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A topological perspective for branching-time logics

Una prospettiva topologica per le logiche del tempo ramificato

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Preface

This work focuses essentially on two main topics. In the first part, we describe *Temporal Logic* and, in particular, some logics for *Branching-Time*. In the second part we analyse from a topological point of view the algebraic structure of *trees*, on which many semantics for temporal logics are based.

The main objective of *Temporal Logic* is the definition of a formal language (and of an associated semantics) capable of expressing *tensed* assertions like "It rained", "I slept in the past", "In the future we will learn how to fly". It must be immediately observed that the truth conditions for assertions of this kind can not be expressed in the context of classical logic, since the truth of these sentences depends on the moment in time in which they are considered. Therefore, the first step towards an adequate definition of truth is the search for suitable mathematical structures representing *time*. In the first two chapters, we will analyse different possible syntactic and semantic choices for temporal logic and different ontological assumptions about time that can variously shape our models.

Then, following [29], we will consider an unusual semantics for temporal logic, based on a natural topological structure added to the (usual) *Ockhamist semantics*. In particular, it turns out that the set of maximal branches (histories) in a tree-like representation of time constitutes a *non-Archimedean* topological space. The relationships between topological properties and Ockhamist semantics will be investigated in detail.

In the final part of the thesis, trees will be considered just as algebraic structures, without any reference to the semantics for branching time. We will try to analyse some set-theoretical properties of trees and to rephrase them into a "topological language".

The structure of the thesis is the following:

- Chapter 1 contains an overview of the development of Temporal Logic and a first formalization of linear-time logic. In the final part, some issues about validity and definability will be considered.
- Chapter 2 contains the description of various types of branching time semantics, which arise from the ontological assumption of *Indeterminism*.

A particular attention is paid to the *bundled-tree* semantics.

- Chapter 3 essentially presents the contents of the paper we mentioned above: *Topological Aspects of Branching-Time Semantics* by M. Sabbadin and A. Zanardo. We completely develop the topological perspective for time logic they analysed, and we show that it is equivalent to bundled-tree semantics.
- Chapter 4, which is partially a research work, contains the analysis of some properties of trees from the topological perspective that was introduced in Chapter 3. We discuss various classical properties, like linearity, finitely branchingness, well-foundedness, and some more particular ones, like ω-cofinality and jointedness. This chapter ends with an analysis of Souslin and special trees, which are linked to the well-known *Souslin's Problem.*

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I want to deeply thank my family, my friends and Note Innate: maybe the great majority of you could not fully understand what this was all about (I know, mathematical language and problems are difficult to be shared), but your support and encouragement were very precious to me throughout my years of study.

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Finally, I must admit that this thesis would not have been written (not in English, for sure) without the fundamental and unconditional help from Benny. She read this entire work multiple times (probably more times than I did myself!) and corrected the enormous amount of grammatical and syntactical mistakes I made. Moreover, her support, encouragement, optimism and reassurance led me towards the end of this work. This accomplishment would not have been possible without her.

Contents

Chapter 1

Time logic and linear-time semantics

1.1 Brief history of temporal logic

Temporality has been a centre of discussion for philosophers since ancient times because of its complexity and theoretical nature: as an example, some of Zeno's paradoxes (5^{th} Century B.C.) refer to the questions about infinite divisibility of time intervals.

The first scientific approach to time reasoning certainly is Aristotle's argument about *future contingents* (statements about possible future events which may or may not occur, such as the famous "There will be a sea-fight tomorrow") in Chapter 9 of his *De Interpretatione* (4^{th} Century B.C.): he asserts that definite truth values should not be ascribed to *future contingents* at the present time. A few decades later, the philosopher Diodorus Cronus defined *possibility* as "what is or will ever be", and *necessity* as "what is and will always be": these concepts are still central in time logic.

During the Middle Ages, some philosophers brought important contribution in the discussion on temporality, often analysing the relation between free will and determinism. Apparently, the most important development is the work of William of Ockham (c. 1287-1347). He argued that propositions about the contingent future can not be known by humans as true or false at the present time. However, he thought that humans still have some freedom of choice amongst different possible futures, thus he suggested a future-branching model of time with many possible time-lines (histories). Hence, the truth of propositions regarding future events is relativized to a possible actual history. This model of time is now called Ockhamist.

The first traces of these problems in mathematical logic can be found in the work of George Boole (1815-1864) and Charles S. Peirce (1839-1914). However, the "official" introduction of time logic in modern logic is commonly linked to the works of Arthur Prior (1914-1969), in particular [26], [27] and [28], published in the second half of the 20^{th} century. His formalization of tensed verbs led to the development of present formal Temporal Logic: it is applied in various fields, such as Philosophy, Computer Science, Artificial Intelligence, Physics and Linguistics.

In the same period, the research of Saul Kripke (1940-) on the semantics for modal logics endowed time logic with a language and a semantics suitable for a logical and mathematical development of this subject. In Kripke's models, time is represented as an arbitrary ordered set of moments with a "before-later" relation, hence they are suitable for a lot of applications, not necessarily linked to the original idea of physical time (for example, the set of states of a computation).

In the recent years, time logic developments follow two main paths: on one hand, semantics are investigated from various points of view (logical, algebraic, topological, ...), on the other hand, time logic is implemented in the context of Computer Science, Artificial Intelligence, where the fact that the truth of a statement may depend on time needs to be formalized and implemented.

The interested reader can find other information about the history of Time Logic in [25] and in [41].

1.2 Syntax

In this section we define a syntax, a language, in order to provide a formalization of sentences like "in the past, it rained" or "in the future, we will be able to fly".

The logic we consider in this thesis is an extension of propositional classical logic and, accordingly, its language is an extension of propositional language. In addition to the usual Boolean connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$, this language for time logic contains the unary operators F and P. The formulae $F\phi$ and $P\phi$ are respectively read as "ϕ will happen" and "ϕ happened"[1].

The following definition gives a first answer to the question about the syntax for temporal logic. Other symbols will be added later.

Definition 1.2.1. The set of the *Priorean formulae*, $\mathcal{L}_{\mathcal{P}}$, is the smallest set containing the propositional variables $p_0, p_1, p_2, \ldots$ and every formula con-

[1] These operators were introduced for the first time by Arthur Prior. Thus, F and P are often referred to as *Priorean* operators.

structed by recursive application of the boolean operators $\neg$, $\wedge$ ($\vee$ and $\Rightarrow$ defined as usual) and the *temporal* operators P, F.

Remark 1.2.1. We could take both F and P as defined operators, starting from other two operators, G and H.

Given a proposition ϕ, we read $G\phi$ as "henceforth, ϕ" (or "from now on, ϕ" or "since now, ϕ") and $H\phi$ as "hitherto, ϕ" (or "until now, ϕ").

Starting from them, we can define F and P by the positions $F\phi = \neg G \neg \phi$ and $P\phi = \neg H \neg \phi$: in fact, in the future ϕ will occur if and only if it is not the case that *from now on* $\neg\phi$ will occur, and similarly, in the past ϕ occurred if and only if it is not the case that *until now* $\neg\phi$ has always occurred.

Dually, we can work the other way around and define G and H starting from F and P: $G\phi = \neg F \neg \phi$ and $H\phi = \neg P \neg \phi$.

Definition 1.2.2. Given a Priorean formula ϕ, the *mirror formula* is obtained replacing every F with P in ϕ, and vice-versa (and, consequently, every G with H and vice-versa).

1.3 Flow of time and semantics

In order to assign truth values to tensed propositions, we need to define a valuation: a function that inductively associates truth values to propositions. The first idea is to make valuation time-dependent, which is to associate a different valuation to each moment.

In everyday life language, the truth of tensed assertions generally depends on the moment in time in which they are considered. For instance, the sentence "The battle of Waterloo was fought" is true now, and it has been true at every moment since June 18th, 1815. On the contrary, it is false if considered at any previous moment. Then, the first step towards a formal definition of truth for tensed propositions is to provide a formal description of the set-theoretical structure of the set of moments in time.

1.3.1 Representation of time

Definition 1.3.1. A *representation of time*, or a *flow of time*, is a pair $\mathcal{T} = (T, <)$ consisting of a set T, whose elements are called *moments*, or *time points*, and a binary relation $<$ on T with the following properties:

- transitivity: $\forall t_1, t_2, t_3 \in T$, $t_1 < t_2$ and $t_2 < t_3$ implies $t_1 < t_3$,
- irreflexivity: $\forall t \in T, t \not< t$.

In this chapter, possibly indexed $\mathcal{T}$ always denotes the flow of time $(T, <)$, indexed in the same way.

Then, the representations of time are particular *Kripke Frames*[2], which consist of a set W (of *possible worlds*) endowed with a set of *accessibility* relations between worlds. In our case, possible worlds are called moments and the accessibility relations are $<$ and its inverse $>$.

Remark 1.3.1. The relation $<$ is meant to represent the "earlier than" relation, so $t_1 < t_2$ is read as "the moment t_1 precedes t_2", "the moment t_1 is in the past of t_2", "the moment t_2 follows t_1", or "the moment t_2 is in the future of t_1". Hence, the transitivity of $<$ expresses the (expected) fact that if t_2 is in the future of t_1 and t_3 is in the future of t_2, then t_3 is in the future of t_1. The irreflexivity expresses the fact that every moment is not in its own past, nor in its own future.

Example 1.3.1. Since the requests expressed in Definition 1.3.1 are not very tight, there are many mathematical structures that can be viewed as flows of time. Here are some examples and remarks:

- natural or integer numbers with the usual order: they are both discrete flows of time, the first one with a beginning, a starting moment.
- rational or real numbers with the usual strict order $<$: they are both dense[3], and the second one is also continuous.
- the examples above are totally ordered sets, but totality is not requested in the definition. A non linear example of flow of time is a set X with an irreflexive and transitive relation $\prec$ such that for every $x_0 \in X$ there are exactly two distinct $x_1, x_2 \in X$, both different from x_0, such that $x_0 \prec x_1$ and $x_0 \prec x_2$.
- a less usual but very important example of flow of time is the four dimensional *Minkowsky spacetime*, $\mathcal{S} = (\mathbb{R}^4, \triangleleft)$, with $(x_0, x_1, x_2, t) \triangleleft (x'_0, x'_1, x'_2, t')$ if $t < t'$ and the spatial distance between the two points is less than $c \cdot (t' - t)$, with c the speed of light. One may observe that this example is very different from the ones presented above: in fact, the past of a given space-time moment is *not* unique. For example, if we consider 1 year in the past of a space-time moment (x_1, x_1, x_2, t), it is a 3 dimensional ball in $\mathbb{R}^3$ with a 1 light-year radius centred in (x_0, x_1, x_2), with temporal coordinate equal $t - 1$ year. In the next chapter we will avoid this kind of situation, and to do so we will require the left-connectedness property for our flows of time.

Further assumptions on the structure of representations of time will be discussed in the next chapter. They will be aimed at a characterization of

[2] They were firstly developed in [17] and [16], and they are analized, for example, in [10].

[3] For every pair of subsets X, Y of $\mathbb{R}$ such that $\forall x \in X, y \in Y,\ x \leq y$, there exists a $z \in \mathbb{R}$ such that $\forall x \in X, y \in Y,\ x \leq z \leq y$.

these structures that suits better the intuition that we have about time. We will also add some remarks about the ontological commitments that those assumptions involve.

Remark 1.3.2. We can notice that our definition of flow of time excludes circular time models: if there were moments $t_1 < t_2 < t_3 < \ldots < t_n < t_1$, by transitivity we would have $t_1 < t_1$, against the irreflexivity property. As Venema says in [34], why should logicians choose to exclude this kind of representation? They (maybe) should not choose between different ontologies, and moreover many cultures have precisely a circular and cyclic idea of time. It seems that circular time simply did not receive great attention in logical literature.

We conclude this section by introducing some new notations, which will be useful later on:

Definition 1.3.2. Given a flow of time $(T, <)$ and a moment t of T, we will call the *future of* t the set of time points in the future of t, which is $F_t = \{t' \in T \mid t < t'\}$. Symmetrically, the *past of* t is the set of time points in the past of t, which is $P_t = \{t' \in T \mid t' < t\}$.

1.3.2 Valuation

Now that we have a basic structure for time, we can give a first definition of valuation in the context of temporal logic, i.e. a valuation for Priorean Formulas: it is a function that maps every moment to possibly different classical valuations, which send propositional variables to truth values (classically, true or false, 0 or 1). From now on the set $\{p_1, p_2, \ldots\}$ of propositional variables is always denoted by Φ, in every language we will consider.

Definition 1.3.3. A *valuation* over a representation of time $\mathcal{T} = (T, <)$ is a function $\mathcal{V} : T \to {}^{\Phi}\{0,1\}$ [4], which associates a classical valuation of propositional variables in Φ to each time point t of T.

Definition 1.3.4. Given a flow of time $(T, <)$ and a valuation $\mathcal{V}$ over it, a *model* is the triple $\mathcal{M} = (T, <, \mathcal{V})$.

We can now give a formal inductive definition of *truth* for Priorean formulae at a moment t in a model $\mathcal{M}$:

Definition 1.3.5. Given a Priorean formula ϕ, a model $\mathcal{M} = (T, <, \mathcal{V})$ and a time point t, we write "ϕ is true, or *holds*, in the model $\mathcal{M}$ at the moment t" as $\mathcal{M}, t \vDash \phi$.

[4] If X and Y are sets, by ${}^{X}Y$ we denote the set of all functions from X into Y.

The formal definition is given by the the following rules, by induction on the complexity of the formula.

$$\begin{array}{lll} \mathcal{M},t \vDash p_i & \text{if and only if} & \mathcal{V}(t)(p_i) = 1 \\ \mathcal{M},t \vDash \neg\phi & \text{if and only if} & \text{not } \mathcal{M},t \vDash \phi \\ \mathcal{M},t \vDash \phi\wedge\psi & \text{if and only if} & \mathcal{M},t \vDash \phi \text{ and } \mathcal{M},t \vDash \psi \\ \mathcal{M},t \vDash P\phi & \text{if and only if} & \exists s \in T, s < t,\ \mathcal{M},s \vDash \phi \quad (1.1) \\ \mathcal{M},t \vDash F\phi & \text{if and only if} & \exists s \in T, t < s,\ \mathcal{M},s \vDash \phi \quad (1.2) \end{array}$$

Remark 1.3.3. It is interesting to consider the common language reading of rows (1.1) and (1.2) in the definition above. These rows describe the meaning of the new non-classical operators, that characterize temporal logic:

- in the model $\mathcal{M}$, the proposition $P\phi$ *is true* at the time t if there exists another time point s, in the past of t, in which the proposition ϕ holds.
- in the model $\mathcal{M}$, the proposition $F\phi$ *is true* at the time t if there exists another time point s, in the future of t, in which the proposition ϕ holds.

It is clear from this reading that the definition conveys the right idea.

Remark 1.3.4. As stated in Remark 1.2.1, the operators G and H can be defined on the basis of P and F. But one can follow the other way around. In this case, the last two rows of Definition 1.3.5 have to be replaced by:

$$\begin{array}{lll} \mathcal{M},t \vDash G\phi & \text{if and only if} & \forall s \in T, s < t,\ \mathcal{M},s \vDash \phi \quad (1.1^*) \\ \mathcal{M},t \vDash H\phi & \text{if and only if} & \forall s \in T, t < s,\ \mathcal{M},s \vDash \phi \quad (1.2^*) \end{array}$$

Remark 1.3.5. If in the representation of time $(T,<)$ there exists a moment t_s with empty past (i.e. a *starting time point*), then for every formula ϕ, $H\phi$ is true in t_s, since the condition $\forall s, s < t\ \mathcal{M}, s \vDash \phi$ is trivially verified.

Similarly, if there exists a moment t_e with empty future (i.e. an *ending time point*), then for every formula ϕ, $G\phi$ is true in t_e, since the condition $\forall s, t < s\ \mathcal{M}, s \vDash \phi$ is trivially verified.

Example 1.3.2. We can consider an example of valuation in a simple time model, as done in [34].

Consider the set $\mathbb{N}$ with the usual order $<$ of natural numbers as flow of time, and let ϕ, ψ be two Priorean formulae. Let $\mathcal{M} = (\mathbb{N}, <, \mathcal{V})$ be a model such that $\mathcal{M}, n \vDash \phi$ for every $n > 1000$, and $\mathcal{M}, m \vDash \psi$ for every even time point m.

With this valuation, we can show that $FG\phi$ holds for every $n \geq 0$. In fact, $G\phi$ is "from now on, ϕ", so it is true for every $n \geq 1000$; hence the sentence $FG\phi$ is "in a future moment, from that moment on, ϕ" is true for every moment in the past of any $n \geq 1000$, which is for every $n \geq 0$. In symbols: $\mathcal{M}, 1000 \vDash G\phi$ implies that for all n, $\mathcal{M}, n \vDash FG\phi$.

Similarly, we can show that $GF\psi$ holds throughout $\mathcal{M}$. In fact, the "translation" into natural language of the sentence evaluated in a time point n would be "for every moment m in the future of n, there is a moment in the future of m in which ψ holds". And this is clearly true for every n, since there will always be even numbers in which ψ holds by construction of $\mathcal{M}$.

It is likewise easy to see that $FG\psi$ is not true at 0, nor at any moment of this model. $\mathcal{M}, n \nvDash G\psi$, because there will always be moments in which ψ is false (the odd ones), hence $\mathcal{M}, n \nvDash FG\psi$.

Some authors, like Sabbadin and Zanardo (of [29], [37], [38]), prefer the following different notation for the valuation:

Definition 1.3.6. A *valuation* is a function $V : \mathcal{L} \to \mathcal{P}(T)$ that associates to every propositional variable p a subset of the set T in the flow of time $\mathcal{T}$. The points of this subset are meant to be the ones in which the proposition is true. For complex formulae ϕ, the valuation is defined recursively by means of the following rules:

$t \in V(\neg\phi)$	if and only if	$t \notin V(\phi)$
$t \in V(\phi \wedge \psi)$	if and only if	$t \in V(\phi) \cap V(\psi)$
$t \in V(P\phi)$	if and only if	$\exists s \in T, s < t$ such that $s \in V(\phi)$
$t \in V(F\phi)$	if and only if	$\exists s \in T, t < s$ such that $s \in V(\phi)$

This "subset notation" for valuation will be slightly more convenient than the "functional" one in the following chapters.

Remark 1.3.6. It is easy to switch from the functional notation of Definition 1.3.5 to the subset one with the position $V(p_i) = \{t \in T \mid \mathcal{V}(t)(p_i) = 1\} = \{t \in T \mid \mathcal{M}, t \vDash p\}$. It turns out also that, for arbitrary formula ϕ, $V(\phi) = \{t \in T \mid \mathcal{M}, t \vDash \phi\}$.

1.4 Validity and definability

We conclude this chapter with a brief description of some problems about validity, satisfiability and definability. They are not the main focus of this thesis, but it is worth discussing them in order to have a wider overview of Temporal Logic. The following material is borrowed from [34] and [38].

In general, the truth of a formula at a given moment in a given model is not very significant in Temporal Logic. The interesting problem is to identify the formulae that maintain their truth throughout the representation of time, independently of the chosen valuation. As said in [34], it is felt that such formulae provide information concerning the structure of the underlying flow of time, as they describe properties of the "earlier-later" relation $<$. Formally:

Definition 1.4.1. Given a formula ϕ and a flow of time $\mathcal{T} = (T, <)$, we say that ϕ is *valid* in $\mathcal{T}$, written $\mathcal{T} \vDash \phi$, if for every valuation $\mathcal{V}$ and every moment $t \in T$, we have $(T, <, \mathcal{V}), t \vDash \phi$.

Definition 1.4.2. Given a formula ϕ and a family $(\mathcal{T}_\lambda)_{\lambda\in\Lambda}$ of flows of time, ϕ is valid in $(\mathcal{T}_\lambda)_{\lambda\in\Lambda}$ if it is valid in every $\mathcal{T}_\lambda$.

Dually, we have the definition of satisfiability: a formula is satisfiable if its negation is not valid, hence if there exists a valuation that verifies its truth. Formally:

Definition 1.4.3. Given a formula ϕ and a flow of time $(T, <)$, we say that ϕ is *satisfiable* in $\mathcal{T}$, if $\neg\phi$ is *not* valid in $\mathcal{T}$.

Definition 1.4.4. Given a formula ϕ and a family of flows of time $(\mathcal{T}_\lambda)_{\lambda\in\Lambda}$, we say that ϕ is *satisfiable* in $(\mathcal{T}_\lambda)_{\lambda\in\Lambda}$, if $\neg\phi$ is *not* valid in $(\mathcal{T}_\lambda)_{\lambda\in\Lambda}$.

Example 1.4.1. Let $\mathscr{L}_\mathscr{D}$ be the class of dense linearly ordered sets. Given any propositional variable p_0, the formula $Fp_0 \to FFp_0$ is valid on $\mathscr{L}_\mathscr{D}$, and it is not valid in the class of *non*-dense flow of time.

To prove this, consider a dense linear order $\mathcal{T}$, an arbitrary valuation $\mathcal{V}$ on it, and a time point $t \in T$ such that $(T, <, \mathcal{V}), t \vDash Fp_0$. Then, by definition of F, there is $s \in T$, such that $t < s$ and $(T, <, \mathcal{V}), s \vDash p_0$. By density, there is $u \in T$, $t < u < s$, so that $(T, <, \mathcal{V}), u \vDash Fp_0$ and $(T, <, \mathcal{V}), t \vDash FFp_0$. Then, we can conclude $\mathcal{T} \vDash Fp_0 \to FFp_0$ because both $\mathcal{V}$ and t were arbitrary.

Moreover, if we take $\mathcal{T} = (\mathbb{Z}, <)$, and a valuation $\mathcal{V}$ such that $\mathcal{V}(z)(p_0) = 0$ for every $z \neq 3$ and $\mathcal{V}(3)(p_0) = 1$, we have that Fp_0 is clearly true at 2, but FFp_0 is not true at 2 because there is no other integer between 2 and 3, and there is no other future point in which ϕ holds.

We can generalize this argument by showing that $Fp_0 \to FFp_0$ can be falsified in every *non*-dense time model. In fact, $\mathcal{T}$ is non-dense if there are two points $t < s$ with nothing else in between, so we can build a valuation which makes p_0 true *only* at s. As above, Fp_0 is true at t, but FFp_0 is not. Hence $\mathcal{T} \nvDash Fp_0 \to FFp_0$.

From this example we understand that the formula $Fp_0 \to FFp_0$ is very informative, since it characterizes dense linear time representation. We can generalize this behaviour with the definition of *definability*:

Definition 1.4.5. A Priorean formula ϕ *defines* a class $\mathscr{C}$ of flows of time in a class $\mathscr{K}$ if for every flow of time $\mathcal{T}$ in $\mathscr{K}$, $\mathcal{T} \vDash \phi$ if and only if $\mathcal{T} \in \mathscr{C}$.

The discussion of Example 1.4.1 contains the proof of the following Proposition:

Proposition 1.4.1. *The Priorean formula $F\phi \to FF\phi$ defines the class $\mathscr{L}_{\mathscr{D}}$ of dense linearly ordered flows of time.*

There are many interesting properties that can be defined using Priorean formulae. The following propositions present some of them. Most of the proofs are straightforward and some of them can be found in [3], Section II.2.2.

Proposition 1.4.2. *The conjunction of the Priorean formula*

$$PFp_0 \to (Pp_0 \lor p_0 \lor Fp_0)$$

with its mirror formula defines the class of non-*branching flows of time.*

Proposition 1.4.3. *Let **Lin** be the class of linear ordered flows of time. The following list contains a number of properties of $(T, <) \in$ **Lin** and the respective defining Priorean formulae:*

having a first point	$\exists x \forall y, x < y \lor x = y$	$H\bot \lor PH\bot$
left-seriality	$\forall x \exists y, y < x$	$P\top$
left-unboundedness	$\forall x \exists y, y < x$	$Hp_0 \to Pp_0$
having a last point	$\exists x \forall y, y < x \lor x = y$	$G\bot \lor FG\bot$
right-seriality	$\forall x \exists y, x < y$	$F\top$
right-unboundedness	$\forall x \exists y, x < y$	$Gp_0 \to Fp_0$
density	$\forall x < y \exists z, x < z < y$	$Fp_0 \to FFp_0$
discretness	$\forall x, \exists y = P(x), \exists z = S(x)$	$[(p_0 \land Hp_0) \to FHp_0] \land$ $\land [(p_0 \land Gp_0) \to PGp_0]$

with $P(x)$ the immediate predecessor of x, i.e. $y < x$ such that $\nexists w, y < w < x$, and $S(x)$ the immediate successor of x, i.e. $x < z$ such that $\nexists w, x < w < z$.

Proposition 1.4.4. *Each of the following formulae defines the class of transitive orderings, i.e. posets $(X, <)$ in which $<$ satisfies the first order formula $\forall t, t'(\exists t''(t < t'' < t') \to t < t')$:*

- $FFp \to Fp$
- $Gp \to GGp$
- $Pp \to GPp$
- $Fp \to HFp$

Thus, they all are tense logic counterparts of transitivity[5].

However, it can be shown that there are some properties of flows of time that can not be tense logically defined. To give a proof of a "negative" result of this kind, we use the important notion of p-morphism.

Definition 1.4.6. Let $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$ be Kripke frames[6]. A *p-morphism* from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a function $f : W_1 \to W_2$ such that

- for every $u, w \in W_1$, if uR_1w, then $f(u)R_2f(w)$,
- for every $s, t \in W_2$, if sR_2t and $s = f(u)$ for some $u \in W_1$, then there exists $w \in W_1$ such that uR_1w and $t = f(w)$.

We will write $f : \mathcal{F}_1 \to \mathcal{F}_2$ to denote that f is a p-morphism from $\mathcal{F}_1$ to $\mathcal{F}_2$.

Proposition 1.4.5 (Preservation under $\boldsymbol{p}$-morphisms)**.** *Let ϕ be a formula in a language $\mathcal{L}$ with a binary relation $<$, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Kripke frames. If $f : \mathcal{F}_1 \to \mathcal{F}_2$ is an onto p-morphism, then $\mathcal{F}_1 \vDash \phi$ implies $\mathcal{F}_2 \vDash \phi$.*

Proof. A detailed proof of this result can be found in the entry "p-morphism" of the on-line mathematical encyclopedia *PlanethMath.org*. It involves the notion of p-morphism between models. □

A frame $\mathcal{F}'$ is said to be a *p-morphic image* of a frame $\mathcal{F}$ if there is an onto p-morphism $f : \mathcal{F} \to \mathcal{F}'$.

Remark 1.4.1. Let $\mathscr{C}_\phi$ be the class of all frames validating a formula ϕ. Then, by the previous proposition, $\mathscr{C}_\phi$ is closed under p-morphic images: if a frame is in $\mathscr{C}_\phi$, so is any of its p-morphic images.

Starting from this remark, we can produce an effective proof of the non-definability of some properties of frames: it suffices to show that there is a p-morphism from a model with the considered property onto a model without it.

Proposition 1.4.6. *Irreflexivity ($\forall t,\ t \not R t$) and asymmetry ($\forall t, t',\ tRt' \to t' \not R t$) can not be expressed by Priorean formulae.*

Proof. Let $\mathcal{F}_1 = (\mathbb{N}, <)$ and $\mathcal{F}_2 = \big(\{0\}, R\big)$, where the relation R is simply $0R0$. Notice that $\mathcal{F}_1$ is in both the class of irreflexive frames and in the class of asymmetric frames, and $\mathcal{F}_2$ is in neither. Let $f : \mathbb{N} \to \{0\}$ be the obvious surjective map, sending $n \mapsto 0$. Clearly, $m < n$ implies $f(m)Rf(n)$,

[5]About this result, in [3] Van Benthem interestingly pointed out that "*Apparently unrelated axioms* [...] *turned out to express the same property of precedence* (<)".

[6]We recall that a *Kripke frame* (W, R) is made by a set W (of worlds) and a binary relation R on it, as said in Section 1.3.1.

which is the first condition of Definition 1.4.6. Moreover, if $f(m)R0$, then $f(m)Rf(m+1)$, which is the second condition of Definition 1.4.6. So, f is a p-morphism.

Let $\mathscr{C}$ be either the class of all irreflexive frames or the class of all asymmetric frames. Let ϕ be a Priorean formula expressing irreflexivity (respectively, asymmetry). Then ϕ is validated by $\mathscr{C}$, hence ϕ is validated by $\mathcal{F}_1$ (since $\mathcal{F}_1$ is in $\mathscr{C}$). So, by the previous proposition, ϕ is validated by $\mathcal{F}_2$ as well, which means $\mathcal{F}_2$ is $\mathscr{C}$ too, which is a contradiction. Therefore, a Priorean formula defining irreflexivity or asymmetry can not exist. □

This result implies that the class of *all* flows of time (in the larger class of frames) can not be defined using Priorean formulae, because it is the class of transitive and irreflexive frames.

Proposition 1.4.7. *The class of properly branching flows of time*[7] *can not be defined by a Priorean formula.*

Proof. Let $\mathcal{F}_1 = (T, <)$ and $\mathcal{F}_2 = (L, \prec)$, where $T = \{a, b, c\}$ with $a < b, a < c$ and $b \not< c$, and $L = \{x, y\}$ with $x < y$. T is a properly branching flow of time, while L is linear, hence non-properly branching. Let $f : T \to L$ be the morphism sending $a \mapsto x$, $b \mapsto y$ and $c \mapsto y$. It is an onto p-morphism, since

- $a < b \to f(a) = x \prec y = f(b)$ and $a < c \to f(a) = x \prec y = f(c)$,
- $x < y$ and $x = f(a)$ and b (or c) satisfies $a < b$ and $y = f(b)$,
- $f(a) = x, f(b) = y$, so it is surjective.

Let $\mathscr{C}$ be the class of properly branching frames. Let ϕ be a Priorean formula expressing properly branchingness. Then ϕ is validated by $\mathscr{C}$, hence ϕ is validated by $\mathcal{F}_1$ (since $\mathcal{F}_1$ is in $\mathscr{C}$). So, by the Proposition 1.4.5, ϕ is validated by $\mathcal{F}_2$ as well, which means $\mathcal{F}_2$ is properly branching, which is a contradiction. Therefore, a Priorean formula defining properly branchingness can not exist, hence the class of properly branching flows of time can not be defined by a Priorean formula in the class of flows of time. □

This kind of research can be included in the branch of Modal Logic called *Correspondence Theory*. A wide discussion on this topic can be found in [3] Chapter II.2.1[8], where some *preservation properties* are fully developed and proved.

[7] See Definitions 2.1.1 and 2.1.3.

[8] Even if there are probably some more complete sources on this topic in the literature.

Chapter 2

Branching-time semantics

2.1 Indeterminism

We now introduce and discuss the *ontological* assumptions that characterizes the structure for time that we introduced in Section 1.3.1.

Indeterminism is the idea that events are not deterministically caused: no event is certain and the outcome of any process is not fixed *a priori*. If we assume this principle, we can describe time in a tree-like fashion: every moment has a unique and determined past, but, in general, many possible futures. In other words, the future causal flow of events is *not* settled, while the causal past is. The philosophical implications of indeterminism can be deepened in the interesting items *Indeterminism* of [44] and *Theories of free will* of [40].

As said in the preface, choosing between different time ontologies is not a logicians' task: they do not have to decide which structure represents time faithfully. They just study the properties of such representations and let others (physicians, programmers, philosophers) decide about the one that fits their applications[1] best.

However, the basic principles of indeterminism, with the consequent tree-like representation of time, will be adopted in the following chapters of this thesis. The set-theoretical and topological properties of this representation, as well as the semantics based on them, will be the main object of our discussion in the next chapters.

