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Elements of a Bimodule with a Semilocal Endomorphism Ring

Relatore:

Prof. Alberto Facchini

Laureando:

Francesco Pagliuca

Matricola: 1178737

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Introduction

The study of block decompositions of matrices is one of the classical themes in Linear Algebra. One of the modern approaches to study this kind of problems is considering the morphisms in the category Mod-R of right modules over a ring R.

In the paper [CEDF19], Campanini, El-Deken and Facchini studied the Grothendieck category Morph(Mod-R) of all morphisms between two right modules over a fixed ring R. In that category, the objects are R-module morphisms $\mu_M\colon M_0 \to M_1$. A morphism $u\colon \mu_M \to \mu_N$ in the category Morph(Mod-R) is a pair of R-module morphisms (u_0, u_1) such that $u_1\mu_M = \mu_N u_0$.

In the category Morph(Mod-R), the study of direct-sum decomposition corresponds to the study of block decompositions of matrices. Isomorphism in this category corresponds to the matrix equivalence $\sim$ defined, for any two rectangular $m \times n$ matrices A and B, by $A \sim B$ if $B = Q^{-1}AP$ for some invertible $n \times n$ matrix P and some invertible $m \times m$ matrix Q.

For fixed right R-modules M_0 and M_1, the objects $\mu_M\colon M_0 \to M_1$ of the category Morph(Mod-R) are the objects of a full subcategory of Morph(Mod-R) whose class of objects is $\mathrm{Hom}_R(M_0, M_1)$. Now $\mathrm{Hom}_R(M_0, M_1)$ is an $\mathrm{End}_R(M_1)$-$\mathrm{End}_R(M_0)$-bimodule. Hence it is natural to ask which results of [CEDF19] remain true for a corresponding suitably defined category $\mathcal{E}$ whose objects are the objects of any R-S-bimodule ${}_RM_S$. This is what we do in this thesis.

In [CEDF19] it was shown that the behavior of morphisms whose endomorphism ring in Morph(Mod-T) is semilocal is very similar to the behavior of modules with a semilocal endomorphism ring. For instance, direct-sum decompositions of a direct sum $\bigoplus_{i=1}^n M_i$, that is, block-diagonal decompositions, where each object M_i of Morph(Mod-T) denotes a morphism $\mu_{M_i}\colon M_{0,i} \to M_{1,i}$ and where all the modules

$M_{j,i}$ have a local endomorphism ring $\mathrm{End}(M_{j,i})$, depend on two invariants. This behavior is very similar to that of direct-sum decompositions of serial modules of finite Goldie dimension, which also depend on two invariants (monogeny class and epigeny class). When all the modules $M_{j,i}$ are uniserial modules, the direct-sum decompositions (block-diagonal decompositions) of a direct-sum $\bigoplus_{i=1}^{n} M_i$ depend on four invariants.

In this thesis, our original aim was to extend the results in [CEDF19] to arbitrary bimodules, giving them a category structure, but this has lead us to the study of some special natural additive decompositions of elements in bimodules. In particular, we define an internal direct sum and we study its relations with the idempotent endomorphisms and with the categorical biproduct. We also characterize when two decompositions of an element are equal and when they are isomorphic instead. In the last chapter, using some natural functors, we see the condition under which this category is semilocal. Finally, we conclude with some embeddings in other categories, in particular in the category $\mathrm{Morph}(\text{Mod-}R)$.

Fix two associative rings R and S with identity and a bimodule ${}_RM_S$. Our category $\mathcal{E}$ has the bimodule ${}_RM_S$ as its class of objects, and, for any two objects $x, y \in {}_RM_S$, $\mathrm{Hom}_{\mathcal{E}}(x,y) = Rx \cap yS$. Thus the set of all morphisms $x \to y$ in $\mathcal{E}$ is also a subset of ${}_RM_S$. For two morphisms $rx = ys\colon x \to y$ and $r'y = zs'\colon y \to z$, we have that $r'rx = r'ys = zs's$, so $r'rx = zs's\colon x \to z$ is a morphism in $\mathcal{E}$.

Chapter 1

The Categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$

1.1 Morphism Category

We begin this first chapter recalling what the *morphism category* is, as studied in [CEDF19].

Definition 1.1. *Let* R *be an associative ring with identity* $1 \neq 0$ *and* $\text{Mod-}R$ *the category of right* R*-modules. Denote by* $\text{Morph}(\text{Mod-}R)$ *the category defined as follows:*

1. *The objects of* $\text{Morph}(\text{Mod-}R)$ *are the* R*-module morphisms* $\mu_M : M_0 \to M_1$ *between right* R*-modules.*

2. *A morphism* $u : \mu_M \to \mu_N$ *in* $\text{Morph}(\text{Mod-}R)$ *is a pair of* R*-module morphisms* (u_0, u_1) *such that* $u_1\mu_M = \mu_N u_0$.

Recall the notion of *preadditive category.*

Definition 1.2. *A category* $\mathcal{A}$ *is a* preadditive *category if*

(a) *The set* $\text{Hom}_{\mathcal{A}}(A, B)$ *is an abelian group for every* A, B *objects of* $\mathcal{A}$.

(b) *The composition* $\circ : \text{Hom}_{\mathcal{A}}(B, C) \times \text{Hom}_{\mathcal{A}}(A, B) \to \text{Hom}_{\mathcal{A}}(A, C)$ *is* $\mathbb{Z}$*-bilinear, that is, for every* $f, f' : A \to B$ *and every* $g, g' : B \to C$*, with* A, B, C *objects of* $\mathcal{A}$*,*

$$g \circ (f + f') = g \circ f + g \circ f' \text{ and } (g + g') \circ f = g \circ f + g' \circ f.$$

For semplicity we will denote each object $\mu_M : M_0 \to M_1$ in Morph(Mod-R) by M. For every pair M, N of objects of Morph(Mod-R) the group

$$\mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M,N)$$

is a subgroup of the cartesian product

$$\mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M_0,N_0) \times \mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M_1,N_1).$$

Then, for M,N objects of Morph(Mod-R), addition on $\mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M,N)$ is defined by

$$u + v = (u_0 + v_0, u_1 + v_1)$$

for every $u = (u_0,u_1)$, $v = (v_0,v_1) \in \mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M,N)$. Therefore, the category Morph(Mod-R) is preadditive.

Definition 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor. For all A, A' objects of $\mathcal{A}$, the functor F induces a mapping*

$$F_{AA'} : \mathrm{Hom}_{\mathcal{A}}(A,A') \to \mathrm{Hom}_{\mathcal{B}}(F(A),F(A')),$$

defined by $F_{AA'}(f) = F(f)$ for every $f : A \to A'$.

The functor F is called a faithful *functor if $F_{AA'}$ is injective for every A, A' objects of $\mathcal{A}$. While it is called a* full *functor if $F_{AA'}$ is surjective for every A, A' objects of $\mathcal{A}$. The functor F: $\mathcal{A} \to \mathcal{B}$ is* essentially surjective *if for every B object of $\mathcal{B}$ there exists A object of $\mathcal{A}$ such that $F(A) \cong B$*

Theorem 1.4. [CEDF19, Theorem 2.1] *The category* Morph(Mod-R) *is equivalent to the category of right modules over the triangular matrix ring* $T := \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$.

Thanks to the equivalence in Theorem 1.4, Morph(Mod-R) is a Grothendieck category.

Let us briefly recall what products and coproducts in a category are and in particular what they are in the category Morph(Mod-R).

Definition 1.5. *Let $\mathcal{A}$ be a category and let A, B be objects of $\mathcal{A}$. A* product

$$\left(A \prod B, \pi_A, \pi_B\right)$$

of A and B in $\mathcal{A}$ consists of an object $A\prod B$ of $\mathcal{A}$ and morphisms

$$\pi_A : A\prod B \to A \qquad \text{and} \qquad \pi_B : A\prod B \to B$$

such that for any pair of morphisms $f : P \to A$, $g : P \to B$ there is a unique morphism $h : P \to A\prod B$ with $\pi_A \circ h = f$ and $\pi_B \circ h = g$.

The definition of coproduct is just the dual definition.

Definition 1.6. *Let $\mathcal{A}$ be a category and let A, B be objects of $\mathcal{A}$. A* coproduct

$$\left(A\coprod B, \epsilon_A, \epsilon_B\right)$$

of A and B in $\mathcal{A}$ consists of an object $A\coprod B$ of $\mathcal{A}$ and morphisms

$$\epsilon_A : A \to A\coprod B \qquad \text{and} \qquad \epsilon_B : B \to A\coprod B$$

such that for any pair of morphisms $f : A \to P$, $g : B \to C$ there is a unique morphism $h : A\coprod B \to P$ with $h \circ \epsilon_A = f$ and $h \circ \epsilon_B = g$.

Definition 1.7. *A category $\mathcal{A}$ is an* additive *category if it is preadditive, has a zero object, and every two objects A and B have a product $A\prod B$ (equivalently, a coproduct $A\coprod B$).*

Following the results in [CEDF19], recall what coproducts and products are in Morph(Mod-R).

Let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a family of objects of Morph(Mod-R), that is, M_λ is an object $\mu_{M_\lambda} : M_{0,\lambda} \to M_{1,\lambda}$ for every λ in an index set Λ. The coproduct of the family $\{M_\lambda \mid \lambda \in \Lambda\}$ is the object $\bigoplus_{\lambda\in\Lambda} M_\lambda$, where

$$\mu_{\bigoplus_{\lambda\in\Lambda} M_\lambda} : \bigoplus_{\lambda\in\Lambda} M_{0,\lambda} \to \bigoplus_{\lambda\in\Lambda} M_{1,\lambda}$$

is defined componentwise, with the canonical embedding $e_{\lambda_0} : M_{\lambda_0} \to \bigoplus_{\lambda\in\Lambda} M_\lambda$ for every $\lambda_0 \in \Lambda$.

The product of the family $\{M_\lambda \mid \lambda \in \Lambda\}$ is the object $\prod_{\lambda\in\Lambda} M_\lambda$, where

$$\mu_{\prod_{\lambda\in\Lambda} M_\lambda} : \prod_{\lambda\in\Lambda} M_{0,\lambda} \to \prod_{\lambda\in\Lambda} M_{1,\lambda}$$

is defined componentwise, with the canonical embedding $p_{\lambda_0} : M_{\lambda_0} \to \prod_{\lambda \in \Lambda} M_\lambda$ for every $\lambda_0 \in \Lambda$.

Moreover, it is possible to define some canonical functors associated to this category. For any ring R, there are several canonical covariant additive functors

$$\text{Morph}(\text{Mod-}R) \to \text{Mod-}R.$$

In particular, we recall:

1. The domain functor $D\colon \text{Morph}(\text{Mod-}R) \to \text{Mod-}R$, which associates to each object M of Morph(Mod-R) the right R-module M_0 and to any morphism (u_0, u_1) in Morph(Mod-R) the right R-module morphism u_0 in Mod-R.

2. The codomain functor $C\colon \text{Morph}(\text{Mod-}R) \to \text{Mod-}R$, which associates to each object M of Morph(Mod-R) the right R-module M_1 and to any morphism (u_0, u_1) in Morph(Mod-R) the right R-module morphism u_1 in Mod-R.

From these two functors it possible to construct the product functor:

$$D \times C\colon \text{Morph}(\text{Mod-}R) \to \text{Mod-}R \times \text{Mod-}R,$$

which associates to every object M of Morph(Mod-R) the pair (M_0, M_1) belonging to Mod-$R\times$Mod-R and to every morphism (u_0, u_1) in Morph(Mod-R) the morphism (u_0, u_1) in Mod-$R \times$ Mod-R.

1.2 Definition of the Category $\mathcal{C}$

We continue defining the category we want to study and presenting its first properties. Let R and S be rings. Let ${}_RM_S$ be an R-S-bimodule and let $\mathcal{C}$ be the category defined as follows:

1. $\text{Ob}(\mathcal{C}) = {}_RM_S$,

2. $\text{Hom}_{\mathcal{C}}(x, y) = \{(r, s) \in R \times S \mid rx = ys\}$.

Define a *composition* between morphisms:

$$\begin{aligned} \circ\colon \text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) &\to \text{Hom}_{\mathcal{C}}(x, z) \\ ((h, k), (r, s)) &\mapsto (hr, ks). \end{aligned}$$

This composition is *associative* because so is the product into the ring structures. The multiplicative identity of any object $x \in {}_RM_S$ is $(1_R, 1_S)$. Notice that $\mathrm{Hom}_{\mathcal{C}}(x, y) \subseteq R \times S$.

Let $x \in {}_RM_S$. Consider the cyclic submodules yS and Rx of ${}_RM_S$. Define $(yS :_R x) := \{r \in R \mid rx \in yS\}$ and $(Rx :_S y) := \{s \in S \mid ys \in Rx\}$. They are called the *idealizer* of yS and Rx respectively. More precisely $\mathrm{Hom}_{\mathcal{C}}(x, y) \subseteq (yS :_R x) \times (Rx :_S y)$. This is a subgroup of the additive group of $R \times S$. When $x = y$, $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a subring of $R \times S$.

1.3 Preadditivity

Define the operation of addition between two morphisms as the one induced by the ring $R \times S$. So, given $(r, s), (h, k) \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, define:

$$(r, s) + (h, k) = (r + h, s + k).$$

This operation is $\mathbb{Z}$-*bilinear* with respect to the composition because of the distributivity on the left-hand side and on the right-hand side between addition and multiplication in the rings R and S,

$$\begin{aligned}((h, k) + (j, t)) \circ (r, s) &= (h + j, k + t) \circ (r, s) = ((h + j)r, (k + t)s) = \\ &= (hr + jr, ks + ts) = (hr, ks) + (jr, ts) = ((h, k) \circ (r, s)) + ((j, t) \circ (r, s)).\end{aligned}$$

In the same way $\mathbb{Z}$-linearity on the left-hand side can be proved. Hence $\mathcal{C}$ is a preadditive category.

Now let us look for initial and terminal objects in order to eventually identify the *zero object*, if it exists.

Definition 1.8. *Let $\mathcal{A}$ be a category. An object $A \in \mathrm{Ob}(\mathcal{A})$ is called an* initial object *of $\mathcal{A}$ if for every $B \in \mathrm{Ob}(\mathcal{A})$ there exists exactly one morphism $f : A \to B$.*

Our initial object should be an element $x \in {}_RM_S$ such that for every other element $y \in {}_RM_S$ it is possible to find a unique pair $(r, s) \in R \times S$ such that $rx = ys$. Observe that the element 0_M is not an initial object (there is not an *unique* morphism, in fact, $r \cdot 0_M = ys$ holds for every morphism of the form $(r, 0_S)$). For a similar reason every other object is not an initial object.

Let us recall the concepts of *ideal* in a preadditive category $\mathcal{A}$ and *factor category* modulo an ideal.

Definition 1.9 (Ideal). *An* ideal *of a preadditive category $\mathcal{A}$ assigns to every pair A, B of objects of $\mathcal{A}$ a subgroup $\mathcal{I}(A,B)$ of the abelian group $\mathrm{Hom}_{\mathcal{A}}(A,B)$ with the property that for every $\phi : C \to A$, $\psi : A \to B$ and $\omega : B \to D$ with $\psi \in \mathrm{Hom}_{\mathcal{A}}(A,B)$, one has that $\omega\psi\phi \in \mathcal{I}(C,D)$.*

Definition 1.10 (Factor Category). *If $\mathcal{I}$ is an ideal of a preadditive category $\mathcal{A}$, the* factor category *$\mathcal{A}/\mathcal{I}$ has the same objects as $\mathcal{A}$ (i.e., $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}/\mathcal{I})$), the group of morphisms $A \to B$ in $\mathcal{A}/\mathcal{I}$ is $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A,B) := \mathrm{Hom}_{\mathcal{A}}(A,B)/\mathcal{I}(A,B)$, and the composition is that induced by the composition of $\mathcal{A}$.*

Let x and y be objects in $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(x,y) = \{(r,s) \in R \times S \mid rx = ys\}$. Define

$$\mathcal{I}(x,y) := \mathrm{l.ann}_R(x) \times \mathrm{r.ann}_S(y).$$

This is a subgroup of $(yS :_R x) \times (Rx :_S y)$ and is a two-sided ideal when $x = y$.

