

# UNIVERSITA' DEGLI STUDI DI PADOVA 

TESI DI LAUREA MAGISTRALE

## Study of the systems of higher spin fields and their interactions

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## 1 Introduction

Classical higher spin field theory has been developed over many years in the last century, starting from work [1] by E. Majorana of 1932 and by P.A.M. Dirac [2] of 1936 in which he generalized his famous spin- $1 / 2$ field equation to the description of free higher spin fields. Dirac's work was followed by such prominent theorists as Fierz and Pauli [3], Wigner [4], Bargmann and Wigner [5], Rarita and Schwinger [6], Weinberg [7], Fronsdal [8] and others. A motivation behind these studies was that, because in Quantum Field Theory it is assumed that elementary particles are described by different irreducible representations of the Poincarè group, there is no apparent reason to not consider higher spin particles as well.

Moreover, composite strongly interacting and short-lived massive particles (resonances) with spin up to $15 / 2$ have been observed experimentally. They were first discovered in the 1960s. The Regge properties (i.e. spin-mass dependence) of the resonances, nicely fitting to a spectrum of states of a one-dimensional relativistic object, led to the original discovery of string theory. So the study of higher spin particles and fields is of interest from both the experimental and purely theoretical perspective.

Though the non-interacting higher spin fields theory is free of any pathology, many problems arise at the interaction level (e.g. with electromagnetism and gravitation). The construction of the consistent interactions of higher-spin fields is the major non-trivial problem in this theory, as we shall demonstrate by trying to couple a massive spin-2 particle to a Maxwell field, and a massless spin- $5 / 2$ particle to gravity. These inconsistencies show up in the classical theory as well as at the quantum level, and demonstrate that the minimal coupling does not work for higher-spin fields, and we will see that, in particular, the problem of coupling of higher spins to gravity (for example the Aragone-Deser problem [9]) can be solved by considering the fields in backgrounds with a non-zero cosmological constant such as a de Sitter or an Anti de Sitter (AdS) spaces. This observation was first made by Fradkin and Vasiliev [10], who constructed consistent cubic interactions of higher spin fields with gravity in an (anti)-de-Sitter background. Their results led, later on, to the development by Vasiliev of a powerful method for the construction of complete non-linear equations describing higher-spin interactions [11, 12, 13].

The aim of this thesis is to study the main features of the theory which describes the dynamics of higher spin particles and corresponding fields, and the problems of their interactions (in particular with the gravitational field). The structure of actions and equations of motions of massless higher spin fields will be studied together with their global and local symmetries, both in flat and AdS space-time. In particular, we will study the so called "triplet" systems which are composed of three totally symmetric tensors (for bosons) or spinor-tensors (for fermions) of rank $s, s-1, s-2$, and they have an interesting field content: they describe particles of spin from $s$ to 1 or 0 (bosons) and from $s$ down to $1 / 2$ (fermions). The importance of the triplet systems is that they furnish a link between higher spin theory and string theory. Two different descriptions of the triplets will be given, the "metric-like" formulation and the "frame-like" formulation with the aim to solve the still open problem of the construction of the "metric-like" action for fermionic triplets in AdS. The particular case which will be studied in detail is the triplet system which describes fields of spins $5 / 2,3 / 2$ and $1 / 2$.

The construction of the metric-like action for the spin-5/2 and spin-3/2 doublet in AdS
space, which is straightforwardly extended to the triplet system by adding the spin-1/2 field action, is the main original result of this thesis that resolves the problem of the construction of such an action encountered previously by two groups of theorists [14, 15].

The thesis is organized as follows.
In Section 2, we will review the procedure followed by Fang and Fronsdal, using the metric-like formalism, to construct the Lagrangian for irreducible higher spin fields, both in the bosonic and fermionic case.

In Section 3 we will present the interaction problem, giving two examples of how inconsistencies show up when one tries to introduce the minimal interaction of higher spin fields with an electromagnetic field and gravity.

In Section 4, we will present the frame-like formulation of massless higher spin fields, and we will see that this kind of formalism proves to be much more powerful and general than the metric-like one. In fact, we will see how this approach naturally allows to relax the conditions on the fields and on the gauge parameters, uncovering a number of features that seemed to be hidden in the old metric-like approach to higher spin field theory. One of these features regards the "triplet" systems of higher spin fields. As we will see in Section 8, string theory contains an infinite tower of massive higher spin particle states, which naturally split into the massless triplet systems in the limit in which string tension goes to zero.

In Sections 5, 6 and 7 we will focus our attention on the doublet system of fields wih spin $-5 / 2$ and spin $-3 / 2$. Starting from their frame-like formulation, we will derive the metric-like action and equations of motions, first in Minkowski and then in AdS spacetime. We will also verify that the action is gauge-invariant and splits into the sum of actions for irreducible spin- $3 / 2$ and spin- $5 / 2$ fields, thus showing that it describes the fields with the correct number of degrees of freedom.

In Section 8, we will briefly review the main features of the open bosonic string in flat space-time, such as the action, conformal symmetry, BRST quantization and appearance of higher-spin triplets in the tensionless string limit.

In the Appendices we summarize our notation and conventions, and give the details of some computations that we skip in the discussion, for brevity.

## 2 Metric-like description of free irreducible higherspin fields

In this section we will consider the construction of higher spin field equations and actions in a flat space-time in the so called "metric-like" formalism, making use of a notation which naturally generalizes that of Einstein for the linearized gravity (i.e. massless spin-2 field). In 1939, Fierz and Pauli [3] studied particles of arbitrary high spin, following a field-theoretical approach, requiring Lorentz invariance and positivity of energy. Then, when Wigner [4] and Bargmann and Wigner [5] published their works on the representations of the Poincarè group, it became clear that the positivity of energy was equivalent to the requirement that the single-particle states are described by irreducible unitary representations of the Poincarè group. Although in space-times with dimension $D>4$ there also exist independent tensor fields with mixed symmetry, in the following we shall restrict our attention to the totally symmetric tensors (and tensor-spinors).

The group-theoretical approach leads us to classify particles in terms of the quantum numbers related to the two Casimir operators ${ }^{1}$ of the Poincarè group, i.e.:

$$
\begin{equation*}
C_{1}=P^{m} P_{m},, \quad C_{2}=W^{m} W_{m}, \tag{2.1}
\end{equation*}
$$

where $P_{m}$ is the 4 -momentum of the particle and $W^{m}=-\frac{1}{2} \varepsilon^{m n r s} J_{n r} P_{s}$ is the PauliLubanski pseudovector. For the particles of mass $m$ and spin $s$, we have $C_{1}=m^{2}$ and $C_{2}=-m^{2} s(s+1)$. When $P^{m} P_{m}=m^{2}=0$, we have either $W^{2} \neq 0$ or $W^{2}=0$. In the first case, the representations do not correspond to physical states, and thus we will not consider them. On the contrary, when $W^{2}=0$, it can be shown that $W^{m}=h P^{m}$ where $h$ is the helicity operator $h=\frac{\mathbf{J} \cdot \mathbf{P}}{|\mathbf{P}|}$. Thus, representations of the Poincarè group can be classified in this case by the helicity eigenvalues. In the following, we separately treat the integer spin and half-integer spin cases.

### 2.1 Massive spins

### 2.1.1 Bosonic case

In the bosonic case ( $s$ integer), the particle states are described by the representation $D\left(\frac{s}{2}, \frac{s}{2}\right)$ of the $D=4$ Lorentz group. The corresponding field is a totally symmetric tensor, which satisfies the tracelesness condition:

$$
\begin{equation*}
\eta^{m_{1} m_{2}} \phi_{m_{1} m_{2} \ldots m_{s}}=0 . \tag{2.2}
\end{equation*}
$$

The Casimir operator $C_{1}$ leads to the following equation of motion:

$$
\begin{equation*}
\left(\square+m^{2}\right) \phi_{m_{1} \ldots m_{s}}=0, \tag{2.3}
\end{equation*}
$$

while the Casimir $C_{2}$ requires all the lower values of spin to vanish. This is achieved, together with (2.2), by imposing the following transversality condition:

$$
\begin{equation*}
\partial^{m_{1}} \phi_{m_{1} \ldots m_{s}}=0 \tag{2.4}
\end{equation*}
$$

[^0]Let us show that the counting of the degrees of freedom reproduces the expected result in $4 D$, i.e. $2 s+1$ for a massive field of $\operatorname{spin} s$. The symmetric field $\phi_{m_{1} m_{2} \cdots s}$ has

$$
\begin{equation*}
C(D+s-1, s)=\frac{(D+s-1)!}{s!(D-1)!} \tag{2.5}
\end{equation*}
$$

independent components. Imposing the tracelessness condition (2.2) we rule out $C(D+s-3, s-2)$ of these components, while the transversality condition (2.4) removes other $C(D-2+s, s-1)$ of them. However, we should remove the trace from the counting of the number of the independent components in the transverality condition, and hence we must add $C(D-4+s, s-3)$ degrees of freedom. Summing up all these contributions we get

$$
\begin{align*}
\mathcal{N}_{b, m \neq 0} & =C(D-1+s, s)-C(D-3+s, s-2)-C(D-2+s, s-1)+ \\
& +C(D-4+s, s-3)=C(D-4+s, s)+2 C(D-4+s, s-1) \tag{2.6}
\end{align*}
$$

which for $D=4$ gives $2 s+1$ degrees of freedom, as expected.

### 2.1.2 Fermionic case

Let us now consider a massive fermionic field of $\operatorname{spin} s=n+\frac{1}{2}$. This is represented by a symmetric tensor-spinor $\psi_{m_{1} \ldots m_{n}}$ of rank $n$, which in $D=4$ transforms as the $D\left(\frac{n+1}{2}, \frac{n}{2}\right) \oplus$ $D\left(\frac{n}{2}, \frac{n+1}{2}\right)$ representation of the Lorentz group. This field satisfies the equations

$$
\begin{gather*}
\gamma^{m} \psi_{m m_{2} \ldots m_{n}}=0,  \tag{2.7}\\
(i \not \partial-m) \psi_{m_{1} \ldots m_{n}}=0,  \tag{2.8}\\
\partial^{m_{1}} \psi_{m_{1} \ldots m_{n}}=0 . \tag{2.9}
\end{gather*}
$$

A counting similar to that done for bosons, leads us to the following number of degrees of freedom propagated by the field $\psi_{m_{1} \ldots m_{n}}$ in $D$ space-time dimensions

$$
\begin{equation*}
\mathcal{N}_{f, m \neq 0}=C(D-3+n, n) \times 2^{[D] / 2}, \tag{2.10}
\end{equation*}
$$

where $[D] \equiv D+\frac{1}{2}\left[(-1)^{D}-1\right]$, and $2^{[D] / 2}$ is the dimension of the representation to which belongs a Lorentz-spinor in $D$ dimensions. Again, the number of degrees of freedom, for $D=4$, reduces to $2 \times(2 s+1)$, as expected.

One may ask if equations (2.3), and conditions (2.2) and (2.4), and the corresponding equations (2.7)-(2.9) for the fermionic half-integer spin fields can be deduced from a Lagrangian. This was done in 1974 by Singh and Hagen [16, 17] who constructed the Lagrangians for bosonic and fermionic massive higher-spin fields generalizing the earlier construction of Fierz and Pauli [3]. In these Lagrangians the transversality condition (2.4) was incorporated with the use of a chain of auxiliary fields of ranks $s-2, s-3, \ldots 0$, which are all symmetric and traceless (in the bosonic case) or $\gamma$-traceless (in the fermonic case) too. When the equations of motion and the subsidiary conditions are satisfied, all the lower spin fields vanish.

Since in this thesis we will only deal with massless fields, we do not present here the Lagrangian for the massive higher spin fields. In what follows we will consider the massless limit of the Singh-Hagen Lagrangians, carried out by Fronsdal in the bosonic case [8] and by Fang and Fronsdal in the fermionic case [18].

### 2.2 Massless higher spins

### 2.2.1 Bosonic case

In the bosonic case, Fronsdal found that all the auxiliary fields of lower spins decouple in the massless limit except for a field of rank $s-2$. Moreover, the two remaining fields of rank $s$ and $s-2$ can be combined together into a symmetric tensor field $\Phi_{m_{1} \ldots m_{s}}$ which is double-traceless (for $s \geq 4$ ), i.e. such that

$$
\eta^{m_{1} m_{2}} \eta^{m_{3} m_{4}} \Phi_{m_{1} m_{2} m_{3} m_{4} \ldots m_{s}}=0 .
$$

The Lagrangian in question has the following form

$$
\begin{align*}
\mathcal{L}= & -\frac{1}{2}\left(\partial_{r} \Phi_{m_{1} \ldots m_{s}}\right)^{2}+\frac{1}{2} s\left(\partial \cdot \Phi_{m_{2} \ldots m_{s}}\right)^{2}+\frac{1}{2} s(s-1)\left(\partial \cdot \partial \cdot \Phi_{m_{3} \ldots m_{s}}\right) \Phi^{\prime m_{3} \ldots m_{s}} \\
& +\frac{1}{4} s(s-1)\left(\partial_{r} \Phi_{m_{3} \ldots m_{s}}^{\prime}\right)^{2}+\frac{1}{8} s(s-1)(s-2)\left(\partial \cdot \Phi_{m_{4} \ldots m_{s}}^{\prime}\right)^{2}, \tag{2.11}
\end{align*}
$$

where $\Phi_{m_{3} \ldots m_{s}}^{\prime} \equiv \eta^{m_{1} m_{2}} \Phi_{m_{1} m_{2} \ldots m_{s}}$, and $\partial \cdot \Phi_{m_{2} \ldots m_{s}} \equiv \partial^{m_{1}} \Phi_{m_{1} m_{2} \ldots m_{s}}$. The Lagrangian (2.11) is gauge-invariant under the following transformations with the symmetric and traceless parameter $\xi_{m_{1} \ldots m_{s-1}}(x)$ :

$$
\begin{equation*}
\delta \Phi_{m_{1} \ldots m_{s}}=\partial_{\left(m_{1}\right.} \xi_{\left.m_{2} \ldots m_{s-1}\right)}, \quad \xi_{m_{4} \ldots m_{s}}^{\prime}=0 \tag{2.12}
\end{equation*}
$$

The equations of motion derived from the Lagrangian (2.11) are

$$
\begin{equation*}
\mathcal{F}_{m_{1} \ldots m_{s}}-\frac{1}{2} \eta_{\left(m_{1} m_{2}\right.} \mathcal{F}_{\left.m_{3} \ldots m_{s}\right)}^{\prime}=0 \tag{2.13}
\end{equation*}
$$

where the so-called Fronsdal tensor $\mathcal{F}_{m_{1} \ldots m_{s}}$ is defined as

$$
\begin{equation*}
\mathcal{F}_{m_{1} \ldots m_{s}} \equiv \square \Phi_{m_{1} \ldots m_{s}}-\partial_{\left(m_{1}\right.} \partial \cdot \Phi_{\left.m_{2} \ldots m_{s}\right)}+\partial_{\left(m_{1}\right.} \partial_{m_{2}} \Phi_{\left.m_{3} \ldots m_{s}\right)}^{\prime}, \quad \square=\partial_{m} \partial^{m} \tag{2.14}
\end{equation*}
$$

Due to the double-tracelessness of $\Phi_{m_{1} \ldots m_{s}}$, we can rewrite equation (2.13) in the following simpler form:

$$
\begin{equation*}
\mathcal{F}_{m_{1} \ldots m_{s}}=0 \tag{2.15}
\end{equation*}
$$

It is worth noting that, if we consider the spin 2 case, equation (2.13) reduces to the Einstein linearized equation, and the Fronsdal tensor is nothing but the linearized Ricci tensor.

We can now exploit the gauge invariance (2.12) to recast the equation (2.15) in a simpler form. To this end we impose the following gauge fixing condition

$$
\begin{equation*}
\mathcal{G}_{m_{2} \ldots m_{s}} \equiv \partial \cdot \Phi_{m_{2} \ldots m_{s}}-\frac{1}{2} \partial_{\left(m_{2}\right.} \Phi_{\left.m_{3} \ldots m_{s}\right)}^{\prime}=0, \tag{2.16}
\end{equation*}
$$

where $\mathcal{G}_{m_{4} \ldots m_{s}}^{\prime}=0$ because $\eta^{m_{1} m_{2}} \eta^{m_{3} m_{4}} \Phi_{m_{1} m_{2} m_{3} m_{4} \ldots m_{s}}=0$. In this gauge the Fronsdal tensor (2.14) and hence eq. (2.15) reduce to

$$
\begin{equation*}
\square \Phi_{m_{1} \ldots m_{s}}=0, \tag{2.17}
\end{equation*}
$$

which implies that we indeed deal with a massless higher spin field.

We can now make the counting of the physical degrees of freedom described by the Fronsdal field. The field $\Phi_{m_{1} m_{2} m_{3} m_{4} \ldots m_{s}}$ with vanishing double-trace has $C(D-1+s, s)-$ $C(D-5+s, s-4)$ indipendent components in $D$ dimensions. With the partial gaugefixing (2.16) we remove $C(D-2+s, s-1)-C(D-4+s, s-3)$ of these components. The residual gauge symmetry (2.12) with a parameter $\xi_{m_{2} \ldots m_{s}}$ which satisfies $\square \xi_{m_{2} \ldots m_{s}}=0$, allows us to remove other $C(D-2+s, s-1)-C(D-4+s, s-3)$ components. Then, we are left with the following number of degrees of freedom:

$$
\begin{equation*}
\mathcal{N}_{b, m=0}=C(D-5+s, s)+2 C(D-5+s, s-1), \tag{2.18}
\end{equation*}
$$

which, in $D=4$, gives exactly two degrees of freedom for all $s$. These degrees of freedom are associated with a positive and negative helicity of the spin-s particle along its light-like momentum.

### 2.2.2 Fermionic case

In the fermionic case, Fang and Fronsdal [18] found that, in the massless limit, all the lower-rank (auxiliary) fields in the Singh-Hagen Lagrangian decouple, except for the two symmetric and $\gamma$-traceless tensor-spinors of highest ranks $n-1$ and $n-2$. The originary field and these two auxiliary fields can be combined into a single rank- $n$ tensor-spinor $\Psi_{m_{1} \ldots m_{n}}$ which is triple $\gamma$-traceless (for $s=n+\frac{1}{2} \geq \frac{7}{2}$ ), i.e. $\gamma^{m_{1}} \gamma^{m_{2}} \gamma^{m_{3}} \Psi_{m_{1} \ldots m_{n}}=0$. The Lagrangian describing the free dynamics of this field has the following form

$$
\begin{align*}
i \mathcal{L}= & \bar{\Psi}_{m_{1} \ldots m_{n}} \not \partial \Psi^{m_{1} \ldots m_{n}}+n \bar{\Psi}_{m_{1} \ldots m_{n}} \not \partial \Psi^{m_{2} \ldots m_{n}}-\frac{1}{4} n(n-1) \bar{\Psi}_{m_{3} \ldots m_{n}}^{\prime} \not \partial \Psi^{\prime m_{3} \ldots m_{n}} \\
& -n\left[\bar{\Psi}_{m_{2} \ldots m_{n}} \partial \cdot \Psi^{m_{2} \ldots m_{n}}-\text { h.c. }\right]+\frac{1}{2} n(n-1)\left[\bar{\Psi}_{m_{3} \ldots m_{n}}^{\prime} \partial \cdot \Psi^{\prime m_{3} \ldots m_{n}}-\text { h.c. }\right] . \tag{2.19}
\end{align*}
$$

The equations of motion derived from this Lagrangian are:

$$
\begin{equation*}
\mathcal{S}_{m_{1} \ldots m_{n}}-\frac{1}{2} \gamma_{\left(m_{1}\right.} \boldsymbol{\phi}_{\left.m_{2} \ldots m_{n}\right)}-\frac{1}{2} \eta_{\left(m_{1} m_{2}\right.} S_{\left.m_{3} \ldots m_{n}\right)}^{\prime}=0 \tag{2.20}
\end{equation*}
$$

where the fermionic Fronsdal tensor-spinor is defined as:

$$
\begin{equation*}
\mathcal{S}_{m_{1} \ldots m_{n}} \equiv i\left[\not \partial \Psi_{m_{1} \ldots m_{n}}-\partial_{\left(m_{1}\right.} \Psi_{\left.m_{2} \ldots m_{n}\right)}\right] . \tag{2.21}
\end{equation*}
$$

Equations (2.20) can be rewritten in the form:

$$
\begin{equation*}
\mathcal{S}_{m_{1} \ldots m_{n}}=0 . \tag{2.22}
\end{equation*}
$$

The Lagrangian and the equations of motion enjoy the following gauge symmetry:

$$
\begin{equation*}
\delta \Psi_{m_{1} \ldots m_{n}}=\partial_{\left(m_{1}\right.} \varepsilon_{\left.m_{2} \ldots m_{n}\right)}, \quad \gamma^{m_{2}} \varepsilon_{m_{2} \ldots m_{n}}=0 \tag{2.23}
\end{equation*}
$$

where $\varepsilon_{m_{2} \ldots m_{n}}$ is a symmetric rank $n-1$ tensor-spinor. Note that equation (2.22) implies that the time component of the $\gamma$-traceless part of the field $\Psi_{m_{1} \ldots m_{n}}$, which we will write $\tilde{\Psi}_{0 m_{2} \ldots m_{n}}$ is not dynamical, since it is constant in time. We can see this by multiplying (2.22) with $\gamma^{m_{1}}$ from the left, thus getting

$$
\begin{equation*}
i\left[-2 \not \partial \tilde{\Psi}_{m_{2} \ldots m_{n}}+2 \partial^{m} \tilde{\Psi}_{m m_{2} \ldots m_{n}}\right]=2 i \partial^{m} \tilde{\Psi}_{m m_{2} \ldots m_{n}}=0 \tag{2.24}
\end{equation*}
$$

The last equality in (2.24) tells us that $\tilde{\Psi}_{0 m_{2} \ldots m_{s}}$ is a conserved charge, thus it does not vary with time.

