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Fourier Analysis on some Compact Lie Groups

Relatore:
Prof. Paolo Ciatti

Laureando: Matteo Trevisan
Matricola: 2052825

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Chapter 1

Introduction

Thanks to Fourier analysis, we know that every real-valued periodic function f can be decomposed into a sum of sines and cosines with integer period. Furthermore, if we take a complex-valued function f of period 2π on $\mathbb{R}$, then

$$f(x) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{imx}$$

with

$$\hat{f}(m) = \frac{1}{2\pi} \int_0^{2\pi} e^{-imy} f(y) dy.$$

Under certain regularity properties, this series converges absolutely and this leads to some useful applications: precisely, every function in the series is orthogonal to the other ones, and they can be studied separately to solve partial differential equations.
The most famous application is the *heat equation*, solution of the following Cauchy problem: for a given (periodic) function f on $\mathbb{R}$,

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) \\ u(x,0) = f(x) \end{cases}$$

with the *Laplace operator* Δ defined as $\Delta f(x) = -\frac{\partial^2}{\partial x^2} f(x)$.
The solution, which is unique, is

$$u(x,t) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{imx} e^{-m^2 t}.$$

This results follows from the observation that the terms of the series are eigenfunctions of the Laplace operator with eigenvalue $-m^2$ (for more details see, for example, Elias M. Stein and Rami Shakarchi, [4]).

The purpose of this thesis is to generalize the problem and, using Fourier analysis, find a solutions and prove the uniqueness.
Furthermore, in the "classical" problem we have a periodic function f, which can be seen as a function over $SO(2)$ (the circumference), a commutative compact Lie Group. We want to study the equation on $SU(2)$, the group of complex 2 x 2 matrices with determinant 1.
This is the simplest non-commutative compact Lie group and, in order to find the eigenspaces of the Laplace operator in $L^2(SU(2))$, we need to study some results on representation theory and on operators.

Then, there will be a generalization of the Laplace operator: we will define formally the operator $(1-\Delta)^{-\alpha}$ for certain $\alpha \in \mathbb{C}$ (and then extend the definition), in order to find the *generalized heat equation*, solution of

$$\begin{cases} \frac{\partial}{\partial_t} u(x,t) = (1-\Delta)^{-\alpha} u(x,t) \\ u(x,0) = f(x) \end{cases}$$

for a given $f \in SU(2)$ that should have certain regularity properties.

There are three main steps in this thesis.
The first one is the proof of the $Peter-Weyl\ theorem$, holding on for every compact Lie groups G, which basically states that $L^2(G)$ is the closure of the sum of some orthogonal functional spaces that are representation spaces for G.

This lead us to prove a generalized form of the $Plancherel's\ theorem$, hence to find a formula for the decomposition of every $f \in L^2(G)$ in orthogonal functions. We will find some hypothesis under whom the convergence is absolute and prove that these functional spaces are eigenspaces of the Laplace (and find the eigenvalues).

We apply these results on $G = SU(2)$, computing the orthogonal decomposition of $L^2(SU(2))$ and the eigenvalues of the Laplace operator. Then we can state and solve the problem, introducing via spectral calculus the complex powers of the operator $(1-\Delta)$, starting from the function $(1+s)^{-\alpha} = \frac{1}{\Delta(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-(1+s)\tau} d\tau$.

This is analogous to what is done in the Fourier analysis on the circle, but more difficoult due to the non-commutativity of the group.

1.1 Prerequisites

A *linear Lie Group* G is a closed subgoup of $GL(n, \mathbb{R})$.
A *Lie Algebra* $\mathfrak{g}$ is a a vector space over $\mathbb{R}$ or $\mathbb{C}$ on which is defined the *commutator*, a linear map

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\mapsto [X, Y] \end{aligned}$$

such that

$$[X, Y] = -[Y, X] \tag{1.1}$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{1.2}$$

This second property is called the *Jacobi identity*.
Every linear Lie Group G has an associated Lie algebra $\mathfrak{g}$: so many problems on the group can be studied in the linear settings of the Lie algebra. In particular,

$$\mathfrak{g} = Lie(G) := \{X \in M(n, \mathbb{R}) | \forall t \in \mathbb{R} \exp(tX) \in G\}.$$

A compact Lie group is a manifold and the associated Lie algebra coincides, as a vector space, with the tangent space at the identity of G.

A derivation of $\mathfrak{g}$ is a linear endomorphism $D \in End(\mathfrak{g})$ such that the Leibniz rule holds:

$$D([X, Y]) = [DX, Y] + [X, DY].$$

The space $Der(\mathfrak{g})$ equipped with the commutator is a Lie algebra itself, and it is equal to $Aut(\mathfrak{g})$.
On $\mathfrak{g}$ we can define, for every $g \in G$, an automorphism
$\mathrm{Ad}(g) : X \mapsto gXg^{-1}$. furthermore, $Ad : G \to Aut(\mathfrak{g})$ is a group morphism.
For a matrix A, we define the map $\mathrm{ad}(A) : X \mapsto AX - XA$; the map

associating to $A \in \mathfrak{g}$ the operator $\mathrm{ad}A$ yields a representation of the Lie algebra. This map is called the *adjoint representation* and it is a derivation of $\mathfrak{g}$.
These two maps are strictly correlated:

$$\frac{d}{dt}\mathrm{Ad}(\exp tX)\bigg|_{t=0} = \mathrm{ad}X.$$

We first describe the theory of representations of compact Lie groups, after that we will point the attention on the special case of $SU(2)$, that is the group

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \alpha, \beta \in \mathbb{C} \middle| |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

This is a simply connected Lie group with Lie algebra $\mathfrak{su}(2)$. A basis for $\mathfrak{su}(2)$ is given by

$$\left\{ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}.$$

While studing its representation, we will need to pass to the complexified Lie algebra: $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) + i\mathfrak{su}(2)$. This passage is harmless, since the representations will be over complex vector spaces. A basis of $\mathfrak{sl}(2, \mathbb{C})$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Chapter 2

Haar measure and Peter-Weyl theorem

2.1 Haar Measure

A measure μ on a locally compact group G is said to be *left invariant* (*right invariant*) if

$$\int_G f(gx)\mu(dx) = \int_G f(x)\mu(dx)$$

$$\left(\int_G f(xg)\mu(dx) = \int_G f(x)\mu(dx)\right)$$

for every g $\in$ G and for every f $\in$ $C_c(G)$ (continuous functions on G with compact support). In particular, for every Borel set E and for every g $\in$ G we have $\mu(gE) = \mu(E)$ (or $\mu(Eg) = \mu(E)$).
It is known that such a (non-zero) measure on a compact group exists and it is unique up to a positive multiplicative constant (see Edwin Hewitt and Kenneth A. Ross, [3] for a proof).
Furthermore, from now on we will use only normalised measures:

$$\int_G \mu(dx) = 1.$$

2.2 Some results on operators

We recall some definitions and simplie results from functional analysis. Our aim is to give a structure to the space of the irreducible representations.

A bounded operator A on a Hilbert spaces H has the norm

$$\|A\| = \sup_{\|u\| \leq 1} \|Au\|.$$

The *adjoint* operator A^* is a linear continuous map such that $(Au|v) = (u|A^*v)$ (it exist because of the Riesz representation theorem, which states that, for v fixed, there exist a unique w such that $(Au|v) = (u|w)$). Furthermore, we know $\|A\| = \|A^*\|$ and $(A^*)^* = A$ (see Brezis, [2]).

A is a *self-adjoint* operator if $A^* = A$. The following statements are valid if A is an operator of this type.
An eigenvalue λ of A is real: $(Au|u) = (u|Au) \Longrightarrow \lambda\|u\|^2 = \bar{\lambda}\|u\|^2$.
The eigenspaces of two different eigenvalues of A are orthogonal ($(Au|v) = (u|Av) \Longrightarrow (\lambda - \mu)(u|v) = 0$).

$\|A\| = \sup_{\|u\| \leq 1} |(Au|u)|$ (Let us call M this number). In fact, while $\leq$ is obtained only using the Cauchy-Schwartz inequality, on the other hand we have that

$$\|A\| = \sup_{\|u\| \leq 1} \|Au\|, \|w\| = \sup_{\|v\| \leq 1} |Re(w|v)|$$

for every $w \in H$, and so

$$\|A\| = \sup_{\|u\|,\|v\| \leq 1} \|Re(Au|v)\|.$$

We conclude using the following identity (the so-called *polarization identity*):

$$\begin{aligned}
&4Re(Au|v) = (A(u+v)|u+v) - (A(u-v)|u-v) \\
&\Longrightarrow |Re(Au|v)| \leq \frac{M}{4}(\|u+v\|^2 + \|u-v\|^2) = \frac{M}{2}(\|u\|^2 + \|v\|^2) \\
&\Longrightarrow |Re(Au|v)| \leq M.
\end{aligned}$$

An operator is *compact* if the image under it of a bounded set is relatively compact (has a compact closure). Equivalently:
- the image of the unitary ball is relatively compact;
- a bounded sequence (u_n) has a subsequence (u_{n_k}) such that (Au_{n_k}) converges.

