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Scalar-Induced Gravitational Waves From Inflation

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The ones who love us never really leave us
In loving memory of
Ilario and Gabriele

Contents

Index **i**

Introduction **3**

1 GWs from single-field slow-roll Inflation **4**
1.1 The physics of Inflation . . . 5
1.2 Slow-Roll conditions . . . 7
1.3 Reheating phase . . . 8
1.4 Quantum fluctuations: origin of cosmological perturbations . . . 10
1.5 Metric Tensor perturbations . . . 11
1.6 Stress-Energy Tensor perturbations . . . 12
1.7 Dynamics of fluctuations . . . 12
1.8 Scalar Perturbations . . . 14
1.9 Gravitational waves from inflation . . . 17
1.10 Consistency relation . . . 20
1.11 Second Order Gravitational Waves . . . 21
1.12 Post-inflationary evolution of GW . . . 21
1.13 Energy-density of gravitational waves . . . 22
1.14 Why are primordial GWs so interesting? . . . 23
1.14.1 Energy scale of inflation . . . 23
1.14.2 Predictions from inflation . . . 24
1.14.3 GWs beyond the SR, as a testing way for the inflationary model . . . 26
1.14.4 Primordial gravitational waves in the EFTI . . . 27

2 Classical GWs production during inflation **29**
2.1 Production of second order GWs . . . 31
2.1.1 Second order GWs sourced by inflaton perturbations . . . 33
2.1.2 GWs equation neglecting scalar and vector metric perturbations 34
2.2 GWs sourced by scalar fluctuations . . . 35
2.2.1 Second order GWs in the curvatone scenario . . . 35

2.2.2 Second-order GWs sourced by spectator scalar fields 37
2.3 Particle production as a source of GWs 39
2.3.1 Inflaton coupled to a scalar field 40
2.3.2 Axion inflation: pseudoscalar inflaton coupled to a gauge field 43
2.4 GWs production during reheating after Inflation 47
2.4.1 Preheating with Parametric Resonance 48
2.4.2 Tachyonic Preheating 50
2.5 GWs from inflation in Modified Gravity 53
2.5.1 Why Modified Gravity? 53
2.5.2 Overview of the main models 54
2.5.3 Primordial GWs in the Scalar-Tensor theories of gravity . . . 55
2.6 Consistency relation and possible violations 61
2.7 GWs as source of information for the thermal history of the Universe 63
2.7.1 Equation of state of the Universe and spectral tilt 65

3 Scalar Induced GWs after Inflation 67
3.1 General formalism 68
3.1.1 General Solutions 72
3.1.2 Inclusion of primordial non-Gaussianity 74
3.2 Analytical Transfer Functions 77
3.2.1 First order solutions 78
3.2.2 General SubHorizon Kernel 81
3.2.3 SuperHorizon Kernel approximation 86
3.2.4 Matching to radiation domination 87
3.3 Typical induced GWs spectra 88
3.3.1 Dirac Delta Peak 89
3.3.2 Log-Normal peak 91
3.3.3 Broken power-law 93

4 Inflationary models with features in the PS 97
4.1 Effective field theory of inflation models with sharp features 97
4.2 PBH and GWs from resonant amplification during Inflation 108
4.2.1 Field fluctuation and induced amplification 114
4.3 Fast Roll Inflation 120
4.4 Induced GWs from slow-roll after an enhanincg phase 129

5 Induced GWs from inflationary models 135
5.1 Scalar GWs induced by Fast-Roll (one-field model) 136
5.1.1 Spectral energy density of GWs from Radiation epoch 138
5.1.2 Final Results 153
5.1.3 Spectral energy density of GWs from Inflationary epoch . . . 157

5.2 Scalar induced GWs from EFTI 168
5.2.1 Spectral energy density of GWs from Radiation epoch 168
5.2.2 Spectral energy density of GWs from Inflationary epoch . . . 174
5.2.3 Cross-talk terms 178
5.3 Scalar induced GWs from two-field Fast-Roll 181
5.3.1 Fast-Roll, two-field model within the Slow-Roll formalism . . 185
5.3.2 Fast-Roll, two-field model beyond the Slow-Roll formalism . 189

Conclusions **195**

Acknowledgements **196**

Bibliography **197**

List of Figures **214**

Introduction

The objective of this thesis is to study the nature of a stochastic background of gravitational waves (GWs) induced by density (scalar) perturbations beyond linear order, at second-order in cosmological perturbation theory.
This thesis also includes the intent to study the difference and similarities between GWs backgrounds induced during the radiation and inflation phases. If we define an inflationary model always stationed in a slow-roll phase, we argue a slow-roll parameter ϵ always constant and, above all, much smaller than one; hence from the formula linking the scalar curvature spectrum to the power-spectrum of the inflationary fluctuation, it is clear that the former is much larger than the latter. Since the curvature power-spectrum is the main element in producing a second-order GWs background during radiation dominance (just as the scalar field fluctuation spectrum induces the GWs background during the Inflation era), one expects a totally negligible induced GWs background during Inflation compared to that induced during the subsequent radiation phase. The hypothesis defined earlier comes into play at this point: moving away from the slow-roll implies a consideration of a larger slow-roll parameter (albeit less than one to have inflation); thus the two power-spectra are placed in a different condition of comparability (in natural units they are completely identical), so, at least in principle, one does not expect a complete domination of one period-induced GWs background over another. This idea match totally well with the practice of this thesis, since its aim is to study the validity of the induced GWs background of the inflationary epoch with respect to that induced during the later radiation epoch.
The Gravitational waves represent a smocking gun of the inflationary phase: we can represent the principal element that lead the primordial accelerated expansion of the Universe with a scalar field, that in a perturbative model can be divided in a background term associated to the observative symmetries of homogeneity and isotropy (Robertson-Walker symmetries), and a perturbative one depending on the spatial coordinates.
This perturbative value of the real scalar field induces a perturbative term on the stress-energy tensor, that in conclusion can be write like the sum of a first background term describing a perfect fluid and a perturbation able to describe the

anisotropy and the imperfection of the fluid. From the mathematical consistency of the Einstein equation, we can argue that this last fluctuation induces a perturbation over the Einstein tensor, so over the metric tensor. In conclusion we can write this last term like the sum of a background tensor metric plus a perturbative tensor term. We can decompose the perturbation of the metric tensor into scalar, vector and tensor perturbations depending on the component of the metric perturbation that we are considering. The tensor perturbations represent the degree of freedom of the gravitational sector, so they describe the Gravitational waves. The study of the dynamics of the scalar and tensorial fluctuations in the various causal regimes leads to the conclusive result of quasi-scale-invariant curvature and tensorial spectra, in full agreement with the observations on large scalar scales.

In addition to the vacuum fluctuations of the metric tensor, related at the first perturbative order, it is possible to address the perturbative problem by going to an order of development subsequent to the linear one. Developing the computations of tensor modes at second-order in perturbation theory, and considering the transverse and trace-free space part of the Einstein equation, we can find the dynamical equation of a second-order background of Gravitational waves. This is a free-wave equation equipped on the RHS with a source term given by quadratic combinations of scalar (density and gravitational) pertubations offered by both the anisotropy term and the Einstein tensor. From solving the equation, the two-point correlation function of the tensor perturbations is derived, which will provide the tensor power spectrum which is a crucial quantity to derive observable predictions for such a stochastic background of gravitational waves. Within this general framework, a given specific model, in agreement with the dynamic phase in which the scalar source terms are active, specifies an appropriate source, thus a specific spectrum. An enhanced comparability between radiation-induced and inflation-induced GWs backgrounds is offered by a condition of departure from the slow-roll. Examples of this approach include three specific inflationary models: a model where a step feature in the inflaton potential is treated in a perturbative way through the so called Effective Field Theory approach on a slow-roll background within a quasi de-Sitter stage, a model of inflation called fast-roll, and a two-field inflationary model that by changing the energetic nature of adiabatic-isoentropic perturbations, and exploiting a dynamic induction mechanism leads to the result of a dominant inflationary induced GWs background.

The aim of this thesis is to calculate the induced GWs background for the first time for a one-field and two-field fast-roll inflationary model, respectively. The idea is to use an analytical computational approach by building simplifying models, and to compare, where possible, this theoretical resolution with a computational matrix solution.

The thesis is organized as follows: in chapter 1 we resume the physics of inflation, talking about the principal inflationary model used in single-field, the Slow-Roll model. In this framework we talk about the gravitational waves produced at the linear perturbative order, and we resume the calculation for the curvature power-spectrum and the tensor power-spectrum. In chapter 2 we introduce the classical production of primordial Gravitational waves during the inflationary epoch, so we go to the second order of perturbations. We resume the general tensor power spectrum related to that theory, so we resume different physical model that produce specific signature in the observable of the spectrum. In the chapter 3 we resume the theory of the induced GWs background during a general post-inflationary epoch: we talk about the general formalism of the perturbative theory at the second order, and we derive the general form of the tensor power-spectrum, that present specific signature with respect to the specific nature of the post-inflationary phase of induction. In the chapter 4 we introduce different inflationary models that define a a condition of departure with respect to the slow-roll dynamic, in order to underline the condition for which a different ratio between the spectra induce a more important production of induced inflationary GWs with respect to the standard ones related to the radiation epoch. In the chapter 5 we calculate the Spectral energy density for the induced GWs background in the two principal epoch of production, for an inflationary model of Fast-Roll with one field. Then we define the counts for a two field inflationary model. Here in particular we obtain different kind of curvature power spectrum for a dynamic system of Fast-Roll and Slow-Roll with respect to the physical complexity of the problem.

Chapter 1

Gravitational waves from single-field slow-roll Inflation

The following chapter is written on the basis of the review [1]. The Inflationary Solution defines an elegant treatment of some of the problems associated with the Hot Big Bang theory, such as the horizon problem, the flatness problem and the magnetic monopole problem [2], [3], [4], [5], [6], [7]. Inflation can be defined as a sufficiently long period in the primordial universe characterised by an accelerated expansion [8]. In addition to providing a careful resolution to the above problems, considering the intrinsic quantum nature of the inflationary treatment, it is possible to define a mechanism for the generation of primordial seeds of all observable structures in the Universe and the anisotropies of CMB radiation: the quantum interpretation from this point of view is of absolute regard since, by considering the quantum fluctuation of the fields associated with the description of the dynamics of the Universe, the above-mentioned seeds are consequentially generated, avoiding the necessity to add them by hand in a classical theory [9], [10], [11], [12], [13]. The fields that can be considered are a scalar field and the metric tensor. From the development of a perturbative theory within the limits of General Relativity, it must be understood that a fluctuation of the scalar field at the dynamical level (and this can be deduced directly from Einstein's equation) implies a natural fluctuation of the metric tensor, which of course in turn can be described as combinations of fluctuations of various types and nature. The tensor-like fluctuation, already destined to be a smocking gun of inflationary theory, describes the background of Gravitational Waves. The beginning of this chapter will focus on a description of the classical characteristics of Inflation, and then move on to a description of the quantum nature of the system, so as to understand how such primordial gravitational waves can be produced.

1.1 The physics of Inflation

Standard Cosmology is described by a homogeneous and isotropic universe, the space-temporal interpretation of which can be mathematically described by a metric of FLRW:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{1.1}$$

where t is the cosmic time, r, θ, ϕ, are the comoving spherical coordinates and K is the curvature of the three-dimensional spatial hypersurface. The metric is identified by the evolution of the scale factor $a(t)$ and the spatial curvature parameter. Since the next objective is to define, via the Einstein equation, the evolution of the scale factor that quantifies the expansion of the universe, it is necessary to specify the energy-momentum tensor in the interspace that permeates the environment. Working under the assumption of a homogeneous and isotropic universe, this tensor can be that associated with a perfect fluid:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{1.2}$$

where ρ is the density of the fluid and P the pressure, while u_μ is the four-speed and $g_{\mu\nu}$ the metric tensor. Substituting (1.1) and (1.2) into the aforementioned Einstein equation, we obtain the Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \quad , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P), \tag{1.3}$$

where H is the Hubble rate. From now on, it will be of appropriate simplicity to define $K = 0$, since current observational constraints require us to think of a universe that has never really felt the effects of its curvature dynamically [14] . Friedmann's equations reveal to us what kind of fluid can drive the inflationary status. The definition of inflation requires only an accelerated expansion of the universe, i.e. $\ddot{a} > 0$. If we transfer this information into the equation just above, it is easy to see how $w < -1/3$, hence how inflation cannot be supported by radiation or normal non-relativistic matter, but by a fluid describing a singular equation of state with the value defined above. The most representative and simple example suitable for the description of such inflationary dynamics is that for which we consider a De-Sitter universe, for which $P \simeq -\rho$, for which the scaling factor evolves as $a(t) = a_0 e^{H_i(t-t_i)}$, where we define the Hubble parameter as a constant. It is now necessary to introduce the concept of the Hubble radius $R_H(t) = 1/H(t)$, which defines the region of causal connection space for any time t for an observable, such as can be a fluctuation, with respect to the observer. In a De Sitter model, as can be easily guessed and reasoned from a physical and non-comoving key, the radius of the Hubble sphere is constant during inflation, while all physical distances are brought to increase due to accelerated expansion, so sooner or later they will exceed this radius by becoming causally disconnected observables at a characteristic horizon-crossing time. The idea of inflation, even in solving the horizon problem, is exactly this: to make all observable fluctuations that initially at the inflationary period are all inside the sphere, therefore all causally connected, slowly by virtue of their width k exit that sphere, only to re-enter it later at the end of inflation thus returning to a causally connected status for which they can therefore interact again, coming out of the super-horizon freezing state. The easiest way to define a stress-impulse source that provides such a researched equation of state is to present a scalar field ϕ, and an appropriate associated potential. The Lagrangian density associated with such a minimally coupled scalar field is described as follows:

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial_\mu \phi - V(\phi). \tag{1.4}$$

By varying the associated action with respect to the field and equating it to zero, we obtain the dynamical equation of motion for such a zero-spin boson, i.e. the Klein-Gordon equation $\Box\phi = \frac{\partial V}{\partial \phi}$. Assuming a RW metric in the idea of a flat universe, it is possible to rework the equation in the following way:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0. \tag{1.5}$$

By deriving the action with respect to the metric tensor, it is possible to find the energy-momentum tensor for the scalar field:

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right]. \tag{1.6}$$

At this point it is necessary to match this last expression with (1.2), in order to define, in terms of the field, a homogeneous and isotropic (perfect) fluid, finding that the latter must have a density and pressure defined as follows

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \quad , \quad P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{1.7}$$

However, the quantity defining the sign of the acceleration of the universe reads

$$\rho_\phi + 3P_\phi = 2[\dot{\phi}^2 - V(\phi)], \tag{1.8}$$

therefore, in order to get an accelerated expansion phase of the universe it is necessary to require that $V(\phi) > \dot{\phi}^2$. Specifically, in the description of an almost de Sitter universe, we need the condition that the potential is much larger than the kinetics of the field. Thus, a scalar field whose energy density is dominant in the universe and whose potential dominates its kinetic energy defines the perfect protagonist to drive the inflationary dynamics. The simplest way to describe such a prospectus is to imagine a field defining a slow-roll phase towards the minimum of its potential.

1.2 Slow-Roll conditions

The simplest way to analyse the configuration described above is by taking a potential that presents a local region in which it is defined as flat. In such a situation, starting from a general initial condition for which the inflaton begins its motion with kinetics greater than the potential, (a condition then reversed in the constant section thanks to the cosmic expansion that makes the velocity of the system progressively smaller) one arrives at the constant section where in sufficiently long times the dynamic evolution of the field will be guided by the friction, that will overcomes the acceleration term (also with the objective of defining the inflationary model in its vision as an attractor) for which $\ddot{\phi} \ll 3H\dot{\phi}$.

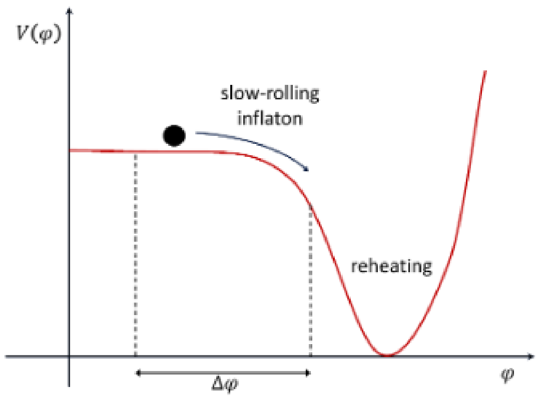

Figure 1.1: Example of an inflationary Slow-Roll potential described by a flat region [1].

By substituting the two introduced slow-roll conditions into Friedmann's first equation and KG's equation of motion, we obtain

$$H^2 = \frac{8\pi G}{3}V \quad , \quad 3H\dot{\phi} + V_\phi = 0, \tag{1.9}$$

assuming that we are working in the background environment, i.e. defining the central background term of the homogeneous field that defines the dynamic inflationary solution in an isotropic homogeneous expanding curved K universe . It is necessary at this point to introduce the first-order slow-roll parameters [15], [16], [17] in order to generalise the treatment exposed to any model and potential associated

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 = \frac{3}{2}\frac{\dot{\phi}^2}{V} = -\frac{\dot{H}}{H^2}, \tag{1.10}$$

$$\eta = M_{pl}^2 \frac{V_{\phi\phi}}{V} = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{1.11}$$

When these two parameters are considered to be much smaller than 1, then one experiences a slow-roll inflationary dynamic. During such inflation, the slow-roll parameters are constant at first order, defining a self-sustaining condition for inflation, in fact it is easy to show how $\dot{\eta}, \dot{\epsilon} = O(\epsilon^2, \eta^2)$. In order for inflation to be successful, thus resolving the shortcomings stated above, it must last long enough for all the observables that were in the Hubble sphere (during the pre-inflation) to exit (an intrinsic condition of the accelerated expansion), and then re-enter in sub-horizon soon after the inflation, depending on the magnitude k of the mode relative to the studied fluctuation. Usually a time estimate in the inflationary period is offered by the number of e-foldings [16], defined by

$$N_{tot} = \int_{t_i}^{t_f} H, dt, \tag{1.12}$$

where t_i and t_f represent the inflation start and end times. The minimum number of N required for the inflationary model to be functioning and consistent is $N \simeq 60$ [18].

1.3 Reheating phase

Among the greatest successes of the standard Hot Big Bang Model there are the phenomena of primordial nucleosynthesis and CMB: it is necessary to require, however, in order to be able to guarantee the description of these phenomena for which we currently have observational evidence, that the universe at some point in its life must be dominated by radiation (at 1 MeV such domination must already be defined) and then switch to non-relativistic matter domination at the appropriate equilibrium condition. Therefore, we know that inflation must end, sooner or later. In single-field slow-roll models, inflation ends when the potential returns to make the inflaton that travels through it feel its curvature, and from there the field goes back to acquiring kinetics that allow it to oscillate around the system's minimum, subsequently beginning to decay by producing radiation, a fundamental condition for initiating the subsequent hot big bang phase. The transition from inflationary

dynamics to the domination of radiation that repopulates the universe in place of the now decayed scalar field, is called Reheating [6], [7], [8], [19].
There are several models that associate GW production during this phase, in addition to the classical GW induced in the inflationary period, so it is necessary to define a state of art of the Reheating period. Furthermore, it is easy to show how some Reheating parameters are connectable with those relating to inflationary power-spectra (think of how the oscillation frequency of the field is directly proportional to the slow-roll parameter η_V), so if there are analytical constraints on such Reheating parameters, these will be reflected in becoming limits for scalar/tensor fluctuations.
At the end of inflation all the energy of the universe is saved in the scalar field (having constant density in de Sitter, and since the other components have been diluted in the cosmic expansion). Reheating must therefore define a conversion of energy into another form, which subsequently develops into a radiation-dominated scenario in a thermal equilibrium condition. In order to build such a transition, many models have been advanced, in some of which the decay of the inflaton is assumed to take place in a perturbative environment (slow decay), while others, which go by the name of parametric resonance decays, make use of a rapid non-perturbative decay. If the oscillations of the scalar field are sufficiently small compared to the minimum, the inflaton can decay into relativistic particles, as soon as the decay becomes efficient: if the decays occurs slowly, these can be described by fermions alone. Due to the slow decay of the inflaton, the products have plenty of time to thermalise, so that their energy distribution function will soon be argued by a black-body function. Subsequently, these fermions will have to decay further, so they convert into radiation. However, the scalar field could decay into bosons, and here, in contrast to the previous case, the decay would occur in a rapid and non-perturbative manner for the characteristic purposes of parametric resonance. The decay process (oscillation on the minimum) is so fast that it could end with very few oscillations. This condition is called the preheating phase [20].

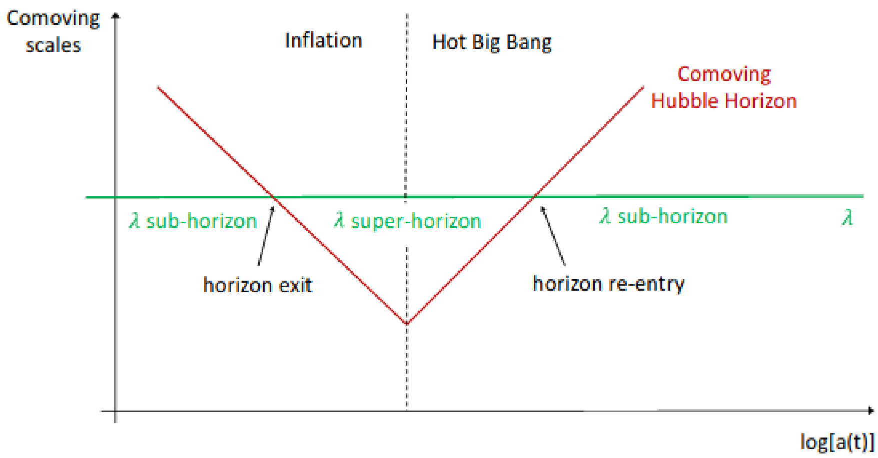

Figure 1.2: Temporal evolution of the comoving Hubble radius during Inflation and other successive epoch [1], [21].

1.4 Quantum fluctuations: origin of cosmological perturbations

The observational patterns associated with the formation of Large Cosmological Structures (LSS) and the study of anisotropy in the temperature of the cosmic background radiation, can be understood by defining the existence of small scalar fluctuations that during the epochs following inflation can re-enter the horizon returning to the causal connection that would lead them to a new interaction of gravitational matrix. Standard cosmology is a classical cosmology that does not provide for the use of such perturbative fluctuations, which would have to be inserted manually, thus significantly affecting the plausibility of the theory. However, granting to the cosmological model a quantum approach, would ensure the presence of such necessary perturbative products as a default condition. With respect to the QFT and a perturbative approach, a field can split into its central background component and a fluctuation to which a second quantization operation must be applied, capable of transforming the initially classical fluctuation into a quantum field operator. Each field is associated with a quantum fluctuation defined as such within the limits of the causal connection sphere. It is necessary to anticipate that such fluctuations under sub-horizon conditions oscillate independently of any frequency, with an associated null mean value calculated over macroscopic time, leading to a null production of particle-antiparticle binary systems. As already stated, the task of inflation is to stretch all the relative lengths of the fluctuations, so they fall outside the causal connection horizon, defining the condition for which $k \gg$

aH, where k represents the width of the fluctuation, transforming the perturbation into a classical matrix object [9], [10], [12], [22]. Here the fluctuations remain frozen in their amplitude value over time, while their length increases exponentially, again in a conventional physical distance approach. When inflation ends, these fluctuations will re-enter the Hubble sphere, starting with the smallest, during successive domination phases, depending on the size associated with the fluctuation. When such fluctuations re-enter in sub-horizon, with the same amplitude value with which they exited, not null, they will again be subjected to the gravitational impact that will shape them, producing the observed LSS and CMB anisotropy fluctuations. In cosmology, there are two fields involved in this discussion: the inflaton and the metric tensor, whose tensor matrix fluctuation identifies the degrees of freedom of gravity, defining precisely the GWs. The next step is therefore to study the dynamics of the fluctuations of these fields.

1.5 Metric Tensor perturbations

Recalling the definition of conformal time $\tau = \int \frac{dt}{a(t)}$ the perturbation of the metric tensor around the usual RW background is understood as follows:

$$g_{00} = -a(\tau)\left(1 + 2\sum_{r=1}^{\infty}\frac{1}{r!}\Psi^{(r)}\right), \tag{1.13}$$

$$g_{0i} = a^2(\tau)\sum_{r=1}^{\infty}\frac{1}{r!}\omega_i^{(r)}, \tag{1.14}$$

$$g_{ij} = a^2(\tau)\left(\left[1 - 2\left(\sum_{r=1}^{\infty}\frac{1}{r!}\Phi^{(r)}\right)\right]\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}h_{ij}^{(r)}\right), \tag{1.15}$$

where the functions $\Phi^{(r)}$, $\Psi^{(r)}$, $\omega_i^{(r)}$, $h_{ij}^{(r)}$, are the perturbations of the metric tensor of order r, and $h_{ij}^{(r)}$ is the traceless-transverse tensor that defines the GWs background. Since maintaining an overall first-order perturbative theory (although this reasoning is valid for any fixed order r) the dynamics of the various scalar, vector and tensorial perturbations remain decoupled, it is convenient to decompose these objects into elements that have well-defined transformation properties under spatial rotation [23], [24]. From Helmoltz's theorem it is possible to decompose each vector into a solenoidal and a longitudinal component, called the vector and scalar part respectively

$$\omega_i = \partial_i\omega^{\parallel} + \omega_i^{\perp}, \tag{1.16}$$

with $\omega_i^{\perp}$ the null divergence vector, and $\omega^{\parallel}$ the longitudinal component. The trace-free and transverse part of the spatial component of the metric can be decomposed in the same way

$$h_{ij} = D_{ij}h^{\parallel} + \partial_i h_j^{\perp} + \partial_j h_i^{\perp} + h_{ij}^{T}, \tag{1.17}$$

where $h^{\parallel}$ is the scalar function, $h_i^{\perp}$ is the solenoidal vector field, while h_{ij}^{T} is the solenoidal symmetric tensor part with null trace. It's important to define the trace-free operator $D_{ij} = \partial_i \partial_j - \delta_{ij}\frac{\nabla^2}{3}$.

1.6 Stress-Energy Tensor perturbations

The generic energy-momentum tensor for a fluid can be written as follows

$$T_{\mu\nu} = (\rho + P_0)u_\mu u_\nu + P_0 g_{\mu\nu} + \pi_{\mu\nu}, \tag{1.18}$$

where ρ is the energy density of the fluid, P_0 the homogeneous pressure component, u_μ the quadri-velocity of the fluid and $\pi_{\mu\nu}$ represents the anisotropic stress tensor, which takes into account all the imperfections of the fluid in question, in fact it cancels at zero for a perfect fluid or in the case of a minimally coupled scalar field. Perturbing the previous expression and decomposing each quantity respecting its symmetries, the components developed to the first order of that tensor can be written as follows:

$$T_0^0 = -\rho_0 + \delta\rho, \tag{1.19}$$

$$T_i^i = 3(P_0 + \delta P) = 3P_0(1 + \pi_L), \tag{1.20}$$

$$T_0^i = T_i^0 = 0, \tag{1.21}$$

$$T_j^i = P_0\left[(1 + \pi_L)\,\delta_j^i + \pi_{T,j}^i\right], \tag{1.22}$$

neglecting the vector perturbations. π_L represents the pressure fluctuation, while π_T represents the transverse and traceless tensor component of the scalar-vector-tensor decomposition of the stress-anisotropy tensor.

1.7 Dynamics of fluctuations

It is necessary to underline the dynamics of the fluctuations described above by perturbing the problem (for now) to the first order, so it is necessary to start from the development of the action of the scalar field, which we assume to be minimally coupled to gravity, assuming precisely that only inflaton and gravity are the only ingredients capable of determining the dynamics of the universe

$$S = \int \sqrt{-g} \left[\frac{1}{2} M_{Pl}^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \tag{1.23}$$

At the first order, tensorial perturbations at the dynamic level remain decoupled from those of a different nature. The result that will soon be achieved describes how scalar and tensorial fluctuations, once exited on superhorizon scales from the sphere of causal connection, will remain frozen for the duration of their sustainment outside the aforementioned sphere, therefore the initial amplitude of such fluctuations upon re-entry will be identical to that of first exit: then they will vary due to the intrinsic nature of the causal relationship.
A relevant observable quantity for interpreting such fluctuations is the Power Spectrum. Given a generic field $g(x,t)$, which can be written in Fourier space as follows

$$g(x,t) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \exp{(ikx)} g_k(t). \tag{1.24}$$

The adimensional Power-Spectrum $P_g(k)$ is written in that way

$$\langle g_{k_1} g_{k_2}^* \rangle = \frac{2\pi^2}{k^3} P_g(k) \delta^{(3)}(k_1 - k_2), \tag{1.25}$$

with Dirac brackets describing a generic average over the ensamble. The sense of such a function is to measure for each value of length k the amplitude of the associated fluctuation. It is also useful to provide the following trivial definition for which

$$\langle g^2(x,t) \rangle = \int \frac{dk}{k} P_g(k). \tag{1.26}$$

Thus, the Power Spectrum per logarithmic unit interval over the frequency of the perturbation, determines the variance over the distribution of the perturbation.
It is useful to define the generic functional trend of the spectrum in relation to frequency by associating the typical spectral index:

$$n_g(k) - 1 = \frac{d \ln P_g}{d \ln k}. \tag{1.27}$$

Let us refer to what has been said so far with respect to a canonically quantized scalar field χ. We divide the scalar field into its homogeneous and classical background component and its fluctuation which is to be canonically quantized $\chi(x,\tau) = \chi(\tau) + \delta\chi(x,\tau)$. Once the convenient rescaling $\widetilde{\delta\chi} = a\delta\chi$ has been defined, we move on to quantize the field $\widetilde{\delta\chi}$ by transforming it into a quantum field operator, which can be written as a linear combination of its creation and annihilation operators (following, moreover, what we know to be the solution of the KG equation associated with the fluctuation of a zero spin bosonic scalar field):

$$\tilde{\delta\chi} = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[u_k(\tau) a_k \exp(ikx) + u_k^* a_k^* \exp(-ikx)\right], \tag{1.28}$$

with u_k and u_k^* satisfying the typical commutation relations derived from those associated with the creation and destruction operators

$$[a_k, a_{k'}] = 0, \quad [a_k, a_{k'}^*] = \delta^{(3)}(k - k'). \tag{1.29}$$

From the definition of $\tilde{\delta\chi}$ and the last two equations, it easily follows that

$$\langle \delta\chi_{k_1} \delta\chi_{k_2}^* \rangle = \frac{|u_k^2|}{a^2} \delta^{(3)}(k_1 - k_2), \tag{1.30}$$

therefore we can write finally a scalar spectrum of the form

$$P_{\delta\chi}(k) = \frac{k^3}{2\pi^2} |\delta\chi|^2 . \tag{1.31}$$

1.8 Scalar Perturbations

In this section, the first-order dynamics of the scalar fluctuations of the physical system will be studied. In order to describe the scalar fluctuations as simply as possible, we attempt to express them as a function of gauge invariant potential quantities, precisely because of the trivial properties of associated gauge transformations. We choose the metric tensor first presented, in first-order linear perturbation with respect to the RW background. We consider the spatial curvature of the spatial hypersurface with fixed conformal time constant at the linear level,

$$R^{(3)} = \frac{4}{a^2} \nabla^2 \hat{\Phi}, \quad \hat{\Phi} = \Phi + \tfrac{1}{6} \nabla^2 \chi^{\parallel}. \tag{1.32}$$

$\hat{\Phi}$, defined as the curvature perturbation, it is not a gauge invariant quantity, in fact if we perform a transformation on a space hypersurface at constant shift time, such a curvature perturbation does not remain invariant, gaining spurious terms that, for the purpose of the notion of invariance, should elide. Therefore it is useful to modify such a curvature perturbation with terms suitable for that purpose. Those terms in one specific gauges allow us to return to the usual starting perturbation. We can write

$$-\zeta = \hat{\Phi} + H \frac{\delta\rho}{\rho'} . \tag{1.33}$$

This quantity is finally gauge invariant, and it is defined as the gauge invariant curvature perturbation for hypersurfaces with uniform energy density. Infact, in that gauge, it comes back to the usual simple curvature perturbation. It is now necessary, in order to calculate the observable related to the power spectrum of the curvature perturbation, to define the evolution of this fluctuation by means of the general KG equation of motion of the inflaton; starting from the action of the previous paragraph, we could write:

$$\delta\varphi^{''} + 2H\delta\varphi^{'} - \nabla^2\delta\varphi + a^2\delta\varphi\frac{\partial^2 V}{\partial\varphi^2}a^2 + 2\Psi\frac{\partial V}{\partial\varphi} - \varphi_0^{'}(\Psi^{'} + 3\Phi^{'} + \nabla^2\omega^{\parallel}) = 0. \quad (1.34)$$

In order to simplify this equation of motion, we introduce the Sasaki-Mukhanov's gauge-invariant [25]

$$Q_\varphi = \delta\varphi + \frac{\varphi^{'}}{H}\hat{\Phi}. \quad (1.35)$$

This quantity is intrinsically linked to the curvature perturbation gauge invariant ζ, so solving the dynamics for one variable implies having solved it for the other as well. Using the canonical quantization process, the field $\tilde{Q_\varphi} = aQ_\varphi$ is introduced, so the previous KG equation is rewritten in a simplified way as follows [26]

$$\tilde{Q_\varphi''} + \tilde{Q_\varphi}\left(k^2 - \frac{a^{''}}{a} + M_\varphi^2 a^2\right) = 0, M_\varphi^2 = \frac{\partial^2 V}{\partial\varphi^2} - \frac{8\pi G}{a^3}\left(\frac{a^3\varphi^2}{H}\right). \quad (1.36)$$

In the SR approximation (the case in question for the development of this chapter) $\frac{M_\varphi^2}{H^2} = 3\eta - 6\epsilon$.
Translating into Fourier space, it is evident to note how expression (1.36) defines the Bessel equation, for which we recognise a default solution given by the linear combination of first and second-order Henkel functions. Defining the space-time conditions for which the modified gauge invariant inflaton fluctuation is on super-horizon scales(a necessary condition in order to be able to observationally define the scalar power spectrum), the solution of motion is defined as follows

$$|Q_\varphi| = \frac{H}{\sqrt{2k^3}}(\frac{k}{aH})^{\frac{3}{2}-\nu_\varphi} \quad (1.37)$$

where $\nu_\varphi = \frac{3}{2} + 3\epsilon - \eta$. Through the mathematical connection between the Sasaki-Mukhanov variable and the curvature perturbation, knowing the scalar superhorizon solution of the former and knowing how to analytically define the power spectrum on the latter (very trivially, it is its square modulus), we arrive at the observable of the curvature perturbation spectrum

$$P_\zeta = \left(\frac{H^2}{2\pi\dot{\varphi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu_\varphi} \simeq \left(\frac{H^2}{2\pi\dot{\varphi}}\right)^2. \quad (1.38)$$

This result shows us that for each frequency k the curvature fluctuation always maintains the same amplitude, even if during the course of the evolution of the universe it will change the domination phase, as long as it remains in the super-horizon condition.
The spectral index reads:

$$n_\zeta - 1 = 3 - 2\nu_\varphi = 2\eta_v - 6\epsilon. \tag{1.39}$$

1.9 Gravitational waves from inflation

The inflationary model, in its perturbative-quantum view, defines natural fluctuations of the scalar field that induce an equally natural fluctuation on the metric tensor, which must therefore be perturbed with respect to the basic version associated with the description of a homogeneous and isotropic universe. Such fluctuation can be understood in terms of scalar, vector, and tensor objects; the latter specifically, called h_{ij} delineates the background profile of a stochastic background of gravitational waves [27], [28], [29], [30] describing the degrees of freedom of the gravitational sector. No constraint equations originating from the continuity equation of the stress-impulse tensor are associated with these modes. The dynamic evolution of such modes is in fact exclusively described by the transverse and trace-free spatial part of the Einstein equation (where in fact h_{ij} is contained), which, in the presence of a perfect fluid, does not provide any information on the energy content of the universe, since no energy density terms are contained in the resulting dynamic equation on the graviton. A possible coupling between the GWs and the content of the universe is only present when we consider the presence of a stress-impulse anisotropy tensor, or more generally, by sending the development forward beyond the first order, unifying tensor terms of degree two, with combinations of linear scalars. By perturbing the previously described action at first order, it is possible to find the perturbed linear action for the tensorial degrees of freedom [31], [32],

$$S_T^{(2)} = \frac{M_{Pl}^2}{8} \int a^2(t) \left[\dot{h_{ij}} \dot{h_{ij}} - \frac{1}{a^2(t)} (\nabla h_{ij})^2 \right] d^4x. \tag{1.40}$$

Deriving this solution with respect to the gauge invariant h_{ij}, the equation of motion of the graviton is obtained as usual:

$$\nabla^2 h_{ij} - a^2 \ddot{h_{ij}} - 3a\dot{a}\dot{h_{ij}} = 0. \tag{1.41}$$

Such tensor fluctuations are called gravitational waves because they solve the wave equation. In principle, the tensor fluctuation enjoys a total of 16 degrees of freedom, which must obviously be reduced by virtue of the properties enjoyed by the tensor. In fact, the symmetry, the null-trace and the transversality of the wave denote a result of only two degrees of freedom, which can be interpreted as the two different and perpendicular states of polarization of the wave. Thus, in general, it is possible to underline the solution of the equation of motion as follows:

$$h_{ij}(x,t) = \sum_{\lambda=+,\times} h^{(\lambda)}(t) e_{ij}^{(+,\times)}, \tag{1.42}$$

where $h(t)$ represents the amplitude of the wave, while $e_{ij}^{(+,\times)}$ describes the polarization tensor, with $+, \times$ the two chosen polarization states of the GW.

In order to obtain a more formal solution for the equation of motion, it is appropriate to define a change of variable

$$v_{ij} = \frac{aM_{Pl}}{\sqrt{2}} h_{ij}. \tag{1.43}$$

We can rewrite the action in the following way

$$S_T^{(2)} = \frac{M_{Pl}^2}{8} \int \left[v_{ij}^{'} v_{ij}^{'} - (\nabla v_{ij})^2 + \frac{a^{''}}{a} v_{ij} v_{ij} \right] d^4x. \tag{1.44}$$

In the Fourier space we can write

$$v_{ij}(x,t) = \int \frac{d^3k}{(2\pi)^3} \sum_{(\lambda=+,\times)} v_k^{(\lambda)}(t) e_{ij}^{(\lambda)}(k) \exp{(ikx)}. \tag{1.45}$$

Using this last writing within the suitably varied action, we obtain the equation of motion for each mode $v_k^{(\lambda)}$:

$$v_k^{(\lambda)''} + v_k^{(\lambda)} \left(k^2 - \frac{a^{''}}{a} \right) = 0. \tag{1.46}$$

This is clearly the equation for an harmonic oscillator, the solution of which is an oscillating wave with a pulsation dependent on the scale factor that governs the expansion of the universe. This solution is equal to that for the scalar field problem, since the dynamic equations in the two cases are quite similar. This equation must be solved within two specific limits, relating to the comparison of the scaling factor, Hubble rate and frequency, which parameters intrinsically determine the causal connection condition of the fluctuation.

The first case is that for which $a^{''}/a << k^2$, i.e. $k >> aH$ associated with the subhorizon condition of the perturbation. Neglecting the term proportional to the scaling factor in the pulsation term in the equation above, the equation on v_k becomes that of a free harmonic oscillator dependent only on the size of the fluctuation. So the solution is a simple free waves that oscillates with the frequence in k. The tensor h_{ij} oscillates consequently, but with a damping factor proportional to the inverse of a, associated with the change of variable between the tensor entities: this approximation justifies and takes into account the expansion of the universe. The formal solution is written as follows:

$$v_k(\tau) = A \exp{(ik\tau)}, \tag{1.47}$$

for which the considerations made above follow. The second regime of the study is for the superhorizon case, so $k^2 << a^{''}/a$, for which the equation holds two solutions (of course, we are talking about a second order differential equation):

$$v_k(\tau) \propto a, \quad v_k(\tau) \propto \tfrac{1}{a^2}, \tag{1.48}$$

which trace a h constant in time and one h decreasing in time. It is of factual interest to study the solution that remains constant in time, since the second in the regime of causal disconnection will decay, and will not be of observational interest due to the expansion of the universe.
By solving the equation of motion more carefully, we can find the exact value of the frozen fluctuation, which will consequently allow us to define the power spectrum of the corresponding tensor modes. Through the standard quantization of the field

$$v_k^{(\lambda)} = v_k(\tau)a_k^{\hat{(\lambda)}} + v_k^*(\tau)a_{-k}^{\hat{(\lambda)}+}, \tag{1.49}$$

where the modes are appropriately normalized and produced for the canonical creation and annihilation operators respectively.
In order to simplify the problem, it is assumed as an initial condition that in the infinite past all the modes were causally connected and were therefore described by a Bunch-Davies [33] vacuum state. Equation (1.46) is still a Bessel equation, which, in a convenient De Sitter space-time, defines the exact solution [27]:

$$v_k(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right], \tag{1.50}$$

with C_1 and C_2 constants of integration, $H_\nu^{(1)}$ e $H_\nu^{(2)}$ first and second order Henkel functions with $\nu \simeq \frac{3}{2} + \epsilon$.
In order to determine the integration constants, we exploit the condition for which on subhorizon scales, due to the principle of equivalence, or more simply using the initial condition set, we already know the solution of the mode, i.e. it must behave like a plane wave $\frac{e^{-ik\tau}}{\sqrt{2k}}$.
Knowing the asymptotic Henkel functions in subhorizon scales

$$H_\nu^{(1)}(x >> 1) \simeq \frac{e^{ix}}{\sqrt{x}}, \tag{1.51}$$

with Henkel's first function equal to the complex conjugate of the second, it becomes trivial to assume $C_2 = 0$, and so connecting the asymptotic solution with the known wave solution we can yield the constant

$$C_1 = \frac{\sqrt{\pi}}{2}e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}, \tag{1.52}$$

by which it is possible to determine the overall value of the tensor fluctuation

$$v_k = \frac{\sqrt{\pi}}{2}e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\sqrt{-\tau}H_\nu^{(1)}(-k\tau). \tag{1.53}$$

In particular, for the purpose of determining the observable tensor power spectrum, the superhorizon solution is required, where the Hankel function shows the following behaviour

$$H_\nu^{(1)}(x << 1) \simeq \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right) x^{-\nu} \tag{1.54}$$

so the fluctuation becomes

$$v_k \simeq \frac{(-k\tau)^{\frac{1}{2}-\nu}}{\sqrt{2k}}. \tag{1.55}$$

Recalling the general definition of Power Spectrum, and considering both polarization states, one can conclude

$$P_T(k) = \frac{k^3}{2\pi^2} \sum_\lambda \left| h_k^{(\lambda)} \right|^2, \tag{1.56}$$

so when the mode is in the superhorizon condition

$$P_T(k) = \frac{8}{M_{Pl}} \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{-2\epsilon}. \tag{1.57}$$

In the Slow Roll hypothesis, the spectrum is again almost scale invariant.

1.10 Consistency relation

In the inflationary model studied (Slow-Roll), it is quite useful to discover a fundamental consistency relation that links quantities associated with fluctuations, especially those of tensor format. We introduce the tensor-to-scalar-ratio parameter

$$r(k_*) = \frac{A_T(k_*)}{A_S(k_*)}, \tag{1.58}$$

which estimates the amplitude of gravitational waves versus scalar perturbations at a fixed pivot scale k_*. From the ratio of the spectra calculated in the previous sections we obtain

$$r = \frac{8}{M_{Pl}^2} \left(\frac{\dot{\varphi}}{H} \right)^2, \tag{1.59}$$

i.e. that $r = 16\epsilon$. In the previous section, the tensor spectral index was defined to be $n_T = -2\epsilon$. Then, at the smallest order of expansion with respect to the usual slow-roll parameters, one can write the consistency relation [34] :

$$r = -8n_T. \tag{1.60}$$

This theoretical identity can be experimentally verified thanks to a measurement of the overall tensor power spectrum, i.e. the spectral index as well as the amplitude must be known. If the relation is true, we would have absolute certainty of the existence of inflation, as this theory is the only one that can support such consistency: however, if this relation is true, it would say that for a large tensor spectral index one would have difficulty in estimating the dependence in the tensor scale. Currently only an upper bound on the value of the tensor-to-scalar-ratio is available, with $r_{0.05} < 0.07$ at 95% C.L [35].

1.11 Second Order Gravitational Waves

So far, only first-order metric tensor fluctuations on a RW background metric have been considered: however, the general metric perturbation had been extended to a generic r-order perturbative, and it makes complete sense to see what happens going precisely beyond the linear level. In the latter, in fact, the evolution of the scalar, vector, and tensor perturbations is governed by equations of motion in which these unknowns are presented decoupled from each other, greatly simplifying the system analytically. At mixed successive orders, things get much more complicated, in fact the combination of scalar perturbations at first order can act as a source of GW defined globally at order two. Thus, even if we assume the absence of a stochastic first-order GWs background, if we are dealing with inflationary fluctuations that will provide a curvature perturbation that will oscillate and decay into a new subhorizon regime, these will naturally provide gravitational waves of order two.

1.12 Post-inflationary evolution of GW

Once the inflation is over, i.e. when the Hubble sphere radius in comoving coordinates reaches its minimum value, the tensor fluctuations describing the gravitational waves will re-enter the horizon in the next domination phases, so it makes sense there to understand their future behaviour. The job of the inflationary epoch is precisely to stretch the perturbations by sending them into superhorizon scales where they will have a constant amplitude over time. Subsequently such fluctuations will re-enter the causal connection domain in sequence, dependent on their magnitude k. On superhorizon scales, however, there is also a decaying solution on the scale factor in addition to the frozen one: however, this result is not observed precisely infact it disappear due to cosmic expansion. Therefore it is the constant solution that re-enters the horizon, which identifies the power spectrum statistic. Such frozen modes, upon re-entry, return to oscillate and decay dumped by a factor of $1/a$. However, during the domination of radiation and NR matter, the scaling factor evolves differently, as $a \simeq \tau$, and $a \simeq \tau^2$, respectively, so the equation of motion becomes the Bessel equation on tensor amplitudes that denotes two different solutions depending on the period of re-entry

$$h_k(\tau) = h_{k,i}\left(j_0(k\tau)\right), \quad h_k(\tau) = h_{k,i}\left(\frac{3j_1(k\tau)}{k\tau}\right), \tag{1.61}$$

where $h_{k,i}$ is the initial condition provided by the previous freezing, while j_0 and j_1 are the Bessel functions. These solutions suggest that the higher the frequency of the wave, the stronger the amplitude dump.
During an epoch of pure domination of the cosmological constant, the universe can be described under de Sitter's key, whereby there is a scaling factor that evolves exponentially, just as in the inflationary case for $\epsilon = 0$. Therefore the solution in that epoch follows the already defined (1.53). Subsequently, it will be shown how the present energy density of the GWs background may have canalized on itself the different modes of oscillation depending on the various epochs through which the wave had to pass.

1.13 Energy-density of gravitational waves

An important definition to provide for the theory is the energy density of the GWs background. Consider the weak-field limit, a limit for which gravitational waves are considered as ripples on the space-time, which is mathematically described by the usual background of fixed RW. Einstein's field equation in the vacuum reads trivially as $G_{\mu\nu} = 0$, or even more simply $R_{\mu\nu} = 0$. Once the Riemann tensor is written as usual as the sum of a fixed background term and small perturbations up to the second order $R_{\mu\nu} = \bar{R_{\mu\nu}} + R^1_{\mu\nu}(h) + R^2_{\mu\nu}(h) + O(h^3)$, thanks to the vacuum equation (not considering additional matter sources for simplicity), it is possible to deduce how the presence of the intrinsic GWs in the perturbative terms can modify the background term, concretely defining the stress-energy tensor term of the stochastic background $t_{\mu\nu}$.
The Riemann tensor can be written as the sum of a background term and a fluctuation, with both terms satisfying the vacuum equation [36]. Since the Background term varies only on large scales with respect to some fixed reference, it identifies together with the small term developed at second order the regular system contribution. The remaining part establishes the fluctuation to the fixed counterpart, and by itself solves the void equation $R^{(1)}_{\mu\nu}(h) = 0$. So we can write the smooth part on the vacuum

$$\bar{R}_{\mu\nu} + \langle R^{(2)}_{\mu\nu}(h) \rangle = 0, \tag{1.62}$$

where the average on the second term quantifies the smooth contribution on the second-order fluctuation. This reasoning can obviously be extended to the overall Einstein equation

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = \langle R^{(2)}_{\mu\nu}(h)\rangle - \frac{1}{2}\langle R^{(2)}\rangle\bar{g}_{\mu\nu}. \tag{1.63}$$

The right-hand term is, as anticipated earlier, the one that modifies the background so it quantifies the energy-momentum tensor $t_{\mu\nu}$, minus a normalisation factor $8\pi G$. In terms of the tensorial mode this stress is written as follows

$$t_{\mu\nu} = \frac{1}{32\pi G}\langle \partial_\mu h_{ij}\partial_\nu h^{ij}\rangle; \tag{1.64}$$

The 00 term of the previous tensor quantifies the observational result

$$\rho_{gw} = \frac{1}{32\pi G a^2}\langle h^{'}_{ij}(x,\tau)h^{'ij}(x,\tau)\rangle. \tag{1.65}$$

However, it is sometimes common to represent this energy density on a logarithmic unit, normalizing to the critical density

$$\Omega_{GW}(k\tau) = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\ln k}. \tag{1.66}$$

1.14 Why are primordial GWs so interesting?

Primordial gravitational waves define a fundamental tool for studying the fundamental physics of the primordial universe in which they were produced.

1.14.1 Energy scale of inflation

From the study of the GWs it is possible to trace the specific mechanism that produced them, in fact the parallelism between the amplitudes of the tensor and scalar spectra (which generate the former) is proof of this. In particular, the measurement of the amplitude of the tensor power spectrum naturally allows for an estimate of the energy scale of the inflationary momentum [37]: this conclusion can be deduced from a better writing of the amplitude defined above:

$$P_T(k) = \frac{16H^2}{\pi M_{Pl}^2}\left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{1.67}$$

This amplitude, thanks to the first Friedmann equation, is totally equal in slow-roll, to the potential of the theory that quantifying the model. Being dimensionally an

energy to the fourth power, the fourth root of this field function manages to give us a more direct estimate of the universe's energy during inflation.
The difficult observability of this amplitude precludes such a measurement, so it might be a good idea to look for this energy information in the amplitude of the scalar spectrum, for which we have strong estimates, at least on the broad scales. However, a similar rewriting of the amplitude of the scalar curvature spectrum leads to the following conclusion

$$P_\zeta(k) = \frac{H^2}{\pi\epsilon M_{Pl}^2}\left(\frac{k}{aH}\right)^{3-2\nu}. \tag{1.68}$$

Here it is simple to note that the measurement of this quantity does not give a direct estimate of the energy, due to the parametrization offered by the slow-roll parameter ϵ: therefore the energy estimate is provided depending on another parameter that quantifies the choice of one inflationary model over another. Working in the scalar environment, it is possible to link the inflationary potential, (and hence the energy), with the amplitude of the scalar background and the slow-roll parameter via the Friedmann equation, finding

$$V = 24\pi^2 M_{Pl}^4 A_S \epsilon. \tag{1.69}$$

Using the relation between the tensor-to-scalar-ratio and the prime parameter of slow-roll, one can simply bind V to r

$$V = \frac{3\pi^2 A_S}{2} M_{Pl}^4 r. \tag{1.70}$$

Knowing the amplitude of the curvature fluctuations thanks to the Planck data [18] from the CMB study, it is possible to link the inflationary energy scale when the pivot scale enters on superhorizon scales, with the observable r

$$V = \left(1.88\times 10^{16} GeV\right)^4 \frac{r}{0.10}. \tag{1.71}$$

Thus, estimating r implies determining the energy scale of inflation.

1.14.2 Predictions from inflation

It is convenient to rewrite the already discussed power spectra of scalar and tensor perturbations in the following power form

$$P_\zeta(k) = A_S(k_*)\left(\frac{k}{k_*}\right)^{n_S-1}, \tag{1.72}$$

$$P_T(k) = A_T(k_*) \left(\frac{k}{k_*}\right)^{n_T}, \tag{1.73}$$

where

$$n_S - 1 = 2\eta_\nu - 6\epsilon, \quad n_T = -2\epsilon, \tag{1.74}$$

$$A_T(k_*) = \frac{16H^2}{\pi M_{Pl}^2}, \tag{1.75}$$

$$A_S(k_*) = \frac{H^2}{\pi\epsilon M_{Pl}^2}. \tag{1.76}$$

It is reasonable to observe the presence of four observables, i.e. the two amplitudes and the two spectral indices: however, these are intrinsically dependent on the slow-roll parameters which depend on the potential defining an appropriate inflationary model. The observables depend on the model chosen, so it makes sense to place constraints on the models themselves (hence on the slow-roll parameters) in order to have a meaningful estimate of the observables.
In order to simplify the system, it makes sense to reduce the number of terms through the functional relationships by which they are bound. It is quite true to observe, at least on the large scale, that $\Delta T/T \simeq \zeta + h$: knowing the scalar value of the anisotropy fluctuation of CMB equal to $\Delta T/T \simeq 10^{-5}$, it is possible to link the scalar fluctuation to the tensor fluctuation, writing one in the terms of the other. Furthermore, it is possible to express this reasoning in terms of the tensor-to-scalar ratio r, which is also due to the cosmological consistency relation related to the tensorial spectral index $r = -8n_T$. Therefore the two amplitudes and an index can be deduced by knowing r. Hence the linearly independent remaining parameters suitable for cataloguing any inflationary model can be (r, n_S). It is therefore possible to define such a plane (r, n_S), within which it is possible to restrict regions of space in order to group models exhibiting the same properties, into three main classes of models: small field models for which $\eta_v < 0$, large field models for which $0 < \eta_v < 2\epsilon$, and hybrid models with $\eta_v > 2\epsilon$. From the system composed of (1.60), (1.74), it is possible to define r again alternatively:

$$r = \frac{8}{3}(1 - n_S) + \frac{2\eta_v}{3\pi}. \tag{1.77}$$

A prediction for small field models, for which $\eta_v < 0$, is a lower value of r, hence a lower production of GW perturbations with respect to scalar ones; on the contrary, in large field models, r increases its value, supporting the opposite solution.
Another determination to reach the same conclusion comes from the definition of scalar field excursion, for which it is worth remembering that

$$\frac{\Delta\varphi}{M_{Pl}} \simeq \sqrt{r} N_{CMB}. \tag{1.78}$$

By definition, large field models are marked by an LHS value greater than 1, leading to a higher r value than in the case of small field models, for which the ratio must be less than unity, resulting in a less consistent production of the stochastic GWs background.

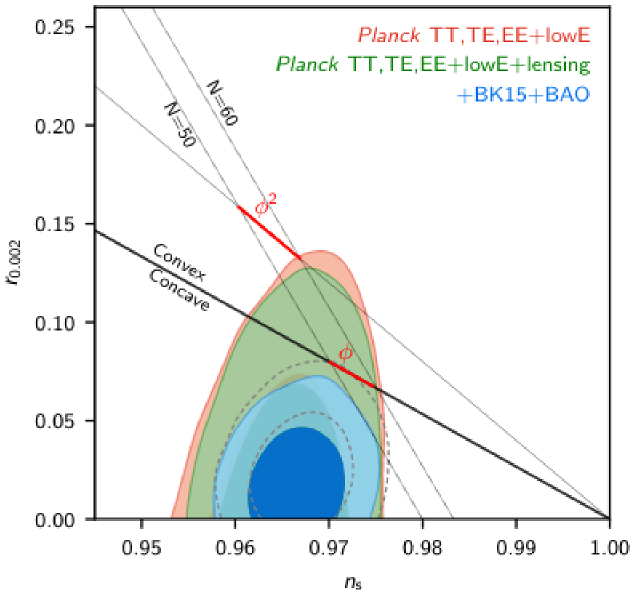

Figure 1.3: Constraints on the tensor-to-scalar ratio r0.002 in the ΛCDM model, using Planck TT,TE,EE+lowE and Planck TT,TE,EE+lowE+lensing (red and green, respectively), and joint constraints with BAO and BICEP2/Keck (blue, including Planck polarization to determine the foreground components) [38].

1.14.3 GWs beyond the SR, as a testing way for the inflationary model

The measurement of the tensor power spectrum suggests a strong way of distinguishing between the various inflationary models (each with its own specific physical laws) capable of producing this GWs background. In fact, beyond the already discussed Slow-Roll there are other important inflationary models that deserve further investigation; in general these can be divided into two macrocategories, those that are built on a GR theory, and those that use a modified theory of gravity (MG). It is logical to think that each model predicts a different tensor power spectrum, with amplitude, spectral index and non-Gaussianity factor characteristic of the system, so the theoretical-observational feedback will determine the model

closest to the description of reality.
If we consider the models subject to a GR view, it is possible to enunciate the existence of theories that predict sufficiently particular tensor spectra, with GW production at the perturbative second order relative to the extra production of fields secondary to the scalar inflaton. We could also cite the interesting predictions offered by solid inflation [39], elastic inflation [40] and warm inflation [41]. MG models, on the other hand, were introduced in order to have a more elegant explanation of the primordial accelerated expansion of the universe. Considering the general action

$$S = \int \sqrt{-g}(\mathcal{L}_{grav} + \mathcal{L}_{mat})d^4x. \tag{1.79}$$

In order to obtain an accelerated expansion of the universe, (so inflation) one can either proceed as done so far, i.e. by adding externally to the metric a scalar field with its dynamics imprinted in the Lagrangian density of matter, or modify the metric itself and thus the Hilbert-Einstein action, by inserting a further scalar degree of freedom beyond those already provided by a perturbative theory.
In the last version described, the dynamics of the system is governed entirely by the metric degrees of freedom of the gravitational field, without the addition of an external field in order to be able to explain the accelerated expansion of the primordial universe.