To carry out the counting of the degrees of freedom, we note that a symmetric triple $\gamma$-traceless rank- $n$ tensor-spinor has $[C(D+n-1, n)-C(D+n-4, n-3)] \times 2^{\frac{[D]}{2}}$ indipendent components. Though, since $\tilde{\Psi}_{0 m_{2} \ldots m_{s}}$ is not dynamical, we already have $[C(D+n-2, n-1)-C(D+n-3, n-2)] \times 2^{[D] / 2}$ constraints. Moreover, we can fix a gauge using the $\gamma$-traceless parameter $\varepsilon_{m_{1} \ldots m_{n-1}}$, e.g. by putting to zero the following combination of the field components

$$
\begin{equation*}
\mathcal{G}_{m_{1} \ldots m_{n}} \equiv \Psi_{m_{2} \ldots m_{n}}-\frac{1}{D-4+2 n} \gamma_{\left(m_{2}\right.} \Psi_{\left.m_{3} \ldots m_{n}\right)}^{\prime}=0 \tag{2.25}
\end{equation*}
$$

Upon this gauge-fixing we are still left with the residual gauge transformations of the form

$$
\begin{equation*}
\not \partial \varepsilon_{m_{2} \ldots m_{n}}-\frac{2}{D-4+2 n} \gamma_{\left(m_{2}\right.} \partial \cdot \varepsilon_{\left.m_{3} \ldots m_{n}\right)}=0 \tag{2.26}
\end{equation*}
$$

The gauge-fixing (2.25) and the residual gauge transformations (2.26) remove [ $C$ ( $D+n-$ $2, n-1)-C(D+n-3, n-2)] \times 2^{[D] / 2}$ components each. Thus, the number of degrees of freedom is found to be:

$$
\begin{equation*}
\mathcal{N}_{f, m=0}=C(D-4+n, n) \times 2^{[D] / 2} . \tag{2.27}
\end{equation*}
$$

Again, in $D=4$ we are left with 4 degrees of freedom, as expected. We stress the fact that equations (2.22) do not reduce to the Dirac equation, under the gauge-fixing (2.25). To obtain the Dirac equations $\not \partial \Psi_{m_{1} \ldots m_{n}}=0$ one should use the residual symmetry (2.26) to set $\Psi_{m_{3} \ldots m_{n}}^{\prime}=0$.

## 3 The interaction problem

The construction of consistent interactions of massless and massive higher-spin fields with electromagnetic fields and gravity, as well as among themselves, is a highly non-trivial and very interesting problem which has been addressed since the first appearance of higherspin fields in works by Dirac, and Fierz and Pauli in the 1930s. The consistency requires, first of all, that interactions do not spoil (but can only modify) symmetries of the free theory and, hence, the number of physical degrees of freedom of the fields remain intact. Below we demonstrate these issues on two simple examples, the problem of coupling a massive spin-2 field to an electro-magnetic field, and the problem of coupling a massless spin $-5 / 2$ field to gravity.

### 3.1 Coupling of massive spin-2 field to a photon

Trying to couple a massive spin 2 field $\phi_{m n}$ to a Maxwell field $A_{m}$, we will show how the minimal coupling procedure gives rise to inconsistencies. As we considered in Section 2, the field $\phi_{m n}=\phi_{n m}$ satisfies eqs. (2.2)-(2.4). We would like to couple the field $\phi_{m n}$ to $A_{m}$ by replacing the partial derivative with the covariant one. Then the equations take the form

$$
\begin{equation*}
\left(\nabla^{2}+m^{2}\right) \phi_{m n}=0, \quad \nabla^{m} \phi_{m n}=0, \quad \phi_{m}^{m}=0 . \tag{3.1}
\end{equation*}
$$

From the equation of motion and the transversality condition, we get:

$$
\begin{equation*}
\left[\nabla^{2}+m^{2}, \nabla^{m}\right] \phi_{m n}=0 \tag{3.2}
\end{equation*}
$$

Since the covariant derivates do not commute, equation (3.2) results in further constraints on the field $\phi_{m n}$. For instance, for a constant electromagnetic field-strength $F_{m n}$, we find that the field $\phi_{m n}$ should satisfy the additional condition

$$
\begin{equation*}
i e F^{m n} \nabla_{m} \phi_{n p}=0, \tag{3.3}
\end{equation*}
$$

a constraint which we did not have in the absence of the electro-magnetic interactions. As a result the field $\phi_{m n}$ does not propagate the correct number of degrees of freedom and hence such a minimal electro-magnetic interaction is inconsistent.

### 3.2 Coupling $s=5 / 2$ to gravity

Following [19], we shall now illustrate the problem arising when one tries to couple a massless spin $5 / 2$ field to gravity, which is known as the Aragone-Deser problem [9]. Such a field is represented by a tensor-spinor $\Psi_{m_{1} m_{2}}^{\alpha}{ }^{2}$ which is symmetric in its tensor indices, and obeys the following equation of motion

$$
\begin{equation*}
\mathcal{S}_{m_{1} m_{2}}=i \gamma^{n}\left(\partial_{n} \Psi_{m_{1} m_{2}}-\partial_{m_{1}} \Psi_{n m_{2}}-\partial_{m_{2}} \Psi_{n m_{1}}\right)=0 \tag{3.4}
\end{equation*}
$$

It is invariant under the following gauge transformations:

$$
\begin{equation*}
\delta \Psi_{m_{1} m_{2}}=2 \partial_{\left(m_{1}\right.} \xi_{\left.m_{2}\right)} \tag{3.5}
\end{equation*}
$$

[^1]Now, in order to describe the minimal coupling of the field $\Psi_{m_{1} m_{2}}$ to gravity, one should replace the ordinary derivative with the covariant one, obtaining:

$$
\begin{equation*}
\mathcal{S}_{m_{1} m_{2}}=i \gamma^{n}\left(\nabla_{n} \Psi_{m_{1} m_{2}}-\nabla_{m_{1}} \Psi_{m m_{2}}-\nabla_{m_{2}} \Psi_{m m_{1}}\right)=0, \tag{3.6}
\end{equation*}
$$

where $\nabla_{n} \Psi_{a b}=\partial_{n} \Psi_{a b}-\Gamma_{n m_{1}}^{m} \Psi_{n m_{2}}-\Gamma_{n m_{2}}^{m} \Psi_{m m_{1}}-\frac{1}{4} \omega_{n a b} \gamma^{a b} \Psi_{m_{1} m_{2}}$ is the covariant derivative, and $\gamma^{n}=\gamma^{a} e_{a}{ }^{n}$ are the "curved" $\gamma$-matrices, $\omega_{n a b}$ is the spin-connection, and $\Gamma_{n p}^{m}$ is the affine connection (Christoffel symbols).

We should now check that the minimal generalization of the gauge transformations (3.5):

$$
\begin{equation*}
\delta \Psi_{m_{1} m_{2}}=2 \nabla_{\left(m_{1}\right.} \xi_{\left.m_{2}\right)} \tag{3.7}
\end{equation*}
$$

leaves the equation of motion (3.6) invariant. The gauge variation of equation (3.6) is

$$
\begin{equation*}
\delta S_{m_{1} m_{2}}=i\left[R_{m_{1} n} \gamma^{n} \xi_{m_{2}}+R_{m_{2} n} \gamma^{n} \xi_{m_{1}}-\left({R_{n m_{1} m_{2}}}^{p}+R_{n m_{2} m_{1}}{ }^{p}\right) \gamma^{n} \xi_{p}\right] \tag{3.8}
\end{equation*}
$$

We see that the variation of the equations of motion is non-vanishing, even when the gravitational field satisfies the "free" Einstein equations, i.e. when the Ricci tensor is zero, $R_{m n}=0$. So in this case the minimal approach to coupling the spin $s=5 / 2$ to gravity seems to be insufficient. Moreover, no way has been found to add non-minimal terms to (3.6) in order to remove the whole Riemann tensor appearing in the variation (3.8). Thus, it has been concluded that it is not possible to consistently couple the spin-5/2 field to gravity, at least in flat space-time.

This issue occurs for consistent coupling of every higher spin $(s>3 / 2)$ fermionic and bosonic field to gravity and is in agreement with the fact that, in flat space-time, any interacting theory involving higher spin particles would lead to a trivial S-matrix [20, 21]. Since this result is related to the trivial geometry of the flat Minkowski space, one may try to overcome the obstacle by constructing a theory, for instance, in an AdS background, which enjoys a space-time symmetry that is different from the Poincarè group ${ }^{3}$ and has a non-zero constant curvature.

Fronsdal in the bosonic case [22], and Fang and Fronsdal in the fermionic case [23], generalized the free higher-spin equations (2.15) and (2.22) to the AdS background. The equations of motion for an $s=5 / 2$ tensor-spinor interacting with gravity with an AdS background are

$$
\begin{equation*}
\mathcal{S}_{m_{1} m_{2}}=i \gamma^{n}\left(\mathcal{D}_{n} \Psi_{m_{1} m_{2}}-\mathcal{D}_{m_{1}} \Psi_{m m_{2}}-\mathcal{D}_{m_{2}} \Psi_{m m_{1}}\right)-2 \sqrt{-\Lambda} \Psi_{m_{1} m_{2}}=0 \tag{3.9}
\end{equation*}
$$

where a "mass-like" term, proportional to the square root of the cosmological constant $\Lambda$, has been added to the left hand side of the equation. Note that $-\Lambda$ is positive in AdS, then the term $\sqrt{-\Lambda}$ actually plays the role of a mass-like term, like the one appearing in the Dirac equation $(i \not \partial-m) \psi=0$. It can be shown that the spin $-5 / 2$ field obeying eq. (3.9) propagates in AdS the same number of degrees of freedom as in the Minkowski space due to the gauge-invariance of (3.9) under the following transformations:

$$
\begin{equation*}
\delta \Psi_{m_{1} m_{2}}=2 \mathcal{D}_{\left(m_{1}\right.} \xi_{\left.m_{2}\right)} \tag{3.10}
\end{equation*}
$$

The derivative in (3.9) and (3.10) is defined as follows

[^2]\[

$$
\begin{equation*}
\mathcal{D}_{m}=\nabla_{m}+i \frac{\sqrt{-\Lambda}}{2} \gamma_{m} \tag{3.11}
\end{equation*}
$$

\]

where the term $i \frac{\sqrt{-\Lambda}}{2} \gamma_{m}$ is meant to act only on spinor indices, and behaves like the affine connection in ordinary covariant derivatives, whose sign changes depending on whether the spinor index is up or down. Remember that the AdS-Riemann tensor takes the following simple form:

$$
\begin{equation*}
R_{m n p}^{A d S} q=\Lambda\left(\delta_{m}^{q} g_{n p}-\delta_{n}^{q} g_{m p}\right), \tag{3.12}
\end{equation*}
$$

and note that this implies that, in AdS:

$$
\begin{equation*}
\left[\mathcal{D}_{m}, \mathcal{D}_{n}\right] \psi^{\alpha}=0 \tag{3.13}
\end{equation*}
$$

for every spinor $\psi^{\alpha}$, while for a vector $A_{p}$ we have:

$$
\begin{equation*}
\left[\mathcal{D}_{m}, \mathcal{D}_{n}\right] A_{p}=\left[\nabla_{m}, \nabla_{n}\right] A_{p}=-R_{m n p}^{A d S}{ }^{q} A_{q} \tag{3.14}
\end{equation*}
$$

Now, if we consider fluctuations of the Riemann curvature around the AdS background, we find the following variation of equation (3.9)

$$
\begin{equation*}
\delta S_{m_{1} m_{2}}=-i\left(R_{n m_{1} m_{2}}^{p}+R_{n m_{2} m_{1}}^{p}\right) \gamma^{n} \xi_{p}+\ldots, \tag{3.15}
\end{equation*}
$$

where dots stand for terms that can be removed by an appropriate modification of the gauge transformations (3.10). This variation can be balanced by introducing the following additional term in equation (3.9) [19]:

$$
\begin{equation*}
\mathcal{S}_{m_{1} m_{2}}^{(1)}=-\frac{i}{2 \Lambda}\left(R_{p m_{1} m_{2} q}+R_{p m_{2} m_{1} q}\right) \not D \Psi^{p q} \tag{3.16}
\end{equation*}
$$

whose gauge variation is

$$
\begin{align*}
\delta S_{m_{1} m_{2}}^{(1)} & =-\frac{2 i}{\Lambda}\left(R_{p m_{1} m_{2} q}+R_{p m_{2} m_{1} q}\right) \gamma_{n} \mathcal{D}^{n} \mathcal{D}^{p} \xi^{q}= \\
& =\frac{i}{\Lambda}\left(R_{p m_{1} m_{2} q}+R_{p m_{2} m_{1} q}\right) \gamma_{n}\left[\mathcal{D}^{n}, \mathcal{D}^{p}\right] \xi^{q}+\ldots \tag{3.17}
\end{align*}
$$

where now ... stand for terms with the anticommutator of $\mathcal{D}_{n}$ which can be canceled by a modification of the gauge transformations and by adding more interaction terms. If we restrict ourselves only to linear terms in the fluctuating graviational field, we find that the variation (3.17) exactly cancels (3.15). It is worth noting that, since the cosmological constant enters (3.16) in a non-polynomial way, we cannot take the limit $\Lambda \rightarrow 0$, and hence we cannot reduce to the Minkowskii space-time.

This example demonstrates that it is possible to construct an interacting theory (at least at the first non-linear order in weak fields) between higher spins and gravity in a space-time with a non-zero cosmological constant. This observation was first made by Fradkin and Vasiliev [10], who constructed consistent cubic interactions of higher spin fields with gravity in an (anti)-de-Sitter background. Their results led, later on, to the development by Vasiliev of a powerful method for the construction of complete nonlinear equations describing higher-spin interactions $[11,12,13]$. The method is based on an alternative, so called frame-like, formulation of the higher-spin theory [24, 25] which we will now review for the free field case.

## 4 Frame-like description of free higher-spin fields

As we have seen in the first section, the metric-like formulation of massless higher-spin fields requires the fields to be double-traceless in the bosonic case, and triple- $\gamma$-traceless in the fermionic one. In the "frame-like" formulation of massless higher-spin gauge fields, the latter are represented by differential one-forms that carry irreducible representations of the local Lorentz group acting in the space tangent to the (generically curved) spacetime.

The frame-like approach has turned out to be more general and powerful than the metric-like one, as far as the higher spin interaction problem is concerned, somewhat similar to how the Cartan approach to the description of general relativity allowed one to couple gravity to fermionic spinor fields. In the Cartan (frame-like) formulation gravity is described by the one-form vielbein (or frame-field) $e^{a}=e_{m}{ }^{a} d x^{m}$, where the index "a" is a Lorentz index and the index " $m$ " is a world index, i.e. an index which has to be lowered or raised with the space-time metric $g_{m n}$. The space-time metric is defined as the "composite" field

$$
g_{m n}=e_{m}^{a} e_{n}^{b} \eta_{a b} .
$$

The Riemann tensor is defined as $R^{a b}[\omega]=d \omega^{a b}+\omega^{a}{ }_{c} \wedge \omega^{c b}$, where $\omega^{a b}=-\omega^{b a}=$ $d x^{m} \omega_{m}^{a b}$ is the spin-connection, which, in the zero-torsion case (or metricity condition), can be expressed as a function of only the vielbein fields, and is thus an auxiliary field.

To describe higher spin fields, by analogy with the frame-field $e^{a}$, one introduces rank $s-1$ symmetric traceless one-form fields $e^{a_{1} \ldots a_{s-1}}=d x^{m} e_{m ;}{ }^{a_{1} \ldots a_{s-1}}$. The spin connection is given by a one-form field $\omega^{a_{1} \ldots a_{s-1}, p}=d x^{m} \omega_{m ;}{ }^{a_{1} \ldots a_{s-1}, p}$ which is totally symmetric in the upper first $s-1$ indices, while the symmetrization on $s$ upper indices gives zero. In the half-integer spin case the fields also carry a spinor index.

### 4.1 Frame-like action for bosonic fields in flat space-time

In flat space-time we will not distinguish world indices from tangent-space ones, and we will label both of them by Latin lower case letters. The form-index will always be separated from the others by a semicolon. So the frame-like bosonic spin- $s$ field (or the higher-spin vielbein) is

$$
\begin{equation*}
e^{n_{1} \ldots n_{s-1}}=d x^{m} e_{m}{ }^{n_{1} \ldots n_{s-1}}, \tag{4.1}
\end{equation*}
$$

and the one-form spin connection is

$$
\begin{equation*}
\omega^{n_{1} \ldots n_{s-1}, p}=d x^{m} \omega_{m}{ }^{n_{1} \ldots n_{s-1}, p} . \tag{4.2}
\end{equation*}
$$

In (4.2) the indices $n_{1} \ldots n_{s-1}$ are totally symmetric, and $\omega^{n_{1} \ldots n_{s-1}, p}$ has the symmetry property of the Young tableau $Y(s-1,1)$, i.e. the symmetrization ${ }^{4}$ of every $s$ indices vanishes:

$$
\begin{equation*}
\omega^{\left(n_{1} \ldots n_{s-1}, p\right)}=0 . \tag{4.3}
\end{equation*}
$$

[^3]In order for the spin connection to be an auxiliary field, we need to impose the zero torsion condition. In the linearized case of the free higher spin fields the torsion constraint has the following form

$$
\begin{equation*}
T^{n_{1} \ldots n_{s-1}} \equiv d e^{n_{1} \ldots n_{s-1}}-(s-1) d x^{q} \omega^{n_{1} \ldots n_{s-1}, p} \eta_{p q}=0 \tag{4.4}
\end{equation*}
$$

The left hand side of (4.4) generalizes the notion of torsion of the graviational field, which is given by $T^{a}=\nabla e^{a}=d e^{a}+\omega^{a}{ }_{b} \wedge e^{b}$.

Under the gauge transformations the higher-spin vielbein (4.1) and the spin connection (4.2) are transformed as follows:

$$
\begin{align*}
& \delta e^{n_{1} \ldots n_{s-1}}=d \xi^{n_{1} \ldots n_{s-1}}-(s-1) d x^{q} \xi^{n_{1} \ldots n_{s-1}, p} \eta_{p q}, \\
& \delta \omega^{n_{1} \ldots n_{s-1}, p}=d \xi^{n_{1} \ldots n_{s-1}, p}-(s-2) d x^{q} \xi^{n_{1} \ldots n_{s-1}, p r} \eta_{r q} \tag{4.5}
\end{align*}
$$

where each of the parameters $\xi^{n_{1} \ldots n_{s-1}}, \xi^{n_{1} \ldots n_{s-1}, p}$ and $\xi^{n_{1} \ldots n_{s-1}, p_{1} p_{2}}$ is symmetric on each group of indices, and the last two have the symmetry properties respectively of the Young tableaux $Y(s-1,1)$ and $Y(s-1,2)$, i.e. the symmetrization of any of their $s$ indices yields zero.

The zero torsion condition (4.4), which is invariant under the gauge variations (4.5), can be deduced from an action, together with the dynamical field equation for the higherspin vielbein $e^{n_{1} \ldots n_{s-1}}$. This action is constructed by analogy with the linearized Cartan action for the general relativity and has the following form [26]

$$
\begin{equation*}
S=\int_{M^{D}} d x^{a_{1}} \ldots d x^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left(d e^{n_{1} \ldots n_{s-2} p}-\frac{s-1}{2} d x_{m} \omega^{n_{1} \ldots n_{s-2} p, m}\right) \omega_{n_{1} \ldots n_{s-2}} q, r . \tag{4.6}
\end{equation*}
$$

For $s=2$ the above action coincides with the linearized gravity action. It is worth noting that the action (4.6) does not describe the spin 1 and the scalar fields. This is due to the fact that these fields do not carry any tangent space indices, while at least one of the tangent space indices is needed to construct (4.6). Thus, to include in this construction also the Maxwell and the scalar field, one should add to the action (4.6) the conventional Maxwell and Klein-Gordon terms.

The action (4.6) is invariant under the gauge transformations (4.5) provided that the connection satisfies the traceless condition

$$
\begin{equation*}
\eta_{n_{1} m} \omega^{n_{1} \ldots n_{s-1}, m}=0 \tag{4.7}
\end{equation*}
$$

which we will call the relaxed traceless condition, because it is weaker than the conventional trace constraint

$$
\begin{equation*}
\eta_{n_{1} n_{2}} \omega^{n_{1} n_{2} \ldots n_{s-1}, m}=0 . \tag{4.8}
\end{equation*}
$$

If (4.8) holds, also the condition (4.7) is satisfied in virtue of the symmetry property (4.3), but not vice versa. Therefore, the action (4.6) describes a system of free higher-spin fields associated with reducible representations of the tangent-space Lorentz group, as we will consider in detail in Section 4.3.

