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REGIME SWITCHES IN ECONOMIC DYNAMIC SYSTEMS

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Pi el studia...

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Abstract

This thesis examines regime shifts within dynamic economic systems, investigating how transitions between distinct states occur under certain conditions. It begins by defining regime shifts conceptually and explores economic scenarios where these shifts are likely, including examples of historical economic events. Core concepts such as thresholds, tipping points, and bifurcations are first discussed descriptively and later analyzed mathematically. To provide a formal understanding, essential mathematical tools are introduced, including dynamic systems, equilibrium stability, stochastic modeling, time series analysis, hazard points, Skiba points, Markov chains, and Poisson processes. In the final section, a recent model addressing adaptation to climate tipping points is analyzed in depth. This model, developed as part of the IIASA Young Scientists Summer Program, provides insights into preparing for and adapting to environmental regime shifts, illustrating practical applications of theoretical concepts explored in this thesis.

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Introduction

Among the vast number of themes related to the subject of economics, one topic particularly intriguing is the matter of *regime switching*. It indicates situations where economic processes shift from an initial model or set of rules to another one with distinct characteristics. These regimes can represent various states of the economy, such as periods of expansion or recession, high or low volatility, and many others. Examples of regime switches in the economy include a multitude of different facts, some of which are presented in the next chapter. It is possible to define the concept as follows: in very few words, a regime shift is a discrete break in a dynamic system, a discrete change in either the objective function or the transition dynamics [25]. In what follows, this definition will be enriched, through the presentation of real-life examples and answering whether it is possible to foresee the shift or not, and how that can be very useful for analysis and policy matters.

Structure

The discussion proceeds as follows.

First, we introduce the concept of regime switch - named indifferently regime *shift* in this work. A detailed definition will be followed by brief literature and historical reviews, in which we will mention various authors who analyzed this concept in their work.

In the second chapter, titled **Economic Regimes**, we will provide some examples of situations in which a regime shift can occur, with a focus on economic and financial systems. The kinds of regimes that are going to be presented and discussed concern monetary policy, fiscal policy, market volatility, and technological investments.

Chapter 3 focuses on the fundamental concepts of thresholds and tipping points and their importance to fully understand the regime shift matter. After a complete definition, we will analyze economic examples of tipping points, taken from financial markets and macroeconomics. **Threshold and Tipping Points** will have a "descriptive" take on the subject, leaving the mathematical features for the following chapter.

In **Mathematical Framework** we will try to give insights on the technical context of this

phenomenon, starting from the general field of study where the regime switching belongs, dynamic systems, with a recall of the concepts of equilibrium points, their stability and bifurcations analysis, passing by some econometrics models, with a focus on time series analysis, to finally introduce examples of stochastic models used in the literature to examine the eventual occurrence of the switch, such as Markov processes.

The last chapter, named **Climate Tipping Points**, consists in the description of the main aspects of a recent article on the subject, *How to prepare for and adapt to a climate tipping point* by M. Muttoni [26], which offers a new version of the DICE model and various interesting intuitions to illustrate and solve an optimal control problem, which features a dynamic system subject to the occurrence of a regime switch in the environmental economy matter.

Definition

In this section, we will delve deeply into this topic, with a particular focus on its economic and financial implications. To facilitate this analysis, we have extensively referenced *Dynamic Economic Problems with Regime Switches*, by Haunschmied et al. [25], which has provided numerous examples and a thorough review of the existing literature.

Using the term regime switching, economists refer to the analysis of situations in which economic or financial processes evolve between states with distinct properties and parameters. The reasons for these transitions are numerous, both at an individual and at a universal level.

The first kind includes occurrences which have a significant impact on individuals' consumption behaviors. Examples can be retirement, divorce, serious illness, or conversion to a new faith. At a broader level, historical events have often acted as triggers for major economic and political changes in society. For instance, the application of steam power in the production system, along with others findings and inventions, catalyzed the Industrial Revolution in Britain between the late 18^{th} and the first half of the 19^{th} century. More generally, the introduction of new technologies has consistently and significantly modified prevailing production paradigms. Other examples comprehend changes in property rights regime, such as the enclosure process in England in the 1700, that created a landless working class, and the emancipation of slave labor in the post-Civil War United States. Political and social revolutions, like the French Revolution (1789-1799), which ended the monarchy and initiated the republic government form, the Russian Revolution (1917), with the creation of the Soviet Union, the Chinese Revolution (1949), resulting in the proclamation of the People's Republic and the implementation of a planned economy and the Arab Spring (2010-2012), are other notable examples. Additionally, decolonization processes similar to the one happened in Africa, that carried the formation of new independent countries; while environmental changes, including human-induced climatic

changes or natural ones, have also been significant sparks.

Usually, regime shifts can be foreseen, allowing economic agents to adapt their behaviors prior to the actual occurrence of the switch. Many economists elaborated models of optimal response to anticipated regime changes. Among the firsts are Kamien and Schwartz, who wrote about a machine failure over time, Long, whit an article about a monopolist's expectation about the nationalization of his mine, and Kemp and Long [3]. The paper by these last two authors addresses optimal control problems where the cost functional involves discontinuities over time, as opposed to traditional methods that often assume continuity. This provides a significant extension to the field, making it applicable to a broader range of real-world problems where discontinuities in time are present. In their analysis, they also pointed out an interesting side-effect of agents' anticipations, particularly relevant in environmental policies: the Green Paradox [16], which suggests that taxes and laws enacted aiming at a reduction carbon emissions in the long term may inadvertently accelerate the extraction and consumption of fossil fuels in the short term. Another related work is by Hillman and Long, who developed a theoretical model to examine how the anticipation of future trade disruptions impacts the pricing and depletion of an exhaustible resource.

In the models cited so far, the date of the shift is outside the control of the planner; whether known, as assumed by Kemp and Long [3], or stochastic, like the one supposed by Hillman and Long. Alternatively, Hung et al., examining the dynamics and economic implications of transitioning from the use of exhaustible resources to inexhaustible substitutes, considered the date of the switch to be optimally chosen by the agents.

Besides the consideration of whether the date of the transformation is given or chosen, another crucial question is how many decision-makers are dealing with an anticipated regime shift. The response to this matter delineates two different approaches: the optimal control theory method, applicable when there is a single agent, and the dynamic games approach, used when multiple agents plan their responses non-cooperatively. The first one is much simpler, at the expense of a completeness which, on the other hand, the second one has. Both the approaches have a wide representation in the literature. For instance King et al., Folke et al., and Rust and Rothwell, in their papers, provided a range of applications - from fisheries management to nuclear energy, passing by forest conservation - using optimal control theory to analyze regime shifts in real-world scenarios. Articles using the dynamic games method are presented, among the others, by Tornell, Doraszelski, and Long et al.

Chapter 1

Economic Regimes

The concept of regime switching is universally present in the real economy. Indeed, regimes can shift from one to another with distinct features, encompassing various economic and financial aspects. In this section, we will present different types of economic regimes, analyzing their definitions and mean features. The focus will be on monetary policy regimes, fiscal policy regimes, market volatility regimes, and technological and innovation regimes.

1.1 Monetary Policy

Monetary policy concerns the decisions taken by central banks to influence the cost and the availability of money in an economy [29]. Central banks' main tool is the control of the policy interest rates, and their purposes are essentially maintaining price stability and fostering economic growth. A monetary regime is a predetermined statement by the central bank about what policies will be followed in various situations. For many different reasons, these regimes can change from one to another, making central banks apply further policies to achieve their objectives.

As maintaining price stability is one of the central banks' fundamental missions, inflation targeting regimes are the first to mention. Inflation targeting takes some advantages, such as transparency and accountability, but also has some weaknesses, such as rigidity, potentially lowering economic growth and causing financial instability. Theoretically, formal adoption of inflation targeting reduces inflation by anchoring expectations and enhanced credibility of the central bank, even if the literature found many alternative explanations for the great moderation in inflation.

The paper *Inflation targeting regimes*, by Carare and Stone [13], offers a clear look on this subject. The authors classify monetary frameworks of countries that use inflation targeting by the clarity and credibility of the commitment to the inflation target. The clarity, a key feature

in the matter, is essentially composed by two elements: the authorities' public description and communication of the objective, that can be found in official reports, and the institutional transparency, indispensable to hold the central bank responsible of its decisions. The researchers measure the credibility using two indicators as well. The first is, unsurprisingly, the actual rate of inflation, while the second, even if it is not under the direct control of the central bank, is the government debt. After checking for these gauges, the founded categories are three. The implicit price stability anchor (IPSA for short) practitioners are industrial countries whose credibility and historical successful inflation allow them to be more flexible and to not completely follow an official target. On the other hand, the full-fledged inflation targeting (FFIT) requires that the country's central bank make an explicit commitment to the target, with full transparency and accountability. Finally, the inflation targeting lite (ITL) countries float their exchange rate and announce an inflation target but are not able to maintain it as the foremost policy objective. Switches between all of them are theoretically possible, but a more consistent analysis leaves only three plausible options: IPSA to FFIT (never happened), FFIT to IPSA (not under consideration, maybe in the future) and ITL to FFIT (interesting for emerging countries).

From an international point of view, four primary exchange rate regimes exist: specie standard, managed fixed exchange rates, free float, and managed float. These regimes differ in their approach to the *Mundell-Fleming's trilemma*, an economic concept positing the impossibility of simultaneously achieving a fixed exchange rate, free capital movement, and independent monetary policy. Each regime offers distinct advantages and disadvantages, shaping a country's economic landscape. A fixed exchange rate regime, where a currency's value is pegged to another currency or a basket of currencies, can provide stability and predictability, facilitating international trade and investment. However, it can limit a country's ability to respond to domestic economic shocks, potentially leading to imbalances. Managed float, a hybrid approach, combines elements of fixed and floating exchange rates. Central banks intervene to influence the exchange rate within a certain band, offering flexibility while maintaining some degree of stability. However, the effectiveness of such an intervention can be limited by market forces. A free-float regime, where exchange rates are determined by market forces, allows for automatic adjustment to economic shocks and can promote price stability. Nevertheless, excessive volatility in exchange rates can create uncertainty and hinder economic growth. Currency boards impose strict monetary discipline by backing the domestic currency with foreign reserves. This can help control inflation and maintain exchange rate stability, but it limits a country's monetary policy autonomy.

A notable historical example of a shift in international monetary regimes is the collapse of the Bretton Woods system. Established in 1944, this fixed exchange rate system linked the currencies of 44 countries to the U.S. dollar, which was, in turn, pegged to gold. However, the

system's demise in 1971, marked by the U.S. decision to suspend the dollar's convertibility to gold, ushered in an era of floating exchange rates. The realization of the European Union can be also considered, among several other things, a monetary regime switch: starting from the '90s, with an improvement of economic convergence and an increased cooperation between the national central banks, to peak in 2002, with the adoption of the euro by the first twelve countries, this revolutionary event marked a transition from a system of multiple national currencies to a single currency union, aiming to enhance economic integration and reduce transaction costs. However, the *eurozone* has faced challenges, including the sovereign debt crisis, highlighting the potential limitations of a currency union in addressing asymmetric shocks.

1.2 Fiscal Policy

Fiscal policy is the use of government spending and taxation to influence the economy, typically to promote strong and sustainable growth and reduce poverty and unemployment. Before the Great Depression in the 1930s, the so-called *lassez-faire* prevailed; after that, governments have played a more proactive role in the economy, until recent years, in which policymakers generally employ strong policies only when there is a worldwide recession threat [28].

Generally speaking, the effectiveness of fiscal policy depends on various factors, including the timing, magnitude, and composition of the measures, as well as the state of the economy. Additionally, the interaction between fiscal and monetary policy, and the potential for international spillovers, can influence the overall impact of fiscal policy decisions.

Fiscal policy regimes are many: for instance, the two opposite of austerity and stimulus. First one's measures involve reducing government spending and increasing taxes to lower budget deficits and public debt, at the cost of potential contraction in economic growth, an higher unemployment and social unrest due to cuts and the lack of investments in welfare programs and public services. Policies related to the austerity regime were implemented in the E.U. coping to the debt crisis in the 2010s. Before hitting the European - and the global - economy, the crisis originated in the U.S. in 2008, with the infamous Lehman Brothers Inc. bankruptcy considered to be the beginning of that. Even if it was in the very same years, the U.S. government response in terms of fiscal policy regime adopted was completely the opposite of the approach followed in some European countries. The Congress, indeed, enacted a stimulus package known as American Recovery and Reinvestment Act 2009 (ARRA), significantly increasing government spending. Policies of the kind are particularly implemented during recessions, and they can lead to higher economic growth and increased employment in the short run, at the cost of raising deficits and public debt. Choose the optimal policy between them may vary depending on specific economic conditions and political constraints, making very likely the possibility of

a regime shift. For instance, taking in consideration the just mentioned world debt crisis in the 2010s, the Eurozone's countries which we have seen initially reacted applying austerity measures, in time decided to concentrate more on supporting the growth. The persistent economic stagnation, which highlighted the limitations of austerity measures, and deflationary risks, with consequent concerns about a potential economic spiral, impacted on the international community, which demanded for a more balanced approach.

