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Unipotent Representations of Finite Groups of Lie Type

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Chapter 1

Algebraic groups

1.1 Algebraic groups

1.1.1 Definitions and basic notions

Let $\mathbb{K}$ be an algebrically closed field. By $\mathbb{A}^n$ we denote the affine space of dimension n endowed with the Zarisky topology (the topology having as closed basis the zero-loci of polynomial systems). With the name "affine algebraic variety" we refer to a closed subspace X of $\mathbb{A}^n$ (for some n). Let $I \leq \mathbb{K}[T_1 \dots T_n]$ be the (radical) ideal of the polynomial vanishing on X; we denote by $\mathbb{K}[X] = {}^{\mathbb{K}[T_1 \dots T_n]}\!/_{I}$ the ring of regular functions on X over $\mathbb{K}$ (sometimes called "coordinate ring" of X over $\mathbb{K}$), that can be identified with the algebra of the polynomial functions on X with coefficients in $\mathbb{K}$.
We say that a map ϕ between affine algebraic varieties X, Y, with $Y \subseteq \mathbb{A}^r$, is a morphism of affine varieties if it can be expressed in coordinates by polynomial functions on X, that is

$$\begin{aligned} \phi : & X \to Y \\ & x \mapsto (\psi_1(x) \dots \psi_r(x)) \end{aligned}$$

where $\psi_i \in \mathbb{K}[X]$ for any i. Morphisms of affine varieties are continuous with respect to the Zarisky topology.
Any morphism of affine varieties $\phi : X \to Y$ induces functorially (contravariantly) a morphism of algebras:

$$\begin{aligned} \phi^* : \mathbb{K}[Y] &\to \mathbb{K}[X] \\ f &\mapsto f \circ \phi \end{aligned}$$

If $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ are algebraic varieties, so is the product $X \times Y \subseteq \mathbb{A}^{m+n}$. We will always consider the product endowed with the Zarisky topology. It holds $\mathbb{K}[X \times Y] \cong \mathbb{K}[X] \times \mathbb{K}[Y]$.
In this thesis, we will assume most of the basic notions of algebraic geometry regarding affine (and sometimes projective) affine varieties.

Now we introduce the notion of "algebraic group", a mathematical object that will play a central role in the present thesis.
For a group G, we call inversion map the map

$$\begin{aligned} \iota : & G \to G \\ & x \to x^{-1} \end{aligned}$$

and multiplication map the map

$$\begin{aligned} \mu : & G \times G \to G \\ & (x, y) \to xy. \end{aligned}$$

Definition 1.1.1. *An (affine) algebraic group G is an affine algebraic variety which has a group structure such that the multiplication and the inversion maps are morphisms of algebraic varieties.*

Example 1.1.2. *We give some examples of algebraic groups*

- *The additive group $(\mathbb{K}, +)$ is an algebraic group (as affine variety, it is the zero locus of the ideal (0)). Its coordinate ring is $K[T]$. We will denote this algebraic group by $\mathbb{G}_a$.*

- *The multiplicative group $(\mathbb{K}^*, \cdot)$ is an algebraic group. Indeed it can be identified as the zero locus of $(xy - 1)$ in $\mathbb{A}^2$, which is an affine variety. Its coordinate ring is given by*
$$\mathbb{K}[x, y] \big/ (xy - 1) \cong \mathbb{K}[T, T^{-1}].$$
We will denote this algebraic group by $\mathbb{G}_m$.

- *The group $GL_n(\mathbb{K})$ of invertible $n \times n$ matrices over $\mathbb{K}$. It is the subset of the space of the $n \times n$ matrices where the determinant (which is a polynomial function) does not vanishes: $GL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) \mid det(A) \neq 0\}$. Hence, identifying the space of the $n \times n$ matrices affine space of dimension n^2, $GL_n(\mathbb{K})$ can be identified with the closed subset of $\mathbb{A}^{n^2+1}$*
$$\{(A, d) \in \mathbb{A}^{n^2+1} \mid d(det(A)) - 1 = 0\}$$
via the polynomial map $A \mapsto (A, det(A)^{-1})$. Its coordinate ring is given by
$$\mathbb{K}[GL_n(\mathbb{K})] = \mathbb{K}[T_{i,j}, S]_{1 \le i,j \le n} \big/ (Sdet(T) - 1) \cong \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \le i,j \le n}.$$
Where $T = (T_{ij})_{1 \le i,j \le n}$.

- *The group $SL_n(\mathbb{K})$ of $n \times n$ matrices with determinant equal to 1 over $\mathbb{K}$. Its coordinate ring is*
$$\mathbb{K}[SL_n(\mathbb{K})] = \mathbb{K}[T_{i,j}]_{1 \le i,j \le n} \big/ (det(T) - 1).$$

Note that we endow the product of affine varieties with the Zarisky topology, not the product topology, hence (although morphism of varieties are continuous) an algebraic group is not, in general, a topological group.

Example 1.1.3. *The field $\mathbb{K}$ is algebraically closed, therefore infinite. The additive group $\mathbb{G}_a$ is an algebraic group, but it is not a topological group endowed with the Zarisky topology. Indeed the Zarisky topology in $\mathbb{K}$ is the co-finite topology, since any polynomial system can have only finitely many zeros. Therefore the product topology on $\mathbb{K} \times \mathbb{K}$ has a closed basis consisting of sets of the form $F \times \mathbb{K}$ and $\mathbb{K} \times F$ with $F \subset \mathbb{K}$ a finite set. Then $\mu^{-1}(0) = \{(a, -a) \mid a \in \mathbb{K}\} \subseteq \mathbb{K} \times \mathbb{K}$ is not closed in the product topology. Therefore the multiplication map is not continuous with respect to the product topology, hence $\mathbb{G}_a$ is not a topological group.*

1.1.2 Connectedness

Generally speaking, for topological spaces (hence, in particular, for affine varieties) we have the following distinct definitions:

Definition 1.1.4.
- *A topological space X is said to be "connected" if it cannot be decomposed as disjoint union of closed proper and non-empty subsets.*
- *A topological space X is said to be "irriducible" if it cannot be decomposed as union of closed proper and non-empty subsets.*

Anyway, for algebraic groups we have the following result:

Proposition 1.1.5. *[9, Proposition 1.13]Let G be an algebraic group. Then the irreducible components of G are pairwise disjoint, hence they are the connected components.*

In general, any affine variety has a finite number of maximal irreducible subsets ([9, Proposition 1.10]), hence an algebraic group has a finite number of maximal connected subsets, that is, of connected components. Moreover, there exists a unique irreducible component containing the identity of the group e (namely the connected component of the identity), that we will denote by G^0. The following holds:

Proposition 1.1.6. *[9, Proposition 1.13]The irreducible component G^0 containing e is a closed normal subgroup of G of finite index.*

Therefore , ${}^{G}\!/_{G^0}$ is always a finite group.

Proposition 1.1.6 allows us to give the following definition:

Definition 1.1.7. *G^0 is the maximal connected closed subgroup of G.*

In the development of this thesis, we will often assume G to be a connected algebraic group, hence $G = G^0$.

As always for affine varieties, connectdness of the variety can be read on the ring of regular function being an integral domain. This is enough to show that the algebraic groups $GL_n(\mathbb{K}), \mathbb{G}_a, \mathbb{G}_m$ of the previous example are connected (also $SL_n(\mathbb{K})$ is connected, but it is not so evident from the ring of functions).

1.1.3 Hopf Algebras

In the case of an algebraic group G, the group structure over G allows some extra structure over its coordinate ring $\mathbb{K}[G]$: it turns out to be an *Hopf Algebra*. To define an Hopf algebra, we need the following definitions.

Definition 1.1.8. *For any $\mathbb{K}$ vector spaces V, W, we define the twist map $\tau : V \otimes W \to W \otimes V$ as the map given by $\tau(v \otimes w) = w \otimes v$ for any $v \in V, w \in W$.*

Definition 1.1.9. *A $\mathbb{K}$-vector space A with two linear maps $m : A \otimes A \to A$, called multiplication, and $u : \mathbb{K} \to A$, called unit, is a $\mathbb{K}$-algebra if the following diagrams commute:*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xleftarrow{id \otimes u} & A \otimes \mathbb{K} \\ {\scriptstyle u \otimes id}\uparrow & \searrow^{m} & \downarrow{\scriptstyle \cong} \\ \mathbb{K} \otimes A & \xrightarrow{\cong} & A \end{array}$$

The commutativity of these diagram is referred as, respectively from left to right, the "associativity" and "unit" property.

Definition 1.1.10. *A linear map $f : A_1 \to A_2$ between two algebras (A_1, m_1, u_1), (A_2, m_2, u_2) is an algebra morphism if*

$$f \circ m_1 = m_2 \circ (f \otimes f)$$

and

$$u_2 = f \circ u_1$$

Remark 1.1.11. *If (A, m, u) is an algebra, $A \otimes A$ is an algebra with*

- *multiplication $m_{A \otimes A}$ defined by $m_{A \otimes A} = (m_A \otimes m_A) \circ (id_A \otimes \tau \otimes id_A)$*
- *unit $u_{A \otimes A}(1_{\mathbb{K}}) = u_A(1_{\mathbb{K}}) \otimes u_A(1_{\mathbb{K}})$ (and extended by linearity)*

Definition 1.1.12. *A $\mathbb{K}$-vector space C with two linear maps $\Delta : C \to C \otimes C$, called comultiplication, and $\epsilon : C \to \mathbb{K}$, called counit, is a $\mathbb{K}$-coalgebra if the following diagrams commute:*

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\Delta \otimes id} & C \otimes C \\ {\scriptstyle id \otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array} \qquad \begin{array}{ccc} C \otimes C & \xrightarrow{id \otimes \epsilon} & C \otimes \mathbb{K} \\ {\scriptstyle \epsilon \otimes id}\downarrow & \nwarrow^{\Delta} & \uparrow{\scriptstyle \otimes 1_{\mathbb{K}}} \\ \mathbb{K} \otimes C & \xleftarrow{1_{\mathbb{K}} \otimes} & C \end{array}$$

The commutativity of these diagram is referred as, respectively from left to right, "coassociativity" and "counit" property.

Definition 1.1.13. *A linear map $f : C_1 \to C_2$ between two coalgebras $(C_1, \Delta_1, \epsilon_1)$, $(C_2, \Delta_2, \epsilon_2)$ is a colgebra morphism if*

$$\Delta_2 \circ f = (f \otimes f) \circ \Delta_1$$

and

$$\epsilon_1 = \epsilon_2 \circ f$$

Remark 1.1.14. *1. If (C, Δ, ϵ) is a coalgebra, $C \otimes C$ is a coalgebra with*

- *comultiplication $\Delta_{C\otimes C}$ defined by $\Delta_{C\otimes C} = (id_C \otimes \tau \otimes id_C) \circ (\Delta_C \otimes \Delta_C)$*
- *counit $\epsilon_{C\otimes C}$ defined by $\epsilon_{C\otimes C} = \epsilon_C \otimes \epsilon_C$*

2. $\mathbb{K}$ is a coalgebra with

- *comultiplication $\Delta_{\mathbb{K}}$ defined by $\Delta_{\mathbb{K}}(\lambda) = \lambda \otimes \lambda$ for ant $\lambda \in \mathbb{K}$*
- *counit $\epsilon_{\mathbb{K}}$ defined by $\epsilon_{\mathbb{K}}(\lambda) = 1_{\mathbb{K}}$ for any $\lambda \in \mathbb{K}$*

Definition 1.1.15. *A $\mathbb{K}$ vector space B with linear maps $m : B \otimes B \to B$, $u : \mathbb{K} \to B$, $\Delta : B \to B \otimes B$ and $\epsilon : B \to \mathbb{K}$ is a bialgebra if*

- *(B, m, u) is an algebra*
- *(B, Δ, ϵ) is a coalgebra*
- *Δ, ϵ are algebra morphisms*

Definition 1.1.16. *A $\mathbb{K}$ vector space B with linear maps $m : B \otimes B \to B$, $u : \mathbb{K} \to B$, $\Delta : B \to B \otimes B$, $\epsilon : B \to \mathbb{K}$ and $S : B \to B$ is a $\mathbb{K}$-Hopf algebra if*

- *$(B, m, u, \Delta, \epsilon)$ is a bialgebra*
- *$S : B \to B$ is a $\mathbb{K}$-linear map such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
B \otimes B & & \xrightarrow{S\otimes id} & & B \otimes B & & \xleftarrow[id\otimes S]{} & & B \otimes B \\
\Delta\uparrow & & & & \downarrow m & & & & \Delta\uparrow \\
B & \xrightarrow{\epsilon} & \mathbb{K} & \xrightarrow{u} & B & \xleftarrow[u]{} & \mathbb{K} & \xleftarrow[\epsilon]{} & B
\end{array}$$

The map S is called antipode.

As already stated above, we have the following result:

Proposition 1.1.17. *[7, §7.6] Let G be an algebraic group over the field $\mathbb{K}$, and let $\mathbb{K}[G]$ be the ring of regular functions. Then $\mathbb{K}[G]$ is an Hopf algebra.*

Proof. The algebra structure on $K[G]$ is the usual one as polynomial ring.
The coalgebra structure is given by the pull-back of the group multiplication μ^* (recalling that $\mathbb{K}[G \times G] \cong \mathbb{K}[G] \otimes \mathbb{K}[G]$) and of the group identity e^* (regarding the identity of the group as the morphism from the trivial group to G, and recalling that $\mathbb{K}[e] \cong \mathbb{K}$). The coassociativity and counit properies follow respectively from the associativity of μ and the definition of e by controvariant functoriality of the pull-back.
Plugging together the two structures we have a bialgebra, since, again by functoriality of the construction, the pullback of morphisms of affine varieties (namely μ and e) are algebra morphisms.
The antipode is given by the pull-back of the group inversion i^*. Indeed, write $j : G \to G \times G$ for the map defined by $j(g) = (g, g)$ for any $g \in G$ and note that, under the isomorphism $\mathbb{K}[G \times G] \cong \mathbb{K}[G] \otimes \mathbb{K}[G]$, the pull back j^* is the usual multiplication of $\mathbb{K}[G]$; then the the following diagram commutes by the properties of the inversion i:

$$\begin{array}{ccccccccc}
G \times G & & \xleftarrow{\quad i \times id \quad} & & G \times G & & \xrightarrow{\quad id \times i \quad} & & G \times G \\
\downarrow{\scriptstyle m} & & & & {\scriptstyle j}\uparrow & & & & \downarrow{\scriptstyle m} \\
G & \longrightarrow & \{e\} & \longrightarrow & G & \longleftarrow & \{e\} & \longleftarrow & G
\end{array}$$

where $\{e\}$ is the trivial group and the bottom maps are the only group morphism existing. Then passing to the coordinate rings an considering the controvariant functoriality of the pull-back this diagram yields the fact that i^* is an antipode.

□

Example 1.1.18. *Consider the algebraic group $G = GL_n(\mathbb{K})$ and its coordinate ring $\mathbb{K}[G] = \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \leq i,j \leq n}$.*
The Hopf algebra structure is then given by the usual algebra structure and the coalgebra one defined by:

- *the counit*
$$e^* : \mathbb{K}[G] \to \mathbb{K}$$
defined by $e * (f) = f(I_n)$, with I_n being the $n \times n$ identity. So $e^*(T_{ij}) = \delta_{ij}$, $e^*(\frac{1}{det(T)}) = 1$

$$\mu^* : \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \leq i,j \leq n} \to \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \leq i,j \leq n} \otimes \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \leq i,j \leq n}$$

defined by $\mu^*(T_{i,j}) = \sum_{k=1}^{n} T_{i,k} \otimes T_{k,j}$ and $\mu^*(\frac{1}{det(T)}) = \frac{1}{det(T)} \otimes \frac{1}{det(T)}$.

The antipode

$$i^* : \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \leq i,j \leq n} \to \mathbb{K}\left[T_{i,j}, \frac{1}{det(T)}\right]_{1 \leq i,j \leq n}$$

is defined by $i^*(\frac{1}{det(T)}) = det(T)$ *(which is a polynomial expression in* $T_{i,j}$*) and* $i^*(T_{i,j}) = (-1)^{i+j}\frac{1}{det(T)}det(T'_{ij})$ *where* $T'_{ij} = (T_{r,s})_{r\neq j, s\neq i}$.

1.1.4 Chevalley theorem and Jordan decomposition

In what follows V will always be a finite dimensional $\mathbb{K}$-vector space. We already introduced some linear groups as example of algebraic groups. Actually, any closed subgroup of $GL_n(\mathbb{K})$ inherits an algebraic groups structure.
Now we present a result that can be seen somehow as the converse of the previous statement. We need the following natural definition.

Definition 1.1.19. *A morphism of algebraic groups is a group morphism*

$$\phi : G_1 \to G_2$$

that is also a morphism of varieties, that is the pull-back

$$\phi^* : \mathbb{K}[G_2] \to \mathbb{K}[G_1]$$

is a morphism of algebras.
It is an isomporphism if it is also bijective and ϕ^{-1} *is an algebraic group morpism.*

We have a strong characterization of algebraic groups: any algebraic group can be embedded as closed subgroup of a general linear group. This result will be a corollary of the following theorem (known also as Chevalley theorem).

Theorem 1.1.20. *[9, Theorem 5.5]Let* G *be an algebraic group,* $H \leq G$ *a closed subgroup. Then there exists a morphism of algebraic groups* $\phi : G \to GL(V)$ *and a one dimensional subspace* $W \leq V$ *such that* $H = stab_\phi(W) = \{g \in G \mid \phi(g)W = W\}$.

Corollary 1.1.21. *Given an algebraic group* G *over* $\mathbb{K}$*, there exist an injective morphism of algebraic groups from* G *into* $GL_n(\mathbb{K})$ *for some* $n \in \mathbb{N}$.

Proof. Is the Theorem 1.1.20 taking as closed subgroup $H = \{e\}$. □

The following result is a decomposition for elements in an algebraic group, the so called "Jordan decomposition". Thanks to the existence of the embedding in $GL_n(\mathbb{K})$, we will be able to define it in a general algebraic group. The course of reasoning is to define it for $GL_n(\mathbb{K})$ and then give a theorem that guarantee that it makes sense also in the general setting.

Definition 1.1.22. *An element* $X \in GL(V)$ *is called*

- *unipotent if* $(X - id)$ *is nilpotent*
- *semisimple if* X *is diagonalizable*

Proposition 1.1.23. *[9, Proposition 2.2]For any $g \in GL(V)$ there exist unique elements $u, s \in GL(V)$ such that u is unipotent, s is semisimple and it holds $g = su = us$.*

Proof. The proof relies on the existence of the classical additive Jordan decomposition (i.e. $g = s+n$ with s semisimple, n nilpotent and s, n commuting elements). □

Theorem 1.1.24. *[9, Theorem 2.5] Let G be an algebraic group. Then*

1. *for any embedding $\rho : G \to GL(V)$ and for any $g \in G$ there exists unique g_u, g_s such that $\rho(g_u)$ is unipotent, $\rho(g_s)$ is semisimple and $g = g_u g_s = g_s g_u$;*
2. *the elements g_u, g_s do not depend on the chosen embedding;*
3. *for any morphism of algebraic groups $\phi : G \to H$, $\phi(g_u) = \phi(g)_u$ and $\phi(g_s) = \phi(g)_s$.*

The decomposition $g = g_u g_s = g_s g_u$ is called "Jordan decomposition" of g.

This last result allows us to define unipotent and semisimple elements also for a general algebraic group G.

Definition 1.1.25. *Let G be an algebraic group, $g \in G$ with Jordan decomposition $g = g_u g_s$. Then we say that:*

- *g is unipotent if $g = g_u$*
- *g is semisimple if $g = g_s$.*

If all the elements in G are unipotent, we say that G is an "unipotent group".

If $\mathbb{K} = \overline{\mathbb{F}_p}$, this decomposition is the analogue of the p, p'-decomposition for finite group. This connection is made clear in the following remark.

Remark 1.1.26. *Let G be a group, p a prime. Any element of G of finite order can be decomposed uniquely as product of an element of order a power of p (p-element) and an element of order coprime with p (p'-element) that commute with each other. Indeed, if $h \in G$ is an element of order $k = p^\alpha s$ (where p^α is the biggest power of p dividing k, hence s is coprime with p), by Bezout identity there exist $c, b \in \mathbb{Z}$ such that $k = ap^\alpha + cs$. Then $h = (h^{ap^\alpha})(h^{cs})$, and since the order of h^{ap^α} divide s it is prime with p, while the order of h^{cs} divides p^α and so it is a power of p. This decomposition is called the p, p' decomposition of h.*
Let now $K = \overline{\mathbb{F}_p}$, G be an algebraic group over $\overline{\mathbb{F}_p}$, $\phi : G \to GL_n(\overline{\mathbb{F}_p})$ a closed embedding. Since any element of $\overline{\mathbb{F}_p}$ lies in some finite extension of $\mathbb{F}_p$, any element of $GL_n(\overline{\mathbb{F}_p})$ is in some $GL_n(\mathbb{F}_{p^d})$, hence it has finite order; so any of its elements has a p, p' decomposition.
In $GL_n(\overline{\mathbb{F}_p})$, the following equivalences hold:

- *u is unipotent $\Leftrightarrow$ u is a p-element*
  Indeed, if u is unipotent, there exists an $m \in \mathbb{N}$ such that $(u-I)^m = 0$, therefore for any $p^n \geq m$, $0 = (u-I)^{p^n} = u^{p^n} - I$, so the order of u divides p^n and hence is a power of p. Conversely, if $u^{p^n} = I$, it holds $(u-I)^{p^n} = u^{p^n} - I = 0$, so u is an unipotent element.

- *s is semisimple $\Leftrightarrow$ s is a p'-element*
  Indeed, if s is semisimple, it is conjugated in $GL_n(\overline{\mathbb{F}_p})$ to a diagonal matrix $d = diag(a_i)_{1\leq i\leq n}$, therefore its order is $\max\{$ order of a_i in $\overline{\mathbb{F}_p}^* \mid i = 1\dots n\}$ and the order of elements in $\overline{\mathbb{F}_p}^*$ is always coprime with p because for any $a \in \overline{\mathbb{F}_p}$ there is a q, power of p, such that $a^{q-1} - 1 = 0$, and $q-1$ is coprime with p. Conversely, if $s^m = I$ with m coprime with p, then s is semisimple: the minimal polynomial of s divides $x^m - 1$, and since p does not divide m this polynomial has all distinct roots in $\overline{\mathbb{F}_p}^*$ because it is coprime with its derivative.

Hence given an element $g \in G$ the g_u and g_s of the Jordan decomposition of g are actually respectively the p-element and the p'-element of its p, p'-decomposition.

Example 1.1.27. *Let $G = GL_n(\overline{\mathbb{F}_p})$, $g \in G$. Then g is conjugated in $GL_n(\overline{\mathbb{F}_p})$ to some matrix of the form* $A = \begin{pmatrix} J_{x_1}^{(k_1)} & & & \\ & J_{x_2}^{(k_2)} & & \\ & & \ddots & \\ & & & J_{x_n}^{(k_n)} \end{pmatrix}$ *with $x_i \in \overline{\mathbb{F}_p}^*$, where $J_{x_i}^{k_i}$ denotes the Jordan block of dimesion $k_i \times k_i$ with eigenvalues equal to x_i:*

$$J_{x_i}^{(k_i)} = \begin{pmatrix} x_i & 1 & & \\ & x_i & 1 & \\ & & \ddots & \ddots \\ & & & x_i \end{pmatrix}.$$

Then

$$A = A_s A_u = \begin{pmatrix} x_1 I_{k_1} & & & \\ & x_2 I_{k_2} & & \\ & & \ddots & \\ & & & x_n I_{k_n} \end{pmatrix} \begin{pmatrix} U_{x_1^{-1}}^{(k_1)} & & & \\ & U_{x_2^{-1}}^{(k_2)} & & \\ & & \ddots & \\ & & & U_{x_n{}^{-1}}^{(k_n)} \end{pmatrix}$$

where for any $i = 1\dots n$, the matrix $U_{x_i^{-1}}^{(k_i)}$ is the upper unitriangular matrix of dimension $k_i \times k_i$ of the form $U_{x_i^{-1}}^{(k_i)} = \begin{pmatrix} 1 & x_i^{-1} & & \\ & 1 & x_i^{-1} & \\ & & \ddots & \ddots \end{pmatrix}$ *is the Jordan decomposition of A, with A_s the semisimple part, A_u the unipotent one and $A_s A_u = A_u A_s$.*

A_s has order coprime with p, since its order is the least common multiple of the orders of the diagonal blocks, and the i-th diagonal block has the same order as x_i because $(x_i I_{k_i})^h = x_i^h I_{k_i}$ for any $h \in \mathbb{N}$. Any element of $\overline{\mathbb{F}_p}^*$ has always order coprime with p, so any diagonal block has order coprime with p an therefore also A_s has order coprime with p.
A_u has order a power of p, since its order is again the least common multiple of the orders of the diagonal blocks $U^{(k_i)}_{x_i^{-1}}$, and any of this block has order a power of p. Indeed a direct computation shows that

$$(U^{(k_i)}_{x_i^{-1}})^p = \begin{pmatrix} 1 & px_i^{-1} & x_i^{-p} & & \\ & 1 & px_i^{-1} & x_i^{-p} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_i^{-p} & & \\ & 1 & 0 & x_i^{-p} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

and iterating the computation it yields $(U^{(k_i)}_{x_i^{-1}})^{p^{k_i}} = I_{k_i}$. So $U^{(k_i)}_{x_i^{-1}}$ has order a power of p for any $i = 1 \dots n$, and therefore also A_u has order a power of p .
So A_s is the p'-part of A and A_u is the p-part of A.

Before ending this section, we state another consequence of Chevalley Theorem

Corollary 1.1.28. *[9, Proposition 5.7]Let G be an algebraic group, $H \leq G$ a normal closed subgroup. Then the coset space ${}^{G}\!/_{H}$ is an affine variety and an algebraic group (with the usual quotient group structure).*

Moreover, if $H \leq G$ is a closed subgroup, there exists by Theorem 1.1.20 a morphism of algebraic groups $\phi : G \to GL(V)$ such that H is the stabilizer of a line, and we can consider the projective space $\mathbb{P}(V)$ and its element $W \in \mathbb{P}(V)$ corresponding to the line stabilized by H; the action of G on V induces an action of G over $\mathbb{P}(V)$ with action map

$$\begin{aligned} G \times \mathbb{P}(V) &\to \mathbb{P}(V) \\ (g, \bar{v}) &\mapsto \overline{\phi(g)(v)} \end{aligned}$$

(where the overlying bar denotes the equivalence class of the element of V in $\mathbb{P}(V)$) that is also a morphism of algebraic variety. Then ϕ induces a bijection between the left cosets of H and the orbit of W in $\mathbb{P}(V)$ under the action of G,

$$\tilde{\phi} : {}^{G}\!/_{H} \to G.W \subseteq \mathbb{P}(V) \ .$$

An algebraic group acting on an algebraic variety with an action map which is also a morphism of varieties has orbits that are open in their closure [9, Proposition 5.4]), and so the bijection $\tilde{\phi}$ allows us to endow ${}^{G}\!/_{H}$ with a structure of quasi-projective variety. The space ${}^{G}\!/_{H}$ is often referred to as "homogeneus space" or as "quotient space of G by H"

1.1.5 Lie algebra of an algebraic group

We can always functorially attach to an algebraic group a Lie algebra as follows. Recall that, given an algebra A over $\mathbb{K}$, a derivation $D \in Der(A)$ of A is an endomorphism of A that respects the Leibniz rule for the product. The $\mathbb{K}$-vector space $End(A)$ of the endomorphisms of A is a Lie algebra structure with Lie bracket given by the commutator [6, §1.2], that is

$$[f, g] = f \circ g - g \circ f \qquad \text{for} f, g \in End(A).$$

The set of derivation of an algebra is a linear subspace of $End(A)$, and moreover f, g are derivations, then their commutator is a derivation as well:

$$[f, g](ab) = f \circ g(ab) - g \circ f(ab) = f(g(a)b + ag(b)) - g(f(a)b + af(b)) = \\ b(f \circ g - g \circ f)(a) + a(f \circ g - g \circ f)(b) = a[f, g](b) + b[f, g](a)$$

. So $Der(A)$ with bracket given by the commutator is a Lie algebra [6, §1.3].

Definition 1.1.29. *We call Lie algebra of G the space of the "left invariant derivations of $\mathbb{K}[G]$":*

$$Lie(G) = \{D \in Der(\mathbb{K}[G])|\ D \circ \lambda_x = \lambda_x \circ D \text{ for any } x \in G\}$$

where $\lambda_x : \mathbb{K}[G] \to \mathbb{K}[G]$ ($x \in G$) is defined as $\lambda_x(f)(g) = f(x^{-1}g)$ for $f \in \mathbb{K}[G], g \in G$.
The Lie bracket is given by the commutator

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \qquad D_1, D_2 \in Lie(G)$$

Note that this is indeed a Lie algebra since it is a linear subspace of the derivation and if two derivations D_1 and D_2 are left invariant, their commutator is left invariant as well:

$$[D_1, D_2]\lambda_x = D_1 \circ D_2 \circ \lambda_x - D_2 \circ D_1 \circ \lambda_x = D_1 \circ \lambda_x \circ D_2 - D_2 \circ \lambda_x \circ D_1 = \\ \lambda_x \circ D_1 \circ D_2 - \lambda_x \circ D_2 \circ D_1 = \lambda_x \circ [D_1, D_2]$$

The tangent space of an affine variety X at $x \in X$ is a finite dimensional $\mathbb{K}$-vector space that can be defined as

$$T_x(X) = \{\delta : \mathbb{K}[X] \to \mathbb{K} \text{ linear } |\delta(fg) = f(x)\delta(g) + \delta(f)g(x) \quad \text{for any} f, g, \in \mathbb{K}[X]\}$$

For an algebraic group G, we get an isomorphism

$$\Theta : Lie(G) \to T_e(G)$$

defined by

$$\Theta(D)(f) = Df(e),$$

so we can identify the Lie algebra of an algebraic group with its tangent space at the identity.
For any morphism of affine varieties $\phi : X \to Y$ we can define the differential at a point $x \in X$ as the linear map $d_x\phi : T_x(X) \to T_{\phi(x)}(Y)$ defined by $d_x\phi(\delta) = \delta \circ \phi^*$ for any $\delta \in T_x(X)$. Thus the assignment of a Lie algebra to an algebraic group is a functorial construction, since for any morphism of algebraic group $\phi : G \to H$ we can define $d\phi : Lie(G) \to Lie(H)$ considering the previous identification $Lie(G) \cong T_e(G)$ and setting $d\phi := d_e\phi$, and composition and identity are preserved when passing to the differential.

