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Tesi di Laurea

Is the monopole Dirac string real?

La stringa di Dirac è reale?

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Abstract

The main aim of this thesis is to address a question, recently raised in the literature, whether the existence of particles carrying a magnetic charge (instead of an electric one) is consistent with physical laws of Nature. Such particles are called monopoles. The possibility that they may exist in Nature was suggested by Dirac. The Dirac showed that for the monopoles to obey the laws of electrodynamics, to each of them there should be attached a magnetic flux string, called the Dirac string. This string must be invisible, i.e. un-physical. This requirement lead to the famous Dirac quantization condition for the electric and magnetic charges. There however appeared claims in a recent literature that the Dirac string cannot be invisible and therefore, according to the authors, the Dirac monopoles cannot exist as elementary particles.

In this thesis the classical theory of magnetic monopoles is studied in both Lagrangian and Hamiltonian formulation of electrodynamics, and its extension to quantum mechanics is also considered. This investigation ultimately confirms that the Dirac string is an auxiliary non-physical tool and the Dirac theory of monopoles provides a consistent description of these exotic particles.

Translation

L'obiettivo principale di questa tesi è esaminare una questione recentemente sollevata nella letteratura scientifica, ovvero se l'esistenza di particelle portatrici di carica magnetica (invece che elettrica) sia conforme alle leggi fisiche della Natura. Tali particelle vengono denominate monopoli. Dirac suggerì la possibilità che potessero esistere in Natura, dimostrando che, affinché i monopoli rispettino le leggi dell'elettrodinamica, a ciascuno di essi deve essere associata una stringa magnetica, detta stringa di Dirac. Questa stringa deve essere invisibile, ovvero non fisica. Questo requisito porta alla famosa condizione di quantizzazione di Dirac per le cariche elettriche e magnetiche. Tuttavia, in articoli più recenti sono emerse affermazioni secondo le quali la stringa di Dirac non può essere invisibile e, quindi, secondo gli autori, i monopoli di Dirac non possono esistere come particelle elementari.

In questa tesi verrà studiata la teoria classica dei monopoli magnetici sia nella formulazione Lagrangiana che Hamiltoniana dell'elettrodinamica, considerandone anche l'estensione alla meccanica quantistica. Questa indagine confermerà in ultima analisi che la stringa di Dirac è uno strumento ausiliario non fisico e che la teoria di Dirac dei monopoli offre una descrizione coerente di queste particelle esotiche.

Introduction

In his 1931 groundbreaking work [1] Dirac aimed to find an explanation why electric charge is always observed in integral multiples of the electron charge e. He believed there must be fundamental reasons in nature for these observations, as well as a specific cause for the exact value of the electric charge e.

Dirac inadvertently laid the foundations for the theoretical existence of magnetic monopoles suggesting that their existence could naturally explain the quantized nature of electric charge. This idea not only provided a theoretical justification for the quantization of electric charge but also hinted at a profound symmetry between electricity and magnetism, suggesting a more unified view of fundamental forces. This idea is reinforced by his second work [2], in which he formulated a relativistic extension of the monopole theory.

The core of this discussion centers on how an electric charge interacts with the vector potential generated by a magnetic monopole, which is connected to an infinitely long and thin solenoid known as the "Dirac string". This string's undetectability hinges on the assumption that the wave function of the electric charge is single-valued under gauge transformations, leading to the necessity of the Dirac quantization condition, $qg = 2\pi N$. If this condition holds then having even a single monopole located anywhere in the universe would justify the quantization observed in electric charges.

Despite the absence of direct experimental evidence for magnetic monopoles, Dirac's theory remains a cornerstone in the field of theoretical physics. It challenges physicists to reconsider the symmetries of nature and continues to inspire searches for magnetic monopoles in high-energy physics experiments and astrophysical observations.

In this work I will study the theory of magnetic monopole as originally formulated by Dirac. The first chapter is dedicated to exploring the electrodynamics of electric charge alone, setting the stage for the natural introduction of magnetic charge to restore the symmetry of duality. First, I will present the fundamental equations of electrodynamics for electric charge in covariant form. Subsequently, I will introduce the Lagrangian and its invariance under gauge transformations, followed by defining the action and formulating the action principle, through which the equations of motion can be derived. From there, I will present the transition to Hamiltonian mechanics using Dirac's constraint theory.

In Chapter 2, the inclusion of magnetic charge will restore electric-magnetic duality. The approach will mirror that of Chapter 1, starting with the presentation of the fundamental equations of this revised electrodynamics, highlighting the challenges presented by the variational principle. Indeed the natural incorporation of a vector potential is non-trivial, necessitating the introduction of the so-called Dirac string. The Lagrangian for the system will be presented, and the action principle defined, from which the equations of motion will be deduced. The transition to Hamiltonian mechanics will also be made through Dirac's theory of constraints. Ultimately, moving to quantum theory will yield the renowned Dirac quantization condition.

In the final chapter, I will focus on the Dirac string. Specifically, I will demonstrate that two vector potentials of the same monopole, but associated with different strings, differ by a gauge transformation. The gauge function in question will be presented and studied, providing an intuitive geometric interpretation. To conclude, I will comment on the article [3] that claims that the string generates a non-zero field momentum, thus suggesting it could be physically detectable. However, I will elucidate how this approach misunderstands Dirac's theory.

Conventions

- **Natural Units:** the system employs natural units, setting $c = \hbar = 1$.
- **Space-Time Metric:** the Minkowski space-time metric with the signature $(-, +, +, +)$ is used.
- **Index Notation:** Greek indices $(\mu, \nu, \lambda, \dots)$ represent space-time dimensions (0, 1, 2, 3), while Latin lower case indices $(i, j, k, \dots)$ denote three-dimensional spatial dimensions (1, 2, 3).
- **World Line Parameterization:** The motion of particles is described through world lines in space-time, parameterized as $y^\mu(\tau) = (y^0(\tau), \mathbf{y}(\tau))$.

Electrodynamics of electric charge

1.1 Equations of electrodynamics

The fundamental equations of electrodynamics for an electric charge under the presence of an electromagnetic field are

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,, \qquad \frac{d\varepsilon}{dt} = e\mathbf{v} \cdot \mathbf{E}\,, \tag{1.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho\,, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}\,, \tag{1.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0\,, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,. \tag{1.3}$$

These equations are respectively *Lorentz equations*, the *work-energy theorem* (1.1) and *Maxwell equations* (1.2)-(1.3).

In these equation ρ, $\mathbf{j}$ and $\mathbf{v}$ are the density, the current and the velocity of the charged particle, and $\mathbf{p}$ is its momentum, ε is the energy of the particle. These may be written in a covariant form introducing the electromagnetic anti-symmetric tensor $F^{\mu\nu} = -F^{\mu\nu}$ and its dual

$$* F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\,, \tag{1.4}$$

such that

$$F^{i0} = -F^{0i} = E^i\,, \qquad * F^{i0} = B^i\,, \tag{1.5}$$

$$F^{ij} = -\varepsilon^{ijk} B^k\,, \qquad * F^{ij} = \varepsilon^{ijk} E^k\,. \tag{1.6}$$

So the fundamental equations of electrodynamics in the manifestly covariant form become

$$\frac{dp^\mu}{d\tau} = eF^{\mu\nu}(y)u_\nu\,, \tag{1.7}$$

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \tag{1.8}$$

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{1.9}$$

In (1.7)-(1.8)-(1.9) τ is the proper time along the world line of a charged particle, $y^\mu(\tau)$ represents the space-time coordinates of the particle, parameterized by τ, u_ν is the four-velocity of the particle, defined as the derivative of y^μ with respect to τ and p^μ is the relativistic four-momentum of the particle.

The set (1.7)-(1.8)-(1.9) are known as *Lorentz equations*, *Bianchi identity* and *Maxwell equation* respectively. Using this more elegant and compact formulation one must introduce the four-current $j^\mu = (\rho, \mathbf{j})$. Equation (1.8) admit a canonical basis of solutions which are constructed by introducing an arbitrary vector field $A_\mu(x)$, called vector potential, and by setting

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{1.10}$$

1.1.1 Gauge invariance

Different vector potentials can give rise to the same field strength. Given an arbitrary scalar field $\Lambda(x)$, called gauge function, one can introduce a new vector potential by setting

$$A'_\mu = A_\mu + \partial_\mu \Lambda \,. \tag{1.11}$$

Gauge transformation constitutes a one-parameter group, with parameter $\Lambda(x)$ depending on a space-time point. Maxwell's equations are invariant under the gauge transformation.

1.2 Lagrangian formulation

The Lagrangian provides a concise and elegant description of the dynamics of a physical system, encapsulating the essential features of system's behavior. In the context of electrodynamics, the Lagrangian not only offers a systematic and mathematically elegant framework for deriving the equations of motion but also lays the foundation for exploring symmetries, conservation laws, and the quantization of fields. By adopting the principle of least action, we may unify the description of particles and fields, allowing us to seamlessly integrate classical and quantum concepts.

I will introduce the principle of least action, first for a classical theory of point particles and then extended to a field theory. I will then analyze the application of the Lagrangian formulation to electrodynamics. As we will see. the set of equations (1.1)-(1.3) and (1.9) can be deduced from the principle of least action.

1.2.1 Principle of least action

Consider first a non-relativistic particle of mass m with kinetic energy $\frac{1}{2}mu^i u_i$ in a potential $V(x^i)$. The Lagrangian is

$$L = T - V = \frac{1}{2}mu^i u_i - V(x^i), \tag{1.12}$$

where this quantity is defined for any path of $x^i(\tau)$ and $u^i(\tau) = dx^i/d\tau$. In general this is true for all the Lagrangian coordinates $q = \{q_n(t)\}$ of a system with N degrees of freedom where $n = 1, ..., N$.

$$L = L[q(t), \dot{q}(t)] \,. \tag{1.13}$$

We can now define the action for any path of $q(t)$ connecting two points $q(t_1)$ and $q(t_2)$, with $t_1 < t_2$.

$$I = \int_{t_1}^{t_2} L[q(t), \dot{q}(t)] dt \,. \tag{1.14}$$

The principle of least action states that the action is stationary under variation of the path about the actual path followed by the particle in motion from $q(t_1)$ to $q(t_2)$. The action is stationary when

$$\delta I = \frac{d}{d\alpha} I\left[q + \alpha\delta q\right]\Big|_{\alpha=0} = 0 \,, \tag{1.15}$$

or alternatively

$$\delta I = I\left[q + \delta q\right] - I\left[q\right] = 0, \tag{1.16}$$

where we consider only the linear term in δq, neglecting second order or higher terms. With this procedure one obtains the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{1.17}$$

which are equivalent to Newtonian equations of motion. For the Lagrangian (1.12) these are

$$m\ddot{x}^i = -\frac{\partial V(x^i)}{\partial x^i}\,. \tag{1.18}$$

In order to extend this procedure to field theories, and in particular to relativistic ones, we must substitute the Lagrangian coordinates with the Lagrangian fields $\varphi = \{\varphi_r(t, \mathbf{x})\}$, where $r = 1, ..., N$. Using Lorentz-covariant formulation we have

$$\varphi_r(t, \mathbf{x}) = \varphi_r(x^\mu)\,. \tag{1.19}$$

These fields are supposed to describe the system from kinematical point of view, every observable physical quantity can be expressed in terms of the fields φ, even though in general the fields themselves are not observables. The fields φ and their derivatives are required to vanish sufficiently fast at spatial infinity

$$\lim_{|x|\to\infty} \varphi(t, \mathbf{x}) = 0\,. \tag{1.20}$$

In field theory the Lagrangian L is the space integral of a Lagrangian density so that the action integral is

$$I = \int dtL = \int dt \int d^3x\mathcal{L} = \int d^4x\mathcal{L}. \tag{1.21}$$

In (1.21) the integration domain is not invariant since the time variable is integrated over a finite interval. To overcome this we should substitute in (1.14) the space-like hyperplanes $t = t_1$ and $t = t_2$ with two infinitely extended and non-intersecting space-like hyper-surfaces Γ_1 and Γ_2. So the generalized action is

$$I[\varphi] = \int_{\Gamma_1}^{\Gamma_2} \mathcal{L}(\varphi(x), \partial\varphi(x))d^4x\,. \tag{1.22}$$

By construction this is a Lorentz invariant integral. The Lagrangian fields φ satisfies the Euler-Lagrange equation

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)} - \frac{\partial\mathcal{L}}{\partial\varphi_r} = 0\,, \tag{1.23}$$

in the region between the hyper-surfaces Γ_1 and Γ_2 under arbitrary variations $\delta\varphi$, vanishing on the hyper-surfaces Γ_1 and Γ_2

$$\delta\varphi_r|_{\Gamma_1} = 0 = \delta\varphi_r|_{\Gamma_2}\,. \tag{1.24}$$

So the general scheme to follow for the formulation of a physical theory via the principle of least action is first identify the expression for the action, then derive the equations of motion with the action principle and finally apply Noether's theorem to

1.2.2 Action of electrodynamics

In the case of electrodynamics the Lagrangian fields are the four components of the vector potential A_μ. The connection between the electric and magnetic field and the potential vector arises from the resolution of the Bianchi identity which is solved as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{1.10}$$

From this relation it follows that, modulo the $U(1)$ gauge invariance $A'_\mu = A_\mu + \partial_\mu \Lambda(x)$, the vector field A_μ should be regarded as an independent physical field of Maxwell's theory, rather than $F_{\mu\nu}$.