A look at (4.5) tells us that the parameters of the gauge transformations which leave the action invariant should satisfy the constraints similar to (4.7), namely

$$
\begin{equation*}
\eta_{n_{1} m} \xi^{n_{1} \ldots n_{s-1}, m}=0, \quad \eta_{n_{1} m} \xi^{n_{1} \ldots n_{s-1}, m l}=0 . \tag{4.9}
\end{equation*}
$$

From (4.6) one deduces the equations of motion, which can be written in the following form:

$$
\begin{align*}
& (s-1) \omega_{\left[n,{ }^{n_{1} \ldots n_{s-1}, b} \eta_{m], b}=\partial_{[m} e_{n]} ;^{n_{1} \ldots n_{s-1}}\right.}{ }_{(b)}^{m} \partial^{c} \omega^{d ;}{ }_{\left.n_{1} \ldots n_{s-2}\right)[c, d]}+\partial_{d} \omega_{\left(b ; n_{1} \ldots n_{s-2}\right.}{ }^{[m, d]}+\partial_{d} \omega^{d ;}{ }_{{ }_{1} \ldots n_{s-2}[d,}{ }^{m]}=0 \tag{4.10}
\end{align*}
$$

where (4.10) is the zero torsion condition, while (4.11) gives the dynamical second order equation for the higher-spin vielbein $e^{n_{1} \ldots n_{s-1}}$, when the higher-spin connection is substituted by its expression in terms of components of $d e^{n_{1} \ldots n_{s-1}}$.

### 4.2 Fronsdal case

The action (4.6) describes a single (irreducible) massless field of spin $s$, when the oneforms and the gauge parameters satisfy the strong traceless conditions ${ }^{5}$ :

$$
\begin{equation*}
\eta_{n_{1} n_{2}} \tilde{e}^{n_{1} \ldots n_{s-1}}=0, \quad \eta_{n_{1} n_{2}} \tilde{\omega}^{n_{1} \ldots n_{s-1}, m}=0 \tag{4.12}
\end{equation*}
$$

as well as on the gauge parameters:

$$
\begin{equation*}
\eta_{n_{1} n_{2}} \tilde{\xi}^{n_{1} \ldots n_{s-1}}, \quad \eta_{n_{1} n_{2}} \tilde{\xi}^{n_{1} \ldots n_{s-1}, m} \quad \eta_{n_{1} n_{2}} \tilde{\xi}^{n_{1} \ldots n_{s-1}, m p} \tag{4.13}
\end{equation*}
$$

It is easy to see that using the transformations (4.5), one can remove the totally antisymmetric part of the higher-spin vielbein. Then, since the higher-spin connection is just an auxiliary field in virtue of equation (4.10), all the degrees of freedom are carried by the symmetric field $\tilde{e}^{\left(n_{1} ; n_{2} \ldots n_{s}\right)} \equiv \Phi^{n_{1} n_{2} \ldots n_{s}}$. Note that, because (4.12) holds, the field $\Phi^{n_{1} \ldots n_{s}}$ is double-traceless and hence of the Fronsdal type. The remaining gauge symmetry is the symmetry (2.12) of the Fronsdal formulation. Thus, in this irreducible case, upon partial gauge fixing and removing auxiliary fields we recover from the action (4.6) the Fronsdal action. This is similar to how one gets the Einstein action for gravity from its Cartan form.

### 4.3 Triplet case

Let us consider now in more detail the case in which the frame-like fields $e^{n_{1} \ldots n_{s-1}}$ are unconstrained and $\omega^{n_{1} \ldots n_{s-1}, m}$ obey the relaxed conditions (4.7). By decomposing the higher-spin vielbein into a sum of traceless tensors of lower rank, one can show that the action (4.6) splits into the sum of the actions for the traceless vielbeins and connections $\tilde{e}^{n_{1} \ldots n_{t-1}}, \tilde{\omega}^{n_{1} \ldots n_{t-1}, m}$, where $t=3,5, \ldots, s$ if $s$ is odd, and $t=2,4, \ldots, s$ if $s$ is even. Thus, the action (4.6) can be rewritten as follows:

$$
\begin{align*}
& S=\sum_{k=1}^{[s / 2]} \alpha(t, D) \int_{M^{D}} d x^{a_{1}} \ldots d x^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left(d \tilde{e}^{n_{1} \ldots n_{t-2} p}\right. \\
&\left.\quad-\frac{t-1}{2} d x_{m} \tilde{\omega}^{n_{1} \ldots n_{t-2} p, m}\right) \tilde{\omega}_{n_{1} \ldots n_{t-2}}^{q, r} \tag{4.14}
\end{align*}
$$

[^4]where $[s / 2]$ is the integral part of $s$ and $k=2 t$ or $k=2 t+1$ depending on whether $s$ is even or odd. Each term of the sum (4.14) is invariant under the transformations (4.5) with the traceless parameters (4.13) and thus describes a single Fronsdal-like field of $\operatorname{spin} t$.

We thus find that the irreducible field content of the unconstrained frame-like fields is similar to that of the higher-spin field triplets which naturally arise in String Theory [14]. The only difference is that the latter also include in their spectra the spin-one and spin-zero fields.

In the metric-like formulation, the higher-spin triplets are represented by the following three totally symmetric tensors of rank $s, s-1$ and $s-2$ :

$$
\begin{equation*}
\Phi_{n_{1} \ldots n_{s}} \quad C_{n_{1} \ldots n_{s-1}}, \quad D_{n_{1} \ldots n_{s-2}}, \tag{4.15}
\end{equation*}
$$

satisfying the equations of motion:

$$
\begin{align*}
& C_{n_{1} \ldots n_{s-1}}=\partial_{m} \Phi^{m}{ }_{n_{1} \ldots n_{s-1}}-(s-1) \partial_{\left(n_{s-1}\right.} D_{\left.n_{1} \ldots n_{s-2}\right)},  \tag{4.16}\\
& \square \Phi_{n_{1} \ldots n_{s}}=s \partial_{\left(n_{s}\right.} C_{\left.n_{1} \ldots n_{s-1}\right)},  \tag{4.17}\\
& \square D_{n_{1} \ldots n_{s-2}}=\partial_{m} C^{m}{ }_{n_{1} \ldots n_{s-2}} . \tag{4.18}
\end{align*}
$$

These equations are gauge-invariant under the following transformations of the fields:

$$
\begin{align*}
& \delta \Phi_{n_{1} \ldots n_{s}}=s \partial_{\left(n_{s}\right.} \xi_{\left.n_{1} \ldots n_{s-1}\right)}  \tag{4.19}\\
& \delta C_{n_{1} \ldots n_{s-1}}=\square \xi_{n_{1} \ldots n_{s-1}},  \tag{4.20}\\
& \delta D_{n_{1} \ldots n_{s-2}}=\partial_{m} \xi^{m}{ }_{n_{1} \ldots n_{s-2}} . \tag{4.21}
\end{align*}
$$

where the parameter $\xi^{n_{1} \ldots n_{s-1}}$ is totally symmetric.
As was shown in [26], the metric-like triplet fields are related to the components of the higher spin vielbein and connection of the frame-like formulation, subject to the relaxed trace constraints (4.7), as follows

$$
\begin{align*}
& \Phi_{n_{1} \ldots n_{s}}=s e_{\left(n_{s} ; n_{1} \ldots n_{s-1}\right)},  \tag{4.22}\\
& C_{n_{1} \ldots n_{s-1}}=(s-1) \omega_{m ; n_{1} \ldots n_{s-1}} m+\partial^{m} e_{m ; n_{1} \ldots n_{s-1}},  \tag{4.23}\\
& D_{n_{1} \ldots n_{s-2}}=e_{p ; n_{1} \ldots n_{s-1}} \eta^{s-1 p} . \tag{4.24}
\end{align*}
$$

It is also possible to show that, upon this identification the zero torsion condition (4.10) and the dynamical equation (4.11) are equivalent to the equations (4.16), (4.17) and (4.18), modulo the absence in the frame-like equations of the equations for spin-1 and spin 0 fields, that can be added "by hand" (as discussed above).

### 4.4 Fermionic frame-like action in flat space-time

The procedure, generalizing the Cartan formulation of gravity to the description of higher spin fields described in Section 3.1 for the bosonic case, can also be extended to fermionic fields. The half-integer spin field is described by a set of tensor-spinor one-forms
$\psi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t}}^{\alpha}=d x^{n} \psi_{n ; a_{1} \ldots a_{s-\frac{3}{2}}^{2}, b_{1} \ldots b_{t}}^{\alpha}$ with $0 \leq t \leq s-\frac{3}{2}$. The starting point is the definition of a set of fermionic Riemann curvatures (one for each value of $t$ ):

$$
\begin{equation*}
R_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t}}=d \psi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t}}-\left(s-t-\frac{3}{2}\right) e^{c} \psi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t} c} . \tag{4.25}
\end{equation*}
$$

The curvatures (4.25) are invariant under the gauge transformations:

$$
\begin{equation*}
\delta \psi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t}}=d \xi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t}}-\left(s-t-\frac{3}{2}\right) e^{c} \xi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t} c} \tag{4.26}
\end{equation*}
$$

We stress here the fact that the fields $\psi_{a_{1} \ldots a_{s-\frac{3}{2}}, b_{1} \ldots b_{t}}^{\alpha}$ with $t \geq 1$ do not contribute to the free action, while they play an important role to preserve gauge invariance when introducing interactions. The action which involves the field $\psi_{a_{1} \ldots a_{s-\frac{3}{2}}}$, has the following form

$$
\begin{equation*}
S=i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left(\bar{\psi}_{d_{1} \ldots d_{s-\frac{3}{2}}} \gamma^{p q r} d \psi^{d_{1} \ldots d_{s-\frac{3}{2}}}-6 \bar{\psi}_{d_{1} \ldots d_{s-\frac{5}{2}}} \gamma^{q} d \psi^{d_{1} \ldots d_{s-\frac{5}{2}} r}\right), \tag{4.27}
\end{equation*}
$$

It is invariant under the gauge transformations (4.26) provided that the gauge parameters satisfy the conditions

$$
\begin{equation*}
\gamma_{b} \xi^{a_{1} \ldots a_{s-\frac{3}{2}}, b}=0, \quad \xi^{a_{1} \ldots a_{s-\frac{5}{2}} b}{ }_{b}=0 \tag{4.28}
\end{equation*}
$$

In analogy with (4.7) for the bosonic case, we shall call (4.28) the relaxed trace conditions. Making the variation of (4.27), we get the following equations of motion

$$
\begin{align*}
\frac{1}{s-\frac{3}{2}} \gamma^{m q r} \partial_{q} \psi_{r ; a_{1} \ldots a_{s-\frac{3}{2}}} & =\gamma^{m} \partial^{r} \psi_{\left(a_{1} ; a_{2} \ldots a_{s-\frac{3}{2}}\right) r}-\gamma^{q} \partial^{r} \psi_{q ; r\left(a_{2} \ldots a_{s-\frac{3}{2}}\right.} \delta_{\left.a_{1}\right)}^{m} \\
& -\gamma^{r} \partial_{r} \psi_{\left(a_{1} ; a_{2} \ldots a_{s-\frac{3}{2}}\right.}{ }^{m}+\gamma^{r} \partial_{r} \psi^{p ;}{ }_{p\left(a_{2} \ldots a_{s-\frac{3}{2}} q\right.} \delta_{\left.a_{1}\right)}^{m} \\
& -\gamma^{m} \partial_{\left(a_{1}\right.} \psi^{q ;}{ }_{\left.a_{2} \ldots a_{s-\frac{3}{2}}\right) q}+\gamma^{q} \partial_{\left(a_{1}\right.} \psi^{m ;}{ }_{\left.{ }_{2} \ldots a_{s-\frac{3}{2}}\right)}{ }^{q} . \tag{4.29}
\end{align*}
$$

This equation will be studied in detail in Section 5, for $s=5 / 2$.

### 4.5 Fang-Fronsdal case

We consider now the case in which all the fields and the parameters satisfy the strong $\gamma$-traceless conditions, i.e.

$$
\begin{array}{ll}
\gamma^{c} \tilde{\psi}_{a_{1} \ldots a_{s-\frac{3}{2}}, c}=0, & \gamma^{c} \tilde{\psi}_{a_{1} \ldots a_{s-\frac{5}{2}}}=0 \\
\gamma^{c} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{3}{2}}, c}=0, & \gamma^{c} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2} c}}=0 \tag{4.30}
\end{array}
$$

It can be shown that the $\gamma$-traceless field $\tilde{\psi}_{m ; n_{1} \ldots n_{s-\frac{3}{2}}}$ can be decomposed into the following irreducible parts: $\tilde{\psi}_{\left(m ; n_{1} \ldots n_{s-\frac{3}{2}}\right)}$, of rank $s, \gamma^{m} \tilde{\psi}_{m ; n_{1} \ldots n_{s-\frac{3}{2}}}$ of $\operatorname{rank} s-1, \eta^{m k} \tilde{\psi}_{m ; n_{1} \ldots n_{s-\frac{5}{2}} k}$ of rank $s-2$, and finally its part that satisfies $\tilde{\psi}_{\left(m ; n_{1} \ldots n_{s-\frac{3}{2}}\right)}=0$. The last one can be gauged away by an appropriate choice of the $\gamma$-traceless parameter $\tilde{\xi}_{n_{1} \ldots n_{s-\frac{3}{2}}, m}$ (Stueckelberg symmetry). Having fixed the gauge, we are left with the three totally symmetric
tensor-spinors, which of course are $\gamma$-traceless. These three tensors can be combined together into a single totally symmetric Fang-Fronsdal tensor-spinor $\Psi^{n_{1} \ldots n_{s-\frac{1}{2}}}$, which is triple-gamma traceless. The residual gauge-symmetry

$$
\begin{equation*}
\delta \tilde{\psi}^{m ; n_{1} \ldots n_{s-\frac{3}{2}}}=\partial^{(m} \tilde{\xi}^{\left.n_{1} \ldots n_{s-\frac{3}{2}}\right)}, \tag{4.31}
\end{equation*}
$$

is the same as the gauge symmetry (2.23) of the Fang-Fronsdal action. We thus established the relation of the frame-like formulation of irreducible half-integer spin fields with the Fang-Fronsdal metric-like formulation.

### 4.6 Triplet case

We consider now the case in which the field $\psi^{a_{1} \ldots a_{s-\frac{3}{2}}}$ and the parameter $\xi^{a_{1} \ldots a_{s-\frac{3}{2}}}$ are unconstrained, while the parameter $\xi^{a_{1} \ldots a_{s-\frac{3}{2}}, b}$ is subject to the constraints (4.28). Like in the Fang-Fronsdal case, by making use of the gauge transformations (4.26), the field $\psi_{a_{1} \ldots a_{s-\frac{3}{2}}}$ can be reduced to the sum of three tensor-spinors of ranks $s-\frac{1}{2}, s-\frac{3}{2}, s-\frac{5}{2}$, which are now unconstrained. These fields can be decomposed into their $\gamma$-traceless parts, obtaining the set of irreducible Fang-Fronsdal fields of spins from $s$ to $\frac{3}{2}$. Note that, as the scalar and vector fields, the spin- $1 / 2$ field is not included in the action (4.27), and should be added separately.

Again, we can relate these frame-like fields to the fields composing the triplet fermionic fields arising in the tensionless limit of the fermionic string spectrum, since their contents are the same, modulo the spin $-1 / 2$ field.

In the metric-like formulation [14], the fermionic triplet consist of the three fields of rank $s-\frac{1}{2}, s-\frac{3}{2}$ and $s-\frac{5}{2}$ which we call, respectively, $\Psi_{m_{1} \ldots m_{s-\frac{1}{2}}}, \chi_{m_{1} \ldots m_{s-\frac{3}{2}}}$ and $\lambda_{m_{1} \ldots m_{s-\frac{5}{2}}}$. Their equations in flat space-time are:

$$
\left\{\begin{array}{l}
\gamma^{n} \partial_{n} \Psi_{m_{1} \ldots m_{s-\frac{1}{2}}}=\left(s-\frac{1}{2}\right) \partial_{\left(m_{1}\right.} \chi_{m_{2} \ldots m_{s-\frac{1}{2}}},  \tag{4.32}\\
\partial^{n} \Psi_{n m_{2} \ldots m_{s-\frac{1}{2}}}-\left(s-\frac{3}{2}\right) \partial_{\left(m_{2}\right.} \lambda_{\left.m_{3} \ldots m_{s-\frac{1}{2}}\right)}=\gamma^{n} \partial_{n} \chi_{m_{2} \ldots m_{s-\frac{1}{2}}}, \\
\gamma^{n} \partial_{n} \lambda_{m_{1} \ldots m_{s-\frac{5}{2}}}=\left(s-\frac{5}{2}\right) \partial^{n} \chi_{n m_{1} \ldots m_{s-\frac{5}{2}}},
\end{array}\right.
$$

which are invariant under the following gauge transformations

$$
\begin{align*}
& \delta \Psi_{m_{1} \ldots m_{s-\frac{1}{2}}}=\left(s-\frac{1}{2}\right) \partial_{\left(m_{1}\right.} \xi_{\left.m_{2} \ldots m_{s-\frac{1}{2}}\right)},  \tag{4.33}\\
& \delta \chi_{m_{1} \ldots m_{s-\frac{3}{2}}}=\gamma^{n} \partial_{n} \xi_{m_{1} \ldots m_{s-\frac{3}{2}}},  \tag{4.34}\\
& \delta \lambda_{m_{1} \ldots m_{s-\frac{5}{2}}}=\partial^{n} \xi_{n m_{1} \ldots m_{s-\frac{5}{2}}} . \tag{4.35}
\end{align*}
$$

Comparing these with the gauge transformations of the frame-like field (4.26) one deduces how the metric-like triplet fields are related to the components of the higher-spin one-form field $\psi_{a_{1} \ldots a_{s-\frac{1}{2}}}$ :

$$
\begin{array}{r}
\Psi_{m_{1} \ldots m_{s-\frac{1}{2}}}=\left(s-\frac{1}{2}\right) \psi_{\left(m_{1} ; m_{2} \ldots m_{s-\frac{1}{2}}\right)} \\
\chi_{m_{1} \ldots m_{s-\frac{3}{2}}}=\gamma^{n} \psi_{n ; m_{1} \ldots m_{s-\frac{3}{2}}} \\
\lambda_{m_{1} \ldots m_{s-\frac{5}{2}}}=\eta^{m l} \psi_{m ; l m_{1} \ldots m_{s-\frac{5}{2}}} \tag{4.38}
\end{array}
$$

In Section 5 for the case $s=\frac{5}{2}$ we will explicitly show how, upon the identifications (4.36)-(4.38), the equations of motion (4.29) reduce to eqs. (4.32).

### 4.7 Fermionic frame-like action in $\operatorname{AdS}_{D}$.

Now let us present the extension to the $\mathrm{AdS}_{D}$ space of the frame-like formulation of fermionic higher spin fields, since we will make use of it in Section 5, when deriving, for the first time, the action for a metric-like fermionic $s=5 / 2$ triplet in $\mathrm{AdS}_{D}$.