Example 1.1.30. *We write $\mathfrak{gl}_\mathfrak{n}(\mathbb{K})$ for the Lie algebra of the $n \times n$ matrices over $\mathbb{K}$ with Lie bracket given by the commutator: $[X, Y] = XY - YX$ for $X, Y \in M_n(\mathbb{K})$ The Lie algebra of $GL_n(\mathbb{K})$ is $\mathfrak{gl}_\mathfrak{n}(\mathbb{K})$; the isomorphism*

$$\phi : Lie(GL_n(\mathbb{K})) \to \mathfrak{gl}_\mathfrak{n}(\mathbb{K})$$

is given by mapping an $X \in \mathfrak{gl}_\mathfrak{n}(\mathbb{K})$ in $D_X \in Lie(GL_n(\mathbb{K}))$ defined by

$$D_X(T_{i,j}) = \sum_{l=1}^{n} T_{i,l}X_{l,j}.$$

Note that under the identification of $Lie(GL_n(\mathbb{K}))$ with $T_{I_n}(GL_n(\mathbb{K}))$ this derivation is identified with

$$\delta_X(T_{i,j}) = D_X(T_{i,j})(I_n) = X_{i,j}.$$

1.2 Reductive Algebraic Groups

1.2.1 Solvable, Semisimple and Reductive Algebraic Groups

Our aim is to study more in detail the structure of algebraic groups, focusing on a class in particular (the one of "reductive" algebraic groups). In order to do this we give now some definitions.

Definition 1.2.1. *A group is said to be "solvable" if its derived series terminates:*

$$G \trianglerighteq [G, G] := G' \trianglerighteq [G', G'] := G^{(2)} \trianglerighteq \cdots \trianglerighteq G^{(d)} \trianglerighteq \{e\}.$$

We have the following inclusion

Proposition 1.2.2. *[9, Corollary 2.10]If G is an unipotent group, then G is solvable.*

Example 1.2.3.

- *$GL_n(\mathbb{K})$ for $n \geq 2$ is not solvable. Indeed $[GL_n(\mathbb{K}), GL_n(\mathbb{K})] = SL_n(\mathbb{K})$ and $[SL_n(\mathbb{K}), SL_n(\mathbb{K})] = SL_n(\mathbb{K})$. [12, Lemma 4 Corollary 2, §8]*
- *The group $T_n \leq GL_n(\mathbb{K})$ of invertible upper triangular matrices is solvable. Indeed a direct computation of the diagonal element of a commutator shows that $[T_n, T_n] = U_n$, the group of the upper triangular matrices with only ones on the diagonal; since U_n is unipotent, it is solvable.*

The group $T_n \leq GL_n(\mathbb{K})$ of invertible upper triangular matrices of the example 1.2.3 it is an important example of solvable group. Indeed, It can be shown that any connected solvable algebraic group can be embedded in a closed subgroup of T_n; this is a consequence of the Chevalley Theorem 1.1.20 and of the following result.

Proposition 1.2.4. *[9, Corollary 4.2]Any connected solvable subgroup of $GL_n(\mathbb{K})$ is conjugated to a closed subgroup of T_n.*

Now we can define the following subgroups that play a key role in the description of the structure of a group G.

Definition 1.2.5. *Let G be an algebraic group, we call:*

- *"radical of G", denoted by $R(G)$, the largest connected normal solvable subgroup of G*
- *"unipotent radical of G", denoted by $R_u(G)$, the largest closed normal connected unipotent subgroup of G.*

The existence of such subgroups is guaranteed by the fact that the product of normal, connected, solvable (respectively unipotent) groups is a normal, connected, solvable (respetively unipotent) group [10, §6.4.14].

Example 1.2.6. *Let T_n be again the group of upper triangular matrices. This is solvable, hence $R(T_n) = T_n$. Its unipotent radical is U_n, the group of upper triangular matrices with all diagonal terms equal to 1.*

To these definitions are related the following ones

Definition 1.2.7. *Let G be a connected non-trivial algebraic group. Then we say that G is*

- *semisimple if $R(G) = \{e\}$*
- *reductive if $R_u(G) = \{e\}$*

Note that from the inclusion $R_u(G) \leq R(G)$ it follows that a semisimple group is always reductive.

Proposition 1.2.8. *[7, §19.5] Let G be a connected algebraic group. Then*

- *$G/R(G)$ is semisimple;*
- *$G/R_u(G)$ is reductive.*

Example 1.2.9.
- *$GL_n(\mathbb{K})$ is reductive, but not semisimple:*
  - *its radical is*

$$R(GL_n(\mathbb{K})) = \{\lambda I_n | \lambda \in \mathbb{K}^*\}$$

– *its unipotent radical is trivial:*

$$R_u(GL_n(\mathbb{K})) = \{I_n\}$$

- *[9, Example 6.17] $SL_n(\mathbb{K})$ is semisimple.*

1.2.2 Tori and Borel subgroups

Now we introduce some particular subgroups of algebraic groups that will be very helpful for understanding the structure of algebraic groups, in particular of reductive ones.

Definition 1.2.10. *An algebraic group T is called "torus" if $T \cong G_m^r$ for some $r \in \mathbb{N}$.*
A torus T contained in an algebraic group G is said to be maximal if it is not a proper subgroup of another torus.

Note that tori are always abelian groups, since G_m is abelian.

Example 1.2.11. *The subgroup $D_n \leq GL_n(\mathbb{K})$ of invertible diagonal matrices is a maximal torus in $GL_n(\mathbb{K})$. Indeed it is isomorphic to G_m^n via*

$$D_n \to G_m^n$$
$$diag(t_1 \dots t_n) \mapsto (t_1 \dots t_n)$$

where $diag(t_1 \dots t_n) = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$.
It is maximal since if $A \in GL_n(\mathbb{K})$ is contained in a torus containing D_n, it commutes with any diagonal matrix and this implies that A is diagonal. Indeed choose $\lambda \in \mathbb{K}$ different from 1 and for any $1 \leq i \leq n$ let $\Lambda_i = diag(\lambda^{\delta_{i,j}})_{1\leq j\leq n}$ be the diagonal matrix with diagonal terms equal to 1 but the $i-th$; then $A\Lambda_i = \Lambda_i A$ implies $a_{ij} = \lambda a_{ij}$ and $a_{ji} = \lambda^{-1} a_{ij}$ for any $j \neq i$, and therefore $a_{ij} = 0 = a_{ji}$ for any $j \neq i$. So D_n is not a proper subgroup of any torus in $GL_n(\mathbb{K})$.
Any closed connected subgroup of D_n is a torus. [10, Corollary 3.2.7]

Note that tori are groups consisting only of semisimple elements. Indeed saying that a group T is isomorphic to the direct product of copies of G_m is equivalent to say that the embedding ρ in Theorem 1.1.24 can be chosen to have image in the subgroup of the diagonal matrices.

Proposition 1.2.12. *[9, Corollary 6.5] All maximal tori of a connected algebraic group G are G-conjugated.*

A consequence of this fact is that any semisimple element of a connected linear algebraic group is contained in a maximal torus [9, Corollary 6.11].

Example 1.2.13. *In $GL_n(\mathbb{K})$ any maximal torus is conjugated to D_n (as defined in Example 1.2.11)*

The Proposition 1.2.12 allows us to define without ambiguity the "rank" of an algebraic group G as follows: if $T \leq G$ is a maximal torus in G and $T \cong G_m^r$, then $rank(G) = r$.
Now we state a property known as "rigidity of tori". As usual, we define the normalizer of a subgroup $H \leq G$ to be the subgroup $N_G(H) = \{g \in G \mid g^{-1}Hg \subseteq H\}$ of G and the centralizer of H to be $C_G(H) = \{g \in G \mid g^{-1}hg = h \text{ for any } h \in H\}$; they are closed and $C_G(H)$ is normal in $N_G(H)$.

Proposition 1.2.14. *[10, Corollary 3.2.9] Let G be an algebraic group and T a maximal torus of G. Then*

$$N_G(T)^0 = C_G(T)^0$$

and the centralizer of T has finite index in the normalizer of T.

For reductive connected groups, something more can be said:

Proposition 1.2.15. *[9, Corollary 8.13] Let G be a reductive and connected algebraic group. If T is a maximal torus of G, then*

$$C_G(T) = T.$$

Proposition 1.2.14, among other uses, guarantees that the group we are going to define (the "Weyl group"), which is central in the study of reductive groups, is always finite.

Definition 1.2.16. *Let G be an algebraic group, T a maximal torus of G. We call "Weyl group" of G the group*

$$W = {}^{N_G(T)}\!\big/_{C_G(T)}.$$

Note that, since maximal tori are all conjugated in G, choosing different maximal tori gives rise to isomorphic Weyl groups.

Remark 1.2.17. *Let G be an algebraic group, T a maximal torus of G. The Weyl group $W = {}^{N_G(T)}\!\big/_{C_G(T)}$ induces an action on T defined for any $\omega \in W$ by*

$$\omega.t = \dot{\omega}t\dot{\omega}^{-1} \qquad \text{for } \dot{\omega} \text{ a representative of } \omega \text{ in } N_G(T),\ t \in T.$$

This is well defined since taking two different representatives of ω, namely $\dot{\omega}$ and $\ddot{\omega}$, it holds $\ddot{\omega} = \dot{\omega}x$ with $x \in C_G(T)$, hence for any $t \in T$

$$\ddot{\omega}t\ddot{\omega}^{-1} = \dot{\omega}xtx^{-1}\dot{\omega}^{-1} = \dot{\omega}t\dot{\omega}^{-1}.$$

Example 1.2.18. *Let $G = GL_n(\mathbb{K})$ and consider its maximal torus D_n. We want to compute its Weyl group. $GL_n(\mathbb{K})$ is reductive, hence $C_G(D_n) = D_n$.*
The normalizer is given by $N_G(D_n) = \{M \in GL_n(\mathbb{K}) | M^{-1}D_nM = D_n\}$, that is matrices that correspond to base changes that maintain the diagonal form of the diagonal matrices; it is easy to see that this matrices con do nothing else than permute the basis vectors and multiply each of them by a non zero scalar, therefore they are the monomial matrices (i.e. matrices with exactly one non zero entry in any row and column).
Hence the group

$$W = {}^{N_{GL_n(\mathbb{K})}(D_n)}\!\diagup_{(D_n)}$$

is the group of the permutation matrices (i.e. matrices with exactly one 1 *in any row and column, and zeros in any other position), that is*

$$W \cong S_n.$$

W acts on D_n permuting the diagonal element For any $\omega \in W$, continue to indicate by ω the corresponding element of S_n; then for any $t = diag(t_1 \dots t_n) \in D_n$,

$$\omega t \omega^{-1} = diag(t_{\omega(1)} \dots t_{\omega(n)})$$

The maximal tori also play an important role in understanding the structure of solvable groups. Indeed it holds the following result.

Proposition 1.2.19. *[7, §19.3]Let G be a solvable algebraic group, $T \in G$ a maximal torus. Then*

$$G = R_u(G) \rtimes T.$$

This means that in a solvable group G any maximal torus is a complement of the unipotent radical, i.e. any element of G can be written as a product of an element of a torus and a unipotent one, and $R_u(G) \cap T = e$. For instance, consider the group T_n of the upper triangular matrices: its unipotent radical is U_n (see 1.2.3) and a maximal torus is D_n. It is easy to see that the intersection of this two subgroups is trivial and that they indeed decompose the whole T_n.
For reductive groups we do not have a decomposition as nice as for solvable ones. Nevertheless, we have some results.

Theorem 1.2.20. *[9, Proposition 6.20], [9, Theorem 8.22] Let G be a connected reductive group. Then*

- *$R(G) = Z(G)^0$ is a torus*
- *$[G, G]$ is semisimple*
- *$[G, G] \cap R(G)$ is finite*

and $G = R(G)[G, G] = Z(G)^0[G, G]$.

Example 1.2.21. *Consider $GL_n(\mathbb{K})$. We already know that its radical is given by the scalar matrices $\mathbb{K}^*I$, and it coincides with its center. Its derived subgroup is $SL_n(\mathbb{K})$ (Example 1.2.3), which is semisimple (Example 1.2.9). Therefore the intersection $\mathbb{K}^*I \cap SL_n(\mathbb{K})$ of the center and the derived subgroup of $GL_n(\mathbb{K})$ is finite because in $\mathbb{K}$ there are finitely many n^{th}-roots of 1 (n if the characteristic of $\mathbb{K}$ is 0, $\frac{n}{gcd(p,n)}$ if the characteristic of $\mathbb{K}$ is p) hence there are finitely many scalar matrices whose determinant is one.*
Given $A \in GL_n(\mathbb{K})$, let a be a zero of $x^n - det(A)$; then A can be decomposed as the product of a scalar matrix and a matrix whose determinant is equal to 1

$$A = (aI)(a^{-1}A);$$

note that the decomposition is not unique in general since it depends on the choice of a.

Now we introduce the so called "Borel subgroups" of a group G.

Definition 1.2.22. *A subgroup $B \leq G$ is called "Borel subgroup" if it is a maximal closed connected and solvable subgroup of G.*

Example 1.2.23. *The group T_n of the Example 1.2.3 is a Borel subgroup of $GL_n(\mathbb{K})$. Indeed it is a closed connected solvable group, and it is maximal by 1.2.4.*

The following proposition stresses out how Borel subgroups behave under conjugation action by G, and give an insight on the structure of these subgroups.

Proposition 1.2.24. *[9, Theorem 6.12][9, Theorem 6.4] All Borel subgroups of a connected algebraic group G are G-conjugated and self normalizing, that is*

$$N_G(B) = B$$

The proof of the fact that Borel subgroups are G-conjugated relies on the fact that a solvable group acting on a projective variety has a fixed point [1, Theorem 10.4] and using the fact that given a Borel subgroup $B \leq G$, the homogeneus space G/B is a projective variety. Indeed, we can consider a rational representation of G on V as in Chevalley Theorem 1.1.20 in a way that B is the stabilizer of a one dimensional subspace $W \subseteq V$; this rational representation induces an action on the flag variety $\mathcal{F}(V) = \{V_1 \subset V_2 \cdots \subset V_{dim(V)-1} \subset V \mid V_i \text{ are linear subspaces } \}$, which is a projective variety, and by Theorem 1.2.4 B, being solvable, stabilizes a flag $f = (W \subset W_2 \cdots \subset W_n)$; the canonical map from G/B to the orbit of f under G in $\mathcal{F}(V)$ is an isomorphism of varieties [1, Theorem 11.1]. By maximality of B this orbit is minimal and therefore (this is a property of algebraic group action that are also morphism of variety [1, Lemma 1.8]) closed; it follows that G/B is a projective variety.
Then letting another Borel subgroup B' act by left translation on G/B, the fact

that it has a fixed points implies that in is contained in a subgroup conjugated to B, and by the maximality properties of Borel subgroups we have the equality.

Remark 1.2.25. *The above discussion allows us also to give a structure of projective varieties on the set of all Borel subgroup of G*

$$\mathcal{B} = \{B \subseteq G \mid B \text{ is a Borel subgroup of } G\}.$$

Indeed thanks to the properties of Borel subgroups stated in Proposition 1.2.24 the following map is a bijection

$$\begin{aligned} {}^{G}\!/_{B} &\to \mathcal{B}. \\ gB &\mapsto gBg^{-1} \end{aligned}$$

Hence $\mathcal{B}$ inherits a projective variety structure from the homogeneus space ${}^{G}\!/_{B}$.

Example 1.2.26. *Consider $G = GL_2(\mathbb{K})$ and its Borel subgroup T_2.*
We have an isomorphism of projective varieties

$$\begin{aligned} \mathcal{F}(\mathbb{K}^2) &\to \mathbb{P}^1(\mathbb{K}) \\ \langle v \rangle \subseteq \mathbb{K}^2 &\to \bar{v} \end{aligned}$$

T_2 stabilizes the standard flag $(\{0\} \subset \mathbb{K}e_1 \subset \mathbb{K}^2)$, which is the same as saying that it stabilizes $\bar{e_1} \in \mathbb{P}^1(\mathbb{K})$; hence ${}^{GL_2(\mathbb{K})}\!/_{T_2}$ has a projective variety structure given by the bijection

$$\begin{aligned} {}^{GL_2(\mathbb{K})}\!/_{T_2} &\to \mathbb{P}^1(\mathbb{K}) \\ gT_2 &\to g.\bar{e_1} \end{aligned}$$

and the collection of Borel subgroups of $GL_2(\mathbb{K})$ is a one dimensional projective space over $\mathbb{K}$.

Remark 1.2.27. *Observe that since a torus is a connected solvable subgroup of G, it is always contained in a Borel subgroup of G. In particular, this holds also for maximal tori, so (taking into account that Borel subgroups are all conjugated) the maximal tori of G are the ones of its Borel subgroups.*
Moreover, the pairs (T, B), with T being a maximal torus, B a Borel subgroup such that $T \leq B$ are all G-conjugated (where G acts by simultaneous conjugation on the elements of such a pair). Indeed Borel subgroups are all conjugate by Proposition 1.2.24 and maximal tori in a Borel subgroup are conjugated via an element of the Borel: take as group the Borel subgroup in Proposition 1.2.12.

Remark 1.2.28. *Le $B \leq G$ be a Borel subgroup, T a maximal torus of B (hence of G), $U = R_u(B)$ its unipotent radical. Since Borel subgroups are solvable, it holds*

$$B = U \rtimes T.$$

Moreover, the unipotent radicals of the Borel subgroups are the maximal closed connected unipotent subgroups of G (and they are all conjugated).

1.2.3 Root systems

In what follows G will be connected and reductive. For any $x \in G$, there is an automorphism $c_x : G \to G$ defined by $c_x(g) = xgx^{-1}$. We define the *adjoint representation of* G to be the following rational representation of G on $\mathfrak{g} = Lie(G)$

$$\begin{aligned} Ad :& G \to GL(\mathfrak{g}). \\ & x \to dc_x \end{aligned}$$

Let $T \leq G$ be a maximal torus. We denote by $X(T)$ the set of the characters of T, that is the set of the morphisms of algebraic groups $\chi : T \to G_m$. $X(T)$ is an abelian group with the group structure given by $(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t)$ for any $t \in T$. The action of the Weyl group W on the maximal torus T induces an action of the Weyl group on the character group, given by

$$(\omega.\chi)(t) = \chi(\omega^{-1}.t) \qquad\qquad \omega \text{ in } W, t \in T.$$

Since T is a torus, it consists of commuting semisimple elements, therefore (since Ad is a rational representation and using Theorem 1.1.24) so does $Ad(T) \leq GL(\mathfrak{g})$, hence the element of $Ad(T)$ can be simultaneously diagonalized [7, Proposition 15.4]. We set for any $\chi \in X(T)$

$$\mathfrak{g}_\chi = \{v \in \mathfrak{g} \mid Ad(t)v = \chi(t)v\}$$

the T-eigenspace in G relative to the character χ.

Definition 1.2.29. *The "set of roots" of G is*

$$\Phi(G) = \{\alpha \in X(T) \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$$

that is the set of characters of T with non-zero eigenspace on $\mathfrak{g}$.

The action of W on $X(T)$ stabilizes $\Phi(G)$, hence there is an action of the Weyl group of G on the set of roots of G [9, Proposition 8.4].

Example 1.2.30. *Consider the group $GL_n(\mathbb{K})$ and its Lie algebra $\mathfrak{gl}_n(\mathbb{K})$, and the maximal torus D_n of diagonal matrices. It can be computed (via the isomorphism between $\mathfrak{gl}_n(\mathbb{K})$ and $T_{I_n}(GL_n(K))$) that the Adjoint representation is given by the standard conjugation ($Ad(g)X = gXg^{-1}$ for $g \in GL_n(\mathbb{K}), X \in \mathfrak{gl}_n(\mathbb{K})$).*
Then considering the matrices $E_{i,j} = (\delta_{i,m}\delta_{j,l})_{1 \leq m,l \leq n}$ with $1 \leq i \neq j \leq n$ we obtain

$$Ad(diag(t_1 \dots t_n))E_{i,j} = t_i t_j^{-1} E_{i,j}.$$

Hence setting $\chi_{ij}(diag(t_1 \dots t_n)) = t_i t_j^{-1}$, the set of roots of $GL_n(\mathbb{K})$ is given by

$$\Phi(GL_n(\mathbb{K})) = \{\chi_{ij} \in X(D_n) \mid 1 \leq i \neq j \leq n\}$$

We have the following structure theorem for connected reductive groups

Theorem 1.2.31. *[9, Theorem 8.17] Let G be a connected reductive algebraic group, $T \leq G$ a maximal torus, $\mathfrak{g} = Lie(G)$ and $\Phi = \Phi(G)$. Then for any $\alpha \in \Phi(G)$, $\mathfrak{g}_\alpha$ is one dimensional and there exists a unique unipotent one-dimensional subgroup U_α of G normalized by T and such that $Lie(U_\alpha) = \mathfrak{g}_\alpha$.*
The group G is generated by these unipotent subgroups together with the chosen maximal torus, that is

$$G = \langle T, U_\alpha \mid \alpha \in \Phi \rangle$$

The U_α of the above theorem are called root subgroups of G (and the eigenspaces $\mathfrak{g}_\alpha$ are called root subspaces of $\mathfrak{g}$).

Example 1.2.32. *Consider $GL_n(\mathbb{K})$ with maximal torus D_n as in Example 1.2.30. Then the root subspace relative to χ_{ij} is*

$$(\mathfrak{gl}_n)_{i,j} = \langle E_{i,j} \rangle_{\mathbb{K}}$$

and the root subgroup is

$$U_{i,j} = I_n + \langle E_{i,j} \rangle_{\mathbb{K}}.$$

The set $\phi(G)$ has the following abstract structure.

Definition 1.2.33. *A subset Φ of a finite dimensional Euclidean vector space E is called abstract root system in E if*

- *Φ is finite, $\Phi \neq 0$, $\langle \Phi \rangle = E$*
- *If $c \in \mathbb{R}$ is such that $c\alpha, \alpha \in \Phi$, then $c \in \{1, -1\}$*
- *For any $\alpha \in \Phi$ the reflection $s_\alpha \in GL(E)$ along α stabilizes Φ.*
- *for any $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$.*

The group $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ is called the Weyl group of Φ

It turns out that $\Phi(G) \subseteq \Phi(G) \otimes_{\mathbb{Z}} \mathbb{R}$ is a root system with Weyl group W isomorphic to the Weyl group of G. Calling $C_\alpha = N_G(ker(\alpha)^0)$, the reflections $s_\alpha \in W$ (with $\alpha \in \Phi(G)$) corresponds to the class in ${N_G(T)}/{C_G(T)}$ of elements in $N_{C_\alpha}(T) \setminus C_{C_\alpha}(T)$ [9, Proposition 9.3].
From the theory on abstract root systems we have notions of base, commonly denoted by Δ, and set of positive roots with respect to Δ, usually denoted by Φ^+. Chosen a base Δ one can show that the Weyl group W is generated by the set $S = \{s_\alpha \mid \alpha \in \Delta\}$, called "simple reflections". We call length of an element $\omega \in W$ (denoted by $l(\omega)$) the minimal length of a word with letters is S that is equal to ω in W.

Example 1.2.34. *Consider $GL_n(\mathbb{K})$ with maximal torus D_n as in Example 1.2.30. Its Weyl group W of $GL_n(\mathbb{K})$, is isomorphic to the symmetric group S_n. Any element $\sigma \in S_n$ acts on the set of roots via*

$$\sigma(\chi_{ij}) = \chi_{\sigma(i),\sigma(j)}.$$

The reflection $s_{i,j}$ relative to the root χ_{ij} is indeed the (i,j) transposition: $ker(\chi_{ij})^0$ is the set of diagonal matrices with same i-th and j-th entries, therefore the elements of its centralizer that normalize but do not centralize D_n are in the same class as the (i,j) transposition matrix in W.

Positive systems are deeply linked to Borel subgroups.

Proposition 1.2.35. *Let G be a connected reductive algebraic group, T a maximal torus of G. Let B be a Borel subgroup of G containing T. Then there exists a base Δ, with respective positive system Φ^+, such that*

$$B = \langle T, U_\alpha \mid \alpha \in \Phi^+ \rangle$$

Note that $R_u(B) = \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$.

Example 1.2.36. *Consider $GL_n(\mathbb{K})$ with maximal torus D_n as in Example 1.2.30. Consider the Borel subgroup T_n of the upper triangular matrices. Then*

$$\Delta = \{\chi_{i,i+1} \mid 1 \leq i \leq n\}$$

is a base with relative positive system

$$\Phi^+ = \{\chi_{ij} \mid 1 \leq i < j \leq n\}$$

that satisfies

$$T_n = \langle D_n, U_{i,j} \mid 1 \leq i < j \leq n \rangle.$$

The unipotent radical of T_n consists of the upper unitriangular matrices and it is indeed generated by the root subaspaces contained in it.
The simple reflections relative to this basis are the permutation matrices relative to the transpositions of adjacent indices.

1.2.4 Root data and dual groups

In the previous section we showed how to any reductive algebraic group can be associated a root system. Differently from what happens for semisimple Lie algebras, this combinatorial datum is not enough to characterize reductive algebraic group; nevertheless, taking a slightly more complete datum (called root datum) allows one to classify algebraic reductive group.

Definition 1.2.37. *A root datum is a quadruple $\Psi = (X, \Phi, Y, \Phi^\vee)$ that satisfies the following conditions:*

1. *X, Y are free abelian groups of finite rank and there exists a non degenerate bilinear map $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$ such that*

$$\langle \cdot, \cdot \rangle : X \to Hom(Y, \mathbb{Z}) \tag{1.1}$$
$$\chi \to \langle \chi, \cdot \rangle \tag{1.2}$$

$$\langle \cdot, \cdot \rangle : Y \to Hom(X, \mathbb{Z}) \tag{1.3}$$
$$\gamma \to \langle \cdot, \gamma \rangle \tag{1.4}$$

are isomorphisms (i.e. $\langle \cdot, \cdot \rangle$ is a perfect pairing between X and Y).

2. *Φ is a finite subset of X, $\Phi^\vee$ is a finite subset of Y and there exists a bijection*

$$\Phi \to \Phi^\vee$$
$$\alpha \to \alpha^\vee$$

such that $\langle \alpha, \alpha^\vee \rangle = 2$

3. *for every $\alpha \in \Phi$, the maps $s_\alpha : X \to X$, $s_{\alpha^\vee} : Y \to Y$ defined by*

$$s_\alpha.\chi = \chi - \langle \chi, \alpha^\vee \rangle \alpha \qquad \text{for all } \chi \in X$$
$$s_{\alpha^\vee}.\gamma = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee \qquad \text{for all } \gamma \in Y$$

are such that $s_\alpha(\Phi) = \Phi$, $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$

Note that this is an enhancement of the concept of root system, since if $\Psi = (X, \Phi, Y, \Phi^\vee)$ is a root datum, Φ is a root system in the subspace in X spanned by Φ, $\Phi^\vee$ is a root system in the subspace in Y spanned by $\Phi^\vee$ [8, Proposition 1.2.5].

Definition 1.2.38. *Let $\Psi = (X, \Phi, Y, \Phi^\vee)$ and $\Psi' = (X', \Phi', Y', \Phi'^\vee)$ be root data with paring respectively $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$. If $\delta : X' \to X$ a group homomorphism, the corresponding transpose map is the map $\delta^\vee : Y \to Y'$ defined by*

$$\langle \delta(\chi'), \gamma \rangle = \langle \chi', \delta^\vee(\gamma) \rangle'$$

for any $\chi' \in X', \gamma \in Y$.

Remark 1.2.39. *Note that if $\Psi = (X, \Phi, Y, \Phi^\vee)$ is a root datum, for any $\alpha \in \Phi$ and for $\chi \in X, \gamma \in Y$ it holds*

$$\langle s_\alpha(\chi), \gamma \rangle = \langle \chi - \langle \chi, \alpha^\vee \rangle \alpha, \gamma \rangle = \langle \chi, \gamma \rangle - \langle \chi, \alpha^\vee \rangle \langle \alpha, \gamma \rangle = \langle \chi, s_\alpha^\vee(\gamma) \rangle$$

That is, $s_{\alpha^\vee}$ as in Definition 1.2.37 is the transpose of s_α.
This implies that the map $^\vee : W_\Phi \to W_{\Phi^\vee}$ that maps any element of the Weyl group of Φ in its transpose, that is an element of the Weyl group of $\Phi^\vee$, is an anti-isomorphism.

Thanks to Remark 1.2.39, we can refer without ambiguity to W_Φ as "the Weyl group of Ψ".

Definition 1.2.40. *Let $\Psi = (X, \Phi, Y, \Phi^\vee)$ and $\Psi' = (X', \Phi', Y', \Phi'^\vee)$ be root data with paring respectively $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$'. A group homomorphism $\delta : X' \to X$ is an homomorphism of root data if it maps bijectively Φ' onto Φ and for any $\alpha' \in \Phi'$ $\delta^\vee(\delta(\alpha')^\vee) = \alpha'^\vee$. If $\delta : X' \to X$ is also an isomorphism of groups, δ is called isomorphism of root data.*

We say that two root data are isomorphic if there exists an isomorphism of root data between them.

Remark 1.2.41. *The definition of isomorphism of root data can be reformulated as follows.*
Let $\Psi = (X, \Phi, Y, \Phi^\vee)$ and $\Psi' = (X', \Phi', Y', \Phi'^\vee)$ be root data with paring respectively $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$'. They are said to be isomorphic if there exist two maps $\delta : X \to X'$, $\epsilon : Y \to Y'$ such that

- *δ, ϵ are isomorphisms of abelian groups;*
- *$\langle \delta(\chi), \epsilon(\gamma) \rangle' = \langle \chi, \gamma \rangle$ for any $\chi \in X, \gamma \in Y$;*
- *$\delta(\Phi) = \Phi'$, $\epsilon(\Phi^\vee) = \Phi'^\vee$;*
- *$\epsilon(\alpha^\vee) = \delta(\alpha)^\vee$.*

Taking $\epsilon = (\delta^\vee)^{-1}$ yields the equivalence between the two definitions.