The action we are searching for must lead to Maxwell's equations of motion. The symmetries of the action are a direct reflection of the symmetries inherent in the system's dynamics. This means that in order to have manifestly covariant equations of motion for a relativistic system we must have an action invariant under Poincaré transformations.

Other conditions we want to have in electromagnetic field theory is that the Lagrangian must preserve locality, signifying that it should be a function of the fields and their time derivatives at the same space-time point x^μ. Consequently, the evolution of a field φ should be determined by the values of the field and its time derivative evaluated at the same space-time point x^μ.

In addition, for the action of a relativistic particle we require reparameterization invariance, which means that the action does not depend on how the particle worldline is parameterized. If the parameterization is changed by regarding the old parameter τ as a function of a new parameter τ', then $I[\tau]$ should remain invariant.

The first step to add these features to the action is to choose $L\,dt$ in (1.14) be proportional to the proper-time measure, which is invariant under Poincaré transformation as well as under reparameterizations

$$ds = \sqrt{\frac{dy^\mu}{d\tau}\frac{dy_\mu}{d\tau}}d\tau\,. \tag{1.25}$$

The total action of electrodynamics, used to describe a charged particle subjected to an electromagnetic field, is made of three parts

$$I_{\mathrm{TOT}} = I_P + I_A + I_I\,. \tag{1.26}$$

The first one takes in to account the action of the free particle, the second one is the action of the electromagnetic field and the last one is the interaction term between the electromagnetic field and the particle. Explicitly the action is

$$I_{\mathrm{TOT}} = -m\int ds - \frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}d^4x - e\int d\tau \int \frac{dy^\mu}{d\tau}\delta^{(4)}(x-y(\tau))A_\mu(x)d^4x\,. \tag{1.27}$$

Where m is the mass and e is the electric charge of the particle. All the terms are Lorentz invariant. Using the principle of least action we get Maxwell's equations (1.2)-(1.3) from $I_\mathrm{A} + I_\mathrm{I}$ and the Lorentz equation (1.1) from $I_\mathrm{P} + I_\mathrm{I}$.

A condition that the action must fulfill is gauge invariance, in particular we require that vector potentials which differ by a gauge transformation yield identical dynamics.

Focusing on the total action (1.27), the vector potential appears in I_A through the field strength, where gauge transformations are nullified in the computation thanks to the Schwartz theorem, and also in I_I. The variation of the coupling term under gauge transformations of the form (1.11) is

$$\delta\left(-e\int dy^\mu A_\mu(y)\right) = -e\int dy^\mu \partial_\mu\Lambda(y) = e(\Lambda(\tau_i) - \Lambda(\tau_f))\,, \tag{1.28}$$

where τ_i and τ_f are initial and final points of the particle trajectory. So, the gauge variation of this term is just a number. Moreover the variation vanishes if one assumes that $\Lambda(\tau_i) = \Lambda(\tau_f)$. So, the action (1.27) is gauge invariant modulo a constant which does not affect the dynamics of the system.

1.3 Hamiltonian formulation

The transition from Lagrangian to Hamiltonian formulation, through the Legendre transformation, is a useful procedure for unraveling the dynamics of physical systems. This transformation introduces canonical coordinates and momenta, providing an alternative perspective on the system's evolution and the ground for the canonical quantization of the system.

However, the assumption of independent momenta and velocities proves too restrictive for many physical systems. This leads to the concept of primary constraints. These constraints are inherent conditions that arise when the momenta cannot be uniquely determined by the velocities due to system's physical properties. Incorporating these primary constraints into the Hamiltonian is not straightforward and requires the use of Lagrange multipliers.

I will define then secondary constraints which adds an additional layer of complexity to the Hamiltonian framework of classical electrodynamics. General techniques of the Hamiltonian description of constrained physical systems was developed by Dirac [13].

1.3.1 Poisson Brackets

In the Hamiltonian approach, the dynamics of a system is formulated with the use of Poisson brackets, which provide a powerful tool for studying the evolution of physical quantities. The canonical Poisson brackets are defined for the phase space variables q_j and p_j, and are expressed as

$$\{q_j, p_k\} = \delta_{jk}\,, \tag{1.29}$$

where δ_{jk} is the Kronecker delta. This relation establishes that the position coordinates q_j and the corresponding momenta p_j are canonically conjugate pairs. Furthermore, the Poisson bracket of any phase space function with itself is zero

$$\{q_j, q_j\} = \{p_j, p_j\} = 0\,. \tag{1.30}$$

The Poisson bracket of two functions f and g, defined on the phase space with coordinates (q_i, p_i), is given by:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \tag{1.31}$$

This bracket has several important properties, such as antisymmetry, linearity, and the Leibniz rule for products. Most notably, it satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{1.32}$$

for any three functions f, g, and h on the phase space. These properties make the Poisson brackets a fundamental aspect of the Hamiltonian mechanics, providing a framework to analyze the time evolution of dynamical variables and to establish relationships between conserved quantities and symmetries. The time evolution of any dynamical variable $A(q_i, p_i, t)$ is governed by its Poisson bracket with the Hamiltonian $H(q_i, p_i, t)$ of the system. Mathematically, this is expressed as:

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}, \tag{1.33}$$

where the first term on the right-hand side represents the evolution due to the system's dynamics, while the second term accounts for any explicit time dependence of the variable A.

Furthermore, this formulation connects to Noether's theorem. If a system has a symmetry, the associated conserved quantity can be identified through a Poisson bracket relation with the Hamiltonian.

$$\{A, H\} = 0 \,. \tag{1.34}$$

This implies that A is a constant of motion. Thus, the explicit relationship between symmetries and conserved quantities in Hamiltonian mechanics is given by the vanishing Poisson bracket of the symmetry generator with the Hamiltonian.

1.3.2 Primary constraints

The departing point for the Hamiltonian formalism is to define canonical momenta for Lagrangian coordinates and fields respectively

$$p_n = \frac{\partial L}{\partial \dot{q}_n} \,, \qquad n = 1, ..., N \,; \tag{1.35}$$

$$\pi_r = \frac{\partial L}{\partial(\partial_0 \varphi_r)} \,, \qquad r = 1, ..., N \,. \tag{1.36}$$

Usually in dynamical theory one assumes that the momenta are independent functions of velocities, however this is too restrictive. In general, when computing the transition from the Lagrangian formulation to the Hamiltonian one, via Legendre transformation, one obtains an equation that is identically zero, then the momenta cannot be expressed in terms of velocities or vice versa. In such cases there exists the following kind of relations that do not involve velocities

$$\Phi_m(q, p, \varphi, \pi) = 0 \,, \qquad m = 1, ..., M. \tag{1.37}$$

Equations (1.37) are called primary constraints and they do not vary in time, i.e. their Poisson bracket with the Hamiltonian must be zero. These constraints emerge when the Legendre transform from the Lagrangian to the Hamiltonian formulation does not lead to a unique definition of conjugate momenta for all generalized coordinates. This situation may, for instance, arise due to gauge symmetries present in the system.

The primary constraints are equations that define a constrained subspace within the full phase space. Specifically, if the system has N degrees of freedom, its phase space is a $2N$-dimensional manifold. However, the presence of primary constraints restricts the system to a lower-dimensional submanifold.

The non-uniqueness in the inverse transformation from Hamiltonian variables (q, p) to Lagrangian variables $(q, \dot{q})$ is due to these constraints. This non-uniqueness can be thought of as the transformation mapping a larger manifold (the $2N$-dimensional phase space) onto a smaller constrained subspace.

To properly formulate the Hamiltonian dynamics within this constrained subspace, the constraints themselves are added to the Hamiltonian with the help of Lagrange multipliers. These multipliers serve as coefficients that adjust the Hamiltonian to ensure adherence to the primary constraints in order to maintain the consistency of the system's description within the reduced phase space.

1.3.3 Total Hamiltonian

The canonical Hamiltonian is define by the Legendre transformation

$$H_{\mathrm{C}} = \dot{q} \cdot p + \dot{\varphi} \cdot \pi - L\,, \tag{1.38}$$

this is a function of the positions and velocities. However this Hamiltonian is not uniquely determined as a function of the p's and the q's, in fact the canonical Hamiltonian is well define only in the submanifold defined by the primary constraints (1.37). The formalism is thus generalized by defining the so called total Hamiltonian H_{T} incorporating all constraints into the canonical Hamiltonian, weighted by arbitrary multipliers λ_m, usually referred as Lagrange multipliers:

$$H_{\mathrm{T}} = H_{\mathrm{C}} + \lambda_m \Phi_m(q, p, \varphi, \pi)\,. \tag{1.39}$$

Since equation (1.37) holds one may think that H_{T} is just H_{C}, actually this is not true. A crucial matter in developing this theory is when to impose the constraints. To make this point more clear we should introduce *weak equalities* denoted by $\approx$. Weak equality between two quantities means that these quantities are equal modulo the constraints. That is they become "strongly" equal when the constraints are satisfied. So we can write

$$H_{\mathrm{T}} \approx H_{\mathrm{C}}\,. \tag{1.40}$$

Hamiltonian of the electromagnetic field

Considering the Lagrangian density for the electromagnetic field alone

$$\mathscr{L}_{\mathrm{A}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}\,. \tag{1.41}$$

We can separate the time and space directions and obtain

$$\mathscr{L}_{\mathrm{A}} = \frac{1}{2}\left(F_{0i}F_{0i} - \frac{1}{2}F_{ij}F_{ij}\right) = \frac{1}{2}\left[\left(\dot{A}_i(t,\mathbf{x}) - \partial_i A_0(t,\mathbf{x})\right)^2 - \frac{1}{2}\left(\partial_i A_j(t,\mathbf{x}) - \partial_j A_i(t,\mathbf{x})\right)^2\right]. \tag{1.42}$$

The canonical momenta are

$$\pi_\mu \equiv \frac{\partial \mathscr{L}}{\partial \dot{A}^\mu(t,\mathbf{x})} = \begin{cases} 0 & (\mu = 0) \\ \dot{A}_i(t,\mathbf{x}) - \partial_i A_0(t,\mathbf{x}) = E_i(t,\mathbf{x}) & (\mu = i) \end{cases} \tag{1.43}$$

Thus at any fixed time t we find the primary constraint for the first equation

$$\pi_0(t,\mathbf{x}) \approx 0\,. \tag{1.44}$$

The Hamiltonian constructed via the Legendre transform is

$$H_{\mathrm{A}} = \sum_{\varphi} \dot{\varphi} \cdot \pi_\varphi - L_{\mathrm{A}} = \tag{1.45}$$

$$= \frac{1}{2}\int d^3x \left(\mathbf{E}^2 + \mathbf{B}^2\right) + \int d^3x \left(\mathbf{E} \cdot \boldsymbol{\nabla} A_0(t,\mathbf{x})\right)\,. \tag{1.46}$$

We can add the primary constraint to the total Hamiltonian where λ_0 with the use of the Lagrangian multiplier $\lambda_0(t,\mathbf{x})$

$$H_{\mathrm{T}}^{\mathrm{A}} = \frac{1}{2}\int d^3x \left(\mathbf{E}^2 + \mathbf{B}^2\right) + \int d^3x \left(\mathbf{E} \cdot \boldsymbol{\nabla} A_0(t,\mathbf{x})\right) + \int d^3x \lambda_0(t,\mathbf{x}) \pi_0(t,\mathbf{x})\,. \tag{1.47}$$

Comparing to (1.46) I set $\dot{A}_0 = \lambda_0$ since $\dot{A}_0$ is undefined by its equation of motion $\dot{A}_0 = \{A_0, H_{\mathrm{T}}^{\mathrm{A}}\} = \lambda_0$. The primary constraint $\pi_0 \approx 0$ leads to a new constraint which is called *secondary constraint*

$$\frac{d\pi_0(t,\mathbf{x})}{dt} = \{\pi_0, H_{\mathrm{T}}^{\mathrm{A}}\} = \boldsymbol{\nabla} \cdot \mathbf{E} \approx 0\,, \tag{1.48}$$

Which is the Gauss law in the vacuum.