Given w, x, y, z objects in $\mathcal{C}$, let $\phi = (r,s)$ be a morphism in $\mathcal{I}(x,y)$, $\psi = (r', s')$ be in $\mathrm{Hom}_{\mathcal{C}}(w,x)$ and $\omega = (r'', s'')$ be in $\mathrm{Hom}_{\mathcal{C}}(y,z)$. Consider the composition $\omega \circ \phi \circ \psi = (r''rr', s''ss')$. We have to check that this pair is in $\mathcal{I}(w,z)$, so $r''rr'w = r''rxs' = r''0_M s' = 0_M$ because of the relation $r'w = xs'$ and the fact that $r \in \mathrm{l.ann}_R(x)$. The same holds for $s''ss'$. In fact, $zs''ss' = r''yss' = r''0_M s' = 0_M$ because of the relation $r''y = zs''$ and the fact that $s \in \mathrm{r.ann}_S(y)$.

Hence the position $\mathcal{I}(x,y) := \mathrm{l.ann}_R(x) \times \mathrm{r.ann}_S(y)$ defines an ideal in the preadditive category $\mathcal{C}$. So we can construct the factor category $\mathcal{C}/\mathcal{I}$. From now on set $\mathcal{D} := \mathcal{C}/\mathcal{I}$.

Let us present our category from a different point of view. Let R and S be rings. Let ${}_R M_S$ be an R-S-bimodule and let $\mathcal{E}$ be the category defined as follows:

1. $\mathrm{Ob}(\mathcal{E}) = {}_R M_S$,
2. $\mathrm{Hom}_{\mathcal{E}}(x,y) = Rx \cap yS$.

Consider the functor $F : \mathcal{C} \to \mathcal{E}$ that associates to each object $x \in {}_R M_S$ the element x itself and to each morphism $(r,s) \in \mathrm{Hom}_{\mathcal{C}}(x,y)$ the element $rx (= ys)$.

The kernel of this functor is the ideal $\mathcal{I}$. The functor F induces an isomorphism between the categories $\mathcal{D} = \mathcal{C}/\mathcal{I}$ and $\mathcal{E}$.

Now that we have passed from the category $\mathcal{C}$ to the category $\mathcal{C}/\mathcal{I}$, we have that the zero object in the category $\mathcal{C}/\mathcal{I}$ exists and it is unique. The same holds for the category $\mathcal{E}$. In fact, in $\mathcal{D}$ the *zero object* is now 0_M and the *zero morphism* is the pair $\overline{(0_R, 0_S)}$. Equivalently in $\mathcal{E}$ the *zero object* is 0_M and the *zero morphism* is 0_M because $R0_M \bigcap xS = \{0_M\}$.

Remark 1.11. It is convenient to describe the endomorphism ring of an object $x \in {}_RM_S$. In the category $\mathcal{C}$, we have that the endomorphism ring of x is

$$\mathrm{End}_{\mathcal{C}}(x) = \{\, (r, s) \in R \times S \mid rx = xs \,\}$$

with the operations induced by the ring direct product $R \times S$. In the category $\mathcal{D}$, the endomorphism ring of x is

$$\mathrm{End}_{\mathcal{D}}(x) = \mathrm{End}_{\mathcal{C}}(x)/(\mathrm{l.ann}_R(x) \times \mathrm{r.ann}_S(x)),$$

with the operations induced by those of $\mathrm{End}_{\mathcal{C}}(x)$. In the category $\mathcal{E}$, the endomorphism ring of x is

$$\mathrm{End}_{\mathcal{E}}(x) = Rx \cap xS,$$

with the addition induced by that of ${}_RM_S$ and the multiplication such that if $rx = xs$ and $r'x = xs'$, then their product is $(r'x = xs')(rx = xs) = (r'rx = xs's)$.

1.4 Additivity

The category $\mathcal{D}$ we have defined is just a preadditive category. Recall that every preadditive category $\mathcal{A}$ can be embedded into an additive category as a full subcategory. In fact, it is possible to construct the *free additive category* $\mathrm{Mat}(\mathcal{A})$ as it is explained in [ML98, pag. 198, es. 6].

Definition 1.12. *For any preadditive category $\mathcal{A}$, let* $\mathrm{Mat}(\mathcal{A})$ *be the additive category whose objects are all n-tuples $(X_1, X_2, \dots, X_n)$ of objects X_i of $\mathcal{A}$ for any integer $n \geq 0$, and whose morphisms from an n-tuple $(X_1, X_2, \dots, X_n)$ to an m-tuple $(Y_1, Y_2, \dots, Y_m)$ are all the $m \times n$ matrices (ϕ_{ij}) of morphisms $\phi_{ij} : X_j \to Y_i$ of $\mathcal{A}$*

In our setting the objects of $\mathrm{Mat}(\mathcal{D})$ are of the form $(x_1,\ldots,x_n)$, where $x_i \in {}_RM_S$ and $n > 0$, because $\mathcal{D}$ has a zero object. Thus $\mathrm{Mat}(\mathcal{D}) = \dot{\bigcup}_{n\geq 1} M^n$. The morphisms are matrices of morphisms of $\mathcal{D}$. Let see how to construct them in a precise way and how to work with them.

If $(x_1,\ldots,x_n),(y_1,\ldots,y_m) \in \mathrm{Ob}(\mathrm{Mat}(\mathcal{D}))$, then

$$\mathrm{Hom}_{\mathrm{Mat}(\mathcal{D})}((x_1,\ldots,x_n),(y_1,\ldots,y_m)) = \begin{pmatrix} \mathrm{Hom}_{\mathcal{D}}(x_1,y_1) & \ldots & \mathrm{Hom}_{\mathcal{D}}(x_n,y_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}_{\mathcal{D}}(x_1,y_m) & \ldots & \mathrm{Hom}_{\mathcal{D}}(x_n,y_m) \end{pmatrix}.$$

So it can be expressed in the following form: $\mathrm{Hom}_{\mathrm{Mat}(\mathcal{D})}((x_1,\ldots,x_n),(y_1,\ldots,y_m))$ $= \{((r_{ij}),(s_{ij})) \in M_{m\times n}(R) \times M_{m\times n}(S) \mid r_{ij}x_j = y_i s_{ij},\ \forall i = 1,\ldots,m,\ \forall j = 1,\ldots,n\}$, where $r_{ij} \in \mathrm{Hom}_{\mathcal{D}}(x_j,y_i)$.

Hence an element of $\mathrm{Hom}_{\mathrm{Mat}(\mathcal{D})}((x_1,\ldots,x_n),(y_1,\ldots,y_m))$ is a pair $(A,B) = ((r_{ij}),(s_{ij}))$ of matrices in $M_{m\times n}(R) \times M_{m\times n}(S)$.

Chapter 2

Internal Direct Sum

2.1 Idempotent Endomorphisms

In this section let us recall some basic notions about idempotent endomorphisms in a category.

Definition 2.1 (Splitting Idempotents). *Idempotents* split *in a category $\mathcal{A}$ if, for every object C of $\mathcal{A}$ and every endomorphism $e : C \to C$ in $\mathcal{A}$ with $e^2 = e$, there exist an object A in $\mathcal{A}$ and two morphisms $f : A \to C$ and $g : C \to A$ such that $e = fg$ and $gf = 1_A$*

Let x be an object of $\mathcal{E}$ and (r, s) be an idempotent in $\mathrm{End}_{\mathcal{E}}(x)$, so that $rx = xs$ and $r^2x = rx$. Consider the object $rx = xs$ and the morphism $f = (1, s) : rx \to x$ and $g = (r, 1) : x \to rx$. Then $fg = (r, s) : x \to x$ and $gf = (r, s) : rx \to rx$ is the identity $(1_R, 1_S)$ of rx because $r(rx) = r^2x = rx = 1 \cdot rx$. This proves that:

Proposition 2.2. *Idempotents split in the category $\mathcal{E}$.*

Recall the next result that holds in a preadditive category.

Proposition 2.3. [Fac19, Proposition 4.17] *The following conditions are equivalent for a preadditive category $\mathcal{A}$*

(a) *Idempotents split in $\mathcal{A}$.*

(b) *For every object A in $\mathcal{A}$, every morphism $e : A \to A$ with $e^2 = e$ has a kernel in $\mathcal{A}$.*

Proof. Assume (a) holds. Let $e : A \to A$ be an idempotent in $\mathcal{A}$. Then $1_A - e$ is also an idempotent. By hypothesis $1_A - e$ splits, then there exist $f : B \to A$ and $g : A \to B$ with $fg = 1_A - e$ and $gf = 1_B$. Then f is a kernel of e, because $ef = (1_A - fg)f = f - fgf = f - f1_B = 0$; and if $t : D \to A$ is morphism such that $et = 0$, then $f(gt) = (1_A - e)t = t$. It remains to prove that such a morphism is unique, in fact, if $t' : D \to B$ is another morphism with $ft' = t$, then $t' = gft' = gt$. Thus $f : B \to A$ is a kernel of e.

Now assume (b) holds. Let $e : A \to A$ be an idempotent in $\mathcal{A}$ and $f : B \to A$ be a kernel of the idempotent $1_A - e$. Then $(1_A - e)e = 0$ so there exists a unique morphism $g : A \to B$ with $e = fg$. It remains to show that $gf = 1_B$, then $(1_A - e)f = 0$ implies that $f = ef = fgf$. But kernels are monomorphisms, hence $1_B = gf$. □

Let (r, s) be an idempotent element of $\text{End}_{\mathcal{E}}(x)$. Then, as in any ring, $(1 - r, 1 - s) \in \text{End}_{\mathcal{E}}(x)$ is also an idempotent endomorphism of x. Moreover, $(1 - r, 1 - s) \in \text{End}_{\mathcal{E}}(x)$ splits, so that there exist $f = (1, 1 - s) : (1 - r)x \to x$ and $g = (1 - r, 1) : x \to (1 - r)x$ such that $(r, s) = fg$ and $gf = 1_x$. According to Proposition 2.3 we have that $(1, 1 - s) : (1 - r)x \to x$ is a kernel of $(r, s) : x \to x$

Recall now the following results about modules.

Definition 2.4. *Let M_R a right R-module over a ring R. Define* $\text{add}(M_R)$ *as the full subcategory of* Mod-R *consisting of all modules isomorphic to direct summands of direct sums M^n of finitely many copies of M.*

Let $E := \text{End}_R(M_R)$ denote the endomorphism ring of M_R, for a fixed right R-module M_R. Then ${}_E M_R$ is a bimodule.

Next Theorem is a fundamental result in the study of decompositions.

Theorem 2.5. [Fac19, Theorem 2.35] *The functors*

$$\text{Hom}_R(M, -)\colon \text{Mod-}R \to \text{Mod-}E \qquad \textit{and} \qquad - \otimes_E M\colon \text{Mod-}E \to \text{Mod-}R$$

induce an equivalence between the full subcategory $\text{add}(M_R)$ *of* Mod-R *and the full subcategory* **proj**-R *of* Mod-E.

It is possible to generalize the previous Definition 2.4 and Theorem 2.5 from the category Mod-R to an arbitrary preadditive category, as follows.

Definition 2.6. *Let A be an object of a preadditive category $\mathcal{A}$. Define* $\operatorname{add}(A)$ *as the subclass of* $\operatorname{Ob}(\mathcal{C})$ *consisting of all objects $B \in \operatorname{Ob}(\mathcal{A})$ for which there exist an integer $n > 0$ and morphisms $f_1, f_2, \dots, f_n : A \to B$ and $g_1, g_2, \dots, g_n : B \to A$ in $\mathcal{A}$ with $\sum_{i=1}^n f_i g_i = 1_B$.*

We denote by $\operatorname{add}(A)$ not only the subclass of $\operatorname{Ob}(\mathcal{A})$, but also the full subcategory of $\mathcal{A}$ whose class of objects is $\operatorname{add}(A)$.

Example 2.7. When $\mathcal{A} = \text{Mod-}R$, then $\operatorname{add}(R_R)$ is the class $\mathbf{proj}\text{-}R$ of all finitely generated projective right R-modules.

Proposition 2.8. [Fac19, Lemma 4.18] *Let A be a non-zero object of a preadditive category $\mathcal{A}$. Set $E := \operatorname{End}_{\mathcal{A}}(A)$. Consider the additive functor*

$$F := \operatorname{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \text{Mod-}E.$$

Then the following properties hold:

(a) *The functor F induces a full and faithful functor* $\operatorname{add}(A) \to \mathbf{proj}\text{-}E$.

(b) *If $\mathcal{A}$ is an additive category with splitting idempotents, then F induces an equivalence* $\operatorname{add}(A) \to \mathbf{proj}\text{-}E$.

Proof. Let B be an arbitrary object of $\operatorname{add}(A)$, by definition there exist

$$f_1, \dots, f_n : A \to B \qquad \text{and} \qquad g_1, \dots, g_n : B \to A$$

such that $\sum_{i=1}^n f_i g_i = 1_B$. Apply F to this identity and get that

$$\sum_{i=1}^{n} F(f_i) F(g_i) = 1_{F(B)},$$

where now $F(f_i) : F(A) \to F(B)$ and $F(g_i) : F(B) \to F(A)$ are right E-module morphisms. Thus the module $F(B)$ turns out to be a direct summand of $F(A)^n \cong E_E^n$, hence a finitely generated projective right modules of Mod-E.

Let us prove that the functor F restricted to $\operatorname{add}(A)$ is a faithful functor, let B' be an object of $\operatorname{add}(A)$ and $f : B \to B'$ be a morphism of $\operatorname{add}(A)$ with $F(f) = 0$, that is, $fh = 0$ for every $h \in \operatorname{Hom}_{\mathcal{A}}(A, B)$. Since $1_B = \sum_{i=1}^n f_i g_i$, we have $f = f 1_B = \sum_{i=1}^n (f f_i) g_i = 0$.

In order to prove that the restriction of F is full, let B, B' two objects in $\mathrm{add}(A)$ and let $\psi : \mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{A}}(A,B')$ be right E-module morphism. Define $f : B \to B'$ by setting $f := \sum_{i=1}^{n} \psi(f_i)g_i$. We need to show that $F(f) = \psi$, i.e., that $F(f)(f') = \psi(f')$ for every $f' \in \mathrm{Hom}_{\mathcal{A}}(A,B)$. Now ψ is a right $\mathrm{End}_{\mathcal{A}}(A)$-module morphism, so that $F(f)(f') = ff' = \sum_{i=1}^{n} \psi(f_i)g_i f' = \psi\big(\sum_{i=1}^{n} f_i g_i f'\big) = \psi(f')$.

Now let $\mathcal{A}$ be an additive category with splitting idempotents. Let P be a finitely generated projective right E-module. Then there are morphisms $\alpha_i : P \to E_E$ and $\beta_i : E_E \to P$ with $1_P = \sum_{i=1}^{n} \beta_i\alpha_i$. Thus the endomorphism of E_E^n given by left multiplication by the matrix $(\alpha_i\beta_j)$ is an idempotent endomorphism with image P. Since $\mathcal{A}$ is additive and the restriction of F to $\mathrm{add}(A)$ is full by (a), there is an endomorphism f of A^n in $\mathcal{A}$ such that $F(f) = (\alpha_i\beta_j)$. Again, the fact that the restriction of F to $\mathrm{add}(A)$ is faithful implies that f must be idempotent, so that f splits. Let $g : A^n \to B$ and $h : B \to A^n$ be morphisms in $\mathcal{A}$ with $hg = f$ and $gh = 1_B$. Then, for the right E-module morphism $F(g) : F(A^n) \to F(B)$ and $F(h) : F(B) \to F(A^n)$, one gets that $F(h)F(g) = F(f)$ and $F(g)F(h) = 1_{F(B)}$. Hence $F(g)$ is onto, so that $F(h)$ and $F(f)$ have the same image. Now the image of $F(f) = (\alpha_i\beta_j)$ is the projective module P, and $F(g)F(h) = 1_{F(B)}$ implies that the image of $F(h)$ is isomorphic to $F(B)$. Thus $P \cong F(B)$, as desired. □

2.2 Internal Direct Sum

Let us recall what a biproduct decomposition in a category is.