[1] For example, the set of real numbers $\mathbb{R}$ is the best choice for Physics, since it allows to use the powerful tools of *calculus*, based on the *continuity* property of the model. Instead, in Computer Science it is useful to represent time moments as phases or steps of a computation. This will produce a discrete model, in which every moment has an immediate successor, such as is $\mathbb{N}$.

2.1.1 New properties for the flow of time

The property of the uniqueness of the past described above can be written as a first order condition, which needs to be satisfied by the elements of $(T, <)$, a flow of time. If two different time points are in the past of a given time point t, then they must be totally ordered, which is, comparable by $<$. This clearly implies that the past is unique and determined, and that there is only one possible flow of time going from a point backwards. Formally:

Definition 2.1.1. A flow of time $\mathcal{T} = (T, <)$ is an *indeterministic flow of time*, or a *tree*, if it fulfills the *tree condition*[2]:

$$\forall\, t, t', t'' \in T,\ [(t' < t\ \wedge\ t'' < t) \rightarrow (t' < t''\ \vee\ t' = t''\ \vee\ t'' < t')] \tag{2.1}$$

Then, in this thesis, trees are transitive and irreflexive orders fulfilling the tree condition. The reader should notice that no connectedness or well-ordering properties are involved in this definition: in some contexts (and in several sources we have used) one of them is often involved in the definition of the tree structure.

Moments with no predecessor are called *roots* and moments with no successor are called *leaves*. A tree with at least a root is called *rooted* tree.

A *subtree* of a tree $(T, <)$ is a structure $(S, <_{\restriction S})$ in which $S \subseteq T$ and $<_{\restriction S}$ is the restriction of the relation $<$ to S: the restriction of $<$ to the subtree is still transitive, irreflexive, and satisfies the tree condition, since they are universal properties.

Given two trees $(T, <)$, $(T', \prec)$, a map $\phi : T \rightarrow T'$ is an *isomorphism of trees* if it is bijective and order-preserving (i.e. $t < t'$ implies $\phi(t) < \phi(t')$). As usual, if there exists an isomorphism between two trees we said that they are said to be *isomorphic*.

From now on, in this thesis, possibly indexed $\mathcal{T}$ will always denote a tree $(T, <)$, with T and $<$ indexed in the same way. For example $\mathcal{T}_\delta$ denotes the tree $(T_\delta, <_\delta)$. Differently from Chapter 1, this symbol will not stand anymore for a generic flow of time, without the tree-property.

Definition 2.1.2. Let $\mathcal{T}$ be a tree and let t and t' be elements of T. We say t and t' are *comparable* moments, and write $t \smile t'$, if either $t < t'$ or $t' < t$ or $t = t'$. We write $t \not\smile t'$ if the two moments are not comparable.

For example, the first order formula for the tree condition (2.1) can be written as $\forall t, t', t'' \in T,\ (t' < t\ \wedge\ t'' < t) \rightarrow (t' \smile t'')$.

[2] This property is also called *left-connectedness* or *subtotality* in some contexts (for example in [19]).

The fact that the flow of time is *properly branching* in the future of a given moment can be expressed by a first order condition too. This says that there exist moments in the future of t that are not in the "early-later" order relation, i.e. they belong to different branches. Formally:

Definition 2.1.3. A tree $\mathcal{T}$ is *properly branching* in the future of $t \in T$ if

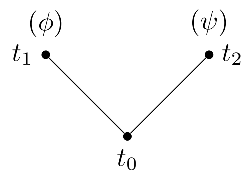
$$\exists\, t', t'' \in T \text{ such that } t < t' \;\wedge\; t < t'' \;\wedge\; t' \not< t'' \tag{2.2}$$

One may also require that there is *just one* flow of time, avoiding the situation in which two or more disjoint "sub"-flows of time coexist. This can be done by requiring the tree to be connected[3]:

Definition 2.1.4. A tree $\mathcal{T}$ is *connected* if, for all $t, t' \in T$, there exists $t'' \in T$ such that $t'' < t$ and $t'' < t'$.

We consider now a problem that this new time structures (and in general non-linear time structures) create when the definition 1.3.5 in the $F\phi$ case (1.2) is applied. This leads to unsatisfactory consequences, as described in the following example, which can be found in many works on this subject:

Example 2.1.1. Consider a time model with three instants, t_0, t_1, t_2, such that $t_0 < t_1$, $t_0 < t_2$ and t_1, t_2 are not in an *"early-later"* relation. We can suppose that a coin is flipped at t_0 and that t_1 and t_2 represent the future of t_0 in which we obtain heads and the future of t_0 in which we obtain tails, respectively.



If we apply the notion of valuation mentioned above, we have counter-intuitive consequences. If $\phi =$ "heads" and $\psi =$ "tails" $= \neg\phi$, we obtain that both $F\phi$ and $F\psi$ (which is equivalent fo $F\neg\phi$) are true when evaluated at t_0, since they refer to *possible* futures.

This result may be confusing, but its "opposite" is not less controversial: we can not accept that just one formula between $F\phi$ and $F\psi$ is true at t_0, since there is complete symmetry in the setting (assuming that the coin is balanced).

[3]Some authors, like Kellerman in [19], prefer to call *forest* any transitive and irreflexive order that fulfills the tree condition, and they call *tree* any connected forest.

This example points out that the semantics defined in Chapter 1 is an interesting notion if the flow of time is *linear*: because of this, it is also referred to as "linear-time semantics". It is clear that we must consider different semantics in case we have a non-linear model, for example a branching-time one.

2.2 Piercean and Ockhamist logics

Arthur Prior firstly tried to overcome the problem described in Example 2.1.1 by investigating it in a three-valued logic framework[4], in which every sentence can be true, false or *undetermined*. For instance, the sentences $F\phi$ and $F\psi$ in the example above turn out to have the undetermined truth value in this framework.

Later, Prior considered two new different approaches, that have been known as *Piercean* and *Ockhamist* semantics, in which the modal notions of *possibility* and *necessity* play a fundamental role. Both semantics introduce a new "reading" of the proposition $F\phi$, and consequently give a new meaning to the operator F. The *Ockhamist* logic, in particular, will lead to our main discussion in the next chapters.

Both logics involve the notion of history, a new idea that will greatly enrich our structure for time. Moreover, a second-order quantification over histories is a crucial aspect of these semantics.

2.2.1 Histories

A history is a course of events in our representation of time, which is a chain of time points maximal under inclusion. Formally:

Definition 2.2.1. Let $\mathcal{T}$ be a tree. A subset h of T is an *history* of T (or a *course of events*) if it has two properties:

1. totality: $\forall t, t' \in h$, if $t \neq t'$, then either $t < t'$ or $t' < t$,
2. maximality: $\forall k \supseteq h$, if k totally ordered, then $k = h$.

Remark 2.2.1. There are different terminologies for moments and histories in a context regarding tree structures (for example in Kellerman [19] and Nyikos [23, 24]): moments are called *nodes*, while histories are called *path*, or *maximal branches*. In this thesis we will maintain our temporal-logic terminology for these objects. However, from now on, we will always use

[4] Prior studied some papers of Jan Łukasiewicz, like "On Three-valued Logic" (1920) in the '50s, and took the *Polish notation* from him.

the term tree instead of "indeterministic flow of time", and roots and leaves instead of starting and ending moments.

Remark 2.2.2. The reader may notice that, given a tree $\mathcal{T}$, the existence of histories is guaranteed by the *Axiom of Choice*, in its *Zorn's Lemma*[5] form: every total subset (chain) of T can be extended to a maximal one. This in particular implies that for every moment t there is at least a history h such that $t \in h$, since the subset $\{t\} \subseteq T$ is obviously totally ordered.

Given a history h of $\mathcal{T}$, if $t \in h$, we say that the history h *passes through* t. The set of histories passing through t is denoted by $H_t(T)$, while the set of *all* histories of T (which is the set of maximal chains of the frame $(T, <)$) will be written as $H(T)$. We will often denote these two sets with H_t and H respectively, if the context makes the dependence on T clear.

Given a time point t, a *possible future* for t is the intersection of a history h passing through t with the future of possibilities of t (see Definition 1.3.2).

Remark 2.2.3. We are now able to rewrite two properties of flow of time structure with this new language of histories, and to prove the equivalence from a set-theoretical point of view:

- the tree condition (Definition 2.1.1) is equivalent to

$$\forall t \in T, \forall h \in H_t, \text{ we have } P_t \subseteq h \tag{2.3}$$

  with P_t the past[6] of t. This means that there might be many histories passing through t, but all of them overlap in the past of t.
- the properly branching condition (Definition 2.1.3) is equivalent to

$$\exists t \in T, \exists h, h' \in H_t \text{ such that } t \in h \ \wedge \ t \in h' \ \wedge \ h \neq h' \tag{2.4}$$

  which means that there are at least two different histories passing through a certain common time point.

We must observe that these properties are second-order conditions on the set T, since histories are particular subsets of T. We prove the equivalence stated above using some hints from [37].

Proposition 2.2.1. *Let $\mathcal{T}$ be a tree. Then, using references from Section 2.1.1 and 2.2.1 we have* (2.3) $\Leftrightarrow$ (2.1) *and* (2.4) $\Leftrightarrow$ (2.2).

[5]**Theorem** (Zorn's Lemma). Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

[6]See Definition 1.3.2.

Proof. Since h is totally ordered and $P_t \subseteq h$, then P_t is obviously totally ordered. Conversely, by (2.1), the past of t is totally ordered, hence it is contained in every maximal totally ordered subset of T passing through t, i.e. $\forall h \in H_t$ we have that $P_t \subseteq h$.

Similarly, from $h \neq h'$ we deduce that there must be some not totally ordered $t' \in h$ and $t'' \in h'$ in the future of t, so $\mathcal{T}$ is properly branching. Conversely, by the Axiom of Choice, every pair of temporally comparable moments can be extended to a history, hence $t < t'$ and $t < t''$ imply that there are two histories h' and h'' containing respectively t and t', and t and t''. From the non-comparability of t' and t'' we have that $h' \neq h''$. □

Remark 2.2.4. Despite the logical complexity of a second-order quantification, histories are often involved in branching time logic. A second-order quantification, for example, is the only way to make sense of propositions that involve the *complete flow of time*, or the set of possible flows of time, for example "it is possible that it will never rain".

2.2.2 Piercean semantics

As said in Section 1.1, Arthur Prior named Piercean logic after Charles Pierce (1839 – 1914), an American logician and philosopher.

Prior's idea to solve the problem briefly explained in Example 2.1.1 was to read $F\phi$ as "*ϕ will inevitably happen*". This means that $F\phi$ is true at a time point t if in every possible future of t (i.e. in the future of every history passing through t) there is a moment in which ϕ is true.

Prior also defined the operator G as primitive, similar to the one of Remark 1.2.1, which is *"in the future, it will always occur"*: $G\phi$ is true in t if ϕ is true in every moment in the future of t of every history passing through t.

Formally:

Definition 2.2.2. The set of the *Piercean formulae*, $\mathcal{L}_{\mathcal{P}i}$, is the smallest set containing the propositional variables $p_0, p_1, p_2, \ldots$ and containing every formula constructed by recursive application of the boolean operators $\neg$, $\wedge$ ($\vee$ and $\Rightarrow$ defined as usual) and of temporal unary operators P, F, and G.

We can now define the evaluation rules in Piercean semantics, according to the new reading of the operators F and G introduced above:

Definition 2.2.3. Given a propositional variable p_i and a Piercean formula ϕ, a model $\mathcal{M} = (T, <, \mathcal{V})$ and a time point t, we have, by induction on the complexity of the formula:

$\mathcal{M}, t \vDash p_i$	if and only if	$\mathcal{V}(t)(p_i) = 1$
$\mathcal{M}, t \vDash \neg\phi$	if and only if	not $\mathcal{M}, t \vDash \phi$
$\mathcal{M}, t \vDash \phi \wedge \psi$	if and only if	$\mathcal{M}, t \vDash \phi$ and $\mathcal{M}, t \vDash \psi$
$\mathcal{M}, t \vDash P\phi$	if and only if	$\exists s \in T, s < t,\ \mathcal{M}, s \vDash \phi$
$\mathcal{M}, t \vDash F\phi$	if and only if	$\forall h \in H_t, \exists s \in h, t < s,\ \mathcal{M}, s \vDash \phi$
$\mathcal{M}, t \vDash G\phi$	if and only if	$\forall h \in H_t, \forall s \in h, t < s,\ \mathcal{M}, s \vDash \phi$

From this definition, it is clear that the past operator P has the same meaning and valuation as in Priorean temporal logic, and consequently the same is true for H.

Prior also introduced a *weak* future operator, called f and defined as $f = \neg G \neg$. It expresses the possibility of a future event and it coincides with the "old" Priorean future operator F (so that the Priorean and Piercean G's coincide as well). The formula $f\phi$ is read as *"it is possible that ϕ will occur in the future"* or *"ϕ may occur in the future"*. Clearly, $F\phi \to f\phi$ is a Piercean validity for every formula ϕ. The valuation rule for f turns out to be

$$\mathcal{M}, t \vDash f\phi \quad \text{if and only if} \quad \exists h \in H_t, \exists s \in h, t < s,\ \mathcal{M}, s \vDash \phi.$$

The operator $g = \neg F \neg$, instead, expresses truth on at least one history: $g\phi$ is true at t whenever there exists a history $h \in H_t$ such that ϕ is true at every time point of h later than t. In fact, $g\phi$ is true if $F\neg\phi$ is false, which is equivalent to "not every history of H_t contains a time point in which $\neg\phi$ holds". Equivalently, there exists $h \in H_t$ such that in every $s \in h$, $t < s$, ϕ is true.

Remark 2.2.5. We are now able to formalize Example 2.1.1 in this new Piercean setting. Given $\phi =$"heads", we simply have that both $f\phi$ and $f(\neg\phi)$ are true in t_0, and both $F\phi$ and $F(\neg\phi)$ are false in t_0.

We do not pay much attention to Piercean logic, since it can be viewed as a fragment of Ockhamist logic, as pointed out in Remark 2.2.8. We conclude this section with one final observation, in which we compare the validity of a formula using linear time semantics and Piercean semantics.

Remark 2.2.6. If we have a linear time model, we can use the valuation of linear-time semantics (Definition 1.3.5), and the first description of the operator F. In this setting, the formula $\phi \to HF\phi$ is true in every t for every formula ϕ: this expresses the fact that if ϕ is true in a moment t, then "ϕ will be true" is true in every moment of the past of t. On the contrary, in the Piercean context outlined above, that same formula is not always true, because of the branching nature of time: using the usual flip-of-the-coin Example 2.1.1, ϕ is true in t_1, but $HF\phi$ is not true in t_1 since in the future t_2, ϕ is false. Of course, if we instead consider the weak future operator, we have that $\phi \rightarrow Hf\phi$ is true at every moment and for every ϕ.

2.2.3 Ockhamist semantics

Prior named the Ockhamist Logic after the English philosopher and theologian William of Ockham (1285-1347). This new approach radically changes one of the ideas we based our previous semantics on. In fact, it denies the principle that the truth of temporal formulae depends (just) on the time point t in which the formulae are evaluated: we must also specify what history passing through t we are considering.

Since we are developing this new semantics with a (properly) branching time model in mind, from now on we will consider trees as time models (see Definition 2.1.1). Moreover, we will write $\langle t, h\rangle$ for pairs $t \in T$, $h \in H_t$, and denote by $\tilde{T} = \{\langle t, h\rangle \mid t \in h\} \subseteq T \times H(T)$.

Given $h \in H_t$, we say that $F\phi$ is true at $\langle t, h\rangle$ if there exists a moment s in the future of t along h in which ϕ holds.

Remark 2.2.7. Since every history is a totally ordered set, the properties of the Ockhamist operators P and F reflect the properties of their linear-time logic counterparts. For example, $\phi \rightarrow HF\phi$, which can be false in the Piercean context (see Remark 2.2.6), is again true for every formula ϕ in every moment t.

Ockhamist logic uses another operator, usually denoted by $\Box$. It is called *historical necessity operator* and it constitutes the syntactic counterpart of the branching property of time. Given a formula ϕ, the proposition $\Box\phi$ is true at $\langle t, h\rangle$ if ϕ is true in $\langle t, h'\rangle$ for every $h' \in H_t$, i.e. regardless of the future evolution of time.

The dual operator $\Diamond = \neg\Box\neg$, called *historical possibility operator*, can be read as "there exists a history $h \in H_t$ such that ... in t". Given a formula ϕ, the proposition $\Diamond\phi$ is true at $\langle t, h\rangle$ if there exists $h' \in H_t$ such that ϕ is true in $\langle t, h'\rangle$.

Formally:

Definition 2.2.4. The set of the *Ockhamist formulae*, $\mathcal{L}_{\mathcal{O}}$, is the smallest set containing the propositional variables $p_0, p_1, p_2, ...$ and containing every formula constructed by recursive application of the boolean operators $\neg$, $\wedge$ ($\vee$ and $\Rightarrow$ defined as usual), the temporal operators P, F and the historical necessity operator $\Box$. Dual tense operators (G, H) are defined in the usual way and the historical possibility operator $\Diamond$ is defined as above.

We can now define a valuation that takes into consideration the new two-sorted dependence on moments *and* histories, and the new modal operators $\Box, \Diamond$. In Ockhamist context, the *Ockhamist valuation* is a function defined on $\tilde{T}$, which maps pairs $\langle t, h\rangle$ (with $t \in h$) to different classical valuations. These valuations send propositional variables to truth values. Since it is a function with a different domain, we change its name into π.

Definition 2.2.5. For any Ockhamist model $\mathcal{M} = (T, <, \pi)$ and a time-history point $\langle t, h\rangle$, the truth conditions at $\mathcal{M}, \langle t, h\rangle$ are defined by the following rules, by induction on the complexity of the arbitrary Ockhamist formula:

$$\begin{array}{lll}
\mathcal{M}, \langle t, h\rangle \vDash p_i & \text{if and only if} & \pi(\langle t, h\rangle)(p_i) = 1 \\
\mathcal{M}, \langle t, h\rangle \vDash \neg\phi & \text{if and only if} & \text{not } \mathcal{M}, \langle t, h\rangle \vDash \phi \\
\mathcal{M}, \langle t, h\rangle \vDash \phi\wedge\psi & \text{if and only if} & \mathcal{M}, \langle t, h\rangle \vDash \phi \text{ and } \mathcal{M}, \langle t, h\rangle \vDash \psi \\
\mathcal{M}, \langle t, h\rangle \vDash P\phi & \text{if and only if} & \exists s \in h, s < t,\ \mathcal{M}, \langle s, h\rangle \vDash \phi \\
\mathcal{M}, \langle t, h\rangle \vDash F\phi & \text{if and only if} & \exists s \in h, t < s,\ \mathcal{M}, \langle s, h\rangle \vDash \phi \\
\mathcal{M}, \langle t, h\rangle \vDash \Box\phi & \text{if and only if} & \forall h' \in H_t,\ \mathcal{M}, \langle t, h'\rangle \vDash \phi \qquad (2.5) \\
\mathcal{M}, \langle t, h\rangle \vDash \Diamond\phi & \text{if and only if} & \exists h' \in H_t,\ \mathcal{M}, \langle t, h'\rangle \vDash \phi \qquad (2.6)
\end{array}$$

Given a propositional variable p_i, we define the "subset" valuation $V(p_i)$ (see Definition 1.3.6) as a set of pairs $\langle t, h\rangle$ with $t \in T, h \in H_t$, which are meant to be the ones in which p_i is true. If we express all the conditions above in this new notation, we have:

Definition 2.2.6. For any Ockhamist model $\mathcal{M} = (T, <, \mathcal{V})$ and a time-history point $\langle t, h\rangle$, the truth conditions at $\mathcal{M}, \langle t, h\rangle$ are defined by the following rules, by induction on the complexity of the arbitrary Ockhamist formula:

$$\begin{array}{lll}
\langle t, h\rangle \in V(\neg\phi) & \text{if and only if} & \langle t, h\rangle \notin V(\phi) \\
\langle t, h\rangle \in V(\phi\wedge\psi) & \text{if and only if} & \langle t, h\rangle \in V(\phi) \cap V(\psi) \\
\langle t, h\rangle \in V(P\phi) & \text{if and only if} & \exists s \in h, s < t,\ \langle s, h\rangle \in V(\phi) \\
\langle t, h\rangle \in V(F\phi) & \text{if and only if} & \exists s \in h, t < s,\ \langle s, h\rangle \in V(\phi) \\
\langle t, h\rangle \in V(\Box\phi) & \text{if and only if} & \forall h' \in H_t,\ \langle t, h'\rangle \in V(\phi) \qquad (2.7) \\
\langle t, h\rangle \in V(\Diamond\phi) & \text{if and only if} & \exists h' \in H_t,\ \langle t, h'\rangle \in V(\phi) \qquad (2.8)
\end{array}$$

Example 2.2.1. We are now able to formalize the flip-of-the-coin Example 2.1.1 in this Ockhamist setting. In our time model with three different time points (t_0, t_1, t_2), we call h_1 the history $\{t_0, t_1\}$ with $t_0 < t_1$ and h_2 the history $\{t_0, t_2\}$ with $t_0 < t_2$; t_1 and t_2 are not in *early-later* relation. Given $\phi =$ "heads", we have that $F\phi$ is true at $\langle t_0, h_1\rangle$ and $F(\neg\phi)$ is true at $\langle t_0, h_2\rangle$: these formulae are true if considered on a suitable history, but none of them is necessary. In fact, both $\Box\phi$ and $\Box(\neg\phi)$ are false in $\langle t_0, h_i\rangle$ for $i = 1, 2$; on the contrary, $\Diamond\phi$ and $\Diamond(\neg\phi)$ are both true.

Remark 2.2.8. Piercean operators F, f, g, G can be expressed in the Ockhamist context by $\Box F, \Diamond F, \Diamond G, \Box G$ respectively, so Piercean language can be viewed as a fragment of Ockhamist language. For this reason, from now on we will only consider Ockhamist logic.

Remark 2.2.9. The second-order quantification in the valuation of $\Box$ and $\Diamond$ makes the truth of $\Box\phi$ and $\Diamond\phi$ history-independent: the truth for this type of formulae depends just on the considered time moment, as in Piercean logic.

Remark 2.2.10. Some authors like Gabbay, Hodkinson and Reynolds require that the Ockhamist valuation does not depend on the chosen history, which means that given a moment t, for all $h, h' \in H_t$, $\pi(t, h) = \pi(t, h')$. This kind of valuation is used in many branching time logics for computation, for example the *Computation Tree Logic* CTL*. See [9] for a complete discussion on this topic.

As observed in [37], this requirement corresponds to the idea that propositional variables should represent atomic, and hence untensed, sentences, so their truth should depend only on the moment we are considering. However, as suggested by Stefan Wölfl, everyday language can express untenced sentences that contain "a trace of futurity" (this expression was coined by Prior himself). As an example, the sentence "the king is dying" is true at a moment-history pair only if the king actually dies along the chosen history. The truth of "the king is dying" at a pair $\langle t, h\rangle$ implies that the sentence "the king *behaves in the same way as* a man who is dying" is true at $\langle t, h'\rangle$ for every h' passing through t. Thus, according to these observations, it seems natural to assume that the valuation of propositional variables contains arbitrary sets of pairs.

This kind of problem was already discussed by Prior in [27]. He suggested to distinguish between two different kinds of propositional variables: variables of the first kind are true at moments, variables of the second kind are true at moment-history pairs.

2.3 Bundled trees

The literature on branching-time presents other semantics in which second-order quantification is somehow avoided, although histories are still deeply involved. An important example is the *bundled-tree semantics*, which we are going to investigate in detail in this section and in the following chapters. In [37], the author presents also the *Kamp frames* and *Ockhamist frames* semantics, which we are not going to investigate in this thesis. In any case, these two semantics can be easily shown to be equivalent to that based on bundled trees.

Definition 2.3.1. A *bundle* on a tree $\mathcal{T}$ is a subset $\mathcal{B}$ of $H(T)$ such that every moment of T belongs to some history in $\mathcal{B}$, i.e. such that $\forall t \in T$, $\exists h \in \mathcal{B}$ with $t \in h$. Given a time point $t \in T$, the set $\mathcal{B} \cap H_t$ of all histories in $\mathcal{B}$ that pass through t, will be denoted by $\mathcal{B}_t$.

Definition 2.3.2. Pairs $(\mathcal{T}, \mathcal{B})$ in which $\mathcal{B}$ is a bundle on the tree $\mathcal{T}$ are called *bundled trees*. $(\mathcal{T}, \mathcal{B})$ will also be written as $\mathcal{T}_{\mathcal{B}}$. We will denote by $\tilde{T}_{\mathcal{B}}$ the set $\{\langle t, h\rangle \mid t \in h \in \mathcal{B}\} \subseteq T \times \mathcal{B}$.

The ontological assumption behind the restriction of the quantification to bundles, is that we assume the existence of a set of *admitted* histories. This set needs to satisfy the adequate closure property described above: every moment has at least one admitted history passing through it.

With this idea in mind, we need to modify conditions (2.5) and (2.6) in Definition 2.2.5. In the bundled trees context, a model $\mathcal{M}$ consists of a tree $(T, <)$, a bundle $\mathcal{B}$ on it, and a valuation π that associates a classical valuation to every moment-history pair $\langle t, h\rangle \in \tilde{T}_{\mathcal{B}}$ (which means $h \in \mathcal{B}_t$), according to the rules described below:

Definition 2.3.3. For any Ockhamist bundled model $\mathcal{M} = (\mathcal{T}_{\mathcal{B}}, \pi)$, with $\mathcal{B}$ a bundle on the tree $\mathcal{T}$, and a time-history point $\langle t, h\rangle$, the truth conditions at $\mathcal{M}, \langle t, h\rangle$ are defined by the following rules, by induction on the complexity of the arbitrary Ockhamist formula:

$$\mathcal{M}, \langle t, h\rangle \vDash p_i \quad \text{if and only if} \quad \pi(\langle t, h\rangle)(p_i) = 1$$

$$\mathcal{M}, \langle t, h\rangle \vDash \neg\phi \quad \text{if and only if} \quad \text{not } \mathcal{M}, \langle t, h\rangle \vDash \phi$$

$$\mathcal{M}, \langle t, h\rangle \vDash \phi \wedge \psi \quad \text{if and only if} \quad \mathcal{M}, \langle t, h\rangle \vDash \phi \text{ and } \mathcal{M}, \langle t, h\rangle \vDash \psi$$

$$\mathcal{M}, \langle t, h\rangle \vDash P\phi \quad \text{if and only if} \quad \exists s \in h, s < t,\ \mathcal{M}, \langle s, h\rangle \vDash \phi$$

$$\mathcal{M}, \langle t, h\rangle \vDash F\phi \quad \text{if and only if} \quad \exists s \in h, t < s,\ \mathcal{M}, \langle s, h\rangle \vDash \phi$$

$$\mathcal{M}, \langle t, h\rangle \vDash \Box\phi \quad \text{if and only if} \quad \forall h' \in \mathcal{B}_t,\ \mathcal{M}, \langle t, h'\rangle \vDash \phi \tag{2.5*}$$

$$\mathcal{M}, \langle t, h\rangle \vDash \Diamond\phi \quad \text{if and only if} \quad \exists h' \in \mathcal{B}_t,\ \mathcal{M}, \langle t, h'\rangle \vDash \phi \tag{2.6*}$$

Moreover, in the “subset” notation of Definition 2.2.6, conditions (2.7) and (2.8) become:

$$\langle t, h\rangle \in V(\Box\phi) \quad \text{if and only if} \quad \forall h' \in \mathcal{B}_t,\ \langle t, h'\rangle \in V(\phi) \tag{2.7*}$$

$$\langle t, h\rangle \in V(\Diamond\phi) \quad \text{if and only if} \quad \exists h' \in \mathcal{B}_t,\ \langle t, h'\rangle \in V(\phi) \tag{2.8*}$$

Remark 2.3.1. Clearly we can consider as a bundle the set of *all* histories $H(T)$. If this is the case, the bundled tree is said to be *complete*, and we come back to the Ockhamist-tree semantics introduced above.

Remark 2.3.2. The technique of replacing trees with bundled trees is often called *a Henkin move*. It can appear in many different contexts to replace a second-order quantification in a simple structure, with a first-order quantification in more a complex structure. This technique was firstly introduced by Leon Henkin in [13], in the context of *general semantics* for the Theory of Types. Henkin's aim was to move from the second-order quantification over the power set $\mathcal{P}(D)$ of the domain D, to a quantification over a suitably closed subset X of $\mathcal{P}(D)$. This new quantification is over *elements* of X, and hence it is a first-order quantification.

Fixing a set of admitted histories is a further (and controversial) ontological assumption, but it produces a great simplification from a technical point of view. On one hand, the ontology is much more complicated, since histories are primitive entities. On the other hand, bundled trees are (equivalent to) first-order definable structures (Theorem 2.3.1 below), so the bundled tree semantics turns out to be simpler than the one based on trees: in fact, as shown below in Theorem 2.4.1 (Proposition (6.3) of [37]), the second-order quantification over branches in the tree semantics can not be mimicked by a first-order quantification in any way.

The changes produced by the *Henkin move* in our context will be studied in Section 2.4.

2.3.1 First-order definability for bundled trees

In order to describe bundled trees in a first-order language, we borrow the following idea from geometry. Several presentations of geometry consider both *points* and *lines* as primitive entities (lines are not viewed as sets of points), and the mutual relation between them is described with suitable axioms. In the same way, we can describe bundled trees as two-sorted first-order structures where moments and histories are two different kinds of individuals. In the following parts of the thesis, opposite to the Ockhamist context, there will be no set-theoretical construction favouring one sort of entities (primitive moments) over another (histories as sets of moments).

In this perspective, the relation involved will be the usual earlier/later relation between moments ($\prec$), and a new binary relation ε between moments and histories: the meaning of $\varepsilon(t, h)$ is *t occurs in h* or *h passes through t*.

This section and Section 2.4.1 are essentially borrowed from [37].

Definition 2.3.4. A two sorted structure $\mathcal{S} = \langle \mathrm{T}, \mathrm{H}, \prec, \varepsilon \rangle$ is a *bundled-tree (first-order) structure* (*BTS*) if the following axioms hold.[7]

(A0) tree axioms[8] for $(\mathrm{T}, \prec)$

(A1) $\forall h \, \exists t \, \varepsilon(t, h) \;\wedge\; \forall t \, \exists h \, \varepsilon(t, h)$

(A2) $\forall h \, \forall t, t' \, [(\varepsilon(t, h) \;\wedge\; \varepsilon(t', h)) \rightarrow (t \prec t' \;\vee\; t' \prec t \;\vee\; t' = t)]$

(A3) $\forall h \, \forall t \, [\varepsilon(t, h) \rightarrow \forall t' \, (t' \prec t \rightarrow \varepsilon(t', h))]$

(A4) $\forall h \, \forall h' \, [\forall t \, (\varepsilon(t, h) \rightarrow \varepsilon(t, h')) \rightarrow h = h']$

Remark 2.3.3. The consistency of these axioms can be readily verified observing that, for every bundled tree $\mathcal{T}_{\mathcal{B}} = (T, \mathcal{B})$, the structure $\mathcal{S}_{(\mathcal{T}, \mathcal{B})} = \langle T, \mathcal{B}, <, \in \rangle$ is a model for *BTS*.

Conversely, the next proposition shows that every *BTS* can be viewed as a bundled-tree-like flow of time, which means that the elements of H can represent histories and that H can be seen as a bundle on T.

Proposition 2.3.1. *Let* $\mathcal{S} = \langle \mathrm{T}, \mathrm{H}, \prec, \varepsilon \rangle$ *be a BTS. Then,*

(1) $(\mathrm{T}, \prec)$ *is a tree;*

(2) for every $h \in \mathrm{H}$*, the set* $\bar{h} = \{t \in \mathrm{T} \mid \varepsilon(t, h)\}$ *is a history in* $(\mathrm{T}, \prec)$*;*

(3) the set $\mathrm{H}_{\mathcal{S}} = \{\bar{h} \mid h \in \mathrm{H}\}$ *is a bundle on* $(\mathrm{T}, \prec)$*.*

Proof. Claim *(1)* is just **(A0)**.