Hamiltonian of the free particle

For the case of a relativistic free particle we have

$$L_{\text{free}} = -m\sqrt{-\frac{dy_\mu}{d\tau}\frac{dy^\mu}{d\tau}} \equiv -m\sqrt{-\dot{y}_\mu \dot{y}^\mu}\,. \tag{1.49}$$

The transition to the Hamiltonian formulation via Legendre transform leads to

$$p_\mu = \frac{\partial L}{\partial \dot{y}^\mu} = m\frac{\dot{y}_\mu}{\sqrt{-\dot{y}_\mu \dot{y}^\mu}}\,, \tag{1.50}$$

$$H_{\text{C}} = p_\mu \dot{y}^\mu - L \equiv 0\,. \tag{1.51}$$

Since this expression is identically zero the system must have a constraint

$$p^\mu p_\mu + m^2 \approx 0, \tag{1.52}$$

and the total Hamiltonian is

$$H_{\text{T}} = H_{\text{C}} + \lambda(p^\mu p_\mu + m^2) = \lambda(p^\mu p_\mu + m^2),$$

where λ is the Lagrangian multiplier.

Another way to evaluate the Hamiltonian is fixing the reparametrization gauge: $y^\mu(\tau) = (\tau, \mathbf{y}(\tau))$, consequently $\dot{y}^0 = 1$, then we have

$$L_{\text{free}} = -m\sqrt{1-\dot{\mathbf{y}}^2}\,, \tag{1.53}$$

$$\mathbf{p} \equiv \frac{\partial L_{\text{free}}}{\partial \dot{\mathbf{y}}} = m\frac{\dot{\mathbf{y}}}{\sqrt{1-\dot{\mathbf{y}}^2}}\,. \tag{1.54}$$

Performing Legendre transform we find the Hamiltonian for the system of the free relativistic particle:

$$\begin{aligned} H_{\text{free}} &= \mathbf{p}\cdot\dot{\mathbf{y}} - L_{\text{free}} = \\ &= m\frac{\dot{\mathbf{y}}^2}{\sqrt{1-\dot{\mathbf{y}}^2}} + m\sqrt{1-\dot{\mathbf{y}}^2} = \\ &= m\frac{1}{\sqrt{1-\dot{\mathbf{y}}^2}} \equiv p_0 = \sqrt{\mathbf{p}^2+m^2}\,. \end{aligned} \tag{1.55}$$

Note that the relation $p_0 = \sqrt{\mathbf{p}^2+m^2}$ solves the constraint (1.52) of the manifestly reparametrization invariant formulation.

Hamiltonian of the charged particle

Introducing the electromagnetic field through the potential vector A^μ and assigning the particle the charge e in the Lagrangian

$$L_{\text{charge}} = -m\sqrt{-\dot{y}_\mu \dot{y}^\mu} - j_\mu A^\mu\,, \tag{1.56}$$

by inserting the explicit expression of the current we can write

$$L_{\text{charge}} = -m\sqrt{-\dot{y}_\mu \dot{y}^\mu} - eA^\mu(y)\dot{y}_\mu\,. \tag{1.57}$$

the conjugate momenta of y^μ are

$$p_\mu \equiv \frac{\partial L}{\partial \dot{y}^\mu} = \begin{cases} m\frac{-\dot{y}_0}{\sqrt{\dot{y}_0 - \dot{y}_i \dot{y}^i}} + eA_0(t, \mathbf{y}) & (\mu = 0) \\ m\frac{\dot{y}_i}{\sqrt{\dot{y}_0 - \dot{y}_i \dot{y}^i}} - eA_i(t, \mathbf{y}) & (\mu = i) \end{cases} \tag{1.58}$$

Performing Legendre transform we find an identically null Hamiltonian which leads to the constraint

$$\varphi_1 \equiv (p^\mu + eA^\mu)^2 + m^2 \approx 0\,. \tag{1.59}$$

If we fix the gauge $y^\mu = (\tau, \mathbf{y})$ like we did for the free particle we find

$$\text{Lagrangian: } L_{\text{charge}} = -m\sqrt{1 - \dot{\mathbf{y}}^2} - e(-A_0(t, \mathbf{y}) + \mathbf{A}(t, \mathbf{y}) \cdot \dot{\mathbf{y}})\,. \tag{1.60}$$

$$\text{Conjugate momenta: } \mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{y}}} = m\frac{\dot{\mathbf{y}}}{\sqrt{1 - \dot{\mathbf{y}}}} - e\mathbf{A}(t, \mathbf{y})\,. \tag{1.61}$$

The Hamiltonian via Legendre transformation is then

$$\begin{aligned} H_{\text{charge}} &= \mathbf{p} \cdot \dot{\mathbf{y}} - L_{\text{charge}} = \\ &= m\frac{1}{\sqrt{1 - \dot{\mathbf{y}}^2}} - eA_0(t, \mathbf{y}) = p_0 = \sqrt{m^2 + (\mathbf{p} + e\mathbf{A}(t, \mathbf{y}))^2} - eA_0(t, \mathbf{y})\,. \end{aligned} \tag{1.62}$$

Introducing $\mathbf{k}$, the kinetic momenta of the particle, as $k^\mu = (k^0, \mathbf{k}) = (p^0 + eA^0(t, \mathbf{y}), \mathbf{p} + e\mathbf{A}(t, \mathbf{y}))$, we can rewrite (1.62)

$$H_{\text{charge}} = \sqrt{m^2 + \mathbf{k}^2} - eA_0(t, \mathbf{y})\,. \tag{1.63}$$

Hamiltonian of electrodynamics

The Lagrangian of electrodynamics, imposing the gauge $y_\mu = (\tau, \mathbf{y})$ and separating time and space directions, is

$$\mathcal{L} = \frac{1}{2}\int d^3x \left(F_{0i}F_{0i} - \frac{1}{2}F_{ij}F_{ij}\right) - m\sqrt{1 - \dot{\mathbf{y}}} - e(-A_0(t, \mathbf{y}) + \mathbf{A}(t, \mathbf{y}) \cdot \dot{\mathbf{y}})\,. \tag{1.64}$$

Here $A^\mu(x^\mu)$ is shorthand notation for $\int d^3x A_\mu(t, \mathbf{x})\delta^{(3)}(\mathbf{x} - \mathbf{y})$. Evaluating now the momenta conjugate to the Lagrangian velocities $\{\dot{A}_\mu(x);\ \dot{y}^i\}$ of the action (1.27) we find respectively (1.43) and (1.61) which I report again for clarity

$$\pi_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu(t, \mathbf{x})} = \begin{cases} 0 & (\mu = 0) \\ E_i(t, \mathbf{x}) & (\mu = i) \end{cases} \tag{1.43}$$

$$\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{y}}} = m\frac{\dot{\mathbf{y}}}{\sqrt{1 - \dot{\mathbf{y}}}} - e\mathbf{A}(t, \mathbf{y})\,. \tag{1.61}$$

The momenta are to be considered as independent variables, and the full phase space has coordinates A_μ, π^μ with fundamental equal-time Poisson brackets $\{\dot{A}_\mu(t, \mathbf{x}_2), \pi^\nu(t, \mathbf{x}_1)\} = \delta^\nu_\mu \delta(\mathbf{x}_2 - \mathbf{x}_1)$. The canonical Hamiltonian (avoiding the constraints) is then the result of the Legendre transform (1.38) of the Lagrangian (1.64). Its explicit form is

$$\begin{aligned} H_{\text{EM}} &= H_{\text{A}} + H_{\text{charge}} = \\ &= \frac{1}{2}\int d^3x(\mathbf{E}^2 + \mathbf{B}^2) + \int d^3x\ eA_0(t, \mathbf{x})\left(e\delta^{(3)}(\mathbf{x} - \mathbf{y}(t)) - \boldsymbol{\nabla} \cdot \mathbf{E}\right) + \sqrt{m^2 + \mathbf{k}^2}\,. \end{aligned} \tag{1.65}$$

However the total Hamiltonian with the primary constraint (1.44) is

$$H_{\mathrm{T}} = H_{\mathrm{C}} + \int d^3x \lambda_0(t,\mathbf{x})\pi_0(t,\mathbf{x})\,. \tag{1.66}$$

To accurately describe the system's dynamics, the constraints must remain null throughout their evolution in time. By requiring $\dot{\pi}^0 = \{\pi_0, H_{\mathrm{T}}\} \approx 0$ we find the new constraint, which differs from (1.48) by the contribution of the electric charge density of the particle

$$\chi(t,\mathbf{x}) \equiv \frac{d\pi^0(t,\mathbf{x})}{dt} = \boldsymbol{\nabla}\cdot\mathbf{E} - e\delta^{(3)}(\mathbf{x}-\mathbf{y}(\tau)) \approx 0\,, \tag{1.67}$$

which is the Gauss law. So the total Hamiltonian is now

$$H_{\mathrm{T}} = H_{\mathrm{C}} + \int d^3x \lambda_0(t,\mathbf{x})\pi_0(t,\mathbf{x}) + \int d^3x \lambda_1(t,\mathbf{x})\chi(t,\mathbf{x})\,. \tag{1.68}$$

Adding the magnetic charge

In empty space Maxwell's equation possess a symmetry under the exchange of the electric and magnetic field, called electromagnetic duality. This symmetry is broken in the presence of the electrically charged particles as can be seen looking at the form of the equations (1.2) and (1.3).

In this chapter I will introduce the theory of electrodynamics in which the magnetic charge, monopole, is also present which restores the electric-magnetic duality symmetry. I will present two consecutives Dirac's approaches: firstly, modifying the vector potential to take in account the monopole, which results in the emergence of the Dirac string as a singularity within the vector potential. Secondly, its relativistic extension. I will specifically focus on the latter approach, exploring its implications for the dynamics of electromagnetic fields through the action principle for magnetic charges.

2.1 Electromagnetic duality

As I mentioned, Maxwell's equations in empty space are symmetric under the replacement

$$\mathbf{E} \to \mathbf{B}\,, \qquad \mathbf{B} \to -\mathbf{E}\,. \tag{2.1}$$

If we want Maxwell's equations to remain invariant under duality not only in empty space but also in presence of non-vanishing sources, we should introduce a magnetic four-vector current j_g^μ, to which we assign the same property of the electric one j_e^μ. The generalized Maxwell equations are then

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho_e\,, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_e\,; \tag{2.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \rho_g\,, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{j}_g\,. \tag{2.3}$$

To formulate these equations in a covariant formalism we use the electromagnetic dual of the Maxwell tensor $F^{\mu\nu}$, or equivalently its Hodge dual. So in terms of electromagnetic tensor the transformation (2.1) can be express as

$$F^{\mu\nu} \to *F^{\mu\nu}\,, \qquad *F^{\mu\nu} \to -F^{\mu\nu}\,. \tag{2.4}$$

Consequently the generalized Maxwell equations in manifestly covariant form are

$$\partial_\mu F^{\mu\nu} = j_e^\nu\,, \qquad \partial_\mu * F^{\mu\nu} = j_g^\nu\,. \tag{2.5}$$

Here j_g and j_e are point-like currents related to the motion of charges and poles

$$j_e^\nu = e \int \delta^{(4)}(x - y(\tau))\dot{y}^\nu d\tau\,, \qquad j_g^\nu = g \int \delta^{(4)}(x - z(\tau))\dot{z}^\nu d\tau\,. \tag{2.6}$$

2.2 Failure of variational principle

In this section, we examine the theoretical consequences of incorporating magnetic monopoles into the electromagnetic framework, focusing on the emergence of the *Dirac string*.

The introduction of a non-vanishing magnetic current at the right hand side of what was originally the Bianchi identity (2.5-b) raises a series of problems.

In the absence of a magnetic current, the Bianchi identity implies the existence of a vector potential, essential for constructing the dynamics of the electromagnetic field and its coupling to sources in both Lagrangian and Hamiltonian formulations. However, when the magnetic current is non-zero, this identity is violated, posing an obstacle in introducing a vector potential in a natural way.

2.2.1 Singular vector potential, the 1931 approach

In his first work [1] Dirac introduce the magnetic monopole by choosing a specific form of the vector potential, this strategy is necessary to be able to use Poincaré lemma. The magnetic field in the presence of a monopole lies along the radial direction and is of magnitude g/r^2 which means the domain we are working on is $\mathbb{R}^3\backslash\{0\}$ and it is not contractible, as instead required by the Poincaré lemma. Restricting the domain to $V_\gamma = \mathbb{R}^3\backslash\gamma$ it becomes contractible, here γ is a curve called *Dirac string* and in [1] is referred as nodal line.

