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Tesi di Laurea

Teorema di Noether e correnti migliorate

Noether's theorem and improved currents

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Abstract

Il teorema di Noether in teoria di campo classica associa una corrente conservata ad ogni simmetria continua. Ci sono spesso delle ambiguità nella definizione della corrente, legate alla possibilità di aggiungere quantità che la cui divergenza è identicamente nulla off-shell, o quantità che si annullano on-shell. Tale ambiguità è talvolta utile per ottenere correnti ”migliorate”, cioè che godono di opportune proprietà: per esempio, si può utilizzare per simmetrizzare il tensore energia impulso ottenuto dall'invarianza per traslazioni. In questa tesi, viene descritta una procedura recentemente proposta che consente di ottenere correnti “migliorate” direttamente attraverso il teorema di Noether e senza necessità di correzioni ad hoc.

Noether's theorem in classical field theory links a conserved current to every continuous symmetry. There are often ambiguities in the definition of the current, related to the possibility of adding quantities whose divergence is identically zero off-shell, or quantities which vanish on-shell. This ambiguity is sometimes useful to obtain improved currents, i.e. currents with desired suitable properties. For example, this ambiguity can be used to symmetrize the stress-energy tensor obtained from the invariance under translations. In this thesis, a recently proposed procedure is described which allows to obtain improved currents directly through Noether's theorem and without guesswork or ad hoc corrections.

Contents

Introduction

Symmetry is a concept of great importance in Physics, both for classical theories (from mechanics to field theory), and quantum ones. As much as the general concept of symmetry fascinates the human mind, it may be non-trivial to give a general definition in words. Richard Feynman, interpreting Weyl's definition of symmetry, asserted[1] that "a thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation". Examples of important symmetries that a physical model can exhibit are those under constant translations, linked to the homogeneity of space, and those under rotations, linked to the isotropy of space. The concept of symmetry can be easily and rigorously formulated within a group-theoretic framework. Even if the very first applications of group theory, albeit limited to crystallography, to describe physical symmetries can be traced back to Christian S. Weiss (1816) and Auguste Bravais (1850), a more general application of group theory methods to the description of symmetries is due to Emmy Noether (1918).

Noether proved a fundamental result that links symmetries of physical systems with conservation laws. Nowadays, her theorem can be considered one of the pillars of Theoretical Physics. Within the context of a Lagrangian formulation of classical field theory, based on Noether's theorem, symmetry implies conservations of currents, one for each independent parameter of the symmetry (group). In particular, the constant translational symmetry gives the (local) conservation law of stress-energy tensor (also called energy-momentum tensor), which is linked with the (global) conservation of energy and momenta (charges associated with currents).

However, as we will see in Chapter 1, the conserved currents derived through standard proofs of the Noether theorem are ambiguous. This ambiguity can be exploited in order to obtain *improved* currents with desired properties. For example, the stress-energy tensor can be made symmetric if the theory has Lorentz symmetry, and both symmetric and traceless if the theory has conformal symmetry. Nevertheless, these procedures are *ad-hoc* and the desired properties hold only along the solution of the equations of motion (i.e. *on-shell*). In this thesis, and in particular in Chapter 2, based on the article by Kourkoulou, Nicolis and, Sun [6], we will show that the desired properties can be achieved, even *off-shell*, with a non-ad hoc procedure.

[1]from *The Character of Physical Law*, 1967, Richard Feynman

Chapter 1

Noether's theorem in Classical Field Theory

In this Chapter, after recalling some basics of group theory, we will review two different proofs of Noether's first theorem in a field-theoretic framework. We will characterize the Poincaré group and prove that Poincaré invariance implies the conservation of canonical stress-energy tensor (also abbr. with SE tensor) and canonical angular momentum tensor. After having pointed out that in general the canonical SE tensor is not symmetric, we will illustrate that Poincaré invariance guarantees that through Belinfante procedure it is possible to obtain a symmetric SE tensor, namely the Belinfante SE tensor, equivalent to the canonical one. Subsequently, we will characterize the conformal group and prove that conformal invariance guarantees that it is possible to derive a traceless and symmetric SE tensor equivalent to the canonical one. The main references used in this Chapter are [8], and [2].

When not otherwise specified, throughout this thesis, we will consider a $d-$dimensional spacetime as pseudo-riemannian manifold $(\mathcal{M},\eta)$ with the mostly minus convention ($\eta_{00}=+1, \eta_{ii}=-1$ with $i=1,...,d-1$) and a generic classical field theory with Lagrangian density $\mathcal{L}(\phi,\partial\phi,x)$ function of the multiplet ϕ of fields (where each field $\varphi:\mathcal{M}\to T^q_r\mathcal{M}$ of the multiplet belongs to $\mathcal{T}^{(q,r)}(\mathcal{M})$, i.e. space of (q,r) tensor fields), its derivative and the spacetime coordinates.

1.1 Basics of group theory applied to field theory

Lie groups of transformations are particularly relevant for Physics. Let us first of all provide the basic definitions that we will need.

Definition 1.1.1. A continuous group $\mathcal{G}$ that is also a manifold is called a Lie group. We denote $(\mathfrak{g}, [\![\cdot\,,\cdot]\!])$, namely the associated Lie algebra, the tangent space at the identity together with an operation which satisfies the properties of bilinearity, antisymmetry and the Jacobi identity.

Definition 1.1.2. A finite-dimensional representation of a matrix Lie group $\mathcal{G}$ is a Lie group homomorphism $\rho:\mathcal{G}\to GL(V)$ where V is finite-dimensional vector space and $GL(V)$ denotes the general linear group of all automorphism of V. In particular, it follows from the definition of homomorphism that:

$$\forall\, g_1,g_2\in\mathcal{G} \qquad \rho(g_1\cdot g_2)=\rho(g_1)\bullet\rho(g_2) \qquad \text{and} \qquad \rho(e)=\mathrm{id}_V \quad . \tag{1.1}$$

Definition 1.1.3. Let $\rho:\mathcal{G}\to GL(V)$ be a finite-dimensional representation of a matrix Lie group $\mathcal{G}$. The subspace of V

$$W=\{w\in V\mid\forall\, g\in\mathcal{G},\quad \rho(g)w\in W\} \tag{1.2}$$

is called *invariant.* An invariant subspace is called nontrivial if $W\neq\{0\}$ and $W\neq V$. A representation that has no nontrivial invariant subspaces is called *irreducible.*

In the domain of field theories, an *infinitesimal* transformation can be considered acting both on the coordinates x^μ and multiplet of fields as

$$x^\mu \to x'^\mu = x^\mu + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \quad \text{and} \quad \phi(x) \to \phi'(x') = \phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x) \ , \tag{1.3}$$

where $\mathcal{F}(\phi(x)) = \phi'(x')$, ϵ_a is an infinitesimal parameter, and $a = 1, ..., n$ (n is the order of Lie group). This way of seeing the action of the group on the fields and on the coordinates is called *passive viewpoint*. Within this viewpoint, the fields transformation can be rewritten as

$$\phi'(x') = \phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x) = \phi(x') - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu \phi(x') + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x') \ . \tag{1.4}$$

Therefore, the infinitesimal generators G_a of the group, can be defined as

$$\epsilon_a G_a \phi(x') := \phi'(x') - \phi(x') = \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x') - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu \phi(x') \ . \tag{1.5}$$

If, instead, the coordinates are immutable (i.e. $\mathcal{M}$ is mapped by the same coordinates before and after the transformations) and only the fields transform, we are adopting an *active viewpoint*.

$$x^\mu \to x^\mu \qquad \text{and} \qquad \phi(x) \to \phi'(x) = \phi(x) - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu \phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x) \tag{1.6}$$

In this perspective, an equivalent definition of the generators of the group is possible:

$$\epsilon_a G_a \phi(x) := \phi'(x) - \phi(x) \ . \tag{1.7}$$

Frequently in literature [7], [9], the generators are defined through the relation $\epsilon_a G_a \phi(x) = i(\phi'(x) - \phi(x))$ to assure hermiticity of operators when addressing quantum field theory.

Definition 1.1.4. Connected Lie groups $\mathcal{T}$ of transformations (of fields and spacetime coordinates) under which the action $S_V[\phi] = \int_V \mathcal{L}(\phi, \partial\phi, x)\, d^d x$ is invariant *up to a boundary term*[1] are called *continuous symmetries* of the classical field theory. Within the passive viewpoint (assuming that the integration region is invariant under the transformation of coordinates) this can be specified in terms of the *infinitesimal* transformations of fields and spacetime coordinates as

$$S_V\left[\phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x)\right] - S_V[\phi(x)] = \int_V \partial_\mu(\epsilon_a \Lambda_a^\mu) d^d x \ , \tag{1.8}$$

for all transformation belonging to $\mathcal{T}$, where $\Lambda_a^\mu(x)$ is an arbitrary function. The equivalent prescription within the active viewpoint is

$$S_V\left[\phi(x) - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu \phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x)\right] - S_V[\phi] = \int_V \partial_\mu \left(\epsilon_a \Lambda_a^\mu - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \mathcal{L}\right) d^d x \ , \tag{1.9}$$

for all transformation belonging to $\mathcal{T}$. Indeed, differentiating the change of spacetime coordinates gives

$$d^d x' = \left| \det \frac{\partial x'}{\partial x} \right| d^d x = [1 + \epsilon_a \partial_\mu \left(\frac{\delta x^\mu}{\delta \epsilon_a}\right) + \mathcal{O}(\epsilon^2)] d^d x \ , \tag{1.10}$$

so that the symmetry conditions within the passive and active viewpoint are equivalent (considering only first-term in ϵ_a):

$$\begin{aligned} &\mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'), x') \left[1 + \epsilon_a \partial_\mu \left(\frac{\delta x^\mu}{\delta \epsilon_a}\right)\right] - \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) = \epsilon_a \partial_\mu \Lambda_a^\mu \\ &\qquad \left[\mathcal{L}(\phi'(x), \partial_\mu \phi'(x), x) + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu \mathcal{L}\right] \left[1 + \epsilon_a \partial_\mu \left(\frac{\delta x^\mu}{\delta \epsilon_a}\right)\right] - \mathcal{L} = \epsilon_a \partial_\mu \Lambda_a^\mu \ , \end{aligned} \tag{1.11}$$

which implies $\delta \mathcal{L} = \epsilon_a \partial_\mu \left(\Lambda_a^\mu - \frac{\delta x^\mu}{\delta \epsilon_a} \mathcal{L}\right)$ within the active viewpoint, as stated in the Equation 1.9.

[1]The first notion of symmetry provided by E. Noether does not include possible boundary term, at least in the passive viewpoint. The generalization to action invariant *up to a boundary term* is due to Bessel-Hagen (1898-1946).

Definition 1.1.5. In particular, connected Lie groups $\mathcal{T}$ of transformations are called (continuous) symmetries *in the strict sense* of the classical field theory if $\partial_\mu \Lambda_a^\mu \equiv 0$.

Definition 1.1.6. The explicit transformation of coordinates and fields within the passive viewpoint suggests a useful distinction between symmetries:

- Internal symmetries if $\frac{\delta x^\mu}{\delta \epsilon_a} = 0$. In this case, passive and active viewpoint are equivalent.
- External or spacetime symmetries if $\frac{\delta x^\mu}{\delta \epsilon_a} \neq 0$.