The gauge transformation (4.26) are generalized in the following way:

$$
\begin{equation*}
\delta \psi_{a_{1} \ldots a_{s-\frac{3}{2}}}=\mathcal{D} \xi_{a_{1} \ldots a_{s-\frac{3}{2}}}-\left(s-\frac{3}{2}\right) e^{b} \xi_{a_{1} \ldots a_{s-\frac{3}{2}, b}}, \tag{4.39}
\end{equation*}
$$

where the covariant derivative $\mathcal{D}$ is the differential-form counterpart of (3.11), i.e.:

$$
\begin{equation*}
\mathcal{D}=\nabla+i \frac{\sqrt{-\Lambda}}{2} e^{a} \gamma_{a} \tag{4.40}
\end{equation*}
$$

This derivative is covariant with respect to transformations of the $\mathrm{AdS}_{D}$ isometry group $\operatorname{Spin}(2, D-1)$, while the covariant derivative with respect to both the general coordinate transformations and the local Lorentz transformations is $\nabla$. From the definition (4.40) it follows that, in $\mathrm{AdS}_{D}$, the following identity holds for every spinor $\psi$ (without any tensor indices), as previously pointed out in (3.13):

$$
\begin{equation*}
\mathcal{D}^{2} \psi=0 \tag{4.41}
\end{equation*}
$$

Moreover since the $i \frac{\sqrt{-\Lambda}}{2} e^{a} \gamma_{a}$ part of (4.40) acts only on the spinor indices, we have the following action on a generic tensor $T^{a_{1} \ldots a_{n}}$ :

$$
\begin{equation*}
\mathcal{D}^{2} T^{a_{1} \ldots a_{n}}=\nabla^{2} T^{a_{1} \ldots a_{n}}=-n \Lambda e^{\left(a_{1}\right.} e_{b} T^{\left.a_{2} \ldots a_{n}\right) b} . \tag{4.42}
\end{equation*}
$$

where the last equality in (4.42) follows from the fact that the Riemann tensor takes the following form in $\mathrm{AdS}_{D}$ :

$$
\begin{equation*}
R_{a b}^{(A d S)}=-\Lambda e_{a} \wedge e_{b} \tag{4.43}
\end{equation*}
$$

Using (4.41) and (4.42), we find that the derivative $\mathcal{D}$ acts on a tensor-spinor $\psi^{a_{1} \ldots a_{n}}$ in the same way as on a tensor $T^{a_{1} \ldots a_{n}}$, thus:

$$
\begin{equation*}
\mathcal{D}^{2} \psi^{a_{1} \ldots a_{n}}=-n \Lambda e^{\left(a_{1}\right.} e_{b} T^{\left.a_{2} \ldots a_{n}\right) b} . \tag{4.44}
\end{equation*}
$$

Note that the action of $\mathcal{D}$ on a matrix $\gamma_{a}$ is:

$$
\begin{equation*}
\mathcal{D} \gamma_{a}=-i \sqrt{-\Lambda} e^{b} \gamma_{b a} . \tag{4.45}
\end{equation*}
$$

### 4.8 Fang-Fronsdal case in $\operatorname{AdS}_{D}$

Now we consider the case when the higher spin field, which we shall call $\tilde{\psi}_{a_{1} \ldots a_{s-\frac{3}{2}}}$ according to our notation, satisfies the strong $\gamma$-traceless condition:

$$
\begin{equation*}
\gamma^{b} \tilde{\psi}_{b a_{1} \ldots a_{s-\frac{5}{2}}}=0 \tag{4.46}
\end{equation*}
$$

which implies, for consistency with (4.26), that the parameters in (4.39) must obey the following conditions:

$$
\begin{equation*}
\gamma^{c} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2}}}=0, \quad\left(s-\frac{3}{2}\right) \gamma^{c} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2}} c, b}=-i \sqrt{-\Lambda} \gamma_{b}{ }^{c} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2}}}=i \sqrt{-\Lambda} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2}}} . \tag{4.47}
\end{equation*}
$$

Using the condition $\tilde{\xi}_{\left(a_{1} \ldots a_{s-\frac{3}{2}}, b\right)}=0$, from the above equality it follows that

$$
\begin{equation*}
\left(s-\frac{3}{2}\right) \gamma^{b} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2}} c, b}=-i \sqrt{-\Lambda} \tilde{\xi}_{a_{1} \ldots a_{s-\frac{5}{2}} c} . \tag{4.48}
\end{equation*}
$$

The action which is gauge invariant under (4.39) is:

$$
\begin{align*}
& S=i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left[\overline{\tilde{\psi}}_{d_{1} \ldots d_{s-\frac{3}{2}}} \nu^{p q r} \mathcal{D} \tilde{\psi}^{d_{1} \ldots d_{s-\frac{3}{2}}}-6\left(s-\frac{3}{2}\right) \overline{\tilde{\psi}}_{d_{1} \ldots d_{s-\frac{5}{2}}}{ }^{p} \gamma^{q} \mathcal{D} \tilde{\psi}^{d_{1} \ldots d_{s-\frac{5}{2}} r}\right. \\
&\left.-\frac{3 i \sqrt{-\Lambda}\left(s-\frac{3}{2}\right)}{D-2}\left(e^{r} \overline{\tilde{\psi}}_{d_{1} \ldots d_{s-\frac{3}{2}}}{ }^{p q} \tilde{\psi}^{d_{1} \ldots d_{s-\frac{3}{2}}}+2\left(s-\frac{3}{2}\right) e^{p} \overline{\tilde{\psi}}_{d_{1} \ldots d_{s-\frac{5}{2}}} \tilde{\psi}^{r d_{1} \ldots d_{s-\frac{5}{2}}}\right)\right] \tag{4.49}
\end{align*}
$$

Note that, in comparison with eq. (4.27), the action (4.49) contains two more terms proportional to the square root of the cosmological constant, which ensure the gauge invariance.

### 4.9 Triplet case in AdS

Fermionic triplets do exist also in AdS space. We will consider here the simplest case of spin $s=5 / 2$, which we shall further study in Section 5 . The fermionic frame-like field is the one-form $\psi^{a}$, and the gauge transformations (4.39) take the form

$$
\begin{equation*}
\delta \psi^{a}=\mathcal{D} \xi^{a}-e_{b} \xi^{a, b} \tag{4.50}
\end{equation*}
$$

where $\xi^{a}$ is $\gamma$-traceful, while the antysimmetric parameter $\xi^{a, b}$ satisfies the following relation:

$$
\begin{equation*}
\gamma_{b} \xi^{b, a}=-i \sqrt{-\Lambda}\left(\gamma^{a b} \xi_{b}\right)=i \sqrt{-\Lambda}\left(\xi^{a}-\gamma^{a} \gamma^{b} \xi_{b}\right) . \tag{4.51}
\end{equation*}
$$

From the form of the gauge transformation (4.50), one gets the following gauge variation of the $\gamma$-trace of $\psi^{a}$ :

$$
\begin{equation*}
\delta\left(\gamma_{a} \psi^{a}\right)=\mathcal{D}\left(\gamma_{a} \xi^{a}\right) \tag{4.52}
\end{equation*}
$$

Using the above relations, we see that the frame-field $\gamma_{a} \psi^{a}$ enjoys the same gauge invariance as a Rarita-Schwinger field in $A d S$.

The action for the field $\psi^{a}$ which is invariant under the transformations (4.50) has the following form

$$
\begin{align*}
& S=i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left[\bar{\psi}_{f} \gamma^{b c d} \mathcal{D} \psi^{f}-6 \bar{\psi}^{b} \gamma^{c} \mathcal{D} \psi^{d}\right. \\
&\left.\quad-\frac{3 i \sqrt{-\Lambda}}{D-2}\left(e^{d} \bar{\psi}_{f} \gamma^{b c} \psi^{f}+2 e^{b} \bar{\psi}^{c} \psi^{d}+2 e^{d} \bar{\psi}_{f} \gamma^{f} \gamma^{b} \psi^{c}-e^{d} \bar{\psi}_{f} \gamma^{f} \gamma^{b c} \gamma_{i} \psi^{i}\right)\right] \tag{4.53}
\end{align*}
$$

In Section 6.2 we will see that upon the following substitution

$$
\begin{equation*}
\psi^{a}=\tilde{\psi}^{a}+\frac{1}{D} \gamma^{a} \tilde{\psi}, \quad \gamma_{a} \tilde{\psi}^{a}=0, \quad \tilde{\psi}=\gamma^{a} \psi_{a} \tag{4.54}
\end{equation*}
$$

the action (4.53) splits into the sum of the Fang-Fronsdal actions (4.49) for the irreducible spins $s=5 / 2$ and $s=3 / 2$. In fact, $\tilde{\psi}^{a}$ is precisely the irreducible $s=5 / 2$ field, and $\tilde{\psi}$ is the irreducible $s=3 / 2$ field. In order to show the gauge invariance of (4.53) (as well as of the all actions in this thesis), we will make use (see Appendix) of the following identity

$$
\begin{equation*}
\varepsilon_{a_{1} \ldots a_{D-p}, b_{1} \ldots b_{p}} e^{a_{1}} \ldots e^{a_{D-p}} e^{f}=\frac{(-1)^{(p-1)(D-p+1)} p}{D-p+1} \delta_{\left[b_{1}\right.}^{f} \varepsilon_{\left.b_{2} \ldots b_{p}\right] a_{1} \ldots a_{D-p+1}} e^{a_{1}} \ldots e^{a_{D-p+1}} \tag{4.55}
\end{equation*}
$$

## 5 Metric-like action for the doublet of spin 5/2 and $3 / 2$ fields

In [26], together with the frame-like formulation of the higher spin fields, it was suggested that, with the identification given by (4.36)-(4.38), one would obtain that (4.29) reduces to the equations of motion for the fermionic triplets. In this section, as a warm-up exercise before addressing the AdS case, we will explicitely derive the metric-like action for the doublet of fields of spin $5 / 2$ and $3 / 2$ in flat space-time from the frame-like action (4.27), which for the reducible $s=5 / 2$ field has the following form

$$
\begin{equation*}
S=i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left(\bar{\psi}_{d} \gamma^{p q r} d \psi^{d}-6 \bar{\psi}^{p} \gamma^{q} d \psi^{r}\right) \tag{5.1}
\end{equation*}
$$

where $e^{a}=\delta_{m}^{a} d x^{m}$ is the flat space-time vielbein. The action (5.1) is gauge-invariant (see Appendix) under the transformations

$$
\begin{equation*}
\delta \psi_{a}=d \xi_{a}-e^{c} \xi_{a, c}, \tag{5.2}
\end{equation*}
$$

where the gauge parameter $\xi_{a}$ is unconstrained, while the parameter $\xi_{a, b}$ is antisymmetric and $\gamma$-traceless

$$
\begin{equation*}
\gamma^{a} \xi_{a, b}=0 . \tag{5.3}
\end{equation*}
$$

To derive the metric-like action from eq. (5.1) we first pass from its differential form expression to the action for the differential form components using the following relations:

$$
\begin{align*}
& d x^{m_{1}} \wedge \cdots \wedge d x^{m_{d}}=d^{D} x \varepsilon^{m_{1} \ldots m_{D}},  \tag{5.4}\\
& \varepsilon_{i_{1} \ldots i_{k} i_{k+1} \ldots i_{n}} \varepsilon^{i_{1} \ldots i_{k}} j_{k+1} \ldots j_{n} \tag{5.5}
\end{align*}=k!(n-k)!\delta_{\left[i_{k+1}\right.}{ }^{j_{k+1}} \ldots \delta_{\left.i_{n}\right]^{\prime}}{ }^{j_{n}}=k!\delta_{i_{k+1} \ldots i_{n}}^{j_{k+1} \ldots j_{n}} .
$$

Since we are considering flat space-time, we have $e^{a}=d x^{a}$, so the gauge transformation (5.2) for the components $\psi_{m ; b}$ of $d x^{m} \psi_{m ; b}$ has the following form ${ }^{6}$

$$
\begin{equation*}
\delta \psi_{m ; b}=\partial_{m} \xi_{b}+\xi_{m, b} \tag{5.6}
\end{equation*}
$$

Using the identities (5.4) and (5.5), we can now easily write down the component action:

$$
\begin{equation*}
S=\int_{M^{D}} d^{D} x\left(\bar{\psi}_{m ; d} \gamma^{m n s} \partial_{n} \psi_{s ;}{ }^{d}-6 \bar{\psi}_{m ;}{ }^{[m} \gamma^{n} \partial_{n} \psi_{s ;}{ }^{s]}\right) \tag{5.7}
\end{equation*}
$$

The equations of motion for the unconstrained fermionic field $\psi_{a ; b}$ are derived from (4.29) by taking $s=5 / 2$ :

$$
\begin{align*}
\gamma^{a n s} \partial_{n} \psi_{s ;}{ }^{b}= & \eta^{a b} \gamma^{n} \partial_{n} \psi_{s ;}{ }^{s}+\gamma^{n} \partial^{b} \psi_{n ;}{ }^{a} \\
& +\gamma^{a} \partial_{n} \psi^{b ; n}-\eta^{a b} \gamma^{s} \partial_{n} \psi_{;}{ }^{n} \\
& -\gamma^{a} \partial^{b} \psi_{s ;}{ }^{s}-\gamma^{n} \partial_{n} \psi^{b ; a} \tag{5.8}
\end{align*}
$$

[^5]and one could verify that these coincide with the equations obtained from the action (5.7). Let us decompose the field $\psi_{a ; b}$ into its symmetric and antisymmetric part
\[

$$
\begin{equation*}
\psi_{a ; b}=\psi_{(a ; b)}+\psi_{[a ; b]} \tag{5.9}
\end{equation*}
$$

\]

We can now make use of the gauge transformations (5.6) and eliminate some components of the field (5.9). To this end, let us split further the antisymmetric part $\psi_{[a ; b]}$ of the field as follows

$$
\begin{equation*}
\psi_{[a ; b]}=\tilde{\psi}_{[a ; b]}+X_{[a ; b]} \tag{5.10}
\end{equation*}
$$

where $\tilde{\psi}_{[a ; b]}$ denotes the $\gamma$-transversal part of $\psi_{[a ; b]}$ i.e. such that $\gamma^{a} \tilde{\psi}_{[a ; b]}=0$. The component $\tilde{\psi}_{[a ; b]}$ can be removed using the gauge transformation (5.6) with an appropriate choice of the "Stueckelberg" parameter $\xi_{a, b}$, which is both $\gamma$-traceless and antisymmetric. Thus, $\tilde{\psi}_{[a ; b]}$ is a pure gauge field that should effectively drop out from the action (5.7). On the contrary, we cannot gauge away the symmetric part $\psi_{(a ; b)}$ and the $\gamma$-traceful part $\gamma^{a} \psi_{a ; b}$ of the field, which will thus constitute the metric-like action.

So, upon gauge fixing $\tilde{\psi}_{[a ; b]}=0$, the field (5.9) reduces to

$$
\begin{equation*}
\psi_{a ; b}=\psi_{(a ; b)}+X_{[a b]} \tag{5.11}
\end{equation*}
$$

Our aim is now to rewrite the field (5.11) in terms of the triplet of fields defined by (4.36), (4.37) and (4.38), which in this case are:

$$
\left\{\begin{array}{l}
\Psi_{a b}=2 \psi_{(a ; b)}  \tag{5.12}\\
\chi_{b}=\gamma^{a} \psi_{a ; b} \\
\lambda=\eta^{a b} \psi_{a ; b}
\end{array}\right.
$$

Note that the field $\lambda$ is here not independent, namely

$$
\begin{equation*}
\lambda=\frac{1}{2} \Psi^{c}{ }_{c} . \tag{5.13}
\end{equation*}
$$

Thus, we are effectively dealing with the doublet of fields $\left\{\Psi_{a b}, \chi_{a}\right\}$. The symmetric part of (5.11) is just $\frac{1}{2} \Psi^{a b}$. What remains is to represent the antisymmetric part $X_{[a b]}$ in terms of (5.12). Since we put the traceless antisymmetric part $\tilde{\psi}_{a ; b}$ to zero, we have

$$
\begin{equation*}
\psi_{[a ; b]}=\psi_{a ; b}-\frac{1}{2} \Psi_{a b}=X_{[a b]} \tag{5.14}
\end{equation*}
$$

Now, since $X_{[a b]}$ is traceful, it should have the following generic form

$$
\begin{equation*}
X_{[a b]}=\alpha\left(\gamma_{a} \gamma^{m} \psi_{[m ; b]}-\gamma_{b} \gamma^{m} \psi_{[m ; a]}\right)+\gamma_{a b} C \tag{5.15}
\end{equation*}
$$

where $\alpha$ is a constant and $C$ is a spinor field. Since $\gamma^{a} X_{[a b]}=\gamma^{a} \psi_{[a ; b]}=\chi_{b}-\frac{1}{2} \gamma^{a} \Psi_{a b}$, one finds that (see Appendix)

$$
\left\{\begin{array}{l}
\alpha=\frac{1}{D-2}  \tag{5.16}\\
C=\frac{1}{(D-1)(D-2)}\left(\gamma^{a} \chi_{a}-\frac{1}{2} \Psi^{c}{ }_{c}\right),
\end{array}\right.
$$

and we finally get the decomposition of $\psi_{a ; b}$ in terms of the fields $\Psi_{a ; b}$ and $\chi_{a}$ :

$$
\begin{align*}
& \psi_{a ; b}=\frac{1}{2} \Psi^{c}{ }_{c}+\frac{1}{D-2}\left(\gamma_{a} \chi_{b}-\gamma_{b} \chi_{a}+\frac{1}{2}\left(\gamma_{b} \gamma^{m} \Psi_{m a}-\gamma_{a} \gamma^{m} \Psi_{m b}\right)\right) \\
&-\frac{1}{(D-1)(D-2)} \gamma_{a b}\left(\gamma^{c} \chi_{c}-\frac{1}{2} \Psi^{c}{ }_{c}\right) \tag{5.17}
\end{align*}
$$

### 5.1 Doublet equations of motion

In principle, in order to recast the action (5.1) in its metric-like form, one could replace the field $\psi_{a ; b}$ with its decomposition given by (5.17). However, it will be sufficient to take into account the definition of the doublet of fields $\left\{\Psi_{a b}, \chi_{a}\right\}$ in terms of the components of $\psi_{a ; b}$, given by (5.12). To simplify the left hand side of (5.8) we use the identity:

$$
\begin{equation*}
\gamma^{a n s}=\gamma^{a n} \gamma^{s}+2 \eta^{s[a} \gamma^{n]} \tag{5.18}
\end{equation*}
$$

Substituting (5.18) and using (5.12) we get

$$
\begin{align*}
& \frac{1}{2} \eta^{a b} \gamma^{n} \partial_{n} \Psi^{c}{ }_{c}-\frac{1}{2} \gamma^{a} \partial^{b} \Psi^{c}{ }_{c}-\eta^{a b} \partial_{n} \chi^{n}  \tag{5.19}\\
+ & 2 \partial^{(a} \chi^{b)}+\gamma^{a} \partial_{n} \Psi^{b n}-\gamma^{n} \partial_{n} \Psi^{a b}-\gamma^{a} \gamma^{n} \partial_{n} \chi^{b}=0
\end{align*}
$$

Let us rewrite the above equation as follows:

$$
\begin{equation*}
\eta^{a b}\left(\frac{1}{2} \gamma^{n} \partial_{n} \Psi^{c}{ }_{c}-\partial_{n} \chi^{n}\right)+\gamma^{a}\left(-\frac{1}{2} \partial^{b} \Psi^{c}{ }_{c}+\partial_{n} \Psi^{b n}-\gamma^{n} \partial_{n} \chi^{b}\right)+\left(2 \partial^{(a} \chi^{b)}-\gamma^{n} \partial_{n} \Psi^{a b}\right)=0 \tag{5.20}
\end{equation*}
$$

and call the terms inside the parenthesis, respectively $S, T^{b}$ and $R^{(a b)}$, so that equation (5.20) takes the rather simple form:

$$
\begin{equation*}
\eta^{a b} S+\gamma^{a} T^{b}+R^{(a b)}=0 . \tag{5.21}
\end{equation*}
$$

We can multiply (5.21) first by $\eta_{a b}$ and then by $\gamma_{a} \gamma_{b}$, and by subtracting and adding the resulting equations, we obtain the following two:

$$
\left\{\begin{array}{l}
\gamma_{b} T^{b}=0  \tag{5.22}\\
D S+R^{a}{ }_{a}=0
\end{array}\right.
$$

Noting that $R^{a}{ }_{a}=-2 S$, we get $S=0$, or explicitly:

$$
\begin{equation*}
\gamma_{n} \partial^{n} \lambda=\partial_{n} \chi^{n} . \tag{5.23}
\end{equation*}
$$

Now we are left with one tensor equation:

$$
\begin{equation*}
\gamma^{a} T^{b}+R^{(a b)}=0 \tag{5.24}
\end{equation*}
$$

Multiplying this before by $\gamma_{a}$ and then by $\gamma_{b}$, and using (5.22) we finally get:

$$
\begin{equation*}
T^{a}=0, \quad R^{(a b)}=0 . \tag{5.25}
\end{equation*}
$$

To summarize, we have obtained the following system of equations:

$$
\begin{align*}
& \frac{1}{2} \gamma^{n} \partial_{n} \Psi^{c}{ }_{c}=\partial_{n} \chi^{n},  \tag{5.26}\\
& \partial_{n} \Psi^{n a}-\frac{1}{2} \partial^{a} \Psi^{c}{ }_{c}=\gamma^{n} \partial_{n} \chi^{a},  \tag{5.27}\\
& \gamma^{n} \partial_{n} \Psi^{a b}=2 \partial^{(a} \chi^{b)}, \tag{5.28}
\end{align*}
$$

which is indeed the system describing the fermionic doublet of spin $s=5 / 2$ and $s=3 / 2$. To this system, one can add the Dirac equation for a massless spin $1 / 2$ field, thus getting the equations of motion for the triplet fermionic fields in the metric-like approach [14, 27]. We would like to stress the fact that equation (5.26) is nothing but the trace of (5.28), so we are left with just two indipendent tensor equations.
It is straightforward to verify that equations (5.26)-(5.28) are invariant under the gauge transformations:

$$
\begin{align*}
\delta \Psi_{a b} & =2 \partial_{(a} \xi_{b)}  \tag{5.29}\\
\delta \chi_{a} & =\gamma^{n} \partial_{n} \xi_{a} \tag{5.30}
\end{align*}
$$

with the $\gamma$-traceful parameter $\xi_{a}$.