A different fiscal policy regime dichotomy is the one between balance budget and deficit spending. The application of the latter has a great example in Franklin D. Roosenvelt's New Deal as a reaction of the Great Depression in the 1930s, which caused severe economic downturn, high unemployment, and widespread poverty in the U.S. The series of programs, public work projects, financial reforms, and regulations included in the New Deal marked a shift from minimal government intervention to extensive deficit spending on public works and social programs, resulting in economic growth and laying the foundation for future social safety nets, even if this coincided with a major increase in public debt [11]. On the other hand, the balance budget regime aims for government revenues to equal expenditures, with no or a very low deficit level. It promotes fiscal discipline and long-term sustainability, but it may limit the government's ability to respond to economic downturns. An example is the balanced budget rule included in the 2009-introduced German Basic Law, which mandates that the federal government must not run a structural deficit exceeding 0.35% of GDP.

1.3 Market Volatility

Volatility is the degree of variation of a series of trading prices over time, usually measured by the standard deviation of logarithmic returns [24]. Market volatility regimes can be considered as periods characterized by varying levels of price fluctuations in financial markets. Shifts from one period to another with different features are observed in reality, essentially from periods with low volatility to periods with high volatility (or vice versa).

A high volatility regime features significant and frequent price fluctuations, often associated with economic uncertainty and market stress. It can lead to reduced investment, slower economic growth, and increased caution among consumers and businesses, with investors becoming risk-averse. Besides the already mentioned 2008 global financial crisis, a perfect example of a period characterized by this type of volatility regime is the one correspondent with the COVID-19 Pandemic (2020). The disease outbreak and the consequential lock-down caused unprecedented economic disruption; financial markets experienced extreme volatility, with rapid declines followed by sharp recoveries [21]. The period of the so-called Great Moderation, from the mid-1980s to the mid-2000s, was, on the contrary, a period of economic stability and steady

growth. It was a period of low volatility: the financial agents had confidence in the market and tended to take more risks and invest more.

A substantial body of research has delved into the phenomenon of regime shifts in market volatility. Employing sophisticated time series analysis and econometric models such as GARCH and ARIMA, researchers have consistently documented the existence of distinct volatility regimes. However, the dynamics of financial markets are intricate and influenced by a multitude of factors. Market microstructure, investor sentiment, macroeconomic conditions, and structural breaks can all contribute to the emergence and persistence of different volatility regimes. A comprehensive understanding of these complex dynamics is crucial for effective risk management, asset pricing, and policymaking. For instance, accurately identifying regime shifts can enable investors to adjust their portfolios to changing market conditions, while policymakers can design more targeted regulatory frameworks.

1.4 Technological Innovations

Technological and innovation regimes refer to frameworks or systems characterized by specific patterns and rates of technological advancement and innovation. These regimes influence economic structures, productivity, and societal development. The switch from one regime to another with different characteristics can take place all at once, when radical innovations hit the current situation, or step by step, with an incremental path.

Radical innovations refer to innovations that are new to the firm, market, and industry; which incorporate a substantially different and new technology; and which provide substantially higher customer benefits relative to current products in the industry. They can disrupt existing industries, leading to the rise of new competitors and the decline of established firms. The next adoption of artificial intelligence in numerous sectors is a clear example of a radical innovation, with an high probability of a future regime shift in various sector.

In contrast, incremental innovations refer to improvements in a firm's existing product offerings that better satisfy the needs of its current and potential customers. Incremental innovations manifest as adaptations, refinements, enhancements, or line extensions, incorporating new features that offer additional benefits. To the extent that incremental innovations entail changes in the underlying technology, those changes in the technological trajectory tend to be relatively small and place limited strains on a firm's existing competencies. The evolution of smartphones is a prime example of incremental innovation, with various companies contributing to continuous improvements in design, functionality, and user experience.

Chapter 2

Thresholds and Tipping Points

The concepts of *thresholds* and *tipping points* are crucial for understanding regime shifts. They help identify critical junctures where small changes can lead to significant and often irreversible transitions in the state of an economy or market. Indeed, quite often a regime switch occurs when a certain endogenous variable crosses some thresholds, the exact value of which may, or may not, be known to a decision-maker. It is possible to make a distinction between two types of thresholds in the state variable of a dynamic system. The first type, once crossed, induces a discrete change in the differential equation (or difference equation) that describes the state dynamics or in the preferences of the decision-makers. Such thresholds are often critical points that signify a transition from one regime to another, potentially resulting in abrupt policy changes or market reactions. The second type only involves, upon crossing it, a change in the basin of attraction [25]. Although the transition may not be as abrupt as with discrete thresholds, it signifies a substantial alteration in the qualitative properties of the optimal policies available to decision-makers. This type emphasizes the stability and resilience of the system, illustrating how small changes in conditions can push the system toward a new equilibrium (figure 2.1).

2.1 Economics Examples

The literature on regime shifts in economic-related systems is quite rich and full of tipping points' examples. In order to present the main aspects of this matter, we focus on three macro-areas: environmental economics, finance, fiscal and monetary policy. This distinguish makes simpler to give a more formal definition of the tipping point notion, which can be slightly different for various "sectors"; moreover, it helps to better analyze: first, the conditions that positively impact the tipping possibility, second, the consequences that the switch has on the system.

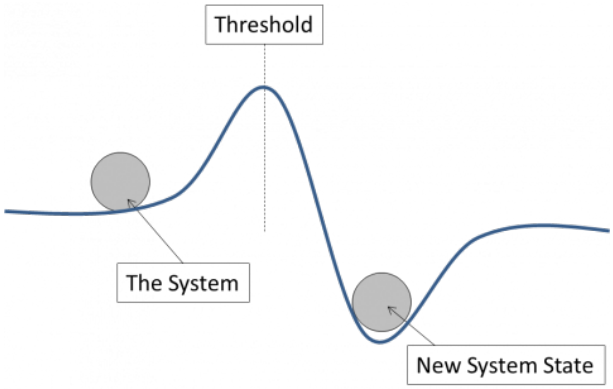


Figure 2.1: The *ball in a basin* metaphor explains how resilient systems work. If the ball is pushed a little bit, it will return to the bottom of the basin, i.e., to its initial state. If the ball is pushed hard enough, it will leave the basin and eventually settle somewhere else, i.e., in an additional state. The height of the basin thus corresponds with resilience capacity: the higher the basin, the harder of a push the ball can withstand and still return to its initial stage [17]

2.1.1 Environmental Economics

An ecological threshold is the point at which there is an abrupt change in an ecosystem's quality, property, or phenomenon, where small changes in an environmental driver produce large responses in the ecosystem. Regime shifts in ecosystems are policy-relevant because they can affect the flow of ecosystem services that societies rely on; they are difficult to predict and often hard, or even impossible, to reverse. Historical examples are many: we can mention the desertification phenomenon that in ancient times hit the Sahara region, which, after a prolonged period of declining vegetation cover, collapsed suddenly into a desert; or even more recent occurrences, such as the significant environmental regime shifts in the North Pacific Ocean in 1976, which followed changes in various climatic variables. Additionally, coral reef bleaching illustrates how small increases in ocean temperature can lead to catastrophic losses in biodiversity, emphasizing the urgent need for effective environmental policies to manage such tipping points. The Muttoni's paper, which we will discuss in the last chapter, is based on this very concept: the stochastic model elaborated in it has indeed the final objective to give practical suggestions to the planner who has to face a climate-related tipping point.

The analysis of tipping points within the context of environmental and economic policies has become increasingly crucial over time. Daily news reports showcase the devastating impacts of events, initially seemingly insignificant, on both society and the economy over the long term. Therefore, all individuals, especially those with the power to influence international policies, must consider the possibility that various tipping elements (figure 3.2) could trigger shifts in current environmental dynamics. Efforts should be made to delay or, if possible, prevent such occurrences, or at least prepare to mitigate their most severe consequences.

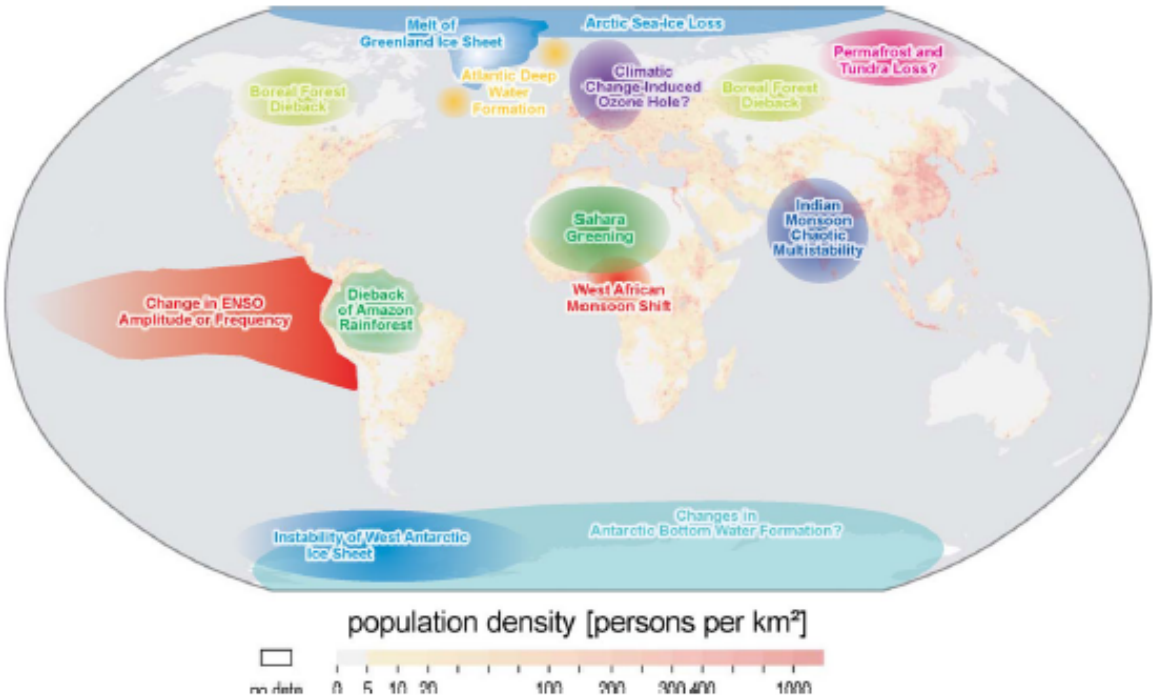


Figure 2.2: Map of potential policy-relevant tipping elements in the climate system [14].

2.1.2 Finance

Financial markets are complex systems where economic agents interact to determine the prices of securities, such as stocks, bonds, and derivatives. These markets are characterized by periods of relative stability, punctuated by episodes of heightened volatility and extreme price movements. These shifts in market dynamics, often referred to as regime shifts or phase transitions, can have profound implications for investors, policymakers, and the broader economy.

A key question in financial economics is how to identify and anticipate such regime shifts. Although traditional economic models often assume rational behavior and efficient markets, recent research has increasingly recognized the role of behavioral factors, such as herd behavior, overconfidence, and fear, in driving market dynamics. These psychological factors can amplify price movements and contribute to the formation of bubbles and crashes. To understand the underlying mechanisms of market instability, researchers have drawn insights from various fields, including physics, mathematics, and complexity theory. One such approach is the log-periodic power law (LPPL) model, which combines elements of rational expectations, behavioral finance, and statistical physics. By analyzing the accelerating price dynamics and oscillatory patterns observed in bubble episodes, the LPPL model can provide early warning signals of impending market crashes.

A bubble is defined as a market regime in which the price accelerates "super-exponentially,"

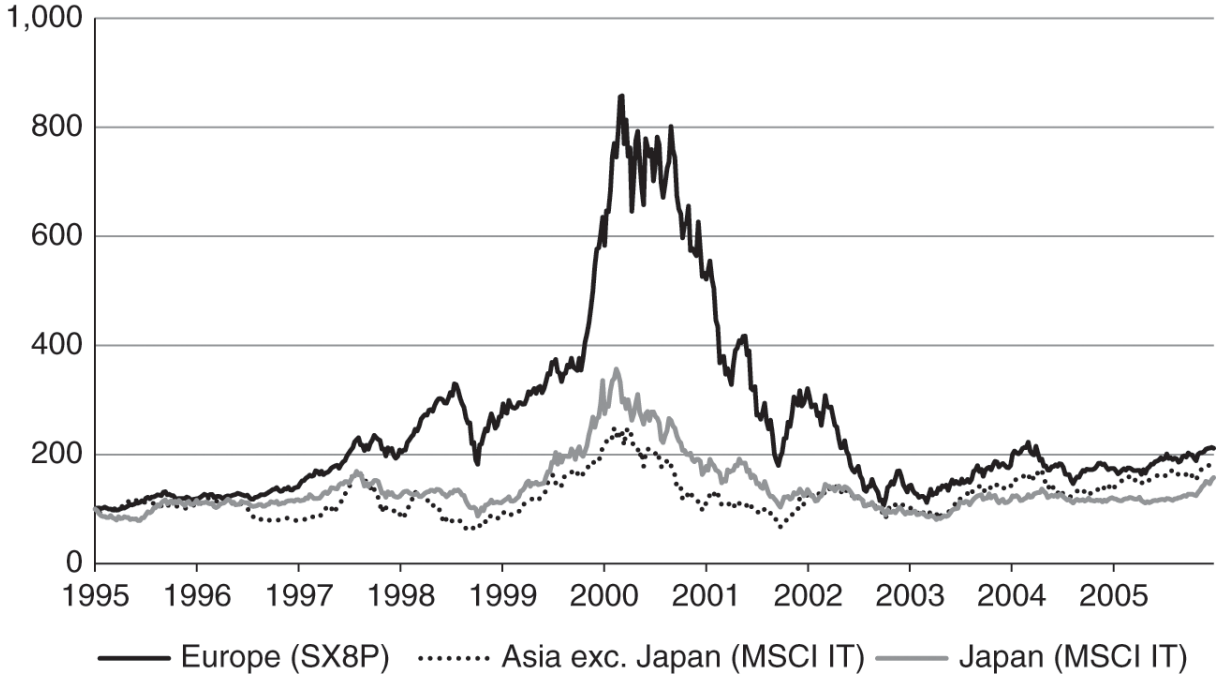


Figure 2.3: Global technology stock price indexes, 1995-2006 [23]

meaning that the growth rate of the price increases itself. A constant price growth rate (also called return) leads to exponential growth, the normal average trajectory of most economic and financial time series. When the growth rate itself grows, the price accelerates hyperbolically, interpreted as a result of progressively increasing market cooperation among investors. As the bubble matures, it approaches a critical point, triggering potential instability in the form of a crash or a regime shift. This critical point is also referred to as a bifurcation, catastrophe, or tipping point. An example is the dot-com boom, a stock market bubble that peaked on March 10, 2000. This period of market growth coincided with the widespread adoption of the World Wide Web and significant venture capital inflows, leading to rapidly increasing valuations for new dot-com startups. When investors began to realize that many of these companies would not deliver on their promises, a mass sell-off crossed a threshold, leading to a sharp decline in stock prices and marking a regime shift from high market optimism to widespread financial distress (figure 3.3). Following such events, regulatory adjustments are often necessary to stabilize markets and prevent future bubbles.