1.14.4 Primordial gravitational waves in the EFTI

Another interesting way to study the inflationary dynamics together with its smocking gun described by gravitational waves, consists in the effective field theory of inflation (EFTI) approach [42], in which it is possible with a single examination, to define a multitude of inflationary models under a single class. The basic idea of the theory is based on the definition of the action of a Goldstone boson particle, which, by definition, defines a spontaneous symmetry breaking by temporal diffeomorphisms. This action enjoys a symmetry shift on this fluctuation, depending on the temporal dependencies that are guaranteed on the multiplicative coefficients of the various operators developed to a certain order. In fact, by defining a specific temporal dependency that is not constant, but rather perturbed by a step-feature, the model theory leads, through the development of the action at the appropriate order for the determination of the fluctuation's equation of motion through which the correlation functions can be found, to the definition of spectra and bispectra that reflect these features with respect to, for example, a typical intrinsic slow-roll trend. It is possible in that approach to define a general action for the induced tensor perturbation [43]:

$$S_T^{(2)} = \frac{M_{Pl}^2}{4} \int a^2(\tau)\alpha \left[h_{ij}^{'} h_{ij}^{'} - c_T^2 (\nabla h_{ij})^2 - m^2 h_{ij}^2 \right] d\tau dx^3, \tag{1.80}$$

where the parameters α, c_T, m, are induced by linear combination of the coefficients present in the starting action [42].
This modelling allows one to think of GWs different from the standard ones, with a propagation speed different from that of light, and a non-trivial mass. It is possible to state how, remaining in a slow-roll inflationary case with an initial condition on fluctuations described by the Bunch Davies vacuum, the tensor power spectrum is defined in the following form

$$P_T = \frac{2H^2}{\pi^2 M_{Pl}^2 c_T} \left(\frac{k}{k_*} \right)^{n_T}, \tag{1.81}$$

with

$$n_T = -2\epsilon + \frac{2m^2}{3\alpha H^2} \left(1 + \frac{4}{3}\epsilon \right). \tag{1.82}$$

It is essential to note that, compared to the slow-roll case in which the tensor power spectrum index was intrinsically negative (leading to an overall red-tilt of the process), here there are additional terms that can positively increase the spectral index, leading to a blue-tilt variation of the system. That condition clearly induce a growth of the observable function, hence, the observation of a greater portion of gravitational wave density.

Chapter 2

Classical production of primordial Gravitational waves during Inflation

In this chapter the interest shifts to the study of primordial gravitational waves outside the previously studied mechanism of vacuum fluctuations of the gravitational field, described of course by the metric tensor. This mechanism of 'Classical' production of gravitational waves can be defined either during the inflationary period, or during the phase after Reheating. During these two phases, in fact, there are two different ways in which a gravitational background can be produced: by vacuum fluctuations in the gravitational sector or by a classical mechanism. The former case has been extensively described in the previous chapter in the study of an inflationary one-field Slow-Roll model. In the latter case, a tensor power spectrum has also been defined as almost scale invariant: however, it is possible to define a zoology for such an observable related to the choice of gravitational theory associated to the inflationary model.
On the other hand, one can speak of the second classical production model of GW when a new source term arises in the equation of graviton dynamics, in the RHS, to generate a new second-order perturbative class of gravitational waves. This source term is defined according to the choosen model, e.g. it can be defined by the presence of a second scalar field beyond the inflaton, but also by the massive production of secondary particles. Depending on the selected model, and thus on the associated source, it is possible to study and observe a different GW power spectrum, which is associated with obvious features that distinguish it from the SR model of reference. Initially, it will be interesting to study different modelling approaches for the production of a classical second-order perturbative GWs background, and then return to the standard production by vacuum oscillation of the metric tensor associated, however, with different gravitational approaches.
In the previous chapter, a linear first-order perturbative approach was considered with respect to a space-time background of FRW: in such a configuration and

speaking in a purely dynamical key, the scalar, vector and tensor fluctuation modes are totally decoupled (think of the free wave equation found for h in the previous section). However, this consideration loses its meaning if one decides to study the problem for higher perturbative orders, in fact already at the second perturbative order it is easy to see how an appropriate combination of scalars can give rise to a vector or tensorial mode. This reasoning is obviously not only valid for the perturbative order r=2, but also for all subsequent ones, with the only constraint that scalar, vector and tensorial modes of the same order remain dynamically decoupled. On the other hand, the combination of two spatially rotationally invariant objects (i.e. two scalars) may no longer be subject to such symmetry.
Considering finally the metric tensor and the stress-energy tensor perturbed up to the second order in Einstein's equation, the equation that for the linear model we had seen to be a free wave equation, now at the second order perturbative, gains in the RHS (the LHS remains the same by shifting the tensorial modes to the second order) a source identified by the appropriate combination of scalar modes at the first order that can come out either from the metric tensor or from the energy tensor. Thus a combination of two first-order scalar (or vector) perturbations represents a generating source of a new class of gravitational waves [44]: this implies that whenever there is a curvature perturbation there is a second-order GWs background, in fact the curvature perturbation during the inflationary period identify the scalar fluctuation that induces the variation of the Einstein equation, leading to the variation of the metric tensor and the stress-energy tensor, from whose perturbation to degree two, we obtain the equation for the induced GWs by an opportune combination of scalars (even if we therefore neglect the presence of the tensorial modes of order 1, which instead define the standard vacuum GWs background).
More precisely, by taking the transverse and traceless spatial part of the Einstein equation perturbed up to second order, it is possible to observe how, in spite of the first-order case, new terms are induced beyond the standard LHS contributions of free waves equation. These terms are defined as a combination of perturbations of scalars from the Einstein tensor and the anisotropic tensor component of the energy-pulse tensor, thus defining the source of the induced GWs background [44] [45], [46], [47], [48], [49], [50], [51].
Focusing on inflationary physics, the second-order source can be structured in different model-dependent ways, defined for example by inflationary fluctuations or extra particle production. However, this mechanism can also be extended to a post-inflationary time, such as radiation domination. When the curvature perturbation (which at the end of inflation and in accordance with the predictions of the Hot Big Bang Model obviously defines a radiation fluctuation) re-enters the Hubble horizon during radiation domination, it becomes causally connected again, so it

return to interact with other scalar modes and to be subject to deformations related to gravity or simply to the expansion of the universe, thus defining a dynamic source capable of generating the same second-order GWs background, but induced in a later post-inflationary phase. In this section, the focus will be on the GWs background induced during inflation. Generally speaking, what will be obtained is either a second-order GWs background negligible in amplitude compared to a standard first-order GWs background, or directly an induced GWs background so small that it falls outside the sensitivity of the interferometers capable of measuring it. However, it can be observed that thanks to the right sources, an important second-order induced GWs background can be obtained from scalars.

2.1 Production of second order GWs

Let us consider the spatial part of Einstein's equation at second order, and then consider only the transverse-traceless part via an appropriate projection operator:

$$\hat{\Pi}_{ij}^{lm} G_{lm}^{(2)} = k^2 \hat{\Pi}_{ij}^{lm} T_{lm}^{(2)}, \tag{2.1}$$

where $\hat{\Pi}_{ij}^{lm} = \Pi_l^i \Pi_m^j - \frac{1}{2}\Pi_{ij}\Pi^{lm}$ is the projection operator, with $\Pi_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\Delta}$ and $k^2 = 8\pi G$. Consider a classical background of FLRW perturbing up to order two (see (1.15)), neglecting for simplicity the first-order vector and tensor perturbations, thus defining $h_{ij} = h_{ij}^{(2)}$.
In this form, the second-order Einstein tensor is written as follows:

$$\begin{aligned} G_j^{i(2)} = a^{-2} \Bigg[& \frac{1}{4}(h_j^{''i} + 2Hh_j^{'i} - \nabla^2 h_j^i) + 2\Psi^{(1)}\partial^i\partial_j\Psi^{(1)} - 2\Phi^{(1)}\partial_i\partial_j\Psi^{(1)} \\ & +4\Phi^{(1)}\partial_i\partial_j\Phi^{(1)} + \partial^i\Psi^{(1)}\partial_j\Psi^{(1)} - \partial^i\Psi^{(1)}\partial_j\Phi^{(1)} - \partial^i\Phi^{(1)}\partial_j\Psi^{(1)} \\ & +3\partial^i\Phi^1\partial_j\Phi^{(1)} + (\Psi^{(2)}, w_i^{(2)}) + (diagonalpart)\delta_j^i \Bigg] . \end{aligned} \tag{2.2}$$

The stress-energy tensor of a second-order perturbed perfect fluid is written [52]:

$$T_j^{i(2)} = (\rho^{(0)} + P^{(0)})v^{(1)i}v_j^{(1)} + P^{(0)}\pi_j^{(2)i} + P^{(1)}\pi_j^{(1)i} + P^{(2)}\delta_j^i. \tag{2.3}$$

Using the expressions for the linear first-order perturbations of the energy-momentum tensor and rewriting them as a function of the linear metric perturbations [53], the compacted starting equation becomes:

$$h_{ij}^{''} + 2Hh_{ij}^{'} - \nabla^2 h_{ij} = -4\hat{\Pi}_{ij}^{lm}\mathcal{S}_{lm}, \tag{2.4}$$

with $\mathcal{S}_{lm}$:

$$\begin{aligned}\mathcal{S}_{lm} = {} & 2\Psi\partial^l\partial_m\Psi - 2\Phi\partial^l\partial_m\Psi + 4\Phi\partial^l\partial_m\Phi + 4\Psi\partial^l\partial_m\Psi \\ & +\partial^l\Psi\partial_m\Psi - \partial^l\Psi\partial_m\Phi - \partial^l\Phi\partial_m\Psi + 3\partial^l\Phi\partial_m\Phi \\ & -\frac{4}{3(1+\omega)H^2}\partial_l(\Phi^{'} + 3H\Psi)\partial_m(\Phi^{'} + 3H\Psi) \\ & -\frac{2c_s^2}{3\omega H^2}\left[3H(H\Psi - \Phi^{'}) + \nabla^2\Phi\right]\partial_l\partial_m(\Psi - \Phi),\end{aligned} \tag{2.5}$$

with $\omega = P^0/\rho^0$, $\Psi = \Psi^{(1)}$, $\Phi = \Phi^{(1)}$ and $c_s = P^1/\rho^1$. It is fundamental to note that the source tensor is actually composed, as anticipated earlier, of linear combinations of scalar fluctuations from both the Einstein tensor and the stress-energy tensor. In order to solve the dynamics of the induced GWs background, the tensor solution is appropriately transformed in the Fourier space

$$h_{ij}(\mathbf{x},\tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \exp{(i\mathbf{k}\mathbf{x})}[h_{\mathbf{k}}(\tau)e_{ij}(\mathbf{k}) + \bar{h}_{\mathbf{k}}(\tau)\bar{e}_{ij}(\mathbf{k})]. \tag{2.6}$$

The two polarization tensors $e_{ij}(\mathbf{k})$, $\bar{e}_{ij}(\mathbf{k})$ can be expressed in terms of two polarization vectors orthogonal to the direction of propagation of the wave marked by vector $\mathbf{k}$:

$$e_{ij}(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[e_i(\mathbf{k})e_j(\mathbf{k}) - \bar{e}_i(\mathbf{k})\bar{e}_j(\mathbf{k})\right], \tag{2.7}$$

$$\bar{e}_{ij}(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[e_i(\mathbf{k})\bar{e}_j(\mathbf{k}) - \bar{e}_i(\mathbf{k})e_j(\mathbf{k})\right]. \tag{2.8}$$

We must now also rewrite the RHS source term in Fourier space in terms of the polarization tensors

$$\hat{\Pi}_{ij}^{lm}\mathcal{S}_{lm} = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \exp{(i\mathbf{k}\mathbf{x})}\left[e_{ij}(\mathbf{k})e^{lm}(\mathbf{k}) + \bar{e}_{ij}(\mathbf{k})\bar{e}^{lm}(\mathbf{k})\right]\mathcal{S}_{lm}(\mathbf{k}). \tag{2.9}$$

Thus, the second-order equation of motion for tensor modes induced by scalars in Fourier space, for each polarization state, is defined as follows:

$$h_{\mathbf{k}}^{''} + 2Hh_{\mathbf{k}}^{'} + k^2h_{\mathbf{k}} = \mathcal{S}(\mathbf{k},\tau), \tag{2.10}$$

where

$$\mathcal{S}(\mathbf{k},\tau) = -4e^{lm}(\mathbf{k})\mathcal{S}_{lm}(\mathbf{k}), \tag{2.11}$$

is the convolution of the two linear scalar perturbations. Equation (2.4) is a wave equation with a source, the solution of which is written:

$$h_{\mathbf{k}}(\tau) = \frac{1}{a(\tau)}\int d\tilde{\tau}G_{\mathbf{k}}(\tau;\tilde{\tau})\left[a(\tilde{\tau})\mathcal{S}(\mathbf{k},\tilde{\tau})\right], \tag{2.12}$$

where the Green's function $G_{\mathbf{k}}$ solves equation (2.4) with a source given by $\frac{1}{a}\delta(\tau-\tilde{\tau})$. Given this solution (2.12), the expression for the two-point correlation function for the GW background can be written in terms of the source as follows

$$\langle h_{\mathbf{k}}(\tau)h_{\mathbf{k}'}(\tau)\rangle = \frac{1}{a^2(\tau)}\int_{\tau_0}^{\tau} d\tilde{\tau}_1 d\tilde{\tau}_2 a(\tilde{\tau}_1)a(\tilde{\tau}_2)G_{\mathbf{k}}(\tau;\tilde{\tau}_1)G_{\mathbf{k}'}(\tau;\tilde{\tau}_2)\langle\mathcal{S}(\mathbf{k},\tilde{\tau}_1)\mathcal{S}(\mathbf{k}^{'},\tilde{\tau}_2)\rangle, \tag{2.13}$$

with τ_0 the turn-on time of the source. Expression (2.13) represents the generic expression for the GW power spectrum due to the induced second-order tensor modes that solve the equation of motion with a generic source. Now it becomes interesting to go into the specifics of some models to find some solution relative to a specific source.

2.1.1 Second order GWs sourced by inflaton perturbations

An immediate application of the second-order perturbative development theory adopted above, can be defined imagining that the source term is directly described by inflationary fluctuations. It is important to note that, regardless of how they are generated, or regardless of what these first-order linear scalar perturbations are, they end up generating second-order tensor modes. Knowing the scalar power spectrum for the inflationary perturbation, it is possible to calculate the tensor power spectrum of the induced GWs: such a consideration arises very trivially from the fact that the source term can be written as a function of the primordial inflationary scalar perturbation evaluated on much earlier timescales $\Phi_{\mathbf{k}}(\tau)$, so the two-point tensor correlator can be written as a function of the primordial spectrum associated with the inflationary fluctuation $P_{\Phi}(k)$, with:

$$\langle\Phi_{\mathbf{k}}\Phi_{\mathbf{k}'}\rangle = \frac{2\pi^2}{k^3}P_{\Phi}(k)\delta(\mathbf{k}+\mathbf{k}^{'}), \tag{2.14}$$

the last, knowing the connection to the observable curvature power spectrum, is strongly constrained by CMB and LSS measurements, at least on the large scales. Scalar perturbations define the source role of the second-order induced GWs background not only during the inflationary period, but also in later phases, when these, like curvature perturbations, re-enter the Hubble sphere after having been frozen for a long time in Super-Horizon. Thus when such curvature perturbations re-enter the horizon of causality during e.g. the radiation-dominated phase, they return to dynamically evolve in time, and thus are ready to interact appropriately defining a source for a new background of second-order induced tensor modes but in the radiation phase [44], [53]. Such a GWs background could define an observable amplitude values in present time, on specific scales, within the interferometric

sensitivities [54]. However, considering a power spectrum for the inflationary fluctuation that draws a power-law on all scales, subject to all the scalar constraints imposed by CMB measurements, would produce an induced background whose present spectral energy density would be much lower than the sensitivity curve of various experimental set-up already constructed. For example, for a power-law scalar power spectrum with a red-tilt of $n_S = 0.95$, a spectral energy density of $\Omega_{GWs} \simeq 10^{-22}(\frac{f}{Hz})^{-0.1}$ would be obtained [54].
The presence of a second-order GWs background induced by curvature perturbations that fall within the causality horizon in the age of matter dominance, however, changes the various predictions about the polarization of the CMB [55]. This effect limits our ability to estimate the inflationary tensor power spectrum, thus making it even more difficult to estimate the energy of this period, as it is totally parallel to the amplitude of the tensor spectrum mentioned earlier. The amount of B modes due to the presence of second-order GWs is estimated to be secondary to those produced by weak-lensing. However, the second-order gravitational contribution becomes non-negligible in several inflationary models, in particular those associated with intense particle production. The main idea of this work, which will be explored in more detail shortly, is to solve the appropriate extra-field equations of motion, thus enabling the source term to be written and thus allowing the graviton dynamics to be solved by the Green's function method. Usually in such designs, only the tensor perturbation is considered within the metric tensor, neglecting scalar and vector fluctuations for the sake of simplicity.

2.1.2 GWs equation neglecting scalar and vector metric perturbations

Let us consider a second-order perturbed FLRW metric neglecting first and second-order scalar and vector perturbations. The equation of motion for GWs is simplified as follows:

$$h_{ij}^{''} + 2Hh_{ij}^{'} - \nabla^2 h_{ij} = \frac{2}{M_{Pl}^2}\hat{\Pi}_{ij}^{lm}T_{lm}, \tag{2.15}$$

with T_{lm} the generic energy-momentum tensor. It is essential to note that the source term is defined only by the latter, infact all scalar perturbation contributions relative to the metric counterpart in the Einstein tensor have been set to zero. The resolution of the equation of motion, therefore, is written as follows:

$$h_{ij}(\mathbf{k},\tau) = \frac{2}{M_{Pl}^2}\int d\tilde{\tau} G_{\mathbf{k}}(\tau,\tilde{\tau})\hat{\Pi}_{ij}^{lm}(\mathbf{k})T_{lm}(\mathbf{k},\tilde{\tau}). \tag{2.16}$$

Proceeding as before, we find the amplitude of the tensor mode on the fixed polarization state

$$h_{\mathbf{k}}(\tau) = \frac{1}{a(\tau)} \int d\tilde{\tau} G_{\mathbf{k}}(\tau, \tilde{\tau}) \left[a(\tilde{\tau}) \mathcal{T}(\mathbf{k}, \tilde{\tau})\right], \tag{2.17}$$

where $\mathcal{T}(\mathbf{k}, \tau)$ is the purely energetic counterpart of the source before given in Fourier space.

To specify the solution, it is necessary to fix the evolution of the scaling factor and the evolution of the appropriate projection of the energy-momentum tensor. An exact solution for the Green's function exists for a de Sitter phase, but also for radiation and matter-dominated phases. Studying the production of the GWs background in an inflationary context, and considering the usual de Sitter background to support inflation, Green's solution assumes the current formulation [56]

$$G_{\mathbf{k}}(\tau, \tilde{\tau}) = \frac{1}{k^3 \tilde{\tau}^2} \left[(1 + k^2 \tau \tilde{\tau}) \sin k(\tau - \tilde{\tau}) + k(\tau - \tilde{\tau}) \cos k(\tau - \tilde{\tau})\right] \Theta(\tau - \tilde{\tau}). \tag{2.18}$$

2.2 GWs sourced by scalar fluctuations

In the following section, two inflationary models will be analysed: here the source for GW's induced background is described by the presence of extra scalar fields (i.e. their perturbation), beyond the inflaton. The spectral GW abundances generated by the perturbations of these extra fields will be studied, without taking into account the source terms associated with the scalar-vector metric perturbations; we neglect them for the sake of simplicity .

2.2.1 Second order GWs in the curvatone scenario

In the curvaton scenario [57], we consider a secondary particle beyond the inflaton, i.e. the curvaton assumed not to modify or influence the inflationary dynamics described by the first scalar field. For simplicity, it is possible to assume that the curvature perturbation associated with the inflationary perturbation of the first field is so small and negligible that it does not originate LSS upon its re-entry into the Hubble sphere. This condition can be achieved by lowering the energy scale of the inflation (hence H), thereby also lowering the amount of GW emitted in the standard first-order perturbative format, since H defines the amplitude of the tensor power spectrum. Specifically, requiring the inflationary curvature perturbation to be much smaller than that required to explain the CMB anisotropy phenomenon (hence much smaller than 10^{-5}) corresponds to considering a value of $H << 10^{-5} M_{Pl}$ [58] . Therefore, with such a value of the Hubble

rate associated with the inflationary period, the spectral energy density of the first-order perturbative GW background associated with a vacuum oscillation of the metric tensor drops significantly, becoming smaller, or at most comparable with the second-order induced GWs background of interest [58], [59], [60] .
Let us now examine the role of the curvaton in more detail, starting with the assumption that a secondary field involves isocurvature perturbations. Once the inflation is over (so we no longer have the scalar inflaton field, but its fluctuation is saved in the value of the curvature perturbation, which henceforth connotes a transposed radiation density perturbation), the curvaton decays (into radiation too), so that only one type of fluctuation energy remains (i.e. that of radiation), and there will be a transition from an isocurvature perturbation to an adiabatic curvature perturbation. When that perturbation re-entered the sphere of causal connection will provide the seed from which the LSS and CMB anisotropy will develop, with a time transition expressed by an appropriate transfer function. However, it is possible to consider the presence of two different GWs backgrounds induced by different scalar perturbations: the first is the one induced by isocurvature perturbations of the curvaton when they re-enter the horizon between the end of inflation and the decay of the curvaton, while the second is the GWs background induced by the final curvature perturbation left after the decay of the curvaton, upon its re-entry into the sphere of causal connection. The latter case is able to define a spectral energy density of the order of $\Omega_{GW} \simeq 10^{-20}$ [58]; the interesting result to evaluate is that such a GWs background induced by the curvature perturbation is subdominant with respect to that generated by an isocurvature source re-entering the horizon before the decay of the secondary field [58]. Consider therefore the first GWs background. The equation of motion of the GW background induced at second order by the curvature isocurvature perturbations is formulated as follows

$$h_{ij}^{''} + 2Hh_{ij}^{'} - \nabla^2 h_{ij} = -\frac{2}{M_{Pl}^2}\hat{\Pi}_{ij}^{lm}\partial_l\delta\sigma\partial_m\delta\sigma, \tag{2.19}$$

where $\delta\sigma$ is the isocurvature perturbation of the curvaton. The solution of this dynamical equation is provided by equation (2.17), where the Green's function associated with a scalar induction in the radiation period is written $G_k(\tilde{\tau},\tau) = \frac{\sin k(\tilde{\tau}-\tau)}{k}$, with the integration that begins when the scalar isocurvature fluctuation falls within the Hubble sphere.
We must observe that the source term in the integrated solution is determined by the fluctuation of the curvaton $\delta\sigma$ and its temporal evolution, so it is crucial to understand whether at the moment of re-entry into the horizon, the curvaton is already fluctuating around its potential minimum marking a decay, or not. In fact, for those modes that re-enter during the decay we have that $\delta\sigma_k \simeq a^{-\frac{3}{2}}$, while for those modes founding the source that re-enter between the end of the inflation and the beginning of the oscillation $\delta\sigma_k \simeq a^{-1}$ [58]. These two trends obviously

lead to two different source, resulting in two different induced GWs backgrounds. Knowing that the spectral energy density depends on the physics of the curvaton, hence on its fluctuation, and that the latter after the aforementioned decay changes from an isocurvature perturbation to an observable curvature perturbation and knowing the connection between the two, it is always possible to redefine the spectral energy density of the GWs background in terms of observable scalar quantities, such as the same curvature perturbation for which we know of the limits imposed by the CMB [61], and f_{NL}. So knowing the connection between the two types of perturbation (iso and curv), it is possible to transform the dependence that the spectral energy density has with respect to the isocurvature power spectrum, into the dependence towards the observable power spectrum of curvature perturbation. Assuming a scale invariant curvature power spectrum with a value of $A_S \simeq 10^{-9}$, and defining k_D as the scale of separation that enters the horizon when the curvaton decays, the current spectral energy density of the induced GWs background results [58]

$$\Omega_{GW} \simeq 10^{-15} \left(\frac{f_{NL}^{local}}{10^2} \right)^2 \left(\frac{k}{k_D} \right)^5 \left(\frac{\Gamma}{m} \right)^{\frac{7}{2}}, \tag{2.20}$$

for $k_D \leq k \leq (m/\Gamma)^{\frac{1}{2}} k_D$ for modes entering during decay, while for the remaining ones

$$\Omega_{GW} \simeq 10^{-15} \left(\frac{f_{NL}^{local}}{10^2} \right)^2 \left(\frac{\Gamma}{m} \right), \tag{2.21}$$

for $k \geq k_D(m/\Gamma)^{\frac{1}{2}}$. From this expression, in the perturbative regime $\Gamma \leq m$ and maximizing the NG constraints [61], one can currently find an induced GWs background of the order of $\Omega \simeq 10^{-19}$, proving what was asserted in the beginning of the section.

2.2.2 Second-order GWs sourced by spectator scalar fields

It is important to define a further model suitable for a conspicuous production of gravitational waves induced by scalars at second-order. We can introduce again an additional scalar field called spectator field beyond the inflaton; however, this secondary field is assumed to be light in order to not modify the inflationary dynamics managed by the first scalar field, i.e. the inflaton. The spectator field plays a crucial role in the additive production of a second-order induced GWs background [62], [63], [64] ; contrary to what was assumed in the previous curvature model, here it is to be expected that even the inflationary perturbation generates a non-negligible curvature perturbation, such that a large production of a standard GWs background is induced. The peculiarity of the model we intend to deal with, consists in observing how assuming a speed of sound for the secondary field $c_S < 1$ produces a more abundant induced GWs background than in contrary cases, such as the one for which the speed of the fluctuation reaches unity. It is possible to trivially define this speed of sound from a simple lagrangian of the field $P(X, \sigma)$, with X a canonical kinetic term, so that $c_S = \partial_X P/(\partial_X P + 2X\partial^2_{XX} P)$. Consider now the action associated with the scalar fluctuations of the spectator field

$$S^{(2)}_{\delta\sigma} = \int d\tau d^3 x a^4 \frac{1}{2a^2} \left[\delta\sigma'^2 - c_S^2(\nabla\delta\sigma)^2\right] - V_{(2)}, \tag{2.22}$$

where $V_{(2)}$ is the potential written at second order. The spectator field leads to the treatment of scalar perturbations, beyond those associated with the inflaton, whose amplitude is determined by the sound velocity of the secondary fluctuation itself; knowing that any scalar perturbation (or rather, appropriate combinations of them) defines a source for an induced GWs background, the spectator perturbations will not be any less. Paying attention therefore to the field fluctuations in question, it is possible to simply rewrite the dynamic equation of the tensor modes generated:

$$h''_{ij} + 2Hh'_{ij} - \nabla^2 h_{ij} = -\frac{2c_S^2}{M_{Pl}^2} \hat{\Pi}^{lm}_{ij} \partial_l \delta\sigma \partial_m \delta\sigma. \tag{2.23}$$

It is important to note that, in contrast to the other dynamic equations discussed above, the sound velocity term is present here, which can make important changes to the final result. Working as proposed in the previous sections we find the solution (2.17), and subsequently integrating over the entire inflationary period under study, we find the second-order GW power spectrum tensor on superHorizon scales originating from the secondary field fluctuations defined as follows [62]

$$P_T = c\left(\frac{H^4}{c_S^{\frac{18}{5}} M_{Pl}^4}\right), \tag{2.24}$$

with c a numerical factor close to three. The significant result found is that the amplitude of the induced GWs background turns out to be inversely proportional to the adiabatic speed of sound of the scalar fluctuation generating it. The total scalar and tensorial power spectrum will therefore be provided by the sum of two terms, the first standard due to the vacuum oscillations, and the second induced by the presence of the spectator field; thus the total tensor-to-scalar-ratio will be strongly sensitive to c_S. The uncovered dependence of r with c_S introduces a degeneracy between the various parameters of the model: the most obvious example is that there is no longer a one-to-one correlation between r and the Energy of the inflationary system. However, Planck's measurements on the CMB scales impose stringent constraints to be followed for the scalar curvature fluctuations, (and hence

for the primordial spectrum), so these observational constraints in turn constrain c_S to similarly specific values, thus reducing the abundance of the generated GWs background [63], [64]. Assuming in fact that the spectator field does not originate a major curvature perturbation, a strong upper limit is defined on the amplitude of GWs originated by the secondary field on the CMB scales, thus drawing a negligible total contribution. On the contrary, if the curvature perturbation is mainly originated by the spectator field fluctuation, the GWs background produced will be greater, and considering the fixed amplitude of the scalar fluctuations on the CMB scales (which mark the denominator of the tensor-to-scalar-ratio), the value of r may be greater than other cases.
Considering a spectator field with a light mass m evolving in a near de-Sitter background, the spectral tilt associated with the power spectrum of $\delta\sigma$ does not vanish, thus affecting the tensor power spectrum with its spectral tilt [62]

$$n_T = 2\left(\frac{2m^2}{3H^2} - 2\epsilon\right) - \frac{18}{5}\frac{\dot{c}_S}{Hc_S}, \tag{2.25}$$

The following expression shows how, in contrast to the first-order GWs associated with a vacuum fluctuation, the induced GWs on superHorizon scales can exhibit a blue tilt, leading to an increase for the overall GWs background abundance.

2.3 Particle production as a source of GWs

There are a multitude of inflationary models associated with quantum production of extra fields during inflation, which can contribute strongly to the generation of an induced GWs background. If the inflaton is minimally (or not minimally) coupled to a secondary scalar or gauge field, there will be an active transfer of energy from the first to the second sector, thus leading to the expected creation of extra quanta [65]. These new particles will make a major contribution to the energy-momentum tensor, in particular to its anisotropic tensor component: we know from the previous sections that a second-order treatment of such tensor energy components is proposed as the source of an induced GWs background, so the more particles are produced, the more the anisotropic term of the stress-energy tensor is enlarged, leading to a consequent increase in the source, hence in the abundance of the GWs background of interest [66], [67], [68], [69], [70].
Various models have been built on this construction, in particular it is necessary to distinguish scenarios in which the production of these extra fields occurs during an inflationary Slow-Roll phase of the inflaton, and scenarios in which this creation occurs during the oscillation of the main scalar field around the minimum, i.e.

during the description of the decay of the inflaton; it will be of interest to analyse both cases. Among these models, it is worth mentioning the one that sees a coupling link between the inflaton and an A_μ [71] gauge field: here a band of modes of the aforementioned produced gauge field is subject to exponential growth, which leads to a strong increase in the energy-momentum tensor, hence in the source, leading to a conspicuous induced GWs background. However, such exponentially increasing extra modes will also produce a curvature perturbation, for which at least on the CMB scales there are stringent observational limitations. In fact, the current data on the amplitude value of the scalar fluctuations severely restrict the parameter space of these models, even though they allow the treatment of an induced GWs background so abundant that it falls within the sensitivity window of known measuring instruments.
The first models analysed in this overall view present non-minimal coupling between the inflaton and extra fields [66], [72], [67], [68]. In particular, couplings of the inflaton with massive scalar fields or with gauge fields are investigated: in the former case a production of extra fields describing a flare is defined, while in the latter case the production occurs continuously during the inflationary dynamics.

2.3.1 Inflaton coupled to a scalar field

Consider a physical system described by the following Lagrangian [56]:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{g^2}{2}(\varphi-\varphi_0)^2\chi^2, \tag{2.26}$$

where φ is the inflaton, $V(\varphi)$ is the potential driving the inflationary dynamics, χ is the scalar extra field, whose self-interaction component we imagine we can neglect. The mass of the secondary field, as clearly deduced from the Lagrangian density, depends on time, being linearly related to the value of the inflationary field, which in the meantime describes a slow-roll dynamics slowly rolling in the constant potential section: when the inflaton φ reaches the value of φ_0, m_χ cancels to zero, making the production of the particle section identified by the secondary field energetically favoured. During such a dynamical period for which the inflaton reaches this non-trivial value, a non-perturbative and explosive production of particles χ is thus described: once dynamically this phase is concluded, it is possible to observe a Universe filled with secondary field particles χ, beyond the usual inflaton. As confirmed above, the presence of this new and intense particle support provides an important contribution to the overall energy-momentum tensor, in particular its spatial part will be read $T_{ab} = \partial_a\chi\partial_b\chi + \delta_{ab}(...)$, where the factor proportional to the Kronecker delta will be swept away by the transverse-traceless projection operator (whose application is necessary to define the background). As done firstly, we promote the scalar field $\chi(k,\tau)$ to a quantum operator via the second quantization $\hat{\chi}(k,\tau)$, and moving in Fourier space:

$$\hat{\chi}(x,\tau) = \frac{1}{a(\tau)} \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \exp{(ikx)} \hat{\chi}(k,\tau). \tag{2.27}$$

If we substitute this transformation into the recently provided definition of the energy-momentum tensor, and subsequently the latter into (2.17) to find the amplitude of the generated tensorial mode, it is possible to define the two-point correlator of the induced GWs background [56] (from which the tensor GW power spectrum, hence the observable spectral energy density, will then be derived)

$$\begin{aligned} \langle h_{ij}(k,\tau)h_{ij}(k^{'}\tau)\rangle = \frac{1}{2\pi^3 M_{Pl}^4} \int \frac{d\tilde{\tau}_1}{a(\tilde{\tau}_1)^2} G_k(\tau,\tilde{\tau}_1)\times \\ \int \frac{d\tilde{\tau}_2}{a(\tilde{\tau}_2)^2} G_{k'}(\tau,\tilde{\tau}_2)\Pi_{ij}^{ab}(k)\Pi_{ij}^{cd}(k^{'})\times \\ \int d^3p d^3p^{'} p_a(k_b - p_b)p_c^{'}(k_d^{'} - p_d^{'})\times \\ \langle \hat{\chi}(p,\tilde{\tau}_1)\hat{\chi}(k-p,\tilde{\tau}_1)\hat{\chi}(p^{'},\tilde{\tau}_2)\hat{\chi}(k^{'}-p^{'},\tilde{\tau}_2)\rangle. \end{aligned} \tag{2.28}$$

Applying Wick's theorem and neglecting the discontiguous terms, it can be seen that the GW power spectrum depends on the two-point correlator of scalar operators $\langle \hat{\chi}(p,\tilde{\tau}_1)\hat{\chi}(q,\tilde{\tau}_2)\rangle$.
We could write this latter quantity only after solving the equation of motion for the scalar field χ . We decompose $\hat{\chi}(k,\tau)$ in terms of the creation and destruction operators

$$\hat{\chi}(k,\tau) = \chi(k,\tau)\hat{a}_k + \chi^*(-k,\tau)\hat{a}_{-k}^{\perp}. \tag{2.29}$$

From the Lagrangian, the equation of motion for χ is derived:

$$\chi^{''}(k,\tau) + \omega^2(k,\tau)\chi(k,\tau) = 0, \tag{2.30}$$

with

$$\omega^2(k,\tau) = k^2 + g^2 a^2(\tau)[\varphi(\tau) - \varphi_0]^2 - \frac{a^{''(\tau)}}{a(\tau)}. \tag{2.31}$$

This expression can be approximated in different ways depending on the behaviour of the system, in fact three main periods can be identified in the treatment of the problem:

— At the beginning of the inflationary phase where the value of the inflationary field in the background still reaches φ_0, the Universe does not yet contain the massive scalar field χ (since it hasn't yet been produced in terms of the perturbative flare), therefore the source of the GWs background does not exist, as well as the GWs background itself.

— At some point the inflaton will reach φ_0, so the mass of the secondary field will fall to zero, therefore energetically speaking the creation of the secondary particle sector will be favoured, so for a period of production Δt_{nad} the evolution of $m_\chi(t)$ will be non-adiabatic, i.e. $\dot{m_\chi} > m_\chi^2$.
In order to have an efficient particle production, the production time must be significantly shorter than the characteristic Hubble expansion time. In a de-Sitter background it is trivial to write the inflaton evolution equation in a linear format $\varphi(t) = \varphi_0 + \dot{\varphi}_0 t$. Conjugating the non-adiabaticity condition with the definition of m_χ we obtain $\Delta t_{nad} \simeq (g\dot{\varphi}_0)^{-\frac{1}{2}}$: applying the efficient production requirement, we obtain $g >> H^2/|\dot{\varphi}_0|$. This condition obviously implies that the accelerated cosmological expansion can be neglected in the intuition of the process occurring so rapidly.

— the end the inflaton will leave the value φ_0, therefore m_χ will go back to adiabatically evolving together with the whole production process: the creation of secondary particles will therefore be stopped, but the Universe is now filled with these particles χ that we know give a large contribution to the energy-momentum tensor, therefore to the source, going to define an important induced GW background.

Let us consider the dynamic nature of the last stage: in order to calculate the abundance of the GWs background, hence the tensor power spectrum, one needs the amount of χ quanta produced during the non-adiabatic central phase, since these will define the source. Thus imposing the non-adiabatic condition and using the linear evolution approximation of the inflaton, expression (2.30) is rewritten as follows:

$$\ddot{\chi} + (k^2 H^2 \tau_0^2 + g^2 \dot{\varphi}_0^2 t^2)\chi = 0. \tag{2.32}$$

From this form it is possible to deduce the amount of extra quanta produced during the non-adiabatic phase, which defines an initial condition for the dynamic treatment of the subsequent adiabatic phase. During the final adiabatic phase, in fact, one can appropriately rewrite the pulsation as $\omega = \left|g[\varphi(\tau) - \varphi_0]/(H\tau)\right|$, so that one can solve (2.30) in the last phase of interest as well. By sending the conformal time to zero, characteristic of the end of inflation, the correlator (2.28) results [56]:

$$\langle h_{ij}(k) h_{ij}(k')\rangle = \frac{\delta^{(3)}(k+k')}{2\pi^5 k^6 |\tau_0|^3} \frac{H^4}{M_{Pl}} (1 + \frac{1}{4\sqrt{2}}) \times (\frac{g\dot{\varphi}_0}{H^2})^{\frac{3}{2}} F_{\Delta_{\tau_{nad}/\tau_0}}(k|\tau_0|), \tag{2.33}$$

where

$$F_\epsilon(y) = \left| \int_0^{1-\epsilon} x \frac{\sin xy - xy \cos xy}{\ln x} dx \right|^2 \simeq [(y\cos y - \sin y)\ln \epsilon]^2, \tag{2.34}$$

where the last approximate expression holds when the parameter ϵ goes to zero.
In order to determine the overall tensor power spectrum, one must add, to the result just found, the spectral contribution of the first-order GWs background associated with a vacuum oscillation of the meric tensor, obtaining, in conclusion:

$$P_h(k) = \frac{2H^2}{\pi^2 M_{Pl}^2}\left[1 + 4.8\times 10^{-4}\frac{(k\tau_0\cos k\tau_0\sin k\tau_0)^2}{|k\tau_0|^3}\times\frac{H^2}{M_{Pl}^2}(\frac{g\dot{\varphi}_0}{H^2})^{\frac{3}{2}}\ln^2\frac{\sqrt{g\dot{\varphi}_0}}{H^2}\right]. \tag{2.35}$$

As can easily be seen, the contribution induced by the production of extra quanta turns out to be the scale dependent term that is added to the standard power spectrum scale invariant relative to the first-order problem. Reasoning on the slow-roll condition for which $\dot{\varphi}_0 = \sqrt{2\epsilon}HM_{Pl}$, and considering a reasonably small value for the first slow-roll parameter, it is possible to evaluate how the correction inflicted on the global power spectrum due to the second-order contribution turns out to be extremely small, of the order of $10^{-2}\sqrt{H/M_{Pl}}$, i.e. an amount much smaller than unity. Therefore, it is reasonable to conclude that the presence of a scalar particle gas during inflationary dynamics does not contribute to an appreciable change in the standard tensor power spectrum originating at first order due to an obvious vacuum fluctuation of the metric tensor induced by a previous inflationary perturbation.
It should be remarked that, if the scalar mode had been calculated during the second non-adiabatic period, the additional second-order spectrum produced would have been quite similar to that found by the third time period of the model. Even if the presence of several separate secondary particle bursts were proposed to the model, the amount of GWs induced by these scalars would still be negligible compared to the main first-order phenomenon provided by the vacuum oscillations [56].

2.3.2 Axion inflation: pseudoscalar inflaton coupled to a gauge field

Consider a new physical system described by this Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\varphi}{4f}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2.36}$$

where V is the potential of the first field driving the slow-roll inflation, f is an estimate indicating the degree of coupling of the pseudo-scalar inflaton φ and the gauge field A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is the field strenght associated with the bosonic field of spin 1, while $\tilde{F}^{\mu\nu}$ is its dual.
The above coupling leads, as seen exactly in the previous model, to a production

of a second-order induced GWs background, since the energy transfer from the inflationary sector to the secondary field leads to a massive production of the latter, feeding the presence of a new current capable of magnifying the energy-momentum tensor, hence the source of the induced GWs background; the consequence is obviously also a conspicuous production of boosted scalar modes, which provide an increase in the curvature perturbation, always within the required observational constraints. However, there is a second factor that must be taken into account here, namely the phenomenon of back-reaction on the background dynamics: indeed, in order to produce gauge quanta one must transfer energy from the inflaton sector to the gauge sector, so this new form of energy in the form of new and secondary particles can counteract the inflaton by modifying its dynamics at the background [67], [68], [69] [73], [74]. The inflaton in fact gains, from this back-reaction, an additional friction term that further slows down the already slow-rolling dynamical mechanism. As in the previous case, therefore, the equation of motion of the produced tensorial modes is (2.15), with solution given by (2.16).
One chooses to work in a convenient Coulomb gauge, so A_μ can conveniently be described by the potential vector $\mathbf{A}(\tau, \mathbf{x})$, defined as $a^2\mathbf{B} = -\nabla \times \mathbf{A}$, $a^2\mathbf{E} = -\mathbf{A}'$, where $\mathbf{E}$ and $\mathbf{B}$ have the usual relationships with the force tensor, so the spatial part of the stress-energy tensor results in the form $T_{ab} = -a^2(E_aE_b + B_aB_b) + (...)\delta_{ab}$. Following the same logic as in the previous section, one must first find the solution to the equation of motion for the gauge field, and then substitute it into the energy-momentum tensor delineating the source inducing GWs background [72], [67]. The equation of motion for the potential vector introduced earlier is:

$$\left(\frac{\partial^2}{\partial\tau^2} - \nabla^2 - \frac{\varphi'}{f}\nabla\right) \times \mathbf{A} = 0. \tag{2.37}$$

We promote the potential vector $\mathbf{A}(\tau, \mathbf{x})$ to a quantum operator $\hat{\mathbf{A}}(\tau, \mathbf{x})$, and then decompose its modes as a combination of the usual creation and destruction operators, taking into account how each mode holds two degrees of freedom associated with two different and perpendicular states of polarization

$$\hat{A}_i(\mathbf{x}, \tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \exp i\mathbf{kx}\hat{A}_i(\tau, \mathbf{k}) = \sum_{s=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[\epsilon_s^i(\mathbf{k})A_s(\tau, \mathbf{k})\hat{a}_s^{\mathbf{k}} \exp i\mathbf{kx} + h.c\right] \tag{2.38}$$

with ϵ_s^i the usual polarization tensor to which the usual transformation properties are associated. Assuming a de-Sitter inflationary background, it is possible to write the equations of motion for the amplitudes $A_\pm$:

$$\frac{d^2A_\pm(k, \tau)}{d\tau^2} + \left[k^2 \pm 2k\frac{\zeta}{\tau}\right] A_\pm(\tau, k) = 0, \tag{2.39}$$

with

$$\zeta = \frac{\dot{\varphi}}{2fH} = \sqrt{\frac{\epsilon}{2}}\frac{M_{Pl}}{f}. \tag{2.40}$$

that expose the coupling term f in a different manner.
Equation (2.39) shows a different behaviour for the two different helicity states of the gauge field; indeed, depending on the sign of the coupling parameter, one mode will be subject to an exponential instability, while the perpendicular one will go to zero. This explicit parity violation will, as will be seen later, also generate a parity violation in the final tensor power spectrum. Assuming positive ζ, the solution of equation (2.39) is written

$$A_+(k,\tau) \simeq \frac{1}{\sqrt{2k}}\left(\frac{k}{2\zeta aH}\right)^{\frac{1}{4}} \exp\left(\pi\zeta - 2\sqrt{2\zeta k/aH}\right), \tag{2.41}$$

while at the same time the polarization mode - goes to zero $A_- \simeq 0$.
We think that a parity violation of this form is also reported in the treatment of induced tensor-polarized modes, since they are precisely generated by a source that is defined via the modes of the gauge field. Therefore, to define the behaviour of the two helicity states of the GWs, one split the general tensorial mode into the sum of the two polarization contributions. So in Fourier space we project h_{ij} onto the two helicity modes

$$h^{ij}(\mathbf{k}) = \sqrt{2}\sum_{s=\pm}\epsilon^i_s(\mathbf{k})\epsilon^j_s(\mathbf{k})h_s(\mathbf{k}). \tag{2.42}$$

We can elevate $h_\pm$ to an operator $\hat{h}_\pm$, the expression of which can again be defined by equation (2.17) [72], [67]

$$\begin{aligned}\hat{h}_\pm(\mathbf{k}) = -\frac{2H^2}{M^2_{Pl}}\int d\tilde{\tau}G_k(\tau,\tilde{\tau})\tilde{\tau}^2\int\frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\hat{\Pi}^{lm}_\pm(\mathbf{k})\times \\ \times\left[\hat{A}'_l(\mathbf{q},\tilde{\tau})\hat{A}'_m(\mathbf{k}-\mathbf{q},\tilde{\tau}) - \varepsilon_{lab}\hat{A}^(_b\mathbf{q},\tilde{\tau})\varepsilon_{mcd}(k_c-q_c)\hat{A}_d(\mathbf{k}-\mathbf{q},\tilde{\tau})\right],\end{aligned} \tag{2.43}$$

with the Green's function obtained in the previous sections for the inflationary resolution.
Substituting (2.41) into the last expression, using Wick's theorem, we are able to define the GWs power spectrum in terms of Green's function and the amplitude of the gauge field modes. Specifically, for $\zeta > 1$, the correlator results

$$\begin{aligned}\langle h_s(\mathbf{k})h_s(\mathbf{k}')\rangle = \frac{H^4\zeta}{4\pi^3M^4_{Pl}}\exp(4\pi\zeta)\delta(\mathbf{k}+\mathbf{k}')\int d\tilde{\tau}_1 d\tilde{\tau}_2|\tilde{\tau}_1|^{\frac{3}{2}}|\tilde{\tau}_2|^{\frac{3}{2}}\,G_k(\tau,\tilde{\tau}_1)G_k(\tau,\tilde{\tau}_2)\times \\ \times\int d^3\mathbf{q}\left|\epsilon^i_{-s}(\mathbf{k})\epsilon^i_+(\mathbf{q})\right|^2\left|\epsilon^j_{-s}(\mathbf{k})\epsilon_+(\mathbf{k}-\mathbf{q})\right|^2\times \\ \times\sqrt{|\mathbf{k}-\mathbf{q}|}\sqrt{q}\exp(-2\sqrt{2\zeta})(\sqrt{|\tilde{\tau}_1|}+\sqrt{|\tilde{\tau}_2|})(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|}).\end{aligned} \tag{2.44}$$

The two terms in the second row are those that show different behaviour for the two polarization states. In the limit for which $k\tau \to 0$, the integrals above can be solved numerically, so [66]

$$\langle h_+(\mathbf{k})h_+(\mathbf{k}')\rangle \simeq 8.6 \times 10^{-7}\frac{H^4}{M_{Pl}^4}\frac{\exp 4\pi\epsilon}{\zeta^6}\frac{\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{k^3}, \tag{2.45}$$

$$\langle h_-(\mathbf{k})h_-(\mathbf{k}')\rangle \simeq 1.8 \times 10^{-9}\frac{H^4}{M_{Pl}^4}\frac{\exp 4\pi\epsilon}{\zeta^6}\frac{\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{k^3}. \tag{2.46}$$

The numerical factors define a difference of at least 3 orders of magnitude between the two scale invariant correlators.
To conclude, it is therefore necessary to acknowledge the final tensor power spectrum by also taking into account the central first-order contribution

$$P_T^+ = \frac{H^2}{\pi^2 M_{Pl}^2}\left(1 + 8.6 \times 10^{-7}\frac{H^2}{M_{Pl}^2}\frac{\exp 4\pi\zeta}{\zeta^6}\right), \tag{2.47}$$

$$P_T^- = \frac{H^2}{\pi^2 M_{Pl}^2}\left(1 + 1.8 \times 10^{-9}\frac{H^2}{M_{Pl}^2}\frac{\exp 4\pi\zeta}{\zeta^6}\right). \tag{2.48}$$

The parity violation can be identify through the chiral parameter [75]:

$$\Delta_\chi = \frac{P_T^+ - P_T^-}{P_T^+ + P_T^-}, \tag{2.49}$$

that can be also written as follows

$$\Delta_\chi = \frac{4.3 \times 10^{-7}\frac{\exp 4\pi\zeta}{\zeta^6}\frac{H^2}{M_{Pl}^2}}{1 + 4.3 \times 10^{-7}\frac{\exp 4\pi\zeta}{\zeta^6}\frac{H^2}{M_{Pl}^2}}. \tag{2.50}$$

When one considers the coupling parameter ζ as small, then inevitably the GWs background generated by the vacuum oscillations of the metric tensor dominates, hence one returns to the restoration of symmetry for parity $\Delta_\chi \to 0$, while vice versa, for large ζ the induced GWs background dominates, hence $\Delta_\chi \to 1$.
The departure of Δ_χ from zero thus marks an innovative feature of parity violation peculiar to the selected model, since, as seen in the first chapter, such a violation is not supposed to occur in the study of the GWs background generated by vacuum fluctuations.

2.4 GWs production during reheating after Inflation

From the previous section, the strong message that must get through is that when there is a strong time-dependent inhomogeneity in the energy density distribution of the Universe, i.e. when feeding the anisotropic sector of the stress-energy tensor, a non-trivial source is defined to produce second-order induced GWs. Such production can occur not only during the inflationary phase, but also subsequently during reheating, where the inflaton decays and produce a set of secondary particles that can describe the source term. The production of GWs during reheating was first theorised by Khlebnikov and Tkachev [76]. Since the reheating is the final phase of any inflationary model, it is easy to see how GWs produced during this phase can represent a fundamental source of information of the physics of inflation and reheating. Since it is well after Planck time, so that as soon as they are produced the GWs are already highly decoupled, they do not interact with the environment and arrive at us at exactly this size. Their spectrum hides fundamental information about the physics of the production period, such as the degree of coupling between the inflaton and other fields present.

At the end of inflation, the field that had driven the accelerated expansion phase falls into the minimum of its potential, beginning to oscillate around this stable minimum and decay into lighter particles, which subsequently begin to interact in order to reach a state of observable and assumed thermal equilibrium. Initially, it was always thought that the inflaton's oscillation phase around the minimum point could be described through a perturbative mathematical approach (slow, small oscillations around the minimum), resulting in a collection of decay products already in thermal equilibrium [7], [77]. However, it is possible that such oscillations may also be large and coherent, thus describing a non-perturbative dynamics for which the overall energy of the system is explosively and rapidly shifted from the scalar starting sector to the producing sector. Such a rapid mechanism is called Parametric Resonance [20], [78]: here an approach of oscillation around the minimum does not work, as the process is rapid and violent. To distinguish this mechanism from the entire reheating process, this phase is called Preheating. Following this explosive production phase, the particle products are not yet in a state of thermal equilibrium since they have not time to reach it, as opposed to a peturbative approach; therefore, after Preheating, a subsequent phase is required in which the rapidly and violently oscillating products (bosons) can interact to thermalize. Preheating is the specific moment of interest for the study of the boson-induced GWs background. Usually the two inflationary models that are put before this Preheating phase are chaotic inflation [6], [79], [80] and hybrid inflation [81], in the latter of which the Preheating process is markedly different, and is called Tachyonic Preheating [81], [82], [83], [84]; nevertheless, in both cases the production process of GWs is essentially the same.

2.4.1 Preheating with Parametric Resonance

We can talk about Parametric Resonance when one expects a coupling between the scalar inflaton field driving the accelerated expansion of the universe and another scalar field, whose mass is assumed to be light such that it does not influence the dynamics of the inflaton during the period of inflationary expansion [20], [78], [85]. Consider then a system with an inflaton φ coupled with a light scalar field χ, via the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \mathcal{V}(\varphi, \chi), \tag{2.51}$$

with

$$\mathcal{V}(\varphi, \chi) = V(\varphi) + \frac{1}{2}g^2\varphi^2\chi^2 - \frac{1}{2}m_\chi^2\chi^2, \tag{2.52}$$

where g represents the coupling constant between the two fields. During inflation the field χ is assumed to be light such that the inflationary dynamics is driven only by φ, without interaction or back-reaction interference. It is therefore possible to make the assumptions for which the mass of the secondary field and the expansion of the Universe are neglected, given the intrinsic non-perturbative rapidity of the model [85]. Thus, ignoring the second field, one can see that the equation of motion for the background component of the inflaton at the end of inflation is described by equation (6), taking however into consideration how here, contrary to what happens for inflation, the field cannot be considered homogeneous and the kinetic energy cannot be neglected: given the interaction in fact, there will be a continuous energy interchange between the parts involved, leading K to vary. Consider a potential for the inflaton of the type

$$V(\varphi) = \frac{1}{2}m_\varphi^2\varphi^2 \tag{2.53}$$

to determine the dynamics of the scalar field. Substituting in (6), and not neglecting this time the gradient term, the dynamical solution for the inflaton will be described by a damped harmonic oscillator $\varphi(t) = \phi(t)\sin(m_\varphi t)$, with ϕ the amplitude of the damped oscillation in time [85].
From the Lagrangian (2.51) it is possible, via Lagrange's equation, to describe the equation of motion relative to the secondary field

$$\ddot{\chi} + 3H\dot{\chi} - \frac{1}{a^2}\nabla^2\chi + g^2\varphi^2\chi = 0. \tag{2.54}$$

so, switching to Fourier space, and with a simple change of variables [85]

$$q = \frac{g^2\phi^2}{4m_\varphi^2}, \quad A_k = \tfrac{k^2}{m_\varphi^2} + \tfrac{g^2\phi^2}{2m_\varphi^2}, \quad z = m_\varphi t \tag{2.55}$$

the equation for each individual mode χ_k becomes a Mathieu equation

$$\frac{d^2\chi_k}{dz^2} + [A_k - 2q\cos 2z]\,\chi_k = 0. \tag{2.56}$$

The solution to this Mathieu equation is given by the following combination

$$\chi_k(z) = f_+(z)\exp\mu_k z + f_-(z)\exp -\mu_k z, \tag{2.57}$$

where $f_\pm$ are oscillating periodic functions and μ_k is a complex number, which depends on both the wavelength k and the parameters of the system, including A_k and q.
If μ_k has an imaginary part then the solution χ_k exhibits an exponential growth (increasing the energy-momentum tensor, hence the source, producing an induced background Gws). Thus for each k it is necessary to calculate A_k and q and establish the band of frequencies associated with stable modes, and those for which there is parametric instability. In order to distinguish these two regimes q must be studied, since if larger than unity, it quantifies the class of growing modes. The amplitude explosion of the secondary field modes can be interpreted as a rapid and violent production of bosonic particles, since the number density of particles is proportional to the energy mode.
Preheating ends when the exponential growth becomes energetically disadvantageous, i.e. when the energy density of the created secondary field begins to equal the energy density of the oscillating field, reasoning that trivially follows from the conservation of energy. The system produced, given the celerity of the process, is certainly not thermalized, so the pumped bosonic modes will interact with each other dissipating heat and energy, leading to the thermalization of the final system.
If a potential of the type is choosen

$$V(\varphi) = \frac{\lambda_c}{4}\varphi^4 \tag{2.58}$$

with φ at null mass, the inflaton at the end of the accelerated expansion phase would not be dynamically described by a damped sinusoid, but by an elliptical cosine [86], [87]. However, all the considerations made above apply almost identically. However, it is worth mentioning how, beyond the assumptions made, one is ignoring the back-reaction capacity of the exponentially enlarged secondary field on the oscillating inflaton, a phenomenon that can complicate the analysis in question [86].

In fact, as soon as the bump-modes begin to be produced, the presence of a bubble production phase [88] must be mentioned. At a certain point, the oscillations of the secondary field χ over time become non-linear (favouring exponential rather than oscillating growth), so that these bumped modes counter-react on the oscillating inflaton field, leading it to modify its oscillations overall, making them larger and of variable amplitude and frequency. Thus, the overall profile of the first scalar field $\varphi(t, \mathbf{x})$, can be seen as the sum of a still oscillating background part and a new inhomogeneity term induced by the back-reaction in the coupling between fields. This new term present peaks in same place of the peaks of the secondary field: these peaks are called bubbles. When the height of these peaks becomes comparable with the central oscillating background value induced by the oscillation itself, the bubbles begin to grow, expand, then collide with each other. This turbulent collision phase leads to a subsequent thermal equilibrium phase. These bubbles, in the short production and collision phase, act as the source of the induced GWs background.