It is obvious that a finite rank operator is compact and, if A is compact and B is bounded, then AB and BA are compact.
We want to state that the set of compact operators is a closed two-sided ideal in $L(H)$, the group of linear operators of H.
In order to do that, we miss only to demonstrate that if (A_n) is a sequence of compact operator and $\lim_{n\to\infty}\|A_n - A\| = 0$, than the operator A is compact.
In fact, Let us take a a sequence (u_k) in H in the unit sphere: because of the compactness of A_1, we can take a subsequence $(u_k^{(1)})$ such that $(A_1 u_k^{(1)})$ converges, and from that a subsequence $(u_k^{(2)})$ such that $(A_2 u_k^{(2)})$ converges and so on. From these subsequences we consider (u'_k):=$(u_k^{(k)})$ and we can conclude if $(Au_k^{(k)})$ is a Cauchy sequence since $L(H)$ is a Banach space: this is true since, for k, l and n large enough and a fixed $\varepsilon > 0$ we have

$$\|Au'_k - Au'_l\| \leq \|Au'_k - A_n u'_k\| + \|A_n u'_k - A_n u'_l\| + \|A_n u'_l - Au'_l\| \leq \varepsilon.$$

In this section our aim is to establish a "Spectral Theorem" for compact operators.
We need the existance of a non-zero eigenvalue of A:

Lemma 2.2.1. *If A is a compact self-adjoint operator it has $\|A\|$ or $-\|A\|$ as an eigenvalue.*

Proof. We already know that $\|A\| = \sup_{\|u\|\leq 1}|(Au|u)|$ and that these numbers are real for every u so, replacing A with -A if needed, we have
$\lambda := \|A\| = \sup_{\|u\|\leq 1}(Au|u)$.

There exists a sequence (u_n) such that

$$\|u_n\| = 1, \lim_{n\to\infty}(Au_n|u_n) = \lambda$$

and since A is compact there exists a subsequence such that (Au_{n_k}) converges to v. Then we take

$$\lim_{k\to\infty}\|Au_{n_k} - \lambda u_{n_k}\|^2 = \lim_{k\to\infty}(\|Au_{n_k}\|^2 - 2\lambda(Au_{n_k}|u_{n_k}) + \lambda^2) = \|v\|^2 - \lambda^2$$

and, using that $\|A\| = \lambda, \|u_{n_k}\| = 1$, we obtain

$$\lim_{k\to\infty} \|Au_{n_k} - \lambda u_{n_k}\|^2 = 0$$
$$\implies \lim_{k\to\infty} u_{n_k} = u, Au = \lambda u.$$

□

We can finally state

Theorem 2.2.2 (Spectral theorem)**.** *Let A be a compact self-adjoint operator, (λ_n) the sequence of the non-zero eigenvalues and P_n the orthogonal projections onto the eigenspaces H_n. Then the H_n are finite-dimensional, the sequence (λ_n) is either finite or convergent to zero and*

$$A = \sum_{n=0}^{\infty} \lambda_n P_n$$

where the right hand side converges in the norm topology (the sum is finite if (λ_n) is a finite sequence).

Proof. The idea is very simple and, with the properties we have already shown, similar to the finite-dimensional case.
Let us start stating that every eigenspace of an eigenvalue $\lambda \neq 0$ is finite dimensional, because A is a compact operator (also if restricted to this eigenspace) and the sphere of a space is compact if and only if it is finite dimensional (see Brezis, [2] for the details).
Then we define $\lambda_1 = \|A\|$, then $A_1 := A - \lambda_1 P_1$ is again a compact self-adjoint operator, hence we can continue this process until $A_N = 0$ or we obtain an infinite sequence, decreasing in modulo by construction (there cannot be an eigenvalue of A greater in modulo than $\|A\|$ by definition).
Let us suppose, by contradiction, that $|\lambda| \geq \alpha > 0$. Then we take, for every n, $v_n \in H_n$, $\|v_n\| = 1$. Since A is compact we can extract a converging subsequence from (Av_n), but this leads to a contradiction since

$$\|Av_p - Av_q\|^2 = \|\lambda_p v_p - \lambda_q v_q\|^2 = \lambda_p^2 + \lambda_q^2 \geq 2\alpha^2.$$

Hence

$$A = \sum_{n=1}^{\infty} \lambda_n P_n.$$

□

2.3 Representations

Given a normed vector space over $\mathbb{R}$ or $\mathbb{C}$, $L(V)$ denotes the algebra of bounded linear operators on V (which are the $n \times n$ real or complex matrices if dim(V)= n $< \infty$). A representation of a topological group G is a map

$$\pi : G \longrightarrow L(V)$$

such that
a) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, $\pi(e) = Id$
b) for every $v \in V$ the map $g \mapsto \pi(g)v$ is continuous on g.

A representation of a Lie algebra $\mathfrak{g}$ is a map

$$\rho : \mathfrak{g} \longrightarrow L(V)$$

which is a Lie algebra morphism:

$$\rho([X, Y]) = [\rho(X), \rho(Y)] = \rho(x)\rho(Y) - \rho(Y)\rho(X).$$

If a subspace $W \subset V$ satisfies $\pi(g)W = W$ for every $g \in G$ we say that it is *invariant*. Those subspaces are crucial in this theory: in particular, we care about *irreducible representation*, the ones with 0 and V as only invariant subspaces.
Instead, if a representation is *reducible*, we have a *quotient representation* π_1 of G on the quotient space V/W, where W is an invariant subspace.
Let (π_1, V_1), (π_2, V_2) be two representations of G and $A : V_1 \to V_2$ a continuous linear map. If

$$A\pi_1(g) = \pi_2(g)A$$

for every $g \in G$ we call A an *intertwinning operator* for these representations. If such a map exists, π_1 and π_2 are said to be *equivalent*.

An operator T on a Hilbert space H is *unitary* if $T^{-1} = T^*$. A representation π of G on H is unitary if $\pi(g)$ is a unitary operator for every $g \in G$ (so if $\forall g \in G, \forall v \in H, \|\pi(g)v\| = \|v\|$).
Unitary representation are useful while working on reducible representations: in fact, if π is unitary and W is invariant and close, the quotient representation on H/W is equivalent to the subrepresentation on $W^\perp$ (which

is invariant as well, because of the orthogonality).

We want to decompose V in a direct sum of irreducible invariant subspaces: to show this is possible, we need a proper inner product on V.
If we choose a representation π on a finite-dimensional space V and a Euclidean inner product $(.|.)_0$ on V we can define, also on V:

$$(u|v) = \int_G (\pi(g)u|\pi(g)v)_0 \mu(dg)$$

where μ is a Haar measure on G (from now on we will always use this measure).
This is also a Euclidean inner product (the positivity and the Cauchy-Schwarz equations are true because we select initially a Euclidean inner product).
The point is that now π is unitary with respect to $(.|.)$:

$$\|\pi(g')v\| = (\pi(g')v|\pi(g')v) = \int_G (\pi(g)\pi(g')v|\pi(g)\pi(g')v)\mu(dg) =$$

$$= \int_G (\pi(gg')v|\pi(gg')v)\mu(dg) = \int_G (\pi(g)v|\pi(g)v)\mu(dg) = (v|v) = \|v\|$$

where we used the properties of the representation and the invariance of the Haar measure.
Now, if V is finite-dimensional, we can decompose, with this inner product and with respect to π:

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_N.$$

We only need to choose V_1 as a non-zero invariant subspace with minimal dimension, then $V_1^\perp$ is also invariant (because π is unitary) and we repeat the process until $(V_1 \otimes \cdots \otimes V_N)^\perp$ becomes zero.

Theorem 2.3.1 (Schur's Lemma). *i) If a linear map $A : V_1 \to V_2$ intertwines two finite dimensional irreducible representations of a topological group G (π_1, V_1) and (π_2, V_2) either $A = 0$ or A is an isomorphism*
ii) If π is a $\mathbb{C}$-linear representation of a topological group G on a finite-dimensional complex vector space V and commutes with a $\mathbb{C}$-linear map $A : V \to V$ then $A = \lambda I$ for λ in $\mathbb{C}$.

These properties holds also with a Lie algebra.

Proof. i)$Ker(A)$ and $Im(A)$ are invariant subspaces since $A\pi_1(g) = \pi_2(g)A$ and so they should be equal to 0 and V, respectevely, or viceversa.
ii) There exist an eigenvalue λ, so $A - \lambda I$ is not invertible and so is equal to zero because of the previous point. □

We define for our purpose a *character* as a one-dimensional representation. By the Schur's lemma, an irreducible $\mathbb{C}$-linear representation of a commutative group G is a character.
The simplest example of this is when $G = SO(2) \simeq U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$ and so the characters are $\chi_m(\vartheta) = e^{im\vartheta}$, $m \in \mathbb{Z}$.