2.1.3 Economics

In this work, we consider policymakers as the primary actors who must, where possible, prevent or, at the very least, respond to regime switches when they lead to negative consequences, as in the case of climate-related tipping points discussed at the beginning of this chapter and

examined in greater depth in the final chapter. However, this is not the only perspective on the issue. Policymakers can also intentionally induce regime switches to harness their potential benefits. Conversely, they may fall prey to misjudgments about economic conditions, leading to the crossing of critical thresholds with potentially devastating economic consequences, as theorized, for instance, by Dornbusch in his model. This author, in his article *Expectations and exchange rate dynamics* [2], although not explicitly referring to regime changes, in a certain sense identifies a threshold when discussing the phenomenon of overshooting, a term he coined to describe the situation in which an economic variable, such as the exchange rate, temporarily exceeds its long-term equilibrium value due to policy actions or market forces. The long-term equilibrium value can indeed be interpreted as a threshold: once this value is surpassed by the system, it is possible for markets to overestimate the depreciation of the exchange rate, potentially driving the system towards (theoretically) infinite inflation.

Moreover, institutions are not the sole actors within the economic landscape capable of initiating, exploiting, managing, or preventing regime switches. Shifting the focus from the macroeconomic to the microeconomic level, the literature abounds with analyses and models addressing regime shifts in contexts such as competition among firms within the same industry, the expansion of specific sectors, and even the sudden introduction of innovative products, often interpreted as tipping points. The already cited textbook by Haunschmied et al. [25], provides descriptions of various of these studies. For instance, a generic model of optimal switching from competition to cooperation can cover a lot of cases which regard two or more parties merging into a single entity, founding an analytical solution that indicates the instant when is optimal starting to cooperate instead continuing to play strategically.

Chapter 3

Mathematical Framework

Regime shifts occur when an economic system transitions between different structural states, often abruptly and irreversibly. These transitions, triggered by various internal dynamics or external shocks, represent critical junctures where small changes in underlying variables can lead to large systemic shifts. In this chapter, we explore the mathematical foundations of regime shifts: from the examination of the concept of dynamic system and its equilibrium, passing by bifurcation analysis and a quick review of the stability notion, we will study models that capture the essence of these phenomena. Mathematical models of regime switching provide frameworks to analyze the transition from different regimes, aiding in both forecasting and policy design.

3.1 Dynamic Systems

The analysis of dynamic systems provides a foundation for understanding how small changes in economic or financial parameters can amplify over time, potentially leading to regime shifts. By examining the stability of equilibria and the onset of bifurcations, we gain insights into the precursors of these shifts.

The mathematical framework of this section is critical to fully understand the concepts, even from an economic-oriented point of view. In order to provide a clear explanation, we have extensively relied on a variety of authoritative sources, particularly textbooks. Among these, the foundational framework and formal definitions, unless otherwise specified, have been drawn from Kuznetsov's *Elements of Applied Bifurcation Theory* [7]. This text has provided us with a robust mathematical foundation upon which to build our analysis.

In the field of systems theory, a dynamic system is a mathematical model that represents an object with a finite number of degrees of freedom evolving over time according to a set of rules which can be deterministic or stochastic, and that primarily depends on the state of the system at any given time. In deterministic systems, the current state fully provides the next state based

on fixed mathematical laws; on the other hand, when randomness is introduced, the next state of the system is influenced by probabilistic factors and stochastic systems are considered [6]. Formally,

Definition 1. A *dynamic system* is a triple $\{T, X, \phi^t\}$ where T is a time set, X is a state space, and $\phi^t : X \to X$ is a family of evolution operators parameterized by $t \in T$ and satisfying two natural properties that reflect the deterministic character of the behavior of dynamic systems. The first implies that the system does not change spontaneously its state, the latter that the law governing the behavior of the system does not change in time: the system is autonomous.

The mathematical representation of a dynamic system varies based on whether we are considering discrete or continuous time. In the first case, it can be modeled as a difference equation like the following, where ϵ_t is a random shock introduced into the system:

$$x_{t+1} = f(x_t; \theta) + \epsilon_t \tag{3.1}$$

Contrarily, the most common way to define a continuous-time dynamic system is by differential equations. A system of n *autonomous ordinary differential equations* (ODEs for short), given the $f_1, \ldots, f_n$ functions of the $n+1$ variables $(t, x_1, \ldots, x_n)$, is displayed in normal form as:

$$\begin{cases} \dfrac{x_1(t)}{t} = \dot{x}_1 = f_1(t, x_1, \ldots, x_n) \\ \cdots = \ldots \\ \dfrac{x_n(t)}{t} = \dot{x}_n = f_n(t, x_1, \ldots, x_n) \end{cases} \tag{3.2}$$

where $x(t)$ is the state of the system at time t. As in the definition 1, the system is represented as a vector in the state space, a conceptual framework used to illustrate all possible states of the system, in which each point corresponds to a unique position of the system at a given time. In economics, the state (or *phase*) space helps visualize how variables evolve over time, and whether they tend towards stable states or exhibit more complex behavior such as oscillations or chaos [10].

3.1.1 Equilibria

In the context of dynamic systems, an *equilibrium point* is a solution in correspondence of which the system does not change over time, meaning that all state variables remain constant when the system is at this point. Mathematically, an equilibrium point x^* of a system defined by the equations $\dot{x} = f(x)$ satisfies:

$$f(x^*) = 0$$

To better analyze the concept of equilibrium, it is helpful to express the system in *matrix form*. This representation is especially straightforward when dealing with a linear, autonomous system with constant coefficients. For a system of two equations in two variables, the matrix form of $\dot{x} = Ax + b$ is typically written as follows:

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} \quad A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.3}$$

If $b = 0$, then the system is homogeneous. The matrix A is non-singular if there exists its inverse A^{-1}, which occurs if and only if its determinant is non-zero [19]. When A is non-singular, the only solution to the homogeneous system is the origin, $x^* = 0$, where $x_1^* = 0$ and $x_2^* = 0$. It is important to mention that it is always possible to transform a non-homogeneous system into an homogeneous one, by expressing it in terms of deviation from the equilibrium point (assuming it exists). The characteristic equation of the matrix A is:

$$\Phi(\lambda) = det(A - \lambda I) = 0 \tag{3.4}$$

where I is the so-called identity matrix. Solving the 3.4, the λ values found are the *eigenvalues* of the matrix A, while the solutions of the system obtained substituting the eigenvalues into the system are the corresponding *eigenvectors* [18]. The numbers λ_1, λ_2 found solving the auxiliary determine a classification of the equilibrium points. Generally, when the matrix A is non-singular as we prefixed, they are of four different kinds: node and saddle, when the solutions of the auxiliary are real numbers, focus and center, when, contrarily, the solutions of the auxiliary have complex parts.

Equilibria	Eigenvalues
Node	$\lambda_1\lambda_2 > 0$
Saddle	$\lambda_1\lambda_2 < 0$
Focus	$Re\lambda_1 = Re\lambda_2 \neq 0$
Center	$Re\lambda_1 = Re\lambda_2 = 0$

Following the general theorems of stability, these equilibria are further divide based on the behavior of the system in the proximity of them. If the real eigenvalues (or real parts of complex eigenvalues) are negative, then the equilibrium point is *asymptotically stable*. Examples of such equilibrium positions are stable node and stable focus . If the real part of at least one eigenvalue is positive, the corresponding equilibrium point is unstable. For example, it may be a saddle [18]. Finally, in the case of purely imaginary roots (when the equilibrium point is

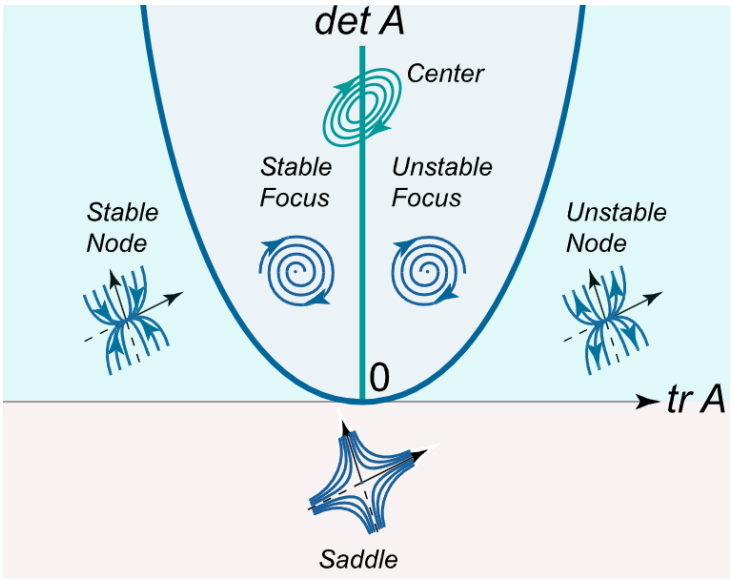


Figure 3.1: Types of stability [27]

a center), we are dealing with the classical stability in the sense of *Lyapunov*. The concepts of asymptotic stability and Lyapunov stability will be studied more deeply at the end of the section, enlightening the importance of them in the matter of regime switching.

Even not computing the eigenvalues λ_1, λ_2, but only knowing the trace and the determinant of the matrix A, it is possible to determine the type of an equilibrium point. The trace of the matrix is the sum of its diagonal elements:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad tr(A) = a_{1,1} + a_{2,2} \quad det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

Writing the auxiliary of the matrix A, 3.4, in terms of these values, we have:

$$\lambda^2 - tr(A)\lambda + det(A) = 0 \tag{3.5}$$

The discriminant of this equation is:

$$\Delta = tr(A)^2 - 4det(A) \tag{3.6}$$

The curve $det(A) = (tr(A)/2)^2$ resulted, plotted on a graph where the horizontal axis represents the trace and the vertical axis represents the determinant of matrix A, simply is a parabola with its vertex at the origin and its axis aligned with the positive half of the vertical axis (figure 3.1). It shows the different stability modes of a system's equilibria based on the values the determinant and the trace assumes in the exact point.

Equilibrium Analysis

The analysis of equilibria within a dynamic system offers insights into how systems evolve and respond to external shocks. In economic contexts, equilibrium points signify steady states where variables such as prices, output, or interest rates tend to remain constant over time. However, real-world economic systems are seldom isolated from disturbances, and shifts can occur as the system encounters changes in parameters or exogenous shocks. This is where regime switching enters: it provides a framework for understanding how the stability of equilibria influences the system's propensity to transition between different structural states.

As seen before, equilibrium points can vary significantly in their stability characteristics. When all eigenvalues of the Jacobian matrix A have negative real parts, the equilibrium is asymptotically stable. This stability suggests resilience: small perturbations will return to equilibrium over time. Conversely, if one or more eigenvalues have positive real parts, the equilibrium becomes unstable, increasing the likelihood of a shift toward a different regime if disturbed. This behavior aligns closely with the concept of tipping points in regime-switching models. Once an unstable equilibrium is reached, the economic system is likely to move away from the current regime, potentially entering a new one.

In this way, regime switches are intimately linked with the stability of equilibrium points. When the system is in the vicinity of an unstable equilibrium, small changes in economic parameters can lead to large, often irreversible shifts in the system's behavior. Understanding the structure of the Jacobian matrix, therefore, provides a mathematical foundation for identifying these critical points and, by extension, predicting regime switches.

3.1.2 Bifurcations

Bifurcations represent points at which the qualitative nature of a dynamic system's equilibria changes as parameters pass critical values. In economic models, bifurcations are particularly relevant for identifying structural changes that correspond to regime switches. For instance, a saddle-node bifurcation might represent a situation where two equilibria, one stable and one unstable, merge and annihilate each other. This phenomenon could signify a transition point in the economy—such as a market collapse or rapid shift in demand.

