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QCD axion couplings at finite density

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Abstract

After reviewing the theoretical and phenomenological aspects of axion physics, the thesis focuses on the modifications of axion properties in systems at finite baryonic density. Those are especially relevant for highly dense stellar environments such as supernovae and neutron stars, which are typically employed to set stringent limits on axion couplings from anomalous stellar cooling. Specifically, we consider finite density modifications of axion couplings to nucleons within the framework of Heavy Baryon Chiral Perturbation Theory and assess their consequences for astrophysical constraints on the axion parameter space.

Moreover, we analyse the effect of finite density corrections on nucleophobic axion models, *i.e.* ultraviolet completions of the axion effective field theory in which the axion couplings to nucleons are suppressed. These models have the advantage of evading some of the astrophysical bounds on the axion parameter space. Since these bounds are obtained from highly dense systems, we question whether the nucleophobia condition is spoiled by finite density effects.

Contents

Chapter 1

Introduction

Despite being one of the most successful theories in the history of Physics, the *Standard Model* (SM) of particle physics is believed to be insufficient to give a complete description of the fundamental laws of Nature. In particular, it fails to explain some experimental evidences, such as matter-antimatter asymmetry, neutrino masses, and dark matter. Moreover, the theory turns out to be burdened by many parameters taking unnaturally small values. Among these *small-value problems* it is worth recalling the ones associated to the value of the cosmological constant in units of the Planck mass $\Lambda \sim 10^{-31}\, m_{\rm Pl}$, to that of the vacuum expectation value of the electroweak Higgs field $v \sim 10^{-17}\, m_{\rm Pl}$, and the so-called *strong CP problem*, *i.e.* the exceedingly small value taken by the CP-violating theta parameter of *Quantum Chromodynamics* (QCD), $\left|\bar{\vartheta}\right| \lesssim 10^{-10}$.

The necessity to give a physical explanation to all these questions has led the scientific community to put much effort on the attempt to extend the SM. Of course, an extension of the SM is deemed more attractive when it is able to account for more than one of these problems at once. This is the case of the *QCD axion*: this pseudoscalar particle was recognised in the late 1970s by Steven Weinberg and Frank Wilczek [1, 2] as the low-energy remnant of the Peccei-Quinn solution to the strong CP problem [3, 4]. The QCD axion is believed to be a very light and feebly interacting particle, thus representing a valid candidate for dark matter [5–7]. Moreover, the axion turns out to appear naturally in the spectrum of string theories, see for example [8–11], which are believed to be one of the most appealing formalisms for a quantum theory of gravity. Finally, differently from particles predicted by other extensions of the SM, the axion is light enough to make its observation be within the possibilities of current experimental facilities.

In order to guide experiments towards the observation of the QCD axion, it is necessary to place constraints on its parameter space. These may stem both from theoretical arguments and from experimental evidence. A class of bounds which turns out to be particularly stringent is given by those obtained by determining how the presence of a cooling channel associated to axion production would modify the evolution of astrophysical systems.

In this work we will focus, in particular, on the bound stemming from core-collapse supernovae, associated to the observation of the neutrino flux originated in the SN1987A event [12–17]. The current value of this bound, obtained in [18, 19], was computed without fully taking into account the fact that supernovae are extremely dense objects, whose baryonic density is of the order of nuclear saturation density $n_0 = 0.16\,{\rm fm}^{-3}$. At such high densities the coupling constants for interactions between axions and nucleons, which are those that determine the supernova bound, are sizeably modified by in-medium effects, heavily affecting the value of the bound. The aim of this work is therefore to obtain a reliable expression of the axion-nucleon couplings as functions of density, following [20], in order to properly take into account the corrections due to the nuclear medium in the calculation of the supernova bound.

We then want to apply this formalism to the specific case of *nucleophobic axion models* [21]. This is a class of axion models in which the axion interactions with nucleons are suppressed, allowing them to relax some of the most stringent astrophysical bounds. In particular, if the axion-electron coupling is also suppressed, which is the case in the so-called *astrophobic axion models*, the allowed region in the axion parameter space gets enlarged by up to an order of magnitude, allowing also for axions with mass $m_a \sim 0.1\,\mathrm{eV}$. Since the main feature of the nucleophobic models is their evasion of astrophysical axion bounds, it is necessary to test whether the nucleophobia conditions are spoiled by finite density effects. We will analyse this issue in Chapter 6, which represents the main original result of this thesis.

The structure of the present work is the following. In Chapter 2 we will give an introduction to the strong CP problem and the Peccei-Quinn mechanism, leading to the construction of the axion effective field theory (EFT). Chapter 3 will be devoted to the description of the main axion models, used as benchmarks in literature. In Chapter 4 we will then analyse the main astrophysical axion bounds, with particular detail on the supernova bound. Then, in Chapter 5, we will build the formalism needed to describe axion physics at finite baryonic density, with detailed calculation of the in-medium corrections to the axion-nucleon couplings. Finally, as anticipated, Chapter 6 will be devoted to the analysis of finite density effects in nucleophobic axion models. We will then sum up our conclusions in Chapter 7, while more technical details on the topics of Baryon Chiral Perturbation Theory, Thermal Field Theory, and the phenomenon of meson condensation are discussed in the appendices A, B, and C, respectively.

Chapter 2

The QCD axion

During the 1970s, the discovery by Adler, Bell, and Jackiw [22, 23] of the chiral anomaly brought to a solution of the so-called $U(1)$ problem, *i.e.* the nonobservation of a light meson associated to a spontaneously broken approximate global $U(1)_A$ symmetry of QCD. However, due to the presence of instantons, it also introduced a source of CP violation in the theory. The nonobservation in experiments of this violation yields a small-value problem known as the strong CP problem.

Our goal in this chapter is to introduce in detail the strong CP problem and discuss its most appealing solution, the Peccei-Quinn mechanism, implying the existence of a new pseudoscalar particle, the axion.

2.1 The strong CP problem

2.1.1 QCD instantons and the QCD vacuum structure

The QCD instantons

QCD is the current theory of the strong interaction. It is a gauge theory with gauge group $SU(3)_C$, known as the *colour group*, describing the dynamics of the gluon and quark fields. In absence of matter fields, QCD is a quantum Yang-Mills theory described by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= -\frac{1}{2}\,\text{Tr}[G_{\mu\nu}G^{\mu\nu}] + \frac{g_s^2\vartheta}{16\pi^2}\,\text{Tr}\Big[G_{\mu\nu}\tilde{G}^{\mu\nu}\Big] \\ &= -\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} + \frac{g_s^2\vartheta}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu},\end{aligned} \tag{2.1}$$

where $G_{\mu\nu} = \frac{\lambda^a}{2}G^a_{\mu\nu}$, with $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^a_\mu A^a_\nu$, is the gluon field strength tensor and $\tilde{G}_{\mu\nu} = \frac{\lambda^a}{2}\tilde{G}^a_{\mu\nu}$, with $\tilde{G}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{a,\rho\sigma}$ and $\varepsilon^{0123} = -1$, is its dual. The first term in eq. (2.1) is the usual Yang-Mills kinetic term, while the second term is known as the *theta term* or *topological term*. It is a CP-odd operator, which introduces some interesting features in the QCD dynamics.