1.2 Standard derivations of Noether's first theorem

Noether's first theorem. For all generator G_a, where $a = 1, ..., n$, of a Lie group $\mathcal{T}$ of order n describing a continuous symmetry of the classical field theory, there exists a current j_a^μ (called Noether current) which is conserved: $\partial_\mu j_a^\mu = 0$ along the solutions of the equations of motion (*on-shell*).

***Passive viewpoint* proof.** In this case, spacetime coordinates and fields transform as Eq. 1.4; to simplify the notation, we define $\omega_a^\mu := \frac{\delta x^\mu}{\delta \epsilon_a}$ and $\Delta_a \phi(x') := \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x')$ so that

$$\phi'(x') = \phi(x') - \epsilon_a \omega_a^\mu \partial_\mu \phi(x') + \epsilon_a \Delta_a \phi(x') \ . \tag{1.12}$$

Here $\bar{\delta}\phi = -\omega_a^\mu \partial_\mu \phi + \Delta_a \phi$ and $\delta\phi = \Delta_a \phi$ are called, respectively, the *total variation* and the *form variation* of the fields. According to the definition of continuous symmetry, the infinitesimal element of action must be invariant up to a boundary term:

$$\mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'), x') d^d x' - \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) d^d x \overset{!}{=} \epsilon_a \partial_\mu \Lambda_a^\mu d^d x \ . \tag{1.13}$$

A straightforward algebraic manipulation gives

$$\begin{aligned} &\epsilon_a(\partial_\mu \omega_a^\mu)\mathcal{L} + \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'), x') - \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) \\ &= \epsilon_a(\partial_\mu \omega_a^\mu)\mathcal{L} + \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'), x') - \mathcal{L}(\phi'(x), \partial_\mu \phi'(x), x) + \mathcal{L}(\phi'(x), \partial_\mu \phi'(x), x) - \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) \ . \end{aligned} \tag{1.14}$$

The second and the third term differ only by the transformation $x^\mu \to x'^\mu = x^\mu + \epsilon_a \omega_a^\mu$, while the the forth and fifth term only by the transformation $\phi(x) \to \phi'(x) = \phi(x) + \epsilon_a \delta\phi(x)$.

$$\begin{aligned} &\epsilon_a(\partial_\mu \omega_a^\mu)\mathcal{L} + \epsilon_a \omega^\mu (\partial_\mu \mathcal{L}) + \epsilon_a \frac{\partial \mathcal{L}}{\partial \phi} \cdot \delta\phi + \epsilon_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \delta \partial_\mu \phi - \epsilon_a \partial_\mu \Lambda_a^\mu \\ &\qquad = \epsilon_a \partial_\mu (\omega_a^\mu \mathcal{L}) + \epsilon_a \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \delta\phi \right) - \epsilon_a \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} \right] \cdot \delta\phi - \epsilon_a \partial_\mu \Lambda_a^\mu \overset{!}{=} 0 \ . \end{aligned} \tag{1.15}$$

The term in squared brackets vanishes because of the equation of motion; factoring out ϵ_a we obtain the statement of Noether's theorem $\partial_\mu j_a^\mu = 0$ with

$$j_a^\mu = \omega_a^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta_a \phi - \Lambda_a^\mu \ . \tag{1.16}$$

The Noether current j_a^μ in the case of invariance of the action *in the strict sense* reduces to

$$j_a^\mu = \omega_a^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta_a \phi = -\omega_a^\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial_\nu \phi - \delta_\nu^\mu \mathcal{L} \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \bar{\delta}\phi \ . \tag{1.17}$$

***Active viewpoint* proof.** In the first instance, we consider internal symmetries for which the fields transform as

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon_a \Delta_a \phi(x) \ , \tag{1.18}$$

according to the notation used in the former proof. In order to guarantee the invariance of action, the Lagrangian must be invariant up to a d-divergence:

$$\mathcal{L} \to \mathcal{L} + \epsilon_a \partial_\mu F^\mu_a \qquad \text{namely} \qquad \delta\mathcal{L} = \epsilon_a \partial_\mu F^\mu_a \ , \tag{1.19}$$

where F^μ_a is some functional of the fields (which coincides with Λ^μ_a of the passive viewpoint). We can explicitly calculate the variation of $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L}(\phi', \partial_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) &= \frac{\partial \mathcal{L}}{\partial \phi} \cdot (\epsilon_a \Delta_a \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial_\mu (\epsilon_a \Delta_a \phi) \\ &= \epsilon_a \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta_a \phi \right) + \epsilon_a \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta_a \phi \ . \end{aligned} \tag{1.20}$$

The last term vanishes due to the Euler-Lagrange equation. Therefore, we obtain the statement of Noether's theorem with:

$$j^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta_a \phi - F^\mu_a \ . \tag{1.21}$$

Within active viewpoint we can also consider spacetime symmetries. In this case, a useful strategy for the proof is to make the infinitesimal parameter ϵ_a suitably spacetime modulated in the following way:

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon_a(x) \Delta_a(\phi) \ . \tag{1.22}$$

In general, to derive the statement of Noether's first theorem it is only necessary that the transformation reduces to $\phi \to \phi + \epsilon_a \Delta_a$ in the limit of constant ϵ_a. Indeed, for constant ϵ_a the Lagrangian density varies as $\delta\mathcal{L} = \epsilon^a \partial_\mu F^\mu_a$ and expanding with Taylor $\epsilon_a(x)$, we obtain:

$$\delta\mathcal{L} = \epsilon^a(x) \partial_\mu F^\mu_a + \partial_\mu \epsilon^a(x)\, G^\mu_a + \partial_\mu \partial_\nu \epsilon^a(x)\, G^{\mu\nu}_a + \ldots \tag{1.23}$$

where $G^\mu_a, G^{\mu\nu}_a, \ldots$ are some functionals of the fields. Considering only functions $\epsilon^a(x)$ that vanish at infinity, we can integrate $\delta\mathcal{L}$ by parts:

$$\delta S = \int \delta\mathcal{L}\, d^d x = \int \epsilon^a(x)\, \partial_\mu \underbrace{\left[F^\mu_a - G^\mu_a - \partial_\nu G^{\mu\nu}_a - \ldots\right]}_{j^\mu_a} d^d x \ . \tag{1.24}$$

The action should be stationary for all field variations that vanish at infinity,
therefore $\partial_\mu j^\mu_a = 0$. □

In practice, to explicitly determine the Noether current within the active viewpoint, the variation of the Lagrangian density $\delta\mathcal{L}$ for a *generic* $\epsilon^\mu_a(x)$ is determined by direct calculation, and j^μ_a is consequently found by comparing Eq. 1.23 and 1.24.

Preposition 1.1. Under the assumption that the configurations of the fields fall off sufficiently rapidly at infinity, for all generator of the Lie group indexed by a there exists a conserved charge Q_a.

Proof. The integration of $\partial_0 j^0_a = -\partial_i j^i_a$ and subsequent the use of the divergence theorem gives

$$\int_V d^{(d-1)}x\, \partial_t j^0_a = -\int_{\partial V} j^i_a \cdot d\Sigma_i = 0 \qquad \text{that implies conservation of} \qquad Q_a = \int_V d^{(d-1)}x\, j^0_a(x) \ . \tag{1.25}$$

Remark. For all conserved j^μ_a associated with a continuous symmetry also $\tilde{j}^\mu_a = j^\mu_a + \partial_\nu X^{\mu\nu}$ is conserved $\forall\, X^{\mu\nu}$ antisymmetric tensor $X^{\mu\nu} + X^{\nu\mu} = 0$, since $\partial_\mu \partial_\nu X^{\mu\nu} \equiv 0$. This note will turn out to be fundamental later in this Chapter to improve the properties of conserved Noether currents.

1.3 Canonical stress-energy tensor

In the first instance, we consider the constant translational spacetime symmetry, $x^\mu \to x^\mu + \epsilon_a \omega^\mu_a$, which is obtained with $\omega^\mu_a = \delta^\mu_a$. If the classical field theory exhibit constant translational symmetry, we say that the fields belongs to a representation of the Lie group $(\mathbb{R}^4, +)$. A basis for generators in this representation is, based on Equation 1.5, $P_a = -\partial_a$ with $a = 1, ..., d$. According to Noether's first theorem, there are in particular d conserved currents (one for each independent parameter of the symmetry group):

$$j^\mu_a = -\delta^\nu_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \ . \tag{1.26}$$

Considering the independence of the d currents, we can define a (2,0) tensor, namely the canonical stress-energy (or energy-momentum) tensor, which is also conserved *on-shell*:

$$T^{\mu\nu}_c = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \qquad \text{with} \qquad \partial_\mu T^{\mu\nu}_c = 0 \ . \tag{1.27}$$

In addition, as stated in the Preposition 1.1 there exist a conserved charge for each conserved currents. It is evident from Eq. 1.27 that, in general, the canonical SE tensor does not have the desired properties of symmetry and tracelessness. Furthermore, for gauge theories the canonical tensor is not necessarily gauge invariant. In particular, the symmetry property is desirable for two reasons. In the first place, as we will see in the following Paragraph, a symmetric SE tensor allows us to define a conserved angular momentum tensor which is analogous to the one definable in the mechanical case. Secondly, in general relativity the source of the gravitational interaction is precisely a *symmetrical* momentum tensor.

1.4 Poincaré invariance

We recall that the Poincaré infinitesimal transformation of spacetime coordinates in the neighborhood of identity is

$$x^{\mu\prime} = x^\mu + \epsilon^\mu(x) = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu + a^\mu \quad \Rightarrow \quad \epsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \qquad \text{with} \qquad \omega^\mu{}_\nu + \omega^\nu{}_\mu = 0 \ . \tag{1.28}$$

Definition 1.4.1. The set of coordinates transformation which preserves the flat metric $\eta_{\mu\nu}$ form the isometry group of $(\mathcal{M}, \eta_{\mu\nu})$, namely the *Poincaré group.*

$$x \to x' \qquad \eta'_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \eta_{\rho\sigma} := \eta_{\mu\nu} \ . \tag{1.29}$$

Since our main interest is continuous symmetries of field theories (that are connected Lie group), we consider only the component connected to identity of Poincaré group: from Eq.1.28, this component is[2] $\mathbf{R}^{1,(d-1)} \rtimes SO(1, d-1)^+$.