### 5.2 Action

Let us now rewrite the action (5.7) in terms of the doublet of fields $\Psi_{a b}$ and $\chi^{a}$. To do this, let us start by calculating the first term in (5.7):

$$
\bar{\psi}_{m ; d} \gamma^{m n s} \partial_{n} \psi_{s ;}{ }^{d}
$$

Using the identity (5.18), we get:

$$
\begin{align*}
\bar{\psi}_{m ; d} \gamma^{m n s} \partial_{n} \psi_{s ;}{ }^{d} & =\bar{\psi}_{m ; d}\left[\gamma^{m n} \partial_{n} \chi^{d}+\left(\gamma^{n} \partial_{n} \psi^{m ; d}-\gamma^{m} \partial_{n} \psi^{n ; d}\right)\right] \\
& =\bar{\psi}_{m ; d} \gamma^{m} \gamma^{n} \partial_{n} \chi^{d}-\bar{\psi}_{m ; d} \eta^{m n} \partial_{n} \chi^{d} \\
& +\bar{\psi}_{m ; d}\left(\gamma^{n} \partial_{n} \psi^{m ; d}-\gamma^{m} \partial_{n} \psi^{n ; d}\right) \tag{5.31}
\end{align*}
$$

Noticing that $\bar{\psi}_{m ; d} \gamma^{m}=\bar{\chi}_{d}$ we can finally write:

$$
\begin{align*}
\bar{\psi}_{m ; d} \gamma^{m n s} \partial_{n} \psi_{s ;}{ }^{d} & =\bar{\chi}_{d} \gamma^{n} \partial_{n} \chi^{d}-\bar{\psi}_{m ; d} \partial^{m} \chi^{d} \\
& +\bar{\psi}_{m ; d} \gamma^{n} \partial_{n} \psi^{m ; d}-\bar{\chi}^{d} \partial_{n} \psi^{n ; d} \tag{5.32}
\end{align*}
$$

We temporarily put aside the above equation and compute the contribution of the second term in (5.7)

$$
\begin{align*}
-6 \bar{\psi}_{m ;}{ }^{[m} \gamma^{n} \partial_{n} \psi_{s ;}{ }^{s]}= & -\left(\bar{\psi}_{m ;}{ }^{m} \gamma^{n} \partial_{n} \psi_{d ;}{ }^{d}+\bar{\psi}_{m ;}{ }^{n} \gamma^{d} \partial_{n} \psi_{d ;}{ }^{m}\right. \\
& +\bar{\psi}_{m ;}{ }^{d} \gamma^{m} \partial_{n} \psi_{d ;}{ }^{n}-\bar{\psi}_{m ;}{ }^{n} \gamma^{m} \partial_{n} \psi_{d ;}{ }^{d} \\
& \left.-\bar{\psi}_{m ;}{ }^{m} \gamma^{d} \partial_{n} \psi_{d ;}{ }^{n}-\bar{\psi}_{m ;}{ }^{d} \gamma^{n} \partial_{n} \psi_{d ;}{ }^{m}\right) \\
= & -\left(\frac{1}{4} \Psi^{c}{ }_{c} \not \partial \Psi^{c}{ }_{c}+\bar{\psi}_{m ; d} \partial^{d} \chi^{m}+\bar{\chi}^{d} \partial^{n} \psi_{d ; n}\right. \\
& \left.-\frac{1}{2} \bar{\chi}^{n} \partial_{n} \Psi^{c}{ }_{c}-\frac{1}{2} \bar{\Psi}_{c}^{c}{ }_{c} \partial_{n} \chi^{n}-\bar{\psi}_{m ;}{ }^{d} \gamma^{n} \partial_{n} \psi_{d ;}{ }^{m}\right) \tag{5.33}
\end{align*}
$$

We now sum up the right hand side of (5.33) which still contains the full field $\psi_{a ; b}$ with the similar terms in (5.32), and get:

$$
\begin{align*}
& -\bar{\psi}_{m ; d}\left(\partial^{m} \chi^{d}+\partial^{d} \chi^{m}\right)+\bar{\psi}_{m ; d} \not \partial\left(\psi^{m ; d}+\psi^{d ; m}\right)-\bar{\chi}^{d} \partial^{n}\left(\psi_{n ; d}+\psi_{d ; n}\right)= \\
& =-\bar{\Psi}_{m d} \partial^{(m} \chi^{d)}+\frac{1}{2} \bar{\Psi}_{m d} \not \partial \Psi^{m d}-\bar{\chi}^{d} \partial^{n} \Psi_{n d} \tag{5.34}
\end{align*}
$$

Summing up all these contributions together, we finally get the action:

$$
\begin{align*}
S=\int_{M^{D}} d^{D} x & \left(\bar{\chi}^{d} \not \partial \chi_{d}-\bar{\Psi}_{m d} \partial^{(m} \chi^{d)}+\frac{1}{2} \bar{\Psi}_{m d} \not \partial \Psi^{m d}-\bar{\chi}^{d} \partial^{m} \Psi_{m d}+\right. \\
& \left.-\frac{1}{4} \bar{\Psi}_{c}^{c} \not \partial \Psi_{c}^{c}+\frac{1}{2} \bar{\chi}^{n} \partial_{n} \Psi_{c}^{c}+\frac{1}{2} \bar{\Psi}_{c}^{c} \partial_{n} \chi^{n}\right) \tag{5.35}
\end{align*}
$$

To verify that the expression above is correct we derive the equations of motion from (5.35)

$$
\begin{align*}
\frac{\delta S}{\delta \chi_{a}} & =\gamma^{n} \partial_{n} \chi^{a}-\partial^{n} \Psi_{m a}-\frac{1}{2} \partial_{a} \Psi^{c}{ }_{c}=0  \tag{5.36}\\
\frac{\delta S}{\delta \Psi_{a b}} & =-\partial^{(a} \chi^{b)}+\frac{1}{2} \not \partial \Psi^{a b}+\eta^{a b}\left(-\frac{1}{4} \not \partial \Psi^{c}{ }_{c}+\frac{1}{2} \partial_{n} \chi^{n}\right)=0 \tag{5.37}
\end{align*}
$$

We now notice that the (5.37) can be rewritten as:

$$
M^{a b}-\frac{1}{2} \eta^{a b} M_{c}^{c}=0
$$

where $M^{a b}=-\partial^{(a} \chi^{b)}+\frac{1}{2} \not \partial \Psi^{a b}$. The contraction of (5.37) with $\eta_{a b}$ results in:

$$
M_{a}^{a}=0 ; \Rightarrow M^{a b}=0
$$

Thus, we are left with the two indipendent equation (5.27) and (5.28). In this way we have verified that the metric-like action (5.35) has the correct form. If we add to this action the kinetic term $i \bar{\psi} \gamma^{m} \partial \psi$ for a spin- $1 / 2$ field $\psi(x)$ we will get (modulo notation and conventions) the metric-like action for the spin-5/2 triplet constructed e.g. in [28, 14].

## 6 Fermionic doublet of $s=5 / 2, s=3 / 2$ in AdS

Let us now consider the case, introduced in Section (4.9), of a reducible frame field $\psi_{a}=\psi_{m ; a} d x^{m}=\psi_{b ; a} e^{b}{ }_{m} d x^{m}{ }^{7}$ in AdS. The action we start with is the one given in (4.53), which enjoys the gauge-invariance (4.50).

We remember here that the gauge parameter $\xi^{a}$ is $\gamma$-traceful, while the antisymmetric parameter $\xi^{a, b}$ satisfies the following relation:

$$
\begin{equation*}
\gamma_{b} \xi^{b, a}=-i \sqrt{-\Lambda}\left(\gamma^{a b} \xi_{b}\right)=i \sqrt{-\Lambda}\left(\xi^{a}-\gamma^{a} \gamma^{b} \xi_{b}\right) . \tag{6.1}
\end{equation*}
$$

As pointed out in Section 3, we see that the frame-field $\gamma_{a} \psi^{a}$ enjoys the same gauge invariance as a Rarita-Schwinger field in $A d S$.
Using again the identities (5.4) and (5.5), we can rewrite (4.53) in its component form

$$
\begin{gather*}
S=i \int_{M^{D}} d^{D} x e\left[\bar{\psi}_{a ; b} \gamma^{a c d} \mathcal{D}_{c} \psi_{d ;}{ }^{b}-6 \bar{\psi}_{a ;}{ }^{[a} \gamma^{b} \mathcal{D}_{b} \psi_{c ;}{ }^{c]}+i \sqrt{-\Lambda}\left(\bar{\psi}_{a ; c} \gamma^{a b} \psi_{b ;}{ }^{c}{ }^{c}{ }^{[a}{ }^{[a}{ }^{[a} \psi_{b ;}{ }^{b]}+2 \bar{\psi}_{a ; b} \gamma^{[ } \gamma^{[a} \psi_{c ;}{ }^{c]}+\bar{\psi}_{a ; b} \gamma^{b} \gamma^{c a} \gamma_{d} \psi_{c ;}{ }^{d}\right)\right] \\
\left.+2 \bar{\psi}_{a}\right) \tag{6.2}
\end{gather*}
$$

where:

$$
e=\frac{1}{D!} e_{m_{1}}^{a_{1}} \ldots e_{m_{D}}^{a_{D}} \varepsilon_{a_{1} \ldots a_{D}} \varepsilon^{m_{1} \ldots m_{D}}=\sqrt{-g}
$$

### 6.1 Gauge Invariance

In this section we shall verify the gauge invariance of action (4.53) under the gauge transformations (4.50). To this aim, we first show that the following identity holds:

$$
\begin{equation*}
\delta\left(\bar{\psi}_{f} \gamma^{b c d} \mathcal{D} \psi^{f}\right)=2 \delta \psi_{f} \gamma^{b c d} \mathcal{D} \psi^{f}+\nabla\left(\delta \bar{\psi}_{f} \gamma^{b c d} \psi^{f}\right) \tag{6.3}
\end{equation*}
$$

where the total derivative does not affect the variation of the action ${ }^{8}$ (4.53), and can thus be neglected. To show that (6.3) holds, we write:

$$
\begin{equation*}
\bar{\psi}_{f} \gamma^{b c d} \mathcal{D} \psi^{f}=\bar{\psi}_{f} \gamma^{b c d} \nabla \psi^{f}+i \frac{\sqrt{-\Lambda}}{2} \bar{\psi}_{f} \gamma^{b c d} \gamma^{a} e_{a} \psi^{f} \tag{6.4}
\end{equation*}
$$

and separately calculate the variation of each of the two terms on the right-hand-side of (6.4). Then we have:

$$
\begin{align*}
\delta\left(\bar{\psi}_{f} \gamma^{b c d} \nabla \psi^{f}\right) & =\delta \bar{\psi}_{f} \gamma^{b c d} \nabla \psi^{f}+\bar{\psi}_{f} \gamma^{b c d} \nabla \delta \psi^{f}= \\
& =\delta \bar{\psi}_{f} \gamma^{b c d} \nabla \psi^{f}-\nabla\left(\bar{\psi}_{f} \gamma^{b c d} \delta \psi^{f}\right)+\left(\nabla \bar{\psi}_{f}\right) \gamma^{b c d} \delta \psi^{f}= \\
& =\delta \bar{\psi}_{f} \gamma^{b c d} \nabla \psi^{f}+\nabla\left(\delta \bar{\psi}_{f} \gamma^{b c d} \psi^{f}\right)+\delta \psi^{f} \gamma^{b c d} \nabla \bar{\psi}_{f}= \\
& =2 \delta \psi_{f} \gamma^{b c d} \nabla \psi^{f}+\nabla\left(\delta \bar{\psi}_{f} \gamma^{b c d} \psi^{f}\right) . \tag{6.5}
\end{align*}
$$

In the lines above we made use of the properties of one-forms, as well as of the Grassman (anticommuting) nature of fermionic variables, and the fact that in $D=4$ in the Majorana

[^6]basis: $\gamma_{\alpha \beta}^{b c d}=-\gamma_{\beta \alpha}^{b c d}$. Now, to obtain (6.3), we consider the second term on the r.h.s. of (6.4). Using the following $\gamma$-matrix identity:
\[

$$
\begin{equation*}
\gamma^{b c d} \gamma^{a}=\gamma^{b c d a}+3 \eta^{a[b} \gamma^{c d]} \tag{6.6}
\end{equation*}
$$

\]

we can write:

$$
\begin{equation*}
\bar{\psi}_{f} \gamma^{b c d} \gamma^{a} e_{a} \psi^{f}=\bar{\psi}_{f} \gamma^{b c d a} e_{a} \psi^{f}+3 \bar{\psi}_{f} e^{[b} \gamma^{c d]} \psi^{f} \tag{6.7}
\end{equation*}
$$

Now, the first term on the right hand side of (6.7), vanishes, because $\gamma_{\alpha \beta}^{b c d l}=-\gamma_{\beta \alpha}^{b c d l}$, and then we have:

$$
\begin{equation*}
\delta\left(\bar{\psi}_{f} \gamma^{b c d} \gamma^{a} e_{a} \psi^{f}\right)=2 \delta \bar{\psi}_{f} \gamma^{b c d} \gamma^{a} e_{a} \psi^{f} \tag{6.8}
\end{equation*}
$$

Hence, (6.3) holds. However, we are not going to directly verify the gauge invariance of (6.2) under the transformations (4.50), but we shall take a simpler path. We will first separately verify the gauge invariance of the actions for the Fang-Fronsdal fields with $s=3 / 2$ and $s=5 / 2$, and then we will show that the action (4.53) decomposes into the direct sum of these two actions, and hence is indeed gauge invariant under (4.50). The decomposition of the unconstrained field $\psi^{a}$ into its $s=5 / 2$ and $s=3 / 2$ parts is:

$$
\begin{equation*}
\psi^{a}=\tilde{\psi^{a}}+\frac{1}{D} \gamma^{a} \tilde{\psi} \tag{6.9}
\end{equation*}
$$

where the $\gamma$-traceless field $\tilde{\psi}^{a}$ (such that $\gamma_{a} \tilde{\psi}^{a}=0$ ) carries spin $s=5 / 2$ and the trace $\tilde{\psi}=\gamma_{a} \psi^{a}$ is the $s=3 / 2$ field. In fact, we have already noticed that the latter transforms as the Rarita-Schwinger field in $A d S$.

Accordingly, the gauge transformations (4.50) split as follows:

$$
\begin{align*}
& \delta \tilde{\psi}=\mathcal{D}\left(\gamma_{a} \xi^{a}\right),  \tag{6.10}\\
& \delta \tilde{\psi}^{a}=\mathcal{D} \tilde{\xi}^{a}-e_{b} \tilde{\xi}^{a, b} \tag{6.11}
\end{align*}
$$

where the gauge parameter $\tilde{\xi}^{a}$ is $\gamma$-traceless, and hence, in view of (6.1), $\tilde{\xi}^{a, b}$ satisfies the following relation:

$$
\begin{equation*}
\gamma_{b} \tilde{\xi}^{a, b}=i \sqrt{-\Lambda} \tilde{\xi}^{a} \tag{6.12}
\end{equation*}
$$

According to equation (4.49), the actions for the irreducible (Fang-Fronsdal) fields of spin $-3 / 2$ and $5 / 2$ have the following form

$$
\begin{align*}
S_{3 / 2}= & i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3 p q r}}\left(\overline{\tilde{\psi}} \gamma^{p q r} \mathcal{D} \tilde{\psi}\right)  \tag{6.13}\\
S_{5 / 2}= & i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left[\tilde{\tilde{\psi}}_{f} \gamma^{p q r} \mathcal{D} \tilde{\psi}^{f}-6 \overline{\tilde{\psi}}^{p} \gamma^{q} \mathcal{D} \tilde{\psi}^{r}\right. \\
& \left.-i \frac{3 \sqrt{-\Lambda}}{D-2}\left(e^{r} \overline{\tilde{\psi}}_{f} \gamma^{p q} \tilde{\psi}^{f}+2 e^{p} \overline{\tilde{\psi}}^{q} \tilde{\psi}^{r}\right)\right] \tag{6.14}
\end{align*}
$$

The $s=3 / 2$ action (6.13) is manifestly invariant under the gauge transformations $\delta \tilde{\psi}=\mathcal{D} \tilde{\xi}$, since in $\operatorname{AdS} \mathcal{D}^{2} \epsilon=0$, for any spinor $\epsilon$, as already stated in (4.41)

Let us now turn to the action for $s=5 / 2$ field, which obeys the gauge transformation (6.11). Since the calculation is rather cumbersome, we will present only the result of the
variation of each term in (6.14). The gauge variation of the first term in (4.53) gives (modulo a factor of 2 ):

$$
\begin{align*}
& e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \overline{\tilde{\psi}}_{f} \gamma^{b c d} \mathcal{D}\left(\mathcal{D} \tilde{\xi}^{f}-e_{l} \tilde{\xi}^{f, l}\right)=e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left[-\Lambda \overline{\tilde{\psi}}_{f} \gamma^{b c d} e^{f} e_{l} \tilde{\xi}^{l}+\overline{\tilde{\psi}}_{f} \gamma^{b c d} e_{l} \mathcal{D} \tilde{\xi}^{f, l}\right]= \\
& =\frac{3}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \overline{\tilde{\psi}}_{f} \gamma^{c} \mathcal{D} \tilde{\xi}^{f, d}\right]+ \\
& +\left(\frac{12}{(D-1)(D-2)}-\frac{6}{D-2}\right) e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}_{f} \gamma^{c} \tilde{\xi}^{f}+\frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, c} \tag{6.15}
\end{align*}
$$

The variation of the second term is:

$$
\begin{align*}
& e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left(6 \Lambda \overline{\tilde{\psi}}^{b} \gamma^{c} e^{d} e_{l} \tilde{\psi}^{l}-6 \overline{\tilde{\psi}} \gamma^{c} e_{l} \mathcal{D} \tilde{\xi}^{d, l}\right)=\frac{6 \Lambda}{D-1} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{b} \gamma^{c} \tilde{\xi}_{b}+ \\
& +\frac{6}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}^{b} \gamma^{c} \mathcal{D} \tilde{\xi}^{d,}{ }_{b}+\frac{6 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \bar{\psi}^{b} \mathcal{D} \tilde{\xi}^{d} \\
& -\frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, d} \tag{6.16}
\end{align*}
$$

Summing (6.15) and (6.16), we get:

$$
\begin{equation*}
\frac{6 \Lambda}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{d} \gamma^{c} \tilde{\xi}_{d}+\frac{3 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[\overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \overline{\tilde{\psi}}^{c} \mathcal{D} \tilde{\xi}^{d}\right] \tag{6.17}
\end{equation*}
$$

The variation of the third term is:

$$
\begin{equation*}
-\frac{3 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}-e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)}\left(\overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, c}+i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{f} \gamma^{c} \tilde{\xi}^{f}\right) \tag{6.18}
\end{equation*}
$$

The variation of the fourth term is:

$$
\begin{align*}
& -e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \frac{6 i \sqrt{-\Lambda}}{D-2} \overline{\tilde{\psi}}^{c}\left[\mathcal{D} \tilde{\xi}^{d}-e_{l} \tilde{\xi}^{d, l}\right]= \\
& =-e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \frac{6 i \sqrt{-\Lambda}}{D-2} \bar{\psi}^{c} \mathcal{D} \tilde{\xi}^{d}+e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)} \overline{\tilde{\psi}}_{b} \tilde{\xi}^{b, c} \tag{6.19}
\end{align*}
$$

Summing up (6.18) and (6.19), we get:

$$
\begin{equation*}
-\left[\frac{6 \Lambda}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{d} \gamma^{c} \tilde{\xi}_{d}+\frac{3 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[\overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \overline{\tilde{\psi}}^{c} \mathcal{D} \tilde{\xi}^{d}\right]\right] \tag{6.20}
\end{equation*}
$$

which exactly cancels (6.17). Thus, the total variation of the action under the gauge transformations (6.11) vanishes. The next step is to verify that the action (4.53) indeed splits into the sum of the actions (6.13) and (6.14).

### 6.2 Action Splitting

In the first place, the cross terms, i.e. those terms containing the both fields $\tilde{\psi}^{a}$ and $\tilde{\psi}$ should vanish. To verify this, we substitute $\mathcal{D}=\nabla+i \frac{\sqrt{-\Lambda}}{2} \gamma^{a} e_{a}$ and rewrite the action (4.53) as follows

$$
\begin{align*}
S=i \int_{M^{D}} e^{a_{1}} \ldots & e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left[\bar{\psi}_{f} \gamma^{b c d} \nabla \psi^{f}-6 \bar{\psi}^{b} \gamma^{c} \nabla \psi^{d}-\frac{3 i \sqrt{-\Lambda}}{D-2} \times\right. \\
& \left.\times\left(\frac{D}{2} e^{d} \bar{\psi}_{f} \gamma^{b c} \psi^{f}+D e^{b} \bar{\psi}^{c} \psi^{d}+2 e^{d} \bar{\psi}_{f} \gamma^{f} \gamma^{b} \psi^{c}-e^{d} \bar{\psi}_{f} \gamma^{f} \gamma^{b c} \gamma_{i} \psi^{i}\right)\right] \tag{6.21}
\end{align*}
$$

In (6.21), the derivative $\tilde{\psi}^{a}-\tilde{\psi}$ mixed terms cancel one another, so let us concentrate on the mass-like mixed terms. Here we separately list their contributions:

1. $\frac{D}{2} e^{d} \bar{\psi}_{f} \gamma^{b c} \psi^{f} \rightarrow-4 D e^{b} \overline{\tilde{\psi}} \gamma^{c} \tilde{\psi}^{d}$;
2. $D e^{b} \bar{\psi}^{c} \psi^{d} \rightarrow+2 D e^{b} \overline{\tilde{\psi}} \gamma^{c} \tilde{\psi}^{d}$;
3. $2 e^{d} \bar{\psi}_{f} \gamma^{f} \gamma^{b} \psi^{c} \rightarrow 2 D e^{b} \overline{\tilde{\psi}} \gamma^{c} \tilde{\psi}^{d}$.