One can show that the topological term can be written as a total derivative

$$G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} = \partial_\mu K^\mu, \quad K^\mu \equiv \varepsilon^{\mu\alpha\beta\gamma}\Big(A^a_\alpha G^a_{\beta\gamma} - \frac{g_s}{3}f^{abc}A^a_\alpha A^b_\beta A^c_\gamma\Big), \tag{2.2}$$

where K^μ is known as the *Chern-Simons current*. This property gives rise to two puzzles regarding the role of this term.

The first puzzle stems from the fact that, by defining the Hamiltonian starting from the Lagrangian eq. (2.1), this turns out to be independent of ϑ. However, quantum dynamics boils down to the time evolution of states, $|\psi\rangle \to e^{itH}|\psi\rangle$, which appears to be unaffected by the presence of the theta term.

The second puzzle, instead, arises in the path integral formulation, where the dynamics is determined by the generating functional. Wick rotating to a Euclidean spacetime this reads, up to a normalisation,

$$\mathcal{Z} = \int \mathcal{D}A\, e^{-S_E[A]}. \tag{2.3}$$

However, we observe that the contribution of the topological term to the Euclidean action S_E,

$$\frac{g_s^2\vartheta}{32\pi^2}\int_{\mathbb{E}_4} \mathrm{d}^4x\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{g_s^2\vartheta}{32\pi^2}\int_{\mathbb{E}_4} \mathrm{d}^4x\, \partial_\mu K_\mu = \frac{g_s^2\vartheta}{32\pi^2}\int_{\partial\mathbb{E}_4} \mathrm{d}\sigma_\mu\, K_\mu, \tag{2.4}$$

vanishes for configurations such that K^μ goes to zero faster than r^{-3} as $r \to +\infty$, which seems to be the case for all configurations with finite static Yang-Mills energy, stemming from the kinetic term in the Euclidean action. In fact, naively one has for finite energy configurations

$$\mathcal{E}_{YM} = \int \mathrm{d}^3\vec{x}\, \mathcal{H} \sim \int \mathrm{d}^3\vec{x}\left(\vec{E}^2 + \vec{B}^2\right) < +\infty. \tag{2.5}$$

But $\left|\vec{E}\right|$ vanishes faster than r^{-2}, implying that the $G\tilde{G}$ term, which has the same spatial dependence as the GG term, must vanish faster than r^{-4}. Then, being $G\tilde{G} \sim \partial_\mu K^\mu \sim K^\mu/r$, the Chern-Simons current must indeed vanish faster than r^{-3}.

We start by trying to solve this latter puzzle. We observe that if we were able to find classical configurations of the gluon field for which the integral in eq. (2.3) does not vanish, then the topological term would actually contribute to the generating functional. To find these configurations, we start by considering those field configurations which minimise the static Yang-Mills energy of QCD [24]. This can be easily shown to take only nonnegative values, so that a minimum is reached for configurations with vanishing field strength. The simplest of them is the trivial configuration in which the gluon field vanishes $A_\mu = 0$. We can then perform a gauge transformation so that our new configuration will not be vanishing, but it will instead be a pure gauge $A'_\mu = \frac{i}{g_s}U^{-1}\partial_\mu U$. Such a configuration still satisfies the classical equations of motion and has a null Euclidean action, since the kinetic term $-\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{Tr}[G_{\mu\nu}G_{\mu\nu}]$ is left invariant by the gauge transformation. These pure gauge configurations are called *classical vacua.*

The gluon field for a pure gauge configuration is uniquely determined by the x-dependent $SU(3)_C$ matrix U. Let us first consider a $SU(2)$ subgroup of the QCD gauge group. The group manifold of $SU(2)$ is the three-sphere S^3. Choosing the temporal gauge $A_0 = 0$, in order for the residual gauge transformations to be time-independent, and assuming the following boundary condition on U, $\lim_{|\vec{x}|\to+\infty} U(\vec{x}) = U_0$, the three-dimensional space gets compactified to S^3 for what concerns the gauge transformations, since the points at infinity cannot be distinguished by means of $U(\vec{x})$. Therefore, U turns out to be a continuous map $U : S^3 \to S^3$.

We now need to introduce a fundamental concept in algebraic topology, *homotopy.* Let X, Y be two topological spaces and let $f, g : X \to Y$ be two continuous maps. f and g are said to be *homotopic* if there exists a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Homotopy defines an equivalence relation between continuous maps; the equivalence classes associated to this equivalence relation are known as *homotopy classes.* Let $\pi_n(X)$ be the set of homotopy classes of maps $f : S^n \to X$, where S^n is the n-sphere. With an appropriate choice of a binary operation, the set $\pi_n(X)$ has the structure of a group, and it is known as the n-th homotopy group of X. A very important result on homotopy groups is that the n-th homotopy group for the n-sphere is isomorphic to the group of integer numbers with the sum operation:

$$\pi_n(S^n) \simeq \mathbb{Z}. \tag{2.6}$$

Applying this result to the U matrices, we understand that gauge transformations, and therefore classical vacua, can be classified in terms of the homotopy class they belong to. Moreover, due to the isomorphism eq. (2.4), different homotopy classes can be uniquely identified by specifying an integer number, known as the *winding number*. Gauge transformations belonging to different homotopy classes cannot be continuously deformed one into the other. This implies that one cannot move from a classical vacuum to another one with a different winding number by keeping the gluon field in a pure gauge configuration throughout the process. It follows that any configuration interpolating between two classical vacua must have a positive Euclidean action $S_E > 0$. However, in general, the U matrices are not elements of $SU(2)$, but of $SU(3)$. Fortunately, a theorem by Bott [25], states that mappings from S^3 to any simple Lie group G, like $SU(3)$, can be deformed continuously, *i.e.* are homotopic, to mappings to a $SU(2)$ subgroup of G; hence, the homotopy classes are exactly the same described above.

Belavin, Polyakov, Schwartz, and Tyupkin [26] showed that it is possible to build topologically nontrivial configurations of the gluon field for which the integral in eq. (2.3) does not vanish: the *BPST instantons*. They are configurations continuously interpolating between classical vacua with different winding number with minimal Euclidean action. Since they minimise the action, they are solutions of the classical equations of motion. When evaluated on an instanton solution $G^{(\nu)}_{\alpha\beta}$ interpolating between two vacua with winding numbers m and n, with $m = n + \nu$, the integral in eq. (2.3) takes the value

$$\int_{\mathbb{E}_4} \mathrm{d}^4x \, G^{(\nu)a}_{\alpha\beta} \tilde{G}^{(\nu)a}_{\alpha\beta} = \frac{32\pi^2}{g_s^2}\nu. \tag{2.7}$$

The kinetic term in the Euclidean action for a $\nu = 1$ BPST instanton takes the finite value

$$-\frac{1}{4}\int_{\mathbb{E}_4} \mathrm{d}^4x \, G^a_{\mu\nu} G^a_{\mu\nu} = \frac{8\pi^2}{g_s^2}. \tag{2.8}$$