Accordingly to Eq. 1.28, the total variations of the field are:

- For a scalar field φ: $\varphi'(x') = \varphi(x)$ hence $\bar{\delta}\varphi = 0$.
- For a vector field A_μ: $A'_\mu(x') = (\delta^\mu_\nu + \omega^\mu_\nu) A^\nu(x)$ hence $\bar{\delta} A_\mu(x) = \omega_{\mu\nu} \eta^{\nu\rho} A_\rho(x)$.
- For a tensor field:

$$T'^{\mu_1 ... \mu_q}_{\nu_1 ... \nu_r} = (\delta^{\mu_1}_{\rho_1} + \omega^{\mu_1}_{\rho_1}) ... (\delta^{\mu_q}_{\rho_q} + \omega^{\mu_q}_{\rho_q})(\delta^{\sigma_1}_{\nu_1} + \omega^{\mu_1}_{\rho_1}) ... (\delta^{\sigma_r}_{\nu_r} + \omega^{\mu_r}_{\rho_r}) T^{\rho_1 ... \rho_q}_{\sigma_1 ... \sigma_r} \ . \tag{1.30}$$

In general for the multiplet of fields $\phi = \{\phi_1, \phi_2, ..., \phi_N\}$, we have:

$$\bar{\delta}\phi_r = \frac{1}{2} \omega_{\mu\nu} (\tilde{S}^{\mu\nu})_r^{\ s} \phi_s \quad \text{so that} \quad \bar{\delta}\phi = \frac{1}{2} \omega_{\mu\nu} S^{\mu\nu} \cdot \phi \ , \tag{1.31}$$

[2] We recall that with $SO(1, d-1)^+$, we are indicating the proper ($\det \Lambda = 1$), orthochronous (${\Lambda^0}_0 \geq 1$) Lorentz group.

where r identifies a single field (scalar, vector or tensor) within the multiplet and s denotes the contraction of the indices within the multiplet. $S^{\mu\nu}$ are the generators for the (irreducible) representation of $SO(1,d-1)^+$ on the vector space of the tensor fields evaluated at the origin $\phi(0)$. Without loss of generality we can consider the tensor $S^{\mu\nu}$ completely antisymmetric in $\mu\nu$: a possible symmetric component would not contribute because it is multiplied by $\omega_{\mu\nu}$ (antisymmetric).

Considering the action of Poincaré group on both the fields and coordinates, we get from Equations 1.5 and 1.4:

- Infinitesimal spacetime translations, $x^\mu \to x'^\mu = x^\mu + \epsilon_a \omega^\mu_a$, with $\omega^\mu_a = \delta^\mu_a$, whose generators are $P_a = -\partial_\alpha$ (Paragraph 1.3).
- Infinitesimal Lorentz transformation, $x^\mu \to x'^\nu = x^\nu + \omega^{\nu\rho} x_\rho$ with $\frac{d(d-1)}{2}$ independent parameter in $\omega^{\nu\rho}$ with $0 \le \nu < \rho \le d$, whose generators are given by

$$\phi'(x') = \phi(x') + \frac{1}{2}\omega^{\mu\nu}\, S_{\mu\nu}\cdot\phi(x') - \omega^{\mu\nu}\frac{x_\nu\partial_\mu - x_\mu\partial_\nu}{2}\phi(x') \tag{1.32}$$

$$J_{\mu\nu} = (x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}\ . \tag{1.33}$$

The invariance of action under $SO(1,d-1)^+$ implies the conservation of $\frac{d(d-1)}{2}$ independent Noether current, one for each parameter of $\omega_{\nu\rho}$, considering that $\bar{\delta}\phi = \frac{\delta\mathcal{F}}{\delta\omega_{\nu\rho}} = \frac{1}{2}S^{\nu\rho}\cdot\phi$ and $\frac{\delta x^\alpha}{\delta\omega_{\nu\rho}} = \eta^{\alpha\nu}x^\rho$:

$$j^\mu_{\omega_{\nu\rho}} = -\eta^{\alpha\nu}x^\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial_\alpha\phi - \delta^\mu_\alpha\mathcal{L}\right) + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}S^{\nu\rho}\cdot\phi = -T^{\mu\nu}_c x^\rho + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}S^{\nu\rho}\cdot\phi\ . \tag{1.34}$$

Recalling that the parameters $\nu\rho$ of the transformation are antisymmetric, the conserved Noether currents can be rewritten as

$$\frac{1}{2}\left(T^{\mu\rho}_c x^\nu - T^{\mu\nu}_c x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}S^{\nu\rho}\cdot\phi\right) = M^{\mu\nu\rho}_c\ . \tag{1.35}$$

The tensor $M^{\mu\nu\rho}_c$ of the $\frac{d(d-1)}{2}$ independent currents is called canonical angular momentum tensor. The *full* Poincaré invariance implies the conservation $\partial_\mu T^{\mu\nu}_c = 0$ and $\partial_\mu M^{\mu\nu\rho}_c = 0$. In particular from the conservation of $M^{\mu\nu\rho}_c$ we obtain:

$$T^{\nu\rho}_c - T^{\rho\nu}_c = \partial_\mu \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\cdot(S^{\nu\rho}\cdot\phi)\right)}_{:=Y^{\mu\nu\rho}} \quad\longrightarrow\quad T^{\mu\nu}_c - T^{\nu\mu}_c = \partial_\rho Y^{\rho\mu\nu}\ . \tag{1.36}$$

Consequently, $T^{\mu\nu}_c$ is symmetric only if the multiplet is composed of only scalar fields (these transform trivially under the Poincare group, $S^{\mu\nu}\equiv 0$).

1.4.1 Symmetric stress-energy tensor: Belinfante procedure

In order to determine, an *improved* SE tensor we can exploit the ambiguity in the definition of conserved Noether current (see Remark 1.2), and consider *equivalent* SE tensors given by

$$T^{\mu\nu} = T^{\mu\nu}_c + \partial_\rho X^{\rho\mu\nu} \quad \text{with} \quad X^{\rho\mu\nu} = -X^{\mu\rho\nu} \tag{1.37}$$

which are conserved *on-shell* ($\partial_\mu T^{\mu\nu} = 0$) iff the canonical one is conserved. A procedure, traced back to Belinfante (1939) [3], exploit this property in the case of a Poincaré-invariant theory, to derive a *symmetric* SE tensor (hereinafter referred to as Belinfante's tensor $T^{\mu\nu}_B$). In the first instance, we proceed with some guesswork: we assume that $X^{\rho\mu\nu}$ is the $\rho\mu$ antisymmetric part of $Y^{\rho\mu\nu}$. Exploiting Eq. 1.36 gives

$$\begin{aligned} T^{\mu\nu}_B &= T^{\mu\nu}_c + \frac{1}{2}(\partial_\rho Y^{\rho\mu\nu} - \partial_\rho Y^{\mu\rho\nu}) \\ &= T^{\mu\nu}_c + \frac{1}{2}(-T^{\mu\nu}_c + T^{\nu\mu}_c) - \frac{1}{2}\partial_\rho Y^{\mu\rho\nu} \\ &= \frac{1}{2}(T^{\mu\nu}_c + T^{\nu\mu}_c) - \frac{1}{2}\partial_\rho Y^{\mu\rho\nu}\ . \end{aligned} \tag{1.38}$$

However, even if the first term is the symmetric component of $T_c^{\mu\nu}$, the last one (and thus $T_B^{\mu\nu}$) is not symmetric in $\mu\nu$. We can replace this last term with

$$-\frac{1}{2}\partial_\rho(Y^{\mu\rho\nu}+Y^{\nu\rho\mu}) \ , \tag{1.39}$$

that is symmetric in $\mu\nu$, and as long as we define

$$X^{\rho\mu\nu}=\frac{1}{2}(Y^{\rho\mu\nu}-Y^{\mu\rho\nu}-Y^{\nu\rho\mu}) \ , \tag{1.40}$$

we obtain a *symmetric* Belinfante tensor, whose full expression is

$$T_B^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial^\nu\phi-\eta^{\mu\nu}\mathcal{L}+\frac{1}{2}\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot S^{\mu\nu}\cdot\phi-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot S^{\rho\nu}\cdot\phi-\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\cdot S^{\rho\mu}\cdot\phi\right) \ . \tag{1.41}$$

1.5 Conformal invariance

Exact or approximate conformal invariance is a property of many interesting models in physics. As we will show shortly, conformal invariance of a field theory is strictly linked with the possibility of making the SE tensor traceless.

A conformal transformation is a coordinate transformation $x\to x'$ under which the flat metric of the manifold $\mathcal{M}$ transform as:

$$\eta_{\rho\sigma}\to\eta'_{\rho\sigma}(x')=\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\eta_{\mu\nu}=\Lambda(x)\,\eta_{\rho\sigma} \ . \tag{1.42}$$

In particular, if $\Lambda(x)\equiv 1$, the transformation corresponds to the Poincaré group; if $\Lambda(x)\equiv$ constant, the transformation corresponds to global scale transformation. We aim to characterize the infinitesimal conformal transformation $x'^\mu=x^\mu+\epsilon^\mu+\mathcal{O}(\epsilon^2)$, exploiting the definition given above.

$$\begin{aligned}\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}&=\eta_{\rho\sigma}\left(\delta^\rho_\mu+\frac{\partial\epsilon^\rho}{\partial x^\mu}+\mathcal{O}(\epsilon^2)\right)\left(\delta^\sigma_\nu+\frac{\partial\epsilon^\sigma}{\partial x^\nu}+\mathcal{O}(\epsilon^2)\right)\\&=\eta_{\mu\nu}+\underbrace{(\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu)}_{:=f(x)\eta_{\mu\nu}}+\mathcal{O}(\epsilon^2) \ .\end{aligned} \tag{1.43}$$

To determine $f(x)$, we apply both side $\eta^{\mu\nu}$ on its definition:

$$f(x)\eta_{\mu\nu}\eta^{\mu\nu}=2(\partial_\mu\epsilon^\mu)\to f(x)=\frac{2}{d}(\partial\cdot\epsilon) \ . \tag{1.44}$$

We apply ∂^ν, then ∂_ν and finally we add the symmetric $\mu\nu$ counterpart:

$$\begin{aligned}d\cdot[\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu]&=2(\partial\cdot\epsilon)\eta_{\mu\nu}\\d\cdot[\partial_\mu(\partial\cdot\epsilon)+\Box\,\epsilon_\mu]&=2\partial_\mu(\partial\cdot\epsilon)\\d\cdot[\partial_\mu\partial_\nu(\partial\cdot\epsilon)+\Box\,\partial_\nu\epsilon_\mu]&=2\partial_\mu\partial_\nu(\partial\cdot\epsilon)\quad+\quad\mu\leftrightarrow\nu\\d\,\Box\,(\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu)+2(d-2)\partial_\mu\partial_\nu(\partial\cdot\epsilon)&=0\\(d-1)\,\Box(\partial\cdot\epsilon)&=0 \ .\end{aligned} \tag{1.45}$$

Given $\Box\,(\frac{\partial\epsilon^\mu}{\partial x^\mu}(x^\mu))=0$, the derivative of ϵ^μ can be at most linear in x^μ. Therefore, infinitesimal conformal transformation are characterized by the following Equation (conformal Killing equation)

$$\epsilon_\mu=a_\mu+b_{\mu\nu}x^\nu+c_{\mu\nu\rho}x^\nu x^\rho\qquad\text{with}\qquad\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu=\frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu} \ , \tag{1.46}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho}$ are (infinitesimal) constants. In particular, individually each term of ϵ_μ corresponds to:

1. Infinitesimal spacetime translation, $\epsilon_\mu = a_\mu$ with corresponding generator $P_\mu = -\partial_\mu$, as previously stated.