Axiom **(A1)** states that every $t \in \mathrm{T}$ occurs along some $\bar{h}$, and every $\bar{h}$ passes at least through one t, and hence it is not empty. Moreover, by Axiom **(A2)**, every $\bar{h}$ is totally ordered by $\prec$. In order to prove that $\bar{h}$ is a history, we have to show that it is a *maximal* chain. Assume by reductio that $\bar{h}$ is not maximal, so there exists $t_0 \notin \bar{h}$ such that $\prec$ totally orders $\bar{h} \cup \{t_0\}$. Then, since every $\bar{h}$ contains the past of all its moments (as a consequence of Axiom **(A3)**), t_0 must be in the future of every t in $\bar{h}$ (i.e. $\forall t \in \bar{h}, \; t \prec t_0$). By **(A1)** (first part), we can consider $\bar{h}_0$ containing t_0, hence, again by **(A3)**, $\bar{h} \subseteq \bar{h}_0$. But, by Axiom **(A4)**, if $\bar{h} \subseteq \bar{h}'$ the $\bar{h} = \bar{h}'$, so $\bar{h} = \bar{h}_0$. Then $t_0 \in \bar{h}$, which contradicts the assumption. Hence $\bar{h}$ is maximal, so it is a history according to Definition 2.2.1. Thus, we have proved claim *(2)*.

Moreover, by Axiom **(A1)** (second part), every $t \in \mathrm{T}$ belongs to some $\bar{h}$, so $\mathrm{H}_{\mathcal{S}}$ is a bundle in the tree $(\mathrm{T}, \prec)$ since it satisfies the Definition 2.3.1, so we have proved claim *(3)*. □

[7] These axioms are meant to be expressed in a two-sorted first order language, $\mathcal{L}_{BTS}$, for *BTS*'s. In order to avoid heavy notation, we use ϵ, $\prec$, $t, t', \ldots$, and $h, h', \ldots$ also as symbols of the language, with their obvious interpretation.

[8] See Definitions 1.3.1 and 2.1.1.

Remark 2.3.4. According to the remark and the proposition above, we have that any BTS $\mathcal{S}$ corresponds to a bundled tree $\mathrm{T}_{\mathrm{H}_\mathcal{S}} = (\mathrm{T}, \mathrm{H}_\mathcal{S})$, and that any bundled tree $\mathcal{T}_\mathcal{B} = (\mathcal{T}, \mathcal{B})$ corresponds to the structure $\mathcal{S}_{(\mathcal{T},\mathcal{B})}$. Then, it is natural to observe that the first map is the inverse of the second and vice-versa, which implies that the following isomorphisms hold:

$$\mathcal{S} \cong \mathcal{S}_{(\mathrm{T},\mathrm{H}_\mathcal{S})} \text{ and } (\mathcal{T}, \mathcal{B}) \cong (\mathcal{T}, \mathrm{H}_{\mathcal{S}_{(\mathcal{T},\mathcal{B})}})$$

Hence, bundled trees turn out to be the best candidates for the correspondence with two-sorted first-order structures for flows of time.

2.4 Trees vs. bundled trees

In this section we want to compare bundled trees and trees from two points of view:

- from the logical point of view, we will prove that the Henkin move from trees to bundled trees is an actual move from second-order logic to first-order logic;
- from the semantics point of view, we will analyse the *descriptive adequacy* of the two structures, and give an example of different results we obtain if we choose one model or the other.

The aim is to show that the bundled tree semantics is really different from the tree semantics, and that an ontological position is involved in a choice between them.

2.4.1 Second-order definability for trees

The tree semantics can be viewed as a particular case of bundled tree semantics, in which only complete bundled trees are considered. Then, we need to extend the axiomatization of Definition 2.3.4 in a way such that the set $\mathrm{H}_\mathcal{S}$ defined in the Proposition 2.3.1 coincides with the set $H(\mathrm{T})$ of all histories of $(\mathrm{T}, \prec)$, in any model $\mathcal{S}$ of the new set of axioms. This can be done resorting to second-order logic as follows:

- we extend the language $\mathcal{L}_{BTS}$ with variables X, X', X'', ... ranging over $\mathcal{P}(\mathrm{T})$, so that also the symbol $\in$ will occur in formulae. Then, the quantifications $\forall X$ and $\exists X$ are second-order quantifications;
- we define the formula (coming from Axiom **(A2)**)

  $$Lin(X) \stackrel{\text{def}}{=} \forall t, t' \, [(t \in X \ \wedge \ t' \in X) \rightarrow (t \prec t' \ \vee \ t' \prec t \ \vee \ t' = t)$$

  with a free variable X, which is true in $\mathcal{S}$ if and only if X is interpreted on a totally ordered subset of T.

- we define the formula

$$His(X) \stackrel{\text{def}}{=} Lin(X) \wedge \forall X' \left[(Lin(X') \wedge X \subseteq X') \to (X = X') \right]$$

  with a free variable X, where $\subseteq$ is defined in the usual way. The formula $His(X)$ is true in $\mathcal{S}$ if and only if X is interpreted on a history of T.
- now, the condition $\mathrm{H}_{\mathcal{S}} = H(\mathrm{T})$ is equivalent to the second-order condition

$$\forall X \left[His(X) \to \exists h \forall t \, (t \in X \to \varepsilon(t, h)) \right] \tag{2.9}$$

Hence, a structure for *trees* can be defined in the language $\mathcal{L}_{BTS} + \{X, X', X'', \ldots, \in, \subseteq\}$ as a structure fulfilling (2.9) in addition to **(A0)**, ..., **(A4)** of Definition 2.3.4.

Definition 2.4.1. A two-sorted structure $\mathcal{S} = \langle \mathrm{T}, \mathrm{H}, \prec, \varepsilon \rangle$ is a *tree structure*, if the following axioms hold (see footnote 7).

(A0) tree axioms for $(\mathrm{T}, \prec)$

(A1) $\forall h \, \exists t \, \varepsilon(t, h) \wedge \forall t \, \exists h \, \varepsilon(t, h)$

(A2) $\forall h \, \forall t, t' \left[(\varepsilon(t, h) \wedge \varepsilon(t', h)) \to (t \prec t' \vee t' \prec t \vee t' = t) \right]$

(A3) $\forall h \, \forall t \left[\varepsilon(t, h) \to \forall t' \, (t' \prec t \to \varepsilon(t', h)) \right]$

(A4) $\forall h \, \forall h' \left[\forall t \, (\varepsilon(t, h) \to \varepsilon(t, h')) \to h = h' \right]$

(A5) $\forall X \left[His(X) \to \exists h \forall t \, (t \in X \to \varepsilon(t, h)) \right]$.

The next theorem shows that there is no set of first-order conditions that can replace **(A5)**.

Theorem 2.4.1. *There exists no set* Σ *of sentences of the first-order language* $\mathcal{L}_{BTS}$ *for BTS such that, for every structure* $\mathcal{S} = \langle \mathrm{T}, \mathrm{H}, \prec, \varepsilon \rangle$*,* $\mathcal{S}$ *is a model of* Σ *if and only if the set* $\mathrm{H}_{\mathcal{S}}$ *coincides with* $H(\mathrm{T})$*.*

Proof. Assume by reductio that Σ exists, and consider any tree $\mathcal{T}_0 = (T_0, <_0)$ in which every moment has at least two not $<_0$-comparable successors. In other words, it contains a copy of a binary tree (see Example 1.3.1). Then the first-order formula

$$\alpha_0 \stackrel{\text{def}}{=} \forall t \, \exists t', t'' [t \prec t' \wedge t \prec t'' \wedge \neg(t' \prec t'' \vee t'' \prec t' \vee t' = t'')] \tag{2.10}$$

is true in every structure of the form $\langle T_0, \mathrm{H}, <_0, \varepsilon \rangle$ (a structure with set of moments equal to a tree as above, with no further assumptions on the set H).

Consider now the particular structure $\mathcal{S}_{T_0} = \langle T_0, H(T_0), <_0, \in \rangle$, with $H(T_0)$ the set of all histories of T_0 and $\in$ the usual membership relation.

Let Σ_0 be the set of all sentences of $\mathcal{L}_{BTS}$ which are true in $\mathcal{S}_{T_0}$. Since $H(T_0)$ is precisely the set of all histories of T_0, we have $H(T_0) = \mathrm{H}_{\mathcal{S}_{T_0}}$. This implies that $\Sigma \subseteq \Sigma_0$, by the assumption on Σ. Moreover, $\alpha_0 \in \Sigma_0$, because the structure has an at-least-binary tree as set of moments. Hence, Σ_0 has an infinite model.

By the Löwenheim-Skolem Theorem[9], Σ_0 has a model $\mathcal{S}' = \langle T', H', \prec', \varepsilon' \rangle$ of size $\aleph_0$. Then T' and H' are denumerable sets. We can now reach a contradiction by observing that:

- $\mathcal{S}'$ is a model for **(A0)**, ..., **(A4)**, because these axioms are written with the language $\mathcal{L}_{BTS}$. In particular $(T', \prec')$ is a tree;
- by the assumption on Σ, the set $\mathrm{H}_{\mathcal{S}'}$ coincides with the set $H(T')$ of all histories of T';
- the tree $(T', <)$ has uncountably many histories ($|\mathrm{H}_{\mathcal{S}'}| = |H(T')| \geq 2^{\aleph_0}$), since α_0 is true in $\mathcal{S}'$;
- by definition of $\mathrm{H}_{\mathcal{S}}$ (see Proposition 2.3.1), we have $\mathrm{H}_{\mathcal{S}'} \subseteq H'$, so $2^{\aleph_0} = |\mathrm{H}_{\mathcal{S}'}| \leq |H'| = \aleph_0$.

This is a contradiction, so we conclude that Σ does not exist. □

The results obtained in this section show that the Henkin move produces an actual shift from second-order logic to the first-order one. So, from a logical perspective, bundled trees count as the first-order counterparts of trees, and there is a deep "technical" difference between these two structures.

2.4.2 Descriptive adequacy

In this section we compare the *descriptive adequacy* of trees and bundled trees, which is their capability of validating or falsifying tensed sentences. The example presented here is borrowed from [2].

For technical reasons, we need some moments in our time model to be marked with a "tick": this is an addition to the structure that allows us to clearly identify "check-times" and easily build histories, but there is no clock, metric or other ontological assumption involved.

Given a tree $\mathcal{T}$ (or a bundled tree $\mathcal{T}_{\mathcal{B}}$), we say that a subset $\Upsilon \subseteq T$ is a set of *ticks*, or of *ticked moments*, if

- $\forall t \in T$, $\forall h \in H_t$ (or in $\mathcal{B}_t$), there exists the minimum (w.r.t. $<$ relation) of the set $h \cap F_t \cap \Upsilon$, which means that starting from t, we can consider the *next* ticked moment in h.

[9] **Theorem** (Löwenheim-Skolem). *If a theory Φ with at most countably many axioms expressed in a first-order language $\mathcal{L}$ has an infinite model $\mathcal{M}$, then for every infinite cardinal number κ greater than $|\mathcal{L}|$ and $|\mathcal{M}|$, Φ has a model of cardinality κ.*

- $\forall h \in H$ (or in $\mathcal{B}$), $\curlyvee \cap h$ has no upper bound, which means that there will always be another ticked moment in every history.

Example 2.4.1. The sentences we consider concern a particular *radium* atom α and its radioactive status: at any given moment, it can be stable or it can decay. Assume that the following sentence holds:

As long as, at a given tick, α has not yet decayed,
1. α *might* decay before the next tick (2.11)
2. α *might not* decay before the next tick.

Then we can build a model with two kinds of histories: for every n, a history in which α decays between the n-th and the $(n+1)$-th tick, and another single history in which α never decays. This setting can be represented by the tree T of Figure 2.1, in which each history is isomorphic to the set of non-negative real numbers, and the subset of ticked moments in each history is a copy of $\mathbb{N}$.

Let ϕ be the sentence "Atom α has not decayed (yet)".

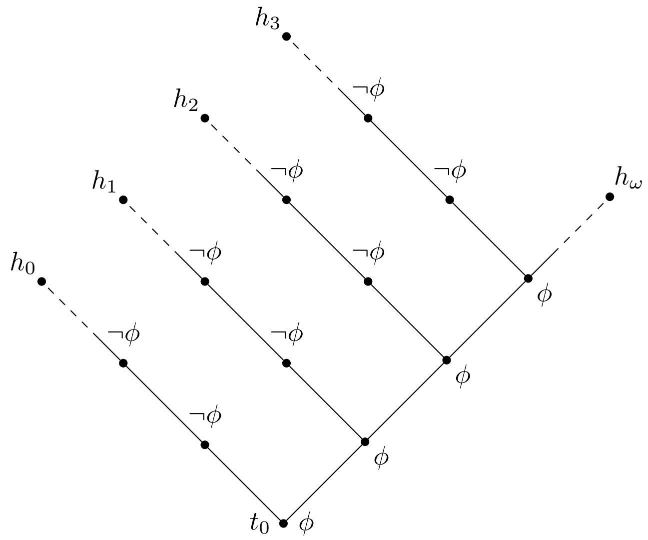


Figure 2.1: Tree model for the decay of the radium atom.

In our model, as visually described in the considered tree of Figure 2.1, ϕ is true in every moment of the history h_ω, and ceases to be true right after the n-th tick on the history h_n. In other words, the atom decays in t with $n < t < n+1$ along the history h_n, and never decays on the history h_ω.

We can now consider two different models. The *tree* model M_T uses a second-order quantification over every history of the tree $\mathcal{T}$, h_ω included; the *bundled-tree* model $M_{\mathcal{B}}$ uses a first-order quantification, considering histories of h_n-type only: the bundle[10] will be $\mathcal{B} = \{h_n \mid n \in \mathbb{N}\} = H(T) \smallsetminus \{h_\omega\}$.

As a consequence of the property stated in (2.11), we have that the following sentence is true both for M_T and $M_{\mathcal{B}}$, i.e. regardless of whether h_ω is an admitted history or not:

> Every no-decay chain of ticks of length n can be extended to a no-decay chain of length $n+1$. (2.12)

This extension is actually possible, for every n, in both the tree model M_T and the bundled-tree model $M_{\mathcal{B}}$. In fact:

- if α has not decayed before the n-th tick, every moment of h_{n+1} is in the future of possibilities of n: in other words, $n \in h_{n+1}$;
- along h_{n+1}, α has not decayed at the $(n+1)$-th tick yet;
- this implies that $\langle n+1, h_{n+1}\rangle \in M_T(\phi)$ and $\langle n+1, h_{n+1}\rangle \in M_{\mathcal{B}}(\phi)$;
- by the definition of the operator $\Diamond$ in the tree model, $h_{n+1} \in H$ is a witness of $\Diamond\phi$ in the future of n, which implies that $\langle n, h\rangle \in M_T(\Diamond\phi)$ for every $h \in H_n$;
- by the definition of the operator $\Diamond$ in the bundled-tree model, $h_{n+1} \in \mathcal{B}$ is a witness of $\Diamond\phi$ in the future of n, which implies that $\langle n, h\rangle \in M_T(\Diamond\phi)$ for every $h \in \mathcal{B}_n$.

Hence, a no-decay chain of length n can be extended to a no-decay chain of length $n+1$, both for the tree and the bundled-tree model.

On the contrary, the truth of the following sentence depends precisely on the admissibility of h_ω:

> At the starting moment t_0, it is inevitable that α will decay after a finite number of ticks. (2.13)

In fact, this sentence is formalized by $\Box F(\neg\phi)$, and must be true when evaluated at t_0. We have that:

- $\langle t_0, h\rangle \in M_T(\Box F(\neg\phi))$ if and only if $\forall h \in H_{t_0}, \langle t_0, h\rangle \in M_T(F(\neg\phi))$, but this is not true since $\langle t_0, h_\omega\rangle \in M_T(F\phi)$: α never decays along h_ω.
- $\langle t_0, h\rangle \in M_{\mathcal{B}}(\Box F(\neg\phi))$ if and only if $\forall h \in \mathcal{B}_{t_0}, \langle t_0, h\rangle \in M_T(F(\neg\phi))$, and this is true since in every history in $\mathcal{B}$, α decays after a finite number of ticks.

This example raises an ontological problem. On one side, the fact that (2.12) and (2.13) can be simultaneously true in the bundled-tree model is used in [2] to conclude that bundled-trees are not suitable as representation of time. On the other side, other authors (for example Øhrstrøm and Hasle)

[10] This is actually the only non-trivial and interesting bundle for this model.

think that the existence of a suitable semantics which simultaneously makes (2.12) and (2.13) true is enough to state that "a person might hold both of these sentences without contradicting himself".

As said in [37], the Henkin move, and branching-time semantics as a particular instance of it, are defended in several works of Johan van Benthem: he affirms that if we consider sets in a particular situation, it is preferable to identify the range of quantification which is relevant for our purposes, rather than considering *all* possible sets. In private correspondence with Alberto Zanardo, van Benthem wrote this illuminating opinion:

> "Formulating things in second-order logic allows us to remain silent on *just which properties of the runs*[11] are crucial for our understanding of events over time. By contrast, putting in a set of runs explicitly invites us at least to state interesting conditions on them, that explain the temporal reasoning practice we want to analyse".

Leaving this kind of matters aside[12], this example proves that M_T and $M_{\mathcal{B}}$ have different descriptive powers: in particular, it shows that the first-order quantification over a selected set can highly change the result of the valuation in the model, even if the variation produced by the bundle is minimal.

[11] In our language: histories.

[12] For further remarks on this topic, see [2].

Chapter 3

A topological perspective

As a starting point for further results, we describe the interesting topological perspective on time logic presented in [29]: the authors enrich bundled tree structures with a natural topology on the set of histories and derive some properties of this set from those of the topological space. The order of presentation of the contents is borrowed from [29].

3.1 Some definitions and results in Topology

We start this chapter with some basic topological definitions and results. Our aim is to create a common base of notations and concepts that will be used later on, and to make this thesis self-contained. We will deepen some aspects with simple propositions and remarks, which will be important in the next sections.

The following definitions are taken from [20], [6] and [7]. We will prove just some results, mainly the ones directly used in the following chapter.

Definition 3.1.1. Given a set X, a *topology* on X is a subset τ of $\mathcal{P}(X)$ such that

1. $\emptyset \in \tau$, $X \in \tau$.
2. given $\{A_i \in \tau \mid i \in I\}$ any collection of elements of τ, $\bigcup_{i\in I} A_i \in \tau$.
3. given $A, B \in \tau$, $A \cap B \in \tau$.

The elements of τ are *open sets* of X.

A topology on a set X is a set of subsets of X closed under arbitrary unions and finite intersections, containing the whole set X and the empty set.

Definition 3.1.2. A *topological space* is a pair (X, τ) in which τ is a topology on the set X.

In this section, (X, τ) always denotes a topological space. This will be tacitly assumed in every definition and result.

Definition 3.1.3. Given $S \subseteq X$, (S, τ_S) is a topological space with the *induced topology* $\tau_S = \{A \cap S \mid A \in \tau\}$.

Definition 3.1.4. A subset C of X is *closed* if its complementary set $X \smallsetminus C$ is open. Subsets of X that are both open and closed are called *clopen* sets.

Proposition 3.1.1. *The intersection of an arbitrary family of closed sets of X is closed. The union of a finite number of closed sets of X is closed.*

Definition 3.1.5. Given a point $x \in X$, a *neighbourhood*[1] of x is an open set $A \in \tau$ containing x.

Definition 3.1.6. Given any subset S of X, the *closure* of S is the smallest closed superset of S, and it will be denoted by $\bar{S}$. Equivalently, $\bar{S}$ is the intersection of all closed supersets of S. Equivalently again, a point $x \in X$ belongs to $\bar{S}$ if and only if for every A (open) neighbourhood of x, $A \cap S \neq \emptyset$.

Proposition 3.1.2. *A subset C of X is closed if and only if $C = \bar{C}$. If C is closed, then $S \subseteq C$ if and only if $\bar{S} \subseteq C$.*

Definition 3.1.7. The *boundary* of S is $\partial S = \bar{S} \smallsetminus S$. The *interior* of S is $S^\circ = S \smallsetminus \partial S$.

Definition 3.1.8. A subset D of X is *dense* if $\bar{D} = X$, or, equivalently, if every open set $A \in \tau$ has not-empty intersection with D.

Definition 3.1.9. A topological space X is *separable* if there exists a subset D of X which is dense and countable.

Definition 3.1.10. Given two topological spaces (X, τ), (Y, τ'), $f : X \to Y$ is *continuous* if $\forall\, V \in \tau'$, $f^{\leftarrow}(V) = \{x \in X \mid f(x) \in V\} \in \tau$: in other words, the preimage of any open subset of Y is open in X.

3.1.1 Connectedness, compactness and separability

Definition 3.1.11. A topological space (X, τ) is *connected* if X is not the union of two disjoint non-empty open subsets. Equivalently, X is connected if the only clopen subsets of X are $\emptyset$ and X itself. A topological space is *disconnected* if it is not connected, i.e. it is the union of a pair of disjoint non-empty open subsets.

[1] Some authors call these sets *open neighbourhood*, and call *neighbourhood* of x any superset of an open neighbourhood of x.

Remark 3.1.1. The *connected components* of a topological space X are its connected subsets, with respect to the induced topology. The collection of the connected components of X is a partition of X.

Definition 3.1.12. A topological space is *totally disconnected* if every connected component has exactly one element.

Definition 3.1.13. Given a subset S of X, $s \in S$ is an *isolated point* of S if there exists a neighbourhood A of x such that $A \cap S = x$. If a subset S of X consists only of isolated points, then S is called *discrete*[2].

Lemma 3.1.3. *A point x is isolated (in X) if and only if the* singleton $\{x\}$ *is an open subset of the topology.*

Remark 3.1.2. If x is an isolated point, then every dense subset of X contains x: in fact, $\{x\}$ is open, and a dense subset has non-empty intersection with every open subset. If x is not isolated, then $X \setminus \{x\}$ is dense: in fact, every open subset A containing x has at least another point, so every (non empty) open subset of τ has non-empty intersection with $X \setminus \{x\}$.

Proposition 3.1.4. *Assume that (X, τ) is a topological space in which singleton sets are closed*[3]*, and let Y be dense in X. Then (X, τ) and (Y, τ_Y) have the same isolated points.*

Definition 3.1.14. Given a subset S of X, an *open cover* of S is a family $\mathcal{C} = \{A_i \mid i \in I\}$ of open subsets of X whose union contains S. A *subcover* of a cover $\{A_i \mid i \in I\}$ is a cover $\{A_i \mid i \in J\}$ with $J \subseteq I$.

Definition 3.1.15. A subset S of X is *compact* if every open cover has a finite subcover. If X itself is compact, it is called a *compact space.*

Proposition 3.1.5. *A closed subset of a compact space is compact.*

Proof. Let X be compact and $\mathcal{C}$ an open cover of the closed set $C \subseteq X$. $X \setminus C$ is open, then $\mathcal{C}' = \mathcal{C} \cup \{X \setminus C\}$ is an open cover of X. By compactness, there exists a finite subcover $\mathcal{V} = \{V_1, \ldots, V_r\}$ of $\mathcal{C}'$, which covers X. It is a finite cover of C too, and (if there exists n such that $V_n = X \setminus C$) it remains a finite cover of C even if we remove $X \setminus C$. Thus, every open cover $\mathcal{C}$ of C has a finite subcover. $\square$

Definition 3.1.16. Two different points x, y of X can be *separated by neighbourhoods* if there exist A, B neighbourhoods of x and y respectively, such that $A \cap B = \emptyset$. A topological space (X, τ) is a *Hausdorff space* if all distinct points in X are pairwise neighbourhood-separable.

[2]Every discrete space is clearly totally disconnected.

[3]Hausdorff spaces, defined below, have this property.

Proposition 3.1.6. *If a topological space (X, τ) is Hausdorff, then every compact subset of X is closed.*

Definition 3.1.17. A topological space is *second countable* if the topology has a countable base.

Definition 3.1.18. A topological space is called *separable* if it contains a countable and dense subset; that is, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the space such that every non-empty open subset of the space contains at least one element of the sequence.

Proposition 3.1.7. *Let (X, τ) be a topological space. If it is second countable, then it is separable.*

Proof. Let $\mathcal{B}$ be a countable base for the topology. For each non-empty set $B \in \mathcal{B}$, pick a point $x_B \in B$. Since $\mathcal{B}$ is countable, the set $\{x_B \mid B \in \mathcal{B}\}$ is countable. Moreover, each open set in τ is a union of elements of $\mathcal{B}$, so each non-empty open set U contains at least one of the sets B, and so $x_B \in U$. Thus $\{x_B \mid B \in \mathcal{B}\}$ is dense in X and countable, so X is separable. □

Definition 3.1.19. A topological space (X, τ) is said to satisfy the *countable chain condition* (*ccc*) if any collection of pairwise disjoint non-empty open subsets of X is countable.

Proposition 3.1.8. *Every separable topological space satisfies the ccc.*

Proof. Let X be a separable space and D be a countable dense subset of X, and suppose that $\mathscr{U}$ is a family of pairwise disjoint non-empty open subsets of X. For each $U \in \mathscr{U}$ there is an $x_U \in D \cap U$, since D is dense. If $U, V \in \mathscr{U}$, with $U \neq V$, then $U \cap V = \emptyset$, so if we take $x_U \in U \cap D$ and $x_V \in V \cap D$, we have that $x_U \neq x_V$. Thus, the function from $\mathscr{U}$ to D mapping $U \mapsto x_U$ is injective, and it follows immediately that $|\mathscr{U}| \leq |D|$. Therefore, $\mathscr{U}$ is countable. □

3.1.2 Bases, subbases, non-Archimedean spaces

Definition 3.1.20. A subset $\mathscr{B}$ of $\mathcal{P}(X)$ is a *base* of (or *generates*) the topology τ if every open set is a union of elements of $\mathscr{B}$.[4]

[4]For the purposes of this thesis, it is convenient to assume that every base does not contain the empty set: as an open set of τ, $\emptyset$ can be obtained as the nullary union of no elements of the base.

Proposition 3.1.9. *A subset $\mathscr{B}$ of $\mathcal{P}(X)$ is a base for τ if and only if every finite intersection of elements of $\mathscr{B}$ is a union of elements of $\mathscr{B}$. Moreover, a subset $\mathscr{B}$ of $\mathcal{P}(X)$ is a base for τ if and only if it is a cover of X such that, for all $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Definition 3.1.21. A subset $\mathscr{P}$ of $\mathcal{P}(X)$ is a *subbase* for τ if the set of all finite intersections of elements of $\mathscr{P}$ is a base for τ.

Proposition 3.1.10. *A subset $\mathscr{P}$ of $\mathcal{P}(X)$ is a subbase for τ if and only if it generates the topology τ. This means that τ is the smallest topology containing $\mathscr{P}$: any topology τ' on X containing $\mathscr{P}$ must also contain τ.*

Definition 3.1.22. A base $\mathscr{B}$ for a topology τ is said to have *rank 1* if, for all $B_1, B_2 \in \mathscr{B}$, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 \cap B_2 = \emptyset$.

We will often write $(X, \mathscr{B})$ or $(X, \mathscr{P})$, meaning the topological space (X, τ) with topology τ generated by the base $\mathscr{B}$ or by the subbase $\mathscr{P}$.

Definition 3.1.23. A topological space is *non-Archimedean* if it is Hausdorff and has a base of rank 1.

Lemma 3.1.11. *Let B be an element of a rank 1 base $\mathscr{B}$ of a non-Archimedean space (X, τ), and let $x \in X \smallsetminus B$. Then there exists $B' \in \mathscr{B}$ such that $x \in B'$ and $B' \cap B = \emptyset$.*

Proof. Consider any $b \in B$. Then, by the Hausdorff property, there exists $B' \in \mathscr{B}$ such that $b \in B', b \notin B'$. Then neither $B \subseteq B'$ nor $B' \subseteq B$, hence $B' \cap B = \emptyset$, since $\mathscr{B}$ has rank 1. □

Corollary 3.1.12. *The elements of a rank 1 base of a non-Archimedean topological space are closed sets, hence they are clopen.*

Proof. Let B be an element of a rank 1 base $\mathscr{B}$ of (X, τ), non-Archimedean. Then, by Lemma 3.1.11, every $x \in X \smallsetminus B$ is contained in an open set disjoint from B. Then $X \smallsetminus B$ is the union of these open sets, hence is open. Thus, B is closed. □

3.2 Adding topology to trees

3.2.1 Topology over trees and bundled trees

We want to equip the set of histories $H(T)$ of a tree $\mathcal{T}$ with a topological structure. The natural choice is to consider the set of all $H_t = \{h \in H \mid t \in$

$h\}$ as a subbase for the topology. As a consequence, the topological space we will consider is $(H(T), \mathcal{O}_\mathcal{T})$, with subbase

$$\mathcal{O}_\mathcal{T} = \{H_t \mid t \in T\}$$

and we will denote the topology as $\tau_\mathcal{T}$.

Remark 3.2.1. Open sets of this topology are closely connected with sets of moments in T: given $S \subseteq T$ we can build an open set $o_S = \bigcup_{s\in S} H_s$, and, conversely, any open set is the union of H_t's[5], so it is o_S for some suitable S. The open set o_S is the set of all histories passing through a certain moment of S; the closed set $H(T) \setminus o_S$ is the set of all histories avoiding all moments of S.

Proposition 3.2.1. *The topological space* $\big(H(T), \mathcal{O}_\mathcal{T}\big)$ *is a non-Archimedean space.*

Proof. We must show that the base has the rank 1 property, and that the space is Hausdorff.

Given $t, s \in T$, different time points, $t < s$, $t > s$, or the two moments are not $<$-comparable. In the first case, $H_t \subseteq H_s$, because the past of s is unique and contains t, hence every history passing through s passes through t too. In the second case, $H_t \supseteq H_s$. In the third case, $H_t \cap H_s = \emptyset$: by contradiction, $h \in H_t \cap H_s$ would contain both t and s, so they would be $<$-comparable. Thus, $\mathcal{O}_\mathcal{T}$ has the rank 1 property.

Now, let h_1, h_2 be different histories of $H(T)$ (which means different points of the topological space $\big(H(T), \mathcal{O}_\mathcal{T}\big)$). Since they are maximal totally ordered subsets of T, there must be two moments t_1, t_2 such that $t_1 \in h_1 \setminus h_2$ and $t_2 \in h_2 \setminus h_1$. Then, $h_1 \in H_{t_1}$ and $h_2 \in H_{t_2}$, but t_1, t_2 are not $<$-comparable moments, so $H_{t_1} \cap H_{t_2} = \emptyset$. Thus, different points belong to disjoint open subsets, so $\big(H(T), \tau_\mathcal{T}\big)$ is Hausdorff. □

Combining this result with Corollary 3.1.12, we can conclude that H_t's are closed in the $\tau_\mathcal{T}$ topology (hence clopen).

Moreover, from Remark 3.2.1, we have that any H_t is $H(T) \setminus o_S$, where $S = \{s \in T \mid s \text{ not } <\text{-comparable with } t\}$: in fact, if $s \neq t$, $s \not< t$, $t \not< s$, then no history in h_t passes through s.

Remark 3.2.2. On the basis of this result, we can define a map that associates a non-Archimedean space to any tree $\mathcal{T}$:

$$\nu : (T, <) \longmapsto \big(H(T), \mathcal{O}_\mathcal{T}\big)$$

[5] In general, by Definition 3.1.21, this sentence should be "Any open set is the union of finite intersections of elements of the prebase", but since $\mathcal{O}_\mathcal{T}$ has rank 1 (as shown below in Proposition 3.2.1) the intersection has no effect.