In his paper Dirac presented a formula for the vector potential in spherical coordinates as $\mathbf{A} = (g/r)\tan(\theta/2)\hat{\phi}$ observing that its curl yields the radial field $g\hat{\mathbf{r}}/r^2$ in all space except at $r = 0$ (where the charge is located) and along the negative semi-axis. He pointed out that "this solution is valid at all points except along the line $\theta = \pi$ where A becomes infinite". This solution can be expressed equivalently as

$$\mathbf{A} = g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}\,. \tag{2.7}$$

Jackson [7] proposes an interpretation for this result: the magnetic monopole is imagined either as a particle situated at the end of a sequence of dipoles or at the extremity of a closely coiled solenoid or string extending indefinitely. He presents an integral form for the total vector potential for a solenoid lying on the curve γ

$$\mathbf{A} = -g\int_\gamma d\boldsymbol{\gamma}' \times \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\,. \tag{2.8}$$

This results in (2.7) when we analyze the scenario where the string is aligned along the negative z-axis, with the magnetic monopole at the origin, so $d\boldsymbol{\gamma}' = dx'\hat{\mathbf{z}}$. It's important to note that integration along this trajectory requires the condition $\sin\theta \neq 0$ so Dirac potential (2.7) is singular for all $\mathbf{x} \in \boldsymbol{\gamma}$. Consequently, across the entire domain, the curl of the vector potential yields

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{g}{r^2}\hat{\mathbf{r}} + \mathbf{B}_{\text{string}}\,. \tag{2.9}$$

The field of the string can be modeled using distributions, with explicit derivation provided in Appendix II.

$$\mathbf{B}_{\text{string}} = 4\pi g\delta(x)\delta(y)\Theta(-z)\hat{\mathbf{z}}\,, \tag{2.10}$$

$$\mathbf{B}_{\text{mon}} = \boldsymbol{\nabla}\times\mathbf{A} - \mathbf{B}_{\text{string}}\,. \tag{2.11}$$

In conclusion, the magnetic field generated by a static monopole admits, locally and in a restricted domain V_γ a vector potential. $\mathbf{A}$ is singular along the curve $\boldsymbol{\gamma}$, which connects the magnetic charge to infinity, while it's regular for all $\mathbf{x} \in V_\gamma$.

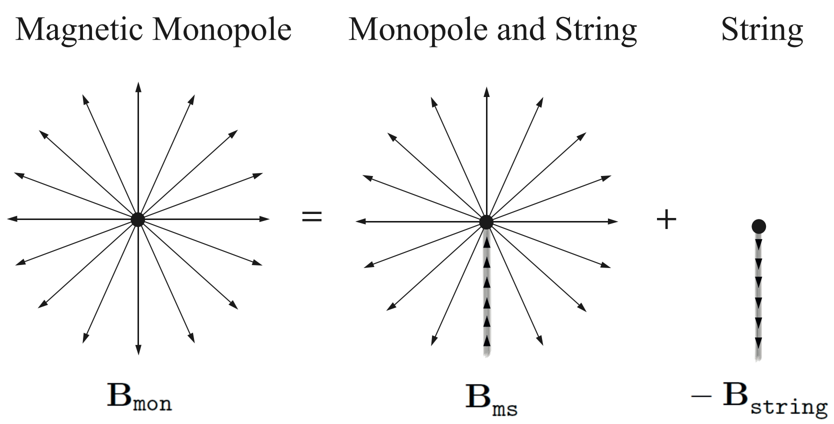


Figure 1: Representation of the monopole field $\mathbf{B}_{\rm mon}$ defined by equation (2.11).

2.2.2 The 1948 relativistic extension

In his second approach [2] Dirac proposed a more general description of magnetic monopole physics providing a complete dynamical theory by bringing a four-dimensional extension of his 1931 work.

In detail, the Bianchi identity demands the total magnetic flux crossing any closed surface to be zero. So equation

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \,. \tag{1.10}$$

must fail somewhere on the surface, since it fails in any closed surface surrounding the pole, equation (1.10) fails on a line of points, a *string* extending outward the pole. The string may be any curved line, extending from the pole to infinity or another pole of equal and opposite strength. The string then sweeps a two dimensional sheet in space-time, with parametrization

$$w^\mu = w^\mu(\tau, \sigma) \,. \tag{2.12}$$

Since one extremity of the string must always coincide with the magnetic pole, it is suitable to introduce a proper string coordinate $u(\tau, \sigma)$ such that

$$w^\mu(\tau, \sigma) = z^\mu(\tau) + u^\mu(\tau, \sigma), \qquad\qquad u^\mu(\tau, 0) = 0. \tag{2.13}$$

In the presence of magnetic monopoles equation (1.10) is not valid, since (1.10) leads to (1.3-a) and thus contradicts (2.5-b). The form of electromagnetic tensor should be replace by an expression of the form

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + {*C_{\mu\nu}} \,, \tag{2.14}$$

where C is a tensor whose support is on the string worldsheet $w(\tau, \sigma)$. So C vanishes everywhere except on the string. C is determined in such a way that (2.5-b) holds

$$\partial_\mu C^{\mu\nu} = j_g^\nu \,. \tag{2.15}$$

A solution for (2.15) has been proposed by Dirac in [2]

$$C_{\mu\nu}(x) = -g \int\!\!\int \left(\frac{\partial w_\mu}{\partial \tau} \frac{\partial w_\nu}{\partial \sigma} - \frac{\partial w_\nu}{\partial \tau} \frac{\partial w_\mu}{\partial \sigma} \right) \delta^{(4)}(x - w(\tau, \sigma)) d\tau d\sigma \,. \tag{2.16}$$

It is possible to show that the string term contained in the new definition of the field strength (2.14) leads also to (2.10)

$$\begin{aligned} B^i_{\text{string}}(t,\mathbf{x}) = C^{i0}(t,\mathbf{x}) = g\int_\gamma d\sigma \frac{\partial w^i(\tau,\sigma)}{\partial\sigma}\delta(\tau-t)\delta^{(3)}(\mathbf{x}-\mathbf{u}(\tau,\sigma)) = \\ = g\int_\gamma d\sigma \frac{\partial u^i(t,\sigma)}{\partial\sigma}\delta^{(3)}(\mathbf{x}-\mathbf{u}(t,\sigma)) = \\ = g\int_\gamma du^i(t,\sigma)\delta^{(3)}(\mathbf{x}-\mathbf{u}(t,\sigma)) \,. \end{aligned} \tag{2.17}$$

And if we consider the stationary case the time dependence disappear. Fixing the string position to the negative semi-axis z we get (2.10) (Explicit derivation in Appendix III). From (2.17) we can also obtain (2.3-a)

$$\nabla_{\mathbf{x}} \cdot \mathbf{B}_{\text{string}} = g\int_\gamma d\mathbf{u}(\sigma)\cdot\nabla_{\mathbf{x}}\delta^{(3)}(\mathbf{x}-\mathbf{u}(\sigma)) = -\int_\gamma d\mathbf{u}(\sigma)\cdot\nabla_{\mathbf{u}}\delta^{(3)}(\mathbf{x}-\mathbf{u}(\sigma)) \,. \tag{2.18}$$

Let $\mathbf{u}_0$ and $\mathbf{u}_f$ be the points where the string γ originates and terminates so (2.18) is equal to

$$\nabla_{\mathbf{x}} \cdot \mathbf{B}_{\text{string}} = g\delta(\mathbf{x}-\mathbf{u}_0) - g\delta(\mathbf{x}-\mathbf{u}_f) \,. \tag{2.19}$$

The right hand side is equal to the magnetic density of the charge g located at $\mathbf{u}_0$ and a magnetic charge $-g$ located at $\mathbf{u}_f$.

2.3 Lagrangian formulation

In this section I will present the Lagrangian of the system involving both electric and magnetic charge, derived through the principle of least action. The procedure is analogous to the system exclusively featuring the electric charge. This formulation will, in turn, yield the equation of motion of the magnetic charge.

2.3.1 Principle of least action for the magnetic charge

The action of a system with electric and magnetic charge is composed of three parts, analogously to the electrodynamics of electrically charged particles, namely $I = I_{\text{P}} + I_{\text{A}} + I_{\text{I}}$.

$$I_{\text{TOT}} = I_{\text{P}} + I_{\text{A}} + I_{\text{I}} \,, \tag{2.20}$$

$$I_{\text{P}} = -m_e\int d\tau\sqrt{\dot{y}_\mu\dot{y}^\mu} - m_g\int d\tau\sqrt{\dot{z}_\mu\dot{z}^\mu} \,, \tag{2.21}$$

$$I_{\text{A}} = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^4x \,, \tag{2.22}$$

$$I_{\text{I}} = -e\int\int \frac{dy_\mu}{d\tau}\delta^{(4)}(x-y(\tau))A^\mu(x)d^4x d\tau \,. \tag{2.23}$$

Where I_{P} is the kinetic action for the (electric and magnetic) particles, I_{A} is the action for the electromagnetic field, where the presence of the monopole is taken into account in the field $F^{\mu\nu}$ as defined in (2.14). I_{I} describes the interaction of the electrically charged particle with the electromagnetic field.

The Lagrangian 'coordinates' here are $y^\mu(\tau)$, $z^\mu(\tau)$, $u^\mu(\tau,\sigma)$ and $A_\mu(x)$. Varying the action with respect to $A_\mu(x)$ we get Equation (2.5-a). Equation (2.5-b) is automatically satisfied from the definition of C. Varying instead the monopole and the charged particle trajectories y^μ and z^μ respectively we obtain

$$m_e \frac{dp_e^\mu}{d\tau} = e[\partial^\mu A^\nu - \partial^\nu A^\mu] u_\nu \,, \qquad\qquad m_g \frac{dp_g^\mu}{d\tau} = g * F^{\mu\nu}(z) u_\nu \,. \tag{2.24}$$

The first of the above equations is equal to the standard Lorentz force induced by the field strength (2.14) everywhere except on the string. So one can replace $\partial^\mu A^\nu - \partial^\nu A^\mu$ with $F_{\mu\nu}$ if the electric particle never passes through the string. So we impose the condition

$$y(\tau) \neq w(\tau, \sigma) \,, \tag{2.25}$$

which is known as *Dirac's veto*. This implies that on the string the electric current is zero and (2.24-a) becomes

$$m_e \frac{dp_e^\mu}{d\tau} = e F^{\mu\nu}(y) u_\nu \,. \tag{2.26}$$

The variation of the action with respect to the string coordinates produce

$$\frac{\partial}{\partial z^\mu} F^{\nu\mu}(y) = 0 \quad \rightarrow \quad \frac{\partial}{\partial w^\mu} F^{\nu\mu}(w) = 0 \,, \tag{2.27}$$

holding at all points on the string worldsheet. (2.27) is not a new equation, since imposing Dirac's veto we get $j_e(w) = 0$, and (2.27) is a consequence of (2.5). By imposing the Dirac's veto we have obtained the generalized Maxwell-Lorentz equations from the variation of the action (2.20). In this treatment the position of the string has remained undetermined.