Finally, we want to study the representations of a compact group G and their link to the function space $L^2(G)$.
Given a compact group G, its normalized Haar measure μ and a unitary representation of G (π, H), we define the operator K_v, for v, $w \in H$:

$$K_v w := \int_G (w|\pi(g)v)\pi(g)v\mu(dg)$$

which implies

$$(K_v w|w') = \int_G (w|\pi(g)v)\overline{(w'|\pi(g)v)}.$$

Obviously $\|K_v w\| \leq \|v\|^2\|w\|$ and so K_v is bounded and K_v is self-adjoint by the last display. Let now $g_0 \in G$:

$$K_v(\pi(g_0)w) = \int_G (w|\pi(g_0^{-1}g)v)\pi(g)v\mu(dg) =$$

$$= \int_G (w|\pi(g)v)\pi(g_0 g)v\mu(dg) = \pi(g_0)K_v w,$$

hence K and π commute. We can also easily show that this operator is compact, since we have demonstrated that the space of compact operators is closed for the norm topology and, given the compact and continuous (on G) operator $P_v w := (w|v)v$ we have

$$K_v = \int_G P_{\pi(g)v}\mu(dg).$$

Let us observe that $(K_v v|v) > 0$, hence $K_v \neq 0$.
With all these information, we can say that these operators have a non-zero eigenvalue ($\|K_v\|$ or$-\|K_v\|$), and the corresponding eigenspace is finite-dimensional and invariant under the representation π.
To summarise, we have proved the following:

Theorem 2.3.2. *i) Every unitary representation of a compact group contains a finite dimensional subrepresentation. ii) Every irreducible unitary representation of a compact group is finite dimensional.*

Now we fix an orthonormal basis $\{e_1, \ldots, e_n\}$ (n=d_π) of H, thus we can see π as a matrix with entries $\pi_{ij}(g) = (\pi(g)e_j|e_i)$. We need another theorem to find the relations among these functions.

Theorem 2.3.3. *Let π be an irreducible unitary $\mathbb{C}$-linear representation of a compact group G on a complex Euclidean vector space H with dimension d_π. Then, for $u, v, u', v' \in H$,*

$$(K_v u|u) = \int_G |(\pi(g)u|v)|^2 \mu(dg) = \frac{1}{d_\pi}\|u\|^2\|v\|^2,$$

$$\int_G (\pi(g)u|v)\overline{(\pi(g)u'|v')}\mu(dg) = \frac{1}{d_\pi}\|u\|^2\|v\|^2.$$

Proof. For $v \in H$, K_v and π commutes: by Schur's Lemma $K_v = \lambda(v)I$. Hence

$$\int_G |(\pi(g)u|v)|^2 \mu(dg) = \lambda(v)\|u\|^2.$$

Interchanging the roles of u and v we get

$$\lambda(u)\|v\|^2 = \lambda(v)\|u\|^2 \Longrightarrow \lambda(u) = \lambda_0\|u\|^2$$

for a constant λ_0. Now, for every $g \in G$:

$$\sum_{i=1}^{n} |(\pi(g)u|e_i)|^2 = \|\pi(g)u\|^2 = \|u\|^2;$$

integrating over G we obtain

$$\|u\|^2 = \sum_{i=1}^{n} \int_G |(\pi(g)u|e_i)|^2 \mu(dg) = n\lambda_0\|u\|^2$$

,

$$\Longrightarrow \lambda_0 = \frac{1}{n} \Longrightarrow \int_G |(\pi(g)u|v)|^2 \mu(dg) = \frac{1}{d_\pi}\|u\|^2\|v\|^2.$$

□

We have just proved the *Schur's orthogonally relations*:

$$\int_G \pi_{ij}(g)\overline{\pi_{kl}(g)} = \frac{1}{d_\pi}\delta_{ik}\delta_{jl},$$

that can be written alternatively as

$$\int_G \mathrm{tr}(A\pi(g))\overline{\mathrm{tr}(B\pi(g))}\mu(dg) = \frac{1}{d_\pi}\mathrm{tr}(AB^*)$$

with A and B two endomorphisms of H.

2.4 $L^2(G)$

Let $M\pi$ denote the subspace of $L^2(G)$ generated by the entries of the representation π, that is by the functions of the following form:

$$g \mapsto (\pi(g)u|v), (u, v \in H).$$

Theorem 2.4.1. *Let (π, H) and (π', H') be two not equivalent irreducible unitary representations of a compact group G. Then M_π and $M_{\pi'}$ are two orthogonal subspaces of $L^2(G)$: if $u, v \in H$, $u', v' \in H'$, then*

$$\int_G (\pi(g)u|v)\overline{(\pi'(g)u'|v')} = 0.$$

Proof. For a fixed map $A : H \longrightarrow H'$ we put

$$\tilde{A} := \int_G \pi'(g^{-1})A\pi(g)\mu(dg).$$

This operator intertwines the representations π and π' and so is equal to zero by the Schur's lemma. Hence (using the fact that π' is unitary)

$$(\tilde{A}u|u') = \int_G (A\pi(g)u|\pi'(g)u')\mu(dg) = 0.$$

The theorem is proved taking A as the rank 1 operator defined by $Au = (u|v)v'$ with $v \in H, v' \in H'$. In this case

$$\begin{aligned}\tilde{A}u &= \int_G \pi'(g^{-1})A\pi(g)\mu(dg) = \\ &= \int_G \pi'(g^{-1})(\pi(g)u|v)v'\mu(dg) = \int_G \pi'(g^{-1})(\pi(g)u|\pi'v).\end{aligned}$$

$\square$

This theorem is crucial: we can now say that 2 representations π_1 and π_2 of a compact group G are equivalent if and only if $M_{\pi_1} = M_{\pi_2}$ (the "$\Rightarrow$" implication was already known).
Let us now choose $H = L^2(G)$ and $\pi = R$, the right regular representation:

$$(R(g)f)(x) = f(xg).$$

Using the same notation as before, for a generic irreducible representation π of G, let $M_\pi^{(1)}$ be the subspace generated by the entries of the first row, that is by the functions $x \mapsto \pi_{1j}(x)$, for $j = 1, \ldots, n = d_\pi$. Let us observe that, because it is a product of matrices,

$$\pi_{1j}(xg) = \sum_{k=1}^{n} \pi_{1k}\pi_{kj}(g).$$

This is a linear combination of the first row's entrances, hence $M_\pi^{(1)}$ is invariant under R. Furthermore, the map from H into $M_\pi^{(1)}$

$$A : \sum_{j=1}^{n} c_j e_j \mapsto \sum_{j=1}^{n} c_j \pi_{1j}(x)$$

is an isomorphism that intertwines the representations π and R: if $u = \sum_{j=1}^{n} c_j e_j$, then

$$\begin{aligned}A\pi(g)u &= A\sum_{j=1}^{n} c_j\pi(g)e_j = A\sum_{j=1}^{n} c_j \left(\sum_{i=1}^{n} \pi_{ij}(g)e_i\right) = \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \pi_{ij}(g)c_j\right)\pi_{1i}(x) = \sum_{j=1}^{n} c_j\pi_{1j}(xg) = R(g)Au.\end{aligned}$$

Furthermore $\|Au\|^2 = \frac{1}{n}\|u\|^2$. We can now state that

$$M_\pi = M_\pi^{(1)} \oplus \cdots \oplus M_\pi^{(n)},$$

with the other subspaces defined in the same way as the first, considering the other rows. Thus the representation R restricted to M_π is equivalent to

$$\pi \oplus \cdots \oplus \pi = n\pi.$$

The same statement for the regular left representation $((L(g)f)(x) = f(g^{-1}x))$ can be obtained using columns instead of rows.
We continue this section with the following fundamental results, which basically justify the use of the Fourier series on compact groups.

Theorem 2.4.2 (Peter-Weyl theorem)**.** *Let Λ be the set of equivalence classes of irreducible unitary representations of the compact group G and, for each $\lambda \in \Lambda$, let M_λ be the space generated by the coefficient of a representation in the class λ (they are all the same, as we just saw). Then*

$$L^2(G) = \widehat{\bigoplus_{\lambda\in\Lambda}} M_\lambda,$$

which is the closure in $L^2(G)$ of the space of finite linear combinations of coefficients of finite dimensional representations of G.