Definition 2.9 (Biproducts). *Let A_1, A_2 be objects of a preadditive category $\mathcal{A}$. A* biproduct *of A_1 and A_2 is a 5-tuple $(B, \pi_1, \pi_1, \epsilon_1, \epsilon_2)$, where $B \in \mathrm{Ob}(\mathcal{A})$ and $\pi_1 : B \to A_1$, $\pi_2 : B \to A_2$, $\epsilon_1 : A_1 \to B$, $\epsilon_2 : A_2 \to B$ are morphisms such that*

$$\pi_1 \circ \epsilon_1 = 1_{A_1}, \quad \pi_2 \circ \epsilon_2 = 1_{A_2}, \quad \epsilon_1 \circ \pi_1 + \epsilon_2 \circ \pi_2 = 1_B.$$

In short, we will say that B is a biproduct *of A_1 and A_2.*

For right R-modules, there is the following nice interplay between splitting of idempotents, direct-sum decompositions and the categorical definition of biproducts.

Proposition 2.10. *Let P_S be a right S-module. There is a one-to-one correspondence η between the set I of all idempotent elements of* $\mathrm{End}(P_S)$ *and the set*

$$D = \{(A, B) \mid A, B \leq P_S,\ P_S = A \oplus B\}$$

of all the pairs (A, B) of submodules of P_S whose sum is direct and equal to P_S. If $e \in I$, the corresponding pair is $\eta(e) = (e(P_S), \ker e)$. If $(A, B) \in D$, the corresponding idempotent is the epimorphism

$$p_A : A \oplus B \to A$$
$$a + b \mapsto a$$

with $a \in A$ and $b \in B$, called the projection of A along B.

In this section, we will show that the same interplay occurs for our bimodule ${}_RM_R$ and our categories $\mathcal{D} \cong \mathcal{E}$.

In our setting, let x, x_1, x_2 be objects in $\mathcal{E}$. Then x is a biproduct of x_1 and x_2 if and only if there exist morphisms $(p_i, q_i) : x \to x_i$ and $(e_i, f_i) : x_i \to x$, for $i = 1, 2$, such that the following conditions hold:

$$p_i e_i x_i = x_i,\ (e_1 p_1 + e_2 p_2)x = x,\ p_i x = x_i q_i \text{ and } e_i x_i = x f_i$$

for $i = 1, 2$.

We now need a concept for a bimodule that is the analogue of the notion of internal direct sum in Mod-R.

Definition 2.11 (Internal direct sum)**.** *Let ${}_RM_S$ be an R-S-bimodule and $x, x_1, x_2 \in {}_RM_S$. We say that x is the* internal direct sum *of x_1 and x_2, and we will write $x = x_1 \oplus x_2$, if $x = x_1 + x_2$, $Rx = Rx_1 \bigoplus Rx_2$ and $xS = x_1S \bigoplus x_2S$, where the last two are internal direct sums of modules.*

In the next proposition we show that every internal direct-sum decomposition of $x \in {}_RM_S$ determines a biproduct decomposition of x in $\mathcal{E}$

Proposition 2.12. *Let ${}_RM_S$ an R-S-bimodule, x, x_1, $x_2 \in {}_RM_S$ and assume $x = x_1 \oplus x_2$. Then x is the biproduct of x_1 and x_2 in the category $\mathcal{E}$.*

Proof. By hypothesis, we have $x = x_1 + x_2$, $x_1 = r_1 x$, $x_2 = r_2 x$, $Rx_1 \bigcap Rx_2 = 0$, $x_1 = xs_1$, $x_2 = xs_2$, $x_1 S \bigcap x_2 S = 0$.

Define the embeddings and the projections as follows: from $x_i = xs_i$ define $\epsilon_i : x_i \to x$ as $\epsilon_i = (1_R, s_i)$ and from $x_i = r_i x$ define $\pi_i : x \to x_i$ as $\pi_i = (r_i, 1_S)$.

Let us check that $(x, \epsilon_1, \epsilon_2, \pi_1, \pi_2)$ is the biproduct of x_1 and x_2. Start with $\pi_i \circ \epsilon_i : x_i \to x_i$. We have that $\pi_i \circ \epsilon_i = (r_i, s_i)$ and $x = x_1 + x_2$, then $r_1 x = r_1 x_1 + r_1 x_2$, so we obtain $(1 - r_1)x_1 = x_1 - r_1 x_1 = r_1 x - r_1 x_1 = r_1 x_2$; using the fact that $Rx_1 \bigcap Rx_2 = 0$, we conclude that $x_1 = r_1 x_1$ and $r_1 x_2 = 0$, hence $r_1 x_1 = x_1$, that is $\pi_1 \circ \epsilon_1 = 1_{x_1}$. It can be done in the same way for the other.

Now take $\epsilon_1 \circ \pi_1 + \epsilon_2 \circ \pi_2 : x \to x$, we have that $\epsilon_1 \circ \pi_1 + \epsilon_2 \circ \pi_2 = (r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$, we want to show that it is equal to the identity of x, that is $(r_1 + r_2)x = x$. But $(r_1 + r_2)x = (r_1 + r_2)(x_1 + x_2) = r_1 x_1 + r_2 x_2$, where the last equality holds because we previously observed that $r_2 x_1 = 0$ and $r_1 x_2 = 0$, so we conclude that $(r_1 + r_2)x = x_1 + x_2 = x$. □

We show now that every idempotent endomorphism of $\mathcal{E}$ determines a biproduct decomposition.

Proposition 2.13. *Let $(r, s) \in \mathrm{End}_{\mathcal{E}}(x)$ be an idempotent endomorphism in $\mathcal{E}$. Then*

$$(x, (r, 1), (1 - r, 1), (1, s), (1, 1 - s))$$

is a biproduct of rx and $(1 - r)x$ in $\mathcal{E}$.

Proof. Let us check that $\pi_1 \circ \epsilon_1 = 1_{rx}$, $\pi_2 \circ \epsilon_2 = 1_{(1-r)x}$ and $\epsilon_1 \circ \pi_1 + \epsilon_2 \circ \pi_2 = 1_x$.

Let us prove the first one. We have $(r, 1) \circ (1, s) = (r, s) : rx \to rx$, with $r^2 x = rx = 1 \cdot rx$ because (r, s) is an idempotent endomorphism.

For the second one we have $(1 - r, 1) \circ (1, 1 - s) = (1 - r, 1 - s) : (1 - r)x \to (1 - r)x$, that is $(1 - r)^2 x = (1 - r)x = 1 \cdot (1 - r)x$.

About the last one we get $(1, s) \circ (r, 1) + (1, 1 - s) \circ (1 - r, 1) = (r, s) + (1 - r, 1 - s) = (1_R, 1_S)$. Hence it is a biproduct of rx and $(1 - r)x$ in $\mathcal{E}$. □

Clearly, given any biproduct

$$(B, \pi_1, \pi_1, \epsilon_1, \epsilon_2)$$

of A_1 and A_2 in a preadditive category $\mathcal{A}$, from the equality $\pi_1 \circ \epsilon_1 = 1_{A_1}$, we can always associate to the biproduct an idempotent endomorphism $\epsilon_1 \circ \pi_1$ of B. In the particular case of our category $\mathcal{E}$, we have:

Proposition 2.14. *Let $(x, (p_1, q_1), (p_2, q_2), (r_1, s_1), (r_2, s_2))$ be a biproduct of y_1 and y_2 in $\mathcal{E}$. Then $(r_i p_i, s_i q_i) : x \to x$ is an idempotent endomorphism of x in $\mathcal{E}$ for each $i = 1, 2$.*

Proof. Recall that $(p_i, q_i) : x \to y_i$ are morphism such that $p_i x = y_i q_i$ and $(r_i, s_i) : y_i \to x$ are morphism such that $r_i y_i = x s_i$, with $i = 1, 2$. If we consider $(r_i, s_i) \circ (p_i, q_i) : x \to x$, then we have that $r_i p_i \cdot r_i p_i x = r_i \cdot p_i r_i y_i \cdot q_i = r_i \cdot y_i q_i = r_i p_i x$. This conclude the proof. □

Therefore, starting from a biproduct x of two elements y_1, y_2 in $\mathcal{E}$, we can associate to it an idempotent endomorphism in $\mathcal{E}$ as in Proposition 2.14, and then we can come back to a biproduct of two different elements x_1, x_2, which is an internal sum, as in Proposition 2.13. A natural question is to determine the relation between y_1, y_2 on the one hand, and x_1, x_2 on the other hand. Our next step is to prove that x_1, x_2 are isomorphic to y_1, y_2 respectively.

Starting from the biproduct $(x, (p_1, q_1), (p_2, q_2), (r_1, s_1), (r_2, s_2))$ of y_1 and y_2 in $\mathcal{E}$ we pass to an idempotent endomorphism $(r_i p_i, s_i q_i) : x \to x$ and from this to the biproduct $(x, (r_1 p_1, 1), (1 - r_1 p_1, 1), (1, s_1 q_1), (1, 1 - s_1 q_1))$ of $r_1 p_1 x$ and $(1 - r_1 p_1) x$ in $\mathcal{E}$. Set $x_1 \coloneqq r_1 p_1 x$ and $x_2 \coloneqq (1 - r_1 p_1) x$.

Proposition 2.15. *In the previous construction $y_i \cong x_i$.*

Proof. Let us construct two morphisms, one from y_i to x_i and one from x_i to y_i such that they are one the inverse of the other.

Let $(r_1, s_1) \circ (r_1 p_1, 1) = (r_1 p_1 r_1, s_1) : y_1 \to x_1$ and $(p_1, q_1) \circ (1, s_1 q_1) = (p_1, q_1 s_1 q_1) : x_1 \to y_1$ be the two morphisms. Let us show that are one the inverse of the other: $(r_1 p_1 r_1, s_1) \circ (p_1, q_1 s_1 q_1) : x_1 \to x_1$ and we have $r_1 p_1 r_1 p_1 x_1 = r_1 p_1 r_1 p_1 r_1 x = r_1 p_1 x = x_1$; $(p_1, q_1 s_1 q_1) \circ (r_1 p_1 r_1, s_1) : y_1 \to y_1$ and we have $p_1 r_1 p_1 r_1 y_1 = p_1 r_1 p_1 x s_1 = p_1 r_1 y_1 q_1 s_1 = y_1$. Then $y_1 \cong x_1$. In the same way it can be done for y_2. □

From the last Propositions we can deduce that:

Proposition 2.16. *Let ${}_RM_S$ an R-S-bimodule, $x, y_1, y_2 \in_R M_S$, consider the biproduct*

$$(x, \epsilon_1, \epsilon_2, \pi_1, \pi_2)$$

of y_1 and y_2 in the category $\mathcal{E}$. Then $x = (\epsilon_1 \circ \pi_1)x \oplus (\epsilon_2 \circ \pi_2)x$ and, in the category $\mathcal{E}$, $(\epsilon_i \circ \pi_i)x \cong y_i$.

We can observe that

$$x_2 = (1 - r_1p_1)x = x - r_1p_1x = r_1p_1x + r_2p_2x - r_1p_1x = r_2p_2x$$

so we conclude that

$$(1 - r_1p_1, 1) = (r_2p_2, 1) : x \to x_2.$$

Similarly for $(1, 1 - s_1q_1) = (1, s_2q_2)$.

Hence, from Proposition 2.13, Proposition 2.14 and the last observation, we can conclude the following.

Theorem 2.17. *Fix an element $x \in {}_RM_S$. Then there is a one to one correspondence between*

$$\{(r, s) \in \mathrm{End}_{\mathcal{E}}(x) \mid (r, s) \text{ is idempotent}\}$$

and

$$\{(x, (p_1, q_1), (p_2, q_2), (r_1, s_1), (r_2, s_2)) \mid q_1 = q_2 = 1_S \text{ and } r_1 = r_2 = 1_R\}.$$

Now we want to show that there is also a one to one correspondence between the finite decompositions in $\mathcal{E}$ of an element x and all the complete finite families of orthogonal idempotents of $\mathrm{End}_{\mathcal{D}}(x)$.

Proposition 2.18. *Let $\{e_1, \dots, e_n\}$ be a complete family of orthogonal idempotent endomorphisms in $\mathcal{E}$, where $e_i = (r_i, s_i) : x \to x$. Then there is an inner decomposition $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n$ of x in $\mathcal{E}$.*

Proof. By hypothesis, we know that $r_ix = xs_i$ and $r_i^2 = r_i$ because each $e_i = (r_i, s_i) : x \to x$ is an idempotent endomorphism. Furthermore from the completeness we get that $\sum_{i=1}^n r_i = 1_R$ and $\sum_{i=1}^n s_i = 1_S$ while frome orthogonality $r_ir_j = 0$ and $s_is_j = 0$ for every $i, j = 1, \dots n$ such that $i \neq j$.

Define $x_i := r_i x$. We need to check that $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n$. First observe that

$$r_1 x + r_2 x + \cdots + r_n x = \Big(\sum_{i=1}^{n} r_i\Big) x = 1_R \cdot x = x.$$

Now we want that $Rx = Rx_1 \bigoplus Rx_2 \bigoplus \cdots \bigoplus Rx_n$ as modules, i.e.,

$$Rx = Rr_1 x \bigoplus Rr_2 x \bigoplus \cdots \bigoplus Rr_n x.$$

It can be trivially seen that $Rr_1x + Rr_2x + \cdots + Rr_nx \subseteq Rx$, for the other inclusion we have that $rx = r \cdot 1_R \cdot x = r(r_1 + r_2 + \cdots + r_n)x = rr_1x + rr_2x + \cdots + rr_nx$, hence $Rx \subseteq Rr_1x + Rr_2x + \cdots + Rr_nx$. It remains to check that the sum is direct. Let $y \in (\sum_{i \neq j} Rr_i x) \cap Rr_j x$, then there exist $h_1, h_2, \ldots, h_n$ such that $y = h_1 r_1 x + h_2 r_2 x + \cdots + \hat{h_j} r_j x = h_j r_j x = h_j r_j^2 x = h_j r_j r_j x = h_j r_j x s_j$, where $\hat{a} = 0$ for every a, hence $h_j r_j x s_j = (h_1 r_1 + \cdots + \hat{h_j} r_j + \ldots h_n r_n) x s_j = (h_1 r_1 + \cdots + \hat{h_j} r_j + \ldots h_n r_n) r_j x = 0.$ □

Proposition 2.19. *Let x be an object of $\mathcal{E}$. If $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is an inner decomposition of x in $\mathcal{E}$, then there exists a complete and orthogonal family $\{(r_1, s_1), \ldots, (r_n, s_n)\}$ of idempotent endmorphisms of x with $x_i = r_i x$ for every $i = 1, .., n$.*

Proof. By hypothesis, from the definition of internal product in $\mathcal{E}$ we have that

$$x = \sum_{i=1}^{n} x_i, \qquad Rx = \bigoplus_{i=1}^{n} Rx_i \qquad \text{and} \qquad xS = \bigoplus_{i=1}^{n} x_i S.$$

From the direct sum decomposition of Rx and xS for every $i = 1, \ldots, n$ there exists a unique (r_i, s_i) such that $x_i = r_i x = x s_i$. Fix as family of endomorphisms

$$E_n = \{(r_1, s_1), (r_2, s_2), \ldots, (r_n, s_n)\}.$$

First let us show that each element is idempotent. Take

$$\begin{aligned} r_i^2 x - r_i x = r_i(r_i x) - r_i x &= r_i x_i - x_i = \\ &= r_i\big(x - \sum_{k \neq i} x_k\big) - x_i = \\ &= r_i x - \big(\sum_{k \neq i} r_i x_k\big) - x_i = \\ &= -\sum_{k \neq i} r_i x_k. \end{aligned}$$

But we also have that $r_i^2 x - r_i x = (r_i - 1_R)x_i$. Then, because $(\sum_{j \neq i} Rx_j) \cap Rx_i = 0$, we obtain $r_i^2 x = r_i x$.
From $x = x_1 + x_2 + \cdots + x_n = r_1 x + r_2 x + \cdots + r_n x = (\sum_{i=1}^n r_i)x$ we deduce that the family is complete.
It remains to prove that the family is orthogonal. For every $i \neq j$ we have

$$r_i r_j x = r_i x_j = r_i(x - \sum_{k \neq j} x_k) = - \sum_{k \neq i,j} r_i x_k.$$

Similarly, $r_i r_j x = 0$. □

Let us conclude this section using the Definition 2.11 of internal direct sum and Proposition 2.8 in order to have an equivalence between categories.