Summing all the contributions above, we find that all mixed terms indeed cancel each other.

Now let us consider the spin $s=5 / 2$ case associated with the gamma-traceless field $\tilde{\psi}^{a}$, putting $\tilde{\psi}=\gamma_{a} \psi^{a}$ to zero. Since the last two terms in (4.53) contain the $\gamma$-trace of the doublet field $\psi^{a}$, we are sure that these disappear as we take only the $\gamma$-traceless field $\tilde{\psi}^{a}$, and we are thus left with the action (6.14). So we are done with this case.

Let us now, by putting the field $\tilde{\psi}^{a}$ to zero, verify that the action (4.53) actually contains the action (6.13) for the spin $s=3 / 2$ field. The spin $3 / 2$ Lagrangian in (6.13) is

$$
\begin{equation*}
\bar{\psi} \gamma^{b c d} \mathcal{D} \psi=\bar{\psi} \gamma^{b c d} \nabla \psi+i \frac{\sqrt{-\Lambda}}{2} \bar{\psi} \gamma^{b c d} \gamma^{a} e_{a} \psi \tag{6.22}
\end{equation*}
$$

Using the identity $\gamma^{b c d} \gamma^{a}=\gamma^{b c d a}+3 \eta^{a[b} \gamma^{c d]}$ we get the following result for (6.22):

$$
\begin{equation*}
\bar{\psi} \gamma^{b c d} \mathcal{D} \psi=\bar{\psi} \gamma^{b c d} \nabla \psi-3 i \frac{\sqrt{-\Lambda}}{2} e^{[b} \bar{\psi} \gamma^{c d]} \psi \tag{6.23}
\end{equation*}
$$

where $\gamma^{b c d a}$ does not contribute due to the antisymmeytric nature of its spinor indices.
Now let us substitute into the action (4.53) the field $\tilde{\psi}$ everywhere. The first term is:

$$
\begin{align*}
& \bar{\psi}_{f} \gamma^{b c d} \mathcal{D} \psi^{f} \rightarrow \overline{\tilde{\psi}} \gamma^{f} \gamma^{b c d} \mathcal{D} \gamma_{f} \tilde{\psi}= \\
& =\tilde{\tilde{\psi}} \gamma^{f} \gamma^{b c d} \nabla \gamma_{f} \tilde{\psi}+i \frac{\sqrt{-\Lambda}}{2} \overline{\tilde{\psi}} \gamma^{b c d} \gamma^{a} e_{a} \tilde{\psi}= \\
& =-(D-6) \overline{\tilde{\psi}} \gamma^{b c d} \nabla \tilde{\psi}+i \frac{\sqrt{-\Lambda}}{2} \tilde{\tilde{\psi}} \gamma_{f} \gamma^{b c d} \gamma^{a} \gamma^{f} \tilde{\psi}= \\
& =-(D-6) \overline{\tilde{\psi}} \gamma^{b c d} \nabla \tilde{\psi}+i \frac{\sqrt{-\Lambda}}{2} \tilde{\tilde{\psi}} \gamma_{f}\left(\gamma^{b c d a}+3 \eta^{a[b} \gamma^{c d]}\right) e_{a} \tilde{\psi}=  \tag{6.24}\\
& =-(D-6) \tilde{\tilde{\psi}} \gamma^{b c d} \nabla \tilde{\psi}+3 i \frac{\sqrt{-\Lambda}}{2} \tilde{\psi} \gamma_{f} e^{[b} \gamma^{c d]} \gamma^{f} \tilde{\psi} \\
& =-(D-6) \overline{\tilde{\psi}} \gamma^{b c d} \nabla \tilde{\psi}-3 i \frac{\sqrt{-\Lambda}}{2}(D-4) e^{[b} \tilde{\psi} \gamma^{c d]} \tilde{\psi}
\end{align*}
$$

The second term is:

$$
\begin{align*}
& -6 \bar{\psi}{ }^{[b} \gamma^{c} \mathcal{D} \psi^{d]} \rightarrow-6 \tilde{\tilde{\psi}} \gamma^{[b} \gamma^{c} \mathcal{D} \gamma^{d]} \tilde{\psi}= \\
& =-6 \tilde{\tilde{\psi}} \gamma^{[b} \gamma^{c} \gamma^{d]} \nabla \tilde{\psi}-3 i \sqrt{-\Lambda} \overline{\tilde{\psi}} \gamma^{[b} \gamma^{c} \gamma_{a} \gamma^{d]} e^{a} \tilde{\psi}= \\
& =-6 \tilde{\tilde{\psi}} \gamma^{[b} \gamma^{c} \gamma^{d]} \nabla \tilde{\psi}-3 i \sqrt{-\Lambda} \overline{\tilde{\psi}}\left(-\gamma^{[b} \gamma^{c} \gamma^{d]} \gamma_{a}+2 \gamma^{[b} \gamma^{c} \eta^{d]}{ }_{a}\right) e^{a} \tilde{\psi}=  \tag{6.25}\\
& =-6 \tilde{\tilde{\psi}} \gamma^{[b} \gamma^{c} \gamma^{d]} \nabla \tilde{\psi}-3 i \sqrt{-\Lambda} \overline{\tilde{\psi}}\left(-3 \eta_{a}{ }^{\left[b \gamma^{c} d\right]}+2 \eta_{a}^{[d} \gamma^{b c]}\right) e^{a} \tilde{\psi}= \\
& =-6 \tilde{\psi} \gamma^{b c d} \nabla \tilde{\psi}-3 i \sqrt{-\Lambda} e^{[b} \tilde{\tilde{\psi}} \gamma^{c d]} \tilde{\psi}
\end{align*}
$$

Summing up (6.24) and (6.25) we get:

$$
\begin{equation*}
-D \overline{\tilde{\psi}} \gamma^{b c d} \nabla \tilde{\psi}-3 i \frac{\sqrt{-\Lambda}}{2}(D-2) e^{[b} \overline{\tilde{\psi}} \gamma^{c d]} \tilde{\psi} \tag{6.26}
\end{equation*}
$$

The mass-like terms in (4.53) give:

$$
\begin{equation*}
-3 i \frac{\sqrt{-\Lambda}}{2} \frac{(D-1)(D-2)}{D-2} e^{[b} \overline{\tilde{\psi}} \gamma^{c d]} \tilde{\psi}=-3 i \frac{\sqrt{-\Lambda}}{2}(D-1) e^{[b} \overline{\tilde{\psi}} \gamma^{c d]} \tilde{\psi} \tag{6.27}
\end{equation*}
$$

Summing up (6.26) and (6.27), we get the Lagrangian which is proportional to (6.22)

$$
\begin{equation*}
-D \overline{\tilde{\psi}} \gamma^{b c d} \nabla \tilde{\psi}+3 i \frac{\sqrt{-\Lambda}}{2} D \overline{\tilde{\psi}} e^{[b} \gamma^{c d]} \tilde{\psi}=-D \overline{\tilde{\psi}} \gamma^{b c d} \mathcal{D} \tilde{\psi} \tag{6.28}
\end{equation*}
$$

Thus, we have shown that the action (4.53) is indeed the direct sum of the actions for irreducible fields of spins $s=3 / 2$ and $s=5 / 2$.

## 7 Metric-like action for the spin $s=5 / 2$ and $s=3 / 2$ doublet in AdS

In order to derive the metric-like action and equations of motion for the doublet of fields of spins $5 / 2$ and $3 / 2$, we start from the frame-like action in (6.21), which is the most suitable for our purposes. Formally, we can rewrite (6.21) in the following way

$$
\begin{equation*}
S=S_{\nabla}+S_{\Lambda} \tag{7.1}
\end{equation*}
$$

where, with straightforward notation, $S_{\nabla}$ is the part of the action which contains only the covariant derivative terms, while $S_{\Lambda}$ is the "mass-like" part, which is proportional to $\sqrt{-\Lambda}$. Now, since for simplicity in what follows we will restrict ourselves to the case of Majorana fermions in the real basis (e.g. in $D=4$ ), we will rewrite (6.21) by simply replacing the field $\bar{\psi}^{a}$ with $\psi^{a}$, and $\bar{\psi}^{f} \gamma_{f}$ with $-\gamma^{f} \psi_{f}$, thus obtaining:

$$
\begin{align*}
S= & i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left[\psi_{f} \gamma^{b c d} \nabla \psi^{f}-6 \psi^{b} \gamma^{c} \nabla \psi^{d}-\right. \\
& \left.\frac{3 i \sqrt{-\Lambda}}{D-2}\left(\frac{D}{2} e^{d} \psi_{f} \gamma^{b c} \psi^{f}+D e^{b} \psi^{c} \psi^{d}+2 e^{d} \psi^{c} \gamma^{b}\left(\gamma^{f} \psi_{f}\right)+e^{d}\left(\gamma^{f} \psi_{f}\right) \gamma^{b c} \gamma_{i} \psi^{i}\right)\right] \tag{7.2}
\end{align*}
$$

Now, $S_{\nabla}$ is the same as the action in the flat space-time case, with the replacement $\partial \rightarrow \nabla$. Furthermore, since $\bar{\chi}_{a}=\bar{\psi}_{b ; a} \gamma^{b}=-\gamma^{b} \psi_{b ; a}=-\chi_{a}$, where $\psi_{a ; b}$ are components of the one-form field $\psi_{b}=d x^{m} e_{m}^{a}(x) \psi_{a ; b}$, from (5.35) we get

$$
\begin{equation*}
S_{\nabla}=i \int_{M^{D}} d^{D} x e\left(-\chi^{a} \not \nabla \chi_{a}+\frac{1}{2} \Psi_{a b} \not \forall \Psi^{a b}+2 \chi^{a} \nabla^{b} \Psi_{a b}-\frac{1}{4} \Psi^{c}{ }_{c} \not \nabla \Psi^{c}{ }_{c}-\chi^{a} \nabla_{a} \Psi^{c}{ }_{c}\right) \tag{7.3}
\end{equation*}
$$

For what concerns $S_{\Lambda}$, it has the following form for the component field $\psi_{a ; b}$

$$
\begin{equation*}
S_{\Lambda}=i \sqrt{-\Lambda} \int_{M^{D}} d^{D} x e\left(\frac{D}{2} \psi_{a ; b} \gamma^{a c} \psi_{c ;}{ }^{b}+D \psi_{a ;}{ }^{[a} \psi_{b}{ }^{b]}+2 \psi_{c ;}{ }^{[c} \gamma^{a]} \gamma^{b} \psi_{a ; b}+\left(\gamma^{b} \psi_{a ; b}\right) \gamma^{a c}\left(\gamma^{d} \psi_{c ; d}\right)\right) \tag{7.4}
\end{equation*}
$$

In order to rewrite the action (7.4) in terms of the doublet (5.12), the following identity has been used

$$
\begin{equation*}
\gamma^{b} \psi_{a ; b}=\gamma^{b} \Psi_{a b}-\chi_{a} \tag{7.5}
\end{equation*}
$$

which is a direct consequence of the definition of the doublet (5.12), and can be further verified by using (5.17). The result is

$$
\begin{gather*}
S_{\Lambda}=\sqrt{-\Lambda} \int_{M^{D}}\left[-\frac{D+2}{2} \chi^{a} \chi_{a}+\frac{D-4}{8} \Psi^{a}{ }_{a} \Psi^{b}{ }_{b}-\frac{D-4}{4} \Psi_{a b} \Psi^{a b}+\right. \\
\left.+3 \chi^{a} \gamma^{b} \Psi_{a b}-\chi_{a} \gamma^{c a} \chi_{c}-\frac{3}{2} \chi_{a} \gamma^{a} \Psi^{c}{ }_{c}+\Psi^{a}{ }_{b} \gamma^{b c} \Psi_{a c}\right] \tag{7.6}
\end{gather*}
$$

Now we are ready to write the full metric-like Lagrangian in AdS for the doublet of spins $s=5 / 2$ and $s=3 / 2$. Summing up (7.3) and (7.6), we get

$$
\begin{align*}
S=\int_{M^{D}} d^{D} x e & {\left[i\left(-\chi^{a} \not \nabla \chi_{a}+\frac{1}{2} \Psi_{a b} \not \nabla \Psi^{a b}+2 \chi^{a} \nabla^{b} \Psi_{a b}-\frac{1}{4} \Psi^{c}{ }_{c} \not \nabla \Psi^{c}{ }_{c}-\chi^{a} \nabla_{a} \Psi^{c}{ }_{c}\right)\right.} \\
& +\sqrt{-\Lambda}\left(-\frac{D+2}{2} \chi^{a} \chi_{a}+\frac{D-4}{8} \Psi^{a}{ }_{a} \Psi^{b}{ }_{b}-\frac{D-4}{4} \Psi_{a b} \Psi^{a b}+\right. \\
& \left.\left.+3 \chi^{a} \gamma^{b} \Psi_{a b}-\chi_{a} \gamma^{c a} \chi_{c}-\frac{3}{2} \chi_{a} \gamma^{a} \Psi^{c}{ }_{c}+\Psi^{a}{ }_{b} \gamma^{b c} \Psi_{a c}\right)\right] \tag{7.7}
\end{align*}
$$

From this action we derive the equations of motion for the doublet of the fields $\left\{\chi^{a}, \Psi^{a b}\right\}:$

$$
\begin{align*}
\frac{\delta S}{\delta \chi^{a}} & =\left(-2 \not \forall \chi_{a}+2 \nabla^{b} \Psi_{a b}-\nabla_{a} \Psi^{c}{ }_{c}\right)+\sqrt{-\Lambda}\left(-(D+2) \chi_{a}+3 \gamma^{b} \Psi_{a b}+2 \gamma_{a}{ }^{c} \chi_{c}-\frac{3}{2} \gamma_{a} \Psi^{c}{ }_{c}\right)=0,  \tag{7.8}\\
\frac{\delta S}{\delta \Psi^{a b}} & =\left[\not \forall \Psi_{a b}-2 \nabla_{(a} \chi_{b)}-\frac{1}{2} \eta_{a b}\left(\not \forall \Psi^{c}{ }_{c}-2 \nabla_{a} \chi^{a}\right)\right]+ \\
& \sqrt{-\Lambda}\left[-\frac{D-4}{2} \Psi_{a b}-3 \gamma_{\left(a \chi_{b)}\right.}-2 \gamma_{(a}{ }^{c} \Psi_{b)}+\frac{1}{2} \eta_{a b}\left(\frac{D-4}{4} \Psi^{c}{ }_{c}+\frac{3}{2} \gamma^{c} \chi_{c}\right)\right]=0 . \tag{7.9}
\end{align*}
$$

Note that, as in the flat case, (7.9) can be rewritten as:

$$
\begin{equation*}
T^{a b}-\frac{1}{2} T^{a}{ }_{a}=0 \Rightarrow T^{a b}=0 \tag{7.10}
\end{equation*}
$$

where, in this case:

$$
\begin{equation*}
T^{a b}=\not \forall \Psi^{a b}-2 \nabla^{(a} \chi^{b)}-\sqrt{-\Lambda}\left(\frac{D-4}{2} \Psi^{a b}+3 \gamma^{(a} \chi^{b)}+2 \gamma^{a}{ }_{c} \Psi^{b) c}\right)=0 \tag{7.11}
\end{equation*}
$$

Thus, we are left with the system of equations:

$$
\begin{align*}
-2 \not \forall \chi_{a}+2 \nabla^{b} \Psi_{a b}-\nabla_{a} \Psi^{c}{ }_{c}=\sqrt{-\Lambda}\left((D+2) \chi_{a}-3 \gamma^{b} \Psi_{a b}-2 \gamma_{a}{ }^{c} \chi_{c}+\frac{3}{2} \gamma_{a} \Psi^{c}{ }_{c}\right)  \tag{7.12}\\
\not \forall \Psi^{a b}-2 \nabla^{(a} \chi^{b)}=\sqrt{-\Lambda}\left(\frac{D-4}{2} \Psi^{a b}+3 \gamma^{(a} \chi^{b)}+2 \gamma^{a}{ }_{c} \Psi^{b) c}\right) \tag{7.13}
\end{align*}
$$

which is the generalization to $A d S$ space of (5.26), (5.27) and (5.28).
Note that the equations (7.12) and (7.13) must be invariant under the following gauge transformations of the metric-like fields $\Psi_{a b}$ and $\chi_{a}$ :

$$
\begin{align*}
\delta \Psi_{a b} & =\mathcal{D}_{(a} \xi_{b)}  \tag{7.14}\\
\delta \chi_{a} & =\gamma^{b} \mathcal{D}_{b} \xi_{a}-i \sqrt{-\Lambda} \gamma^{a b} \xi_{b} \tag{7.15}
\end{align*}
$$

At a first glance, we see that the covariant derivative terms in equations (7.12) and (7.13) are very similar to the ordinary derivative terms from the respective flat space-time equations (5.27) and (5.28). Indeed, they are identical upon the substitution $\partial \leftrightarrow \nabla$. In addition, the right hand side of (7.12) and (7.13) contain cosmological constant terms, which are needed to ensure gauge invariance. Thus, in the flat space-time (i.e. at zero cosmological constant limit), equations (7.12) and (7.13) reduce to (5.27) and (5.28). Taking the same limit, the gauge transformations (7.14) and (7.15) reduce to (5.29) and (5.30).

## 8 Triplets and String-Theory

In this section we will review the main features of the free open bosonic string in Minkowski space-time. After presenting the string action, we will examine its global and local symmetries, as well as the form of the string energy-momentum tensor. The latter is of crucial importance for the derivation of the triplet system equations, since it naturally leads to the Virasoro algebra, which is the algebra of the worldsheet conformal symmetry of string theory. This in turn allows one to perform the standard BRST quantization of string dynamics, from which the triplet system of higher spin fields appears in the tensionless limit, as we will see.

### 8.1 Free bosonic string in flat space-time

The action for a free bosonic string in flat space-time is [29]

$$
\begin{equation*}
S=-\frac{T}{2} \int_{M} d^{2} \sigma \sqrt{h} h^{\alpha \beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu} \tag{8.1}
\end{equation*}
$$

where the integral in $d^{2} \sigma$ is taken over the two-dimensional worldsheet (swept by the string moving in space-time) parametrized by the coordinates ( $\sigma^{\alpha}=\tau, \sigma$ ), $\tau$ is the proper time of the string, $\sigma$ parametrizes points of the string ${ }^{9}$, and $T$ having dimension of (length) ${ }^{-2}$ is identified with the string tension. For open strings, the tension is related to the Regge parameter $\alpha^{\prime}$ as follows $T=\left(2 \pi \alpha^{\prime}\right)^{-1} . h^{\alpha \beta}(\tau, \sigma)$ is the metric which characterizes the geometry of the worldsheet. The functions $X^{\mu}(\tau, \sigma)$ are the string coordinates in a $D$ dimensional flat space-time $(\mu=0,1, \cdots, D-1)$. Let us note that in this section we are using Greek letters from the beginning of the alphabet $(\alpha, \beta, \ldots)$ to denote the 2 dimensional vector indices, while we use the Greek indices starting from the middle of the alphabet $(\mu, \nu, \ldots)$ to indicate physical space-time indices. ${ }^{10}$
The only terms one could add to action (8.1) are the following two

$$
\begin{align*}
& S_{1}=\Lambda \int d^{2} \sigma \sqrt{h}  \tag{8.2}\\
& S_{2}=\frac{1}{2 \pi} \int d^{2} \sigma \sqrt{h} R^{(2)}(h) \tag{8.3}
\end{align*}
$$

However, both of them are excluded from the consideration because of the following reasons. If we consider the action $S+S_{1}$, the trace of the resulting equations of motion of $h_{\alpha \beta}=0$ would imply that $\Lambda=0$. The second term $S_{2}$, in which $R^{(2)}$ denotes the curvature of the 2 -dimensional manifold, is an Einstein-Hilbert term. In two dimensions, the combination $\sqrt{h} R^{(2)}$ is a total derivative and, hence, does not contribute to the equations of motion.