Now we see how this concepts are related to algebraic groups.
Let T be a torus. We denote by $Y(T)$ the set of the cocharacters of T, that is the set of the morphisms of algebraic groups $\gamma : G_m \to T$. $Y(T)$ is an abelian group with the group structure given by $(\gamma_1 + \gamma_2)(\lambda) = \gamma_1(\lambda)\gamma_2(\lambda)$ for $\gamma_1, \gamma_2 \in Y(T), \lambda \in G_m$. There is an action of the Weyl group W of G on the cocharacter group, given by

$$(\omega.\gamma)(\lambda) = \omega.(\gamma(\lambda)) \qquad\qquad \omega \text{ in } W, \lambda \in G_m.$$

Definition 1.2.42. *We denote by $\langle \cdot, \cdot \rangle$ the bilinear map*

$$\langle \cdot, \cdot \rangle : X(T) \times Y(T) \to \mathbb{Z}$$

defined as follows: for any $\chi \in X(T), \gamma \in Y(T)$ and for any $\lambda \in G_m$,

$$\chi \circ \gamma(\lambda) = \lambda^{\langle \chi, \gamma \rangle}.$$

This definition makes sense since the composition of a character and a cocharacter is an automorphism of G_m, and the only automorphisms of G_m are the maps that act by rising the elements to some integer power [2, §1.9].

Proposition 1.2.43. *[9, Proposition 3.6] Let T be a torus, $X(T)$ the character group and $Y(T)$ the cocharacter group. Then $\langle\cdot,\cdot\rangle$ is a perfect pairing between $X(T)$ and $Y(T)$*

$$\langle\cdot,\cdot\rangle : X(T) \to Hom(Y(T), \mathbb{Z}) \tag{1.5}$$
$$\chi \to \langle\chi,\cdot\rangle \tag{1.6}$$

$$\langle\cdot,\cdot\rangle : Y(T) \to Hom(X(T), \mathbb{Z}) \tag{1.7}$$
$$\gamma \to \langle\cdot,\gamma\rangle \tag{1.8}$$

Are isomorphisms.

Lemma 1.2.44. *[2, Proposition 3.1.1] Let T be a torus. The following maps are isomorphisms of abelian groups*

$$\begin{aligned} Y(T)\otimes_{\mathbb{Z}}\mathbb{K}^* &\to Hom(X(T),\mathbb{K}^*) \\ \gamma\otimes\lambda &\mapsto (\chi\mapsto\lambda^{\langle\chi,\gamma\rangle}) \\ X(T)\otimes_{\mathbb{Z}}\mathbb{K}^* &\to Hom(Y(T),\mathbb{K}^*) \\ \chi\otimes\lambda &\mapsto (\gamma\mapsto\lambda^{\langle\chi,\gamma\rangle}) \end{aligned}$$

Proposition 1.2.45. *[2, §1.9] Let G be a connected reductive group, $T \leq G$ a maximal torus, $\Phi(G)$ be the set of roots of G. For any $\alpha \in \Phi(G)$ there exists an unique $\alpha^\vee \in Y(T)$ such that for any $\chi \in X(T)$, $s_\alpha.\chi = \chi - \langle\chi,\alpha^\vee\rangle\alpha$.*

The element $\alpha^\vee$ of Proposition 1.2.45 is called *coroot corresponding to the root* α. Note that is particular

$$\langle\alpha,\alpha^\vee\rangle\alpha = \alpha - s_\alpha.\alpha = 2\alpha$$

that is $\langle\alpha,\alpha^\vee\rangle$=2.

Definition 1.2.46. *Let G be a connected reductive algebraic group with root system $\Phi(G)$. The "set of coroots" of G is*

$$\Phi^\vee(G) = \{\alpha^\vee \in Y(T) \mid \alpha \in \Phi(G)\}$$

Proposition 1.2.47. *[8, Lemma 1.2.15] Let G be a connected reductive algebraic group, $T \leq G$ a maximal torus. Then the quadruple*

$$\Psi(G) = (X(T), \Phi(G), Y(T), \Phi^\vee(G))$$

is a root datum, with Weyl group the Weyl group of G.

Example 1.2.48. *Let $G = GL_n(\mathbb{K})$, $T = D_n$ the maximal torus of diagonal matrices.*

We saw in Example 1.2.30 that a root system for G with respect to the maximal torus $T = D_n$ is given by

$$\Phi = \{\chi_{ij} | 1 \geq i \neq j \leq n\}$$

where $\chi_{ij}(t) = t_i t_j^{-1}$ for $t = diag(t_1, \ldots, t_n) \in T$, and it has Weyl group isomorphic to the symmetric group S_n, with reflection relative to the root χ_{ij} given by the permutation $s_{ij} = (i, j)$.
Since the map

$$T \to G_m^n$$
$$diag(t_1, \ldots t_n) \to (t_1, \ldots t_n)$$

is an isomorphism, the character group of G is

$$X(T) = \langle \chi_i \mid 1 \leq i \leq n \rangle_{\mathbb{Z}}$$

the free abelian group generated by

$$\chi_i : T \to G_m.$$
$$diag(t_1 \ldots t_n) \to t_i$$

With respect for this basis, $\chi_{ij} = \chi_i - \chi_j$. For analogous reasons the cocharacters group of G is

$$Y(T) = \langle \gamma_i \mid 1 \leq i \leq n \rangle_{\mathbb{Z}},$$

where the γ_i are the cocharacter mapping $\lambda \in G_m$ in the diagonal matrices on which diagonal appear only ones but for the $(i, i) - th$ position, where there is λ (e.g. $\gamma_1(\lambda) = diag(\lambda, 1, 1, \ldots, 1)$, $\gamma_2 = diag(1, \lambda, 1, \ldots, 1)$ etc). Denoting as usual by $\langle \cdot, \cdot \rangle$ the paring between charcater and cocharacter, we have

$$\chi_i \circ \gamma_j(\lambda) = \lambda^{\delta_{ij}} = \lambda^{\langle \chi_i, \gamma_j \rangle}$$

Hence

$$\langle \chi_i, \gamma_j \rangle = \delta_{ij}$$

(i.e., $\{\chi_i \ |i = 1 \ldots n\}$, $\{\gamma_j \ |j = 1 \ldots n\}$ are dual base).
Hence the coroots of Φ are given by

$$\Phi^\vee = \{\gamma_{ij} | 1 \leq i \neq j \leq n\}$$

where $\gamma_{ij} = \gamma_i - \gamma_j$ (that is, for $\lambda \in G_m$ $\gamma_{ij}(\lambda)$ is the diagonal matrix having λ as $(i, i) - th$ component, λ^{-1} as $(j, j) - th$ component and ones everywhere else on the diagonal). Indeed

$$\langle \chi_k, \gamma_{ij} \rangle = (-1)^{\delta_{ik}} - (-1)^{\delta_{jk}} = \begin{cases} 0 \text{ for } k \neq i, k \neq j \\ 1 \text{ for } k = i \\ -1 \text{ for } k = j \end{cases}$$

for any $1 \leq k, i, j \leq n$*, therefore it holds*

$$s_{ij}\chi_k = \begin{cases} \chi_k, & \textit{for } k \neq i, k \neq j \\ \chi_j, & \textit{for } k = i \\ \chi_i, & \textit{for } k = j \end{cases}$$
$$= \chi_k - \langle \chi_k, \gamma ij \rangle \chi_{ij}.$$

Hence the root datum of G *is*

$$\Psi = (\{\chi_i - \chi_j | 1 \leq i \neq j \leq n\}, \ \langle \chi_k | 1 \leq k \leq n \rangle_{\mathbb{Z}}, \\ \{\gamma_i - \gamma_j | 1 \leq i \neq j \leq n\}, \ \langle \gamma_k | 1 \leq k \leq n \rangle_{\mathbb{Z}}).$$

By Proposition 1.2.47, to any connected reductive group G with a maximal torus $T \leq G$ can be associated a root datum $\Psi(G)$. This assignment depends in fact only on the isomorphism class of the algebraic group. Indeed let G_1, G_2 be connected reductive algebraic groups with root data $\Psi(G_1), \Psi(G_2)$ with respect to the maximal tori $T_1 \leq G_1$ and $T_2 \leq G_2$, and assume that $\phi : G_2 \to G_1$ is an isomorphism of algebraic groups mapping T_2 in T_1. Then ϕ induces an isomorphism of groups $f : X(T_1) \to X(T_2)$ between the character groups given by $f(\chi) = \chi \circ \phi$, and this defines an isomorphism of root data $\Psi_1 \to \Psi_2$ [10, §9.6.1]. In particular, it follows that the root datum of a group G does not depend on the choice of the maximal torus (since by Proposition 1.2.12, maximal tori in an algebraic group are all conjugated). Conversely, if two algebraic groups have isomorphic root datum, they are isomorphic.

Theorem 1.2.49. *[10, Theorem 9.6.2] Let* G_1*,* G_2 *be connected reductive algebraic groups with isomorphic root data respectively* $\Psi(G_1), \Psi(G_2)$*. Then* G_1 *and* G_2 *are isomorphic as algebraic groups.*

Furthermore, the following existence result holds

Theorem 1.2.50. *[10, Theorem 10.1.1]Le* Ψ *be a root datum. Then there exists a connected reductive algebraic group* G *with maximal torus* T *with root datum* $\Psi(G) = (X(T), \Phi, Y(T), \Phi^\vee)$ *isomorphic to* Ψ*.*

Adding up the previous results, we have seen that root data classify completely connected reductive algebraic groups: there is a bijection

$$\{\text{connected reductive algebraic groups}\}/\sim \longrightarrow \{\text{root data}\}/\sim \qquad (1.9)$$

Where by $\sim$ we mean the equivalence relation given by isomorphisms of the objects (algebraic groups on the left, root data on the right).
This classification allows us to define the *dual group* of a reductive connected algebraic group.

Definition 1.2.51. *If* $\Psi = (X, \Phi, Y, \Phi^\vee)$ *is a root datum, the quadruple* $\Psi^* = (Y, \Phi^\vee, X, \Phi)$ *is called the root datum dual to* Ψ*.*

The fact that in the definition of root datum the roles of (X, ϕ) and $(Y, \phi^\vee)$ can be interchanged yields the fact that if Ψ is a root datum, so is Ψ^*, therefore the Definition 1.2.51 is coherent. Note that the dual of a root datum is defined up to isomorphism: any root datum isomorphic to Ψ^* is a root datum dual to Ψ.

Remark 1.2.52. *If $\Psi = (X, \Phi, Y, \Phi^\vee)$ is a root datum with perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$, the dual $\Psi^* = (Y, \Phi^\vee, X, \Phi)$ root datum has perfect pairing $\langle \cdot, \cdot \rangle^\vee : Y \times X \to \mathbb{Z}$ given by $\langle \gamma, \chi \rangle^\vee = \langle \chi, \gamma \rangle$ for any $\chi \in X, \gamma \in Y$*

Definition 1.2.53. *Two connected reductive algebraic groups G, G^* are said to be dual if their root data are dual to each other.*

Therefore by the correspondence (1.9) any algebraic group G has a dual group G^*, and it is unique up to isomorphism.

Remark 1.2.54. *In a more explict way, G and G^* are dual to each other if, given $T \leq G, T^* \leq G^*$ maximal tori, there exists an isomorphism $\delta : X(T) \to Y(T^*)$ between the character group of G and the cocharacter group of G^* mapping the roots of G in the coroots of G^* and satisfying $\delta^\vee(\delta(\alpha)^\vee) = \alpha^\vee$ for any α root of G.*

Remark 1.2.55. *[2, Proposition 4.2.3] Let G and G^* be connected reductive groups with Weyl groups respectively W, W^* and assume G, G^* are dual to each other. Then there exists an isomorphism of root data between the root datum $\Psi(G)$ of G and the dual root datum $\Psi(G^*)^*$ of G^* , therefore there exists an isomorphism of groups $\delta : X^* \to Y$ inducing this root data isomorphism. This induces an isomorphisms between the Weyl group of the root system (X^*, Φ^*) and the Weyl group of the (co)root system $(Y, \Phi^\vee)$, and by Remark 1.2.39 there is an anti-isomorphism given by taking the transpose between the Weyl group of the root system (X, Φ) and the one of $(Y, \Phi^\vee)$. Hence we can induce an anti-isomorphism of the Weyl group of G and the one of G^*, denoted by $\delta : W \to W^*$, mapping s_α in $s_{\delta(\alpha)}$ and such that*

$$\delta(\omega.\chi) = \delta(\omega).\delta(\chi)$$

for any $\omega \in W, \chi \in X$. .

Example 1.2.56. *The algebraic group $G = GL_n(\mathbb{K})$ is the dual of itself. Indeed recall from Example 1.2.48 that the root datum $\Psi = (\Phi, X(T), \Phi^\vee, Y(T))$ of $GL_n(\mathbb{K})$ is given by*

$$\begin{aligned} \Phi &= \{\chi_i - \chi_j | 1 \leq i \neq j \leq n\}, \\ X(T) &= \langle \chi_k | 1 \leq k \leq n \rangle_{\mathbb{Z}}, \\ \Phi^\vee &= \{\gamma_i - \gamma_j | 1 \leq i \neq j \leq n\}, \\ Y(T) &= \langle \gamma_k | 1 \leq k \leq n \rangle_{\mathbb{Z}}. \end{aligned}$$

Where $\chi_i(t) = t_i$ for $t = diag(t_j)_{1 \leq j \leq n} \in T$, $\gamma_i(\lambda) = diag(\lambda^{\delta_{i,j}})_{1 \leq j \leq n}$.
Then the dual datum is given by

$$\Psi^* = (\Phi^\vee, Y(T), \Phi, X(T))$$

With $\Phi^\vee, Y(T), \Phi, X(T)$ *defined as above. But the group isomorphism*

$$\delta : Y(T) \to X(T)$$
$$\gamma_i \to \chi_i$$

defines an isomorphism between these root data.
Indeed, its transpose $\delta^\vee : Y(T) \to X(T)$ *is defined by*

$$\langle \delta(\gamma_i), \gamma_j \rangle = \langle \gamma_i, \delta^\vee(\gamma_j) \rangle^\vee,$$

hence $\delta^\vee = \delta$, *and* $\delta(\gamma_i - \gamma_j) = \chi_i - \chi_j \in \Phi$ *for any* $\gamma_i - \gamma(j) \in \Phi^\vee$.

1.2.5 Bruhat decomposition

We now give a decomposition of a connected reductive algebraic group G in double cosets of a Borel subgroup $B \leq G$. In order to do this, we introduce the following group-theoretic definition:

Definition 1.2.57. *A pair* (B, N) *of subgroups of a group* G *is called* BN*-pair for* G *(or "Tits system") if it satisfies the following conditions:*

1. $G = \langle B, N \rangle$.
2. $B \cap N$ *is normal in* N.
3. *the group* $W = {}^{N}\!/_{B \cap N}$ *is generated by a set of involutions* S.
4. $BnB \cdot B\dot{s}B \subseteq Bn\dot{s}B \bigcup BnB$, *where* $\dot{s} \in N$ *is a lifting of an involution* $s \in S \subseteq W$ *in* N *and* $n \in N$.
5. $B \neq \dot{s}B\dot{s}$ *with* $\dot{s}$ *as before.*

The group W *is called "the Weyl group" of the* BN*-pair.*

Definition 1.2.58. *Let* G *be a group and let* (B, N) *be a* BN*-pair for* G *with Weyl group* W *and set of generating involution* S*. For any* $\omega \in W$*, the length of* ω *is the minimal number of elements in* S *needed to write* ω *as product of elements in* S *and it is denoted by* $l(\omega)$.

Theorem 1.2.59. *[3, Proposition 3.1.2] If* G *is a connected reductive algebraic group,* $B \leq G$ *is a Borel subgroup and* $T \leq B$ *is a maximal torus, then the pair* $(B, N_G(T))$ *is a* BN*-pair for* G*, with Weyl group* ${}^{N_G(T)}\!/_{T}$ *and as set of generating involutions the simple reflection relative to the base associated to the chosen Borel subgroup.*

This structure allows us to get the so-called Bruhat decomposition for reductive algebraic groups.

Theorem 1.2.60. *Let G be a group with a BN-pair (B,N). Then*

$$G = \bigsqcup_{\omega \in W} B\dot{\omega}B$$

where $\dot{\omega} \in N$ is any lifting of $\omega \in W$.

Proof. Two double cosets are identical or disjoint; we prove that the double cosets are the same if and only if $\dot{\omega}, \dot{\upsilon}$ are representatives of the same element in W.
First, observe that if $\dot{\omega}$ and $\ddot{\omega}$ are representatives of the same element in W, then $\dot{\omega} = \ddot{\omega}t$ for some $t \in B \cap N$; therefore, since $B \cap N \subseteq B$, $B\dot{\omega}B = B\ddot{\omega}tB = B\ddot{\omega}B$. Hence the coset is independent of the chosen representative, so we can write just $B\omega B$ with $\omega \in W$.
Now we prove that if $B\omega B = B\upsilon B$ with $\omega, \upsilon \in W$, then $\omega = \upsilon$. We assume $l(\upsilon) \leq l(\omega)$, and we proceed by induction on $l(\upsilon)$.
If $l(\upsilon) = 0$, then $\upsilon = 1$. If $B = B\omega B$, then any representative of ω must lie in B, so $\omega = 1$ in W.
If $l(\upsilon) > 0$, then we can write $\upsilon = \upsilon' s$ with $s \in S$ and $l(\upsilon') \leq l(\upsilon)$. Then since $B\upsilon' s = B\upsilon \subseteq B\upsilon B = B\omega B$, we have by the fourth conditions on BN-pairs that

$$B\upsilon' \subseteq B\omega Bs \subseteq B\omega B \cdot BsB \subseteq B\omega sB \cup B\omega B$$

hence $B\upsilon' B = B\omega sB$ or $B\upsilon' B = B\omega B$. Since $l(\upsilon') < l(\upsilon)$ it follows by induction that either $\upsilon' = \omega s$ or $\upsilon' = \omega$, but since $l(\upsilon') < l(\upsilon) \leq l(\omega)$ we get $\upsilon' = \omega s$, hence $\upsilon = \upsilon' s = \omega s^2 = \omega$.
On the other hand, again with an induction on the length it can be proved that the union of these double cosets is closed under multiplication; then it is the whole group G, since B and N generates G and they both are contained in the double cosets union. □

In particular, the previous decomposition holds for connected reductive algebraic groups, with B being a chosen Borel subgroup, $N = N_G(T)$ the normalizer in G of a maximal torus contained in B and $W = {}^{N_G(T)}\!/_{T}$ the Weyl group of G.

Remark 1.2.61. *If $G = GL_n(\mathbb{K})$, the Bruhat decomposition is obtained by Gaussian elimination: by left multiplication with a upper triangular matrix b_1, any invertible matrix g can be made into a matrix with all the pivot in different rows, and the latter can be made upper triangular by left multiplication by some permutation matrix $\dot{\omega}$. Hence $\dot{\omega}b_1 g \in T_n$, so $g \in T_n\dot{\omega}T_n$ for some $\omega \in W$.*

This allows us to parameterize by W the orbits of a G-action on the projective variety of the Borel subgroups $\mathcal{B}$ for G connected reductive group. Indeed, remember that the structure on $\mathcal{B}$ is given via the bijection ${}^{G}\!/_{B} \to \mathcal{B}$, $gB \mapsto gBg^{-1}$. Consider then the action of G on ${}^{G}\!/_{B}$ given by left multiplication, meaning $x.gB = xgB$. Note that, like for double cosets (see the proof of Theorem 1.2.60), the coset $\dot{\omega}B \in {}^{G}\!/_{B}$ does not depend on the choice of the representative $\dot{\omega} \in N_G(T)$ of $\omega \in W$: if $\dot{\omega}$ and $\ddot{\omega}$ are two representatives in $N_G(T)$ of the same element $\omega \in W$, there is an $x \in T$ such that $\ddot{\omega} = \dot{\omega}x$ and therefore since $T \subseteq B$ we have $\ddot{\omega}B = \dot{\omega}xB = \dot{\omega}B$. So the coset ωB with $\omega \in W$ is a well defined set.

Theorem 1.2.62. *Let G be a connected reductive algebraic group, $B \leq G$ be a Borel subgroup. Consider the G- action via left multiplication on both components of ${}^{G}\!/_{B} \times {}^{G}\!/_{B}$. Then there is a bijection*

$$\begin{aligned} W &\to \{G - \textit{orbits on } {}^{G}\!/_{B} \times {}^{G}\!/_{B}\} \\ \omega &\longmapsto G.(B, \omega B) \end{aligned}$$

Proof. The map is surjective. Let $(g_1B, g_2B) \in {}^{G}\!/_{B} \times {}^{G}\!/_{B}$; it lies in the same G-orbit as $(B, g_1^{-1}g_2B)$. By Bruhat decomposition, there exists a $\omega \in W$ such that $g_1^{-1}g_2$ lies in $B\omega B$. Then $(B, g_1^{-1}g_2B)$ lies in the same G orbit as $(B, \omega B)$; hence any G-orbit contains an element of the form $(B, \omega B)$ for $\omega \in W$.
The map is injective. Suppose $\omega, \omega' \in W$ and assume that $(B, \omega B)$ lies in the same G-orbit as $(B, \omega' B)$. Then there exists a $g \in G$ such that $(B, \omega B) = (gB, g\omega' B)$ that means $gB = B$, hence $g \in B$, and therefore $g\omega' B = \omega B$ implies $\omega' \in B\omega B$, hence by disjointedness of the double cosets in the Bruhat decomposition it follows $\omega = \omega'$. Hence different $\omega \in W$ give rise to different G-orbits. □

The previous action can be read, thanks to the bijection we recalled before, instead that on the homogeneous space ${}^{G}\!/_{B}$, on variety $\mathcal{B}$ of the Borel subgroups; hence we can restate the previous result as follows

Corollary 1.2.63. *Let G be acting on $\mathcal{B} \times \mathcal{B}$ via $g.(B_1, B_2) = (gB_1g^{-1}, gB_2g^{-1})$ for $g \in G$, $B_1, B_2 \in \mathcal{B}$. Then the following map*

$$\begin{aligned} W &\to \{G - \textit{orbits on } \mathcal{B} \times \mathcal{B}\} \\ \omega &\longmapsto G.(B, \omega B \omega^{-1}) \end{aligned}$$

is a bijection

Definition 1.2.64. *In the notation of Corollary 1.2.63, the G-orbit $G.(B, \omega B \omega^{-1})$ is denoted by $\mathcal{O}(w)$.*
If $(B_1, B_2) \in \mathcal{O}(w)$, we say that B_1 and B_2 are in "relative position ω", and we write $B_1 \xrightarrow{\omega} B_2$

The above definition is intrinsic, meaning it does not depend on the choice of the Borel subgroup B and the maximal torus T (hence from the explicit realization of the Weyl group) accordingly to the following result

Proposition 1.2.65. *Let $B, \tilde{B}$ be Borel subgroups of G and $T, \tilde{T}$ be maximal tori contained respectively in B and $\tilde{B}$, and let $g \in G$ such that*

$$\tilde{T} = gTg^{-1} \ , \ \tilde{B} = gBg^{-1}.$$

Then

1. *The induced isomorphism between the realizations of the Weyl group*

$$\sim : {}^{N_G(T)}\!/_{T} \to {}^{N_G(\tilde{T})}\!/_{\tilde{T}}$$

does not depend on the choice of g in point (1).

2. *$B_1, B_2 \in \mathcal{B}$ are in relative position $\omega \in {}^{N_G(T)}\!/_{T}$ with respect to (T, B) if and only if they are in relative position $\tilde{\omega} \in {}^{N_G(\tilde{T})}\!/_{\tilde{T}}$ with respect to $(\tilde{T}, \tilde{B})$.*

Proof. 1. Note that there exists a $g \in G$ as in the statement by Remark 1.2.27. To show that the isomorphism $\sim$ does not depend on the choice of g it suffices to prove it in the case $B = \tilde{B}, T = \tilde{T}$. Then $N_G(T) \cap B = T$, hence $g \in T$ and so it acts trivially via conjugation on the Weyl group.

2. Suppose $(B_1, B_2) = x.(B, \omega B \omega^{-1})$. By definition of $\sim$, we have

$$(\tilde{B}, \tilde{\omega}\tilde{B}\tilde{\omega}^{-1}) = (gBg^{-1}, g\omega B\omega^{-1}g^{-1}) = g.(B, \omega B\omega^{-1}).$$

Then $(\tilde{B}, \tilde{\omega}\tilde{B}\tilde{\omega}^{-1})$ is in the same G- orbit as $(B, \omega B\omega^{-1})$, hence as (B_1, B_2). □

Example 1.2.66. *Let $G = GL_2(\mathbb{K})$ be the group of 2×2 invertible matrices, $B = T_2$ the Borel subgroup of the upper triangular matrices, $T = D_2$ the maximal torus consisting of the diagonal matrices; the Weyl group is given by $W = {}^{N(D_2)}\!/_{D_2} \cong S_2 = \{1, s\}$. The Borel subgroup variety is $\mathcal{B} \cong \mathbb{P}^1(\mathbb{K})$, with isomorphism given by $gT_2g^{-1} \xrightarrow{\cong} g.\bar{e_1}$ (see Example 1.2.26; recall $\bar{e_1} = [1, 0], \bar{e_2} = [0, 1]$). Denote by $T_2^- = sT_2s$ the Borel subgroup of the lower triangular matrices. Then*

- *$\mathcal{O}(1) = \Delta(\mathbb{P}^1(\mathbb{K}))$ is the orbit containing all the elements of the form $(gT_2g^{-1}, gT_2g^{-1}) \xrightarrow{\cong} (g.\bar{e_1}, g.\bar{e_1})$, for $g \in G$.*
- *$\mathcal{O}(s) = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta(\mathbb{P}^1(\mathbb{K}))$ is the orbit containing all the elements of the form $(gT_2g^{-1}, gT_2^-g^{-1}) \xrightarrow{\cong} (g.\bar{e_1}, g.\bar{e_2})$, for $g \in G$.*

1.2.6 Levi and parabolic subgroups

We end this first chapter introducing parabolic subgroups. These groups can be seen somehow as a generalization of the Borel subgroups, since they have some property in common with the latter.

Let G be a connected reductive algebraic group, $B \leq G$ a Borel subgroup and $T \leq B$ a maximal torus with relative Weyl group W generated by $S = \{s_{\alpha|\alpha\in\Delta}\}$, with Δ basis of the root system $\Phi(G)$ relative to B.

Definition 1.2.67. *We call standard parabolic subgroup of W the subgroups of W of the form*

$$W_I = \langle s_\alpha \in S_I \rangle$$

for some $I \subseteq \Delta$, where $S_I := \{s_\alpha \in S \mid \alpha \in I\}$. A subgroup of W is said to be "parabolic" if it is conjugated to W_I for some $I \subseteq \Delta$.

The set $\Phi_I := \Phi(G) \bigcap \sum_{\alpha\in I} \mathbb{Z}\alpha$ is called parabolic subsystem of roots. For W_I standard parabolic subgroup of W, let P_I be the set $P_I := BW_IB = \bigsqcup_{\omega\in W_I} B\omega B$. It can be shown that P_I is actually a subgroup using properties of BN-pairs and that W_I is generated by the set S_I of simple reflections [9, Lemma 11.14]. This allows us to give the following definition.

Definition 1.2.68. *We call standard parabolic subgroups of G the subgroups of G of the form*

$$P_I = BW_IB = \bigsqcup_{\omega\in W_I} B\omega B.$$

for W_I standard parabolic subgroup of W. A subgroup of G is said to be parabolic if it is conjugated to P_I for some $I \subseteq \Delta$.

Remark 1.2.69. *[9, Proposition A.25]The parabolic subsystem of roots Φ_I is a root system with Weyl group W_I*

Using the notation of the above definition, we now give a characterization of the standard parabolic subgroups

Proposition 1.2.70. *[9, Proposition 12.1] The standard parabolic subgroups P_I, $I \subseteq \Delta$, are closed connected self normalizing subgroups of G containing B. They are not mutually conjugated. In addition*

$$P_I = \langle T, U_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle.$$

Moreover, any closed connected subgroup of G containing B is a standard parabolic subgroup.

Corollary 1.2.71. *The parabolic subgroup of G are precisely the closed connected subgroups of G containing a Borel.*

Proof. By Proposition 1.2.70 any closed subgroup of G containing B is a standard parabolic subgroup, and any standard parabolic subgroup of G contains B. Any Borel $\tilde{B}$ in G is conjugated to B by Proposition 1.2.24, so since conjugation is a group automorphism, any subgroup of G containing $\tilde{B}$ is conjugated to a subgroup of G containing B, hence to a standard parabolic subgroup of G. So any subgroup of G containing a Borel is a parabolic subgroup. Conversely, any parabolic subgroup is obtained by conjugation from a standard one by definition; then conjugating B by the same element yields a Borel subgroup contained in the parabolic subgroup we are considering. □

Remark 1.2.72. *[1, §11.2] The parabolic subgroups of G are precisely the ones such that the homogeneous space G/P is a complete (hence projective, since homogeneous spaces are always quasi-projective) variety.*

Example 1.2.73. *Let $G = GL_n(\mathbb{K})$, $D_n \leq T_n$ be the usual maximal torus and Borel subgroup. The root system base relative to T_n is given by the roots $\Delta = \{\chi_{i,i+1} \in X(D_n) | \ 1 \leq i \leq n-1\}$, where $\chi_{i,i+1}(diag(t_1 \dots t_n)) = t_i t_{i+1}^{-1}$.*
Fix $j \in \{1 \dots n\}$ and let $I = \{\chi_{i,i+1} \in \Delta \ | i \neq j+1\}$. Therefore (identifying $W \cong S_n$) $W_I = \{\sigma \in S_n \mid \sigma(i) \leq j \text{ for any } i \leq j\} \cong S_j \times S_{n-j}$. Then

$$P_I = \bigsqcup_{\sigma \in S_j \times S_{n-j}} T_n \sigma T_n$$

is the subgroups of the block upper triangular matrices with diagonal blocks $j \times j$ and $n-j \times n-j$, that is

$$\begin{aligned} P_I &= \{A \in GL_n(\mathbb{K}) \ | A_{k,l} = 0 \ \ when \ \ k > j \ and \ l \leq (n-j)\} \\ &= \{\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \ | A_1 \in GL_j(\mathbb{K}), \ A_2 \in GL_{n-j}(\mathbb{K}), \ A_2 \in M_{n-j,j}(\mathbb{K})\}. \end{aligned}$$

The associated parabolic subsystem of roots Φ_I is given by

$$\Phi_I = \{\chi_{i,k} | i,k \leq j \ or \ i,k > j\}.$$

Then

$$\Phi_I \cup \Phi^+ = \{\chi_{i,k} \mid i,k \leq j \ or \ i,k > j\} \cup \{\chi_{i,k} \mid i < k\} = \{\chi_{i,k} \mid i \leq j \ if \ k \leq j\}$$

and indeed

$$P_I = \langle D_n, U_{i,k} \mid i \leq j \ if \ k \leq j \rangle$$

where the root subspaces are $U_{i,k} = I_n + \mathbb{K}E_{i,k}$.