QCD vacuum structure

The existence of instanton solutions in classical QCD has deep implications on the structure of the corresponding quantum theory. To show this, we start by observing that, in the semiclassical approximation, we can associate to each classical vacuum a corresponding quantum vacuum state. The amplitude for the tunnelling between two vacuum states of winding numbers m and n, with $m = n+\nu$, is

$$\langle m|n\rangle \sim e^{-S_\nu}, \tag{2.9}$$

where S_ν is the Euclidean action corresponding to the instanton solution interpolating between the corresponding classical vacua. So we understand that the complex topological structure of the QCD gauge group introduces a nontrivial structure for the ground state of the theory, which is characterised by the presence of infinite degenerate vacuum states each identified by an integer index. These vacua are connected by gauge transformations of nontrivial winding number. In particular calling $U^{(n)}$ the gauge transformations belonging to the homotopy class of winding number n, one has

$$U^{(1)}\left|n\right\rangle = \left|n+1\right\rangle, \quad U^{(n)} = \left[U^{(1)}\right]^n. \tag{2.10}$$

The noninvariance of these states under gauge transformations is, however, a clear indicator of the fact that these states cannot be used to describe the actual vacua of the theory. To find the correct expression of the vacuum states we exploit a fundamental property of relativistic quantum field theories: *cluster decomposition*. This is just the requirement that distant enough experiments must yield uncorrelated results. Let us consider a large Euclidean spacetime region Ω which is split into two large regions Ω_1 and Ω_2. Let $\mathcal{O}$ be a local operator with support on Ω_1. Denoting by ϕ the set of fields of the theory, so as to generalise the discussion to the case in which we introduce also quark fields, one has in the path integral formulation that the vacuum expectation value of $\mathcal{O}$ is

$$\begin{aligned}\langle\mathcal{O}\rangle_\Omega &= \frac{\sum_\nu f(\nu)\int_\nu \mathcal{D}\phi\, e^{-S_\Omega[\phi]}\mathcal{O}[\phi]}{\sum_\nu f(\nu)\int_\nu \mathcal{D}\phi\, e^{-S_\Omega[\phi]}} \\ &= \frac{\sum_{\nu_1,\nu_2=\nu-\nu_1} f(\nu_1+\nu_2)\int_{\nu_1}\mathcal{D}\phi\, e^{-S_{\Omega_1}[\phi]}\mathcal{O}[\phi]\int_{\nu_2}\mathcal{D}\phi\, e^{-S_{\Omega_2}[\phi]}}{\sum_{\nu_1,\nu_2=\nu-\nu_1} f(\nu_1+\nu_2)\int_{\nu_1}\mathcal{D}\phi\, e^{-S_{\Omega_1}[\phi]}\int_{\nu_2}\mathcal{D}\phi\, e^{-S_{\Omega_2}[\phi]}},\end{aligned} \tag{2.11}$$

where we summed over all the topological sectors, denoted by ν, each weighed by an appropriate factor $f(\nu)$. Cluster decomposition requires

$$f(\nu_1+\nu_2) = f(\nu_1)f(\nu_2). \tag{2.12}$$

This equation has solution

$$f(\nu) = e^{i\vartheta\nu}, \tag{2.13}$$

where ϑ is an arbitrary parameter. But now we have

$$\begin{aligned}\langle\mathcal{O}\rangle_\Omega &= \frac{\sum_\nu e^{i\vartheta\nu}\int_\nu \mathcal{D}\phi\, e^{-S_\Omega[\phi]}\mathcal{O}[\phi]}{\sum_\nu e^{i\vartheta\nu}\int_\nu \mathcal{D}\phi\, e^{-S_\Omega[\phi]}} \\ &= \frac{\sum_{n,m=n+\nu} e^{i\vartheta m}e^{-i\vartheta n}\left\langle m\right|\mathcal{O}\left|n\right\rangle}{\sum_{n,m=n+\nu} e^{i\vartheta m}e^{-i\vartheta n}\left\langle m|n\right\rangle}.\end{aligned} \tag{2.14}$$

This is just the vacuum expectation value taken between vacuum states defined as the following linear combinations of the topological vacua

$$\left|\vartheta\right\rangle \equiv \sum_{n=-\infty}^{+\infty} e^{i\vartheta n}\left|n\right\rangle, \tag{2.15}$$

known as the *theta vacua*. These kets are eigenvectors of the $U^{(1)}$ operator defined above, with a phase as eigenvalue

$$U^{(1)}\left|\vartheta\right\rangle = \sum_{n=-\infty}^{+\infty} e^{i\vartheta n}U^{(1)}\left|n\right\rangle = \sum_{n=-\infty}^{+\infty} e^{i\vartheta n}\left|n+1\right\rangle = e^{-i\vartheta}\left|\vartheta\right\rangle. \tag{2.16}$$

This, together with the relation $U^{(n)} = \left[U^{(1)}\right]^n$, implies that the theta vacua are left invariant by gauge transformations, and therefore can describe the actual vacua of QCD.

Moreover, theta vacua introduce a *superselection rule* in the Hilbert space of the theory. In fact, if $\mathcal{O}_1, \mathcal{O}_2, \dots$ are gauge invariant operators, the matrix element of their time-ordered product between two different theta vacua $|\vartheta\rangle\,, |\vartheta'\rangle$ is given by

$$\langle\vartheta'|T(\mathcal{O}_1\mathcal{O}_2\cdots)|\vartheta\rangle = \sum_{m,n} e^{-im\vartheta'+in\vartheta}\,\langle m|T(\mathcal{O}_1\mathcal{O}_2\cdots)|n\rangle\,. \tag{2.17}$$

The matrix element in the last expression just depends on the difference $\nu = m-n$ since, due to gauge invariance, $U^{(1)-1}T(\mathcal{O}_1\mathcal{O}_2\cdots)U^{(1)} = T(\mathcal{O}_1\mathcal{O}_2\cdots)$, so that m and n change in the same way under a gauge transformation. Thus, we can write it as a function $F = F(\nu)$. Then

$$\langle\vartheta'|T(\mathcal{O}_1\mathcal{O}_2\cdots)|\vartheta\rangle = \sum_n e^{in(\vartheta-\vartheta')}\sum_\nu e^{i\frac{\nu}{2}(\vartheta+\vartheta')}F(\nu) = 2\pi\delta(\vartheta-\vartheta')\sum_\nu e^{i\nu\vartheta}F(\nu), \tag{2.18}$$

which vanishes for $\vartheta \neq \vartheta'$. Hence, gauge invariant operators cannot connect vacua with a different value of ϑ. This implies that each value of ϑ characterises a different theory: only a specific value of ϑ corresponds to our universe.

Theta vacua and topological term

We are now ready to show the relation between the nontrivial vacuum structure of QCD and the topological term in eq. (2.1). In particular, we want to show that the latter can be obtained as a result of the former. With this aim, let us consider the standard Yang-Mills Lagrangian for QCD

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}. \tag{2.19}$$

Fixing the value of ϑ, the vacuum-to-vacuum amplitude in presence of external sources can be written as

$$\langle\vartheta|\vartheta\rangle_J = \sum_\nu e^{i\vartheta\nu}\sum_m \langle m|m+\nu\rangle_J\,. \tag{2.20}$$

In the path integral formalism, making use of eq. (2.5), this can be rewritten as

$$\begin{aligned}\langle\vartheta|\vartheta\rangle_J &= \sum_\nu e^{i\vartheta\nu}\int \mathcal{D}A\, e^{-\int \mathrm{d}^4x\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}+\int\mathrm{d}^4x J\cdot A} \\ &= \sum_\nu \int \mathcal{D}A\, e^{-\int \mathrm{d}^4x\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}+i\vartheta\frac{g_s^2}{32\pi^2}\int\mathrm{d}^4x G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}+\int\mathrm{d}^4x J\cdot A}\delta\left(\nu-\frac{g_s^2}{32\pi^2}\int \mathrm{d}^4x\, G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}\right).\end{aligned} \tag{2.21}$$

Hence, the theta parameter appearing in eq. (2.1) can be identified with the theta angle specifying the vacuum of our theory. This gives a solution to our first puzzle: even though the Hamiltonian does not depend on the topological term, this fixes the specific vacuum state of QCD, giving a fundamental contribution to the dynamics of the gluon field.

