions preserve right Kan extensions, this will be implied by our final observation:

$$\mathcal{F}_{|(\mathcal{U}_e \cap \mathrm{Open}(V))^{op}} \simeq \mathrm{Ran}(\mathcal{F}_{|(\mathcal{U}'')^{op}})$$

In order to see this, we need to show that the canonical natural transformation induced by the universal property of right Kan extensions induces a point-wise equivalence, i.e. that, for each $W \in \mathcal{U}_e \cap \mathrm{Open}(V)$, the map $\phi : \mathcal{F}(W) \to \lim \mathcal{F}_{|(\mathcal{U}'' \cap \mathrm{Open}(W))^{op}}$ is an equivalence.
But this follows from the assumption that $\mathcal{F}$ satisfies the sheaf condition over $\mathcal{U}_e$:

$$\mathcal{U}'' \cap \mathrm{Open}(W) = \mathcal{U}_e \cap (\mathcal{U}'(J) \cap \mathrm{Open}(W)) = \mathcal{U}_e \cap \mathcal{U}'(J \cap W)$$

where $J \cap W := \{j \cap W \hookrightarrow W\}_{j \in J}) \in K(W)$ generates the covering $\mathcal{U}'(J \cap W)$ of W. ■ □

Proof. (of C.5.4.3) We will freely adopt the notation of the previous Construction and of Lemma C.5.4.5.

($\Longrightarrow$) : Let's prove that a sheaf $\mathcal{F} \in \mathrm{Sh}_C(X)$ satisfies the sheaf condition over $\mathcal{U}_e$.
Given any finite $I \subseteq \mathcal{U}_e$, let $\mathcal{U}' := \mathcal{U}'(I) \in \tau(\cup I)$ be a covering of $\cup I$ as in the Construction above. Let $\gamma : P_f(I) \to \mathcal{U}'(I)^{op}$ be the right-cofinal map of C.5.4.5,ii.
Thus, we can express the sheaf condition C.5.3.3 of $\mathcal{F}$ at $\cup I$ by the following chain of equivalences:

$$\mathcal{F}(\cup I) \xrightarrow{\simeq} \lim \mathcal{F}_{|\mathcal{U}'^{op}} \xrightarrow{\simeq} \lim \mathcal{F} \circ \gamma$$

where the second equivalence follows from the aforementioned right-cofinality of γ (see C.5.4.5,ii).

Now we are left to show the $\mathcal{F}$-density of $\mathcal{U}_e$ into $\mathrm{Open}(X)$, namely that the canonical comparison map $\mathcal{F} \to \mathrm{Ran}(\mathcal{F}_{|\mathcal{U}_e^{op}})$ induced by the universal property of right Kan extensions is an equivalence; by [20],2.2.2, it suffices to check that point-wise. In other words, we wish that, for each $U \in \mathrm{Open}(X)$, the canonical map $\mathcal{F}(U) \to \lim \mathcal{F}_{(\mathcal{U}_e \cap \mathrm{Open}(U))^{op}}$ be an equivalence.
Notice first that $\mathcal{U}_e \cap \mathrm{Open}(U) = (\mathcal{U}_e)_{/U}$ induces a quasi-compact basis of U, so we can simplify the notation and assume wlog $U = X$ and $\mathcal{U}_e \cap \mathrm{Open}(U) = \mathcal{U}_e$.
Being $\mathcal{U}_e$ a basis for X, it contains some open cover of X, namely there exists some (possibly infinite) collection $I \subseteq \mathcal{U}_e$ for which $\cup I = X$. Suppose I to be finite. Define $\mathcal{U}' := \mathcal{U}'(I) = \cup_I \mathrm{Open}(i)$ as in the Construction above, and consider the following commutative diagram in the ∞-category C:

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r] \arrow[d, "\simeq"', "(a)"] & \lim \mathcal{F}_{|\mathcal{U}_e^{op}} \arrow[d, "(b)"', "\simeq"] \\
\lim \mathcal{F}_{|\mathcal{U}'(I)^{op}} \arrow[r, "\simeq", "(c)"'] & \lim \mathcal{F}_{(\mathcal{U}_e \cap \mathcal{U}'(I))^{op}}
\end{tikzcd}$$

where the decorated arrows are equivalences by the following arguments:

- (a) : This is the sheaf condition of $\mathcal{F}$ at the covering $\mathcal{U}'(I) \in \tau(X)$;
- (b) : We proved that F satisfies the sheaf condition over $\mathcal{U}_e$, so $\mathcal{F}_{|\mathcal{U}_e^{op}} \simeq \mathrm{Ran}(\mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}'(I))^{op}})$ by C.5.4.5,iii.
- (c) : Since $I \subseteq \mathcal{U}_e$, the inclusion $\mathcal{U}_e \cap \mathcal{U}'(I) \hookrightarrow \mathcal{U}'(I)$ is left-cofinal in sSet by C.5.4.5,i.