Definition 2.20. *Let ${}_R M_S$ be an R-S-bimodule. Define $\mathcal{A}_x := \{ y \in {}_R M_S \mid$ there exists $z \in {}_R M_S$ such that $x = y \oplus z \}$ and define $\mathcal{P}_x$ as the class of all cyclic projective right modules over the ring* $\mathrm{End}_{\mathcal{D}}(x)$.

Proposition 2.21. *The functor* $\mathrm{Hom}_{\mathcal{D}}(x, -) \colon \mathcal{D} \to \text{Mod-}\mathrm{End}_{\mathcal{D}}(x)$ *induces an equivalence between the full subcategory of $\mathcal{D}$ with class of objects $\mathcal{A}_x$ and the full subcategory of* $\text{Mod-}\mathrm{End}_{\mathcal{D}}(x)$ *with class of objects $\mathcal{P}_x$. Hence in $\mathcal{D}$ there is a one-to-one correspondence between internal direct summands of x up to isomorphisms and projective cyclic modules over the ring* $\mathrm{End}_{\mathcal{D}}(x)$.

2.3 Examples

Example 2.22. Consider the $\mathbb{Z}$-$\mathbb{Z}$-bimodule ${}_{\mathbb{Z}}\mathbb{Z}_{\mathbb{Z}}$, take $z \in \mathbb{Z}$. The endomorphism ring $\mathrm{End}_{\mathcal{E}}(z) = \{\overline{(a,b)} \in \mathbb{Z} \times \mathbb{Z} \mid az = zb\}$ is equal to 0 if $z = 0$ and equal to $\mathbb{Z}$ if $z \neq 0$. Hence in this case we can conclude that the idempotents are only the trivial ones, i.e., 0 and 1. Hence, all non-zero objects are indecomposable.

Example 2.23. Let R be a ring, with its natural R-R-bimodule structure ${}_R R_R$. It is clear that, for any idempotent element $e \in R$, $1 = e \oplus (1-e)$ is an internal direct-sum decomposition of the identity 1 of R. Conversely, if $1 = x_1 \oplus x_2$ is an internal direct sum, then ${}_R R = Rx_1 \bigoplus Rx_2$, so that there exists an idempotent $e \in R$ such that $Rx_1 = Re$ and $Rx_2 = R(1-e)$. The unique way of writing 1 has a sum of an element of Re and an element of $R(1-e)$ is $1 = e + (1-e)$. Therefore $x_1 = e$ and $x_2 = 1-e$. This shows that the internal direct-sum decompositions of 1 in ${}_RR_R$ are exactly those of the form $1 = e \oplus (1-e)$ for some idempotent $e \in R$. More generally [AF92, Corollary 6.20], the internal direct-sum decompositions $1 = x_1 \oplus \cdots \oplus x_n$ correspond exactly to the complete n-tuples of pairwise orthogonal idempotents $(e_1, \ldots, e_n)$.

Example 2.24. In [FHLV95, Example 1.6] it was shown that, for every integer $n \geq 2$, there exists an artinian module A_T over a suitable ring T which is a direct sum of t indecomposable submodules for every $t = 2, 3, \ldots, n$. In the ring $R := \mathrm{End}(A_T)$, there are therefore complete sets of pairwise orthogonal primitive idempotents of cardinality t for every $t = 2, 3, \ldots, n$. By Example 2.23, the identity of R is therefore an internal direct sum of t indecomposable elements of the bimodule ${}_RR_R$ for every $t = 2, 3, \ldots, n$.

Example 2.25. If R is a commutative ring, every R-module M_R is an R-R-bimodule. Let x be an element of M_R. Then there is a one-to-one correspondence between the set $\{(S_1, \ldots, S_n) \mid S_i$ is an R-submodule of xR for every $i = 1, 2, \ldots, n$ and $xR = S_1 \oplus \cdots \oplus S_n\}$ and the set $\{(x_1, \ldots, x_n) \mid x = x_1 \oplus \cdots \oplus x_n$ is an internal decomposition of $x \in {}_RM_R\}$. The proof of this is similar to the proof given in Example 2.23.

Example 2.26. Let us apply what we have seen in Example 2.25 to the particular case of a vector space V_k over a field k, so that our bimodule ${}_RM_S$ is now the k-k-bimodule ${}_kV_k$. It is easy to see that in this category $\mathcal{D}$, the indecomposable objects are the $x \in {}_kV_k$ with $x \neq 0$. The morphisms $x \to y$ in $\mathcal{D}$ are the morphisms $\overline{(r,s)}\colon x \to y$, so that $\mathrm{Hom}_{\mathcal{D}}(x,y) \cong k$ if x and y generate the same one-dimensional vector space, and $\mathrm{Hom}_{\mathcal{D}}(x,y) = 0$ otherwise. Two elements x, y of V_k are isomorphic if and only if they generate the same vector space (either one-dimensional or zero-dimensional).

If $(x_1, \ldots, x_n)$ is an object of $\mathrm{Mat}(\mathcal{D})$ with $x_i \neq 0$ for every $i = 1, 2, \ldots, n$, then $\mathrm{End}_{\mathcal{D}}(x_1, \ldots, x_n)$ is the ring $\mathbb{M}_{n\times n}(k)$ of $n \times n$ matrices. It is easy to see that idempotents split in $\mathrm{Mat}(\mathcal{D})$. It can be easily seen that the only indecomposable biproduct decompositions of such an $(x_1, \ldots, x_n)$ in $\mathrm{Mat}(\mathcal{D})$ is $x_1 \coprod \cdots \coprod x_n$. (To see this, notice that every object $(y_1, \ldots, y_m)$ of $\mathrm{Mat}(\mathcal{D})$ has a degree, the number of its non-zero elements y_j. The degree of a biproduct is the sum of the degrees of the summands. The objects of $\mathrm{Mat}(\mathcal{D})$ indecomposable as a biproduct are those of degree one.)

Example 2.27. Let us apply what we have seen in Example 2.25 to the particular case of the ring $\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$. This ring has a finitely generated indecomposable projective module P_R such that $R_R^3 \cong R_R \oplus P_R$. The bimodule ${}_RM_S$ is always the R-R-bimodule ${}_RR_R$. Since R is an integral domain, all non-zero elements $x \in {}_RR_R$ have an endomorphism ring $\mathrm{End}_{\mathcal{D}}(x)$ that is a subring of R, hence has no non-trivial idempotent, so that all non-zero elements x of ${}_RR_R$ are indecomposable. The object $(1_R, 1_R, 1_R)$ of $\mathrm{Mat}(\mathcal{D})$ has endomorphism ring $\mathrm{End}_{\mathrm{Mat}(\mathcal{D})}(1_R, 1_R, 1_R) \cong \mathbb{M}_{3\times 3}(R)$. Because of the indecomposable decomposition $R_R^3 \cong R_R \oplus P_R$, there is an idempotent endomorphism $e \in \mathrm{End}_{\mathrm{Mat}(\mathcal{D})}(1_R, 1_R, 1_R)$ that doesn't have a kernel in $\mathcal{D}$. Therefore $\mathcal{D}$ does not have splitting idempotents, and the decomposition $R_R^3 \cong R_R \oplus P_R$ in $\mathbf{proj} - \mathrm{End}_{\mathcal{D}}(1_R)$ does not lift to biproduct decomposition of $(1_R, 1_R, 1_R)$ in $\mathrm{Mat}(\mathcal{D})$.

Example 2.28. If R is a commutative noetherian integral domain of Krull dimension 1 (for instance, a Dedekind domain), then we have uniqueness of decomposition as an internal direct-sum of any element $x \in R$ into indecomposables up to isomorphism, because if x is a torsion-free element, then $xR \cong R_R$ is indecomposable as an R-module because its endomorphism ring is isomorphic to R, hence is a domain, hence has no non-trivial idempotents. If x is a torsion element, then xR has its endomorphism ring isomorphic to $R/\mathrm{ann}(x)$, which is an artinian commutative ring, hence is a finite direct product of local artinian rings. Hence by the Krull-Schmidt-Azumaya Theorem xR is a direct-sum of indecomposables up to isomorphism in a unique way.

Example 2.29. Let G be any abelian group, so that it is naturally a $\mathbb{Z}$-$\mathbb{Z}$-bimodule ${}_{\mathbb{Z}}G_{\mathbb{Z}}$. Fix an element $x \in G$. There are three cases. If $x = 0$, then the unique internal decomposition of x is the trivial decomposition $x = 0$. If $x \neq 0$ and x is not a torsion element, then the unique internal decomposition is the trivial decomposition $x = x$, that is, x is internally indecomposable, because its endomorphism ring $\mathrm{End}_{\mathcal{D}}(x)$ is isomorphic to the integral domain $\mathbb{Z}$, hence x has no nontrivial idempotent endomorphisms. The third case is for $x \neq 0$, x torsion. Let n be the order of the torsion element x of G. Decompose n as a product of powers of distinct primes: $n = p_1^{n_1} \dots p_t^{n_m}$, with the p_i distinct primes. Then the unique internal decomposition of x into indecomposable elements is $x = \bigoplus_{i=1}^{t} p_1^{n_1} \dots \widehat{p_i^{n_i}} \dots p_t^{n_m} x$. It corresponds to the direct sum decomposition $t(G) = \bigoplus_p t_p(G)$ of the torsion part $t(G)$ of G into its p-torsion parts components $t_p(G)$.

2.4 Isomorphic elements, isomorphic internal direct sums

The study of block decompositions of matrices is one of the classical themes in Linear Algebra. We refer to the description of matrices up to the matrix equivalence $\sim$ defined, for any two rectangular $m \times n$ matrices A and B, by $A \sim B$ if $B = Q^{-1}AP$ for some invertible $n \times n$ matrix P and some invertible $m \times m$ matrix Q. The equivalence relation $\sim$ on the set of $m \times n$ matrices corresponds to the isomorphism relation in the category Morph(Mod-R). See [CEDF19]. More generally, this also applies to our category $\mathcal{C}$:

Proposition 2.30. *Two objects x, y of $\mathcal{C}$ are isomorphic in $\mathcal{C}$ if and only if there exist an element $r \in R$ invertible in R and an element $s \in S$ invertible in S such that $rx = ys$.*

Proof. Assume that there exist two elements $r \in R$, invertible in R, and $s \in S$, invertible in S, such that $rx = ys$. Then $(r, s) : x \to y$ is a morphism in $\mathcal{C}$. Let r^{-1}, s^{-1} be the inverses of r, s, respectively. Multiplying the equality $rx = ys$ by r^{-1} on the left and s^{-1} on the right, we get that $r^{-1}y = xs^{-1}$. Hence $(r^{-1}, s^{-1}) : y \to x$ is a morphism in $\mathcal{C}$, which is clearly the inverse of $(r, s) : x \to y$. □

In the categories $\mathcal{D}$ and $\mathcal{E}$ the situation is a little more complicate. In fact we have that:

Proposition 2.31. *Two objects x, y of $\mathcal{D}$ are isomorphic in $\mathcal{D}$ (equivalently, in $\mathcal{E}$) if and only if there exist $r \in (yS :_R x)$ and $r' \in (xS :_R y)$ such that $r'r - 1 \in \mathrm{l.ann}_R(x)$ and $rr' - 1 \in \mathrm{l.ann}_R(y)$, if and only if there exist $s \in (Rx :_S y)$ and $s' \in (Ry :_S x)$ such that $ss' - 1 \in \mathrm{r.ann}_R(y)$ and $s's - 1 \in \mathrm{r.ann}_R(x)$.*

In view of the usual definition of unique factorization domain in Commutative Algebra, it is natural to consider the following definition:

Definition 2.32. *Two internal decompositions*

$$y_1 \oplus \cdots \oplus y_m = x = x_1 \oplus \cdots \oplus x_n$$

of an element $x \in {}_RM_S$ are equal *if $n = m$ and there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $x_i = y_{\sigma(i)}$ for every $i = 1, 2, \dots, n$.*

On the other hand recall that two direct-sum decompositions

$$N_1 \oplus \cdots \oplus N_n = M_R = M_1 \oplus \cdots \oplus M_m$$

of a right R-module M_R are said to be *isomorphic* if $n = m$ and there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \dots, n$. Thus let us consider the following definition:

Definition 2.33. *Two internal decompositions*

$$y_1 \oplus \cdots \oplus y_m = x = x_1 \oplus \cdots \oplus x_n$$

of an element $x \in {}_RM_S$ are isomorphic *if $n = m$ and there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $x_i \cong y_{\sigma(i)}$ in $\mathcal{D}$ for every $i = 1, 2, \dots, n$.*

Notice that, in an internal decomposition $x = x_1 \oplus \cdots \oplus x_n$, one has that if $x_i \neq 0$, then $x_i \notin Rx_j \cup x_jS$ for every index $j \neq i$.

Recall now the following useful result:

Lemma 2.34. ([AF92, Exercise 7.2(c)] and [Coh03, p. 144]) *Let R be any ring and e, f idempotent elements of R. Then:*

(a) *$Re = Rf$ if and only if $f = e + (1 - e)xe$ for some $x \in R$.*

(b) *$Re \cong Rf$ if and only if $eR \cong fR$, if and only if there exist $x \in eRf$ and $y \in fRe$ with $xy = e$ and $yx = f$. In this case, e and f are said to be* isomorphic *idempotents.*

(c) *e is isomorphic to f and $1-e$ is isomorphic to $1-f$ if and only if there exists an invertible element $u \in R$ such that $f = u^{-1}eu$. In this case, e and f are said to be* conjugate *idempotents.*

Remark 2.35. Furthermore conjugate idempotents are isomorphic, in fact, if $f = u^{-1}eu$, take $a = eu$ and $b = u^{-1}e$, so that $a = eu = eeu = euf \in eRf$ and $b = u^{-1}e = u^{-1}ee = fu^{-1}e \in fRe$ with $ab = e$ and $ba = f$.

Hence, as far as idempotents are concerned, there are three equivalence relations on the set of all idempotent elements of a ring R that are noteworthy to our aims: being equal, being isomorphic and being conjugate.

Lemma 2.36. *Let $x = x_1 \oplus \cdots \oplus x_n = y_1 \oplus \cdots \oplus y_m$ be two internal decompositions of an element $x \in {}_RM_S$. Fix an index $i = 1, 2, \ldots, n$ and an index $j = 1, 2, \ldots, m$. Suppose $x_i = r_i x = x s_i$ and $y_j = r'_j x = x s'_j$. Then the following conditions are equivalent:*

(a) $x_i = y_j$.

(b) $Rx_i = Ry_j$ *and* $\bigoplus_{k \neq i} Rx_k = \bigoplus_{\ell \neq j} Ry_\ell$.

(c) $Rx_i = Ry_j$ *and* $R(x - x_i) = R(x - y_j)$.

(d) $x_i S = y_j S$ *and* $\bigoplus_{k \neq i} x_k S = \bigoplus_{\ell \neq j} y_\ell S$.

(e) $x_i S = y_j S$ *and* $(x - x_i)S = (y - y_j)S$.