Proof. We already saw that the subspaces M_λ are two by two orthogonal. Let us define

$$H := \widehat{\bigoplus_{\lambda\in\Lambda}} M_\lambda, H_0 := H^\perp.$$

Our aim is to show that $H_0 = 0$. Let us assume the opposite by contradiction.
H_0 is invariant under the representation R and closed, thus it contains a closed, finite-dimensional subspace $Y \neq 0$, which is invariant under R and irreducible. The restriction of R to Y belongs to one of the classes λ. Let $F \in Y$, $f \neq 0$, and put

$$F(g) := \int_G f(xg)\overline{f(x)}\mu(dx) = (R(g)f|f).$$

The function F belongs to M_λ. We will see that it is also orthogonal to M_λ, thus it is equal to zero.
Let us choose a representation (π, V) of the class λ, and u, $v \in V$. Then

$$\int_G F(g)\overline{(\pi(g)u|v)}\mu(dg) = \int_G \int_G f(xg)\overline{f(x)(\pi(g)u|v)},$$

and putting $xg = g'$ (change permitted by the Haar measure) we obtain

$$\int_G F(g)\overline{(\pi(g)u|v)}\mu(dg) = \int_G \overline{f(x)} \left(\int_G f(g')\overline{\pi(g')u|\pi(x)v}\mu(dg') \right) \mu(dx) = 0.$$

Since

$$F(e) = \int_G |f(x)|^2 \mu(dx) = 0$$

it follows $f = 0$, a contraddiction. □

Let us now define the $Hilbert - Schmidt\ norm$ on $L(H)$, where H is a finite dimensional Hilbert spaces:

$$|||A|||^2 = \operatorname{tr}(AA^*) = \sum_{i,j=1}^{n} |a_{ij}|^2$$

if we write A as a matrix, with an orthonormal basis on H.
Then, given an integrable function on G and a representative (π_λ, H_λ) for every class $\lambda \in \Lambda$, the Fourier coefficient $\hat{f}(\lambda)$, which action on H_λ is defined as

$$\hat{f}(\lambda) = \int_G f(g)\pi_\lambda(g^{-1})\mu(dg).$$

The following theorem is just an application of the previous results:

Theorem 2.4.3 (Plancherel's theorem)**.** *If $f \in L^2(G)$, then*
i) f is equal (in $L^2(G)$) to the sum of its Fourier coefficients

$$f(g) = \sum_{\lambda \in \Lambda} d_\lambda \operatorname{tr}(\hat{f}(\lambda)\pi_\lambda(g)).$$

ii)

$$\int_G |f|^2 \mu(dg) = \sum_{\lambda \in \Lambda} d_\lambda \left|\left|\left| \hat{f}(\lambda) \right|\right|\right|^2.$$

iii) If f_1 and $f_2 \in L^2(G)$, then

$$\int_G f_1(g)\overline{f_2(g)}\mu(dg) = \sum_{\lambda\in\Lambda} d_\lambda \mathrm{tr}(\hat{f}_1(\lambda)\hat{f}_2(\lambda)^*).$$

iv) The map $f \mapsto \hat{f}$ is a unitary isomorphism from $L^2(G)$ onto the space of sequences of operators $A = (A_\lambda) \in L(H_\lambda)$, for which

$$\|A\| = \sum_{\lambda\in\Lambda} d_\lambda |||A_\lambda|||^2 < \infty.$$

We want to state another powerful result:

Theorem 2.4.4. *i) Given $A = (A_\lambda) \in L(H_\lambda)$, if*

$$\sum_{\lambda\in\Lambda} d_\lambda^{3/2} |||A_\lambda||| < \infty,$$

then the Fourier series

$$\sum_{\lambda\in\Lambda} d_\lambda \mathrm{tr}(A(\lambda)\pi_\lambda(g))$$

converges absolutely and uniformly on G.
ii) Given a continuous function f such that

$$\sum_{\lambda\in\Lambda} d_\lambda^{3/2} \left|\left|\left| \hat{f}(\lambda) \right|\right|\right| < \infty,$$

then $f(g)$ is equal to its Fourier series, which converges absolutely and uniformly on G.

Proof. i) Since

$$|\mathrm{tr}(AB)| \le |||A||| |||B|||, \; |||\pi_\lambda(g)||| = \sqrt{d_\lambda},$$

we have

$$d_\lambda |\mathrm{tr}(A(\lambda)\pi_\lambda(g))| \le |||A_\lambda|||.$$

ii) We define

$$h(g) = \sum_{\lambda\in\tilde{G}} d_\lambda \mathrm{tr}(\hat{f}(\lambda)\pi_\lambda(g)).$$

Thanks to i), we know that the convergence is uniform and thus h is continuous.

$$\hat{h}(\lambda) = \int_G h(g)\pi_\lambda(g^{-1})\mu(dg) = \int_G \left(\sum_{\lambda' \in \tilde{G}} d_{\lambda'} \mathrm{tr}(\hat{f}(\lambda')\pi_{\lambda'}(g)) \right) \pi_\lambda(g^{-1})\mu(dg).$$

Knowing the Schur's orthogonality relations and switching the integral and the series we find

$$\hat{h}(\lambda) = d_\lambda \int_G \mathrm{tr}(\hat{f}(\lambda)\pi_\lambda(g))\pi_\lambda(g^{-1}) = \hat{f}(\lambda).$$

Thus, by the Plancherel theorem, $f = g$. □

We end this section introducing the *Casimir operator* and proving some results on it.

Let us take a Lie algebra $\mathfrak{g}$ with a basis $X_1, \ldots, X_n$ and a bilinear form β on g which is symmetric, non-degenerate and invariant ($\beta([X,Y],Z) = -\beta(Y,[X,Z])$). Such a map exists (a proof of this can be found on Faraut, [1]).

We put $g_{ij} = \beta(X_i, X_j)$, $g^{ij} = (g_{ij})^{-1}$.

If ρ is a representation of g on a vector space V, its Casimir operator is

$$\Omega_\rho = \sum_{i,j=1}^{n} g^{ij}\rho(X_i)\rho(X_j).$$

In particular, one can take β as positive definite and an orthonormal basis in order to have

$$\Omega_\rho = \sum_{i=1}^{n} \rho(X_i)^2.$$

One can easily show (as in Faraut, [1]) that this operator commute with ρ, and that this property does not depend on the choice of the basis.

By the Schur's lemma, this implies that there exists $\kappa_\rho \in \mathbb{C}$ such that

$$\Omega_\rho = -\kappa_\rho I.$$

If $\mathfrak{g}$ is the Lie algebra associated to a compact linear Lie Group G, then for a representation π on G we can define Ω_π as the Casimir operator of the representation $d\pi$ on g.

If π is not trivial, then we saw that exists on V a Euclidean inner product for which π is unitary, hence

$$d\pi(X)^* = -d\pi(X)$$

and, if $v \neq 0$,

$$(\Omega_\pi v|v) = -\sum_{i=1}^{n} \|d\pi(X_i)v\|^2 < 0.$$

We just proved that $\kappa_\pi > 0$. That will be crucial while talking about Fourier analysis on $SU(2)$.

Chapter 3

SU(2)

3.1 Bases and Haar Measure

Now we will focus on a particular compact Lie Group: $SU(2)$, which is the simplest non-commutative compact linear Lie-groups together with $SO(3)$. It is known that these two groups are "almost" the same: there is a surjective homomorphism which maps two elements of $SU(2)$ into one of $SO(3)$, so its kernel is ± 1 (a proof of this can be found on Faraut, [1]).
Let us start defining $SU(2)$: it consists of the matrices

$$\left\{ g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The inverse of these matrices is

$$g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}.$$

This group is homeomorphic to the unit sphere of $\mathbb{C}^2$ and therefore is compact, connected and simply connected.
Its Lie algebra $\mathfrak{su}$ has the following matrices as elements of a basis:

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The commutation relations are the following:

$$[X_1, X_2] = 2X_3, [X_2, X_3] = 2X_1, [X_3, X_1] = 2X_2.$$

Now we need to find the Haar measure. Since $SU(2) \simeq S^3$, the unit sphere of $\mathbb{R}^4$, we just want a measure on S^3 which is invariant under $SO(4)$.

Let ω be the differential form of degree 3 on $\mathbb{R}^n$ defined by

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_4.$$

At every point x, for vectors $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^4$,

$$\omega_x(\xi_1, \xi_2, \xi_3) = det(x, \xi_1 \xi_2, \xi_3).$$

This differential form is invariant under $SL(4, \mathbb{R})$, thus its restriction to the unit sphere is invariant under $SO(4)$.
Let μ be the normalised Haar measure on $SU(2) \simeq S^3$, that is

$$\int_{SU(2)} f(x)\mu(dx) = \frac{1}{\overline{\omega_4}} \int_{SU(2)} f\omega,$$

with

$$\overline{\omega_4} = \int_{SU(2)} \omega.$$

In order to find this constant, Let us write an element of $SU(2)$ as

$$x = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

Since $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ we can put $x_1 = \cos\vartheta$, $x_2 = \sin\vartheta\cos\varphi$, $x_3 = \sin\vartheta\sin\varphi\cos\psi$, $x_4 = \sin\vartheta\sin\varphi\sin\psi$.
Let Φ denote the map $(\vartheta, \varphi, \psi) \mapsto x = (x_1, x_2, x_3, x_4)$.