2.1.2 The strong CP problem

As stated before, the topological term

$$\mathcal{L}_{\rm QCD} \supset \frac{g_s^2 \vartheta}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \tag{2.22}$$

is CP-odd. Therefore, it introduces a source of CP violation in the theory of the strong interaction.

Another source of CP violation appears when we consider quarks in our description:

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} + \frac{g_s^2 \vartheta}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} + \sum_q \bar{q}\Big(i\not{D} - e^{i\vartheta_q \gamma_5} m_q\Big) q, \tag{2.23}$$

where $D_\mu = \partial_\mu - i g_s \frac{\lambda^a}{2} A^a_\mu$, $\not{D} = \gamma^\mu D_\mu$, and we introduced a CP-violating phase depending on an angle ϑ_q in the quark masses.

Let us consider the redefinition of a single quark flavour through an axial transformation of the form

$$q \to e^{i\gamma_5 \alpha} q. \tag{2.24}$$

Under this redefinition, the angle ϑ_q gets shifted as

$$\vartheta_q \to \vartheta_q + 2\alpha. \tag{2.25}$$

The transformation in eq. (2.24) is however anomalous in QCD. In the Fujikawa approach [27], the anomaly manifests itself in a noninvariance of the path integral measure in the generating functional. In particular, this varies as

$$\mathcal{D}q\,\mathcal{D}\bar{q} \to e^{-i\alpha \frac{g_s^2}{16\pi^2} \int {\rm d}^4x G^a_{\mu\nu} \tilde{G}^{a,\mu\nu}} \mathcal{D}q\,\mathcal{D}\bar{q}\,, \tag{2.26}$$

yielding a shift of the QCD ϑ parameter of the form

$$\vartheta \to \vartheta - 2\alpha. \tag{2.27}$$

Since, by the S-matrix equivalence theorem [28, 29], physical quantities must be invariant under the field redefinition eq. (2.24), we understand that neither ϑ nor ϑ_q are observable. Only the combination

$$\bar{\vartheta} = \vartheta + \vartheta_q \tag{2.28}$$

is observable. This object completely describes CP-violation in QCD. Within the SM, the CP violating phase in the mass terms finds its origin in the diagonalisation of the Yukawa sector, and therefore the observable theta parameter reads

$$\bar{\vartheta} = \vartheta + \arg\det(Y_U Y_D), \tag{2.29}$$

$Y_{U,D}$ being the Yukawa matrices, respectively, of up and down quark flavours.

A bound on this parameter can be found experimentally, by exploiting some observable which is particularly sensitive to CP violation. The observable of choice is the *neutron electric dipole moment* (nEDM). It is defined in terms of the nonrelativistic Hamiltonian

$$H = -d_n \vec{E} \cdot \hat{S}, \tag{2.30}$$

which can be written in terms of a relativistic-invariant operator as

$$\mathcal{L} = -d_n \frac{i}{2} \bar{n} \sigma_{\mu\nu} \gamma_5 n F^{\mu\nu}. \tag{2.31}$$

The most accurate theoretical prediction for the value of d_n is obtained from QCD sum rules [30]:

$$d_n = 2.4(1.0) \times 10^{-16} \bar{\vartheta}\, \mathrm{e\,cm} = 1.2(0.5) \times 10^{-2} \bar{\vartheta}\, \mathrm{e\,GeV}^{-1}. \tag{2.32}$$

Comparing this result with the current experimental bound [31]

$$|d_n^{\mathrm{exp}}| < 3.0 \times 10^{-26}\, \mathrm{e\,cm} = 1.5 \times 10^{-12}\, \mathrm{e\,GeV}^{-1} \;\; (90\% \text{ C.L.}), \tag{2.33}$$

one obtains a very stringent upper bound on the value of the observable theta parameter

$$|\bar{\vartheta}| \lesssim 10^{-10}. \tag{2.34}$$

This bound gives us an extremely small value for the CP-violating parameter, compared to the naive expectation $\bar{\vartheta} \sim \mathcal{O}(1)$. This small value problem is known as the *strong CP problem*.

The strong CP problem has some characteristics which make it qualitatively different from other small value problems, like the cosmological constant problem or the neutrino mass problem. First of all, the value of $\bar{\vartheta}$ turns out to be radiatively stable to a very good degree. In fact, since $\bar{\vartheta}$ contains is also sensititve to the CKM CP-violating phase, its radiative corrections must be proportional to the Jarlskog invariant $J_{\mathrm{CKM}} = \mathrm{Im}\, V_{ud} V_{cd}^* V_{cs} V_{us}^*$ [32], and, just by considering spurionic properties, they should be given by

$$\mathrm{Im}\det\left[Y_U Y_U^\dagger, Y_D Y_D^\dagger\right] = \left(\frac{2^6}{v^{12}}\right) \prod_{i>j=u,c,t} (m_i^2 - m_j^2) \prod_{k>\ell=d,s,b} (m_k^2 - m_\ell^2) J_{\mathrm{CKM}} \approx 10^{-20}, \tag{2.35}$$

which would diagrammatically correspond to a 6-loop diagram with 12 Yukawa insertions connected by 6 Higgs propagators [33]. However, an accidental symmetry of the SM Yukawa sector under $H \leftrightarrow \tilde{H}$, $u_R \leftrightarrow d_R$, $Y_U \leftrightarrow Y_D$ makes all 6-loop contribution vanish, so that radiative corrections only start at 7 loops. They are of the form $\delta\bar{\vartheta}_{\mathrm{div.}} \approx 10^{-33} \log \Lambda_{\mathrm{UV}}$, so that if the observable theta parameter is small at some heavy energy scale Λ_{UV} it remains small under renormalisation group (RG) evolution.

Another typical feature of the strong CP problem is the seemingly absent anthropic explanation of the bound in eq. (2.34). In fact, even if this bound were relaxed by various orders of magnitude, as long as $\bar{\vartheta} < 1\%$ [34, 35], nothing catastrophic would happen, differently from cases like that of the cosmological constant.

2.1.3 The Peccei-Quinn mechanism

A mechanism to solve the strong CP problem was proposed by Peccei and Quinn in [3, 4]. The main idea is to exploit an anomalous global symmetry in the theory to rotate away the topological term, and thus fix $\bar{\vartheta} = 0$. We first observe that no such symmetry exists within the SM. In fact the global symmetry group of the SM is $U(1)_B \times U(1)_{L_1} \times U(1)_{L_2} \times U(1)_{L_3}$, which is not anomalous under QCD, and so it cannot generate the correct anomaly operator to rotate away the theta term. The symmetry required by the Peccei-Quinn (PQ) mechanism must therefore be introduced by extending the SM. However, the SM gives us a correct description of particle physics phenomena at the scales of current experimental facilities, so we understand that such a new symmetry must be spontaneously broken at low energies in order to account for its nonbservation. We call this anomalous and spontaneously broken $U(1)_{\mathrm{PQ}}$ symmetry the *Peccei-Quinn symmetry.*

Weinberg and Wilczek [1, 2] soon realised that the spontaneous symmetry breaking (SSB) of the PQ symmetry implied, through the Goldstone theorem [36, 37], the existence of a pseudo Nambu-Goldstone boson (pNGB), which due to the fact that its existence "washed away" the strong CP problem was given by Wilczek the name *axion*, after a known brand of detergents. The presence of the axion makes it possible to test the PQ mechanism at low energies through the observation of a remnant, the QCD axion itself.