2. Infinitesimal rigid Lorentz rotation and scale transformations, $\epsilon_\mu = b_{\mu\nu}x^\nu$. Substituting in the Eq. 1.46, we obtain
$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}(\eta^{\rho\sigma}b_{\rho\sigma})\eta_{\mu\nu} = 2\alpha\eta_{\mu\nu} \quad . \tag{1.47}$$
For a generic tensor b of rank (0,2) the former Equation fix a constrain on its symmetric component:
$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \quad . \tag{1.48}$$
   - $m_{\mu\nu}$ corresponds to the infinitesimal Lorentz rotations $x'_\mu = (\delta_{\mu\nu} + m_{\mu\nu})x^\nu$, whose generators are given by Eq. 1.32.
   - $\alpha\eta_{\mu\nu}$ corresponds to the infinitesimal scale transformations $x'^\mu = (1+\alpha)x^\mu$. If we denote with Δ the action of scale transformations on fields evaluated at the origin, we can determine the generator, namely D, as follow:
$$D\phi = \frac{\delta\mathscr{F}}{\delta\alpha}\phi - \frac{\delta x^\mu}{\delta\alpha}\partial_\mu\phi \qquad \text{so that} \qquad D = -x^\mu\partial_\mu + \Delta \quad . \tag{1.49}$$

3. Infinitesimal special conformal transformations (SCTs), $\epsilon_\mu = c_{\mu\nu\rho}x^\nu x^\rho$. Firstly, we have to derive another important identity in order to characterize in the explicit form SCTs; from Eq. 1.46 it follows that
$$\partial_\rho\partial_\mu\epsilon_\nu + \partial_\rho\partial_\nu\epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon) \quad . \tag{1.50}$$
Applying a cyclic permutation of indexes in the former Equation ($\nu\to\mu, \mu\to\rho, \rho\to\nu$) gives
$$\partial_\nu\partial_\rho\epsilon_\mu + \partial_\mu\partial_\rho\epsilon_\nu = \frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon) \tag{1.51}$$
$$\partial_\mu\partial_\nu\epsilon_\rho + \partial_\nu\partial_\mu\epsilon_\rho = \frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon) \quad . \tag{1.52}$$
Adding member by member Equations 1.51 + 1.52 − 1.50, we obtain
$$\partial_\mu\partial_\nu\epsilon_\rho + \partial_\nu\partial_\mu\epsilon_\rho = \frac{2}{d}\left[\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon) + \eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon) - \eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon)\right] \tag{1.53}$$
$$\partial_\mu\partial_\nu\epsilon_\rho = \frac{1}{d}\left[\eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu - \eta_{\mu\nu}\partial_\rho\right](\partial\cdot\epsilon) \quad . \tag{1.54}$$
Applying the former Equation to the variation that characterize the SCTs $\epsilon_\mu = c_{\mu\nu\rho}x^\nu x^\rho$ gives
$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \quad , \tag{1.55}$$
where we have defined:
$$b_\nu = \frac{1}{d}\partial_\nu(\partial^\mu\epsilon_\mu) = \frac{1}{d}\partial^\mu c_{\mu\nu\rho}x^\rho = \frac{1}{d}\partial_\mu c^\mu{}_{\nu\rho}x^\rho = \frac{1}{d}\delta^\rho_\mu c^\mu{}_{\nu\rho} = \frac{c^\mu{}_{\nu\mu}}{d} \quad . \tag{1.56}$$
Finally the explicit expression for SCTs results
$$x'^\mu = x^\mu + 2(x\cdot b)x^\mu - (x^2)b^\mu \quad . \tag{1.57}$$
We can define a vector k_μ, called the *Killing vector*, as the generator of SCTs for spinless fields:
$$k_\mu = \frac{\delta x^\mu}{\delta b^\nu}\partial_\nu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu \quad . \tag{1.58}$$
If we want to consider, instead, fields that transform according to a non-trivial representations under SCTs, and defining κ_μ as the generators acting on the fields evaluated at the origin,

the generators for transformation acting on fields evaluated at generic x can be determined translating

$$K_\mu \phi(0) = \kappa_\mu \phi(0) \tag{1.59}$$

$$K_\mu \phi(x) = [\kappa_\mu + 2x_\mu \Delta - x_\nu S_{\mu\nu} - 2x_\mu x^\nu \partial_\nu + x^2 \partial_\mu]\phi(x) \ , \tag{1.60}$$

as can be checked using the Hausdorff formula, $\exp(-x^\rho P_\rho) K_\mu \exp(x^\rho P_\rho) = K_\mu + [K_\mu, x^\rho P_\rho] + \frac{1}{2!}[[K_\mu, x^\rho P_\rho], x^\rho P_\rho] + ...$, once the commutator $[K_\mu, x^\rho P_\rho]$ has been determined by explicit calculation.

In summary, full conformal invariance of a classical field theory is associated with

$$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+2)(d+1)}{2} \tag{1.61}$$

generators, and consequently the same number of conserved Noether currents.

1.5.1 Traceless and symmetric stress-energy tensor: virial currents

In this paragraph, we are going to show that if a classical field theory has *full* conformal invariance, we can find an equivalent symmetric *and* traceless SE tensor. In the first instance, we consider the infinitesimal scale transformation

$$x'^\mu = (1+\alpha)x^\mu \qquad \text{and} \qquad \phi'(x') = (1+\alpha\Delta)\phi(x) \ . \tag{1.62}$$

Exploiting $T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho X^{\rho\mu\nu}$, the conserved Noether current under scale invariance may be rewritten as

$$\begin{aligned} j^\mu &= -\left[T_B^{\mu\nu} - \partial_\rho X^{\rho\mu\nu}\right] x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta\phi \\ &= -T_B^{\mu\nu} x_\nu + \partial_\rho (X^{\rho\mu\nu} x_\nu) - X^{\rho\mu}{}_\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta\phi \ , \end{aligned} \tag{1.63}$$

that is equivalent to

$$\begin{aligned} j^\mu &= -T_B^{\mu\nu} x_\nu - \frac{1}{2}\left(\eta_{\rho\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \cdot S^{\mu\nu} \cdot \phi - \eta_{\rho\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \cdot S^{\rho\mu} \cdot \phi\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \Delta\phi \\ &= -T_B^{\mu\nu} x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot (\Delta - \eta_{\rho\mu} S^{\rho\mu}) \cdot \phi \ . \end{aligned} \tag{1.64}$$

The last term, hereinafter V^μ, is called *virial* current. We assume that it is writable as the divergence of a generic (2,0) tensor field:

$$V^\mu = \partial_\nu \sigma^{\mu\nu} \ . \tag{1.65}$$

We denote respectively $\sigma_+^{\mu\nu}$ and $\sigma_-^{\mu\nu}$the symmetric and antisymmetric component of σ. We define our candidate traceless SE tensor, equivalent to the Belinfante one

$$\Theta^{\mu\nu} = T_B^{\mu\nu} - \frac{1}{2}\partial_\lambda \partial_\rho \Xi^{\lambda\rho\mu\nu} \ , \tag{1.66}$$

with Ξ antisymmetric in $\mu\rho$. With some guess work [4], we can found the right Ξ, in order to achieve tracelessness of $\Theta^{\mu\nu}$

$$\Xi^{\lambda\rho\mu\nu} = \frac{2}{d-2}\left[\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} - \eta^{\rho\nu}\sigma_+^{\lambda\mu} + \frac{1}{d-1}\left(\eta^{\lambda\mu}\eta^{\rho\nu} - \eta^{\lambda\rho}\eta^{\mu\nu}\right)\sigma_{+\ \alpha}^{\alpha}\right] \ . \tag{1.67}$$

Indeed, with this choice we can rewrite the Noether current associated with scale invariance as

$$j^\mu = -\Theta^{\mu\nu} x_\nu - \frac{1}{2}\partial_\lambda \partial_\rho \left(\Xi^{\lambda\rho\mu\nu}\right) x_\nu + \partial_\rho \sigma^{\mu\rho} \ . \tag{1.68}$$

We now compute the divergence of this expression in order to demonstrate the proposed $\Theta^{\mu\nu}$ tracelessness.

$$\partial_\mu j^\mu = -\Theta^\mu_{\ \mu} - x_\nu \partial_\mu T_B^{\mu\nu} - \frac{1}{2}(\partial_\lambda \partial_\rho \Xi^{\lambda\rho\mu}{}_\mu) + \partial_\mu \partial_\rho \sigma_+^{\mu\rho} = 0 \tag{1.69}$$

The second term vanishes because the Belinfante tensor is conserved. We have considered only the symmetric part of $\sigma^{\mu\nu}$ because the antisymmetric part vanishes when derived $\partial_\mu \partial_\rho$. It remains to be verified, and we do this by explicit calculation below, that

$$\begin{aligned} \partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} &= \frac{1}{2} \partial_\lambda \partial_\rho \Xi^{\lambda\rho\mu}{}_\mu \\ &= \frac{1}{2} \partial_\lambda \partial_\rho \left\{ \frac{2}{d-2} \left[\eta^{\lambda\rho} \sigma_{+\ \alpha}^{\ \alpha} - 2\sigma_+^{\lambda\rho} + d\sigma_+^{\lambda\rho} - \eta^{\lambda\rho} \sigma_{+\ \alpha}^{\ \alpha} \right] \right\} \\ &= \partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} \ . \end{aligned} \tag{1.70}$$

Finally, the explicit expression of the improved SE tensor is

$$\begin{aligned} \Theta^{\mu\nu} = &\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} + \frac{1}{2} \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \cdot S^{\mu\nu} \cdot \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot S^{\rho\nu} \cdot \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \cdot S^{\rho\mu} \cdot \phi \right) \\ &- \frac{1}{d-2} \left[\Box \sigma_+^{\mu\nu} - \partial_\rho \partial^\mu \sigma_+^{\rho\nu} + \eta^{\mu\nu} \partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} - \partial_\lambda \partial^\nu \sigma_+^{\lambda\mu} + \frac{1}{d-1} \left(\partial^\mu \partial^\nu \sigma_{+\ \alpha}^{\ \alpha} - \eta^{\mu\nu} \Box \sigma_{+\ \alpha}^{\ \alpha} \right) \right] \ . \end{aligned} \tag{1.71}$$

It can be proved that, when the action is invariant under the *full* conformal symmetry (in particular under SCTs), our assumption (Equation 1.65) on the virial current holds.

Chapter 2

Non-ad hoc improved Noether currents

As stated in the Chapter 1, the standard derivation of a SE tensor with the desired properties of symmetry and tracelessness from Noether's first theorem is *ad-hoc*, involving guesswork to some extent. Another well-known approach and definition of the SE tensor is possible. It follows from the coupling of the action functional with a non-flat metric, and the subsequent variation of the action with respect to this metric, evaluated for $g^{\mu\nu}(x) \equiv \eta^{\mu\nu}$. This SE tensor is usually referred to as Einstein-Hilbert SE tensor, and it has been showed that is equivalent *on-shell* to the one derived through Belinfante procedure [5]. However, it is not guaranteed that these tensors are equivalent *off-shell*. In this Chapter, we are going to expose a recent finding (Jan 2022) by Kourkoulou, Nicolis and Sun [6], where a new derivation of Noether's first theorem applied to translational symmetry (subsequently extended to generic spacetime symmetry) is provided. In particular, the authors derive without guesswork a symmetric (for Lorentz invariant theory), traceless (for scale invariant theory) and both symmetric and traceless (for conformal invariant theory) SE tensor. This method appears natural, it guarantees symmetry and tracelessness of SE tensor even *off-shell*, and it shows that it is possible to obtain a traceless (but non-symmetric) tensor with scale invariance (without fully conformal invariance). We will also illustrate this result with an example for a simple scalar field theory. However, in general the procedure of Kourkoulou, Nicolis and Sun does not guarantee gauge invariance of SE tensor. Consequently, we are going to expose a frequently neglected derivation (1921) by Bessel-Hagen (exposed in [1]) through which it is possible to derive the correct gauge-invariant SE tensor with a *non-ad hoc* procedure in the case of free electrodynamics.