Theorem 3.1.1. *If f is an integrable function on SU(2), then*

$$\int_{SU(2)} f(x)\mu(dx) = \frac{1}{2\pi^2} \int_0^{\pi} d\vartheta \int_0^{\pi} d\varphi \int_0^{2\pi} d\psi f \circ \Phi(\vartheta, \varphi, \psi) \sin^2(\vartheta) \sin(\psi).$$

Proof. We just need to prove that $\Phi^*\omega = \sin^2\vartheta \sin\varphi d\vartheta \wedge d\varphi \wedge d\psi$: if this is true, then for $f = 1$ we find $\overline{\omega_4} = \int_{SU(2)} \omega = 2\pi^2$.

The differentials in the new coordinates are:

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix} = \begin{pmatrix} -\sin\vartheta \\ \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi\cos\psi \\ \cos\vartheta\sin\varphi\sin\psi \end{pmatrix} d\vartheta +$$
$$+ \begin{pmatrix} 0 \\ -\sin\varphi \\ \cos\varphi\cos\psi \\ \cos\varphi\sin\psi \end{pmatrix} \sin\vartheta d\varphi + \begin{pmatrix} 0 \\ 0 \\ -\sin\psi \\ \cos\psi \end{pmatrix} \sin\vartheta\sin\varphi d\psi.$$

The vector $x = \Phi(\vartheta, \varphi, \psi)$ and the three columns of the above right-hand side are orthogonal unit vectors , hence they form an orthonormal basis. This basis is positively orientated: if we put $\vartheta = \varphi = \psi = 0$ the 4 vectors became the canonical basis, which has determinant 1, and since the determinant can only be ± 1 and it changes continuously it is equal to 1 in each point.
Therefore $\det\left(\Phi, \frac{\partial\Phi}{\partial\vartheta}, \frac{\partial\Phi}{\partial\varphi}, \frac{\partial\Phi}{\partial\psi}\right) = \sin^2\vartheta\sin\varphi.$ □

3.2 Irreducible representation of SU(2)

We want to study the irreducible representations of $SU(2)$: in order to do that, we start considering the irreducible representation of its complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. A basis of $\mathfrak{g}$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the commutation relations are

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H.$$

We introduced this Lie algebra because it is the complexification of the real Lie algebra $\mathfrak{su}(2)$: every $Z \in \mathfrak{sl}(2, \mathbb{C})$ can be written uniquely as $Z = X + iY$, with $X, Y \in \mathfrak{su}(2)$.
We choose, as a vector space on which are made the representations, the space P_m of complex polynomials in two variables of degree m. Written

$g \in SL(2,\mathbb{C})$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the representation π_m of $SL(2,\mathbb{C})$ on P_m is defined by

$$(\pi_m(g)f)(x,y) = f(ax+cy, bx+dy) = f\left(\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

Given a representation π of G, it exists the derived representation of $\mathfrak{g}$:

$$d\pi(X) = \frac{d}{dt}\pi(\exp(tX))|_{t=0}.$$

The derived representation $\rho_m = d\pi_m$ of $\mathfrak{sl}(2,\mathbb{C})$ on P_m can be obtain as follows:

$$\begin{aligned}
\pi_m(\exp(tH))f(x,y) &= f(e^t u, e^{-t}v) \Longrightarrow \rho_m(H)f = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}, \\
\pi_m(\exp(tE))f(x,y) &= f(x, tx+y) \Longrightarrow \rho_m(E)f = x\frac{\partial f}{\partial y}, \\
\pi_m(\exp(tF))f(x,y) &= f(x, tx+y) \Longrightarrow \rho_m(E)f = y\frac{\partial f}{\partial x}.
\end{aligned}$$

A basis of P_m, which has dimension $m+1$, is given by the monomials $f_j(x,y) = x^j y^{m-j}$ for $j \in 0, 1, \dots, m$.

Let us observe how the image of the basis of $\mathfrak{g}$ under ρ_m acts on them (using the relations just found):

$$\begin{aligned}
\rho_m(H)f_j &= (2j-m)f_j \\
\rho_m(E)f_j &= (m-j)f_{j+1} \\
\rho_m(F)f_j &= jf_{j-1}.
\end{aligned}$$

Hence we can write them as matrices with respect to the basis $f_0, \dots, f_m$:

$$\rho_m(H) = \begin{pmatrix} -m & & & & \\ & -m+2 & & & \\ & & \ddots & & \\ & & & m-2 & \\ & & & & m \end{pmatrix}$$

$$\rho_m(E) = \begin{pmatrix} 0 & & & & \\ m & 0 & & & \\ & \ddots & \ddots & & \\ & & 2 & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

$$\rho_m(F) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & m \\ & & & & 0 \end{pmatrix}.$$

Every representation ρ_m is irreducible. In fact, if we suppose that exists an eigenspace W which is non-zero and invariant, then we can find at least one eigenvalue of the restriction of the operator $\rho_m(H)$ to W because this matrix is diagonal. Therefore, since the vectors $f_0, \ldots, f_m$ are the eigenvectors, one of them belongs to W. But since W is stable, when we act on the eigenvector with $\rho_m(E)$ and $\rho_m(F)$ we obtain other vectors of W, so it is clear that we can obtain all the basis using iteratively these two operators on the first eigenvector. Hence $W = P_m$. These representations are perfect for our purpose, since

Theorem 3.2.1. *An irriducible finite dimensional $\mathbb{C}$-linear representation of $\mathfrak{sl}(2,\mathbb{C})$ is equivalent to ρ_m for a positive integer m.*

Proof. Let ρ be such a representation on V. The aim of this proof is to find a map that intertwines ρ and ρ_m (for the correct index m).
Because it is a (non-zero) complex valued matrix, $\rho(H)$ has at least one non-zero eigenvalue.
We take as λ_0 the eigenvalue of $\rho(H)$ with the minimal real part and φ_0 an associeted eigenvector.
We have to show that $\varphi_1 := \rho(E)\varphi_0$ is also an eigenvector of $\rho(H)$, if it is non-zero:

$$\begin{aligned} \rho(H)\varphi_1 = \rho(H)\rho(E)\varphi_0 = \rho(E)\rho(H)\varphi_0 + \rho([H,E])\varphi_0 = \\ = \lambda_0\rho(E)\varphi_0 + 2\rho(E)\varphi_0 = (\lambda_0 + 2)\varphi_1 \end{aligned}$$

(we used that $[H,E] = HE - EH$ and the commutation relations showed at the beginning of the chapter: we will use them again).
Hence we can put $\varphi_k = \rho(E)^k\varphi_0$ and find iteratively that

$$\rho(H)\varphi_k = (\lambda_0 + 2k)\varphi_k.$$

Until these vectors are nonzero, they are aigenvectors of $\rho(H)$ for different eigenvalues, hence they are linearly independent. This happens until a

certain index k, let us put it equal to m (so $\varphi_i = 0$ for $i \neq 0$, and non-zero otherwise).
We call W the space generated by these vector (with dimension $m+1$), and we need to demonstrate that this is invariant. For sure it is invariant under $\rho(H)$, since the generators are eigenvectors for it, and under $\rho(E)$, because $\rho(E)\varphi_k = \varphi_{k+1} \in V\ \forall k \leq m$.
We are now interested in the action of $\rho(F)$ on these vectors,

$$\begin{aligned}\rho(H)\rho(F)\varphi_0 &= \rho(F)\rho(H)\varphi_0 + \rho([H,F])\varphi_0 \\ &= \lambda_0\rho(F)\varphi_0 - 2\rho(F)\varphi_0 \\ &= (\lambda_0 - 2)\rho(F)\varphi_0\end{aligned}$$

and since λ_0 was chosen as the eigenvalue of $\rho(H)$ with minimal real part, it follows that $\rho(F)\varphi_0 = 0$.
Then, for $1 \leq k \leq m$, $\rho(F)\varphi_k = \alpha_k\varphi_{k-1}$ with $\alpha_k = -k(\lambda_0 + k - 1)$. We will show it by induction:
for k=1,

$$\rho(F)\varphi_1 = \rho(F)\rho(E)\varphi_0 = \rho(E)\rho(F)\varphi_0 + \rho([F,E])\varphi_0 = -\rho(H)\varphi_0 = -\lambda_0\varphi_0.$$

Then, if the statement is true for $k \leq l$, we have

$$\begin{aligned}rho(F)\varphi_{l+1} &= \rho(F)\rho(E)\varphi_k = \rho(E)\rho(F)\varphi_k + \rho([F,E])\varphi_k = \\ &= -k(\lambda_0 + k - 1)\rho(E)\varphi_{k-1} - \rho(H)\varphi_k \\ &= (-k(\lambda_0 + k - 1) - (\lambda_0 + 2k))\varphi_k = \\ &= -(k+1)(\lambda_0 + k)\varphi_k.\end{aligned}$$