2.4.2 Tachyonic Preheating

The simplest way to describe a tachyonic preheating phenomenon [81], [82], [83], [84] is to consider a usual scalar field ϕ starting from the maximum of its potential $V(\phi)$, and then rolling towards the minimum of the same potential, starting to oscillate around it. In fact, near the maximum of the potential, where the second derivative of the potential by definition is negative, the quadratic mass of the field (which is precisely the second derivative of V) becomes negative, and this condition induces an exponential growth on the modes of first-order fluctuations of the field. In fact, the dynamic equation for such modes is written as follows:

$$\ddot{\phi}_k(t) + E_k^2(t)\phi_k(t) = 0, \tag{2.59}$$

with

$$E_k^2(t) = k^2 + m^2(t), \tag{2.60}$$

where m is the mass of the field, i.e. $m^2(t) = V_{\phi\phi}$.
When the scalar field is close to the maximum, as stated above, the second derivative of the potential is negative, so the quadratic mass of the field will also be negative, thus E_k^2 may become negative; substituting this condition into the dynamic equation of the first-order fluctuations we find a solution for them which predicts a strong exponential growth. The value of ϕ for which there is a turn in the potential, i.e. the inflection point for which there is cancellation of the quadratic mass of the field, establishes the moment at which this function returns definitively positive together with the quadratic energy, thus marking the end of tachyonic preheating. Subsequently, the scalar field rolls towards the minimum oscillating on it, tending to rise above the inflection point and to originate a successive phases of Tachyonic

Preheating: the entire process is only concluded when this rising condition is denied by the damping of the overall amplitude of the oscillation, followed by the decay of the field. Such a mechanism is expected after an hybrid inflation phase: the advantage of these models lies in the fact that they can take place on a large variable energy scale, as opposed to large field models; moreover, hybrid inflation does not require small couplings to explain the observed CMB anisotropies, contrary to the New Inflation. Such models therefore lead to the production of an induced GWs background at frequencies and amplitudes accessible to current observational tools. In the case in question, the field that starts to fall from the maximum of the potential and then oscillates around its minimum is a secondary scalar field called the waterfall field σ. Due to the spinoidal instability, some fluctuations of the secondary field will gain a strong exponential growth trend, producing, as in the previous case, bubbles [89], [90], whose interaction leads to a subsequent thermalization of the Universe as well as to the definition of an anisotropic scalar source term devolved to induce a GWs background. Consider the model described by the following potential:

$$\mathcal{V}(\varphi, \sigma) = \frac{1}{4}\lambda_t(\sigma^2 - v^2)^2 + \frac{1}{2}g^2\varphi^2\sigma^2 + V(\varphi), \tag{2.61}$$

where $|\sigma|^2 = \sigma_1^2 + \sigma_2^2$ with σ_1 and σ_2 the two scalar field, is the waterfall field, while φ describe the inflaton field.
The minimum energy of the system is allocated at the points $\sigma = \pm v$ and $\varphi = 0$. The critical point at which the potential curvature changes is given by $\varphi_c = v\sqrt{\lambda_t/g}$. When $\varphi > \varphi_c$ the inflaton decreases rolling the potential where $\sigma = 0$ and the masses are quadratically positive. The inflaton ends when φ reaches φ_c, or at any time when the attached Slow-Roll conditions are broken. When the critical point is reached, the curvature of the effective potential with respect to the secondary field becomes negative (as can be seen in the figure), so the waterfall field acquires, as seen above, a negative quadratic mass inducing an exponential growth on the fluctuation modes; subsequently, the field rolls towards its minima.

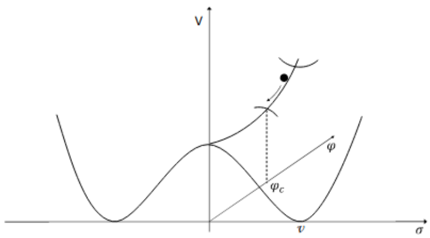

Figure 2.1: Typical potential for the hybrid inflation [1].

When σ reaches the maximum of its effective potential, the modes of the field fluctuations grow exponentially, leading to an overproduction of the field itself with a consequent inhomogenization of the homogeneous energy field. Identical to the previous model, a sequence of events can be provided: the σ fluctuations grow exponentially, formation and collision of bubbles associated with the fluctuation peaks, turbulent regime and thermalization, with the interaction of bubbles originating the source of the GWs background [91].

2.5 GWs from inflation in Modified Gravity

2.5.1 Why Modified Gravity?

The idea to move away from a standard gravitational theory offered by the formalism of General Relativity, thus arriving at a treatment of Modified Gravity (MG), arises from a growing desire to solve problems concerning the fundamental physics of interactions and concerning cosmological observables. In fact, it is well known that using GR as the theory for studying gravity [92], a complete quantum description of metric space-time and fundamental gravitational interactions is not possible, i.e. a quantum theory of gravity is missing. Moreover, when one tries to unify all possible interactions, (via superstring or supergravity approaches), one runs into the modelling of effective actions with minimal couplings that somehow bear problems of a geometric nature, or related to the presence of higher order terms associated with the presence of curvature invariants. Reasoning in the cosmological sphere instead [93], it is simple to note how the combination of GR with the Standard Model fails when try to resolve some serious inconsistencies of the Standard Hot Big Bang Model, such as the problems of flatness, horizon or magnetic monopole. Therefore it is necessary to introduce, in order to interpret the primordial expansion of the universe during inflation, always in the GR modelling, the presence of a scalar field with zero spin not contemplated by the SM. In addition, recent cosmological observations claim that even at this time the universe is experiencing an accelerated expansion phase, and in order to dynamically explain this process in the GR, a cosmological constant to represent dark energy had to be forced into Einstein's cosmological equation. However, this idea fails when try to explain certain cosmological inconsistencies, including the coincidence problem.
It is therefore sufficiently clear that GR cannot efficiently explain the accelerated expansion mechanism of the universe in its criticality regimes.
The simplest and most intuitive way to try to overcome these problems is to extend Einstein's theory with corrections and modifications, which must then be able to reproduce the same GR results in the observable ranges. The simplest way to define such extension terms is to add higher order curvature invariants and minimally or non-minimally coupled scalar fields. The greatness of the MG consists in its simplistic ability to describe the primordial accelerated expansion of the universe without the presence of a scalar field external to the gravitational sector described by the metric tensor, but rather directly due to the gravitational sector itself.
In such a scenario we will use only first-order GWs originating from quantum vacuum fluctuations of the metric tensor, appropriately treated in the first chapter: here the difference lies precisely in a different gravitational approach, of MG precisely, and it will be interesting there to see the new tensor power spectra produced with different specific features that will characterise them with respect to the tensor

spectra obtained in GR.

2.5.2 Overview of the main models

When we think the inflation on a gravitational theory of MG, we expect however the studied GWs to be produced via a quantum mechanism associated with the GR models; however such GWs could acquire several features, such as a propagation velocity different from c $c_T \neq c$ and a possibly non-negligible mass m_T : in general one imagines that GWs are subject to a dynamical equation of motion significantly different from that frequently seen, therefore one can expect a different tensor mode solution, hence a different observable tensor GWs power spectrum.
In particular several works have been advanced to consider the effects that these new and modified tensorial modes may have on the CMB, considering the features of different propagation velocity c_T [94], [95], [96], [97], [98], [99], different mass m_T [100], [101] or with a non-standard friction term [95], [96], [97] in the dynamic equation of the graviton. [102] studied the relationship between c_T and the NG of the primordial tensor perturbations in an inflationary approach of EFTI.
There are a wide variety of MG models that link so well with an inflationary dynamics framework; it is possible to distinguish all these models into three broad macro-categories

- Theories involving extra scalar fields in the gravitational metric tensor:
  in GR the gravitational degrees of freedom are entirely contained in the metric tensor, which is a tensor of rank 2. However, in order to describe a matter-gravity coupling, no one denies us the possibility of adding within the metric tensor the presence of a scalar field, the effect of which obviously must vanish on the scales in which GR correctly work. However, vector or tensor fields can also be added within the metric tensor. However, between all these theories, the most interesting and productive is the scalar-tensor theory, which will be discussed shortly.

- Theories in which action terms are introduced relative to higher-order derivatives. By means of the GR theory inscribed in the definition of the metric tensor, one arrives to the definition of the equation of motion for the various fields studied that at most reach the second order of derivation; one way to go beyond GR theory would be to ensure the presence of higher orders of derivation in the field equations. Such an approach could still lead to the presence of instability in the theory, however there are several theories that start from a Lagrangian with higher orders of derivation which (through games relating to appropriate symmetry breaking) eventually involve only

dynamic field equations (scalar and tensorial) at the second order of derivation to which we therefore associate a stable dynamics.
The most trivial way around this practice is to replace the curvature invariants with functions of the latter: we therefore speak of the $f(R)$ theories [103]. As an example, it is well known that an R^2 term leads to an equation of motion of fourth order derivation: correcting the curvature term implies adding an additional scalar degree of freedom to the system. Specifically theories in which R is replaced by $R + R^2$ are the so-called scalar-tensor theories.

- Theories built on higher-dimensional spaces.

2.5.3 Primordial GWs in the Scalar-Tensor theories of gravity

Such theories, as mentioned above, add a scalar degree of freedom in the gravitational sector appropriately described by $g_{\mu\nu}$. The simplest Lagrangian density suitable to describe such a formalism that includes an extra scalar degree of freedom is the following [104], [105], [106]:

$$\mathcal{L} = \frac{1}{16\pi G}\sqrt{-g}\left[f(\Phi)R - g(\Phi)\nabla_\mu\Phi\nabla^\mu\Phi - 2\Lambda(\Phi)\right], \tag{2.62}$$

where f, g, h, Λ are arbitrary functions of the additional scalar degree of freedom Φ. This Lagrangian is so general that it describes several models including the Brans-Dicke [107] model, which constitutes the first MG model to replace GR.
In the last period, these models have been carefully modified taking into account strong considerations on the general symmetry imposed on the system. The basic idea is to define Lagrangians with higher derivation terms, which, however, since they do not satisfy the imposed symmetry requirements, are eliminated when drawing up the equation of motion; therefore, from a $\mathcal{L}$ constructed with high derivation terms we arrive at an equation of motion for fields at the second-order. These models are based on Galilean symmetry [108], i.e. the invariance on a Minkowski space-time under the Galilean field transformation $\Phi \to \Phi + b_\mu x^\mu + c$, where c is a constant and b_μ is a constant vector. Subsequent models [109], [110], [111] extended this symmetry developing it on a dynamical background, resulting in the Covariant Galileon Inflationary model. Subsequently this theory was further generalized to the more general scalar-tensor theory leading to equations of motion for fields on an expanding curved space-time, called Generalized G-Inflation. This model is fully equivalent to the Lagrangian proposed by Horndeski [112], which is proposed to be the most general scalar-tensor gravitational theory leading to an equation of motion for fields of the second derivative order starting from a Lagrangian with higher derivative order (with respect to the second).

Generalized G-Inflation

In 1974, Horndeski presented a paper [112] in which the most general Lagrangian of a scalar-tensor theory defined in a four-dimensional space-time is presented, which although contain terms with order of development greater than the second, nevertheless generates an equation of motion for the associated fields at the second-order, without any kind of dynamical instabilities. This is written as follows [111], [112]

$$S = \sum_{i=2}^{5} \int d^4x \sqrt{-g} \mathcal{L}_i, \tag{2.63}$$

with

$$\mathcal{L}_2 = K(\Phi, X), \mathcal{L}_3 = -G_3(\Phi, X)\Box\Phi, \tag{2.64}$$

$$\mathcal{L}_4 = G_4(\Phi, X)R + G_{4X}\left[(\Box\Phi)^2 - (\nabla_\mu\nabla_\nu\Phi)^2\right], \tag{2.65}$$

$$\mathcal{L}_5 = G_5(\Phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\Phi - \frac{G_{5X}}{6}\left[(\Box\Phi)^3 - 3(\Box\Phi)(\nabla_\mu\nabla_\nu\Phi)^2 + 2(\nabla_\mu\nabla_\nu\Phi)^3\right], \tag{2.66}$$

where $G_{\mu\nu}$ is the Einstein tensor, $G_{iX} = \frac{\partial G_i}{\partial X}$, and K e G_i are generic function of the scalar degree Φ, while $X = -\partial_\mu\Phi\partial^\mu\Phi/2$.
It should be noted that the system is described by four arbitrary functions of Φ and X, and that the addition of the default Hilbert-Einstein term is not required. It is important to see how the system is described only by the gravitational sector, since no external scalar matter field is introduced. The action just described is able to describe a variety of models, including single-field slow-roll inflation, k inflation [113], Higgs G inflation [114], and Galileon inflation.
It is possible to observe how this action [115] can dynamically explain the primordial accelerated expansion phase of the universe, favouring the development of the known Hot Big Bang Model, which, once the accelerated expansion phase is over, would like a precise domination in radiation.
In order to describe the equations of motion in the background of the scalar field, a unitary gauge is assumed for which $\Phi = \Phi(t)$, and the metric is of the form $ds^2 = -dt^2 + a^2(t)dx^2$ and we have to substitute it into the action.
In order to calculate the GWs tensor power spectrum instead, one needs a perturbative theory beyond the background, so in the usual unitary gauge one considers a metric of the form

$$ds^2 = -N^2dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{2.67}$$

where

$$N = 1 + \alpha, \quad N_i = \partial_i\beta, \quad \gamma_{ij} = a^2(t)\exp(2\zeta)\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj}\right), \tag{2.68}$$

with α, β e ζ scalar perturbations, while h_{ij} is the transverse-tracefree tensor which specify the GWs.
Perturbing the overall action to the second order (necessary to calculate the two-point correlation function), and leaving only the tensor degrees of freedom suitable for describing gravitational modes, we write [115]

$$S_T^{(2)} = \frac{M_{Pl}^2}{8} \int dt d^3 x a^3(t) \left[\mathcal{G}_T \dot{h}_{ij} \dot{h}_{ij} - \frac{\mathcal{F}_T}{a^2(t)} (\nabla h_{ij})^2 \right], \tag{2.69}$$

where

$$\mathcal{F}_T = \frac{2}{M_{Pl}^2} \left[G_4 - X(\ddot{\Phi} G_{5X} + G5\Phi) \right], \tag{2.70}$$

$$\mathcal{G}_T = \frac{2}{M_{Pl}^2} \left[G_4 - 2X G_{4X} - X(H\dot{\Phi} G_{5X} - G_{5\Phi}) \right]. \tag{2.71}$$

It is trivial to observe that this action is entirely identical to the classical second-order action seen in chapter one describing freely propagating gravitational waves, however there is a substantial difference in the multiplicative coefficients. Specifically, these new multiplicative amplitudes will bring the GWs background of the first type to a different propagation velocity than the usual c, i.e. $c_T^2 = \mathcal{F}/\mathcal{G}$, and will change the amplitude of the GWs background with respect to the standard GR case. In order to avoid instability, it is required that $\mathcal{F}, \mathcal{L} > 0$ [115], while to simplify the model appropriately we write

$$\epsilon = -\frac{\dot{H}}{H^2} \simeq const, \quad f_T = \frac{\dot{\mathcal{F}}_T}{H\mathcal{F}_T} \simeq const, \quad g_T = \frac{\dot{\mathcal{G}}_T}{H\mathcal{G}_T} \simeq const. \tag{2.72}$$

We now perform a shift in Fourier space, and a rescaling of h_{ij} and time t in order to obtain an equation of motion quite similar to the standard GR equation [115]

$$dy_T = \frac{c_T}{a} dt, \quad z_T = \tfrac{a}{2} M_{Pl} (\mathcal{F}\mathcal{G})^{\frac{1}{4}}, \quad v_{ij} = z_t h_{ij}. \tag{2.73}$$

These rescaling operations are analogous to those provided in the first chapter, defining an appropriate one this time also on the time scale in order to put ourselves in a convenient reference frame where $c = c_T$. Following these useful transformations, the action is rewritten

$$S_T^{(2)} = \frac{M_{Pl}^2}{2} \int dy_T d^3 x \left[(v_{ij}^{'})^2 + \frac{z_T^{''}}{z_T} v_{ij}^2 - (\nabla v_{ij})^2 \right], \tag{2.74}$$

where the prime derivative is placed with respect to the new time variable. Deriving the action with respect to the tensor modes as done in chapter one, we find the standard free wave motion equation, of which we already know the two solutions:

on superhorizon scales one will decay, while the other of observational interest will remain frozen by defining the constant h_{ij}. Being interested in the latter solution, once the usual canonical quantization is performed, we define the frozen superhorizon solution for the tensor mode GWs

$$k^{\frac{3}{2}} h_{ij} \simeq 2^{\nu_T - 2} \frac{\Gamma(\nu_T)}{\Gamma(3/2)} \frac{(-y_T)^{1/2-\nu_T}}{z_T} k^{3/2-\nu_T} e_{ij}, \tag{2.75}$$

from which we can derive the observable of the tensor power spectrum scale invariant [115]

$$P_T = \frac{8\gamma_T}{M_{Pl}^2} \frac{\mathcal{G}_T^{1/2}}{\mathcal{F}_T^{3/2}} \left(\frac{H}{2\pi}\right)^2, \tag{2.76}$$

with a tensor spectral index $n_T = 3 - 2\nu_T$.
A first difference to the tensor power spectrum of a single-field slow roll model in GR is in the description of the amplitude, which suffers from the fact that here the GWs are not calculated from the geometric degrees of freedom alone, but also from the added extra scalar degrees of freedom. The system is thus affected by an unusual starting Lagrangian density. Furthermore, here, in contrast to the GR case, there is a spectral index that can be blue-tilted.

Potential-Driven G-Inflaton

It is interesting to investigate a specific instance of the general model just shown, where only the first two Lagrangian terms of (2.63) are taken into account [115]; the basic idea remains the same: to start from a generic Lagrangian density with high degrees of derivation embedded in some symmetry, and then conclude with a general equation of motion for the extra scalar field and the tensor GWs metric perturbation at the right order. Such a model is interesting not so much for the features related to the tensor power spectrum (one can anticipate it to be the same as a single-field slow-roll inflationary theory in GR), but rather because it is able to state a different consistency equation between the tensor-to-scalar-ratio and the tensor spectral tilt, which can thus be compared with the standard GR one found. The action is rewritten as follows [115]:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{Pl}^2 R + \mathcal{L}_\Phi\right], \tag{2.77}$$

with

$$\mathcal{L}_\Phi = K(\Phi, X) - G(\Phi, X)\Box\Phi, \tag{2.78}$$

where K and G are the generic function of the extra scalar field Φ and $X = \nabla_\mu \Phi \nabla^\mu \Phi/2$.

It is important to note that from $\mathcal{L}_\Phi$ one can derive the known Einstein-Hilbert term thanks to an appropriate choice of G_4. For the model, the following choice is imposed in order to derive an appropriate consistency relation

$$K(\Phi, X) = X - V(\Phi), \quad G(\Phi, X) = -g(\Phi)X. \tag{2.79}$$

One of the most resonant models written with such analytical choice is the Higgs G-Inflation model, the purpose of which is to explain the primordial accelerated expansion of the universe using only Standard Model particles.
As far as the evolutionary study of the scalar field background is concerned, it is worth mentioning how, under the construction of slow-roll inflation, an accelerated primordial expansion can be achieved. To achieve this construction we require [115]

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = -\frac{\ddot{\Phi}}{H\dot{\Phi}}, \quad \alpha = \frac{g_\Phi \dot{\Phi}}{gH}, \beta = \frac{g_{\Phi\Phi}X^2}{V_\Phi}, \tag{2.80}$$

with all these parameters in modulus taken much less than one under the slow-roll assumptions. One chooses the regime in which the Galileon effect is dominant, i.e. $\left|gH\dot{\Phi}\right| >> 1$, so that the equation of motion of the background in slow-roll becomes $gH^2\dot{\Phi}^2 + V_\Phi \simeq 0$ [115], the solution of which can be interpreted as follows

$$\dot{\Phi} = -sgn(g)M_{Pl}\left(\frac{V_\Phi}{3gV}\right)^{1/2}, \tag{2.81}$$

with the obvious requirement that the field rolls slowly over the flat potential. The solution is one of constant velocity, thus in line with a slow-roll model in GR. However here an additional factor is found to further decrease the constant velocity by a term of $1/\sqrt{gV_\Phi}$.
As far as the calculation of the tensor power spectrum is concerned, it is sufficient to simply take the general observational solution found previously (2.76) and substitute it with the settings of the studied model; we will obtain

$$P_T(k) = \frac{8}{M_{Pl}^2}\left(\frac{H^2}{2\pi}\right)^2, \quad n_T = -2\epsilon. \tag{2.82}$$

There is a complete lack of features compared to the single-field slow-roll GR model in vacuum fluctuations of the metric tensor, so much to induce a degenerative character for the diffrent kind of GWs . However, one main difference between the cases is again the tensor spectral tilt, which can blue-tilt if the quadratic sound speed of the scalar fluctuations associated with the extra first-order scalar degree of freedom is positive.

In order to obtain the tensor-to-scalar-ratio, however, it is also necessary to define the scalar power spectrum associated with the curvature perturbation relative to the fluctuation not of the inflaton, but of the new intrinsic scalar degree of freedom always in the gravitational sector, hence in the metric tensor; from the model, at the Horizon crossing where the curvature perturbation is defined, we obtain

$$P_\zeta = \left(\frac{3\sqrt{6}}{64\pi^2}\right)^2 \frac{H^2}{M_{Pl}^2\epsilon}. \tag{2.83}$$

One can observe a clear distinction with respect to the multiplicative amplitude coefficient: this observation allows one to understand how r therefore changes the consistency relation.

GW PRODUCTION	**Discriminant**	**Specific discriminant**	**Examples of specific models**	**Produced GW**
Vacuum oscillations quantum fluctuations of the gravitational field stretched by the accelerated expansion	**theory of gravity**	General Relativity	*single-field slow-roll*	*broad spectrum*
			all other models in GR	broad spectrum
		MG/EFT approach	G-Inflation	broad spectrum
			Potential-driven G-Inflation	broad spectrum
			EFT approach	broad spectrum
Classical production second-order GW generated by the presence of a source term in GW equation of motion	**source term**	vacuum inflaton fluctuations	*all models*	*broad spectrum*
		fluctuations of extra scalar fields	inflaton+spectator fields	broad spectrum
			curvaton	broad spectrum
		gauge particle production	pseudoscalar inflaton+gauge field	broad spectrum
			scalar infl.+pseudoscalar+gauge	broad spectrum
		scalar particle production	scalar inflaton+ scalar field	peaked
		particle production during preheating	chaotic inflation	peaked
			hybrid inflation	peaked

Figure 2.2: Table summarising the main GWs production mechanisms studied so far, relating to the inflationary and subsequent reheating period [1].

2.6 Consistency relation and possible violations

The measure of the primordial gravitational waves treated so far would provide a vast understanding of the fundamental physics of the primordial universe, and could allow for the testing of the consistency relation, which, if experimentally ascertained, would confirm the existence of an inflationary period. In the first chapter we studied how, from an inflationary slow-roll model in a de-Sitter background in single field GR, there exists a very specific consistency relation between the tensor-to-scalar-ratio and the spectral tensor tilt, at the first order in the development of the slow-roll parameters and for each k perturbation scale:

$$r = -8n_T, \tag{2.84}$$

where in general these two parameters are usually scale-dependent. A similar relationship connecting tensorial and scalar features also exists in the characterization of the running spectral tensor index, indeed [15]

$$\frac{dn_T}{d\ln k} \simeq \frac{r}{8}\left[\frac{r}{8} + (n_S - 1)\right]. \tag{2.85}$$

Nevertheless, it is legitimate to think that such tensor running is suppressed in slow-roll, so it is useful to think that the tensor spectral tilt is described only by n_T.

The expression (2.84) is related to the model that formulates it, and binds two potential observables. The experimental link that can be constructed between them determines the winner among all theoretical models that set out to study inflationary dynamics, predicting different modifications of the standard consistency relation. It is important to remember that the standard connectivity between the two parameters is offered by the fact that both are writable in the terms of ϵ, which offers a scale for measuring energy during inflation. Therefore, (as already seen), it is possible to infer how the knowledge of r, or n_T, allows us to quantify the energy status during accelerated expansion. Nevertheless, a deviation from this consistency relation would allow the loss of this connection, so knowledge of either observational tenor would not guarantee the energy knowledge.

The choice of rewarding different inflationary models (including those exposed so far), would lead to an intrinsically different choice of non-standard tensor and scalar power spectrum, leading to different values of r and n_T, thus a violation of consistency. For example, some proposed inflationary examples evaluated the possibility of a blue tensor tilt, obviously an assumption in total disagreement with (2.84), which, associated with a model of GWs production by vacuum fluctuation, predicted red-tilt relations on the various observable spectra. However, it is worth mentioning how such models of extra particle production, or of MG also predict

a different amplitude approach for the tensor power spectrum, destroying the standard connection created between the latter and the energy of the inflationary system.
For the sake of completeness, a table summarizing all the main models with their modifications of consistency laws is left on .

	Model	Tensor power-spectrum	Tensor spectral index		Consistency relation
Background	Standard infl.	$P_{\rm T}=\frac{8}{M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2$	$n_{\rm T}=-2\epsilon$	red	$r=-8n_{\rm T}$
	EFT inflation$^{(a)}$	$P_{\rm T}=\frac{8}{c_{\rm T}M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2$	$n_{\rm T}=-2\epsilon+\frac{2}{3}\frac{m_{\rm T}^2}{\alpha H^2}\left(1+\frac{4}{3}\epsilon\right)$	r/b	-
	EFT inflation$^{(b)}$	$P_{\rm T}=\frac{8}{c_{\rm T}M_{\rm pl}^2}\frac{2^{\frac{-p}{1+p}}}{\pi}\Gamma^2\left(\frac{1}{2(1+p)}\right)\left(\frac{H}{2\pi}\right)^2$	$n_{\rm T}=\frac{p}{1+p}$	blue	violation
	Gen. G-Infl.	$P_{\rm T}=\frac{8}{M_{\rm pl}^2}\gamma_{\rm T}\frac{\mathcal{G}_{\rm T}^{1/2}}{\mathcal{F}_{\rm T}^{3/2}}\left(\frac{H}{2\pi}\right)^2$	$n_{\rm T}=3-2\nu_{\rm T}$	r/b	-
	Pot.-driv. G-Infl.	$P_{\rm T}=\frac{8}{M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2$	$n_{\rm T}=-2\epsilon$	r/b	$r\simeq-\frac{32\sqrt{6}}{9}n_{\rm T}$
Extra background	Particle prod.	$P_{\rm T}^{+}=8.6\times10^{-7}\frac{4H^2}{M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2\frac{e^{4\pi\xi}}{\xi^6}$	-	blue	violation
	Spectator field	$P_{\rm T}\simeq3\frac{H^4}{c_S^{18/5}M_{\rm pl}^4}$	$n_{\rm T}\simeq2\left(\frac{2m^2}{3H^2}-2\epsilon\right)-\frac{18}{5}\frac{\dot{c}_S}{Hc_S}$	r/b	violation

Figure 2.3: Summary table for the main GWs production models of the overall tensor power spectrum amplitude and spectral index. The last column shows for each model the associated consistency relation [1].

2.7 GWs as source of information for the thermal history of the Universe

The gravitational waves treated so far are produced during the inflationary era and initially have a mode width such that they persist in the sub-horizon, inflation itself enlarging these dimensions leads to the freezing of the modes, which, at the end of inflation (and therefore during a subsequent domination phase) will re-enter the causally connected regime. It has already been analyzed how, on re-entry, they return to oscillate in a dumped way, and this dump is described precisely by the scale factor indicating the cosmological expansion factor. Hence the present abundance of the GWs background will be observed, and will reflect the expansion history of the universe [116]. Therefore, measuring such a background of inflationary GWs would allow the possibility of tracing the history of the thermal evolution of the universe, including the reheating phase.

GWs transfer function

In the usual FLRW spacetime, tensor modes obey the free waves equation if there are no sources of the second type. During inflation the tensor modes are stretched into superhorizon scales where they remain frozen in time; however, these will sooner or later re-enter the Hubble regime, so they will again vary in time subject to causal interaction influences: this temporal variation from a regime of constant nature is analytically expressed in terms of a temporal transfer function

$$h_k(\tau) = h_{k,prim} T_h(\tau, k). \tag{2.86}$$

T_h is the transfer function describing the evolution of the tensor mode GWs once it re-enters the horizon at the end of inflation. It is appropriately normalized such that $T_h(\tau, k) \to 1$ when $k \to 0$ (the obvious superhorizon condition for which the mode remains frozen in time at its primordial value).
We define

$$\Delta^2_{h,prim}(k) = \frac{d\langle h_{ij} h^{ij} \rangle}{d \ln k}, \tag{2.87}$$

therefore linking the equations (2.86) with the (1.66) we can obtain the spectral energy density for the GWs background

$$\Omega_{GW}(k, \tau) = \frac{1}{12} \left(\frac{1}{aH} \right)^2 \Delta^2_{h,prim}(k) T'^2_h(k, \tau). \tag{2.88}$$

For modes within the horizon, this formula can be re-approximated as follows [116]

$$\Omega_{GW}(k, \tau) = \frac{1}{12} \left(\frac{k}{aH} \right)^2 \Delta^2_{h,prim}(k) T^2_h(k, \tau). \tag{2.89}$$

It is known that the primordial tensor power spectrum in a slow-roll single field inflationary model can be written in terms of the slow-roll parameters and the Hubble parameter during inflation as follows

$$\Delta^2_{h,prim}(k) = 64\pi G\left(\frac{H^2}{2\pi}\right)\left[1 - 2\epsilon\ln\frac{k}{k_*} + 2\epsilon\left(\eta - \epsilon\right)\left(\ln\frac{k}{k_*}\right)^2\right]. \tag{2.90}$$

Solving (1.46) during radiation or matter domination, on subhorizon scales, we find, as already anticipated, that the amplitude of the tensorial mode solution obviously depends on k and is modulated by the inverse of the scaling factor (which in turn depends on time), while the oscillatory trend is guaranteed by the Bessel solutions. The dumping factor is the protagonist of this section. It should be pointed that each tensor mode has a dimension k, therefore the present GWs background consists of a superposition of waves that re-enter the horizon at times of different historical evolution: in particular different classes of k will undergo different dumping dependent on a scaling factor that presents distinct temporal evolutions according to the domination phase in question.
Within the matter domination, the solution of (1.46) results:

$$h_k(\tau) = h_{k,prim}\left(\frac{3j_1(k\tau)}{k\tau}\right). \tag{2.91}$$

This solution has to be averaged over time, so that the dumping amplitude $1/a$ can be extracted. Therefore, the GWs spectrum looks like this

$$\Omega_{GW}(k,\tau_0) = \frac{1}{12}\left(\frac{k}{aH}\right)^2\Delta^2_{h,prim}(k)\left(\frac{3\bar{j_1}(k\tau)}{k\tau_0}\right)^2(...)\,, \tag{2.92}$$

where the last factor defines all the terms associated with the change in the scaling factor from the re-entry of the mode in the horizon to the present observation for each mode k: a first dumping factor comes from the change in the relativistic degrees of freedom going from one domain phase to the next. Another function takes into account the various temporal transitions of the dumped scaling factor associated with the GWs amplitude when it passes from one domain phase to another (from reheating to radiation, and from this to matter). To sum up, the entire transfer function can be rewritten

$$T_h^2(k) = \Omega_m^2\left(\frac{g_*(T_{in})}{g_{*s}}\right)\left(\frac{g_{*s0}}{g_{*s}(T_{in})}\right)^{4/3}\left(\frac{3\bar{j_1}(k\tau)}{k\tau_0}\right)^2 T_1^2(x_{eq})T_2^2(x_R). \tag{2.93}$$

This expression suggests how, once the various degrees of freedom are defined in the various epochs and Ω_m, the GWs spectrum turns out to be a function of only r and the reheating temperature T_R.

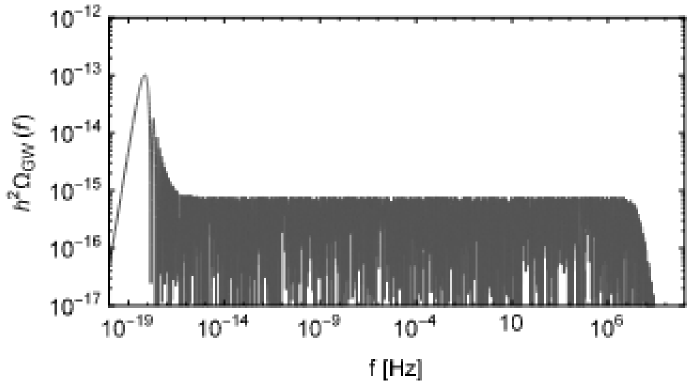

Figure 2.4: Actual GWs spectral energy density [1].

2.7.1 Equation of state of the Universe and spectral tilt

From a direct observation of the present GWs spectral background, it is possible to reconstruct the evolution of the equation of state [117], [118] that characterized the universe in its primordial phase. Assuming, as in the standard inflationary frame, that the primordial tensor power spectrum is scale invariant, then the dependence of the present GWs spectrum on frequency is completely inscribed in the transfer function, so that $\Omega_{GW} \propto k^2 T_T^2(k)$. Recalling that the tensor modes return to vary only into the Hubble sphere, oscillating and decaying due to a dumping factor of $1/a$, it can be guessed that $T_T(k) = |h_{k,0}| / |h_{k,prim}| = (a_0/a_{in})^{-1}$, hence that $\Omega_{GW} \propto k^2 a_{in}^2$.
If it is possible to write the equation of state describing the dynamics of the universe at the 'in' when the tensor mode cross on subhorizon scales as $\omega = p/\rho$, then the Hubble rate is $H^2 \propto a^{-3(1+\omega)}$. Therefore for the mode of width k, which falls within the horizon and for which therefore $k = aH$ we obtain in conclusion $a_{in} \propto k^{-2/(1+3\omega)}$. If we assume a successive adiabatic evolution of the universe, for each mode that enters the horizon when the universe is described by ω, we get

$$\Omega_{GW}(f) = \Omega_{gw,F}(f/F)^{[2(3\omega-1)/(1+3\omega)]}. \tag{2.94}$$

Thus it is trivial to observe that for modes re-entering during the matter domination $\Omega \propto f^{-2}$, while for modes that re-enter during the radiation domination one obtains $\Omega \propto f^0$. Thus a change in domination in the universe will be seen in the final present spectrum as a change in slope at the scale of the modes that cross the horizon at that time. Therefore from the slope analysis it is possible to infer the equation of state that the universe had at the time the frequencies of the modes associated with that slope were crossing the horizon, also identifying the exact moment of switch of the different phases of dominations, where there is a net change of slope [117].

Chapter 3

Scalar induced Gravitational waves in the Post-Inflationary epoch

This chapter is based on the reviews [119], [120], [121], [122], [123] . The aim of this chapter consists in the description of the state of art of induced gravitational waves from primordial scalar fluctuations, making explicit the general analytical formulation of the problem. The key idea of this section consists in imagining a different evolution of the primordial universe with respect to the standard one. This idea makes it possible to generalize the study of an induced GWs background to a general post-inflationary phase, which is not necessarily the radiation phase, which is the one always considered. From the generalization of the problem, it is possible to derive an observable related to the tensor power spectrum that is affected by specific features associated with the dynamics of the generic post-inflationary phase of induction. Therefore, an observation of such an induced GWs spectrum could give us an idea of the presence of an initial post-inflationary phase other than the radiation phase that is typically taken into account. We have to imagine that before the radiation-dominated phase (which must necessarily be thought within the limits of the Hot Big Bang Model) there is an earlier phase with a generic constant equation of state and a generic adiabatic speed of sound as a consequence. This adiabatic speed of sound is the adiabatic velocity related to the physics fluctuations that interact during the sub-horizon period of re-entry, and that generate the source term for the scalar induced GWs background.

3.1 General formalism

It is necessary to introduce a general formulation related to the production of such a second-order induced GWs background, thus deriving the equation of motion for the aforementioned tensor perturbations of the second order in perturbation theory; in order to obtain this result, a different approach is chosen with respect to the previous chapter: a study associated with the writing of the action formalism will follow, taking into account all contributions, including those relating to a non-Gaussian distribution of the scalar system.

Derivation from the action

This section will be mainly based on the articles [121], [54], [124] [120]. The ultimate goal of this section is to write the equation of motion for the transverse-traceless component of the second-order perturbation of the metric tensor in the spatial region, i.e. for h_{ij}. Although the physics is for a second-order perturbative cosmological system, the action formalism works well enough, if one chooses an appropriate working gauge and focuses only on the correct tensor-scalar-scalar interactions (necessary to have a second-order problem). We can decide to work with a Newton gauge using an exponential notation of Misner, Throne and Wheeler [36]. Assuming to construct the perturbation metric on a background of FLRW, one finds an overall metric of the form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\exp{(2\Psi)}dt^2 + a(t)^2 \exp{(2\Phi)} Y_{ij} dx^i dx^j, \tag{3.1}$$

where $g_{\mu\nu}$ is the space-time metric, $i = 1, 2, 3$ are the space component, $a(t)$ is the scale factor; the conformal decomposition notation of the spatial metric was used in the writing, whereby

$$\frac{\partial}{\partial t} detY = Y^{ij} \frac{\partial}{\partial t} Y_{ij} = 0. \tag{3.2}$$

Note how for a flat FLRW background in Cartesian coordinates we have that $Y_{ij} = \delta_{ij}$. The conformal decomposition of the spatial component of the metric tensor is very useful for the treatment of the system, as it splits the degrees of freedom related to the trace (thus to the overall volume change of the system) from the degrees of freedom of the null trace (related to volume preservation). In Newton's Gauge, as can be easily observed, such a decomposition splits the scalar modes from the tensorial ones in the definition of spatial metric perturbation, in fact Y_{ij} contains only transverse and trace-free tensorial degrees of freedom (h_{ij}). This decomposition is also very useful for the action in fact it leaves a clear split between Y_{ij} and Φ.
Consider, for simplicity of the system, a perturbed metric on the FLRW on which

we insert a scalar field φ, which will later be generalized to the description of a perfect fluid. The action is presented in its generality as follows:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right), \tag{3.3}$$

where g is the determinant of the tensor metric $g_{\mu\nu}$, R is the Ricci scalar 4D, $\frac{\partial}{\partial x_\mu} = \partial_\mu$ and $V(\varphi)$ is the potential for the scalar field. In the conformal decomposition (3+1), after writing several algebraic steps and integrations by part, we can rewrite the action in the following way

$$\begin{aligned} S = \int d^3x dt \Bigg[& ae^{\Psi+\Phi}\left(\frac{1}{2}R^{(3)}[Y_{ij}] - 2Y^{ij}D_iD_j\Phi - Y^{ij}D_i\Phi D_j\Phi - \frac{1}{2}Y^{ij}D_i\varphi D_j\varphi\right) \\ & +a^3e^{3\Phi-\Psi}\left(\frac{1}{8}Y^{ij}Y^{kl}\dot{Y}_{ik}\dot{Y}_{jl} - 3(H+\dot{\Phi})^2 + \frac{1}{2}\dot{\varphi}^2\right) - a^3e^{3\Phi+\Psi}V(\varphi)\Bigg], \end{aligned} \tag{3.4}$$

with $R^{(3)}[Y_{ij}]$ and D_i respectively the Ricci scalar 3D and the covariant derivative applied to the tensor mode Y_{ij}. Working in Cartesian coordinates we have that $detY = 1$. Now we should derive the action for the tensor modes Y_{ij}, knowing that we obtain a result that obviously must preserve the transverse-traceless symmetry offered by the modes, and thus obtain the equation of motion for the graviton, i.e. the transverse and traceless space part of Einstein's equations.
However, before performing the action derivation, it is useful to exploit a perturbative cosmological approach with a relative action expansion. An elegant and useful way of decomposing the spatial metric, remaining within the assumptions of the problem, consists in considering the exponential matrix

$$Y_{ij} = (e^h)_{ij} = \delta_{ij} + h_{ij} + \frac{1}{2}\delta^{kl}h_{ik}h_{jl} + O(h^3), \tag{3.5}$$

with $h_{ij} << 1$ the transverse and tracefree tensor. From now on, the spatial indices will be contracted with the spatial background metric δ_{ij}, so $\delta^{ij}\delta^{kl}h_{ik}h_{jl} = h^{ij}h_{ij}$. Instead, the inverse metric is written $Y^{ij} = \delta^{ik}\delta^{jl}(e^{-h})_{kl}$. With such an expansion, we will write the second-order 3D Ricci scalar as follows

$$R^{(3)}[e^h] = -\frac{1}{4}\partial_i h_{kl}\partial^i h^{kl} + O(h^3). \tag{3.6}$$

Obviously, the whole problem is developed to the second perturbative order with respect to the equation of motion, while we are up to the third order if we consider the problem with respect to the action. The other perturbative variables can be written as follows

$$\Psi = \Psi(t, \mathbf{x}), \quad \Phi = \Phi(t, \mathbf{x}), \quad \varphi = \bar{\varphi}(t) + \delta\varphi(t, \mathbf{x}) \tag{3.7}$$

with $\bar{\varphi}(t)$ the background solution for the scalar field Φ, Ψ e $\delta\varphi$ are instead the scalar perturbations for the system scalar field/metric tensor.
Using these expansion terms and retaining only the third-order terms (with two scalars and a tensor), we arrive at a simplified version of the action

$$\begin{aligned} S = \int d^3x dt \Bigg[\frac{a^3}{8} \dot{h}^{ij} \dot{h}^{ij} - \frac{a}{8} \partial_i h_{kl} \partial^i h^{kl} - 2a h^{ij} \partial_i (\Phi + \Psi) \partial_j \Phi + \\ + a h^{ij} \partial_i \Phi \partial_j \Phi + \frac{a}{2} h^{ij} \partial_i \delta\varphi \partial_j \delta\varphi \Bigg], \end{aligned} \tag{3.8}$$

where the first two terms are the second-order action values of the Lagrangian for the tensor modes, so the variation of these two terms will provide the LHS of the famous graviton free dynamics equation. Now varying with respect to h_{ij} we obtain the overall equation of motion at the second order of the GWs induced by scalars

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - a^{-2}\Delta h_{ij} = a^{-2}\mathcal{P}_{ij}^{ab}[-8\partial_a(\Phi + \Psi)\partial_b\Phi + 4\partial_a\Phi\partial_b\Phi + 2\partial_a\delta\varphi\partial_b\delta\varphi] \tag{3.9}$$

with $\mathcal{P}_{ij}^{ab}$ the transverse traceless projector, so the equation is consistent with such degrees of freedom, thus presenting an important analytical consistency. In the present case under study we do not assume the presence of stress-anisotropy tensor sources at the linear order of scalar perturbations, therefore we have, from the spatial component of the null-trace Einstein equations, the condition for which

$$\Psi + \Phi = 0. \tag{3.10}$$

This condition in the primordial Universe is not so true, given the presence of neutrinos with a large mean free path, the latter of which generates a non-trivial stress-anisotropy component in the energy-momentum tensor of the overall fluid [53], [125]. Before defining the general GWs solution, let us come back to the description of a scalar field described by a perfect fluid. Such a perfect fluid with energy density ρ and pressure P predicts an energy-momentum tensor of the following form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{3.11}$$

with u_μ the four-velocities of the fluid. A perfect fluid is characterised by the simple equation of state for which $\omega = P/\rho$. In the perturbative case, we have that $\rho = \bar{\rho} + \delta\rho$ and $u_i = \partial_i v$, with v the velocity perturbation; we neglect the vector modes brought to decay with the expansion of the Universe. The previous tensor valid for a perfect fluid must be properly compared with the generic tensor of a scalar field

$$T^{\varphi}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V(\varphi)\right). \tag{3.12}$$

The description of a scalar field described as a perfect fluid follows from the following comparison

$$u_\mu = \frac{\partial_\mu\varphi}{\sqrt{-\partial_\alpha\varphi\partial^\alpha\varphi}}. \tag{3.13}$$

Moving on to the spatial component u_i we have that

$$\delta\varphi \iff v\sqrt{\rho + P}. \tag{3.14}$$

In the terms of a perfect fluid, the equation of motion is very trivially rewritten as [53], [54]

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - a^{-2}\Delta h_{ij} = a^{-2}\mathcal{P}^{ab}_{ij}\left[4\partial_a\Phi\partial_b\Phi + 2(\rho + P)\partial_a v\partial_b v\right]. \tag{3.15}$$

3.1.1 General Solutions

This section is written on the basis of the articles [126] and [127] [120] [121], [122], [123]. In order to solve the conclusive equation of motion for the induced GW background, it is necessary to solve the first-order dynamics of the scalar fluctuations that generate it. From the conservation of momentum it is possible to link the scalar field fluctuation $\delta\varphi$ to the scalar curvature metric fluctuation Φ [128]

$$\delta\varphi = -\sqrt{\frac{2}{\epsilon}}\left(\Phi + \frac{\Phi'}{\mathcal{H}}\right). \tag{3.16}$$

Therefore, it is sufficient to find a single scalar solution, in function of Φ. We can define the Bardeen potential as the product of an evolutionary transfer function in time for a constant value representing the curvature perturbation originating on super-horizon scales during inflation

$$\Phi(\mathbf{k},\tau) = T_\Phi(k,\tau)\Phi_k. \tag{3.17}$$

Considering primordial fluctuations, we have that Φ_k is brought into Super-horizon scales by accelerated expansion and comes from quantum fluctuations during inflation. In general, as explored in the previous chapter, any source of fluctuation capable of generating a curvature perturbation will define a source for the induced GWs background.
Thanks to the last two equations, it is possible to write the dynamics of the induced GWs background, which in Fourier space reads [53], [54], [121], [122], [123]

$$h^{''}_{\mathbf{k},\lambda} + 2\mathbf{H}h^{'}_{\mathbf{k},\lambda} + k^2 h_{\mathbf{k},\lambda} = \mathcal{S}_{\mathbf{k},\lambda}, \tag{3.18}$$

with

$$\mathcal{S}_{\mathbf{k},\lambda} = 4\int \frac{d^3q}{(2\pi)^{3/2}} e^{ij}_\lambda(k) q_i q_j \Phi_{\mathbf{q}} \Phi_{|\mathbf{k}-\mathbf{q}|} f(\tau, q, |\mathbf{k}-\mathbf{q}|), \tag{3.19}$$

and

$$\begin{aligned} f\left(\tau, q, |\mathbf{k}-\mathbf{q}|\right) &= T_\Phi\left(q\tau\right) T_\Phi\left(|\mathbf{k}-\mathbf{q}|\,\tau\right) \\ +\frac{1+b}{2+b}\left(T_\Phi(q\tau) + \frac{T'_\Phi(q\tau)}{\mathcal{H}}\right)&\left(T_\Phi(|\mathbf{k}-\mathbf{q}|\,\tau) + \frac{T'_\Phi\left(|\mathbf{k}-\mathbf{q}|\,\tau\right)}{\mathcal{H}}\right). \end{aligned} \tag{3.20}$$

Where $b = (1-3\omega)/(1+3\omega)$. If we want to switch from equation (3.15) to equation (3.18), the Fourier transform for h_{ij} in terms of the polarization tensor $e_{ij}(k)$ was used. The solution to the Fourier dynamics expressed in (3.18) can be found through Green's method,

$$h_{\mathbf{k},\lambda}(\tau) = \int_{\tau_i}^{\tau} d\tilde{\tau} G_h(\tau,\tilde{\tau})\mathcal{S}_{\mathbf{k},\lambda}(\tilde{\tau}), \tag{3.21}$$

where G_h is the Green's function defined on the homogeneous solution of (3.18). This formal solution (3.21) is defined in such a way that at the initial time τ_i there are no solutions of GWs induced during the inflationary phase, which have to be added separately, in fact we have that $h_{\mathbf{k},\lambda}(\tau_i) = h'_{\mathbf{k},\lambda}(\tau_i) = 0$. Since we are interested in particular to the tensor power-spectrum of the induced GWs background, it is legitimate to calculate the two-point correlation function for the induced GWs, i.e.

$$\langle h_\lambda(k,\tau) h_\lambda(k',\tau)\rangle = \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 G(\tau,\tau_1) G(\tau,\tau_2) \langle \mathcal{S}_\lambda(k,\tau_1) \mathcal{S}_\lambda(k',\tau_2)\rangle. \tag{3.22}$$

From the Fourier definition of the source (and neglecting any induced input of primordial origin), the two-point contribution of the RHS term is calculated

$$\begin{aligned}\langle \mathcal{S}_\lambda(k,\tau_1) \mathcal{S}_\lambda(k',\tau_2)\rangle = 16 \int \frac{d^3q}{(2\pi)^{3/2}} \int \frac{d^3q'}{(2\pi)^{3/2}} e^{ij}_\lambda(k) q_i q_j e^{ij}_\lambda(k') q'_i q'_j \\ \times f\left(\tau_1, q, |\mathbf{k}-\mathbf{q}|\right) f\left(\tau_2, q', \left|\mathbf{k}'-\mathbf{q}'\right|\right) \langle \Phi_q \Phi_{|\mathbf{k}-\mathbf{q}|} \Phi_{q'} \Phi_{\left|\mathbf{k}'-\mathbf{q}'\right|}\rangle.\end{aligned} \tag{3.23}$$

It is clear to observe how the induced GWs spectrum depends on the four-point function of the scalar perturbation. This statistical function could be decomposed into a disconnected term, i.e. the product of two two-point functions, and a connected term [129]; the latter, however, assuming a Gaussian-distributed curvature perturbation, goes to zero, so the overall four-point function becomes

$$\begin{aligned}\langle \Phi_q \Phi_{|\mathbf{k}-\mathbf{q}|} \Phi_{q'} \Phi_{\left|\mathbf{k}'-\mathbf{q}'\right|}\rangle = \frac{2\pi^2}{q^3} \mathcal{P}_\Phi(q) \frac{2\pi^2}{|\mathbf{k}-\mathbf{q}|^3} \mathcal{P}_\Phi\left(|\mathbf{k}-\mathbf{q}|\right) \\ \times (2\pi)^6 \delta^{(3)}(\mathbf{q}+\mathbf{q}') \delta^{(3)}(\mathbf{k}+\mathbf{k}'-\mathbf{q}-\mathbf{q}').\end{aligned} \tag{3.24}$$

In the next section we will study the case for which the primordial fluctuations are non-Gaussian. Reworking (3.22), integrating on an internal momentum using the definition of delta and rewriting the remaining integral in spherical momentum coordinates, we find [126]

$$\bar{\mathcal{P}}_h = 8 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2 - (1-u^2+v^2)^2}{4uv}\right)^2 \bar{I}^2(\tau,k,u,v) \mathcal{P}_\Phi(ku) \mathcal{P}_\Phi(kv). \tag{3.25}$$

This expression takes into account the sum over the two polarization states, and also considers an appropriate average over the oscillation, taking into account how the observation of an SGWB always measures an average over different length scales λ. For convenience, a reparametrization of the momenta was introduced

$$v = \frac{q}{k}, \quad u = \frac{|\mathbf{k}-\mathbf{q}|}{k}. \tag{3.26}$$

All time dependencies have been appropriately placed in a special kernel function defined as follows

$$I(\tau, k, u, v) = \int_{\tau_i}^{\tau} d\tilde{\tau} G(\tau, \tilde{\tau}) f(\tilde{\tau}, k, u, v), \tag{3.27}$$

The tensor power spectrum averaged in (3.25) defines the main quantity needed in order to study the scalar-induced GW background, in fact its knowledge allows us to calculate the spectral energy density observable. It is essential to remember how this formulation holds as long as we assume that the curvature perturbation, whatever its origin, has a clear statistic, relative to a Gaussian distribution. Failing this assumption changes the current result.

3.1.2 Inclusion of primordial non-Gaussianity

The primordial fluctuations, generated by the quantum scalar field fluctuations during inflation, are very close to represent a Gaussian distribution statistic; however, a departure condition is expected due to the gravitational interactions [128] . This perturbative departure from a Gaussian distribution is called non-Gaussianity (NG) [130], [131], [132], [133], [52], [134]. Interestingly, the observation of such NG on primordial fluctuations can provide information on the particle content during the inflationary phase. For practical convenience, we express predictions about the primordial fluctuations generated during inflation as a function of the curvature perturbation $\mathcal{R}$, due to the well-established and convenient gauge properties of time conservation on Super-Horizon scales [135]. In the Newtonian working gauge one has the possibility to connect the curvature quantities Φ and $\mathcal{R}$ [136]

$$\mathcal{R} = \frac{5 + 3\omega}{3(1 + \omega)} \Phi = \frac{2b + 3}{b + 2} \Phi. \tag{3.28}$$

The interest of this section relates to the study of NGs generated during inflation, thus relating to primordial NGs [130] [137], [138], [139].
Such primordial NGs are statistically analyzed by the magnitude and shape of the three-point correlation function [140], [141], [130] (or equivalently by the bispectrum). We consider a type of local NG that can be expressed as a local perturbative expansion around the already defined Gaussian-style curvature perturbation $\mathcal{R}^g$

$$\mathcal{R}(x) = \mathcal{R}^g + \frac{3}{5} f_{NL} (\mathcal{R}^g(x))^2, \tag{3.29}$$

where the fraction $\frac{3}{5}$ comes from the equation (3.28). It is important to note, however, that this is only a specific choice of local writing of the curvature perturbation, and that it is also possible to choose different ones. However there is no study on the impact of other shape of primordial NGs on the induced GWs spectrum. Translating into Fourier space, one can see trivially how the square in (3.29) becomes a convolution. Therefore by changing variable, in Fourier space we obtain

$$\Phi_q = \Phi_q^g + F_{NL} \int \frac{d^3 l}{(2\pi)^3} \Phi_l^g \Phi_{|\mathbf{q}-\mathbf{l}|}^g, \tag{3.30}$$

with

$$F_{NL} = \frac{3}{5}\left(\frac{2b+3}{b+2}\right) f_{NL}. \tag{3.31}$$

It is now necessary to reconsider (3.23) in the light of the new definition of the curvature perturbation, and to rewrite its four-point correlation function appropriately

$$\begin{aligned}
\langle \Phi_q \Phi_{|\mathbf{k}-\mathbf{q}|} \Phi_{q'} \Phi_{\left|\mathbf{k}'-\mathbf{q}'\right|} \rangle &= \langle \Phi_q^g \Phi_{|\mathbf{k}-\mathbf{q}|}^g \Phi_{q'}^g \Phi_{\left|\mathbf{k}'-\mathbf{q}'\right|}^g \rangle \\
+F_{NL}^2 \int \frac{d^3 l}{(2\pi)^3} \int \frac{d^3 l'}{(2\pi)^3} &\langle \Phi_q^g \Phi_l^g \Phi_{|\mathbf{k}-\mathbf{q}-\mathbf{l}|}^g \Phi_{q'}^g \Phi_{l'}^g \Phi_{\left|\mathbf{k}'-\mathbf{q}'-\mathbf{l}'\right|}^g \rangle \\
(|\mathbf{k}-\mathbf{q}| \iff q) + \left(\left|\mathbf{k}^{'}-\mathbf{q}^{'}\right| \iff q^{'}\right) &+ \left(q \iff q^{'}; |\mathbf{k}-\mathbf{q}| \iff \left|\mathbf{k}^{'}-\mathbf{q}^{'}\right|\right) \\
F_{NL}^2 \int \frac{d^3 l}{(2\pi)^3} \int \frac{d^3 l'}{(2\pi)^3} &\langle \Phi_l^g \Phi_{|\mathbf{q}-\mathbf{l}|}^g \Phi_{l'}^g \Phi_{\left|\mathbf{k}-\mathbf{q}-\mathbf{l}'\right|}^g \Phi_{q'}^g \Phi_{\left|\mathbf{k}'-\mathbf{q}'\right|}^g \rangle \\
&\left(q \iff q^{'}; |\mathbf{k}-\mathbf{q}| \iff \left|\mathbf{k}^{'}-\mathbf{q}^{'}\right|\right) + O(F_{NL}^4).
\end{aligned} \tag{3.32}$$

The first term of the RHS is the Gaussianity term already calculated in the past section. The second line (i.e., the six-point correlation function of the primordial curvature perturbation), which highlights NG's first contribution, delineates a total of 6 possible non-zero Wick contractions, which, counted together with the four possible permutations expressed in the third subsequent line, provides a net contribution of 24 terms for the leading-order contribution of NG. The fourth line, on the other hand, preserves eight possible non-zero contractions, which, counted together with the two possible permutations (found in line 5), provides a second contribution of 16 terms. NG contributions can be classified into three different classes: the "H", "C" and "Z" classes [142]. These terms, used appropriately in (3.22) guarantee a perturbative NG contribution to the induced tensor power spectrum of the following form [142]:

$$\bar{\mathcal{P}}_h^H = 2^5 F_{NL}^2 k^3 \sum_\lambda \int \frac{d^3q}{2\pi} (e_\lambda^{ij}(k) q_i q_j)^2 \bar{I}^2(\tau, q, |\mathbf{k}-\mathbf{q}|) \times \frac{\mathcal{P}_\Phi(q)}{q^3} \int \frac{d^3l}{2\pi} \frac{\mathcal{P}_\Phi(l)}{l^3} \frac{\mathcal{P}_\Phi(|\mathbf{k}-\mathbf{q}-\mathbf{l}|)}{|\mathbf{k}-\mathbf{q}-\mathbf{l}|^3}, \tag{3.33}$$

$$\bar{\mathcal{P}}_h^C = 2^6 F_{NL}^2 k^3 \sum_\lambda \int \frac{d^3q}{2\pi} \int \frac{d^3l}{2\pi} e_\lambda^{ij}(k) q_i q_j e_\lambda^{ij}(k) l_i l_j \bar{I}(\tau, q, |\mathbf{k}-\mathbf{q}|) \bar{I}(\tau, l, |\mathbf{k}-\mathbf{l}|) \times \frac{\mathcal{P}_\Phi(l)}{l^3} \frac{\mathcal{P}_\Phi(|\mathbf{k}-\mathbf{l}|)}{|\mathbf{k}-\mathbf{l}|^3} \frac{\mathcal{P}_\Phi(|\mathbf{q}-\mathbf{l}|)}{|\mathbf{q}-\mathbf{l}|^3}, \tag{3.34}$$

$$\bar{\mathcal{P}}_h^C = 2^6 F_{NL}^2 k^3 \sum_\lambda \int \frac{d^3q}{2\pi} \int \frac{d^3l}{2\pi} e_\lambda^{ij}(k) q_i q_j e_\lambda^{ij}(k) l_i l_j \bar{I}(\tau, q, |\mathbf{k}-\mathbf{q}|) \bar{I}(\tau, l, |\mathbf{k}-\mathbf{l}|) \times \frac{\mathcal{P}_\Phi(l)}{l^3} \frac{\mathcal{P}_\Phi(q)}{q^3} \frac{\mathcal{P}_\Phi(|\mathbf{k}-\mathbf{q}-\mathbf{l}|)}{|\mathbf{k}-\mathbf{q}-\mathbf{l}|^3}. \tag{3.35}$$

Working in a perturbative way, it is simple to assume that these contributions offered by the non-Gaussian character of the problem, are simple perturbations of the central term found previously under conditions of pure Gaussian formalism expressed in formula (3.25). The first NG contribution (3.33) is also called hybrid correction. It should be noted that the second line of (3.33) can be absorbed into a new definition of the primordial curvature spectrum including the NG contributions. In fact one can write

$$\mathcal{P}_\mathcal{R}^{NL}(q) = \mathcal{P}_\mathcal{R}(q) + F_{NL}^2 q^3 \int \frac{d^3l}{2\pi} \frac{\mathcal{P}_\mathcal{R}(l)}{l^3} \frac{\mathcal{P}_\mathcal{R}(|\mathbf{q}-\mathbf{l}|)}{|\mathbf{q}-\mathbf{l}|^3}, \tag{3.36}$$

with that term following from the generic calculation of the two-point function $\langle \Phi(k)\Phi(k')\rangle$ using (3.30). Using such a formulation in the definition of the tensor power spectrum, it is easy to derive an induced GWs background that also takes into account the possible non-Gaussian nature of the scalar fluctuations generating it [143].