Moreover, given a bundled tree $\mathcal{T}_\mathcal{B} = (\mathcal{T}, \mathcal{B})$, we can repeat the same construction used above, and build a topology $\tau_\mathcal{B}$ on $\mathcal{B}$, generated by the subbase

$$\mathcal{O}_\mathcal{B} = \{\mathcal{B}_t \mid t \in T\}$$

The generated topological space $(\mathcal{B}, \mathcal{O}_\mathcal{B})$ turns out to be non-Archimedean, since the topology is induced by $\tau_\mathcal{T}$: in fact, $\mathcal{B}_t = H_t \cap \mathcal{B}$, so $\mathcal{O}_\mathcal{B} = \{b \cap \mathcal{B} \mid b \in \mathcal{O}_\mathcal{T}\}$. Thus, it inherits the non-Archimedean property from $\mathcal{O}_\mathcal{T}$.

Remark 3.2.3. As above, we can construct a map that associates a non-Archimedean space to any bundled tree $\mathcal{T}_\mathcal{B} = (\mathcal{T}, \mathcal{B})$:

$$\mu : (\mathcal{T}, \mathcal{B}) \longmapsto (\mathcal{B}, \mathcal{O}_\mathcal{B})$$

3.2.2 Turnaround: from topological spaces to trees

We want to investigate the existence of “reverse” maps of the ones defined in Remarks 3.2.2 and 3.2.3, that are maps that associates trees or bundled trees to non-Archimedean spaces.

Lemma 3.2.2. *Given a non-Archimedean space $(X, \mathcal{O})$, the poset $\mathcal{T}_\mathcal{O} = (\mathcal{O}, \supset)$ is a tree.*

Proof. The $\supset$ relation is trivially irreflexive and transitive, and the rank 1 property of $\mathcal{O}$ implies the tree condition (Definition 2.1.1): in fact, if $B' \supset B$ and $B'' \supset B$, then $B' \cap B'' \supseteq B \neq \emptyset$, so $B' \supset B''$, $B'' \supset B'$ or $B' = B''$. □

Remark 3.2.4. Notice that the elements of the subbase $\mathcal{O}$ are the moments of the tree, and that we need to use the *superset* relation $\supset$ in the opposite direction from the $<$ of the definition of tree.

Remark 3.2.5. On the basis of the previous lemma, we can build a map that associates a tree to every non-Archimedean space:

$$\alpha : (X, \mathcal{O}) \longmapsto (\mathcal{O}, \supset)$$

We can give a refinement of this map in the case of bundled trees, explained in the next lemma and remark:

Lemma 3.2.3. *Let $(X, \mathcal{O})$ be a non-Archimedean space. Then, for every $x \in X$, the set*

$$\mathcal{C}_x = \{B \in \mathcal{O} \mid x \in B\}$$

is a maximal chain in $\mathcal{O}$ (with respect to the inclusion $\supset$), and the set

$$\mathcal{B}_X = \{\mathcal{C}_x \mid x \in X\}$$

is a bundle in $\mathcal{T}_\mathcal{O}$.

Proof. Consider $B_1, B_2 \in \mathcal{C}_x$: they are in relation $\supset$ or $\subset$, since they are elements of a base of rank 1 with non-empty intersection ($x \in B_1 \cap B_2$), so $\mathcal{C}_x$ is an $\subseteq$-chain. Moreover, if $B' \notin \mathcal{C}_x$, by Lemma 3.1.11 it has empty intersection with some $B \in \mathcal{C}_x$, so every proper extension of $\mathcal{C}_x$ is not a $\subseteq$-chain, hence $\mathcal{C}_x$ is maximal. So, in our time-logic language, it is a history.

Now, every element of $\mathcal{B}_X$ is a history, so if we prove the closure property (Definition 2.3.1), we can conclude that it is a bundle on $T_{\mathcal{O}}$. Given $B \in \mathcal{O}$, it is not empty[6], so for every $x \in B$, $B \in \mathcal{C}_x$, hence $B \in \mathcal{C}_x \in \mathcal{B}_X$. Thus, $\mathcal{B}_X$ is a bundle. □

Remark 3.2.6. This result suggests that we can consider a map from non-Archimedean spaces to bundled trees, namely

$$\beta : (X, \mathcal{O}) \longmapsto (\mathcal{T}_{\mathcal{O}}, \mathcal{B}_X)$$

Clearly, if the bundled tree turns out to be complete, this map is the map α defined above.

Given the notion of isomorphism between trees (see Section 2.1.1), we can consider the composition of the maps defined above in Remarks 3.2.5 and 3.2.6 with the ones produced in Remarks 3.2.2 and 3.2.3. The following are natural questions:

- is the tree $(T, <)$ always isomorphic to the tree $\mathcal{T}_{\mathcal{O}_T}$, which comes from the following composition?

$$(T, <) \overset{\nu}{\longmapsto} \big(H(T), \mathcal{O}_T\big) \overset{\alpha}{\longmapsto} (\mathcal{O}_T, \supset) = \mathcal{T}_{\mathcal{O}_T}$$

- are there any relations (homeomorphisms?) between a non-Archimedean topological space $(X, \mathcal{O})$ and $\big(H(T_{\mathcal{O}}), \mathcal{O}_{T_{\mathcal{O}}}\big)$[7], which comes from the following composition?

$$(X, \mathcal{O}) \overset{\alpha}{\longmapsto} (\mathcal{O}, \supset) \overset{\nu}{\longmapsto} \big(H(T_{\mathcal{O}}), \mathcal{O}_{T_{\mathcal{O}}}\big)$$

- is the bundled tree $(\mathcal{T}, \underline{\mathcal{B}})$[8] isomorphic to the bundled tree $(\mathcal{T}_{\mathcal{O}_{\underline{\mathcal{B}}}}, \mathcal{B}_{\underline{\mathcal{B}}})$, which comes from the following composition?

$$(\mathcal{T}, \underline{\mathcal{B}}) \overset{\mu}{\longmapsto} (\underline{\mathcal{B}}, \mathcal{O}_{\underline{\mathcal{B}}}) \overset{\alpha}{\longmapsto} (\mathcal{T}_{\mathcal{O}_{\underline{\mathcal{B}}}}, \mathcal{B}_{\underline{\mathcal{B}}})$$

[6] We have assumed that the empty set is not contained in any base, and it is an element of the generated topology because of the nullary union.

[7] $T_{\mathcal{O}}$ is simply $\mathcal{O}$, the set of moments of the space $\mathcal{T}_{\mathcal{O}} = (\mathcal{O}, \supset)$.

[8] We underlined the symbol for the bundle $\underline{\mathcal{B}}$ just to differentiate it from the $\mathcal{B}_X$ used in the costruction of the map above.

- are there any relations (homeomorphism?) between a non-Archimedean topological space $(X, \mathcal{O})$ and $(\mathcal{B}_X, \mathcal{O}_{\mathcal{B}_X})$, which comes from the following composition?

$$(X, \mathcal{O}) \xmapsto{\beta} (\mathcal{T}_{\mathcal{O}}, \mathcal{B}_X) \xmapsto{\mu} (\mathcal{B}_X, \mathcal{O}_{\mathcal{B}_X})$$

In order to answer to these questions, we need to prove some results regarding the presentation of topological spaces: the complete solution to these problems is given in Propositions 3.2.6 and 3.2.8.

3.2.3 Presentation of non-Archimedean spaces

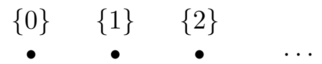
Almost every topological argument involves the notion of homeomorphism: given two topological spaces (X, τ), (Y, τ'), an *homeomorphism* between them is a bijective continuous[9] map $\phi : X \to Y$ with continuous inverse function. This notion, which is also called *topological equivalence*, is widely used: if two topological spaces are homeomorphic, their topologies share a lot of properties.

Unfortunately, this notion is not relevant to our context: in fact, different bases can generate the same non-Archimedean space, but they correspond to different trees, as seen in the following example:

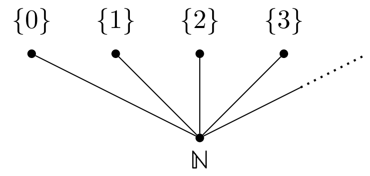
Example 3.2.1. Consider the topological space $(\mathbb{N}, \tau)$, where τ is the discrete topology over the set of natural numbers. $(\mathbb{N}, \tau)$ is a non-Archimedean space: for example, the set of the singletons $\mathcal{O}_1 = \big\{\{n\} \mid n \in \mathbb{N}\big\}$ is a rank 1 base for the topology. It is easy to verify also that $\mathcal{O}_2 = \mathcal{O}_1 \cup \{\mathbb{N}\}$, $\mathcal{O}_3 = \mathcal{O}_1 \cup \big\{\{n \geq n_0\} \mid n_0 \in \mathbb{N}\big\}$, $\mathcal{O}_4 = \mathcal{O}_1 \cup \big\{\{0, \ldots, n_0\} \mid n_0 \in \mathbb{N}\big\}$ are rank 1 bases for the discrete topology over $\mathbb{N}$, so the topological spaces $(\mathbb{N}, \mathcal{O}_i)$ are pairwise homeomorphic. On the contrary, the correspondent trees $(\mathcal{O}_i, \supset)$ are clearly not isomorphic:

- the tree $(\mathcal{O}_1, \supset)$ consists of infinite (disjoint) connected components, each containing just one moment;
- the tree $(\mathcal{O}_2, \supset)$ is a rooted tree (with root $\{\mathbb{N}\}$) with infinite uncomparable moments above the root and nothing above;
- the tree $(\mathcal{O}_3, \supset)$ is rooted, has a linear subtree $\big(\big\{\{n \geq n_0\} \mid n_0 \in \mathbb{N}\big\}\big)$ and a leaf $(\{n_0\})$ starting from every moment of the subtree;
- the tree $(\mathcal{O}_4, \supset)$ is not rooted, has a linear subtree $\big(\big\{\{0, \ldots, n_0\} \mid n_0 \in \mathbb{N}\big\}\big)$ and a leaf $(\{n_0\})$ attached to every moment of the subtree.

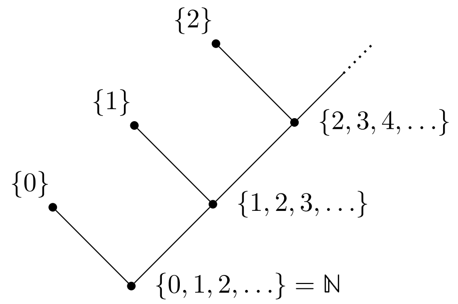
[9]See Definition 3.1.10.



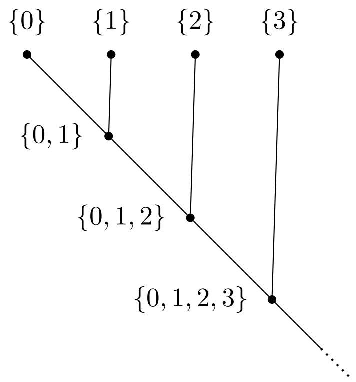
Representation of the tree $(\mathcal{O}_1, \supset)$.



Representation of the tree $(\mathcal{O}_2, \supset)$.



Representation of the tree $(\mathcal{O}_3, \supset)$.



Representation of the tree $(\mathcal{O}_4, \supset)$.

Moreover, each pair $(\mathbb{N}, \mathcal{O}_i)$ determines the bundled tree $\mathcal{T}_i = (\mathcal{T}_{\mathcal{O}_i}, \mathcal{B}_\mathbb{N})$ by means of the definitions in Lemma 3.2.3: $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4$ are complete bundled trees, while $\mathcal{T}_3$ is not, so also the completeness of bundled trees that come from non-Archimedean spaces depends on the chosen base. In fact:

- for $i = 1$, $\mathcal{C}_n = \{B \in \mathcal{O}_1 \mid n \in B\} = \big\{\{n\}\big\}$, thus $\mathcal{B}_{\mathbb{N},1} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ contains every possible chain, hence the bundled tree $\mathcal{T}_1$ is complete;
- for $i = 2$, $\mathcal{C}_n = \{B \in \mathcal{O}_2 \mid n \in B\} = \big\{\{n\}, \mathbb{N}\big\}$, thus $\mathcal{B}_{\mathbb{N},2}$ contains every possible chain, hence the bundled tree $\mathcal{T}_2$ is complete;
- for $i = 3$, $\mathcal{C}_n = \{B \in \mathcal{O}_3 \mid n \in B\} = \big\{\{m \geq n_0\} \mid n_0 \leq n\big\} \cup \{n\}$; but $\big\{\{m \geq n_0\} \mid n_0 \in \mathbb{N}\big\}$ is a chain different from every $\mathcal{C}_n$ (before it does not contain any $\{n\}$), so $\mathcal{B}_{\mathbb{N},3}$ does not contain every possible chain, hence the bundled tree $\mathcal{T}_3$ is *not* complete;
- for $i = 4$, $\mathcal{C}_n = \{B \in \mathcal{O}_4 \mid n \in B\} = \big\{\{0, 1, \dots, n_0\} \mid n_0 \geq n\big\} \cup \{n\}$, thus $\mathcal{B}_{\mathbb{N},4}$ contains every possible chain, hence the bundled tree $\mathcal{T}_4$ is complete.

On the basis of this example, it makes sense to consider pairs $(X, \mathcal{O})$ and $(X, \mathcal{O}')$ as different objects when $\mathcal{O} \neq \mathcal{O}'$, even if $\mathcal{O}$ and $\mathcal{O}'$ generate the same topology.

Definition 3.2.1. A pair $(X, \mathcal{O})$, consisting of a set X and a (sub)base for a topology on X, is called a *presentation of the topological space*.

Definition 3.2.2. The presentations $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ of two topological spaces are *isomorphic* if there exists a bijection $\phi : X \to X'$ such that the induced map on the power sets $\hat{\phi} : \mathcal{P}(X) \to \mathcal{P}(X')$, defined by $\hat{\phi}(A) = \{\phi(a) \mid a \in A\}$, is a bijection from $\mathcal{O}$ onto $\mathcal{O}'$.

Lemma 3.2.4. *If two presentations of topological spaces $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ are isomorphic, then the correspondent trees $\mathcal{T}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}'}$ are isomorphic, and $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ are homeomorphic as topological spaces.*

Proof. Consider the map $\hat{\phi} : \mathcal{P}(X) \to \mathcal{P}(X')$ induced by ϕ as in Definition 3.2.2. It clearly preserves inclusion, hence it is an order-preserving map with respect to $\supset$. Moreover, it is a bijection of $\mathcal{O}$ onto $\mathcal{O}'$, by definition. Hence, $\mathcal{T}_{\mathcal{O}} \cong \mathcal{T}_{\mathcal{O}'}$. Finally, isomorphic presentations generate the same topological space, up to homeomorphism. □

The previous example is a counterexample to the converse of the last implication: we can have homeomorphic topological spaces with non-isomorphic presentations.

Now, recall that $\mathcal{O}_{\mathcal{B}} = \{\mathcal{B}_B \mid B \in \mathcal{O}\}$ and that $\mathcal{B}_B = H_B \cap \mathcal{B}$.

Lemma 3.2.5. *For every non-Archimedean space $(X, \mathcal{O})$ and every bundle $\mathcal{B}$ in $\mathcal{T}_{\mathcal{O}}$, the function $\psi : \mathcal{O} \to \mathcal{O}_{\mathcal{B}}$ mapping $B \mapsto \mathcal{B}_B$ is a bijection.*

Proof. Consider $B_1, B_2 \in \mathcal{O}$, and assume $B_1 \nsubseteq B_2$: the symmetric case can be dealt with exchanging B_1 and B_2. Consider $x \in B_1 \setminus B_2$: by Lemma 3.1.11, there exists $B' \in \mathcal{O}$ such that $B' \cap B_2 = \emptyset$. Let B'' be $B_1 \cap B'$. Since both B_1 and B' contain x, by the rank 1 property of the base, B'' is either B_1 or B', and, clearly, $B'' \cap B_2 = \emptyset$. Now, consider any $h \in \mathcal{B}_{B''}$: $h \in \mathcal{B}_{B_1}$ because $B'' \subseteq B_1$, and $h \notin \mathcal{B}_{B_2}$ because $B'' \cap B_2 = \emptyset$. Then ψ is injective because different elements of the domain have different images. Moreover, by definition, every element of $\mathcal{O}_{\mathcal{B}}$ is $\mathcal{B}_B$ for some $B \in \mathcal{O}$, and hence ψ is surjective. □

Proposition 3.2.6. *Every presentation of a non-Archimedean space $(X, \mathcal{O})$ is isomorphic to $(\mathcal{B}_X, \mathcal{O}_{\mathcal{B}_X})$.*

Proof. Consider the function $\phi : X \to \mathcal{B}_X$ defined by $x \mapsto \mathcal{C}_x$. It is a bijection: in fact, X is Hausdorff, hence if $x_1 \neq x_2$, there exist disjoint open neighbourhoods of x_i (of the form $H_{x_i} = \mathcal{C}_{x_i}$), so ϕ is injective; moreover every $\mathcal{C}_x$ is clearly $\phi(x)$ for some x, hence ϕ is surjective.

Now, consider the restriction to $\mathcal{O}$ of the map $\hat{\phi}$ induced by ϕ on $\mathcal{P}(X)$. Given $B \in \mathcal{O}$ and $x \in X$, the definition of $\mathcal{C}_x$ implies that $B \in \mathcal{C}_x$ if and only if $x \in B$, so $\hat{\phi}(B) = \{\mathcal{C}_x \mid x \in B\} = (\mathcal{B}_X)_B$. Thus, the function induced by ϕ on $\mathcal{O}$ coincides with the function ψ of the previous Lemma 3.2.5, so it is a bijection from $\mathcal{O}$ to $\mathcal{O}_{\mathcal{B}_X}$. □

Remark 3.2.7. In the particular case of complete bundled trees (i.e. $\mathcal{B}_X = H(T_\mathcal{O})$), this Proposition yields that $(X, \mathcal{O}) \cong \big(H(T_\mathcal{O}), \mathcal{O}_{\mathcal{T}_\mathcal{O}}\big)$, as presentations of non-Archimedean spaces.

This concludes the analysis of the compositions $\nu \circ \alpha$ and $\mu \circ \beta$: they are isomorphisms of presentations of non-Archimedean spaces.

Regarding the compositions $\alpha \circ \nu$ and $\beta \circ \mu$, starting from trees or bundled trees, Lemma 3.2.7 below states that they are isomorphisms only for a particular class of trees:

Definition 3.2.3. A tree $\mathcal{T}$ is said to be *totally branching* if $H_t \neq H_{t'}$, for all $t \neq t'$ in T. A bundled tree $(\mathcal{T}, \mathcal{B})$ is totally branching if $\mathcal{T}$ is totally branching.

Lemma 3.2.7. *For every non-Archimedean space $(X, \mathcal{O})$, the tree $\mathcal{T}_\mathcal{O}$ is totally branching.*

Proof. Consider $B, B' \in \mathcal{O}$, $B \neq B'$, and assume that $B \nsubseteq B'$, so we can consider $x \in B \setminus B'$ (the other case is symmetric). Then $\mathcal{C}_x$ is a history in $\mathcal{T}_\mathcal{O}$, which contains B and does not contain B', hence $H_B \neq H_{B'}$. Thus, $\mathcal{T}_\mathcal{O}$ is totally branching. □

This lemma shows that a tree $\mathcal{T}$ can not be isomorphic to the associated $\mathcal{T}_{\mathcal{O}_\mathcal{T}}$, unless it is totally branching. The next proposition proves that this condition is sufficient for the isomorphism to be verified.

Proposition 3.2.8. *Every totally branching tree $\mathcal{T}$ is isomorphic to the tree $\mathcal{T}_{\mathcal{O}_\mathcal{T}}$. Moreover, every totally branching bundled tree $(\mathcal{T}, \underline{\mathcal{B}})$ is isomorphic to the bundled tree $(\mathcal{T}_{\mathcal{O}_{\underline{\mathcal{B}}}}, \mathcal{B}_{\underline{\mathcal{B}}})$.*

Proof. The function $\phi : t \mapsto H_t$ clearly maps T onto $\mathcal{O}_\mathcal{T}$, and is injective because $\mathcal{T}$ is totally branching, so different t's are mapped to different H_t's.

Moreover, since $\mathcal{B}_t = \mathcal{B} \cap H_t$, the totally branching condition implies $\mathcal{B}_t \neq \mathcal{B}_{t'}$ for all $t \neq t'$. So, the map $\phi : t \mapsto \mathcal{B}_t$ is an isomorphism (same proof as above). Finally, for every $h \in \mathcal{B}$, $\hat{\phi}(h) = \{\phi(t) \mid t \in h\} = \{\mathcal{B}_t \mid h \in \mathcal{B}_t\} = \mathcal{C}_h$, hence the function ϕ induces a bijection between $\mathcal{B}$ and $\mathcal{B}_\mathcal{B}$, so it is an isomorphism of bundled trees. □

This concludes the analysis of the compositions $\alpha \circ \nu$ and $\beta \circ \mu$: they are isomorphisms of (bundled) trees only if the starting (bundled) tree is totally branching; if it is not the case, $\alpha \circ \nu$ and $\beta \circ \mu$ send the (bundled) tree $\mathcal{T}$ to an "associated" totally branching (bundled) tree, which is properly "smaller" than $\mathcal{T}$. This relation will be investigated in the following section.

3.2.4 Condensation of trees

We conclude this section rephrasing the definition of totally branching trees and the results above using the notions of *furcation*, *bridge* and *condensation*. These terms have been introduced by Ruaan Kellerman in his PhD dissertation [19], which, to my knowledge, is one of the most exhaustive sources regarding the algebraic structure of trees.

The aim of this section is to prove Theorem 3.2.22, which completes the "translation" of the results of Section 3.2.3 in this newly introduced terminology. Therefore, the reader can skip the proofs of Propositions 3.2.9, 3.2.10, 3.2.11 and 3.2.13. We decided to prove those results in order to complete the description of this topic, but they are technical results related to the tree structure, and so they are not directly linked to our main objective.

In the following, we assume that every tree is connected, which *is* a restrictive hypothesis. In any case, all the definitions and results below can be extended to the case of non-connected trees (*forests*, in Kellerman's terminology) by considering the connected components.

In this section we will often use the comparability relation $\smile$ between moments, defined in Definition 2.1.2.

Definition 3.2.4. Given a tree $\mathcal{T} = (T, <)$,

- a *segment* on $\mathcal{T}$ is every subset A of T, which is totally ordered and closed, which means that if $t < t'$ are elements of A and $t < s < t'$, then $s \in A$[10];
- a *bridge* on $\mathcal{T}$ is a segment B such that for every history h, either $h \cap B = \emptyset$ or $h \cap B = B$;
- a *furcation* on $\mathcal{T}$ is a segment F, which is not a bridge: in other words, F is a furcation if there exists a history h such that $\emptyset \neq F \cap h \neq F$ [11].

[10] The notion of closeness we are using here, which is borrowed from [19], is obviously different from the topological one. It is used only in this Section 3.2.4, so the two meanings can not be mistaken.

[11] Equivalently, in our time-logic terminology, a segment is a furcation if it contains a moment with at least two distinct possible futures and a moment of one of them. A tree has a furcation if the flow of time is properly-branching (see Definition 2.1.3).

Example 3.2.2. Consider the tree $\mathcal{T}$ of Figure 3.1: $\{a,b,c\},\{d,e\},\{f,g\}$ (and every singleton, see Remark 2.2.2) are bridges of $\mathcal{T}$. $\{b,c,d\}$, $\{c,f\}$, $\{f,g,h\}$, $\{g,i\}$ are furcations.

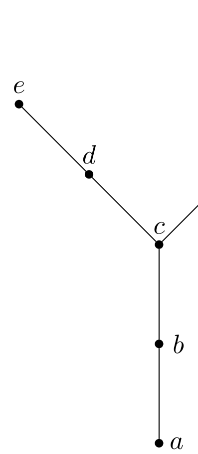


Figure 3.1: Example of a tree with some bridges and furcations.

Proposition 3.2.9. *Let $\mathcal{T}$ be a tree and let A and B be non disjoint bridges. Then $A \cup B$ is a bridge.*

Proof. Let h be a history, with $A \cap B \subseteq h$. Since A and B are bridges, then $A, B \subseteq h$, hence $A \cup B \subseteq h$, thus $A \cup B$ is totally ordered. Moreover $A \cup B$ is closed. In fact:

- if $a, b \in A$ and $a < c < b$, $c \in A$ since A is a segment.
- if $a, b \in B$ and $a < c < b$, $c \in B$ since B is a segment.
- if $a \in A \smallsetminus B, b \in B \smallsetminus A$ and $a < c < b$, consider $d \in A \cap B \neq \emptyset$. If $b \leq d$ then $b \in A$, which is a contradiction. Hence $d < b$. Since both c and d are in the past of b, by tree property we have $c \smile d$. Thus, either $a < c \leq d$, in which case $c \in A$, or $d < c < b$, in which case $c \in B$.
- if $a \in B \smallsetminus A, b \in A \smallsetminus B$ and $a < c < b$, symmetric proof.

Hence $A \cup B$ is a segment.

Now, let h be a history such that $h \cap (A \cup B) \neq \emptyset$. Then, without loss of generality, suppose that $h \cap A \neq \emptyset$. Then, since A is a bridge, $h \cap A = A \supseteq A \cap B$. Thus, $h \cap B \neq \emptyset$, from which $h \cap B = B$. Then $h \cap (A \cup B) = (h \cap A) \cup (h \cap B) = A \cup B$. Thus, $A \cup B$ is a bridge. □

Proposition 3.2.10. *Let $\mathcal{T}$ be a tree and A be a bridge in $\mathcal{T}$. Then A is contained in a unique a maximal bridge.*

Proof. Let $\mathcal{A}$ be the set of bridges of $\mathcal{T}$ containing A. Consider a $\subset$-chain $\mathcal{C}$ in $\mathcal{A}$: we want to show that it has an upper bound in $\mathcal{A}$, then from Zorn's Lemma (see footnote 5 of Chapter 2) we have that A is contained in a maximal bridge.

Let $C_0 = \bigcup \mathcal{C}$. It is totally ordered (since it is the union of a chain of bridges) and closed, hence it is a segment. Let now h be a history in $\mathcal{T}$, and suppose that $h \cap C_0 \neq \emptyset$. Then there exists $C_1 \in \mathcal{C}$ such that $h \cap C_1 \neq \emptyset$, but C_1 is a bridge, hence $h \cap C_1 = C_1$. Then $A \subseteq h$, since $C_1 \supseteq A$ by definition. It implies that for every $C \in \mathcal{C}$, $h \cap C = C$.

Then, $h \cap C_0 = h \cap (\bigcup \mathcal{C}) = \bigcup \{h \cap C \mid C \in \mathcal{C}\} = \bigcup \{C \mid C \in C_0\} = C_0$. It implies that C_0 is a bridge containing A, and it is an upper bound for $\mathcal{C}$. Then $\mathcal{A}$ has maximal elements, which are maximal bridges containing A.

In order to prove uniqueness, let B_1, B_2 be maximal bridges containing A. By Proposition 3.2.9, $B_1 \cup B_2$ is a bridge containing A, hence, by maximality, $B_1 = B_1 \cup B_2 = B_2$. □

Remark 3.2.8. If A and B are maximal bridges in a tree $\mathcal{T}$, then they are disjoint or equal. In fact, given every h history such that $h \cap A = A$, we have that $h \cap B = \emptyset$ or $h \cap B = B$: in the first case, $A \cap B = \emptyset$. In the second case, by maximality, $A = B$. Hence, the set of maximal bridges forms a partition of the tree. In the example of Figure 3.1, the partition is made by $\{a, b, c\}$, $\{d, e\}$, $\{f, g\}$, $\{h\}$, $\{i\}$.

Starting from this partition, we can define a relation $\sim$ between moments by $t \sim t'$ if t and t' belong to the same maximal bridge. It is readily verified that $\sim$ is an equivalence relation on $\mathcal{T}$. The equivalence classes are obviously the maximal bridges themselves.

Given $t \in T$, the unique maximal bridge of $\mathcal{T}$ containing t will be denoted by $[t]$. The set of all maximal bridges can be endowed with the relations $<, >$, and $\smile$ by requiring that every element of the first bridge is related to every element of the second. For instance, $[a] \smile [b]$ holds when $t \smile t'$ for all $t \in [a]$ and $t' \in [b]$.

Proposition 3.2.11. *Let $\mathcal{T}$ be a tree and let $a, b \in T$.*

- *If $a < b$ and $[a] \neq [b]$, then $[a] < [b]$.*
- *If $a \not\smile b$, then $[a] \not\smile [b]$.*

Proof. Let h be a history in $\mathcal{T}$ with $a, b \in h$. Since $[a]$ and $[b]$ are bridges, $[a], [b] \subseteq h$, hence $[a] \cup [b]$ is linear. Since $[a] \neq [b]$, then $[a] \cap [b] = \emptyset$. Assume by contradiction that there exists $a' \in [a], b' \in [b]$ with $b' < a'$. Then, since $[a] \cup [b]$ is linear, the relations between a, a', b, b' are described by one of the following situations:

- $b' < a' \leq a < b$, which implies that $a \in [b]$
- $b' \leq a < a' \leq b$, which implies that $a \in [b]$
- $b' \leq a < b \leq a'$, which implies that $a \in [b]$
- $a < b' < a' \leq b$, which implies that $b' \in [a]$
- $a < b' \leq b < a'$, which implies that $b' \in [a]$
- $a < b < b' < a'$, which implies that $b' \in [a]$

Each of these cases violates the fact that $[a] \cap [b] = \emptyset$. Then $[a] < [b]$.

Moreover, assume by contradiction $[a] \smile [b]$. Then, by definition $\forall x \in [a], \forall y \in [b]$, $x \smile y$, but this violates the hypothesis $a \not\smile b$. □

Corollary 3.2.12. *Let $\mathcal{T}$ be a tree and let $a, b \in T$. Then, $a \smile b$ if and only if $[a] \smile [b]$.*

Proposition 3.2.13. *Let $\mathcal{T}$ be a tree and let $a, b \in T$. The following conditions are equivalent:*

1. *a and b belong to the same bridge;*
2. *$[a] = [b]$;*
3. *for every history h in $\mathcal{T}$, $a \in h$ if and only if $b \in h$;*
4. *for every moment $t \in T$, $t \smile a$ if and only if $t \smile b$.*

Proof. (*1.* $\Leftrightarrow$ *2.*) Immediate.

(*2.* $\Rightarrow$ *3.*) Suppose $[a] = [b]$. Let h be a history, $a \in h$. Then $[a] \subseteq h$, and so $[b] \subseteq h$, which gives $b \in h$. From this we deduce that for every history h, if $a \in h$, then $b \in h$. Similarly we can show necessity.

(*3.* $\Rightarrow$ *2.*) Since $a = b$ immediately implies $[a] = [b]$, we may assume $a \neq b$. Let h be a history passing through a: by condition *3.* we also have that $b \in h$, so $a \smile b$. Without loss of generality, we may assume that $a < b$. Consider the segment $[a, b] = \{t \in T \mid a \leq t \leq b\}$. Let h' be a history with non-empty intersection with $[a, b]$. Then, by the tree condition, $a \in h'$, thus, because of what we proved above, $b \in h'$. Hence $[a, b] \in h'$. Then $[a, b]$ is a bridge. By Proposition 3.2.10, $[a, b]$ is contained in a unique maximal bridge, hence $[a] = [b]$.

(*3.* $\Rightarrow$ *4.*) Let $c \in T$, $c \smile a$, and let h be a history with $a, c \in h$. By condition *3.* we also have that $b \in h$, so $c \smile b$. So, for every $t \in T$, if $t \smile a$, then $t \smile b$. Similarly we can show necessity.