2.1 Non-ad hoc improvements *à la* Kourkoulou, Nicolis and Sun

As stressed out by Kourkoulou, Nicolis and Sun, the limits of the standard proof of Noether's first theorem within the active viewpoint for spacetime symmetry exposed in Chapter 1, is the arbitrariness of the field transformation chosen (Eq. 1.22). We consider the fields transformation for a spacetime modulated translation of coordinates that is a symmetry for constant ϵ^μ

$$\phi(x) \to \phi(x) - \epsilon^\mu(x)\partial_\mu\phi(x) - \partial_\mu\epsilon_\nu(x)\Psi^{\mu\nu}(x) \tag{2.1}$$

$$\delta\phi = -\epsilon^\mu(x)\partial_\mu\phi(x) - \partial_\mu\epsilon_\nu(x)\Psi^{\mu\nu}(x) \tag{2.2}$$

where $\Psi^{\mu\nu}(x)$ is an arbitrary functional of the fields multiplet. Under this transformation, for a *generic* variation $\epsilon^\mu(x)$ the Lagrangian density varies as

$$\begin{aligned}\delta\mathcal{L} = &-\frac{\partial\mathcal{L}}{\partial\phi}\cdot\epsilon^\mu\partial_\mu\phi - \frac{\partial\mathcal{L}}{\partial\phi}\cdot(\partial_\mu\epsilon_\nu)\Psi^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot(\partial_\rho\epsilon^\mu)(\partial_\mu\phi)\\ &- \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot(\epsilon_\mu\partial_\rho\partial_\mu\phi) - \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot[(\partial_\rho\partial_\mu\epsilon_\nu)\Psi^{\mu\nu} + (\partial_\mu\epsilon_\nu)(\partial_\rho\Psi^{\mu\nu})]\\ &= -\epsilon^\mu\left[\frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\mu\partial_\rho\phi\right] - (\partial_\mu\epsilon_\nu)\frac{\partial\mathcal{L}}{\partial\phi}\cdot\Psi^{\mu\nu}\\ &-(\partial_\mu\epsilon_\nu)\left[-\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\right)\cdot\Psi^{\mu\nu} + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot\Psi^{\mu\nu}\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial^\nu\phi\right] - (\partial_\rho\partial_\mu\epsilon_\nu)\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot\Psi^{\mu\nu}\ .\end{aligned} \tag{2.3}$$

The action must be invariant for constant translations $\epsilon^\mu(x) \equiv \epsilon^\mu$, and if we consider a symmetry in the *strict* sense, we obtain, according to respectively the Equation 2.3 and 1.9, that the variation of Lagrangian density must be *exactly* equal to

$$\delta\mathcal{L} = -\epsilon^\mu\left[\frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\mu\partial_\rho\phi\right] = \epsilon^\mu\partial_\mu\mathcal{L}\ . \tag{2.4}$$

This relation is satisfied iff the Lagrangian does not explicitly depend on the spacetime coordinates. Considering again a *generic* displacement $\epsilon^\mu(x)$, we have

$$\begin{aligned}\delta\mathcal{L} = -\partial_\mu(\epsilon^\mu\,\mathcal{L}) - (\partial_\mu\epsilon_\nu)\left[\frac{\delta S}{\delta\phi}\cdot\Psi^{\mu\nu} + \partial_\rho\,\mathscr{S}^{\rho\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}\right] - (\partial_\rho\partial_\mu\epsilon_\nu)\,\mathscr{S}^{\rho\mu\nu}\\ = -\partial_\mu(\epsilon^\mu\,\mathcal{L}) - (\partial_\mu\epsilon_\nu)\,\mathcal{T}^{\mu\nu} - (\partial_\rho\partial_\mu\epsilon_\nu)\,\mathscr{S}^{\rho\mu\nu}\ ,\end{aligned} \tag{2.5}$$

where we have defined

$$\mathcal{T}^{\mu\nu} = T_c^{\mu\nu} + \frac{\delta S}{\delta\phi}\cdot\Psi^{\mu\nu} + \partial_\rho\,\mathscr{S}^{\rho\mu\nu} \tag{2.6}$$

$$T_c^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{2.7}$$

$$\mathscr{S}^{\rho\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot\Psi^{\mu\nu} \qquad \text{and finally} \qquad \frac{\delta S}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\right)\ . \tag{2.8}$$

For a *generic* displacement $\epsilon^\mu(x)$, the expression of the variation of the Lagrangian density can provide a definition of the SE tensor.

$$\delta\mathcal{L} = -(\partial_\mu\epsilon_\nu(x))T^{\mu\nu} + \text{ total derivatives.} \tag{2.9}$$

Now, we integrate over the spacetime region V the former equation. Under the assumption that the fields configuration fall off sufficiently rapidly at the boundary of the integration region, the total derivatives vanish. An integration by parts of the first term, assuming that also $\epsilon^\mu(x)$ vanishes at infinity, gives

$$\delta S[\phi] = \int_V d^dx\ \epsilon_\nu(x)\partial_\mu T^{\mu\nu} = 0 \qquad \text{on-shell.} \tag{2.10}$$

Considering the arbitrariness of $\epsilon^\mu(x)$, we can conclude the *on-shell* conservation $\partial_\mu T^{\mu\nu} = 0$ because of the fundamental lemma of calculus of variations. Now, we consider the last term of Eq. 2.5: since $\partial_\rho\partial_\mu\epsilon_\nu$ is symmetric in $\rho\mu$, we can add to $\mathscr{S}^{\rho\mu\nu}$ anything that is antisymmetric in $\rho\mu$. For this reason, the former can be rewritten as

$$-(\partial_\rho\partial_\mu\epsilon_\nu)\mathscr{S}^{\rho\mu\nu} = -(\partial_\rho\partial_\mu\epsilon_\nu)(\mathscr{S}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu}) = (\partial_\mu\epsilon_\nu)\partial_\rho(\mathscr{S}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu}) + \text{ total derivatives.} \tag{2.11}$$

Implementing this rewriting in Eq. 2.3, and comparing it with Eq. 2.9 gives

$$T^{\mu\nu} = \mathcal{T}^{\mu\nu} + \Delta T^{\mu\nu} = \mathcal{T}^{\mu\nu} - \partial_\rho(\mathscr{S}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu}) \qquad \text{with} \qquad \Sigma^{\rho\mu\nu} = -\Sigma^{\mu\rho\nu}\ . \tag{2.12}$$

If we choose suitably the functional $\Psi^{\mu\nu}$, so that the field transformations in Eq. 2.1 is a symmetry of the action for a characterizing displacement $\epsilon^\mu(x)$ (e.g. for a Lorentz-invariant theory we chose $\Psi^{\mu\nu}$ such that for $\epsilon^\mu(x) = \omega^{\mu\nu} x_\nu$ the transformation 2.1 is symmetry), the tensor $\mathcal{T}^{\mu\nu}$ will automatically have the properties guaranteed by the exploited symmetry. Subsequently, to obtain the conserved SE tensor $T^{\mu\nu}$ of Eq. 2.9, we have to correct $\mathcal{T}^{\mu\nu}$ through Eq. 2.12. It may seem that the procedure exposed involves guesswork, in particular in the choice of the correct $\Sigma^{\rho\mu\nu}$ to obtain the desired property for $T^{\mu\nu}$; but it is not the case, as we will show explicitly in the next paragraphs, following what found by Kourkoulou, Nicolis and Sun. Indeed, the tensor $\Sigma^{\rho\mu\nu}$ can be parameterized as the general linear combination of tensor at one disposal to give rise to the desired property for $T^{\mu\nu}$.

2.1.1 Lorentz invariant theories

In this Paragraph, we consider Poincaré-invariant theories. In these theories, the transformations of fields given by Eq. 2.1 are a symmetry *in the strict sense* for all $\epsilon^\mu(x) = \omega^{\mu\nu} x_\nu$, such that $\omega^{\mu\nu} + \omega^{\nu\mu} = 0$. According to Chapter 1, called $S^{\mu\nu}$ the generators of Lorentz group in the fields representation, the comparison of Eq. 1.32 with Eq. 2.1 gives

$$\Psi_L^{\mu\nu} = \frac{1}{2} S^{\mu\nu} \cdot \phi \qquad \text{and} \qquad \mathscr{S}^{\rho\mu\nu} \equiv \mathscr{S}_L^{\rho\mu\nu} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\mu\nu} \cdot \phi \ . \tag{2.13}$$

Since $\partial_\mu \epsilon_\nu = \omega_{\nu\mu}$ is constant, from Eq. 2.3 it follows that

$$\delta \mathcal{L} = -\partial_\mu(\epsilon^\mu \mathcal{L}) - \omega_{\mu\nu} \mathcal{T}^{\mu\nu} \ . \tag{2.14}$$

Since Lorentz transformations are a symmetry of the field theory, the Lagrangian must be invariant up to *exactly* the divergence term $-\partial_\mu(\epsilon^\mu \mathcal{L})$. It follows, from the arbitrariness and antisymmetry of $\omega_{\mu\nu}$ that $\mathcal{T}^{\mu\nu}$ must be symmetric, even *off-shell.* We choose $\Sigma^{\rho\nu\mu}$, so that the correction term of the SE tensor, given by Eq. 2.12, is also symmetric in $\mu\nu$.

$$\partial_\rho(\mathscr{S}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu}) \overset{!}{=} \partial_\rho(\mathscr{S}^{\rho\nu\mu} + \Sigma^{\rho\nu\mu}) \tag{2.15}$$

The most general linear combination of the tensors $\mathscr{S}^{\rho\mu\nu}$ and $\eta^{\mu\nu}$, considering that $\mathscr{S}_L^{\rho\mu\nu} = -\mathscr{S}_L^{\rho\nu\mu}$ (resulting from the antisymmetry of generators of Lorentz group in the fields representation) and considering also that $\eta^{\mu\nu} = \eta^{\nu\mu}$, that is antisymmetric in its first two indexes is

$$\Sigma^{\rho\mu\nu} = \alpha \mathscr{S}^{\nu\rho\mu} + \beta(\mathscr{S}^{\rho\mu\nu} - \mathscr{S}^{\mu\rho\nu}) + \gamma(\mathscr{S}_\sigma{}^{\sigma\rho}\eta^{\mu\nu} - \mathscr{S}_\sigma{}^{\sigma\mu}\eta^{\rho\nu}) \ . \tag{2.16}$$

Applying the condition 2.15 to the former Equation gives

$$\begin{aligned} 2\mathscr{S}^{\rho\mu\nu} =& \alpha \mathscr{S}^{\mu\rho\nu} + \beta(\mathscr{S}^{\rho\nu\mu} - \mathscr{S}^{\nu\rho\mu}) + \gamma(\mathscr{S}_\sigma{}^{\sigma\rho}\eta^{\mu\nu} - \mathscr{S}_\sigma{}^{\sigma\nu}\eta^{\rho\mu}) \\ &- \alpha \mathscr{S}^{\nu\rho\mu} - \beta(\mathscr{S}^{\rho\mu\nu} - \mathscr{S}^{\mu\rho\nu}) - \gamma(\mathscr{S}_\sigma{}^{\sigma\rho}\eta^{\mu\nu} - \mathscr{S}_\sigma{}^{\sigma\mu}\eta^{\rho\nu}) \ . \end{aligned} \tag{2.17}$$

This relation is verified iff $\alpha = 1$, $\beta = -1$ and $\gamma = 0$. Therefore, the improved symmetric SE tensor is given by

$$\begin{aligned} T_L^{\mu\nu} =& \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} + \frac{1}{2}\frac{\delta S}{\delta \phi} \cdot S^{\mu\nu} \cdot \phi \\ &+ \frac{1}{2}\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\mu\nu} \cdot \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\rho\nu} \cdot \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} S^{\rho\mu} \cdot \phi \right) \ . \end{aligned} \tag{2.18}$$

We note that this results coincide *on-shell* with the one found through Belinfante procedure, but, in addition, the $\mu\nu$ symmetry of 2.18 is also verified off-shell.