3.2 Analytical Transfer Functions

This section is mainly based on the articles [127] and [144]. In the most general case the kernel (3.27) and the tensor power spectrum (3.25) must be calculated numerically. However, in the situation of causally connected modes of interest for which $k\tau >> 1$, the time integration of the kernel is very difficult, in fact the integrand is the product of three oscillating functions with frequencies dependent on the tensor and scalar wave numbers. The region for which $k\tau >> 1$ is the one of observational interest, in the sense that for GWs to be defined as such, they must necessarily fall within the causal horizon. Extending the integral so far to time τ_0 and focusing on scales much smaller than the horizon we have trivially that $k\tau_0 >> 1$. It should be mentioned that the accessible scales for future GWs interferometers are in the range between $k \simeq 10^7 - 10^{18}\text{Mpc}^{-1}$. However, these observational scales are very small, so much so that they fall within the horizon much earlier than the BBN (scales that enter the horizon at the BBN epoch correspond to $k \simeq 10^3\text{Mpc}^{-1}$). This means the possibility of calculating such a kernel at a time much earlier than the BBN, i.e. when the GWs modes have already re-entered far inside the horizon in a Universe dominated by radiation: in this context we assume for simplicity such GWs as a radiation fluid with $\omega = 1/3$. It is crucial to note, however, that such observational modes of interest re-enter the horizon considering time much before the epoch of BBN, leaving one possibility to have a phase in the early Universe prior to a complete radiation dominated epoch that could have therefore $w_{new} \neq 1/3$; in such a case, one will necessarily have to follow the GWs up to the Universe transition in a later radiation-dominated phase. This shows the potentiality of exploiting GWs to infer information about the evolution of the Universe at very early times to reveal a possibly non-standard evolution during those phases.

In the model under consideration, it is considered that immediately after inflation there is a period of domination of the Universe prior to radiation, in which ω (hence b) is taken as the free parameter, and second-order waves are induced at this stage. It should be noted that when the equation-of-state ω and the adiabatic speed of sound of the associated fluid c_s^2 are constant (therefore during an epoch that is not a thermal transition phase), it is possible to analytically calculate the kernel (3.27). With these assumptions, it is first necessary to calculate the solution of the scalar fluctuation defining the source term generating the induced GWs background, in a suitable Newtonian gauge. In the situation in which $\omega \neq 1/3$, the time kernel can be computed in two different regimes: for modes that re-enter the horizon before the transition and for modes that re-enter after the transition during the radiation dominated epoch; the kernel will therefore be computed for modes that before the reheating transition (so the instantaneous transition that divide the new post-inflationary phase and the radiation epoch) will be on sub-horizon or super-horizon scales. Then we have to match these solutions with kernel associated to the radiation epoch, when the related tensor modes re-ntry on subhorizon scales. These kernel in fact will give us the observable spectral energy density.

3.2.1 First order solutions

In order to have a complete solution for the induced tensor modes, it is necessary, as already stated, to find a dynamic solution of the scalar fluctuations that induces this GWs background by defining the source. After a series of simplifications, the equation of motion for the Newtonian potential for generic values of c_s^2 and in the absence of isocurvature fluctuations, reads [136]

$$\Phi'' + (2\epsilon - \eta)\mathcal{H}\Phi' - \left(\eta + 2s\left(1 + \epsilon - \eta - 2s + \frac{\dot{s}}{Hs}\right)\right)\mathcal{H}^2\Phi + c_s^2 k^2 \Phi = 0, \quad (3.37)$$

where

$$\eta = \frac{\dot{\epsilon}}{H\epsilon} \quad , \quad s = \frac{\dot{c_s^2}}{Hc_s^2}. \quad (3.38)$$

This notation with ϵ, η and s is typical of inflationary models, and is used to greatly simplify the form of the equation of motion. The solution to the dynamics identified above for generic cosmological parameters, is expressed in terms of a linear combination of Bessel functions of the first and second kind

$$\Phi(k\tau) = (c_s k\tau)^{-b-3/2}(C_1 \mathcal{J}_{b+3/2}(c_s k\tau) + C_2 \mathcal{Y}_{b+3/2}(c_s k\tau)). \quad (3.39)$$

This solution deserves a correct initial condition, which is provided by the fact that this curvature perturbation, before returning to oscillation if it's on subhorizon scales again, was in a frozen condition. Therefore this value is the starting condition when the system returns to oscillate

$$\Phi(k\tau) = \Phi_{\mathbf{k}} 2^{b+3/2}\Gamma[b + 5/2](c_s k\tau)^{-b-3/2}\mathcal{J}_{b+3/2}(c_s k\tau), \quad (3.40)$$

with $\Phi_{\mathbf{k}}$ the frozen superhorizon-value defined by the inflationary phase. Note how this general solution cannot hold in the case where $c_s = 0$ as the gradient term in the general starting equation would disappear. Such a system predicts a solution that does not decay, thus defining a constant source suitable for the production of induced GWs of the second-order in perturbation theory.
Given knowledge of the scalar solution, one can now finally find the tensor solution using Green's integration techniques. First, the two independent and homogeneous

solutions associated with (3.18), called h_1 and h_2, must be structured. These homogeneous solutions can be written, for a constant ω, in the terms of the Bessel functions

$$h_1(k\tau) = (k\tau)^{-b-1/2}\mathcal{J}_{b+1/2}(k\tau), \quad h_2(k\tau) = (k\tau)^{-b-1/2}\mathcal{Y}_{b+1/2}(k\tau). \tag{3.41}$$

These solutions are an increasing and a decaying mode respectively. Obviously, the solution does not predict the presence of c_s since such modes propagate at the speed of light c. Therefore it is possible to write the Green's solution

$$G(\tau,\tilde{\tau}) = \frac{\pi}{2k}\frac{(k\tilde{\tau})^{b+3/2}}{(k\tau)^{b+1/2}}\left(\mathcal{J}_{b+1/2}(k\tilde{\tau})\mathcal{Y}_{b+1/2}(k\tau) - \mathcal{J}_{b+1/2}(k\tau)\mathcal{Y}_{b+1/2}(k\tilde{\tau})\right). \tag{3.42}$$

With such a result, one has all the necessary inputs for the study of the GWs background. It is most convenient to introduce the variable

$$x = k\tau, \tag{3.43}$$

to be used from now on, considering also that $\tilde{x} = k\tilde{\tau}$.
It is possible to overcome the difficulty related to a time integration of a product of at least three oscillating functions by suitably simplifying the source term of the induced background GWs; in fact, by exploiting the properties of the Bessel functions we have

$$\begin{aligned} f(x,u,v) = \frac{2^{2b+3}\Gamma^2[b+5/2]}{(2b+3)(b+2)}(c_s x)^{-2b-1}(uv)^{-b-1/2} \\ \times\left(\mathcal{J}_{b+1/2}(c_s xv)\mathcal{J}_{b+1/2}(c_s ux) + \frac{b+2}{b+1}\mathcal{J}_{b+5/2}(c_s vx)\mathcal{J}_{b+5/2}(c_s ux)\right). \end{aligned} \tag{3.44}$$

The work done consists in writing the Bessel functions $\mathcal{J}_{b+3/2}$ present in the definition of the scalar fluctuation Φ as a linear combination of $\mathcal{J}_{b+1/2}$ and $\mathcal{J}_{b+5/2}$: this rewriting has no specific physical meaning, but only the mathematical purpose of simplifying the analyticity of the computations in question as much as possible. Replacing (3.44) and (3.43) in the definition of the kernel (3.27) we obtain

$$I(x,u,v) = \pi 4^b\Gamma^2[b+3/2]\frac{2b+3}{b+2}(c_s^2 uvx)^{-b-1/2}(\mathcal{J}_{b+1/2}(x)\mathcal{I}_{\mathcal{Y}} - \mathcal{Y}_{b+1/2}(x)\mathcal{I}_{\mathcal{J}}), \tag{3.45}$$

where

$$\begin{aligned} \mathcal{I}_{\mathcal{J}/\mathcal{Y}} = \int_0^x d\tilde{x}\tilde{x}^{1/2-b}(\mathcal{J}_{b+1/2}(\tilde{x})/\mathcal{Y}_{b+1/2}(\tilde{x})) \\ \times\left(\mathcal{J}_{b+1/2}(\tilde{x}c_s v)\mathcal{J}_{b+1/2}(\tilde{x}uc_s) + \frac{b+2}{b+1}\mathcal{J}_{b+5/2}(\tilde{x}vc_s)\mathcal{J}_{b+5/2}(\tilde{x}uc_s)\right). \end{aligned} \tag{3.46}$$

Unfortunately, we are not aware of a general analytical solution in time, unless b is a complex number, in which case the Bessel functions are simplified into spherical Bessel functions, so that the whole problem can be rewritten in terms of transcendental functions. A practical example is that offered by radiation, for which $b = 0$ [126]. Now (3.46) must be integrated in the two relevant regimes of subhorizon ($x >> 1$) and superhorizon scales ($x << 1$). In fact we have to consider that in the first post inflationary phase we study the induced GWs that could be on superhorizon scales or on subhorizon scales before the reheating period. Since we need the definition of the kernel during the radiation epoch (for an observational interest), we have to follow the tensor modes until we reach the radiation phase. Facing this problem we have to divide the calculation for these two kind of tensor modes (so the problem will be reflected on the associated kernel). In the model under consideration, it is considered that immediately after inflation there is a period of domination of the Universe prior to radiation, in which ω (hence b) is taken as the free parameter, and second-order waves are induced at this stage. However, in order to join to the correct cosmological solutions of the Standard Hot Big Bang Model, one must subsequently and necessarily go through a subsequent phase of standard radiation domination, so whatever tensor solution is found in the first phase must be matched and followed appropriately until the radiation phase is reached and beyond. For simplicity we assume an instantaneous transitive phase in a time called reheating τ_{rh}. The magnitude of the Hubble sphere marked at that time is called k_{rh}. This approximation of instantaneous transition is particularly positive for the study of those intermediate modes that fall within the horizon at the time of reheating, for which therefore $k \simeq k_{rh}$; in this sense a smooth transition would accentuate the difficulty of the problem.

Let us further assume that the scalar scale associated with the peak of the primordial spectrum enters the horizon long before the instantaneous reheating phase, such that during or after such reheating there is no longer a source term for the induced GWs background, since it has already decayed because of the oscillation, long ago; therefore these tensorial modes, once temporally transited in the radiation-dominated phase and obviously in a causally connected condition, will define a free-waves dynamic, so they oscillate and decay (whereas in the previous phase there was a source term able to change the damping of the oscillating phase, so in the transitive moment we must define a mathcing condition between the two distinct dynamics).

3.2.2 General SubHorizon Kernel

In this section we will explore the case where all tensor modes of observational interest are already within the horizon, well before reheating, so that we have that $k >> k_{rh}$. This condition allows the integration extremum in (3.46) to be extended to $x >> 1$, since when one needs to perform a solution match at the moment of reheating with a subsequent solution of free GWs propagating in a radiation-dominated Universe, the upper integration limit will be $x_{rh} \simeq k/k_{rh} >> 1$. With this mathematical approximation, it is possible to rewrite the complete kernel solution, already taking into account the integration over time, expressed in equation (3.45) [145]

$$\begin{aligned} I(x >> 1, u, v) = x^{-b-1} 4^b \Gamma^2[b+3/2] \frac{2b+3}{b+2} \frac{\left|1-y^2\right|^{b/2}}{c_s^2 uv} \\ \times \Bigg[\cos\left(x - \frac{b\pi}{2}\right) \left(P_b^{-b}(y) + \frac{b+2}{b+1} P_{b+2}^{-b}(y)\right) \Theta[c_s(u+v)-1] \\ \frac{2}{\pi} \sin\left(x - \frac{b\pi}{2}\right) \left(Q_b^{-b}(y) + \frac{b+2}{b+1} Q_{b+2}^{-b}(y)\right) \Theta[c_s(u+v)-1]] \\ -\frac{2}{\pi} \sin\left(x - \frac{b\pi}{2}\right) \left(\mathcal{Q}_b^{-b}(-y) + \frac{b+2}{b+1} \mathcal{Q}_{b+2}^{-b}(-y)\right) \Theta[-c_s(u+v)+1] \Bigg], \end{aligned} \tag{3.47}$$

where $P_b^{-b}(y)$ e $Q_b^{-b}(y)$ are the one type and second type Ferrer function, valid for $|y| < 1$ while $\mathcal{Q}_b^{-b}(y)$ is the Legendre function of second kind, correct for $|y| > 1$. We can define for sake of simplicity the parameter

$$y = 1 - \frac{1 - c_s^2(u-v)^2}{2c_s^2 uv} = -1 - \frac{1 - c_s^2(u+v)^2}{2c_s^2 uv}. \tag{3.48}$$

It is necessary to perform a series expansion of the Bessel functions for large arguments of interest. Note how the term y is associated with the area of the triangle composed of the three main momenta of the problem, $c_s|\mathbf{k}-\mathbf{q}|$, $c_s|\mathbf{q}|$, and k.

The presence of the Heaviside function indicates the presence of a possible resonance, when the frequency of the tensorial modes equals the sum of the two typical scalar frequencies $k \simeq 2c_s k_p$ where k_p describes the peak value for the spectrum . In more direct terms, if $c_s(u+v) \to 1$ the three Bessel functions defined in (3.46) for $x \to \infty$ lead to a diverging integral. It might also be of interest to take into account the NG contributions studied above, but this would complicate the problem exponentially, so the problem will only be investigated in the Gaussian framework. The goal is to define the mean of the kernel square, so if we square (3.47) and then we calculate the oscillation (time) average we conclude

$$\begin{aligned}\bar{I}^2(x,u,v) = x^{-2(b+1)}4^{2b}\Gamma^4[b+3/2]\left(\frac{2b+3}{b+2}\right)^2\frac{\left|1-y^2\right|^b}{2c_s^4u^2v^2} \\ \times\left(P_b^{-b}(y)+\frac{b+2}{b+1}P_{b+2}^{-b}(y)\right)^2\Theta[c_s(u+v)-1] \\ +\frac{4}{\pi^2}\left(Q_b^{-b}(y)+\frac{b+2}{b+1}Q_{b+2}^{-b}(y)\right)^2\Theta[c_s(u+v)-1] \\ +\frac{4}{\pi^2}\left(\mathcal{Q}_b^{-b}(-y)+2\frac{b+2}{b+1}\mathcal{Q}_{b+2}^{-b}(-y)\right)^2\Theta[-c_s(u+v)+1]).\end{aligned} \tag{3.49}$$

With such a solution, we are ready to calculate the overall induced GWs background with the specifications for the starting assumptions. It will remain only the integral on momenta, which can also be performed computationally. The following objective is to connect this kernel to the radiation domination kernel, and look at the behaviour of the latter in typical IR and resonance regimes for the GWs background.

Matching with Radiation Domination

It is now necessary to discuss the connection between the two thermal phases defined earlier, the second of which is the radiation phase. First we match the background quantities in the reheating time τ_{rh}, i.e. the scaling factor and the Hubble parameter. Conclusively it is found from the matching condition that these parameters in radiation domination follow a typical solution related to that period, but with an additional time shifting that is written

$$\tilde{\tau} = \tau - \frac{b}{1+b}\tau_{rh}, \tag{3.50}$$

where b is related to the equation of state of the previous epoch . Now the tensor modes, or equivalently the associated kernel, must be matched: the continuity condition on the reheating time of the modes and its derivative imply equivalent continuity conditions on the kernel in question. Working on scales already in subhorizon, we have that the kernel before the transition defines itself as follows:

$$I(x >> 1, u, v) \simeq x^{-b-1}\left(A_{1,b}\sin\left(x-\frac{b\pi}{2}\right)+A_{2,b}\cos\left(x-\frac{b\pi}{2}\right)\right), \tag{3.51}$$

where the coefficients are known from before from (3.47). The kernel after the radiation transition must still be associated with tensor modes on subhorizon scales. These ones from the assumptions made earlier must behave as free GWs propagating in a radiation-dominated Universe. Hence we have that the kernel has the following form

$$I_{RD}(x >> 1, u, v) = A_{1,RD}\frac{\sin\bar{x}}{\bar{x}} + A_{2,RD}\frac{\cos\bar{x}}{\bar{x}}. \tag{3.52}$$

By setting the reheating time τ_{rh} and defining the continuity conditions, we find the unknowns $A_{1,RD}$ and $A_{2,RD}$, which allow us to have the correct form of the kernel in successive subhorizon radiation domains. Therefore we have that

$$\Omega_{GWs}(k >> k_{rh}, \tau >> \tau_{rh}) = \frac{k^2}{12a^2H^2}\bar{\mathcal{P}}_h^{RD}, \tag{3.53}$$

with

$$\bar{\mathcal{P}}_h^{RD} = 8\int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2 - (1-u^2+v^2)^2}{4uv}\right)^2 \bar{I}^2_{RD}(\tau, k, u, v)\mathcal{P}_\Phi(ku)\mathcal{P}_\Phi(kv). \tag{3.54}$$

This spectral energy density solution is, moreover, totally equivalent to

$$\Omega_{GWs}(k >> k_{rh}, \tau >> \tau_{rh}) = \frac{k^2}{12a^2H^2}\bar{\mathcal{P}}_h|_{\tau=\tau_{rh}}, \tag{3.55}$$

with

$$\bar{\mathcal{P}}_h = 8\int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2 - (1-u^2+v^2)^2}{4uv}\right)^2 \bar{I}^2(\tau, k, u, v)\mathcal{P}_\Phi(ku)\mathcal{P}_\Phi(kv). \tag{3.56}$$

where the kernel is that of (3.47), obviously computed at the time of continuity. However, it must be remembered that this solution is only valid for tensorial modes on subhorizon scales at the time of reheating, i.e. for $k\tau_{rh} >> 1$. Then a solution must also be found for all the remaining complementary modes, which only go into subhorizon scales after the reheating time, i.e. for $k\tau_{rh} << 1$. From the match of the two solutions, one can also find the correct evolution of the system for all those intermediate modes that re-enter the horizon in a period approximately close to reheating, which will therefore be characterized by the condition $k\tau_{rh} \simeq 1$. Such scales will be called k_m. Connecting the two solutions is not so easy, however. By writing the scale that last crosses the horizon in reheating

$$k_{rh} = \frac{1+b}{\tau_{rh}} \tag{3.57}$$

it is possible to make a comparison between this and the k_m scale as follows

$$\frac{k_m}{k_{rh}} = \frac{1}{1+b}. \tag{3.58}$$

Therefore if $b > 0$ we have that $k_m < k_{rh}$, then the approximation in subhorizon approach is extendable up to the scale of k_{rh}, where one expects a good matching between the two causal connection theories (since the scale k_m at the time of reheating is in superhorizon). For $b < 0$ we can find the opposite solution.

Resonances

It is necessary to devote a section to the study of the kernel seen in formula (3.49) with respect to its resonance point that we have discussed earlier. We discuss the possibility to study a peak in the spectral energy density for the specific condition for which $k \to 2c_s k_p$. The theory involved allows us to consider three interacting vectors, one for the produced tensorial modes $\mathbf{k}$, and two for the two scalar modes $\mathbf{q}$ and $|\mathbf{k}-\mathbf{q}|$; these vectors obviously satisfy the conservation of momentum, implying a triangular inequality that can be defined as follows:

$$|k-q| < |\mathbf{k}-\mathbf{q}| < k+q \quad , \quad \big||\mathbf{k}-\mathbf{q}|-q\big| < k < |\mathbf{k}-\mathbf{q}|+q \tag{3.59}$$

which in terms of u and v can also be read as

$$|1-v| < u < |1+v| \quad , \quad |u-v| < 1 < u+v. \tag{3.60}$$

These three momenta identify a triangle whose area is defined through the angle between $\mathbf{k}$ and $\mathbf{q}$, in fact

$$\sin(\theta_k) = \frac{2A}{kq}. \tag{3.61}$$

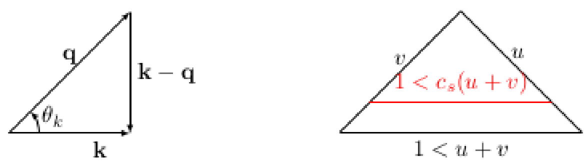

Figure 3.1: Illustration of the triangular inequality and the vector relationship involved in the GWs integral convolution [119].

This consideration implies that the projection performed with the polarization tensor on the scalar momenta is literally proportional to that area

$$e_\lambda^{ij} q_i q_j \propto \sin^2(\theta_k) \propto A^2. \tag{3.62}$$

This result leads to the reasoning according to which if the triangular inequality should be saturated, therefore for example $1 = u + v$, there would be a zero area, so the integral (3.25) would go to zero locally. The theory allows one to expect resonance when

$$c_s(u+v) = 1. \tag{3.63}$$

Logic allows one to think that if $c_s^2 = 1$, then the point of resonance would coincide with the saturation of the triangular inequality, so that there would be a zero overall contribution. As a consequence (due to conservation of momentum), in the case of $c_s^2 = 1$, there would be no resonance in the induced spectrum. Conversely, if instead $c_s^2 < 1$, there would be a resonance effect on the final spectrum. This condition can be proven by expanding the kernel (3.49) around the point $c_s(u+v) \simeq 1$, which corresponds to choose $y \simeq -1$, where the Legendre functions of the theory present a divergence. Assuming a first case for which $b < 0$, it is possible to see that the kernel diverges as:

$$\bar{I^2}_{res}(x,u,v,b<0) \simeq x^{-2(b+1)} 4^{2b} \Gamma^4[b+5/2] \frac{8}{c_s^4 u^2 v^2} \frac{\csc^2 b\pi}{\Gamma^2[3+b]} \big|1-|y|\big|^{2b}. \tag{3.64}$$

So it's possible to observe that a very peaked primordial spectrum leads to $\Omega^{res}_{GWs}(b<0) \propto \left|k - 2c_s k_p\right|^{2b}$. If instead we consider $b > 0$, we should observe

$$\bar{I^2}_{res}(x,u,v,b<0) \simeq x^{-2(b+1)} 4^{3b} \Gamma^4[b+5/2] \frac{32}{\pi^2 c_s^4 u^2 v^2} \left(\frac{(1+b+b^2)}{(b+1)(b+2)} \frac{\Gamma[b]}{\Gamma[2b+3]} \right)^2, \tag{3.65}$$

the result of which, being independent of y, has no divergence. However, it is possible to show that for $1 > b > 0$, there is still a peak in the GWs spectrum at $k = 2c_s k_p$, if we consider a very peaked scalar spectrum, finding in fact

$$\Omega^{res}_{GWs}(1>b>0) \propto constant - \left|k - 2c_s k_p\right|^{b}. \tag{3.66}$$

Infrared regime

It is useful to identify the behaviour of the kernel where $u \simeq v >> 1$, a condition that best represents the IR regime for the GWs spectrum being $v \simeq k_p/k$, with $k << k_p$. This condition implies that, whenever there will be a peak k_p that can be studied in the scalar fluctuation spectrum, one would study the system for scales much larger than this peak. The kernel studied on subhorizon scales should therefore be restricted for frequencies for which $k_{rh} << k << k_p$ (putting together the IR and subhorizon condition of the system). Again, the cases where $b < 0$ and $b > 0$ exhibit different behaviour. In the limit $y \to 1$ (from (3.48) if we consider the conditions $u \simeq v >> 1$), the averaged kernel (3.49) becomes

$$\bar{I^2}_{IR}(x>>1,u,v,b<0) \simeq x^{-2(b+1)} 4^{2b} \Gamma^4[b+5/2] \frac{8}{c_s^4 u^2 v^2} \frac{\csc^2 b\pi}{\Gamma^2[3+b]} \big|1-|y|\big|^{2b}. \tag{3.67}$$

when $b < 0$ and

$$\bar{I^2}_{res}(x, u, v, b < 0) \simeq x^{-2(b+1)} 4^{3b} \Gamma^4[b + 5/2] \frac{32}{\pi^2 c_s^4 u^2 v^2} \left(\frac{(1 + b + b^2)}{(b + 1)(b + 2)} \frac{\Gamma[b]}{\Gamma[2b + 3]} \right)^2, \tag{3.68}$$

when $b > 0$.
These solutions are only quite similar to those previously seen for the resonance study given in formulae (3.64) and (3.65); however the fundamental difference lies in the fact that now these formulae are to be studied in the IR limits in which, $u \simeq v >> 1$, therefore $1 - y \simeq v^{-2}$. Calculating these results at reheating and adding the k^2 term present in (3.25), one obtains a GWs spectrum that in the infrared view scales proportionally to a factor of $k^{3-2|b|}$ [146].

3.2.3 SuperHorizon Kernel approximation

In the previous section the kernel was studied in subhorizon approximation; however, at the time of reheating there exist several tensor modes which are still on superhorizon scales, so $k << k_{rh}$. Therefore in that section we cannot consider the extension to the upper integration limit of the kernel at $x >> 1$. In fact we have to consider the opposite range according to which $x << 1$. We recall another key approximation in the model, namely that the scalar modes re-enter the horizon long before reheating, in order to have in that period a GWs background similar with the radiation, subjected to a free-wave condition; therefore it is imposed that $k_p >> k_{rh} >> k$. Therefore, in order to have the largest integral contribution to the system, one must concentrate in the region of integration of the generic kernel around the peak in P_Φ, where therefore one has that $v \simeq k_p/k >> 1$ and hence also $u \simeq v >> 1$.
With these assumptions we arrive at a kernel of the form

$$I(x << 1, u, v) \simeq B_{1,b} + B_{2,b} x^{-2b}, \tag{3.69}$$

with the exact coefficients

$$B_{1,b} = -\frac{(3 + 2b)^2(1 + b + b^2)}{4b(1 + b)^2(2 + b)} (c_s v)^{-2}, B_{2,b} = \frac{4^{1+b}\Gamma^2[b + 5/2]}{b(1 + b)(2 + b)\pi} (c_s v)^{-2(1+b)}. \tag{3.70}$$

This solution describes the kernel for those modes that are on superhorizon scales before reheating. These modes are therefore not yet actual GWs, so this result must be followed until after re-entry into the horizon during the next radiation domination phase.

3.2.4 Matching to radiation domination

The kernel (3.69) is valid for modes in superhorizon during an arbitrary pre-radiation domination phase in which $b = const.$ After this period there will be reheating, so the source term will go to zero in oscillatory decay and thereafter the tensor modes will start to move as freely propagating massless tensor modes. The continuity of the modes h_{ij} and its derivative at the reheating boundary extends to the continuity condition on the kernel. Therefore we have to go from the superhorizon kernel for modes in the first domination phase, to a kernel for modes still on superhorizon scales but experiencing the next domination phase of radiation, the latter of which must read

$$I_{RD}(\bar{x} << 1, u, v) \simeq B_{1,RD} + B_{2,RD}(k\bar{\tau})^{-1}, \tag{3.71}$$

with $\bar{\tau}$, the conformal time shifted. From the matching to reheating of the two kernels and derivatives, we obtain

$$B_{1,RD} = B_{1,b} + \frac{1-b}{1+b} B_{2,b}(k\tau_{rh})^{-2b} \quad , \quad B_{2,RD} = \frac{2b}{(1+b)^2} B_{2,b}(k\tau_{rh})^{1-2b}. \tag{3.72}$$

The operation just performed describes the passage of a superhorizon kernel between the two domination stages. Now, however, it is necessary to follow the kernel (or the induced tensorial modes) until it re-enters on subhorizon scales, where it will begin to describe freely oscillating tensorial modes in radiation without of any scalar source. Therefore, if we define the average of the square of this last term, we read

$$\bar{I^2}_{RD}(k << k_{rh}, \tau >> \tau_{rh}) \simeq \frac{1}{2\bar{x}^2}(B_{1,RD}^2 + B_{2,RD}^2). \tag{3.73}$$

Gathering the most significant contributions, offered by the condition for which $v >> 1$, we arrive at the following final form of kernel for tensor modes finally on subhorizon scales during the radiation epoch

$$\begin{aligned} \bar{I^2}_{RD}(k << k_{rh}, \tau >> \tau_{rh}) &\simeq \frac{1}{2\bar{x}^2}\left(\frac{(3+2b)^2}{4b(1+b)^2(2+b)}\right)^2 \\ \times \left((1+b+b^2)(c_s v)^{-2} - 4^{1+b}\Gamma^2[b+3/2]\frac{1-b}{\pi}(c_s v)^{-2(1+b)}\left((1+b)\frac{k}{k_{rh}}\right)^{-2b}\right)^2 &. \end{aligned} \tag{3.74}$$

This averaged squared kernel will be used in the definition of the induced GWs background (3.25), complementing the previous case, where one working only on specific tensor modes for which $k >> k_{rh}$. The union of the two solutions in subhorizon and superhorizon scales quantifies the overall result of the GWs background, while their matching in the intermediate condition for which $k \simeq k_{rh}$

turns out to be particularly good if reheating occurs in instantaneous terms.
We therefore conclude by calculating the observable of the problem

$$\Omega_{GWs}(k << k_{rh}, \tau >> \tau_{rh}) = \frac{k^2}{12a^2H^2}\bar{\mathcal{P}}_h^{RD}, \tag{3.75}$$

with

$$\bar{\mathcal{P}}_h^{RD} = 8\int_0^{\infty} dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2-(1-u^2+v^2)^2}{4uv}\right)^2 \bar{I}^2{}_{RD}(x >> 1, u, v)\mathcal{P}_\Phi(ku)\mathcal{P}_\Phi(kv). \tag{3.76}$$

processed using the latest kernel estimated in (3.74).

3.3 Typical induced GWs spectra

This section is mainly based on the articles [127] and [144].
While in the last section we studied the kernel for a GW background induced in the radiation period, it is now also important to calculate, given the latter, some characteristic examples of spectral energy densities of induced GW backgrounds: it is clear that in order to define that function, it is necessary to integrate in convolution on the momenta the kernel found in the desired domination regime with the product of the scalar spectra; Now, such functions will be necessary for the determination of the final result. It is convenient, therefore, to consider standard input power spectra, such as a Dirac delta, a broken power law, an almost scale-invariant function, to see what induced GW background are able to produce, knowing from now on the kernel, calculated earlier.
It is clear to think that the induced GW background that can be measured today is defined as follows

$$\Omega_{GWs,0} = \frac{1}{3M_{pl}^2H_0^2}\frac{d\rho_{GWs,0}}{d\ln k}. \tag{3.77}$$

Nevertheless, the solution found earlier defines the GWs background in the radiation period. Now inside the horizon the GWs behave like a radiation fluid, therefore after reheating it will be seen that $\Omega_{GW} = \Omega_{GW,rh} = const.$ To correlate $\Omega_{GW,rh}$ with the actual $\Omega_{GWs,0}$ one can exploit the behaviour of GWs as radiation and therefore write that

$$\begin{aligned}\Omega_{GWs,0}h^2 &= \Omega_{r,0}h^2\frac{1}{\rho_{r,0}}\frac{d\rho_{GW,0}}{d\ln k}\\ \Omega_{GWs,0}h^2 &= 1.62\times10^{-5}\left(\frac{\Omega_{r,0}h^2}{4.18\times10^{-5}}\right)\left(\frac{g_*(T_{rh})}{106.75}\right)\left(\frac{g_{*,s}(T_{rh})}{106.75}\right)^{-4/3}\Omega_{GW,rh},\end{aligned} \tag{3.78}$$

where $\Omega_{r,0}h^2 \simeq 4.18 \times 10^{-5}$ is the current radiation density fraction given by Planck. In the case where the induced GW are generated during the radiation phase, the index "rh" should be replaced by "c", which indicates the time point at which the induced tensor modes enter on subhorizon scales and thus begin to behave like perfect GWs.

3.3.1 Dirac Delta Peak

This one is perhaps the most trivial and analytically simple example of a scalar spectrum inducing a GW background; such a scalar spectrum cannot be produced by a single-field inflationary model [147] [148] [149], so it will be necessary to introduce the presence of a multi-field system into the associated model [150]. If the peak is extremely steep, the model of a Dirac delta can be used

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{A}_{\mathcal{R}}\delta\left(\ln k/k_p\right). \tag{3.79}$$

In the integral convolution it can be rewritten as

$$\mathcal{P}_{\mathcal{R}}(kv) = \mathcal{A}_{\mathcal{R}}\delta\left(\ln kv/k_p\right) = \mathcal{A}_{\mathcal{R}}\frac{k_p}{k}\delta\left(v - \frac{k_p}{k}\right), \tag{3.80}$$

same for u. Recalling the boundaries for u arising from the conservation of momentum, and integrating over the momenta using the definition of Dirac Delta, we can adapt the range over the frequencies of tensor modes to $0 < k < 2k_p$. Considering this range and evaluating the problem for $v = u = k_p/k$, we arrive at

$$\Omega_{GW,rh}(k) = \frac{2}{3}\left(\frac{k_p}{k_{rh}}\right)^2\left(1 - \frac{k^2}{4k_p^2}\right)^2 \bar{I}^2{}_{RD}(k/k_{rh}, k/k_p)\Theta(2k_p - k). \tag{3.81}$$

Note that this solution is written under the more general condition for which there is a domination phase at $b = constant$ before radiation, so the intermediate scale k_{rh} must be taken into account in the problem. If, on the other hand, in a more specific case one wishes to consider the induced GWs background directly in radiation, it is sufficient to remove k_{rh} in favour of k_p [126]. It makes sense to evaluate the specific trend of the GWs background in the infrared regime by expanding this solution for $k << k_p$. For $b < 0$ one finds

$$\begin{aligned}\Omega_{GW,rh}(b<0, k << k_p) = &\frac{\mathcal{A}_{\mathcal{R}}^2}{12}\left(\frac{2^{1+b}(2+b)\Gamma^2[3/2+b]}{\pi c_s^{2(1+b)}(1+b)^{1+b}}\right)^2\left(\frac{k_{rh}}{k_p}\right)^{2+4b}\\ &\times\begin{cases}\frac{2^{3+2b}}{\pi(1+b)^{2b}}\left(\frac{k}{k_{rh}}\right)^2 & ,(k \leq k_{rh}/(1+b))\\ \left(\frac{\pi}{\sin(b\pi)\Gamma[2+b]}\right)^2\left(\frac{k}{k_{rh}}\right)^{2+2b} & ,(k \geq k_{rh}/(1+b)).\end{cases}\end{aligned} \tag{3.82}$$

A broken power law has been set in the constraint for which $k \simeq k_m$, where for $b < 0$ there is a transition between the subhorizon kernel approximation and the superhorizon kernel approximation, so the two complementary kernels found must be used in these two frequency ranges. For $b > 0$ instead one finds

$$\Omega_{GWs,rh}(b > 0, k << k_p) = \frac{\mathcal{A}_{\mathcal{R}}^2}{24\pi}\left(\frac{(2+b)(1+b+b^2)}{c_s^2 b(1+b)^2}\right)^2\left(\frac{k_{rh}}{k_p}\right)^2 \times \begin{cases} \left(\frac{k}{k_{rh}}\right)^2, & (k \leq k_{rh}) \\ \frac{1}{2}\left(\frac{2^{1+b}\Gamma[b+3/2]}{(1+b)^{1+b}}\right)^2\left(\frac{k}{k_{rh}}\right)^{2-2b} & (k \geq k_{rh}), \end{cases} \tag{3.83}$$

where this time the match occurs at $k \simeq k_{rh}$. The most important information that these two results are able to bring is that the IR tilt has dependence on the equation-of-state parameter in the period in which the induced GWs were generated. Therefore, an observational study of this index can give clear indications of the thermal history of the expanding primordial Universe.

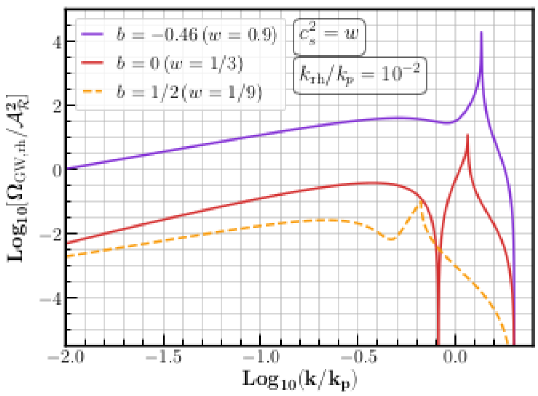

Figure 3.2: Induced GWs spectral energy density for a primordial scalar Dirac spectrum with different values of b [119].

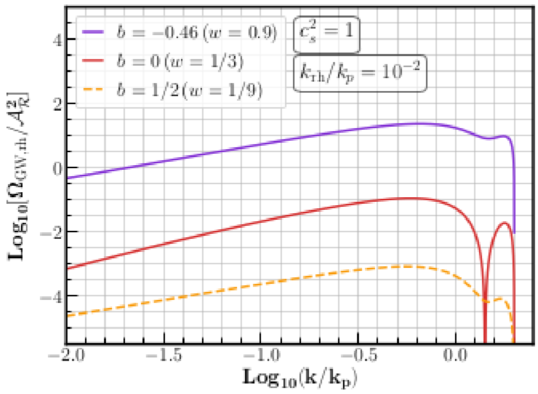

Figure 3.3: Induced GWs spectral energy density for a primordial scalar Dirac spectrum with assumed value $c_s^2 = 1$ [119].

3.3.2 Log-Normal peak

We must complete the previous discussion by imagining that we are once again describing a scalar peak, which will now, however, have a finite width. This scalar spectrum, which can be modelled by multi-field inflationary models, can be parameterized by a log-norm function [151], of the type

$$\mathcal{P}_\mathcal{R}(k) = \frac{\mathcal{A}_\mathcal{R}}{\sqrt{2\pi}\Delta} \exp\left(-\frac{\ln^{(k/k_p)}}{2\Delta^2}\right), \tag{3.84}$$

with Δ the dimensionless width of the peak. Note how such a curvature spectrum is perfectly normalized. We can consider the limit of the highly peaked spectrum, for which $\Delta << 1$ [150] [152] [153], on the other hand, if $\Delta \simeq 1$, then it will be said to be a broad peak [154] [155] . In the first sharp peak case, the following induced GWs background is found

$$\Omega_{GWs,\Delta}(k) = erf\left(\frac{1}{\Delta}\sinh^-1\left(\frac{k}{2k_p}\right)\right)\Omega_{GW,\delta}(k), \tag{3.85}$$

where $erf(x)$ is the error function and $\Omega_{GW,\delta}(k)$ is the spectrum induced by a scalar spectrum function equal to a Dirac delta, so this solution in the background induces a simple corrective factor with respect to (3.81) due to the Δ width. This allows us to evaluate a typical behaviour of k^3 in IR under the domination of radiation [146]. The largest corrective effects occur when the ratio k/k_p is smaller than the width Δ. Thus for $k << k_p$ we obtain

$$\Omega_{GW,\Delta}(k) = erf\left(\frac{k}{2k_p\Delta}\right)\Omega_{GW,\delta}(k), \tag{3.86}$$

which provides a correction of $erf(k/2k_p\Delta) \simeq k/2k_p\Delta$ when $\Delta >> k/2k_p$. Therefore, due to the presence of the thickness Δ, the IR tail calculated in the previous section is modified in the cases $2k_p\Delta < k_{rh}$ and $2k_p\Delta > k_{rh}$, respectively, in

$$\Omega_{GWs}(k << k_p, 2k_p\Delta < k_{rh}) \propto \mathcal{A}_{\mathcal{R}}^2 \begin{cases} \left(\frac{k}{k_p}\right)^3, & k << 2k_p\Delta << k_{rh} \\ \left(\frac{k}{k_p}\right)^2, & 2k_p\Delta << k << k_{rh} \\ \left(\frac{k}{k_p}\right)^{2-2|b|}, & k << 2k_p\Delta << k_p \end{cases} \tag{3.87}$$

or

$$\Omega_{GWs}(k << k_p, 2k_p\Delta > k_{rh}) \propto \mathcal{A}_{\mathcal{R}}^2 \begin{cases} \left(\frac{k}{k_p}\right)^3, & k << k_{rh} \\ \left(\frac{k}{k_p}\right)^{3-2|b|}, & k_{rh} << k << 2k_p\Delta \\ \left(\frac{k}{k_p}\right)^{2-2|b|}, & 2k_p\Delta << k << k_p \end{cases} \tag{3.88}$$

It is essential to note how the introduction of a finite peak width induces an increase in the richness of the induced GWs spectrum. Note also how if $b = 0$, as is the case in radiation dominance, there will only be two distinct types of slope, with k_{rh} losing its meaning completely. If, on the other hand, the peak has a significant width, thus $\Delta = 1$ [151], it is possible to define the radiation-induced GWs background in the vicinity of the scalar peak as follows

$$\Omega_{GWs}^{peak}(\Delta > 1) \simeq 0.125 \frac{\mathcal{A}_{\mathcal{R}}^2}{\Delta^2} e^{-\frac{\ln^2 k/k_p}{\Delta^2}}. \tag{3.89}$$

This result implies how a background induced GWs from a log-norm scalar spectrum, is itself of the log-norm form, however presenting a narrower peak given by $\Delta/\sqrt{2}$. Factor two comes into play as an indirect reflection of the secondary perturbative nature of the proposed GWs [151].

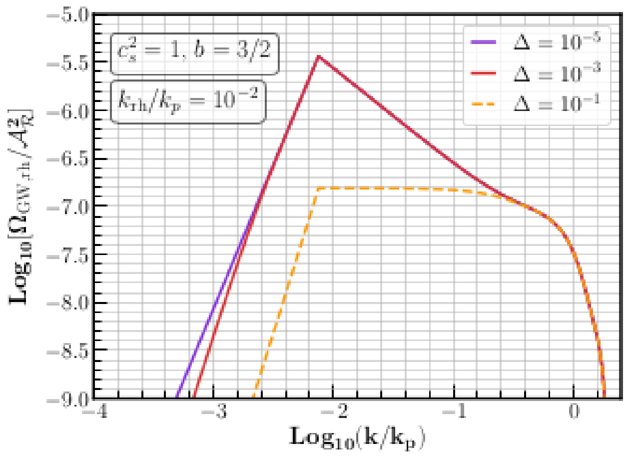

Figure 3.4: Induced GWs spectral energy density for a primordial log-normal scalar spectrum [119].

3.3.3 Broken power-law

If the curvature perturbation during a single-field inflationary phase is boosted, then a curvature power spectrum relative to a broken power-law function can be defined [147] [148] [149] [156]. It is possible to parameterize the primordial curvature power spectrum as a broken power-law in the following way

$$\mathcal{P}_\mathcal{R} = \mathcal{A}_\mathcal{R} \begin{cases} \left(\frac{k}{k_p}\right)^{n_{IR}} & k < k_p, \\ \left(\frac{k_p}{k}\right)^{-n_{UV}} & k > k_p, \end{cases} \tag{3.90}$$

where n_{IR}, $n_{UV} > 0$ are respectively the infrared and ultraviolet scalar tilts relative to a peaked spectrum in k_p. In single-field models it is often possible to find that $n_{UV} = 4$ [147]. If, on the other hand, the bump in the curvature perturbation occurs by virtue of a bump in the scalar field potential [156], then n_{UV} will be related to the second derivative of that potential in the bump, as well as to the non-Gaussianity parameter f_{NL} [156]. It is necessary to study the problem with the aim of working the scalar-induced GWs background during the radiation-dominated phase. From the curvature power spectrum one can therefore find such GWs spectrum, whose behaviour in the IR follows the following pattern

$$\Omega_{GWs,rh}(k << k_p) \simeq 12\mathcal{A}_\mathcal{R}^2 \left(\frac{1}{2n_{IR}-3} + \frac{1}{2n_{UV}+3}\right)\left(\frac{k}{k_p}\right)^3 \ln^2\left(\frac{k}{k_p}\right). \tag{3.91}$$

This equation only holds true if $n_{IR} > 3/2$ (condition related to the requirement of convergence of the integral for large internal momenta $u \simeq v >> 1$) and $n_{UV} > 0$.

The UV tail of the induced background GWs, on the other hand, exhibits two possible behaviours. If $0 < n_{UV} < 4$ then the integral also converges in the limit for which $v \to 0$. Thus this implies that one can take the scaling dependence of the curvature power spectrum out of the integral convolution on momenta, so that one can define a solution of the form $\Omega_{GWs,rh} \simeq \mathcal{P}_{\mathcal{R}}^2$, unless numerical multiplicative factors characteristic of the integral on momenta. Therefore in the UV limit, the induced background GWs reads

$$\Omega_{GW,rh}(k >> k_p, n_{UV} < 4) \simeq \frac{\mathcal{A}_{\mathcal{R}}^2}{12} F(n_{UV}) \left(\frac{k}{k_p}\right)^{-2n_{UV}} \tag{3.92}$$

where $F(n_{UV})$ is given by

$$F(n_{UV}) = 8 \int_0^{\infty} dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2 - (1-u^2+v^2)^2}{4uv}\right)^2 \bar{I}^2(n_{UV}, k, u, v)(uv)^{n_{UV}} \tag{3.93}$$

This form was found by noticing that the integral shows a divergence in the exact limit in which $n_{UV} = 4$; in fact, in the case in which $n_{UV} = 4$ the induced GWs spectrum shows a logarithmic type of divergence. Concluding, if instead we take the UV case for which $n_{UV} > 4$, we would have a rapidly decreasing primordial power spectrum, so we wouldn't expect a solution for which $\Omega_{GWs,rh} \simeq \mathcal{P}_{\mathcal{R}}^2$, but rather a result expressing, due to conservation of momentum, a strong fall-off for frequencies $k \simeq 2k_p$. In fact, for $n_{UV} > 4$, we observe

$$\Omega_{GW,rh}(k >> k_p, n_{UV} > 4) \simeq \frac{16}{3} \mathcal{A}_{\mathcal{R}}^2 \left(\frac{1}{n_{UV} - 4} + \frac{1}{n_{IR} + 4}\right) \left(\frac{k}{k_p}\right)^{-4-n_{UV}}. \tag{3.94}$$

Combining all the results together, it is possible to conclude that

$$\Omega_{GWs,rh}(k) \propto \mathcal{A}_{\mathcal{R}}^2 \begin{cases} \left(\frac{k}{k_p}\right)^3, & k << k_p \\ \left(\frac{k}{k_p}\right)^{-\Delta}, & k >> k_p \end{cases} \tag{3.95}$$

with

$$\Delta = \begin{cases} 2n_{UV}, & 0 < n_{UV} < 4 \\ 4 + n_{UV}, & n_{UV} > 4. \end{cases} \tag{3.96}$$

So it is easy to see how the knowledge of the UV tilt of the GWs background leads to a consequent knowledge of the UV tilt of the scalar induction spectrum. If, on the other hand, one had a GWs background induction phase different from that of radiation (which precedes it), one would have a variation of the sensitive (3.96), which can be defined as follows [157]

$$\Omega_{GWs,rh}(k) \propto \mathcal{A}_{\mathcal{R}}^2 \begin{cases} \left(\frac{k}{k_p}\right)^3, & k << k_{rh} \\ \left(\frac{k}{k_p}\right)^{3-2|b|}, & k_{rh} << k << k_p \\ \left(\frac{k}{k_p}\right)^{-\Delta-2b}, & k >> k_p \end{cases} \tag{3.97}$$

where b quantifies the dominance fluid in the pre-radiation phase, in which the GWs background is induced; this time the UV tilt changes as follows

$$\Delta = \begin{cases} 2n_{UV}, & 0 < n_{UV} + b < 4 \\ 4 + n_{UV}, & n_{UV} + b > 4. \end{cases} \tag{3.98}$$

A limiting case of the broken power-law is that of quasi-scale-invariant curvature power spectrum for which $n_{UV} = n_{IR} = 0$, a typical result obtainable from an inflationary slow-roll model in a single field. An induced GWs background will be found equal to

$$\Omega_{GW,rh} = \mathcal{A}_{\mathcal{R}}^2 \mathcal{F}(b, c_s) \left(\frac{k}{k_{rh}}\right)^{-2b} \tag{3.99}$$

with

$$\mathcal{F}(b, c_s) = \frac{2}{3} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2 - (1 - u^2 + v^2)^2}{4uv}\right)^2 \bar{I}^2(b, c_s, u, v), \tag{3.100}$$

a numerical factor that always converges.

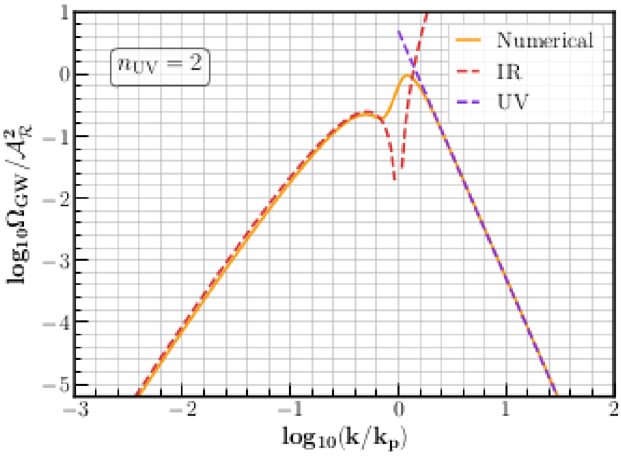

Figure 3.5: Induced GWs spectral energy density for a primordial scalar broken-power law spectrum with $n_{IR} = 4$ and $n_{UV} = 2$ [119].

Chapter 4

Inflationary models with features in the Power-Spectrum

4.1 Effective field theory of inflation models with sharp features

This section is written on the basis of the article [158]. The focus of this model is to face a slow-roll single-field inflation system with small, steep step features in the potential (and sound velocity) of the inflaton in an EFTI environment [42] [159] [160]. This solution makes it possible to study the correlation functions between curvature perturbations and to associate a usual quasi-scale invariant trend due to the de-Sitter background with features that allow for an observable modification of it. In conclusion it will therefore be possible to find a modified power-spectrum self-consistent and an elevated NG both dependent on the main parameter of step features β that quantify the steepness of the step in the potential. This approach is so simple that it is possible to translate and generalize it by subjecting to features not only the potential (hence H), but also the other multiplicative coefficients present in the action of the linear inflaton perturbation.
One of the key goals of standard cosmology is to explain the origin of the currently observable cosmic structures of LSS: taking advantage of what has been said in the previous chapters, it is worth recalling how the standard cosmological approach predicts the existence of an inflationary period that can be read from a perturbative point of view, whereby the inflationary fluctuation originates a curvature perturbation on superhorizon scales, and this on re-entry, returns to oscillate generating such cosmological seeds. The simple single-field slow-roll model, as already seen, predicts a quasi-scale-invariant curvature perturbation spectrum that completely fits Planck's observational data [161], which predicts an amplitude for the CMB anisotropy fluctuations on the instrument's large sensitivity scales.

However, one has to imagine how even a slight modification of the model in question, i.e. a perturbation on the slow-roll approach, could lead to a modification of the curvature spectrum that could still fit well with the observational data. The EFTI is an example among them, which reduces a perturbative single-field theory in SR developed therefore around an expanding metric background, to a theory concerning the physics of a Goldstone Boson that breaks symmetry by time diffeomorphism. The idea of this section is therefore to study a model of single field slow-roll inflation, with the addition of a modification concerning the presence of a small sharp step in the inflationary potential assumed trivially to be flat (as the standard theory predicts), in an EFTI context. This approach, as we shall see, leads to significant modifications of the curvature power spectrum, which will no longer remain scale invariant, but will be modified by the presence of slight damped oscillations; this model will also lead to the raising of the NG.

Effective field theory of single field inflation

It is fundamental to introduce the concept of EFTI by describing the main ideas of this approach. The main idea of EFTI [42] [159] [160] is that instead of considering a standard lagrangian density of a scalar field and studying the perturbations of the same living in an expanding metric background of FRW, one studies the more general action of such fluctuations (around a quasi-de-Sitter background, in a slow-roll model) rewritten in the physical terms of a Goldstone boson, which explicitly breaks symmetry by temporal diffeomorphisms. Hence an initial quasi-de-Sitter universe is assumed to experience an accelerated inflationary expansion by slow-roll, thus with constant V, H, ϵ, which are then at a certain point in time modified and 'perturbed' by the arrival of a feature on the potential (which, however, is also reflected in the other parameters mentioned above). Inflationary fluctuations will enjoy a generic Bunch-Davies trend as usual initial condition written on the SR; however, if such fluctuations have a magnitude k such that they remain in the Hubble sphere even during such a feature step, this results in modifying the oscillating and decaying perturbation. Such modified modes will subsequently come out of the horizon with their degree of modification and define a different curvature perturbation, whose square modulus will quantify a slighly modified power spectrum. By choosing a perturbative variation approach on the SR (we will vary ϵ, V and H by very little) it is easy to see how this deviation from the scale invariant PS of slow roll is indeed minimal. The goldstone boson field π, under diffeomorphism transforms nonlinearly, since for $t \to t + \zeta_0(x)$, we have that $\pi \to \pi - \zeta_0(x)$; moreover as mentioned before, this field parametrizes the linear adiabatic inflationary perturbation $\delta\phi(x)$. At the linear level therefore, π makes explicit the comoving curvature perturbation

$$\zeta = -H\pi. \tag{4.1}$$

The most general and effective action for π can be constructed geometrically using a gauge in which the metric results in $\delta\phi = 0$. Then we can reintroduce the goldstone boson via the Stuckeldberg trick [42]. We obtain finally

$$\begin{aligned} S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{Pl}^2 R - M_{Pl}^2 \dot{H}(t+\pi)\left(\dot{\pi}^2 - \frac{(\partial_i \pi)^2}{a^2}\right)\right. \\ \left. +2M_2^4(t+\pi)\left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i \pi)^2}{a^2}\right) - \frac{4}{3}M_3^4(t+\pi)\dot{\pi}^3 + ...\right]. \end{aligned} \tag{4.2}$$

An important physical simplification comes from the fact that we are interested in making predictions at very high energy scales, of the order of H, so assuming that the Golstone boson gravitational interaction mixting terms with metric fluctuations are associated with much lower energy scales, they can be safely neglected. The key reasoning in the approach consists in the time dependence of the multiplicative coefficients of the field operators: the basic approach is to take such coefficients, including H and $\dot{H}$ constant over time, so as to have an inflationary attractor solution that persists in the background status of the theory inscribed in a slow-roll / quasi-de-Sitter stage. Doing so we have an approximately invariant Lagrangian for shift symmetry on π. This reasoning therefore leads to the elimination of any development after the zero background term of the coefficients in question, since

$$f(t+\pi) \simeq f(t) + \dot{f}(t)\pi + ... \tag{4.3}$$

We have to see that this shift symmetry on π is nothing but the translation of an inflationary background in quasi-de-Sitter in Slow-roll. So assuming a slow-roll Universe in quasi-de-Sitter, H and $\dot{H}$ are weakly time-dependent, so there is invariance on the shift symmetry for the Lagrangian density as long as temporally one remains in the inflationary phase for which precisely H (or V of all consequence), is constant. When, however, the step features on these parameters arrive temporally, the coefficients of S will acquire a non-trivial dependence in time that will lead to an obvious break in shift symmetry, ϵ will therefore not be at least momentarily constant and could therefore vary to the point of no longer being much smaller than one, causing the system to fall out of a slow-roll inflationary model and thus producing a curvature spectrum no longer scale invariant, since $n_S - 1 \propto \epsilon$. Therefore the feature must occur temporally at a sufficiently distant time to guarantee the formation of a quasi scale invariant spectrum. We further imagine, in order to avoid the problems outlined above, that the slow-roll parameter $\epsilon = -\dot{H}/H^2$ controlling the symmetry breaking assumes a temporary modification for which however $\epsilon << 1$ [162], remaining in the slow-roll classification that guarantees the observable (and hence sought-after) quasi-scale-invariant power

spectrum of curvature.
It is necessary to anticipate how, remaining with the multiplicative parameters constant (thus remaining in SR), the typical scale-invariant slow-roll spectrum would be found from the action. Adding the developments of these coefficients over time (a factor related to the introduction of a step feature that make these coefficients no longer constant), modifies this result by adding damped oscillations to the perturbations.