2.2 Axion EFT

We have introduced the QCD axion as the pNGB associated to the spontaneous breaking of a $U(1)_{\mathrm{PQ}}$ anomalous symmetry. Since there is only one broken generator, from the Goldstone theorem we find that the axion must have only one degree of freedom, *i.e.* it must be a real spin-0 field. Moreover, its Lagrangian must be endowed with a quasi-shift symmetry $a \to a+\kappa f_a$, where f_a is a mass scale known as the *axion decay constant* and, as we are going to see, is closely related to the order parameter of the PQ symmetry spontaneous breaking. This shift transformation should leave the action invariant up to a term

$$\delta S = \frac{g_s^2 \kappa}{32\pi^2} \int \mathrm{d}^4 x \, G^a_{\mu\nu} \tilde{G}^{a,\mu\nu}. \tag{2.36}$$

An appropriate choice of κ can thus be used to remove the topological term of the QCD Lagrangian, implementing the PQ mechanism.

In an EFT approach, the axion is therefore described by a nonrenormalisable Lagrangian of the form [38]

$$\mathcal{L}_a = \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \frac{g_s^2}{32\pi^2}\frac{a}{f_a} G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + \frac{1}{4} g^0_{a\gamma} a F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{2 f_a}\sum_f \bar{f} c^0_f \gamma^\mu \gamma_5 f. \tag{2.37}$$

A few comments are now in order regarding the terms appearing in eq. (2.37). We first observe that the only necessary interaction term is the axion-gluon one, which stems from the anomaly of the $U(1)_{\mathrm{PQ}}$ under QCD to solve the strong CP problem. We note however, that this term could potentially be problematic: in fact if $\langle a \rangle \neq 0$ this term would generate again CP violation in the strong sector. Fortunately, the Vafa-Witten theorem [39] implies $\langle a \rangle = 0$ in a vector-like theory like QCD, so that the PQ solution is not spoiled. Since $G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$ is a P-odd object, in order for the whole operator to be invariant under parity the axion must be a pseudoscalar field.

The axion-photon interaction is again due to an anomaly, *i.e.* that of $U(1)_{\mathrm{PQ}}$ under Quantum Electrodynamics (QED). This term is model-dependent, in the sense that the specific value of the coupling constant $g^0_{a\gamma}$ depends on the specific UV completion of the axion EFT. Also the axion interactions with SM fermions are model-dependent. Moreover, we observe that the axion must appear only through its derivatives in these interaction terms, due to the pseudo-shift symmetry. Finally, the axion only couples to the axial fermion currents, because of the axial nature of the $U(1)_{\mathrm{PQ}}$.

Let us consider for simplicity two-flavour QCD, with $q = (u, d)^T$. Including also the quark mass term, one has

$$\mathcal{L}_a = \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \frac{g_s^2}{32\pi^2}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + \frac{1}{4}g^0_{a\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{2f_a}\bar{q}c^0_q\gamma^\mu\gamma_5 q - \bar{q}_L\mathcal{M}_q q_R + \text{h.c.}, \tag{2.38}$$

where $c^0_q = \text{diag}(c^0_u, c^0_d)$ and $\mathcal{M}_q = \text{diag}(m_u, m_d)$. It is convenient to rotate away the axion-gluon interaction term by means of the axion-dependent field redefinition

$$q \to e^{i\gamma_5\frac{a}{2f_a}Q_a}q, \tag{2.39}$$

where Q_a is an arbitrary 2×2 matrix with unit trace: $\text{Tr}\,Q_a = 1$. The noninvariance of the path integral measure under this transformation cancels the $aG\tilde{G}$ term, while modifying also the $aF\tilde{F}$ operator. Moreover, also the quark kinetic and mass terms change under this field redefinition. It follows that the couplings of our theory are modified as

$$g^0_{a\gamma} \quad\to\quad g_{a\gamma} = g^0_{a\gamma} - (2N_c)\frac{\alpha}{2\pi f_a}\,\text{Tr}\big(Q_a Q^2\big), \quad Q = \text{diag}(2/3, -1/3), \tag{2.40a}$$

$$c^0_q \quad\to\quad c_q = c^0_q - Q_a \tag{2.40b}$$

$$\mathcal{M}_q \quad\to\quad \mathcal{M}_a = e^{i\frac{a}{2f_a}Q_a}\mathcal{M}_q e^{i\frac{a}{2f_a}Q_a}, \tag{2.40c}$$

and the effective Lagrangian in the new basis reads

$$\mathcal{L}_a = \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \frac{1}{4}g_{a\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{2f_a}\bar{q}c_q\gamma^\mu\gamma_5 q - \bar{q}_L\mathcal{M}_a q_R + \text{h.c.} \tag{2.41}$$

We note that in this basis the axion-photon and axion-quark interaction terms acquire a model-independent contribution.

Working at even lower energies, one can describe the axion EFT in terms of interactions with hadrons. A very useful way to do this is by exploiting *chiral perturbation theory* (ChPT) techniques. We map the Lagrangian in eq. (2.41) to the corresponding axion-dressed ChPT Lagrangian

$$\mathcal{L}_a = \frac{f_\pi^2}{4}\,\text{Tr}\Big[(D_\mu\Sigma)^\dagger(D^\mu\Sigma)\Big] + \frac{Bf_\pi^2}{2}\,\text{Tr}\Big[\Sigma\mathcal{M}_a^\dagger + \mathcal{M}_a\Sigma^\dagger\Big] + \frac{\partial_\mu a}{2f_a}\frac{1}{2}\,\text{Tr}[c_q\sigma^a]\,J^a_\mu, \tag{2.42}$$

where $f_\pi = 92.3\,\text{MeV}$, B is related to the quark condensate, and $\Sigma = e^{i\frac{\pi^a\sigma^a}{f_\pi}}$. Here we have kept only the iso-triplet part of the axion coupling to the axial current, since the iso-singlet part would be associated to the heavy η', which we do not include in the ChPT description. The current is given by

$$J^a_\mu = \frac{i}{2}f_\pi^2\,\text{Tr}\Big[\sigma^a\Big(\Sigma D_\mu\Sigma^\dagger - \Sigma^\dagger D_\mu\Sigma\Big)\Big]. \tag{2.43}$$

Finally, up to now we have worked in the implicit assumption that the energy scale of interest was below that of the electroweak phase transition. However, as we are going to see, there are valid reasons to assume that the spontaneous breaking of the PQ symmetry happens at energies much higher than 1 TeV. Therefore the axion should be present also in the unbroken phase of the SM. This is encoded in the axion EFT Lagrangian [40]

$$\begin{aligned}\mathcal{L}_a = & \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \frac{g_s^2}{32\pi^2}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + c_W\frac{g^2}{32\pi^2}\frac{a}{f_a}W^i_{\mu\nu}\tilde{W}^{i,\mu\nu} + c_B\frac{g'^2}{32\pi^2}\frac{a}{f_a}B_{\mu\nu}\tilde{B}^{\mu\nu} \\ & + \frac{\partial_\mu a}{f_a}\left[\sum_f \left(\bar{f}_L c_{f_L}\gamma^\mu f_L + \bar{f}_R c_{f_R}\gamma^\mu f_R\right) + c_H H^\dagger i\overleftrightarrow{D}^\mu H\right],\end{aligned} \tag{2.44}$$

where $H^\dagger \overleftrightarrow{D}^\mu H \equiv H^\dagger(D_\mu H) - (D_\mu H)^\dagger H$.