2.4 Hamiltonian formulation

In order to derive the Hamiltonian of the system by performing the Legendre transformation we should first of all derive the canonically conjugated momenta. The Lagrangian coordinates are the electric charge and magnetic pole coordinates, the four-potential A^μ and the coordinate of the string. Choosing a parametrization that singles out the time: $t = y^0 = z^0 = w^0 = \tau$ we find the following momenta

$$p_e^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}^i} = m_e \frac{\dot{y}^i}{\sqrt{1-\dot{\mathbf{y}}^2}} + e A^i(t, \mathbf{y}) \,, \tag{1.58}$$

$$p_g^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{z}^i} = m_g \frac{\dot{z}^i}{\sqrt{1-\dot{\mathbf{z}}^2}} + g \int_0^\infty d\sigma \left(\boldsymbol{\pi}(w(\sigma)) \times \frac{\partial \mathbf{u}}{\partial \sigma} \right)^i \,, \tag{2.28}$$

$$\pi_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu(t, \mathbf{x})} = \begin{cases} 0 & (\mu = 0) \\ \dot{A}_i(t, \mathbf{x}) - \partial_i A_0(t, \mathbf{x}) = E_i(t, \mathbf{x}) & (\mu = i) \end{cases} \,, \tag{1.43}$$

$$p_u^i \equiv \frac{\partial L}{\partial \dot{u}^i(\sigma)} = g \left(\mathbf{E}(w(\sigma)) \times \frac{\partial \mathbf{u}}{\partial \sigma} \right)^i \,. \tag{2.29}$$

The canonical Hamiltonian is then

$$H_{\rm C} = \frac{1}{2}\int d^3x (\mathbf{E}^2 + \mathbf{B}^2) + \sqrt{m_e^2 + \mathbf{k}_e^2} + \sqrt{m_g^2 + \mathbf{k}_g^2} + \int d^3x \, e A_0(t, \mathbf{x}) \left(e\delta^{(3)}(\mathbf{x} - \mathbf{y}) - \boldsymbol{\nabla} \cdot \mathbf{E} \right) . \tag{2.30}$$

Where $\mathbf{k}_e$ and $\mathbf{k}_g$ are the kinetic momenta of the particles

$$k_e^i = p_e^i - e A^i(\mathbf{y}) = m_e \dot{y}^i \,, \qquad k_g^\mu = p_g^i - g \int_0^\infty d\sigma \left(\mathbf{E}(w(\sigma)) \times \frac{\partial \mathbf{u}}{\partial \sigma} \right)^i = m \dot{z}^i \,. \tag{2.31}$$

Since in the canonical momenta not all the velocities $\dot{q}$ can be expressed as function of the $q's$ and the $p's$ the total Hamiltonian must be of the form of (1.66). The two primary constraints which follow from (1.43) and (2.29) are

$$\pi^0(\mathbf{x}) \approx 0\,, \tag{1.44}$$

$$\varphi^i(\sigma) \equiv p_u^i(\sigma) - g\left(\mathbf{E}(w(\sigma)) \times \frac{\partial \mathbf{u}}{\partial \sigma}\right)^i \approx 0\,. \tag{2.32}$$

The Hamiltonian plus primary constraints is

$$H^* = H_\mathrm{C} + \int d^3x \lambda_0(t,\mathbf{x})\pi_0(t,\mathbf{x}) + \int d^3x\; \boldsymbol{\lambda}(\sigma)\cdot \boldsymbol{\varphi}(\mathbf{x})\,, \tag{2.33}$$

where λ_0 and $\boldsymbol{\lambda}$ are Lagrange multipliers.

To make a consistent description we need to verify if the constraints are conserved during the time evolution of the system as we did in Section 1.3.3 via the Poisson bracket. First consistency condition $\chi(\mathbf{x}) = \{\pi^0, H\}$ is (1.67) which we discussed in the previous chapter, the second one $\{\boldsymbol{\varphi}(\sigma), H\}$ is

$$\begin{aligned}&\boldsymbol{\chi}(\sigma) \equiv \frac{d\boldsymbol{\varphi}(\sigma)}{dt} = \\ &= eg\frac{\mathbf{k}_e}{\sqrt{m_e^2+\mathbf{k}_e^2}} \times \frac{\partial \mathbf{u}}{\partial \sigma}\delta(\mathbf{w}(\sigma)-\mathbf{y}) - g\left[\left(\frac{\mathbf{k}_g}{\sqrt{m_g^2+\mathbf{k}_g^2}} + \mathbf{v}_0\right) \times \frac{\partial \mathbf{u}}{\partial \sigma}\right] \boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{w}(\sigma)) \approx 0_\mathrm{D}\,.\end{aligned} \tag{2.34}$$

The symbol 0_D denotes a quantity which is zero if Dirac's veto (2.25) holds, in fact equation (2.34) actually vanishes independently of the multiplier $\boldsymbol{\lambda}(\sigma)$ in (2.33) after imposing (2.25). So, by requiring $\dot{\pi}^0(\mathbf{x}) \approx 0$ and $\dot{\boldsymbol{\varphi}}(\sigma) \approx 0$ we find two secondary constraints one of which is the Gauss law. In order to have a correct formulation we should also check also the Poisson bracket algebra for the constraints. All the brackets vanish identically and independently of the veto except

$$\{\xi^r, \xi^s\} = -eg\varepsilon^{rks}\frac{\partial u^k}{\partial \sigma}\delta(\sigma-\sigma')\delta^{(3)}(\mathbf{y}-\mathbf{y}(\sigma)) = 0_\mathrm{D}\,. \tag{2.35}$$

Then all our constraints are first class constraints when the veto holds. The total Hamiltonian is thus

$$H_\mathrm{TOT} = H^* + \int d^3x \lambda_1(t,\mathbf{x})\chi(t,\mathbf{x})\,. \tag{2.36}$$

2.5 Quantization condition

This chapter will focus on the Dirac quantization condition. The first study of the quantization of electric charge, first discussed in 1931 [1], led him to hypothesize the existence of magnetic monopoles. Dirac had two objectives, firstly he wanted to explain the electric charge quantization and, secondly, he wanted to find the reason for the observed experimental value of the elementary electric charge. Dirac explicitly expressed these goals in [10]: "I was not searching for anything like monopoles at the time. What I was concerned with was the fact that electric charge is always observed in integral multiples of the electronic charge e, and I wanted some explanation for it. There must be some fundamental reason in nature why that should be so, and also there must be some reason why the charge e should have just the value that it does have. It has the value that makes $\hbar c/e^2$ approximately 137. And I was looking for some explanation of this 137." In his paper Dirac introduced the concept of semi-infinite magnetized lines (strings) terminating in a monopole, which he used to establish the quantization condition $eg = \frac{1}{2}n\hbar c$, where n is an integer.

The theoretical studies before Dirac's proposal largely dismissed the existence of magnetic monopoles. Maxwell's equations, foundational to classical electrodynamics, were formulated under the assumption of the nonexistence of free magnetic charges. Quantum mechanics, in its early development, reinforced this point, with the necessity of incorporating the vector potential in its formulation.

However, Dirac's approach ingeniously reconciled these theoretical frameworks, as I showed reporting his formulation in the previous sections. He succeeded in demonstrating that magnetic monopoles could indeed be integrated into the formulation of both classical and quantum electrodynamics.

Following Dirac's proposal, numerous derivations and interpretations have emerged, further exploring and validating the quantization condition. These include semi-classical derivations by Saha [11] and Wilson [12], and contributions by Fierz and Schwinger [9], which underscore the robustness of Dirac's theory within the broader spectrum of theoretical physics.

In this chapter, we delve into a detailed analysis of the Dirac quantization condition, exploring its mathematical support. The focus will be on how Dirac's hypothesis of magnetic monopoles serves not merely as a theoretical curiosity but as a fundamental element in understanding the nature of electric charge quantization.

Formulating an appropriate quantization condition

The integer coefficient N in Dirac's quantization rule $eg = 2\pi N$ is derived from the trigonometric property that $\cos(2\pi N)$ equals one for N being an integer. This can also be represented as:

$$\mathrm{e}^{i2\pi N} = 1, \tag{2.37}$$

which follows from Euler's relation. In spherical coordinates, the azimuthal angle φ and $\varphi + 2\pi$ denote the same spatial point, which allows for the definition of function $F(\varphi)$ that is single-valued under 2π periodicity, i.e. $F(\varphi) = F(\varphi + 2\pi)$. However, a function like $F(\varphi) = \varphi$ does not satisfy this periodicity condition.

For the complex function $F(\varphi) = \mathrm{e}^{i2k\varphi}$, k must be confined to half-integer values to maintain the function to be single-valued, leading to:

$$k = \frac{N}{2}, \quad N \in \mathbb{Z}. \tag{2.38}$$

2.5.1 Transition to Quantum Mechanics

The transition to quantum mechanical theory is made by promoting the dynamical coordinates and momenta to operators and their Poisson bracket to i times the corresponding commutators. First class constraints (2.32) become now subsidiary conditions imposed on the state vector ψ, explicitly

$$\varphi^j(\sigma)\psi = 0\,, \tag{2.39}$$

which in the space of coordinates corresponds to

$$\left[-i\frac{\partial}{\partial u^j(\sigma)} - g\left(\mathbf{E}(w(\sigma)) \times \frac{\partial \mathbf{u}}{\partial \sigma}\right)^j\right]\psi(\mathbf{u}) = 0. \tag{2.40}$$

This can be integrated to give

$$\psi(\mathbf{u}) = \exp\left(-ig\int_S d\mathbf{S}\cdot\mathbf{E}(\mathbf{x})\right)\psi(\mathbf{u}_0), \qquad d\mathbf{S} = \frac{\partial \mathbf{u}}{\partial \sigma}d\sigma \times d\mathbf{u}(\sigma)\,. \tag{2.41}$$

where $\mathbf{u}_0$ corresponds to some fixed string position, S is a surface spanned by the string $\mathbf{w}_0 = \mathbf{z} + \mathbf{u}_0$ to $\mathbf{w}$ and $d\mathbf{S}$ is the infinitesimal surface element. If we rotate the string $\mathbf{u}$ back to $\mathbf{u}_0$ position, S is a closed surface. Since the argument of the exponential in (2.41) is purely imaginary, we can use the condition that the wave function must be a single-valued (2.38) function of the string position to obtain

$$g\int_S d\mathbf{S}\cdot\mathbf{E}(\mathbf{x}) = 2\pi N\,, \tag{2.42}$$

with N an integer. The integral (2.42) can be solved thanks to the Gauss law, indicating with V the volume enclosed by S we can write

$$g\int_V d^3x\,\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{x}) = eg = 2\pi N \Rightarrow \boxed{\frac{eg}{2\pi} = N} \tag{2.43}$$

Which is known as *Dirac quantization condition.*

Reality of the string

In this chapter, I will show that the Dirac string is non-physical when the quantization condition (2.43) is satisfied. The inability to detect the string arises from the irrelevance of its spatial location and the principle of gauge invariance. Indeed, two vector potentials associated with different strings originating from the same point differ by a gauge transformation.

3.1 Fields generated by the monopole

We should first calculate the electric and magnetic fields for the system including the magnetic charge alone in order to show that the string term does not contribute in the result. Using (1.5) for the field strength (2.14) and considering only the string contribution we find

$$E^i_{\text{string}} = *C^{i0} = 0\,, \qquad\qquad B^i_{\text{string}} = C^{i0}\,. \tag{3.1}$$

So for the total system we have

$$\mathbf{E} = 0\,, \qquad\qquad \hat{\mathbf{B}} = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}_{\text{mon}} + \mathbf{B}_{\text{string}}\,. \tag{3.2}$$

The curl of $\mathbf{A}$ is (2.9), substituting it into the expression for the magnetic field we find $\mathbf{B}_{\text{mon}} = g/r^2\hat{\mathbf{r}}$. So, the electric and magnetic fields do not depend on the location of the string, therefore the trajectory of a magnetic or electric charged particle will not be affected by the Dirac string. However, the string is present in the vector potential as can be seen from the Figure 1. In the next section I will show how a transformation of the string position affects this quantity and its consequences.