(*4.* $\Rightarrow$ *3.*) Let h be a history passing through a. For every $t \in h$, $t \smile a$, so by condition *4.* we have $t \smile b$. Since histories are maximal totally ordered subsets of T, we have $b \in h$. Hence, for every history h' of $\mathcal{T}$, $a \in h'$ implies $b \in h'$. Similarly we can prove the converse implication. □

Definition 3.2.5. Let $\mathcal{T} = (T, <)$ be a tree. We will denote the set of maximal bridges of $\mathcal{T}$ by $[T] = \{[t] \mid t \in T\}$. The structure $[\mathcal{T}] = ([T], <)$ is called the *condensation* of the tree $\mathcal{T}$.

Remark 3.2.9. The condensation of the tree is the quotient structure of the tree generated by the relation $\sim$. It is straightforward to prove that $<$ on $[T]$ is irreflexive, transitive, and satisfies the tree condition, hence $([T], <)$ is a tree.

Example 3.2.3. Consider the tree $\mathcal{T}$ of Figure 3.1: its condensation is described in the following picture. We want to highlight the fact that the condensation of a tree with the "same form" of $\mathcal{T}$, but with infinitely many moments in every bridge, would have the same condensation as $\mathcal{T}$ (with nothing between different "condensed" moments).

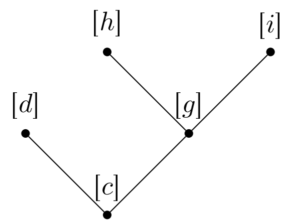


Figure 3.2: Condensation $[\mathcal{T}]$ of the tree $\mathcal{T}$ of Figure 3.1.

Remark 3.2.10. If $\mathcal{S} = (S, <)$ is a subtree of $\mathcal{T}$, then $[\mathcal{S}]$ is a subtree of $[\mathcal{T}]$. Clearly, if $\mathcal{S}$ is a proper subtree of $\mathcal{T}$, the same does *not* apply to $[\mathcal{S}]$: in fact, $[\mathcal{S}]$ is a proper subtree of $[\mathcal{T}]$ if and only if $\mathcal{S}$ and $\mathcal{T}$ have different furcations, i.e. if they *ramify* differently.

As an example, consider the tree $\mathcal{T}$ of Figure 3.1: on one hand, every subtree $\mathcal{S}$, which is obtained by cutting some moments from the bridges that are not singletons, has the same condensation of $\mathcal{T}$. On the other hand, if we cut an entire bridge we obtain a different condensation: in the following Figure we have the condensation of the subtree $\mathcal{S}$ of $\mathcal{T}$ obtained by cutting the bridge $\{d, e\}$.

Definition 3.2.6. A tree $\mathcal{T} = (T, <)$ is called *condensed* if $\mathcal{T} \cong [\mathcal{T}]$.

Lemma 3.2.14. *Let $\mathcal{T}$ be a tree. Then every non-empty bridge in $[\mathcal{T}]$ consists of a single moment.*

Proof. Let $[a], [b] \in [T]$ with $[a] \neq [b]$. Then a and b belong to different maximal bridges in $\mathcal{T}$. Hence, by Proposition 3.2.13, we can conclude that there exists $c \in T$ such that $c \smile a$ and $c \not\smile b$. Then, by Corollary 3.2.12 we

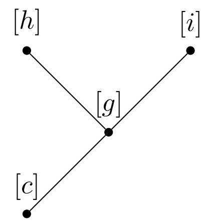


Figure 3.3: Condensation $[S]$ of the subtree $S = \{a, b, c, f, g, h, i\}$.

have $[c] \smile [a]$ and $[c] \not\smile [b]$. Thus, $[a]$ and $[b]$ belong to different maximal bridges in $[\mathcal{T}]$ (again by Proposition 3.2.13 applied to the tree $[\mathcal{T}]$). □

Proposition 3.2.15. *Let $\mathcal{T}$ be a tree. The following conditions are equivalent:*

1. *$\mathcal{T}$ is condensed;*
2. *$\mathcal{T} \cong [\mathcal{S}]$ for a tree $\mathcal{S} = (S, <)$;*
3. *$[t] = \{t\}$ for every $t \in T$.*

Proof. (*1.* ⇒ *2.*) Let $\mathcal{T}$ be condensed. Then $\mathcal{T} \cong [\mathcal{T}]$.

(*2.* ⇒ *3.*) Let $\mathcal{T} \cong [\mathcal{S}]$ for a certain tree $\mathcal{S}$, and $\phi : T \to [S]$ an isomorphism. Let $a, b \in T$, $a \neq b$. Then $\phi(a) \neq \phi(b)$, so they belong to different maximal bridges in $[\mathcal{S}]$, by Lemma 3.2.14. Hence, without loss of generality, we may conclude that there exists $c \in [S]$ such that $c \smile \phi(a)$ and $c \not\smile \phi(b)$ (by Proposition 3.2.13). Thus, $a \smile \phi^{-1}(c)$ and $b \not\smile \phi^{-1}(c)$, so $[a] \neq [b]$. Hence, different moments can not be in the same maximal bridge.

(*3.* ⇒ *1.*) Assume that $\{t\} = [t]$ for every moment $t \in T$. This position defines an isomorphism $\phi : T \to [T]$, by means of $t \mapsto [t]$: in fact, if $t < t'$, clearly $[t] < [t']$, because we need to verify the $<$ relation just on t and t'. □

Now we can link the notion of condensation to the contents of Section 3.2.3.

Lemma 3.2.16. *Let $\mathcal{T}$ be a tree. Then for every $t, t' \in T$, $[t] = [t']$ if and only if $H_t = H_{t'}$.*

Proof. If $[t] = [t']$, then $t' \in [t]$. For every $h \in H_t$, $t \in h \cap [t] \neq \emptyset$, so $h \cap [t] = [t]$, hence $t' \in h$. Thus, $h \in H_{t'}$, which implies that $H_t \subseteq H_{t'}$. By applying the same argument to t', we have $H_t \supseteq H_{t'}$. Hence, $H_t = H_{t'}$.

Conversely, if $H_t = H_{t'}$, then $t \in h$ if and only if $t' \in h$. Hence, if $h \cap [t] \neq \emptyset$, $h \cap t = [t]$ and contains t'. So $t' \in [t]$, hence $[t'] \subseteq [t]$. By maximality, we have the equality. □

Corollary 3.2.17. *Let $\mathcal{T}$ be a tree. Then it is totally branching if and only if $\forall t \in T$, $[t] = \{t\}$, i.e. if and only if maximal bridges in $\mathcal{T}$ consist of a single moment*[12].

Proof. By definition, a tree $\mathcal{T}$ is totally branching if, for all $t \neq t'$ in T, $H_t \neq H_{t'}$. Assume by contradiction that there exists t such that $[t] \neq \{t\}$. Then there exists $t' \in [t], t' \neq t$. Then $[t] = [t']$, so, by Lemma 3.2.16 we have $H_t = H_{t'}$, which is a contradiction.

Conversely, assume that for all $t \in T$, $[t] = \{t\}$. Then, given $t' \neq t$, $[t'] = \{t'\} \neq \{t\} = [t]$. Hence, by Lemma 3.2.16, $H_t \neq H_{t'}$. Thus, the tree $\mathcal{T}$ is totally branching. □

Corollary 3.2.18. *A tree $\mathcal{T}$ is totally branching if and only if it is condensed, i.e if and only if $\mathcal{T} \cong [\mathcal{T}]$.*

Proof. By Proposition 3.2.15 and Lemma 3.2.17. □

The following results are aimed to investigate the properties of maximal totally branching subtrees of a given tree. As a consequence of these results, we will have Theorem 3.2.22 below. A simpler proof of this theorem will also be given.

Proposition 3.2.19. *Let $\mathcal{T}$ be a tree and let $\mathcal{S}$ be a totally branching subtree of $\mathcal{T}$, $\subset$-maximal with respect to this property. That is, either $S = T$, or, for every $t \in T \setminus S$, $(S \cup \{t\}, <)$ is not totally branching. Then S contains exactly one element for every maximal bridge of $\mathcal{T}$.*

Proof. If $\mathcal{T}$ is totally branching, then $\mathcal{S} = \mathcal{T}$, so we can apply Corollary 3.2.17 and complete the proof.

Let $\mathcal{T}$ be a non-totally branching tree. Suppose by contradiction that S does *not* contain exactly one moment for every maximal bridge. Then, either there exists a maximal bridge that contains more than one element of S, or there exists a maximal bridge disjoint from S.

In the first case, let A be the maximal bridge and $s, s' \in S \cap A$ with $s \neq s'$. Then for every history h such that $h \cap A \neq \emptyset$, we have that $h \cap A = A$, because A is maximal. Hence h contains both s and s'. So every history passing through s contains s' too, which means that $H_s = H_{s'}$. Thus, $\mathcal{S}$ is not totally branching, which is a contradiction.

In the second case, let B be a maximal bridge such that $B \cap S = \emptyset$, and $b \in B$. Then $(S \cup \{b\}, <)$ is a totally branching subtree of $\mathcal{T}$. In fact, for

[12]Equivalently, this occurs if every moment belongs to a furcation. Moreover, in our time-logic language, a tree is totally branching if it is properly branching in every moment, so, we could say, if it is a *totally-indeterministic* flow of time.

every $s \in S$, s and b belong to different maximal bridges, so there exists at least one history passing through s and not through b, or vice versa. Then $H_s \neq H_b$, which proves that $(S \cup \{b\}, <)$ is a totally branching subtree of $\mathcal{T}$. Moreover, it is clearly strictly bigger than $\mathcal{S}$, which is a contradiction because $\mathcal{S}$ is maximal. □

Corollary 3.2.20. *Let $\mathcal{T}$ be a tree and let $\mathcal{S}$ be a totally branching subtree of $\mathcal{T}$, $\subset$-maximal with respect to this property. Then, $\mathcal{S}$ is isomorphic to $[\mathcal{T}]$.*

Proof. The isomorphism is the condensation map $[\cdot] : s \mapsto [s]$. The fact that this map is invertible is a consequence of the result shown in the previous proposition. □

The property of $\alpha \circ \nu : T \mapsto T_{\mathcal{O}_\mathcal{T}}$ we state in this remark and in the following proposition will be useful in the proof of Theorem 3.2.22:

Remark 3.2.11. Let $\mathcal{T}$ be a tree and $\mathcal{S}$ a subtree of $\mathcal{T}$. At a fist glance, because of the shortening of the notation, we could think that $\alpha \circ \nu(S) = \mathcal{O}_\mathcal{S}$ is a subtree of $\alpha \circ \nu(T) = \mathcal{O}_\mathcal{T}$, but this is not the case. In fact, it is not even a subset: $\mathcal{O}_\mathcal{S} = \{H_s(S) \mid s \in S\} \subseteq H(S)$ and $\mathcal{O}_\mathcal{T} = \{H_t(T) \mid t \in T\} \subseteq H(T)$, and $H_s(S) \subseteq H_s(T)$, so $\mathcal{O}_\mathcal{S}$ and $\mathcal{O}_\mathcal{T}$ consist of different elements. An example in which the strict subset relation holds, can be produced considering the tree of Figure 3.1 and its subtree $S = T \setminus \{h\}$: $H_g(S)$ consists of a single history, $H_g(T)$ contains two different histories.

However, we can clearly build an injective order morphism from $\mathcal{O}_\mathcal{S}$ to $\mathcal{O}_\mathcal{T}$, sending $H_s(S)$ to $H_s(T)$, because if $H_s(S) \subset H_{s'}(S)$, then $H_s(T) \subset H_{s'}(T)$, and if $H_s(S) \neq H_{s'}(S)$, then $H_s(T) \neq H_{s'}(T)$.

Proposition 3.2.21. *Let $\mathcal{T}$ be a tree, and $\mathcal{S}$ a totally branching subtree of $\mathcal{T}$ $\subset$-maximal with respect to this property. Then $\mathcal{O}_\mathcal{S} \cong \mathcal{O}_\mathcal{T}$.*

Proof. The map considered in the remark above, namely $H_s(S) \mapsto H_s(T)$, is an injective morphism, so it is an isomorphism if it is surjective. Therefore, we must show that for every $t \in T$ there exists $s \in S$ such that $H_s(T) = H_t(T)$. Assume by *reductio* that such an s does not exist. Then for every $s \in S$, $H_s(T) \neq H_t(T)$, but then $(S \cup \{t\}, <)$ is a totally branching subtree of $\mathcal{T}$ and it strictly contains $\mathcal{S}$, which contradicts its maximality. □

Theorem 3.2.22. *Let $\mathcal{T}$ be a tree. Then the condensation $([T], <)$ is isomorphic to $(T_{\mathcal{O}_\mathcal{T}}, \supset)$, defined in Section 3.2.2.*

Proof. We just need to show that $(T_{\mathcal{O}_\mathcal{T}}, \supset)$ is isomorphic to a totally branching subtree of $\mathcal{T}$ $\subset$-maximal with respect to this property. If we prove this claim, we can apply Corollary 3.2.20 and complete the proof.

Let $\mathcal{S}$ be a totally branching subtree of $\mathcal{T}$ $\subset$-maximal with respect to this property. Then, by Proposition 3.2.8, $\mathcal{S} \cong \mathcal{S}_{\mathcal{O}_\mathcal{S}}$. Moreover, by Proposition 3.2.21, $\mathcal{O}_\mathcal{S} \cong \mathcal{O}_\mathcal{T}$, thus $\mathcal{S}_{\mathcal{O}_\mathcal{S}} \cong \mathcal{T}_{\mathcal{O}_\mathcal{T}}$. Hence, by transitivity, we conclude that $\mathcal{S} \cong \mathcal{T}_{\mathcal{O}_\mathcal{T}}$. □

As a consequence of this theorem, the map $\alpha \circ \nu : T \mapsto T_{\mathcal{O}_\mathcal{T}}$ can be re-defined by using the condensation map $[\,\cdot\,] : T \mapsto [T]$ for every tree $\mathcal{T}$, up to isomorphisms.

As announced above, we now give a direct and simpler proof of Theorem 3.2.22.

Proof 2 of Theorem 3.2.22. Let $f : [T] \to \mathcal{T}_{\mathcal{O}_\mathcal{T}}$ the map defined by $[t] \mapsto H_t$. It is an isomorphism. To prove this claim, we must prove that it is a well-defined bijective morphism (which reverses the order).

- Assume that $[t] = [s]$, and consider $h \in H_t$. Then $h \cap [t] = [t] = [s] \ni s$, so $h \in H_s$. Thus, $H_t \subseteq H_s$. With the same argument we prove the other inclusion, and conclude that f is well-defined.
- Assume that $[t] \neq [s]$. Then, by Proposition 3.2.13 ($\neg(2) \Leftrightarrow \neg(3)$), there exists $h \in H_t$, $h \notin H_s$, or viceversa. In particular, $H_t \neq H_s$, and we can conclude that f is injective.
- Clearly, every element of $\mathcal{O}_\mathcal{T}$ is of the form H_t, hence $f([t]) = H_t$ for some $[t] \in [T]$. So f is surjective.
- If $[t] < [s]$, then $t < s$, so $H_t \supset H_s$. Moreover, if $H_t \supset H_s$, then $t < s$ and there exists $h \in H_t \setminus H_s$. Then $t \in h$ and $s \notin h$. Hence, $[t] \neq [s]$, otherwise $h \cap [t] = [t] \ni s$, which is a contradiction. Then $t < s$ and $[t] \neq [s]$, so, by Proposition 3.2.11 we conclude that $[t] < [s]$. Therefore, both f and f^{-1} are morphisms. □

3.3 Topological validity

The results presented in Section 3.2.2 show that presentations of non-Archimedean spaces correspond to totally branching bundled trees in a natural way. This suggests that we can define a new notion of Ockhamist validity based on our topological construction:

Definition 3.3.1. An Ockhamist formula ϕ is *topologically valid* if for every presentation of non-Archimedean space $(X, \mathcal{O})$, ϕ is valid with respect to the associated bundled tree, which means that $(\mathcal{T}_\mathcal{O}, \mathcal{B}_X) \vDash \phi$.

An Ockhamist formula ϕ is *topologically weakly valid* if, for every presentation of non-Archimedean space $(X, \mathcal{O})$, ϕ is valid with respect to the associated complete bundled tree, which means that $\big(\mathcal{T}_{\mathcal{O}_\mathcal{T}}, H(T_{\mathcal{O}_\mathcal{T}})\big) \vDash \phi$.

Clearly, from the definition above, we have that validity with respect to bundled trees implies topological validity. We are going to show that the converse implication holds too, which will be a proof of the following:

Theorem 3.3.1. *Validity with respect to bundled trees coincides with topological validity.*

The proof of the claim above is divided into three steps:

- construction of the *branching extension* of a bundled tree;
- proof of two properties of the branching extension: it is totally branching and it preserves the completeness of the bundle;
- proof that satisfiability with respect to bundled trees implies satisfiability with respect to totally branching bundled trees.

Once we have completed these three steps, we can conclude the proof because, from what we proved in Proposition 3.2.8, totally branching trees correspond to non-Archimedean spaces.

We will adopt two standard notations in the next construction: 2 will stand for $\{0,1\}$, and ${}^{I}J$ is the set of all functions from I to J.

3.3.1 Branching extension

Let $(\mathcal{T},\mathcal{B})$ be a bundled tree. In order to simplify our notation, given $t \in T$, we will denote by $\hat{t}$ the past P_t plus t, so $\hat{t} = \{u \in T \mid u \leq t\}$.

We start our construction of the *branching extension* defining the set T^* made with every map from the past of any moment to $\{0,1\}$:

$$T^* = \bigcup\{{}^{\hat{t}}2 \mid t \in T\} \tag{3.1}$$

It is clear that every moment $t \in T$ has $2^{|\hat{t}|}$ representatives in T^*.

Lemma 3.3.2. *T^* is a tree with the strict subset relation*[13]*: $\mathcal{T}^* = (T^*, \subset)$.*

Proof. The strict subset relation $\subset$ is irreflexive and transitive, so we just need to prove the tree condition. Let $t_1, t_2 \in T$ and consider $f \in {}^{\hat{t_1}}2, g \in {}^{\hat{t_2}}2$. $f \subset g$ if and only if $t_1 < t_2$ and f is the restriction[14] $g_{\restriction_{\hat{t_1}}}$ of g to $\hat{t_1}$. Thus, if h is another element of T^* in the past of g (i.e. $h \subset g$), with $h \in {}^{\hat{t_3}}2$, we have that $f = h$, $f \subset h$, or $h \subset f$. The relation between f and h depends on the mutual relation between t_1 and t_3, which are surely compatible since they both belong to the past of t_2. □

[13] We see a function $f : A \to B$ as a subset of the cartesian product $A \times B$, so the subset relation between functions is well defined.

[14] Given $f : A \to B$ and $A' \subseteq A$, the *restriction of f to A'* is $f_{\restriction_{A'}} : A' \to B$ defined by $f_{\restriction_{A'}}(a) = f(a)$ for every $a \in A'$, hence $f_{\restriction_{A'}} = \{(a, f(a)) \mid a \in A'\}$.

Now, given a history h of $\mathcal{T}$, and any $\chi \in {}^h2$, we set

$$\chi^* = \{\chi_{\restriction_{\hat{t}}} \mid t \in h\}. \tag{3.2}$$

Lemma 3.3.3. *The set χ^* is a history on $\mathcal{T}^*$.*

Proof. The set χ^* is totally ordered by $\supset$. In fact, its elements are restrictions of a function on a subset of a history, hence they are totally ordered as functions, since $t_1 < t_2$ implies $\hat{t}_1 < \hat{t}_2$. Then, if $f \in T^*$ such that $\chi^* \cup \{f\}$ is totally ordered by $\supset$, and $f \in {}^{\hat{t}}2$ for a given t, we have that $t \smile s$ for all $s \in h$. Then, by maximality of h, $t \in h$, hence $f \in \chi^*$. So χ^* is a totally ordered subset of T^* which is maximal with respect to the inclusion, hence it is a history of $\mathcal{T}^*$. □

Now, starting from the bundle $\mathcal{B}$, we set

$$\mathcal{B}^* = \{\chi^* \mid \chi \in {}^h2, h \in \mathcal{B}\}. \tag{3.3}$$

Lemma 3.3.4. *The set $\mathcal{B}^*$ is a bundle on the tree $\mathcal{T}^*$.*

Proof. Let f be an element of T^*, in particular $f \in {}^{\hat{t}}2$ for a suitable $t \in T$. We can consider a history h of $\mathcal{T}$, $h \in \mathcal{B}$ and h containing t, since $\mathcal{B}$ is a bundle on $\mathcal{T}$. Then $f = \chi_{\restriction_{\hat{t}}}$, so $f \in \chi^* \in \mathcal{B}^*$. Thus, for every moment f of T^* there is a history χ^* of $\mathcal{B}^*$ passing through it, hence $\mathcal{B}^*$ is a bundle. □

So, with definitions (3.1), (3.2) and (3.3), we have constructed the *branching extension* $(\mathcal{T}^*, \mathcal{B}^*)$ of a given bundled tree $(\mathcal{T}, \mathcal{B})$.

3.3.2 Properties of the branching extension

Now we can show that for every tree $\mathcal{T}$ the branching extension $\mathcal{T}^*$ is totally branching and it contains an isomorphic copy of $\mathcal{T}$: the name of this new object is explained with these properties.

Proposition 3.3.5. *Let $\mathcal{T}$ be a tree. Then $(T^*, \subset)$ is totally branching. Moreover, there exists an injective order-morphism from $\mathcal{T}$ into $\mathcal{T}^*$.*

Proof. Let f, g be moments of T^*. If $f \not\smile_\subset g$, then clearly $H_f(T^*) \neq H_g(T^*)$. If $f \smile_\subset g$, without loss of generality we may assume that $f \subset g$. Let $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ be the respective domains, and $t \in \mathrm{dom}(g) \smallsetminus \mathrm{dom}(f)$. We define a new $g' : \mathrm{dom}(g) \to 2$ such that $g'(u) = g(u)$ for every $u \in P_t$, and $g'(t) = 1 - g(t)$. Then g and g' are $\subset$-uncomparable, and $f \subset g'$. Hence, $g' \in H_f(T^*) \smallsetminus H_g(T^*)$. Thus, $\mathcal{T}^*$ is totally branching.

Finally, we can construct an injective order morphism $\phi_0 : T \to T^*$ with the position $\phi_0 : t \mapsto f_t^0$, defining $f_t^0 \in {}^{\hat{t}}2$ as $f_t^0(u) = 0$ for every $u \in \hat{t}$. It is an order morphism, since $t < t'$ implies $f_t^0 = (f_{t'}^0)_{\restriction_{\hat{t}}}$. Moreover, ϕ_0 is injective for cardinality reasons. □

Finally, we can prove that the branching extension operator respects the completeness of bundles:

Proposition 3.3.6. *Let $(\mathcal{T}, \mathcal{B})$ be a bundled tree. If $(\mathcal{T}, \mathcal{B})$ is a complete bundled tree, then its branching extension $(\mathcal{T}^*, \mathcal{B}^*)$ is also a complete bundled tree.*

Proof. We need to prove that for every history $h^* \in H(T^*)$ there exists $h \in H(T)$ and $\chi \in {}^h2$ such that $h^* = \chi^*$ defined in (3.2). This means that every history of $\mathcal{T}^*$ comes from a history of $\mathcal{T}$. Hence, if $\mathcal{B}$ is complete, $\mathcal{B}^*$ is complete too.

Consider $h^* \in H(T^*)$. It is a set of functions totally ordered by $\subset$, so $\bigcup h^*$ is a function defined on a subset of T onto 2. We set $\chi = \bigcup h^*$, and $D = \mathrm{dom}(\chi) \subseteq T$ (hence $\chi \in {}^D2$, clearly). If we show that D is a history of $\mathcal{T}$, we have that $\chi^* = \{\chi_{\restriction_{\hat{t}}} \mid t \in D\} = h^*$, by definition of h^* , and this concludes the proof.

D is totally ordered by $<$, because it is the union of a collection of linear subsets of T, one included in the another. Moreover, if $u \in D$ and $v < u$, $v \in D$: in fact, by maximality of h^*, if $f \in h^*$ and $u \in \mathrm{dom}(f)$, then $f_{\restriction_{\hat{u}}} \in h$, so every $v < u$ belongs to the domain of some function in h^*, hence it belongs to D. Now, assume as a *reductio* that D is not $<$-maximal in T, which means that there exists $t \notin D$ such that $\{t\} \cup D$ is totally ordered. Then, by what we proved above, $u < t$ for every $u \in D$, otherwise t would belong to D. Thus, we can consider a function $f \in {}^{\hat{t}}2$ that extends χ properly. Then $f \notin h^*$, but $g \subseteq f$ for all $g \in h^*$, which contradicts the maximality of h^*. □

Gathering all the results of the previous section together, we prove the following:

Theorem 3.3.7. *Let $(\mathcal{T}, \mathcal{B})$ be a bundled tree. Then its branching extension $(T^*, \mathcal{B}^*)$ is a totally branching bundled tree, and it is complete if and only if $(\mathcal{T}, \mathcal{B})$ is complete.*

We end this section with a tangible example of the action of the *branching extension* operator.

On the top of Figure 3.4 we have a finite and not totally branching tree $\mathcal{T}$, and below its branching extension $\mathcal{T}^*$. It is the union of two disjoint connected trees, since the root a of $\mathcal{T}$ is mapped into two non-comparable

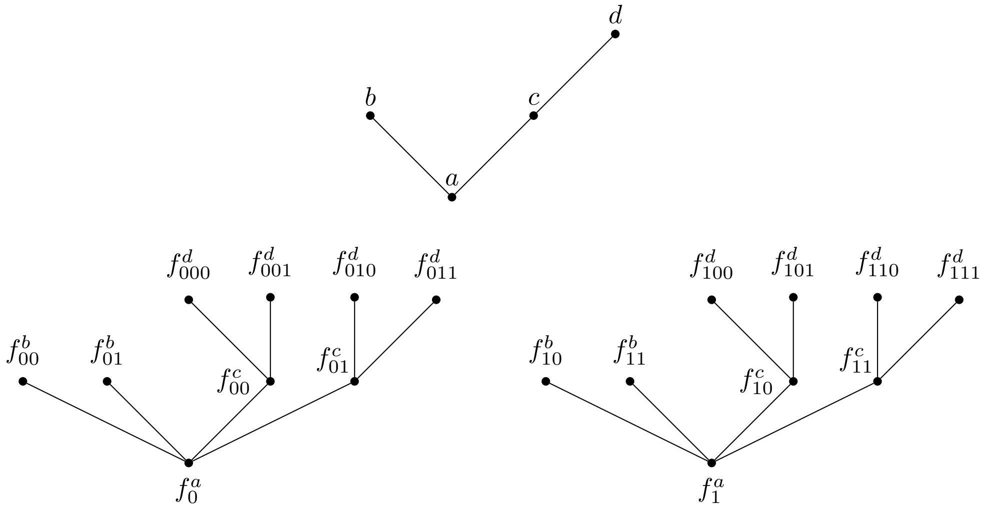


Figure 3.4: A tree $\mathcal{T}$ and its branching extension $\mathcal{T}^*$.

functions: $f_0 : a \mapsto 0$ and $f_1 : a \mapsto 1$. Nodes in $\mathcal{T}^*$ have the sequence of their values on their subscript, and the correspondent moment in the starting tree $\mathcal{T}$ on their superscript.

3.3.3 Satisfiability in the branching extension

Now we need to prove that satisfiability in a bundled tree $\mathcal{T}_\mathcal{B} = (\mathcal{T}, \mathcal{B})$ implies satisfiability in its branching extension $(\mathcal{T}^*, \mathcal{B}^*)$. In order to simplify the notation, we define the function $\varphi : T^* \to T$ by

$$\varphi(f) = \max(\mathrm{dom}(f)), \tag{3.4}$$

so, if $f \in {}^{\hat{t}}2$, then $\varphi(f) = t$. We denote by φ also the natural extension of this function to a map from $\mathcal{B}^*$ to $\mathcal{B}$, which is

$$\varphi(\chi^*) = \{\varphi(f) \mid f \in \chi^*\}. \tag{3.5}$$

Now, given a bundled tree valuation V on $\tilde{T}_\mathcal{B} = \{\langle t, h\rangle \mid t \in h \in \mathcal{B}\}$ (see Definition 2.3.3), we define a new valuation V^* on $\tilde{T}^*_{\mathcal{B}^*} = \{\langle f, \chi^*\rangle \mid f \in \chi^* \in \mathcal{B}^*\}$ by setting, for every propositional variable p,

$$V^*(p) = \big\{ \langle f, \chi^*\rangle \in \tilde{T}^*_{\mathcal{B}^*} \mid \langle \varphi(f), \varphi(\chi^*)\rangle \in V(p)\big\}, \tag{3.6}$$

and by extending the valuation to any compound formula by means of the rules of Definition 2.3.3.

Proposition 3.3.8. *Let* $(\mathcal{T}^*, \mathcal{B}^*)$ *be the branching extension of the bundled tree* $(\mathcal{T}, \mathcal{B})$*, and let* φ *be the function defined by* (3.4) *and* (3.5)*. Then, for every valuation* V *on* $\tilde{T}_{\mathcal{B}}$*, and for every Ockhamist formula* A*, and every pair* $\langle f, \chi^* \rangle \in \tilde{T}^*_{\mathcal{B}^*}$*, we have*

$$\langle f, \chi^* \rangle \in V^*(A) \textit{ if and only if } \langle \varphi(f), \varphi(\chi^*) \rangle \in V(A) \tag{3.7}$$

where V^* *is defined by* (3.6)*.*

Proof. We now prove the statement with an induction on the complexity of the formula A [15].

- If A is a propositional variable p_0, we have that $\langle f, \chi^* \rangle \in V^*(p_0)$ if and only if $\langle \varphi(f), \varphi(\chi^*) \rangle \in V(p_0)$ by definition of V^*, (3.6). So, we can inductively assume that the thesis holds for every subformula of A. We need to analyse only the cases in which A is obtained with the application of $\neg$, $\wedge$, F, P and $\Diamond$, because $\vee, \rightarrow$, $\Box$, G and H are defined connectives and operators.
- Assume that A is $\neg B$. Then $\langle f, \chi^* \rangle \in V^*(\neg B)$ if and only if $\langle f, \chi^* \rangle \notin V^*(B)$, if and only if $\langle \varphi(f), \varphi(\chi^*) \rangle \notin V(B)$, and finally it is true if and only if $\langle \varphi(f), \varphi(\chi^*) \rangle \in V(\neg B)$.
- Assume that A is $B \wedge C$. Then $\langle f, \chi^* \rangle \in V^*(B \wedge C)$ if and only if $\langle f, \chi^* \rangle \in V^*(B) \cap V^*(C)$, if and only if $\langle \varphi(f), \varphi(\chi^*) \rangle \in V(B) \cap V(C)$, and finally it is true if and only if $\langle \varphi(f), \varphi(\chi^*) \rangle \in V(B \wedge C)$.
- Assume that A is FB. Let $\langle f, \chi^* \rangle \in \tilde{T}^*_{\mathcal{B}^*}$, where χ^* comes from χ by means of (3.2), and assume that $f \in {}^{\hat{t}}2$ and $\chi \in {}^{h}2$ (with $h \in \mathcal{B}$), so we have $\varphi(f) = t$, $\varphi(\chi^*) = h$, $t \in h$, and $f \subset \chi$. Recall that $f \in \chi^*$ if and only if $f = \chi\restriction_{\varphi(f)}$, which is equivalent to $f \subset \chi$.
  Then, statements from (i_F) to (viii_F) below are equivalent to one another: some equivalences are justified below.