Generally speaking, the standard parabolic subgroups of $GL_n(\mathbb{K})$ (taking as usual diagonal and upper triangular matrices as torus and Borel) are block upper triangular matrices, that means that they are stabilizer of flags in the natural n-dimensional representation (the Borel subgroups of GL_n are the stabilizer of a *complete* flag, since they are all conjugated to T_n).
Now we present the so called Levi decomposition of a parabolic subgroup.

Definition 1.2.74. *Let H be an algebraic group. We say that H has a Levi decomposition if there exists a connected reductive subgroup $L \leq H$ such that*

$$H = R_u(H) \rtimes L.$$

Such an L is called "Levi subgroup" of H.

Now let G be a connected reductive algebraic group, T a maximal torus contained in a Borel subgroup B. Let $I \subseteq \Delta$, where Δ is the base of the root system associated to B, and consider the standard parabolic subgroup P_I. Let

$$U_I := \langle U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_I \rangle$$
$$L_I := \langle T, U_\alpha \mid \alpha \in \Phi_I \rangle$$

Then we have the following result

Proposition 1.2.75. *[9, Proposition 12.6] Let P_I be a standard parabolic subgroup of a connected reductive group G. Then $U_I = R_u(P_I)$ and $P_I = U_I \rtimes L_I$ is a Levi decomposition for P_I, and any other closed complement to U_I is conjugated to L_I in P_I. Moreover, L_I is reductive with maximal torus T and root system Φ_I, and $L_I = C_G(Z(L_I)^0)$.*

L_I is called standard Levi complement of P_I. Note that, in contrast with what happens for standard parabolic subgroups, different Levi standard complements may be G-conjugated. The result can be extended to all parabolic subgroups of G. Note that in the case the parabolic subgroup is a Borel subgroup B, the Levi complement is just the torus and we find again the usual semidirect product decomposition $B = R_u(B) \rtimes T$.

Theorem 1.2.76. *[3, Proposition 3.4.2] Let P be a parabolic subgroup of a connected reductive group G, and let $T \leq P$ a maximal torus contained in P. Then there exists a unique Levi subgroup of P containing T. Moreover all Levi subgroups of P are $R_u(P)$ conjugated.*

Note that, since any parabolic subgroup is conjugated to some standard Parabolic subgroup P_I, the Theorem 1.2.76 implies in particular that any Levi subgroup of of a Parabolic subgroup is G-conjugated to a standard Levi subgroup. By abuse of terminology, we usually call the Levi subgroup of a parabolic subgroup P of G a Levi subgroup of G.

Example 1.2.77. *Consider the standard parabolic subgroup of $P_I \leq GL_n(\mathbb{K})$ of Example 1.2.73.Then its unipotent radical is*

$$U_I = \langle U_{ik} \mid i \leq j \text{ if } k \leq j \rangle = \left\{ \begin{pmatrix} I_j & A \\ 0 & I_{n-j} \end{pmatrix} | A \in M_{n-j,j}(\mathbb{K}) \right\}$$

while its standard Levi complement is

$$\begin{aligned} L_I &= \langle D_n, U_{i,k} \mid i,k \leq j \textit{ or } i,k > j \rangle \\ &= \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} | A_1 \in GL_j(\mathbb{K}), A_2 \in GL_{n-j}(\mathbb{K}) \right\} \end{aligned}$$

Chapter 2

Finite Groups of Lie type

2.1 The Frobenius morphism

2.1.1 Frobenius morphisms on algebraic varieties

In what follows we will work with algebraic groups over $\mathbb{K} = \overline{\mathbb{F}_p}$, with $p > 0$ a prime. For any p power $q = p^e$ with $e \in \mathbb{N}$, there exists a unique finite subfield $\mathbb{F}_q \subseteq \overline{\mathbb{F}_p}$ which contain exactly q elements; this field can be defined as the fixed point of the morphism $\sigma : \overline{\mathbb{F}_p} \to \overline{\mathbb{F}_p}$ defined by $\sigma(x) = x^q$, for $x \in \overline{\mathbb{F}_p}$.
The aim of this section is to generalize this kind of argument to algebraic groups, coming to define the so-called finite groups of Lie type.
Before working with algebraic groups, we need to define the concept of "being defined over $\mathbb{F}_q$" (or "having an $\mathbb{F}_q$-rational structure") for affine varieties. We start with an example:

Example 2.1.1. *Let $X = \overline{\mathbb{F}_p}^n$. Define the map*

$$
\begin{aligned}
F_q : \overline{\mathbb{F}_p}^n & \to \overline{\mathbb{F}_p}^n \\
(x_i)_{1 \le i \le n} & \mapsto (x_i^q)_{1 \le i \le n}
\end{aligned}
$$

We call this map "standard Frobenius morphism" over $\overline{\mathbb{F}_p}^n$. It is an $\mathbb{F}_q$- linear map and a bijective morphism of affine varieties; yet, it is not an automorphism of algebraic varieties (the inverse is not a morphism). The fixed points set of this map, denoted by $(\overline{\mathbb{F}_p}^n)^{F_q}$, is $\mathbb{F}_q^n$.
If $V \subseteq \overline{\mathbb{F}_p}^n$ is a closed subset whose vanishing ideal I is contained in $\mathbb{F}_q[x_1 \dots x_n]$, then $F_q(V) \subseteq V$. Hence F_q restricts to a morphism $V \to V$ and $V^{F_q} = V \cap \mathbb{F}_q^n$.

Our intent is to define a morphism with similar properties as the standard Frobenius morphism for general affine varieties. In order to do this, we study the associated algebra morphism F_q^*. We have

$$F_q^* : \overline{\mathbb{F}_p}[x_1, \dots, x_n] \to \overline{\mathbb{F}_p}[x_1 \dots x_n]$$
$$\sum_{i=1}^{n} a_i x_i \to \sum_{i=1}^{n} a_i x_i^q$$

and it has the following properties

- it is injective (since F_q is surjective) and $F_q^*(\overline{\mathbb{F}_p}[x_1, \dots, x_n]) \subseteq \overline{\mathbb{F}_p}[x_1^q, \dots, x_n^q] = \{f^q \mid f \in \overline{\mathbb{F}_p}[x_1, \dots, x_n]\}$.
- Since every element in $\overline{\mathbb{F}_p}$ belongs to some $\mathbb{F}_{p^e}$, hence to some $\mathbb{F}_{q^m}$, for any $f = \sum_{i=1}^{n} a_i x_i \in \overline{\mathbb{F}_p}[x_1, \dots, x_n]$ there exists an $m \in N$ such that all the coefficients a_i belong to $\mathbb{F}_{q^m}$; this implies that $a_i = a_i^{q^m}$, hence $F_q^m(f) = \sum_{i=1}^{n} a_i x_i^{q^m} = \sum_{i=1}^{n} a_i^{q^m} x_i^{q^m} = f^{q^m}$.

This lead to the following ("intrinsic") definition.

Definition 2.1.2. *Let X be an affine variety over $\overline{\mathbb{F}_p}$. A morphism*

$$F : X \to X$$

is called "(geometric) Frobenius Morphism" if there exists a $q = p^e$ with $e \in \mathbb{N}_{\geq 1}$ such that the associated ring homomorphism

$$F^* : \overline{\mathbb{F}_p}[X] \to \overline{\mathbb{F}_p}[X]$$

satisfies the following conditions:

- *It is injective and $F^*(\overline{\mathbb{F}_p}[X]) = \{f^q \mid f \in \overline{\mathbb{F}_p}[X]\}$*
- *for any $f \in \overline{\mathbb{F}_p}[X]$ there exists an $m \in \mathbb{N}$ such that $(F^*)^m(f) = f^{q^m}$*

If such a morphism exists, we say that X is defined over $\mathbb{F}_q$ (or that it admits an $\mathbb{F}_q$-rational structure) and F is the (geometric) Frobenius morphism associated to this structure.
The set of F-fixed points

$$X^F = \{x \in X \mid F(x) = x\}$$

is called set of $\mathbb{F}_q$-rational points in X and denoted by $X(\mathbb{F}_q)$.

Remark 2.1.3. *Let X be an affine variety over $\overline{\mathbb{F}_p}$ endowed with a Frobenius morphism F, that determines a $\mathbb{F}_q$-rational structure. We define the "arithmetic Frobenius map" as the map*

$$\tau : \overline{\mathbb{F}_p}[X] \to \overline{\mathbb{F}_p}[X] \quad s.t. \quad \tau(f) = {F^*}^{-1}(f^q)$$

that is well defined and bijective since F^* is injective and has as image $\{f^q | f \in \overline{\mathbb{F}_p}[X]$.
Since F^* is a $\overline{\mathbb{F}_p}$-algebra morphism and $f \mapsto f^q$ is a ring morphism of $\overline{\mathbb{F}_p}[X]$, τ is is a ring morphism and it satisfies $\tau(\lambda f) = (F^*)^{-1}(\lambda^q f^q) = \lambda^q (F^*)^{-1}(f^q) = \lambda^q \tau(f)$ for any $\lambda \in \overline{\mathbb{F}_p}$. Hence τ is $\mathbb{F}_q$-linear, but not $\overline{\mathbb{F}_p}$-linear: it is an $\mathbb{F}_q$-algebra morphism, but not an $\overline{\mathbb{F}_p}$-algebra morphism. Note also that τ commutes with F^*, since $\tau \circ F^*(f) = (F^*)^{-1}(F^*(f)^q) = (F^*)^{-1}(F^*(f^q)) = f^q = F^* \circ \tau(f)$. Moreover for any $f \in \overline{\mathbb{F}_p}[X]$ there exists an $m \in \mathbb{N}$ such that $\tau^m(f) = f$ [5, Remark 4.1.2]. The fixed point set of τ on $\overline{\mathbb{F}_p}[X]$ is

$$(\overline{\mathbb{F}_p}[X])^\tau := \{f \in \overline{\mathbb{F}_p}[X] \mid \tau(f) = f\} = \{f \in \overline{\mathbb{F}_p}[X] | F^*(f) = f^q\}.$$

This set can be proved to be a finitely generated $\mathbb{F}_q$-subalgebra of $\overline{\mathbb{F}_p}[X]$ that contains a set of algebra generators for $\overline{\mathbb{F}_p}[X]$ [5, Lemma 4.1.3], and this implies that the natural map

$$(\overline{\mathbb{F}_p}[X])^\tau \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p} \to \overline{\mathbb{F}_p}[X]$$

is an isomorphism. Note that reading the action of τ and F^* on $(\overline{\mathbb{F}_p}[X])^\tau \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p}$, we have for any $f \otimes \lambda \in (\overline{\mathbb{F}_p}[X])^\tau \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p}$

$$F^*(f \otimes \lambda) = f^q \otimes \lambda \qquad\qquad \tau(f \otimes \lambda) = f \otimes \lambda^q.$$

Actually it can be shown that the following are equivalent,:to give a decomposition $\overline{\mathbb{F}_p}[X] \cong A \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p}$ (the isomorphism given by the product) where A is an $\mathbb{F}_q$-subalgebra, to give a (geometric) Frobenius morphism or to give an arithmetic Frobenius map. Indeed, any of them uniquely determines the other two. [3, Proposition 4.1.8]

Note that from Definition 2.1.2 it is clear that if F is a Frobenius morphism relative to an $\mathbb{F}_q$-structure, then a power F^n with $n \in \mathbb{N}_{\geq 1}$ is still a Frobenius morphism, relative to an F_{q^n}-structure.

There is also a more "concrete" characterization of a Frobenius morphism, that relates it in a strong way with the standard Frobenius morphism.

Proposition 2.1.4. *Let X be an affine variety over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with associated Frobenius morphism F. Then there exists an $n > 0$ and a closed embedding $\iota : X \to \overline{\mathbb{F}_p}^n$ such that $\iota \circ F = F_q \circ \iota$.*
Moreover, the vanishing ideal of $\iota(X)$ is generated by polynomials in $\mathbb{F}_q[x_1 \dots x_n]$

Proof. Consider $\overline{\mathbb{F}_p}[X]^\tau = \{f \in \overline{\mathbb{F}_p}[X] \mid F^*(f) = f^q\}$ as in Remark 2.1.3; it is a (reduced) finitely generated $\mathbb{F}_q$-algebra, hence there exists a (radical) ideal $\mathcal{J} \subseteq \mathbb{F}_q[x_1 \dots x_n]$ such that

$$\overline{\mathbb{F}_p}[X]^\tau = {}^{\mathbb{F}_q[x_1 \dots x_n]}\!/_{\mathcal{J}}.$$

Then, since $\overline{\mathbb{F}_p}[X] \cong (\overline{\mathbb{F}_p}[X])^\tau \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p}$, we have

$$\overline{\mathbb{F}_p}[X] = {}^{\overline{\mathbb{F}_p}[x_1 \dots x_n]}\!\diagup_{\mathcal{I}}$$

with $\mathcal{I}$ being the ideal generated by $\mathcal{J}$ in $\overline{\mathbb{F}_p}[x_1 \dots x_n]$ (note that $\mathcal{J} = \mathcal{I} \cap \mathbb{F}_q[x_1 \dots x_n]$). Then the canonical projection $\pi : \overline{\mathbb{F}_p}[x_1 \dots x_n] \to \overline{\mathbb{F}_p}[X]$ is a surjective morphism with kernel $\mathcal{I}$, hence the affine variety morphism $\iota : X \to \overline{\mathbb{F}_p}^n$ defined by $\iota^* = \pi$ is a closed embedding having as image $\iota(X) = \mathbb{V}(\mathcal{I})$. Therefore the defining ideal of the variety $(\iota(X))$ is $\mathcal{I} = (\mathcal{J})$.
Moreover, π restricted to $\mathbb{F}_q[x_1 \dots x_n]$ is the canonical projection onto $\overline{\mathbb{F}_p}[X]^\tau$, hence for any $f \in \mathbb{F}_q[x_1 \dots x_n]$, it holds

$$(F_q \circ \iota)^*(f) = \pi \circ F_q^*(f) = \pi(f^q) = \pi(f)^q = F^* \circ \pi(f) = (\iota \circ F)^*(f)$$

Hence, since $\overline{\mathbb{F}_p}[X] \cong (\overline{\mathbb{F}_p}[X])^\tau \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p}$, $(F_q \circ \iota)^* = (\iota \circ F)^*$ and this yields the equality of the morphisms of affine varieties. □

Example 2.1.5.

- *The standard Frobenius morphism F_q on the affine space $X = \overline{\mathbb{F}_p}^n$ defined in Example 2.1.1 is a Frobenius morphism. The corresponding arithmetic Frobenius morphism $\tau : \overline{\mathbb{F}_p}[x_1 \dots x_n] \to \overline{\mathbb{F}_p}[x_1 \dots x_n]$ as in Remark 2.1.3 is defined on $f = \sum_{i=1}^n a_i x_i \in \overline{\mathbb{F}_p}[x_1 \dots x_n]$ by*

$$\tau(\sum_{i=1}^n a_i x_i) = (F_q^*)^{-1}((\sum_{i=1}^n a_i x_i)^q) = (F_q^*)^{-1}(\sum_{i=1}^n a_i^q x_i^q) = \sum_{i=1}^n a_i^q (F_q^*)^{-1}(x_i^q) = \sum_{i=1}^n a_i^q x_i.$$

It follows that $\overline{\mathbb{F}_p}[x_1 \dots x_n]^\tau = \mathbb{F}_q[x_1 \dots x_n]$, and the decomposition of Remark 2.1.3 is given by $\overline{\mathbb{F}_p}[x_1 \dots x_n] = \mathbb{F}_q[x_1 \dots x_n] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_p}$.
The embedding of Theorem 2.1.4 is the identity.

- *Let $X = \overline{\mathbb{F}_p}^n$, let $\sigma \in S_n$ be the permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$ and consider the morphism*

$$\overline{\mathbb{F}_p}^2 \to \overline{\mathbb{F}_p}^2.$$
$$(x_1, x_2 \dots x_n) \mapsto (x_n^q, x_{n-1}^q, \dots x_1^q).$$

This is a Frobenius morphism because the associated algebra morphism is injective and with image $\{f^q | f \in \overline{\mathbb{F}_p}[x, y]\}$; furthermore for any $f \in \overline{\mathbb{F}_p}[x, y]$ there exists an $m \in \mathbb{N}$ such that all the coefficients of f lie in $\mathbb{F}_{q^m}$, so $(F^*)^{2m}(f) = f^{q^{2m}}$ (we need the 2 in the exponent since it is the order of the permutation σ) . We do not give more details since denoting by $\tilde{\sigma} : \overline{\mathbb{F}_p}^n \to \overline{\mathbb{F}_p}^n$ the morphism permuting the entries according to σ, it holds $F = F_q \circ \tilde{\sigma}$ and since $\tilde{\sigma}$ has finite order

and commutes with F_q, we will prove in Corollary 2.1.16 that this is enough to guarantee that F is a Frobenius morphism.
The associated arithmetic Frobenius morphism $\tau : \overline{\mathbb{F}_p}[x_1 \dots x_n] \to \overline{\mathbb{F}_p}[x_1 \dots x_n]$ is defined by $\tau(x_i) = x_{n-i+1}$ for $1 \leq i \leq n$, $\tau(a) = a^q$ for any $a \in \overline{\mathbb{F}_p}$.
Choosing ζ to be a generating element of $\mathbb{F}^*_{q^2}$, the $\mathbb{F}_q$-subalgebra of the fixed points of τ is

$$\overline{\mathbb{F}_p}[x_1, x_2 \dots x_n]^\tau = \mathbb{F}_q[\zeta x_1 + \zeta^q x_n, \zeta x_2 + \zeta^q x_{n-1}, \dots, \zeta x_n + \zeta^q x_1].$$

A closed embedding as in Theorem 2.1.4 is given by

$$\begin{aligned} \iota : \overline{\mathbb{F}_p}^n &\to \overline{\mathbb{F}_p}^n \\ (x_i)_{1\leq i\leq n} &\mapsto ((\zeta x_i + \zeta^q x_{n-i})_{1\leq i\leq n}) \end{aligned}$$

Indeed this is a closed embedding and $F_q \circ \iota(x_1 \dots x_n) = (\zeta^q x_i^q + \zeta^{q^2} x_{n-i}^q)_{1\leq i\leq n} = (\zeta^q x_i^q + \zeta x_{n-i}^q)_{1\leq i\leq n}) = \iota(x_n^q, x_{n-1}^q \dots x_1^q) = \iota \circ F(x_1 \dots x_n)$.

Corollary 2.1.6. *Let X be an affine variety over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with associated Frobenius morphism F. Then F is a bijective map and the set X^F of the $\mathbb{F}_q$-rational points of X is finite.*

Proof. We use the notation of Proposition 2.1.4. The fixed point set of the standard Frobenius F_q lies in $\mathbb{F}_q^n$, hence it is finite; since $\iota \circ F = F_q \circ \iota$ and ι is injective, this implies that also the $\mathbb{F}_q$-rational points of X are a finite set.
The bijectivity of F follows, again, from the fact that $\iota \circ F = F_q \circ \iota$ and from the bijectivity of F_q. □

Remark 2.1.7. *As already pointed out in Remark 1.1.26, any element of $\overline{\mathbb{F}_p}$ lies in $\mathbb{F}_{p^e}$ for some $e \in \mathbb{N}_{\geq 1}$. Moreover, for any q power of p, $(F_q)^2 = F_{q^2}$, the standard Frobenius morphism for the q^2 structure of $\overline{\mathbb{F}_p}^n$; note that for any q power of p, for any $e \in \mathbb{N}_{\geq 1}$ there exists an $n \in \mathbb{N}_{\geq 1}$ such that $q^n \geq p^e$. Hence for any q power of p*

$$\overline{\mathbb{F}_p}^n = \bigcup_{n\geq 1} (\overline{\mathbb{F}_p}^n)^{F_{q^n}}.$$

From the characterization given in Proposition 2.1.4, we can transfer this observation on the structure of $\overline{\mathbb{F}_p}^n$ on an algebraic variety X defined over $\mathbb{F}_q$, that is

$$X = \bigcup_{n\geq 1} (X)^{F^n}$$

If an affine variety is endowed with a Frobenius morphism F inducing a rational structure, this structure is inherited by its F-stable closed subsets.

Lemma 2.1.8. *[5, Corollary 4.1.5]Let X be an affine variety over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with associated Frobenius morphism F. Let X' be an F-stable closed subset of X, that is $F(X') \subseteq X'$. Then $F|_{X'}$ is a Frobenius morphism.*

To complete the picture, we give the definition of morphism of varieties defined over $\mathbb{F}_q$.

Definition 2.1.9. *Let X, X' be affine varieties over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with associated Frobenius morphisms F, F' respectively. Then a morphism of varieties $\phi : X \to X'$ is called rational, or defined over $\mathbb{F}_q$, if $\phi \circ F = F' \circ \phi$.*

This definition has an equivalent characterization using the associated ring of regular functions.

Proposition 2.1.10. *Let X, X' be affine varieties over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with associated Frobenius morphisms F, F' respectively, and let $\phi : X \to X'$ be a morphism of varieties. Then ϕ is defined over $\mathbb{F}_q$ if and only if $\phi^*(\overline{\mathbb{F}_p}[X']^{\tau'}) \subseteq \overline{\mathbb{F}_p}[X]^{\tau}$*

Proof. If ϕ is defined over $\mathbb{F}_q$, then for any $f \in \overline{\mathbb{F}_p}[X']^{\tau'}$ it holds $F^* \circ \phi^*(f) = \phi^* \circ F'^*(f) = \phi^*(f^q) = (\phi^*(f))^q$ that means $\phi^*(f) \in \overline{\mathbb{F}_p}[X]^{\tau}$.
Conversely, $\phi^*(\overline{\mathbb{F}_p}[X']^{\tau'}) \subseteq \overline{\mathbb{F}_p}[X]^{\tau}$ means that for any f in $\overline{\mathbb{F}_p}[X']^{\tau}$ it holds $\phi^* \circ F'(f) = \phi^*(f^q) = (\phi^*(f))^q = F \circ \phi^*(f)$. Hence since $\overline{\mathbb{F}_p}[X']^{\tau'}$ generates $\overline{\mathbb{F}_p}[X']$ and $\overline{\mathbb{F}_p}[X]_0$ generates $\overline{\mathbb{F}_p}[X]$, it follows $\phi^* \circ F' = F \circ \phi^*(f)$. □

Example 2.1.11. *Let $X = X' = \overline{\mathbb{F}_p}^n$. The morphism $\iota : X \to X'$ of Example 2.1.5, defined as $(x_i)_{1\le i\le n} \mapsto (\zeta x_i + \zeta^q x_{n-i})_{1\le i\le n}$ with $\zeta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, is a rational morphism considering X endowed with the rational structure given by the Frobenius F defined by $(x_i)_{1\le i\le n} \mapsto (x^q_{n-i+1})_{1\le i\le n}$, and X' endowed with the rational structure given by the standard Frobenius morphism F_q. Indeed by construction of ι we have $\iota \circ F = F_q \circ \iota$.*
Denoting by τ_q the arithmetic Frobenius morphism relative to the standard Frobenius morphism F_q and by τ the arithmetic Frobenius morphism relative to the Frobenius morphism F, the rational structures read on the rings of regular functions as $\overline{\mathbb{F}_p}[X']^{\tau_q} = \mathbb{F}_q[x_1 \dots x_n]$ and $\overline{\mathbb{F}_p}[X]^{\tau} = \mathbb{F}_q[\zeta x_1 + \zeta^q x_n \dots \zeta x_n + \zeta^q x_1]$. Hence $\iota^*(\overline{\mathbb{F}_p}[X']^{\tau}_q) = \iota^*(\mathbb{F}_q[x_1 \dots x_n]) = \mathbb{F}_q[\zeta x_1 + \zeta^q x_n, \dots, \zeta x_n + \zeta^q x_1]$.
Note that, more in general, in the Proposition 2.1.4 we required exactly the closed embedding to be rational.

The notion of Frobenius morphism can be slightly generalized in the following useful way.

Definition 2.1.12. *Let X be an algebraic group defined over $\overline{\mathbb{F}_p}$. An automorphism of algebraic groups $F : X \to X$ is called generalized Frobenius morphism if some power of F is a Frobenius morphism.*

Example 2.1.13. *Consider the field $\overline{\mathbb{F}_2}$, and consider the algebraic variety $X = \mathbb{V}(xy-1) \subseteq \overline{\mathbb{F}_2}^2$; the coordinate ring of X is $\overline{\mathbb{F}_2}[X] \cong \overline{\mathbb{F}_2}[x, x^{-1}]$. Then the following map is a generalized Frobenius morphism on $X \times X$:*

$$F : X \times X \to X \times X$$
$$((x, x^{-1}), (y, y^{-1})) \mapsto ((xy, x^{-1}y^{-1}), (xy^{-1}, x^{-1}y))$$

indeed $F^2 = F_2|_{X \times X}$.
Anyway this is not a Frobenius morphism: the corresponding algebra morphism

$$\overline{\mathbb{F}_2}[x, x^{-1}, y, y^{-1}] \to \overline{\mathbb{F}_2}[x, x^{-1}, y, y^{-1}]$$

is defined by $F^*(x) = xy, F^*(y) = xy^{-1}$ *but* $xy \notin \{f^2 \mid f \in \overline{\mathbb{F}_2}[X \times X]\}$

Remark 2.1.14. *Note that, also if Proposition 2.1.4 does not properly hold for generalized Frobenius morphisms, loosening the assumption on* F *and requiring it to be just a generalized Frobenius morphism the properties stated in Corollary 2.1.6 and Remark 2.1.7 still hold true. Indeed also if in the case of a Frobenius we derived these properties from the existence of as embedding as in Proposition 2.1.4 , for a generalized Frobenius* F *one can proceed as follows:*

- *If* F *is a generalized Frobenius morphism such that* F^n *is a Frobenius morphism, then*
$$X^F \subset X^{F^n}.$$
Since F^n *is a Frobenius morphism, by Remark 2.1.6 it follows that* X^{F^n} *is finite and so also the* F*-fixed points set of* X *is finite.*

- *Since* F^n *is a Frobenius morphism*
$$X = \bigcup_{e \geq 1} X^{(F^n)^e} \subseteq \bigcup_{i \geq 1} X^{F^i} \subseteq X,$$
hence it still holds for generalized Frobenius morphisms
$$X = \bigcup_{i \geq 1} X^{F^i}.$$

Moreover the restriction of a generalized Frobenius morphism F *to a closed* F*-stable subset* $X' \subseteq X$ *is still a generalized Frobenius morphism. Indeed since a power of* F *is a Frobenius morphism, therefore bijective,* F *is bijective, and hence* $(F|_{X'})^n = F^n|_{X'}$*, and for an appropriate* n *the latter is a Frobenius morphism for* X' *from Lemma 2.1.8.*

We conclude this section with a lemma that allows us to construct new (generalized) Frobenius morphisms from a given one, composing it with an automorphism of affine varieties.

Lemma 2.1.15. *Let* X *be an affine variety over* $\overline{\mathbb{F}_p}$*,* F *a (generalized) Frobenius morphism, and* $\phi : X \to X$ *an automorphism of affine varieties . If there exists a positive integer* $n \geq 1$ *such that* $(\phi \circ F)^n = F^n$*, then* $\phi \circ F$ *is a (generalized) Frobenius morphism.*

Proof. Assume that F is a Frobenius morphism. Since ϕ is an automorphism of affine varieties, ϕ^* is an isomorphism of algebras. It follows that $(\phi \circ F)^* = F^* \circ \phi^*$ is injective (since F^* and ϕ^* are so) and has the same image as F^*. Moreover since F is a Frobenius morphism ,for any $f \in \overline{\mathbb{F}_p}[X]$ there exists an $m \in \mathbb{N}$ such that $(F^*)^m(f) = f^{q^m}$, so $((\phi \circ F)^*)^{nm}(f) = (((\phi \circ F)^n)^*)^m(f) = ((F^n)^*)^m(f) = (F^*)^{mn}(f) = (f^{q^m})^n = f^{q^{nm}}$.
If F is a generalized Frobenius morphism, then exists an l such that F^l is a Frobenius morphism, Hence

$$(\phi \circ F)^{nl} = ((\phi \circ F)^n)^l = (F^n)^l = (F^l)^n$$

and the latter is a power of a Frobenius morphism, hence a Frobenius morphism. So $\tau \circ F$ is a generalized Frobenius morphism. □

Remark 2.1.16. *If $\phi : X \to X$ is an automorphism of finite order of affine varieties commuting with F, it satisfies the assumptions of the Lemma 2.1.15.*
Indeed since ϕ^* is of finite order, let's say $n \in \mathbb{N}$, and it commutes with F, it holds $((\phi \circ F)^*)^n(F) = (F^*)^n \circ (\phi^*)^n = (F^*)^n$.

2.1.2 Frobenius morphisms of algebraic groups

Now we can specialize the notion of Frobenius morphism, and hence of rational structure, to the situation in which the affine variety is also an algebraic group.

Definition 2.1.17. *An algebraic group G over $\overline{\mathbb{F}_p}$ is said to be defined over $\mathbb{F}_q$ if it defined over $\mathbb{F}_q$ as algebraic variety with a Frobenius morphism F which is a morphism of algebraic groups.*

So in the case of algebraic groups, we require the Frobenius morphism to be also a group morphism; this can be rephrased by saying that the multiplication and the inversion morphisms of the algebraic groups have to be defined over $\mathbb{F}_q$ with respect to the rational structure induced by the Frobenius morphism [11, §11].