3.2 Moving the string

As I mentioned in the previous chapter, Section 2.3.1, in both Lagrangian and Hamiltonian formulation the position of the string is indeterminate. However when we change the Maxwell equation in order to include the magnetic charge we are forced to introduce a vector potential which is singular along the string. The vector potential (2.7), hence, takes in account the position of the string through its form. We should then verify if two different strings, γ_1 and γ_2, intersecting only at the origin lead to equivalent vector potentials which, for instance, differs by a gauge transformation. According to Equation (2.8), in [8] it is shown that the Dirac's vector potential is defined in the domain $V_\gamma = \mathbb{R}^3\backslash\gamma$, considering two strings we can use a common domain $V_0 = V_1 \cap V_2 = \mathbb{R}^3\backslash\Gamma$ where $\Gamma = \gamma_1 \cup \gamma_2$ is a curve extending from minus infinity to infinity passing through the position of the monopole, this can be parameterized as

$$\Gamma = \gamma_1 \cup \gamma_2 \qquad \leftrightarrow \qquad \mathbf{y}(\sigma) = \begin{cases} \mathbf{y}_1(-\sigma) & -\infty < \sigma < 0 \\ \mathbf{y}_2(\sigma) & 0 \leq \sigma < \infty \end{cases} \tag{3.3}$$

Since both potentials $\mathbf{A}_1$ and $\mathbf{A}_2$, in V_0, lead to the same magnetic and electric field as argued in the previous section they fulfill the identities

$$\boldsymbol{\nabla} \times \mathbf{A}_1 = \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}_2 \qquad \Rightarrow \qquad \boldsymbol{\nabla} \times (\mathbf{A}_2 - \mathbf{A}_1) \equiv 0\,. \tag{3.4}$$

Hence in V_0 the form $\mathbf{A}_2 - \mathbf{A}_1$ is closed, however it is not exact since the domain is not contractible so it's not possible to use the Poincaré lemma. To restore this feature it is possible to restrict the domain introducing an infinitely extended surface Σ whose boundary is the curve Γ. A possible parametrization of the surface Σ is

$$\mathbf{y}(\sigma, u) \qquad -\infty < \sigma < \infty \qquad 0 \leq u < \infty \tag{3.5}$$

And the boundary condition reads $\mathbf{y}(\sigma, 0) = \mathbf{y}(\sigma)$. With this restriction we obtain a contractible domain, Σ corresponds in fact to an infinitely extended half-plane, so the Poincaré lemma ensures that in $\mathrm{I\!R}^3 \backslash \Sigma$ the form $\mathbf{A}_2 - \mathbf{A}_1$ is exact, there exists then a scalar function $\Lambda(\mathbf{x})$ such that

$$\mathbf{A}_2 - \mathbf{A}_1 = \boldsymbol{\nabla}\Lambda \qquad \text{for all } \mathbf{x} \in \mathrm{I\!R}^3 \backslash \Sigma\,. \tag{3.6}$$

In the restricted domain $\mathrm{I\!R}^3 \backslash \Sigma$ the Dirac potentials $\mathbf{A}_1$ and $\mathbf{A}_2$, differs by a gauge transformation, so *the change of the Dirac string is equivalent to a gauge transformation.*

3.3 Gauge function

A possible gauge function for (3.6) is given by Lechner [8] and it is the result of the two integrals of the form (2.8) over the surface Σ

$$\Lambda(\mathbf{x}) = g \int_\Sigma \frac{x^k - x'^k}{|\mathbf{x} - \mathbf{x}'|^3} d\Sigma^k\,. \tag{3.7}$$

This function is regular in domain $\mathrm{I\!R}^3 \backslash \Sigma$ and it is singular for $\mathbf{x} \in \Sigma$, in particular the singularity of the gauge function corresponds to a finite discontinuity when $\mathbf{x}$ crosses the surface Σ. In order to determine this discontinuity we should consider an arbitrary closed loop G passing through $\mathbf{x}$ and intersecting Σ only at one point $\mathbf{x}$. In this way, the loop G circles the curve Γ just once. The discontinuity can be therefore evaluated through the integral

$$\Delta\Lambda(\mathbf{x}) = \int_G d\Lambda = \int_G \boldsymbol{\nabla}\Lambda \cdot d\mathbf{x} = \int_G (\mathbf{A}_2 - \mathbf{A}_1) \cdot d\mathbf{x}\,. \tag{3.8}$$

Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are well defined in $\mathrm{I\!R}^3 \backslash \Gamma$, the integral can be considered as the integral along the whole loop G and therefore can be evaluated by Stokes' theorem. By the way there exists no surface S with boundary G such that on S both $\mathbf{A}_1$ and $\mathbf{A}_2$ are well defined, S necessarily intersects at least one of the Dirac strings γ_1 and γ_2. However it is possible to defined two distinct surfaces S_1 and S_2, such that $\partial S_1 = G = \partial S_2$, in which, respectively, $\mathbf{A}_1$ and $\mathbf{A}_2$ are regular. Under these hypothesis it is possible to apply Stokes' theorem

$$\begin{aligned}\Delta\Lambda(\mathbf{x}) &= \int_G \mathbf{A}_2 \cdot d\mathbf{x} - \int_G \mathbf{A}_1 \cdot d\mathbf{x} = \int_{S_2} \boldsymbol{\nabla} \times \mathbf{A}_2 \cdot d\boldsymbol{\Sigma} - \int_{S_1} \boldsymbol{\nabla} \times \mathbf{A}_1 \cdot d\boldsymbol{\Sigma} = \\ &= \int_{S_2} \mathbf{B} \cdot d\boldsymbol{\Sigma} - \int_{S_1} \mathbf{B} \cdot d\boldsymbol{\Sigma} = \int_{S_1 \cup S_2} \mathbf{B} \cdot d\boldsymbol{\Sigma} = g\,.\end{aligned} \tag{3.9}$$

Where in the last step the Gauss theorem for the magnetic charge has been used. We can conclude that the discontinuity of the gauge function $\Lambda(\mathbf{x})$ is independent of the transition point $\mathbf{x} \in \Sigma$ and is equal to the charge of the monopole.

3.3.1 General case

If we do not restrict the domain to $\mathrm{I\!R}^3 \backslash \Sigma$ the difference of the two vector potentials referred respectively to γ_1 and γ_2 strings is different, namely

$$\mathbf{A}_2 - \mathbf{A}_1 = \boldsymbol{\nabla}\Lambda - 4\pi g \int_\Sigma \delta^{(3)}(\mathbf{x} - \mathbf{x}')\, d\boldsymbol{\Sigma}'\,. \tag{3.10}$$

If we momentarily neglect the δ term, so basically we stick for a moment to the restriction of the domain $\mathbb{R}^3 \backslash \Sigma$, and we focus on the first term $\boldsymbol{\nabla}\Lambda$ we notice that equation (3.7) can be written as

$$\Lambda(\mathbf{x}) = g\Omega_\Sigma(\mathbf{x}). \tag{3.11}$$

In (3.11) Ω_Σ is the solid angle subtended by the surface Σ, the complete derivation of the solid angle is done in Appendix V, this allows us to give a geometrical interpretation to the gauge function (3.7)

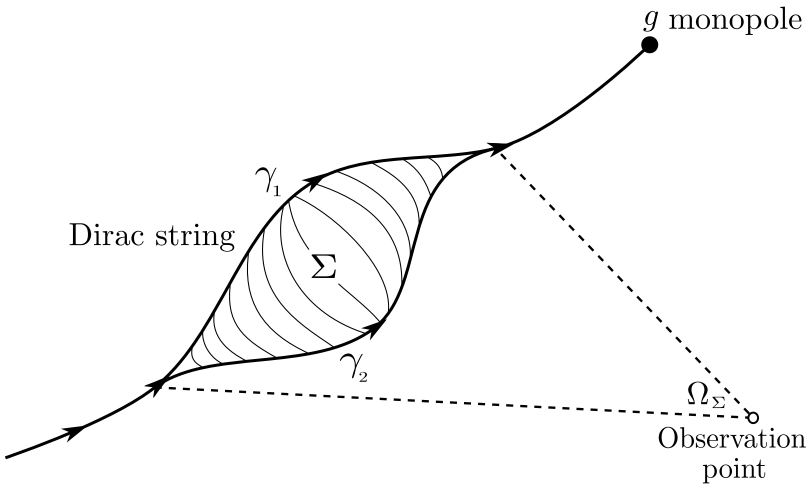


Figure 2: Representation of a magnetic monopole g as the end of a line of dipoles or as the end of a tightly wound solenoid that stretches off to infinity. The potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ correspond to the strings γ_1 and γ_2

Comparing now Equation (3.10) and Equation (3.6), in addition to interpreting Λ with the solid angle Ω, a new term involving the delta function appears due to the choice of the domain. Jackson [7] notes that, when the observation point crosses the surface Σ the solid angle Ω_Σ suddenly changes by 4π. This discontinuity of Ω_Σ at the surface Σ gives rise to a δ-function of magnitude 4π whenever $\boldsymbol{\nabla}\Omega_\Sigma$ is evaluated at a point $\mathbf{x}$ located on the surface Σ. Nevertheless, the resulting δ-function within $\boldsymbol{\nabla}\Omega_\Sigma$ is effectively cancelled out by the δ term in eq. (3.10). This means that $\mathbf{A}_2 - \mathbf{A}_1$ is a well-defined, continuous function across all $\mathbf{x}$ except along the closed loop Γ. Thus the inclusion of δ term is essential in (3.10).

3.3.2 Connection to the quantization condition

A strong point of the theory is that we can connect this result to the quantization condition (2.43). Considering the Schrödinger equation in natural units for a non-relativistic particle of mass m and electric charge q coupled to a time-independent Dirac potential $\mathbf{A}(\mathbf{x})$ given by (2.8)

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left(-i\boldsymbol{\nabla} - q\mathbf{A}\right)^2 \Psi. \tag{3.12}$$

This equation is known to be invariant under the gauge transformations of the form $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda$ with a corresponding change of the phase of the state vector

$$\Psi' = \mathrm{e}^{iq\Lambda}\,\Psi, \tag{3.13}$$

where $\Lambda(\mathbf{x})$ is a time-independent gauge function. Considering the case in which the gauge function represents the change of the string location from γ_1 and γ_2 we can substitute the gauge function with equation (3.11). Consider now the value Ω_1 corresponding to one side of the surface S and the value Ω_2 corresponding to the other side. They are related by $\Omega_1 = \Omega_2 + 4\pi$. It follows that $\mathrm{e}^{iqg\Omega_1} = \mathrm{e}^{iqg(\Omega_2+4\pi)}$. This means that the wave function of the charge q differs by the quantity e^{iqg}, and this would make the Dirac string observable as the charge crosses the surface, unless we impose the quantization condition (2.43), i.e, to ensure the un-observability of the string we are force to impose the Dirac quantization condition.

3.4 Field momentum

This chapter delves into the examination of "Field momentum and the reality of the Dirac String" by Singleton and Gonuguntla [3]. Their work, through the analysis of electromagnetic field momentum in a system involving magnetic monopoles and the Dirac string, aimed to show the reality of the Dirac string. However, through an accurate analysis, I aim to uncover some fundamental inaccuracies in their interpretation that invalidate their conclusions.

The two authors claim that even if, thanks to the Dirac quantization condition, the string is unobservable as I discussed in the previous sections, it is instead detectable when an electric field is present. The definition of the field momentum given in [3] is

$$\mathbf{P}_{\mathrm{EM}} = \frac{1}{4\pi}\int (\mathbf{E}\times\hat{\mathbf{B}})d^3x\,, \tag{3.14}$$

where $\hat{\mathbf{B}} = \boldsymbol{\nabla}\times\mathbf{A}$ is defined in (2.9). The system under consideration is an electric charge at the location $\mathbf{r}_0$ plus a magnetic monopole in the origin. The form of the electric field is $\mathbf{E} = q(\mathbf{r}-\mathbf{r}_0)/r^3$ and the form of the physical magnetic field of the monopole was assumed by Singleton and Gonuguntla to be given by Equation (2.9), so it can be divided into the monopole contribution and the string contribution.

The field momentum, considering only the contribution of $\mathbf{B}_{\mathrm{mon}}$ is

$$\mathbf{P}_{\mathrm{EM}} = \frac{1}{4\pi}\int (\mathbf{E}\times\mathbf{B}_{\mathrm{mon}})d^3x = \frac{qg}{4\pi}\int\left(\frac{\mathbf{r}-\mathbf{r}_0}{r^3}\times\frac{\hat{\mathbf{r}}}{r^2}\right)d^3x = 0\,. \tag{3.15}$$

As expected it is zero. The discussion of the two authors continues showing that the string term (2.10) in Equation (3.14) does contribute a non-zero part to the field momentum. This is actually true, in fact the integral

$$\mathbf{P}_{\mathrm{EM}}^{\mathrm{string}} = \frac{1}{4\pi}\int (\mathbf{E}\times\mathbf{B}_{\mathrm{string}})d^3x = \frac{qg}{4\pi}\int\left(\frac{\mathbf{r}-\mathbf{r}_0}{r^3}\times 4\pi g\delta(x)\delta(y)\Theta(-z)\hat{\mathbf{z}}\right)d^3x \tag{3.16}$$

is not zero. This does not mean that the center-of-energy theorem from special relativity is violated nor that there is some hidden momentum in the system [3] that balances the electromagnetic field momentum from (3.16). The total field momentum of the system is (3.15) because as remarked in Section 3.1, the *physical* magnetic field of the system involving a magnetic charge is (2.11) (in accordance with the definition of the field strength (2.14)) and not (2.9), so the string contribution does not appear.

This becomes more evident when we consider the presence of the magnetic charge in the classical Maxwell equations where the vector potential never appears. Thus, we can construct the dynamical theory simply by employing the modified Maxwell equations (2.3), which alone are sufficient for determining the field momentum. The difficulty emerges when employing the Lagrangian and Hamiltonian formalisms, which account for the vector potential. As reported in this work, the method to incorporate the vector potential, reconciling thus $\boldsymbol{\nabla}\cdot\mathbf{B}\neq 0$ and $\hat{\mathbf{B}} = \boldsymbol{\nabla}\times\mathbf{A}$, is to introduce a singularity in the form of $\mathbf{A}$ associated with the Dirac string that bears no physical significance.