  (i_F) $\langle f, \chi^* \rangle \in V^*(FB)$;

  (ii_F) there exists a $g \supset f$ such that $\langle g, \chi^* \rangle \in V^*(B)$;

  (iii_F) by inductive hypothesis, there exists $g \supset f$ such that $\langle \varphi(g), \varphi(\chi^*) \rangle \in V(B)$;

  (iv_F) there exist $u \in h$ with $u > t$, and $g \in {}^{\hat{u}}2$ such that $g \in \chi^*$, $g\restriction_{\hat{t}} = f$ and $\langle \varphi(g), \varphi(\chi^*) \rangle \in V(B)$;

  (v_F) by definition of φ, there exist $u \in h$ with $u > t$, and $g \in {}^{\hat{u}}2$ such that $g \in \chi^*$, $g\restriction_{\hat{t}} = f$ and $\langle u, h \rangle \in V(B)$;

[15] In this proof we use upper-case letters for formulae in order to avoid the usage of ϕ and its variation φ in the same context.

(vi$_F$) there is $u \in h$ with $u > t$ such that $\langle u, h\rangle \in V(B)$;

(vii$_F$) $\langle t, h\rangle \in V(FB)$;

(viii$_F$) $\langle \varphi(f), \chi^*\rangle \in V(FB)$.

The equivalences (i$_F$) $\Leftrightarrow$ (ii$_F$) and (vi$_F$) $\Leftrightarrow$ (vii$_F$) hold by Definition 2.3.3, $F\phi$ case. The equivalence (iii$_F$) $\Leftrightarrow$ (iv$_F$) is the definition of $\supset$ between functions (see footnote 14). The implication (vi$_F$) $\Rightarrow$ (v$_F$) follows by choosing $g = \chi_{\restriction \hat{u}}$.

Clearly, the equivalence (i$_F$) $\Leftrightarrow$ (viii$_F$) completes the proof for the inductive case $A = FB$.

- Assume that A is PB and let f, χ, χ^*, t, h as above. Then, statements from (i$_P$) to (viii$_P$) below are equivalent to one another: some equivalences are justified below.

(i$_P$) $\langle f, \chi^*\rangle \in V^*(PB)$;

(ii$_P$) there exists a $g \subset f$ such that $\langle g, \chi^*\rangle \in V^*(B)$;

(iii$_P$) by inductive hypothesis, there exists $g \subset f$ such that $\langle \varphi(g), \varphi(\chi^*)\rangle \in V(B)$;

(iv$_P$) there are $u \in h$ with $u < t$, and $g \in {}^{\hat{u}}2$ such that $g \in \chi^*$, $f_{\restriction \hat{u}} = g$ and $\langle \varphi(g), \varphi(\chi^*)\rangle \in V(B)$;

(v$_P$) by definition of φ, there are $u \in h$ with $u < t$, and $g \in {}^{\hat{u}}2$ such that $g \in \chi^*$, $f_{\restriction \hat{u}} = g$ and $\langle u, h\rangle \in V(B)$;

(vi$_P$) there is $u \in h$ with $u < t$ such that $\langle u, h\rangle \in V(B)$;

(vii$_P$) $\langle t, h\rangle \in V(PB)$;

(viii$_P$) $\langle \varphi(f), \chi^*\rangle \in V(PB)$.

The equivalences (i$_P$) $\Leftrightarrow$ (ii$_P$) and (vi$_P$) $\Leftrightarrow$ (vii$_P$) hold by Definition 2.3.3, $P\phi$ case. The equivalence (iii$_P$) $\Leftrightarrow$ (iv$_P$) is the definition of $\supset$ between functions (see footnote 14). The implication (vi$_P$) $\Rightarrow$ (v$_P$) follows by choosing $g = \chi_{\restriction \hat{u}}$.

Clearly, the equivalence (i$_P$) $\Leftrightarrow$ (viii$_P$) completes the proof for the inductive case $A = PB$.

- Assume that A is $\Diamond B$ and let f, χ, χ^*, t, h as above. Then, statements from (i$_\Diamond$) to (viii$_\Diamond$) below are equivalent to one another: some equivalences are justified below.

(i$_\Diamond$) $\langle f, \chi^*\rangle \in V^*(\Diamond B)$;

(ii$_\Diamond$) there exists a $\xi^* \in \mathcal{B}^*$ with $f \in \xi^*$ such that $\langle f, \xi^*\rangle \in V^*(B)$;

(iii$_\Diamond$) by inductive hypothesis, there exists $\xi^* \in \mathcal{B}^*$ with $f \in \xi^*$ such that $\langle \varphi(f), \varphi(\xi^*) \rangle \in V(B)$;

(iv$_\Diamond$) there are $h' \in \mathcal{B}$ with $t \in h'$, and $\xi \in {}^{h'}2$ such that $\xi \supset f$ and $\langle \varphi(f), \varphi(\xi^*) \rangle \in V(B)$;

(v$_\Diamond$) by definition of φ, there are $h' \in \mathcal{B}$ with $t \in h'$, and $\xi \in {}^{h'}2$ such that $\xi\restriction_{\hat{t}} = f$ and $\langle t, h' \rangle \in V(B)$;

(vi$_\Diamond$) there is $h' \in \mathcal{B}$ with $t \in h'$ such that $\langle t, h' \rangle \in V(B)$;

(vii$_\Diamond$) $\langle t, h \rangle \in V(\Diamond B)$;

(viii$_\Diamond$) $\langle \varphi(f), \chi^* \rangle \in V(\Diamond B)$.

The equivalences (i$_\Diamond$) $\Leftrightarrow$ (ii$_\Diamond$) and (vi$_\Diamond$) $\Leftrightarrow$ (vii$_\Diamond$) hold by Definition 2.3.3, $\Diamond\phi$ case (2.6*). The equivalence (iii$_\Diamond$) $\Leftrightarrow$ (iv$_\Diamond$) is the Definition 3.2 of χ^* applied to the function ξ. The implication (vi$_\Diamond$) $\Rightarrow$ (v$_\Diamond$) holds because we can always extend f to a $\xi \in {}^{h'}2$.

Clearly, the equivalence (i$_\Diamond$) $\Leftrightarrow$ (viii$_\Diamond$) completes the proof for the last inductive case $A = \Diamond B$. □

This result concludes the proof of the equivalence between bundled tree semantics and topological semantics for Ockhamist logics (Theorem 3.3.1). The weak topological validity part of that Theorem is a consequence of the second part of Proposition 3.3.6.

3.3.4 Concluding remarks

In this chapter we have shown that the tree semantics for branching-time logics (in particular the Ockhamist semantics) can be considered from a topological perspective, and we have also shown that totally branching bundled trees correspond to presentations of non-Archimedean spaces in a natural way. Moreover, this last section showed that the restriction to totally branching trees does not cause any loss of generality.

In the conclusion of [29] (the main source for the contents of this chapter), the authors state what follows: "we think that topology really offers a deeper insight into the structure of trees and bundled trees". This sentence was a starting point for our work: we are going to develop this idea in the next chapter, in which we will analyse some properties of trees from this rather new topological perspective.

Chapter 4

Topological characterization of properties of trees

In this chapter, we analyse some properties of trees from a topological perspective. We will also connect natural topological properties with some (possibly strange or unnatural) properties of trees.

We will adopt the notation introduced in Sections 3.2.1 and 3.2.2, which we will now recall here:

- given a tree $\mathcal{T} = (T, <)$, the non-Archimedean space associated is $\big(H(T), \mathcal{O}_{\mathcal{T}}\big)$;
- given a bundled tree $\mathcal{T}_{\mathcal{B}} = (\mathcal{T}, \mathcal{B})$, the non-Archimedean space associated is $(\mathcal{B}, \mathcal{O}_{\mathcal{B}})$;
- given a non-Archimedean topological space $(X, \mathcal{O})$, the associated tree is $\mathcal{T}_{\mathcal{O}} = (\mathcal{O}, \supset)$;
- given a non-Archimedean topological space $(X, \mathcal{O})$, the associated bundled tree is $(\mathcal{T}_{\mathcal{O}}, \mathcal{B}_X)$.

Moreover, we will occasionally use the condensed tree $[\mathcal{T}]$, whose definition and properties are described in Section 3.2.4.

4.1 Preliminary notions on ordinal numbers

Definition 4.1.1. A partially ordered set (poset) is a *well-ordered* set if every non-empty subset has a least element.

Remark 4.1.1. Every well-ordered poset is totally ordered: if x, y are different elements of the poset, $\{x, y\}$ has a least element, hence either $x < y$ or $y < x$.

Definition 4.1.2. Given two posets $(X, <)$, $(Y, \prec)$, $f : X \to Y$ is an *order-preserving* map if for all $x_1, x_2 \in X$, $x_1 < x_2$ implies $f(x_1) \prec f(x_2)$. If f is an order-preserving bijection, it is called *order-isomorphism*. Moreover, if such a map exists, $(X, <)$ and $(Y, \prec)$ are *of the same order-type*, or they are *order-isomorphic*.

Having the same order-type clearly is an equivalence relation, and some definitions of ordinal numbers[1] involve exactly this idea: they are equivalence classes of well-ordered sets. This definition, however, is abandoned in Zermelo-Fraenkel Set Theory, because these equivalence classes are too large to form a set[2]. J. Von Neumann introduced another approach to this problem in his *Zur Einführung der transfiniten Zahlen* in 1923, which leads to the following definition:

Definition 4.1.3. A set S is an *ordinal number* (often shortened as *ordinal*) if it is well-ordered with respect to set membership relation $\in$, and every element of S is also a subset of S.

For instance, we can construct some ordinals, starting from 0: $0 = \{\} = \emptyset$, $1 = \{\emptyset\} = \{0\}$, $2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$, ..., $\omega = \{0, 1, 2, \dots\}$ [3], $\omega + 1 = \{0, 1, 2, \dots, \omega\}$, and so on. In general, for every ordinal α, we set $\alpha + 1 = \alpha \cup \{\alpha\}$: $\alpha + 1$ is called the *successor* of α.

In this way, we define a unique representative of each equivalence class of well-ordered sets up to order-isomorphism (this fact can be shown by transfinite induction).

An equivalent modern definition of ordinal number involves the notion of $\in$-transitive set:

Definition 4.1.4. A set S is called *$\in$-transitive* if for all x, y, $x \in y \in S$ implies $x \in S$.

Definition 4.1.5. A set S is an *ordinal number* (often shortened as *ordinal*) if it is $\in$-transitive and each element of S is $\in$-transitive. We call **Ord** the class of all ordinals.

Remark 4.1.2. We do not prove the equivalence of the Definitions 4.1.3 and 4.1.5. Nonetheless, we want to underline the fact that the condition "each

[1] For example, the one that can be found in *Principia Mathematica* written by Alfred North Whitehead and Bertrand Russell and published in 1910, 1912, and 1913.

[2] The interested reader may deepen these topics in the Chapter *Zermelo-Fraenkel System and von Neumann Ordinals* in [8].

[3] This is the first infinite ordinal: it is order-isomorphic to the set of natural numbers with the usual order $<$. It can be obtained as the union of all finite ordinals.

element of S is $\in$-transitive" and the condition "every element of S is also a subset of S" are equivalent.

The proof of the following results can be found in any textbook on set-theory, for example [22], Chapter 2.

Proposition 4.1.1. *If S is an ordinal and $A \in S$, then A is an ordinal too. Moreover, every transitive set of ordinals is an ordinal.*

Proposition 4.1.2. *If S is a set of ordinals, $\bigcup S$ is an ordinal.*

This proposition shows that the "successor" operation is not the only way for producing new ordinals. If an ordinal is not a successor, it is called *limit* ordinal.

Proposition 4.1.3. *The class of ordinals* **Ord** *is totally ordered by $\in$.*

Proposition 4.1.4. *If A is a non-empty set of ordinals, then $\bigcap A$ is an ordinal, and in fact $\bigcap A \in A$. Hence, any set of ordinals is well-ordered, with its intersection as least element.*

The proof of this fact can be found in [22], Theorem 9.10 at page 72. This result leads us to a Corollary which will be used later in this chapter.

Corollary 4.1.5. *Every decreasing chain of ordinal numbers is finite.*

Proof. A chain of ordinals is a set of ordinals, hence it is well-ordered, by the previous proposition. So, it has a least element. Moreover, it is reached in a finite number of downward steps. Otherwise, the chain deprived of that least element (and of its immediate successors until a limit step is reached) would be a subset of a well-ordered set without a least element. □

We conclude this section with two definitions that establish a link between ordinal numbers and well-founded trees.

Definition 4.1.6. Given a well-founded tree $\mathcal{T}$ (i.e. a tree with at least a minimal element), we denote the set of the roots of the tree with $T(0)$. Moreover, given an ordinal α, assuming that $T(\beta)$ has been defined for all $\beta < \alpha$, $T(\alpha)$ is the set of minimal members of $T \setminus \bigcup_{\beta<\alpha} T(\beta)$. Then $T(\alpha)$ is the *α-level* of the tree.

Definition 4.1.7. The *height* of a well-founded tree $\mathcal{T}$ is the least ordinal number α, such that $T(\alpha) = \emptyset$.

4.2 Linear trees and properly branching trees

We start with two simple examples in order to point out what the approach we want to use in the following is.

Proposition 4.2.1. *A tree is linear if and only if $\tau_{\mathcal{T}}$ is the trivial topology*[4] *on $H(T)$.*

Proof. Let $\mathcal{T}$ be a linear tree. It contains a single history h, which by maximality coincides with the whole tree. Then $H(T) = \{h\}$, and $H_t = h$ for every $t \in T$. Then, the base $\mathcal{O} = \{H_t \mid t \in T\} = \{\{h\}\} = \{H(T)\}$, so the generated topology is trivial.

Conversely, if $H(T)$ is equipped with the trivial topology, the only possible base is $\mathcal{O} = \{H(T)\}$ (because we have assumed that no base contains the empty set). However, the base is the collection of *all* the H_t's, hence, given t, s in T, $H_t = H_s = H(T)$. Thus, there can not be $t, s \in T$ with $t \not\smile s$, because any two moments belong to the same histories. So the tree is linear. □

Remark 4.2.1. Using the condensation, we have that a tree $\mathcal{T}$ is linear if and only if the condensed tree $[\mathcal{T}]$ consists of just one point: since the unique maximal bridge of $\mathcal{T}$ is the tree $\mathcal{T}$ itself, it is clear that it will condense in a single point.

Corollary 4.2.2. *A tree is properly branching or not connected (i.e. there exist moments $t, s \in T$ such that $t \not\smile s$) if and only if the base $\mathcal{O}_{\mathcal{T}}$ of the non-Archimedean space contains at least two different elements.*

Proof. This is essentially the contrapositive of the previous statement. However, we can give a direct proof for this statement too.

Consider a properly branching or not connected tree, and let t, s be non-comparable moments. Then $H_t \neq H_s$, so the topology $\tau_{\mathcal{T}}$ contains at least two different open sets.

Conversely, if $\tau_{\mathcal{T}}$ contains two different open sets, they are generated by different open sets of the base. Then, there exist t, s such that $H_t \neq H_s$, so there exists $h \in H_t$ such that $h \notin H_s$ (or vice-versa). Hence, there exists $t' \in h$ with $t' \not\smile s$. Thus, $\mathcal{T}$ is properly branching, or not connected. □

Now we can extend this result by characterizing trees *with a first ramification in a moment* t_0. Having a first ramification in t_0 means that for all $t_1, t_2 < t_0$, $t_1 \smile t_2$, and that for all $t_3 > t_0$ there exists t_4 such that $t_3 \not\smile t_4$. In other words, t_0 is the maximum of a totally ordered starting sequence in

[4]The *trivial* or *indiscrete* topology for a set X is $\tau = \{\emptyset, X\}$: it is the minimal topology on every set X.

the tree, which is a chain contained in the tree whose elements are $\leq$ than every other element of the tree.

Proposition 4.2.3. *If a connected tree $\mathcal{T}$ has a first ramification at a moment t_0, then the base $\mathcal{O}_{\mathcal{T}}$ of the non-Archimedean space contains the whole space and other two different elements.*

Proof. Since the tree firstly ramifies at t_0, then t_0 is the maximum of a totally ordered starting sequence in the tree. Then, the condensation of this sequence is the maximal bridge $[t_0]$. Moreover, using the notation of the definition above, $t_3 \not\leq t_4$ implies $[t_3] \not\leq [t_4]$. So the condensed tree $[\mathcal{T}]$ contains at least three different points, $[t_0]$, $[t_3]$, $[t_4]$. Because of the isomorphism of Theorem 3.2.22, they correspond to different elements of the base. Moreover, every history passes through t_0, so H_{t_0} coincides with the whole space. □

Remark 4.2.2. The converse is not true. As a counterexample, we can build a tree like the one shown in Figure 4.1, with three copies of the natural numbers $\mathbb{N}_l, \mathbb{N}_r, \mathbb{N}_d$ (left, right, down). The relations between them are easily understandable from the picture. It does not have a *first* ramification (even if it ramifies), but its condensation is single rooted and different from its root. Thus, the associated base contains the whole space and two different elements. The reader should notice that the tree of the example is *not* jointed (see Definition 4.8.1).

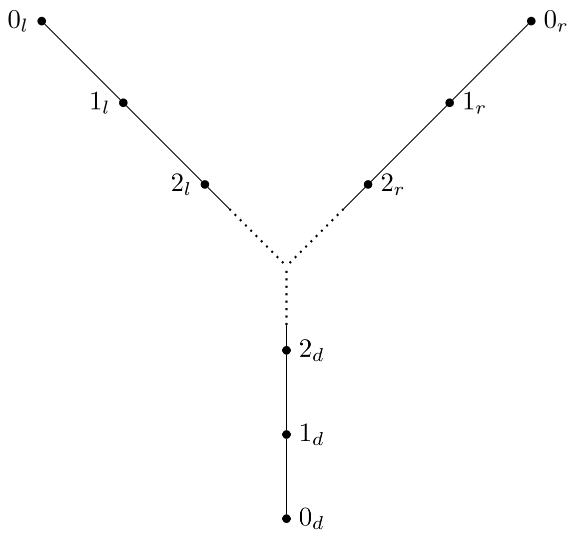


Figure 4.1: Example of a tree that ramifies without a *first* ramification.

4.3 Connected trees

Many authors define trees as (our) connected trees (see footnote on 15). So, it is interesting to characterize them from a topological perspective.

One might naively hope that a tree is connected if and only if the topological space $\big(H(T), \mathcal{O}_\mathcal{T}\big)$ is connected, but this is false. In fact, we proved that $\big(H(T), \mathcal{O}_\mathcal{T}\big)$ is non-Archimedean, and that every element of the base is clopen, so in general the space can not be connected (see Definition 3.1.11).

Proposition 4.3.1. *Let $\mathcal{T}$ be a tree. Then, it is connected if and only if the union of any two elements of the base $\mathcal{O}_\mathcal{T}$ is contained in another element of the base.*

Proof. Let $\mathcal{T}$ be a connected tree and let H_{t_1} and H_{t_2} be two elements of $\mathcal{O}_\mathcal{T}$. If $H_{t_1} \subseteq H_{t_2}$ or $H_{t_2} \subseteq H_{t_1}$, the thesis is trivially verified. Otherwise, $H_{t_1} \cap H_{t_2} = \emptyset$ and $t_1 \not\sim t_2$. Since $\mathcal{T}$ is connected, there exists s such that $s < t_1$, $s < t_2$, hence $H_{t_1} \subseteq H_s$, $H_{t_2} \subseteq H_s$. Thus, $H_{t_1} \cup H_{t_2} \subseteq H_s$.

Conversely, consider a tree such that the union of every pair of elements of the base $\mathcal{O}_\mathcal{T}$ is contained in another element of the base. Consider two uncomparable moments $t \not\sim t'$. By the hypothesis, there must be $s \in T$ such that $H_s \supseteq H_t \cup H_{t'}$, which implies that $H_s \supseteq H_t$, and $H_s \supseteq H_{t'}$, hence $s < t$, $s < t'$. Thus, $\mathcal{T}$ is connected. □

4.4 Minimal bundles

Proposition 4.4.1. *A subset K of $H(T)$ is a bundle in $\mathcal{T}$ if and only if K is dense in $\big(H(T), \mathcal{O}_\mathcal{T}\big)$.*

Proof. Assume that K is a bundle. Then, by definition (2.3.1), for every moment $t \in T, \exists h_t \in K$ such that $t \in h_t$. This implies that for every t, $h_t \in K \cap H_t$, hence $K \cap H_t \neq \emptyset$. Thus, every open set of $\mathcal{O}_\mathcal{T}$ intersects K, which means that K is dense.

Now, let K be a dense subset of $H(T)$. Then, for every t, $K \cap H_t \neq \emptyset$, so there exists $h \in H_t \cap K$. But $h \in H_t$ implies $t \in h$, hence the bundle condition holds true for K. □

Combining this proposition with the Remark 3.1.2 about isolated points, we obtain the following:

Corollary 4.4.2. *The set of isolated histories in $\big(H(T), \mathcal{O}_\mathcal{T}\big)$ is the set of all histories belonging to every bundle on $\mathcal{T}$.*

Proof. If h is isolated, then, by Remark 3.1.2, h belongs to every dense subset of $(H(T), \mathcal{O}_{\mathcal{T}})$. However by the previous proposition, dense subsets of $(H(T), \mathcal{O}_{\mathcal{T}})$ are bundles of $\mathcal{T}$. Hence, h belongs to every bundle. Moreover, if h is not isolated, then $H(T) \smallsetminus \{h\}$ is dense (again by Remark 3.1.2), hence it is a bundle. Thus, h does not belong to every bundle on $\mathcal{T}$. □

It is worth noticing that the set of isolated histories considered in the corollary above arises naturally from this topological perspective: on the contrary, the notion of minimal bundle may not be that immediate or "natural" in the algebraic context of bundled trees.

Once more, combining Proposition 4.4.1 with Proposition 3.1.4, we obtain directly the following:

Corollary 4.4.3. *For every bundle $\mathcal{B}$ on a tree $\mathcal{T}$, h is isolated in $\tau_{\mathcal{B}}$ if and only if it is isolated in $\tau_{\mathcal{T}}$.*

Now we can produce another characterization of a specific class of bundles, which comes naturally from this topological perspective, as before:

Proposition 4.4.4. *A bundle $\mathcal{B}$ on a tree $\mathcal{T}$ is minimal (with respect to $\subset$) if and only if τ_B is the discrete topology on $H(T)$.*

Proof. Consider $\mathcal{B}$ a minimal bundle and $h \in \mathcal{B}$. Then, by minimality, $\mathcal{B} \smallsetminus \{h\}$ is not a bundle, hence it is not dense (by Proposition 4.4.1), so there exists an open H_t (of the base $\mathcal{O}_{\mathcal{T}}$) such that $(\mathcal{B} \smallsetminus \{h\}) \cap H_t = \emptyset$. However, $\mathcal{B}$ is dense, so $\mathcal{B} \cap H_t \neq \emptyset$, hence $\mathcal{B} \cap H_t = \{h\}$. This implies that $\{h\}$ is a finite intersection of open subsets in τ_B, so it is open, thus h is isolated. As a consequence, every point of $\mathcal{B}$ is isolated, hence τ_B is the discrete topology.

Conversely, assume that the topology $\tau_{\mathcal{B}}$ generated by a bundle $\mathcal{B}$ is discrete. Then, every $h \in \mathcal{B}$ is isolated in $\tau_B\mathcal{B}$, hence in $\tau_{\mathcal{T}}$ (by Corollary 4.4.3). So, by Corollary 4.4.3, for every $h \in \mathcal{B}$, $\mathcal{B} \smallsetminus \{h\}$ is not a bundle, hence $\mathcal{B}$ is minimal. □

Gathering together all the above results, we obtain the following topological characterization for trees with a minimal bundle:

Corollary 4.4.5. *A tree $\mathcal{T}$ has a minimal bundle if and only if the set $\mathcal{I}$ of isolated points of $\big(H(T), \tau_{\mathcal{T}}\big)$ is dense. If this is the case, $\mathcal{I}$ is the unique minimal bundle for $\mathcal{T}$.*

We want to draw attention once again to the fact that this new perspective has led us to a new result: a purely algebraic characterization of trees with a minimal bundle may be difficult. On the contrary, this result arises in a very natural way in this topological context.

4.5 Finitely branching ω-trees

The aim of the next steps is to topologically characterize a subclass of finitely branching trees, which are flows of time that have a finite number of possible "immediate futures" for every moment. This section is borrowed from [29].

Definition 4.5.1. A tree $\mathcal{T}$ is an *ω-tree* if every history $h \in H(T)$ is isomorphic to the set of natural numbers.

For an ω-tree $\mathcal{T}$, and for every h history of $\mathcal{T}$, we denote by ϕ_h the isomorphism from ω to h. For any $h \in H(T)$, the moment $\phi_h(0)$ is a *root* of the tree: clearly, a tree is connected if and only if it is *single-rooted*, which means that it has a unique root.

For any given moment t of an ω-tree, and for every $h \in H_t$, we call the moment $\phi_h\big(\phi_h^{-1}(t)+1\big)$ the *immediate successor*[5] *of t along h.* We call the set $S(t) = \big\{\phi_h\big(\phi_h^{-1}(t)+1\big) \mid h \in H_t\big\}$ the *set of the immediate successors of t.*

Definition 4.5.2. An ω-tree $\mathcal{T}$ is *finitely branching* if, for every $t \in T$, the set $S(t)$ is finite.

Definition 4.5.3. We denote by T_n the subtree of $\mathcal{T}$ obtained by considering only the first n levels: formally, this set is obtained recursively by setting: $T_0 = \{\phi_h(0) \mid h \in H(T)\}$ (the set of roots in T) and $T_{n+1} = T_n \cup \bigcup_{t \in T_n} S(t)$.

It is easy to prove that an ω-tree is finitely branching if and only if every T_n is finite.

Theorem 4.5.2 below uses the König's tree lemma: since it is a fundamental result about trees and its proof is not so difficult, we will now state and prove it.

Theorem 4.5.1 (König's tree lemma)**.** *Let $\mathcal{T}$ be a finitely branching rooted tree with at least countably many moments. Then there exists an infinite history through $\mathcal{T}$.*

Proof. In order to build an infinite history h, we start from the root t_0. We have finitely many choices for the first successor of t_0 along h, and we select a moment t_1 with infinitely many successors. Such a moment exists because otherwise T would be a finite union of finite sets of moments, but it is infinite. Now we can repeat the process starting from t_1 and we can look for a moment t_2 at level 2 with infinitely many successors. Thus, repeating

[5] We will give the definition of immediate successor of a moment in an arbitrary tree in Section 4.6.

the argument for every level n, we have inductively constructed a history of countable height: the fact that it is a set is granted by the *Axiom of Choice*[6]. $\square$

Remark 4.5.1. Clearly, every finitely branching rooted tree of height ω satisfies the hypothesis of the König Lemma, since it has at least countably many moments.

Theorem 4.5.2. *Let $\mathcal{T}$ be a connected ω-tree. Then $\mathcal{T}$ is finitely branching if and only if $\big(H(T), \tau_{\mathcal{T}}\big)$ is compact.*

Proof. Assume by *reductio* that $H(T)$ is compact and that $S(t_0)$ has infinitely many elements for a certain $t_0 \in T$. For every $t \in T$, $H_t = \bigcup_{s \in S(t)} H_s$, and for every $S' \subsetneq S(t)$, $H_t \supsetneq \bigcup_{s \in S'} H_s$, because every history passing through t contains an immediate successor, and vice-versa. As observed after the Proposition 3.2.1, H_{t_0} is closed, hence compact (by Proposition 3.1.5). $\{H_s \mid s \in S(t_0)\}$ is a cover for H_{t_0} consisting of infinitely many open subsets, but every proper subfamily does not cover H_{t_0}, hence H_{t_0} is not compact, which is a contradiction.

Conversely, assume by *reductio* that $\mathcal{T}$ is finitely branching and that $H(T)$ is *not* compact. Since every cover of open subsets can be made of elements of a base $\mathcal{O}_{\mathcal{T}}$, the assumption that $H(T)$ is not compact is equivalent to the assumption that there exists an infinite set $S \subseteq T$ such that $H(T) = \bigcup_{s \in S} H_s$ and for every finite subset $S' \subseteq S$, $H(T) \supsetneq \bigcup_{s \in S'} H_s$.

Now, let S_n be the set $S \cap T_n$, the first $n+1$ levels of S. Since the tree is finitely-branching and connected ($\Rightarrow$ single-rooted), then T_n is finite, and so is S_n. Thus, $H(T) \supsetneq \bigcup_{s \in S_n} H_s$, then there is $h^* \notin \bigcup_{s \in S_n} H_s$. For every n, let $t_0 < t_1 < \ldots < t_n$ be the starting sequence of n moments of h^*: by construction, it has empty intersection with S. Now, consider $T' = \{t \in T \mid \text{if } s \leq t \text{ then } s \notin S\}$. Since for every n there exists a sequence $t_0 < t_1 < \ldots < t_n$ with empty intersection with S, T' has infinitely many moments and $(T', <)$ is a single-rooted subtree of $\mathcal{T}$. By *König's tree lemma*, T' contains an infinite history which does not intersect with S. Hence, $H(T) \neq \bigcup_{s \in S} H_s$, which is a contradiction. $\square$

[6] The relationship between the König Tree Lemma and the *Axiom of Choice* is an interesting topic. It can be shown that it is equivalent to the principle that every countable set of finite sets has a choice function, which is sometimes called the axiom of countable choice for finite sets. For further reading on this topic, see [33] or [11].

4.6 Well-founded trees

4.6.1 Definition and properties

In this section we will first consider three equivalent algebraic characterizations and some properties of well-founded trees. Then their topological characterization will be investigated. They are quite important: a great number of articles and applications regarding the structure of trees consider only well-founded trees from the beginning.

Definition 4.6.1 (Kellerman, [19]). A tree $\mathcal{T}$ is *well-founded* if every non-empty set of moments contains a minimal moment.

Proposition 4.6.1. *A tree is well-founded if and only if every history is well-ordered.*

Proof. A history is a particular totally ordered non-empty set of moments, so the right to left implication is immediate.

Conversely, suppose that $\mathcal{T}$ is not well-founded and let $\{a_i\}_{i \in \mathbb{N}}$ be an infinite strictly descending chain in $\mathcal{T}$. Then, by Remark 2.2.2, $\{a_i\}_{i \in \mathbb{N}}$ can be extended to a history which is not well ordered. □

Definition 4.6.2 (Nyikos, [23]). A tree $\mathcal{T}$ is *well-founded* if the past P_t of every moment t is well-ordered.

Proposition 4.6.2. *Definitions* 4.6.1 *and* 4.6.2 *of well-ordered trees are equivalent.*

Proof. Call (1) the Definition 4.6.1, and (2) the Definition 4.6.2: (1) easily implies (2). In fact, given $t \in T$, if we consider any subset of P_t, this is a set of moments, hence it has a minimal element. The past of a given point is linear (see tree condition, Definition 2.1.1), so a minimal element is the least element.

Now, assume (2). We want to show that every history is well-ordered, and so we deduce (1) using the equivalence of Proposition 4.6.1. Let h be a history and $B \subseteq h$. Consider $t \in B$: $B \cap P_t \subseteq P_t$, so it has a least element $\bar{t}$ since P_t is well-ordered. Consider now $b \in B$, $b \leq \bar{t}$. $b \leq \bar{t} \leq t$, so b belongs to the past of t, hence it must coincide with $\bar{t}$. □

We want to emphasize that the request of well-foundedness is quite strong. It excludes dense trees (for example no copy of $\mathbb{Q}$ or $\mathbb{R}$ can be contained in our tree), and we can not have a quite simple countable linear order either, in which an infinite descending chain is involved, as seen in Figure 4.2. In this example, the subset $\{n_u \mid n \in \mathbb{N}\}$ of the past of 0_u is not well-ordered.

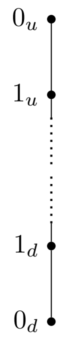


Figure 4.2: A tree that is not well-founded.