For an arbitrary I the argument is the same: the sheaf condition of $\mathcal{F}$ at X still forces (a) to be an equivalence (consider a finite partition of I), while for (b) and (c) the cardinality of I is irrelevant.

($\Longleftarrow$) : Conversely, let it be given a presheaf $\mathcal{F} \in \mathcal{P}(\mathrm{Open}(X), C)$ which satisfies the two conditions in the statement; let's show that F is a C-sheaf on $\mathrm{Open}(X)$, i.e. that, for each $U \in \mathrm{Open}(X)$ and for each open cover $I \subseteq \mathrm{Open}(U)$, the canonical map $\mathcal{F}(U) \to \lim \mathcal{F}_{|\mathcal{U}'(I)^{op}}$ is an equivalence.
Again, let us simplify the notation and assume wlog $U = X$. The $\mathcal{F}$-density of $\mathcal{U}_e$ into $\mathrm{Open}(X)$ allows us to express $\mathcal{F} \simeq \mathrm{Ran}(\mathcal{F}_{|\mathcal{U}_e^{op}})$ as a right Kan extension. As a consequence, we can infer what follows:

- by restricting to $\mathcal{U}'$, one obtains an equivalence $\mathcal{F}_{|\mathcal{U}'^{op}} \simeq \mathrm{Ran}(\mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}')^{op}})$, so that the statements amounts to showing that the composite $\mathcal{F}(X) \to \lim \mathcal{F}_{|(\mathcal{U}')^{op}} \to \lim \mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}')^{op}}$ is an equivalence in C. Indeed, recall that the canonical map $\mathrm{Ran}(\mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}')^{op}}) \to \mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}')^{op}}$ is right-cofinal by the properties of right Kan extensions.
- $\mathcal{F}(X) \simeq \lim \mathcal{F}_{|\mathcal{U}_e^{op}}$.

So, finally, we can reduce the statement to proving that the canonical map $\lim \mathcal{F}_{|\mathcal{U}_e^{op}} \to \lim \mathcal{F}_{|(\mathcal{U}_e \cap \mathcal{U}')^{op}}$ be an equivalence. But this is a consequence of C.5.4.5,iii. $\square$

Therefore, the previous Proposition allows us to define C-valued sheaves on a locally compact topological space X by defining them on their restriction to a locally compact basis $\mathcal{U}_e$ of X. This is made precise in the following Corollary.

Corollary C.5.4.6. *(Sheaves on a basis, [26],1.1.4.5-6) Let* $X \in \mathrm{Top}$ *be a (locally quasi-compact) topological space with a quasi-compact basis* $\mathcal{U}_e$ *and let* $C \in \mathrm{Cat}_\infty$ *be a complete* ∞*-category.*
Then, there is a fully faithful embedding $\mathrm{Sh}_C(X) \hookrightarrow \mathcal{P}(\mathcal{U}_e) = \mathrm{Fun}(\mathcal{U}_e^{op}, C)$ *with essential image spanned by those presheaves* $\mathcal{F} : \mathcal{U}_e^{op} \to C$ *which satisfy the sheaf condition over* $\mathcal{U}_e$*.*
Moreover, the open cover site induces a topology on $\mathcal{U}_e$*, so that we can write* $\mathrm{Sh}_C(X) \simeq \mathrm{Sh}_C(\mathcal{U}_e)$*.*

Proof. This is a consequence of the previous Proposition C.5.4.3 and of the properties of right Kan extensions in [24],4.3.2.15. The description of the essential image follows by the following application of the latter result: let $C_0 := \mathcal{U}_e^{op} \subseteq_{f.f.} \mathrm{Open}(X)^{op} =: C$ and $D = D' := C$; consider the following full sub-categories of C-valued presheaves:

- $\mathcal{K} := \langle \mathcal{F} : \mathrm{Open}(X)^{op} \to C \mid \mathcal{F} \simeq \mathrm{Ran}(\mathcal{F}_{|\mathcal{U}_e^{op}}) \rangle \subseteq_{f.f.} \mathcal{P}(X, C)$
- $\mathcal{K}' := \langle \mathcal{F}_0 : \mathcal{U}_e^{op} \to C \mid \forall U \in \mathcal{U}_e \,, \exists \lim (\mathcal{F}_0)_{|((\mathcal{U}_e)_{/U})^{op}} \rangle \subseteq_{f.f.} \mathcal{P}(\mathcal{U}_e, C)$ which are those presheaves satisfying the sheaf condition over $\mathcal{U}_e$.

Then, the restriction functor $\mathcal{K} \to \mathcal{K}'$ is (in particular) an equivalence. $\square$

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