Now we just need to show $\lambda_0 = -m$: this is true because on the one hand

$$\mathrm{tr}\rho(H) = \mathrm{tr}[\rho(E), \rho(F)] = \mathrm{tr}(\rho(EF) - \rho(FE)) = \mathrm{tr}O = 0,$$

and on the other hand

$$\mathrm{tr}\rho(H) = \sum_{i=0}^{m} \lambda_0 + 2i = (m+1)\lambda_0 + (m)(m+1) = (m+1)(\lambda_0 + m).$$

To summarise, we found out that

$$\begin{aligned}\rho(H)\varphi_k &= (2k - m)\varphi_k \\ \rho(E)\varphi_k &= \varphi_{k+1} \\ \rho(F)\varphi_k &= k(m - k + 1)\varphi_{k-1}.\end{aligned}$$

Hence we can construct a linear map $A : V \to P_m$ such that

$$A\varphi_0 = f_0$$

$$A\varphi_k = m(m-1)\cdots(m-k+1)f_k$$

for $1 \le k \le m$.
A intertwines ρ and ρ_m:

$$A \circ \rho(X) = \rho_m(X) \circ A$$

for every $X \in \mathfrak{sl}(2, \mathbb{C})$.
We check this out for the basis we have chosen:

$$\begin{aligned}
A \circ \rho(H)\varphi_k &= (2k-m)A\varphi_k = (2k-m)m \ldots (m-k+1)f_k \\
\rho_m(H) \circ A\varphi_k &= m \ldots (m-k+1)\rho_m(H)f_k = (2k-m)(m) \ldots (m-k+1)f_k, \\
A \circ \rho(E)\varphi_k &= A\varphi_{k+1} = m \ldots (m-k+1)(m-k)f_{k+1} \\
\rho_m(E) \circ A\varphi_k &= m \ldots (m-k+1)\rho_m(E)f_k = m \ldots (m-k+1)(m-k)f_{k+1}, \\
A \circ \rho(F)\varphi_k &= k(m-k+1)A\varphi_{k-1} = k(m-k+1)m \ldots (m-k+2)f_{k-1} \\
\rho_m(E) \circ A\varphi_k &= m(m-1) \ldots (m-k+1)\rho_m(E)f_k = m \ldots (m-k+1)kf_{k-1}.
\end{aligned}$$

□

From now on, when we will talk about a representation π_m, we will refer to its restriction to $SU(2)$.
Let us state (on Faraut, [1] there is a deep study of facts like this) that a subspace invariant under π_m is also invariant under its derived representation ρ_m: hence, since these ones are irreducible, so is π_m for every index $m \in \mathbb{N}$.
Now we can use the properties of representations and of the Lie algebra:

Theorem 3.2.2. *Every irreducible representation π of $SU(2)$ on a finite dimensional complex vector space V is equivalent to one of the π_m.*

Proof. We extend the derived representation $d\pi$ as a $\mathbb{C}$-linear representation ρ of $\mathfrak{sl}(2, \mathbb{C})$ and we prove that this is irreducible. But this is a consequence of π being irreducible: $SU(2)$ is connected, hence generated by $\exp(\mathfrak{su}(2))$, therefore every subspace W of V invariant under ρ, which is also invariant under $\mathrm{Exp}(\rho(X)) = \pi(\exp(X))$, is invariant under $SU(2)$, and so $W = V$.

Then, by the last theorem, ρ is equivalent to one of the ρ_m: for every $X \in \mathfrak{su}(2)$

$$A\rho(X) = \rho_m(X)A \Longrightarrow A\pi(\exp(X)) = \pi_m(\exp(X))A$$

and again, since $SU(2)$ is generated by $\exp(\mathfrak{su}(2))$, for every $g \in SU(2)$,

$$A\pi(g) = \pi_m(g)A$$

where we kept the same index m. □

Let us see two easy example. There is a natural representation of $SU(2)$ on $\mathbb{C}^2$, which obviously consist in seeing an element $g \in SU(2)$ as a linear transformation, is clearly equivalent to π_1, since the space of the homogeneous complex polynomials in two variables of degree one is isomorphic to $\mathbb{C}^2$.
Then there is the adjoint representation on the Lie algebra $Ad(g) : X \mapsto g^{-1}Xg$, $X \in \mathfrak{su}(2)$, $g \in SU(2)$. This is equivalent to π_2, since the space of the homogeneous complex polynomials in two variables of degree 2 is generated by x^2, y^2, xy and one can find a homomorphism between the linear endomorphisms of P_2 and $\mathfrak{su}(2)$, generated by the matrices H, E, F.

3.3 Laplace operator

Let us put $X_1, \dots X_n$ be an orthonormal basis of $\mathfrak{g}$, the Lie algebra of a compact Lie group G. We define the *Laplace operator* as

$$\Delta f(x) = \sum_{i=1}^{n} \frac{d^2}{dt^2} f(x \exp(tX_i))|_{t=0},$$

where f is a C^2 function on G, and so the operator can be written as

$$\Delta = \sum_{i=1}^{n} \rho(X_i)^2.$$

With respect to the inner product on $L^2(G)$, for $\varphi, \psi \in C^2(G)$,

$$(\rho(X)\psi|\varphi) = -(\psi|\rho(X)\varphi)$$

since

$$\int_G \frac{d}{dt}|_{t=0}\psi(g\exp(tX))\overline{\varphi(g)}\mu(dg) = \frac{d}{dt}|_{t=0}\int_G \psi(g\exp(tX))\overline{\varphi(g)}\mu(dg) =$$
$$= \frac{d}{dt}|_{t=0}\int_G \psi(g)\overline{\varphi(g\exp(-tX))}\mu(dg) = \int_G \psi(g)\frac{d}{dt}|_{t=0}\overline{\varphi(g\exp(-tX))}\mu(dg).$$

Thus, with the same hypothesis

$$(\Delta\psi|\varphi) = (\psi|\Delta\varphi)$$

(Δ is symmetric). Moreover, Δ is negative since

$$-(\Delta f|f) = \int_G \sum_{i=1}^n |\rho(X_i f(g))|\mu(dg).$$

Let us now recall, from the previous chapter, that for an irreducible representation π of a connected compact Lie Group G on H, we defined the Casimir Operator as $\Omega_\pi = \sum_{i=1}^n (d\pi(X_i))^2$ and proved $\Omega_\pi = -\kappa_\pi I$ for a positive number κ_π.

Lemma 3.3.1. *If $f \in M_\pi$, the subspace of $L^2(G)$ generated by the entries of π, then f is an eigenfunction of the Laplace operator Δ, with eigenvalue $-\kappa_\rho$.*

Proof. If $f \in M_\pi$ it can be seen as $f(x) = \mathrm{tr}(A\pi(x))$ with A an endomorphism of H, therefore

$$\Delta f(x) = \mathrm{tr}(\Omega_\pi A\pi(x)) = -\kappa_\pi f(x).$$

□

Let us now focus again on $SU(2)$, where we select the following Euclidean inner product:

$$(X|Y) = \frac{1}{2}\mathrm{tr}(XY^*) = -\frac{1}{2}\mathrm{tr}(XY),$$

for which an orthonormal basis is

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We will find an explicit form of the Casimir Operator of a representation ρ of $\mathfrak{su}(2)$, which again extends $\mathbb{C}$-linearly to a representation of $\mathfrak{sl}(2,\mathbb{C})$. Using the same notation of the previous section, we have

$$X_1 = iH, X_2 = E - F, X_3 = i(E + F),$$

hence

$$\begin{aligned}
&\rho(X_1)^2 = -\rho(H)^2, \\
&\rho(X_2)^2 = \rho(E)^2 + \rho(F)^2 - \rho(E)\rho(F) - \rho(F)\rho(E), \\
&\rho(X_3)^2 = -\rho(E)^2 - \rho(F)^2 - \rho(E)\rho(F) - \rho(F)\rho(E), \\
&\Longrightarrow \Omega_\rho = -\rho(H)^2 - 2\rho(E)\rho(F) - 2\rho(F)\rho(E).
\end{aligned}$$

Finally, recalling that $[E, F] = H$ we get

$$-\Omega_\rho = -\rho(H)^2 + 2\rho(H) + 4\rho(F)\rho(E).$$

Furthermore, we again consider the representation ρ_m in the space of complex polynomials in 2 variables (it is not restrictive, since all the representations are equivalent to one of these). To simplify the notation, Let us put $\Omega_m = \Omega_{\rho_m}$. We want to find κ_m.
But, since we know Ω_ρ as a function of H, E, F, we only need to calculate it against a polynomial, and the easiest one is $f_m(x, y) = x^m$.
From the previous analysis we know $\rho_m(H)f_m = mf_m$ and $\rho_m(E)f_m = 0$,

$$\Longrightarrow -\Omega_m f_m = (m^2 + 2m)f_m.$$

To summarise, we know from the Peter-Weyl theorem that

$$L^2(G) = \widehat{\bigoplus_{m\in\mathbb{N}}} M_m,$$

where in the right side we got the orthogonal eigenspaces of the Laplace operator with eigenvalue $-m(m + 2)$.
The Fourier coefficients $\hat{f}(m)$ of an integrable function f on $SU(2)$ ($m \in \mathbb{N} = \Lambda$, the set of equivalence classes of representations of $SU(2)$) are

$$\hat{f}(m) = \int_{SU(2)} f(x)\pi_m(x^{-1})\mu(dx).$$

Thus, the Fourier series of f is

$$\sum_{m=0}^{\infty}(m+1)\mathrm{tr}(\hat{f}(m)\pi_m(x)),$$

which is equal to f if it is an $L^2(SU(2))$ function.
Moreover, we have the convergence in $L^2(SU(2))$ and the Plancherel formula:

$$\int_{SU(2)}|f(x)|^2\mu(dx) = \sum_{m=0}^{\infty}(m+1)\left|\left|\left|\hat{f}(m)\right|\right|\right|^2.$$

We want also the uniform convergence.