Remark 4.6.1. Every well-ordered tree is rooted, but not necessarily single rooted. In fact, if we consider the entire past of a given moment t, it has a least element, which is a root. Moreover, if we have a non-connected tree, it is well-founded if and only if every connected component is well-founded. In this case, we have more than one root.

Definition 4.6.3. A tree $\mathcal{T}$ is *upwards discrete* if every moment $t \in T$ (which is not a leaf) has an immediate successor along every history passing through it. With *immediate successor* of t we mean a moment $t' \in T$, $t < t'$, such that there are no $s \in T$, $t < s < t'$.

Proposition 4.6.3. *A well-founded tree is upwards discrete.*

Proof. Let t be a moment of a well-founded tree $\mathcal{T}$, t not a leaf, and $h \ni t$. Consider the set $B = \{x \in h \mid t < x\}$. It has a least element, which is the immediate successor of t along h. Hence, $\mathcal{T}$ is upwards discrete. □

4.6.2 Topological characterization

In [23], Nyikos suggests that if a tree $\mathcal{T}$ is well-founded, the base $\mathcal{O}_\mathcal{T}$ is Nötherian, but he gives no proof of this claim. We give a detailed proof of this result below

Definition 4.6.4. A base of a topological space is *Nötherian* if every strictly ascending sequence of its elements is finite.

Remark 4.6.2. The reader may notice that we have not required the whole space to be Nötherian. This is a stronger condition[7], which would "ruin" our topological space: in fact, every Hausdorff and Nötherian topological space is finite, and our non-Archimedean spaces always have the Hausdorff property.

[7]To be precise, the open sets satisfy the *Ascending Chain Condition*, often shortened with *ACC*: every increasing chain of open sets stabilizes after a finite number of steps.

Proposition 4.6.4. *Let $\mathcal{T}$ be well-founded. Then $\mathcal{O}_{\mathcal{T}}$ is a Nötherian base*[8].

Proof. Consider an increasing sequence $\{H_{t_n} \mid n \in \mathbb{N}\}$ of elements of the base, with $H_{t_n} \subseteq H_{t_{n+1}}$. Then, for all $n, m \in \mathbb{N}$ with $n \leq m$, we have $H_{t_n} \subseteq H_{t_m}$, so the set $\{t_0, t_1, \ldots\}$ is totally ordered by the "reverse" inclusion, $t_0 \geq t_1 \geq \ldots$. Then, by Remark 2.2.2, there is a history containing $\{t_0, t_1, \ldots\}$. Thus, by the definition of well-founded trees, $\{t_0, t_1, \ldots\}$ has a least element, which means that there exists $i \in \mathbb{N}$ such that $t_i \leq t_n$ for every $n \in \mathbb{N}$. This implies that $H_{t_i} \supseteq H_{t_n}$ for every $n \in \mathbb{N}$. Hence, H_{t_i} is the greatest element for the chain $\{H_{t_n} \mid n \in \mathbb{N}\}$, and it is reached after a finite number i of steps. □

Unfortunately, the converse is not true for the usual reason: the tree $(H(T), \mathcal{O}_{\mathcal{T}})$ is the condensation of the tree $\mathcal{T}$, so it "can not see" the bridges of $\mathcal{T}$. With this idea, we can produce the following counterexample:

Example 4.6.1. Consider the ordinal ω with the reverse order, as in Figure 4.3. It is not well-founded (and not rooted either), but its base is $\mathcal{O}_{\mathcal{T}} = \{H_0\}$, which is clearly Nötherian.

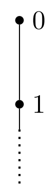


Figure 4.3: Example of a not well-founded tree with a Nötherian base.

This suggests to focus our attention on the class of totally branching trees, whose subclasses are the only one that can be significantly characterized in our topological context.

We start by proving that any tree with a Nötherian base is upwards discrete. We will prove this result by *reductio*, so we need to understand what a non-upwards discrete tree is: a tree is non-upwards discrete if there exist $t \in T$ and $h \in H_t$ such that t does not have an immediate successor along h. In other words, for every $t' \in h$, $t < t'$, there exists $s \in h$, $t < s < t'$.

Proposition 4.6.5. *Let $\mathcal{T}$ be a totally branching tree such that $\mathcal{O}_{\mathcal{T}}$ is Nötherian. Then $\mathcal{T}$ is upwards discrete.*

[8]From now on, sometimes we will say that a tree has a Nötherian base meaning that $\mathcal{O}_{\mathcal{T}}$ is Nötherian.

Proof. Suppose by reductio that $\mathcal{T}$ is *not* upwards discrete. Then, as we said above, there exist $t \in T$, and $h \in H_t$ such that for all $t' \in h$, $t < t'$, there exists $s \in h$, $t < s < t'$. So there is an infinite descending chain of moments pointing towards t along h. Let $t_0, t_1, \ldots, t_n, \ldots$ be a subsequence of this chain (i.e. we must select a countable subset of the chain with the same "dense near t" property): it exists because the chain above is infinite and strictly descending.

Then, $H_{t_0}, H_{t_1}, \ldots, H_{t_n}, \ldots$ is a strictly ascending sequence of elements of the base, whose existence contradicts the Nötherian property requested in the hypothesis. □

Proposition 4.6.6. *Let $\mathcal{T}$ be a totally branching tree such that $\mathcal{O}_\mathcal{T}$ is a Nötherian base. Then $\mathcal{T}$ is well-founded.*

Proof. Let h be a history in $\mathcal{T}$, and $A \subseteq h$. We want to find a least element of A, which leads us to the conclusion using Proposition 4.6.1.

Let a_0 be an element of A, and for every $n \in \mathbb{N}$, given a_n choose a_{n+1} among the predecessors of a_n, i.e. such that $a_n \geq a_{n+1}$. Then $(a_n)_{n \in \mathbb{N}}$ is a descending sequence of moments coming down along A.

Let $\tilde{A} = \{a_n \mid n \in \mathbb{N}\}$, the subset of A that collects all the elements of this sequence. Then the set $\{H_{a_n} \mid a_n \in \tilde{A}\}$ is an ascending sequence of elements of the base, hence it must stabilize after a finite number of steps because of the Nötherian property requested. In other words, this sequence has a greatest element, call it $H_{\tilde{t}}$. Then, $\tilde{t}$ is the least element of $\tilde{A}$, since the tree is totally branching and so there can not be a bridge below $\tilde{t}$. □

Remark 4.6.3. In the previous proof, we could conclude that $\tilde{t}$ is the least element of A, even if the history h is isomorphic to an uncountable ordinal number, by Corollary 4.1.5: there can not be an infinite descending chain in an ordinal, hence we can not properly extend the sequence defined above.

4.7 ω-cofinal trees

In this section we consider trees in which every history has cofinality ω, which are called *ω-cofinal trees*. We start with the definition of cofinality.

Definition 4.7.1. Let $(A, <)$ be a poset and $B \subseteq A$. B is *cofinal* in A if for every $a \in A$, there exists $b \in B$ such that $a < b$.

Remark 4.7.1. The cofinality relation over partially ordered sets is reflexive and transitive. In fact, every poset is cofinal in itself. Moreover, if B is a cofinal subset of a poset A, and C is a cofinal subset of B (with the induced partial ordering), then C is also a cofinal subset of A.

A cofinal subset contains every maximal element of the poset: otherwise, a maximal element would fail to be less than some other element of the cofinal subset. This implies that if there are disjoint cofinal subsets of a poset, then it does not have maximal elements. If a poset has a greatest element, then it belongs to every cofinal subset.

Definition 4.7.2. Let $(X, <)$, $(Y, \prec)$ be posets. We say that $f : Y \to X$ is a *cofinal embedding* if it is an injective order morphism with the property $\forall x \in X$, $\exists y \in Y$ such that $x < f(y)$. In other words, the image of the function f is unbounded from above.

Definition 4.7.3. Given a cardinal number λ, we say that a poset X has *cofinality* λ ($cf(X) = \lambda$) if λ is the cardinality of the least ordinal number such that there exists a cofinal embedding from it into X.

Remark 4.7.2. From Proposition 4.1.4, every non-empty set of cardinal numbers has a least member, in fact, every cardinal is an ordinal. This allows us to give the following equivalent definition of cofinality: given a poset A, the cofinality of A is the least of the cardinalities of the cofinal subsets of A. Given this result, it is clear that the cofinality of a set with a greatest element is 1.

4.7.1 Motivation

The main aim of this section is to characterize trees in which every history has cofinality ω. We are interested in these trees for two main reasons:

- there are definability results that hold in the class of ω-cofinal trees. For example, as stated in [39], no Ockhamist formula is true in a bundled tree if and only if it is complete, but such a formula exists for bundled trees in which every history has cofinality ω. We suggest the article cited above for further results about this topic.
- a generalization of the *König tree lemma* (Theorem 4.5.1) can be easily proved for this class of trees (assuming that the tree is rooted too), as shown in the following remark and theorem, both taken from [42], with some corrections and extensions.

Remark 4.7.3. In order to extend the validity of the *König tree lemma*, the first step is to weaken some hypothesis. We start our discussion from the statement suggested by Remark 4.5.1: a finitely branching rooted tree of height ω has an infinite history.

We can start by dropping the finitely branching hypothesis (which is equivalent to the fact that every level $T(\alpha)$ is finite), but unfortunately the lemma does not work anymore. In fact, if we have a tree of countable height with levels of cardinality ω, it does not necessarily have a history of countable length. Consider for example the tree of Figure 4.4, where the first history has height two, the second height three, and so on. In this tree every history has a finite length.

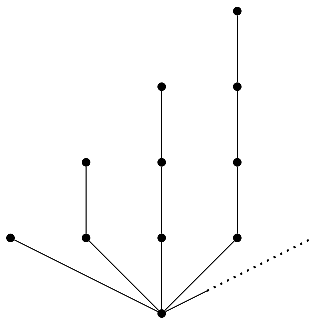


Figure 4.4: Example of a tree with levels of countable cardinality, and no countable history.

Another natural way in which we could generalize the König tree lemma is to extend the height of the tree, and so to consider a tree of uncountable height with countable levels, and to ask whether they have histories with uncountably many moments. As said in [1], it is easy to obtain a tree of uncountable height with no uncountable history: the tree consisting of all one-to-one countable functions from a countable ordinal into ω_1 with the function extension ordering is such a tree. But, its ω level is already uncountable (there are continuum many permutations of ω), so this tree clearly does not satisfy the hypothesis on the size of the levels.

However, and strangely enough, there are trees of height ω_1 with no uncountable history and no uncountable levels: they are called *Aronszajn trees*[9].

As said in [42], the problem we run into with Aronsjazn trees is that the proof of König's lemma fails at the limit step: we may have collected a set of comparable moments that have infinitely many successors, but we cannot guarantee that there that there are moments above all of them.

[9]They are named after Nachman Aronszajn, a Polish matematician, who constructed an Aronszajn tree in 1934; its construction was described in [21]. Since this book was written in French, we found another construction in [1]: in this article, the author shows two different approaches: the classical one, due to Aronszajn, and a newer one, due to S. Todorčević, who uses some of the methods of today's combinatorial set theory. This second method is employed in the proof of the existence of an Aronszajn tree found in [15], Lemma 9.16. Further remarks on this topic can be found in Section 4.9 of this thesis.

However, the proof of König's lemma relies on the fact that the size of the levels is less than the cofinality of the height, and there are no limit steps involved. This fact suggests to consider trees with height of cofinality ω (ω-cofinal trees have this property), and to prove the following generalized König's Lemma:

Theorem 4.7.1. *Let κ be an ordinal with cofinality ω. Then any finitely branching rooted tree of height κ has a history of length κ.*

Proof. Since κ has cofinality ω, then there exists a cofinal embedding $f : \omega \to \kappa$, which is an order morphism such that for every $\alpha \in \kappa$ (or, equivalently, $\alpha < \kappa$) there exists $i \in \omega$ such that $f(i) > \alpha$. Then for every i, $\kappa > f(i)$, so $\kappa \geq \bigcup_{i \in \omega} f(i) \geq \bigcup_{\alpha \in \kappa} \alpha = \kappa$. Thus $\kappa = \bigcup_{i \in \omega} f(i)$.

To build a history of height κ, we start from a root t_0 of our tree with infinitely many successors: such a moment exists because, otherwise, T would be a finite union of finite sets of moments, but it is infinite. Then the level $T(f(1))$ can not be empty because $f(1) < \kappa$, so there is a chain of moments connecting the root to some moments of $T(f(1))$. Again, there is a moment of $T(f(1))$ with infinitely many successors, otherwise the tree would have height $f(1) + n$ for some $n \in \omega$, so it would have a leaf at the end of every history, so the cofinality of the height would be 1. Moreover, the level $T(f(2))$ can not be empty, because $f(2) < \kappa$, so there is a chain of moments connecting some moments of $T(f(1))$ to some other moments of $T(f(2))$. Again, there is a moment of $T(f(2))$ with infinitely many successors, otherwise the tree would have height $f(2) + m$ for some $m \in \omega$, so it would have a leaf at the end of every history, so the cofinality of the height would be 1.

Therefore, we can repeat the same argument with the level $T(f(n))$, for every n, so we have inductively shown that a history of height greater than any $f(i)$ exists. However since f is a cofinal embedding, the only ordinal that is strictly greater than any $f(i)$ (and smaller than or equal to κ), is κ itself. So such a history has height κ. □

Remark 4.7.4. The online source [42] that inspired this section proved this generalization of the König lemma in this way (in our notation: $\delta_i = f(i)$):

> Let us write $\kappa = \bigcup_{i \in \omega} \delta_i$ where δ_i is an ordinal. Then starting at the root of our tree, we can find a node on level δ_1 with at least κ many successors. From that node we can find one of its successors on δ_2 with at least κ many successors, and so on. Such a node exists since each level only has finitely many nodes. Since there is no limit step involved, we have, after some induction, successfully constructed a history of height κ.

However, his proof is not correct, or at least its correctness depends on the chosen cofinal embedding. As an example, if $\kappa = \omega_1 + \omega$, and we set $f(0) = 0$ and $f(i) = \delta_i = \omega_1 + (i-1)$, there is no moment of $T(\delta_1) = T(\omega_1)$ with κ successors, since we have already "jumped above" ω_1.

4.7.2 Topological characterization

Definition 4.7.4. Let (X, τ) be a topological space and $x \in X$ a point. A subset $\mathcal{B}$ of τ is a *local base at x* if, for every open neighbourhood U of x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. A base of the topology *can be localized* if there exists a subcollection which is a local base at some point x.

Remark 4.7.5. Let X be an Hausdorff topological space, and $\mathcal{B}$ a local base at a point $x \in X$. Then $\bigcap \mathcal{B} = x$. In fact, assume $y \in X$, $y \neq x$. Hausdorff property implies that there exists U neighbourhood of x such that $y \notin U$. Hence, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, which implies $y \notin B$. Thus $y \notin \bigcap \mathcal{B}$. Moreover, given a base $\mathcal{B}$ for the topological space, $\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$ is clearly a local base at x.

Definition 4.7.5. Let (X, τ) be a topological space and $\mathcal{B}$ a base for the topology. $\mathcal{B}$ is a *base of countable order* if every strictly decreasing sequence of its elements either has empty intersection or it is a local base at some point.

Remark 4.7.6. It is worth noticing that, if $\{U_n \mid n \in \mathbb{N}\}$ is a countable local base at a point x, then a new local base $\{V_n \mid n \in \mathbb{N}\}$ can be found with the additional property $V_n \supset V_{n+1}$ for each n. The construction is simple: let $V_n = \bigcap_{k \leq n} U_k$.

Lemma 4.7.2. *Let $\mathcal{T}$ be any tree, and h a history on $\mathcal{T}$. Then $\bigcap_{t \in h} H_t = \{h\}$, which means that the base $\mathcal{O}_\mathcal{T}$ can be localized for every h.*

Proof. Let A be $\bigcap_{t \in h} H_t$. Clearly $h \in A$, since $h \in H_t$ for all $t \in h$. Consider $h' \in A$. For all $t \in h$, $h' \in H_t$, hence $t \in h'$. Then $h \subseteq h'$. As a consequence, we have that $h = h'$, since histories are maximal chains.

Moreover, the base $\mathcal{O}_\mathcal{T} = \{H_t \mid t \in T\}$ can be easily localized at a point h of the space $H(T)$. The subfamily $\mathcal{O}_h = \{H_t \mid t \in h\}$ is a local base: in fact, $h \in H_t$ for every $t \in h$, and every open set U containing h is union of open subsets of the base, so U contains H_t for some $t \in h$. □

Corollary 4.7.3. *Let $\mathcal{T}$ be an ω-cofinal tree, and, for every history h, let f_h be the cofinal embedding of $\mathbb{N}$ into h. Then, for every history h, $\{h\} = \bigcap_{n \in \mathbb{N}} H_{f_h(n)}$.*

Proof. Let h be a history in $\mathcal{T}$. Then for all $t \in h$, there is $n \in \mathbb{N}$ such that $t < f_h(n)$. So $H_t \supseteq H_{f_h(n)}$ and $\bigcap_{t\in h} H_t \supseteq \bigcap_{n\in\mathbb{N}} H_{f_h(n)}$. By Lemma 4.7.2, $\bigcap_{t\in h} H_t = \{h\}$. Moreover, $h \in \bigcap_{n\in\mathbb{N}} H_{f_h(n)}$ because, for all $n \in \mathbb{N}$, $f_h(n) \in h$ and hence $h \in H_{f_h(n)}$. Thus, $\{h\} = \bigcap_{t\in h} H_t \supseteq \bigcap_{n\in\mathbb{N}} H_{f_h(n)} \supseteq \{h\}$, so every inclusion holds as an equality. □

Corollary 4.7.4. *Let $\mathcal{T}$ be an ω-cofinal tree. Then the base $\mathcal{O}_\mathcal{T}$ is a base of countable order.*

Proof. In the case of trees in which every history has cofinality ω, Corollary 4.7.3 implies that the local base $\mathcal{O}_h = \{H_t \mid t \in h\}$, as constructed in Lemma 4.7.2, is indeed a countable local base, which means that the base $\mathcal{O}_\mathcal{T}$ is a base of countable order. □

However, the converse implication is not true: the fact that $\mathcal{O}_\mathcal{T}$ is a base of countable order does *not* imply that every history in $\mathcal{T}$ has cofinality ω. Consider for instance the following example.

Example 4.7.1. Let $\mathcal{T}$ be the tree described in Figure 4.5, in which a branch isomorphic to ω starts from every natural number included in the ordinal $\omega+1$ (whose greatest element is called M instead of ω, in order to avoid conflicts in the notation).

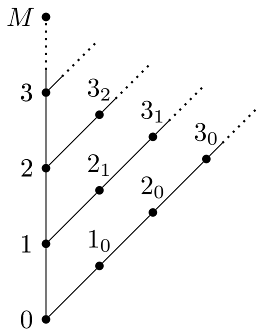


Figure 4.5: Example of a tree in which a history does *not* have cofinality ω.

Clearly, every history of the type $\{0, 1, 2, \ldots, n, (n+1)_n, (n+2)_n, \ldots\}$ is isomorphic to ω, thus it has cofinality ω. However the vertical history $\{0, 1, 2, 3, \ldots, M\}$ has a greatest element, hence it has cofinality 1. Call this history h_M.

Now we observe that the history $h_0 = \{0, 1_0, 2_0, \ldots\}$ ramifies only in 0, so we have that $H_{1_0} = H_{2_0} = H_{3_0} = \ldots = \{h_0\}$. In general, for every n and for every $m > n$, the set H_{m_n} is equal to $H_{(n+1)_n}$ and it contains the single history $h_n = \{0, 1, 2, \ldots, n, (n+1)_n, (n+2)_n, \ldots\}$. Moreover, the set H_M consists just of the vertical history h_M, and h_M belongs to every H_n.

The base of the topology of this tree is

$$\mathcal{O}_\mathcal{T} = \{H_n \mid n \in \mathbb{N}\} \cup \{H_{(n+1)_n} \mid n \in \mathbb{N}\} \cup \{H_M\},$$

and the only infinite strictly decreasing sequence in this base is $\{H_n \mid n \in \mathbb{N}\}$. In fact, $H_n = H_{n+1} \cup \{h_n\}$, so $H_n \supset H_{n+1}$. Moreover $\bigcap\{H_n \mid n \in \mathbb{N}\} = h_M$, so it is a local base at h_M. So this base has the countable order property, but the tree is *not* an ω-cofinal tree.

We could think that the problem of the tree above is the presence of the greatest element of a history, but we can produce a variation of the previous example in which there is a history of given cofinality, and such that $\mathcal{O}_\mathcal{T}$ still is a base of countable order. Given an ordinal λ which is the cofinality of another ordinal α, we draw the tree of Figure 4.6 starting from the tree of the previous example, using M as the starting point of α. Then the vertical history h_M contains the whole ordinal, hence it has cofinality λ. Moreover, we have not produced any variation from the topological point of view (in particular, $\mathcal{O}_\mathcal{T}$ is a base of countable order), since the condensation of this tree is the same as the condensation of the tree of Figure 4.5.

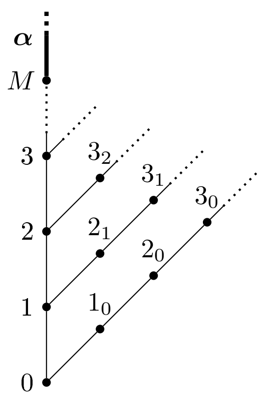


Figure 4.6: Example of a tree in which a history has cofinality λ.

This second example suggests that non-totally branching trees can not be considered in a setting in which cofinality matters. Unfortunately, even requiring the tree to be totally branching is not enough for our characterization of ω-cofinal trees to work. In fact, the condensation of both trees we considered above is a totally branching tree, in which the usual base is a base of countable order, and every history has a greatest element and so has cofinality 1.

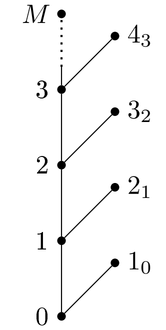


Figure 4.7: Condensation of the previous trees.

As a final remark, a tree in which $\mathcal{O}_\mathcal{T}$ is a base of countable order is *not* necessarily well-founded (see Definition 4.6.1). As an example, we can consider another variation of the tree described above, represented in the following Figure 4.8: in this case, there are a lot of strictly descending sequences of elements of the base ($\{H_n \mid n > z\}$ for every chosen $z \in \mathbb{Z}$), but all of them still intersect in $\{h_M\}$. This is also a totally branching tree.

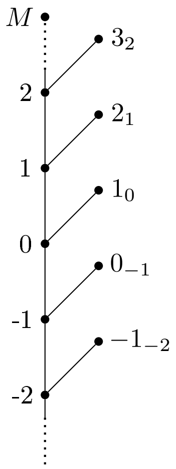


Figure 4.8: A not well-founded tree with base of countable order.

The examples considered above suggest the following result:

Proposition 4.7.5. *Let $\mathcal{T}$ be a totally branching well-founded tree in which the base $\mathcal{O}_\mathcal{T}$ is a base of countable order. Then, every history is isomorphic to an ordinal number up to $\omega + 1$ included.*

Proof. Assume by *reductio* that there is a history h which is isomorphic to an ordinal number greater than or equal to $\omega + 2$. This history contains a copy of the natural numbers, a moment above all the "natural" moments, t_0, and another one which follows, t_1. Since the tree is totally branching, there must be $t_2 > t_0$, $t_2 \not< t_1$.

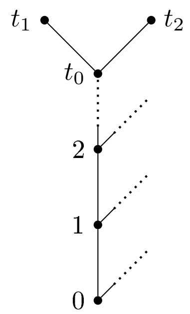


Figure 4.9: Representation of the history considered in the proof.

Then, we have that $\{H_n \mid n \in \mathbb{N}\}$ is a strictly descending chain whose intersection consists of two histories: one ends with t_1 and the other one ends with t_2. Hence, it is *not* a local base for h (see Remark 4.7.5). Moreover, it clearly has non-empty intersection, hence $\mathcal{O}_\mathcal{T}$ is not a base of countable order, which is a contradiction. □

So, we have that in a totally branching well-founded tree in which $\mathcal{O}_\mathcal{T}$ is a base of countable order, every history is either finite, or countably infinite (with or without a greatest element). So, in order to characterize trees in which every history is isomorphic to ω (which we called ω-trees in Section 4.5: they clearly form a subclass of ω-cofinal trees), we just need to require that no history has a greatest element.

In our setting, this property can be simply characterized from a topological point of view: to do so, we must require the space to be perfect.

Definition 4.7.6. A topological space (X, τ) is *perfect* if it has no isolated points.

Theorem 4.7.6. *Let $\mathcal{T}$ be a totally branching tree. Then, every history does not have an ending moment (leaf) if and only if the space $H(T)$ is perfect.*

Proof. Assume that $H(T)$ is perfect. There is no isolated point, which means that for every h history in $\mathcal{T}$, $\{h\}$ is not open in $\tau_\mathcal{T}$, which, again, means that there is no finite intersection of elements H_t of the base that is only $\{h\}$. Suppose by *reductio* that there is a history h_e with an ending moment (i.e. a greatest element) t_e. Then $H_{t_e} = \{h_e\}$, which is a contradiction.

Conversely, assume that the space is not perfect. From the hypothesis we have that there exists $h \in H(T)$ such that $\{h\}$ is open, which means that there is a finite collection $\{t_0, \dots, t_n\}$ of moments of T such that $\{h\} = \bigcap_{i=0}^n H_{t_i}$. However, if $H_t \ni h$, then $t \in h$, so the collection $\{t_0, \dots, t_n\}$ consists of pairwise comparable moments. From the fact that it is finite we deduce that there is a greatest moment: t_g. Hence, $H_{t_i} \supseteq H_{t_g}$ for every i, so $\{h\} = \bigcap_{i=0}^n H_{t_i} = H_{t_g}$. Hence, h is the only history passing through t_g. Moreover, since the tree is totally branching, there can not be a linear bridge above t_g along h, hence t_g is the ending moment of h. □

Gathering together Proposition 4.7.5 and Theorem 4.7.6, we have the following characterization of ω-trees.

Proposition 4.7.7. *Let $\mathcal{T}$ be a totally branching well-founded tree in which the base $\mathcal{O}_\mathcal{T}$ is a base of countable order, and the space $H(T)$ is perfect. Then $\mathcal{T}$ is a (totally branching) ω-tree.*

However, this result is not fully satisfying, since we greatly reduced the class of trees we were able to characterize: we started from ω-cofinal trees and ended up with ω-trees.

Can we improve this result? Which property needs to be weakened in order to characterize the whole class of totally branching ω-cofinal trees? These questions are left as an open problem.

4.8 Jointed trees

We are interested in the class of *jointed trees* because the definability result briefly discussed in the first part of Section 4.7.1 holds for this class too[10]. Moreover, we used trees that are *not* jointed as counterexamples for some statements in the previous sections.

Definition 4.8.1. A tree $\mathcal{T}$ is *jointed* if the intersection of any two different histories is either empty or has a greatest element. Formally, if $\forall h, h' \in H(T), h \neq h'$, $h \cap h' = \emptyset \ \vee \ \exists t = \max\{h \cap h'\}$.

Clearly, every finite tree is jointed. Actually, every tree of finite height is jointed: in fact, the intersection of two different histories (if not empty) is linear, and it has cardinality less or equal to the height of the tree.

So, in order to produce a tree that is not jointed, we have to design a tree with infinite height: for example, after a infinite linear starting segment, we can add two uncomparable moments (l, r), as seen in Figure 4.10. In fact, the

[10]The authors of the cited article [39] prove that the result we mentioned holds for the class of *cofinally jointed* trees, which contains both jointed and ω-cofinal trees.

"left" history $\{0, 1, 2, \ldots, l\}$ and the "right" history $\{0, 1, 2, \ldots, r\}$ intersect in a copy of ω, which does not have a greatest element. A variation of this example is contained in Figure 4.1.

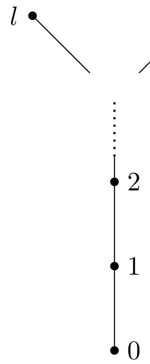


Figure 4.10: Example of a tree that is not jointed: for every $n \in \mathbb{N}$, $n < l$, $n < r$, and $l \not< r$.

We observe that the tree of Figure 4.10 is well-founded, while the one of Figure 4.1 is not: this means that well-foundedness and jointedness are unrelated properties.

Now we will give the topological characterization for jointed trees: we will start with the easier case of connected jointed trees, then we will extend the result to the non-connected case, and finally we will prove a form of the converse. Like in other cases, the restriction to totally branching trees will be necessary.

Proposition 4.8.1. *Let $\mathcal{T}$ be a connected jointed tree. Then, for all $h, h' \in H(T)$ with $h \neq h'$, there exists a minimal $B \in \mathcal{O}_{\mathcal{T}}$ such that $h, h' \in B$: this means that for every $B' \in \mathcal{O}_{\mathcal{T}}$ such that $h, h' \in B'$, $B \subseteq B'$.*[11]

Proof. Let h, h' be different histories of $\mathcal{T}$. Since the tree is connected, $h \cap h' \neq \emptyset$. $\mathcal{T}$ is jointed, so there exists $m = \max(h \cap h')$. H_m clearly contains both h and h', since they both pass through m. Moreover, if $h, h' \in H_t$ for some t, then $t \in h \cap h'$. Hence $t \leq m$, by maximality of m, so $H_m \subseteq H_t$. Finally, if $s > m$, then $s \notin h \cap h'$, hence $H_m \supsetneqq H_s$, and one of the following situations happens:

- $s \in h$ and $s \notin h'$, hence $H_s \not\ni h'$;
- $s \in h'$ and $s \notin h$, hence $H_s \not\ni h$.

[11]We do not know of a possible standard name of this property in the literature. We could not find any reference to it in classical texts in topology, like [20] or the *Hand book of Set Theoretic Topology* edited by K. Kunen and J. E. Vaugan.

So every element of the base strictly contained in H_m can not contain both h and h'. Thus, H_m is the minimal element of the base containing both h and h'. □

Proposition 4.8.2. *Let $\mathcal{T}$ be a jointed tree. Then, for all $h, h' \in H(T)$ with $h \neq h'$, there either exists a minimal $B \in \mathcal{O}_{\mathcal{T}}$ such that $h, h' \in B$, or there is not an element of the base containing both h and h'.*

Proof. If we consider h_1 and h_2 in the same connected component of the tree, we can apply the previous argument. If T_1 and T_2 are different components of T and $h_i \in H(T_i)$, then we can apply the characterization of connected trees (Proposition 4.3.1) to each of them and conclude that for every $t \in h$, $t' \in h'$, $H_t \cup H_{t'}$ is not contained in an element of the base, hence there can not be an element of the base containing both h and h'. □

Proposition 4.8.3. *Let $\mathcal{T}$ be a totally branching connected tree with the property that for all $h, h' \in H(T)$ with $h \neq h'$, there exists a minimal $B \in \mathcal{O}_{\mathcal{T}}$ such that $h, h' \in B$. Then it is a (connected) jointed tree.*

Proof. Let $h_1, h_2 \in H(T)$, $h_1 \neq h_2$. Since the tree is connected, $h_1 \cap h_2 \neq \emptyset$. We want to show that it has a greatest element.

Since every element of the base is H_t for some $t \in T$, let $b \in T$ be a moment such that $H_b = B$, the minimal element of the base containing h_1 and h_2. Clearly, from $h_1, h_2 \in H_b$, we have $b \in h_1 \cap h_2$. Our claim is that b is the greatest element we are looking for.

Let $s \in h_1 \cap h_2$, $s \geq b$. Then $H_s \subseteq H_b$, and $h_1, h_2 \in H_s$. But H_b is $\subseteq$-minimal, hence $H_s = H_b$. But $\mathcal{T}$ is totally branching, hence $H_s = H_b$ implies $s = b$. Thus, $b = \max(h_1, h_2)$, hence $\mathcal{T}$ is jointed. □

Remark 4.8.1. We conclude this section by observing that the tree of Figure 4.10 has the property of the previous proposition (in fact, its condensation is $\{[0], [l], [r]\}$, which is clearly jointed and connected), but it is *not* totally branching and *not* jointed. This remark shows again that the totally branching hypothesis is often necessary in order to properly characterize properties of a tree from a topological perspective.