Conclusion

In this work we have studied magnetic monopoles in the electrodynamics theory. At first we have introduced the theory for only the electric charge, then we add the magnetic charge according to Dirac works [1] and [2].

In this derivation, the magnetic monopole is attached to a Dirac string for which we have presented a complete dynamical theory via the Lagrangian and Hamiltonian formalism. By considering an appropriate quantum mechanics description of monopoles we obtained the quantization condition for the electric and magnetic charge.

Moreover requiring the location of the string to be irrelevant, it is shown that the two arbitrary positions of the string are connected with two gauge potentials, implying that the change of a string to another string is equivalent to a gauge transformation involving a multi-valued gauge function.

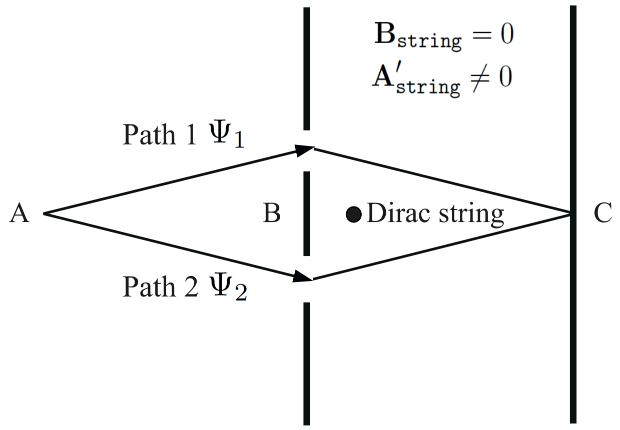
At the end we have analyzed an assertion that the string could be detected by an experiment. A recent paper [3] claimed that there should be a non-zero contribution to the field momentum of the system of an electric and magnetic charge. However, this is incorrect since the Dirac string contribution does not appear in the expressions of the *physical* magnetic and electric field of the system.

Appendix

A. Aharanov-Bohm effect

According to the Aharanov-Bohm effect [16] the vector potential can affect the motion of charged particle in regions where the **B** field vanishes. Since we can interpret the Dirac string as a tightly wound solenoid that stretches off to infinity it is interesting to study whether an Aharanov-Bohm experiment would eventually detect the string.

Considering a double-slit AB experiment with a Dirac string placed between the slits, when charged particles are emitted and passed through the slits, we should be able to detect the presence of the vector potential through the phase introduced in the wave function. This phase manifests as an interference pattern on a screen positioned in front of the slits.



In a region where there is no vector potential the wave function is just the sum of the wave functions of the charges passing through the slits 1 and 2, so $\Psi = \Psi_1 + \Psi_2$. Since the Dirac string is situated between the two slits, it becomes evident that the wave functions acquire a phase shift due to the string potential

$$\begin{aligned}\Psi &= \mathrm{e}^{iq\int_1 \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_1}\,\Psi_1 + \mathrm{e}^{iq\int_2 \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_2}\,\Psi_2 = \\ &= \left(\Psi_1 + \mathrm{e}^{iq\oint_C \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_\mathrm{C}}\,\Psi_2\right)\mathrm{e}^{iq\int_1 \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_1} = \\ &= \left(\Psi_1 + \mathrm{e}^{iqg}\,\Psi_2\right)\mathrm{e}^{iq\int_1 \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_1}\,.\end{aligned} \tag{A.1}$$

Where we used

$$\oint_C \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_\mathrm{C} = \int_2 \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_2 - \int_1 \mathbf{A}_\mathrm{s}\cdot d\mathbf{l}_1 \tag{A.2}$$

and (3.9) to set the first integral equal to g. In (A.1), the relative phase between Ψ_1 and Ψ_2 is of interest, as it is what would be detectable by the experiment. The effect of the Dirac string would be unobservable if $\mathrm{e}^{iqg} = 1$ and this implies the Dirac quantization condition $qg = 2\pi N$. Under this condition, the probability density becomes $P = |\Psi_1 + \Psi_2|^2$, meaning that no change in the interference pattern would be observed due to the Dirac string.

B. Avoiding Dirac string

The implementation of the magnetic charge and the derivation of the Dirac quantization condition involves some unpleasant features like singular gauge transformations and singular potentials. Wu and Yang [17] propose a theory that includes magnetic monopoles without necessitating these singularities, thereby maintaining the symmetry of Maxwell's equations.

The approach used by Wu and Yang is to use different vector potentials in different regions of space. By adopting this method, magnetic monopoles can be incorporated into electrodynamics without the need to introduce the Dirac string into the vector potential. This approach is compatible with the observations made from the Aharonov-Bohm effect, as the vector potential is not visible in the experiment and resolves the issue of non-integrable phase factors in (A.1).

The vector potential (2.7) is non-singular if it is defined in different domains as follows

$$\mathbf{A}_{\mathrm{N}} = g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi} \qquad R_{\mathrm{N}} : 0 \leq \theta \leq \frac{\pi}{2} + \delta\,,\ r > 0\,,\ 0 \leq \phi \leq 2\pi\,,\ \text{all } t \tag{B.1}$$

$$\mathbf{A}_{\mathrm{S}} = -g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi} \qquad R_{\mathrm{S}} : \frac{\pi}{2} - \delta\ \ \leq \theta \leq 0,\ r > 0\,,\ 0 \leq \phi \leq 2\pi\,,\ \text{all } t \tag{B.2}$$

with an overlap extending throughout $\pi/2-\delta<\theta<\pi/2+\delta$ and assuming $0 \leq \delta \leq \pi/2$. Furthermore $\mathbf{A}_{\mathrm{N}}$ and $\mathbf{A}_{\mathrm{S}}$ are non-global functions since they are defined in their respective domain. The region R_{N} removes the negative semi-axis $\theta=\pi$, viceversa, R_{S} removes the positive semi-axis $\theta=0$, the total domain $\mathbb{R}^3$ is then divided into the two overlapping hemispheres, north R_{N} and south R_{S}.

In general the four-potentials A^{N}_μ and A^{S}_μ are well-defined, and the field strength is

$$\partial_\mu A^{\mathrm{N}}_\nu - \partial_\nu A^{\mathrm{N}}_\mu = F_{\mu\nu} = \partial_\mu A^{\mathrm{S}}_\nu - \partial_\nu A^{\mathrm{S}}_\mu\,. \tag{B.3}$$

In the overlapping region $R_\cap := R_N \cap R_S$, the potentials can only differ by a gauge transformation, implying:

$$A^{\mathrm{N}}_\mu - A^{\mathrm{S}}_\mu = \alpha_\mu \ \Rightarrow\ \partial_\mu\alpha_\nu - \partial_\nu\alpha_\mu = 0 \text{ in } R_\cap. \tag{B.4}$$

Moreover, when considering a closed surface surrounding the monopole and performing a line integral over a loop Γ on this surface within $R_\cap$, we find:

$$\oint_\Gamma \alpha_\mu d\gamma^\mu = g\,, \tag{B.5}$$

Since the line integral is equal to the outward magnetic flux across $R = R_{\mathrm{N}} \cup R_{\mathrm{S}}$. This integral serves as a consistency condition for α_μ. A consequence of this formalism is the equation

$$\partial_\mu * F^{\mu\nu} = j^\nu_g, \tag{B.6}$$

which mirrors the analogous characteristic in Dirac's formulation.

It is crucial to emphasize that within the Wu and Yang monopole framework, we deal with two distinct vector potentials, A_{N} and A_{S} which are defined in two separate regions R_{N} and R_{S}. These regions are subjected to condition (B.1-b) and (B.2-b).

As argued in [18], by the same authors of [3], there is an assertion that these two vector potentials can be unified into a single expression. This assumption leads to the calculus of non-zero field momentum analogously to the argument in [3] for the Dirac string. However, as explained in [19], this assertion is incorrect. According to [17] the vector potential can only be properly defined in each of many overlapping regions of spacetime. In (B.1-b)-(B.2-b) it is essential that $\delta > 0$; hence taking the limit $\delta \to 0$ is not possible. Therefore defining a non-singular vector potential for the domain $R = R_{\mathrm{N}} \cup R_{\mathrm{S}}$ is not feasible.

C. Calculations

I Derivation of Equation 1.62

(i) Lagrangian

$$L = -m\sqrt{-\dot{y}_\mu \dot{y}^\mu} - eA_\mu \dot{y}^\mu \tag{I.1}$$

(ii) Conjugate momenta p_μ

$$p_\mu \equiv \frac{\partial L}{\partial \dot{y}^\mu} = \begin{cases} m\frac{\dot{y}_0}{\sqrt{\dot{y}_0\dot{y}^0 - \dot{y}_i\dot{y}^i}} - eA_0 & (\mu = 0) \\ m\frac{\dot{y}_i}{\sqrt{\dot{y}_0\dot{y}^0 - \dot{y}_i\dot{y}^i}} - eA_i & (\mu = i) \end{cases} \tag{I.2}$$

(iii) Hamiltonian via Legendre transformation

$$\begin{aligned} H &= p_\mu \dot{y}^\mu - L = \\ &= m\frac{\dot{y}_\mu \dot{y}^\mu}{\sqrt{-\dot{y}_\mu \dot{y}^\mu}} - eA_\mu \dot{y}^\mu + m\sqrt{-\dot{y}_\mu \dot{y}^\mu} + eA_\mu \dot{y}^\mu = 0 \end{aligned} \tag{I.3}$$

(iv) Primary constraint from the definition of conjugate momenta

$$\begin{aligned} &p_\mu + eA_\mu = m\frac{\dot{y}_\mu}{\sqrt{-\dot{y}_\mu \dot{y}^\mu}} \\ &\text{squaring both sides} \\ &(p_\mu + eA_\mu)(p^\mu + eA^\mu) = -m^2 \\ &(p_\mu + eA_\mu)(p^\mu + eA^\mu) + m^2 = 0 \\ &(p_0 + eA_0)^2 = m^2 + (\mathbf{p} + e\mathbf{A})^2 \end{aligned} \tag{I.4}$$

(v) Gauge fixing $y_\mu = (\tau, \mathbf{y})$

$$\text{Lagrangian: } L = -m\sqrt{1 - \dot{\mathbf{y}}^2} - e(-A_0 + \mathbf{A}\cdot\dot{\mathbf{y}}) \tag{I.5}$$

$$\text{Conjugate momenta: } \mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{y}}} = m\frac{\dot{\mathbf{y}}}{\sqrt{1 - \dot{\mathbf{y}}}} - e\mathbf{A} \tag{I.6}$$

(vi) Hamiltonian via Legendre transformation

$$\begin{aligned} H &= \mathbf{p}\cdot\dot{\mathbf{y}} - L = \\ &= m\frac{\dot{\mathbf{y}}^2}{\sqrt{1 - \dot{\mathbf{y}}}} - e\mathbf{A}\cdot\dot{\mathbf{y}} + m\sqrt{1 - \dot{\mathbf{y}}^2} - eA_0 + e\mathbf{A}\cdot\dot{\mathbf{y}} = \\ &= m\frac{\dot{\mathbf{y}}^2}{\sqrt{1 - \dot{\mathbf{y}}}} - e\mathbf{A}\cdot\dot{\mathbf{y}} + m\frac{(1 - \dot{\mathbf{y}}^2)}{\sqrt{1 - \dot{\mathbf{y}}^2}} - eA_0 + e\mathbf{A}\cdot\dot{\mathbf{y}} = \\ &= m\frac{1}{\sqrt{1 - \dot{\mathbf{y}}^2}} - eA_0 = p_0 = \underset{-}{\boxed{+}}\sqrt{m^2 + (\mathbf{p} + e\mathbf{A})^2} - eA_0 \end{aligned} \tag{I.7}$$

Where I choose + because the Hamiltonian must be positive defined.