2.1.2 Scale invariant theories

In this paragraph, we consider a scale transformation of coordinates, $x^{\mu\prime} = (1+\alpha)x^\mu$, namely $\epsilon^\mu(x) = \alpha x^\mu$, under which the fields transform as

$$\phi(x) \to \phi(x) - \alpha x^\mu \partial_\mu \phi(x) - \Delta \cdot \phi(x) \qquad \text{that suggests} \qquad \Psi_S^{\mu\nu} = \frac{1}{d}\eta^{\mu\nu}\Delta \cdot \phi \tag{2.19}$$

$$\mathscr{S}^{\rho\mu\nu} \equiv \mathscr{S}_S^{\rho\mu\nu} = \frac{1}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\eta^{\mu\nu}\Delta \cdot \phi \ . \tag{2.20}$$

Under this transformation, the Lagrangian varies as

$$\delta\mathcal{L} = -\partial_\mu(\epsilon^\mu \mathcal{L}) - \alpha \mathcal{T}^\mu{}_\mu \ . \tag{2.21}$$

In order to this transformation to be a symmetry of the field theory, the variation of the Lagrangian must be zero up to *exactly* $-\partial_\mu(\epsilon^\mu\mathcal{L})$. This implies that $\mathcal{T}^{\mu\nu}$ is traceless, even *off-shell*. We have to correct the tensor $\mathcal{T}^{\mu\nu}$ via Eq. 2.12:

$$T_S^{\mu\nu} = \mathcal{T}^{\mu\nu} - \partial_\rho(\mathscr{S}^{\rho\mu\nu} + \Sigma^{\rho\mu\nu}) \qquad \text{with} \qquad \mathscr{S}^{\rho\mu\nu} = \frac{1}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \cdot \eta^{\mu\nu}\Delta \cdot \phi \ . \tag{2.22}$$

We choose $\Sigma^{\rho\mu\nu}$ as the most general combination of $\mathscr{S}_S^{\rho\mu\nu}$ (proportional to $\eta^{\mu\nu}$) that is antisymmetric in $\rho\mu$ and that preserve the tracelessness of $\mathcal{T}^{\mu\nu}$:

$$\Sigma^{\rho\mu\nu} = \delta(\mathscr{S}^{\rho\mu\nu} - \mathscr{S}^{\mu\rho\nu}) \ . \tag{2.23}$$

Under this choice of $\Sigma^{\rho\mu\nu}$, the trace results

$$\begin{aligned}&\partial_\rho(\mathscr{S}^{\rho\mu}{}_\mu + \delta\mathscr{S}^{\rho\mu}{}_\mu - \delta\mathscr{S}^{\mu\rho}{}_\mu) \\ &= \partial_\rho((1+\delta)\frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)} \cdot \Delta\cdot\phi - \frac{\delta}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \cdot \eta^\rho{}_\mu \Delta\cdot\phi) = \left(1 + \frac{d-1}{d}\delta\right)\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\Delta\cdot\phi\right) \ .\end{aligned} \tag{2.24}$$

To get tracelessness, we have to impose $\delta = -\frac{d}{d-1}$ and consequently the improved SE tensor for a scale invariant theory is:

$$T_S^{\mu\nu} = T_c^{\mu\nu} + \frac{1}{d}\eta^{\mu\nu}\frac{\delta S}{\delta\phi}\cdot\Delta\cdot\phi + \frac{1}{d-1}\partial_\rho\left(\eta^{\mu\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot\Delta\cdot\phi - \eta^{\rho\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\Delta\cdot\phi\right) \ . \tag{2.25}$$

This result is not in contrast with what we have stated in Chapter 1, namely that to obtain a SE tensor symmetric *and* traceless it is necessary full conformal invariance. Indeed, even if the SE tensor provided by Eq. 2.25 is traceless, it is not in general symmetric.

2.1.3 Conformal invariant theories

We will show in this Paragraph that, following the approach of Kourkoulou, Nicolis and Sun (as we will see shortly applicable when $d \neq 2$), it is necessary that the action is also *fully* conformal invariant to derive a symmetric *and* traceless SE tensor.
First of all, we notice that the mere combination of Lorentz and scale invariance does not imply a SE tensor that is both symmetric and traceless. For a Lorentz and scale invariant theory, if one chose (in agreement with the notation used in Eq. 2.14 and 2.19)

$$\Psi^{\mu\nu} = \Psi_L^{\mu\nu} + \Psi_S^{\mu\nu} \qquad \text{and} \qquad \mathscr{S}^{\rho\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot\Psi_L^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot\Psi_S^{\mu\nu} = \mathscr{S}_L^{\rho\mu\nu} + \mathscr{S}_S^{\rho\mu\nu} \ , \tag{2.26}$$

the transformation of fields reduces to a constant translation if ϵ^μ is constant, a Lorentz transformation if $\epsilon^\mu(x) = \omega^{\mu\nu}x_\nu$ and a scale transformation if $\epsilon^\mu(x) = \alpha x^\mu$. This implies that $\mathcal{T}^{\mu\nu}$ is automatically symmetric and traceless. The most general combination of available tensors, $\mathscr{S}_L^{\rho\mu\nu}$, $\mathscr{S}_S^{\rho\mu\nu}$, $\eta^{\mu\nu}$ which guarantees the $\rho\mu$ antisymmetry of $\Sigma^{\rho\mu\nu}$, is

$$\Sigma^{\rho\mu\nu} = \alpha\mathscr{S}_L^{\nu\rho\mu} + \beta(\mathscr{S}_L^{\rho\mu\nu} - \mathscr{S}_L^{\mu\rho\nu}) + \gamma(\mathscr{S}_{L\,\sigma}^{\ \ \sigma\rho}\eta^{\mu\nu} - \mathscr{S}_{L\,\sigma}^{\ \ \sigma\mu}\eta^{\rho\nu}) + \delta(\mathscr{S}_S^{\rho\mu\nu} - \mathscr{S}_S^{\mu\rho\nu}) \ . \tag{2.27}$$

Requiring $\Delta T^{\mu\nu} = \Delta T^{\nu\mu}$ gives:

$$\begin{aligned} &\partial_\rho(\mathscr{S}_L^{\rho\mu\nu} + \mathscr{S}_S^{\rho\mu\nu} + \alpha\mathscr{S}_L^{\nu\rho\mu} + \beta(\mathscr{S}_L^{\rho\mu\nu} - \mathscr{S}_L^{\mu\rho\nu}) + \gamma(\mathscr{S}_{L\,\sigma}^{\ \ \sigma\rho}\eta^{\mu\nu} - \mathscr{S}_{L\,\sigma}^{\ \ \sigma\mu}\eta^{\rho\nu}) + \delta(\mathscr{S}_S^{\rho\mu\nu} - \mathscr{S}_S^{\mu\rho\nu})) \\ &= \partial_\rho(\mathscr{S}_L^{\rho\nu\mu} + \mathscr{S}_S^{\rho\nu\mu} + \alpha\mathscr{S}_L^{\mu\rho\nu} + \beta(\mathscr{S}_L^{\rho\nu\mu} - \mathscr{S}_L^{\nu\rho\mu}) + \gamma(\mathscr{S}_{L\,\sigma}^{\ \ \sigma\rho}\eta^{\mu\nu} - \mathscr{S}_{L\,\sigma}^{\ \ \sigma\nu}\eta^{\rho\mu}) + \delta(\mathscr{S}_S^{\rho\nu\mu} - \mathscr{S}_S^{\nu\rho\mu})) \ . \end{aligned} \tag{2.28}$$

Exploiting the properties of $\mathscr{S}_L$ and $\mathscr{S}_S$ we get

$$2\mathscr{S}_L^{\rho\mu\nu} - (\alpha+\beta)\mathscr{S}_L^{\mu\rho\nu} + 2\beta\mathscr{S}_L^{\rho\mu\nu} + (\alpha+\beta)\mathscr{S}_L^{\nu\rho\mu} - \gamma(\mathscr{S}_{L\,\sigma}^{\ \ \sigma\mu}\eta^{\rho\nu} - \mathscr{S}_{L\,\sigma}^{\ \ \sigma\nu}\eta^{\rho\mu}) + \delta(\mathscr{S}_S^{\nu\rho\mu} - \mathscr{S}_S^{\mu\rho\nu}) = 0 \ , \tag{2.29}$$

which implies $\delta = 0, \gamma = 0, \beta = -1, \alpha = +1$. Requiring instead $\Delta T^\mu{}_\mu = 0$ gives

$$\partial_\rho(\mathscr{S}_S^{\rho\mu}{}_\mu + \alpha\mathscr{S}_L^{\mu\rho}{}_\mu - \beta\mathscr{S}_L^{\mu\rho}{}_\mu + \gamma(d\mathscr{S}_{L\,\sigma}^{\ \ \sigma\rho} - \mathscr{S}_{L\,\sigma}^{\ \ \sigma\mu}\eta^\rho{}_\mu) + \delta(\mathscr{S}_S^{\rho\mu}{}_\mu - \mathscr{S}_S^{\mu\rho}{}_\mu)) = 0 \ , \tag{2.30}$$

which implies $\delta = -\frac{d}{d-1}$ and $\alpha - \beta - (d-1)\gamma = 0$. The two solutions are inconsistent, therefore we cannot make the SE tensor automatically both symmetric and traceless, if the theory is only Lorentz and scale invariant.
As anticipated, we have to consider the case of field theories that exhibit *full* conformal invariance, namely the infinitesimal special conformal transformations

$$x^{\mu\prime} = x^\mu + b^\mu\, x^2 - 2(b\cdot x)x^\mu \tag{2.31}$$

are also a symmetry, where b^μ is a constant vector. We can interpret this special conformal transformation as combination of spacetime modulated Lorentz and scale transformations, with $\omega^\mu{}_\nu(x) = b^\mu x_\nu$ and $\alpha(x) = -2b\cdot x$, so we assume that Equation 2.26 holds. Under these assumptions, we have

$$\partial_\mu\epsilon_\nu = 2(b_\nu x_\mu - b_\mu x_\nu) - 2(b\cdot x)\eta_{\mu\nu} \ , \tag{2.32}$$

and therefore the Lagrangian varies as:

$$\delta\mathcal{L} = -\partial_\mu(\epsilon^\mu\mathcal{L}) + [2(b_\nu x_\mu - b_\mu x_\nu) - 2(b\cdot x)\eta_{\mu\nu}]\mathcal{T}^{\mu\nu} - [\partial_\alpha(2(b_\nu x_\mu - b_\mu x_\nu) - 2(b\cdot x)\eta_{\mu\nu})]\mathscr{S}^{\alpha\mu\nu} \ . \tag{2.33}$$