4.9 Special, Souslin and Aronszajn trees

In this section we will analyse the notions of *Souslin tree* and of *special tree*, which arise from the study of the Souslin's Problem and are also linked to the extension of the König Tree Lemma discussed in Section 4.7.1. The reasons why we are interested in this class of trees will be clarified in Subsection 4.9.2.

Moreover, we will not characterize special trees completely, so we will show a further limit of the topological approach and leave some open problems.

Many sources are involved in this section. Some of them are [1], [5], [14], [32], [15], and [11].

4.9.1 Antichains

The notion of antichain is fundamental in the definition of special and Souslin trees, and it is also connected to finitely branchingness (Theorem 4.5.2) in a way that we will deepen in this section.

Recall that a *chain* of a poset $(P, <)$ is a totally ordered subset of the poset, i.e. a subset whose elements are pairwise comparable (in symbol, $\smile$).

Definition 4.9.1. Let $\mathcal{T}$ be a tree. A set $X \subseteq T$ is an *antichain* if for all $x_1 \neq x_2 \in X$, we have $x_1 \not\smile x_2$.

Remark 4.9.1. While chains are "vertical" subsets of the trees, antichains are non-vertical subsets: they are made by pairwise non-comparable moments. Notice that there is no request about the cardinality of the antichain, hence $\emptyset$ and every singleton $\{x\}$ are antichains (and chains too).

Using chains and antichains, we can define the height and the width of a (finite) poset[12]: the *height* of a poset is the cardinality of the greatest (longest) chain, the *width* is the cardinality of the greatest (largest) antichain.

These two notions are linked by the important Dilworth Theorem (1948) and by its dual statement, that we transcribe here to show the relevance of this connection: for a complete proof of these results and for some other remarks about chains and antichains, the interested reader can refer to [36].

Theorem 4.9.1 (Dilworth)**.** *If A is the largest antichain in a finite poset $(X, <)$, then there is a partition of S into chains $S = C_1 \cup C_2 \cup \ldots \cup C_n$ such that $n = |A|$. Furthermore, each C_i contains exactly one element of A, and there is no partition of S into less than n chains.*

Theorem 4.9.2 (Dual of Dilworth)**.** *If C is the largest chain in a finite poset $(X, <)$, then there is a partition of S into antichains $S = A_1 \cup A_2 \cup \ldots \cup A_n$ such that $n = |C|$. Furthermore, each A_i contains exactly one element of C, and there is no partition of S into less than n antichains.*

[12] Notice that the definition of *height* is not equivalent to the one for trees contained in Definition 4.1.7, even if trees are particular kind of posets.

Remark 4.9.2. In the context of trees, we can easily prove the second part of the dual theorem above: given an antichain X and a history h of a tree $\mathcal{T}$, $X \cup h$ is either empty or a singleton. In fact, antichains are not requested to intersect every history, and in this case we can get $\emptyset$ as intersection. Moreover, if the intersection is not empty, it must be a singleton, since two moments belonging to it are simultaneously comparable and uncomparable, hence equals.

A link between the notion of antichain and the topological perspective of this thesis is contained in the following Theorem 4.9.6, which is stated in [23] without proof. We developed the whole proof and we split it in a definition and three preliminary results:

Definition 4.9.2. An antichain X is said to be *maximal* if for every $t \in T$, $X \cup \{t\}$ is not an antichain, i.e. there exists $x \in X$ such that $t \smile x$.

The reader should notice that this has nothing to do with the cardinality of the antichain (hence, with the width of the tree): in fact, the set of all minimal moments of the tree is a maximal antichain, and it can be a singleton if the tree is single-rooted.

Lemma 4.9.3. *If X is a maximal antichain of a tree $\mathcal{T}$, then for every history $h \in H(T)$ there is a unique moment t belonging to $X \cap h$. Moreover, $\bigcup_{x \in X} H_x = H(T)$.*

Proof. To show existence, let h be a history, and suppose by contradiction that $X \cap h = \emptyset$. Then for all $t \in h$, and for all $x \in X$, $t \not\smile x$, otherwise $x \in h$. Thus, $X \cup \{t\}$ is an antichain strictly greater than X, against the maximality of X, which is a contradiction. To show uniqueness, suppose by contradiction that $h \cap X \supseteq \{x_1, x_2\}$. Then $x_1 \smile x_2$, because they belong to the same history, hence X is not an antichain, which is a contradiction. The second claim is a straightforward consequence of the first one. □

Lemma 4.9.4. *Let X be an antichain of a tree $\mathcal{T}$. Then the function $f : x \mapsto H_x$ is injective.*

Proof. We can prove something more: if $x, y \in X$ and $x \neq y$, then $H_x \cap H_y = \emptyset$, which clearly implies $H_x \neq H_y$.

Suppose by contradiction that there exists $h \in H_x \cap H_y$. Then $x \in h$, $y \in h$, so $x \smile y$. Thus, X is not an antichain, which is a contradiction. □

Proposition 4.9.5. *Let $\mathcal{T}$ be a well-founded tree and $S \subseteq T$. Then there exists a maximal antichain included in S. Moreover, we can build a maximal antichain X whose elements are minimal in the following sense: if Y is another antichain, for every $x \in X$, $y \in Y$ such that $x \smile y$, then $x \leq y$.*

Proof. The existence of this antichain can be proved by choosing, in every history $h \in H(T)$, exactly the minimal element of $h \cap S$, so the antichain is the set $\{\min(S \cap h) \mid h \in H(T)\}$.

However, in order to avoid a quantification over the set of histories, we can explicitly construct this maximal antichain with a transfinite induction over the height[13] of the tree $\mathcal{T}$.

Set $\text{height}(\mathcal{T}) = \alpha$. In every step, we use the fact that every level $T(\beta)$ of a well-founded tree is an antichain, because all moments in it belong to different histories. In order to simplify the notation, let $S(\beta) = T(\beta) \cap S$ be the set of the elements of S at level β.

- Set $S_0 = S$
- For every ordinal $\beta < \alpha$, set

$$S_{\beta+1} = \left(S_\beta \smallsetminus \bigcup_{t \in S(\beta)} H_t \right) \cup S(\beta).$$

  Removing every history passing through every moment of level β, and adding level β back, we just remove every moment at higher levels, comparable with some moment of $S(\beta)$.
- For every limit ordinal $\gamma < \alpha$, set

$$S_\gamma = \left(\bigcap_{\beta < \gamma} S_\beta \smallsetminus \bigcup_{t \in S(\gamma)} H_t \right) \cup S(\gamma).$$

  This inductive step is more complicated than the previous one because it involves a limit ordinal number, but the idea is the same.

Clearly, the induction terminates because there is a boundary for the ordinal number, namely the height of the tree. The last step S_α of the induction gives the maximal antichain we were looking for.

Moreover, its elements are minimal in the way explained in the statement, since we "started from the bottom". To be precise, suppose by *reductio* that Y was an antichain, $y \in Y$, $x \in S_\alpha$ and $y < x$. Then, there would be ordinal numbers $\beta_1 < \beta_2$ such that $y \in S(\beta_1)$ and $x \in S(\beta_2)$. Then, in an inductive step $\leq \beta_1$, our process would have selected y (or a moment in its past) and deleted the rest of H_y from S: by doing so, it would have removed x, which belongs to the future of y. And this is a contradiction. □

Example 4.9.1. In order to visualize the inductive process described above, we apply it to a finite tree (even if, clearly, it is interesting only when applied

[13] $\mathcal{T}$ is a well-founded tree, hence we can use the notion of height given in Definition 4.1.7.

to infinite trees). Let S be the subset $\{1, 3, 5, 8, 9, 10, 12\}$ of the tree $\mathcal{T}$, highlighted in figure 4.11. On the first inductive step, no moment is removed. On the second step, we remove the set $\bigcup H_1$ consisting of $\{1, 3, 4\}$ from S, and add back 1, so we remove 3. Similarly, in step 3 we remove 8 and 10 (above 5), and in step 4 we remove 12 (above 9). There is nothing left to be removed in step 5. At the end, the only subset that is left is $S_5 = \{1, 5, 9\}$, which clearly is a maximal antichain contained in S whose elements are minimal.

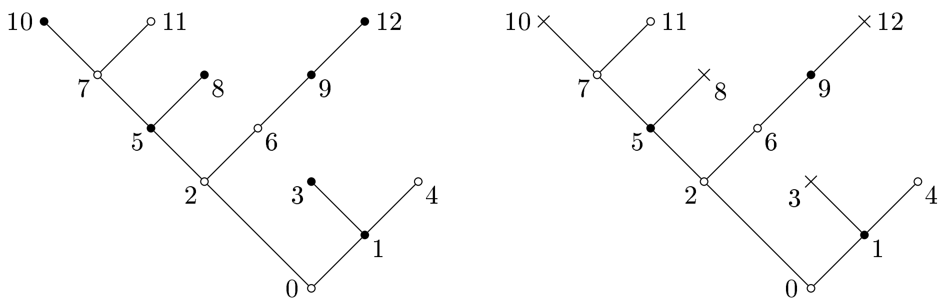


Figure 4.11: Example of the inductive process of Proposition 4.9.5.

Theorem 4.9.6. *Let $\mathcal{T}$ be a well-founded tree. $H(T)$ is compact if and only if every antichain is finite.*

Proof. We will now prove the two implications separately:

($\Rightarrow$) Let X be an antichain of $\mathcal{T}$. There are two possibilities: X is either a maximal antichain, or is not maximal.

In the first case, we have that $\bigcup_{x\in X} H_x = H(T)$, by Lemma 4.9.3. Moreover, $H_X = \{H_x \mid x \in X\}$ is an open covering of $H(T)$, thus, by compactness, it admits a finite subcovering. However, if we remove any x_0 from X, the set $\{H_x \mid x \in X \setminus \{x_0\}\}$ is not a cover any more, since x_0 was the only representative moment in X for each history $h \in H_{x_0}$, again by Lemma 4.9.3. Hence, H_X is itself its finite subcovering, so it is finite. So X is finite since the map $x \mapsto H_x$ is injective on antichains by Lemma 4.9.4.

In the second case, we have that $\bigcup_{x\in X} H_x \neq H(T)$, by Lemma 4.9.3. Then, we can replace X with $\tilde{X}$, a maximal antichain containing X. It can be built using the set $T(0)$ of the roots of the tree (it is well defined since the tree is well-founded), by adding to X the set of the roots which do not already belong to some history passing through some moment of X. Formally:

$$\tilde{X} = X \cup \left[T(0) \setminus \bigcup \big(\bigcup_{x\in X} H_x \big) \right]$$

Then, $\tilde{X}$ is a maximal antichain because the histories not already belonging to H_X pass through some root, hence $\bigcup_{x \in \tilde{X}} H_x = H(T)$ by Lemma 4.9.3. Now we can repeat the argument used above with $\tilde{X}$, and so we have that $\tilde{X}$ is finite. Hence, X is finite, since it is a proper subset of $\tilde{X}$.

($\Leftarrow$) Let $\mathcal{C} = \{H_x \mid x \in I\}$ be an open covering of $H(T)$. We need to "reduce" I to an antichain in order to use the hypothesis of finiteness: to do so, we consider the maximal antichain included in I, whose existence is granted by Proposition 4.9.5. Call it J. Then, for every $x \in I \setminus J$, there exists a unique $y \in J$, $y \smile x$. Moreover, by the last part of Proposition 4.9.5, $y \leq x$. Hence, $H_y \supseteq H_x$, so the set $\{H_y \mid y \in Y\}$ is a subcovering of $\mathcal{C}$. Moreover, by hypothesis, J is finite since it is an antichain. Then $\mathcal{C}$ has a finite subcovering. Thus, $H(T)$ is compact. □

This result generalizes Theorem 4.5.2 to every well-founded tree: in fact, if every antichain is finite, then every level (which is an antichain, as shown in the proof of Proposition 4.9.5) is finite, which is equivalent to the fact that the tree is finitely branching.

4.9.2 Motivation: the Souslin's Problem

This section is an overview on some classical results on this topic. Its aim is to show how the notion of special tree was firstly developed, and to investigate the notion of Souslin tree. Further (and advanced) developments of this topic can be found in [5] and [32].

The class of special trees is well known and studied in the literature because it is strictly linked to the *Souslin problem*, posed by Mikhail Y. Souslin (1849-1919), and published posthumously in M. Souslin, *Probléme 3*, Fundamenta Mathematicae, vol. 1 (1920). It investigates whether some properties of real numbers describe a subset of the real line isomorphic to the real line itself[14]. This problem can be "translated" in the topological language, and it becomes the converse of Proposition 3.1.8. Hence, it states:

Is every topological space satisfying the countable chain condition (ccc) necessarily separable[15] *?*

[14] The translation in English of the original formulation is "Let P be a complete dense unbounded linearly ordered set that satisfies the countable chain condition. Is P isomorphic to the real line?". In order to deepen this topic, the interested reader may refer to [15], pages 114 and following.

[15] See Definitions 3.1.18 and 3.1.19.

A counterexample would be called a *Souslin line*, while the conjecture that a Souslin line does not exist is called *Souslin's Hypothesis*.

D. Kurepa showed in 1935 (in [21]) that this problem can be rephrased in terms of trees, without any reference to topological considerations: Souslin's problem is reduced to a problem of combinatorial set theory, as follows.

Definition 4.9.3. A well-founded tree $\mathcal{T}$ is a *Souslin tree* if it has height ω_1, every chain is countable, and every antichain is countable.

Theorem 4.9.7 (Kurepa). *There exists a Souslin line if and only if there exists a Souslin tree.*

The statement of this theorem (in French) can be found in [21], §12.D.2, with the proof given by the equivalence $P2 \Leftrightarrow P5$ of the Fundamental Theorem, contained in the Appendix §C.3. A direct proof (in English), can be found in [15], Lemma 9.14, or in [11], Lemma 20.9.

Moreover, it can be proved that the Souslin's problem is independent of ZFC: it is a statement which can not be decided within the standard Zermelo-Fraenkel axiomatization for Set Theory with the *Axiom of Choice*. However, if we weaken the definition, we obtain what today is known as an *Aronszajn tree*, whose existence can be proved in ZFC, as we said in Section 4.7.1:

Definition 4.9.4. A well-founded tree is an *Aronszajn tree* if it has height ω_1, every chain is countable, and every *level* is countable.

One could ask whether every Aronszajn tree is Souslin, and the answer is no. This is a straightforward consequence of the independence of the Souslin problem. However, some particular Aronszajn trees which can be actually defined can be represented as a countable union of antichains, and an uncountable tree can not be countable union of countable sets, so it can not be Souslin.

Hence, the property of being decomposable as countable union of antichains seems to be interestingly linked to the Souslin problem, but was firstly considered only for Aronszajn trees: the contribution of S. Todorčević to this field of research was to examine this property for generic well-founded trees. In fact, as Brodsky says in [5], "Being Aronszajn is mainly a condition on the width of the tree, the cardinality of its levels; being special or non-special is a distinction in the number of its antichains, in some sense related to the height of the tree. We can consider one without the other". Thus, the approach introduced by Todorčević leads us to the definition and properties of Subsection 4.9.4.

4.9.3 Topological characterization of Souslin trees

We want to underline the fact that Theorem 4.9.7 by D. Kurepa shares our general attitude: it essentially suggests a translation from the topological language into the one of trees. P. J. Niykos states Theorem 4.9.9 below in [23]: this result is strongly related to the Souslin's Problem, and we will prove it in detail using the following preliminary Lemma.

Lemma 4.9.8. *Let $\mathcal{T}$ be a totally branching well-founded tree. Then, it is countable if and only if $H(T)$ is separable.*

Proof. ($\Rightarrow$) Let T be countable. Then, clearly, $\mathcal{O}_\mathcal{T}$ is countable, hence $H(T)$ is second countable[16]. Thus, by Proposition 3.1.7, $H(T)$ is separable.

($\Leftarrow$) Assume that $H(T)$ is separable. Then there exists a dense countable subset of $H(T)$: call it D. By Proposition 4.4.1, D is a bundle on $\mathcal{T}$, so for every $t \in T$ there exists $h \in D$ such that $t \in h$. In particular, T is a countable union of histories.

Suppose by contradiction that T is uncountable. Then there exists $h \in H(T)$ with uncountably many moments[17]. This implies that there is a subset of h consisting of different moments indexed by ω_1, namely $\{t_\alpha \mid \alpha \in \omega_1\} \subseteq h$, with $t_\alpha < t_\beta$ if $\alpha \in \beta$. Thus $H_{t_\beta} \subset H_{t_\alpha}$, because the tree is totally branching. Then, for every $\alpha, \beta \in \omega_1$ with $\alpha \in \beta$, there exists $h' \in H_{t_\alpha} \setminus H_{t_\beta}$. Call t'_α any moment in the future of t_α along h' such that $t'_\alpha \not\succ t_\beta$. In particular, t'_α and every moment in the future of t_β are uncomparable. Now, if we select for every $\alpha \in \omega_1$ a moment t'_α as above (in order to visualize this selection, see Figure 4.12), we have an uncountable set of pairwise uncomparable moments, so every history passing through one of them does not contain all the others. Then every bundle must be uncountable, and this contradicts the separability hypothesis. Thus, T is countable. □

Theorem 4.9.9. *A well-founded totally branching tree is a Souslin tree if and only if $H(T)$ is non separable and satisfies the ccc.*

Proof. ($\Rightarrow$) Assume that $\mathcal{T}$ is a Souslin tree.

Consider a collection $\mathscr{U}$ of pairwise disjoint non-empty open subsets of $H(T)$. If one of them is a union of elements of the base $\mathcal{O}_\mathcal{T}$, we replace it with all of these elements, possibly dropping some of them if they are contained in some others. By the rank 1 property, an open subset

[16] See Definition 3.1.17.

[17] Otherwise, T would be a countable union of countable sets, which is countable.

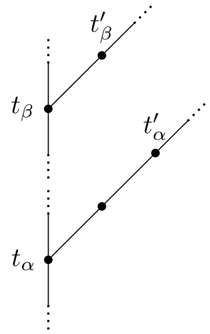


Figure 4.12: Example of the choices for t'_α and t'_β. The vertical line represents the history h: the remaining parts of the tree are not drawn.

of $H(T)$ can be always decomposed as a disjoint union of element of the base $\mathcal{O}_\mathcal{T}$. Then $\mathscr{U} = \{H_t \mid t \in U\}$ for a suitable subset U of T. Moreover, U is an antichain: given t, s in U, $H_t \cap H_s = \emptyset$ by construction, hence $t \not\leq s$. Then, U is countable because the tree is Souslin. The set $\mathscr{U}$ is countable too, because the tree is totally branching, so $H_t \neq H_s$ whenever $t \neq s$. Hence, $H(T)$ satisfies the *ccc*.

Moreover, let D be a dense subset of $H(T)$. Then, by Proposition 4.4.1, D is a bundle on $\mathcal{T}$, so for every $t \in T$ there exists $h \in D$ such that $t \in h$. Then, clearly, $\bigcup D = \bigcup_{h\in D} h = T$. Assume by contradiction that D is countable. By hypothesis, every chain is countable, so every history is countable. Thus, T is a countable union of countable sets, hence it is countable. This is a contradiction because the height of $\mathcal{T}$ is ω_1, so T can not be countable. So $H(T)$ can not be separable: in fact, every dense subset is not countable.

($\Leftarrow$) Suppose that $H(T)$ is not separable and satisfies the *ccc*.

Let A be an antichain of $\mathcal{T}$. Then, the set $H_A = \{H_t \mid t \in A\}$ is a collection of pairwise disjoint non-empty open subsets, so it is countable by hypothesis. Since $\mathcal{T}$ is totally branching, $H_t \neq H_s$ whenever $t \neq s$, hence the map $t \mapsto H_t$ from A to H_A is a bijection. Thus, A is countable.

Let C be a chain of $\mathcal{T}$, and suppose by contradiction that C is uncountable. We can proceed as in the proof of Lemma 4.9.8: we index the elements of C with ordinals below ω_1, so $C = \{t_\alpha \mid \alpha \in \omega_1\}$, with $t_\alpha < t_\beta$ if $\alpha \in \beta$. Then, if we select for every α a certain t'_α defined as above, we have that $\{t'_\alpha \mid t_\alpha \in C\}$ is a set consisting of pairwise uncomparable moments. Thus, $H_{t'_\alpha} \cap H_{t'_\beta} = \emptyset$ if $\alpha \neq \beta$. Then, $\mathscr{U} = \{H_{t'_\alpha} \mid t_\alpha \in C\}$ is an uncountable collection of pairwise disjoint non-empty open subsets of $H(T)$. And this is a contradiction, because $H(T)$ satisfies the *ccc* by hypothesis.

Finally, we show that the height of $\mathcal{T}$ is ω_1. We start by observing that T is uncountable, by Lemma 4.9.8 above, since $H(T)$ is not separable. Now, since every history of $\mathcal{T}$ is countable, the height of $\mathcal{T}$ is at most ω_1. Moreover, since every level of $\mathcal{T}$ is an antichain and therefore countable, and T is uncountable, then the height of $\mathcal{T}$ must be at least ω_1. Hence, the height of $\mathcal{T}$ is precisely ω_1.

Thus, $\mathcal{T}$ is a Souslin tree. □

This result is really interesting: it is an alternative proof of the fact that the existence of a Souslin tree implies the existence of a Souslin line, which is a really important result of Combinatorial Set Theory and Topology. In fact, if $\mathcal{T}$ is Souslin, the space $(H(T), \mathcal{O}_{\mathcal{T}})$ is non-separable and satisfies the *ccc*, hence it is a Souslin line.

4.9.4 Definition and properties of special trees

Definition 4.9.5. A well-founded tree $\mathcal{T} = (T, <)$ is said to be *special* if and only if it is a countable union of antichains, i.e. if there exists a collection $\{A_i \mid i \in \omega\}$ of antichains such that $T = \bigcup_{i\in\omega} A_i$.

In some works, like [32], there is the definition of κ-special tree (and of κ-Souslin, and of κ-Aronszajn), with κ any cardinal: they are trees that can be represented as a union of κ antichains. We will not pay further attention to this generalized definition, and just analyse $\aleph_0$-special trees, which are the special trees defined above.

Remark 4.9.3. We collect some easy results and remarks:

- There is no mention of uniqueness of the decomposition in the definition: in general, a special tree has many different decompositions into antichains.
- The collection of antichains does not form a partition of the tree, *a priori*: in general, there can be many common moments between two different antichains. This will be clarified in the examples below.
- If $\{A_i \mid i \in \omega\}$ is a collection of antichains whose union is the whole special tree, we can build a collection of disjoint (and possibly empty) antichains by collecting $B_i = A_i - \bigcup_{j>i} A_j$: their union is, again, the whole tree.

In Lemma 4.9.11 and in Theorem 4.9.12 we give two equivalent characterizations of special trees, both regarding the existence of particular maps from T to suitable countable sets.

Lemma 4.9.10. *Let $\mathcal{T}$ be a special tree, with $T = \bigcup_{i\in\omega} A_i$ an antichains decomposition. Then, the restriction to every chain of the function $f : T \to \omega$ sending $t \mapsto \min\{i \in \omega \mid t \in A_i\}$ is injective.*

Proof. Let C be a chain of $\mathcal{T}$. Let $x, y \in X$, $x \neq y$. Clearly, if $x \in A_i$, $y \notin A_i$, otherwise A_i would not be an antichain. Hence, f is injective. □

Lemma 4.9.11. *A tree $\mathcal{T}$ is special if and only if there exists $f : T \to \omega$ which is injective on chains.*

Proof. Sufficiency is a consequence of the previous lemma. Suppose now that such an f exists. Then the preimage $f^{-1}(i)$ is an antichain for every i: in fact, if by contradiction $x, y \in f^{-1}(i)$ with $x < y$, then f would not be injective on the chain $C = \{x, y\}$. Then, $T = \bigcup_{i\in\omega} f^{-1}(i)$ is an antichains decomposition for T. Then $\mathcal{T}$ is special. □

Definition 4.9.6. A poset $(P, <)$ (hence, a tree) is $\mathbb{Q}$-*embeddable* if there exists an order morphism from P to $\mathbb{Q}$ with the usual strict order[18].

Theorem 4.9.12. *A tree $\mathcal{T}$ is special if and only if it is $\mathbb{Q}$-embeddable.*

Proof. This result is a corollary of a more general statement proved in two different ways in [32], Theorem 9.1 page 284, and in [14], Théorème 1 page 172. It affirms that a poset $(P, <)$ is $\mathbb{Q}$-embeddable if and only if it is a countable union of antichains. □

The next Lemma is a link between special trees and Aronszajn trees: every special tree satisfies one of the conditions to be Aronszajn:

Lemma 4.9.13. *A special tree has no uncountable history.*

Proof. Let $T = \bigcup_{i\in\omega} A_i$ be a decomposition of T into antichains. Consider a history h. For every $i \in \omega$, $|h \cap A_i| \leq 1$, by Remark 4.9.2, hence $|h| = |\bigcup_{i\in\omega} A_i \cap h| \leq \sum_{i\in\omega} |A_i \cap h| \leq \sum_{i\in\omega} 1 = \aleph_0$. □

We conclude this section with some examples of special and non special trees:

Example 4.9.2. Many of the trees we drew in the previous chapters and sections are special trees: for example, the tree of Figure 4.10 can be represented as a countable union of antichains by $\{l, r\}$ and $\{n\}$ for every n.

Moreover, we have some entire classes which are contained in the class of special trees, namely:

[18] As stated in [5], "this is an unfortunate use of the term embeddable, as we do not require the morphism to be injective, so that it is not an embedding in the usual sense".

- Every finite tree is special: a decomposition that always works is made by all the singletons, plus infinitely many copies of the empty antichain.
- Every countable linear tree is special. Let α be the countable ordinal isomorphic to the tree, ϕ the isomorphism from α to the tree, and $f : \alpha \to \omega$ the invertible map that witnesses the countability of α. Then, the decomposition of the ordinal (hence, of the tree) into antichains is $\bigcup_{i\in\omega} \phi\big(f^{\leftarrow}(i)\big)$. Clearly, every antichain must be a singleton.
- Every well-founded tree which has a countable ordinal number as height is special, and a decomposition into antichains is made by the collection of the levels: they are maximal antichains of the tree.

It is interesting to consider examples of trees that are *not* special. Any uncountable ordinal number is a trivial example of a tree of this kind. In fact, the only possible decomposition into antichains is the one made by all the singletons, since it is a linear tree, but any countable union of singletons can not be the whole tree.

Two more interesting examples of a special and a non-special tree are provided by the following maps, which associate a tree to any poset[19]. Let $(P, <)$ be a poset, and denote by σP the set of all well-ordered subsets of P ordered by end-extension, i.e. $s \prec p$ if and only if s is an initial segment of p. Moreover, let $\sigma' P = \{t \in \sigma P \mid t \text{ has a maximal element}\}$, which is the collection of all well-ordered bounded sequences in P, again with end-extension. Notice that the map $\max(\cdot)$ is a strictly increasing function mapping from $\sigma' P$ to P. Hence, by Theorem 4.9.12, $\sigma'\mathbb{Q}$ is special: in fact, the map $\max(\cdot)$ shows that $\sigma'\mathbb{Q}$ is $\mathbb{Q}$-embeddable. However, $\sigma\mathbb{Q}$ is not special: this fact is proved in [32], Corollary 9.9 page 286. Moreover, both these trees are of height ω_1 with no uncountable chain. Finally, it can be shown that there is a subtree of $\sigma'\mathbb{Q}$ which is Aronszajn (and special): this proof can be found in [32], Theorem 5.2 page 257.

4.9.5 Topological characterization of special trees

Lemma 4.9.14. *Let $\mathcal{T}$ be a tree and A an antichain of $\mathcal{T}$. Then $[A]$ is an antichain of $[\mathcal{T}]$.*

Proof. By Proposition 3.2.11, $a \not\prec b$ implies $[a] \not\prec [b]$. □

Corollary 4.9.15. *Let $\mathcal{T}$ be a special tree. Then its condensation $[\mathcal{T}]$ is special.*

[19] This was originally defined by D. Kurepa (in [14], pages 236-237, which refers to the original work [21]), but we use the notation introduced by S. Todorčević in [32], page 245.

Proof. Let $T = \bigcup_{i\in\omega} A_i$ be a decomposition of T into antichains. By the previous Lemma, $[A_i]$ is an antichain for every i, thus $[T] = \bigcup_{i\in\omega}[A_i]$ is a decomposition of $[T]$ into a countable collection of antichains. □

Clearly, the inverse implication does not hold. In fact, if we consider a linear tree isomorphic to ω_1, its condensation consists of a single moment (hence, vacuously special), and ω_1 is clearly not special. Thus, as we have already observed many times in this work, we must focus our attention to the class of totally branching trees.

Let $\mathcal{T}$ be a special totally branching tree and $T = \bigcup_{i\in\omega} A_i$ be a decomposition of T into antichains. As observed in Remark 4.9.3, we can always assume that $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

If $t, s \in A_i$, then $t \not\smile s$, so $H_t \cap H_s = \emptyset$. In fact, if $h \in H_t \cap H_s$, both t and s would belong to h, and they would be comparable. Thus, if we call $H_{A_i} = \{H_t \mid t \in A_i\}$, we would have that H_{A_i} is a subset of the base $\mathcal{O}_\mathcal{T}$ consisting of pairwise disjoint open sets.

For any index k and $s \in T$, we have that $s \notin A_k$ if and only if $s \in A_j$ for some $j \neq k$, because ($T = \bigcup_{i\in\omega} A_i$ and) we are assuming that the A_i's are pairwise disjoint. Unfortunately, nothing more can be said about the relationship between $s \in A_j$ and any $t \in A_k$ for $j \neq k$: s and t can be either comparable or uncomparable.

As an example, consider the tree of Figure 4.13: it is totally branching and finite, hence we can consider the levels as (pairwise disjoint) antichains whose union is the whole tree. Clearly $b \in T(1)$, $d \in T(2)$ and $b \not\smile d$, but also $c \in T(1)$ and $c \smile d$. Moreover, $H_{T(0)} = H_{T(1)} = H(T)$, but $H_{T(2)}$ does not contain the history $\{a, b\}$.

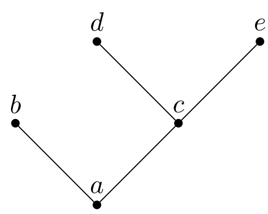


Figure 4.13

Thus, the property described above is a way to rearrange the elements of $\mathcal{O}_\mathcal{T}$, collecting them in a countable number of subfamilies of $\mathcal{O}_\mathcal{T}$ consisting of pairwise disjoint open sets. This can be done in many topological spaces: we just need to be able to separate the open sets of the base that have some common point into a countable number of different subfamilies.

However this can not be done if the base contains an uncountable subfamily of open subsets with a common point (but we are not sure this is the only case in which the property above is *not* verified). As a concrete example, consider the set ω_1 with the base of open subsets $\{\{1, \alpha\} \mid \alpha \in \omega_1\}$. We can not "separate" the base into a countable quantity of families of disjoint open subsets, because there are uncountable many of them which intersect in 1.

Moreover, this last condition is interestingly linked to special trees even in the other "direction". In fact, if we start with a tree $\mathcal{T}$ such that $\mathcal{O}_{\mathcal{T}}$ contains uncountably many open subsets having non empty intersection, then there exists a history h belonging to uncountably many H_t's, which implies that h contains uncountably many moments. Then the tree has uncountable height, so it can not be special, by Lemma 4.9.13.

All the remarks we collected above suggest that this is the right direction to reach the topological characterization of totally branching special trees: we leave this as an open problem for further studies on this topic.

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