In order to fully understand the concept, we start by introducing the notion of *topological equivalence*.

Definition 2. A dynamical system $\{T, \mathbb{R}^n, \phi_t\}$ is called topologically equivalent to a dynamical system $\{T, \mathbb{R}^n, \psi_t\}$ if there is a *homeomorphism* $h : \mathbb{R}^n \to \mathbb{R}^n$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

A homeomorphism is an invertible map such that both the map and its inverse are continuous. It preserves continuity, meaning it is smooth and reversible, and maps entire paths in the first system onto paths in the second system in a way that respects time flow, so each trajectory in one system has a counterpart in the other. Intuitively, two dynamical systems that are topologically equivalent among them, even if they are not exactly the same, behave in a qualitatively similar way. Imagine two systems with states that evolve over time in different ways. Although their exact paths may look different, they can still be considered *equivalent* if there is a way to match up their paths/orbits in a consistent way that preserves the direction of time. Essentially, topological equivalence explains that, although two systems might evolve with different exact numbers or positions, they share the same types of patterns, like cycles or spirals, and they have the same stable and unstable behaviors.

Considering the phase portrait of a dynamic system that depends on parameters, as the parameters vary, the phase portrait makes the same. There are two possibilities: either the system remains topologically equivalent to the original one, or its topology changes.

Definition 3. The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a *bifurcation*.

Thus, a bifurcation is a change in the topological type of the system as its parameters pass through a critical value. In other words, in correspondence of this bifurcation value, the system definitely changes its behavior with respect to the original one. Understanding these bifurcation points equips economists to predict when small changes in parameters — like an interest rate or policy variable — could trigger large-scale regime changes in economics.

In dynamic systems, bifurcations are categorized based on how the equilibrium points change and the nature of this change as parameters cross critical thresholds. In this paragraph, we mention the more economic modeling-related types, which are: saddle-node, transcritical, Hopf and pitchfork.

A saddle-node bifurcation occurs when two equilibria — a stable and an unstable one — merge and annihilate each other as a parameter changes. In economics, this could represent a threshold in the market where, for instance, two competing equilibrium points (e.g., stable growth and rapid decline) collide, resulting in a critical point like a market collapse. Transcritical bifurcations involve two equilibria that exchange their stability as the parameters vary. They represent scenarios where one equilibrium changes from stable to unstable, while another does the opposite, modeling systems that shift from one regime to another (e.g., from low inflation to high inflation). A Hopf bifurcation occurs when a pair of complex conjugate eigenvalues cross the imaginary axis, causing a stable equilibrium to become unstable and leading to periodic oscillations or limit cycles around the equilibrium. This bifurcation is valuable for modeling cyclical behavior in economies such as business cycles, where stable growth transitions into

periodic booms and recessions. A Hopf bifurcation might, for instance, represent periodic cycles of boom and bust in an economy, illustrating how a once stable growth equilibrium can start oscillating due to changes in policy parameters. Finally, in a pitchfork bifurcation, a single equilibrium splits into multiple branches, typically one stable and two unstable or vice versa. This bifurcation is relevant in scenarios with symmetry, where small parameter changes can cause symmetry-breaking behavior.

In summary, bifurcation points mark critical junctures where small parameter changes lead to structural shifts, enabling us to understand the thresholds at which regime switching may occur in economic models. Each of the types of bifurcations we just analyzed, provides insights into regime shifts by illustrating how small changes in parameters can lead to large-scale structural changes in the system. By identifying these types, economists can potentially anticipate and respond to upcoming regime shifts by adjusting the relevant parameters or applying policies to stabilize desired equilibrium points.

3.1.3 Stability

Before in the section, we have introduced the notions of asymptotic and Lyapunov stability. These concepts are crucial for predicting the behavior of a system near equilibrium points and the likelihood that a regime shift would take place. The formulation of Lyapunov stability is the following.

Theorem 1 (Lyapunov, 1982)**.** *Consider a dynamical system defined by* $\dot{x} = f(x), x \in \mathbb{R}^n$*, where* f *is* smooth [*]*. Suppose that it has an equilibrium* x^0 *(i.e.,* $f(x^0) = 0$*), and denote by* A *the Jacobian matrix of* $f(x)$ *evaluated at the equilibrium,* $A = f_x(x^0)$*. Then* x^0 *is stable if all eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of* A *satisfy Re* $\lambda < 0$*.*

In economic modeling, Lyapunov functions are utilized to assess the stability of various system states, particularly in economic model predictive control (EMPC) for nonlinear systems. EMPC frameworks can combine Lyapunov stability analysis with predictive modeling (such as recurrent neural networks) to ensure both optimal economic performance and system stability over time. This approach is widely applied in fields like process control, where EMPC optimizes profitability while maintaining system stability—illustrated in scenarios like energy management in chemical processing, ensuring safe, profitable operations across fluctuating conditions. In the context of regime switches, Lyapunov stability offers a means to evaluate whether slight deviations from an equilibrium state will result in convergence back to equilibrium or in departure toward a new regime. This type of stability implies that while the system may oscillate

[*]A function is considered smooth if it has derivatives of all orders, meaning it's infinitely differentiable. Often referred as C^∞

near an equilibrium, it does not tend toward large deviations, thereby avoiding regime switches. On the other hand, asymptotic stability implies a stronger condition: deviations not only remain bounded but also diminish over time. More formally, if the equilibrium x^* is Lyapunov stable, and all solutions that start out near x^* converge to x^*, then the point is said to be asymptotically stable. Like we said earlier in the dissertation, this is the case when the eigenvalues of the Jacobian matrix all have negative real parts. Asymptotic stability implies suggests resilience to small perturbations. However, if parameter variations push the system toward a bifurcation point, even this "stronger" kind of stability may be compromised, leading to a regime switch. In such scenarios, *Lyapunov functions* can be constructed to analyze the conditions under which stability is lost, thus predicting potential shifts in the economic system's structure.

3.2 How to anticipate the Switch

In the previous section we provided the mathematical context where we conduct the regime switch study: the dynamic systems' framework, with the analysis of bifurcations and the concept of equilibrium points' stability. Shaped the foundations; now, we can complete the building, starting from the unfinished discussion over the concepts of threshold and tipping points of the previous chapter. We did analyze these notions through some examples and applications, highlighting the importance of identifying tipping points in the environmental, financial and macroeconomics sectors. In this section, we will point out some mathematical and econometric methods to do it, along with some strategies to take in order to face the regime shift occurrence. We will focus on key concepts such as the *hazard function* technique, typically used in the literature on regime shifts to develop model when the switching time is unknown, and on the time series analysis. We are going to conclude this part with a few words about a specific threshold known as the *Skiba point*.

3.2.1 Identifying Tipping Points

Managing tipping points effectively requires early identification, which enables decision-makers to mitigate potential risks or harness favorable transitions. Indeed, governments and central banks can use early warning signals to implement stabilizing measures, such as liquidity injections or interest rate adjustments, to preempt crises. Even simple investors, recognizing rising systemic risks, might diversify their portfolios to hedge against potential losses. For example, during periods of financial instability, shifting assets to low-risk bonds or diversifying into commodities can help manage risk. Moreover, in environmental and ecological contexts, policies can be adapted based on early warnings, such as restricting resource extraction to prevent ecosystem collapse. In the financial, environmental, and policy sectors, each type of system

displays unique early warning indicators that reflect increasing vulnerability within the system and signal the approach of a threshold.

For instance, volatility spikes in financial markets can signal instability, particularly when fluctuations in asset prices or interest rates deviate significantly from historical norms. An example can be the situation just before the 2008 financial crisis, when volatility in credit markets began rising sharply and the S&P 500 lost about 56% of its value from the October 2007 peak to the March 2009. Another indicator is the increased correlation between past and present values within a system, known as rising autocorrelation. It suggests diminished resilience and may indicate that the system is closer to a tipping point. It can be observed in an overvalued stock market, where rising autocorrelation in returns can signify increased systemic risk and an impending correction. Changes in system structure are important factors too: indicators like shifts in market sentiment, drastic policy adjustments, or structural economic changes can reflect a system preparing to switch regimes. Policy shifts, such as unexpected tightening by a central bank, often signal impending changes in economic conditions. A final example is the phenomenon known as *critical slowing down*, which occurs when a system takes longer to return to equilibrium after small disturbances. In economic terms, if a market requires increasingly longer recovery periods after minor shocks, it suggests proximity to a critical threshold. The study of this subject is often associated with the analysis of bifurcations that we mentioned early in this document, and the literature evidenced how the critical slowing down is observed in any continuous model approaching a fold bifurcation (also known as saddle-node, figure 3.2). On this particular matter focuses, among a plenty of other sources, the Diks et al's article *Critical slowing down as an early warning for financial crises?* [20], which gives an answer to the title-question through a four historical financial crises' examination: the already cited 2000 Dot-com bubble burst and 2008 Financial Crisis, plus the 1997 Asian Crisis and the 1987 Black Monday. Using a methodology composed by different steps - from the detrending technique to achieve mean-stationarity of the data, to a thorough evaluation of the resulted time-series - the authors found mixed evidences: critical slowing down effectively anticipated the first crisis, while the most recent ones are much less correlated with that, probably indicating that this phenomenon can be used as an early indicator more for natural systems than for social systems, due to presence of smart agents.

Besides that, policymakers or economic actors in general still have at their disposal several quantitative tools, from statistical-econometrics techniques to stochastic methods, passing by the implementation of machine learning in the analysis, to detect early warning indicators. First, to observe for time-varying volatility, volatility clustering or the increasing correlation we mentioned as one of the typical indicators, come in hand statistical models for time series data analysis such as ARIMA (autoregressive integrated moving average) or GARCH (gener-

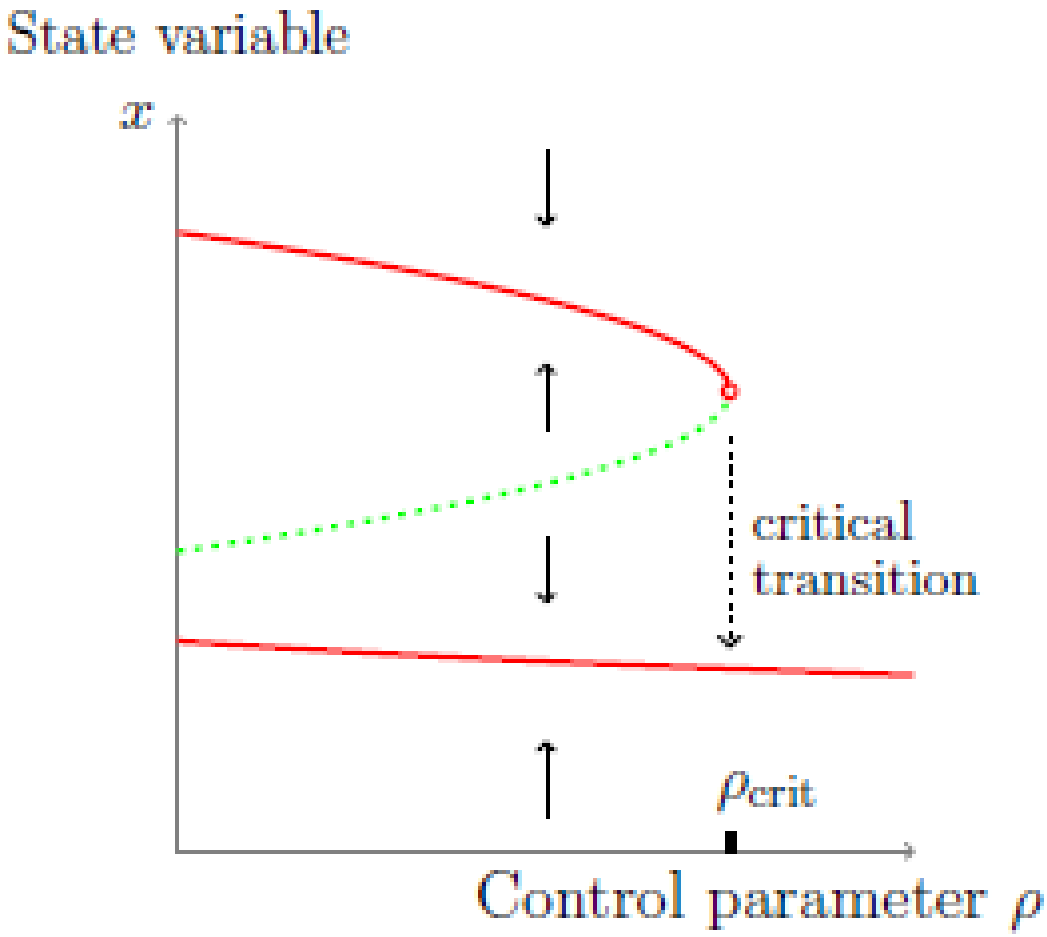


Figure 3.2: A saddle-node bifurcation. The red lines indicate stable equilibria of the state variable x as a function of the control parameter ρ and the green dotted line an unstable equilibrium. The solid arrows indicate whether the state variable is moving up or down in the state space, depending on the control parameter and the state x. The stable top equilibrium branch meets the unstable equilibrium branch at a critical parameter value ρ_{crit} . If the system starts in the upper stable equilibrium, and the control parameter is subsequently increased, slowly or in steps, to just beyond the critical parameter value ρ_{crit}, this induces a critical transition of the state variable to the lower stable equilibrium. [20]

alized autoregressive conditional heteroskedasticity) models. Indeed, by identifying abnormal increases in volatility or other changes in trend, these methods can highlight periods of rising risk [22]. Then, recent literature has explored machine learning techniques, particularly clustering and anomaly detection, for identifying patterns and potential tipping points in complex systems. Techniques such as clustering and neural networks are applied in high-frequency financial markets and other complex datasets to enhance predictive capabilities. Very important in the matter is also the already mentioned bifurcation study, because numerical methods can simulate dynamic systems and perform bifurcation analysis to predict qualitative shifts in system behavior, helping to identify critical thresholds that may not be immediately apparent. At the end, stochastic models account for randomness in system behavior, especially when predicting unknown tipping points. An instance is the hazard function, which we will discuss in the following paragraph, through a formalization taken by the cited Muttoni's paper [26].