2.2.1 The axion potential

We now exploit the ChPT description of the axion EFT, eq. (2.42), to find some information on the properties of the axion. In particular we are interested in studying the scalar potential, *i.e.* (minus) the nonderivative part of the Lagrangian

$$V(a, \pi^a) = -\frac{B f_\pi^2}{2}\operatorname{Tr}\left[\Sigma\mathcal{M}_a^\dagger + \mathcal{M}_a\Sigma^\dagger\right]. \tag{2.45}$$

By expanding one finds

$$V(a, \pi^a) = -B f_\pi^2(m_u + m_d) + \frac{1}{2}B(m_u + m_d)\pi^2 + \frac{i}{4}\frac{B f_\pi^2}{f_a}a\operatorname{Tr}[\Sigma\{Q_a, \mathcal{M}_q\}] + \text{h.c.} + \ldots, \tag{2.46}$$

where $\pi = \sqrt{(\pi^0)^2 + 2\pi^+\pi^-}$. The third term introduces a mass mixing between the axion and the neutral pion. This can be removed my making an appropriate choice of the matrix Q_a: the form we use is $Q_a = \mathcal{M}_q^{-1}/\operatorname{Tr}\mathcal{M}_q^{-1}$. This choice removes also any interaction term linear in a and with an arbitrary number of pion fields: for an odd number of pions that is because $\operatorname{Tr}[\sigma^a] = 0$, while for an even number of pions there is a cancellation with the hermitian conjugate.

Then our potential, expanded for $a/f_a \ll 1$, reads

$$V(a, \pi^a) = -m_\pi^2 f_\pi^2 \cos\left(\frac{\pi}{f_\pi}\right) + \frac{1}{2}\frac{m_u m_d}{(m_u + m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2}a^2\cos\left(\frac{\pi}{f_\pi}\right) + \mathcal{O}\left(\frac{a^3}{f_a^3}\right). \tag{2.47}$$

where $m_\pi^2 = B(m_u + m_d)$, as one can read from eq. (2.46).

By setting the pion field to its ground state $\langle\pi\rangle = 0$, we find the value of the axion mass

$$m_a^2 = \frac{m_u m_d}{(m_u + m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} \implies m_a \simeq 5.7\left(\frac{10^{12}\,\text{GeV}}{f_a}\right)\mu\text{eV}. \tag{2.48}$$

The relation $m_a^2 f_a^2 \sim m_\pi^2 f_\pi^2$ is a result of the fact that the QCD axion solves the strong CP problem. Other pseudoscalar particles with properties similar to those of the axion, but that do not solve the strong CP problem, and therefore do not satisfy this constraint, are known as *axion-like particles* (ALPs).

Finally, we observe that the potential in eq. (2.48) is minimised for $\langle a\rangle = 0$, in agreement with the Vafa-Witten theorem.

2.2.2 Axion-pion kinetic mixing and axion-pion coupling

Let us now examine the current part of the Lagrangian in eq. (2.42). By expanding, we obtain

$$\frac{\partial_\mu a}{2f_a}\frac{1}{2}\,\mathrm{Tr}[c_q\sigma^a]\,J^a_\mu \approx -\frac{1}{2}\left(\frac{m_u-m_d}{m_u+m_d}+c^0_d-c^0_u\right)\frac{f_\pi}{f_a}\partial_\mu a\partial^\mu\pi^0$$
$$+\frac{1}{3}\left(\frac{m_u-m_d}{m_u+m_d}+c^0_d-c^0_u\right)\frac{1}{f_\pi f_a}\partial_\mu a\big(2\partial^\mu\pi^0\pi^+\pi^- - \pi^0\partial^\mu\pi^+\pi^- - \pi^0\pi^+\partial^\mu\pi^-\big). \tag{2.49}$$

The first term gives a kinetic mixing between the axion and the neutral pion. Since we want kinetic terms to be canonically normalised, we need to diagonalise this mixing. To do that we consider the quadratic part of the axion-pion Lagrangian

$$\mathcal{L}_a^{\text{quad.}} = \frac{1}{2}\begin{pmatrix}\partial_\mu a & \partial_\mu\pi^0\end{pmatrix}\begin{pmatrix}1 & \varepsilon\\ \varepsilon & 1\end{pmatrix}\begin{pmatrix}\partial^\mu a\\ \partial^\mu\pi^0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}a & \pi^0\end{pmatrix}\begin{pmatrix}m_a^2 & 0\\ 0 & m_\pi^2\end{pmatrix}\begin{pmatrix}a\\ \pi^0\end{pmatrix}, \tag{2.50}$$

with

$$\varepsilon \equiv -\frac{1}{2}\left(\frac{m_u-m_d}{m_u+m_d}+c^0_d-c^0_u\right)\frac{f_\pi}{f_a} \tag{2.51}$$

and $m_a/m_\pi \sim \mathcal{O}(\varepsilon)$. We perform in sequence:

(i) an orthogonal transformation to diagonalise the kinetic term;

(ii) a rescaling to make the kinetic terms canonical;

(iii) another orthogonal transformation to diagonalise again the mass term, which does not affect the kinetic term.

The net effect of this procedure are the shifts $a \to a - \varepsilon\pi^0$ and $\pi^0 \to \pi^0 + (m_a^2/m_\pi^2)\varepsilon a$. We observe that the axion component in π^0 is suppressed at the level of ε^3, making this diagonalisation completely negligible when considering experimental sensitivities and astrophysical bounds.

The second term in eq. (2.49), instead, gives the axion-pion interaction term

$$\mathcal{L}_a \supset \frac{c_{a\pi}}{f_\pi f_a}\partial_\mu a\big(2\partial^\mu\pi^0\pi^+\pi^- - \pi^0\partial^\mu\pi^+\pi^- - \pi^0\pi^+\partial^\mu\pi^-\big), \tag{2.52}$$

where

$$c_{a\pi} \equiv \frac{1}{3}\left(\frac{m_u-m_d}{m_u+m_d}+c^0_d-c^0_u\right). \tag{2.53}$$

Such a coupling is especially relevant for the axion thermalisation rate in the early universe [41–43]