Effective approach for models with step features in the inflaton potential

Models with features, as expressed earlier, define a breaking condition on the scale invariance of the curvature spectrum and they increase the multi-point curvature correlators, which are strongly momentum-dependent. The idea is to specify a potential model with step features, study the background evolution of the scalar field, and calculate modified versions of the slow-roll parameters to define the multi-point correlators of the curvature perturbation. With an EFTI approach, one initially considers a system in which H and $\dot{H}$, (hence also $V(\phi)$) present a small step feature. At least in the initial phase it is convenient to bring all other coefficients to zero, and then eventually reintroduce them with a feature condition identical to the one we are considering for the Hubble parameter. Consider a potential of the following form [163]:

$$V(\phi) = V_0(\phi)\left[1 + cF\left(\frac{\phi - \phi_f}{d}\right)\right], \tag{4.4}$$

which describes a step c high, d wide and centred in ϕ_f with a generic step function F. When the field crosses the feature, the potential energy $\Delta V = cV$ is converted into kinetic energy for the field $\dot{\phi}^2 = 2\dot{H}$. As long as the step is small, the kinetic conversion is minimal, so the evolution of the inflationary background is not ruined and its effect can be considered as a perturbation on a fixed background.
The idea of the approach is to assume $\dot{H}$ as a time-dependent function, as well as V and H. We therefore consider

$$\dot{H}(t) = \dot{H}_0(t)\left[1 + \epsilon_{step}F\left(\frac{t - t_f}{b}\right)\right], \tag{4.5}$$

which expression indicates a rewriting of the slow-roll parameter as follows

$$\epsilon(t) = \epsilon_0(t)\left[1 + \epsilon_{step}F\left(\frac{t - t_f}{b}\right)\right]. \tag{4.6}$$

Here ϵ_{step} represents the height of the step, which in order not to ruin the slow-roll inflation must necessarily be $\epsilon_{step} << 1$. t_f defines the time of the feature while b

denotes the width of the step, with F the step function defined in a very general way. It is assumed that the background parameters $\dot{H}_0(t)$, $\epsilon_0(t)$ are time-independent, and thus all the time dependence of the problem is inscribed in the step features. With $\left|\epsilon_{step}\right| << 1$ it is possible to expand each parameter with ϵ_{step} parameter of smallness, around a generic constant and dominant background of order zero

$$\epsilon = \epsilon_0 + \epsilon_1 + ...O(\epsilon_{step}^2) \tag{4.7}$$

where the dots point a term of greater degree than the first one with respect to the smallness parameter in ϵ_{step}. Although ϵ is very small, this does not imply that the other higher-order slow-roll parameters are also very small, in fact these, on the contrary, may also be very large. This is the case for the parameter

$$\delta = \frac{1}{2}\frac{d\ln\epsilon}{d\ln\tau} = -\frac{\dot{\epsilon}}{2\epsilon H}. \tag{4.8}$$

Since the first parameter of slow-roll is a function close to a Heaviside theta, the second parameter of slow-roll, which is trivially its derivative, could easily be close to a Dirac delta.
Expanding again in powers of ϵ_{step} we find

$$\delta = \delta_0 + \delta_1 + O(\epsilon_{step}^2). \tag{4.9}$$

The largest contribution to δ_1 is given by

$$\delta_1 \simeq -\frac{1}{2}\frac{\epsilon_{step}}{H}\dot{F}\left(\frac{t-t_f}{b}\right), \tag{4.10}$$

This term is the dominant one among all δ, since for steep steps for very small values of b, the variation of the step function becomes giant. Transforming everything into conformal time we find

$$\delta_1 \simeq -\frac{1}{2}\epsilon_{step}\beta F'\left(-\beta\ln\frac{\tau}{\tau_f}\right), \tag{4.11}$$

with

$$\beta = \frac{1}{bH} \tag{4.12}$$

the steepness parameter.
The idea of the process now is to substitute in the Goldstone Boson action (3.2) the Taylor series developments (3.3) of the multiplicative coefficients taken as functions having steps: there will thus be a break in shift symmetry, since these coefficients no longer being constant in time will have a development in series that will continue even on successive orders; since the derivative at the first order of

step development is a peaked function, this in its corresponding order will make a non-trivial additive contribution to the entire action, leading to the creation of a modified power-spectrum.
The advantage of this approach lies in the simplicity of studying its validity regime and in its ability to extend to all the other coefficients that we have ignored.

Power spectrum

In order to obtain the curvature power-spectrum, in the case of a very steep step $\beta >> 1$, we must go first through the Goldston Boson equation of motion, obtained from the variation of the second-order development of the action; expanding the Hubble parameter around $\pi = 0$ [162]

$$S_2 = \int d^4x a^3 \left[-M_{Pl}^2 \dot{H} \left(\dot{\pi}^2 - \frac{(\nabla \pi)^2}{a^2} \right) + 3M_{Pl}^2 \dot{H}^2 \pi^2 \right]. \tag{4.13}$$

From the variation of this action written at second order, the equation of motion for π is derived:

$$\ddot{\pi} + \left(3H + \frac{\ddot{H}}{\dot{H}} \right) \dot{\pi} - \frac{\nabla^2 \pi}{a^2} = \ddot{\pi} + H(3 - 2\delta)\dot{\pi} - \frac{\nabla^2 \pi}{a^2} = 0, \tag{4.14}$$

where all terms suppressed by the first slow-roll parameter have been deleted. It is straightforward to study the dynamics by rewriting it using the conformal time. The action is rewritten as follows:

$$S_2 = \frac{1}{2} \int d^3x d\tau z^2 \left[\pi'^2 - (\nabla \pi)^2 - 3a^2 \dot{H} \pi^2 \right], \tag{4.15}$$

where the prime derivative marks the derivative with respect to time conforming τ while

$$z^2 = -2a^2 M_{Pl}^2 \dot{H}. \tag{4.16}$$

By redefining $\pi = u/z$ we find

$$S_2 = \frac{1}{2} \int d^3x d\tau \left[u'^2 - (\nabla u)^2 + \left(\frac{z''}{z} + 3a^2 H^2 \epsilon \right) u^2 \right]. \tag{4.17}$$

It should be noted that here the dominant term lives in the second derivative of z, as it is proportional to the second derivative of ϵ, thus to the first derivative of δ, which can be rewritten as

$$\frac{\dot{\delta}}{H} = -\frac{d\delta}{d \ln \tau}. \tag{4.18}$$

This term is proportional to β^2 as can be deduced from (4.11), so it represents the most important contribution. In order to study the perturbation, we must go from the equation of motion for u in the simplistic terms of the variable $x = -k\tau$

$$\partial_x^2 u - \frac{2}{x^2}u + u = \frac{\dot{\delta}}{Hx^2}, \tag{4.19}$$

where the small slow-roll parameters were eliminated under the assumption of small sharp step features. This solution is solved by the Green's function technique, treating the RHS of (4.19) as the source for the LHS [163] [164]. By the usual calculation of the dimensionless spectrum at large times $\tau \to 0$ is found,

$$\ln P_\zeta = \ln P_{\zeta,0} + \frac{2}{3}\int_{-\infty}^{+\infty} d\ln\tau W(k\tau)\frac{d\delta}{d\ln\tau}, \tag{4.20}$$

where

$$W(x) = \frac{3\sin 2x}{2x^3} - \frac{3\cos 2x}{x^2} - \frac{3\sin 2x}{2x}, \tag{4.21}$$

is a window function.

The zero-order power spectrum is that which is found from the constant background terms of H, ϵ, V substituted in the action, finding as already anticipated the background result (for the almost de-Sitter in Slow-roll) almost scale-invariant system

$$P_{\zeta,0} = \frac{H^2}{8\pi^2\epsilon M_{Pl}^2}. \tag{4.22}$$

By integrating (4.20) and substituting (4.11) we find

$$\ln P_\zeta = \ln P_{\zeta,0} - \frac{1}{3}\epsilon_{step}\beta\int_{-\infty}^{+\infty} d\ln\tau W^{'}(k\tau)F^{'}(-\beta\ln\tau/\tau_f), \tag{4.23}$$

where

$$W^{'}(x) = \left(-3 + \frac{9}{x^2}\right)\cos 2x + \left(15 - \frac{9}{x^2}\right)\frac{\sin 2x}{2x}, \tag{4.24}$$

is the derivative of the window function with respect to $\ln x$. It is trivial to note that if $\beta \to \infty$ the function $F^{'}$ would become a Dirac delta, then its integration in (4.23) would provide a spectrum relative to an oscillation of constant amplitude with frequency $2k\tau_f$ up to $k \to \infty$. However, this limit is highly unphysical, since the step must necessarily have a dimension of finite width. Therefore the integral (4.23) can be analytically evaluated for $\beta >> 1$ [163], arriving at

$$\ln P_\zeta = \ln P_{\zeta,0} - \frac{2}{3}\epsilon_{step}W^{'}(k\tau_f)\mathcal{D}\left(\frac{k\tau_f}{\beta}\right), \tag{4.25}$$

with $\mathcal{D}(y)$ being the normalized damping function. The function $W'(x)$ oscillates between -1 and 1 until $k \to \infty$, while $\mathcal{D}$ defines the damping for the oscillation. When $x \to 0$ also $W'(x) \to 0$, so no extra spurious superhorizon contribution is produced. The dumping term localizes in the space of frequencies the range of modes that are perturbed and varied by the features, confirming the idea that modes too far prior or posterior to the temporal occurrence of the step do not turn out to be modified by the initial Bunch-Davies state. Such observations can be deduced in the figure, where it is easy to note that the modes subject to variation in oscillation are those for which $1 < k\tau_f < \beta$, i.e. those that at the time of the feature are on subhorizon scales but with a moment not larger than the temporal inverse $b = 1/(\beta H)$, characterizing the width of the step. It is necessary to emphasize that as β increases, the space-momentum in which observe the dumped oscillations increase as well.

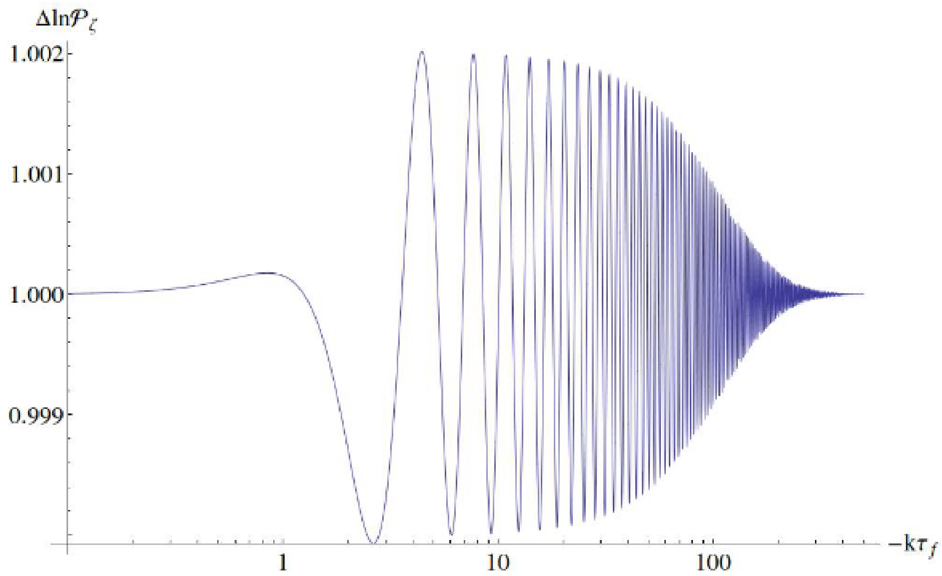

Figure 4.1: Curvature Power-spectrum of the EFTI theory in a step features approach for the potential [158].

For the calculation of the bispectrum, on the other hand, it is necessary to start from a Goldstone boson action developed at the third order; from this, the solution of the equation of motion of the fluctuation is defined, which is used to calculate the three-point correlation function in an in-in formalism [165]. Using at leading order the bunch-davies solution for simplicity, the bispectrum is calculated, which can be expressed in the convenient and reduced quantity f_{NL} [166]

$$\tilde{f}_{NL}(k_1, k_2, k_3) = -\frac{10}{3}\frac{k_1 k_2 k_3}{k_1^3 + k_2^3 + k_3^3}\frac{\mathcal{G}(k_1, k_2, k_3)}{k_1 k_2 k_3}, \tag{4.26}$$

with [166]

$$\frac{\mathcal{G}(k_1, k_2, k_3)}{k_1 k_2 k_3} = \frac{1}{4}\epsilon_{step}\mathcal{D}\left(\frac{K\tau_f}{2\beta}\right)\left[\left(\frac{(k_1^2 + k_2^2 + k_3^2)}{k_1 k_2 k_3 \tau_f} - K\tau_f\right) K\tau_f \cos\left(K\tau_f\right) - \right. \tag{4.27}$$

$$\left. - \left(\frac{k_1^2 + k_2^2 + k_3^2}{k_1 k_2 k_3 \tau_f} - \frac{\sum_{i\neq j} k_i^2 k_j}{k_1 k_2 k_3} K\tau_f\right) \sin\left(K\tau_f\right)\right], \tag{4.28}$$

with $K = k_1 + k_2 + k_3$.

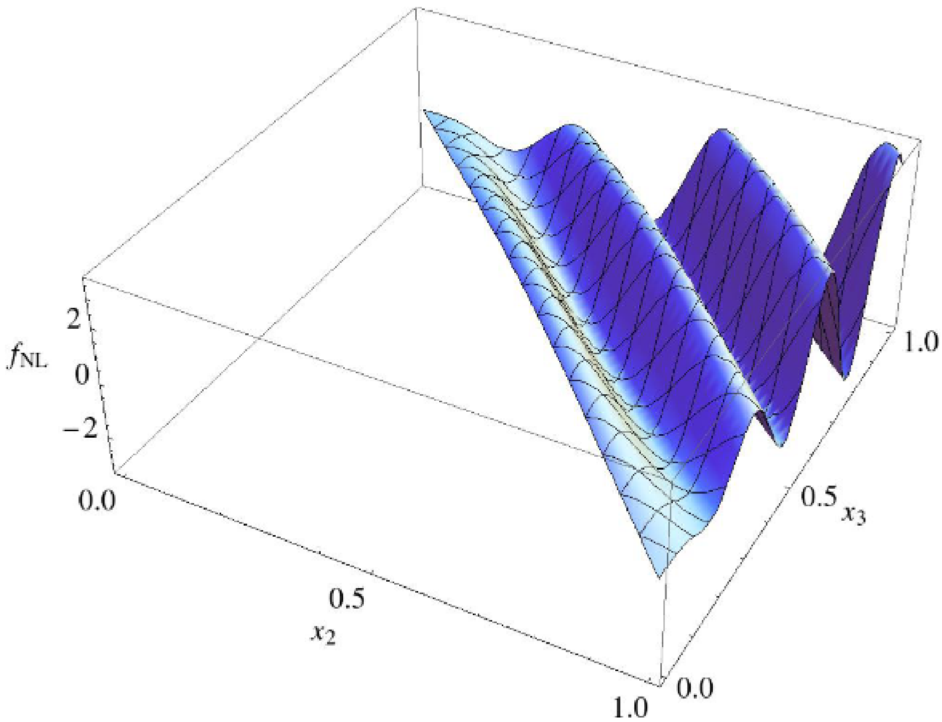

Figure 4.2: Bispectrum in the EFTI Theory [158].

Generalizations

It is possible to generalize the discussion so far by importing the step features seen only on the potential, hence on the Hubble parameters, also on the other multiplicative coefficients of the Goldstone Boson operators present in the action, among which the speed of sound is noteworthy [167] [168] [169] [170] [171] . It is therefore necessary to reactivate these previously reset multiplicative coefficients, and to define them with a time format of step features, as for H. As in the previous section, if one admits a time dependence for ϵ, hence its deviation from the constant course, one must also assume that its perturbation is very small, in order not to spoil the inflation and still guarantee a curvature spectrum that is still almost scale-invariant, although the presence of possible dumped oscillations. In EFTI, the shift symmetry invariance relative to the Lagrangian density supports the thesis that the various multiplicative coefficients associated with it are constant, so their hypothetical development in a Taylor series would only guarantee as a domination term the one of order zero; from such values the classical SR scale-invariant power spectrum without any features is deduced. However, breaking this symmetry guarantees a temporal dependence (in the analytical form of a step) to these coefficients, which in the development will altogether guarantee the presence of terms of the next degree, which generate precisely the damped oscillation above the central background term. This breaking, however, must be a soft-symmetry-breaking: therefore ϵ, i.e. the intrinsic symmetry-breaking parameter, can vary from its constant value, but only imperceptibly, so that the SR condition for which $\epsilon << 1$ persists. The generalization of this process import a temporal step features in all coefficients of the perturbation action. Therefore we write

$$M_n(t) = M_n^{(0)} \left[1 + m_n F_n \left(\frac{t - t_f}{b_n} \right) \right], \tag{4.29}$$

where the several parameters introduced have exactly the same meaning as those seen in the previous sections, with $m_n << 1$. As we have seen before, the operators associated to the derivative of the step will give the best perturbative contribution on the term of degree 0, rather than the step itself, which is suppressed in slow-roll. The derivative term of the step instead involves a term of the type β, which in the theory in question is the dominant one. Looking at the series development (4.3), we see that the n-th derivative of the coefficient M_n is always multiplied by the term of π^n. If the coefficient M_n appears for the first time in the action developed at order m (so it will be multiplied by an operator π^m), then its derivative will only appear in the development of order (m+1) in the action. For example, the coefficient M_3 appears for the first time in the action written at third order, so it is multiplied by the operator $\dot{\pi}^3$. If we develop this parameter in series, to which we guarantee a stepwise progression of the type (3.28), it can be seen that the

term of order 0 will guarantee only the constant background value $M_3^{(0)}$, (since the perturbative term proportional to m_3 will be suppressed), while the first-order development term associated with the derivative of the coefficient (thus of one step, guaranteeing a piqued function) will guarantee the total major perturbative contribution. Nevertheless this term must be read in the action of order 4, since this first derivative naturally carries with it an order 1 to be multiplied by the base of degree 3. Therefore this effect must be sought in the trispectrum.
If we want to write in EFTI the action at second order, we observe how only the parameters $\dot{H}$, M_2, $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ are present [172]; therefore since these present the form of a step features, it is possible to see such feature effects only at the level of the bispectrum.
The specific treatment of the term $M_2^4(t)$ and its trend described by (3.28), induces a trend of step-features also in the defined adiabatic speed of sound c_s^2. Using the same exact analytical steps as in the last sub-section, the same power-spectrum can be found [158]

$$P_\zeta = P_{\zeta,0}\left[1 - \frac{2}{3}\sigma_{step} W'(ks_f)\mathcal{D}\left(\frac{ks_f}{\beta}\right)\right]. \tag{4.30}$$

4.2 PBH and GWs from resonant amplification during Inflation

This section is based on the article [173].
In this section, an inflationary model consisting of two distinct stages dominated, dynamically speaking, by two scalar fields, respectively, will be discussed: the first stage is handled by the scalar ϕ with a potential of the axion-monodromy-like periodic structure, while the second and final stage is defined by the second field χ with a hilltop-like potential. This inflationary mechanism will be able to produce a mechanism of resonance production not only of PBHs, but also of GWs induced at the second-order in perturbation theory: the basic concept is that by increasing the curvature perturbation in resonance, the latter, on re-entry into the horizon, will have a higher density amplitude value, which will be subject to gravitational deformations again due to the causality. Such a re-entry fluctuation could therefore defines a density value above a certain critical threshold, abruptly collapse in on itself, inducing the creation of a Primordial Black Hole. Thus, the more the mechanism involved in resonance raises the curvature perturbation value, the greater is the probability of the creation of such primordial objects. However, knowing that the abundance of the induced GWs background is nothing more than a double integral of convolution of the curvature spectrum, if the latter increases, given the linear relationship, it will cause the induced GWs background itself to increase.
The parametric resonance under discussion originates from the periodic trend of the potential of the first inflationary phase, and brings a boost to the perturbations of both fields as long as they are inside the Hubble sphere (and therefore can vary). At a certain point in time, these fluctuations in the potential will become so high, and therefore no longer surmountable by the first field classically, that a turn on the background trajectory will be performed: consequently the boosted values of the fluctuation of the second field (at this point the dynamically dominant field) will individually contribute to the determination of the final curvature perturbation. The curvature spectrum on the small scales will perform a resonance peak that will lead to the abundant production of PBHs and GWs induced at the second order, both during the inflationary phase and in the post-inflationary phase of re-entry of the scalar modes defining the source term, i.e. the radiation-dominated phase.
PBHs have recently become a very important element of study in modern cosmology [174] [175] [176], as well as being one of the main candidates for representing dark matter [177] [178]. They represent a decisive tool for the study of the universe in its primordial form; furthermore, with the advent of GWs astronomy [179], it came to be thought that the observation of a GWs background could be explained by the phenomenology of PBHs. There are different theoretical approaches to the creation of PBHs [180] [181] [182] [183] , where the most credited is the one

described earlier above, associated with the curvature perturbation boost. Nevertheless, in an inflationary model with a single field (which therefore produces a single adiabatic perturbation mode), it is known that such a curvature perturbation on SuperHorizon scales is frozen [135] [184], blocking any growth. Therefore, if one wants to generate a controlled instability in order to handle the boost of this gauge invariant quantity, it is simply necessary to add a second field that defines, at least in the first inflationary part, the entropic mode. The idea is to boost the first resonance fluctuation, and through a dynamical mechanism that induces this parallel growth, also on the fluctuation of the second field. During the first phase $\delta\phi$ quantifies the adiabatic curvature fluctuation on superhorizon scales, while $\delta\chi$ quantifies the isocurvature fluctuation on superhorizon scales. When the first inflationary phase ends, only the second field will remain and its previously boosted fluctuation on superhorizon scales will become curvature fluctuation (and no longer isocurvature fluctuation), eventually defining a final power spectrum of curvature perturbations with an important peak on small scales.

In order to describe this construction, the idea of an inflationary model with two stages of domination is necessary: the first stage dominated by the first ϕ field, which has a specific oscillatory character in its potential, while the second stage is managed by the second ϕ field with a hill-top potential. The first part of the first inflationary stage is not characterized in the potential by any oscillatory trend: thus the linear potential induces a first slow-roll inflationary dynamics that therefore predicts a flat curvature spectrum on the large scales (i.e. those that go on superhorizon scales first during inflation), in full agreement with the observational data of the CMB offered by Planck. The oscillatory features of the potential comes out gradually and only afterwards, until the motion of ϕ stops at one of the minima of the potential itself. During this first phase χ remains massless, hence has no dynamics, and only begins to move at the beginning of the second phase, when the first field has now stopped at the minimum of its potential.

In the last e-folding of the first phase, the oscillation of the potential on ϕ, via Mathieu's dynamical equation, induces an exponential growth on $\delta\phi$, which subsequently induces a further exponential growth on $\delta\chi$ (as long as the modes are on subhorizon scales).

In the field space the background evolution first goes along the direction of ϕ, and then performs a turn in the trajectory towards the direction identified by χ. This implies that the initial isocurvature perturbation $\delta\chi$ (such in the first phase) will become an adiabatic curvature perturbation (once on superhorizon scales), in the second phase, when it will remain the only energetic fluctuation left. Therefore the boost induced on $\delta\chi$, now an adiabatic fluctuation, will show up on the final (in time) power spectrum with a peak on the small scales.

This mechanism, in addition to strongly generating PBH, also generates a second-order induced GWs background during inflation and during the radiation domination; in the first case the inducing source is represented by the two subhorizon field fluctuations during inflation, while the second GWs background is induced by the evolution of the curvature perturbation once it re-enters the horizon during radiation. The aim of this model is to verify how (contrary to what is usually obtained), the GWs background induced during inflation is higher than that induced during the radiation-dominated phase.

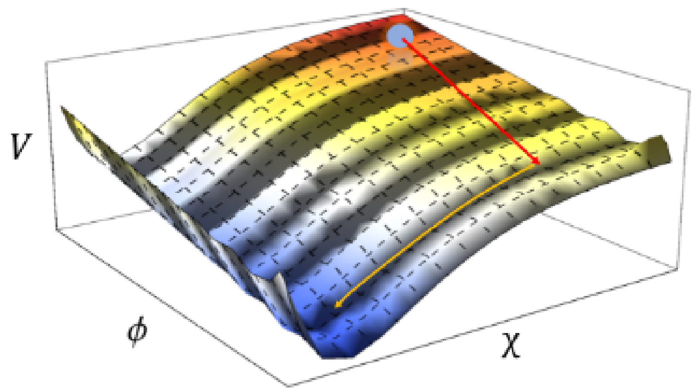

Figure 4.3: Summary of the two-field potential model [173].

Resonant Amplification of Cosmological perturbation

It is important to remember how the choice of using two fields is made in order to dramatically raise the effect of the oscillations. The first inflationary phase is dominated by ϕ with the effect of the oscillations on the potential that become important later on. Initially, the oscillatory character is assumed to be negligible, leading to the definition of a quasi-scale invariant power spectrum curvature on the broad CMB scales, in complete agreement with the observational data [137].
In the very last e-folds of the first phase, the oscillatory term is activated by inducing a Mathieu equation of motion for $\delta\phi$ on the dynamics of the fluctuations, so the latter results in an exponentially boost when it is on subHorizon scales. This fluctuation represents the dynamical source term on the fluctuation of the second field, so when $\delta\phi$ increases exponentially on subHorizon scales, it induces a consequent (in time) exponential growth on $\delta\chi$ as well. Subsequently ϕ will stop at one of the potential minima that it will dynamically not be able to bypass, hence the second phase dominated by χ alone will begin. Therefore, $\delta\chi$ generated during

the first phase will be converted into an adiabatic curvature perturbation (once it enters on superhorizon scales), whose square modulus will mark the curvature power spectrum (according to the standard model approximation).
At the end of inflation, it is to be expected that the χ field will decay, giving rise to the subsequent reheating phase.
It is possible to define the potential $V(\phi, \chi)$ of the model using the following figure (4.4). It is assumed for simplistic evidence that ϕ has a linear trend with small increasing periodic oscillations, while χ has a purely linear potential. The Lagrangian of the model is as follows

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\partial_\mu \chi)^2 - V(\phi, \chi), \tag{4.31}$$

with a two-field potential given by

$$V(\phi, \chi) = g\Lambda_0^3 \phi + \Lambda^4(\phi) \cos \frac{\phi}{f_a} + \varepsilon \Lambda_0^3 \chi + V_0. \tag{4.32}$$

The dimensional coefficient Λ_0 determines the energy scale associated with the background evolution. The dimensionless coefficients g and ε are coupling constants that generate the slope of the potential. The massive scale f_a determines the period of oscillation of the first potential, while $\Lambda(\phi)$ describes the amplitude of the field-dependent barrier defined as follows

$$\Lambda(\phi) = \Lambda_0 \left(1 + \alpha \frac{\phi}{M_p}\right). \tag{4.33}$$

The modulation of the potential barrier is defined by the monotonicity parameter b_*

$$b_*(\phi) = \frac{\Lambda^4(\phi)}{|g|\, \Lambda_0^3 f_a}. \tag{4.34}$$

The ϕ component of the potential is again shown in the figure: it is trivial to see how on its left-hand side the periodic barrier is small enough to define the potential as a linear inflationary potential where only higher-order slow-roll conditions are violated. This dynamic slow-roll initial condition guarantees a quasi-scale invariant power spectrum on the broad CMB scales.

Subsequently the field will continue to roll in its potential reaching the ϕ_0 point: from this moment, the oscillating barrier becomes important and no longer negligible, so that the perturbation modes of sufficiently small scale that stay on subhorizon when this condition is reached, are found to fall in the resonance band.
Finally the field ϕ will reach the end point defined by $\phi = \phi_e$.

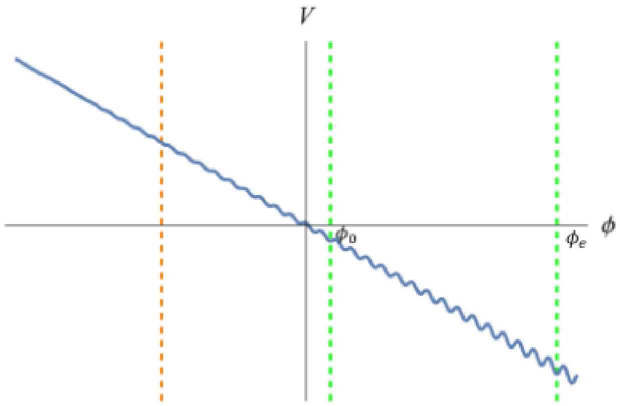

Figure 4.4: Potential for the first dynamical field [173].

After that moment, the inflationary dynamics will be driven by χ, until its decay into reheating. It is necessary to assume that the field excursion $\Delta\phi = \phi_e - \phi_0$ is small enough such that $|g|\,\Lambda_0^3\Delta\phi << V_0$, so that V_0 plays the role of the cosmological constant. Beyond these approximations, new constraints with the following form are required in order to simplify the model under consideration:

- The ϕ evolution dominate first:
  $|g| >> |\varepsilon|$
- In the first phase the potential should be flat: $b_*(\phi << \phi_0) << 1$
- Later on a mechanism must stopϕ: $b_*(\phi_e) \simeq 1$
- With a gauge flat-slicing the effective mass of $\delta\phi$ is governed by $V''(\phi)$ when the parametric resonance occur: $V_0 >> 2|g|\,\Lambda_0^3 f_a$.

Background evolution

The Friedmann equations for the background of the fields are as follows:

$$H^2 = \frac{\rho}{3M_p^2} \quad , \quad \frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}\,(\rho + 3p) \tag{4.35}$$

with energy density and background pressure defined as follows

$$\rho = \frac{1}{2}\dot{\chi}^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi,\chi) \quad , \quad P = \frac{1}{2}\dot{\chi}^2 + \frac{1}{2}\dot{\phi}^2 - V(\phi,\chi). \tag{4.36}$$

The standard Klein-Gordon equations for the two fields are

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \quad , \quad \ddot{\chi} + 3H\dot{\chi} + \frac{\partial V}{\partial \chi} = 0. \tag{4.37}$$

It is convenient to introduce a set of slow-roll parameters

$$\epsilon_{\phi\phi} = \frac{\dot{\phi}^2}{2H^2M_P^2} \quad , \quad \epsilon_{\chi\chi} = \frac{\dot{\chi}^2}{2H^2M_P^2}, \tag{4.38}$$

$$\eta_{\phi\phi} = \frac{\dot{\epsilon}_{\phi\phi}}{H\epsilon} \quad , \quad \eta_{\chi\chi} = \frac{\dot{\epsilon}_{\chi\chi}}{H\epsilon}, \tag{4.39}$$

with $\epsilon = -\dot{H}/H^2 = \epsilon_{\phi\phi} + \epsilon_{\chi\chi}$.
It is assumed true that these slow-rolls parameters are small, in order to have the dynamic description of the main body of the first inflationary moment: hence, the expressions of velocity and acceleration of the fields, under slow-roll approximations, can be found as follows:

$$\begin{aligned} \ddot{\phi} &= 3Hb_*(\phi)\dot{\phi}_0 \sin{(\phi/f_a)}, \\ \dot{\phi} &= \dot{\phi}_0 - 3Hf_a b_*(\phi)\cos{(\phi/f_a)}, \\ \dot{\chi} &= \dot{\chi}_0 = -\frac{\varepsilon\Lambda_0^3}{3H}, \end{aligned} \tag{4.40}$$

with

$$\dot{\phi}_0 = -\frac{g\Lambda_0^3}{3H}. \tag{4.41}$$

During the first phase dominated by ϕ the potential is taken flat given the very small value agreed to the monotonicity parameter b_*, therefore it results that $\dot{\phi} = \dot{\phi}_0$, and therefore that $\dot{\chi} << \dot{\phi}$. Therefore, at least in this phase, the evolution of χ should be ignored. Given the natural growth of ϕ in time, in (4.39.2) it is clear to observe how the second term outlining the oscillatory trend becomes increasingly important until it equals the first velocity term, thus leading to an overall cancellation of the same. The field therefore stops at a minimum of the highly oscillatory potential where $\phi = \phi_e$. Henceforth the domination of the field χ begins.

4.2.1 Field fluctuation and induced amplification

In terms of a gauge choice of flat slicing, the following coupled equations of motion for field fluctuations are chosen:

$$\begin{aligned}
\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k + m^2_{\chi\chi}\delta\chi_k + m^2_{\chi\phi}\delta\phi_k &= 0,\\
\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k + m^2_{\phi\phi}\delta\phi_k + m^2_{\chi\phi}\delta\chi_k &= 0,\\
m^2_{\chi\chi} = \frac{\partial^2 V}{\partial\chi^2} - \frac{1}{M_p^2}\left(3\dot{\chi}^2 + \frac{2\dot{\chi}\ddot{\chi}}{H} - \frac{\dot{H}\dot{\chi}^2}{H^2}\right)&,\\
m^2_{\phi\phi} = \frac{\partial^2 V}{\partial\phi^2} - \frac{1}{M_p^2}\left(3\dot{\phi}^2 + \frac{2\dot{\phi}\ddot{\phi}}{H} - \frac{\dot{H}\dot{\phi}^2}{H^2}\right)&,\\
m^2_{\chi\phi} = \frac{\partial^2 V}{\partial\chi\partial\phi} - \frac{1}{M_p^2}\left(3\dot{\chi}\dot{\phi} + \frac{\dot{\chi}\ddot{\phi} + \dot{\phi}\ddot{\chi}}{H} - \frac{\dot{H}\dot{\chi}\dot{\phi}}{H^2}\right)&.
\end{aligned} \tag{4.42}$$

We have to observe that the previously imposed conditions require the generation of a scale of magnitude on the masses involved with $m^2_{\phi\phi} >> m^2_{\chi\phi} >> m^2_{\chi\chi}$. With this approximation, the equations of motion are greatly simplified

$$\begin{aligned}
\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k &\simeq \frac{\dot{\chi}\ddot{\phi}}{M_p^2 H}\delta\phi_k,\\
\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} - \frac{\Lambda^4(\phi)}{f_a^2}\cos\left(\frac{\phi}{f_a}\right)\right)\delta\phi_k &= 0.
\end{aligned} \tag{4.43}$$

Consider the second of the two equations presented above. By introducing the new variable $\delta\Phi_k = a^{3/2}(t)\delta\phi_k$, one can rewrite the dynamics on the fluctuation of the first field as follows

$$\delta\ddot{\Phi}_k + \omega_k^2(t)\delta\Phi_k = 0, \tag{4.44}$$

where

$$w_k^2(t) = \frac{k^2}{a(t)^2} - \frac{\Lambda^4(\phi)}{f_a^2}\cos\left(\frac{\phi}{f_a}\right) - \frac{9}{4}H^2 - \frac{3}{2}\dot{H}. \tag{4.45}$$

It is well known that at sub-horizon scales and for scaling times much smaller than the characteristic expansion period, the scaling factor can be considered approximately constant. This further assumption leads to redefining the dynamics expressed above in terms of a Mathieu equation defined on an expanding universal background of FRW. The solution that this equation produces for the first field is an exponential growth, as anticipated

$$|\delta\phi_k| \propto \exp\left(\lambda_k H t\right) \tag{4.46}$$

where we can define the growning rate as follows

$$\lambda_k = \mu_k \frac{|g|\,\Lambda_0^3}{6H^2 f_a} - \frac{3}{2}, \tag{4.47}$$

with μ_k the Floquet number for the mode k.
Now it is time to focus on the first equation of (4.42), making explicit the dynamics of $\delta\chi_k$. For the sake of simplicity it is important to focus on the mode k_*, which comes out of the horizon at the moment of switch between the two inflationary periods, i.e. when $\phi = \phi_e$. Since for the first long period of the first phase, i.e. until ϕ reaches ϕ_0, the source term proportional to $\delta\phi_k$ is negligible, in fact $\left|\frac{\dot{\chi}\ddot{\phi}}{M_p^2 H}\delta\phi_{k_*}\right| << \left|\frac{k^2}{a^2}\delta\chi_{k_*}\right|$, we have that $\delta\chi_{k_*}$ decays as a decreasing exponential

$$|\delta\chi_{k_*}| \propto \exp\left(-Ht\right). \tag{4.48}$$

However $\delta\phi_{k_*}$ increases successively in time due to Mathieu's oscilation growth, so the source term for the fluctuation of the second field will also increase, becoming more and more important in the latter's dynamics. Therefore, when $\frac{\dot{\chi}\ddot{\phi}}{M_p^2 H}\delta\phi_{k_*}/\frac{k^2}{a^2}\delta\chi_{k_*}$ reaches an $O(1)$, then $\delta\chi_{k_*}$ will also grow due to the source term that grows exponentially, and we obtain dynamically

$$|\delta\chi_{k_*}| \propto |\delta\phi_{k_*}| \propto \exp\left(\lambda_{k_*} Ht\right). \tag{4.49}$$

As long as the first mode $\delta\phi_{k_*}$ remains in the resonance band growing, $\delta\chi_{k_*}$ will also continue to grow. Nevertheless, it is known that at a certain moment in the dynamics, the first field will stop: from that moment on, the fluctuation modes of the first field that have already gone into superhorizon scales, will be transformed into perturbations of the next field, while the modes that are still inside the sphere at that specific moment will go to zero, since from that moment on, the first field no longer exists. Therefore from that moment the source term of $\delta\chi$ goes to zero, so this fluctuation, after having grown in the previous phase in induction, decays again as in the first phase. All this quantifies a peak in the final spectrum. The mode k_* represents the peak of such a spectrum, since at the exact moment when k_* exits the sphere of causal connection, the first field decays, therefore the fluctuation associated with this quantity has no time to decay.
In conclusion, we can state how $\delta\chi_k$ follows an exponential decay for those small k (wide scales) that are not in the resonance band. Those that are present in will decay first, then grow exponentially in an induced way, and then decay again from the beginning of the second phase, if those modes have a width k such that they are still on the horizon when this condition occurs. The mode k_* is the last mode to come out of the horizon and freeze when exactly this condition occurs, so it has no time (as opposed to subsequent modes) to decay.

Curvature power spectrum

Given the resolution of the dynamics of the fluctuations up to their exit from the horizon and given the known dynamics in the background, it is possible to calculate the curvature power spectrum P_ζ for the model in question through the use of the δN formalism [135] [185] [186] [187]. This formalism asserts that the final value assumed by the conserved comoving curvature perturbation $\mathcal{R}_c(x)$ in the adiabatic limit and on superhorizon scales is given by

$$\mathcal{R}_c(x) = N(x,t,t_f) - \bar{N}(t,t_f) = \delta N(x,t), \tag{4.50}$$

with N the number of e-folds computed from t to t_f determined locally by the solution relative to the dynamics of the background, with t_f time greater than t, the exit time of the scale in question. In the model we have:

$$\delta N(x,t_i) = \left(\frac{\partial N}{\partial \phi}\delta\phi + \frac{\partial N}{\partial \chi}\delta\chi + \frac{1}{2}\frac{\partial^2 N}{\partial \phi^2}\delta\phi^2 + \frac{\partial^2 N}{\partial \chi \partial \phi}\delta\chi\delta\phi + \frac{1}{2}\frac{\partial^2 N}{\partial \chi^2}\delta\chi^2 \right)_{t=t_i}, \tag{4.51}$$

where the fluctuations $\delta\phi$ and $\delta\chi$ are calculated at time t_i in a flat slicing gauge. The hypothesis of the model specifies that having constructed most of the first inflationary period to be fundamentally slow-roll (of which we know the curvature spectrum), it is possible to concentrate the study in a time period after the decay of the first field; in fact the simplicity of this argument lies in understanding that at this time t_i (of exit from the horizon of the scales involved) the fluctuations of the first field $\delta\phi$ have gone to zero, simplifying the problem

$$\delta N(x,t_i) = \left(\frac{\partial N}{\partial \chi}\delta\chi + \frac{1}{2}\frac{\partial^2 N}{\partial \chi^2}\delta\chi^2 \right)_{t_i}, \tag{4.52}$$

if we integrate we can obtain the right number of e-fold for the theory

$$N = \frac{\chi^2}{2M_p^2} + \chi\frac{V_0 + g\Lambda_0^3\phi_e}{\varepsilon M_p^2 \Lambda_0^3} + O((\phi-\phi_e)^2/M_p^2). \tag{4.53}$$

From this result, the curvature power spectrum is defined in the model:

$$P_\zeta(k) = \frac{k^3}{2\pi^2}\left|\frac{\partial N}{\partial \chi}\right|^2 |\delta\chi_k|^2 (t_i) \simeq \frac{H^2}{8\pi^2 M_p^2 \epsilon_{\chi\chi}}\mathcal{A}^2(k), \tag{4.54}$$

with

$$\mathcal{A}^2(k) = 1 + \mathcal{A}^2(k*)\exp\left(-\frac{\ln^2 k/k_*}{2\Delta^2}\right). \tag{4.55}$$

Trivially, such a script is inaccurate, as it would still lack the contribution to the spectrum of the first fluctuation, as the general formula of formalism would have it. However, since the first is basically a slow-roll contribution (and one that is known), it can be ignored initially and added later.
It is possible to calculate the NG associated with the curvature perturbation

$$\frac{3}{5} f_{NL}^{local} = \frac{\partial_{\chi\chi} N}{2(\partial_\chi N)^2} = \epsilon_{\chi\chi}. \tag{4.56}$$

therefore this quantity is thought, at least initially, to be small. However, since the fluctuation involved $\delta\chi$ is boosted exponentially during the induction, the development of a non-trivial non-Gaussianity term is expected. It is crucial to note that only the fluctuation $\delta\chi$ makes any contribution to the spectrum, with $\delta\phi$ not appearing explicitly in the result.

Induced GWs during radiation

We write the perturbative metric, choosing a Newton's Gauge for scalar perturbations, as follows:

$$ds^2 = -a^2(1+2\Psi)d\tau^2 + a^2[(1+2\Phi)\delta_{ij} + h_{ij}]dx^i dx^j, \tag{4.57}$$

where Ψ is the Newton potential, Φ the curvature perturbation and h_{ij} the transverse and traceless tensor perturbation. In the absence of an anisotropic stress component we have trivially $\Psi = -\Phi$. It is now possible to study the induced second-order perturbative GWs originating from the interaction of first-order scalar perturbations. In Fourier space we decompose the tensor perturbation

$$h_{ij}(\tau, \mathbf{x}) = \sum_{\lambda=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \exp(i\mathbf{k}\mathbf{x}) h_{\mathbf{k}}^{\lambda}(\tau) e_{ij}^{\lambda}(\mathbf{k}), \tag{4.58}$$

with e_{ij}^{λ} the polarization tensor, with $\lambda = +, \times$ the two polarization-state. We could write the dynamics for the tensor modes in the usual way

$$h_{\mathbf{k}}^{\lambda''}(\tau) + 2\mathcal{H} h_{\mathbf{k}}^{\lambda'}(\tau) + k^2 h_{\mathbf{k}}^{\lambda}(\tau) = S_{\mathbf{k}}^{\lambda}(\tau), \tag{4.59}$$

while the source term is written like

$$S_{\mathbf{k}}^{\lambda}(\tau) = 2\int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \mathbf{e}^{\lambda}(\mathbf{k}, \mathbf{p}) \left[2\Phi_{\mathbf{p}}(\tau)\Phi_{\mathbf{k}-\mathbf{p}}(\tau) + \left(\Phi_{\mathbf{p}}(\tau) + \frac{\Phi'_{\mathbf{p}}(\tau)}{\mathcal{H}}\right)\left(\Phi_{\mathbf{k}-\mathbf{p}}(\tau) + \frac{\Phi'_{\mathbf{k}-\mathbf{p}}(\tau)}{\mathcal{H}}\right)\right]. \tag{4.60}$$

where $\mathbf{e}^{\lambda}(\mathbf{k},\mathbf{p}) = e^{\lambda}_{lm}(\mathbf{k}p_l p_m)$. We can write the solution using the retarded Green function $g_{\mathbf{k}}(\tau,\tau')$,

$$h^{\lambda}_{\mathbf{k}}(\tau) = \frac{1}{a(\tau)}\int_{-\infty}^{\tau} d\tau_1 a(\tau_1) g_{\mathbf{k}}(\tau,\tau_1) S^{\lambda}_{\mathbf{k}}(\tau_1). \tag{4.61}$$

During radiation domination, the perturbation term Φ can be expressed in terms of the conserved comoving curvature perturbation

$$\Phi_{\mathbf{k}}(\tau) = \frac{2}{3}T(k\tau)\mathcal{R}_{c,\mathbf{k}}, \tag{4.62}$$

with $T(k\tau)$ transfer function.
Using Green's typical function for the radiation domination period, we find [54] [188]

$$P_h(k,\tau) = \int_0^{+\infty} dy \int_{|1-y|}^{1+y} dx \left[\frac{4y^2-(1+y^2-x^2)}{4xy}\right]^2 \times P_\zeta(kx)P_\zeta(ky)F(k\tau,x,y), \tag{4.63}$$

The resulting energy density of the induced GWs background at the present time $\Omega_{GW}(f)h_0^2$ is shown in the figure below (4.5).

Induced GWs during Inflation

During the inflationary period, the two fluctuations associated with the two scalar fields act as the generating source of an induced GWs background in the accelerated expansion phase: since these fluctuations are exponentially boosted, a large source term is expected, hence a large induced GWs background [189].
In Sub-Horizon scales during inflation a convenient gauge choice is flat slicing. The source term mentioned earlier in such a phase can be written as follows

$$S^{\lambda}_{\mathbf{k}}(\tau)\frac{2}{M_p^2}\int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\mathbf{e}^{\lambda}(\mathbf{k},\mathbf{p})\delta\phi_{\mathbf{p}}(\tau)\delta\phi_{\mathbf{k}-\mathbf{p}}(\tau) + (\phi \iff \chi). \tag{4.64}$$

Being interested in studying GWs on SubHorizon scales in inflation, we can approximate the background to a De-Sitter space, with $a(\tau) = -1/(H\tau)$ and $\mathcal{H} = a'/a = -1/\tau$, $(-\infty < \tau < 0)$. Knowing the delayed Green's function during inflation we find conclusively [190]

$$\begin{aligned} P_h(k,\tau_{end}) &= \frac{4}{\pi^4 M_p^4}k^3\int_0^{+\infty} dp p^6 \int_{-1}^{+1} d\cos(\theta)\sin^4(\theta) \\ &\times\left|\int_{\tau_0}^{\tau_{end}} d\tau_1 g_k(\tau_{end},\tau_1)(\delta\phi_p(\tau_1)\delta\phi_{|k-p|}(\tau_1) + \delta\chi_p(\tau_1)\delta\chi_{|k-p|}(\tau_1)\right|^2, \end{aligned} \tag{4.65}$$

where $\tau_{end} \simeq 0$ defines the conformal time of the end of inflation. A numerical integration of the following conclusion yields the present energy density spectrum, also represented in the same figure as before together with the radiation GWs background and the sensitivity curve of LISA.
It is the key message of the model to recognize that the GWs background induced during inflation is fundamentally induced by $\delta\phi$, as this is much larger than the second fluctuation, which does indeed grow exponentially, but it must be remembered that this growth is developed by induction through the first-field fluctuation, which has a longer exponential growth time-frame. If in an inflationary model there are no features on a slow-roll basis (thus remaining in such basic dynamics), we will have a very small slow-roll parameter. This condition deliberately destroys the inflation-induced GWs background with respect to the radiation-induced one, and the reason for this is quite simple: the former GWs background is induced in convolution by the fluctuation spectrum of the inflationary field, while the latter is generated in convolution by the curvature spectrum. These functions are trivially connected by the following script

$$P_\Phi^2(k)/M_p^4 \simeq \epsilon^2 P_\zeta^2(k). \tag{4.66}$$

Therefore, by not assuming a receding feature from the slow-roll, from its almost scale invariant spectrum with an infinitesimal ϵ value, it is observed that P_Φ is completely negligible with respect to P_ζ, therefore the argument concerning the induced GWs background follows. In the model in question, since $\delta\phi$ is much larger than $\delta\chi$, there is a much larger induced GWs background in inflation than in radiation, since

$$\frac{\Omega_{GW}^{inf}}{\Omega_{GW}^{rad}} \simeq \epsilon_{\chi\chi} \frac{P_\Phi^2(k_*)}{P_\chi^2(k_*)} >> 1. \tag{4.67}$$

expressing a comparable relative width factor of the order of $O(10^5)$.

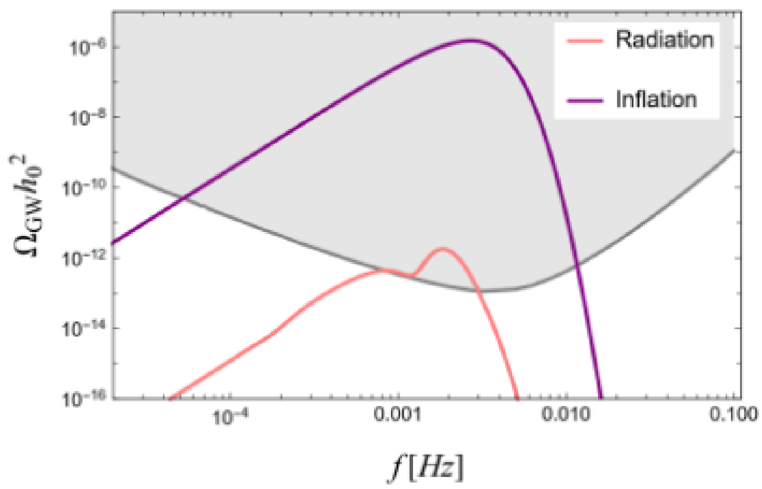

Figure 4.5: Induced GWs spectral energy density in the two dominance phases studied [173]. The grey area determines the sensitivity range of Lisa [191].

4.3 Fast Roll Inflation

This section will be based on the reading of the article [192].
It is the goal of the following section to show the adaptability of a simple theory of spontaneous symmetry breaking with a fast roll inflation phase, where the classical slow-roll conditions $|m|^2 << H^2$ are violated. Nevertheless, it is possible to see how for sufficiently small mass values it is possible to turn this new inflationary phase into a long period. This inflationary phase can be taken to generate the initial conditions for a subsequent and observationally expected slow-roll phase, again in the primordial universe. The fast roll could even justify and explain the current accelerated expansion phase in the universe. It is important to summarize the idea that the Universe, after a long inflationary phase by fast roll (or slow-roll) cannot reach an anti-de-Sitter regime even if the cosmological constant turns out to be negative. Furthermore, for theories with a negative stable minimum on the potential at $V(\phi) < 0$, at the cosmological background level, they exhibit the same instabilities as theories with an unlimited potential from below. Such instabilities lead to the development of singularities with properties that are thus completely independent of the potential theory. The most famous and realistic versions of inflationary models, such as new inflation, chaotic inflation or hybrid inflation are based on the existence of a scalar field ϕ that drives the accelerated expansion dynamics by satisfying the known slow-roll conditions. The simplest is the one for which $|m|^2 << H^2$, where m is the mass of the inflaton and H is the Hubble constant. Using the convenient system of natural units $M_p = 1$, we have that

$H^2 = V/3$, while the condition $|m|^2 << H^2$ can be expressed as $\eta = \frac{|V''|}{|V|} << 1$. The second slow-roll condition can be defined as $\epsilon = \frac{1}{2}(\frac{V'}{V})^2 << 1$ [34].
The consequences of these and applied slow-roll conditions are to make the accelerated expansion stage sufficiently long and to guarantee an approximately scale-invariant curvature power spectrum. The density perturbations with which such a spectrum is associated are obviously the products of the primordial scalar inflationary perturbations, defining the seeds of subsequent large-scale structure-forming processes in the known universe. Under slow-roll conditions one finds how the tilt of such a scalar spectrum is of the form $n - 1 \simeq 2\eta - 6\epsilon$ [34].
Recent observations on the anisotropy of the CMB agree with such inflationary hypotheses for which an almost null tilt would result in the definition of a scale-invariant spectrum.
However, there are several theories for which $|m|^2 \simeq O(H^2)$: for example, if one has a scalar field ϕ not minimally coupled to gravity, this can acquire a correction on the mass terms of the form $\Delta m^2 = \zeta R$. R is the curvature constant and during inflation is $R = 12H^2$. From the choice of a conformal coupling for which $\zeta = 1/6$, we find that $\Delta m^2 = 2H^2$. A similar situation occurs in theories of super-gravity $N = 1$ [193].
We could ask ourselves whether such slow-roll violation theories can describe the accelerated expansion of the Universe.
The aim of this section is to show how a fast-roll inflationary phase could be fundamental in generating consistent initial conditions for a subsequent slow-roll inflation phase in the primordial universe. Such an inflationary moment could further last long enough to justify the current accelerated expansion of the Universe. In the last case scenario there are four distinct possibilities: the scalar field potential could have a minimum for $V(\phi) > 0$, at $V(\phi) = 0$, at $V(\phi) < 0$ or it could be unbounded from below. It is to be expected that the scalar field will naturally tend towards its minimum, stopping there. If the potential has a positive minimum then the Universe will behave as a de-Sitter space; if the minimum is found for $V = 0$, then once reached the minimum the Universe will behave describing a Minkowski regime, while in the case of a negative minimum the Universe will behave as an anti de-Sitter space with a negative cosmological constant.
However, it is possible to show how, at the level of the cosmological background, the latter case turns out to be more complicated. First of all, and this is a trivial consequence of the Friedmann equation for a flat Universe, the Universe cannot reach an anti-de-Sitter phase with negative cosmological constant after a long inflationary phase. Furthermore, theories with a negative stable minimum exhibit the same types of instability in the cosmological background as theories with unlimited potential from below. Such instability leads to the development of singularities with properties therefore independent of the potential $V(\phi)$, so

the existence of a minimum does not guarantee too much security. On the other hand, the development of such instabilities could occur so slowly that theories with unlimited potential from below (as well as those with a stable negative minimum) can describe the present phase of accelerated expansion of the Universe even if this acceleration is due to a previous phase of fast-roll inflation.