The invariance of the action functional under Lorentz and scale transformations implies that the second term of the former Equation must vanish, in agreement with the symmetry and tracelessness of $\mathcal{T}^{\mu\nu}$.

$$\begin{aligned} \delta\mathcal{L} &= -\partial_\mu(\epsilon^\mu\mathcal{L}) - 2(b_\mu\mathscr{S}_\alpha{}^{\alpha\mu} - b_\mu\mathscr{S}^{\alpha\mu}{}_\alpha - b_\mu\mathscr{S}^{\mu\alpha}{}_\alpha) \\ &= -\partial_\mu(\epsilon^\mu\mathcal{L}) + 2b_\mu(\mathscr{S}^{\mu\alpha}{}_\alpha + \mathscr{S}_\alpha{}^{\mu\alpha} - \mathscr{S}_\alpha{}^{\alpha\mu}) \\ &= -\partial_\mu(\epsilon^\mu\mathcal{L}) + 2b_\rho(\mathscr{S}^{\rho\alpha}{}_\alpha + 2\mathscr{S}_\alpha{}^{[\rho\alpha]}) \end{aligned} \tag{2.34}$$

As stressed in Chapter 1, it is not necessary that the Lagrangian is invariant *in the strict sense* under a transformation for this to be a symmetry of the theory: it is sufficient that the variation of the Lagrangian is, within the passive viewpoint, zero *up to a boundary term*. Therefore, we can define the virial current, proportional to the last term of Eq. 2.34, to be a $d-$divergence:

$$V^\rho = \mathscr{S}^{\rho\alpha}{}_\alpha + 2\mathscr{S}_\alpha{}^{[\rho\alpha]} = \partial_\alpha\sigma^{\alpha\rho} \ . \tag{2.35}$$

We recall that $\mathscr{S}_L^{\rho\mu\nu}$ is antisymmetric in $\mu\nu$, $\mathscr{S}_S^{\rho\mu\nu}$ is proportional to $\eta^{\mu\nu}$ and, in particular, symmetric in $\mu\nu$. This given, we can rewrite the definition of virial current as:

$$\mathscr{S}_L^{\rho\alpha}{}_\alpha + \mathscr{S}_S^{\rho\alpha}{}_\alpha + 2\mathscr{S}_{L\,\alpha}^{\ \ [\rho\alpha]} + 2\mathscr{S}_{S\,\alpha}^{\ \ [\rho\alpha]} = \partial_\alpha\sigma^{\alpha\rho} \ . \tag{2.36}$$

The first and last term of the LHS vanishes due to, respectively, antisymmetry and symmetry. We can rewrite $\mathscr{S}_S{}^{\rho\alpha}{}_{\alpha} = \mathscr{S}_S^{\rho\mu\nu}\eta_{\mu\nu}$, and multiplying both side by $\eta^{\mu\nu}$, we get:

$$\mathscr{S}_S^{\rho\mu\nu} = \frac{1}{d}\left[\partial_\alpha \sigma^{\alpha\rho}\eta^{\mu\nu} - 2\mathscr{S}_L{}^{\rho\alpha}{}_{\alpha}\eta^{\mu\nu}\right] \quad . \tag{2.37}$$

This implies that the correction to the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ can be rewritten as

$$\Delta T^{\mu\nu} = -\partial_\rho\left(\mathscr{S}_L^{\rho\mu\nu} - \frac{2}{d}\mathscr{S}_L{}^{\rho\alpha}{}_{\alpha}\eta^{\mu\nu} + \Sigma^{\rho\mu\nu}\right) - \frac{1}{d}\partial_\rho\partial_\alpha\sigma^{(\alpha\rho)}\eta^{\mu\nu} \quad . \tag{2.38}$$

It is important to stress that the Eq. 2.37 holds only if conformal transformations, and in particular SCTs given by Eq. 2.26, are a symmetry of the theory. Under this assumption, similarly to the previous Paragraphs, the tensor $\Sigma^{\rho\mu\nu}$ can be found as the most general linear combination of $\mathscr{S}^{\rho\mu\nu}$ and $\eta^{\mu\nu}$ that makes the correction $\Delta T^{\mu\nu}$ both symmetric and traceless. If $\mathscr{S}_L^{\rho\mu\nu}$ and $\mathscr{S}_S^{\rho\mu\nu}$ are instead independent as assumed in Eq. 2.27, in general, this is not possible.

If $\sigma^{\alpha\rho}$ vanishes, the correct $\Sigma^{\rho\mu\nu}$ in Eq. 2.38 is simply achieved similarly to the case of Lorentz invariance:

$$\Sigma^{\rho\mu\nu} = \Sigma_L^{\rho\mu\nu} = \mathscr{S}_L^{\nu\rho\mu} - (\mathscr{S}_L^{\rho\mu\nu} - \mathscr{S}_L^{\mu\rho\nu}) \quad . \tag{2.39}$$

With this choice, for vanishing $\sigma^{\mu\nu}$ the correction $\Delta T^{\mu\nu}$ is both symmetric and traceless ($\Delta T^{\mu}_{\mu} = -\partial_\rho(0 - 2\mathscr{S}_L{}^{\rho\alpha}{}_{\alpha} + 2\mathscr{S}_L{}^{\rho\alpha}{}_{\alpha}) = 0$). Instead, if we consider a non-vanishing $\sigma^{\mu\nu}$, we can add to $\Sigma_L^{\rho\mu\nu}$ a $d-$divergence term $\partial_\alpha\Xi^{\rho\mu\nu\alpha}$. The tensor Ξ must be antisymmetric in $\rho\mu$, as $\Sigma_L^{\rho\mu\nu}$, in order to not spoil the symmetry of $\Delta T^{\mu\nu}$, and any $\rho\alpha$ antisymmetric component would vanish considering $\partial_\rho\partial_\alpha\Xi^{\rho\mu\nu\alpha}$. Therefore, in general the right $\Sigma^{\rho\mu\nu}$ is

$$\Sigma^{\rho\mu\nu} = \Sigma_L^{\rho\mu\nu} + \partial_\alpha\Xi^{[\rho\mu][\alpha\nu]} \ , \tag{2.40}$$

where, in addition, Ξ must be symmetric with respect to the exchange $\rho\mu \leftrightarrow \alpha\nu$. Indeed, in this way the trace of the correction term results

$$\Delta T^{\mu}{}_{\mu} = -\partial_\alpha\partial_\rho\left(\eta_{\mu\nu}\Xi^{[\rho\mu][\alpha\nu]} + \sigma^{(\alpha\rho)}\right) \quad . \tag{2.41}$$

The tensor $\Xi^{[\rho\mu][\alpha\nu]}$ must be chosen as the most general linear combination of the available tensors, $\eta^{\mu\nu}$ and $\sigma^{(\alpha\rho)}$, to obtain the desired properties

$$\Xi^{[\rho\mu][\alpha\nu]} = A(\eta^{\mu\nu}\sigma^{(\alpha\rho)} + \eta^{\alpha\rho}\sigma^{(\mu\nu)} - \eta^{\mu\alpha}\sigma^{(\nu\rho)} - \eta^{\nu\rho}\sigma^{(\mu\alpha)}) + B(\eta^{\alpha\rho}\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\rho})\sigma^{\beta}_{\beta} \tag{2.42}$$

We can determine through explicit calculation the values of the constant A, B

$$\begin{aligned}\Delta T^{\mu}{}_{\mu} &= -\partial_\alpha\partial_\rho\left\{A\eta_{\mu\nu}\eta^{\mu\nu}\sigma^{(\alpha\rho)} + \sigma^{(\alpha\rho)} + A\eta^{\alpha\rho}\sigma^{\beta}_{\beta} - 2A\sigma^{\alpha\rho} + dB\eta^{\alpha\rho}\sigma^{\beta}_{\beta} - B\sigma^{\beta}_{\beta}\right\} \\ &= -\partial_\alpha\partial_\rho\{(A(d-2)+1)\eta^{\alpha\rho}\sigma^{(\alpha\rho)} + (A + B(d-1))\eta^{\alpha\rho}\sigma^{\beta}_{\beta}\} \\ &\stackrel{!}{=} 0 \ .\end{aligned} \tag{2.43}$$

This implies

$$A = -\frac{1}{d-2} \qquad \text{and} \qquad B = \frac{1}{(d-2)(d-1)} \quad , \tag{2.44}$$

where, as cited above, $d \neq 2$. The final expression for the stress-energy tensor for a fully conformal invariant theory is

$$\begin{aligned}T_C^{\mu\nu} =& \mathcal{T}^{\mu\nu} + \Delta T^{\mu\nu} \\ =& T_c^{\mu\nu} + \frac{\delta S}{\delta\phi}\cdot\left(\frac{1}{2}S^{\mu\nu} + \frac{1}{d}\eta^{\mu\nu}\Delta\right)\cdot\phi + \\ & \frac{1}{2}\partial_\rho\left[\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\cdot S^{\mu\nu}\cdot\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot S^{\rho\nu}\cdot\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\cdot S^{\rho\mu}\cdot\phi\right] \\ & + \frac{1}{d-2}\left[\eta^{\mu\nu}\partial_\alpha\partial_\rho\sigma^{(\alpha\rho)} + \Box\sigma^{(\mu\nu)} - \partial_\rho\partial^\mu\sigma^{(\nu\rho)} - \partial_\alpha\partial^\nu\sigma^{(\mu\alpha)}\right] \\ & - \frac{1}{(d-2)(d-1)}\left[\eta^{\mu\nu}\Box\sigma^{\beta}{}_{\beta} - \partial^\mu\partial^\nu\sigma^{\beta}{}_{\beta}\right] \quad .\end{aligned} \tag{2.45}$$

This expression coincides *on-shell* with the one found in Chapter 1, but in addition it has the desired properties of symmetry and tracelessness even *off-shell*.