Example 2.1.18. *Let $G = GL_n(\overline{\mathbb{F}_p})$, and let $q = p^m$ for some $m \in \mathbb{N}$. Define the map*

$$\begin{aligned} F_q : &GL_n(\overline{\mathbb{F}_p}) \to GL_n(\overline{\mathbb{F}_p}) \\ &(a_{i,j})_{1 \le i,j \le n} \to (a^q_{i,j})_{1 \le i,j \le n} \end{aligned}$$

We call this map the "standard Frobenius morphism" of $GL_n(\overline{\mathbb{F}_p})$. It is a group morphism, hence it makes $GL_n(\overline{\mathbb{F}_p})$ into a group defined over $\mathbb{F}_q$.
The set of fixed points of this map, denoted by $GL_n(\overline{\mathbb{F}_p})^{F_q}$ is the finite general linear group $GL_n(\mathbb{F}_q)$ of non singular matrices or $\mathbb{F}_q$.

The notion of generalized Frobenius morphism for algebraic groups is the analogous in algebraic group context of the definition we gave for algebraic varieties.

Definition 2.1.19. *Let G be an algebraic group defined over $\overline{\mathbb{F}_p}$. A morphism of algebraic groups $F : G \to G$ is called generalized Frobenius morphism if some power of F is a Frobenius morphism for some $\mathbb{F}_q$-rational structure on G.*

Example 2.1.20. *Consider the morphism F in Example 2.1.13. The variety $X = \mathbb{V}(xy-1) \subset \overline{\mathbb{F}_2}$ can be identified with the multiplicative group G_m of $\overline{\mathbb{F}_2}$. Then $F : G_m \times G_m \to G_m \times G_m$ is defined by $(x, y) \mapsto (xy, xy^{-1})$ and so it is a group morphism, thus it is a generalised Frobenius morphism on the rank two torus.*

More generally, generalized Frobenius morphisms which are not Frobenius morphism on reductive connected algebraic groups are the ones that allow to build the so called Suzuki and Ree groups. Details about their existence and classification can be found in [11, §11].

We now show as the results of the previous section translate in the situation of algebraic groups. In particular the following proposition is the equivalent for algebraic groups of Proposition 2.1.4. It states, essentially, that for any algebraic group G defined over $\mathbb{F}_q$ with Frobenius morphism F, there exists a realization of G as group of matrices with respect to which the Frobenius morphism is the standard one.
The idea of the proof is to show that a closed embedding as in Theorem 1.1.20 can be chosen to be defined over $\mathbb{F}_q$ (considering $GL_n(\overline{\mathbb{F}_p})$ endowed with the $\mathbb{F}_q$-rational structure defined by the standard Frobenius morphism F_q).

Proposition 2.1.21. *[5, Proposition 4.1.11]Let G be an algebraic group over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with Frobenius morphism F. There exists an $n > 0$ and a closed embedding of algebraic groups $\iota : G \to GL_n(\overline{\mathbb{F}_p})$ such that*

$$\iota \circ F = F_q \circ \iota$$

.

Corollary 2.1.22. *Let G be an algebraic group over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with generalized Frobenius morphism F. Then*

1. *the set G^F of the $\mathbb{F}_q$-rational points is a finite subgroup of G.*
2. *$G = \bigcup_{n \geq 1} G^{F^n}$*

Proof. G^F is a subgroup of G because F is a group morphism. The other claims follow directly from Remark 2.1.14 □

So any algebraic group G with a generalized Frobenius morphism F produce a finite subgroup G^F. The finite groups of this kind are referred to as finite algebraic groups. We will be interested in studying the finite groups arising in this way from a connected reductive algebraic group. These are called *finite groups of Lie type* or *finite reductive groups*.

Definition 2.1.23. *Let G be a connected reductive algebraic group over $\overline{\mathbb{F}_p}$, let F be a generalized Frobenius morphism. The finite group of F-fixed points G^F is said to be a finite group of Lie type.*

We will be interested in recovering properties of these groups from the ones of the algebraic group from which they arise, and this will be the main focus of the next section. Before starting, we give some results that can be useful to construct (generalized) Frobenius morphisms and therefore to give some examples.
First, we point out that (generalized) Frobenius morphisms restrict to (generalized) Frobenius morphisms on closed subgroups which are F-stable.

Lemma 2.1.24. *Let G be an algebraic group, F a (generalized) Frobenius morphism. Let $H \leq G$ be a closed and F-stable subgroup. Then $F|_H$ is a (generalized) Frobenius morphism.*

Proof. Since H is a subgroup, a group automorphism of G that stabilizes H restricts to a morphism of groups on H. Then the statement follows from Corollary 2.1.8, and Remark 2.1.14 for what concerns generalized Frobenius morphism. □

Lemma 2.1.15 (and Remark 2.1.16) give us the following result.

Lemma 2.1.25. *Let G be an algebraic group, F a (generalized) Frobenius morphism, and let $\tau : G \to G$ be an automorphism of algebraic groups. If there exists a positive integer $n \geq 1$ such that $(\tau \circ F)^n = F^n$, then $\tau \circ F$ is a (generalized) Frobenius morphism.*
In particular, this is the case if τ has finite order and commutes with F.

The following example is an application of Lemma 2.1.25 to the case in which τ is an inner automorphism.

Example 2.1.26. *Let G be an algebraic group over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ with Frobenius morphism F. For any $h \in G$ denote the automorphism of algebraic groups of G given by the conjugation by h with c_h (that is, $c_h(x) = hxh^{-1}$ for $x \in G$). Then the composition $c_h \circ F$, denoted often by hF, is still a Frobenius morphism.*
Indeed by Corollary 2.1.22, there exists an $n \geq 1$ such that $h \in G^{F^n}$. Then

$$(c_h \circ F)^n = c_{hF(h)\cdots F^{n-1}(h)} \circ F^n.$$

But $F^n(hF(h)\cdots F^{n-1}(h)) = hF(h)\cdots F^{n-1}(h)$, i.e. $hF(h)\cdots F^{n-1}(h)$ is F^n-stable and that means that $c_{hF(h)\cdots F^{n-1}(h)}$ commutes with the Frobenius morphism F^n. Moreover it has finite order, say $k \geq 1$ (because any element of an algebraic group over $\overline{\mathbb{F}_p}$ lies in a finite subgroup by Remark 2.1.22, so in particular it has finite order). Then proceeding as in Remark 2.1.25 we see $(c_h \circ F)^{nk} = (c_{hF(h)\cdots F^{n-1}(h)} \circ F^n)^k = F^{nk}$, hence $c_h \circ F$ is a Frobenius morphism.

Now we give some examples of reductive algebraic groups endowed with generalized Frobenius morphisms, and we point out the resulting finite group of Lie type. We have already seen in Example 2.1.18 the case of $GL_n(\overline{\mathbb{F}_p})$ with standard Frobenius morphism.

Example 2.1.27. *1. The special linear group $SL_n(\overline{\mathbb{F}_p})$ of ddegree n over $\overline{\mathbb{F}_p}$is a closed subgroup of $GL_n(\overline{\mathbb{F}_p})$, and it is stable under the standard Frobenius morphism F_q defined in Example 2.1.18. Hence by Remark 2.1.24, the restriction of F_q to $SL_n(\overline{\mathbb{F}_p})$ define an $\mathbb{F}_q$-structure over it, and the group of F_q-rational points is*

$$SL_n(\overline{\mathbb{F}_p})^{F_q} = SL_n(\mathbb{F}_q),$$

the invertible matrices with entries in $\mathbb{F}_q$ and determinant one.

2. Consider again the general linear group $GL_n(\overline{\mathbb{F}_p})$, and let τ be the endomorphism $\tau : GL_n(\overline{\mathbb{F}_p}) \to GL_n(\overline{\mathbb{F}_p})$ defined by $\tau(A) = A^{-T}$. Then τ is an automorphism of order 2 commuting with the standard Frobenius morphism F_q, hence the morphism $F' := \tau \circ F_q$ is again a Frobenius morphism by lemma 2.1.25. Observe that $(F')^2 = F_{q^2}$, hence $GL_n(\overline{\mathbb{F}_p})^{F'} \subseteq GL_n(\mathbb{F}_{q^2})$. In particular $GL_n(\overline{\mathbb{F}_p})^{F'} = GU_n(q)$, the general unitary group over $\mathbb{F}_{q^2}$, that is the group of invertible $n \times n$ matrices in $\mathbb{F}_{q^2}$ preserving the sesquilinear form defined by $\langle (v_i)_{1\leq i\leq n}, (w_i)_{1\leq i\leq n}\rangle = \sum_{i=1}^n v_i w_i^q$ with $(v_i)_{1\leq i\leq n}, (w_i)_{1\leq i\leq n} \in (\mathbb{F}_{q^2})^n$ written in coordinates with respect to the standard basis.
Note that in this example F' is not the standard Frobenius morphism (indeed we have seen that the group of point fixed by F' is a different group from $GL_n(\mathbb{F}_q)$), also if , somehow confusing, by Proposition 2.1.21 there exists a closed embedding ι of $GL_n(\overline{\mathbb{F}_p})$ in some $GL_m(\overline{\mathbb{F}_p})$ (note that does not hold necessarily $m = n$) such that $\iota \circ F' = F_q \circ \iota$.
For instance, consider $g \in GL_{2n}(\overline{\mathbb{F}_p})$ such that $g^{-1}F_q(g) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. The fact that such a g exists in $GL_{2n}(\overline{\mathbb{F}_p})$ will be a consequence of Theorem 2.2.2. Then consider the closed embedding

$$\begin{aligned} \iota : GL_n(\overline{\mathbb{F}_p}) &\to GL_{2n}(\overline{\mathbb{F}_p}) \\ A &\mapsto g \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} g^{-1} \end{aligned}$$

This is rational since for any $A \in GL_n(\overline{\mathbb{F}_p})$

$$\begin{aligned} \iota \circ F'(A) &= \\ g \begin{pmatrix} F_q(A^{-T}) & 0 \\ 0 & F_q(A) \end{pmatrix} g^{-1} &= \\ g \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_q(A) & 0 \\ 0 & F_q(A^{-T}) \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} g^{-1} &= \\ gg^{-1}F_q(g) \begin{pmatrix} F_q(A) & 0 \\ 0 & F_q(A^{-T}) \end{pmatrix} F_q(g^{-1})gg^{-1} &= \\ F_q(g) \begin{pmatrix} F_q(A) & 0 \\ 0 & F_q(A^{-T}) \end{pmatrix} F_q(g^{-1}) &= \\ F_q \circ \iota(A) \end{aligned}$$

2.2 Lang map and rational structure

In this section, G will always be an algebraic group over $\overline{\mathbb{F}_p}$ defined over $\mathbb{F}_q$ and F will be a generalized Frobenius morphism.

2.2.1 Lang map and Lang-Steinberg Theorem

The main philosophy of what follows in this thesis will be to study finite groups of Lie type by gaining information from the reductive group from which they arise. To pursue this goal, it is very useful to define the following map, known as *Lang map.*

Definition 2.2.1. *The morphism of algebraic varieties*

$$\begin{aligned} \mathcal{L} :& G \to G \\ & g \to g^{-1}F(g) \end{aligned}$$

is called the Lang map.

We have that $\mathcal{L}(x) = 1$ if and only if x is a F-stable point. Moreover if $\mathcal{L}(g) = \mathcal{L}(h)$ then there exists a F-stable element $x \in G^F$ such that $g = xh$; furthermore if G is connected, we will see in Theorem 2.2.2, known as *Lang-Steinberg Theorem* that $\mathcal{L}$ is surjective.
As we already said, now we are going to prove surjectivity of the Lang map in the case in which G is connected. This is a key result for everything that follows.

Theorem 2.2.2. *(Lang-Steinberg Theorem) Let G be connected. Then the Lang map $\mathcal{L} : G \to G$ is surjective.*

Proof. Let $x \in G$, and define the map $\mathcal{L}_x : G \to G$ by $\mathcal{L}_x(g) = g^{-1}xF(g)$. The proof articulates in two steps:

- The image of $\mathcal{L}_x$ contains an open subset of G.
The key point is that the differential of a generalized Frobenius morphism at the identity is nilpotent, because the differential of a Frobenius morphism at the identity vanishes. Indeed let as usual F_q be the standard Frobenius morphism over $GL_n(\overline{\mathbb{F}_p})$, $\delta \in T_{I_n}(GL_n(\overline{\mathbb{F}_p}))$ be a derivation and recall that $\overline{\mathbb{F}_p}[GL_n(\overline{\mathbb{F}_p})] = \overline{\mathbb{F}_p}[T_{ij}, \frac{1}{detT}]$ with $i, j = 1 \dots n$. We have the following computation:

$$(d_1 F_q)(\delta)(T_{ij}) = \delta \circ F_q^*(T_{ij}) = \delta(T_{ij}^q) = qT_{ij}^{q-1}\delta_X(T_{ij}) = 0.$$

By Proposition 2.1.21, this implies the vanishing of the differential at the identity for any Frobenius morphism.
Then denoting by ι the inversion map $g \mapsto g^{-1}$ of G, $d_1\mathcal{L}$ is given by

$$d_1\mathcal{L} = d_1\iota + d_1F = -id_{T_1(G)} + d_1F$$

that is surjective since d_1F is nilpotent.
Now take $F' = xFx^{-1} = c_x \circ F$. By Example 2.1.26 it is still a generalized Frobenius morphism (so d_1F' is nilpotent), so defining $\mathcal{L}' : G \to G$ as $\mathcal{L}'(g) = g^{-1}F'(g)$ we still have that its differential at the identity $d_1\mathcal{L}'$ is surjective. Hence the image of $\mathcal{L}'$ contains an open set. Writing $r_x : G \to G$ for the right translation for x ($r_x(g) = gx$ for $g \in G$) we have that $\mathcal{L}_x = r_x \circ \mathcal{L}'$ and hence, since right translation by an element of the group are open maps $\mathcal{L}_x$ contains an open set as well.

- The Lang map $\mathcal{L}$ is surjective.
We proved that the image of the map $\mathcal{L}_x$ contains an open set of G. Then since G is connected, hence irreducible, this open set is dense (that is, $\mathcal{L}_x$ is dominant). Note that the same argoument holds for $x = 1$, hence $\mathcal{L} = \mathcal{L}_1$ is dominant as well. It follows that $\mathcal{L}(G) \cap \mathcal{L}_x(G) \neq \emptyset$, so there exist $g, h \in G$ such that $g^{-1}F(g) = h^{-1}xF(h)$ that yields $x = \mathcal{L}(gh^{-1})$.

□

Corollary 2.2.3. *Let G be connected, H be a closed normal connected and F-stable subgroup of G. Then*

$$G^F / H^F \cong (G/H)^F$$

Proof. Let $\pi : G \to G/H$ be the canonical projection map. Then π maps G^F to $(G/H)^F$ with kernel H^F, so there is an injective map $\tilde{\pi} : G^F/H^F \to (G/H)^F$. Surjectivity follows from the Lang Steinberg Theorem. Indeed let $gH \in (G/H)^F$, i.e. $F(gH) = gH$. Since H is F-stable, this means that $g^{-1}F(g) \in H$. But since H is connected, by Lang Steinberg Theorem the Lang map $\mathcal{L} : H \to H$ is surjective, hence there exists $h \in H$ satisfying $h^{-1}F(h) = g^{-1}F(g)$. Then $gh^{-1} \in G^F$ and so $\pi(gh^{-1}) = gh^{-1}H = gH$. □

Note that the above corollary does not hold if the subgroup H is not connected, as shown in the following example.

Example 2.2.4. *Let the characteristic p of the base field be odd. Consider the connected reductive algebraic group $SL_2(\overline{\mathbb{F}_p})$ and the standard Frobenius morphism F_q, with q a power of p. We have $SL_2(\overline{\mathbb{F}_p})^{F_q} = SL_2(\mathbb{F}_q)$.*
The center of $SL_2(\overline{\mathbb{F}_p})$ is $Z = \{I, -I\}$, so it is F stable but not connected, and the map

$$\pi : SL_2(\mathbb{F}_q) \to \left({SL_2(\overline{\mathbb{F}_p})}/{\{I, -I\}} \right)^{F_q}$$

is not surjective.
For instance, let $\zeta \in \overline{\mathbb{F}_p}^*$ such that $\zeta^q = -\zeta$. Since the characteristic p is odd, $\zeta \neq -\zeta$ and hence $\zeta \in \overline{\mathbb{F}_p}^* \setminus \mathbb{F}_q$. Then for

$$A = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

the coset AZ lies in $\left({SL_2(\overline{\mathbb{F}_p})}/{\{I, -I\}} \right)^{F_q}$, since $F_q(A) = -A$, but since $\pm A$ do not have entries in $\mathbb{F}_q$, there is no matrix in $SL_2(\mathbb{F}_q)$ whose image is AZ.

2.2.2 Existence of F-stable objects

Now we state another fundamental consequence of Lang Steinberg Theorem, that will ensure the existence of F-stable objects (such as tori or Borel subgroups) and will allow us to deduce structural results about groups of Lie type from the knowledge we have of algebraic groups.

Proposition 2.2.5. *Let G be connected. Let X be a non empty set endowed with a G-action, that we write as $(g, x) \to g.x$ ($g \in G, x \in X$), and with a map $F' : X \to X$ compatible with the action and with F, that is $F'(g.x) = F(g)F'(x)$. Then any F'-stable orbit $\mathcal{O}$ of the G-action on X contains a F'-fixed point.*

Proof. Let $\mathcal{O}$ be an orbit of G in X, and assume it F'-stable. Let $x \in \mathcal{O}$; since $\mathcal{O}$ is F'-stable, $F'(x) \in \mathcal{O}$, so there exists $g \in G$ satisfying $F'(x) = g.x$. Since G is connected, the Lang map $\mathcal{L}$ is surjective, hence there exists $h \in G$ satisfying $h^{-1}F(h) = g^{-1}$. Then $F'(h.x) = F(h).F'(x) = F(h)g.x = h.x$, so $h.x \in \mathcal{O}$ is F' stable. □

The above theorem has some fundamental consequences, about the existence of F-stable "substructures" of G.

Corollary 2.2.6. *Let G be connected. Then*

1. *There always exists an F-stable Borel subgroup B_0.*

2. *Any F-stable Borel subgroup B contains an F-stable maximal torus T_0.*

3. *For any standard parabolic subgroup P_I ($I \leq \Delta$, Δ base for the root system of G) there exists an F-stable parabolic subgroup P_0 conjugated to P_I.*
   Moreover, any F-stable parabolic subgroup admits an F-stable Levi decomposition.

Proof. 1. Consider the set

$$\mathcal{B} = \{B \subseteq G | B \text{ is a Borel subgroup of } G\}.$$

The (generalized) Frobenius morphism F induces a map

$$\begin{aligned} F' : & \mathcal{B} \to \mathcal{B} \\ & B \to F(B) \end{aligned}$$

This is well defined since F is a morphism of algebraic groups and so $F(B)$ is still a closed solvable connected subgroup, and the maximality follows from the bijectivity of F.
Then G acts on $\mathcal{B}$ by conjugation on the Borel subgroups:

$$g.B = gBg^{-1} \quad \text{for any } B \in \mathcal{B},$$

and F' is compatible with this action and with F. The action is transitive by Remark 1.2.24, so there is just one orbit in $\mathcal{B}$. Therefore it is F'-stable and we can conclude that there is a F'-fixed point in $\mathcal{B}$, that is an F-stable Borel subgroup of G.

2. Take as group B_0 and let it act by conjugation on the set of its maximal tori. A reasoning analogous to the previous one shows that there exists an F-stable maximal torus.

3. Similarly to the previous points [3, Corollary 4.2.15]

□

In what follows, we will often need to fix or use an F-stable torus or Borel subgroup, and we will usually denote them by T_0 and B_0 respectively.

Note that in the Remark 2.2.6 we stated that any F-stable Borel subgroup contains a maximal torus, but the converse is not true: in general, not any F-stable maximal torus is contained in an F-stable Borel subgroup.

Definition 2.2.7. *An F-stable torus contained in an F-stable Borel subgroup is called maximally split.*

Example 2.2.8. • *Consider the group $GL_n(\overline{\mathbb{F}_p})$ with the standard Frobenius F_q. Then T_n (the subgroup of the upper triangular matrix) is an F-stable Borel subgroup, and D_n (the subgroup of the diagonal matrices) is an F-stable maximal torus contained in it.*

• *Consider again the group $GL_n(\overline{\mathbb{F}_p})$ but with Frobenius morphism $F' = \tau \circ F_q$ as in Example 2.1.27.*
Then D_n is an F-stable maximal torus, but T_n is not an F-stable Borel subgroup: $F'(T_n)$ is the Borel subgroup of the lower triangular matrices. In other words, using the notation of Definition 1.2.64, $T_n \xrightarrow{\omega} F'(T_n)$, where $\omega \in W$ denotes the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ (recall that the Weyl group W of $GL_n(\overline{\mathbb{F}_p})$ is isomorphic to S_n and can be represented by permutation matrices).

2.2.3 Parameterization of F-stable objects

We are now going to give a refinement of Proposition 2.2.5 that allows us to parameterize the orbits of the finite group of Lie type G^F. It will be particularly useful, among other applications, to investigate conjugacy classes of G^F.

Definition 2.2.9. *We say that two elements of $x, y \in G$ are F-conjugate if exists $g \in G$ such that $y = g^{-1}xF(g)$.*

Note that with this terminology the Lang Steinberg Theorem can be rephrased saying that if G is connected, then it has just one F-conjugacy class, since surjectivity of the Lang map is equivalent to the fact that the F-conjugation orbit containing the identity is in fact the whole group.
Adopting this point of view, it follows that connected closed subgroups of a not necessarily connected group G are always all contained in the F-conjugacy class of the identity, from which we can deduce the following Lemma, that will be needed for Theorem 2.2.11

Lemma 2.2.10. *Let H be a closed normal connected F-stable subgroup of G. Then the natural projection induces a bijection between F-conjugacy classes of G and F-conjugacy classes of ${}^{G}\!/_{H}$*

Proof. Clearly F-conjugate elements are mapped into F-conjugated elements.
Now assume $gH, g'H$ to be F-conjugated in ${}^{G}\!/_{H}$. This means that there exists an $h \in H$ and an $x \in G$ satisfying $hg = x^{-1}g'F(x)$. Then consider the morphism $gF = c_g \circ F$: it is a (generalized) Frobenius morphism by Example 2.1.26. Moreover since H is normal in G, gF can be restricted to a Frobenius morphism on H (Lemma 2.1.24). Then since H is connected by Lang-Steinberg Theorem applied to gF there

exists an $y \in H$ such that $h = y^{-1}gF(y)g^{-1}$. Therefore $y^{-1}gF(y) = hg = x^{-1}g'F(x)$ from which we obtain $g = (xy)^{-1}g'F(xy)$, that is g and g' are F-conjugated in G. □

Theorem 2.2.11. *Let G,X,$F' : X \to X$ be as in Proposition 2.2.5, and let $\mathcal{O}$ be an F'-stable G-orbit in X. Let $x_0 \in \mathcal{O}^F$ and assume $Stab_G(x_0)$ is closed. Then the following holds.*

1. *Let $g \in G$. Then $g.x_0 \in \mathcal{O}^F$ if and only if $\mathcal{L}(g) \in Stab_G(x_0)$.*
2. *The Lang map induces a bijection*

$$\begin{aligned} \{G^F\text{-orbits on } \mathcal{O}^F\} &\to \{F\text{-conjugacy classes in } {}^{Stab_G(x_0)}\!/\!_{(Stab_G(x_0))_0}\} \\ G^F.(g.x_0) &\to \overline{\mathcal{L}(g)} \end{aligned}$$

(where the overlying bar on the right hand side denotes the equivalence class in the quotient ${}^{Stab_G(x_0)}\!/\!_{Stab_G(X_0)^0}$)

Proof. Note that an $x_0 \in \mathcal{O}^F$ as in the assumptions exists by Proposition 2.2.5.

1. Observe that

$$\begin{aligned} &g.x_0 \in \mathcal{O}^F && \text{if and only if} \\ &g.x_0 = F(g.x_0) = F(g)F'(x_0) = F(g).x_0 && \text{if and only if} \\ &x_0 = g^{-1}F(g).x_0 = \mathcal{L}(g).x_0 && \text{if and only if} \\ &\mathcal{L}(g) \in Stab_G(x_0) \end{aligned}$$

2. By Lemma 2.2.10 it is enough to prove that the map

$$\begin{aligned} \{G^F\text{-orbits on } \mathcal{O}^F\} &\to \{F\text{-conjugacy classes in } Stab_G(x_0)\} \\ G^F.(g.x_0) &\to \overline{\mathcal{L}(g)} \end{aligned}$$

is a well defined bijection. Indeed $Stab_G(x_0)$ is a closed subgroup by assumption, hence it inherits the structure of algebraic group and then its connected component containing the identity $Stab_G(x_0)^0$ is always a closed normal connected subgroup. Moreover $Stab_G(x_0)$ is F-stable, since $x_0 \in \mathcal{O}^F$ and so $g.x_0 = x_0$ implies $F(g).x_0 = F(g).F(x_0) = F(g.x_0) = F(x_0) = x_0$. Therefore in particular also the connected component of the identity $Stab(x_0)^0$ is F-stable (since F, being a rational morphism, is continuous).
The map is well-defined. Indeed we have a well defined map

$$\mathcal{O}^F \to \{F\text{-conjugacy class of } Stab_G(x_0)\} \tag{2.1}$$

induced by $g \mapsto \mathcal{L}(g)$: if $g, h \in G$ are such that $g.x_0 = h.x_0 \in \mathcal{O}^F$ then $y = h^{-1}g \in Stab_G(x_0)$, so (since $g = hy$)

$$\mathcal{L}(g) = g^{-1}F(g) = y^{-1}h^{-1}F(h)F(y) = y^{-1}\mathcal{L}(h)F(y)$$

that is, $\mathcal{L}(g)$ and $\mathcal{L}(h)$ are in the same F-conjucagy class in $Stab_G(x_0)$.
Moreover if $h \in G^F$, the elements $g.x_0, gh.x_0 \in \mathcal{O}^F$ have the same image in (2.1): $\mathcal{L}(gh) = h^{-1}g^{-1}F(g)F(h) = h^{-1}\mathcal{L}(g)F(h)$, that is $\mathcal{L}(gh), \mathcal{L}(g)$ are in the same F-conjugacy class in $Stab_G(x_0)$.
The we can consider the map

$$\begin{aligned} \{G^F\text{-orbits on } \mathcal{O}^F\} &\to \{F\text{-conjugacy classes in } Stab_G(x_0)\} \\ G^F.(g.x_0) &\to \overline{\mathcal{L}(g)} \end{aligned}$$

as in the statement.
This map is injective. Indeed, take $h.x, g.x \in \mathcal{O}^F$ such that $\mathcal{L}(g), \mathcal{L}(h)$ are F-conjugate in $Stab_G(x_0)$, that is, there exists an $n \in Stab_G(x_0)$ such that $g^{-1}F(g) = n^{-1}h^{-1}F(h)F(n)$. Then $gn^{-1}h^{-1}$ is F-stable and $gn^{-1}h^{-1}.(h.x_0) = gn^{-1}.x_0 = g.x_0$ (because $n^{-1} \in Stab_G(x_0)$); hence $h.x_0$ and $g.x_0$ lie in the same G^F orbit.
The map is surjective. Since G is connected, by the Lang-Steinberg Theorem $\mathcal{L}$ is surjective, so in particular any element of $Stab_G(x_0)$ can be written as $\mathcal{L}(g)$ for $g \in G$, and $g.x_0 \in \mathcal{O}^F$ by point 1.

□

Theorem 2.2.11 has a number of relevant applications. Indeed it allows to parameterize up to G^F-conjugation several F-stable objects, whose existence was shown in Corollary 2.2.6.

Corollary 2.2.12. *Let G be connected and reductive.*

1. *The F-stable Borel subgroups of G are all G^F conjugated*
2. *The pairs (T, B) given by a maximally split torus T and a F-stable Borel subgroup in G containing T are all G^F-conjugated.*
3. *Fixing an F-stable maximal torus T_0 and writing $W = N_G(T_0)/T_0$ for the Weyl group, the assignment $gT_0g^{-1} \mapsto \mathcal{L}(g)$ induces a bijection*

$$\left\{ \begin{array}{c} G^F\text{-conjugacy classes of} \\ F\text{-stable maximal tori in } G \end{array} \right\} \to \{F\text{-conjugacy classes in } W\}$$

Proof. 1. By Remark 1.2.24, G acts transitively on $\mathcal{B}$. This action satisfies the assumption of Theorem 2.2.11: taking B_0 an F-stable Borel subgroup we have $Stab_G(B_0) = N_G(B_0) = B_0$, that is closed. Moreover it was already discussed in Corollary 2.2.6 that it satisfies the assumption of Proposition 2.2.5. Hence there is a bijection between G^F-conjugacy classes of F-stable Borel subgroups and F-conjugacy classes of $B_0/(B_0)^0$. But B_0 is connected, hence there is just one G^F-conjugacy class of F-stable Borel subgroups.

2. By Remark 1.2.27, G acts transitively by simultaneous conjugation on the pairs (T, B) with T maximal torus, B Borel subgroup and $T \leq B$. The Frobenius morphism F induces a map on the set of such pairs (it maps tori in tori, Borel in Borel and preserves the inclusion) compatible with F and the G-action. Taking an F-stable pair $(T_0; B_0)$, $Stab_G((T_0, B_0)) = N_G(B_0) \cap N_G(T_0) = B_0 \cap N_G(T_0) = T_0$ which is closed. Moreover also in this case the stabilizer is connected, hence applying Theorem 2.2.11 there is just one G^F-conjugacy class of F-stable pairs (T, B).

3. Again, G acts transitively (by Proposition 1.2.12) by conjugation on the set $\mathcal{T}$ of all maximal tori in G, and this action satisfies the assumptions of Theorem 2.2.11. Then fixing an F-stable torus T_0 the assignment $g.T_0 = gT_0g^{-1} \mapsto \mathcal{L}(g)$ induces a bijection

$$\{G^F\text{-orbits on } \mathcal{T}^F\} \to \left\{ \begin{array}{c} F\text{-conjugacy classes in} \\ Stab_G(T_0) \big/ (Stab_G(T_0))^0 \end{array} \right\}$$

and $Stab_G(T_0) = N_G(T_0)$; by Proposition 1.2.14 $N_G(T_0)^0 = C_G(T_0) = T_0$, so in this case $Stab_G(T_0) \big/ Stab_G(T_0)^0 = W$. Hence the F-stable maximal tori can be parameterized, up to G^F-conjugation, by F-conjugacy classes of the Weyl group of G.