Dynamics of spontaneous symmetry breaking and fast-roll inflation

Consider a theory for a scalar field ϕ with a potential $V(\phi)$ and energy density $\rho(\phi) = V(\phi) + \frac{\dot{\phi}^2}{2} + \frac{(\partial_i \phi)^2}{2}$. In natural units for which $M_p = 1$, the generic Friedmann equation is written

$$H^2 + \frac{k}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\rho(\phi)}{3}. \tag{4.68}$$

where k defines the constant curvature of the Universe, (which can, in good observational approximation, be thought flat). Then

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho(\phi)}{3}. \tag{4.69}$$

Consider a potential that has a maximum at $\phi = 0$. This potential is generally represented as follows

$$V(\phi) = V_0 - \frac{m^2\phi^2}{2}, \tag{4.70}$$

at least in the vicinity of the maximum. From chapter two we know how in the vicinity of the maximum the second derivative of the potential associated with the quadratic field mass, is defined to be negative, therefore perturbative cosmological modes of tachyonic form develop. We assume that this form of potential holds as long as $|\phi| = \phi_*$, where $V(\phi_*) = V_0/2$, i.e.

$$\phi_* = \frac{\sqrt{V_0}}{m}. \tag{4.71}$$

Suppose that such a field rolls starting from the specific initial condition ϕ_0 until it reaches ϕ_* in a time frame Δt with a small initial kinetics and energy gradient, i.e. $\frac{\dot{\phi}^2}{2}, \frac{(\partial_i\phi)^2}{2} << V_0$. If the kinetics should have to decrease due to the present expansion of the universe, the energy density $\rho(\phi) \simeq V(\phi)$ would decrease by a factor of two upon reaching $\phi = \phi_*$, and the Hubble factor would decrease by a factor of $1/\sqrt{2}$. If this field does not slow down instead, then ρ will remain approximately equal to V_0. In both cases H remains approximately constant, and this information helps us to estimate the total growth in accelerated expansion of the Universe at that stage

$$\frac{a(t_*)}{a_0} \simeq \exp\left(Ht_*\right) \tag{4.72}$$

where $H^2 = \frac{V_0}{3}$ and t_* is the arrival time of the field at the critical point.
In theories where $\phi_* >> 1$, the Universe can continue to expand exponentially even if $\phi >> \phi_*$, as for example happens in chaotic inflation,. However this is not our case, in which the arrival of ϕ at ϕ_* defines the end of inflation. Assuming the background component of the scalar field to be sufficiently homogeneous, in $H = const$, it is possible to write the dynamics of this field component as follows

$$\ddot{\phi} + 3H\dot{\phi} = -V^{'} = m^2\phi. \tag{4.73}$$

You should then look for a solution answer of the form $\phi = \phi_0 \exp{(i\omega t)}$. We get

$$\omega^2 - 3iH\omega + m^2 = 0, \tag{4.74}$$

finding that

$$\omega = i\left(\frac{3H}{2} \pm \sqrt{\frac{9H^2}{4} + m^2}\right). \tag{4.75}$$

While the + sign leads to an exponentially decreasing background solution that is likely to disappear, the - sign leads to an exponentially increasing background field ϕ.

$$\phi(t) = \phi_0 \exp{[(HtF(m^2/H^2))]}, \tag{4.76}$$

with

$$F(m^2/H^2) = \sqrt{\frac{9}{4} + \frac{m^2}{H^2}} - \frac{3}{2}. \tag{4.77}$$

Equation (4.75) provides a simple estimate of the growth rate of the Universe during the period of Inflation

$$\exp{(Ht_*)} = \left(\frac{\phi_*}{\phi_0}\right)^{1/F}. \tag{4.78}$$

It is possible to make this result explicit in the two diametrically opposite inflationary limits for which $m << H$ (SR) and $m >> H$ (FR). In the first case we have:

$$F(m^2/H^2) = \frac{m^2}{3H^2}, \tag{4.79}$$

$$\phi = \phi_0 \exp\left(\frac{m^2 t}{3H}\right), \tag{4.80}$$

and

$$\exp{(Ht_*)} = \left(\frac{\phi_*}{\phi_0}\right)^{3H^2/m^2} = \left(\frac{\phi_*}{\phi_0}\right)^{V/|V|''}. \tag{4.81}$$

This result set in SR quantifies what was said earlier, namely that the more true it is that $m^2 << H^2$ the more giant is the growth of the universe during such inflation.
In the opposite case, then in FR with $m >> H$ we read

$$F(m^2/H^2) = \frac{m}{H}, \tag{4.82}$$

$$\phi = \phi_0 \exp{(Ht)}, \tag{4.83}$$

so

$$\exp Ht_* = \left(\frac{\phi_*}{\phi_0}\right)^{H/m}, \tag{4.84}$$

arriving at the exact opposite solution. We must now think in terms of initial conditions. The formula expressed above would allow the initial condition value of the field ϕ_0 to be taken as a very small term, guaranteeing a long and indefinite inflationary phase. Therefore, why doesn't we take $\phi_0 = 0$ and obtain an inflation that persists even during the electroweak spontaneous symmetry breaking [83]? The answer is in the natural perturbative approach, since ϕ_0, at null time, cannot be smaller than the corresponding fluctuation value at $m > k$, which itself is defined in an exponential growth given the tachyonic nature of the problem. When $m >> H$, we have:

$$\delta\phi_k(t) \simeq \delta\phi_k(0) \exp{(\sqrt{m^2 - k^2}t)}. \tag{4.85}$$

Typically the exponential growth of such fluctuations, (rather than the value of the background field itself), gives rise to the subsequent phenomenon of spontaneous symmetry breaking [83].
Choosing the initial conditions for an inflationary process (or SSB problem), a typical initial amplitude for scalar field fluctuations with $m > k$, it is possible to assume a plane wave SR condition when $t = 0$ in the symmetric potential condition in which $\phi = 0$. At the next time the massive term will turn on, hence the tachyonic feature of the fluctuation problem [83]. The initial dispersion of all exponentially increasing modes having $m > k$ will be defined as [194]

$$\langle \delta\phi^2 \rangle = \int_0^m \frac{dk^2}{8\pi^2} = \frac{m^2}{8\pi^2}, \tag{4.86}$$

therefore the initial mean value of the amplitude of all fluctuations is given by $\delta\phi \simeq m/2\pi$. This averaged value, however, represents a slight overestimation. As an example, note how for fluctuations with $k = m/2$ there is a less exponential subhorizon growth than for fluctuations with $k = m/4$. This difference on long time scales, becomes more and more important, in fact requiring the field to roll along the potential in virtue of a time much greater than m^{-1} (condition for having FR), it is clear to observe that the fluctuations that will make the main contribution to the distribution of growing modes will be those related to the modes for which $m >> k$. Therefore such strong exponential growth must be somehow damped by a smaller initial condition in amplitude. We therefore assume that the smallest value that ϕ_0 can ever take is of the form m/C, with $C = O(10)$. One can then substitute this solution in place of ϕ_0, in order to estimate the maximum expansion growth rate of the universe

$$\exp\left(Ht_*\right) \simeq \left(\frac{10\phi_*}{m}\right)^{1/F}. \tag{4.87}$$

It is now possible to develop this result in different models. In the theories associated with the potential discussed so far, with $m = O(H)$ we have that $\phi_* \simeq M_p = 1$, thus

$$\exp\left(Ht_*\right) = \left(\frac{10}{m}\right)^{1/F}. \tag{4.88}$$

In the even simpler case in which $m = H$, we have that $F^{-1}(1) = 3.3$, so

$$\exp\left(Ht_*\right) = \left(\frac{10}{m}\right)^{3.3}. \tag{4.89}$$

Such a solution for $m << 1$ (i.e. when $m << M_p$) can be significantly large. Assume in one improbable construction that such a model is responsible for the present accelerated expansion of the Universe with Hubble constant $H \simeq 10^{-60}$. Fast-Roll inflation in such a case would be able to drive the expansion of the Universe by a factor

$$\exp\left(Ht_*\right) \simeq (10^{61})^{3.3} \simeq 10^{200} \simeq \exp\left(460\right). \tag{4.90}$$

At the same time, taking $m \simeq 10^2$ GeV,corresponding to the electro-weak scale would result in an expansion by fast roll by a factor of

$$\exp\left(Ht_*\right) \simeq (10^{17})^{3.3} \simeq \exp\left(130\right). \tag{4.91}$$

Fast-Roll Inflation and scalar field perturbation in the very early Universe

It is now necessary to analyze the fast-roll inflation as a possibility of interpreting the primordial Universe, assuming a mass sufficiently smaller than the Planck mass; in the light of what has been said before, in such a case, inflation may be very short and inefficient, however it may still have its place. Suppose that in addition to the scalar field ϕ with $\left|m^2\right| = O(H^2)$, there is a second scalar field χ that during the fast roll remains without dynamics, i.e. light $\left|m_\chi^2\right| << \left|m^2\right| = O(H^2)$. Therefore it follows that the perturbations of these broad-wavelength fields are generated: since the FR is a very short expansive phase, the short modes are not even affected by the resulting exponential growth, since they remain in the Hubble sphere for a long time, thus experiencing the subsequent slow-roll phase that causes the overall oscillation and decay of the short modes. On the contrary, the long modes are strongly affected by the boost provided by the FR and have no way of balancing this growth with the subsequent SR decay since, given their width, they are able to exit and freeze immediately. If inflation occurs during time t the average amplitude of these fluctuations becomes [194]

$$\sqrt{\langle\chi^2\rangle} = \frac{H}{2\pi}\sqrt{Ht}, \tag{4.92}$$

term that can in certain occurrences even reach large values close to 10^{18} GeV. Such an effect can be very useful in defining the correct initial conditions for a subsequent slow-roll phase, if the light χ field decides to play the role of the inflaton. In fact, it is possible to imagine a first FR inflationary phase in which the ϕ field energetically dominates over the second χ field: the dynamical coupling on the field fluctuations leads to an exponential growth of the same. Once the FR is over, when ϕ falls into its hole and decays, the $\delta\chi$ fluctuation remains alone and therefore dominant (from entropic to adiabatic): this through the previous phase has reached a high amplitude value that could define the initial condition (i.e. the height point of $\delta\chi_k$ from which the perturbation starts due to SR to oscillate and decay) for the subsequent slow-roll phase.
There are several inflationary models, including chaotic inflation with its polynomial format potential, which predicts that inflation can start during the Planck era and continue until $V(\phi)$ becomes sufficiently small. This last condition is necessary knowing the analytical link between V and H and knowing that the latter determines the amplitude of scalar density perturbations and tensor modes. Therefore this condition becomes necessary since one does not want an overproduction of such observables. However, such models can enter into a definition of eternal inflationary reproduction, if the Universe starts such a phase in a spatial domain on sufficiently homogeneous Planck scales [194].
However, there are other inflationary models for which inflation can only actually start long after the Planck conditions, i.e. when H becomes much smaller than the respective Planck mass. For models in which inflation arises near the maximum of the potential (an example are the various modifications of hybrid inflation) the Hubble constant doesn't change its initial value. Therefore, in order to avoid an exaggerated production of GWs a restriction on V (thus on H) must be required, so that during inflation it must be true that $V < 10^{-10}M_p^4$. Therefore inflation starts much later, for times in which $t \simeq H^{-1} >> M_p^{-1}$. This statement leads to the problem with initial conditions, since if inflation is delayed in starting, appro-

priate initial conditions from an earlier stage will be needed to define it. Moreover, it is worth mentioning the difficulty in explaining an excessive homogeneity of the Universe at the beginning of inflation, i.e. long after Planck time, when the increasing scalar fluctuations will have produced an important inhomogenisation of the Universe. If the Universe were closed and dominated by non-relativistic matter, it would collapse before the inflationary phase could begin.

However, such problems end with the assumption of an initial "bad inflation" phase producing unacceptable density and tensor perturbations, which are then, at least on the smallest scales, readjusted by the subsequent SR phase that decays and oscillates these solutions in the Hubble sphere. The idea is to assume that this pre-inflation is described by the fast-roll, which can begin in the Planck era. Such a phase, (which has generally been described as short and inefficient) can however last a longer time if one forces the scalar field to remain bound at the top of its effective potential. There are different techniques in order to build such a system: firstly, one can require a coupling condition in interaction between the scalar field ϕ and other fields present in the Universe, the latter of which, in order to force ϕ to remain on the maximum in zero, must preserve a thermal equilibrium condition [194]. Nevertheless, this condition goes against the determination of small but observable density perturbations, which are achievable only if the interaction is extremely weak, thus negating the thermal equilibrium condition.

Here lies the problem with the initial conditions, as we would like an inflation that begins long after the Planck time, and so the previous FR must be required to last a proper time limit in order to arrive at such a phase. Another solution, adopted in cases of Hybrid Inflation, consists in the creation of interaction with other classical fields. A last chance is offered by considering a possible quantum creation from nothing. Such a process is plausible given that the period of development of the FR is that of the Planck scales, and a creation of the particle systems defining the Universe directly on top of the potential is assumed, describing the condition of maximal elongation of the FR, in order to reach the correct starting conditions of the actual Slow-Roll Inflation. Even in the latter case the Fast-Roll stage may be short-lived, but long enough to define in its conclusion a wide and relatively homogeneous Universe that avoids the possibility of collapsing before inflation.

Due to inflation and the subsequent phase of accelerated expansion of the Universe, the fluctuations of the secondary field χ reach high amplitude values and will be brought to become classical once they leave the horizon. However, these fluctuations take on different values depending on the region of the Universe in which one finds them. Those spaces in which the aforementioned fluctuations define optimal initial conditions for a necessary SR-driven inflationary start, will expand exponentially, while regions with poor initial conditions will be suppressed. The pre-inflationary phase solves the problems on initial conditions. However, even a slow-roll-pre-phase

would work discretely from this perspective. It must be concluded that a short FR phase could help in solving the SSB problem in SUSY GUT, justifying the stronger version of the anthropic principle.
The new inflation model is able to justify a Universe composed of several exponentially large parts with different laws of physics in each, providing one of the first interpretations of the anthropic principle. It is possible in such a view to discuss the spontaneous symmetry breaking of a supersymmetric model described by the SU(5) group. This theory predicts at extraordinarily high temperatures a single SU(5) minimum: as the temperature decreases, the system may collapse to the minimum of an SU(4)$\times$ U(1) theory or to the minimum of an SU(3)$\times$SU(2)$\times$ U(1) theory. Nevertheless these minima have the same GUT height, hence the same probability of approach. Since the primordial universe is initially hot and defined by a potential with only one minimum of SU(5) therefore the SSB is neglected, although it is not clear how to choose, when the temperature goes down, the minimum in which to fall with respect to the two theories defined above. The answer is in the value of the large perturbations of the scalar field ϕ produced by inflation: there will be regions in the universe in which the value of such field perturbations are such as to make the system fall on a minimum of a theory SU(3)$\times$SU(2)$\times$ U(1), and regions in which the same will happen relative to the minimum of the complementary theory, thus descending from the minimum of SU(5), to the minimum of SU(4) $\times$ U(1). Therefore, to justify a first form of the anthropic principle, there are different regions with different physical laws; we live in the one in which the SSB drives towards the minimum of the known and valid SM theory.

4.4 Induced GWs from slow-roll after an enhanincg phase

This section is based on the reference [195].
The presence of features during inflation can cause a boost on inflationary scalar perturbations, and result in the presence of large peaks in the primordial spectrum of these fluctuations. Examples of features may include bumps in the potential, ultra slow-roll phases or trajectory shifts in a space of multiple fields [196] [147] [197] [155] [152] [173]. However, it must be noted that an inflationary model does not necessarily have to end with the description of a boost feature, but the latter can continue with a subsequent, secondary slow-roll phase. Potentially such a second slow-roll inflationary phase should not produce, at second order, an amount of induced GWs background considerable in amplitude. Nevertheless, by constructing a model in which the primordial spectrum is written as a combination of several pieces (among which the second SR), one will not only have the GWs background induced by the peak-to-peak convolution, or SR-SR, but the mathematics of the problem will also bring mixed convolution terms associated to the interactions of modes experiencing the two distinct inflationary moments of the system. We are able to study a source-induced background defined by the interaction of a scalar mode of the peak that undergoes the increasing features with the mode of the subsequent SR that decays.
GWs provide a promising window in order to study the physics of the primordial universe, and they could be generated by induction from primordial scalar fluctuations in the primordial universe. Such primordial density perturbations are strongly studied and constrained on the broad CMB scales, and originate from the quantum-derived fluctuations associated with the inflationary period. However, while on the large scales one has a clear idea of what value such density perturbations should have, the same cannot be said on the smaller scales, where the spectrum of perturbation curvatures can take on the most disparate forms as long as one does not have stronger observational constraints.
Therefore, the presence of peaks on the small scales in the scalar spectrum can, in the convolution product, also give rise to a peak on the induced GWs background during the radiation phase. An increase in the primordial curvature perturbation spectrum also leads, as already explained, to the further overproduction of PBH. However, the interest of this section shifts to underline how a peak in the scalar spectrum (for which a convolution is applied in order to determine the induced GWs background) induces an increase in the abundance of the gravitational wave background. However, the inflation does not necessarily have to end with such a peak induced by a sharp feature, but can continue with a second, subsequent slow-roll phase, in which the corresponding SR parameter is assumed to be a free

parameter (it therefore follows that the amplitude of this spectral part is also a free parameter since it is defined by the second ϵ).
The aim of this section is to study the induced GWs background generated by such a second slow-roll phase: as mentioned before, in the scalar convolution the interaction case between the peak spectrum and the free amplitude spectrum associated to the second slow-roll can be considered, identifying a crossing GWs background induced by the interaction of the radiation-scalar modes (in radiation) associated to a boosting phase and a different phase of oscillation with decay in the cosmic expansion. Thus the final solution will consist in the study of a GW background induced in part by a flat scalar contribution, with such a background that will not carry over from the convolution the expected (and observable for an interaction of SR-SR modes) plateau.

Induced GWs from two stages of inflation

In this section we consider an inflationary solution composed of two distinct slow-roll stages joined by a special boosting feature on the primordial scalar perturbations. The first slow-roll phase is standard and is used to justify the correct amplitude of the primordial fluctuations to explain the CMB anisotropies, and will be denoted by $\mathcal{A}_{CMB}$. Then an inflationary feature will be defined that will increase the scalar modes of primordial fluctuation by defining a peak in the curvature spectrum, with amplitude $\mathcal{A}_{peak}$. Thereafter, inflation continues and ends with a secondary slow-roll phase, and the spectrum designated to this phase is assumed to be quasi-scale-invariant with an arbitrary amplitude called $\mathcal{A}_{flat}$. This function represents a free parameter in the theory, so we don't fix the value for the first slow-roll parameter ϵ. The smaller it is chosen, the larger is the value of $\mathcal{A}_{flat}$. It is assumed in the model that $\mathcal{A}_{CMB} << \mathcal{A}_{flat}$, so this idea allows to neglect the induced contribution provided by the first inflationary phase.
In light of what has been said so far, it is possible to define the approximate primordial power spectrum of curvature fluctuation

$$P_\zeta(k) = \mathcal{A}_{peak} P_{\zeta,peak}(k/k_p) + \mathcal{A}_{flat} P_{\zeta,flat}(k/k_p), \tag{4.93}$$

where $P_{\zeta,peak}(k/k_p)$ is a sharp function in the scale $k = k_p$, and $P_{\zeta,flat}(k/k_p)$ is a step function having step in $k = k_p$, assuming non-zero and unit values after the peak scale. It is expected that this formulation can approximate well models where a sharp feature defines a sharp transition between a first and a second slow-roll phase. Where, on the other hand, the transition between the two phases is more gradual, (4.92) is no longer as accurate, although it still allows the system to be solved in a good approximation.
The assumption made in equation (4.92) makes it possible to split the primordial scalar power spectrum into two contributions, and this approximation allows a

clear and mathematical separation of the total induced GWs background: in fact there will exist the GWs background induced by the interaction in convolution of two $P_{\zeta,peak}$, the one associated with the convolution of two $P_{\zeta,flat}$, but also the one associated with the interaction convolution between the two distinct terms of peak and flat, which we will call the crossing term.
This crossing term indicates the GWs background induced by the interaction of two distinct scalar modes associated with the two different dynamical moments characteristic of the model, that of boosting and that of oscillation in decay offered by the second slow-roll. It is to be expected that the more the width of the second scalar mode k, close to the large one having length k_p, is assumed at the horizon re-entry, the more the decay time of the larger mode is limited, promoting the development of a larger source term able to produce a larger GWs background: therefore a smooth decreasing trend in frequency is expected for such a crossing background. The Universe after inflation is assumed to be dominated by a radiation phase but with an arbitrary adiabatic speed of sound. Therefore the resulting induced GWs spectrum produced during the radiation phase evaluated today is written

$$\Omega_{GW,0}h^2 = \Omega_{r,0}h^2 \left(\frac{g_*(T_c)}{g_{*,0}}\right)\left(\frac{g_{*s}(T_c)}{g_{*s,0}}\right)^{-4/3} \Omega_{GW,c}, \tag{4.94}$$

where $\Omega_{r,0}$ is the current radiation density today and $g_*(T)$ and $g_{*s}(T)$ are the effective degrees of freedom in the energy and entropy density at temperature T. Thanks to the data offered by the Planck satellite we observe $\Omega_{r,0}h^2 \simeq 4.18 \times 10^{-5}$ [38]. It is also found that $g_{*,0} = 3.36$ and that $g_{*s,0} = 3.91$. It is also understandable how for $T > 100$ GeV and assuming only standard model particles, we have $g_*(T) = g_{*s}(T) = 106.75$. The 'c' term [198] instead indicates the evaluation at the time when the spectral energy density is constant, i.e. at the time when the tensor modes are well contained within the Hubble horizon. Therefore we read [119] [144]

$$\Omega_{GW,c} = \int_0^\infty dv \int_{|1-v|}^{1+v} du \mathcal{T}(u,v,c_s)\mathcal{P}_\zeta(ku)\mathcal{P}_\zeta(kv), \tag{4.95}$$

with u and v being dimensionless variables related to the internal scalar momenta q and $|\mathbf{k}-\mathbf{q}|$ via the relations $v = q/k$ and $u = |\mathbf{k}-\mathbf{q}|/k$;in the convolution is also present the transfer function, also called kernel, $\mathcal{T}(u,v,c_s)$ defined as

$$\begin{aligned} \mathcal{T}(u,v,c_s) = \frac{y^2}{3c_s^4}\left(\frac{4v^2-(1-u^2+v^2)^2}{4u^2v^2}\right)^2 \times \\ \left[\frac{\pi^2}{4}y^2\Theta[c_s(u+v)-1] + \left(1-\frac{1}{2}y\ln\left|\frac{1+y}{1-y}\right|\right)^2\right], \end{aligned} \tag{4.96}$$

with

$$y = \frac{u^2 + v^2 - c_s^{-2}}{2uv}. \tag{4.97}$$

The transfer function (4.95) [119] [199] [199] is assumed for a primordial universe dominated by a radiation fluid, i.e. having a state relation for which $\omega = 1/3$, but with an arbitrary adiabatic sound speed for scalar fluctuations c_s. For a perfect adiabatic fluid we have that $c_s^2 = \omega = 1/3$, but for a canonical scalar field having a potential of exponential format we have that $\omega = 1/3$ but $c_s^2 = 1$ [200]. It is permissible to consider c_s^2 as a free parameter of the theory, and it will become clear later how in certain frequency regimes the velocity term will have an important influence on the trend of the final induced GWs background spectral energy density. Substituting in the formulation of the spectral energy density (4.94) the definition of the primordial curvature spectrum (4.92), the total GWs abundance can be defined through the sum of several mathematical contributions

$$\Omega_{GWs,c} = \mathcal{A}_{peak}^2 \Omega_{GW,peak} + 2\mathcal{A}_{peak}\mathcal{A}_{flat}\Omega_{GW,cross} + \mathcal{A}_{flat}^2 \Omega_{GW,flat}. \tag{4.98}$$

It is essential to recall how this separation is only possible due to the original assumption of breaking the scalar curvature spectrum into several complementary components.
In order to define $\Omega_{GW,peak}$, it suffices to substitute in the overall spectral energy density (4.94) $\mathcal{P}_{\zeta,peak}$, the same for the totally flat case. The study of the cross term, as mentioned before, evaluates the scalar interaction of convolution of the peak and flat regime modes in the following way

$$\Omega_{GW,cross} = \int_0^\infty dv \int_{|1-v|}^{1+v} du \mathcal{T}(u, v, c_s) \mathcal{P}_{\zeta,peak}(v/v_p) \mathcal{P}_{\zeta,flat}(u/v_p), \tag{4.99}$$

where $v_p = k_p/k$; Furthermore, the exchange symmetry between the momenta u and v has been exploited, marked by the number 2 in (3.97). The aim of the section lies in calculating the new crossing contribution as the free parameter c_s^2 varies, since the other integral contributions have already been calculated [157] [126] [151]. It is necessary at this point to interpret the problem by studying the trend of the GWs background in an infrared regime where $k << k_p$, and an ultraviolet one where $k >> k_p$.

Low-frequency (IR) approximation

In the IR limit, i.e. $k << k_p$, or equivalently $v_p >> 1$, it is safe to assume that since $\mathcal{P}_{\zeta,peak}$ is strongly spiked, only momentum values v very close to the peak value v_p will actively contribute to the integral for v. Therefore it must be assumed that $v >> 1$, so u being constrained by the law of conservation of momentum by $|1-v| < u < 1+v$, we have that $u \simeq v >> 1$. In such a regime the transfer function approximates as follows

$$\mathcal{T}(u \simeq v >> 1, c_s) \simeq \frac{1}{c_s^4} v^{-4} \ln^2 v. \tag{4.100}$$

Taking advantage of the mean value theorem of the integral, we evaluate the integral result in u for $u \simeq v$, producing a minimum error. This results in summary:

$$\Omega^{IR}_{GW,peak/cross} \simeq \frac{1}{c_s^4}\left(\frac{k}{k_p}\right)^3 \ln^2\left(\frac{k}{k_p}\right) \times \left(\mathcal{P}_{\zeta,peak/flat}(k=k_p)\right) \int_0^{\infty} dV \mathcal{P}_{\zeta,peak}(V) V^{-4}, \tag{4.101}$$

where $V = v/v_p$ has been defined. It is therefore inferred that both contributions to the peak and crossing background decay in the infrared as $k^3 \ln^2 k$, defining the typical universal IR behaviour of an induced GWs background during the domination of radiation [199]. Since $\mathcal{A}_{peak} >> \mathcal{A}_{flat}$, the contribution of the cross term to the total, will always be subdominant to that offered by the scalar peak convolution in the IR. This cross term will indeed become more interesting for high frequency limits.

High frequency approximation (UV)

The ultraviolet limit on the external frequency of the GWs background is written through the condition $k >> k_p$, i.e. $v_p << 1$. Thinking as before about the peak of the scalar function under consideration, we conclude that the only analytic region to make an active contribution to the integral on the momenta is for $v << 1$, hence $u \simeq 1$ (and the opposite, by symmetry, which can be accounted for by a factor of 2) [157].

In contrast to the IR case, the approximate behaviour of the transfer function in UV tends to vary according to the choice of c_s^2 (taken as an example equal to 1, or the usual value of 1/3). This makes sense in light of the contrast between the resonance induced in $u + v = c_s^{-2}$ and the limits of integration for $u = 1 + v$ and $u = |1 - v|$ where the integral vanishes. Specifically, if $c_s^2 = 1$ the resonance is completely annihilated by the boundaries $u = 1 - v$. The kernel has a different analytical approximation in the UV in the two cases expressed above, and this result is seen thanks to the Taylor expansion of the variable y for $v << 1$ and $u \to 1$

$$y \simeq \frac{1 - c_s^{-2}}{2v} + \frac{v}{2}. \tag{4.102}$$

When $c_s^2 = 1$ the first term cancels perfectly, therefore $y << 1$ for $v << 1$. By contrast, for $c_s^2 < 1$ this consideration no longer exists, so for $v < 1$ we have that

$|y| >> 1$. The cases we wish to discuss are those for which $c_s^2 = 1/3$ and $c_s^2 = 1$. We find in conclusion

$$\Omega^{UV}_{GW,peak/cross}(c_s^2 < 1) \simeq \frac{8}{27}\frac{1}{(1-c_s^2)^2} \left(\frac{k}{k_p}\right)^{-4} (\mathcal{P}_{\zeta,peak/flat}(k/k_p)) \int_0^\infty dV \mathcal{P}_{\zeta,peak}(V) V^3, \tag{4.103}$$

and

$$\Omega^{UV}_{GW,peak/cross}(c_s^2 = 1) \simeq 2\left(\frac{35+24\pi^2}{8505}\right)\left(\frac{k}{k_p}\right)^{-2} (\mathcal{P}_{\zeta,peak/flat}(k/k_p)) \int_0^\infty dV \mathcal{P}_{\zeta,peak}(V) V. \tag{4.104}$$

The most significant conclusion of these results is to observe how the second case defines a smaller decrease than the first in the UV, thus creating an increased crossing GWs background in its tail that could define a higher observable abundance.

Chapter 5

Induced Gravitational waves from inflationary models

In the previous chapter, an examination was made of different inflationary models capable of deviating from a slow-roll inflationary phase; Indeed, Sasaki's model [173], Linde's Fast-Roll model [192], the model in EFTI [158] despite their profound differences, can be united with each other precisely by this characteristic of departure from an initial (or concluding) quasi scale-invariant spectrum phase. It is important to remember the strength of this hypothesis, as defining an inflation model perpetually stationed in a slow-roll phase, defines a slow-roll parameter ϵ always constant and, above all, much smaller than one; hence from the formula linking the scalar curvature spectrum to the power-spectrum of the inflationary fluctuation, it is clear that the former is much larger than the latter. Since the curvature power-spectrum is the main element in producing a second-order GWs background during radiation dominance (just as the scalar field fluctuation spectrum induces the GWs background during the Inflation era), one expects a totally negligible GWs background during Inflation compared to that induced during the subsequent radiation phase. The hypothesis defined earlier comes into play at this point: moving away from the slow-roll implies a consideration of a larger slow-roll parameter (albeit less than one to have inflation); thus the two power-spectrums are placed in a different condition of comparability (in natural units they are completely identical), so, at least in principle, one does not expect a complete domination of one period-induced GWs background over another. This idea match totally well with the practice of this thesis, since its aim is to study the validity of an induced GWs background during inflation, given that the latter is always neglected and therefore not considered in the light of the reasoning proposed earlier.
The goal of this last chapter is to take up the theory of the inflationary models proposed in the last section, and to calculate, for the first time, the spectral energy density induced by such schemes both during radiation and during the domination

phase of the scalar field. For this study, an analytical-theoretical approach will be used, supported, where possible, also by a computational comparison provided by a program for the analytical calculation of the problem (https://github.com/Lukas-T-W/SIGWfast).
The first model to be described will be the one-field fast-roll phase, followed by the EFTI model.

5.1 Scalar GWs induced by Fast-Roll (one-field model)

In this section, the radiation and inflation-induced GWs from a one-field Fast-Roll model will be calculated. The idea of such a model has already been described extensively in the previous chapter; it suffices to recall how such a model is entirely suitable for providing the initial conditions for a subsequent (and necessary) slow-roll phase, describing modes that in sub-horizon grow brutally as exponentials.
However, as the scale k increases, the residence time in the sphere increases, but the exponential growth factor due to the relationship between the mass of the scalar field and the scale itself decreases. It is important to remember how such an inflationary model is built on large scales, such as those of the CMB. Knowing that on such scales the various measuring apparatuses force us to have a spectral value of the order of $\mathcal{P}_{\mathcal{R}} \simeq 10^{-9}$, it is normal to assume that the fast-roll phase must then relax subsequently with a slow-roll phase, which latter guarantees the scalar values observed on such scales. It is important to remember how precisely the subsequent slow-roll phase serves to deflate the modes that remained on subhorizon scales throughout the FR phase, which in fact grew a lot, avoiding the creation of unobservable GWs or PBH. However, the latest empirical data released by Planck in 2018 [14], allow us to define a useful variation from the flat phase on the large scale of the problem. It is therefore possible to construct theoretical inflationary models that can fit such curvature data, and one of these, is precisely the Fast-Roll model; recall how this defines a strong red-tilt phase on the scalar spectrum, on the CMB scales, followed by the observed quasi-scale-invariant section produced by the slow-roll. This Planck review [14] reports a parameterization of the curvature spectrum due to such an inflationary construction thus writable [14]

$$\mathcal{P}_{\mathcal{R}} = \mathcal{P}_{\mathcal{R}}^{(0)} \gamma_{kin} \left(\frac{k}{k_c} \right). \tag{5.1}$$

Here we define $\mathcal{P}_{\mathcal{R}}^{(0)}$ the curvature power-spectrum of the next slow-roll phase, assumed scale-invariant in the modelling; k_c is the dip scale in which the change of inflationary regime from FR to SR takes place. It is therefore the last scale that freezes in the moment concluding the FR and is therefore the last scale that is subject to the red-tilt. The subsequent scales in fact, remaining even more in the Hubble sphere will experience the slow-roll phase where they will be made to oscillate and decay in order to define a flat scalar spectrum compatible with observations on typical values mentioned earlier. It is important to note how this formulation is modified when the condition for which $k >> k_c$ occurs (due to the reasoning just described). Therefore we can rewrite

$$\mathcal{P}_{\mathcal{R}} \to \mathcal{P}_{\mathcal{R}}^{(0)} \quad , (k >> k_c). \tag{5.2}$$

The function γ expresses the parametrization for the fast-roll, thus describing the inflationary phase of strong red tilt. In Planck's review [14] is considered an initially generic function, as other models besides the fast-roll can provide scalar values that can be adapted to the observations. However, in the problem addressed, the kinetic theory of the fast-roll is considered, whereby

$$\gamma_{kin}(y) = \frac{\pi}{16} y \left| C_c(y) - D_c(y) \right|^2 , \tag{5.3}$$

with:

$$C_c(y) = e^{-iy} \left[H_0^{(2)}(\frac{y}{2}) - (\frac{1}{y} + i) H_1^{(2)}(\frac{y}{2}) \right] , \tag{5.4}$$

$$D_c(y) = e^{iy} \left[H_0^{(2)}(\frac{y}{2}) - (\frac{1}{y} - i) H_1^{(2)}(\frac{y}{2}) \right] , \tag{5.5}$$

with $H_0^{(2)}(\frac{y}{2})$ e $H_1^{(2)}(\frac{y}{2})$ the Henkel function at the second order.

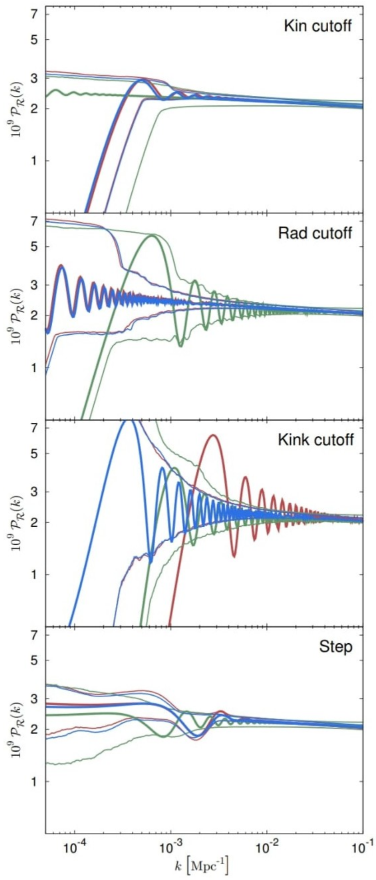

Figure 5.1: Power-spectrum curvature of the problem presented in equation (5.1). This results in an excellent parametrization of the Fast-Roll problem. Clearly, it should be read in the scales to which the problem belongs, i.e. the CMB scales ($10^{-3}\mathrm{Mpc}^{-1}$); therefore we have to consider the function from the red-tilt onset. [14].

5.1.1 Spectral energy density of GWs from Radiation epoch

Recall how the induced GWs background during radiation is written as [119]:

$$\Omega_{GWs,c}(k) = \int_0^\infty dv \int_{|1-v|}^{1+v} du \mathcal{T}(u, v, c_s)\mathcal{P}_\mathcal{R}(ku)\mathcal{P}_\mathcal{R}(kv), \tag{5.6}$$

with

$$v = \frac{q}{k}, \tag{5.7}$$

and

$$u = \frac{|\mathbf{k} - \mathbf{q}|}{k}. \tag{5.8}$$

It is important to remember that the c in the definition of the induced GWs background follows from the Inomata nomenclature [198]; therefore from now on we are calculating the induced GWs background during radiation in the period in which this abundance is constant, i.e. during radiation, where GWs behave exactly like radiation.
The radiation-dominated kernel is the one addressed in Chapters 3 and 4, and is written [119] [195]:

$$\mathcal{T}(u,v,c_s) = \frac{y^2}{3c_s^4}\left(\frac{4v^2-(1-u^2+v^2)^2}{4u^2v^2}\right)^2 \times \\ \left[\frac{\pi^2}{4}y^2\Theta[c_s(u+v)-1] + (1-\frac{1}{2}y\ln\left|\frac{1+y}{1-y}\right|)^2\right]. \tag{5.9}$$

Here we have to remember that

$$y = \frac{u^2+v^2-c_s^{-2}}{2uv}. \tag{5.10}$$

The scale invariant curvature spectrum proposed by the slow-roll phase is defined instead:

$$\mathcal{P}_{\mathcal{R}}^{(0)} \simeq \frac{H^2}{\pi\epsilon M_{Pl}^2}. \tag{5.11}$$

The exact way to analytically calculate the induced GWs background during radiation would be to substitute the curvature spectrum in the definition of the GWs background (5.6); however, it is easy to see that, for the purposes of an analytical-theoretical approach, this road is particularly difficult, given the complexity of the scalar spectrum. Therefore what is best to do is to define a decomposition of the power-spectrum curvature function into its infrared (IR), intermediate (IM) and ultraviolet (UV) limits. The idea is to calculate the double integral of convolution on the power spectrum using these approximations; we will therefore calculate pieces of the induced GWs background, which, when combined at the end, will allow us to have a complete picture of the solution. So the calculation will be performed by substituting, in the spectral convolution, first the IR-IR contribution, then the IM-IM contribution, and finally the UV-UV contribution. However, these are not the only contributions that can be studied; in fact, it is also permissible to have modes adhering to different periods of evolution interact in the convolution, so the last cross-talk contributions from the interaction of distinct, unpaired scalar regimes will also be calculated.
Following this reasoning, it is possible to approximate the scalar curvature spectrum. In the infrared limit, i.e. for $y << 1$,(i.e. $k << k_c$), we obtain

$$H_0^{(2)}\left(\frac{y}{2}\right) \simeq 1 - \frac{2i\log\frac{y}{2}}{\pi}, \tag{5.12}$$

$$H_1^{(2)}\left(\frac{y}{2}\right) \simeq \frac{4i}{\pi y}. \tag{5.13}$$

Then in the approximation of the complex functions of the kinetic spectrum, we find that

$$C_c(y) \simeq 1 - \frac{2i\log\frac{y}{2}}{\pi} - \left(\frac{1}{y}+i\right)\frac{4i}{\pi y}, \tag{5.14}$$

$$D_c(y) \simeq 1 - \frac{2i\log\frac{y}{2}}{\pi} - \left(\frac{1}{y}-i\right)\frac{4i}{\pi y}. \tag{5.15}$$

Trivially, their difference becomes:

$$C_c(y) - D_c(y) = \frac{8}{\pi y}. \tag{5.16}$$

Therefore, in the infrared scalar limit, i.e. for $y << 1$ we get

$$\mathcal{P}_{\mathcal{R}}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)}\frac{4}{\pi|k|}. \tag{5.17}$$

The accuracy of this approximation is reasonable, since the infrared part of the scalar spectrum, as is also evident from figure (5.1), must define the red-tilt phase. Therefore we first calculate the IR-IR contribution, so we can write:

$$\Omega_{GWs,c}(k) = \int_0^\infty dv \int_{|1-v|}^{1+v} du\mathcal{T}(u,v,c_s)\frac{16}{\pi^2}\left(\frac{1}{\frac{ku}{k_c}}\right)\left(\frac{1}{\frac{kv}{k_c}}\right). \tag{5.18}$$

From now on we will define $v_c = \frac{k_c}{k}$. It is worth mentioning the use of the infrared approximation for the curvature spectrum inscribed in the scalar convolution. Therefore, the arguments of the two convolution spectra must be much smaller than one in order to guarantee a formulation like the one written above.
Obviously, the arguments are written for the purpose of the convolution, so they will not be defined by the frequency k (i.e. the external frequency associated with the induced GWs background) but rather by $\frac{q}{k_c}$ and $\frac{|\mathbf{k}-\mathbf{q}|}{k_c}$. We can write:

$$\begin{cases} \frac{q}{k_c} << 1 \\ \frac{|\mathbf{k}-\mathbf{q}|}{k_c} << 1. \end{cases} \tag{5.19}$$

From the decomposition of the absolute value, staying within the analytical limits of interest, one solves the system arriving at the solution for which $k << k_c$. A very important result is then defined: the convolution between two curvature spectra in the respective infrared convolution regimes determines the production of the infrared component of the scalar-induced GWs background. Therefore, let us

calculate such a solution, in which $k << k_c$.
In order to define the problem analytically, it is necessary to introduce the modelling of a toy-model able to reproduce the system theory. Since the input curvature spectrum turns out to be strongly red-tilted, we are allowed (on the large scales characteristic of the problem) to assume that this spectrum is imaginable as a Dirac-delta; the dip of this delta is marked by the switch scale k_c, (or by v_c) in the rewriting of the function $\mathcal{P}_{\mathcal{R}}(v/v_c)$. Therefore in the following limit of the problem, we have that $v_c >> 1$ and that consequently $v << v_c$; in the terms of the moment v and the definition of delta, we choose to remain sufficiently distant from the dip v_c, remaining close to the peak: here in fact we will have the maximum integral contribution to the GWs background.
Since v is, in theory, much smaller than a very large quantity, it is easy to define two domains for the internal momentum:

$$\begin{cases} v >> 1, & u \simeq v \\ v << 1, & u \simeq 1. \end{cases} \tag{5.20}$$

In the first case of (5.20), i.e. for $v >> 1$ and $u \simeq v$, we have a very simple radiation kernel approximation [195]

$$\mathcal{T}(u, v, c_s) \simeq v^{-4} \ln^2 v. \tag{5.21}$$

If we substitute this information in the definition of the GWs background (5.6):

$$\Omega_{GWs,c}(k) = \mathcal{P}_{\mathcal{R}}^{(0)2} \int_0^{\infty} dv \int_{|1-v|}^{1+v} du v^{-4} \ln^2(v) \frac{16}{\pi^2} \times \frac{v_c^2}{uv}. \tag{5.22}$$

Using the similarity of momenta, together with the mean value theorem of the integral, we find

$$\Omega_{GWs,c}(k) = \mathcal{P}_{\mathcal{R}}^{(0)2} \frac{16}{\pi^2} \int_0^{\infty} dv v^{-6} \ln^2(v) \times v_c^2, \tag{5.23}$$

$$\Omega_{GWs,c}(k) = \mathcal{P}_{\mathcal{R}}^{(0)2} \frac{16}{\pi^2} v_c^{-3} \ln^2(v_c) \int_0^{\infty} \frac{dV}{V}. \tag{5.24}$$

In the last step, the variable change from v to $V = v/v_c$ was defined. The idea for this substitution comes from the necessity, in solving the integral, to extract all contributions of external frequency k out of the integral. Such a change of variable, since V is also rewritable as $V = q/k_c$, best handles this requirement. Note how the result found, of the form $\Omega_{GWs} \simeq (k/k_c)^3 \times \ln^2(k/k_c)$, defines an increasing solution (the typical increasing solution in radiation-induced IR); this result should obviously be seen in its defining limit, i.e. it should be considered as long as $k << k_c$. It is easy to see how this result marks the first IR trend, the latter of which is to be divided into two phases: an infrared at lower frequencies (i.e. the latter just found), and an infrared at significantly higher frequencies for which $v << 1$ and $u \simeq 1$. Let us move on to the second and last infrared case. In this limit, it is analytically important to remember the scripture for which

$$y \simeq \frac{1 - c_s^{-2}}{2v} + \frac{v}{2}. \tag{5.25}$$

Therefore, the value of c_s quantifies the possibility of having a different kernel development of the system, since if $c_s < 1$, then it will be the case that $v << 1$, so $|y| >> 1$, since $y = \frac{1-c_s^{-2}}{2v}$, so the kernel is rewritten as:

$$\mathcal{T}(u, v, c_s) \simeq y^{-\alpha} \simeq \left(\frac{2v}{1 - c_s^{-2}}\right)^{\alpha}, \tag{5.26}$$

with the value of α taken positive and even, in order to respect the symmetry of the theory. We decided to parametrize the kernel approximation in order to guarantee the most general conclusive result possible. However, a possible choice of parameter that well describes the kernel approximation is $\alpha = 6$. Exploiting this regime on the adiabatic speed of sound of the scalar fluctuation that acts as a source for the system, we find

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \left(\frac{2}{1 - c_s^{-2}}\right)^{\alpha} \int_0^{\infty} dv v^{\alpha} \int_{|1-v|}^{1+v} du \frac{16}{\pi^2} \times \frac{v_c^2}{uv}, \tag{5.27}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \left(\frac{2}{1 - c_s^{-2}}\right)^{\alpha} \frac{16}{\pi^2} \int_0^{\infty} dv v^{\alpha-1} v_c^2, \tag{5.28}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \left(\frac{2}{1 - c_s^{-2}}\right)^{\alpha} \frac{16}{\pi^2} \left(\frac{k_c}{k}\right)^{\alpha+2} \int_0^{\infty} dV V^{\alpha-1}. \tag{5.29}$$

Note how the trend that follows the logarithmic growth is defined, again in the same infrared connotation, by a steep fall of the induced GWs background. The analytical case just studied is associated with the choice of a general and typical c_s^2 for the classical radiation period (where $c_s^2 = 1/3$). However, as anticipated earlier and also discussed in Chapter 4, there is the possibility of considering the limit in which during the radiation there is a $c_s^2 = 1$ [195] [200]. Thus this condition implies that $v << 1$, with $|y| << 1$ since $y \simeq v/2$, so the kernel is rewritten

$$\mathcal{T}(u, v, c_s) \simeq \frac{y^2}{3} \simeq \frac{v^2}{12}. \tag{5.30}$$

Note how the course of this kernel differs only in the spectral index from the standard case; both will lead to a decreasing induced GWs background. In this situation, and similarly to before, the GWs background is calculated:

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \int_0^\infty dv \int_{|1-v|}^{1+v} du \frac{v^2}{12} \times \frac{v_c^2}{uv}, \tag{5.31}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \frac{3}{36} \int_0^\infty dvv \times v_c^2, \tag{5.32}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \frac{3}{36} \left(\frac{k_c}{k}\right)^4 \int_0^\infty dVV. \tag{5.33}$$

It is important to stress that the integrals written in V are all convergent, and make a unitary contribution to the amplitude of the problem.

Intermediate regime ($k \to k_c$)

Having calculated the convolutional contribution provided by the paired IR-IR, it is necessary to proceed with the calculation of the GWs background contributions. With this logic we proceed by requiring a development of the scalar spectrum in the convolution when the argument tends to unity; an interaction between modes is therefore defined at the boundary between FR and SR. As before we could write

$$\begin{cases} \frac{q}{k_c} \to 1 \\ \frac{|\mathbf{k}-\mathbf{q}|}{k_c} \to 1. \end{cases} \tag{5.34}$$

Solving this system, decomposing the modulus as before, it is easy to find two distinct induced ranges. In fact, if we require that $k >> q$, then $k/k_c \to 1$, with $v << 1$ and $u \to 1$. This writing is reconfirmed by the same definition of delta, since $v < v_c \to 1$.

However in the opposite limit, in which $q >> k$, we find that $q/k_c \to 1$, but since $q/k_c >> k/k_c$, we trivially arrive at the limit in which $k << k_c$ if we consider $v >> 1$ with $u \simeq 1$. The scalar IM-IM convolution therefore induces the presence of two distinct induced GWs backgrounds. The first characteristic of the intermediate section of the external frequency, and one again in the IR regime, which will be added to those calculated above.

Let us therefore proceed to calculate the first intermediate contribution; it will be necessary, as was done for the IR case, to define a series development of the curvature power spectrum function for the argument close to unity. It is recalled that

$$\gamma_{kin}(y) = \frac{\pi}{16} y \left|C_c(y) - D_c(y)\right|^2. \tag{5.35}$$

Henkel functions of the second type are developed for $y \to 1$ up to first order:

$$H_0^{(2)}\left(\frac{y}{2}\right) \simeq H_0^{(2)}\left(\frac{1}{2}\right) - \frac{1}{2}H_1^{(2)}\left(\frac{1}{2}\right)(y-1)\,, \tag{5.36}$$

$$H_1^{(2)}\left(\frac{y}{2}\right) \simeq H_1^{(2)}\left(\frac{1}{2}\right) + \frac{1}{4}\left(H_0^{(2)} - H_2^{(2)}\right)(y-1)\,, \tag{5.37}$$

where we have that:

$$H_0^{(2)}\left(\frac{1}{2}\right) = 0.93 + 0.44i, \tag{5.38}$$

$$H_1^{(2)}\left(\frac{1}{2}\right) = 0.24 + 1.47i, \tag{5.39}$$

$$H_2^{(2)}\left(\frac{1}{2}\right) = 0.03 + 5.44i. \tag{5.40}$$

Exponential functions are therefore also developed

$$e^{-iy} = e^{-i} - ie^{-i}(y-1), \tag{5.41}$$

$$e^{iy} = e^{i} + ie^{-i}(y-1). \tag{5.42}$$

Using these definitions, it is possible to write the square modulus of the difference in the definition of the kinetic spectrum:

$$\begin{aligned} C_c(y) - D_c(y) = &\left[e^{-i} - ie^{-i}(y-1)\right]\Bigg[0.93 + 0.44i - \left(\frac{1}{y} + i\right)[0.24 + 1.47i+ \\ &+\frac{1}{4}[0.93 + 0.44i - 0.03 - 5.44i](y-1)\Bigg]\Bigg] + \\ &-\left[e^{i} + ie^{i}(y-1)\right]\Bigg[0.93 + 0.44i - \left(\frac{1}{y} - i\right)[0.24 + 1.47i+ \\ &+\frac{1}{4}[0.93 + 0.44i - 0.03 - 5.44i](y-1)\Bigg]\Bigg]\,. \end{aligned} \tag{5.43}$$

This solution must then be raised to modulus square, and multiplied by a frequency term in order to define the curvature spectrum; however, it is easy to see the complexity of the last formula. It is therefore necessary to perform an analytical simplification of the spectrum. This simplification is provided by the program

Mathematica, which allows us to write:

$$\gamma_{kin}(y) = \left|(5.59 - 0.44i) - \left(\frac{1.63 - 0.009i}{y}\right) - (6.20 + 1.35i)y + (2.10 + 0.37i)y^2\right|^2 \times \frac{\pi}{16}y. \tag{5.44}$$

It should be noted that the plot of this function, observed in the limit of regard, allows us to observe a function that starts to decrease, then it observes a flattening when the argument reaches unity. Therefore this result found is completely suitable, from a graphical point of view, to approximate the curvature spectrum in the limit of argument tending to one. This function must therefore be substituted in (5.6),

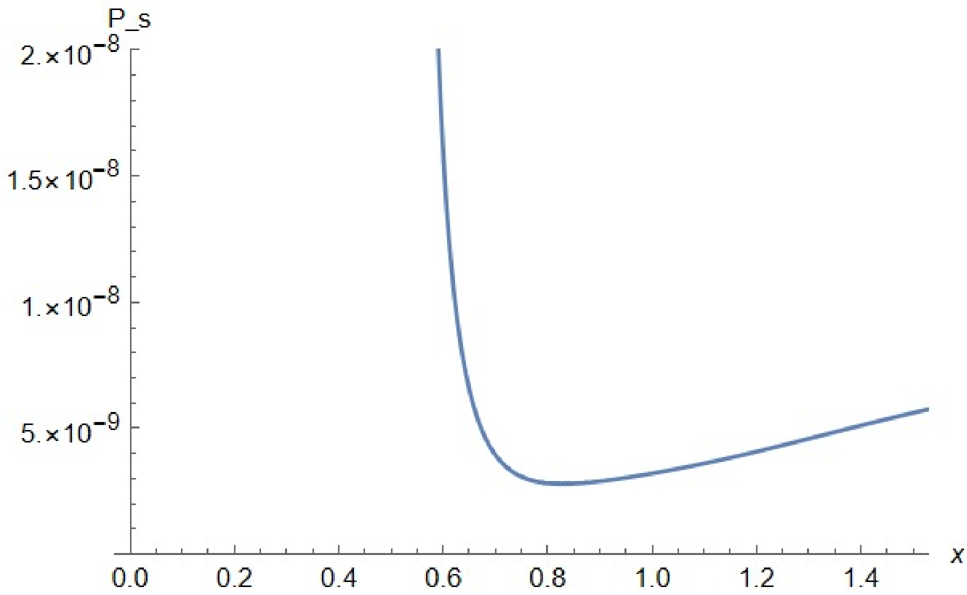

Figure 5.2: Curvature Power-spectrum of the problem presented in equation (5.44), approximated in the intermediate frequency limit of the problem. The value of x in the x-axis determines the y-value of the formulation.

together with the appropriate kernel associated with the corrected limit on internal momenta. We then calculate the IM-IM contribution to the intermediate region of the GWs background as follows (using the condition that $c_s^2 < 1$)

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}$$
$$\times\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right)-(6.20+1.35i)\frac{k}{k_c}+(2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2\times$$
$$\frac{k}{k_c}\times\int_0^\infty dv\frac{kv}{k_c}\left|\frac{1.63-0.009i}{\frac{kv}{k_c}}\right|^2\times\frac{v^\alpha}{(1-c_s^{-2})^\alpha}. \quad (5.45)$$

It is important to highlight that, within the square modulus inscribed in the integral, only the contribution $1/y$ has been considered, in q, in the limits adopted, $k \to k_c$, with $v << 1$; therefore this contribution is indeed the dominant one. Further developing the above formula, we obtain

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}$$
$$\times\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right)-(6.20+1.35i)\frac{k}{k_c}+(2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2\times$$
$$\int_0^\infty dv v^{\alpha-1}, \quad (5.46)$$

therefore

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}$$
$$\times\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right)-(6.20+1.35i)\frac{k}{k_c}+(2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2\times$$
$$\left(\frac{2}{1-c_s^{-2}}\right)^\alpha\times\left(\frac{k_c}{k}\right)^\alpha\int_0^\infty dVV^{\alpha-1}. \quad (5.47)$$

We have to plot this solution. The plot observes a behaviour quite similar to that of the input scalar function: near the unity of the argument (understood as k/k_c) the GWs background first descends, and then flattens out for successive larger values of the argument (to be precise, as soon as the descent is over, the function undergoes a slight ascent before flattening out completely).

It is entirely logical to understand how the solutions found so far in the IR and IM external frequency regimes report an obviously expected analytical continuity. If we instead decide to choose a model for which $c_s = 1$, with the same accounts as

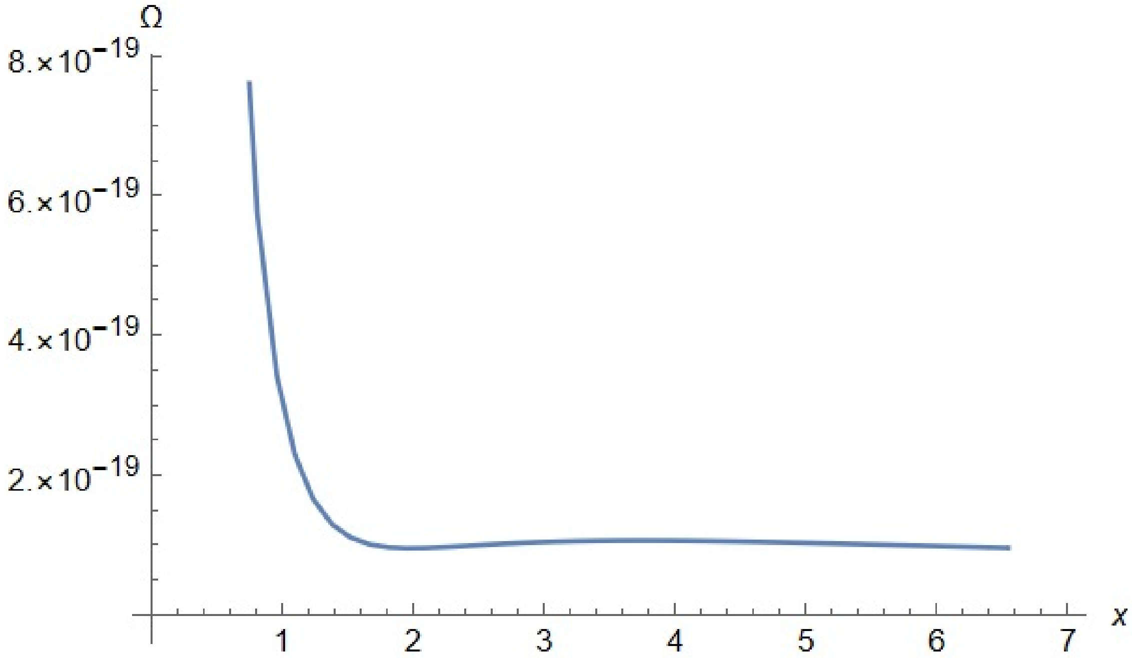

Figure 5.3: Spectral energy density of the problem exposed in equation (5.47).

above, but with a different kernel (seen in the IR section), one finds the following induced GWs background:

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}$$
$$\times\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right)-(6.20+1.35i)\frac{k}{k_c}+(2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2\times$$
$$\left(\frac{k}{k_c}\right)\left(\frac{k_c}{k}\right)^3\int_0^\infty dVV. \quad (5.48)$$

Unlike the previous plot, the latter does not admit a subsequent flattening; in the appropriate argument limits, the function first descends and then rises again.
This result defines a GWs background that can be studied in the intermediate region of external frequencies ($k \to k_c$), induced by the double convolution of the two intermediate scalar contributions. Nevertheless, it was seen earlier how such a double convolution between two intermediate scalar contributions is also capable of inducing an infrared regime GWs background, to be added to that calculated

earlier from the IR-IR scalar convolution. It is therefore written:

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \times \int_0^\infty dv v^{-4} \ln^2 v \left(\frac{kv}{k_c}\right)^2 \left|\frac{1.63 - 0.009i}{\frac{kv}{k_c}}\right|^4, \tag{5.49}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \times \int_0^\infty dv v^{-4} \ln^2 v \left(\frac{kv}{k_c}\right)^{-2}, \tag{5.50}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \times \left(\frac{k}{k_c}\right)^3 \ln^2 \left(\frac{k}{k_c}\right) \int_0^\infty \frac{dV}{V^6}. \tag{5.51}$$

In (5.49) we still only considered the dominant term, deduced through the delta definition assumed in the study model.
It is important to note that this contribution is completely identical to the first one found through IR-IR convolution in the first infrared region; the similarity is total, both in terms of amplitude and analytical trend.

Ultraviolet regime ($k >> k_c$)

Identically to the previous sections, the scalar spectrum for argument $y >> 1$ should be approximated. In such a case, the approximation is entirely trivial, and already presented at the beginning of the chapter; in fact, for $y >> 1$, only the flat section defined by $\mathcal{P}_{\mathcal{R}}^{(0)}$ will remain of the scalar spectrum. The double scalar convolution between the two UV spectra is the convolution between constant functions, so the final result will be a number. Again, Mathematica allows us to evaluate this solution

$$\Omega_{GWs,c} \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \times 0.82. \tag{5.52}$$

Requiring both arguments of the scalar spectra studied in convolution to be much greater than one one finds, (by means of the same analytical procedure used so far), how such a UV-UV scalar convolution interaction induces a GWs background in the external UV frequency limit, whereby $k >> k_c$. It is important once again to note the analytical continuity between the GWs background solutions found in the various topological frequency regimes.