2.1.4 When does scale invariance imply full conformal invariance?

We are going to prove that a scalar classical field theory with Lagrangian density given by (with $d \geq 3$)

$$\mathcal{L}(\phi, \partial_\mu \phi) = \phi^{\frac{2d}{d-2}} f\left(\frac{\partial_\mu \phi \partial^\mu \phi}{\phi^{\frac{2d}{d-2}}}\right) \tag{2.46}$$

is scale-invariant for each possible function f. In addition, we are going to exploit the result of Kourkoulou, Nicolis and Sun exposed, to prove that if the function f is linear in its argument, it is possible to derive a symmetric *and* traceless SE tensor. This suggests that, in the particular case of the Lagrangian given by 2.46 with f linear in its argument, the theory is also fully conformal invariant.
We can determine which specific choice of Δ, known as scaling dimension of the field ϕ, the transformations $x^\mu \to (1+\alpha)x^\mu$, $\phi(x) \to (1+\alpha)^{-\Delta}\phi(x)$ with $\alpha \in \mathbb{R} \setminus \{-1\}$, make the action scale-invariant. In the first instance, we have to prove that if $\mathcal{L}$ transform as a scalar field with scaling dimension $\Delta_{\mathcal{L}} = d$, the action is scale-invariant. This is trivial, because

$$d^d x \mathcal{L}(x) \to (1+\alpha)^d d^d x \mathcal{L}'(x') = (1+\alpha)^{(d-d)} d^d x \mathcal{L}(x) \ . \tag{2.47}$$

Considering that $\partial_\mu \to \frac{1}{1+\alpha}\partial_\alpha$, $\partial^\mu \to \frac{1}{1+\alpha}\partial^\alpha$, $\phi^n(x) \to (1+\alpha)^{-n\Delta}\phi^n(x)$, with Δ, scaling dimension of ϕ, to be determined, the Lagrangian in the Equation 2.46 varies as

$$\mathcal{L}'(x') = (1+\alpha)^{-\Delta\frac{2d}{d-2}} \phi^{\frac{2d}{d-2}} f\left((1+\alpha)^{\left(\frac{4\Delta}{d-2}-2\right)} \frac{\partial_\mu \phi \partial^\mu \phi}{\phi^{\frac{2d}{d-2}}}\right) = (1+\alpha)^{-d}\mathcal{L}(x) \ . \tag{2.48}$$

The last equality holds iff $\Delta = \frac{d-2}{2}$. Defining $f' = \partial_X f$, with $X = \frac{(\partial\phi)^2}{\phi^{\frac{2d}{d-2}}}$, the Equation 2.25 associated with a traceless SE gives

$$\begin{aligned}
T_S^{\mu\nu} &= 2f'\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\phi^{\frac{2d}{d-2}} f + \eta^{\mu\nu}\left(\phi^{\frac{2d}{d-2}} f - f'(\partial\phi)^2\right) - \frac{d-2}{d}\eta^{\mu\nu}\phi\partial_\rho\left(f'\partial^\rho\phi\right) \\
&\quad + \frac{d-2}{d-1}\eta^{\mu\nu}\partial_\rho\left(\phi f'\partial^\rho\phi\right) - \frac{d-2}{d-1}\partial^\nu\left(\phi f'\partial^\mu\phi\right) \\
&=2f'\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}f'\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{d(d-1)}\eta^{\mu\nu}\phi\partial_\rho(f'\partial^\rho\phi) + \frac{d-2}{d-1}\eta^{\mu\nu}f'\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{d-1}\partial^\nu(\phi f'\partial^\mu\phi) \\
&=\frac{d}{d-1}f'\partial^\mu\partial^\nu\phi - \frac{1}{d-1}\eta^{\mu\nu}f'\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{d(d-1)}\eta^{\mu\nu}\phi\partial_\rho(f'\partial^\rho\phi) - \frac{d-2}{d-1}\phi\partial^\nu(f'\partial^\mu\phi) \ .
\end{aligned} \tag{2.49}$$

We note that, in addition to the fact that, as expected, the trace vanishes, all terms, except the last one, are symmetric in $\mu\nu$. Using the Leibniz rule on the last term, we observe that if $\partial^\nu f' \equiv 0$, the tensor derived is both symmetric *and* traceless. The condition stated is equivalent to require:

$$f\left(\frac{(\partial\phi)^2}{\phi^{\frac{2d}{d-2}}}\right) = A + B\frac{(\partial\phi)^2}{\phi^{\frac{2d}{d-2}}} \qquad \text{with} \qquad A, B \in \mathbb{R} \text{ or } \mathbb{C} \ . \tag{2.50}$$

2.2 Non-ad hoc improvements *à la* Bessel-Hagen

As stated in the Introduction of the Chapter, the procedure proposed by Kourkoulou, Nicolis and Sun exposed above has totally general applications, and, importantly, returns SE tensors that have the desired properties even *off-shell*. However, it does not consider possible gauge symmetries, i.e. symmetries of the mathematical formalism of the classical field theory. As a result, in general the derived SE tensors (Eq. 2.18, 2.25, 2.45) are not gauge invariant, although this is a desired property for many physical theories.

In this section we are going to briefly expose a different procedure due to Bessel-Hagen (exposed by Baker in [1]), to correctly apply Noether's theorem in the case of electrodynamics in the light of the gauge symmetry of the theory and consequently derive the correct momentum-energy tensor. Subsequent works (see reference in [1]) have reproduced the method of Bessel-Hagen and extended to the case of arbitrary gauge field theory. A thorough discussion of gauge symmetries is beyond the scope of this thesis; for this reason, we will limit ourselves in this Paragraph to classical electrodynamics, that is a particular abelian gauge theory having $U(1)$ as gauge group.

2.2.1 Free classical electrodynamics

The general theory of free (without currents) electrodynamics can be formulated in $d-$dimensional space in terms of an antisymmetric field strength tensor (also called Faraday tensor) $F_{\mu\nu} + F_{\nu\mu} = 0$ that satisfies

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \qquad \text{equation of motion} \\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= 0 \qquad \text{Bianchi identity.} \end{aligned} \tag{2.51}$$

In particular, solving the Bianchi identity in terms of a vector potential A_μ, the Lagrangian of free electrodynamics results

$$\mathcal{L}(A_\nu, \partial_\mu A_\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \;\; . \tag{2.52}$$

We consider an infinitesimal change of coordinates $x'^\mu = x^\mu + \epsilon^\mu(x)$, under which the field A_ν transform as

$$A_\nu \to A_\nu - \epsilon^\mu(x)\partial_\mu A_\nu - \partial_\mu \epsilon_\nu(x) \Delta^\mu \;\; , \tag{2.53}$$

where Δ^μ is a functional of the covector field A_ν. This is a symmetry in the strict sense $\delta\mathcal{L} = -\partial_\mu(\epsilon^\mu \mathcal{L})$ for some specific choice of $\epsilon^\mu(x)$, in particular for constant ϵ^μ we will obtain the conserved currents associated with the SE tensor.

The non-ad hoc improvement of Bessel-Hagen has its foundations in an *off-shell* identity (also called Noether identity) we already stated during the standard derivation of Noether's theorem within the passive viewpoint (Eq. 1.15). Recalling that the Lagrangian of free electrodynamics is explicitly independent of A_ν, and that $\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = F^{\mu\nu}$, the application of the cited identity to the case of the field and coordinates transformation given by Eq. 2.53 of the vector potential gives

$$-\partial_\mu F^{\mu\nu} \delta A_\nu \equiv \partial_\mu \left(\epsilon^\mu \mathcal{L} + F^{\mu\nu} \delta A_\nu \right) \;\; . \tag{2.54}$$

If a classical field theory exhibits a gauge symmetry this must be considered appropriately in the field transformation to derive a SE tensor that is also gauge-invariant. In particular, the transformation 2.53 is not gauge invariant, and we can expect that the resulting SE tensor is not in general gauge invariant. Considering the fact that gauge symmetries, that are symmetry of the mathematical formalism, do not spoil the currents associated with physical symmetries, we can promote the transformation 2.53 of the field A_ν to

$$A_\nu \to A_\nu + \delta A_\nu = A_\nu - \epsilon^\mu(x)\partial_\mu A_\nu - A_\mu \partial_\nu \epsilon^\mu(x) + \partial_\nu \varphi(A_\alpha, \epsilon^\alpha) \; , \tag{2.55}$$

where $\varphi(A_\alpha, \epsilon^\alpha)$ is a generic scalar gauge parameter, linear in A_α. We want determine the specific scalar function φ which guarantees a gauge invariant SE tensor. Requiring the gauge invariance of the currents in LHS (and the consequently of the SE tensor) is equivalent to require the gauge invariance of the variation of δA_ν. If we consider the gauge transformation $A_\nu \to A_\nu + \partial_\nu C$ with $C(x)$ generic scalar function, imposing the gauge invariance of δA_ν we get

$$\begin{aligned} -\epsilon^\mu \partial_\mu A_\nu - \epsilon^\mu \partial_\mu \partial_\nu C - A_\mu \partial_\nu \epsilon^\mu - (\partial_\mu C)\partial_\nu \epsilon^\mu + \partial_\nu \varphi(A_\alpha, \epsilon^\alpha) + \partial_\nu \varphi(\partial_\alpha C, \epsilon^\alpha) \\ \stackrel{!}{=} -\epsilon^\mu \partial_\mu A_\nu - A_\mu \partial_\nu \epsilon^\mu + \partial_\nu \varphi(A_\alpha, \epsilon^\alpha) \;\; . \end{aligned} \tag{2.56}$$

Therefore finally, $\partial_\nu(\varphi(\partial_\alpha C, \epsilon^\alpha)) = \partial_\nu(\partial_\alpha C\epsilon^\alpha)$ provides an obvious solution for $\varphi(A_\alpha, x^\alpha)$ that makes the expression gauge invariant: $\varphi(A_\alpha, \epsilon^\alpha) = A_\alpha\epsilon^\alpha(x)$. In particular, the non-ad hoc *improved* transformation of the field results

$$\begin{aligned} A_\nu \to A_\nu - \epsilon^\mu\partial_\mu A_\nu - A_\mu\partial_\nu\epsilon^\mu + \partial_\nu(A_\mu\epsilon^\mu) &= A_\nu - \epsilon^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= A_\nu + F_{\nu\alpha}\epsilon^\alpha(x) \ . \end{aligned} \tag{2.57}$$

Applying this improved transformation of A_ν to Eq. 2.54, we obtain *on-shell* a SE tensor for free electrodynamics that is gauge-invariant as initially claimed

$$T^\mu{}_\nu = \left[F^{\mu\rho}F_{\rho\nu} + \frac{1}{4}\delta^\mu{}_\nu F^{\alpha\beta}F_{\alpha\beta}\right] \ . \tag{2.58}$$

It can be proved that, for $d = 4$, the improved transformation determined leaves the Lagrangian invariant up to the boundary term $-\partial_\mu(\epsilon^\mu\mathcal{L})$ if the displacement $\epsilon^\mu(x)$, hitherto assumed general, instead satisfies the conformal Killing equation (cfr. Chapter 1), namely the gauge improvement does not spoil the conformal symmetry of free electrodynamics. Indeed, we can be prove easily by exploiting the antisymmetry of $F^{\mu\nu}$ and the Bianchi identities that

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu}\delta\partial_\mu A_\nu = F^{\mu\nu}\partial_\mu(\epsilon^\alpha F_{\nu\alpha}) \\ &= -\frac{1}{2}F^{\mu\nu}F^\rho{}_\nu\left(\partial_\mu\epsilon_\alpha + \partial_\alpha\epsilon_\mu - \frac{1}{2}\eta_{\mu\alpha}(\partial\cdot\epsilon)\right) - \partial_\alpha(\epsilon^\alpha\mathcal{L}) \ . \end{aligned} \tag{2.59}$$

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