3.2.2 The Hazard Function

The hazard function is a probabilistic tool widely used in reliability engineering, environmental sciences, and economics to model the likelihood of imminent change, particularly when the exact tipping point is unknown. It's essential in situations where a system might suddenly shift due to the accumulation of risk or stress over time, even if the precise moment of shift cannot be predetermined. In economic models, the hazard function helps estimate the probability that a regime change, such as a financial crisis or market crash, might occur within a given time-frame, conditioned on the system having survived until that point without such a shift.

Formally, let indicate with τ the instant in which the switch happens, taking values in $[0, \infty)$, while the programming interval is $[0, T]$. The value which describes the probability distribution of τ is known as *hazard rate*, and it is found by the equation:

$$\lim_{dt \to 0^+} \frac{\mathbb{P}(\tau \leq t + dt \,|\, \tau > t)}{dt} = h(t,\, x(t),\, u(t)) \tag{3.7}$$

where $h(t, x, \mu) > 0$ is the hazard rate function, while x and μ are endogenous variables of the system.

The hazard function's utility lies in capturing the dynamics of systems that face increasing (or decreasing) likelihood of regime change over time. In reliability engineering, for instance, a rising hazard rate might indicate that a machine is more likely to fail as it ages, due to wear and tear. In financial markets, a rising hazard rate could reflect the mounting probability of a market correction following prolonged overvaluation. Or, the function can also model the probability of events like bond defaults or sudden drops in asset prices. By analyzing the hazard rate over time, economists can assess how factors like mounting debt or inflationary pressures contribute

to the risk of a regime switch from economic stability to crisis. In ecological systems, the hazard function is used to estimate the likelihood of critical transitions, such as the desertification of an ecosystem or the collapse of a fishery. For example, as environmental stressors accumulate—say, through pollution or deforestation—the hazard rate can indicate the rising probability of an irreversible regime shift to a degraded state. In industrial operations, the hazard function helps predict the time to system failure based on historical failure data, enabling better maintenance and risk management strategies. This concept is transferable to economic and policy planning, where understanding the hazard rate for regime shifts can guide intervention timing to prevent or mitigate crises: consider for instance a market experiencing sustained growth in asset prices. The hazard function can model the probability that this growth will culminate in a sudden correction or crash. As the market continues to rise, the hazard rate might increase, reflecting the greater likelihood of a tipping point where the trend reverses. Investors and policymakers monitoring this rate can potentially adjust their strategies to mitigate risk or capitalize on the approaching shift. Despite its utility, the hazard function assumes independence of events, which can be a significant limitation in systems with interdependent components or where historical data might not be indicative of future outcomes. For example, external shocks or systemic changes can influence the likelihood of events in ways that a standard hazard model might not capture.

3.2.3 Skiba Points

Skiba showed that if an optimal control problem exhibits two steady states that are saddle points, then there exists in the state space a threshold that separates the two basins of attraction [4]. This threshold is now known as a *Skiba point*, where the decision-maker is indifferent between two trajectories, each converging to a different steady state. Such indifference can lead to significant strategic implications, particularly in situations where small changes in parameters can drastically alter the path of the system.

These points can be classified by focusing on the type of unstable steady state they represent, forming three categories: focus, continuous node, or discontinuous node [12]. The concept of Skiba points is frequently used in the literature, often assuming a single decision-maker, but it can also apply to models featuring multiple agents interacting in a dynamic game setting. The latter introduces complexity, making it suitable for analyzing scenarios such as research and development races in industrial organization.

The strategic importance of Skiba points lies in their ability to influence decision-making. Being at or near a Skiba point can result in significant policy shifts based on minor parameter changes, affecting the overall stability of the system. This necessitates careful monitoring and analysis by decision-makers to navigate potential transitions effectively.

To provide a more formal definition, consider the optimization problem consisting in maximizing the objective function:

$$\hat{U}(x,c) := \int_0^\infty e^{-\theta t} U(x(t), c(t))\, dt \tag{3.8}$$

subject to:

$$\dot{x}(t) = f(x(t), c(t)), t \in \mathbb{R}_+ \quad x(0) = x_0 \tag{3.9}$$

where $x(t) \in \mathbb{R}^n$ are the state variables and $c(t) \in \mathbb{R}^n$ are the control variables which are assumed to be piecewise continuous. We further assume that there exists at least one optimal solution of the problem (3.8)-(3.9). A Skiba point x_s is a state at which multiple stable equilibria exist, and from which the decision-maker is indifferent in terms of long-term utility between alternative paths or basins of attraction. If the utility function has multiple local maxima depending on the trajectory chosen from x_s then, this is identified as a Skiba point. At this point, the value functions corresponding to each trajectory intersect, leading to indifference between those trajectories for the decision-maker.

They can be identified and analyzed in various situations. In environmental economics policymakers may face a Skiba point when deciding whether to implement stricter pollution controls at a cost to economic output or allow emissions to grow for short-term economic benefits. For example, in a region with an industrial sector highly dependent on pollution-intensive processes, the Skiba point represents the pollution level at which the long-term benefits of a clean environment equal the short-term economic costs of stricter regulations. In the same field, consider a model in which a government decides between preserving a natural resource (like a forest) or converting it for economic use (such as agriculture). In this case a Skiba point might represent a threshold in the resource stock. At this point, the government is indifferent between conserving the resource, leading to sustainable use, or exploiting it, leading to depletion and economic gain. Each path has different economic and ecological equilibria, and the point (or curve) marks where both choices yield equal utility or profit in the long run. They can be studied in models of industrial organization too, where firms often choose between investing in research and development to innovate or focusing on current production to maximize short-term profits. A Skiba point might occur where firms are indifferent between investing in R&D to potentially dominate the market later or avoiding R&D costs and sticking to current profitable production. At this Skiba point, the utility of both strategies—future market leadership versus immediate gains—would be equal. Finally, in macroeconomic growth models, individuals or firms decide between saving to accumulate capital and enhance future production or consuming to increase present satisfaction. A Skiba point can mark a critical level of capital where the agent is indifferent between further investment for long-term growth or consumption for immediate utility.

This point often arises in models where one path leads to high future returns from capital accumulation, while the other offers immediate but less sustainable satisfaction [8].

3.3 Stochastic Models for Regime Shifts

We have seen that many variables undergo episodes in which the behavior of the series seems to change quite dramatically, or, in other words, the regimes switch from one to another. In the previous section, we underlined the importance of paying attention at the early warning indicators that these events "lay" before happening, in order to contrast the potential bad effects or simply adapt the policies to new situations. Now, we focus on the analysis of stochastic modeling methods that capture the probabilistic nature of regime shifts in economic and financial systems and are particularly useful for understanding systems influenced by random fluctuations, where sudden changes in state can occur without a clear deterministic trigger.

The question is: how are we going to forecast a time series that has had changes? If the process has changed in the past, clearly it could also change again in the future, and this prospect should be taken into account in forming a forecast. Moreover, the change in regime surely should not be regarded as the outcome of a perfectly foreseeable, deterministic event. Rather, itself it is a random variable. A complete time series model would therefore include a description of the probability law governing the change from state 1 to state 2. These observations suggest that we might consider the process to be influenced by an unobserved random variable s_t^*, which will be called the state or *regime* that the process was in at date t. Thus, we need a description of the time series process for this unobserved variable. Examples of random processes with discrete states in discrete or continuous time are several, among them we mention the followings.

3.3.1 Poisson Process

This paragraph collects formal definitions and equations which, if not specified otherwise, are taken from Pinsky's textbook *An introduction to stochastic modeling* [15].

Definition 4. A Poisson process of intensity or rate $\lambda > 0$ is an integer-valued stochastic process $\{X(t); t \geq 0\}$ for which:

1. for any points $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, the process increments:

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent random variables;

2. for $s \geq 0$ and $t \geq 0$, the random variable $X(s+t) - X(s)$, for $k = 0, 1, \ldots$ has the Poisson distribution:

$$Pr\{X(s+t) - X(s) = k\} = \frac{(\lambda_t)^k e^{-\lambda t}}{k!}$$

3. $X(0) = 0$

In particular, if $X(t)$ is a Poisson process of rate $\lambda > 0$, then the moments are:

$$E[X(t)] = \lambda t \quad Var[X(t)] = \sigma^2_{\chi(t)} = \lambda t$$

Poisson processes are suitable for various applications. They are primarily used for modeling rare, discrete events that can trigger regime shifts, such as sudden market crashes, geopolitical shocks, credit defaults, or major policy changes; moreover, they are valuable for counting transitions or shocks within a time frame: for instance, if the rate of occurrence of a regime-switching event is constant, we could assume transitions follow a Poisson process, capturing the likelihood of observing multiple changes within a period. Then, in finance, Poisson processes can be paired with Brownian motion in jump-diffusion models, which capture both continuous fluctuations and random, discrete jumps. This framework is particularly relevant in financial markets to represent rapid shifts due to unexpected events, a common aspect of regime-switching models. Finally, they also combine well with Markov processes to form Markov-modulated Poisson processes, where the intensity of the Poisson process changes based on the current regime. This allows modeling of situations where the probability of shocks (e.g., interest rate hikes, asset defaults) varies depending on the economic environment.

An example to demonstrate the suitability of Poisson processes to explain the occurrences of rare events is in the figure 3.4, taken from Cipra's *Time Series in Economics and Finance*, which plots the number n_t of mortgage defaults in a bank credit portfolio from the beginning of accounting year to time t.

3.3.2 Markov Process

In the field of stochastic modeling and time series analysis for examining regime shifts, Professor J.D. Hamilton is undoubtedly a leading authority. His work, particularly in "Time Series Analysis," dedicates a substantial portion to Markov chains, providing a solid foundation for the more formal aspects of the following sections. Other sources have contributed to the development of this overview, and they will be appropriately highlighted.

A Markov process $\{s_t\}$ is a stochastic process with the property that, given the value of s_{t-1}, the values of s_t are not influenced by the values of s_u for $u < t - 1$. In words, the probability

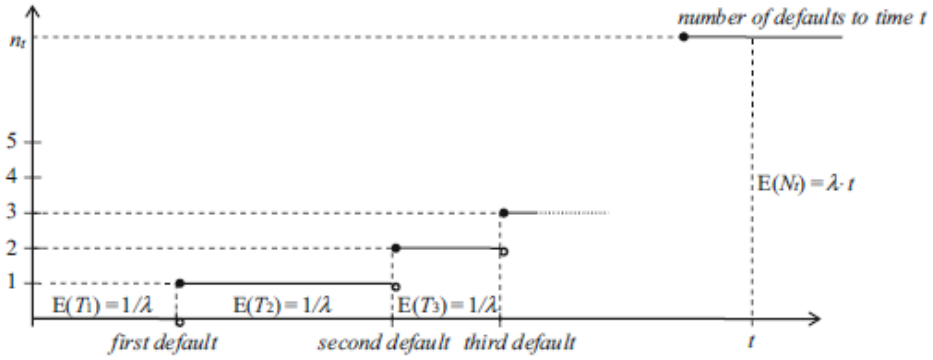


Figure 3.3: Numbers of mortgage defaults in a bank credit portfolio from the beginning of year to time t modeled as Poisson process (T_1, T_2, ... are random periods between defaults) [22].

of any particular future behavior of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behavior. A discrete time Markov chain is a Markov process whose state space is a finite or countable set, and whose time index set is $T = (0, 1, 2, \dots)$.

Formally, let $s_t = 1, 2, \dots$ be a random variable, and its probability to be equal to a certain value j depends only on the most recent value s_{t-1}:

$$P\{s_t = j \,|\, s_{t-1}, s_{t-2} = k, \dots\} = P\{s_t = j \,|\, s_{t-1} = i\} = p_{ij} \tag{3.10}$$

This is known as a *N-state Markov chain* with transition probabilities $\{p_{ij}\}_{i,j=1,2,\dots,N}$, and the property just formalized is the so-called Markov property, which, in words, indicates that the probability of any particular future behavior of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behavior. Indicating by s_t the state or regime of the system in time t, the transition probability value indicates the probability the system has to be in a particular state (j) knowing it just was in another (i). The sum of the probabilities of p_ij, with $j = 1, 2, \dots N$ is 1. The important special case is when p_{ij} is independent of the time variable: the Markov chain is called *homogeneous* and has stationary transition probabilities:

$$p_{ij} = P(s_t = j \,|\, s_{t-1} = i) \tag{3.11}$$

Typically, these numbers p_{ij} are presented in a matrix known as transition probability matrix

of the process and referred as:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{21} & \dots & p_{N1} \\ p_{12} & p_{22} & \dots & p_{N2} \\ \vdots & \vdots & \dots & \vdots \\ P_{1N} & P_{2N} & \dots & p_{NN} \end{bmatrix} \tag{3.12}$$

where its row j's, column i's element is indeed the transition probability p_ij.