Cross-Talk term in Radiation

So far, the double scalar convolution of approximate pieces of curvature power spectrum has been calculated. This scalar function has been broken into the three main internal frequency regimes (IR, IM, UV) and a convolution in the double integral on the momenta has been performed, only between equal terms. In the previous sections, the IR-IR, IM-IM and UV-UV contributions were calculated, and it was seen how these specific convolutions lead to the generation of an induced GWs background that must respect a well-defined frequency range. However, as already mentioned, these are not the only possible convolutions, as one must also consider those associated with unpaired terms. From a physical point of view, it is possible to interpret this dynamics by saying that it is necessary to consider the interaction of modes that belong to different moments of inflationary evolution (upon re-entry into the Hubble sphere).
Therefore, the purpose of this section is to provide such final convolution calculations, so that we will have, by putting together all the results found, a final and complete picture of the scalar-induced GWs background.
We must first recall the shape of the scalar spectrum in its three frequency approximations

$$\mathcal{P}_{\mathcal{R}}(y)^{(IR)} \simeq \mathcal{P}_{\mathcal{R}}^{(0)} \frac{4}{\pi|y|}, \tag{5.53}$$

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(y)^{(IM)} &\simeq \mathcal{P}_{\mathcal{R}}^{(0)} \\ \times &\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{y}\right)-(6.20+1.35i)y+(2.10+0.37i)y^2\right|^2 \times \frac{\pi}{16}y,\end{aligned} \tag{5.54}$$

$$\mathcal{P}_{\mathcal{R}}(y)^{(UV)} \simeq \mathcal{P}_{\mathcal{R}}^{(0)}. \tag{5.55}$$

At this point it is possible to continue with the convolution counts; it should be noted that the latter is not symmetrical, so it will be necessary to calculate all permutations of the combinations that can be studied.
The IR-UV convolution defines the following spectra:

$$\mathcal{P}_{\mathcal{R}}\left(\frac{kv}{k_c}\right) \simeq \mathcal{P}_{\mathcal{R}}^{(0)}\left(\frac{4}{\pi\frac{kv}{k_c}}\right), \tag{5.56}$$

$$\mathcal{P}_{\mathcal{R}}\left(\frac{ku}{k_c}\right) \simeq \mathcal{P}_{\mathcal{R}}^{(0)}. \tag{5.57}$$

To define such a script, it is necessary to declare the state of the scalar function arguments

$$\begin{cases} \frac{q}{k_c} << 1 \\ \frac{|\mathbf{k}-\mathbf{q}|}{k_c} >> 1. \end{cases} \tag{5.58}$$

Splitting the modulus as done so far, within the limits of interest on the momenta, it is easy to solve the system; however, it is observed that only for $k >> q$, (thus with $v << 1$ and $u \simeq 1$) a finite solution is obtained, defining the production of a GWs background living in the ultraviolet region (it is found in the resolution of the system that $k >> k_c$). Therefore the IR-UV convolution induces a GWs background in the external UV frequency regime. Moving on to the calculations we obtain, following the same logic as before

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{2v}{1-c_s^{-2}} \right)^\alpha \times \left(\frac{4}{\pi \frac{kv}{k_c}} \right) \times 1, \tag{5.59}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \int_0^\infty dv v^{\alpha-1} v_c, \tag{5.60}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \left(\frac{k_c}{k} \right)^{\alpha+1} \int_0^\infty dV V^{\alpha-1}. \tag{5.61}$$

The contribution found to be valid in the ultraviolet is, as we can easily observe, a decreasing contribution that must be added to the flatness term found by the UV-UV convolution. Thus for smaller UV frequencies, the latter result will give a non-zero contribution in the sum, which, however, as the frequency increases with respect to the dip momentum, will tend to zero, leaving only the constant term.

Since no symmetry can be exploited in the definition of the scalar problem, the permutation of the case just given must also be calculated; we therefore calculate the UV-IR convolution. In such a case, we rewrite the same system as before (obviously inverting the inequalities): it is easy to see how this system, for no value of the internal momenta, presents a solution. This convolution therefore physically makes no contribution to the GWs background. The same can also be said for IM-IR, UV-IM couplings. A non-zero term is that offered by the IR-IM interaction. It is therefore written

$$\mathcal{P}_{\mathcal{R}} \left(\frac{kv}{k_c} \right) \simeq \mathcal{P}_{\mathcal{R}}^{(0)} \left(\frac{4}{\pi \frac{kv}{k_c}} \right), \tag{5.62}$$

$$\mathcal{P}_{\mathcal{R}}\left(\frac{ku}{k_c}\right) \simeq \mathcal{P}_{\mathcal{R}}^{(0)} \times\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{ku}{k_c}}\right)-(6.20+1.35i)\frac{ku}{k_c}+(2.10+0.37i)\left(\frac{ku}{k_c}\right)^2\right|^2 \times\frac{\pi}{16}\frac{ku}{k_c}. \tag{5.63}$$

The system related to that combination result

$$\begin{cases} \frac{q}{k_c} << 1 \\ \frac{|\mathbf{k}-\mathbf{q}|}{k_c} \simeq 1. \end{cases} \tag{5.64}$$

Again, in the case for which $k >> q$, there is a solution by finding a GWs background that is defined in the intermediate external frequency region ($k \to k_c$). For the rest of the momenta, no solution is allowed.
We proceed to the usual calculation:

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \int_0^\infty dv v^\alpha \int_{|1-v|}^{1+v} du \left(\frac{4}{\pi\frac{kv}{k_c}}\right) \times \frac{\pi}{16}\frac{ku}{k_c} \times \left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{ku}{k_c}}\right)-(6.20+1.35i)\frac{ku}{k_c}+(2.10+0.37i)\left(\frac{ku}{k_c}\right)^2\right|^2 \times \frac{\pi}{16}\frac{ku}{k_c}, \tag{5.65}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right)-(6.20+1.35i)\frac{k}{k_c}+(2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2 \int_0^\infty dv v^{\alpha-1}, \tag{5.66}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right)-(6.20+1.35i)\frac{k}{k_c}+(2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2 \times (\frac{k_c}{k})^\alpha \int_0^\infty dV V^{\alpha-1}. \tag{5.67}$$

This result is the same as that found by the IM-IM convolution. Since two identical solutions live in the same external frequency regime, they can simply be added together, thus doubling the amplitude value.
The last non-zero contribution of unpair convolution is offered by the IM-UV convolution. It is therefore written

$$\mathcal{P}_{\mathcal{R}}\left(\frac{kv}{k_c}\right) \simeq \mathcal{P}_{\mathcal{R}}^{(0)}$$
$$\times\left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{\frac{kv}{k_c}}\right)-(6.20+1.35i)\frac{kv}{k_c}+(2.10+0.37i)\left(\frac{kv}{k_c}\right)^2\right|^2 \times$$
$$\frac{\pi}{16}\frac{kv}{k_c}, \quad (5.68)$$

$$\mathcal{P}_{\mathcal{R}}\left(\frac{ku}{k_c}\right) \simeq \mathcal{P}_{\mathcal{R}}^{(0)}. \quad (5.69)$$

The system for the argument follows:

$$\begin{cases} \frac{q}{k_c} \simeq 1 \\ \frac{|\mathbf{k}-\mathbf{q}|}{k_c} >> 1. \end{cases} \quad (5.70)$$

The system again presents solution only for $v << 1$ and $u \to 1$, inducing a non-trivial background living in the UV, exactly like the first calculated cross-talk. Calculating the GWs abundance we obtain

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\int_0^\infty dv v^\alpha \int_{|1-v|}^{1+v} du \frac{\pi}{16}\frac{kv}{k_c}\left|\frac{1}{\frac{kv}{k_c}}\right|^2, \quad (5.71)$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\int_0^\infty dv v^{\alpha-1} v_c, \quad (5.72)$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \times \left(\frac{k_c}{k}\right)^{\alpha+1}\int_0^\infty dV V^{\alpha-1}. \quad (5.73)$$

This result is identical to that offered in the first IR-UV scattered convolution, so the associated physical considerations are the same.

5.1.2 Final Results

Having reached this point in the discussion, it is possible to conclude by pointing the results of the physical model. The theoretical amplitude of the model can be calculated by calculating the amplitude of each individual contribution. Since the remaining convolution integrals on V are of the order of unity, the amplitude of each result is based on the general trend of $\mathcal{P}_{\mathcal{R}}^{(0)2} \simeq 10^{-18}$. After all, this reasoning is very intuitive: we are calculating the convolution of an input spectrum that has a fixed amplitude scale; one can therefore expect that the induced GWs background, in every frequency range, also respects this numerical property. The final abundance, on the other hand, as can be easily seen in an analytical reconstruction, defines a rising trend followed by a decreasing phase, defined by the IR-IR convolution (also supported by a part of the IM-IM convolution); subsequently, near the reference frequency k_c, a continuous decrease is observed at the previous limit, ending with a slight rise. The latter then naturally tends to flatten out. This flattening is entirely in accordance with the next and last constant trend, defined by the UV-UV convolution and the unpaired convolutions defined above. The figure thus illustrates the analytical-theoretical result found by using the idea of a Dirac Delta toy-model on the dip of the system.

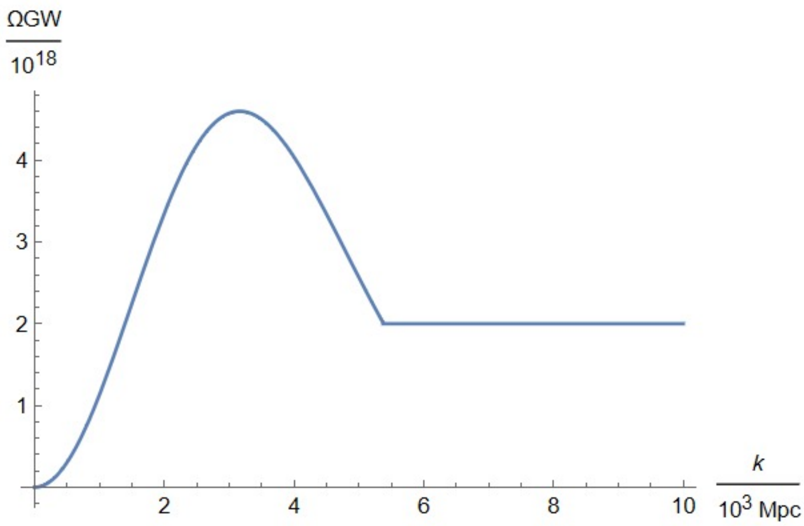

Figure 5.4: Analytical qualitative plot that resume all the theoretical trend in the overall frequency range.

A computational confirmation of the work done is offered by a program developed by Domenech (https://github.com/Lukas-T-W/SIGWfast), which, in radiation domination, calculates the same GWs background. The operation of the script is very simple: it involves as an input the curvature power-spectrum of the problem, and then it defines the analytical computational result. Therefore, substituting our scalar spectrum offered by Planck [14], the solutions found are shown in the figures (5.4), (5.5).

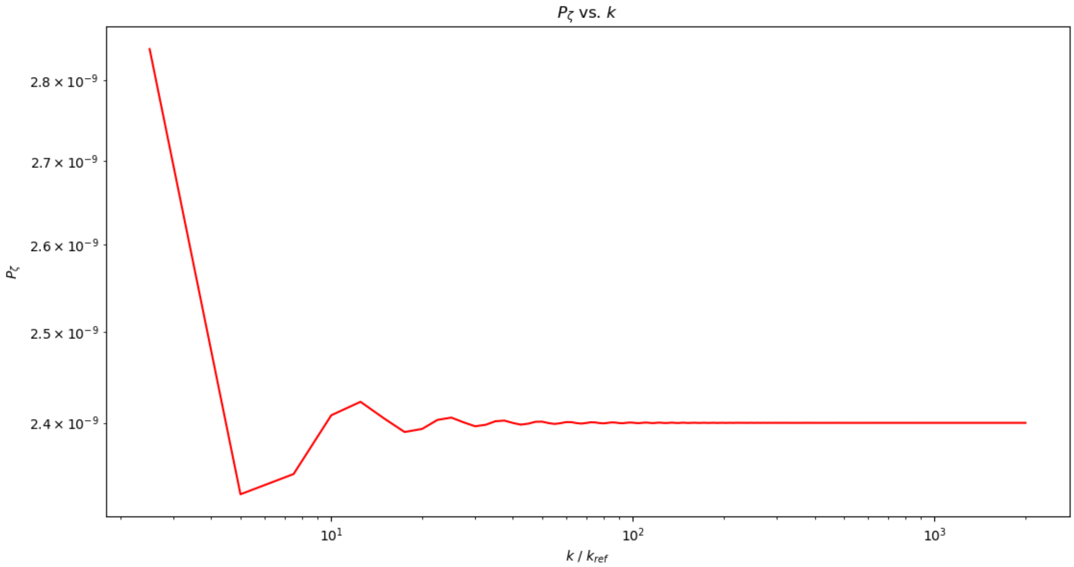

Figure 5.5: Curvature Power Spectrum (https://github.com/Lukas-T-W/SIGWfast).

As can be seen, the theoretical amplitude found coincides with that shown in the graphical solution. The analytical trend of both constructions coincides optimally. The only discrepancy can be seen in the first section of the spectral energy density: the growth trend defined by the IR-IR convolution in the first internal frequency limit is missing from the computational calculation. However, this lack can be justified by the fact that the reference scale of the k_c problem is very small, of the order mentioned before, on the scales of the CMB. Therefore there will exist few scales that can be involved in the FR problem smaller than the aforementioned, and they will all be very close to the value of the latter. This reasoning allows us to assert that therefore in the infrared, the value of v_c is large (greater than one), but not so large as to justify the existence of frequencies in the first range. Therefore, considering only the last IR range for $v << 1$, there is a total correspondence between the analytical-modelling model and the computational solution.
It is important to remember that the calculated induced GWs background, in both cases, is still not the GWs background we observe today; in fact it is the one calculated in the radiation epoch.

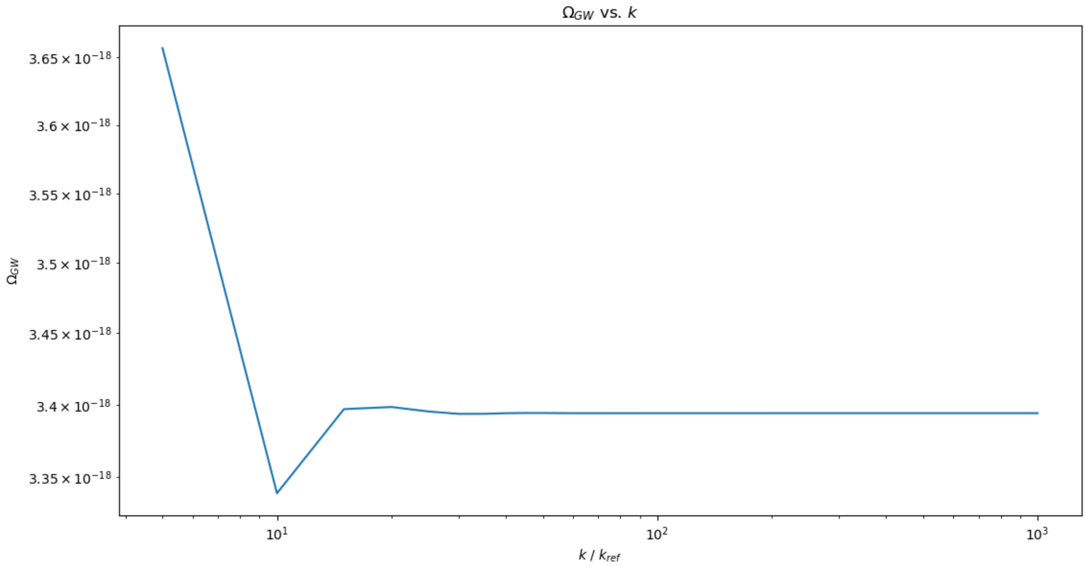

Figure 5.6: Induced GWs spectral energy density during the final radiation domination period. The plot presents the spectral energy density calculated in the conformal time η_c with respect to the frequency ratio k/k_{ref}. k_{ref} is the reference frequency of the system, and was chosen to be of the order of 10^3, in order to have a positive match with the frequency values involved in the theory. In fact, as Planck's graph itself shows us, the fast-roll problem occurs on scales k_c of the order of $5 \times 10^{-3}\text{Mpc}^{-1}$. Both spectra therefore have to be processed on such values (https://github.com/Lukas-T-W/SIGWfast).

The following formula is used to switch from one to the other:

$$\Omega_{GW,0}h^2 = \Omega_{r,0}h^2 \left(\frac{g_*(T_c)}{g_{*,0}}\right)\left(\frac{g_{*s}(T_c)}{g_{*s,0}}\right)^{-4/3} \Omega_{GW,c}, \tag{5.74}$$

where $\Omega_{r,0}$ is the current radiation density today and $g_*(T)$ and $g_{*s}(T)$ are the effective degrees of freedom in the energy and entropy density at temperature T. Thanks to the data offered by the Planck satellite we observe $\Omega_{r,0}h^2 \simeq 4.18 \times 10^{-5}$ [38]. It is also found that $g_{*,0} = 3.36$ and that $g_{*s,0} = 3.91$. It is also understandable how for $T > 100$ GeV and assuming only standard model particles, we have $g_*(T) = g_{*s}(T) = 106.75$. The 'c' term [198] instead indicates the evaluation at the time when the spectral energy density is constant, i.e. at the time when the tensor modes are well contained within the Hubble horizon. The multiplicative pre-factor to the radiation density is of the order of 10^{-7}, so the final abundance observable today, induced during radiation, defines a minimal amplitude value, of the order of

$\Omega_{GW,0}h^2 \simeq 10^{-25}$.
Assuming that the induced tensor power spectrum has the same order of magnitude as the abundance found (since they imply the same physical observable), it is possible to give an estimate of the tensor-to-scalar-ratio, relating the abundance to the known curvature power spectrum. The k_* for which the maximum of the background is observed coincides with $k_* \simeq 5.03512 \times 10^{-3}\text{Mpc}^{-1}$, and at this external frequency it is found that $\Omega_{GWs}(k_*) \simeq 4.95334 \times 10^{-18}$.
It is observed, again from the above plot, that $\mathcal{P}_{\mathcal{R}}(k_*) \simeq 2.32 \times 10^{-9}$. It is therefore possible to give an estimate of r by relating the two spectral values just described. The order of magnitude is of the type $r^{(*)} \simeq 2.32 \times 10^{-9}$; the value is definitely very small, but it is an expected value, being the problem related to the production of a GWs background at the second perturbative order.

5.1.3 Spectral energy density of GWs from Inflationary epoch

It is now necessary to conclude the entire treatment of the studied model. So far, the scalar-induced gravitational wave background during the radiation-dominated phase has been calculated. Now we want to calculate the GWs background during the inflaton domination instead: we would like to proof a superiority of the latter over the former GWs background.
Inomata's article [201] gives us the formulation of the conformal time-dependent tensor power spectrum associated with tensor modes induced during the inflationary phase [201]

$$P_h(k,\eta) = \frac{4}{M_{pl}^4}\sum_j \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2-(1+v^2-u^2)^2}{4uv}\right]^2 \times \tag{5.75}$$

$$\left|I_j(u,v,k,\eta)\right|^2 \mathcal{P}_{\delta\phi_j}(kv)\mathcal{P}_{\delta\phi_j}(ku). \tag{5.76}$$

In such a formulation, the sum is over the scalar field j; in our one-field case, this term will be transparent. We define [201]

$$I_j(u,v,k,\eta) = k^2\int_{-\infty}^{\eta} d\tilde{\eta} g_k(\eta;\tilde{\eta}) f_j(ku,kv,\tilde{\eta}), \tag{5.77}$$

with

$$g_k(\eta;\tilde{\eta}) = \frac{\Theta(\eta-\tilde{\eta})}{k^3\tilde{\eta}^2}\left[k(\tilde{\eta}-\eta)\cos\left(k(\tilde{\eta}-\eta)\right) - (1+k^2\eta\tilde{\eta})\sin\left(k(\tilde{\eta}-\eta)\right)\right], \tag{5.78}$$

the Green's solution in inflaton domination (already encountered in Chapter 2). The source term, on the other hand, is written [201]:

$$f_j(k_1,k_2,\eta) = T_j(k_1,\eta)T_j(k_2,\eta), \tag{5.79}$$

where [201]

$$T_j(k,\eta) = \frac{U_j(k,\eta)}{U_j(k,\eta\to 0)}, \tag{5.80}$$

with U_j the causally connected inflationary solution of the model. The value of this function is taken from the third chapter, in the section on the fast-roll, thus [192]

$$T_j(k,\eta) = \frac{\frac{m}{C}\times e^{\sqrt{m^2-k^2}\ln\left(\frac{1}{1-\eta H}\right)\frac{1}{H}}}{\frac{m}{C}}, \tag{5.81}$$

$$T_j(k,\eta) = \left(\frac{1}{1-\eta H}\right)^{\sqrt{\frac{m^2-k^2}{H^2}}}, \tag{5.82}$$

using conformal time notation inscribed in a de-Sitter inflationary model. Knowing the rescaling formula between the curvature power-spectrum and the power spectrum of the scalar field fluctuation, we write

$$\mathcal{P}_{\delta\phi}(k) \simeq \mathcal{P}_{\mathcal{R}}\epsilon M_{Pl}^2 \simeq \epsilon M_{Pl}^2 \mathcal{P}_{\mathcal{R}}^{(0)} \gamma_{kin}\left(\frac{k}{k_c}\right). \tag{5.83}$$

We rewrite the kernel term in the problem

$$I\left(u, v, \frac{k}{k_c}, \eta\right) = \left(\frac{k}{k_c}\right)^2 \int_{-\infty}^{\eta} \frac{d\tilde{\eta}\Theta(\eta-\tilde{\eta})}{\left(\frac{k}{k_c}\right)^3 \tilde{\eta}^2}$$
$$\left[\frac{k}{k_c}(\tilde{\eta}-\eta)\cos\left(\frac{k}{k_c}(\tilde{\eta}-\eta)\right) - \left(1+\left(\frac{k}{k_c}\right)^2 \eta\tilde{\eta}\right)\sin\left(\frac{k}{k_c}(\tilde{\eta}-\eta)\right)\right] \times \left(\frac{1}{1-\tilde{\eta}H}\right)^{\alpha_u+\alpha_v}, \tag{5.84}$$

where

$$\alpha_u + \alpha_v = \sqrt{\frac{m^2 - \frac{k^2u^2}{k_c^2}}{H^2}} + \sqrt{\frac{m^2 - \frac{k^2v^2}{k_c^2}}{H^2}}. \tag{5.85}$$

The problem continues with the resolution of the integral over the kernel time; it is easy to see how this represents the most complex point of the model, given the analytical complexity of the integral. In principle, it would be convenient to send the conformal time η at the upper limit of the integral to zero, since we conventionally assume finite inflation for this limit; moreover, we are observationally bound to know the value of $P_h(k, \eta \to 0)$. However, this simplification will only be made later. In the analytical spirit of the radiation section, it is appropriate to study the kernel in its infrared approximation ($k << k_c$), sending the conformal time to zero only at the end. This kernel will then be replaced in the integral on the momenta and will allow us to calculate, as done in radiation, the IR trend of the induced inflationary GWs background. Obviously, to calculate the GWs background in other external frequency regimes, such an approximate kernel loses its value, and must be calculated in its entirety form.

Infrared Regime ($k << k_c$)

In the infrared limit, the kernel is simplified as follows

$$I \simeq \left(\frac{k_c}{k}\right) \int_{-\infty}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta}^2} \times$$
$$\left[\frac{k}{k_c}(\tilde{\eta}-\eta)\left(1-\frac{k^2(\tilde{\eta}-\eta)}{k_c^2}\right)-\left(1+\frac{k^2}{k_c^2}\eta\tilde{\eta}\right)\times\left(\frac{k}{k_c}(\tilde{\eta}-\eta)-\frac{k^3}{k_c^3}\frac{(\tilde{\eta}-\eta)^3}{6}\right)\right]\times\left(\frac{1}{1-\tilde{\eta}H}\right)^{\omega}, \tag{5.86}$$

with

$$\omega = \alpha_u + \alpha_v, \tag{5.87}$$

and with the series development of the oscillating functions by small argument, given the regime of interest. By simplifying the integral algebraically, we obtain

$$I \simeq \left(\frac{k}{k_c}\right)^2 \times \int_{-\infty}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta}^2}\left[-\frac{1}{3}(\tilde{\eta}-\eta)^3-(\tilde{\eta}-\eta)\eta\tilde{\eta}\right]\times\left(\frac{1}{1-\tilde{\eta}H}\right)^{\omega}. \tag{5.88}$$

The Mathematica program manages to solve this integral, finding the solution of the approximate kernel in the infrared limit

$$\begin{aligned} I \simeq \left(\frac{k}{k_c}\right)^2 \times \frac{1}{3H^2(\omega-1)} \\ \Bigg[-\frac{(1-\eta H)^{1-w}(H\eta(\omega-1)-1))}{\omega-2} \\ + H^3\eta^3[-((\omega-1)\omega H^{-\omega}(-H)^{\omega}\pi\csc(\pi\omega)) \\ +(1-\eta H)^{1-\omega} HyperG2F1(2,\omega-1,2-\omega,1-H\eta)]\Bigg]. \end{aligned} \tag{5.89}$$

Now the square modulus of this solution must be calculated; this must then be entered into the double scalar convolution integral over internal momenta. For the sake of simplicity, we will rename the kernel function in the following way

$$I \simeq \left(\frac{k}{k_c}\right)^2 f(\eta). \tag{5.90}$$

This simplification leads us to consider the term ω as a function independent of the momentum, and which will therefore not be integrated later. This analytical simplification is implicit in the IR treatment as well as in the fast-roll model in its most general form.

We now turn to the double integral on momenta. Let us recall the form of the spectrum in the IR-IR convolution

$$\mathcal{P}_{\delta\phi}\left(\frac{ku}{k_c}\right) \simeq \frac{H^2}{\pi} \times \frac{4}{\pi \frac{ku}{k_c}}, \tag{5.91}$$

$$\mathcal{P}_{\delta\phi}\left(\frac{kv}{k_c}\right) \simeq \frac{H^2}{\pi} \times \frac{4}{\pi \frac{kv}{k_c}}. \tag{5.92}$$

Since the fluctuation spectrum of the scalar field, is the same, one can use again the toy-model of the Dirac delta introduced in the first calculation. In the infrared $v_c >> 1$, $v << v_c$, therefore the two infrared limits will again be obtained for which

$$\begin{cases} v >> 1, u \simeq v \\ v << 1, u \simeq 1. \end{cases} \tag{5.93}$$

We can call B the term

$$B = \left[\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv}\right]^2 \tag{5.94}$$

for analytical simplicity. In the first case of the system described above, i.e. for $v >> 1$, we have that $B \to 1$. Therefore the tensor power spectrum will be written:

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4} \int_0^\infty dv \int_{|1-v|}^{1+v} du 16 \times \left|f(\eta)\right|^2 \left(\frac{k}{k_c}\right)^4 \times \left(\frac{H^4}{\pi^4}\right) \frac{v_c}{uv}, \tag{5.95}$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{64}{M_{Pl}^4} \frac{H^4}{\pi^4} \left|f(\eta)\right|^2 \int_0^\infty \frac{dv}{v^2} v_c^2 \left(\frac{k}{k_c}\right)^4, \tag{5.96}$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{64}{M_{Pl}^4} \frac{H^4}{\pi^4} \left|f(\eta)\right|^2 \left(\frac{k}{k_c}\right)^3 \int_0^\infty \frac{dV}{V^2}. \tag{5.97}$$

The analytical trend found at this stage is an increasing function as the third power of the external frequency. This growth, collinear to that of radiation domination ($k^3 \ln^2 k$) is greater than the latter, demonstrating, at least as far as the analytical trend is concerned, a first domination of inflation over radiation.
In the second consecutive infrared limit, when $v << 1$, we have that $B \simeq v^2$. We get

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4} \int_0^\infty dv \int_{|1-v|}^{1+v} du 16 \times \left|f(\eta)\right|^2 \left(\frac{k}{k_c}\right)^4 \times \left(\frac{H^4}{\pi^4}\right) \frac{v_c}{uv} \times v^2, \tag{5.98}$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{64}{M_{Pl}^4} \frac{H^4}{\pi^4} \left|f(\eta)\right|^2 \int_0^\infty dv \frac{v^2}{v} v_c^2 \left(\frac{k}{k_c}\right)^4, \tag{5.99}$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{64}{M_{Pl}^4} \frac{H^4}{\pi^4} \left|f(\eta)\right|^2 \int_0^\infty dV V. \tag{5.100}$$

The clear result from this last IR limit is a constant behaviour.

Intermediate regime ($k \to k_c$)

Proceeding with the same scheme as in the last section, we now calculate the IM-IM convolution contribution that provides the creation of a GWs background in IM. Recall how, identical to the radiation case, it is required in the space of momenta that $v < 1$, $u \to 1$.
The curvature power spectrum IM of the problem is recalled:

$$\mathcal{P}_{\mathcal{R}}(x) = \mathcal{P}_{\mathcal{R}}^{(0)} \times$$
$$\left|(5.59 - 0.44i) - \left(\frac{1.63 - 0.009i}{x}\right) - (6.20 + 1.35i)x + (2.10 + 0.37i)x^2\right|^2 \times \frac{\pi}{16}x. \tag{5.101}$$

Recalling the parametrization on the radiation and inflation power spectra, the tensor power-spectrum is calculated. For the moment, we choose, in an approximate key, to still use the approximate kernel, and see if the final result will be more or less close to the rigorous one calculated with the generic kernel.
We can write:

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4}\int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{k}{k_c}\right)^4 \left|f(\eta)\right|^2 \times v^2 \left(\frac{H}{\pi}\right)^4 \frac{kv}{k_c}\frac{ku}{k_c}$$
$$\left|(5.59 - 0.44i) - \left(\frac{1.63 - 0.009i}{\frac{kv}{k_c}}\right) - (6.20 + 1.35i)\frac{kv}{k_c} + (2.10 + 0.37i)\left(\frac{kv}{k_c}\right)^2\right|^2$$
$$\left|(5.59 - 0.44i) - \left(\frac{1.63 - 0.009i}{\frac{ku}{k_c}}\right) - (6.20 + 1.35i)\frac{ku}{k_c} + (2.10 + 0.37i)(\frac{ku}{k_c})^2\right|^2, \tag{5.102}$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4}\left(\frac{k}{k_c}\right)^4 \left|f(\eta)\right|^2 \times \left(\frac{H}{\pi}\right)^4 \times$$
$$\left|(5.59 - 0.44i) - \left(\frac{1.63 - 0.009i}{\frac{k}{k_c}}\right) - (6.20 + 1.35i)\frac{k}{k_c} + (2.10 + 0.37i)\left(\frac{k}{k_c}\right)^2\right|^2$$
$$\int_0^\infty dv v^3 \left(\frac{k}{k_c}\right)^2 \times \frac{1}{\left|\frac{kv}{k_c}\right|^2}, \tag{5.103}$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4}\left|f(\eta)\right|^2 \times \left(\frac{H}{\pi}\right)^4 \times$$
$$\left|(5.59-0.44i) - \left(\frac{1.63-0.009i}{\frac{k}{k_c}}\right) - (6.20+1.35i)\frac{k}{k_c} + (2.10+0.37i)\left(\frac{k}{k_c}\right)^2\right|^2$$
$$\left(\frac{k}{k_c}\right)^2 \int_0^\infty dVV. \quad (5.104)$$

The plot of the latter describes a curve which, (read in the correct range, i.e. near k_c) first starts out flat, and then rises and flattens immediately thereafter.

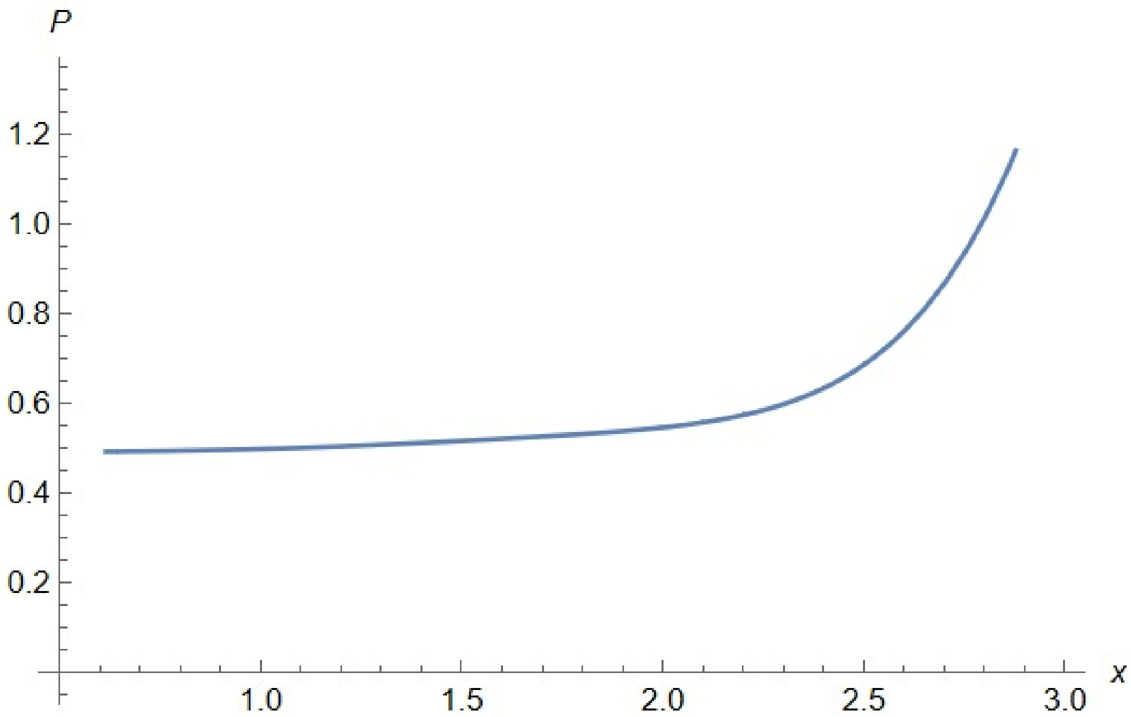

Figure 5.7: Inflationary spectral energy density of the problem set out in equation (5.103), to be read in the frequency limit of interest,

As mentioned earlier, this result uses the approximation of a kernel that is not properly suited to the frequency range of the calculation. It is therefore convenient to give the complete definition of the kernel under consideration. Taking (5.83), and sending the conformal time $\eta \to 0$ primordially, it is possible, with the help of

Mathematica, to solve the integral over the times, finding the following solution

$$
\begin{aligned}
I \simeq\; & \frac{1}{z}\frac{1}{3(\omega-2)(\omega-1)\Gamma(\omega+5)}H^{-3-\omega} \\
& (\frac{3}{2}H\pi\omega(16-12\omega-8\omega^2+3\omega^3+\omega^4)z|z|^{\omega}\csc(\frac{\pi\omega}{2})(H^2(3+\omega)\cos(\frac{z}{H}) \\
& + z^2 HypergeometricPFQ[(\frac{3}{2}+\frac{\omega}{2}),(\frac{3}{2},\frac{5}{2}+\frac{\omega}{2}),-\frac{z^2}{4H^2}])+ \\
& + H^{1+\omega}z^3\Gamma(5+\omega)HypergeometricPFQ[(1,\frac{3}{2}),(\frac{5}{2},\frac{3}{2}-\frac{\omega}{2},2-\frac{\omega}{2}),-\frac{z^2}{4H^2}]- \\
& \frac{1}{2}\pi\omega(6-\omega-7\omega^2+\omega^3+\omega^4)|z|^{1+\omega}\sec(\frac{\pi\omega}{2}) \\
& (z^3 HypergeometricPFQ[(2+\frac{\omega}{2}),(\frac{5}{2},3+\frac{\omega}{2}),-\frac{z^2}{4H^2}]+3H^2(4+\omega)\sin(\frac{z}{H}))).
\end{aligned}
\tag{5.105}
$$

This is the generic solution of the kernel in a complete external frequency range. In such writing z represents the term k/k_c. Let us call the term $Iz = A$ for simplicity. It is therefore possible to recalculate the IM-IM contribution with the same analytical procedures as above, the only difference being to use this kernel, rather than the one approximated in IR. The final solution is written:

$$
\begin{aligned}
\mathcal{P}_h(k,\eta) \simeq\; & \frac{4}{M_{Pl}^2}\left(\frac{H}{\pi}\right)^4 \frac{1}{z^4}|A|^2 \\
& \left|(5.59-0.44i)-\left(\frac{1.63-0.009i}{z}\right)-(6.20+1.35i)z+(2.10+0.37i)(z)^2\right|^2 .
\end{aligned}
\tag{5.106}
$$

Having chosen appropriate values of H and ω, and looking at the function in the vicinity of $z \simeq 1$ (i.e. for $k \to k_c$), we could find a solution that analytically behaves very similarly to that found in the IR kernel approximation; it is therefore logical to think how this approximation is still valid in order to study the contribution of intermediate external frequencies.

It should be recalled that the IM-IM convolution also produces a GWs background contribution in the IR, if one chooses a set of momenta for which $v >> 1$, and $u \simeq v$.

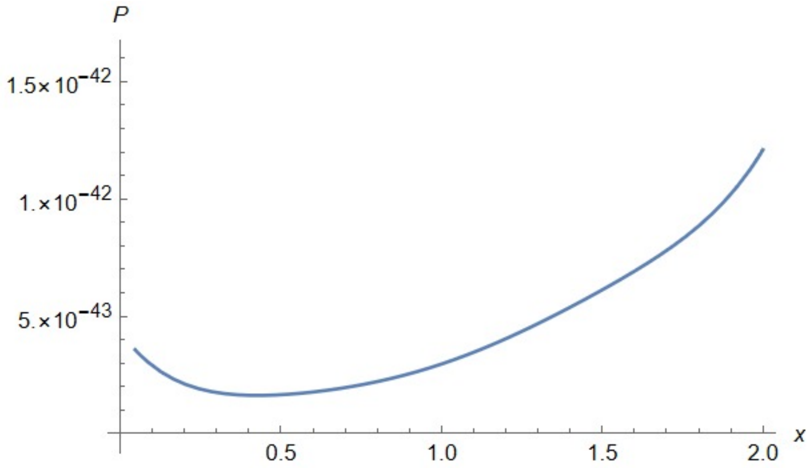

Figure 5.8: Inflationary spectral energy density of the problem set out in equation (5.105), to be read in the frequency limit of interest, neglecting the inaccurate amplitude values, as the amplitude over-all factor was not graphed for analytical simplicity. Unlike the previous plot, here the generic kernel was considered, choosing a value of $H \simeq 10^{16} GeV$ and a value of $\omega \simeq 20$, both of which conform to the model studied.

It is therefore written:

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{k}{k_c}\right)^4 \left|f(\eta)\right|^2 \times \left(\frac{H}{\pi}\right)^4 \left(\frac{kv}{k_c}\right)^2 \left|\frac{kv}{k_c}\right|^{-4}, \quad (5.107)$$

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4} \left|f(\eta)\right|^2 \times \left(\frac{H}{\pi}\right)^4 \left(\frac{k}{k_c}\right)^3. \quad (5.108)$$

The plot reflects that found by the IR-IR convolution, both in terms of amplitude and analytical trend. Therefore, in the IR, in the outer frequencies, these two GWs background contributions are safely added together.
As far as the last paired UV-UV convolution is concerned, identical to the radiation case, the double convolution of constant functions is evaluated, which releases a number that follows the general amplitude trend of the inflation problem.
It is therefore possible to plot all the trends found so far in the overall external frequency bands. Leaving aside for a moment the amplitude character of the problem, which will be addressed below, it is possible to note that, functionally speaking, the inflation GWs background grows more than the radiation GWs background, and is always greater; this is a proof, albeit partial, of the initial intent that this thesis had set.

Cross-Talk terms in Inflation

Once again, in order to conclude the discussion, it is necessary to study the final convolutional terms; doing so, one has a complete picture of all the possible interactions between the modes associated with the scalar spectrum in question, which create the source term. The terms that contribute to the final GWs background are the same as in the radiation case (as well as those that do not provide an analytical contribution). Therefore, the IR-UV, IR-IM, IM-UV convolution contributions will be calculated. The IR-UV convolution induces a background living in the ultraviolet, so the use of the general kernel will be necessary; it is written

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4}\int_0^\infty dv \int_{|1-v|}^{1+v} du \times \left(\frac{4}{\pi\frac{kv}{k_c}}\right) v^2 \times \left(\frac{H^4}{\pi^4}\right)|A|^2 \left(\frac{1}{\frac{k}{k_c}}\right)^2 . \tag{5.109}$$

Using the same analytical procedures up to now, the following solution is found

$$\mathcal{P}_h(k,\eta) \simeq \frac{4}{M_{Pl}^4} \times \left(\frac{H^4}{\pi^4}\right)\left(\frac{1}{z}\right)^5 |A|^2 . \tag{5.110}$$

Choosing appropriate values of H and ω, and choosing an high range of study for z, one can read the plot of a function that remains substantially flat; this term should therefore be added to the constant contribution induced by the UV-UV convolution.
It can be verified that the IM-UV contribution is identical to that just written, so the same conclusions follow. The final term to be calculated IR-IM carries a contribution identical to that studied in the IM-IM convolution, therefore these terms should be added together.

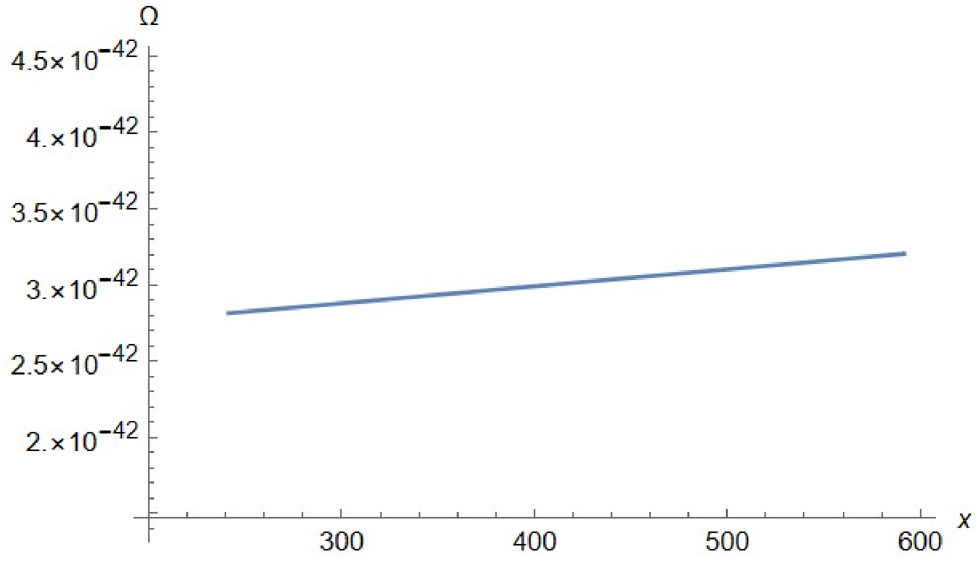

Figure 5.9: Inflationary spectral energy density of the problem set out in equation (5.105), to be read in the frequency limit of interest, neglecting the inaccurate amplitude values, as the amplitude over-all factor was not graphed for analytical simplicity. Note how for large frequency scales the curve tends to remain flat, bringing a constant contribution to the general UV solution.

Final results and comments

The overall graph of the GWs background induced during inflation shows, from an analytical point of view, a substantially higher growth than that described in the case of radiation.

The generic amplitude of the inflationary problem, as can be deduced from the IR solutions in particular, is of the order of unity; it therefore describes an extraordinarily high value, yielding a tensor-to-scalar-ratio value of the order of $r \to 10^9$. Such a value would indeed exclude a Fast-Roll model from inflationary phenomenology. It is necessary to switch from the inflationary tensor-power spectrum, for observational reasons, to the inflation-induced spectral energy density studied during the time of radiation domination. It can be shown that [201]

$$\Omega_{GW}(k, \eta_c) \simeq \frac{1}{48}\mathcal{P}_h(k, \eta \to 0). \tag{5.111}$$

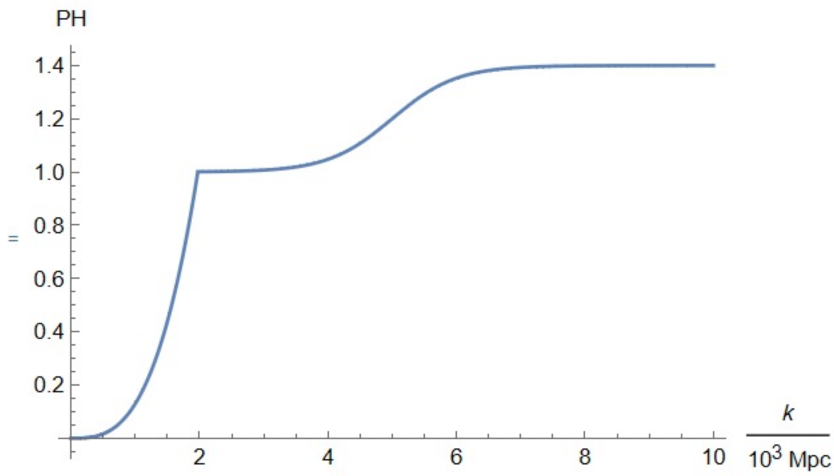

Figure 5.10: Analytical qualitative plot that resume all the theoretical trend in the overall frequency range.

Taking into account the era of subsequent matter domination, the current energy density parameter [201] is found:

$$\Omega_{GW}(k,\eta_0)h^2 = 0.39\left(\frac{g_{*,c}}{106.75}\right)^{-\frac{1}{3}}\Omega_{r,0}h^2\Omega_{GW}(k,\eta_c), \tag{5.112}$$

$$\Omega_{GW}(k,\eta_0)h^2 = 3.4\times 10^{-7}\left(\frac{g_{*,c}}{106.75}\right)^{-\frac{1}{3}}\mathcal{P}_h(k,\eta\to 0), \tag{5.113}$$

where $g_{*,c}$ represents the effective number of degrees of freedom at time η_c.
It is therefore possible to verify that the current inflation-induced GWs background observable is of the order of $\Omega_{GW}(k,\eta_0)h^2 \simeq 10^{-9}$.

5.2 Scalar induced GWs from EFTI

In this section, the focus is to take the curvature perturbation solution offered by the EFTI theory seen in the previous chapter, and to calculate the spectral energy density of the second-order induced GWs background. A main difference between this model and the fast-roll model can already be defined. While the Fast-Roll defines a strong departure from the slow-roll in terms of ϵ, thus defining a strong comparability between the two power spectra of the theory, the EFTI model defines a perturbative departure condition from the slow-roll: ϵ in fact always remains very small, so here we will never expect a higher inflationary GWs background than the radiation background. In fact, the curvature power spectrum overpowers the scalar field perturbation power spectrum, which induces the induced inflationary GWs background.
Another difference between the two models is certainly related to the analytical approach. It is easy to see how a double convolution integral on the momenta of a spectrum oscillating in a damped way in certain frequency ranges does not represent a case of simple analytical handling. Therefore, the idea to deal with such a problem is to develop the curvature power-spectrum around the limit points of the argument's range. Therefore we have to develope around the points of $k\tau_f \to 1$, and $k\tau_f \to \beta$, where $k\tau_f \to \beta$ is remembered to be the sharpness parameter of the features induced in the model. It is obvious to think that this approximation will only allow us to have a part of the GWs background induced in specific external frequency ranges. However, it will be possible to exploit the symmetry of the input function to try to understand the background even in ranges where the analytical development remains forbidden.

5.2.1 Spectral energy density of GWs from Radiation epoch

Recall the formula that defines the induced GWs background in radiation [119] [195]

$$\Omega_{GWs,c}(k) = \int_0^\infty dv \int_{|1-v|}^{1+v} du \mathcal{T}(u, v, c_s)\mathcal{P}_{\mathcal{R}}(ku)\mathcal{P}_{\mathcal{R}}(kv), \tag{5.114}$$

with

$$v = \frac{q}{k}, \tag{5.115}$$

and

$$u = \frac{|\mathbf{k} - \mathbf{q}|}{k}. \tag{5.116}$$

The radiation-dominated kernel is the one addressed in Chapters 3 and 4, and it is written [119] [195]:

$$\mathcal{T}(u,v,c_s) = \frac{y^2}{3c_s^4}\left(\frac{4v^2-(1-u^2+v^2)^2}{4u^2v^2}\right)^2 \tag{5.117}$$

$$\times\left[\frac{\pi^2}{4}y^2\Theta[c_s(u+v)-1]+\left(1-\frac{1}{2}y\ln\left|\frac{1+y}{1-y}\right|\right)^2\right]. \tag{5.118}$$

We have to remember that

$$y = \frac{u^2+v^2-c_s^{-2}}{2uv}. \tag{5.119}$$

The scale invariant curvature spectrum proposed by the slow-roll phase is defined instead:

$$\mathcal{P}_{\mathcal{R}}^{(0)} \simeq \frac{H^2}{\pi\epsilon M_{Pl}^2}. \tag{5.120}$$

The curvature power spectrum that we want to use as input is:

$$\mathcal{P}_{\mathcal{R}}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)}\left(1-\frac{3}{2}\epsilon_{step}W'(k\tau_f)D\left(\frac{k\tau_f}{\beta}\right)\right), \tag{5.121}$$

with $1 < k\tau_f < \beta$.
For the following problem we choose a step function F of the form $F = \tanh x$. Therefore it is found that $D(x) = \frac{\pi x}{\sinh \pi x}$ with $x = \frac{k\tau_f}{\beta}$.
We therefore choose to compute the contribution offered in the convolution by the first oscillating scalar terms; we then proceed to approximate the various analytic terms in the limit for which $k\tau_f \to 1$ (obviously this limit will have to be reconverted with the right convolution arguments). We recall that

$$W^{'}(x) = \left(-3+\frac{9}{x^2}\right)\cos 2x + \left(15-\frac{9}{x^2}\right)\frac{\sin 2x}{2x}. \tag{5.122}$$

Developing for $x \to 1$ we obtain:

$$\sin(2x) \simeq \sin(2) + 4\cos(2)(x-1) - 8\sin(2)(x-1)^2, \tag{5.123}$$

$$\cos(2x) \simeq \cos(2) - 4\sin(2)(x-1) - 8\cos(2)(x-1)^2. \tag{5.124}$$

It is simple to observe how for $k\tau_f \to 1$, considering that in the theory $\beta >> 1$, the argument of the dumping function tends to zero; hence in that limit the dumping function will tend to unity. As in the Fast-Roll model, we write the system of arguments in the convolution that tend to the value of the development

$$\begin{cases} q\tau_f \to 1 \\ |\mathbf{k}-\mathbf{q}|\,\tau_f \to 1. \end{cases} \tag{5.125}$$

The decomposition of the modulus leads to two main limits of interest on momenta. If we assume the condition that $k >> q$, then we have that $k\tau_f \to 1$, hence that $v << 1$ and $u \simeq 1$. Therefore, this limit allows us to induce a GWs background living in the outer frequency range for which $k\tau_f \to 1$. In the opposite limit, however, we have that $q >> k$, hence that $q\tau_f >> k\tau_f$, so we observe that $k\tau_f << 1$ for $v >> 1$, $u \simeq v$. This limit defines the induction of a GWs background in the "infrared" interval, i.e. for $k\tau_f << 1$. Let us proceed to the calculation of the latter infrared contribution. We could write

$$\begin{aligned}\Omega_{GW,c}(k) \simeq \int_0^\infty dv \int_{|1-v|}^{1+v} duv^{-4} \ln^2 v \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \\ \left(\left[-3+\frac{9}{(kv\tau_f)^2}\right]\left[\cos(2) - 4\sin(2)(kv\tau_f-1) - 8\cos(2)(kv\tau_f-1)^2\right] + \right. \\ \left.\left[15-\frac{9}{(kv\tau_f)^2}\right] \times \frac{1}{2kv\tau_f} \times \left[\sin(2) + 4\cos(2)(kv\tau_f-1) - 8\sin(2)(kv\tau_f-1)^2\right]\right) \times \\ same(ku\tau_f). \end{aligned} \tag{5.126}$$

We rename the round brackets of rows two and three of (5.124) with the help of the function $A(kv\tau_f)$. It should be explained how this calculation relates only to the product of the oscillating terms that cause a variation on the constant slow-roll value. This term is in fact the dominant perturbation contribution, and therefore the most interesting to study. In simpler form it is rewritten

$$\Omega_{GW,c}(k) \simeq \int_0^\infty dvv^{-4} \ln^2 v \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \times A(kv\tau_f)^2. \tag{5.127}$$

By developing the term A in such a way as to consider within it only the dominant terms of order 0, considering only the positive contributions in order to have the maximum numerical contribution to the integral, and considering the limit on the intrinsic momentum of the case, it is possible to approximate and suppose the function $A \simeq n$, with n a number independent of the momentum (or rather, weakly dependent on it) and positive. With this idea, the integral is greatly simplified

$$\Omega_{GW,c}(k) \simeq \int_0^\infty dv v^{-4} \ln^2 v \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \times n^2, \tag{5.128}$$

$$\Omega_{GW,c}(k) \simeq \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \times n^2 \int_0^\infty dv v^{-4} \ln^2 v, \tag{5.129}$$

$$\Omega_{GW,c}(k) \simeq \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \times n^2 \int_0^\infty \frac{dq}{k}\frac{k^4}{q^4} \ln^2 q, \tag{5.130}$$

$$\Omega_{GW,c}(k) \simeq \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \times n^2 \frac{(k\tau_f)^3}{\tau_f^3} \int_0^\infty \frac{dq}{q^4} \ln^2 q. \tag{5.131}$$

The solution expresses a cubic power growth of the GWs background in the regime of external frequencies for which $k\tau_f << 1$.
We now proceed to the calculation of the second convolution contribution offered by the interaction of the first oscillating modes, i.e. the one defining an induced background in the limit of external frequencies for which $k\tau_f \to 1$. In the standard case of $c_s^2 < 1$, we write

$$\Omega_{GW,c}(k) \simeq \int_0^\infty dv \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \left(\frac{2v}{1-c_s^{-2}}\right)^\alpha \times A(kv\tau_f) \times same(ku\tau_f). \tag{5.132}$$

Then, simplifying using the same mathematical techniques seen in the Fast-Roll model, we obtain

$$\begin{aligned}\Omega_{GWs,c}(k) \simeq{}& \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}} \left(1-c_s^{-2}\right)^{-\alpha} \\ &\left(\left[-3+\frac{9}{(k\tau_f)^2}\right][\cos(2) - 4\sin(2)(k\tau_f - 1) - 8\cos(2)(k\tau_f-1)^2] + \right. \\ &\left.\left[15-\frac{9}{(k\tau_f)^2}\right] \times \frac{1}{2k\tau_f} \times [\sin(2) + 4\cos(2)(k\tau_f-1) - 8\sin(2)(k\tau_f-1)^2]\right) \\ &\int_0^\infty dv v^\alpha n.\end{aligned} \tag{5.133}$$

We rename the round bracket just outside the integral with the function γ. Thus:

$$\Omega_{GWs,c}(k) \simeq \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}}(1-c_s^{-2})^{-\alpha} \times n \times \gamma(k\tau_f) \int_0^\infty dv v^\alpha, \tag{5.134}$$

$$\Omega_{GWs,c}(k) \simeq \frac{9}{4}\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}}(1-c_s^{-2})^{-\alpha} \times n \times \gamma(k\tau_f) \frac{(k\tau_f)^{-\alpha-1}}{\tau_f^{-\alpha-1}} \int_0^\infty dq q^{\alpha+1}. \tag{5.135}$$

The plot of such a solution is to be studied in the external frequency range for which $k\tau_f \to 1$. In this range the function defines a first undamped oscillation condition, which precisely follows from the power growth calculated earlier. Having calculated the GWs background induced by the scalar convolution of the first oscillation modes, it is now necessary to define the second GWs background induced by the convolution of approximate curvature power spectrum on the upper limit of the moment; hence we proceed to define the GWs background induced by scalars associated with the last part of the damped oscillation in the convolution.
We rewrite the system for the curvature power-spectrum arguments

$$\begin{cases} q\tau_f \to \beta \\ |\mathbf{k}-\mathbf{q}|\, \tau_f \to \beta. \end{cases} \tag{5.136}$$

The decomposition of the modulus leads to two main limits of interest on momenta. If we assume the condition for which $k >> q$, then we have that $k\tau_f \to \beta$, hence that $v << 1$ and $u \simeq 1$. Thus this limit allows us to induce a GWs background that lives in the external frequency range for which $k\tau_f \to \beta$. In the opposite limit, on the other hand, we have that $q >> k$, hence that $q\tau_f >> k\tau_f$, therefore we observe that $k\tau_f << \beta$ for $v >> 1$, $u \simeq v$. This limit defines the induction of a GWs background in the "infrared" range, i.e. for $k\tau_f << \beta$, which overlaps with those calculated above; these terms must therefore be added together.
In this case, the dumping function will have a non-trivial development. Developing the hyperbolic sine function around the value of π, we write

$$\sinh(\pi x) \simeq \sinh(\pi) + \pi^2 \cosh(\pi)(x-1) + ... \tag{5.137}$$

The new infrared contribution is now calculated. We write

$$\begin{aligned} \Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2} \frac{9}{4}\epsilon^2_{step} \int_0^\infty dv \int_{|1-v|}^{1+v} du v^{-4} \ln^2 v \times \\ \left(\left[-3 + \frac{9}{(kv\tau_f)^2} \right] [\cos(2\beta) - 4\sin(2\beta)(kv\tau_f - \beta) - 8\cos(2\beta)(kv\tau_f-\beta)^2] + \right. \\ \left. \left[15 - \frac{9}{(kv\tau_f)^2} \right] \times \frac{1}{2kv\tau_f} \times [\sin(2\beta) + 4\cos(2\beta)(kv\tau_f-\beta) - 8\sin(2\beta)(kv\tau_f-\beta)^2] \right) \times \\ \frac{\pi k v\tau_f/\beta}{\sinh(\pi) + ...} same(ku\tau_f). \end{aligned} \tag{5.138}$$

With the same approximations as in the previous case, we write

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2\int_0^\infty dvv^{-4}\ln^2 v \times n^2 \times (\frac{(\pi kv\tau_f/\beta)}{\sinh{(\pi)}})^2, \tag{5.139}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2\left(\frac{n\pi}{\beta\sinh\pi}\right)^2(k\tau_f)^2\int_0^\infty dvv^{-2}\ln^2 v, \tag{5.140}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2\left(\frac{n\pi}{\beta\sinh\pi}\right)^2\frac{(k\tau_f)^3}{\tau_f}\int_0^\infty \frac{dq}{q^2}\ln^2 q. \tag{5.141}$$

As can easily be seen, this contribution reflects the first infrared result found. The only difference is in the definition of the amplitude which is significantly smaller due to the presence of the large sharpness term β in the denominator. It is now necessary to calculate the last UV contribution offered by the second convolution interaction studied. In the standard case in which $c_s^2 < 1$, we write

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2\int_0^\infty dv\int_{|1-v|}^{1+v} du\left(\frac{2v}{1-c_s^{-2}}\right)^\alpha \times$$
$$\left(\left[-3+\frac{9}{(kv\tau_f)^2}\right][\cos{(2\beta)} - 4\sin{(2\beta)}(kv\tau_f-\beta) - 8\cos{(2\beta)}(kv\tau_f-\beta)^2] + \right.$$
$$\left.\left[15-\frac{9}{(kv\tau_f)^2}\right]\times\frac{1}{2kv\tau_f}\times[\sin{(2\beta)} + 4\cos{(2\beta)}(kv\tau_f-\beta) - 8\sin{(2\beta)}(kv\tau_f-\beta)^2]\right)\times$$
$$\frac{\pi kv\tau_f/\beta}{\sinh{(\pi)}+...} same(ku\tau_f). \tag{5.142}$$

Solving the integral we get

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2(1-c_s^{-2})^{-\alpha}\times\gamma(k\tau_f)\times\frac{\pi k\tau_f/\beta}{\sinh\pi}\int_0^\infty dvv^\alpha n\times\frac{\pi kv\tau_f/\beta}{\sinh{(\pi)}}, \tag{5.143}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2(1-c_s^{-2})^{-\alpha}\times\gamma(k\tau_f)\frac{n\pi}{2\beta\sinh{(\pi)}}\times\frac{\pi k\tau_f}{\beta\sinh{(\pi)}}\int_0^\infty dvv^\alpha kv\tau_f, \tag{5.144}$$

$$\Omega_{GWs,c}(k) \simeq \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2(1-c_s^{-2})^{-\alpha}\times\gamma(k\tau_f)n\left(\frac{\pi}{\beta\sinh\pi}\right)^2\left(\frac{(k\tau_f)^{-\alpha}}{\tau_f^{-2-\alpha}}\right)\int_0^\infty dqq^{\alpha+1}. \tag{5.145}$$

The plot of such a function must take into account a β^2 in the denominator leading to a total flattening to zero of the GWs background induced in the relevant range, i.e. for $k\tau_f \to \beta$.