(f) *The two idempotent endomorphisms* $e_i := \overline{(r_i, s_i)}$ *and* $e'_j := \overline{(r'_j, s'_j)} \in \mathrm{End}_{\mathcal{D}}(x)$ *coincide.*

(g) *The two idempotent endomorphisms* $\lambda_{r_i}, \lambda_{r'_j} \in \mathrm{End}_S(xS)$ *coincide.*

(h) *The two idempotent endomorphisms* $\rho_{s_i}, \rho_{s'_j} \in \mathrm{End}_R(Rx)$ *coincide.*

Proof. First of all, let us show that $\bigoplus_{k \neq i} Rx_k = R(x - x_i)$. The inclusion $\bigoplus_{k \neq i} Rx_k \supseteq R(x - x_i)$ is trivial, because $x = x_1 + \cdots + x_n$. For the inclusion $\bigoplus_{k \neq i} Rx_k \subseteq R(x - x_i)$, notice that, from $R(x - x_i) \subseteq \bigoplus_{k \neq i} Rx_k$, the sum $Rx_i + R(x - x_i)$ is direct. It follows that $Rx \subseteq Rx_i \oplus R(x - x_i) \subseteq Rx_i \oplus (\bigoplus_{k \neq i} Rx_k) = Rx$. Therefore $R(x - x_i) = \bigoplus_{k \neq i} Rx_k$, as desired. The proof for the direct summands $y_k S$ is similar.

(a) $\Rightarrow$ (b) now follows trivially from the remark in the previous paragraph.

(b) $\Rightarrow$ (a) The only way to write x as a sum of an element of Rx_i and an element of $\bigoplus_{k\neq i} Rx_k$ is $x = x_i + (x - x_i)$. Similarly, the only way to write x as a sum of an element of Ry_j and an element of $\bigoplus_{\ell\neq j} Ry_\ell$ is $x = y_j + (x - y_j)$. From condition (b), it follows that $x_i = y_j$.

(b) $\Leftrightarrow$ (c) follows trivially from the remark in the first paragraph of this proof.

The proof that (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) is similar.

(a) $\Rightarrow$ (f) If $x_i = y_j$, then $r_i x = r'_j x$, i.e., we have that the two morphisms $r_i x = r'_j x \colon x \to x$ coincide in the category $\mathcal{E}$, so that the two morphisms

$$e_i := \overline{(r_i, s_i)} \colon x \to x \qquad \text{and} \qquad e'_j := \overline{(r'_j, s'_j)} \colon x \to x,$$

in the equivalent category $\mathcal{D}$, coincide.

(f) $\Rightarrow$ (a) The fact that the endomorphisms $e_i := \overline{(r_i, s_i)}$ and $e'_j := \overline{(r'_j, s'_j)} \in \mathrm{End}_{\mathcal{D}}(x)$ coincide in $\mathcal{D}$ means that $r_i - r'_j \in \mathrm{l.ann}(x)$, that is, $r_i x - r'_j x = 0$. It follows that $x_i = r_i x = r'_j x = y_j$.

(f) $\Leftrightarrow$ (g) follows immediately from the existence of the faithful functor $F_S \colon \mathcal{D} \to$ Mod-S, which maps the endomorphisms $e_i := \overline{(r_i, s_i)}$ and $e'_j := \overline{(r'_j, s'_j)}$ of x in $\mathcal{D}$ to the two endomorphisms $\lambda_{r_i}, \lambda_{r'_j}$ of xS, respectively.

The proof of (f) $\Leftrightarrow$ (h) is similar. □

It follows that:

Proposition 2.37. *Two internal decompositions*

$$x = x_1 \oplus \cdots \oplus x_n \qquad \text{and} \qquad x = y_1 \oplus \cdots \oplus y_m$$

of an element $x \in {}_R M_S$ are equal if and only if $n = m$ and there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that any of the following equivalent conditions is satisfied:

(a) *$Rx_i = Ry_{\sigma(i)}$ for every $i = 1, 2, \ldots, n$.*

(b) *$x_i S = y_{\sigma(i)} S$ for every $i = 1, 2, \ldots, n$.*

(c) *The corresponding complete sets of pairwise orthogonal idempotents in $\mathcal{D}$ coincide: $\{e_1, \ldots, e_n\} = \{e'_1, \ldots, e'_n\}$.*

(d) *The corresponding complete sets of pairwise orthogonal idempotents of $\mathrm{End}_S(xS)$ coincide: $\{\lambda_{r_1}, \ldots, \lambda_{r_n}\} = \{\lambda_{r'_1}, \ldots, \lambda_{r'_n}\}$.*

(e) *The corresponding complete sets of pairwise orthogonal idempotents of* $\mathrm{End}_R(Rx)$ *coincide:* $\{\rho_{s_1},\dots,\rho_{s_n}\} = \{\rho_{s'_1},\dots,\rho_{s'_n}\}$.

Here $e_i := \overline{(r_i, s_i)}, e'_j := \overline{(r'_j, s'_j)}$, *where* $x_i = r_i x = x s_i$ *and* $y_i = r'_i x = x s'_i$.

Proposition 2.38. *Let* $x = x_1 \oplus\dots\oplus x_n = y_1\oplus\dots\oplus y_m$ *be two internal decompositions of an element* $x \in {}_RM_S$. *Fix an index* $i = 1,2,\dots,n$ *and an index* $j = 1,2,\dots,m$. *Suppose* $x_i = r_i x = x s_i$ *and* $y_j = r'_j x = x s'_j$. *Then the following conditions are equivalent:*

(a) x_i *and* y_j *are isomorphic objects in* $\mathcal{D}$.

(b) *The two idempotent endomorphisms* $e_i := \overline{(r_i, s_i)}, e'_j := \overline{(r'_j, s'_j)}$ *are isomorphic idempotents of the ring* $\mathrm{End}_{\mathcal{D}}(x)$.

Proof. Set $E := \mathrm{End}_{\mathcal{D}}(x)$. Because of the category equivalence induced by the additive functor $\mathrm{Hom}_{\mathcal{D}}(x,-)\colon \mathcal{A}_x \to \mathcal{P}_x$, see Proposition 2.21, two objects x_i and y_j of $\mathcal{A}_x$ are isomorphic in $\mathcal{D}$ if and only if the corresponding right ideals Ee_i, Ee'_j are isomorphic in Mod-E, that is, if and only if e_i, e'_j are isomorphic idempotents. □

Proposition 2.39. *Let* $x = x_1 \oplus\dots\oplus x_n = y_1\oplus\dots\oplus y_m$ *be two internal decompositions of an element* $x \in {}_RM_S$. *Suppose* $x_i = r_i x = x s_i$ *and* $y_j = r'_j x = x s'_j$ *for all indices* i *and* j, *so that the corresponding idempotents are* $e_i = \overline{(r_i, s_i)}$ *and* $e'_j = \overline{(r'_j, s'_j)}$. *The following conditions are equivalent:*

(a) *The two internal decompositions of* x *are isomorphic.*

(b) *The two complete sets* $\{e_1,\dots,e_n\}$, $\{e'_1,\dots,e'_n\}$ *of pairwise orthogonal idempotents of* $\mathrm{End}_{\mathcal{D}}(x)$ *are conjugate, that is,* $n = m$ *and there exist a permutation* σ *of* $\{1,2,\dots,n\}$ *and an invertible element* α *of* $\mathrm{End}_{\mathcal{D}}(x)$ *such that* $\alpha e_i \alpha^{-1} = e'_{\sigma(i)}$ *for every* $i = 1,2,\dots,n$.

(c) $n = m$ *and there exist an automorphism* $\overline{(r,s)}\colon x \to x$ *in* $\mathcal{D}$ *and a permutation* σ *of* $\{1,2,\dots,n\}$ *such that* $r x_i = y_{\sigma(i)} s$ *for every* $i = 1,2,\dots,n$.

Proof. (a) ⇒ (c) Suppose that (a) holds, so that $x = x_1 \oplus\dots\oplus x_n = y_1\oplus\dots\oplus y_n$ and that there exists a permutation σ of $\{1,2,\dots,n\}$ with $x_i \cong y_{\sigma(i)}$ for every $i = 1,2,\dots,n$. Set $E := \mathrm{End}_{\mathcal{D}}(x)$. Applying $\mathrm{Hom}_{\mathcal{D}}(x,-)\colon \mathcal{D} \to$ Mod-E, we get two direct-sum decompositions $E_E = e_1E \oplus e_nE = e'_1E \oplus e'_nE$ with $e_iE \cong e'_{\sigma(i)}E$ for every $i = 1, 2, \dots, n$. It follows that there is an automorphism α of E_E such that $\alpha(e_iE) = e'_{\sigma(i)}E$ for every $i = 1, 2, \dots, n$. Automorphisms of E_E are left multiplication by an invertible element of E, so that there exists an invertible element $u \in E$ such that $ue_iE = e'_{\sigma(i)}E$ for every i. Since we have a direct sum of right ideals, for the identity 1 of E we have that $1 = e_1 + \dots + e_n$, so $1 = u1u^{-1} = ue_1u^{-1} + \dots + ue_nu^{-1}$. Therefore, from $ue_iE = e'_{\sigma(i)}E$, we get that $ue_nu^{-1} = e'_{\sigma(i)}$ for every i. Thus $ue_n = e'_{\sigma(i)}u$ for every $i = 1, 2, \dots, n$. Now u is of the form $\overline{(r,s)}$, and if $x_i = r_ix = xs_i$, then $e_i = \overline{(r_i, s_i)}$. Similarly for the elements y_i. Hence $\overline{(r,s)(r_i,s_i)} = \overline{(r'_{\sigma(i)}, s'_{\sigma(i)})(r,s)}$ for every i, so that $rr_ix = xs'_{\sigma(i)}s$. Hence $rx_i = rr_ix = xs'_{\sigma(i)}s = y_{\sigma(i)}s$.

(c) ⇒ (b) Suppose that (c) holds. Let α be the automorphism $\overline{(r,s)}$. It suffices to show that $rx_i = y_{\sigma(i)}s$ implies $\alpha e_i\alpha^{-1} = e'_{\sigma(i)}$. Now $\alpha e_i\alpha^{-1} = e'_{\sigma(i)}$ is equivalent to $\alpha e_i = e'_{\sigma(i)}\alpha$, that is, that $rr_ix = r'_{\sigma(i)}rx$. But if $rx_i = y_{\sigma(i)}s$, then $rr_ix = rx_i = y_{\sigma(i)}s = r'_{\sigma(i)}xs = r'_{\sigma(i)}rx$.

(b) ⇒ (a) Suppose that (b) holds. Let r, r' and s, s' be such that $\alpha = \overline{(r,s)}$ and $\alpha^{-1} = \overline{(r',s')}$. To prove (a) it suffices to show that $x_i \cong y_{\sigma(i)}$ for every i. To this end, it suffices to that $\overline{(r,s)}: x_i \to y_{\sigma(i)}$ and $\overline{(r',s')}: y_{\sigma(i)} \to x_i$ are well defined, mutually inverse morphisms in $\mathcal{D}$, that is, that $rx_i = y_{\sigma(j)}s$, $r'y_{\sigma(i)} = x_is'$, $r'rx_i = x_i$ and $rr'y_{\sigma(i)} = y_{\sigma(i)}$.

To see that $rx_i = y_{\sigma(j)}s$, notice that the equality $\alpha e_i\alpha^{-1} = e'_{\sigma(i)}$ in (b) is equivalent to $\alpha e_i = e'_{\sigma(i)}\alpha$, that is, to the equality $rr_ix = r'_{\sigma(i)}rx$. Thus $rx_i = rr_ix = r'_{\sigma(i)}rx = r'_{\sigma(i)}xs = y_{\sigma(i)}s$. Similarly for the equality $r'y_{\sigma(i)} = x_is'$.

To prove that $r'rx_i = x_i$, multiply the equality $r'rx = x$ by s_i, getting that $r'rxs_i = xs_i$, that is, $r'rx_i = x_i$. Similarly for the equality $rr'y_{\sigma(i)} = y_{\sigma(i)}$. □

Chapter 3

Semilocal Categories

3.1 The Jacobson radical

In this section we recall the basic notions and properties of the *Jacobson radical* of a ring.

Definition 3.1 (Radical)**.** *Let M_R a right R-module. The* radical $\mathrm{rad}(M_R)$ *of a module is the intersection of all maximal submodules of M_R.*

Definition 3.2 (Jacobson radical)**.** *Let R be any ring and consider the regular right R-module R_R. The radical of R_R is called the* Jacobson radical *of the ring R. Thus* $\mathrm{J(R)} := \mathrm{rad(R_R)}$ *is the intersection of all maximal right ideals of R.*

Let us give some classical descriptions of the *Jacobson radical.*

Proposition 3.3. [Fac17, Lemma 29.1] *The Jacobson radical* $\mathrm{J}(R)$ *of any R is the intersection of the right annihilators* $\mathrm{r.ann}_R(S_R)$ *of all simple right R-modules S_R.*

Definition 3.4. *Let M_R be a right R-modules and P be a submodule of M_R. The submodule P is* superfluous *in M_R if, for every submodule K of M_R, $P + K = M_R$ implies $L = M_R$.*

Proposition 3.5. *The Jacobson radical* $\mathrm{J(R)}$ *of a ring can also be described as:*

(i) *The unique largest superfluous right ideal of R.*

(ii) *The set of all $x \in R$ such that $1 - xr$ is right invertible for every $r \in R$.*

(iii) *The set of all $x \in R$ such that $1 - rxs$ is invertible for every $r, s \in R$.*

Proposition 3.6. *Let R be a semisimple artinian ring. Then the Jacobson radical of R is zero.*

Proof. Let R be a semisimple artinian ring, the regular module R_R is semisimple, then it is a direct sum of finitely many simple right R-modules $S_1, S_2, \dots, S_n$. The right modules $S_1 \oplus \dots \oplus \hat{S}_i \oplus \dots S_n$ are n maximal right submodules of R_R, and the intersection of these n maximal right submodules is zero. Hence $\mathrm{J}(R) = 0$ □

3.2 Semilocal Rings

In this section we will recall the main properties of *semilocal rings*. In Commutative Algebra, a commutative ring is *semilocal* if it has only finitely many maximal ideals. For arbitrary rings there is the following definition.

Definition 3.7 (Semilocal Ring)**.** *A ring R is* semilocal *if $R/\mathrm{J}(R)$ is a semisimple Artinian ring. Here $\mathrm{J}(R)$ denotes the Jacobson radical of R.*

The two definitions agree for commutative rings. Moreover it can be proved that:

Proposition 3.8. *The following conditions are equivalent for a ring R:*

(a) *The ring R is semilocal.*

(b) *The ring $R/\mathrm{J}(R)$ is a right Artinian ring.*

(b') *The ring $R/\mathrm{J}(R)$ is a left Artinian ring.*

(c) *The ideal $\mathrm{J}(R)$ is the intersection of finitely many maximal right ideals of R.*

(c') *The ideal $\mathrm{J}(R)$ is the intersection of finitely many maximal left ideals of R.*

Let us now recall some examples of known semilocal rings.

Examples 3.9. [Fac19, Examples 3.13] The following are semilocal rings.

(1) Every right (or left) Artinian ring is semilocal.

(2) Every local ring is semilocal.

(3) The direct product $R_1 \times R_2 \times \cdots \times R_n$ of finitely many semilocal rings R_1, R_2, ..., R_n is semilocal.

(4) If R is a semilocal ring, the ring $\mathbb{M}_n(R)$ of all $n \times n$ matrices over R is semilocal.

(5) A commutative ring is semilocal if and only if it has finitely many maximal ideals.

Proposition 3.10. [Fac98, Exemple (5), pag. 7] *Every homomorphic image of a semilocal ring is semilocal.*

Proof. Let I be an ideal of a semilocal ring R. Since every simple R/I-module is a simple R-module, if $\pi : R \to R/I$ is the canonical projection, then $\pi(\mathrm{J}(R)) \subseteq \mathrm{J}(R/I)$. Hence π induces a surjective homomorphism $R/\mathrm{J}(R) \to (R/I)/\mathrm{J}(R/I)$. But every homomorphic image of a semisimple artinian ring is a semisimple artinian ring, and thus R/I is semilocal. □

The property of being semilocal is a finiteness condition on the R. Furthermore recall the following result.