A Markov process is completely defined once its transition probability matrix and initial state s_i are specified.

Flexibility is one of the most important features of Markov chains, indeed it makes that common examples of their real-problems applications are plenty: from models of weather forecasts, whit a rich literature especially focusing on the hidden Markov processes (HMP) application[†], to simply analyzing the throwing the dice in board-games night, passing even through learning a new language. In the context of this thesis, the importance of Markov chains lies in the large number of natural physical, biological, and economic phenomena that can be described by them and is enhanced by the amenability of Markov chains to quantitative manipulation. While the Poisson processes discussed before have their utility in model especially rare and abrupt occurrences, the fundamental property of Markov processes, the so-called "memorylessness", and their probability transition matrix structure, are very handful in trying to interpret and forecast gradual regime shifts over-time: Markov regime-switches models carry the uncertainty of the macroeconomic indicators.

To analyze the focus of this work, we define a Markov regime-switching model as a continuous-time process with discrete components: while the continuous-time process evolves according to a stochastic differential equation, the discrete component belongs to a finite or a countable set.

There are several models which describe the hybrid nature of regimes mathematically, studied first by Quandt, and later by Goldfeld, Hamilton and many others, who founded lots of ways to use them in financial analysis. Generally, regime-switching models are seen as proxies of the different states of the economy: a discrete shift among different regimes can be observed in a multitude of economical situations, from a day-by-day change in a financial market to some majore events (as we mentioned in previous chapters). Among these models, recent analysis of regime shifts move from the semimartingale[‡] representation of Markov chains established by

[†]A hidden Markov process (HMP) is a discrete-time finite-state homogeneous Markov chain observed through a discrete-time memoryless invariant channel, characterized by a finite set of transition densities indexed by the states of the Markov chain [9]

[‡]In probability theory, a martingale is a stochastic process for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values. The

Elliot et al..

Let $\{X(t) : t \in [0,T]\}$ be a continuous-time, finite-state, and observable Markov chain, with $T > 0$ time-horizon and $t \in [0,T]$. The finite-state space of the homogeneous and irreducible[§] Markov chain $X(t)$, $S = \{e_1, e_2, \ldots, e_D\}$, is called a canonical state space, where $D \in \mathbb{N}, e_i \in \mathbb{R}^D$ and the j-th component of e_i is the Kronecker[¶] delta $\delta_{i,j}$ for each pair of $i, j = 1, 2, ..., D$. The generator of the chain under $\mathbb{P}$ is defined by:

$$\Lambda(t) := [\lambda_{ij}(t)]_{i,j=1,2,\ldots,D} \tag{3.13}$$

For each $i, j = 1, 2, ..., D$, $\lambda_{i,j}(t)$ is the transition intensity of the chain from each state e_i to state e_j at time t.

$$i \neq j \quad \lambda_{ij}(t) = 0 \Rightarrow \lambda_{ii}(t) \leq 0 \tag{3.14}$$

We suppose that for each $i, j = 1, 2, ..., D$:

$$i \neq j \Rightarrow \lambda_{ij}(t) > 0 \quad \lambda_{ii}(t) < 0 \quad t \in [0,T] \tag{3.15}$$

Elliot proved that a Markov chain α can be represented by:

$$X(t) = X(0) + \int_0^t \Lambda^T(u) X(u) du \tag{3.16}$$

where $M(t) : t \in [0,T]$ is an $\mathbb{R}^D$ -valued $(\mathbb{F}, \mathbb{P})$[‖] martingale and Λ^T represents the transpose of the matrix [25].

process is called semimartingale if it is real valued and can be decomposed as the sum of a local martingale and a càdlàg adapted finite-variation process [5].

[§]A Markov chain in which every state can be reached from every other state is known as *irreducible*

[¶]In mathematics, the Kronecker delta, named after Leopold Kronecker, is a function of two variables, usually just non-negative integers. The function is 1 if the variables are equal, and 0 otherwise [1].

[‖]In probability theory and stochastic processes, $\mathbb{F}$ denotes a *filtration*, which represents the information available up to time t in a stochastic process, capturing how knowledge about the system evolves over time. $\mathbb{P}$ specifies the likelihood of various outcomes within the system and governs the stochastic behavior of the process.

Chapter 4

Climate Tipping Points

In this chapter, we analyze a recent application of the regime switches concept in a fundamental field of study, the climate change economics. The work is a research report, titled *"How to prepare for and adapt to a climate tipping point"*, by M. Muttoni under the supervision of S. Wrzaczek, which was written as the final report of a research project conducted over the three-month Young Scientists Summer Program at IIASA and published in the IIASA repository in 2023 [26].

The concept of tipping point was already defined in our work, but the report we are going to present specifically considers *climate* tipping points, which happen in the Earth's climate system and are now more than ever a strong preoccupation for the scientific community. The possibility that increasingly frequent and severe climate-related occurrences result in drastic negative effects on human life and especially on the global economic situation is more than concrete, and the policymakers have to take that into account and consequently prepare. To try and find an optimal response to that issue, the author moved from a continuous-time version of the renown DICE (Dynamic Integrated Climate-Economy) model by Nordhaus to formulate a finite-horizon, two-stage optimal control problem with a stochastic switching time τ (representing the tipping point). In the context of the previous literature on the matter, this study «aims to bridge the existing gap between these two streams (continuous-time consumption/pollution models and discrete-time integrated-assessment models based on DICE) by adapting a continuous-time formulation of DICE to include a tipping point».

4.1 The Problem

The author assumes the existence of one single policymaker, who has in his hands the whole power over two control variables: world's emission abatement and economic investments. The planner is supposed to be aware of the risk of a tipping point, and he knows its positive corre-

lation with global temperature and its effect (on earth's climate, world's economy, or on both). This theoretical framework has two main limitations, underlined in the premise of the work even by the researcher: first, it admits the existence of only one tipping point, even if feedback mechanisms and domino effects in the system are likely to start a tipping point *cascade*; second, in a fragmented world such as the ours, include a single policymaker with a single set of preferences may be unrealistic, and so the solution has to be considered as a benchmark which indicates the best possible *aggregate* behavior, more than an actual formal solution.

The evolution of the state variables describing the climate and the population's goods and services is set to be governed by a dynamical system, which is a continuous-time version of the DICE model before the occurrence of tipping point, and enters a new regime after.

4.1.1 No Tipping

The dynamical system is characterized by a rich set of state variables and by two control variables, which are presented in the following tabular.

Variable	**Description**
$s \in [0,1]$	savings rate: a control that represents the fraction of the net economic output that is re-invested in capital
$\mu \in [0,1]$	abatement rate: a control that represents the fraction of industrial carbon emissions that is cut
K	capital stock
M^{AT}	carbon (atmosphere)
M^{UP}	carbon (shallow ocean and biosphere)
M^{LO}	carbon (deep ocean)
T^{AT}	temperature change (atmosphere)
T^{OC}	temperature change (ocean)

At every moment, the policymaker sets the controls' values in order to optimize a social welfare function, that non only includes the traditional market goods and services, but other items, like leisure or environmental services, too. The objective function is:

$$\max_{s,\,\mu\in[0,1]} \quad \int_0^T e^{-\rho t}\, L(t)\, u(c(t))\, dt \tag{4.1}$$

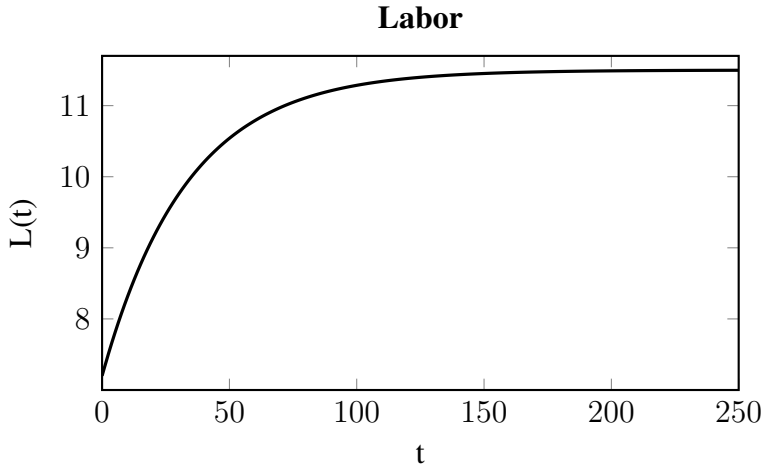
where:

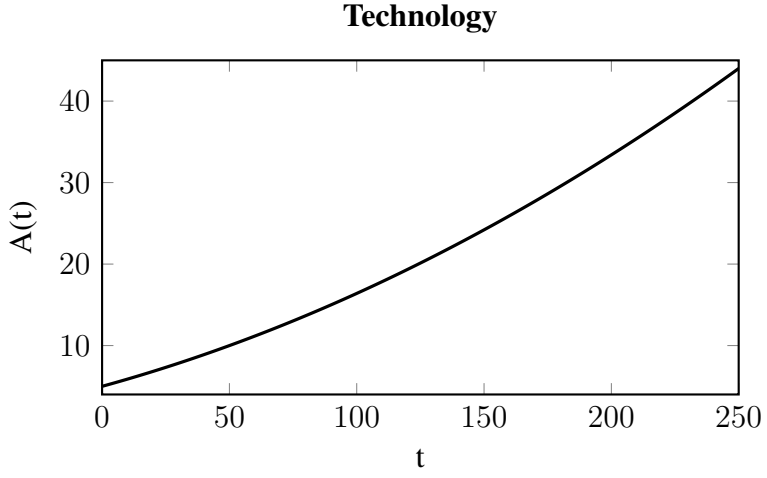
- $\rho > 0$ is the rate of time preference;

- $L(t)$ is the exogenous population, expressed in billions, with a growth rate that declines over time and an asymptotic level of 11.5 billions;
- $u(c)$ is the isoelastic consumption's utility;
- $c(t) = C(t)/L(t)$ is the consumption per-capita and the total consumption $C(t) = (1 - s(t))Q(t)$ is the fraction of the net production function $Q(t)$ which is not re-invested in capital.

The author assumes the global production function to be a Cobb-Douglas with constant returns to scale in capital K, exogenous labor $L(t)$, Hicks-neutral exogenous technological change $A(t)$ and capital elasticity $\gamma \in (0, 1)$:

$$\Upsilon(t, K) = A(t)\, K^{\gamma}\, L(t)^{1-\gamma} \tag{4.2}$$





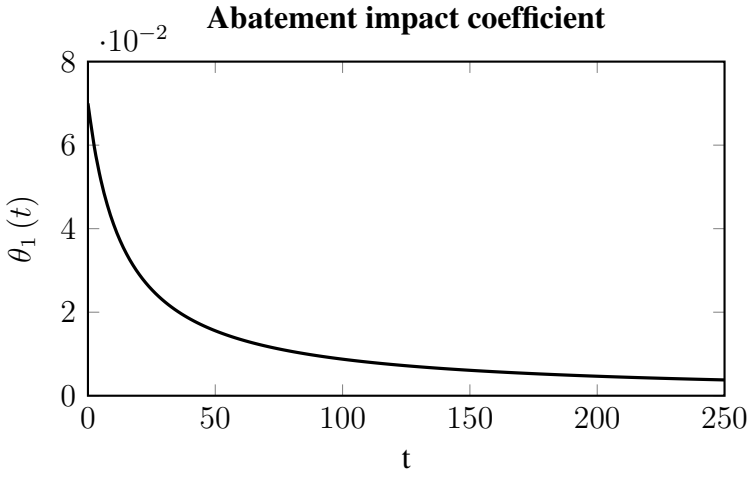
Production is negatively impacted by the rising atmospheric temperature, and, even if the abatement of carbon emissions is the main way to contrast this phenomenon, itself represents a cost in the production function. The policymaker has then to resize the production by two factors, which are both assumed to be proportional to global output. The temperature damages are polynomial functions of global mean temperature change:

$$\Omega(T^{AT}) = \psi_1 \, (T^{AT})^{\psi_2} \tag{4.3}$$

with ψ_1, ψ_2 parameters, while the abatement costs, proportional to a strictly convex polynomial function of μ, are represented by:

$$\Lambda(t, \mu) = \theta_1(t) \, \mu^{\theta_2} \tag{4.4}$$

where $\theta_2 > 1$ is a parameter and the coefficient $\theta_1 \, (t)$ is the price, assumed to be initially high and declining over-time, of a completely carbon-free technology named backstop technology.