2.2.3 Axion-nucleon couplings

Nucleons were not included in our description of the axion EFT up to now, the reason being that a consistent description of nucleons within a ChPT framework, although possible, encounters some technical difficulties due to the small difference between the nucleon mass m_N and the scale at which ChPT breaks down Λ_χ, see Appendix A. In order to overcome these problems, we work in the non-relativistic limit for Baryon ChPT, where the axion-dressed Lagrangian for two light quark flavours takes the form (see Appendix A)

$$\begin{aligned}\mathcal{L}_{aN} &= \bar{N}v^\mu\partial_\mu N + 2g_A c_- \frac{\partial_\mu a}{2f_a}\bar{N}S^\mu\tau^3 N + 2g_0^{ud}c_+\frac{\partial_\mu a}{2f_a}\bar{N}S^\mu N + \dots \\ &= \bar{N}v^\mu\partial_\mu N + 2g_A c_- \frac{\partial_\mu a}{2f_a}(\bar{p}S^\mu p - \bar{n}S^\mu n) + 2g_0^{ud}c_+\frac{\partial_\mu a}{2f_a}(\bar{p}S^\mu p + \bar{n}S^\mu n) + \dots,\end{aligned} \tag{2.54}$$

where $N = (p, n)^T$, $c_\pm = (c_u \pm c_d)/2$, v^μ is the four-velocity of the nucleon,

$$S^\mu = \frac{i}{2}\gamma_5\sigma^{\mu\nu}v_\nu = -\frac{1}{2}\gamma_5(\gamma^\mu \not{v} - v^\mu) \tag{2.55}$$

is the spin operator, g_A, g_0^{ud} are low-energy constants and the axion enters as an axial external current, with components both in the isotriplet and isosinglet directions.

By matching eq. (2.54) with the axion effective Lagrangian written in terms of quarks over a one-nucleon matrix element

$$\frac{\partial_\mu a}{2f_a}c_u\langle N|\bar{u}\gamma^\mu\gamma_5 u|N\rangle + \frac{\partial_\mu a}{2f_a}c_d\langle N|\bar{d}\gamma^\mu\gamma_5 d|N\rangle = \frac{\partial_\mu a}{2f_a}g_A c_-\langle N|2\bar{p}S^\mu p|N\rangle + \frac{\partial_\mu a}{2f_a}g_0^{ud}c_+\langle N|2\bar{p}S^\mu p|N\rangle \tag{2.56}$$

and by exploiting the relation $2\bar{p}S^\mu p \approx \bar{p}\gamma^\mu\gamma_5 p$ in the nonrelativistic limit, see Appendix A, so that

$$\frac{\partial_\mu a}{2f_a}c_u s^\mu\Delta u + \frac{\partial_\mu a}{2f_a}c_d s^\mu \Delta d = \frac{\partial_\mu a}{2f_a}g_A\frac{c_u - c_d}{2}s^\mu + \frac{\partial_\mu a}{2f_a}g_0^{ud}\frac{c_u + c_d}{2}s^\mu, \tag{2.57}$$

where s^μ is the spin of the nucleon at rest and we defined $s^\mu\Delta q = \langle p|\bar{q}\gamma^\mu\gamma_5 q|p\rangle$, one finds

$$g_A = \Delta u - \Delta d, \quad g_0^{ud} = \Delta u + \Delta d. \tag{2.58}$$

Therefore, the axion-nucleon Lagrangian can be written as

$$\mathcal{L}_{aN} = \frac{\partial_\mu a}{2f_a}\bar{N}c_N\gamma^\mu\gamma_5 N, \tag{2.59}$$

where the coupling constant matrix $c_N = \mathrm{diag}(c_p, c_n)$ has elements

$$c_p = g_A c_- + g_0^{ud}c_+ = -\left(\frac{m_d}{m_u+m_d}\Delta u + \frac{m_u}{m_u+m_d}\Delta d\right) + c_u^0\Delta u + c_d^0\Delta d, \tag{2.60a}$$

$$c_n = -g_A c_- + g_0^{ud}c_+ = -\left(\frac{m_u}{m_u+m_d}\Delta u + \frac{m_d}{m_u+m_d}\Delta d\right) + c_u^0\Delta d + c_d^0\Delta u. \tag{2.60b}$$

Chapter 3

Benchmark QCD axion models

As we have seen in the previous chapter, QCD axion physics is described in terms of a nonrenormalisable EFT Lagrangian. This, of course, calls for a UV completion of the axion theory, *i.e.* what is commonly known as an *axion model*. During the years there have been many proposals of specific axion models, some of which have already been dismissed by experimental evidence. Here, we aim to give a brief introduction to the main axion models present in literature.

3.1 The anomaly coefficients

Before describing the benchmark QCD axion models, we need to analyse more in detail the coefficients due to the anomaly of $U(1)_{\rm PQ}$ under QCD and QED. Let us consider the most general case in which the anomaly is generated by a set of fermion fields $\mathcal{Q}$, of PQ charge $X_\mathcal{Q}$, each transforming under the $(\mathcal{C}_\mathcal{Q}, \mathcal{I}_\mathcal{Q}, \mathcal{Y}_\mathcal{Q})$ irreducible representation of the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. We recall that, given a generic anomalous global continuous symmetry group G in a gauge theory with gauge group $\mathcal{G}$, the four-divergence of the Noether current associated to this group is given by

$$\partial_\mu J_G^{a,\mu} = \frac{g^2}{16\pi^2} \,\mathrm{Tr}\left[T_G^a \left\{t_\mathcal{G}^b, t_\mathcal{G}^c\right\}\right] F_{\alpha\beta}^b \tilde{F}^{c,\alpha\beta}, \tag{3.1}$$

where T_G^a are the generators of G, $t_\mathcal{G}^a$ the generators of $\mathcal{G}$, g is the gauge coupling of $\mathcal{G}$ and $F_{\mu\nu}$ the field strength of the corresponding gauge boson. The trace is taken both in flavour space and on gauge group indices. In the specific cases of QCD and QED this expression becomes

$$\partial_\mu J_G^{a,\mu} = \frac{g_s^2 N}{16\pi^2} G_{\alpha\beta}^b \tilde{G}^{b,\alpha\beta}, \tag{3.2a}$$

$$\partial_\mu J_G^{a,\mu} = \frac{e^2 E}{16\pi^2} F_{\alpha\beta} \tilde{F}^{\alpha\beta}. \tag{3.2b}$$

The value of the coefficients N and E can be obtained by representation theory arguments. They both receive contributions by each fermion irreducible representation, which we assume to be all left-handed, with $X_{\mathcal{Q}_L^c} = -X_{\mathcal{Q}_R}$:

$$N = \sum_\mathcal{Q} N_\mathcal{Q}, \quad E = \sum_\mathcal{Q} E_\mathcal{Q}, \tag{3.3}$$

where [38]

$$N_\mathcal{Q} = X_\mathcal{Q} d(\mathcal{I}_\mathcal{Q}) T(\mathcal{C}_\mathcal{Q}), \tag{3.4a}$$

$$E_\mathcal{Q} = X_\mathcal{Q} d(\mathcal{C}_\mathcal{Q}) \operatorname{Tr} q_\mathcal{Q}^2 = X_\mathcal{Q} d(\mathcal{C}_\mathcal{Q}) d(\mathcal{I}_\mathcal{Q}) \left(\frac{1}{12}\big(d(\mathcal{I}_\mathcal{Q})^2 - 1\big) + \mathcal{Y}_\mathcal{Q}^2 \right). \tag{3.4b}$$

Here $d(\cdot)$ indicates the dimension of the representation, $T(\mathcal{C}_\mathcal{Q})$ is the colour Dynkin index, and $q_\mathcal{Q}$ is the electromagnetic charge matrix of the fermion.