Proposition 3.11. *Let R be a semilocal ring, then:*

(a) *Every finitely generated projective R-module has only finitely many direct summands up to isomorphism.*

(b) *The number of direct-sum decompositions of any nonzero finitely generated projective module as a sum of nonzero submodules is finite up to isomorphism.*

(c) *Every finitely generated projective R-module is a direct sum of finitely many indecomposable modules.*

(d) *Every finitely generated projective R-module is not a direct sum of infinitely many nonzero modules.*

In particular, for modules, it is interesting to study the case of the modules M_R whose endomorphism ring $\mathrm{End}(M_R)$ is semilocal. In fact, having a semilocal endomorphism ring is a finiteness condition on the module M_R. In order to explain what it means, recall the following.

Proposition 3.12. *Let M_R be a right R-module and let* $\mathrm{End}(M_R)$ *be a semilocal ring, then:*

(a) *M_R is a direct sum of finitely many indecomposable modules.*

(b) *M_R is not a direct sum of infinitely many nonzero submodules.*

(c) *If N_R and N'_R are right R-modules, then $M_R \oplus N_R \cong M_R \oplus N'_R$ implies $N_R \cong N'_R$.*

(d) *M_R has only finitely many direct summands up to isomorphism.*

3.3 Local Morphisms

Recall now a very useful tool in the theory of semilocal rings, that is *local morphisms*. As for semilocal rings, we will recall the definition for commutative rings.
Let R and S be local commutative rings with maximal ideals I and J respectively. In Commutative Algebra, a ring morphism $\phi\colon R \to S$ is called a *local morphism* if $\phi(I) \subseteq J$. The definition for arbitrary rings is the following.

Definition 3.13. *Let R and S be arbitrary associative rings with identity. A ring morphism $\phi : R \to S$ is said to be a* local *morphism if, for every $r \in R$, $\phi(r)$ invertible in S implies r invertible in R.*

This definition concides with the one in commutative case, when the rings R and S are local commutative rings.

Example 3.14. Let R be a ring. The canonical projection $\pi : R \to R/\mathrm{J}(R)$ is a local morphism. More generally, if I is a two-sided ideal of R and $I \subseteq \mathrm{J}(R)$, then the canonical projection $\pi : R \to R/I$ is a local morphism.

Lemma 3.15. [Fac19, Lemma 3.23] *Let R, S and T be rings and let $\phi : R \to S$, $\psi : S \to T$ be ring morphisms. If ϕ and ψ are local morphisms, then the composite mapping $\psi \circ \phi$ is a local morphism.*

The reason why local morphisms are so useful is the following characterization of semilocal rings.

Theorem 3.16. [CD93, Theorem 1] *A ring R is semilocal if and only if there exists a local morphism of R into a semilocal ring.*

From now on our aim is to show how local morphisms appear in our setting.

Proposition 3.17. *For every $x \in {}_R M_S$ the ring morphism $\phi : \mathrm{End}_{\mathcal{C}}(x) \hookrightarrow R \times S$, defined by $\phi(r,s) = (r,s)$, is a local morphism.*

Proof. Let (r,s) be an element of $\mathrm{End}_{\mathcal{C}}(x)$ such that $\phi((r,s)) = (r,s) \in R \times S$ is an invertible element, so there exists a unique pair $(r',s') \in R \times S$ such that $(rr',ss') = (1_R,1_S)$ and $(r'r,s's) = (1_R,1_S)$. We must show that (r',s') is an element of $\mathrm{End}_{\mathcal{C}}(x)$. Starting from the equation $rx = xs$, multiply on the left by r' and on the right by s'. We obtain $r'rxs' = r'xss'$, so $xs' = r'x$. □

Our next step is to describe the endomorphism ring of a cyclic module. Let us recall a description of the endomorphism ring of a cyclic right module xS over an arbitrary ring S. Recall that if a right S-module N_S is cyclic and x is a generator of N_S, then $N_S \cong S/A$ where $A = \mathrm{r.ann}_S(x)$.

We are interested in $\mathrm{Hom}_S(S/A,S/A) = \mathrm{End}_S(S/A)$. Each S-module morphism $f : S/A \to S/A$ is uniquely determined by the its image on $1 + A$. Let us suppose that $f(1+A) = a+A$, so $f(r+A) = f(1+A)r = ar$. Furthermore, $f(0+A) = 0+A$ so if we change the representative for the zero, we have $f(i+A) = ai + A = 0 + A$. That is, $ai \in A$. Then $\mathrm{End}_S(xS)$ is isomorphic to the *idealizer*

$$\begin{aligned}\mathbb{I}(\mathrm{r.ann}_S(x)) &= \{\, a \in S \mid a\mathrm{r.ann}_S(x) \subseteq \mathrm{r.ann}_S(x) \,\} = \\ &= \{\, a \in S \mid \forall t \in S(xt = 0 \Rightarrow xat = 0) \,\}\end{aligned}$$

in S of $\mathrm{r.ann}_S(x)$ modulo $\mathrm{r.ann}_S(x)$:

$$\mathrm{End}_S(xS) \cong \mathbb{I}(\mathrm{r.ann}_S(x))/\mathrm{r.ann}_S(x).$$

This isomorphism associates to any endomorphism f of xS the element $a+\mathrm{r.ann}_S(x)$, where $a \in S$ is such that $f(x) = xa$.

Similarly for homomorphisms of $xS \to yS$: we have that if

$$\begin{aligned}H_{x,y} &= \{\, a \in S \mid a\mathrm{r.ann}_S(x) \subseteq \mathrm{r.ann}_S(y) \,\} = \\ &= \{\, a \in S \mid \forall t \in S(xt = 0 \Rightarrow yat = 0) \,\},\end{aligned}$$

then

$$\mathrm{Hom}_S(xS, yS) \cong H_{x,y}/\mathrm{r.ann}_S(y).$$

The isomorphism associates to any morphism $f{:}\, xS \to yS$ the element $a + \mathrm{r.ann}_S(x)$, where $a \in S$ is such that $f(x) = ya$.

Furthermore, let $\phi : xS \to xS$ be an idempotent element of $\mathrm{End}_S(xS)$ with $\phi(x) = xa$ for some a in $\mathbb{I}(A)/A$. Notice that ϕ is well defined if and only if for every $t \in S$ $xt = 0 \Rightarrow xat = 0$. Then $\phi(xt) = xat$ for every $t \in S$. The same holds for $1 - \phi : xS \to xS$. The endomorphism $1 - \phi$ corresponds to the right multiplication by $1 - a$, in fact $(1 - \phi)(x) = x - xa = x(1 - a)$. The endomorphism ϕ is idempotent if and only if $\phi^2(x) = \phi(x) \Leftrightarrow \phi(xa) = \phi(x) \Leftrightarrow xa^2 = xa \Leftrightarrow a^2 - a \in ann_S(x)$

Hence the direct sum decomposition is

$$xS = xaS \bigoplus x(1-a)S$$

Consider now the subring $\frac{(Rx :_S x)}{\mathrm{r.ann}_S(x)}$ of $\frac{\mathbb{I}(\mathrm{r.ann}_S(x))}{\mathrm{r.ann}_S(x)} \cong \mathrm{End}_S(xS)$. Take an element $\overline{s} = s + \mathrm{r.ann}_S(x)$ where $xs \in Rx$, so that there exists $r' \in R$ such that $xs = r'x$. Then, for every $s' \in S$ such that $xs' = 0$, we have $xss' = r'xs' = 0$. Hence we have

$$\frac{(Rx :_S x)}{\mathrm{r.ann}_S(x)} \subseteq \frac{\mathbb{I}(\mathrm{r.ann}_S(x))}{\mathrm{r.ann}_S(x)} \cong \mathrm{End}_S(xS).$$

Similarly,

$$\frac{(xS :_R x)}{\mathrm{l.ann}_R(x)} \subseteq \frac{\mathbb{I}(\mathrm{l.ann}_R(x))}{\mathrm{l.ann}_R(x)} \cong \mathrm{End}_R(Rx).$$

Proposition 3.18. *The morphism*

$$\begin{aligned} \psi : \mathrm{End}_{\mathcal{D}}(x) &\longrightarrow \frac{(xS :_R x)}{\mathrm{l.ann}_R(x)} \times \frac{(Rx :_S x)}{\mathrm{r.ann}_S(x)} \\ \overline{(r,s)} &\longmapsto (\overline{r}, \overline{s}) \end{aligned}$$

is an injective local homomorphism.

Proof. The mapping ψ is well defined because for every $(a, b) \in \mathrm{l.ann}_R(x) \times \mathrm{r.ann}_S(x)$, i.e., such that $ax = xb = 0_M$ we have $r + a \in \overline{r}$ and $(r + a)x = rx$.

It is also a homomorphism, because $\psi(\overline{(r',s')} \circ \overline{(r,s)}) = \psi(\overline{(r'r, s's)}) = (\overline{r'r}, \overline{s's}) = (\overline{r'}, \overline{s'})(\overline{r}, \overline{s}) = \psi(\overline{(r',s')}) \circ \psi(\overline{(r,s)})$. The homomorphism ψ is injective because $\mathrm{Ker}(\psi) = \mathrm{l.ann_R}(x) \times \mathrm{r.ann_S}(x)$.

Moreover, ψ is a local morphism: take $\overline{(r,s)} \in \mathrm{End}_{\mathcal{D}}(x)$, suppose $\psi(\overline{(r,s)}) = (\overline{r}, \overline{s})$ is invertible, then $\overline{r}$ is invertible in $\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)}$ and $\overline{s}$ is invertible in $\frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)}$. Hence there exist $\tilde{r} \in R$ and $\tilde{s} \in S$ such that $\overline{\tilde{r}r} = \overline{1_R}$ and $\overline{\tilde{s}s} = \overline{1_S}$, i.e., the pair $\overline{(\tilde{r},\tilde{s})}$ is the inverse of $\overline{(r,s)}$. Now it remains to prove that $\overline{(\tilde{r},\tilde{s})}$ is an element of $\mathrm{End}_{\mathcal{D}}(x)$. Starting from the equation $rx = xs$, multiply on the left by $\tilde{r}$ and on the right by $\tilde{s}$. We obtain $\tilde{r}rx\tilde{s} = \tilde{r}xs\tilde{s}$, so passing to the quotient we get $x\tilde{s} = \tilde{r}x$. □

Consider the morphism $\alpha : \mathrm{End}_{\mathcal{D}}(x) \to \mathrm{End}_S(xS)$ that sends an element $\overline{(r,s)}$ of $\mathrm{End}_{\mathcal{D}}(x)$ with $rx = xs$ to $\lambda_r : xS \to xS$ (notice that $\lambda_r(xS) = rxS = xsS \subseteq xS$), and $\beta : \mathrm{End}_{\mathcal{D}}(x) \to \mathrm{End}_R(Rx)$ that sends an element $\overline{(r,s)}$ of $\mathrm{End}_{\mathcal{D}}(x)$ with $rx = xs$ to $\rho_s : Rx \to Rx$. We have that $\rho_s(Rx) = Rxs = Rrx \subseteq Rx$. Both morphisms are well defined. Notice that on $\mathrm{End}_R(Rx)$ we usually write mappings on the right, because Rx is a left R-module. So β is therefore a ring homomorphism, because $\beta(\overline{(r',s')} \circ \overline{(r,s)}) = \beta(\overline{(r'r,s's)}) = \rho_{s's} = \rho_{s'} \circ \rho_s = \beta(\overline{(r',s')}) \circ \beta(\overline{(r,s)})$.

Hence we can deduce the following.

Proposition 3.19. *The morphism*

$$\xi : \mathrm{End}_{\mathcal{D}}(x) \to \mathrm{End}(Rx)^{\mathrm{Op}} \times \mathrm{End}(xS)$$
$$\overline{(r,s)} \longmapsto (\overline{r}, \overline{s})$$

is an injective local homomorphism.

The proof is similar to that of Theorem 3.18.

So we obtain the following results.

Theorem 3.20. *If R and S are semilocal rings, then $\mathrm{End}_{\mathcal{D}}(x)$ is also a semilocal ring.*

Proof. If R and S are semilocal rings, their direct product is also semilocal and we know from Proposition 3.17 that $\mathrm{End}_{\mathcal{C}}(x) \hookrightarrow R \times S$ is a local morphism, then, because of Theorem 3.16, $\mathrm{End}_{\mathcal{C}}(x)$ is a semilocal ring and from Proposition 3.10 $\mathrm{End}_{\mathcal{C}/\mathcal{I}}(x)$ is semilocal. □

Theorem 3.21. *If $\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)}$ and $\frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)}$ are semilocal rings, then $\mathrm{End}_{\mathcal{D}}(x)$ is also a semilocal ring.*

Proof. It follows directly from Proposition 3.18 and Theorem 3.16. □

Since any semilocal ring has only finitely many finitely generated indecomposable projective R-modules up to isomorphism (Fuller and Shutters, [Fac19, Corollary 3.31], it follows that:

Corollary 3.22. *If R and S are semilocal rings, every $x \in {}_RM_S$ has only finitely many internal direct-sum decompositions in ${}_RM_S$ up to isomorphism.*

Definition 3.23. [Fac19, Definition 4.61] *Let $\mathcal{A}$ be a category. It is a* semilocal *category if it is a preadditive category with a non-zero object such that the endomorphism ring* $\mathrm{End}_{\mathcal{A}}(A)$ *of every non-zero object A of $\mathcal{A}$ is a semilocal ring.*

Theorem 3.24. *If R and S are semilocal rings, then the categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are semilocal categories.*

We will now give an example of a finitely generated module over a semilocal ring whose endomorphism ring is not semilocal.
Recall that a *semiprimary* ring is a semilocal ring whose Jacobson ideal is nilpotent.

Example 3.25. [FH06, Example 3.5]

Let K be a field with a non-onto endomorphism $\alpha : K \to K$. Let $K_0 = \alpha(K)$. Let ${}_KV$ be a vector space different from zero. View ${}_KV$ as a K-K-bimodule taking the scalar product by K as left action and setting as right action $v \cdot k = \alpha(k)v$ for every $v \in V$ and every $k \in K$.
Let $R = \begin{pmatrix} K & {}_KV_K \\ 0 & K \end{pmatrix}$. Then $\mathrm{J}(R) = \begin{pmatrix} 0 & {}_KV_K \\ 0 & 0 \end{pmatrix}$, $R/\mathrm{J}(R) \cong K \times K$ and $\mathrm{J}(R)^2 = 0$, so that R is semiprimary. Fix $a \in K/K_0$ and $0 \neq w \in V$. Consider the right ideal

$$I = \sum_{n \geq 0} \begin{pmatrix} 0 & a^n w \\ 0 & o \end{pmatrix} R = \begin{pmatrix} 0 & K_0[a]w \\ 0 & 0 \end{pmatrix}$$

of R. Then $E := \mathrm{End}_R(R/I) \cong \mathbb{I}(I)/I$, where $\mathbb{I}(I)$ is the idealizer of I in R.
Let $\begin{pmatrix} k_1 & v \\ 0 & k_2 \end{pmatrix} \in \mathbb{I}(I)$. As

$$\begin{pmatrix} k_1 & v \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & k_1 w \\ 0 & 0 \end{pmatrix} \in I,$$

we deduce that $\mathbb{I}(I) = \begin{pmatrix} K_0[a] & V \\ 0 & K \end{pmatrix}$. Hence $E/\mathrm{J}(E) \cong K_0[a] \times K$.
If we choose K, α and a such that a is trascendental over K_0, then $K_0[a] \times K$ is not semisimple artinian. Hence, E is not semilocal.

3.4 Some Natural Functors

It is possible to generalize the notion of local morphism between rings to the notion of local functor between categories in the following way.