Including the just seen functions of costs in the total production function, the net economic output amounts to:

$$Q(t, \mu, K, T^{AT}) = \frac{1 - \Lambda(t, \mu)}{1 + \Omega(T^{AT})} \, \Upsilon(t, K) \tag{4.5}$$

Indicating with δ_K the capital depreciation and assuming that a fraction s of the net output is invested, the capital evolution over time is:

$$\dot{K} = s \, Q(t, \mu, K, T^{AT}) - \delta_K \, K \tag{4.6}$$

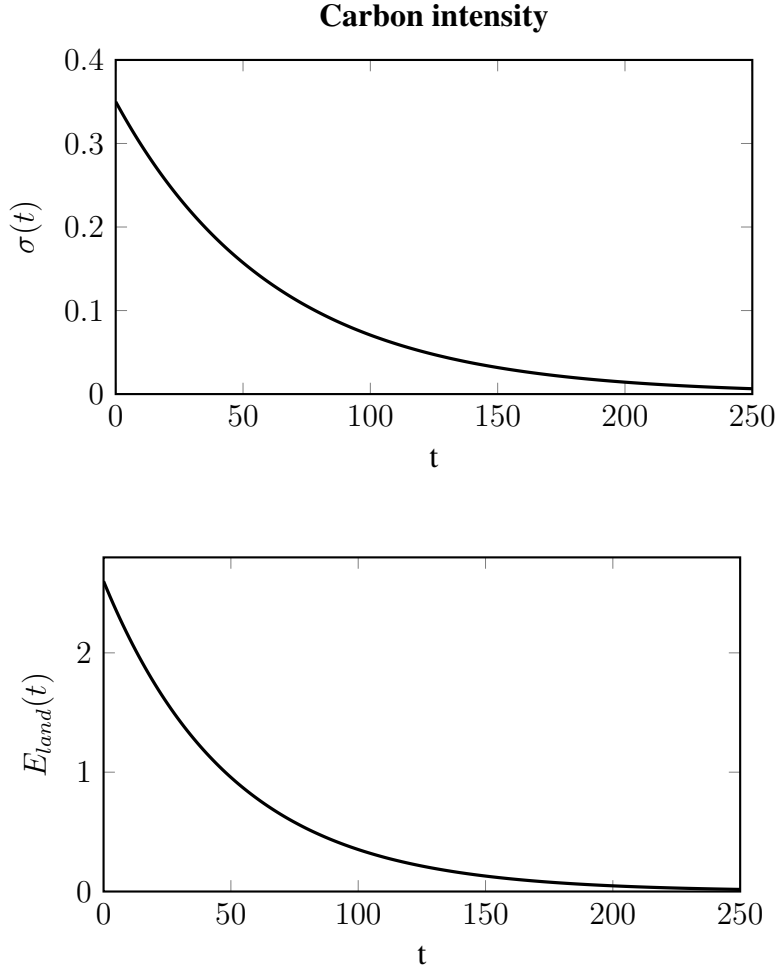
while the consumption function:

$$C(t) = (1 - s(t))\, Q(t, \mu(t), K(t), T^{AT}(t)) \tag{4.7}$$

The last strictly economic dynamics that the researcher includes in this model is the total flow of CO_2 emissions, which comprehends actual industrial emissions and the CO_2 arising from land-use changes. While the second is supposed to be exogenous and it is represented by $E_{land}\,(t)$, the first one is endogenous and calculated as:

$$E_{ind}(t, \mu, K) = (1 - \mu)\, \sigma(t)\, \Upsilon(t, K) \tag{4.8}$$

where $\sigma(t)$ is an exogenous function of time representing the carbon intensity, which depends on the technological advancement.



Finally, to reach the complete formulation of the DICE model in continuous time, the researcher presents some *geophysical equations*, that explain the dynamics of the carbon cycle and of the earth's temperature.

In this framework, $\alpha = 3/11$ of the initial mass of carbon is released in the atmosphere, which at the same time "gives" its CO_2 to the biosphere at a rate δ_1, only to take it back at a rate δ_2. In the meantime, the shallow ocean and the deep ocean make a similar exchange, with respective rates δ_3 and δ_4. The carbon cycle is illustrated by the following system of differential equation:

$$\begin{cases} \dot{M}^{AT} = -\delta_1 M^{AT} + \delta_2 M^{UP} + \alpha\, E(t, \mu, K) \\ \dot{M}^{UP} = \delta_1 M^{AT} - (\delta_2 + \delta_3)\, M^{UP} + \delta_4 M^{LO} \\ \dot{M}^{LO} = \delta_3 M^{UP} - \delta_4 M^{LO} \end{cases} \tag{4.9}$$

that, once introduced vector notation, it is possible to reformulate as:

$$\dot{M} = \Phi\, M + \alpha\, E(t, \mu, K)\, e_1 \tag{4.10}$$

where:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The atmospheric CO_2 has also a clear impact on the increase in atmospheric temperature, through a phenomenon known as radiative forcing. But even if the raising of the CO_2 released is its main driver, this effect depends also on exogenous sources. The total effect of the greenhouse effect (i.e., the radiative forcing) is given by:

$$F(t, M^{AT}) = \eta\, log_2 \left(\frac{M^{AT}}{\tilde{M}} \right) + F^{EX}(t) \tag{4.11}$$

where $\tilde{M}$ is the pre-industrial level of atmospheric carbon and η a parameter. To formulate the complete temperature dynamics, the author also included other important features of the atmosphere: the self-cooling ability and the heat release in the ocean.

$$\begin{cases} \dot{T}^{AT} = \xi_1 \Big[-\xi_2\, T^{AT} + \xi_3\, (T^{OC} - T^{AT}) + F(t, M^{AT}) \Big] \\ \dot{T}^{OC} = \xi_4\, (T^{AT} - T^{OC}) \end{cases} \tag{4.12}$$

Analogously as before, through a vector notation it is possible to reformulate the temperature dynamics as:

$$\dot{T} = \Xi\, T + \xi_1\, F(t, M^{AT})\, e_1 \tag{4.13}$$

Finally, we can collect all the equations presented up to now and formulate the DICE model

in continuous time. The following optimal problem is the problem that the policymaker has to solve in a no tipping point scenario, or that he solves if he does not expect that a regime switch would actually take place (i.e., the *myopic* problem).

$$\max_{s,\mu\in[0,1]} \int_0^T e^{-\rho t}\, L(t)\, u(c(t))\, dt \tag{4.14}$$

$$\text{subject to} \begin{cases} \dot{K} = s\, Q(t,\mu,K,T^{AT}) - \delta_K\, K & K(0) = K_0 \\ \dot{M} = \Phi\, M + \alpha\, E(t,\mu,K)\, e_1 & M(0) = M_0 \\ \dot{T} = \Xi\, T + \xi_1\, F(t,M^{AT})\, e_1 & T(0) = T_0 \end{cases}$$

$$\text{where} \qquad c(t) = \frac{(1-s(t))}{L(t)}\, Q(t,\mu(t),K(t),T^{AT}(t))$$

4.1.2 The Switch

The possible occurrence of a stochastic climate tipping point is integrated in the previous system as follows. Modeled as a positive random variable and denoted by τ the so-called *switching time*, the exact moment at which the tipping point eventually happens, the time horizon is split into two intervals: stage 1 $[0, \tau^T)$ and stage 2 $[\tau, T]$ if $\tau < T$. The switching time's hazard rate is positive correlated with the global mean temperature:

$$\lim_{dt\to 0^+} \frac{\mathbb{P}\left(\tau \le t + dt \,|\, \tau > t\right)}{dt} = h(T(t)) \qquad h_{T^{AT}}, h_{T^{OC}} > 0 \tag{4.15}$$

The planner is assumed to know that a tipping point could occur and which effects it would have on the system, but the exact switching time is unsure. Considered that, in stage 1, he needs to set the values $s_1(t)$ and $\mu_1(t)$ for all the time horizon $[0, T]$. Once the regime shift has happened at the time τ, the policymaker can plan the stage 2 strategy adjusted by this value: for all $\tau \in [0, T]$ and for all $t \in [\tau, T]$, he sets $s_2(\tau, t)$ and $\mu_2(\tau, t)$.

Everything considered, the optimization problem becomes:

$$\max_{s_1,\mu_1,s_2,\mu_2\in[0,1]} \mathbb{E}_\tau \left[\int_0^{\tau^T} e^{-\rho t}\, L(t)\, u(c_1(t))\, dt + \int_{\tau^T}^T e^{-\rho t}\, L(t)\, u(c_2(\tau,t))\, dt \right] \tag{4.16}$$

$$
\text{subject to } \begin{cases}
\dot{K}_1(t) = s_1\, Q_1(t, \mu_1, K_1, T_1^{AT}) - \delta_K\, K_1 & K_1(0) = K_0 \\
\dot{M}_1(t) = \Phi_1\, M_1 + \alpha\, E(t, \mu_1, K_1)\, e_1 & M_1(0) = M_0 \\
\dot{T}_1(t) = \Xi_1\, T_1 + \xi_1\, F(t, M_1^{AT})\, e_1 & T_1(0) = T_0 \\
\frac{d}{dt}\mathbb{P}(\tau > t) = -h(T_1(t))\mathbb{P}(\tau > t) & \mathbb{P}(\tau > 0) = 1 \\
\dot{K}_2(\tau, t) = s_2\, Q_2(t, \mu_2, K_2, T_2^{AT}) - \delta_K\, K_2 & K_2(\tau, \tau) = \varepsilon(T_1^{AT}(\tau))\, K_1(\tau) \\
\dot{M}_2(\tau, t) = \Phi_2\, M_2 + \alpha\, E(t, \mu_2, K_2)\, e_1 & M_2(\tau, \tau) = M_1(\tau) \\
\dot{T}_2(\tau, t) = \Xi_2\, T_2 + \xi_1\, F(t, M_2^{AT})\, e_1 & T_2(\tau, \tau) = T_1(\tau)
\end{cases}
$$

$$
\text{where} \qquad c_1(t) = \frac{(1 - s_1(t))}{L(t)}\, Q_1(t, \mu_1(t), K_1(t), T_1^{AT}(t))
$$

$$
\text{and} \qquad c_2(\tau, t) = \frac{(1 - s_2(t))}{L(t)}\, Q_2(t, \mu_2(\tau, t), K_2(\tau, t), T_2^{AT}(t))
$$

4.2 The Solution

For the first time in the context of a climate/economy model with random switching time, this work employs a new methodology to solve the maximization problem. This approach, developed by Wrzaczek et al., requires the introduction of two auxiliary state variables, Z_1 in stage 1 and Z_2 in stage 2, which are tied to the switching time τ. They are represented by the following equations:

$$
Z_1(t) = \mathbb{P}(\tau > t) \qquad Z_2(\tau, t) = h(T_1(\tau)) Z_1(\tau) \tag{4.17}
$$

and their dynamics are explained by the systems:

$$
\begin{cases}
\dot{Z}_1(t) = -h(T_1(t))\, Z_1(t) \\
Z_1(0) = 1
\end{cases}
\qquad
\begin{cases}
\dot{Z}_2(\tau, t) = 0 \\
Z_2(\tau, \tau) = h(T_1(\tau))\, Z_1(\tau)
\end{cases}
\tag{4.18}
$$

After some basic computations and integral manipulations, the researcher writes the complete vintage structure formulation of the problem:

$$
\max_{s_1, \mu_1, s_2, \mu_2 \in [0,1]} \int_0^T e^{-\rho t} \left[Z_1(t)\, L(t)\, u(c_1(t)) \; + \int_0^t Z_2(\tau, t)\, L(t)\, u(c_2(\tau, t))\, d\tau \right] dt \tag{4.19}
$$

$$\text{subject to} \begin{cases} \dot{K}_1(t) = s_1\, Q_1(t, \mu_1, K_1, T_1^{AT}) - \delta_K\, K_1 & K_1(0) = K_0 \\ \dot{M}_1(t) = \Phi_1\, M_1 + \alpha\, E(t, \mu_1, K_1)\, e_1 & M_1(0) = M_0 \\ \dot{T}_1(t) = \Xi_1\, T_1 + \xi_1\, F(t, M_1^{AT})\, e_1 & T_1(0) = T_0 \\ \dot{Z}_1(t) = -h(T_1(t))\, Z_1(t) & Z_1(0) = 1 \\ \dot{K}_2(\tau, t) = s_2\, Q_2(t, \mu_2, K_2, T_2^{AT}) - \delta_K\, K_2 & K_2(\tau, \tau) = \varepsilon(T_1^{AT}(\tau))\, K_1(\tau) \\ \dot{M}_2(\tau, t) = \Phi_2\, M_2 + \alpha\, E(t, \mu_2, K_2)\, e_1 & M_2(\tau, \tau) = M_1(\tau) \\ \dot{T}_2(\tau, t) = \Xi_2\, T_2 + \xi_1\, F(t, M_2^{AT})\, e_1 & T_2(\tau, \tau) = T_1(\tau) \\ \dot{Z}_2(\tau, t) = 0 & Z_2(\tau, \tau) = h(T_1(\tau))\, Z_1(\tau) \end{cases}$$

Finally, to solve the problem, the author applies the Maxi Principle for heterogeneous systems presented by Veliov. We will not fully analyze this last part of the report, because we were more interested in the practical formulation of a model with a regime switch, more than the climate issue itself. But, in order to fully appreciate the innovative method proposed by Muttoni in the monograph, we will briefly summarize the results and how they managed to reach them.