□

2.2.4 Conjugacy classes in finite groups of Lie type

Theorem 2.2.11 has another important consequence: it allows us to parameterize the conjugacy classes of the finite group G^F knowing the F-stable conjugacy classes of G.

Corollary 2.2.13. *Let G be connected. Let $\mathcal{C}$ be an F-stable G-conjugacy class in G. Take a F-stable point $x_0 \in \mathcal{C}^F$. Then the assigment $g.x_0 \to \mathcal{L}(g)$ induces a bijection*

$$\left\{ \begin{array}{c} G^F\text{-conjugacy classes} \\ \text{contained in } \mathcal{C}^F \end{array} \right\} \to \left\{ \begin{array}{c} F\text{-conjugacy classes of} \\ C_G(x_0) \big/ (C_G(x_0))_0 \end{array} \right\}$$

Proof. G acts transitively on $\mathcal{C}$ by conjugation. The Frobenius morphism can be restricted to $\mathcal{C}$ since it is F-stable, and it is compatible with the action of G and with F. Since G is acting by conjugation, $Stab_G(x_0) = C_G(x_0)$ and it is closed. So we can apply Theorem 2.2.11 □

Remark 2.2.14. *Any F-stable point $x \in G^F$ lies in an F-stable conjugacy class $\mathcal{C}$ of G, since $F(gxg^{-1}) = F(g)xF(g^{-1}) \in \mathcal{C}$ for any $g \in G$, and clearly any G^F-conjugacy class is all contained in one G-conjugacy class. Moreover, by Proposition 2.2.5 any F-stable conjugacy class contains a F-stable point. Hence in order to know* *the conjugacy classes of G^F it is enough to know F-stable conjugacy classes of G, and then use Corollary 2.2.13*

Example 2.2.15.
- *In $G = GL_n(\overline{\mathbb{F}_p})$, centralizers are connected. Indeed, for any $A \in GL_n(\overline{\mathbb{F}_p})$, $C(A) := \{A \in M_n(\overline{\mathbb{F}_p}) | gAg^{-1} - A = 0\}$ is a linear subspace of $M_n(\overline{\mathbb{F}_p})$, hence irreducible, and so $C_G(A) = C(A) \cap G$ is a principal open set in $C(A)$ and hence irreducible.*
  Hence any F-stable conjugacy class of $GL_n(\overline{\mathbb{F}_p})$ contains exactly one conjugacy class of $GL_n(\mathbb{F}_q)$.

- *Consider $G = SL_n(\overline{\mathbb{F}_p})$ with p odd, and take $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Its centralizer in $GL_2(\overline{\mathbb{F}_p})$ consists of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a \in \overline{\mathbb{F}_p}^*, b \in \overline{\mathbb{F}_p}$, and it is connected. Intersecting it with G yields the centralizer of u in G, that is therefore*

$$C_G(u) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \mid a \in \overline{\mathbb{F}_p} \right\}.$$

  However, this is not connected: it has two connected components, namely the one consisting of matrices with 1 on the diagonal and the one consisting of matrices with -1 on the diagonal. It follows that the conjugacy class of u in $SL_2(\mathbb{F}_q)$ contains two distinct conjugacy classes of $SL_2(\mathbb{F}_q)$.
  In particular, elements of the form $u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{F}_q$ belong to one of the two classes according to the class of a in ${\mathbb{F}_q}^* / ({\mathbb{F}_q}^*)^2$. Indeed taking $a \in \mathbb{F}_q$, let $\alpha \in \overline{\mathbb{F}_p}$ such that $\alpha^2 = a$. Then u_a is obtained by u by conjugation for $d_\alpha = diag(\alpha, \alpha^{-1})$, and $\mathcal{L}(d_\alpha) = diag(\alpha^{q-1}, \alpha^{1-q})$ that in $C_G(u) / C_G(u)_0$ belongs to the class of I if $\alpha \in \mathbb{F}_q^*$, i.e. if a is a square in $\mathbb{F}_q^*$, and in the class of $-I$ otherwise.

2.3 The action of the Frobenius Morphisms

In this section G will always be a connected reductive algebraic group over $\overline{\mathbb{F}_p}$ with p prime, F a generalised Frobenius morphism for G.

2.3.1 F-stable Tori

Thanks to Corollary 2.2.12, we can always fix a pair (T_0, B_0) in G consisting of an F-stable Borel subgroup B_0 and an F-stable maximal torus T_0 in it. Such a pair is uniquely determined up to G^F-conjugacy. In particular all maximally split tori of G lie in the same G^F conjugacy class as T_0. Recall that by Corollary 2.2.12, the assignment $gT_0g^{-1} \mapsto \mathcal{L}(g)$ induces a bijection

$$\left\{ \begin{array}{c} G^F\text{-conjugacy classes of} \\ F\text{-stable maximal tori in } G \end{array} \right\} \to \{F\text{-conjugacy classes in } W\}$$

so if we fix T_0 to be a maximally split torus, the identity of the Weyl group $W(T_0)$ corresponds under this bijection to the class of maximally split tori.
We introduce the following notation.

Definition 2.3.1. *Let G be connected and reductive, $T_0 \leq G$ be a maximally split torus, let $\omega \in W$. We denote by T_ω any maximal torus contained in the G^F-conjugacy class which correspond to the F-conjugacy class of $\omega \in W(T_0)$ via the bijection induced by $gT_0g^{-1} \mapsto \mathcal{L}(g)$. We say that T_ω is obtained from T_0 by twisting with ω.*

More explicitly, a torus obtained from T_0 by twisting with $\omega \in W$ is of the form

$$T_\omega = \{gtg^{-1} \mid t \in T_0\} \qquad \text{with } g \in G \text{ such that } \mathcal{L}(g) = \dot{\omega}$$

for some $\dot{\omega}$ lift of ω in $N_G(T_0)$. By letting g range in $\mathcal{L}^{-1}(\dot{\omega})$, we obtain the whole G^F conjugacy class of tori denoted by T_ω. Note that by Corollary 2.2.12 (point 3) choosing a different representative of ω in $N_G(T)$ or choosing another element $\omega' \in W$ F-conjugated to ω gives rise to the same G^F-conjugacy class of F-stable tori, hence writing gT_0g^{-1} such that $\mathcal{L}(g) = \omega$ does not give rise to any ambiguity. If $T_\omega = gT_0g^{-1}$ with $\mathcal{L}(g) = \omega$, conjugation c_g induces an isomorphism of algebraic groups $T_0 \to T_\omega$ satisfying

$$c_g \circ \omega F = F \circ c_g.$$

Indeed,

$$c_g \circ \omega F(t) = c_g(c_{g^{-1}F(g)}(F(t))) = c_{F(g)}(F(t)) = F(c_g(t)).$$

In other words, conjugation by g is a morphism of algebraic groups defined over $\mathbb{F}_q$, where T_0 is endowed with the rational structure given by ωF and T_ω with the one given by F. In particular, to study the F-fixed points in T_ω, it can be observed that

$$gtg^{-1} = F(gtg^{-1}) \quad \text{ if and only if } \quad F(t) = \omega^{-1}.t$$

So conjugation by g restricts to an isomorphism of finite groups

$$T_\omega^F \cong T_0^{\omega F} = \{t \in T_0 \mid \omega.F(t) = t\}.$$

Since T_0 is F-stable, the Frobenius morphism F induces a map on the Weyl group $W = {N_G(T_0)}/{T_0}$; by Corollary 2.2.3, the F-stable points of W are given by

$$W^F = {N_G(T_0)^F}/{T_0^F}.$$

Conjugation by g induces an isomorphism between the F-stable points of the Weyl group relative to a torus obtained from T_0 by twisting with $\omega \in W$, $W(T_\omega) = {}^{N_G(T_\omega)}\!/_{T_\omega}$, and the ωF-fixed points of W:

$$W_G(T_\omega)^F \cong W^{\omega F} = \{x \in W \mid \omega.F(x) = x\}.$$

So, in particular, the Weyl groups obtained from different F-stable tori have non-isomorphic groups of fixed points points.

Example 2.3.2.

- *Consider $GL_2(\overline{\mathbb{F}_p})$ with the $\mathbb{F}_q$-rational structure given by the standard Frobenius F_q. As seen in Corollary 2.2.12, the $GL_2(\mathbb{F}_q)$ conjugacy classes of F-stable maximal tori in $GL_2(\overline{\mathbb{F}_p})$ are parameterized by the Weyl group $W \cong S_2 = \{I, \sigma\}$.*

  The identity I corresponds to the class of maximally split tori. A torus in this class is D_2, the subgroup of diagonal matrices (see Example 2.2.8). Hence the maximally split tori in $GL_2(\overline{\mathbb{F}_p})$ with respect to this $\mathbb{F}_q$-rational structure are all and only the tori of the shape

  $$T = gD_2g^{-1} \qquad\qquad \text{with } g \in GL_2(\mathbb{F}_q).$$

  Now we find a representative for the class of the tori that can be obtained from D_2 by twisting with σ. This is the class of F-stable tori that are not contained in any F-stable Borel.

  Taking $\alpha \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q$, the matrix

  $$x = \begin{pmatrix} 1 & 1 \\ \alpha & \alpha^q \end{pmatrix}$$

  satisfies

  $$\mathcal{L}(x) = x^{-1}F_q(x) = \frac{1}{\alpha^q - \alpha}\begin{pmatrix} \alpha^q & -1 \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \alpha^q & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  that is a representative of σ in $N_{GL_2(\overline{\mathbb{F}_p})}(D_2)$. So a maximal F-stable torus in $GL_2(\overline{\mathbb{F}_p})$ non-maximally split will be given by

  $$T_\sigma = xD_2x^{-1} = \left\{ s_{a,b} = \frac{1}{\alpha^q - \alpha}\begin{pmatrix} a\alpha^q - b\alpha & b - a \\ (a-b)\alpha^{q+1} & b\alpha^q - a\alpha \end{pmatrix} \mid a, b \in \overline{\mathbb{F}_p}^* \right\}$$

  where $s_{a,b}$ has been computed as $s_{a,b} = x\mathrm{diag}(a,b)x^{-1}$ for any $a, b \in \overline{\mathbb{F}_p}$.

  Then the F-stable maximal tori in $GL_2(\overline{\mathbb{F}_p})$ that are not contained in a F-stable Borel subgroup will be all and only the ones in the shape

  $$T = gT_\sigma g^{-1} \qquad\qquad \text{with } g \in GL_2(\mathbb{F}_q).$$

A direct computation shows that the Frobenius morphism acts on $s_{a,b} \in T_\sigma$, with $a, b \in \overline{\mathbb{F}_p}$, by

$$F(s_{a,b}) = s_{b^q,a^q}$$

hence the fixed points are given by

$$T_\sigma^F = \{s_{a,b} | a, b \in \mathbb{F}_{q^2}^*, b = a^q\}.$$

We observe that the morphism σF acts on D_2 by $\sigma F(diag(a,b)) = diag(b^q, a^q)$. So we have

$$F(xdiag(a,b)x^{-1}) = F(s_{a,b}) = s_{b^q,a^q} = xdiag(b^q,a^q)x^{-1} = x(\omega F(diag(a,b)))x^{-1},$$

and indeed from the theory we expected c_x to be defined over $\mathbb{F}_q$ with respect to the rational structures on T_0 and T_σ given respectively from σF and F;

hence in particular the fixed points are given by

$$D_2^{\sigma F} = \{diag(a,b) \mid a, b \in \mathbb{F}_{q^2}, b = a^q\},$$

and the conjugation by x yields indeed an isomorphism $c_x : D_2^{\sigma F} \to T_\sigma^F$.

- *Let $G = GL_n(\overline{\mathbb{F}_p})$, $F = F_q$ the standard Frobenius morphism and let $T = D_n$ the maximally split torus of the diagonal matrices.*
  G^F-conjugacy classes are parameterized by F-conjugacy classes in $W \cong S_n$ (the symmetric group). Since the action induced by F_q on S_n is trivial, F_q-conjugacy classes of S_n are the same as usual conjugacy classes, hence they are given by the possible lengths of the disjoint cycles in the permutations factorization (therefore in bijection with the partitions of n). Let the parameterization be such that the G^F-conjugacy class of T corresponds to the identity of S_n.
  Let $\omega_n = (1, 2, \ldots, n) \in S_n$, T_{ω_n} a maximal torus obtained from T by twisting with ω_n. Then

$$T_{\omega_n}^F \cong T^{\omega_n F} = \{t \in T \mid t = \omega_n F(t)\omega_n^{-1}\}.$$

Let $t = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \in T$*; then*

$$\omega_n F(t)\omega_n = \begin{pmatrix} t_n^q & & & \\ & t_1^q & & \\ & & \ddots & \\ & & & t_{n-1}^q \end{pmatrix},$$

hence $t \in T^{\omega F}$ if and only if $t_i = t_{\omega(i)}^q$ for any $i \geq 1$, that is

$$\begin{cases} t_1 = t_1^{q^n}, \\ t_i = t_{i-1}^q = t_1^{q^i} \text{ for any } i \geq 2. \end{cases}$$

So $T^F_{\omega_n} \cong T^{\omega_n F} \cong \mathbb{F}^*_{q^n}$ *as an abelian group.*
In general, let $\omega \in S_n$ *and assume that the disjoint cycles in its factorization have lengths* $n_1, n_2, \dots n_k$*. Then* w *is in the same conjugacy class as the permutation* $\sigma_1\sigma_2\cdots\sigma_k = (1, 2, \dots, n_1)(n_1+1, n_1+2 \dots n_1+n_2)\cdots(\sum_{j=1}^{k-1} n_j, \dots n_k)$*; then any of these cycles* σ_i *is of the same kind of the permutation* ω_n *considered above, and any of them acts on a different block of the matrices of* T*. Hence*

$$T^F_\omega \cong T^{\omega F} \cong \begin{pmatrix} D^{\sigma_1 F}_{n_1} & & & \\ & D^{\sigma_2 F}_{n_2} & & \\ & & \ddots & \\ & & & D^{\sigma_k F}_{n_k} \end{pmatrix} \cong \mathbb{F}^*_{q^{n_1}} \times \mathbb{F}^*_{q^{n_2}} \times \cdots \times \mathbb{F}^*_{q^{n_k}}.$$

In particular, the order of T^F_ω *is given by the minimal polynomial of* ω *(in this case,* $p_\omega = (x^{n_1} - 1)(x^{n_2} - 1)\cdots(x^{n_k} - 1)$*) evaluated on* q*,*

- *Consider* $GL_n(\overline{\mathbb{F}_p})$ *with Frobenius morphism* $F' = \tau \circ F_q$ *as in example 2.1.27. We have seen in Example 2.2.8 that* D_n *is* $F-$ *stable and* $T_n \xrightarrow{\omega} F(T_n)$ *with* $\omega = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$.
Then take $g \in GL_n(\overline{\mathbb{F}_p})$ *such that* $\mathcal{L}(g) = \dot\omega$ *and set* B_0 *to be the Borel* $B_0 = gT_ng^{-1}$*. Since* $\omega = \omega^{-1} = \mathcal{L}(g)^{-1} = F(g^{-1})g)$ *we have*

$$F(B_0) = F(g)F(T_n)F(g^{-1}) = F(g)\omega T_n\omega^{-1}F(g^{-1}) = gT_ng^{-1} = B_0,$$

that is, $B_0 = g^{-1}T_ng$ *is a* F*-stable Borel subgroup. Therefore set* $T_0 = gD_ng^{-1}$*. It is* F*-stable because* $g^{-1}F(g) = \dot\omega \in W(D_n)$*, so in particular* $g \in N_G(D_n)$ *(we are using point 1 of Theorem 2.2.11). Moreover* T_0 *is maximally split since it is contained in the* F*-stable Borel* B_0*. Hence* $D_n = g^{-1}T_0g$*, and* $\mathcal{L}(g^{-1}) = gF(g^{-1}) = gF(g^{-1})gg^{-1} = g\dot\omega^{-1}g^{-1}$ *that is a representative of* $\omega^{-1} = \omega$ *in* $W(T_0)$ *(see Proposition 1.2.65); so* D_n *is obtained from the maximal split torus* T_0 *by twisting with* ω*.*

2.3.2 F-action on characters

Now Let T be an F-stable maximal torus. Then F induces maps on the character and cocharacter groups defined by:

$$F(\chi) = \chi \circ F \qquad \text{for } \chi \in X(T) \tag{2.2}$$
$$F(\gamma) = F \circ \gamma \qquad \text{for } \gamma \in Y(T) \tag{2.3}$$

These actions satisfy

$$\langle F(\chi), \gamma\rangle = \langle \chi, F(\gamma)\rangle$$

because $F(\chi) \circ \gamma = \chi \circ F \circ \gamma = \chi \circ F(\gamma)$.

Example 2.3.3. • *Let $G = GL_n(\overline{\mathbb{F}_p})$, $F = F_q$ the standard Frobenius morphism, $T = D_n$ the maximally split torus of diagonal matrices. Character and cocharacter groups are given by*

$$X(T) = \langle \chi_k | 1 \leq k \leq n \rangle_{\mathbb{Z}} \qquad \text{with } \chi_i(t) = t_i \text{ for } t = diag(t_j)_{1\leq j\leq n} \in T$$
$$Y(T) = \langle \gamma_k | 1 \leq k \leq n \rangle_{\mathbb{Z}} \qquad \text{with } \gamma_i(\lambda) = diag(\lambda^{\delta_{i,j}})_{1\leq j\leq n} \text{ for } \lambda \in \overline{\mathbb{F}_p}^*.$$

Then

$$F(\chi_i)(t) = F(t_i) = (t_i)^q = (\chi_i(t))^q$$
$$and F(\gamma_i)(\lambda) = F(diag(\lambda^{\delta_{i,j}})_{1\leq j\leq n}) = diag(\lambda^{q\delta_{i,j}})_{1\leq j\leq n} = \gamma_i(\lambda)^q$$

This shows that

$$F(\chi) = q\chi \qquad \text{for any } \chi \in X(T) \tag{2.4}$$
$$F(\gamma) = q\gamma \qquad \text{for any } \gamma \in Y(T) \tag{2.5}$$

• *Let $G = GL_n(\overline{\mathbb{F}_p})$, $F = F_q$ the standard Frobenius morphism, $T = T_\omega$ a F-stable torus obtained from D_n by twisting with $\omega \in W(D_n)$. If $\mathcal{L}(x) = \omega$, then $c_x : D_n \to T$ is a isomorphism of algebraic groups satisfying $c_x \circ \omega F_q = F_q \circ c_x$. Recall that conjugation c_x induces an isomorphism on the Weyl groups $c_x : N_G(D_n)/D_n \to N_G(T)/T$; we continue to denote by ω the image of ω under this isomorphism. Then, the action of F_q on the character group $X(T)$ can be described as*

$$F_q(\chi) = q\omega^{-1}.\chi$$

The action of F on the character group $X(T)$ can be described as follows.

Proposition 2.3.4. *Let T be a torus defined over $\mathbb{F}_q$ with Frobenius morphism F, $X(T)$ its character group. Then the action of F on $X(T)$ can be written as $F = qF_0$, with $F_0 : X(T) \to X(T)$ group automorphism of finite order.*

Proof. A character $\lambda : T \to \overline{\mathbb{F}_p}^*$ can be regarded as a regular function on T (postcomposing with the natural inclusion $\overline{\mathbb{F}_p}^* \to \overline{\mathbb{F}_p}$). This define an inclusion $X(T) \to \overline{\mathbb{F}_p}[T]$, therefore by definition of Frobenius morphism we can define

$$\psi : X(T) \to \overline{\mathbb{F}_p}(T)$$

such that

$$F^*(\psi(\lambda)) = \lambda^q.$$

(that is, ψ is the restriction to $X(T)$ of the arithmetic Frobenius map defined in Remark 2.1.3).
Note that for any $t \in T$

$$\psi(\lambda)(F(t)) = F^*(\psi(\lambda))(t) = \lambda(t)^q \tag{2.6}$$

that is, $\psi(\lambda)$ is defined by the following commutative diagramm

$$\begin{array}{ccc} T & \xrightarrow{F} & T \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \psi(\lambda)} \\ \overline{\mathbb{F}_p}^* & \xrightarrow{F_q} & \overline{\mathbb{F}_p}^* \end{array}$$

and since F is a bijective group morphism this equation determine uniquely $\psi(\lambda)$. It follows that for any $\lambda \in X(T)$ it holds $\psi(\lambda) \in X(T)$. Indeed $\psi(\lambda)$ is a group morphism $T \to \overline{\mathbb{F}_p}^*$ because all the arrows in the diagram are group morphisms. Hence we can write $\psi : X(T) \to X(T)$, and since (2.6) can be written as $F \circ \psi = q$ in $X(T)$. By bijectivity of F and the fact that q is a group automorphism of $X(T)$ ψ is a group automorphism of $X(T)$.
Furthermore, (2.6) implies that for all $m \geq 1$ we have $(\psi^m(\lambda))(F^m(t)) = \lambda(t^{q^m})$, that is $F^m \circ \psi^m = q^m$ on $X(T)$.
The group morphism ψ has finite order. Indeed by Proposition 2.1.21 T can be embedded as algebraic group in $GL_n(\overline{\mathbb{F}_p})$ for some $n \in \mathbb{N}$ with $\mathbb{F}_q$-rational structure given by the standard Frobenius morphism F_q.Since the embedding is a rational morphism of algebraic groups, T will be embedded in some F_q-stable torus $T' \leq GL_n(\overline{\mathbb{F}_p})$. Then in $GL_n(\overline{\mathbb{F}_p})$ by Example 2.3.3 the action of F_q on $X(T')$ can be written as $F_q = q\omega$ with ω an element of the Weyl group of $GL_n(\overline{\mathbb{F}_p})$ (relative to T'). Since the Weyl group is finite, it follows that there exists an $m \in \mathbb{N}$ such that $F_q^m = q^m id$ on $X(T')$, and so for such an m we have $F^m = q^m$ on $X(T)$, hence $F^m = q^m = F^m \circ \psi^m$ on $X(T)$, that is (by bijectivity of F^m) $\psi^m(\lambda) = \lambda$, so ψ has finite order. In particular ψ is invertible, so setting $F_0 = \psi^{-1}$ we obtain that F acts on $X(T)$ as qF_0. □

Remark 2.3.5. *[2, §1.18] In particular if F is a Frobenius morphism and $T \leq G$ is an F-stable maximal torus, the group morphism $F_0 : X(T) \to X(T)$ permutes the roots of the root system Φ of G relative to T.*
Note moreover that since T is F-stable, F permutes the root subgroups U_α because they are the minimal non trivial subgroups of G normalized by T, and it holds $F(U_\alpha) = U_{F_0(\alpha)}$.

Corollary 2.3.6. *[8, Proposition 1.4.19] Let F be a generalised Frobenius morphism for G such that F^d is a Frobenius morphism defining G over $\mathbb{F}_{q_0}$. Let q be the positive real number defined by $q^d = q_0$. Then for any F-stable maximal torus T of G the action induced by F on the character group, $F : X(T) \to X(T)$, satisfies*

$$|det(F)| = q^{rank_{\mathbb{Z}}(X(T))}.$$

Moreover, the extension of F on $X_{\mathbb{R}} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ can be written as

$$F = qF_0 \qquad \text{with} \quad F_0 \in GL(X_{\mathbb{R}}),$$

with F_0 of finite order and inducing a permutation on the root system of G.

In particular, the number q in Corollary 2.3.6 does not depend on the possible choices of d and q_0 in its definition (since it can be deduced uniquely from the determinant of F and the rank of the character group). Therefore we can give the following definition

Definition 2.3.7. *The F-rank of a F-stable torus T in G is the dimension of the q-eigenspace, (q as in Corollary 2.3.6) of the extension of F on $X_{\mathbb{R}} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. If the F-rank of T is equal to the rank of T, we say that T is F-split*

Remark 2.3.8. *Note that if $T \leq G$ is a F-stable torus with respect to F, then there exists an $n \in \mathbb{N}$ such that T is F^n-split: it is enough to take n as the order of $F_0 \in GL(X_{\mathbb{R}})$ in Corollary 2.3.6.*
Observe that T is F^n-split, it implies that the isomorphism $T \to (\overline{\mathbb{F}_p}^*)^r$, with $r = rank(T)$, that can be regraded as an element of $(X(T))^r$, is rational considering T endowed with the rational structure given by F^n and $(\overline{\mathbb{F}_p}^*)^r$ endowed with the rational structure given by the standard Frobenius morphism F_{q^n} (q as in Corollary 2.3.6) [3, Proposition 7.1.3]. Therefore in particular q^n is a power of p and F^n is a Frobenius morphism inducing an $\overline{\mathbb{F}}_{q^n}$-structure.

Note moreover that all maximally split maximal tori of G are G^F-conjugated, so they are isomorphic through a rational morphism of variety and therefore F acts in the same way on all maximally split tori, and in particular it induces the same action on the character groups of maximally split tori. So we can give the following definition:

Definition 2.3.9. *The F-rank of G is the F-rank of the maximally split tori of G. We use the following notation:*

$$\varepsilon_G = (-1)^{(F\text{-rank of } G)}$$

2.3.3 Duality for finite groups of Lie type

Now we wish to define what it means to be dual for group with a rational structure, that is: given G and G^* connected reductive algebraic groups dual to each other and endowed with Frobenius morphisms F and F^* respectively, we wish to give "compatibility" conditions between the isomorphism inducing the duality and the Frobenius morphisms.

Definition 2.3.10. *Let G, G^* be connected reductive algebraic groups with generalised Frobenius morphisms F, F^*. We say that (G, F) and (G^*, F^*) are dual to each other if there is a maximally split torus $T \leq G$ and a maximally split torus $T^* \leq G^*$ such that*

1. *The root data of the two groups $\Psi(G) = (X(T), \Phi(G), Y(T), \Phi^\vee(G))$ and $\Psi(G^*) = (X(T^*), \Phi(G^*), Y(T^*), \Phi^\vee(G^*))$ are dual to each other, with isomorphism*
$$\delta : X(T) \to Y(T^*)$$

2. It holds

$$\delta \circ F = F^* \circ \delta$$

Theorem 2.3.11. *[8, §1.5.18]For any pair (G, F) consisting of a connected reductive group with a generalised Frobenius morphism there exists a corresponding dual pair $(G; F^*)$ as in Definition 2.3.10.*

Example 2.3.12. *Consider $GL_n(\overline{\mathbb{F}_p})$ with the standard Frobenius morphism F_q. We show that it is self dual with respect to the maximally split torus $T = D_n$ Recall (Example 1.2.48) that the root datum of $GL_n(\overline{\mathbb{F}_p})$ is given by*

$$\Psi = (\Phi, X(T), \Phi^\vee, Y(T)) :$$

$$\begin{aligned} \Phi &= \{\chi_i - \chi_j | 1 \le i \ne j \le n\}, \\ X(T) &= \langle \chi_k | 1 \le k \le n \rangle_{\mathbb{Z}}, \\ \Phi^\vee &= \{\gamma_i - \gamma_j | 1 \le i \ne j \le n\}, \\ Y(T) &= \langle \gamma_k | 1 \le k \le n \rangle_{\mathbb{Z}}). \end{aligned}$$

Where

$$\begin{aligned} \chi_i(t) &= t_i && \text{for } t = diag(t_j)_{1 \le j \le n} \in T \\ \gamma_i(\lambda) &= diag(\lambda^{\delta_{i,j}})_{1 \le j \le n} && \text{for } \lambda \in \overline{\mathbb{F}_p}^* \end{aligned}$$

We already saw in Example 1.2.56 that there is an isomorphism between the root datum of G an its dual induced by

$$\begin{aligned} \delta : X(T) &\to Y(T). \\ \chi_i &\to \gamma_i \end{aligned}$$

It remains to check that δ satisfies the condition $\delta \circ F_q = F_q^* \circ \delta$ By example 2.3.3,

$$\begin{aligned} F_q(\chi_i) &= q\chi_i && \text{for any } 1 \le i \le n \\ F_q(\gamma_i) &= q\gamma_i && \text{for any } 1 \le i \le n. \end{aligned}$$

It follows

$$\delta \circ F_q(\chi_i) = \delta(q\chi_i) = q\gamma_i = F_q(\gamma_i) = F_q \circ \delta(\chi_i).$$

Remark 2.3.13. *[2, Proposition 4.3.2] Let (G, F), (G^*, F^*) be dual to each other with maximally split tori T, T^* as in Definition 2.3.10; let W the Weyl group of G relative to T, W^* the Weyl group of G^* relative to T^*. Then the anti-isomorphism*

$$\delta : W \to W^*$$

given in Remark 1.2.55 *satisfies*

$$\delta(F(\omega)) = {F^*}^{-1}(\delta(\omega))$$

Proposition 2.3.14. *Let (G, F), (G^*, F^*) be dual to each other with maximally split tori T, T^* and let W the Weyl group of G relative to T, W^* the Weyl group of G^* relative to T^*. The assignment $\omega \mapsto \delta(\omega^{-1})$ where δ is the anti-isomorphism $\delta : W \to W^*$ from Remark 2.3.13 induces a bijection between the F-conjugacy classes of W and the F^*-conjugacy classes of W^*.*

Proof. Let ω, ω' be F-conjugated in W: $\omega = x^{-1}\omega' F(x)$ for some $x \in W$. Then applying δ we have

$$\delta(\omega) = \delta(x^{-1})\delta(\omega')\delta(F(x)) = \delta(x^{-1})\delta(\omega'){F^*}^{-1}(\delta(x)).$$

Therefore

$$\delta(\omega')^{-1} = {F^*}^{-1}(\delta(x)^{-1})\delta(\omega)^{-1}\delta(x)$$

and applying F^* it yields

$$F^*(\delta(\omega')^{-1}) = \delta(x)^{-1}F^*(\delta(\omega)^{-1})F^*(\delta(x))$$

that is, $F^*(\delta(\omega)^{-1})$ and $F^*(\delta(\omega')^{-1})$ are F^*-conjugated. But since any element is F^*-conjugate to its F^* image (because $F^*(y) = y^{-1}yF^*(y)$) this implies that $\delta(\omega)^{-1}$ and $\delta(\omega')^{-1}$ are F-conjugated. The converse can be proved in an analogous way. □

Corollary 2.3.15. *There assignment $T_\omega \to T_{\delta(\omega)^{-1}}$ induces a bijection*

$$\left\{ \begin{array}{c} G^F\text{-conjugacy classes} \\ \text{of } F\text{-stable maximal tori in } G \end{array} \right\} \to \left\{ \begin{array}{c} {G^*}^{F^*}\text{-conjugacy classes} \\ \text{of } F^*\text{-stable maximal tori in } G^* \end{array} \right\}$$

Proof. It follows from Corollary 2.2.12 and Proposition 2.3.14 □

Chapter 3

Representation Theory for finite groups of Lie type

From now on, G will always be a connected reductive algebraic group over $\overline{\mathbb{F}_p}$, with p prime, F will be a generalized Frobenius morphism for G and $\mathcal{L}$ will denote the the Lang map as in Definition 2.2.2.
In this chapter we are concerned with the representation theory of the finite groups of Lie type, and in particular we deal with linear representations on spaces defined over a field of characteristic zero. A quite standard technique to gain knowledge about the irreducible representations of a group is to detect a suitable family of subgroups and to build representation of the group by induction starting from the ones of the subgroups. Our approach will focus mostly on characters, and for our purpose we will need the concept of *generalized character*.