Definition 3.26. *if $\mathcal{A}$ and $\mathcal{B}$ are categories, a functor $F\colon \mathcal{A} \to \mathcal{B}$ is* local *if for every morphism $f\colon a \to a'$ in $\mathcal{A}$, if $F(f)$ is an isomorphism in $\mathcal{B}$, then f is an isomorphism in $\mathcal{A}$.*

Let us introduce two functors:

(1) The covariant functor $F_S : \mathcal{D} \to$ Mod-S which associates to each $x \in M$ the cyclic right S-module xS and to any morphism $\overline{(r,s)} : x \to y$ the left multiplication

$$\lambda_r : xS \to yS$$
$$x \mapsto rx.$$

This functor is well defined on morphisms because if $\overline{(r,s)} : x \to y$ is a morphism in $\mathcal{D}$ and $\overline{(r,s)} = \overline{(r',s')}$, then $r - r' \in \mathrm{l.ann}_R(x)$, so $\lambda_r = \lambda_{r'}$.

(2) The contravariant functor $F_R : \mathcal{D} \to R$-Mod which associates to each $x \in M$ the cyclic left R-module Rx and to any morphism $\overline{(r,s)} : x \to y$ the right multiplication

$$\rho_s : Ry \to Rx$$
$$y \mapsto ys.$$

This functor is well defined on morphisms because if $\overline{(r,s)} : x \to y$ is a morphism in $\mathcal{D}$ and $\overline{(r,s)} = \overline{(r',s')}$, then $s - s' \in \mathrm{r.ann}_S(x)$, so $\rho_s = \rho_{s'}$.

Observe that the ring homomorphisms

$$\alpha : \mathrm{End}_{\mathcal{D}}(x) \to \mathrm{End}_S(xS) \qquad \text{and} \qquad \beta : \mathrm{End}_{\mathcal{D}}(x) \to \mathrm{End}_R(Rx)$$

defined before the Proposition 3.19 are respectively $(F_S)_{xx}$ and $(F_R)_{xx}$ in the notetion of Definition 1.3.

Proposition 3.27. *The functors* $F_S : \mathcal{D} \to \text{Mod-}S$ *and* $F_R : \mathcal{D} \to R\text{-Mod}$ *are faithful functors.*

Proof. Let us begin with the functor F_S. Following the Definition 1.3 consider the mapping

$$(F_S)_{xy} : \text{Hom}_{\mathcal{D}}(x,y) \to \text{Hom}_S(xS, yS).$$

Let $\overline{(r_1,s_1)} : x \to y$ and $\overline{(r_2,s_2)} : x \to y$ be morphisms of $\text{Hom}_{\mathcal{D}}(x,y)$, so that, $r_1x = ys_1$ and $r_1x = ys_1$. Suppose $\overline{(r_1,s_1)} \neq \overline{(r_2,s_2)}$, then we have either $(r_1-r_2)x \neq 0$ or $y(s_1 - s_2) \neq 0$. In the first case we have that $r_1x - r_2x \neq 0$ then $r_1x \neq r_2x$, that is the same of $\lambda_{r_1}(x) \neq \lambda_{r_2}(x)$, thus $\lambda_{r_1} \neq \lambda_{r_2}$. In the second case we have that $ys_1 - ys_2 \neq 0$ then $ys_1 \neq ys_2$, but from $r_1x = ys_1$ and $r_1x = ys_1$ we obtain the same conclusion.
For the functor F_R we can proceed in the same way as before. □

Remark 3.28. The functors F_S and F_R are not full functors, e.g., let R be a ring with an element x that is right-invertible but non left-invertible and let $s \in R$ such that $xs = 1$. Consider the bimodule ${}_RM_S = {}_RR_R$. Then $\lambda_s \colon R_R \to R_R$ is a morphism between right R-modules, but in this case $xR = R_R$, because x is right-invertible, thus we have a morphism $\lambda_s \colon xR \to xR$ but for every $r \in R$, we have $rx \neq 1 = xs$.

For the direct product ring $R \times S$, let $C_{R\times S}$ be the preadditive category with one object $*$ with endomorphism ring $\text{End}_{C_{R\times S}}(*) = R \times S$.

Proposition 3.29. *The functor* $F \colon \mathcal{D} \to C_{R\times S}$ *that sends any morphism* $\overline{(r,s)} \colon x \to y$ *in* $\mathcal{D}$ *to the element* (r,s) *of* $R \times S$ *is a faithful local functor.*

Proof. Let $\overline{(r,s)} \colon x \to y$ be a morphism in $\mathcal{D}$, so that $rx = ys$. Suppose (r,s) invertible in $R\times S$. Let (r',s') be the inverse of (r,s) in $R\times S$. Multiplying the equality $rx = ys$ by r' on the left and s' on the right, one gets that $r'rxs' = r'yss'$, so $r'y = xs'$. Hence $\overline{(r',s')} \colon y \to x$ is a morphism in $\mathcal{D}$ and is the inverse of

$$\overline{(r,s)} \colon x \to y.$$

□

Proposition 3.30. *The covariant functor* $F_R \times F_S: \mathcal{D} \to (R\text{-Mod})^{\text{op}} \times \text{Mod-}S$ *is a local functor.*

Proof. Recall that the covariant faithful additive functor

$$F_R \times F_S: \mathcal{D} \to (R\text{-Mod})^{\text{op}} \times \text{Mod-}S$$

associates to any morphism $\overline{(r,s)}: x \to y$ the morphism $(\rho_s, \lambda_r): (Rx, xS) \to (Ry, yS)$.

In order to prove that the functor is local, fix a morphism $\overline{(r,s)}: x \to y$ and suppose that $(\rho_s, \lambda_r): (Rx, xS) \to (Ry: yS)$ is an isomorphism in $(R\text{-Mod})^{\text{op}} \times \text{Mod-}S$. Then the two module morphisms $\rho_s: Ry \to Rx$, defined by $\rho_s(ay) = ays = arx$ for every $a \in R$, and $\lambda_r: xS \to yS$, defined by $\lambda_r(xb) = rxb = ysb$ for every $b \in S$, are isomorphisms. Let $\rho': Rx \to Ry$ and $\lambda': yS \to xS$ be their inverse morphisms in R-Mod, S Mod-, respectively.

From the description of the morphisms between cyclic modules in Section 3.3, we get that there exists an element $r' \in R$ such that $\text{l.ann}_R(x)r' \subseteq \text{l.ann}_R(y)$ and $\rho'(x) = r'y$, and there exists an element $s' \in S$ such that $s'\text{r.ann}_S(y) \subseteq \text{r.ann}_S(x)$ and $\lambda'(y) = xs'$. From the four equalities

$$\rho\rho' = 1_{Rx}, \qquad \rho'\rho = 1_{Ry}, \qquad \lambda\lambda' = 1_{xS}, \qquad \lambda'\lambda = 1_{yS},$$

we get that

$$x = r'rx, \qquad y = rr'y, \qquad x = xss', \qquad y = ys's$$

respectively (for instance, from $\rho\rho' = 1_{Rx}$, we get that $x = \rho\rho'(x) = \rho(r'y) = r'\rho(y) = r'ry\text{b})$. Thus $\overline{(r,s)}: x \to y$ morphism in $\mathcal{D}$ implies $rx = ys$, so $r'rxs' = r'yss'$, hence $xs' = r'y$. Hence we have a morphism $\overline{(r',s')}: y \to x$ in $\mathcal{D}$. It is immediate to check that this morphism is an inverse of $\overline{(r,s)}: x \to y$ in $\mathcal{D}$. □

Definition 3.31. *Let* $\mathcal{A}$ *be a category,* A_1, A_2 *and* B *objects of* $\mathcal{A}$*,* $f_1: A_1 \to B$ *and* $f_2: A_2 \to B$ *be morphisms of* $\mathcal{A}$*. A* pullback *of* f_1 *and* f_2 *in* $\mathcal{A}$ *is a triple* (P, α_1, α_2)*, where* P *is an object of* $\mathcal{A}$ *and* $\alpha_1: P \to A_1$*,* $\alpha_2: P \to A_2$ *morphisms such that* $f_1 \circ \alpha_1 = f_2 \circ \alpha_2$ *and for every other triple* (X, β_1, β_2) *with the same property, there exists an unique morphism* $\gamma: X \to P$ *such that* $\beta_1 = \alpha_1 \circ \gamma$ *and* $\beta_2 = \alpha_2 \circ \gamma$*.*

$$
\begin{array}{ccccc}
X & & \xrightarrow{\quad\beta_1\quad} & & \\
& \overset{\gamma}{\searrow} & & & \\
& & P & \xrightarrow{\alpha_1} & A_1 \\
& \beta_2 \searrow & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle f_1} \\
& & A_2 & \xrightarrow{f_2} & B.
\end{array}
$$

The functors A, B also allow to describe our present setting in terms of pullbacks. Consider two elements $x, y \in {}_RM_S$ and the free modules of rank one ${}_RR$ and S_S. From the universal property of free modules there exist unique morphisms $\rho_x\colon {}_RR \to {}_RM_S$ and $\lambda_x\colon S_S \to {}_RM_S$ such that $\rho_x(r) = rx$ and $\lambda_x(s) = xs$ for every $r \in R$, $s \in S$. We have the following diagram

$$
\begin{array}{ccc}
{}_RR & \xrightarrow{\rho_x} & {}_RM_S \\
& & \uparrow{\scriptstyle \lambda_y} \\
& & S_S.
\end{array}
$$

Our claim is that the following is a pullback of abelian groups

$$
\begin{array}{ccc}
{}_RR & \xrightarrow{\quad\rho_x\quad} & {}_RM_S \\
{\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle \lambda_y} \\
\mathrm{Hom}_{\mathcal{C}}(x, y) & \xrightarrow{p_2} & S_S.
\end{array}
$$

where $p_2\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to {}_SS$, $(r, s) \mapsto s$ and $p_1\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to R_R$, $(r, s) \mapsto r$.
In fact, $(\lambda_y \circ p_2)(r, s) = ys = rx = (\rho_x \circ p_1)(r, s)$, so the first property of pullbacks is satisfied. Let X be an abelian group and let $\alpha\colon X \to {}_RR$ and $\beta\colon X \to S_S$ morphisms such that $\rho_x \circ \alpha = \lambda_y \circ \beta$. Define $\phi\colon X \to \mathrm{Hom}_{\mathcal{C}}(x, y)$ such that $a \mapsto (\alpha(a), \beta(a))$. It is easy to see that $p_1 \circ \phi = \alpha$ and $p_2 \circ \phi = \beta$. It remains to show the uniqueness, let $\psi\colon X \to \mathrm{Hom}_{\mathcal{C}}(x, y)$, $a \mapsto (r_a, s_a)$ such that $p_1 \circ \psi = \alpha$ and $p_2 \circ \psi = \beta$, then $s = p_2 \circ \phi(a) = \beta(a) = p_2 \circ \psi(a) = s_a$ and $r = p_1 \circ \phi(a) = \alpha(a) = p_a \circ \psi(a) = r_a$ imply $\phi = \psi$.
Factoring out modulo the kernels, we get a pullback of abelian groups in which all morphisms are monomorphisms:

$$
\begin{array}{ccc}
{}_RR/\mathrm{l.ann}_R(x) & \longrightarrow & {}_RM_S \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{D}}(x, y) & \longrightarrow & S_S/\mathrm{r.ann}_S(y),
\end{array}
$$

or, equivalently, the pullback

$$\begin{array}{ccc} Rx & \longrightarrow & {}_RM_S \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{\mathcal{E}}(x,y) & \longrightarrow & yS, \end{array}$$

where all morphisms are abelian group embeddings.

3.5 Embedding into other categories

In this section our aim is to present some category embeddings from the category $\mathcal{D}$ to other categories.

Proposition 3.32. *Let ${}_RM_S$ be an R-S-bimodule. Then:*

(1) *There is a covariant functor $H\colon \mathcal{D} \to \mathrm{Morph}(\text{Mod-}S)$, which associates to any object x of $\mathcal{D}$ the embedding $\varepsilon_x\colon xS \hookrightarrow M_S$. It associates to any morphism $\overline{(r,s)}\colon x \to y$ in $\mathcal{D}$ the pair of morphisms (λ_r, λ'_r), where $\lambda'_r\colon M_S \to M_S$ is left multiplication by r, and $\lambda_r\colon xS \to yS$ is the restriction of λ'_r.*

(2) *M_S is R-balanced, that is, the canonical ring morphism $\lambda\colon R \to \mathrm{End}(M_S)$ is surjective, then the functor H is full.*

(3) *Conversely, if the functor H is full and the module M_S is cyclic, then M_S is R-balanced.*

Proof. (1) For a morphism $\overline{(r,s)}\colon x \to y$ in $\mathcal{D}$, we have that $rx = ys$, so that $\lambda'_r(xS) = rxS = ysS \subseteq yS$ and the square

$$\begin{array}{ccc} xS & \overset{\varepsilon_x}{\hookrightarrow} & M_S \\ \downarrow{\scriptstyle \lambda_r} & & \downarrow{\scriptstyle \lambda'_r} \\ yS & \overset{\varepsilon_y}{\hookrightarrow} & M_S \end{array}$$

commutes. It is now easy to check that H is a covariant functor. In order to prove that it is a faithful functor, we must show that the mapping

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(x,y) &\to \mathrm{Hom}_{\mathrm{Morph(Mod\text{-}}S)}(\varepsilon_x, \varepsilon_y) \\ \overline{(r,s)} &\mapsto (\lambda_r, \lambda'_r) \end{aligned} \tag{3.1}$$

is injective. But given $rx = ys$ and $r'x = ys'$ such that $(\lambda_r, \lambda'_r) = (\lambda_{r'}, \lambda'_{r'})$, we have that $\lambda'_r = \lambda'_{r'}$, so that $rx = r'x$. This proves that the functor is faithful.

(2) Suppose that M_S is R-balanced. In order to prove that H is full, we must show that the mapping in (3.1) is surjective. Now an arbitrary element of $\mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}S)}(\varepsilon_x, \varepsilon_y)$ is a pair $(f|_{xS}, f)$, where f is an endomorphism of M_S that induces by restriction a left R-module morphism $f|_{xS}: xS \to yS$. Therefore the diagram

$$\begin{array}{ccc} xS & \xhookrightarrow{\varepsilon_x} & M_S \\ \big\downarrow{\scriptstyle f|_{xS}} & & \big\downarrow{\scriptstyle f} \\ yS & \xhookrightarrow{\varepsilon_y} & M_S \end{array}$$

is commutative. Since M_S is R-balanced, there exists $r \in R$ such that $\lambda'_r = f$. Now $f(xS) \subset yS$, so $rx = f(x) \in yS$. It follows that there exists an element $s \in S$ with $rx = ys$. Therefore $\overline{(r,s)}: x \to y$ is the morphism in $\mathcal{D}$ that proves that the mapping in (3.1) is surjective.

(3) Now assume H full and M_S cyclic. In order to prove that the canonical mapping $\lambda: R \to \mathrm{End}(M_S)$ is surjective, fix an endomorphism $f: M_S \to M_S$. Let z be a generator of M_S. The commutative diagram

$$\begin{array}{ccc} zS & \xhookrightarrow{\varepsilon_z} & M_S \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} \\ zS & \xhookrightarrow{\varepsilon_z} & M_S \end{array}$$

shows that the pair (f, f) is a morphism of ε_z into ε_z. Since H is full, there exists a morphism $\overline{(r,s)}: z \to z$ in $\mathcal{D}$ such that $f = \lambda_r$. This proves that M_S is R-balanced. □

From Proposition 3.32 we get that, the category $\mathcal{D}$ is isomorphic to a subcategory of the category Morph(Mod-S).

Remark 3.33. The functor of Proposition 3.32 is faithful but not full. To see this it suffices to take any endomorphism $\varphi: M_S \to M_S$ that is not left multiplication by an element of R for the bimodule ${}_RM_S$, and consider (φ, λ_s).