Lemma 3.3.2. $f \in C^2(SU(2)) \Longrightarrow \widehat{\Delta f}(m) = -m(m+2)\hat{f}(m)$.

Proof. If we put $\varphi(x) = (\pi(x)v|u)$, with $u, v \in H_{\pi_m}$, then

$$\begin{aligned}(\hat{f}(m)u|v)_{H_{\pi_m}} &= (\left(\int_{SU(2)} f(x)\pi_m(x^{-1})\mu(dx)\right) u|v) = \\ &= \int_{SU(2)} f(x)(\pi_m(x^{-1})u|v)\mu(dx) = \\ &= \int_{SU(2)} f(x(u|\pi_m(x)v)\mu(dx) = \\ &= \int_{SU(2)} f(x)\overline{\varphi(x)}\mu(dx) = (f|\varphi)_{L^2(SU(2))},\end{aligned}$$

and

$$\begin{aligned}(\widehat{\Delta f}(m)u|v)_{H_\pi} &= (\Delta f|\varphi)_{L^2(SU(2))} = (f|\Delta\varphi)_{L^2(SU(2))} = \\ &= -m(m+2)(f|\varphi)_{L^2(SU(2))} = -m(m+2)(\hat{f}(m)u|v)_{H_{\pi_m}}.\end{aligned}$$

□

Theorem 3.3.3. *If $f \in C^2(SU(2))$, then we got uniform and absolute converges on*

$$f(x) = \sum_{m=0}^{\infty}(m+1)\mathrm{tr}(\hat{f}(m)\pi_m(x)).$$

Proof. By the previous lemma,

$$\hat{f}(m) = -\frac{1}{m(m+2)}\widehat{\Delta f}(m),$$

hence

$$\sum_{m=1}^{\infty}(m+1)^{3/2}\left|\left|\left|\hat{f}(m)\right|\right|\right| = \sum_{m=1}^{\infty}\frac{(m+1)^{3/2}}{m(m+2)}\left|\left|\left|\widehat{\Delta f}(m)\right|\right|\right| \leq$$

$$\leq \left(\sum_{m=1}^{\infty}\frac{(m+1)^2}{m^2(m+2)^2}\right)^{1/2}\left(\sum_{m=1}^{\infty}(m+1)\left|\left|\left|\widehat{\Delta f}(m)\right|\right|\right|^2\right)^{1/2} < \infty;$$

we used the Shwarz inequality and the fact that

$$\sum_{m=0}^{\infty}(m+1)\left|\left|\left|\widehat{\Delta f}(m)\right|\right|\right|^2 = \int_{SU(2)}|\Delta f(x)|^2\mu(dx) \leq \infty.$$

The statement follows from the theorem of the previous chapter, which requires exactly this hypothesis to state the uniform and absolute convergence. □

Chapter 4

Generalized Heat equation

4.1 Generalized Heat Kernel

The following cauchy problem, usually studied in $\mathbb{R}^n$ or in $SO(2)$, is the *Heat equation* on $SU(2)$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) \\ u(x,0) = f(x) \end{cases}$$

for a given function f in $L^2(SU(2))$. Firstly, we can note that the functions like $u(x,t) = e^{-m(m+2)}v(x)$, with $v \in M_m$, are solution of the first equation since $\Delta v = -m(m+2)v$ as we proved before.
Thus, a solution of the Cauchy problem, if $f \in C^2(SU(2))$ and $t \geq 0$, is $u(x,t) = \sum_{m\in\mathbb{N}} e^{-m(m+2)}v_m(x)$, with $\sum_{m\in\mathbb{N}} v_m(x) = f(x)$, and this solution is unique.
These results are proved defining the *Heat Kernel*, which has the right properties:

$$H_t(x) = \sum_{m_0}^{\infty}(m+1)e^{-m(m+2)t}\mathrm{tr}(\pi_m(x)).$$

In that way we have

$$u(t,x) = \int_{SU(2)} H_t(xy^{-1})f(y)\mu(dy) = H_t * f(x).$$

The definition of the convolution in a compact group should be done in that way, since the Haar measure is invariant under the action of an element $g \in G$ as well as the Lebesgue measure is invariant under the sum of

$x \in \mathbb{R}^n$.
Let us now introduce a generalized case.
Suppose we have a function ψ on $\mathbb{R}$ such that, for appropriate functions Φ and ω on $\mathbb{R}$,

$$\psi(s) = \int_0^\infty \Phi(\tau) e^{-s\omega(\tau)} d\tau.$$

We can define formally

$$\psi(-\Delta) := \int_0^\infty \Phi(\tau) e^{\omega(\tau)\Delta},$$

an operator which act as

$$\psi(-\Delta) f(x) = \int_{SU(2)} f(y) K_\psi(xy^{-1}) \mu(dy) = f * K_\psi(x),$$

with

$$K_\psi(x) = \int_0^\infty \Phi(\tau) H_{\omega(t)}(x) d\tau.$$

We want to use this with the function $\psi(s) = (1+s)^{-\alpha} = \frac{1}{\Delta(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-(1+s)\tau} d\tau$. Recalling that we call $\lambda_m = -m(m+2)$, in that case we have the *generalized Heat kernel*

$$\begin{aligned}
K_\psi(x) = K_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-\tau} H_\tau(x) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-\tau} \sum_{m\in\mathbb{N}} (m+1) e^{\lambda_m \tau} \mathrm{tr}(\pi_m(x)) d\tau \\
&= \sum_{m\in\mathbb{N}} (1-\lambda_m)^{-\alpha} (m+1) \mathrm{tr}(\pi_m(x)).
\end{aligned}$$

This diverges if $\mathrm{Re}(\alpha) \le 0$.
Furthernmore, applying the Tonelli's theorem we get

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}e^{-\tau}\sum_{m\in\mathbb{N}}(m+1)e^{\lambda_m\tau}\mathrm{tr}(\pi_m(x))d\tau d\tau\right| \le \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\mathrm{Re}(\alpha)-1}\sum_{m\in\mathbb{N}}(m+1)e^{\lambda_m\tau-\tau}|\mathrm{tr}(\pi_m(x))|d\tau \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{m\in\mathbb{N}}\int_0^\infty \tau^{\mathrm{Re}(\alpha)-1}e^{-m(m+2)\tau-\tau}(m+1)^2 d\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\mathrm{Re}(\alpha))}\sum_{m\in\mathbb{N}}(m^2+2m+1)^{-\mathrm{Re}(\alpha)}(m+1)^2 \\
&\sim \sum_{m\in\mathbb{N}}(m+1)^{-2\mathrm{Re}(\alpha)+2},
\end{aligned}$$

hence there is absolute convergence if and only if $\mathrm{Re}(\alpha) > \frac{3}{2}$.
We used the fact that $|\mathrm{tr}(\pi_m(x))| \le m+1$. If we are in this hypothesis, we can use Fubini's theorem in order to compute

$$\begin{aligned}
\widehat{K_\alpha}(n) &= \int_{SU(2)} K_\alpha(x)\pi_n(x^{-1})\mu(dx) \\
&= \frac{1}{\Gamma(\alpha)}\int_{SU(2)}\left(\int_0^\infty \tau^{\alpha-1}\sum_{m\in\mathbb{N}}(m+1)e^{\lambda_m\tau-\tau}\mathrm{tr}(\pi_m(x))d\tau\right)\pi_n(x^{-1})\mu(dx) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}(n+1)e^{\lambda_n\tau-\tau}\left(\int_{SU(2)}\mathrm{tr}(\pi_n(e))\mu(dx)\right)d\tau \\
&= (n+1)^2(1-\lambda_n)^{-\alpha},
\end{aligned}$$

where we used the invariance of the Haar measure, the fact that the functions are orthogonal if $m \neq n$ and, when $m = n$, $\pi_n(xx^{-1}) = \pi_n(e) = I$.
From now on we will assume $f \in C^2(SU(2))$ (which implies $f \in L^2(SU(2))$ hence we can get the decomposition in the orthogonal spaces).