Definition 3.0.1. *A generalized character of a group is a linear combination of characters of the group with coefficient in* $\mathbb{Z}$

In particular, we build generalized characters of the group (called Deligne-Lusztig generalized characters) inducing them from characters of the F-fixed points groups of F-stable tori. This allows to parameterize and study the irreducible representations of the group.

3.1 The Deligne-Lusztig generalized characters

3.1.1 l-adic cohomology

In order to define the generalized characters we are interested in, we need the concept of l-adic cohomology.
Let l be a prime and let $\mathbb{Z}_l$ denote the ring of the l-adic integers. We denote by $\mathbb{Q}_l$ the fraction field of $\mathbb{Z}_l$, and $\overline{\mathbb{Q}_l}$ is the algebraic closure. This is an algebraically closed field of characteristic zero [2, §7.1].

Remark 3.1.1. *We will need to consider complex characters of finite groups. Note that these characters take values in the subring of the algebraic integers, therefore it makes no difference to consider this characters as taking values in* $\mathbb{C}$ *or* $\overline{\mathbb{Q}_l}$*, since the algebraic integers are a common subring.*
For what concerns the operation of complex conjugation (that we will need for the scalar product on class functions), on complex roots of unity it coincides with taking the inverse and the latter is a well defined operation for roots of unity in $\overline{\mathbb{Q}_l}$ *as well, and it can be extended on elements of* $\overline{\mathbb{Q}_l}$ *that are sums of roots of unity. These are the only elements in* $\overline{\mathbb{Q}_l}$ *on which generalized characters of finite groups take value, and therefore these are the only elements of* $\overline{\mathbb{Q}_l}$ *for which we need to define the conjugation.*

From now on, let X be an algebraic variety over $\overline{\mathbb{F}_p}$, ad let l be a prime different from p.
For any $i \in \mathbb{Z}$ one can associate to X a finite dimensional vector space $H^i_c(X, \overline{\mathbb{Q}_l})$ over $\overline{\mathbb{Q}_l}$, called the *i-th l-adic cohomology group of X with compact support*. The construction of these spaces can be found in [2, Appendix].
For any algebraic variety, only finitely many l-adic cohomology groups are non trivial.

Proposition 3.1.2. *[2, Property 7.1.1] If* $i \notin \{0, 1, \ldots, 2dimX\}$ *then* $H^i_c(X, \overline{\mathbb{Q}_l}) = 0$

The correspondence between an algebraic variety and its l-adic cohomology groups is functorial (contravariantly), by [3, Proposition 8.1.2]. In particular, the following result holds.

Proposition 3.1.3. *[2, Property 7.1.3] Any automorphism* $g : X \to X$ *of* X *induces linear automorphism* $g^* \in GL(H^i_c(X, \overline{\mathbb{Q}_l}))$ *for any* $i \in \mathbb{N}$*, in such a way to make* $H^i_c(X, \overline{\mathbb{Q}_l})$ *a module over the group* $Aut(X)$ *of automorphisms of* X*.*

So for any element $g \in Aut(X)$, we can consider is action on the i-th cohomology group of X for any $i \in \mathbb{Z}$. We denote the trace of this linear automorphism by $Tr(g|H^i_c(X, \overline{\mathbb{Q}_l}))$. Note that this will be possibly non-zero for finitely many i.

Definition 3.1.4. *Let* $g : X \to X$ *be an automorphism of* X *of finite order. We define the Lefschetz number of* g *on* X *to be*

$$\mathscr{L}(g, X) = \sum_i (-1)^i Tr(g|H^i_c(X, \overline{\mathbb{Q}_l}))$$

We now state some properties of the Lefschetz numbers that we will need in the sequel.

Proposition 3.1.5. *[2, Property 7.1.4] Let* $g \in Aut(X)$ *of finite order. Then* $\mathscr{L}(g, X)$ *is an integer and is independent of the choice of* l*.*

Proposition 3.1.6. *[2, Property 7.1.5] Let* X, Y *be algebraic varieties over* $\overline{\mathbb{F}_p}$*, let* $f : X \to Y$ *be a morphism of algebraic varieties with fibers isomorphic to an affine* *space of fixed dimension. Let $g : X \to X$, $g' : Y \to Y$ be automorphisms of finite order such that $g' \circ f = f \circ g$. Then*

$$\mathscr{L}(g, X) = \mathscr{L}(g', Y)$$

Proposition 3.1.7. *[3, Proposition 8.1.10] Let H be a finite subgroup of $Aut(X)$ such that the quotient variety ${}^{X}\!/_{H}$ exists. Then*

$$H_c^i(X, \overline{\mathbb{Q}_l})^H \cong H_c^i({}^{X}\!/_{H}, \overline{\mathbb{Q}_l})$$

Moreover let $g \in Aut(X)$ be an automorphism of X of finite order that commutes with all the elements of H. Then

$$\mathscr{L}(g, {}^{X}\!/_{H}) = \frac{1}{|H|} \sum_{h \in H} \mathscr{L}(gh, X).$$

Note that if X is an affine variety and H is a finite group, the quotient variety ${}^{X}\!/_{H}$ exists [4, Theorem 5.52], hence in this case the assumptions of Proposition 3.1.7 are satisfied.

3.1.2 The Deligne-Lusztig generalized characters

Let T be an F-stable maximal torus of G. By Remark 1.2.27, T is contained in some Borel subgroup B of G, not necessarily F-stable. Denote by $U = R_u(B)$, the unipotent radical of B. We consider the set $\mathcal{L}^{-1}(U)$. It is an algebraic subset of G, hence an affine algebraic variety over $\overline{\mathbb{F}_p}$.

Lemma 3.1.8. *Let T be an F-stable maximal torus contained in a Borel subgroup B, let U be the unipotent radical of B. Then G^F acts by left translations and T^F acts by right translations on $\mathcal{L}^{-1}(U)$, and these actions commute with each other. This defines an action of $G^F \times T^F$ on $\mathcal{L}^{-1}(U)$*

$$\begin{aligned} G^F \times T^F &\to Aut(\mathcal{L}^{-1}(U)) \\ (g, t) &\mapsto (x \mapsto gxt^{-1}) \end{aligned}$$

Proof. Let $x \in \mathcal{L}^{-1}(U)$.
For any $g \in G^F$, it holds $\mathcal{L}(gx) = x^{-1}g^{-1}F(g)F(x) = x^{-1}F(x) = \mathcal{L}(x) \in U$, therefore $gx \in \mathcal{L}^{-1}(U)$.
For any $t \in T^F$, it holds $\mathcal{L}(xt) = t^{-1}x^{-1}F(x)F(t) = t^{-1}\mathcal{L}(x)t \in t^{-1}Ut = U$, where we can write the last equality because T normalizes U. Therefore $gt \in \mathcal{L}^{-1}(U)$
The actions commutes since left translation and right translation always commute in a group. □

Corollary 3.1.9. *Let T be an F-stable maximal torus contained in a Borel subgroup B, let U be the unipotent radical of B. For any $i \in \mathbb{N}$, the i-th cohomology group of $\mathcal{L}^{-1}(U)$ is a left G^F-module and a right T^F-module such that for any $g \in G^F$, $t \in T^F$ and for any $v \in H_c^i(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})$ it holds $(gv)t = g(vt)$.*

Proof. It is a consequence of Lemma 3.1.8 and Proposition 3.1.3 □

We can now define the generalized characters of Deligne and Lusztig. Let T be an F-stable maximal torus and $\theta : T^F \to \mathbb{C}$ be an irreducible character of T^F; we will write $\theta \in Irr(T^F)$. We already observed in Remark 3.1.1 that since θ takes values in the ring of algebraic integers, it makes no difference whether to consider it as taking values in $\mathbb{C}$ or in $\overline{\mathbb{Q}_l}$.

Definition 3.1.10. *(Deligne-Lusztig generalized characters) Let T be an F-stable maximal torus contained in a Borel subgroup B and let U be the unipotent radical of B. Let $\theta \in Irr(T^F)$ be an irreducible character of T^F. We define the function $R^G_{T,\theta} : G^F \to \overline{\mathbb{Q}_l}$ to be*

$$R^G_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \mathscr{L}((g,t), \mathcal{L}^{-1}(U))\theta(t^{-1})$$

for any $g \in G^F$.

When the algebraic group G we are considering is clear from the context (as it will be in the broad majority of our discussion) we drop the G in the notation and we simply write $R_{T,\theta}$ in place of $R^G_{T,\theta}$.

Proposition 3.1.11. *The functions $R_{T,\theta}$ defined in Definition 3.1.10 are generalized characters of G^F.*

Proof. We use the same notation as in Definition 3.1.10; we regard θ as taking values in $\overline{\mathbb{Q}_l}$.
Let e be the idempotent element of the group algebra of T^F over $\overline{\mathbb{Q}_l}$ defined by

$$e = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1})t.$$

It acts on $H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})$ (extending the action of T^F to its group algebra) as a projector on the submodule on which T^F acts by the character θ:

$$H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})e = H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})_\theta := \{v \in H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l}) \mid vt = \theta(t)v\}$$

(see [2, Proposition 7.2.1]). Therefore, for any $g \in G^F$, it holds

$$Tr(g|H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})_\theta) = Tr(g|H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})e)$$

Observe that since $e^2 = e$, $H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})$ can be decomposed as

$$H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})e \bigoplus H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})(1-e).$$

These spaces are both (g,e)-stable and (g,e) acts as g on $H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})e$, while it annihilates $H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})(1-e)$. Therefore

$$Tr(g|H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l})e) = Tr((g,e)|H^i_c(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}_l}))$$

So, since

$$Tr((g,e)|H_c^i(\mathcal{L}^{-1}(U),\overline{\mathbb{Q}_l})) = \frac{1}{|T^F|}\sum_{t\in T^F} Tr((g,t)|H_c^i(\mathcal{L}^{-1}(U),\overline{\mathbb{Q}_l}))\theta(t^{-1}),$$

we can conclude that

$$\begin{aligned} R_{T,\theta}(g) &= \frac{1}{|T^F|}\sum_{t\in T^F} \mathscr{L}((g,t),\mathcal{L}^{-1}(U))\theta(t^{-1}) \\ &= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\sum_i(-1)^i Tr((g,t)|H_c^i(\mathcal{L}^{-1}(U),\overline{\mathbb{Q}_l})) \\ &= \sum_i(-1)^i\frac{1}{|T^F|}\sum_{t\in T^F} Tr((g,t)|H_c^i(\mathcal{L}^{-1}(U),\overline{\mathbb{Q}_l}))\theta(t^{-1}) \\ &= \sum_i(-1)^i Tr(g|H_c^i(\mathcal{L}^{-1}(U),\overline{\mathbb{Q}_l})_\theta) \end{aligned}$$

So the $R_{T,\theta}$ are an alternating sum of characters of G^F, hence they are generalized characters. □

Note that according to the definition we gave, a Deligne-Lusztig generalized character $R_{T,\theta}$ seems to depend on the choice of the Borel subgroup B containing the F-stable maximal torus T, so we should write $R_{T\leq B,\theta}$. Anyway, we drop the B in our notation because the $R_{T,\theta}$ turn out to be independent of the choice of the Borel subgroup. This will be a consequence of the scalar product formula we are about to give in Theorem 3.1.12 (whose proof does not use the independence from the choice of B of the $R_{T,\theta}$).
We denote the scalar product of complex characters of G^F by

$$\langle f,f'\rangle_{G^F} = \frac{1}{|G^F|}\sum_{g\in G^F} f(g)\overline{f'(g)} \qquad \text{for } f,f' \text{ complex characters of } G^F$$

where the overlying bar denotes complex conjugation. This operation makes sense as well for characters of G^F taking values in $\overline{\mathbb{Q}_l}$ by Remark 3.1.1. This scalar product can be extended by linearity to generalized characters.

Theorem 3.1.12. *[2, Theorem 7.3.4] (Scalar Product Formula) Let T,T' be F-stable tori and let $\theta\in Irr(T^F)$, $\theta'\in Irr(T'^F)$. Then*

$$\langle R_{T,\theta},R_{T',\theta'}\rangle_{G^F} = \frac{1}{|T^F|}|\{g\in G^F \mid gTg^{-1}=T' \text{ and } \theta'\circ c_g=\theta\}|$$

The scalar product formula has some important consequences. In order to state it in clearer way, we introduce the following useful notation

Definition 3.1.13. *Let T,T' be F-stable tori and let $\theta\in Irr(T^F)$, $\theta'\in Irr(T'^F)$. If there exists $g\in G^F$ such that $gTg^{-1}=T'$ and $\theta'\circ c_g=\theta$, we say that (T,θ) and (T',θ') are G^F-conjugated.*

Corollary 3.1.14. *Let T, T' be F-stable tori and let $\theta \in Irr(T^F)$, $\theta' \in Irr(T'^F)$. Then*

1. *The generalized character $R_{T,\theta}$ is independent on the choice of the Borel subgroup B containing T.*

2. *The generalized characters $R_{T,\theta}$ and $R_{T',\theta'}$ are either equal or orthogonal, and they are equal if and only if (T, θ) and (T', θ') are G^F-conjugated.*

Proof. 1. Recall that the proof of Theorem 3.1.12 in [2] does not make use of the independence of $R_{T,\theta}$ from the Borel. Let B, B' be two Borel subgroups containing T. Let U be the unipotent radical of B and U' be the unipotent radical of B'. Write

$$R_{T\leq B,\theta} = \frac{1}{|T^F|}\sum_{t\in T^F} \mathscr{L}((g,t), \mathcal{L}^{-1}(U))\theta(t^{-1})$$
$$R_{T\leq B',\theta} = \frac{1}{|T^F|}\sum_{t\in T^F} \mathscr{L}((g,t), \mathcal{L}^{-1}(U'))\theta(t^{-1})$$

Then

$$\langle R_{T\leq B,\theta}, R_{T\leq B,\theta}\rangle_{G^F} = \langle R_{T\leq B,\theta}, R_{T\leq B',\theta}\rangle_{G^F} = \langle R_{T\leq B',\theta}, R_{T\leq B',\theta}\rangle_{G^F}.$$

This implies

$$\langle R_{T\leq B,\theta} - R_{T\leq B',\theta}, R_{T\leq B,\theta} - R_{T\leq B',\theta}\rangle_{G^F} = 0$$

that is, $R_{T\leq B,\theta} - R_{T\leq B',\theta}$ has norm zero and therefore $R_{T\leq B,\theta} = R_{T\leq B',\theta}$

2. Suppose (T, θ) and (T', θ') are not G^F conjugated, so there is no $g \in G^F$ such that $gTg^{-1} = T'$ and $\theta' \circ c_{g^{-1}} = \theta$, then by Theorem 3.1.12

$$\langle R_{T,\theta}, R_{T,\theta}\rangle_{G^F} = 0.$$

On the other hand, suppose that (T, θ) and (T', θ') are G^F conjugated. We want to prove $R_{T,\theta} = R_{T',\theta'}$.
Recall (see the beginning of §2.3.1) that since $g \in G^F$, conjugation by g induces a rational morphism between T and T', that restrict to an isomorphism $c_g : T^F \to T'^F$. Now let B be a Borel subgroup containing T, and let $B' = gBg^{-1}$, which is a Borel subgroup containing T'. If U is the unipotent radical of B, then $U' = gUg^{-1}$ is the unipotent radical of B'. Then for any $y \in G$, $\mathcal{L}(y) \in U' = gUg^{-1}$ if and only if $\mathcal{L}(yg) = g^{-1}\mathcal{L}(y)F(g) = g^{-1}\mathcal{L}(y)g \in U$. Therefore

$$\mathcal{L}^{-1}(U') = \mathcal{L}^{-1}(U)g$$

and we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}^{-1}(U') & \xrightarrow{\cdot g} & \mathcal{L}^{-1}(U) \\ \big\downarrow{\scriptstyle \cdot c_g(t)} & & \big\downarrow{\scriptstyle \cdot t} \\ \mathcal{L}^{-1}(U') & \xrightarrow{\cdot g} & \mathcal{L}^{-1}(U) \end{array}$$

So we can apply Proposition 3.1.6 and we get for any $x \in G^F$, $t \in T^F$

$$\mathscr{L}((x,t), \mathcal{L}^{-1}(U)) = \mathscr{L}((x, c_g(t)), \mathcal{L}^{-1}(U')).$$

So for any $x \in G^F$

$$R_{T',\theta'}(x) = \frac{1}{|T^F|} \sum_{t' \in T'^F} \mathscr{L}((x,t'), \mathcal{L}^{-1}(U'))\theta'(t'^{-1}) =$$
$$\frac{1}{|T^F|} \sum_{t \in T^F} \mathscr{L}((x, c_g(t)), \mathcal{L}^{-1}(U'))\theta'(c_g(t^{-1})) =$$
$$\frac{1}{|T^F|} \sum_{t \in T^F} \mathscr{L}((x,t), \mathcal{L}^{-1}(U))\theta(t^{-1}) = R_{T,\theta}(x)$$

□

So taking pairs (T, θ) consisting of an F-stable maximal torus and an irreducible character of T^F up to G^F conjugacy, we obtain all the Deligne-Lusztig characters $R_{T,\theta}$, and they form an orthogonal basis of the space they span. This subspace of class functions of G^F is called the space of the uniform functions.

Definition 3.1.15. *A class function of G^F is called a uniform function if it is a linear combination of Deligne-Lusztig characters.*

In particular, the character of the regular representation is an uniform function.

Proposition 3.1.16. *[2, Corollary 7.5.6]The character of the regular representation of G^F is given by*

$$\chi_{reg} = \frac{1}{|G^F|_p} \sum_{T} \sum_{\theta \in Irr(T^F)} \varepsilon_T \varepsilon_G R_{T,\theta}$$

Where the first sum runs over the set of the F-stable maximal tori of G, $|G^F|_p$ is the largest power of p dividing $|G^F|$ and ε_G is as in Definition 2.3.9.

Corollary 3.1.17. *Let χ be an irreducible character of G^F. Then there exists a pair (T, θ) consisting of a maximal F-stable torus and an irreducible character of T^F such that*

$$\langle \chi, R_{T,\theta} \rangle_{G^F} \neq 0$$

Proof. If χ were not an irreducible component of any generalized Deligne-Lusztig character, since the regular representation is uniform by Proposition 3.1.16, χ would not be a component of the regular representation of χ either. But this is a contradiction, since any irreducible character of G^F appears as component of the regular representation of G^F. □

So any irreducible character appears as irreducible component in some generalized Deligne-Lusztig character.
However note that since the $R_{T,\theta}$ are just generalized characters, the fact that they are orthogonal to each other (Corollary 3.1.14) does not imply that different Deligne-Lusztig generalized characters do not have any irreducible component in common. Indeed, it is not true, in general, that the sets of irreducible characters appearing in different generalized Deligne-Lusztig characters are disjoint. However, disjointedness of the sets of of the irreducible components can be achieved considering classes of pairs (T, θ) that keep track of the action by conjugation of the algebraic group G, called *geometric conjugacy classes*, and grouping up the Deligne-Lusztig generalized characters relative to pairs (T, θ) in the same geometric conjugacy class. To study and describe these classes will be the main focus of Section 3.2.1.
Before passing to study these classes, we give an alternative description of the generalized Deligne-Lusztig characters, that allows in particular to give an easier and more intuitive description of $R_{T,\mathbf{1}_{T^F}}$, the generalized characters obtained by inducing the trivial one.

3.1.3 An alternative description for $R_{T,\mathbf{1}_{T^F}}$

Let T_0 be a maximally split maximal torus of G, and let B_0 be an F-stable Borel subgroup of G containing T_0, let U_0 denote the unipotent radical of B_0.
If the F-stable torus T is obtained by T_0 by twisting with ω, with $\omega \in W$, there exists a $g \in G$ such that $\mathcal{L}(g) = \dot{\omega}$ ($\dot{\omega} \in N_G(T)$ representative of ω) that induces an isomorphism $c_g : T_0 \to T$ which is rational considering T_0 endowed with the rational structure induced by ωF and T with the one induced by F (see the begin of § 2.3.1). We can give an alternative description of the Deligne-Lusztig generalized character using this isomorphism.
For any irreducible character θ of $T_0^{\omega F}$, let $R_\omega^\theta : G^F \to \overline{\mathbb{Q}_l}$ be

$$R_\omega^\theta(g) = \frac{1}{|T_0^{\omega F}|} \sum_{t \in T_0^{\omega F}} \mathscr{L}((g,t), \mathcal{L}^{-1}(\dot{\omega}U))\theta(t^{-1}) \tag{3.1}$$

for any $g \in G^F$.
Using techniques similar to the ones used in Corollary 3.1.14, one can prove the following result, that shows that equation (3.1) is just an alternative description for the Deligne-Lusztig generalized characters.

Proposition 3.1.18. *[8, Lemma 2.3.19] The functions R_ω^θ defined in (3.1) are independent of the choice of the representative $\dot{\omega} \in N_G(T_0)$ of $\omega \in W$.*
Moreover, if T is an F stable maximal torus of type ω and g is an element of G such that $T = gT_0g^{-1}$, then $\theta \circ c_{g^{-1}}$ is an irreducible character of T^F and

$$R_\omega^\theta = R_{T,\theta \circ c_{g^{-1}}}.$$

This description is particularly useful when we consider the trivial character $\mathbf{1}_{T^F}$ of the F-fixed points set of an F-stable torus T. Indeed if T is of type ω, with $\omega \in W$, then

$$R_{T,\mathbf{1}_{T^F}} = R_\omega^{\mathbf{1}_{T_0^{\omega F}}}$$

because $\mathbf{1}_{T^F} \circ c_g = \mathbf{1}_{T_0^{\omega F}}$. This allows to give a more intuitive description of the characters $R_{T,\mathbf{1}_{T^F}}$.

We will need the following definition, where we use notation from Definition 1.2.64.

Definition 3.1.19. *For any $\omega \in W$ we denote by X_ω the set given by*

$$X_\omega = \{B \in \mathcal{B} \mid (B, F(B)) \in \mathcal{O}(\omega)\}.$$

Remark 3.1.20. *We claim that the set X_ω is an algebraic variety. In order to prove it, we investigate its structure.*
Given B_0, an F-stable Borel subgroup containing T_0, for any $\omega \in W$ the G-orbit $\mathcal{O}(\omega)$ of pairs of Borel subgroups of G in relative position ω is

$$\mathcal{O}(\omega) = G.(B_0, \omega B_0 \omega^{-1}) \subseteq \mathcal{B} \times \mathcal{B}.$$

We also define Γ to be the graph of F on $\mathcal{B} \times \mathcal{B}$, that is

$$\Gamma = \{(B, F(B)) \mid B \in \mathcal{B}\} \subseteq \mathcal{B} \times \mathcal{B}.$$

Then X_ω is the image of the projection of $\mathcal{O}(\omega) \cap \Gamma$ on the first component. Since $\mathcal{O}(\omega)$ is open in its closure (being an orbit of the action of an algebraic group on an algebraic variety whose action map is also a morphism of varieties [9, Proposition 5.4]) and Γ closed (being a graph), $\mathcal{O}(\omega) \cap \Gamma$ is an algebraic variety. Then since $\mathcal{B}$ is a projective variety, it is complete, hence the projection on the first component is a closed map and therefore the set X_ω is an algebraic variety.

Example 3.1.21. *Let G be $GL_2(\overline{\mathbb{F}_p})$, F be the standard Frobenius morphism F_q. Recall that D_2 denotes the maximally split maximal torus consisting of diagonal matrices, T_2 is the Borel subgroup consisting of upper triangular matrices and $T_2^- = sT_2s^{-1}$ denotes the Borel subgroup of the lower triangular matrices. The Weyl group is $W \cong S_2 = \{1, s\}$. We have*

$$X_1 = \{B \in \mathcal{B} | F(B) = B\} = \{gT_2g^{-1} \,|g \in GL_2(\overline{\mathbb{F}_p}) \text{ s.t. } g^{-1}F(g) \in D_2\}$$
$$X_s = \{B \in \mathcal{B} | F(B) \neq B\} = \{gT_2g^{-1} \,|g \in GL_2(\overline{\mathbb{F}_p}) \text{ s.t. } g^{-1}F(g) \text{ is anti-diagonal}\}$$

Indeed we have

$$\mathcal{O}(1) = \{(gT_2g^{-1}, gT_2g^{-1}) \mid g \in GL_2(\overline{\mathbb{F}_p})\} \subseteq \mathcal{B} \times \mathcal{B}$$
$$\mathcal{O}(s) = \{(gT_2g^{-1}, gT_2^-g^{-1}) \mid g \in GL_2(\overline{\mathbb{F}_p})\} \subseteq \mathcal{B} \times \mathcal{B}\}$$

while the graph of F_q is given by

$$\Gamma = \{(gT_2g^{-1}, F(g)T_2F(g)^{-1}) \mid g \in GL_2(\overline{\mathbb{F}_p})\} \subseteq \mathcal{B} \times \mathcal{B}$$

Therefore

$$\mathcal{O}(1) \cap \Gamma = \{(gT_2g^{-1}, gT_2g^{-1}) \mid g \in GL_2(\overline{\mathbb{F}_p}) \text{ s.t. } g^{-1}F_q(g) \in T_2\}$$
$$\mathcal{O}(s) \cap \Gamma = \{(gT_2g^{-1}, gT_2^{-}g^{-1}) \mid g \in GL_2(\overline{\mathbb{F}_p}) \text{ s.t } g^{-1}F_q(g) \in T_2sT_2\}$$

since $g^{-1}F_q(g) \in N_{GL_2(\overline{\mathbb{F}_p})}(D_2)$, we have that $g^{-1}F_q(g) \in T_2$ if and only if $g^{-1}F_q(g)$ is a diagonal matrix, $g^{-1}F_q(g) \in T_2sT_2$ if and only if $g^{-1}F_q(g)$ is an anti diagonal matrix. So the projection of the first component of these two sets are the varieties X_1 and X_s.
Moreover, by Example 1.2.26 there is an isomorphism $\mathcal{B} \cong \mathbb{P}^1(\overline{\mathbb{F}_p})$ given by $gT_2g^{-1} \mapsto \overline{g.e_1}$, where $\overline{g.e_1}$ denotes the line of $g.e_1$ in $\mathbb{P}^1(\overline{\mathbb{F}_p})$. By Example 1.2.66 we have that through this isomorphism it holds:

$$\mathcal{O}(1) \cong \Delta(\mathbb{P}^1(\overline{\mathbb{F}_p})) = \{([a,b],[a,b]) \mid [a,b] \in \mathbb{P}^1(\overline{\mathbb{F}_p})\}$$
$$\mathcal{O}(s) \cong \mathbb{P}^1(\overline{\mathbb{F}_p}) \times \mathbb{P}^1(\overline{\mathbb{F}_p}) \setminus \Delta(\mathbb{P}^1(\overline{\mathbb{F}_p})) =$$
$$= \{([a,b],[c,d]) \mid ([a,b],[c,d]) \in \mathbb{P}^1(\overline{\mathbb{F}_p}) \times \mathbb{P}^1(\overline{\mathbb{F}_p}) \text{ s.t. } ad - bc \neq 0\}\}.$$

where the square brackets denote the homogeneous coordinates in $\mathbb{P}^1(\overline{\mathbb{F}_p})$),
The graph of F_q can be described as

$$\Gamma \cong \{([a,b],[a^q,b^q]) \mid [a,b] \in \mathbb{P}^1(\overline{\mathbb{F}_p})\}.$$

To give a description of X_1 and X_s through this isomorphism, consider the morphism of projective varieties induced by the standard Frobenius morphism on $\mathbb{P}^1(\mathbb{F}_q)$, that is the morphism given by (writing it in homogeneous coordinates)

$$F_q : \mathbb{P}^1(\overline{\mathbb{F}_p}) \to \mathbb{P}^1(\overline{\mathbb{F}_p})$$
$$[a,b] \mapsto [a^q, b^q]$$

and denote by $\mathbb{P}^1(\mathbb{F}_q)$ the set

$$\mathbb{P}^1(\mathbb{F}_q) := \mathbb{P}^1(\overline{\mathbb{F}_p})^{F_q} = \{[a,b] \in \mathbb{P}^1(\overline{\mathbb{F}_p}) \mid a^qb - ab^q = 0\}$$

Then

$$X_1 \cong \mathbb{P}^1(\mathbb{F}_q)$$
$$X_s \cong \mathbb{P}^1(\overline{\mathbb{F}_p}) \setminus \mathbb{P}^1(\mathbb{F}_q)$$

Definition 3.1.22. *An irreducible character χ of G^F is called unipotent if there exists an $\omega \in W$ such that*

$$\langle \chi, \mathscr{L}(\cdot, X_\omega) \rangle_{G^F} \neq 0$$

where $\mathscr{L}(\cdot, X_\omega) : G^F \to \mathbb{Z}$ is the generalized character $g \mapsto \mathscr{L}(g, X_\omega)$.

We now use the varieties X_ω to give an alternative description of the Deligne-Lusztig generalized characters induced by the trivial character of T^F.