4.2.1 Results

First it has to be defined an Hamiltonian function for each stage. Then, denoted the co-state trajectories of the two stages and the optimal process $((s_i^*, \mu_i^*, K_i^*, M_i^*, T_i^*), i = 1, 2)$, it is possible to apply the maximality condition from the Maximum Principle, which yields that the stage 2 co-states enter the stage 1 adjoint equation, implying an anticipating effect of the tipping point on the stage 1 behavior; and $s_1^*,\ s_2^*$ are stationary points for the respective Hamiltonians, independently whether they belong or not to the inner part of the control set $[0, 1]$. Finally, obtained the necessary conditions, the author gains insight about the analytical composition of the Social Cost of Carbon, which is fully analyzed even through background numerical integration.

The relevant changes the policymaker has to make in order to adapt the optimal policies in anticipation of the tipping point (compared to the myopic scenario) and in response to the tipping point (in the new regime), are investigated using numerical simulations in four different tipping point scenarios: melting of sea ice or ice sheets, destruction of capital due to an environmental catastrophe, transformation of a carbon sink into a carbon source, increase in the impact of temperature on production. To explain the approach followed by the researcher, we consider only the first of these scenarios.

Numerical Simulation

Once set a time horizon of 250 years, she used the continuous-time version of the DICE model earlier presented to parametrize the myopic scenario and the stage 1 of a problem with regime

switch, while for the stage 2, in order to better underline the *qualitative* adjustments to the optimal behavior more than the *quantitative* values of the optimal policies, she chose to overstate the tipping point's effects. The hazard rate is assumed to be linear with respect to the temperature:

$$h(t) = \frac{T^{AT} + T^{OC}}{60}$$

Numerical results were generated using a gradient descent algorithm informed by the Maximum Principle for heterogeneous systems. The iterative process involved forward and backward sweeps to calculate state and co-state trajectories, followed by a line search to determine a descent direction for the Hamiltonians. The controls were then updated to maximize the objective function.

We have already touched upon the "revolution" introduced by the author of this monograph in terms of the strictly mathematical analysis and stochastic modeling of the climate tipping point phenomenon. However, it is also worth emphasizing the highly interesting graphical representation of the numerical simulations' results for the four different proposed scenarios. Essentially, two types of figures are presented: the first highlights the difference in the behavior of a single variable in the *myopic* case (in gray) and in the alternative case where the policymaker is aware of the possibility of a tipping point (in black); the second illustrates the behavior of the same variable in stage 1 (in black) and in stage 2 after the switch (in color). The curves for stage 2 start at the switching time τ and end at $T = 250$, and although the tipping point could occur at any moment, of course only a finite number of curves (precisely ten) have been represented. So, if somebody is interested in investigating the effect that the anticipation of the tipping point has on a policy level, to look to the first kind of the graphs gives an immediate response.

Consider the first scenario: the melting of sea ice or ice sheets. In this research is supposed that a tipping point may consist in the full destruction of an important ice reservoir, given that such occurrence has drastic effects on the atmosphere's cooling rate, making the system to cross the threshold. Analyzing the behavior of the atmosphere temperature variable T^{AT} (figure 5.1 [26]), the effect turns out to be catastrophic, even considering that the outcome are voluntarily exaggerated. To follow the behavior of this state variable, one has to consider the black curve until the switching time $t = \tau$ is reached, in correspondence of which starts the stage 2 curve that represents the atmosphere temperature dynamics up to $T = 250$. The same methodology can be applied for studying all the variables, even a control such as the abatement rate μ (figure 5.2 [26]). Compared to the myopic scenario, in anticipation of the tipping point we observe a precautionary increase of the variable, in order to postpone the tipping point by keeping the atmospheric temperature as low as possible.

This paper highlight a fundamental lesson for the policymaker: the importance of prevention, which can postpone the regime shift as far as possible, ideally at the end or even after the

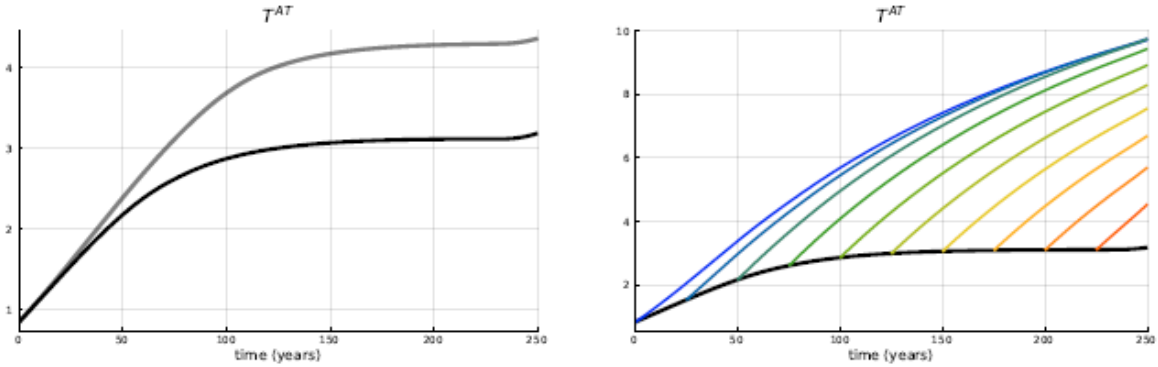


Figure 4.1: Ice melting effects on atmospheric temperature

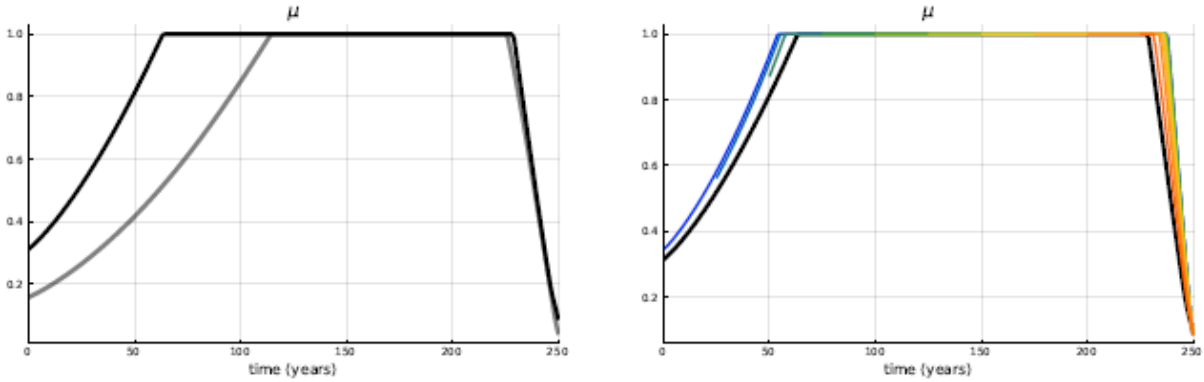


Figure 4.2: The abatement rate to prevent or adapt to the ice melting

end of the time horizon.

Conclusions

In this paper we tried to present and analyze the phenomenon of regime shifts within economic dynamic systems, with the aim of providing a comprehensive understanding of their occurrence, implications, and manageability. By analyzing the transitions between distinct economic states, the work has shed light on the mechanisms that underpin these shifts, offering both theoretical insights and practical applications.

From the conceptual foundation laid in the introduction to the in-depth analysis of examples in monetary policy, fiscal policy, market volatility, and technological innovations, the thesis has illustrated the multifaceted nature of regime shifts. The discussion on thresholds and tipping points further underscored their critical role in anticipating systemic changes, while the mathematical framework provided a robust toolkit for examining these phenomena. The application of stochastic models and the incorporation of concepts such as Skiba points had the scope to more deeply analyze the matter of regime switches.

The exploration of climate tipping points in the final chapter demonstrated the practical significance of the theoretical and mathematical constructs discussed earlier. By adapting the DICE model to environmental regime shifts, this study not only advanced academic understanding, but also contributed to the pressing discourse on adaptation and resilience to climate change.

Despite these contributions, certain limitations must be acknowledged. The inherent complexity and uncertainty of regime shifts mean that no single model or approach can fully capture their dynamics. In addition, the interdisciplinary nature of the topic calls for the further integration of insights from fields such as behavioral economics, political science, and systems ecology. We chose to focus on only some points, the ones that, in our opinion, were the fundamental to appreciate the matter itself.

Looking ahead, future research could expand on this work by applying the presented methodologies to other domains, such as technological disruptions, global trade dynamics, or socio-political transitions. Investigating the role of non-linear feedback loops and exploring new econometric tools for real-time detection of tipping points could further enhance our ability to foresee and manage regime shifts.

In conclusion, the study of regime shifts represents a vital intersection of theory, practice, and policy. By deepening our understanding of these critical transitions, we equip ourselves to navigate the uncertainties of complex systems more effectively. This thesis serves as a step in that direction, fostering a more nuanced approach to economic analysis and decision making in an era of rapid change.

Bibliography

[1] Wilfred Kaplan. *Advanced calculus*. Pearson Education India, 1952.

[2] Rudiger Dornbusch. "Expectations and exchange rate dynamics". In: *Journal of political Economy* 84.6 (1976), pp. 1161–1176.

[3] Murray C. Kemp and Ngo Van Long. "Optimal Control Problems with Integrands Discontinuous with respect to Time". In: *Economic Record* (1977).

[4] Aleksandr K. Skiba. "Optimal growth with a convex-concave production function". In: *Econometrica: Journal of the Econometric Society* (1978), pp. 527–539.

[5] D. Williams and L. Chris G. Rogers. *Diffusions, Markov processes, and martingales*. Vol. 2. John Wiley & Sons, 1979.

[6] Carl P. Simon and Lawrence E. Blume. *Mathematics for economists*. New York London: Norton, 1994.

[7] Yuri A. Kuznetsov. *Elements of applied bifurcation theory*. Vol. 112. Springer, 1998.

[8] Wolf-Jürgen Beyn, Thorsten Pampel, and Willi Semmler. "Dynamic optimization and Skiba sets in economic examples". In: *Optimal Control Applications and Methods* 22.5-6 (2001), pp. 251–280.

[9] Y. Ephraim and N. Merhav. "Hidden Markov processes". In: *IEEE Transactions on Information Theory* 48.6 (2002), pp. 1518–1569.

[10] Ronald Shone. *Economic dynamics: phase diagrams and their economic application*. eng. 2nd ed. Cambridge: Cambridge University Press, 2002.

[11] Arthur M. Schlesinger. *The Coming of the New Deal: The Age of Roosevelt, 1933-1935*. Vol. 2. Houghton Mifflin Harcourt, 2003.

[12] Richard F. Hartl et al. "Multiple equilibria and thresholds due to relative investment costs". In: *Journal of Optimization Theory and Applications* 123 (2004), pp. 49–82.

[13] Alina Carare and Mark R. Stone. “Inflation targeting regimes”. In: *European Economic Review* 50.5 (2006), pp. 1297–1315. ISSN: 0014-2921. DOI: https://doi.org/10.1016/j.euroecorev.2005.02.003. URL: https://www.sciencedirect.com/science/article/pii/S0014292105000541.

[14] Timothy M. Lenton et al. “Tipping elements in the Earth’s climate system”. In: *Proceedings of the national Academy of Sciences* 105.6 (2008), pp. 1786–1793.

[15] Mark Pinsky and Samuel Karlin. *An introduction to stochastic modeling*. Academic press, 2010.

[16] Hans-Werner Sinn. *The Green Paradox: A Supply-Side Approach to Global Warming*. The MIT Press, 2012.

[17] Carlos Fiorentino and Carlos Montana Hoyos. “The emerging discipline of biomimicry as a paradigm shift towards design for resilience”. In: *The International Journal of Designed Objects* 8.1 (2014), p. 10.

[18] Shair Ahmad and Antonio Ambrosetti. *A textbook on ordinary differential equations*. Vol. 88. Springer, 2015.

[19] L. V. Ballestra. *Matematica per l’economia. Elementi di teoria ed esercizi.* Apogeo Education, 2015.

[20] Cees Diks, Cars Hommes, and Juanxi Wang. “Critical slowing down as an early warning signal for financial crises?” In: *Empirical Economics* 57 (2019), pp. 1201–1228.

[21] Rashmi Chaudhary, Priti Bakhshi, and Hemendra Gupta. “Volatility in international stock markets: An empirical study during COVID-19”. In: *Journal of Risk and Financial Management* 13.9 (2020), p. 208.

[22] Tomas Cipra et al. *Time series in economics and finance*. Springer, 2020.

[23] William Quinn and John D Turner. *Boom and bust: A global history of financial bubbles*. Cambridge University Press, 2020, p. 161.

[24] James H Stock and Mark W Watson. *Introduction to econometrics*. Pearson, 2020.

[25] Josef Haunschmied et al. *Dynamic economic problems with regime switches*. Springer, 2021.

[26] Maddalena Muttoni. *How to prepare for and adapt to a climate tipping point*. Tech. rep. IIASA, 2022.

[27] Alex Svirin. URL: https://math24.net/linear-autonomous-systems-equilibrium-points.html.

[28] *International Monetary Fund*. URL: https://www.imf.org/en/Publications/fandd/issues/Series/Back-to-Basics/Fiscal-Policy.

[29] *EUR-Lex, Access to European Union Law*. URL: https://eur-lex.europa.eu/EN/legal-content/glossary/monetary-policy.html.

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I am no good at big speeches. Not that kind of movies no-good, the one that are, in fact, incredible at them. "I don't know how can i say thanks to all of you" only to proceed to give an articulate, fantastic moment to the spectators. I really am no good.

But, some things I have to write. It's only right.

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Lento ma sicuro