Dually, we can state the following:

Proposition 3.34. *Let ${}_RM_S$ be an R-S-bimodule. Then:*

(1) *There is a contravariant functor* $L: \mathcal{D} \to \mathrm{Morph}(R\text{-Mod})$, *which associates to any object* x *of* $\mathcal{D}$ *the embedding* $\eta_x: Rx \hookrightarrow {}_RM$. *It associates to any morphism* $\overline{(r,s)}: x \to y$ *in* $\mathcal{D}$ *the pair of morphisms* (ρ_s, ρ'_s), *where* $\rho'_s: {}_RM \to_R M$ *is right multiplication by* s, *and* $\rho_s: Ry \to Rx$ *is the restriction of* ρ'_s.

(2) ${}_RM$ *is* S*-balanced, that is, the canonical ring morphism* $\lambda: S \to \mathrm{End}({}_RM)$ *is surjective, then the functor* L *is full.*

(3) *Conversely, if the functor* L *is full and the module* ${}_RM$ *is cyclic, then* ${}_RM$ *is* S*-balanced.*

A similar category embedding appears in relation to the Eilenberg-Watts Theorem:

Theorem 3.35. (Eilenberg [Eil60], Watts [Wat60]) *Let* R *and* S *be rings and* $F: \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}S$ *be a right exact additive functor that preserves direct sums. Then* $F(R_R)$ *is an* R*-*S*-bimodule and the two functors,* $F, - \otimes_R M_S$ *are naturally isomorphic.*

This correspondence between R-S-bimodules and right exact additive functors that preserve direct sums, that is, colimit-preserving functors, is an equivalence between the category R-BiMod-S of R-S–bimodules and the category

$$\mathrm{Func}_{\mathrm{coc}}(\mathrm{Mod}\text{-}R, \mathrm{Mod}\text{-}S)$$

of all additive colimit-preserving functors $\mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}S$.

For every bimodule ${}_RM_S$, let $- \otimes_R M_S: \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}S$ be the corresponding functor in the Eilenberg-Watts Theorem. For any ring T, let $U_T: \mathrm{Mod}\text{-}T \to \mathsf{Ab}$ denote the forgetful functor. Thus we have two functors $U_R: \mathrm{Mod}\text{-}R \to \mathsf{Ab}$ and $U_S \circ (- \otimes_R M_S): \mathrm{Mod}\text{-}R \to \mathsf{Ab}$. Both functors U_R and $U_S \circ (- \otimes_R M_S)$ are right exact additive functors that preserve direct sums.

Proposition 3.36. *For every* R*-*S*-bimodule* ${}_RM_S$*, there is a canonical mapping*

$${}_RM_S \to \mathrm{Nat}(U_R, U_S \circ (- \otimes_R M_S))$$

into the class of all natural transformations $U_R \to U_S \circ (- \otimes_R M_S)$. *It associates to each element* x *of* ${}_RM_S$ *the natural transformation* $\eta_x: U_R \to U_S \circ (- \otimes_R M_S)$ *defined, for every right* R*-module* A_R, *by* $\eta_{x,A}: A_R \to A \to A \otimes_R M$, $\eta_{x,A}: a \in A_R \to A \mapsto a \otimes x$.

The proof is easy.

Now "morphisms" between two natural transformations are just commutative squares of natural transformations. Therefore, to give a coherent presentation, it is now convenient to determine, for any fixed left R-module ${}_RM$ the natural transformations $\eta\colon G_M \to G_M$ of the functor

$$G_M := - \otimes_R M\colon \text{Mod-}R \to \mathsf{Ab}, \quad G_M\colon A_R \to A \otimes_R M,$$

into itself:

Proposition 3.37. *For every left R-module ${}_RM$, there is a one-to-one correspondence between its endomorphism ring $\text{End}_R({}_RM)$ and the class of all natural transformations $\eta\colon G_M \to G_M$, where $G_M := - \otimes_R M\colon \text{Mod-}R \to \mathsf{Ab}$.*

Proof. This is an exercise which we leave to the reader. For any left R-module morphism $f\colon {}_RM \to {}_RM$, the corresponding natural transformations $\eta_f\colon G_M \to G_M$ associates to any right R-module A_R the abelian group morphism

$$\eta_{f,X} := 1_X \otimes f\colon X \otimes_R M \to X \otimes_R M.$$

Conversely, if $\eta\colon G_M \to G_M$ is any natural transformations, then η associates to the object R_R of Mod-R the abelian group morphism $\eta_R\colon R \otimes_R M \cong M \to R \otimes_R M \cong M$. It is easy to check that $\eta_R\colon M \to M$ is a left R-module morphism and that $\eta = \eta_{\eta_R}$ (for every right R-module A_R and every element $a \in A$, consider the right R-module morphism $\lambda_a\colon R_R \to A_R$, $\lambda_a\colon 1 \to a$). □

Corollary 3.38. *For every ring R, there is a one-to-one correspondence between the ring R and the class of all natural transformations $\eta\colon U_R \to U_R$, where $U_R\colon \text{Mod-}R \to \mathsf{Ab}$ is the forgetful functor.*

Proof. Take ${}_RM := {}_RR$ in the previous proposition. □

Let ${}_RM_S$ be a fixed R-S-bimodule. Let us show that the mapping ${}_RM_S \to \text{Nat}(U_R, U_S \circ (- \otimes_R M_S)), x \mapsto \eta_x$ described in the statement of Proposition 3.36 is the object mapping of a functor $\mathcal{D} \to \text{Nat}(U_R, U_S \circ (- \otimes_R M_S))$. This functor associates to any morphism $\overline{(r,s)}\colon x \to y$ in $\mathcal{D}$ the pair $(\rho_r, - \otimes \rho_s)$ consisting of the natural transformation $\rho_r\colon U_R \to U_R$ (where ρ_r associates to the right R-module A_R

the group morphism $\rho_{r,A}\colon A \to A,\ a \mapsto ar$) and the natural transformation $-\otimes\rho_s\colon (U_S\circ(-\otimes_R M_S)) \to (U_S\circ(-\otimes_R M_S))$ (where $-\otimes\rho_s$ associates to the right R-module A_R the group morphism $(-\otimes\rho_s)_A\colon A\otimes_R M \to A\otimes_R M,\ a\otimes m \mapsto a\otimes(ms)$).

From $rx = ys$ we get that $(ar)\otimes x = (a\otimes y)s$ for every $a \in A_R$, so that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\eta_{y,A}} & A\otimes_R M \\ {\scriptstyle \rho_{r,A}}\big\downarrow & & \big\downarrow{\scriptstyle 1_A\otimes\rho_s} \\ A & \xrightarrow{\eta_{x,A}} & A\otimes_R M \end{array}$$

of abelian groups and group morphisms commute, hence

$$\begin{array}{ccc} U_R(-) & \xrightarrow{\eta_{y,-}} & U_S\circ(-\otimes_R M_S) \\ \big\downarrow{\scriptstyle \rho_r} & & \big\downarrow{\scriptstyle -\otimes\rho_s} \\ U_R(-) & \xrightarrow{\eta_{x,-}} & U_S\circ(-\otimes_R M_S) \end{array}$$

is a commutative square of functors Mod-$R \to \mathsf{Ab}$ and natural transformations. Therefore the pair $(\rho_r, -\otimes\rho_s)$ is a morphism $\eta_y \to \eta_x$ in the category $\mathrm{Nat}(U_R, U_S\circ(-\otimes_R M_S))$. It is now easy to show that:

Theorem 3.39. *For every R-S-bimodule ${}_RM_S$, there is a faithful contravariant functor*

$$\mathcal{D} \to \mathrm{Nat}(U_R, U_S\circ(-\otimes_R M_S)),$$

$$x \mapsto \eta_x,$$

$$(\overline{(r,s)}\colon x\to y) \mapsto ((\rho_r, -\otimes\rho_s)\colon \eta_y \to \eta_x.$$

For the covariant version of Theorem 3.39, we must state the Eilenberg-Watts Theorem 3.35 in its varian for left module: Let R and S be rings and $F\colon S\text{-Mod} \to R\text{-Mod}$ be a right exact additive functor that preserves direct sums. Then $F({}_SS)$ is an R-S-bimodule and the two functors, $F, {}_RM\otimes_S -$ are naturally isomorphic.

The theorem corresponding to Theorem 3.39 is the following:

Theorem 3.40. *For every R-S-bimodule ${}_RM_S$, there is a faithful covariant functor*

$$\mathcal{D} \to \mathrm{Nat}(U_S, U_R\circ({}_RM\otimes_S -)),$$

$$x \mapsto \zeta_x,$$

$$(\overline{(r,s)}\colon x\to y) \mapsto ((\lambda_s, \lambda_r\otimes -)\colon \zeta_x \to \zeta_y.$$

The natural transformation $\zeta_x : U_S \to (U_R \circ ({}_R M \otimes_S -)$, is such that, for every left S-module ${}_S B$, $\zeta_{x,B} : b \in B \mapsto x \otimes b \in M \otimes_S B$. For every left S-module ${}_S B$ and every $b \in {}_S B$, we have a commutative square

$$\begin{array}{ccc} b & \xmapsto{\zeta_x} & x \otimes b \\ \downarrow{\scriptstyle \lambda_s} & & \downarrow{\scriptstyle \lambda_r} \\ sb & \xmapsto{\zeta_y} & ys \otimes b = r(x \otimes b). \end{array}$$

Hence there is a commutative square

$$\begin{array}{ccc} U_S(-) & \xrightarrow{\zeta_{x,-}} & U_S \circ ({}_R M \otimes_S -) \\ \downarrow{\scriptstyle \lambda_s} & & \downarrow{\scriptstyle \lambda_r \otimes -} \\ U_S(-) & \xrightarrow{\zeta_{y,-}} & U_R \circ ({}_R M \otimes_S -) \end{array}$$

of natural transformations between functors S-Mod $\to$ Ab, that is, a morphism $(\lambda_s, \lambda_r \otimes -) : \zeta_x \to \zeta_y$ in the category $\mathrm{Nat}(U_S, U_R \circ ({}_R M \otimes_S -))$.

3.6 Rings of Finite Type

Definition 3.41. *Let S be an arbitrary ring and $n \geq 1$ be an integer. The ring S has* type n *if the ring $S/\mathrm{J}(S)$ is a direct product of n division rings, and S is a* ring of finite type *if it has type n for some integer $n \geq 1$.*

Proposition 3.42. *Let S be a ring and $n \geq 1$ an integer. The following conditions are equivalent:*

(a) *The ring S has type n.*

(b) *n is the smallest of the positive integers m for which there is a local morphism of S into a direct product of m division rings.*

(c) *The ring S has exactly n distinct maximal right ideals, and they are all two-sided ideals in S.*

(d) *The ring S has exactly n distinct maximal left ideals, and they are all two-sided ideals in S.*

Proposition 3.43. *Let x be an object of $\mathcal{D}$. If $\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)}$ and $\frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)}$ are rings of type m and n, respectively, then $\mathrm{End}_{\mathcal{D}}(x)$ has type $\leq m+n$. Moreover, if $I_1, I_2, \dots, I_m$ are the m maximal ideals of $\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)}$ and $K_1, K_2, \dots, K_n$ are the n maximal ideals of $\frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)}$, then the at most $n+m$ maximal ideals of $\mathrm{End}_{\mathcal{D}}(x)$ are among the completely prime ideals $(I_t \times \frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)}) \cap \mathrm{End}_{\mathcal{D}}(x)$, where $t = 1, \dots, m$ and $(\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)} \times K_q) \cap \mathrm{End}_{\mathcal{D}}(x)$, where $q = 1, \dots, n$.*

Proof. Let I_t, with $t = 1, 2, \dots, m$ be the m maximal ideals of the ring $\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)}$ of type m. Then the canonical projection

$$\pi_R : \frac{(xS:_R x)}{\mathrm{l.ann}_R(x)} \longrightarrow \frac{(xS:_R x)}{\mathrm{l.ann}_R(x)} / \mathrm{J}\Big(\frac{(xS:_R x)}{\mathrm{l.ann}_R(x)}\Big) \cong \prod_{t=1}^{m} \frac{(xS:_R x)}{\mathrm{l.ann}_R(x)} / I_t$$

is a local morphism. Similarly for the canonical projection

$$\pi_S : \frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)} \longrightarrow \frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)} / \mathrm{J}\Big(\frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)}\Big) \cong \prod_{q=1}^{n} \frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)} / K_q.$$

Therefore, composing these projection with $\xi : \mathrm{End}_{\mathcal{D}}(x) \to \mathrm{End}(Rx) \times \mathrm{End}(xS)$ and using Lemma 3.15, we obtain that

$$\mathrm{End}_{\mathcal{D}}(x) \longrightarrow \prod_{t=1}^{m} \frac{(xS:_R x)}{\mathrm{l.ann}_R(x)} / I_t \times \prod_{q=1}^{n} \frac{(Rx:_S x)}{\mathrm{r.ann}_S(x)} / K_q$$

is a canonical local morphism into the direct product of $m+n$ division rings. Hence by Proposition 3.42 the ring $\mathrm{End}_{\mathcal{D}}(x)$ is a ring of type $m+n$. Furthermore, from the proof of Proposition 3.42, one can see that the maximal ideals of $\mathrm{End}_{\mathcal{D}}(x)$ are among the kernels of the $m+n$ canonical projections, which concludes the proof. □

Bibliography

[AF92] Frank W. Anderson and Kent R. Fuller. *Rings and categories of modules*. Springer-Verlag, 1992.

[CD93] Rosa Camps and Warren Dicks. On semilocal rings. *Israel Journal of Mathematics*, 81:203–211, 1993.

[CEDF19] Federico Campanini, Susan F. El-Deken, and Alberto Facchini. Homomorphisms with semilocal endomorphism rings between modules. *Algebras and Representation Theory*, pages 1–20, 2019. Published online on 13 Dec. 2019.

[Coh03] Paul M Cohn. *Further algebra and applications*. Springer-Verlag London, 2003.

[Eil60] Samuel Eilenberg. Abstract description of some basic functors. *J. Indian Math. Soc*, 24:231–234, 1960.

[Fac98] Alberto Facchini. *Module theory: endomorphism rings and direct sum decompositions in some classes of modules*, volume 167. Birkhäuser, 1998.

[Fac02] Alberto Facchini. Direct sum decompositions of modules, semilocal endomorphism rings, and krull monoids. *Journal of Algebra*, 256:280–307, 2002.

[Fac04] Alberto Facchini. Geometric regularity of direct-sum decompositions in some classes of modules. *Fundamentalnaya i Prikladnaya Matematika*, 10(3):231–244, 2004. English translation in Journal of Mathematical Sciences, 139(4):6814-6822, 2006.

[Fac12] Alberto Facchini. Direct-sum decompositions of modules with semilocal endomorphism rings. *Bulletin of Mathematical Sciences*, 2:225–279, 2012.

[Fac17] Alberto Facchini. *Introduction to Ring and Module Theory.* Libreria Progetto, 2017.

[Fac19] Alberto Facchini. *Semilocal Categories and Modules with Semilocal Endomorphism Rings*, volume 331 of *Progress in mathematics.* Birkhäuser, 2019.

[FH06] Alberto Facchini and Dolors Herbera. Local morphisms and modules with a semilocal endomorphism ring. *Algebras and representation theory*, 9(4):403–422, 2006.

[FHLV95] Alberto Facchini, Dolors Herbera, Lawrence S. Levy, and Peter Vámos. Krull-schmidt fails for artinian modules. *Proceedings of the American Mathematical Society*, 123(12):3587–3592, 1995.

[Lam91] Tsit Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics.* Springer-Verlag, first edition, 1991.

[ML98] Saunders Mac Lane. *Categories for Working Mathematicians*, volume 5 of *Graduate texts in mathematics.* Springer-Verlag, second edition, 1998.

[Wat60] Charles E. Watts. Intrinsic characterizations of some additive functors. *Proceedings of the American Mathematical Society*, 11(1):5–8, 1960.