Lemma 3.1.23. *[2, Proposition 7.7.7]Let $\omega \in W$, $\dot\omega$ a representative of ω in $N_G(T_0)$. Then*

1. *The group $(U \cap \dot\omega U \dot\omega^{-1})T_0^{\omega F}$ acts on $\mathcal{L}^{-1}(\dot\omega U)$ by right translation*
2. *The map*

$$\begin{aligned} \mathcal{L}^{-1}(\dot\omega U) &\to X_\omega \\ x &\mapsto xBx^{-1} \end{aligned}$$

is a surjective morphism of varieties with as fibers the orbits of $(U \cap \dot\omega U \dot\omega^{-1})T_0^{\omega F}$ on $\mathcal{L}^{-1}(\dot\omega U)$

Theorem 3.1.24. *Let T be an F-stable maximal torus of G obtained from a maximally split torus T_0 by twisting with ω. Then*

$$R_{T,\mathbf{1}_{T^F}}(g) = \mathscr{L}(g, X_\omega)$$

Proof. By Proposition 3.1.18,

$$R_{T,\mathbf{1}_{T^F}} = R_\omega^{\mathbf{1}_{T_0^{\omega F}}}$$

and by definition for any $g \in G^F$

$$R_\omega^{\mathbf{1}_{T_0^{\omega F}}}(g) = \frac{1}{|T_0^{\omega F}|} \sum_{t \in T_0^{\omega F}} \mathscr{L}((g,t), \mathcal{L}^{-1}(\dot\omega U)).$$

Now, $\mathcal{L}^{-1}(\dot\omega U)$ is an affine variety acted upon by the finite group $T^{\omega F}$, so by Proposition 3.1.7

$$\mathscr{L}(g, \mathcal{L}^{-1}(\dot\omega U)/T_0^{\omega F}) = \frac{1}{|T_0^{\omega F}|} \sum_{t \in T_0^{\omega F}} \mathscr{L}((g,t), \mathcal{L}^{-1}(\dot\omega U)) = R_\omega^{\mathbf{1}_{T_0^{\omega F}}}(g).$$

Moreover by Lemma 3.1.23 there is a projection $\mathcal{L}^{-1}(\dot\omega U) \to X_\omega$ whose fibers are the orbits of $(U \cap \dot\omega U \dot\omega^{-1})T_0^{\omega F}$ on $\mathcal{L}^{-1}(\dot\omega U)$, therefore it factorizes as

$$\mathcal{L}^{-1}(\dot\omega U) \to \mathcal{L}^{-1}(\dot\omega U)/T_0^{\omega F} \to X_\omega.$$

The latter morphism is compatible with the action of G^F on both varieties, and has as fibers orbits isomorphic to $(U \cap \dot\omega U \dot\omega^{-1})$, which is isomorphic as variety to an affine space. Then by Proposition 3.1.6,

$$\mathscr{L}(g, X_\omega) = \mathscr{L}(g, \mathcal{L}^{-1}(\dot\omega U)/T_0^{\omega F}) = R_\omega^{\mathbf{1}_{T_0^{\omega F}}}(g).$$

So we proved

$$R_{T,\mathbf{1}_{T^F}}(g) = \mathscr{L}(g, X_\omega)$$

□

Remark 3.1.25. *By Theorem 3.1.24, an irreducible character χ of G^F is unipotent (as in Definition 3.1.22) if and only if there exists an F stable torus T such that*

$$\langle \chi, R_{T,\mathbf{1}_{T^F}} \rangle_{G^F} \neq 0.$$

3.2 Lusztig classification of irreducible characters

We saw in Proposition 3.1.17 that any irreducible character of G^F appears in at least one Deligne-Lusztig generalized character. We now wish to investigate under which conditions two Deligne-Lusztig generalized characters have no irreducible component in common. The answer to this question relies on the notion of geometric conjugacy classes.

3.2.1 Geometric conjugacy

If T and T' are two F-stable maximal tori of G they are G-conjugated, so we can write $T' = gTg^{-1}$ for some $g \in G$. However, in general this conjugation is not a rational morphism with respect to the rational structures induced by F on both the maximal tori. So in particular conjugation by elements of G does not, in general, conjugate T^F to T'^F, hence we cannot use it in a naive way to compare $Irr(T^F)$ and $Irr(T'^F)$. Nevertheless, we now introduce a device (the norm morphism) that will allow us to do this comparison.

Definition 3.2.1. *For any $n \in \mathbb{N}$, we denote by $N_{F^n/F}$ the homomorphism of T given by*

$$\begin{aligned} N_{F^n/F} : T &\longrightarrow T \\ t &\mapsto tF(t)F^2(t)\cdots F^{n-1}(t) \end{aligned}$$

The morphisms $N_{F^n/F}$ are called norm morphisms.

Remark 3.2.2. *Note that any $N_{F^n/F}$ norm morphism on T restricts to*

$$N_{F^n/F} : T^{F^n} \to T^F.$$

Therefore each character θ of T^F determines a character $\theta \circ N_{F^n/F}$ of T^{F^n}.

Definition 3.2.3. *Two pairs (T, θ) and (T', θ') consisting of an F-stable torus and an irreducible character of the F-fixed point group of the torus are geometrically conjugated by $g \in G$ if*

$$T' = gTg^{-1}$$

and for any $n \in \mathbb{N}$ such that $g \in G^{F^n}$

$$\theta \circ N_{F^n/F} = \theta' \circ N_{F^n/F} \circ c_g$$

as characters of T^{F^n}.

Example 3.2.4. *The set* $\{(T, \mathbf{1}_{T^F}) \mid T$ *is* F*-stable*$\}$ *is a geometric conjugacy class. Indeed two* F*-stable tori* T, T' *are conjugated in* G*, and if* $T' = gTg^{-1}$ *for* $g \in G$*, then for any* n *such that* $g \in G^n$ *it holds*

$$\mathbf{1}_{T^F} \circ N_{F^n/F}(x) = 1 = \mathbf{1}_{T'^F} \circ N_{F^n/F}(gxg^{-1})$$

for any $x \in T^{F^n}$*.*

Theorem 3.2.5. *[3, Proposition 11.1.3] Let* T *and* T' *be* F*-stable tori,* $\theta \in Irr(T^F)$*,* $\theta' \in Irr(\theta')$*. If* $R_{T,\theta}$*,* $R_{T',\theta'}$ *have a common irreducible component, then* (T, θ) *and* (T', θ') *are geometrically conjugated in* G*.*

Theorem 3.2.5 stresses out the importance of geometric conjugacy classes in order to study the irreducible representations of the finite group G^F by mean of the Deligne-Lusztig generalized characters. We therefore spend some thoughts to investigate them; we will find a description of geometric conjugacy classes in terms of the group of rational cocharacters of the torus, that will lead to a parameterization of these classes using the dual group.
In order to do that, we need some preliminary results. Let $\mathbb{Q}_{(p)}$ denote the localization of $\mathbb{Z}$ away from the ideal generated by p, that is

$$\mathbb{Q}_{(p)} = \left\{ \frac{r}{s} \mid r, s \in \mathbb{Z} \text{ , } p \text{ does not divide } s \right\}$$

.

Lemma 3.2.6. *1. The group* $\overline{\mathbb{F}_p}^*$ *is isomorphic to the additive group* $\mathbb{Q}_{(p)}/\mathbb{Z}$*.*

2. There is an embedding $\overline{\mathbb{F}_p}^* \to \overline{\mathbb{Q}_l}$

Proof. 1. (Sketch, for details see [2, Proposition 3.1.3]) Recall $\overline{\mathbb{F}_p}^* = \bigcup_{q=p^n, n>0} \mathbb{F}_q^*$. It can be shown that for any power q of p it is possible to choose a $\zeta_q \in \overline{\mathbb{F}_p}^*$ to be a $q-1$ primitive root of 1 (that is a generator of $\mathbb{F}_q^*$) in a way that this elements satisfies $(\zeta_{q^n})^n = \zeta_q$. Then, mapping $\zeta_q \in \overline{\mathbb{F}_p}^*$ to $\frac{1}{q-1} \in \mathbb{Q}_{(p)}$ induces the desired isomorphism.

2. The group $\mathbb{Q}_{(p)}/\mathbb{Z}$ can be embedded into the complex group of the roots of 1 by mapping $\frac{r}{s} \mapsto e^{\frac{2\pi i r}{s}}$. The roots of 1 are algebraic integers and so they lie in a common subfield of $\mathbb{C}$ and $\overline{\mathbb{Q}_l}$, as in Remark 3.1.1. So since $\overline{\mathbb{F}_p}^* \cong \mathbb{Q}_{(p)}/\mathbb{Z}$ by point 1, we have an embedding as in the statement.

□

Note that the isomorphism $\overline{\mathbb{F}_p}^* \cong \mathbb{Q}_{(p)}/\mathbb{Z}$ and the embedding $\overline{\mathbb{F}_p}^* \to \overline{\mathbb{Q}_l}$ of Lemma 3.2.6 are not canonical. From now on we shall assume these two maps chosen once and for all.

Proposition 3.2.7. *[3, Proposition 11.1.7] Let T be a torus with a generalized Frobenius morphism F. Let $n \in \mathbb{N}$ be such that T is F^n-split, and let q be the positive real nuber associated to F as in Corollary 2.3.6. Let $X(T)$ and $Y(T)$ be respectively the characters and the cocharacters group of T. Then the following holds:*

1. *The sequence*
$$0 \to X(T) \xrightarrow{F-id} X(T) \xrightarrow{res|_{T^F}} Irr(T^F) \to 1$$
is exact, where the map $res|_{T^F}$ is given by taking the restriction of characters of T on T^F and regarding the restriction as irreducible characters by taking the embedding of $\overline{\mathbb{F}_p}^*$ into $\overline{\mathbb{Q}_l}$.

2. *Let $\zeta \in \overline{\mathbb{F}_p}^*$ denote the image of $\frac{1}{q^n-1}$ under the chosen isomorphism ${}^{\mathbb{Q}_{(p)}}\!\diagup_{\mathbb{Z}} \to \overline{\mathbb{F}_p}^*$, and let $\nu_\zeta : Y(T) \to T^{F^n}$ be the map defined by $\gamma \mapsto \gamma(\zeta)$. Let the norm morphism $N_{F^n/F}$ act on $Y(T)$ by $\gamma \mapsto N_{F^n/F} \circ \gamma$. Then the following diagram*
$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y(T) & \xrightarrow{F^n-id} & Y(T) & \xrightarrow{\nu_\zeta} & T^{F^n} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N_{F^n/F}} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle N_{F^n/F}} & & \\ 0 & \longrightarrow & Y(T) & \xrightarrow{F-id} & Y(T) & \xrightarrow{N_{F^n/F}\circ\nu_\zeta} & T^F & \longrightarrow & 1 \end{array}$$
is commutative with exact rows.

Point 2 of Proposition 3.2.7 in particular yields an isomorphism
$$T^F \cong {}^{Y(T)}\!\diagup_{(F-id)Y(T)}.$$
Therefore we can consider a character $\theta \in Irr(T^F)$ as a character of $Y(T)$ such that $(F-id)Y(T)$ lies in its kernel. Observe moreover that under this isomorphism, again by point 2 of Proposition 3.2.7, θ and $\theta \circ N_{F^n/F}$ are identified with the same character of $Y(T)$. This yields the following result:

Proposition 3.2.8. *[2, Proposition 4.1.3] Let T and T' be F-stable tori, $\theta \in Irr(T^F)$, $\theta' \in Irr(\theta')$. The following are equivalent*

- *(T, θ) and (T', θ') are geometrically conjugated by $g \in G$;*
- *$T' = gTg^{-1}$ and conjugation by g transforms θ', regarded as character of $Y(T')$ in θ regarded as character of $Y(T)$*

With this new description, we can parameterize geometric conjugacy classes. Since F-stable maximal tori are all G-conjugated,fixing a maximal torus T by Proposition 3.2.8 we can find representatives of the shape (T, θ), $\theta \in Irr(T^F)$ for any geometric conjugacy class. Hence it is enough to consider elements of this kind. With this reduction, it is enough to consider the $N_G(T)$ action, i.e. the W action. Then considering the group isomorphism between characters of $Y(T)$ and $X(T) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!\diagup_{\mathbb{Z}}$ of Lemma 1.2.44 we have the following bijection:

Lemma 3.2.9. *[2, Proposition 4.1.4]Fix an F-stable maximal torus T. There is a bijection*

$$\left\{ \begin{array}{c} \textit{Geometric conjugacy classes of} \\ \textit{pairs } (T', \theta') \textit{ with } T' \textit{ an } F\textit{-stable} \\ \textit{maximal torus, } \theta' \in Irr(T') \end{array} \right\} \to \left\{ \begin{array}{c} F\textit{-stable } W\textit{-orbits on} \\ X(T) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}}\} \end{array} \right\}$$

3.2.2 Relations with the dual group

The parameterization of geometric conjugacy classes in Lemma 3.2.9 can be better described in terms of the dual group of G. From now on, (G^*, F^*) will be a pair consisting of an algebraic group and a generalized Frobenius morphism that is in duality with (G, F) as in Definition 2.3.10.
Let $T \leq G$, $T^* \leq G^*$ be the maximally split tori posed in duality. So there is an isomorphism

$$\delta : X(T) \to Y(T^*)$$

that can be extended to

$$\delta : X(T) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}} \to Y(T^*) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}}$$

by letting it act as the identity on ${}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}}$. Let W be the Weyl group of G relative to T, W^* the Weyl group of G^* relative to T^*. By Remark 1.2.55 there is an anti-isomorphism $\delta : W \to W^*$ respecting the action of the Weyl group on $X(T)$ and satisfying, by Remark 2.3.13,

$$\delta \circ F = F^* \circ \delta.$$

Therefore δ induces a bijection

$$\left\{ \begin{array}{c} F\text{-stable } W\text{-orbits on} \\ X(T) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}} \end{array} \right\} \to \left\{ \begin{array}{c} F^*\text{-stable } W^*\text{-orbits on} \\ Y(T^*) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}} \end{array} \right\}. \tag{3.2}$$

Moreover under the fixed isomorphism ${}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}} \to \overline{\mathbb{F}_p}^*$ of Lemma 3.2.6, there is an isomorphism of abelian groups

$$Y(T^*) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}} \to T^*$$

given by evaluation, that is $\gamma \otimes \lambda \mapsto \gamma(\lambda)$ for any $\gamma \otimes \lambda \in Y(T^*) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}}$ [2, Proposition 3.1.2].
So we can rephrase bijection (3.2) as

$$\left\{ \begin{array}{c} F\text{-stable } W\text{-orbits on} \\ X(T) \otimes_{\mathbb{Z}} {}^{\mathbb{Q}_{(p)}}\!/\!_{\mathbb{Z}} \end{array} \right\} \to \left\{ F^*\text{-stable } W^*\text{-orbits on } T^* \right\}$$

It can be observed [2, Corollary 3.7.2] that the F^*-stable W^*-orbits on T^* are the intersections of the F^*-stable semisimple G^*-orbits with T^*. Therefore the previous discussion can be summarized in the following statement

Theorem 3.2.10. *[3, Proposition 11.1.5] Let (G^*, F^*) be in duality with (G, F). There is a bijective correspondence between geometric conjugacy classes of pairs (T, θ) consisting of an F-stable maximal torus of G and an irreducible character of T^F, and F^*-stable conjugacy classes of semisimple elements in G^*.*

We will sometimes denote by $\lambda_{(s)}$ the geometric conjugacy class posed in bijection with s in Theorem 3.2.10.

Example 3.2.11. *Consider the geometric conjugacy class consisting of pairs of the shape $(T, \mathbf{1}_{T^F})$ with T an F-stable torus of Example 3.2.4. Under the bijection of Theorem 3.2.10, this class corresponds always to 1, the semisimple conjugacy class of G^* consisting only of the identity.*
Indeed according to Proposition 3.2.7 the trivial character of T^F can be identified with the trivial character of $Y(T)$. Therefore, since the various bijections we considered were induced by group isomorphisms, the class of $(T, \mathbf{1}_{T^F})$ corresponds to the identity element of T^*.

There is also a parameterization for G^F-conjugacy classes of pairs (T, θ) consisting of an F-stable torus and an irreducible character of T^F in terms of the dual group, that can be regarded as a refinement of Theorem 3.2.10.
Indeed if $T \leq G$ and $T^* \leq G^*$ are maximal tori posed in duality, Proposition 3.2.7 yields an isomorphism $Irr(T^F) \cong T^{*F^*}$. Moreover by Corollary 2.3.15 we have a bijective correspondence between G^F-classes of F-stable maximal tori in G and G^{*F^*}-classes of F^*-stable maximal tori in G^*. Using these facts, one can show the following result.

Proposition 3.2.12. *[3, Proposition 11.1.16] There is a bijection*

$$\left\{ \begin{array}{c} G^F\text{-conjugacy classes of pairs} \\ (T, \theta) \text{ with } T \text{ an } F\text{-stable} \\ \text{maximal torus and } \theta \in Irr(T) \end{array} \right\} \to \left\{ \begin{array}{c} G^{*F^*}\text{-conjugacy classes of pairs} \\ (T^*, s) \text{ with } T^* \text{ an } F^*\text{-stable} \\ \text{maximal torus and } s \in T^{*F^*} \end{array} \right\}.$$

Note that the bijection in Proposition 3.2.12 is canonical once we choose the maps in Lemma 3.2.6.
Now, recall by point 2 of Corollary 3.1.14 that the G^F-conjugacy classes of pairs (T, θ) with T an F-stable maximal torus and $\theta \in Irr(T)$ parameterize the Deligne-Lusztig generalized characters $R^G_{T,\theta}$. For this reason, we will sometimes write $R^{G^*}_{T^*}(s)$ in place of $R^G_{T,\theta}$ (or simply $R_{T^*}(s)$ in place of $R_{T,\theta}$, when the algebraic group G is clear from the context) where (T, θ) and (T^*, s) correspond to each other under the bijection in Proposition 3.2.12.

3.2.3 Lusztig series

The parameterization introduced in the last section allows us to identify a partition for the irreducible characters of the finite group of Lie type G^F. The definition of this partition will enable us to state a powerful result, the *Jordan decomposition for characters* (Theorem 3.2.17).

Definition 3.2.13. *The Lusztig (geometric) series associated to a semisimple F^*-stable semisimple conjugacy class (s) of G^* is the set of irreducible characters of G^F that are irreducible components of an $R_{T,\theta}$ for some pair (T,θ) consisting of an F-stable torus and an irreducible character of T^F that is in the geometric conjugacy class posed in bijection with s in Theorem 3.2.10.*
We will denote it by $\mathscr{E}(G^F,(s))$

Theorem 3.2.14. *We have*

$$Irr(G^F) = \sqcup_{(s)} \mathscr{E}(G^F,(s))$$

with (s) running over the F^*-stable semisimple conjugacy classes of G^*.

Proof. By Corollary 3.1.17 for any χ irreducible character of G^F there exists a (T,θ) such that χ is an irreducible component of $R_{T,\theta}$, so χ belongs to some Lusztig series. Moreover by Theorem 3.2.5 two Deligne-Lusztig generalized characters $R_{T,\theta}$ and $R_{T',\theta'}$ have no irreducible component in common if (T,θ) and (T',θ') are not geometrically conjugated, so Lusztig series are disjoint. □

Remark 3.2.15. *Note that the definition of Lusztig series associated to (s), with (s) an F^*-stable semisimple conjugacy class of G^*, can be reformulated as follows:*

$$\mathscr{E}(G^F,(s)) = \{\chi \in Irr(G^F) \mid \langle \chi, R_{T,\theta}\rangle_{G^F} \neq 0 \text{ for some } (T,\theta) \in \lambda_{(s)}\}.$$

Indeed if $\langle \chi, R_{T,\theta}\rangle_{G^F} \neq 0$ for some $(T,\theta) \in \lambda_{(s)}$, then χ is an irreducible component of $R_{T,\theta}$ and so $\chi \in \mathscr{E}(G^F,(s))$
Conversely, suppose that $\chi \in \mathscr{E}(G^F,(s))$, so χ is an irreducible component of $R_{T,\theta}$ for some $(T,\theta) \in \lambda_{(s)}$. By Corollary 3.1.17 we know that there exists a pair (T',θ') such that $\langle \chi, R_{T',\theta'}\rangle_{G^F} \neq 0$ and since χ is an irreducible component of both $R_{T,\theta}$ and $R_{T',\theta'}$, by Theorem 3.2.5 it follows that (T,θ) and (T',θ') are in the same geometric conjugacy class $\lambda_{(s)}$.

Remark 3.2.16. *By Example 3.2.11 the identity element of G^* corresponds to the geometric conjugacy class consisting of pairs of the shape $(T, \mathbf{1}_{T^F})$ where T runs over the F-stable tori of G. Then by Remark 3.2.15 we have*

$$\mathscr{E}(G^F,1) = \{\chi \in Irr(G^F) \mid \langle \chi, R_{T,\mathbf{1}_{T^F}}\rangle_{G^F} \neq 0 \text{ for some } F\text{-stable torus } T\}.$$

Hence by Remark 3.1.25 we have

$$\mathscr{E}(G^F,1) = \{\text{unipotent characters of } G^F\}$$

The unipotent characters of G^F are an important class of irreducible characters because of the following result, known as *Jordan decomposition for characters*. It enlightens a correspondence between the Lusztig series of a finite group of Lie type G^F and the unipotent characters of other finite groups of Lie type, so it allows to classify irreducible characters of G^F by means of a semisimple conjugacy class of the dual group and of a unipotent character (of another finite group of Lie type).

Theorem 3.2.17. *[8, Theorem 2.6.4](Jordan decomposition for character) Assume $Z(G)$ to be connected. For any semisimple element $s \in {G^*}^{F^*}$, let (s) be the conjugacy class of s in G^*. Then there is a bijection*

$$\mathscr{E}(G^F,(s)) \to \mathscr{E}(C_{G^*}(s)^{F^*},1)$$

such that if $\chi \mapsto \rho_\chi$, it holds

$$\langle \chi, R_{T^*}^{G^*}(s)\rangle_{G^F} = \varepsilon_G \varepsilon_{C_{G^*}(s)} \langle \rho_\chi, R_{T^*,\mathbf{1}_{T^*F^*}}^{C_{G^*}(s)} \rangle_{C_{G^*}(s)}$$

for any maximal torus T^* of G^* containing s.

Remark 3.2.18. *We stated Theorem 3.2.17 for groups with connected centre because if $Z(G)$ is connected then all the centralizers of a semisimple element in the dual group are connected [2, Theorem 4.5.9], and through all this chapter we assumed connectdness of the algebraic group.*
Nevertheless, the Theorem can be extended to groups whose center is not connected, reformulating definitions and results in an appropriate way. For instance, see [3, Theorem 11.5.1].

Appendix A

Lusztig and Harish-Chandra Inductions

In this appendix G is a connected reductive algebraic group over $\overline{\mathbb{F}_p}$, with p prime, F is a generalized Frobenius morphism for G, $\mathcal{L}$ denotes the the Lang map as in Definition 2.2.2, and $\mathscr{L}$ denotes the Lefschetz number as in Definition 3.1.4.
The purpose of this appendix is to give a quick introduction to the Harish Chandra induction theory, a theory deeply linked to the one of the Deligne-Lusztig generalized characters set out in Chapter 3.
The Deligne-Lusztig generalized characters introduced in Definition 3.1.10 can be considered as part of a more broad construction, that allows to construct generalized characters starting from irreducible characters of the fixed point set of the Levi subgroups of parabolic subgroups of G.
Let L be an F-stable Levi subgroup of a parabolic subgroup P of G and let U be the unipotent radical of P. As we did in Lemma 3.1.8 for a maximal torus T contained in a Borel subgroup B, we can in this more general setting define an action of $G^F \times L^F$ on $\mathcal{L}^{-1}(U)$

$$G^F \times L^F \to Aut(\mathcal{L}^{-1}(U)).$$
$$(g, l) \mapsto (x \mapsto gxl^{-1})$$

This allows to give the following Definition.

Definition A.0.1. *(Lusztig induction) Let L be an F-stable Levi subgroup of a parabolic subgroup P of G and let U be the unipotent radical of P. Let $\lambda \in Irr(L^F)$ be an irreducible character of L^F. We define the function $R_{L \leq P, \lambda} : G^F \to \overline{\mathbb{Q}_l}$ to be*

$$R_{L \leq P, \lambda}(g) = \frac{1}{|L^F|} \sum_{l \in L^F} \mathscr{L}((g, l), \mathcal{L}^{-1}(U))\lambda(l^{-1})$$

for any $g \in G^F$.

It can be shown with the same proof of Proposition 3.1.11 that the functions $R_{L \leq P, \lambda}$ defined in Definition A.0.1 are generalized characters of G^F.

However, it is not known in general if the generalized character introduced in Definition A.0.1 are independent of the choice of the parabolic subgroup. Nevertheless, it is proved that this property still holds in several cases (see [8, Theorem 3.3.7]). In Proposition 3.1.14 we have seen that the independence holds in the situation where the Levi subgroup is a torus (and hence the parabolic subgroup is a Borel subgroup), and this allowed us to develop the theory of Deligne-Lusztig generalized characters in Chapter 3.
A different direction that can be taken and guarantees independence of the generalized character $R_{L\leq P,\lambda}$ of the parabolic subgroup P is to restrict ourselves to study the case in which the parabolic subgroups are required to be F-stable.

Proposition A.0.2. *[3, Theorem 5.3.1] Let L be an F-stable Levi subgroup of two different F-stable parabolic subgroups P and Q. Then*

$$R_{L\leq P,\lambda} = R_{L\leq Q,\lambda}$$

for any $\lambda \in Irr(L^F)$.

In this setting, the fact that the Levi subgroup is F-stable and also the parabolic P is required to be F-stable ensures that also the unipotent radical U of P is F-stable. Therefore if $x \in \mathcal{L}^{-1}(U)$, then $xu \in \mathcal{L}^{-1}(U)$, so we can define a map

$$\pi : \mathcal{L}^{-1}(U) \to {}^{G}\!/_{U}$$
$$x \mapsto xU$$

whose image is $({}^{G}\!/_{U})^F \cong {}^{G^F}\!/_{U^F}$, and whose fibers are isomorphic to the affine variety U. The action of G^F by left translation satisfies $g \circ \pi = \pi \circ g$ (for any $g \in G^F$), therefore by Proposition 3.1.6 it holds

$$\mathscr{L}(g, \mathcal{L}^{-1}(U)) = \mathscr{L}(g, {}^{G^F}\!/_{U^F}).$$

Moreover ${}^{G^F}\!/_{U^F}$ is a discrete variety. This implies (by [3, Proposition 8.1.8]) that the l-adic cohomology of ${}^{G^F}\!/_{U^F}$ is all concentrated in degree zero, and

$$H^0_c({}^{G^F}\!/_{U^F}, \overline{\mathbb{Q}_l}) = \overline{\mathbb{Q}_l}\left[{}^{G^F}\!/_{U^F}\right]$$

is a permutation module for G^F (where $\overline{\mathbb{Q}_l}\left[{}^{G^F}\!/_{U^F}\right]$ denotes the algebra of the functions over $\overline{\mathbb{Q}_l}$ of the finite variety ${}^{G^F}\!/_{U^F}$) . The fact that the l-adic cohomology is all concentrated in one degree guarantees that in the situation where the parabolic subgroup P is F-stable, the generalized character defined in Definition A.0.1 is an actual character.
Now, if L is an F-stable Levi subgroup of an F-stable parabolic subgroup P with

unipotent radical U, since $P = U \rtimes L$ we can consider L^F as the quotient ${P^F}/{U^F}$, and given a character λ of L^F we define *inflation* of λ to be the natural lifting of λ to P^F through the quotient, that is

$$Infl_{L^F}^{P^F}(\lambda)(lu) = \lambda(l) \qquad\qquad \text{for any } u \in U^F.$$

Recall that if H is a subgroup of a group K, and χ is a character of H, Frobenius induction allows to construct a character of K as follows:

$$Ind_H^K(\chi)(x) = \frac{1}{|H|} \sum_{\{k \in K \ | \ kxk^{-1} \in H\}} \chi(kxk^{-1}).$$

With these two constructions, we can describe the characters $R_{L \leq P, \lambda}$ of Definition A.0.1 in the case where the parabolic subgroup P is F-stable in a more elementary way.

Proposition A.0.3. *Let L be an F-stable Levi subgroup of an F-stable parabolic subgroup P with unipotent radical U, let $\lambda \in Irr(L^F)$. Then*

$$R_{L \leq P, \lambda} = Ind_{P^F}^{G^F} \circ Infl_{L^F}^{P^F}(\lambda)$$

Proof. Let $g \in G^F$. Then

$$\begin{aligned}
R_{L \leq P, \lambda}(g) &= \frac{1}{L^F} \sum_{l \in |L^F|} Tr\left(g, l^{-1} \middle| \overline{\mathbb{Q}_l}\left[{G^F}/{U^F}\right]\right) \lambda(l) \\
&= \frac{1}{|L^F|} \sum_{l \in L^F} |\{xU^F \in {G^F}/{U^F} \mid gxl^{-1}U^F = xU^F\}| \lambda(l) \\
&= \frac{1}{|L^F|} \sum_{l \in L^F} \frac{1}{|U^F|} |\{x \in G^F \mid x^{-1}gx = lU^F\}| \lambda(l) \\
&= \frac{1}{|P^F|} \sum_{l \in L^F} \sum_{\{x \in G^F \ | \ x^{-1}gx \in lU^F\}} \lambda(l) = \\
&= \frac{1}{|P^F|} \sum_{l \in L^F} \sum_{\{x \in G^F \ | \ x^{-1}gx \in lU^F\}} Infl_{L^F}^{P^F} \lambda(x^{-1}gx) \\
&= Ind_{P^F}^{G^F} \circ Infl_{L^F}^{P^F}(\lambda)(g)
\end{aligned}$$

□

Definition A.0.4. *(Harish-Chandra induction) Let L be an F-stable Levi subgroup of an F-stable parabolic subgroup P with unipotent radical U, let $\lambda \in Irr(L^F)$. The character of G^F*

$$Ind_{P^F}^{G^F} \circ Infl_{L^F}^{P^F}(\lambda)$$

is called Harish-Chandra induction of λ.

Example A.0.5. *Let* T_0 *be an* F*-stable maximally split maximal torus of* G*, and let* B_0 *be an* F*-stable Borel subgroup containing* T*. Then the Deligne-Lusztig generalized character* $R_{T_0,\mathbf{1}_{T_0^F}}$ *is an actual character and is equal to*

$$R_{T_0,\mathbf{1}_{T_0^F}} = Ind_{B_0^F}^{G^F} \circ Infl_{T_0^F}^{B_0^F}(\mathbf{1}_{T_0^F}) = Ind_{B_0^F}^{G^F}(\mathbf{1}_{B_0^F})$$

The theory of Harish-Chandra induction is broadly studied and yields some deep and complete results for representations of finite groups of Lie type. A reference for this is [3, Chapters 5-6].

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