We are now ready to define (and then rewrite using Tonelli's theorem):

$$
\begin{aligned}
&(1-\Delta)^{-\alpha} f(x) = f * K_\alpha(x) = \int_{SU(2)} f(y) K_\alpha(xy^{-1}) \mu(dy) \\
&= \frac{1}{\Gamma(\alpha)} \int_{SU(2)} f(y) \left(\int_0^\infty \tau^{\alpha-1} \sum_{m\in\mathbb{N}} e^{\lambda_m \tau - \tau} (m+1) \mathrm{tr}(\pi_m(xy^{-1})) d\tau \right) \mu(dy) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} \sum_{m\in\mathbb{N}} e^{\lambda_m \tau - \tau} (m+1) \mathrm{tr} \left(\pi_m(x) \int_{SU(2)} f(y) (\pi_m(y^{-1})) \mu(dy) \right) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} \sum_{m\in\mathbb{N}} e^{\lambda_m \tau - \tau} (m+1) \mathrm{tr}(\hat{f}(m) \pi_m(x)) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \sum_{m\in\mathbb{N}} (m+1) \mathrm{tr}(\hat{f}(m) \pi_m(x)) \int_0^\infty \tau^{\alpha-1} e^{\lambda_m \tau - \tau} d\tau \\
&= \sum_{m\in\mathbb{N}} (1-\lambda_m)^{-\alpha} (m+1) \mathrm{tr}(\hat{f}(m) \pi_m(x)).
\end{aligned}
$$

4.2 Analytic extension

Our purpose is now to obtain a meromorphic extension for all $\alpha \in \mathbb{C} \setminus \{0, -1, -2 \dots\}$. It is sufficient to note that $\partial_\tau \tau^\alpha = \alpha \tau^{\alpha-1}$ and integrate by parts (assume $\mathrm{Re}(\alpha) > 0$):

$$
\begin{aligned}
K_\alpha(x) &= \frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty \partial_\tau \tau^\alpha e^{-\tau} H_\tau(x) d\tau \\
&= \frac{1}{\alpha\Gamma(\alpha)} \left[\tau^\alpha e^{-\tau} H_\tau(x) \Big|_0^\infty - \alpha\Gamma(\alpha) \int_0^\infty \tau^\alpha \partial_\tau (e^{-\tau} H_\tau)(x) d\tau \right] \\
&= -\frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty \tau^\alpha \partial_\tau (e^{-\tau} H_\tau)(x) d\tau \\
&= -\frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty \tau^\alpha e^{-\tau} (-H_\tau + \partial_\tau H_\tau)(x) d\tau \\
&= \frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty \tau^\alpha e^{-\tau} (H_\tau - \Delta H_\tau)(x) d\tau \\
&= \frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty \tau^\alpha e^{-\tau} \sum_{m\in\mathbb{N}} (1-\lambda_m) e^{\lambda_m \tau} (m+1) \mathrm{tr}(\pi_m(x)) d\tau.
\end{aligned}
$$

The integral now converges for $\mathrm{Re}(\alpha) > -1 \setminus \{0\}$ and now we can take

$$
\begin{aligned}
(1-\Delta)^{-\alpha} f(x) &= \int_{SU(2)} f(y) K_\alpha(xy^{-1}) \mu(dy) \\
&= \frac{1}{\alpha\Gamma(\alpha)} \int_{SU(2)} f(y) \left(\int_0^\infty \tau^\alpha e^{-\tau} \sum_{m\in\mathbb{N}} (1-\lambda_m) e^{\lambda_m \tau} (m+1) \mathrm{tr}(\pi_m(xy^{-1})) d\tau \right) \mu(dy) \\
&= \frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty \tau^\alpha \sum_{m\in\mathbb{N}} (1-\lambda_m) e^{\lambda_m \tau - \tau} (m+1) \mathrm{tr} \left(\pi_m(x) \int_{SU(2)} f(y) \pi_m(y) \mu(dy) \right) d\tau \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \tau^\alpha \sum_{m\in\mathbb{N}} (1-\lambda_m) e^{\lambda_m \tau - \tau} (m+1) \mathrm{tr}((\pi_m(x)) \hat{f}(m)) d\tau.
\end{aligned}
$$

We just need to repeat this procedure n times to have the operator defined for $\mathrm{Re}(\alpha) > -n$, except for the values $\{0, -1, \dots, -n+1\}$.

4.3 The Cauchy problem

The *generalized heat equation* is the solution of the following Cauchy problem, which is the same as the "usual" one except for the fact that we use the operator we just defined insted of Δ:

$$
\begin{cases}
\frac{\partial}{\partial t} u(x,t) = (1-\Delta)^{-\alpha} u(x,t) \\
u(x,0) = f(x)
\end{cases}
$$

with given $\alpha \in \mathbb{C} \setminus \{0, -1, -2 \dots\}$ and a C^2 function $f \in L^2(SU(2))$.
We will first show the uniqueness of the solution using the *maximum principle*:

Lemma 4.3.1. *If u is a solution of the previous Cauchy problem, we have*

$$
\min_{x\in SU(2)} f(x) = \min_{x\in SU(2)} u(x,0) \le u(x,t) \le \max_{x\in SU(2)} u(x,0) = \max_{x\in SU(2)} f(x)
$$

Proof. Let us define

$$
u_\varepsilon(t,x) := u(t,x) + \varepsilon t.
$$

Let us take an interval $[0,T]$ and $(x_0,t_0) \in SU(2) \times [0,T]$ such that

$$
u_\varepsilon(x_0,t_0) = \min \{ u_\varepsilon(x,t) | (x,t) \in SU(2) \times [0,T] \}.
$$

Note that a minimum exists since this set, being the product of two compact sets, is compact.
We want to prove that $t_0 = 0$. Suppose by contradiction that $t_0 \neq 0$. Hence, we have that

$$\frac{\partial}{\partial t} u_\varepsilon(x_0, t_0) = 0,$$
$$\Delta u_\varepsilon(x_0, t_0) \geq 0$$

and this gives a contradiction since

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \varepsilon > 0.$$

Therefore there is the minimum at $t = 0$ and, since $u_\varepsilon(x, 0) = u(x, 0)$,

$$u_\varepsilon(x, t) = u(x, t) + \varepsilon t \geq \min_{x \in SU(2)} u(x, 0)$$

and, since the inequality holds for every $\varepsilon > 0$, $u(x, t) \geq \min_{x \in SU(2)} u(x, 0)$.
On the other hand, we can replace u with $-u$ and, using the previous result, get

$$-u_\varepsilon(x, t) = -u(x, t) - \varepsilon t \geq \min_{x \in SU(2)} (-u(x, 0))$$

and again, with the limit $\varepsilon \to 0$,

$$u(x, t) \leq \max_{x \in SU(2)} u(x, 0).$$

□

Theorem 4.3.2. *If there exists a solution to the previous Cauchy problem, this is unique.*

Proof. Suppose that u_1 and u_2 are two distinct solutions. Then $u_1 - u_2$ is a solution of another Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) = (1 - \Delta)^{-\alpha} u(x, t) \\ u(x, 0) = 0 \end{cases}.$$

From the previous lemma,

$$0 = \min_{x \in SU(2)} f(x) \leq (u_1 - u_2)(x, t) \leq \max_{x \in SU(2)} f(x) = 0,$$

hence $u_1 = u_2$. □

Recalling that

$$(1-\Delta)^{-\alpha}u(x) = \sum_{m\in\mathbb{N}}(1-\lambda_m)^{-\alpha}(m+1)\mathrm{tr}(\hat{u}(m)\pi_m(x))$$

we can note that the functions like

$$u(x,t) = e^{\frac{(1-\lambda_m)^{-\alpha}}{\Gamma(\alpha)}t}v(x),$$

with $v(x) \in M_m$, are solutions of the Cauchy problem.
We just have to recall that we can decompose $f(x)$ in those orthogonal subspaces

$$f(x) = \sum_{m=0}^{\infty}(m+1)\mathrm{tr}(\hat{f}(m)\pi_m(x)),$$

to state that the solution of the problem is

$$u(x,t) = \sum_{m=0}^{\infty}(m+1)\mathrm{tr}(\hat{f}(m)\pi_m(x))e^{\frac{(1-\lambda_m)^{-\alpha}}{\Gamma(\alpha)}t}$$

which converges absolutely and uniformly on $G \times [0,\infty[$ and it is a C^∞ function because so are the exponential and the functions in M_m for every m.

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