The action (8.1) enjoys the invariance under the following reparametrization of the worldsheet variables

[^7]\[

$$
\begin{align*}
& \delta \sigma^{\alpha}=-\xi^{\alpha}(\tau, \sigma)  \tag{8.4}\\
& \delta X^{\mu}=\xi^{\alpha} \partial_{\alpha} X^{\mu}  \tag{8.5}\\
& \delta h^{\alpha \beta}=\xi^{\gamma} \partial_{\gamma} h^{\alpha \beta}-\partial_{\gamma} \xi^{\alpha} h^{\gamma \beta}-\partial_{\gamma} \xi^{\beta} h^{\alpha \gamma}  \tag{8.6}\\
& \delta(\sqrt{h})=\partial_{\alpha}\left(\xi^{\alpha} \sqrt{h}\right) \tag{8.7}
\end{align*}
$$
\]

together with the so called Weyl scaling invariance

$$
\begin{equation*}
\delta h^{\alpha \beta}=\phi(\tau, \sigma) h^{\alpha \beta} \tag{8.8}
\end{equation*}
$$

Further, the action (8.1) is invariant under rigid Poincaré transformations of the spacetime coordinates.

$$
\begin{equation*}
\delta X^{\mu}=a^{\mu}{ }_{\nu} X^{\nu}+b^{\mu} \tag{8.9}
\end{equation*}
$$

The two parameters $\xi^{\alpha}$ of the local reparametrizations (8.5) and (8.6), and the Weyl re-scaling (8.8) allow us to make the gauge choice

$$
h^{\alpha \beta}=\eta^{\alpha \beta}=\left(\begin{array}{rr}
-1 & 0  \tag{8.10}\\
0 & 1
\end{array}\right)
$$

Thus, the metric $h^{\alpha \beta}$ can always be reduced to the Minkowski metric. Then, action (8.1) takes the following form

$$
\begin{equation*}
S=-\frac{T}{2} \int_{M} d^{2} \sigma \eta^{\alpha \beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu} \tag{8.11}
\end{equation*}
$$

Now we easily get that the equation of motion coming from (8.11) is the two dimensional wave equation:

$$
\begin{equation*}
\square X^{\mu}=\left(\frac{\partial^{2}}{\partial \sigma^{2}}-\frac{\partial^{2}}{\partial \tau^{2}}\right) X^{\mu}=0 \tag{8.12}
\end{equation*}
$$

To ensure that the action (8.11) is invariant under a general variation of the coordinates $X^{\mu}$, we should require that the following boundary term vanishes

$$
\begin{equation*}
-T \int d \tau\left[\left.X_{\mu}^{\prime} \delta X^{\mu}\right|_{\sigma=\pi}-\left.X_{\mu}^{\prime} \delta X^{\mu}\right|_{\sigma=0}\right]=0 \tag{8.13}
\end{equation*}
$$

where $X_{\mu}^{\prime}=\frac{\partial X^{\mu}}{\partial \sigma}$. Equation (8.13) gives a boundary condition for an open bosonic string, while in the case of the closed string it is replaced by the periodicity condition $X^{\mu}(\sigma=$ $0)=X^{\mu}(\sigma=\pi)$.

### 8.2 The energy-momentum tensor

In the Lagrangian formalism, the energy momentum tensor of a matter field can be derived as the functional derivative of the action integral with respect to the space-time metric field to which the matter field is coupled. Thus, in order to derive the energy-momentum tensor of the worldsheet matter fields $X^{\mu}(\tau, \sigma)$ we should use the action (8.1) (before imposing the gauge (8.10)). The energy-momentum tensor is

$$
\begin{equation*}
T_{\alpha \beta}=-\frac{2}{T} \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^{\alpha \beta}} \tag{8.14}
\end{equation*}
$$

which, using (8.1) becomes

$$
\begin{equation*}
T_{\alpha \beta}=\partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu}-\frac{1}{2} h_{\alpha \beta} h^{\alpha^{\prime} \beta^{\prime}} \partial_{\alpha^{\prime}} X^{\mu} \partial_{\beta^{\prime}} X_{\mu} \tag{8.15}
\end{equation*}
$$

The above tensor is symmetric and traceless, i.e. $h^{\alpha \beta} T_{\alpha \beta}=0$, as a consequence of the Weyl symmetry. The tensor $T_{\alpha \beta}$ actually vanishes since it coincides with the equation of motion of $h_{\alpha \beta}$ which is the auxiliary (non-dynamical) field in this formulation

$$
\begin{equation*}
\frac{\delta S}{\delta h^{\alpha \beta}}=0, \quad \Rightarrow \quad T_{\alpha \beta}=\partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu}-\frac{1}{2} h_{\alpha \beta} h^{\alpha^{\prime} \beta^{\prime}} \partial_{\alpha^{\prime}} X^{\mu} \partial_{\beta^{\prime}} X_{\mu}=0 \tag{8.16}
\end{equation*}
$$

Now if we define the so called induced worldsheet metric $G_{\alpha \beta}=\partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu}$ and $G=$ $\left|\operatorname{det} G_{\alpha \beta}\right|$, the equation (8.16) takes the form

$$
\begin{equation*}
G_{\alpha \beta}=\frac{1}{2} h_{\alpha \beta} h^{\alpha^{\prime} \beta^{\prime}} G_{\alpha^{\prime} \beta^{\prime}} \tag{8.17}
\end{equation*}
$$

Using the properties of the determinant, we get the following equation

$$
\begin{equation*}
G=\frac{1}{4} h\left(h^{\alpha \beta} G_{\alpha \beta}\right)^{2} \tag{8.18}
\end{equation*}
$$

Using the above relation we can rewrite the string action in the so-called Nambu-Goto form that does not contain the auxiliary metric $h_{\alpha \beta}$

$$
\begin{equation*}
S=\frac{1}{2} \int_{\Sigma} d^{2} \sigma \sqrt{h} h^{\alpha \beta} G_{\alpha \beta}=\int_{\Sigma} d^{2} \sigma \sqrt{G} \tag{8.19}
\end{equation*}
$$

The form of equation (8.19) implies that the action integral (8.1) is nothing but the area of the worldsheet $\Sigma$ swept by the string, and hence it is the generalization of the action for a massive point particles $-m \int d s$, as we could have guessed from the beginning.

### 8.3 Solutions of the equations of motion

Let us now turn to the consideration of the solution of the equations of motion (8.12). This can be written as a superposition of right and left-moving waves

$$
\begin{equation*}
X^{\mu}=X_{R}^{\mu}\left(\sigma^{-}\right)+X_{L}^{\mu}\left(\sigma^{+}\right), \tag{8.20}
\end{equation*}
$$

where $\sigma^{+}=\tau-\sigma$ and $\sigma^{-}=\tau+\sigma$ are the light cone coordinates. In this basis, the Minkowski tensor $\eta^{\alpha \beta}$ becomes

$$
\begin{equation*}
\eta_{+-}=\eta_{-+}=\frac{1}{2} \quad \eta_{++}=\eta_{--}=0 \tag{8.21}
\end{equation*}
$$

In the gauge (8.10) the constraints (8.16) take the form

$$
\begin{align*}
& T_{10}=T_{01}=\dot{X} \cdot X^{\prime}=0  \tag{8.22}\\
& T_{00}=T_{11}=\frac{1}{2}\left(\dot{X}^{2}+X^{\prime 2}\right)=0 \tag{8.23}
\end{align*}
$$

where $\dot{X}=\frac{d X}{d \tau}, X \cdot Y=X^{\mu} Y_{\mu}$. In the light-cone coordinates, we have

$$
\begin{align*}
& T_{++}=\frac{1}{2}\left(T_{00}+T_{11}\right)=\partial_{+} X \cdot \partial_{+} X  \tag{8.24}\\
& T_{--}=\frac{1}{2}\left(T_{00}-T_{01}\right)=\partial_{-} X \cdot \partial_{-} X \tag{8.25}
\end{align*}
$$

and $T_{+-}=T_{-+}=0$ identically. Using the equations above, the constraint equations $T_{++}=T_{--}=0$ become

$$
\begin{equation*}
\dot{X}_{R}^{2}=\dot{X}_{L}^{2}=0 \tag{8.26}
\end{equation*}
$$

In two-dimensional quantum field theory, the energy-momentum conservation takes the form $\partial_{-} T_{++}+\partial_{+} T_{-+}=0$, and the corrsponding equation with $+\leftrightarrow-$. In the case under consideration, i.e. in the gauge (8.10), $T_{+-}=0$, and then we are left with the current conservation law

$$
\begin{equation*}
\partial_{-} T_{++}=0 . \tag{8.27}
\end{equation*}
$$

The property (8.27) is very important, because it implies the existence of an infinite set of conserved quantities. In fact, if $f\left(x^{+}\right)$is a function of only $x^{+}$, we have $\partial_{-}\left(f\left(x^{+}\right) T_{++}\right)=0$, and thus the charge $Q_{f}=\int d \sigma f\left(x^{+}\right) T_{++}$is conserved. This conservation law is due to the fact that the covariant gauge choice (8.10) does not completely fix the symmetry. Indeed, this gauge is preserved by any combined reparametrization (8.6) and Weyl re-scaling (8.8) such that

$$
\begin{equation*}
\partial^{\alpha} \xi^{\beta}+\partial^{\beta} \xi^{\alpha}=\phi \eta^{\alpha \beta} \tag{8.28}
\end{equation*}
$$

In terms of the light-cone gauge parameters $\xi^{ \pm}=\xi^{0} \pm \xi^{1}$, this means that $\xi^{+}$may be an arbitrary function of $\sigma^{+}$and $\xi^{-}$an arbitrary function of $\sigma^{-}$. If the general reparametrizations (8.6) are generated by the operator $V=\xi^{\alpha} \partial / \partial \sigma^{\alpha}$, then the generators of the residual symmetries are

$$
\begin{equation*}
V^{+}=\xi^{+}\left(\sigma^{+}\right) \partial / \partial \sigma^{+}, \quad V^{-}=\xi^{-}\left(\sigma^{-}\right) \partial / \partial \sigma^{-} \tag{8.29}
\end{equation*}
$$

The symmetry generated by (8.29) is the conformal symmetry of the two-dimensional worldsheet.

Going back to the condition (8.13) on the surface term in the action variation, we impose the (so called Neumann) boundary condition

$$
\begin{equation*}
X^{\prime \mu}=0, \quad \text { for } \sigma=0 \text { and } \sigma=\pi \tag{8.30}
\end{equation*}
$$

These boundary conditions prevent momentum from flowing off the ends of the string. The general solution of (8.12) that satisfies the boundary condition (8.30) is given by

$$
\begin{equation*}
X^{\mu}(\sigma, \tau)=x^{\mu}+l^{2} p^{\mu} \tau+i l \sum_{n \neq 0} \frac{1}{n} \alpha_{n}^{\mu} e^{-i n \tau} \cos (n \sigma) \tag{8.31}
\end{equation*}
$$

where $x^{\mu}$ and $p^{\mu}$ are the center of mass position and momentum of the string, and $l$ is a fundamental length. Using (8.11), the computation of the Poisson brackets of the $X^{\mu}$ and $\dot{X}^{\mu}$ at equal time yields

$$
\begin{array}{r}
{\left[X^{\mu}(\sigma), X^{\nu}\left(\sigma^{\prime}\right)\right]_{P . B .}=\left[\dot{X}^{\mu}(\sigma), \dot{X}^{\nu}\left(\sigma^{\prime}\right)\right]_{P . B .}=0} \\
{\left[\dot{X}^{\mu}(\sigma), X^{\nu}\left(\sigma^{\prime}\right)\right]_{P . B .}=T^{-1} \delta\left(\sigma-\sigma^{\prime}\right) \eta^{\mu \nu}} \tag{8.33}
\end{array}
$$

which provide the following Poisson brackets for the fourier modes $\alpha_{n}^{\mu}$

$$
\begin{equation*}
\left[\alpha_{m}^{\mu}, \alpha_{n}^{\nu}\right]_{\text {P.B. }}=i m \delta_{m,-n} \eta^{\mu \nu} \tag{8.34}
\end{equation*}
$$

Comparing the above Poisson bracket with equation (8.31), we find that

$$
\begin{equation*}
\left[p^{\mu}, x^{\nu}\right]_{\text {P.B. }}=\eta^{\mu \nu}, \tag{8.35}
\end{equation*}
$$

and hence the center-of-mass position and momentum are canonically conjugate variables.
Now we consider the mode expansion of the constraint $T_{\alpha \beta}=0$. In order to write this Fourier transform, we extend the definition of $X_{L}$ and $X_{R}$ beyond the interval $0 \leq \sigma \leq \pi$, by imposing that $X_{R}(\sigma+\pi)=X_{L}(\sigma), X_{L}(\sigma+\pi)=X_{R}(\sigma)$. These imply that $X_{R}$ and $X_{L}$ are periodic functions of $\sigma$ with period $2 \pi$. Thus, the constraint equations are equivalent to imposing the vanishing of their Fourier components

$$
\begin{align*}
L_{m} & =T \int_{0}^{\pi}\left(e^{i m \sigma} T_{++}+e^{-i m \sigma} T_{--}\right) d \sigma= \\
& =\frac{T}{4} \int_{-\pi}^{\pi} e^{i m \sigma}\left(\dot{X}+X^{\prime}\right)^{2} d \sigma=\frac{1}{2} \sum_{-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_{n} \tag{8.36}
\end{align*}
$$

where $\alpha_{0}^{\mu}=l p^{\mu}$. It can be shown that the constraint $L_{0}=0$ is linked to the mass of the string $M^{2}=p^{\mu} p_{\mu}$ in a given state of oscillation by the following relation

$$
\begin{equation*}
M^{2}=\frac{1}{\alpha^{\prime}} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_{n} \tag{8.37}
\end{equation*}
$$

Using (8.36), after some Poisson bracket algebra we get

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]_{\text {P.B. }}=\frac{1}{4} \sum_{k, l}\left[\alpha_{m-k} \cdot \alpha_{k}, \alpha_{n-l} \cdot \alpha_{l}\right]=i(m-n) L_{m+n} \tag{8.38}
\end{equation*}
$$

Equation (8.38) defines the so called Virasoro algebra. It can be shown that this algebra is isomorphic to the algebra obeyed by the generators (8.29) of the residual symmetry, which is the classical $d=2$ conformal algebra.

### 8.4 The bosonic triplets in the BRST quantization of the string

It is now possible to illustrate how the triplet systems arise in the framework of string theory, by analyzing the BRST approach to the quantization of the string. So far, in this section we have been concerned only with a classical construction of string theory, without worrying about quantization. Without giving the details here, we say that there is a number of problems to take into account when switching from the classical to the quantum theory, i.e. passing from the Poisson brackets of dynamical variables to the commutators of operators on a Fock space. The most significant one, is that the Virasoro algebra (8.38) is slightly modified, because some issues are met with the normal ordering of the operators $\alpha_{j}^{\mu}$. The result is that the algebra (8.38) acquires an anomalous term, which is a non unusual fact in quantum field theory. The new algebra is

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=(m-n) L_{m+n}+\frac{1}{12} D\left(m^{3}-m\right) \delta_{m+n} \tag{8.39}
\end{equation*}
$$

where $D$ denotes the number of space-time dimensions and the last (anomalous) term is called the central charge.

According to the standard BRST method, we define the ghost modes $C_{k}$ of ghost number one, and the antighost modes $B_{k}$ of ghost number minus one, satisfying the anticommutation relations:

$$
\begin{equation*}
\left\{C_{k}, B_{l}\right\}=\delta_{k,-l} \tag{8.40}
\end{equation*}
$$

The ghost number operator is defined as:

$$
\begin{equation*}
U=\sum_{k}: C_{k} B_{k}: \tag{8.41}
\end{equation*}
$$

The BRST charge is given by:

$$
\begin{equation*}
\mathcal{Q}=\sum_{-\infty}^{+\infty}\left[C_{-k} L_{k}-\frac{1}{2}(k-l): C_{-k} C_{-l} B_{k+l}:\right]-C_{0} \tag{8.42}
\end{equation*}
$$

which provides the string field equation by imposing the condition that every physical state be annihilated by $Q$

$$
\begin{equation*}
\mathcal{Q}|\Phi\rangle=0 . \tag{8.43}
\end{equation*}
$$

Since, by construction $Q^{2}=0$, eq. (8.43) is gauge invariant under the field variations of the form

$$
\begin{equation*}
\delta|\Phi\rangle=\mathcal{Q}|\Xi\rangle \tag{8.44}
\end{equation*}
$$

In order to take the tensionless limit $\left(T \rightarrow 0, \alpha^{\prime} \rightarrow \infty\right)$, it is useful to consider the following re-scaled generators:

$$
\begin{equation*}
l_{k}=\frac{1}{\sqrt{2 \alpha^{\prime}}} L_{k}, \quad l_{0}=\frac{1}{\alpha^{\prime}} L_{0}, . \tag{8.45}
\end{equation*}
$$

Then in the limit $\alpha^{\prime} \rightarrow \infty$ one can define the reduced generators we are left with

$$
\begin{equation*}
l_{0}=p^{2}, \quad l_{m}=p \cdot \alpha_{m} \quad(m \neq 0) \tag{8.46}
\end{equation*}
$$

The new generators satisfy the simpler algebra:

$$
\begin{equation*}
\left[l_{k}, l_{m}\right]=k \delta_{k,-m} l_{0} \tag{8.47}
\end{equation*}
$$

without the central charge $\frac{D}{12} m\left(m^{2}-1\right)$.
In order to have a non-degenerate BRST operator in the tensionless limit, we define the re-scaled ghost and antighost operators:

$$
\begin{equation*}
c_{k}=\sqrt{2 \alpha^{\prime}} C_{k}, \quad b_{k}=\frac{1}{\sqrt{2 \alpha^{\prime}}} B_{k} \tag{8.48}
\end{equation*}
$$

for $k \neq 0$ and

$$
\begin{equation*}
c_{0}=\alpha^{\prime} C_{0}, \quad b_{0}=\frac{1}{\alpha^{\prime}} B_{0} \tag{8.49}
\end{equation*}
$$

for $k=0$. Upon the redefinitions above, the BRST operator (8.42) takes the following form when $\alpha^{\prime} \rightarrow \infty$ :

$$
\begin{equation*}
Q=\sum_{-\infty}^{+\infty}\left[c_{k} l_{k}-\frac{k}{2} b_{0} c_{-k} c_{k}\right] . \tag{8.50}
\end{equation*}
$$

and is now identically nilpotent. Let us rewrite (8.50) singling out the zero mode components in the following way:

$$
\begin{equation*}
Q=c_{0} l_{0}-b_{0} M+\tilde{Q} \tag{8.51}
\end{equation*}
$$

where $\tilde{Q}=\sum_{k \neq 0} c_{-k} l_{k}$ and $M=\frac{1}{2} \sum_{-\infty}^{+\infty} k c_{-k} c_{k}$. We also decompose the string field and the gauge parameter in a similar way:

$$
\begin{align*}
|\Phi\rangle & =\left|\Phi_{1}\right\rangle+c_{0}\left|\Phi_{2}\right\rangle,  \tag{8.52}\\
|\Xi\rangle & =\left|\Xi_{1}\right\rangle+c_{0}\left|\Xi_{2}\right\rangle . \tag{8.53}
\end{align*}
$$

We now focus on totally symmetric states and parameters that are built from the vacuum by application of the creation operator $\alpha_{-1}^{\mu}$ :

$$
\begin{align*}
\left|\Phi_{1}\right\rangle & =\sum_{s=0}^{\infty} \frac{1}{s!} \Phi_{\mu_{1} \ldots \mu_{s}}(x) \alpha_{-1}^{\mu_{1}} \ldots \alpha_{-1}^{\mu_{s}}|0\rangle  \tag{8.54}\\
& +\sum_{s=2}^{\infty} \frac{1}{(s-2)!} D_{\mu_{1} \ldots \mu_{s-2}}(x) \alpha_{-1}^{\mu_{1}} \ldots \alpha_{-1}^{\mu_{s-2}} c_{-1} b_{-1}|0\rangle  \tag{8.55}\\
\left|\Phi_{2}\right\rangle & =\sum_{s=1}^{\infty} \frac{-i}{(s-1)!} C_{\mu_{1} \ldots \mu_{s-1}}(x) \alpha_{-1}^{\mu_{1}} \ldots \alpha_{-1}^{\mu_{s-1}} b_{-1}|0\rangle  \tag{8.56}\\
|\Xi\rangle & =\sum_{s=1}^{\infty} \frac{i}{(s-1)!} \xi_{\mu_{1} \ldots \mu_{s-1}}(x) \alpha_{-1}^{\mu_{1}} \ldots \alpha_{-1}^{\mu_{s-1}} b_{-1}|0\rangle \tag{8.57}
\end{align*}
$$

Substituting the above fields into the equations (8.43) and (8.44), the $s$-th terms in the sums above result in the triplet equations of motion (4.16), (4.17) and (4.18), and we also recover the gauge transformations (4.19), (4.20) and (4.21).

## 9 Conclusion

In this thesis we have reviewed some of the main features of higher spin fields, with a particular attention to the massless theory. As we have shown String Theory contains massive particle states with arbitrary spin, yielding massless higher spin states in the zero-tension limit, which combine into the higher spin triplets. Thus, it seems natural to conjecture that the tension in String Theory is generated through a mechanism of spontaneous symmetry breaking of a huge gauge symmetry in a certain Higher Spin Gauge Theory.

The results of Sections 8 and 9 of this thesis may be seen as a first step towards a technically much more complicated task of obtaining the metric-like action for fermionic triplets of arbitrary high spin in AdS. We have shown that this is possible in the simplest case of the fermionic triplet with $s=5 / 2$ both in flat and AdS background. In the latter case we have seen how the gauge symmetries are modified in a suitable way, in order for the action and the equations of motion to be invariant. In fact, as pointed out in [26], it was an incomplete assumption on the form of the gauge transformation of the triplet fields that prevented other authors from obtaining the fermionic triplet Lagrangians in AdS space.