(vii) Introduction of kinetic moneta $k_\mu = (k_0, \mathbf{k})$

$$H_{\text{charge}} = \sqrt{m^2 + \mathbf{k}^2} - eA_0 \tag{I.8}$$

II Derivation of Equation (2.10)

The curl of Equation (2.7) gives

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A}_\gamma &= \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left\{\int_\gamma \frac{g\,d\boldsymbol{\gamma}'}{|\mathbf{x}-\mathbf{x}'|}\right\}\right)\\ &= \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\left\{\int_\gamma \frac{g\,d\boldsymbol{\gamma}'}{|\mathbf{x}-\mathbf{x}'|}\right\}\right)-\boldsymbol{\nabla}^2\left\{\int_\gamma \frac{g\,d\boldsymbol{\gamma}'}{|\mathbf{x}-\mathbf{x}'|}\right\}\\ &= g\boldsymbol{\nabla}\int_\gamma \boldsymbol{\nabla}\cdot\left(\frac{d\boldsymbol{\gamma}'}{|\mathbf{x}-\mathbf{x}'|}\right)-g\int_\gamma \boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d\boldsymbol{\gamma}'\end{aligned} \tag{II.1}$$

Using the result $\boldsymbol{\nabla}\cdot(d\boldsymbol{\gamma}'/|\mathbf{x}-\mathbf{x}'|) = d\boldsymbol{\gamma}'\cdot\boldsymbol{\nabla}(1/|\mathbf{x}-\mathbf{x}'|)$, the first integral becomes

$$\begin{aligned}\int_\gamma \boldsymbol{\nabla}\cdot\left(\frac{d\boldsymbol{\gamma}'}{|\mathbf{x}-\mathbf{x}'|}\right) &= \int_\gamma \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\boldsymbol{\gamma}'\\ &= -\int_\gamma \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot d\boldsymbol{\gamma}'\\ &= -\frac{1}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{II.2}$$

Considering Equation (II.2), the first term of Equation (II.1), yields the field of the magnetic monopole

$$g\boldsymbol{\nabla}\int_\gamma \boldsymbol{\nabla}\cdot\left(\frac{d\boldsymbol{\gamma}'}{|\mathbf{x}-\mathbf{x}'|}\right) = g\boldsymbol{\nabla}\left(-\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{g}{r^2}\hat{\mathbf{r}}, \tag{II.3}$$

where we have used $\boldsymbol{\nabla}(1/|\mathbf{x}-\mathbf{x}'|) = -\hat{\mathbf{r}}/r^2$. The second term of Equation (II.1) yields the magnetic field of the Dirac string

$$-g\int_\gamma \boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d\boldsymbol{\gamma}' = 4\pi g\int_\gamma \delta(\mathbf{x}-\mathbf{x}')\,d\boldsymbol{\gamma}', \tag{II.4}$$

where we have used $\boldsymbol{\nabla}^2(1/|\mathbf{x}-\mathbf{x}'|) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. To derive Equation (2.11), we first take the curl of Equation (2.7) written in rectangular coordinates,

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A}_\gamma =&\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\left\{\hat{\mathbf{z}}\int_{-\infty}^0 \frac{g\,dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right\}\right)\\ =&\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\left\{\hat{\mathbf{z}}\int_{-\infty}^0 \frac{g\,dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right\}\right)-\boldsymbol{\nabla}^2\left\{\hat{\mathbf{z}}\int_{-\infty}^0 \frac{g\,dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right\}\\ =&\boldsymbol{\nabla}\int_{-\infty}^0 \frac{\partial}{\partial z}\left(\frac{dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right)-g\,\hat{\mathbf{z}}\int_{-\infty}^0 \boldsymbol{\nabla}^2\left(\frac{dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right)\end{aligned} \tag{II.5}$$

To simplify the first term we may write

$$\frac{\partial}{\partial z}\left(\frac{1}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right) = -\frac{z-z'}{\left(x^2+y^2+(z-z')^2\right)^{3/2}}, \tag{II.6}$$

so that

$$\int_{-\infty}^0 \frac{\partial}{\partial z}\left(\frac{dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right) = -\int_{-\infty}^0 \frac{z-z'}{\left(x^2+y^2+(z-z')^2\right)^{3/2}}\,dz'. \tag{II.7}$$

Consider the substitution $u(z') = x^2 + y^2 + (z - z')^2$. Hence, $du = -2(z - z')dz'$, and the right-hand side of the integral in Equation (II.7) takes the form

$$\frac{1}{2} \lim_{\beta \to \infty} \int_{u(z'=-\beta)}^{u(z'=0)} \frac{du}{u^{3/2}} = \lim_{\beta \to \infty} \left. \frac{-1}{\sqrt{u}} \right|_{u(z'=-\beta)}^{u(z'=0)} = -\frac{1}{|\mathbf{x}|} + \lim_{z' \to -\infty} \frac{1}{|\mathbf{x} - z'\hat{\mathbf{z}}|} = -\frac{1}{r}. \tag{II.8}$$

Using this result in the first term in Equation (II.5) we obtain the monopole field

$$g\boldsymbol{\nabla} \int_{-\infty}^{0} \frac{\partial}{\partial z} \left(\frac{dz'}{|\mathbf{x} - z'\hat{\mathbf{z}}|} \right) = g\boldsymbol{\nabla} \left(-\frac{1}{r} \right) = \frac{g}{r^2} \hat{\mathbf{r}}. \tag{II.9}$$

To simplify the second term in Equation (II.5) consider

$$\boldsymbol{\nabla}^2 \left(\frac{1}{|\mathbf{x} - z'\hat{\mathbf{z}}|} \right) = -4\pi\delta(\mathbf{x} - z'\hat{\mathbf{z}}) \tag{II.10}$$

$$= -4\pi\delta(x)\delta(y)\delta(z - z'). \tag{II.11}$$

Using this equation in the second term of Equation (II.5) we obtain the string field

$$-g\hat{\mathbf{z}} \int_{-\infty}^{0} \boldsymbol{\nabla}^2 \left(\frac{dz'}{|\mathbf{x} - z'\hat{\mathbf{z}}|} \right) = 4\pi g\delta(x)\delta(y) \left\{ \int_{-\infty}^{0} \delta(z - z')dz' \right\} \hat{\mathbf{z}} \tag{II.12}$$

$$= 4\pi g\delta(x)\delta(y)\Theta(-z)\hat{\mathbf{z}} \tag{II.13}$$

where in the last step we have used the integral representation of the step function $\Theta(\xi - \alpha) = \int_{-\infty}^{\xi} \delta(\tau - \alpha)d\tau$ to identify the quantity within the brackets $\{ \}$ in Equation (II.13) as $\Theta(-z) = \int_{-\infty}^{0} \delta(z - z')dz'$.

III Derivation of Equation (2.17)

$$B^i_{\text{string}}(x) = C^{i0}(x) = -g \int \int \left(\frac{\partial w^i}{\partial \tau} \frac{\partial w^0}{\partial \sigma} - \frac{\partial w^0}{\partial \tau} \frac{\partial w^i}{\partial \sigma} \right) \delta^{(4)}(x - w(\tau, \sigma))d\tau d\sigma \tag{III.1}$$

Choosing a parametrization for the string that singles out the time $w^\mu = (\tau, \mathbf{w}(\tau, \sigma))$ we have $\frac{\partial w^0}{\partial \tau} = 1$ and $\frac{\partial w^0}{\partial \sigma} = 0$. In addition, since $w^\mu(\tau, \sigma) = z^\mu(\tau) + u^\mu(\tau, \sigma)$, $\frac{\partial w^i}{\partial \sigma} = \frac{\partial u^i}{\partial \sigma}$

$$\mathbf{B}_{\text{string}}(t, \mathbf{x}) = g \int \int \frac{\partial \mathbf{u}(\tau, \sigma)}{\partial \sigma} \delta(t - \tau)\delta^{(3)}(\mathbf{x} - \mathbf{u}(\tau, \sigma))d\tau d\sigma = \tag{III.2}$$

$$= g \int \frac{\partial \mathbf{u}(t, \sigma)}{\partial \sigma} \delta^{(3)}(\mathbf{x} - \mathbf{u}(t, \sigma))d\sigma \tag{III.3}$$

We can drop the time dependence assuming the system to be stationary. If we require that the string lies along the negative z-axis an easy parametrization is $\mathbf{u}(\sigma) = (0, 0, \sigma)$ with $\sigma \in (0, -\infty)$, whoose derivatives respect to σ is $\hat{\mathbf{z}}$. Then

$$\mathbf{B}_{\text{string}}(\mathbf{x}) = g \int_{-\infty}^{0} \hat{\mathbf{z}}\delta(x)\delta(y)\delta(z - \sigma)d\sigma = g\hat{\mathbf{z}}\delta(x)\delta(y)\Theta(-z) \tag{III.4}$$

IV Derivation of Equation (2.24-a)

The action regarding the electric charge is

$$I_e = -\int j^\nu A_\nu d^4x - m_e \int d\tau \tag{IV.1}$$

Varying the action with respect y^μ

$$\begin{aligned}
&\delta_y \left(\int j^\mu A_\mu d^4x - m_e \int d\tau \right) \\
&= \int \left(\frac{dp_e^\mu}{d\tau} - e(\partial^\mu A^\nu - \partial^\nu A^\mu) v_\nu \right) \delta y_\mu d\tau = \\
&= \int \left(\frac{dp_e^\mu}{d\tau} - e(\partial^\mu A^\nu - \partial^\nu A^\mu + *C^{\mu\nu}) v_\nu \right) \delta y_\mu d\tau + e \int *C^{\mu\nu} v_\nu \delta y_\mu d\tau \\
&= \int \left(\frac{dp_e^\mu}{d\tau} - eF^{\mu\nu} v_\nu \right) \delta y_\mu d\tau + \delta_y \left(\frac{1}{2} \int d^4x * C_{\mu\nu} C_e^{\mu\nu} \right)
\end{aligned} \tag{IV.2}$$

Where $C_e^{\mu\nu}$ is associated with a formal Dirac string worldsheet attached to the electric current. The last term is eg times the number of intersections between the Dirac string worldsheet of the electric particle and the monopole. Under infinitesimal variations of the electric particle worldline it is zero. So the the variation of the action (IV.1) leads to the standard Lorentz force (2.24-a)

$$m_e \frac{dp_e^\mu}{d\tau} = eF^{\mu\nu}(y) u_\nu \tag{IV.3}$$

V Derivation of Equation (3.10)

Starting from (2.8)

$$\mathbf{A}_2 - \mathbf{A}_1 = g \boldsymbol{\nabla} \times \oint_\Gamma \frac{d\boldsymbol{\gamma}'}{|\mathbf{x} - \mathbf{x}'|}. \tag{V.1}$$

Using Stoke's theorem and $\boldsymbol{\nabla}(1/|\mathbf{x} - \mathbf{x}'|) = -\boldsymbol{\nabla}'(1/|\mathbf{x} - \mathbf{x}'|)$,

$$\begin{aligned}
\mathbf{A}_2 - \mathbf{A}_1 &= -g \boldsymbol{\nabla} \times \int_\Sigma \boldsymbol{\nabla}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times d\boldsymbol{\Sigma}' \\
&= \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \left\{ \int_\Sigma \frac{g\, d\boldsymbol{\Sigma}'}{|\mathbf{x} - \mathbf{x}'|} \right\} \right) \\
&= \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \left\{ \int_\Sigma \frac{g\, d\boldsymbol{\Sigma}'}{|\mathbf{x} - \mathbf{x}'|} \right\} \right) - \boldsymbol{\nabla}^2 \left\{ \int_\Sigma \frac{g\, d\boldsymbol{\Sigma}'}{|\mathbf{x} - \mathbf{x}'|} \right\}
\end{aligned} \tag{V.2}$$

Making use of $\boldsymbol{\nabla} \cdot (d\boldsymbol{\Sigma}'/|\mathbf{x} - \mathbf{x}'|) = d\boldsymbol{\Sigma}' \cdot \boldsymbol{\nabla}(1/|\mathbf{x} - \mathbf{x}'|)$ Equation (V.2) reads

$$\begin{aligned}
\mathbf{A}_2 - \mathbf{A}_1 &= g \boldsymbol{\nabla} \int_\Sigma \boldsymbol{\nabla} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot d\boldsymbol{\Sigma}' - g \int_\Sigma \boldsymbol{\nabla}^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d\boldsymbol{\Sigma}' \\
&= g \boldsymbol{\nabla} \int_\Sigma \frac{(\mathbf{x}' - \mathbf{x}) \cdot d\boldsymbol{\Sigma}'}{|\mathbf{x} - \mathbf{x}'|^3} + 4\pi g \int_\Sigma \delta^{(3)}(\mathbf{x} - \mathbf{x}')\, d\boldsymbol{\Sigma}'
\end{aligned} \tag{V.3}$$

where we have used $\boldsymbol{\nabla}(1/|\mathbf{x} - \mathbf{x}'|) = -(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3$ and $\boldsymbol{\nabla}^2(1/|\mathbf{x} - \mathbf{x}'|) = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}')$. The integral in the first term is the solid angle

$$\Omega_\Sigma(\mathbf{x}) = \int_\Sigma \frac{(\mathbf{x}' - \mathbf{x}) \cdot d\boldsymbol{\Sigma}'}{|\mathbf{x} - \mathbf{x}'|^3}, \tag{V.4}$$

and therefore we obtain (3.10).

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