5.2.2 Spectral energy density of GWs from Inflationary epoch

Identical to how it was done in the Fast-Roll section, we will now calculate the spectral energy density of the second-order induced GWs background during the inflationary phase. Let us recall the main formulae to be used in that section. The tensor-power spectrum is defined as follows [201]

$$P_h(k,\eta) = \frac{4}{M_{pl}^4}\sum_j \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2-(1+v^2-u^2)^2}{4uv}\right]^2 \times \tag{5.146}$$

$$\left|I_j(u,v,k,\eta)\right|^2 \mathcal{P}_{\delta\phi_j}(kv)\mathcal{P}_{\delta\phi_j}(ku). \tag{5.147}$$

In such a formulation, the sum is over the scalar field j; in our one-field case, this term will be transparent. We define [201]

$$I_j(u,v,k,\eta) = k^2\int_{-\infty}^{\eta} d\tilde{\eta} g_k(\eta;\tilde{\eta}) f_j(ku,kv,\tilde{\eta}), \tag{5.148}$$

with

$$g_k(\eta;\tilde{\eta}) = \frac{\Theta(\eta-\tilde{\eta})}{k^3\tilde{\eta}^2}\left[k(\tilde{\eta}-\eta)\cos\left(k(\tilde{\eta}-\eta)\right) - (1+k^2\eta\tilde{\eta})\sin\left(k(\tilde{\eta}-\eta)\right)\right], \tag{5.149}$$

the Green's solution in inflaton domination (already encountered in Chapter 2). The source term, on the other hand, is written [201]:

$$f_j(k_1,k_2,\eta) = T_j(k_1,\eta)T_j(k_2,\eta), \tag{5.150}$$

where [201]

$$T_j(k,\eta) = \frac{U_j(k,\eta)}{U_j(k,\eta\to 0)}, \tag{5.151}$$

with U_j the causally connected inflationary solution of the model.
In the EFTI model in question, it is convenient to choose a definition of causally connected modes that describes a Bunch-Davies vacuum state, (since the departure from the slow-roll condition is minimal). Hence we write

$$T(k,\eta) \simeq (1+ik\eta)e^{-ik\eta}. \tag{5.152}$$

Recall the shape of the inflationary power spectrum:

$$\mathcal{P}_{\delta\phi} \simeq \epsilon M_{Pl}^2\mathcal{P}_{\mathcal{R}} \simeq \epsilon M_{Pl}^2\mathcal{P}_{\mathcal{R}}^{(0)}\left(1-\frac{3}{2}\epsilon_{step}W'(k\tau_f)D\left(\frac{k\tau_f}{\beta}\right)\right), \tag{5.153}$$

with

$$W'(x) = \left(-3 + \frac{9}{x^2}\right)\cos(2x) + \left(15 - \frac{9}{x^2}\right)\frac{\sin(2x)}{2x}, \tag{5.154}$$

and the dumping function defined as before

$$D(x) = \frac{\pi x}{\sinh \pi x}. \tag{5.155}$$

It is crucial now to solve the integral over the times that defines the kernel function; in this problem, the kernel is written as

$$I \simeq k^2 \int_{-\infty}^{\eta \to 0} d\eta' g_k(\eta, \eta')(1 + ikv\eta')(1 + iku\eta')e^{-ik\eta'(u+v)}. \tag{5.156}$$

In such a model, we choose to send the conformal time to zero, and thus find the analytical form of the kernel valid in every frequency region. The Mathematica program succeeds in solving this integral, providing the solution:

$$I \simeq -\frac{u^2 + 4uv + v^2 - 1}{(u + v - 1)^2(1 + u + v)^2 k^2}k^2. \tag{5.157}$$

Note how this problem, compared to the Fast-Roll problem, offers a much simpler, analytically tractable solution. It is now possible to write down the tensor-power-spectrum of the system:

$$\begin{aligned}\mathcal{P}_h(k, \eta \to 0) \simeq \frac{4}{M_{Pl}^4} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv}\right)^2 \\ \left|\frac{u^2 + 4uv + v^2 - 1}{(u + v - 1)^2(u + v + 1)^2}\right|^2 \times \mathcal{P}_{\mathcal{R}}^{(0)2}\frac{9}{4}\epsilon_{step}^2\epsilon^2 M_{Pl}^4 \times \\ \left(\left(-3 + \frac{9}{(kv\tau_f)^2}\right)\cos(2kv\tau_f) + \left(15 - \frac{9}{(kv\tau_f)^2}\right)\frac{\sin(2kv\tau_f)}{2kv\tau_f}\right) \times \frac{\pi kv\tau_f/\beta}{\sinh(\pi kv\tau_f/\beta)} \times \\ same(ku\tau_f).\end{aligned} \tag{5.158}$$

Now, exactly as in the case of radiation, we study the GWs background induced by the scalar convolution of the first oscillating scalar modes: thus it is necessary to develop the spectra with the convolution argument that tend to one (as done in radiation), and to calculate the double convolution on the momenta. Again, from the system on the arguments, a bifurcation of the induced GWs background is evident: for $q >> k$, then for $v << 1$ and $u \simeq 1$ a valid GWs background is induced for $k\tau_f \to 1$. In the opposite case, for $v >> 1$ and $u \simeq v$ the convolution in question induces an infrared background for which $k\tau_f << 1$. Let us first concentrate on the calculation of this last infrared term. In the limits of $v >> 1$ and $u \simeq v$ we have that

$$\left(\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv}\right)^2 \simeq 1, \tag{5.159}$$

$$\left|\frac{u^2 + 4uv + v^2 - 1}{(u + v - 1)^2(u + v + 1)^2}\right|^2 \simeq v^{-4}. \tag{5.160}$$

Therefore we can write the tensor-power spectrum

$$\mathcal{P}_h(k, \eta \to 0) \simeq 4\frac{9}{4}\epsilon^2\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}}\int_0^\infty dv \int_{|1-v|}^{1+v} duv^{-4}$$
$$\left(\left[-3 + \frac{9}{(kv\tau_f)^2}\right][\cos(2) - 4\sin(2)(kv\tau_f - 1) - 8\cos(2)(kv\tau_f - 1)^2] +\right.$$
$$\left.\left[15 - \frac{9}{(kv\tau_f)^2}\right] \times \frac{1}{2kv\tau_f} \times [\sin(2) + 4\cos(2)(kv\tau_f - 1) - 8\sin(2)(kv\tau_f - 1)^2]\right) \times$$
$$same(ku\tau_f), \tag{5.161}$$

$$\mathcal{P}_h(k, \eta \to 0) \simeq 4\frac{9}{4}\epsilon^2\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}}\int_0^\infty dvA(kv\tau_f)^2v^{-4}, \tag{5.162}$$

$$\mathcal{P}_h(k, \eta \to 0) \simeq 4\frac{9}{4}\epsilon^2\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}}n^2\int_0^\infty \frac{dv}{v^4}, \tag{5.163}$$

$$\mathcal{P}_h(k, \eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon^2_{step}\mathcal{P}^{(0)2}_{\mathcal{R}}n^2\left(\frac{k\tau_f}{\tau_f}\right)^3\int_0^\infty \frac{dq}{q^4}. \tag{5.164}$$

The solution grows as a third power, parallel to the homologous result found in the radiation. In the second and last case induced by the first scalar convolution, then for $v << 1$ and $u \simeq 1$ a background is induced for $k\tau_f \to 1$. With such limits one can approximate:

$$\left(\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv}\right)^2 \simeq v^2, \tag{5.165}$$

$$\left|\frac{u^2 + 4uv + v^2 - 1}{(u + v - 1)^2(u + v + 1)^2}\right|^2 \simeq v^{-2}. \tag{5.166}$$

where we culd write the tensor-power spectrum in that way

$$\mathcal{P}_h(k,\eta \to 0) \simeq \int_0^\infty dv \int_{|1-v|}^{1+v} du \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2} \times \gamma(k\tau_f)n, \tag{5.167}$$

where $\gamma(k\tau_f)$ is the same as used above in the case of radiation. Developing as done so far, we obtain:

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2} \times \gamma(k\tau_f)n \int_0^\infty dv, \tag{5.168}$$

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2} \times \gamma(k\tau_f)n\frac{\tau_f}{k\tau_f}. \tag{5.169}$$

This solution, in the limit of interest $k\tau_f \to 1$, once the third power growth is complete, starts to oscillate, as can be deduced from a plot of it.
At this point, the last contribution to the GWs background offered by the convolution of the last oscillating modes can also be calculated; the spectrum in the maximum limit of argument, when $kv\tau_f \to \beta$, is studied again. As in the radiation case, there are two subcontributions given by the splitting of the modulus in the resolution of the system on the convolution arguments. For $v << 1$ and $u \simeq 1$ we induce a GWs background for $k\tau_f \to \beta$, while for $v >> 1$ and $u \simeq v$ we induce a readable induced GWs background for $k\tau_f << \beta$, which will be added to the previous infrared contributions (favouring, as in radiation, only an upward shift of the term seen just before). For the first contribution we write

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2} \times \int_0^\infty \frac{dv}{v^4}A^2(kv\tau_f)\left(\frac{\pi kv\tau_f/\beta}{\sinh\pi}\right)^2, \tag{5.170}$$

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2}n^2\left(\frac{\pi}{\beta\sinh(\pi)}\right)^2\frac{(k\tau_f)^3}{\tau_f}\int_0^\infty \frac{dq}{q^2}. \tag{5.171}$$

This GWs background solution holds for $k\tau_f << \beta$ and provides a contribution parallel to the infrared one found by the paired convolution of the first oscillating modes, but with an amplitude value strongly damped by the sharpness term β^2. The last contribution is instead written

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2} \times \int_0^\infty dvn \times \left(\frac{\pi kv\tau_f/\beta}{\sinh(\pi)}\right) \times \gamma(k\tau_f)_{k\tau_f-\beta} \times \frac{\pi k\tau_f}{\sinh(\pi)}, \tag{5.172}$$

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2}n\left(\frac{\pi}{\beta\sinh(\pi)}\right)^2\gamma(k\tau_f)_{k\tau_f-\beta}\tau_f^2\int_0^\infty dqq. \tag{5.173}$$

Again, the function, by virtue of the additional factor of β^2 in the denominator, is damped to a flat trend in the relevant ultraviolet limit ($k\tau_f \to \beta$). As in radiation, it is only possible to evaluate the induced GWs background in three selective zones, in the external limits of the problem. A principle of oscillation is evident for small frequencies, while for higher values of k there will be flattening of the oscillation that is damped altogether. It can easily be assumed that in the central frequency region (which cannot be calculated analytically due to the difficulty of approximation in the central region) there is a continuation of the first oscillation which undergoes increasingly stronger damping.
It is fundamental to note that the dominant amplitude term in the radiation is handled by the term $\mathcal{P}_{\mathcal{R}}^{(0)2}$. The same applies to the study of the inflationary amplitude, but with the addition of the slow-roll term $\epsilon^2 << 1$, which makes this amplitude much smaller. This result fully respects the general thought outlined at the beginning of the section: the GWs background induced during inflation is totally negligible compared to the radiation GWs background.

5.2.3 Cross-talk terms

Also in this model it is necessary to make different modes, belonging to different topological regions of the spectrum, communicate in the convolution. In the present case it is only possible to calculate the convolution between the first oscillating mode and the last one (and vice versa, given the non-symmetry of the convolution itself). We calculate these last terms in both domination regimes.

Cross-talk term in Radiation era

We start with the IR-UV convolution count; then we write the system on the momenta:

$$\begin{cases} kv\tau_f \to 1 \\ ku\tau_f \to \beta. \end{cases} \tag{5.174}$$

The modulus splitting shows how only for $k >> q$, then for $v << 1$ with $u \simeq 1$ a concrete induced GWs background is produced, for $k\tau_f \to \beta$. The corresponding spectral energy density is then written

$$\Omega_{GWs,c}(k) \simeq \frac{9}{4}\epsilon_{step}^2 \mathcal{P}_{\mathcal{R}}^{(0)2} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left(\frac{2v}{1-c_s^{-2}}\right)^\alpha \times$$

$$\left(\left[-3+\frac{9}{(kv\tau_f)^2}\right][\cos(2) - 4\sin(2)(kv\tau_f - 1) - 8\cos(2)(kv\tau_f-1)^2] + \right.$$

$$\left.\left[15-\frac{9}{(kv\tau_f)^2}\right] \times \frac{1}{2kv\tau_f} \times [\sin(2) + 4\cos(2)(kv\tau_f-1) - 8\sin(2)(kv\tau_f-1)^2]\right)$$

$$\left(\left[-3+\frac{9}{(ku\tau_f)^2}\right][\cos(2\beta) - 4\sin(2\beta)(ku\tau_f - \beta) - 8\cos(2\beta)(ku\tau_f-\beta)^2] + \right.$$

$$\left.\left[15-\frac{9}{(ku\tau_f)^2}\right] \times \frac{1}{2ku\tau_f} \times [\sin(2\beta) + 4\cos(2\beta)(ku\tau_f-\beta) - 8\sin(2\beta)(ku\tau_f-\beta)^2]\right) \times$$

$$\frac{ku\tau_f/\beta}{\sinh(\pi)}. \quad (5.175)$$

So we find that

$$\Omega_{GWs,c}(k) \simeq \frac{9}{4}\epsilon_{step}^2 \mathcal{P}_{\mathcal{R}}^{(0)2} \gamma_\beta(k\tau_f)\frac{k\tau_f}{\beta\sinh(\pi)}\int_0^\infty dv v^\alpha n, \quad (5.176)$$

$$\Omega_{GWs,c}(k) \simeq \frac{9}{4}\epsilon_{step}^2 \mathcal{P}_{\mathcal{R}}^{(0)2} \gamma_\beta(k\tau_f)\frac{1}{\beta\sinh(\pi)}\frac{(k\tau_f)^{-\alpha}}{\tau_f^{-\alpha-1}}\int_0^\infty dq q^\alpha. \quad (5.177)$$

The nature of this GWs background underlines a total flattening to zero in the region of interest, quantifying an additional negligible sum contribution in the external UV frequency region $k\tau_f \to \beta$. It is possible to verify how, on the other hand, the permutation of the convolution just studied does not lead to any physical results.

Cross-Talk term in Inflationary era

Also in the inflationary period the unpaired convolution contribution is the same as that seen in radiation, as can easily be expected. Therefore in the internal momentum limit for which $v << 1$ and $u \simeq 1$ the IR-UV scalar convolution induces a GWs background for $k\tau_f \to \beta$. We could write

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2}\int_0^\infty dv \int_{|1-v|}^{1+v} du$$
$$\left(\left[-3+\frac{9}{(kv\tau_f)^2}\right][\cos(2) - 4\sin(2)(kv\tau_f - 1) - 8\cos(2)(kv\tau_f - 1)^2] + \right.$$
$$\left.\left[15-\frac{9}{(kv\tau_f)^2}\right] \times \frac{1}{2kv\tau_f} \times [\sin(2) + 4\cos(2)(kv\tau_f - 1) - 8\sin(2)(kv\tau_f - 1)^2]\right)$$
$$\left(\left[-3+\frac{9}{(ku\tau_f)^2}\right][\cos(2\beta) - 4\sin(2\beta)(ku\tau_f - \beta) - 8\cos(2\beta)(ku\tau_f - \beta)^2] + \right.$$
$$\left.\left[15-\frac{9}{(ku\tau_f)^2}\right] \times \frac{1}{2ku\tau_f} \times [\sin(2\beta) + 4\cos(2\beta)(ku\tau_f - \beta) - 8\sin(2\beta)(ku\tau_f - \beta)^2]\right) \times$$
$$\frac{ku\tau_f/\beta}{\sinh(\pi)}, \quad (5.178)$$

finding the following solution

$$\mathcal{P}_h(k,\eta \to 0) \simeq \frac{36}{4}\epsilon^2\epsilon_{step}^2\mathcal{P}_{\mathcal{R}}^{(0)2}\frac{n\tau_f}{2\beta\sinh(\pi)}\gamma(k\tau_f)_\beta\int_0^\infty dq. \quad (5.179)$$

This solution brings a further oscillation around the value of β, to be added to the others found by the paired contributions.

5.3 Scalar induced GWs from two-field Fast-Roll

The goal of this section is to introduce a Fast-Roll model described by the presence of two scalar fields. The background idea of this section is similar to that proposed in the article by Sasaki [173], described in chapter four.
In that article, we want to make it clear that one way to have a strong growth (a variation, in general) of the curvature perturbation is to rely on the presence of two fields. In fact, if there were only one, this would produce a unique and natural adiabatic curvature perturbation that tends to freeze on superhorizon scales, blocking all growth. With the addition of a field there will be adiabatic curvature perturbation and isocurvature perturbation, so it is generally possible to see how the curvature perturbation on superhorizon scales does not remain constant over time, but can vary.
Controlled growth of the curvature perturbation, on the other hand, leads to increased formation of PBH and an induced background at the second order of GWs, both in radiation and during the inflationary phase. Two fields ϕ and χ are thus introduced, with the former initially dominating over the latter, until its decay. The first field, due to the shape of its potential, initially describes a long slow-roll phase. Subsequently a growth in the oscillatory character of the potential induces a differential Mathieu equation on the fluctuation of ϕ, leading to an exponential growth of $\delta\phi$. Given the coupling between the fields, the fluctuation of the second field $\delta\chi$ will also be affected by this phase, and will also grow exponentially over time and frequencies by induction.
At a certain point in time, the first field will decay, so that only the second field will remain, which will finally become the dynamically dominant one, and will concretely begin to move on its potential. At this time there will remain only the fluctuation of the only remaining field, $\delta\chi$, which from isocurvature perturbation will become curvature perturbation, once on SuperHorizon scales. Therefore at this time it is possible to write the curvature power spectrum as parallel to the power-spectrum of $\delta\chi$, since $\delta\phi$ has either gone to zero if on subhorizon scales, or has been replaced by the modes of the now dominant fluctuation already gone out of the Hubble sphere. So, in summary, the first scalar field defines exponential growth on its fluctuation; this dynamically induces growth on the isocurvature fluctuation of the second field. At the time of the decay of the first, the second field will dominate, and its fluctuation will become of adiabatic curvature, defining the curvature power spectrum, once this fluctuation is on superhorizon scales. It is safe to assume that this fluctuation alone originates the curvature perturbation in the time studied, since it remains the only single adiabatic scalar fluctuation. From here we can assume the reasoning offered in equation (4.50), with solution in (4.53).
However, this result does not contain all the contributions of curvature perturbation

that the theory has to offer: the formula (4.50) in fact, in its entirety, must take into account all the temporal instants t_* in which a scalar fluctuation exits on superhorizon scales as an adiabatic perturbation, originating the curvature perturbation. In summary, the approximate formula (4.51) forgets the curvature contributions offered by the first field in its first domination phase. However, this approximation remains possible, as the model was meant. In fact, the first field, temporally speaking, does practically only slow-roll (whose contribution in curvature is considered in the final moment of the discussion). The exponential growth phase of the primary fluctuation is so rapid in time that it can be critically assumed to make no contribution to the formation of a curvature perturbation (and associated spectrum).

The convenience of such line lies in the parallelism between the curvature power-spectrum and the power-spectrum of $\delta\chi$. Assuming that the latter induces the radiation-induced GWs background, and the power spectrum on the fluctuation of the former field induces the inflationary GWs background (with $P_{\delta\phi} >> P_{\delta\chi}$) one finds how the latter GWs background is extraordinarily greater than the former.

The idea of this section is to take up this system, imagining a first field ϕ that does Fast-Roll, dynamically dominating a second scalar field χ that does Slow-Roll instead. Initially only the first one moves; when this decays it will leave for the actual dynamics of χ which will propose the slow-roll phase on the scales of the CMB. The idea is to be able to demonstrate a trend in the fluctuation of the first field, right from the start, of the form of an exponential increasing in time and decreasing in frequencies (as in the case of single field, of course); the latter will induce a parallel growth on the fluctuation of the second field. In the decay time the boosted $\delta\chi$ fluctuation will remain alone, and will generate the curvature power spectrum of the system, identically as seen before. This applies for an exit-time of the scalar modes following the decay, to ensure the disappearance of the first field fluctuations. The same reasoning is generated as before: this approach doesn't contain the contribution to the curvature perturbation by the first field fluctuation. With good reason we can accept this lack, given the temporal rapidity with which the Fast-Roll occurs. In summary, the Fast-Roll induced growth phase replaces the oscillatory growth phase induced with the Mathieu equation. As the two phases are identical and extremely short, in terms of time (compared to the subsequent or preceding dynamic phases), their contribution to the definition of the perturbation of the e-folding number can be neglected. Thus, in this FR-SR model, with a good approximation we will have a curvature power spectrum defined by the fluctuation of the secondary field alone, during an exit-time following the decay.

As in Sasaki's model [173], since the exponential growth in time of the second fluctuation is induced by the first, which therefore begins this process earlier, we will have the same relationship between the spectra, with the same conclusion.

This would be another way of defining a GWs background induced in inflation greater than that of radiation.
Let us therefore consider, the two-field model, ϕ the first FR field that dominates dynamically in the first phase of an inflationary model composed of two moments, and χ the SR field that begins to dominate in the second phase, i.e. once ϕ has decayed.
It's now possible to build the potential for the following theory:

$$V(\phi,\chi) = V_0 - \frac{m^2\phi^2}{2}g + V_0^{'}\left(1 - \cos\left(\frac{\chi}{\mu}\right)\right)\alpha. \tag{5.180}$$

Writing the system with separable potentials, we take for ϕ a classical Fast-Roll potential, as studied in chapter four, and for χ a typical Small-field-model Slow-Roll potential. The terms g and α represent the coupling constants of the system. In order to satisfy the dynamic condition between the fields, it will be necessary to require that $g >> \alpha$. It is important to define the solution for the background of the scalar fields. Recall how

$$\ddot{\phi} + 3H\dot{\phi} + U^{'} = 0, \tag{5.181}$$
$$\ddot{\chi} + 3H\dot{\chi} + W^{'} = 0, \tag{5.182}$$

with U and W the separable potential for the two field, respectively.
For the second field that describe a Slow-Roll phase we can easily find

$$3H\dot{\chi} \simeq W^{'}. \tag{5.183}$$

The Fast-Roll is a pre-inflationary period for which the slow-roll parameter ϵ is considered close to 1, not higher, in order to have an accelerated expansion anyway. Recall the link between the slow-roll parameters

$$\eta = \eta_v - \epsilon, \tag{5.184}$$

where η quantifies the ratio between the acceleration of the field and the friction term, while η_v represents the ratio between the second derivative of the potential and the potential itself. Given the form of U, and under the assumption that the mass of m_ϕ is close to the starting value V_0, it is easy to see that η_v is a highly increasing function, and so the same follows for η. This reasoning, which is not necessary for the purposes of the discussion, allows us to neglect the friction term (related to velocity) with respect to the acceleration term. Therefore

$$\ddot{\phi} + U^{'} \simeq 0. \tag{5.185}$$

The solution for the background of the first field has already been seen in chapter four

$$\phi(t) = \phi_0 e^{HtF\left(\frac{m_\phi^2}{H^2}\right)}, \tag{5.186}$$

with

$$F\left(\frac{m_\phi^2}{H^2}\right) = \sqrt{\left(\frac{9}{4} + \frac{m_\phi^2}{H^2}\right)} - \frac{3}{2}, \tag{5.187}$$

in the Fast-Roll hypothesis for which $\left|m_\phi^2\right| >> H^2$.
This background solution, however, is expressed in its most general form, including the dynamic friction term. From now on, the calculation of the curvature power spectrum will be divided into three categories. The first (0) involves the consideration of only the contribution of the fluctuation of the secondary field, following the decay. As seen above, this reasoning, although plausible, does not consider the contributions made by the fluctuation of the first field at earlier times. Therefore the second category (1) will also consider this addition, arriving at a more formal description of the problem.
It is now necessary to make a theoretical remark on the nature of the formalism to be used in the theory, namely the formalism δN [202]. The latter, in the multi-field treatment, allows us to study the total curvature perturbation of the system in the following way [202]

$$\zeta(t_c, \mathbf{x}) \simeq \delta N(t_c, t_*, \mathbf{x}) = \sum_i N_{,i}\delta\phi_*^i + \frac{1}{2}\sum_{i,j} N_{,ij}\delta\phi_*^i\delta\phi_*^j. \tag{5.188}$$

where

$$N_{,i} = \frac{\partial N}{\partial \phi_*^i} \tag{5.189}$$

$$N_{,ij} = \frac{\partial^2 N}{\partial \phi_*^i \partial \phi_*^j}. \tag{5.190}$$

Here N is the number of e-folds of an unperturbed theory, with ϕ_* the value of the field in the background definition. The summation is exhibited over the number of fields of the theory. According to the formalism δN, ζ, computed at some time t_c is equivalent, on the large scales, to the perturbation of the e-folding number $\mathcal{N}(t_c, t_*, \mathbf{x})$ from an initial flat hypersurface in $t = t_*$ to a final hypersurface of uniform density in the final time $t = t_c$ [202]. We consider, during inflation, t_* as the time of exit from the Hubble sphere of the mode, while t_c some later time during or after inflation. The statement exhibited in (5.185) is possible by splitting the field value into the sum of the background with the perturbation value, and

serially expanding $\delta N(t_c, t_*, \mathbf{x})$ with respect to the small value of the perturbation of the i-th field.
This formalism, however, is built in the view in which all fields make Slow-Roll [202]. In fact generally $\mathcal{N}$ would depend on the value of the field $\phi_i(t)$, but also on the value of the derivative $\dot{\phi}_i(t)$ [202]. In the above formalism, this last dependence is omitted precisely because of the Slow-Roll. In fact

$$3H\dot{\phi}_i \simeq -V_{,i}, \tag{5.191}$$

in t_*; it can be seen that the velocity feature is embedded in a function of a totally positional character: hence the approximation. However, in general, this kinetic contribution must be fully taken into account (in our fast-roll case in particular). Therefore, while sections (0) and (1) will define a calculation of the curvature power-spectrum via the approximation of the δN formalism for which the fields indicatively make SR, the last section (2) will evaluate the final spectrum also taking into account the non-negligible kinetic contributions of the theory, relative to the first field ϕ.
To summarize, count (0) only evaluates the scalar contribution of the second field fluctuation at the second time moment of domination. Count (1) also considers the scalar contributions of curvature made by the first field at earlier times. Both theories, however, rest on a model constructed to ignore kinetic contributions in defining the curvature perturbation. Model (2) will rewrite the definition of the formalism in a more general form, thus allowing the kinetic terms, which a Fast-Roll phase inevitably defines, to be taken into account.

5.3.1 Fast-Roll, two-field model within the Slow-Roll formalism

In this section, as argued above, the contributions to the curvature perturbation offered initially by the second field's fluctuation alone (at a time moment of domination of the latter) in case (0) will be studied; later, in case (1) the contributions related to the first field's fluctuation at its previous time of domination will also be added. Everything will take place in the approximate Slow-Roll version of the formalism δN [202].
We therefore recall the form of the perturbation on the number of e-folds on large scales [202]:

$$\zeta(t_c, \mathbf{x}) \simeq \delta N(t_c, t_*, \mathbf{x}) = \sum_i N_{,i}\delta\phi_*^i + \frac{1}{2}\sum_{i,j} N_{,ij}\delta\phi_*^i\delta\phi_*^j. \tag{5.192}$$

We could remember the general definition for the number of e-folds

$$N = \int H dt, \tag{5.193}$$

so we can write, in this case with two field

$$N = \int \frac{H d\varphi}{\dot{\varphi}}, \tag{5.194}$$

$$N = -\frac{1}{m_p^2} \int \frac{V}{V'} d\chi - \frac{1}{m_p^2} \int \frac{U + \dot{\phi}^2}{U' + \ddot{\phi}} d\phi. \tag{5.195}$$

Now we have to introduce this integration constant in order to continue with our calculation

$$C = m_p^2 \int \frac{d\chi}{V'} - m_p^2 \int d\phi (\ddot{\phi} + U'). \tag{5.196}$$

The physical meaning of this constant is to uniquely identify the particle trajectory of the binary system in field-space [202]. This constant is calculated by exploiting the respective symmetries relative to the dynamics of the two fields, directly from their equation of motion in a Lagrangian formalism.
It is possible to write the variation of the number of e-folds, simply by deriving (5.192)

$$\begin{aligned} dN = \frac{1}{m_p^2} & \left[\left(\frac{V}{V'} \right)_* - \frac{\partial \chi_c}{\partial \chi_*} \left(\frac{V}{V'} \right)_c - \frac{\partial \phi_c}{\partial \chi_*} \left(\frac{U + \dot{\phi}^2}{U' + \ddot{\phi}} \right)_c \right] d\chi_* \\ & \frac{1}{m_p^2} \left[\left(\frac{U + \dot{\phi}^2}{U' + \ddot{\phi}} \right)_* - \frac{\partial \phi_c}{\partial \phi_*} \left(\frac{U + \dot{\phi}^2}{U' + \ddot{\phi}} \right)_c - \frac{\partial \chi_c}{\partial \phi_*} \left(\frac{V}{V'} \right)_c \right] d\phi_*. \end{aligned} \tag{5.197}$$

This analytic construction is based on the reasoning that (5.192) depends on χ_* and ϕ_*, obviously. Furthermore, in the derivation, we take into account that the terms χ_c and ϕ_c depend on C which depends on ϕ_* and χ_*, in fact $C = C(\phi_*, \chi_*)$. We now derive the values of the fields calculated on the last hypersurface of the problem

$$d\phi_c = \frac{d\phi_c}{dC} \left(\frac{\partial C}{\partial \phi_*} d\phi_* + \frac{\partial C}{\partial \chi_*} d\chi_* \right), \tag{5.198}$$

$$d\chi_c = \frac{d\chi_c}{dC} \left(\frac{\partial C}{\partial \phi_*} d\phi_* + \frac{\partial C}{\partial \chi_*} d\chi_* \right). \tag{5.199}$$

From the form of C that identifies the classical trajectory of the problem, we can write:

$$\frac{\partial C}{\partial \phi_*} = -m_p^2 (\ddot{\phi} + U')_*, \tag{5.200}$$

$$\frac{\partial C}{\partial \chi_*} = \frac{m_p^2}{V'_*}. \tag{5.201}$$

Energy considerations must now be consider for the problem. On the hypersurface with uniform energy density in time t_c we have that

$$U(\phi_c) + W(\chi_c) + \frac{1}{2}\dot{\phi}_c^2 = const. \tag{5.202}$$

Deriving with respect to C, and remaining within the exclusively positional working hypothesis of the problem (typical of the 0,1 models), we obtain

$$\frac{d\phi_c}{dC}U'_c + \frac{\chi_c}{dC}V'_c = 0. \tag{5.203}$$

A second condition to be imposed comes from the derivation of the integration constant. We obtain

$$1 = -m_p^2\frac{d\phi_c}{dC}(\ddot{\phi} + U')_c + m_p^2\frac{d\chi_c}{dC}\frac{1}{V'_c}. \tag{5.204}$$

It is logical to see how the last two equations (5.200) and (5.201) define a closed system for the variables $\frac{d\phi_c}{dC}$ e $\frac{d\chi_c}{dC}$, whose solution is written

$$\begin{cases} m_p^2\frac{d\phi_c}{dC} = \frac{-1}{(\ddot{\phi}+U')_c + \frac{U'_c}{W'^2_c}}, \\ m_p^2\frac{d\chi_c}{dC} = \frac{1}{(\ddot{\phi}+U')_c + \frac{U'_c}{W'^2_c}} \times \frac{U'_c}{W'_c}. \end{cases} \tag{5.205}$$

Now we have to define the following variation:

$$\frac{d\phi_c}{d\phi_*} = \frac{d\phi_c}{dC}\frac{dC}{d\phi_*} \simeq \frac{(\ddot{\phi} + U')_*}{(\ddot{\phi} + U')_c + \frac{U'_c}{W'^2_c}}, \tag{5.206}$$

$$\frac{d\phi_c}{d\chi_*} = \frac{d\phi_c}{dC}\frac{dC}{d\chi_*} \simeq \frac{-1}{(\ddot{\phi} + U')_c + \frac{U'_c}{W'^2_c}} \times \frac{1}{V'_*}, \tag{5.207}$$

$$\frac{d\chi_c}{d\chi_*} = \frac{d\chi_c}{dC}\frac{dC}{d\chi_*} \simeq \frac{1}{(\ddot{\phi} + U')_c + \frac{U'_c}{W'^2_c}} \times \frac{U'_c}{W'_c} \times \frac{1}{W'_*}, \tag{5.208}$$

$$\frac{d\chi_c}{d\phi_*} = \frac{d\chi_c}{dC}\frac{dC}{d\phi_*} \simeq \frac{(\ddot{\phi} + U')_*}{(\ddot{\phi} + U')_c + \frac{U'_c}{W'^2_c}} \times \frac{U'_c}{W'_c}. \tag{5.209}$$

These derivations are essential and must be substituted in the variation of the e-folding number N set out in formula (5.194). We therefore find

$$\frac{dN}{d\phi_*} = \frac{1}{m_p^2}\left[\left(\frac{U+\dot{\phi}^2}{\ddot{\phi}+U'}\right)_* - \frac{(\ddot{\phi}+U')_*}{(\ddot{\phi}+U')_c + \frac{U'_c}{W_c'^2}}\left(\frac{U+\dot{\phi}^2}{\ddot{\phi}+U'}\right)_c + \right.$$
$$\left. + \frac{(\ddot{\phi}+U')_*}{(\ddot{\phi}+U')_c + \frac{U'_c}{W_c'^2}} \times \frac{U'_c}{W'_c}\left(\frac{W}{W'}\right)_c\right], \quad (5.210)$$

$$\frac{dN}{d\chi_*} = \frac{1}{m_p^2}\left[\left(\frac{W}{W'}\right)_* - \frac{1}{(\ddot{\phi}+U')_c + \frac{U'_c}{W_c'^2}} \times \frac{U'_c}{W'_c} \times \frac{1}{W'_*}\left(\frac{W}{W'}\right)_c + \right.$$
$$\left. + \frac{1}{(\ddot{\phi}+U')_c + \frac{U'_c}{W_c'^2}} \times \frac{1}{W'_*}\left(\frac{U+\dot{\phi}^2}{\ddot{\phi}+U'}\right)_c\right]. \quad (5.211)$$

These terms are those necessary for the calculation of the curvature perturbation on large scales, as easily observed through equation (5.189).
In the case (0), as also reported in Sasaki's paper [173], only the contribution of the second scalar field in $t_i = t_* > t_{decay}$ to the spectrum is considered, since the Fast-Roll is so fast in time that it provides a minimal resource to the spectrum. Therefore

$$\mathcal{P}_\zeta \simeq \mathcal{P}_{\delta\chi} = \frac{k^3}{2\pi^2}\left|\frac{dN}{d\chi_*}\right|^2 |\delta\chi_*|^2 . \quad (5.212)$$

In case (1), instead, the contributions of the first field are also taken into account, for a more complete result

$$\mathcal{P}_\zeta = \frac{k^3}{2\pi^2}|\delta N|^2 \simeq \frac{k^3}{2\pi^2}\left|\frac{dN}{d\phi_*}\delta\phi_* + \frac{dN}{d\chi_*}\delta\chi_*\right|^2 . \quad (5.213)$$

The only unknowns for the complete solution of curvature power-spectrum remain the perturbations on the fields calculated in the exit time from the horizon, namely $\delta\phi_*$ and $\delta\chi_*$. We will eventually calculate the value of these functions.

5.3.2 Fast-Roll, two-field model beyond the Slow-Roll formalism

In case (2), the kinetic contribution is also taken into account, since, at least the field ϕ, does not do Slow-Roll, but Fast-Roll. Therefore, as explained before, the formalism must be modified and generalized, since now δN will depend not only on the position, but also on the velocity term $\dot{\phi}$.
We can generalize the theory of the formalism δN as follows

$$\zeta(t_c, \mathbf{x}) = \delta N(t_*, t_c, \mathbf{x}) = \sum_i N_{,i}\delta\phi_*^i + \sum_i \frac{\partial N}{\partial \dot{\phi}_*^i}\delta\dot{\phi}_*^i, \tag{5.214}$$

We choose to remain in a first-order perturbative theory for analytical simplicity. The form of the integration constant of the classical trajectory is also modified. It is possible, for the purposes of the theory, to think that this variable also depends on the kinetic contribution of the first field. It can be written

$$C = m_p^2 \int \frac{d\chi}{V'} - m_p^2 \int d\phi(\ddot{\phi} + U') - m_p^2 \int d\dot{\phi}(\ddot{\phi} + U'). \tag{5.215}$$

The function N will obviously remain of the same form as expressed in (5.192). Nevertheless now the functions ϕ_c and χ_c will have the dependence of C which in turn depends on $C = C(\phi_*, \chi_*, \dot{\phi}_*)$. Therefore now ϕ_c and χ_c will be functions that also depend on the new term $\dot{\phi}_*$. Thus, the variation of N will include new derivative contributions

$$dN = dN_{(0,1)} + \frac{1}{m_p^2}\left[-\frac{\partial\phi_c}{\partial\dot{\phi}_*}\left(\frac{U + \dot{\phi}^2}{\ddot{\phi} + U'}\right)_c - \left(\frac{W}{W'}\right)_* \frac{\partial\chi_c}{\partial\dot{\phi}_*}\right] d\dot{\phi}_*. \tag{5.216}$$

We have to write the derivatives of the fields calculated on the last hypersurface, taking into account the new dependence of the integration constant with respect to the kinetic term of the first field

$$d\phi_c = \frac{d\phi_c}{dC}\left[\frac{\partial C}{\partial\phi_*}d\phi_* + \frac{\partial C}{\partial\chi_*}d\chi_* + \frac{\partial C}{\partial\dot{\phi}_*}d\dot{\phi}_*\right], \tag{5.217}$$

$$d\chi_c = \frac{d\chi_c}{dC}\left[\frac{\partial C}{\partial\phi_*}d\phi_* + \frac{\partial C}{\partial\chi_*}d\chi_* + \frac{\partial C}{\partial\dot{\phi}_*}d\dot{\phi}_*\right]. \tag{5.218}$$

Now, as before, the variations of the constant with respect to its defining variables must be defined

$$\frac{\partial C}{\partial \phi_*} = -m_p^2(\ddot{\phi} + U')_*, \tag{5.219}$$

$$\frac{\partial C}{\partial \chi_*} = \frac{m_p^2}{V_*'}, \tag{5.220}$$

$$\frac{\partial C}{\partial \dot{\phi}_*} = -m_p^2(\ddot{\phi} + U')_*. \tag{5.221}$$

At this point, the energetic condition on the uniform energy density spatial-hypersurface must be introduced again

$$U(\phi_c) + W(\chi_c) + \frac{1}{2}\dot{\phi}_c^2 = const. \tag{5.222}$$

Nevertheless, this time, the kinetic dependence of the integration constant must be considered, so the derivation brings a new term

$$\frac{d\phi_c}{dC}U_c' + \frac{d\chi_c}{dC}W_c' + \dot{\phi}_c\frac{d\dot{\phi}_c}{dC} = 0. \tag{5.223}$$

The latter must be coupled to the equation of variation of the integration constant, as before

$$1 = -m_p^2\frac{d\phi_c}{dC}(\ddot{\phi} + U')_c + m_p^2\frac{d\chi_c}{dC}\frac{1}{V_c'} - m_p^2\frac{d\dot{\phi}_c}{dC}(\ddot{\phi} + U')_c. \tag{5.224}$$

It is logical to observe that in this case the system is not closed: a new energy condition must be introduced, and a system of three independent equations must be defined for three unknowns.
Such a new condition can be found in the definition of the first slow-roll parameter, which we imagine tends to 1 in the time for which the system is studied. Thus

$$\epsilon_{t_c,\phi} \simeq 1 \simeq \left(\frac{\dot{\phi}^2}{U}\right)_c. \tag{5.225}$$

We can write

$$\frac{d\dot{\phi}_c}{dC} \simeq \frac{-m^2\phi_c}{\sqrt{2U_c}}\frac{d\phi_c}{dC}. \tag{5.226}$$

At this point we are able to write the closed system

$$\begin{cases} \frac{d\phi_c}{dC}U_c' + \frac{d\chi_c}{dC}W_c' + \dot{\phi}_c\frac{d\dot{\phi}_c}{dC} = 0 \\ 1 = -m_p^2\frac{d\phi_c}{dC}(\ddot{\phi} + U')_c + m_p^2\frac{d\chi_c}{dC}\frac{1}{V_c'} - m_p^2\frac{d\dot{\phi}_c}{dC}(\ddot{\phi} + U')_c \\ \frac{d\dot{\phi}_c}{dC} \simeq \frac{-m^2\phi_c}{\sqrt{2U_c}}\frac{d\phi_c}{dC}. \end{cases} \tag{5.227}$$

This system must be solved algebraically, leading to the following solutions on the derivatives

$$m_p^2 \frac{d\phi_c}{dC} \simeq \frac{1}{-(\ddot{\phi}+U')_c \frac{m^2\phi_c}{\sqrt{2U_c}} + \frac{1}{W_c'^2} \times \left(\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'\right) - (\ddot{\phi}+U')_c}, \tag{5.228}$$

$$m_p^2 \frac{d\chi_c}{dC} \simeq \frac{\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'}{-(\ddot{\phi}+U')_c \frac{m^2\phi_c}{\sqrt{2U_c}} + \frac{1}{W_c'^2} \times \left(\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'\right) - (\ddot{\phi}+U')_c} \times \frac{1}{W_c'}, \tag{5.229}$$

$$m_p^2 \frac{d\dot{\phi}_c}{dC} \simeq \frac{\frac{-m^2\phi_c}{\sqrt{2U_c}}}{-(\ddot{\phi}+U')_c \frac{m^2\phi_c}{\sqrt{2U_c}} + \frac{1}{W_c'^2} \times \left(\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'\right) - (\ddot{\phi}+U')_c}. \tag{5.230}$$

For the sake of analytical simplicity, let us rename the common denominator of the three unknowns defined above with the term B.
It is now possible to calculate the derivatives of the fields

$$\frac{d\phi_c}{d\phi_*} \simeq \frac{d\phi_c}{dC}\frac{dC}{d\phi_*} \simeq \frac{-(\ddot{\phi}+U')_*}{B}, \tag{5.231}$$

$$\frac{d\phi_c}{d\chi_*} \simeq \frac{d\phi_c}{dC}\frac{dC}{d\chi_*} \simeq \frac{1}{W_*' \times B}, \tag{5.232}$$

$$\frac{d\chi_c}{d\phi_*} \simeq \frac{d\chi_c}{dC}\frac{dC}{d\phi_*} \simeq \frac{-(\ddot{\phi}+U')_*}{W_c' \times B} \times \left[\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'\right], \tag{5.233}$$

$$\frac{d\chi_c}{d\chi_*} \simeq \frac{d\chi_c}{dC}\frac{dC}{d\chi_*} \simeq \frac{\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'}{B \times W_c'W_*'}, \tag{5.234}$$

$$\frac{d\phi_c}{d\dot{\phi}_*} \simeq \frac{d\phi_c}{dC}\frac{dC}{d\dot{\phi}_*} \simeq \frac{-(\ddot{\phi}+U')_*}{B}, \tag{5.235}$$

$$\frac{d\chi_c}{d\dot{\phi}_*} \simeq \frac{d\chi_c}{dC}\frac{dC}{d\dot{\phi}_*} \simeq \frac{-(\ddot{\phi}+U')_*}{W_c' \times B} \times \left[\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'\right]. \tag{5.236}$$

Now, as a final step, it is necessary to replace these terms in the definition of the perturbation of the e-folding number expressed in formula (5.213).
We therefore write

$$\frac{dN}{d\chi_*} \simeq \frac{1}{m_p^2}\left[\left(\frac{W}{W'}\right)_* - \left(\frac{W}{W'}\right)_c \frac{\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'}{B \times W_c'W_*'} - \left(\frac{U+\dot{\phi}^2}{U'+\ddot{\phi}}\right)_c \times \frac{1}{W_*' \times B}\right], \tag{5.237}$$

$$\frac{dN}{d\phi_*} \simeq \frac{1}{m_p^2}\left[\left(\frac{U+\dot{\phi}^2}{U'+\ddot{\phi}}\right)_* + \left(\frac{U+\dot{\phi}^2}{U'+\ddot{\phi}}\right)_c \frac{(\ddot{\phi}+U')_*}{B} + \left(\frac{W}{W'}\right)_c \frac{(\ddot{\phi}+U')_*}{B \times W_c'} \frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U_c'\right], \tag{5.238}$$

$$\frac{dN}{d\dot{\phi}_*} \simeq \frac{1}{m_p^2}\left[\left(\frac{U+\dot{\phi}^2}{U'+\ddot{\phi}}\right)_c \frac{(\ddot{\phi}+U')_*}{B} + \left(\frac{W}{W'}\right)_c \frac{(\ddot{\phi}+U')_*}{B\times W'_c}\frac{m^2\phi_c\dot{\phi}_c}{\sqrt{2U_c}} - U'_c\right]. \quad (5.239)$$

Therefore we can conclude:

$$\mathcal{P}_\zeta \simeq \frac{k^3}{2\pi^2}|\delta N|^2 \simeq \frac{k^3}{2\pi^2}\left|\frac{dN}{d\phi_*}\delta\phi_* + \frac{dN}{d\chi_*}\delta\chi_* + \frac{dN}{d\dot{\phi}_*}\delta\dot{\phi}_*\right|^2. \quad (5.240)$$

In order to get a final solution of the curvature spectrum, the dynamic solution of the field fluctuations must be defined. This can be found by solving the given system of differential equations associated to a theory of two coupled fields

$$\begin{aligned}
\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k + m^2_{\chi\chi}\delta\chi_k + m^2_{\chi\phi}\delta\phi_k = 0,\\
\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k + m^2_{\phi\phi}\delta\phi_k + m^2_{\chi\phi}\delta\chi_k = 0,\\
m^2_{\chi\chi} = \frac{\partial^2 V}{\partial\chi^2} - \frac{1}{M_p^2}\left(3\dot{\chi}^2 + \frac{2\dot{\chi}\ddot{\chi}}{H} - \frac{\dot{H}\dot{\chi}^2}{H^2}\right),\\
m^2_{\phi\phi} = \frac{\partial^2 V}{\partial\phi^2} - \frac{1}{M_p^2}\left(3\dot{\phi}^2 + \frac{2\dot{\phi}\ddot{\phi}}{H} - \frac{\dot{H}\dot{\phi}^2}{H^2}\right),\\
m^2_{\chi\phi} = \frac{\partial^2 V}{\partial\chi\partial\phi} - \frac{1}{M_p^2}\left(3\dot{\chi}\dot{\phi} + \frac{\dot{\chi}\ddot{\phi}+\dot{\phi}\ddot{\chi}}{H} - \frac{\dot{H}\dot{\chi}\dot{\phi}}{H^2}\right).
\end{aligned} \quad (5.241)$$

Under the hypothesis of the theory for which the first field ϕ dynamically dominates over the second field χ, thus for $g >> \alpha$, a singular simplification occurs for which $m_{\phi\phi} >> m_{\chi\phi} >> m_{\chi\chi}$. Therefore, the system of differential equations on fluctuations is simplified as follows

$$\begin{cases}
\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k \simeq \frac{\dot{\chi}\ddot{\phi}}{M_p^2 H}\delta\phi_k,\\
\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + (\frac{k^2}{a^2} - m^2)\delta\phi_k = 0.
\end{cases} \quad (5.242)$$

The differential equation on the first field fluctuation leads to the following solution:

$$\begin{aligned}
\delta\phi \simeq \frac{1}{20\sqrt{-4k^2+a^2(9H^2+4m^2)}}\times\\
\Bigg[e^{-\frac{(3aH+\sqrt{-4k^2+a^2(9H^2+4m^2)})t}{2a}} m(3a(-1+e^{\frac{(3aH+\sqrt{-4k^2+a^2(9H^2+4m^2)})t}{a}})H+\\
+(1+e^{\frac{(3aH+\sqrt{-4k^2+a^2(9H^2+4m^2)})t}{a}})\sqrt{-4k^2+a^2(9H^2+4m^2)})\Bigg]. \quad (5.243)
\end{aligned}$$

It is possible to read such a solution analytically: as expected, this solution grows exponentially in time and decreases exponentially in frequency space. This solution is completely parallel to the one described by Linde in his one-field Fast-Roll study [192]. Therefore it is logical to think how these two produce a totally parallel curvature spectrum: a strong red-tilt (intrinsic to FR theory) followed by a flattening typical of SR. The latter is appropriately initialized by the previous inflationary phase.
This solution must now be included in the source term of the differential equation on the scalar fluctuation of the second field. Solving the equation, as expected, we find a solution parallel to the input solution, decreasing in frequency and increasing in time in the sphere.
This result is fully expected, given the complete symmetry with the case offered by Sasaki. Therefore the conclusion offered earlier follows. Such fluctuations induce a curvature power-spectrum totally parallel to that seen in single-field theory. Therefore, if the input function is analytically similar, the induced GWs background solutions will also follow this line.

Conclusions

The purpose of this thesis is to study the scalar-induced gravitational wave background at different phases of cosmological domination. Understanding the parametric formula that links the curvature and inflationary perturbation spectra explains how a phase of departure from the slow-roll is necessary, thus ensuring hypothetical numerical comparability between the GWs backgrounds. It is certainly a viable numerical option to have comparable scalar spectra, so the condition of negligibility of the inflationary GWs background with respect to the radiation-induced background becomes absolutely to be reformulated.

The one-field fast-roll model aims to study both the backgrounds produced by the system, evaluating their analytical comparability. The complex analytical form that the source spectrum guarantees, leads to a mathematical operation of approximating the function in the internal frequency ranges of interest. The convolution count between approximated scalar spectrum terms in the respective ranges, (both paired and unpaired), shows how a GWs background of external frequency living in a region of specific momenta is produced. Therefore, the cohesion of all results provides a complete picture of the problem. In the radiation computation, the overall trend of the GWs induced background well fit that found by computational solution. This result confirms the validity of the approximations used, both mathematical and physical, in having used appropriate theoretical modeling. However, the overall amplitude of the problem is extremely low in terms of observation in the respective frequency ranges. Nevertheless, this solution is acceptable, given the nature of the problem at the second-order perturbative.

The result of the inflationary induction phase cannot be compared with a computational solution, since the program used does not work in appropriate dynamic de-Sitter ranges. Nevertheless, having used the same physical modeling techniques as for the radiation account, there is a good hope that this calculation is also correct. This problem, however, is extremely more complex because of the analytical form of the inflationary kernel. An amplitude estimate of the total system is therefore possible only through the infrared convolution account, given the strong analytical complexity of the generalized kernel. The amplitude value found is extremely large, showing not only an analytical but also a numerical augmentation of that GWs background, but one that goes far beyond initial expectations. In fact, the value of the tensor to scalar ratio far exceeds empirical limits, so we would be inclined to exclude the model from inflationary phenomenology. The two-field model, on the other hand, gives us confidence that we would be able to find a curvature spectrum totally parallel to that for the single-field theory. Therefore it would be possible to reproduce the exact same accounts, with the same approximations, and a totally similar result would be found. The only difference would be in the strong analytical complexity associated with the introduction of a secondary field. Moreover, account (0), relating to the contribution to the curvature of the secondary field alone in the double slow-roll regime guarantees a higher inflation fund than that produced in the later epoch.

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List of Figures

1.1 Example of an inflationary Slow-Roll potential described by a flat region [1]. 7
1.2 Temporal evolution of the comoving Hubble radius during Inflation and other successive epoch [1], [21]. 10
1.3 Constraints on the tensor-to-scalar ratio r0.002 in the ΛCDM model, using Planck TT,TE,EE+lowE and Planck TT,TE,EE+lowE+lensing (red and green, respectively), and joint constraints with BAO and BICEP2/Keck (blue, including Planck polarization to determine the foreground components) [38]. 26

2.1 Typical potential for the hybrid inflation [1]. 51
2.2 Table summarising the main GWs production mechanisms studied so far, relating to the inflationary and subsequent reheating period [1]. 60
2.3 Summary table for the main GWs production models of the overall tensor power spectrum amplitude and spectral index. The last column shows for each model the associated consistency relation [1]. 62
2.4 Actual GWs spectral energy density [1]. 65

3.1 Illustration of the triangular inequality and the vector relationship involved in the GWs integral convolution [119]. 84
3.2 Induced GWs spectral energy density for a primordial scalar Dirac spectrum with different values of b [119]. 90
3.3 Induced GWs spectral energy density for a primordial scalar Dirac spectrum with assumed value $c_s^2 = 1$ [119]. 91
3.4 Induced GWs spectral energy density for a primordial log-normal scalar spectrum [119]. 93
3.5 Induced GWs spectral energy density for a primordial scalar broken-power law spectrum with $n_{IR} = 4$ and $n_{UV} = 2$ [119]. 96

4.1 Curvature Power-spectrum of the EFTI theory in a step features approach for the potential [158]. 104

4.2 Bispectrum in the EFTI Theory [158]. 105
4.3 Summary of the two-field potential model [173]. 110
4.4 Potential for the first dynamical field [173]. 112
4.5 Induced GWs spectral energy density in the two dominance phases studied [173]. The grey area determines the sensitivity range of Lisa [191]. 120

5.1 Power-spectrum curvature of the problem presented in equation (5.1). This results in an excellent parametrization of the Fast-Roll problem. Clearly, it should be read in the scales to which the problem belongs, i.e. the CMB scales (10^{-3}Mpc^{-1}); therefore we have to consider the function from the red-tilt onset. [14]. 138
5.2 Curvature Power-spectrum of the problem presented in equation (5.44), approximated in the intermediate frequency limit of the problem. The value of x in the x-axis determines the y-value of the formulation. 145
5.3 Spectral energy density of the problem exposed in equation (5.47). . 147
5.4 Analytical qualitative plot that resume all the theoretical trend in the overall frequency range. 153
5.5 Curvature Power Spectrum (https://github.com/Lukas-T-W/SIGWfast). 154
5.6 Induced GWs spectral energy density during the final radiation domination period. The plot presents the spectral energy density calculated in the conformal time η_c with respect to the frequency ratio k/k_{ref}. k_{ref} is the reference frequency of the system, and was chosen to be of the order of 10^3, in order to have a positive match with the frequency values involved in the theory. In fact, as Planck's graph itself shows us, the fast-roll problem occurs on scales k_c of the order of $5 \times 10^{-3}\text{Mpc}^{-1}$. Both spectra therefore have to be processed on such values (https://github.com/Lukas-T-W/SIGWfast). 155
5.7 Inflationary spectral energy density of the problem set out in equation (5.103), to be read in the frequency limit of interest, ... 162
5.8 Inflationary spectral energy density of the problem set out in equation (5.105), to be read in the frequency limit of interest, neglecting the inaccurate amplitude values, as the amplitude over-all factor was not graphed for analytical simplicity. Unlike the previous plot, here the generic kernel was considered, choosing a value of $H \simeq 10^{16} GeV$ and a value of $\omega \simeq 20$, both of which conform to the model studied. 164

5.9 Inflationary spectral energy density of the problem set out in equation (5.105), to be read in the frequency limit of interest, neglecting the inaccurate amplitude values, as the amplitude over-all factor was not graphed for analytical simplicity. Note how for large frequency scales the curve tends to remain flat, bringing a constant contribution to the general UV solution. 166
5.10 Analytical qualitative plot that resume all the theoretical trend in the overall frequency range. 167