## A Notation and conventions

Minkowskii space-time signature is mostly minus:

$$
\begin{equation*}
\eta^{a b}=(+,-, \cdots,-,-) \tag{A.1}
\end{equation*}
$$

Clifford algebra

$$
\left\{\gamma^{a}, \gamma^{b}\right\}=2 \eta^{a b}
$$

The totally antisymmetric $\gamma$-matrices are defined as:

$$
\begin{equation*}
\gamma^{a_{1} \ldots a_{p}}=\gamma^{\left[a_{1}\right.} \ldots \gamma^{\left.a_{p}\right]} \tag{A.2}
\end{equation*}
$$

where the antisymmetrization is taken with unit weight, i.e.

$$
\begin{equation*}
\varepsilon_{a_{1} \ldots a_{p}} \gamma^{a_{1} \ldots a_{p}}=\varepsilon_{a_{1} \ldots a_{p}} \gamma^{a_{1}} \ldots \gamma^{a_{p}} \tag{A.3}
\end{equation*}
$$

Gamma-matrix identities

$$
\begin{align*}
& \gamma^{a} \gamma^{d_{1} \ldots d_{n}} \gamma_{a}=(-1)^{n}(D-2 n) \gamma^{d_{1} \ldots d_{n}}  \tag{A.4}\\
& \gamma^{a_{1} \ldots a_{p}} \gamma^{a_{p+1}}=\gamma^{a_{1} \ldots a_{p+1}}+p \eta^{a_{p+1}\left[a_{1}\right.} \gamma^{\left.a_{2} \ldots a_{p}\right]}  \tag{A.5}\\
& \gamma^{d} \gamma^{a_{1} \ldots a_{p}}=(-1)^{p} \gamma^{a_{1} \ldots a_{p}} \gamma^{d}+2 p \eta^{d\left[a_{1}\right.} \gamma^{\left.a_{2} \ldots a_{p}\right]} \tag{A.6}
\end{align*}
$$

Starting from Section 6, we restricted ourselves to the case of Majorana spinors. In $D=4$ a Majorana spinor has a general form

$$
\Psi_{M}=\binom{\psi}{-i \sigma^{2} \psi^{*}}
$$

It coincides with its charge conjugated spinor

$$
\Psi_{M}^{C} \equiv-i \gamma^{2} \Psi_{M}^{*}=-i\left(\begin{array}{cc}
0 & -\sigma^{2}  \tag{A.7}\\
\sigma^{2} & 0
\end{array}\right)\binom{\psi^{*}}{-i \sigma^{2} \psi}=\Psi_{M}
$$

where $\gamma^{2}$ is purely imaginary, since from (A.1) it follows that $\left(\gamma^{2}\right)^{2}=-1$.

$$
\begin{equation*}
\bar{\Psi}_{M} \Psi_{M}=\Psi_{M}^{T}\left(\gamma^{2} \gamma^{0}\right) \Psi_{M} \equiv\left(\Psi_{M}\right)^{\alpha} C_{\alpha \beta}\left(\Psi_{M}\right)^{\beta} \tag{A.8}
\end{equation*}
$$

where $\left(\gamma^{2} \gamma^{0}\right)_{\alpha \beta} \equiv C_{\alpha \beta}=-C_{\beta \alpha}$, i.e. the charge-conjugation matrix, can be defined with both lower indices. In general we have

$$
\begin{equation*}
\bar{\Psi}_{M} \gamma^{a_{1} \ldots a_{p}} \Psi_{M}=\Psi_{M}^{\alpha} C_{\alpha \gamma}\left(\gamma^{a_{1} \ldots a_{p}}\right)^{\gamma}{ }_{\beta} \Psi_{M}^{\beta}=\Psi_{M}^{\alpha}\left(\gamma^{a_{1} \ldots a_{p}}\right)_{\gamma \beta} \Psi_{M}^{\beta} \tag{A.9}
\end{equation*}
$$

The previous definition implies that, in the Majorana basis, the matrices $\gamma_{\alpha \beta}^{a}$ satisfy the Clifford algebra

$$
\begin{equation*}
\left\{\gamma^{a}, \gamma^{b}\right\}_{\alpha \beta}=2 \eta^{a b} C_{\alpha \beta} \tag{A.10}
\end{equation*}
$$

In Sections 6, 7 and 8 we have made wide use of the following symmetry properties of the gamma-matrices in Majorana basis in $D=4$ :

$$
\begin{align*}
& \left(\gamma^{a}\right)_{\alpha \beta}=\left(\gamma^{a}\right)_{\beta \alpha},  \tag{A.11}\\
& \left(\gamma^{a b}\right)_{\alpha \beta}=\left(\gamma^{a b}\right)_{\beta \alpha},  \tag{A.12}\\
& \left(\gamma^{a b c}\right)_{\alpha \beta}=-\left(\gamma^{a b c}\right)_{\beta \alpha},  \tag{A.13}\\
& \left(\gamma^{a b c d}\right)_{\alpha \beta}=-\left(\gamma^{a b c d}\right)_{\beta \alpha} . \tag{A.14}
\end{align*}
$$

## B Appendices

## B. 1 Decomposition of the field $\psi_{a ; b}$

Let us sketch the derivation of the decomposition of $\psi_{a ; b}$ given in equation (5.17) . From (5.10), we have that:

$$
\begin{equation*}
\gamma^{a} X_{[a b]}=\gamma^{a} \psi_{[a ; b]} \tag{B.1}
\end{equation*}
$$

Applying the condition above to $X_{[a b]}$ as defined in (5.15), we get

$$
\begin{align*}
\gamma^{a} X_{[a b]} & =\alpha\left(D \gamma^{m} \psi_{[m ; b]}-\gamma^{a} \gamma_{b} \gamma^{m} \psi_{[m ; a]}\right)+(D-1) \gamma_{b} C \\
& =\alpha\left(D \gamma^{m} \psi_{[m ; b]}-\gamma^{a} \gamma_{b} \gamma^{m} \psi_{[m ; a]}-2 \eta^{a}{ }_{b} \gamma^{m} \psi_{[m ; b]}\right)+(D-1) \gamma_{b} C= \\
& =\alpha(D-2) \gamma^{m} \psi_{[m ; b]}+\gamma_{b}\left(\alpha \gamma^{a m} \psi_{m ; a}+(D-1) C\right)= \\
& =\gamma^{a} \psi_{[a ; b]} . \tag{B.2}
\end{align*}
$$

Thus we get

$$
\left\{\begin{array}{l}
\alpha=\frac{1}{D-2} ;  \tag{B.3}\\
C=-\frac{1}{(D-1)(D-2)}\left(\gamma^{c} \chi_{c}-\frac{1}{2} \Psi^{c}{ }_{c}\right),
\end{array}\right.
$$

where, we remind that $\chi_{c}=\gamma^{a} \psi_{a ; c}$ and $\Psi_{a b}=2 \psi_{(a ; b)}$.
As a result, the component field $\psi_{a ; b}$ decomposes into the symmetric and antisymmetric parts as follows

$$
\begin{align*}
\psi_{a ; b}=\frac{1}{2} \Psi_{a b}+\tilde{\psi}_{[a ; b]}+\frac{1}{D-2}\left(\gamma_{a} \chi_{b}-\gamma_{b} \chi_{a}\right. & \left.+\frac{1}{2}\left(\gamma_{b} \gamma^{m} \Psi_{m a}-\gamma_{a} \gamma^{m} \Psi_{m b}\right)\right) \\
& -\frac{1}{(D-1)(D-2)} \gamma_{a b}\left(\gamma^{c} \chi_{c}-\frac{1}{2} \Psi^{c}{ }_{c}\right) \tag{B.4}
\end{align*}
$$

where $\tilde{\psi}_{[a ; b]}$ is $\gamma$-traceless.

## B. 2 Gauge invariance of the $s=5 / 2$ and $3 / 2$ fermionic doublet action in flat space-time

In order to check the gauge invariance of the action (5.1), we make use of the identity (4.55) for $p=3$, namely

$$
\begin{equation*}
\varepsilon_{a_{1} \ldots a_{D-3} p q r} e^{a_{1}} \ldots e^{a_{D-3}} e^{c}=\frac{3}{D-2} \delta_{[b}^{f} \varepsilon_{c d]} a_{1} \ldots a_{D-2} e^{a_{1} \ldots a_{D-2}} \tag{B.5}
\end{equation*}
$$

The variation of the first term in (5.1) under the transformations (5.2) is (to simplify things, we only explicitely write last one-form vielbein $e^{a_{D-3}}$ )

$$
\begin{align*}
& e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \delta\left(\bar{\psi}_{f} \gamma^{b c d} d \psi^{f}\right)=2 e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \bar{\psi}_{f} \gamma^{b c d} d\left(\delta \psi^{f}\right)= \\
= & 2 e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \bar{\psi}_{f} \gamma^{b c d} d\left(d \xi^{d}-e^{l} \xi^{f}{ }_{l}\right) \tag{B.6}
\end{align*}
$$

Using the general fact that $d^{2} \omega=d d \omega=0$ for every $p$-form $\omega$, as well as the fact that, in flat space-time, $e^{c}=\delta_{m}^{c} d x^{m}$, we have

$$
\begin{equation*}
2 e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \bar{\psi}_{f} \gamma^{b c d} d\left(d \xi^{f}-e^{l} \xi^{f}{ }_{l}\right)=-2 e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} e^{l} \bar{\psi}_{f} \gamma^{b c d} d \xi^{f,}{ }_{l} \tag{B.7}
\end{equation*}
$$

Exploiting (B.5), the last line becomes

$$
\begin{equation*}
-\frac{6}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \bar{\psi}_{f} \gamma^{b c d} d \xi^{f,}{ }_{b} \tag{B.8}
\end{equation*}
$$

Now, since $\gamma^{b c d}=\gamma^{c d} \gamma^{b}+2 \gamma^{[d} \eta^{c] b}$ and $\gamma^{b} \xi^{f,}{ }_{b}=0$, we get

$$
\begin{equation*}
-\frac{12}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \bar{\psi}_{f} \gamma^{c} d \xi^{d, f} \tag{B.9}
\end{equation*}
$$

The gauge variation of the second term in (5.1) yields

$$
\begin{align*}
& -6 e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \delta\left(\bar{\psi}^{b} \gamma^{c} d \psi^{d}\right)=-12 \bar{\psi}^{b} \gamma^{c} d\left(\delta \psi^{d}\right) \\
& =-12 e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \bar{\psi}^{b} \gamma^{c} d\left(d \xi^{d}-e^{l} \xi^{d,}{ }_{l}\right) \\
& =+\frac{36}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2}[d d} \bar{\psi}^{b} \gamma^{c} d \xi^{d,}{ }_{b]}=\frac{12}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \bar{\psi}^{f} \gamma^{c} \xi^{d,} f \tag{B.10}
\end{align*}
$$

Thus, the two terms (B.8) and (B.10) indeed cancel each other, and the action (5.1) is gauge-invariant.

## B. 3 Gauge invariance of the fermionic irreducible $s=5 / 2$ action in AdS

We shall now check the gauge invariance of the following action for the irreducible framelike field $\tilde{\psi}^{a}\left(\right.$ such that $\left.\gamma_{a} \tilde{\psi}^{a}=0\right)$ of $\operatorname{spin} s=5 / 2$

$$
\begin{equation*}
S_{5 / 2}=i \int_{M^{D}} e^{a_{1}} \ldots e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} p q r}\left[\overline{\tilde{\psi}}_{f} \gamma^{p q r} \mathcal{D} \tilde{\psi}^{f}-6 \overline{\tilde{\psi}}^{p} \gamma^{q} \mathcal{D} \tilde{\psi}^{r}-i \frac{3 \sqrt{-\Lambda}}{D-2}\left(e^{r} \overline{\tilde{\psi}}_{f} \gamma^{p q} \tilde{\psi}^{f}+2 e^{p} \overline{\tilde{\psi}}^{q} \tilde{\psi}^{r}\right)\right] \tag{B.11}
\end{equation*}
$$

under the gauge transformations:

$$
\begin{equation*}
\delta \tilde{\psi}^{a}=\mathcal{D} \tilde{\xi}^{a}-e_{b} \tilde{\xi}^{a, b} \tag{B.12}
\end{equation*}
$$

with the $\gamma$-traceless parameters $\tilde{\xi}^{a}$ e $\tilde{\xi}^{a, b}$. In order to check the gauge invariance of (B.11), we use the identity (4.55), that we rewrite here:

$$
\begin{equation*}
\varepsilon_{a_{1} \ldots a_{D-p}, b_{1} \ldots b_{p}} e^{a_{1}} \ldots e^{a_{D-p}} e^{f}=\frac{(-1)^{(p-1)(D-p+1)} p}{D-p+1} \delta_{\left[b_{1}\right.}^{f} \varepsilon_{\left.b_{2} \ldots b_{p}\right] a_{1} \ldots a_{D-p+1}} e^{a_{1}} \ldots e^{a_{D-p+1}} . \tag{B.13}
\end{equation*}
$$

In the following, we will write the variation of each term of the action (4.53) (modulo a factor 2).

The gauge variation of the first term in (B.11) gives:

$$
\begin{align*}
& e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d} \overline{\tilde{\psi}}_{f} \gamma^{b c d} \mathcal{D}\left(\mathcal{D} \tilde{\xi}^{f}-e_{l} \tilde{\xi}^{f, l}\right)=e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left[-\Lambda \overline{\tilde{\psi}}_{f} \gamma^{b c d} e^{f} e_{l} \tilde{\xi}^{l}+\overline{\tilde{\psi}}_{f} \gamma^{b c d} e_{l} \mathcal{D} \tilde{\xi}^{f, l}\right]= \\
& =\frac{3}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[\Lambda \overline{\psi_{b}} \gamma^{b c d} e_{l} \tilde{\xi}^{l}-\overline{\tilde{\psi}}_{f} \gamma^{b c d} \mathcal{D} \tilde{\xi}^{f,}{ }_{b}\right]= \\
& =\frac{3}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[-2 \Lambda \overline{\tilde{\psi}}^{c} \gamma^{d} e_{l} \tilde{\xi}^{l}+i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{f} \gamma^{c c} \tilde{\xi}^{f, d}-\Lambda \overline{\tilde{\psi}}_{f} \gamma^{c d} e_{l} \gamma^{l} \tilde{\xi}^{f}+\right. \\
& \left.-i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{d} \gamma^{c d} e_{b} \tilde{\xi}^{f, b}+2 \tilde{\tilde{\psi}}_{f} \gamma^{c} \mathcal{D} \tilde{\xi}^{f, d}\right]= \\
& =\frac{6 \Lambda}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \tilde{\psi}^{d} \gamma^{c} \tilde{\xi}_{d}+\frac{3}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \tilde{\tilde{\psi}}_{f} \gamma^{c} \mathcal{D} \tilde{\xi}^{f, d}\right] \\
& -\left(\frac{6 \Lambda}{D-2}-\frac{6 \Lambda}{(D-2)(D-1)}\right) e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}_{f} \gamma^{c} \tilde{\xi}^{f}+\frac{6 i}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, c}= \\
& =\frac{3}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \overline{\tilde{\psi}}_{f} \gamma^{c} \mathcal{D} \tilde{\xi^{f, d}}\right]+ \\
& +\left(\frac{12}{(D-1)(D-2)}-\frac{6}{D-2}\right) e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}_{f} \gamma^{c} \tilde{\xi}^{f}+\frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, c} \tag{B.14}
\end{align*}
$$

The variation of the second term in (B.11) is:

$$
\begin{align*}
& e^{a_{D-3}} \varepsilon_{a_{1} \ldots a_{D-3} b c d}\left(6 \Lambda \overline{\tilde{\psi}}^{b} \gamma^{c} e^{d} e_{l} \tilde{\psi}^{l}-6 \overline{\tilde{\psi}} \gamma^{c} e_{l} \mathcal{D} \tilde{\xi}^{d, l}\right)= \\
& -6 \Lambda e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \bar{\psi}^{b} \gamma^{c} e_{l} \tilde{\xi}^{l}+\frac{18}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}^{[b} \gamma^{c} \mathcal{D} \tilde{\xi}^{d]}{ }_{b}= \\
& \frac{6 \Lambda}{D-1} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{b} \gamma^{c} \tilde{\xi}_{b}+\frac{6}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}^{b} \gamma^{c} \mathcal{D} \tilde{\xi}^{d,}{ }_{b}-\frac{6}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}^{c} \gamma^{b} \mathcal{D} \tilde{\xi}^{d,}{ }_{b}= \\
& \frac{6 \Lambda}{D-1} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{b} \gamma^{c} \tilde{\xi}_{b}+\frac{6}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}^{b} \gamma^{c} \mathcal{D} \tilde{\xi}^{d,}{ }_{b}+\frac{6 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2} \varepsilon_{a_{1} \ldots a_{D-2} c d}} \overline{\tilde{\psi}}^{b} \mathcal{D} \tilde{\xi}^{d} \\
& -\frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, d} \tag{B.15}
\end{align*}
$$

Summing (B.14) and (B.15), we get:

$$
\begin{equation*}
\frac{6 \Lambda}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{d} \gamma^{c} \tilde{\xi}_{d}+\frac{3 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[\overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \overline{\tilde{\psi}}^{c} \mathcal{D} \tilde{\xi}^{d}\right] \tag{B.16}
\end{equation*}
$$

The variation of the third term is:
$-\frac{3 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \overline{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}-e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)}\left(\overline{\tilde{\psi}}_{f} \tilde{\xi}^{f, c}+i \sqrt{-\Lambda} \overline{\tilde{\psi}}_{f} \gamma^{c} \tilde{\xi}^{f}\right)$

The variation of the fourth term is:

$$
\begin{align*}
& -e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \frac{6 i \sqrt{-\Lambda}}{D-2} \overline{\tilde{\psi}}^{c}\left[\mathcal{D} \tilde{\xi}^{d}-e_{l} \tilde{\xi}^{d, l}\right]= \\
& =-e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d} \frac{6 i \sqrt{-\Lambda}}{D-2} \overline{\tilde{\psi}}^{c} \mathcal{D} \tilde{\xi}^{d}+e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \frac{6 i \sqrt{-\Lambda}}{(D-2)(D-1)} \overline{\tilde{\psi}}_{b} \tilde{\xi}^{b, c} \tag{B.18}
\end{align*}
$$

Summing up (B.17) and (B.18), we get:

$$
\begin{equation*}
-\left[\frac{6 \Lambda}{(D-2)(D-1)} e^{a_{D-1}} \varepsilon_{a_{1} \ldots a_{D-1} c} \overline{\tilde{\psi}}^{d} \gamma^{c} \tilde{\xi}_{d}+\frac{3 i \sqrt{-\Lambda}}{D-2} e^{a_{D-2}} \varepsilon_{a_{1} \ldots a_{D-2} c d}\left[\tilde{\tilde{\psi}}_{f} \gamma^{c d} \mathcal{D} \tilde{\xi}^{f}+2 \overline{\tilde{\psi}}^{c} \mathcal{D} \tilde{\xi}^{d}\right]\right] \tag{B.19}
\end{equation*}
$$

which exactly cancels (B.16). Thus, the total variation of the action under the gauge transformations (6.11) vanishes.

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[^0]:    ${ }^{1}$ To be precise, this two operators are the only Casimir operators just in $D=4$. For $D>4$ there are $\frac{D}{2}$ ( $D$ even) or $\frac{D-1}{2}$ ( $D$ odd) Casimir operators for the Poincarè group, and its representations should be classified by the eigenvalues of all these operators.

[^1]:    ${ }^{2}$ For the sake of simplicity, from now on, we shall drop the spinor index $\alpha$.

[^2]:    ${ }^{3}$ The isometry group of AdS space-time is $\operatorname{Spin}(2, D-1)$.

[^3]:    ${ }^{4}$ In this thesis, symmetrization or antisymmetrization over $n$ indices is meant to be taken with unit weight, i.e.:

    $$
    A^{\left(m_{1} \ldots m_{n}\right)}=\frac{1}{n!}\left(A^{m_{1} \ldots m_{n}}+A^{m_{2} m_{1} \ldots m_{n}}+(\text { permutations })\right) .
    $$

[^4]:    ${ }^{5}$ Following [26], we shall always put a tilde ~ over objects that obey the strong traceless condition in the bosonic case and the $\gamma$-traceless condition in the fermionic case.

[^5]:    ${ }^{6}$ As in Section 4, the world index, placed to the left, is separated from the tangent-space indices by the semicolon. Here, we do not need to distinguish it from tangent-space indices, since we are in flat space-time.

[^6]:    ${ }^{7}$ Note that $\psi_{a ; b}$ carries again only (locally) flat space-time indices.
    ${ }^{8}$ In fact, we remember that in a curved space the ordinary rule for integration by parts $\int d^{D} x \partial(\ldots)=0$ is replaced with its covariant form $\int d^{D} x \sqrt{-g} \nabla(\ldots)=0$.

[^7]:    ${ }^{9}$ For simplicity, we will take $0 \leq \sigma \leq \pi$.
    ${ }^{10}$ In this section we use a notation which is different from the one used in the rest of this thesis. This notation is here more convenient, since we have to deal with many different kind of indices, i.e. world-sheet, space-time and number indices.