In the case in which the anomaly is generated by the SM quarks with a generation-independent PQ charge assignment, the QCD anomaly coefficient takes the form

$$N = n_g \frac{2X_{q_L} - X_{u_R} - X_{d_R}}{2}, \tag{3.5}$$

where $n_g = 3$ is the number of quark generations, and with $T(3) = 1/2$ being the Dynkin index for the fundamental representation.

3.2 Origin of model-dependent axion couplings

We have already observed that the coupling constants in the effective Lagrangian of eq. (2.41) receive both model-dependent and model-independent contributions. We discussed the origin of the model independent part in the previous chapter, and now we are ready to analyse the model-dependent terms.

The axion is the pNGB associated to the spontaneous breaking of an anomalous $U(1)_{\mathrm{PQ}}$. If we denote by J_μ^{PQ} the Noether current associated to the PQ symmetry, we know that this is conserved up to anomalous terms:

$$\partial^\mu J_\mu^{\mathrm{PQ}} = \frac{g_s^2 N}{16\pi^2} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} + \frac{e^2 E}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{3.6}$$

where N and E are, respectively, the QCD and QED anomaly coefficients. The Goldstone theorem implies that $\langle 0|J_\mu^{\mathrm{PQ}}|a\rangle = i v_a p_\mu$, where v_a is the order parameter of the SSB of the PQ symmetry. By anomaly matching, we find that the axion effective Lagrangian must contain terms of the form

$$\mathcal{L}_a \supset \frac{a}{v_a}\frac{g_s^2 N}{16\pi^2} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} + \frac{a}{v_a}\frac{e^2 E}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{v_a} J_\mu^{\mathrm{PQ}}, \tag{3.7}$$

where we also introduced a term describing the coupling of the axion to the PQ conserved current, which depends on the charges of the fields under PQ transformations. Choosing the same normalisation for the $aG\tilde{G}$ that was chosen in eq. (2.37), we find

$$f_a = \frac{v_a}{2N}, \tag{3.8}$$

which relates the axion decay constant to the order parameter v_a. Moreover, we can rewrite our Lagrangian: considering for simplicity just two chiral fermions $f_{L,R}$ charged under PQ, so that $J_\mu^{\mathrm{PQ}} = \bar{f}_L X_{f_L} \gamma_\mu f_L + \bar{f}_R X_{f_R} \gamma_\mu f_R$, eq. (3.2) becomes

$$\mathcal{L}_a \supset \frac{g_s^2}{32\pi^2}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + \frac{e^2}{32\pi^2}\frac{E}{N}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{f_a}\frac{1}{2N}\left[\bar{f}_L X_{f_L}\gamma_\mu f_L + \bar{f}_R X_{f_R}\gamma_\mu f_R\right] \equiv$$
$$\equiv \frac{g_s^2}{32\pi^2}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + \frac{1}{4}g^0_{a\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{2f_a}\bar{f}c^0_f\gamma^\mu\gamma_5 f, \tag{3.9}$$

where in the second step we neglected the coupling with the conserved vector current since it vanishes under integration by parts. Then the model-dependent axion couplings to photons and fermions have the form

$$g^0_{a\gamma} = \frac{\alpha}{2\pi f_a}\frac{E}{N}, \tag{3.10a}$$
$$c^0_f = \frac{X_{f_R} - X_{f_L}}{2N} = -\frac{X_{H_f}}{2N}, \tag{3.10b}$$

where in the last step we assumed a Yukawa interaction term with a PQ-charged scalar H_f of the kind $\bar{f}_L H_f f_R$, yielding the relation $X_{f_L} - X_{f_R} = X_{H_f}$.

3.3 PQWW axion model

Historically, the first axion model was proposed in the very papers where the axion itself was introduced [1, 2]. In this model, known as the *Peccei-Quinn-Weinberg-Wilczek model* (PQWW), the $U(1)_{\rm PQ}$ anomaly is generated by the usual SM quarks, and the scalar sector is extended by the introduction of a second Higgs doublet. The model is therefore an example of a *two Higgs doublet model* (2HDM). The necessity of extending the scalar sector can be understood by looking at the Yukawa terms. In the PQWW model these read

$$-\mathcal{L}_Y = \bar{q}_L Y_U H_u u_R + \bar{q}_L Y_D H_d d_R + \text{h.c.}, \tag{3.11}$$

where u_R, d_R are three-dimensional vectors whose components are the right-handed quarks, q_L is again a three-dimensional vectors, whose components are the left-handed quark doublets, $Y_{U,D}$ are the Yukawa matrices, and $H_{u,d}$ the two Higgs doublets. Denoting by X_{q_L}, X_{u_R}, X_{d_R}, and $X_{u,d}$ respectively the PQ charges of the left-handed quark doublets, of the up and down right handed quark fields, and of the two Higgs doublets, invariance of eq. (3.6) under PQ transformations yields

$$X_{q_L} - X_{u_R} = X_u \tag{3.12a}$$
$$X_{q_L} - X_{d_R} = X_d \tag{3.12b}$$

By summing the two expressions

$$X_u + X_d = 2X_{q_L} - X_{u_R} - X_{d_R} = \frac{2N}{n_g}, \tag{3.13}$$

which allows for a nonzero value of the QCD anomaly coefficient. The same is not true if we only have a single Higgs doublet H, yielding a Yukawa sector

$$-\mathcal{L}_Y = \bar{q}_L Y_U H u_R + \bar{q}_L Y_D \tilde{H} d_R + \text{h.c.}, \tag{3.14}$$

where $\tilde{H} = i\sigma^2 H^*$. In fact, since $X_H = -X_{\tilde{H}}$, PQ invariance would imply

$$0 = X_H + X_{\tilde{H}} = 2X_{q_L} - X_{u_R} - X_{d_R} = \frac{2N}{n_g}, \tag{3.15}$$

so that the PQ symmetry could not be anomalous.

Let us now assume that below some energy scale the two Higgs doublets acquire a nontrivial vacuum expectation value (VEV)

$$H_u \supset \frac{v_u}{\sqrt{2}} e^{i\frac{a_u}{v_u}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H_d \supset \frac{v_d}{\sqrt{2}} e^{i\frac{a_d}{v_d}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.16}$$

where we neglected the radial modes, since they do not play an important role in our discussion. The contribution of these fields to the PQ current is given by

$$J_\mu^{\rm PQ} \supset -X_u H_u^\dagger \overleftrightarrow{\partial}_\mu H_u - X_d H_d^\dagger \overleftrightarrow{\partial}_\mu H_d \supset \sum_{i=u,d} X_i v_i \partial_\mu a_i. \tag{3.17}$$

We can now define the axion field as

$$a = \frac{1}{v_a}(X_u v_u a_u + X_d v_d a_d), \quad v_a^2 = X_u^2 v_u^2 + X_d^2 v_d^2, \tag{3.18}$$

so that, in agreement with the Goldstone theorem $\langle 0|J_\mu^{\rm PQ}|a\rangle = i v_a p_\mu$. Moreover, we observe that under a PQ transformation $a_i \to a_i + \kappa X_i v_i$ the axion field transforms as $a \to a + \kappa v_a$.

One can then define the electroweak Higgs boson as a linear combination of H_u and H_d. In particular, the order parameter of electroweak symmetry breaking v can be written as

$$v^2 = v_u^2 + v_d^2. \tag{3.19}$$

In order to avoid a kinetic mixing between the axion and the Z boson, which would not allow us to integrate out the heavy Z boson when writing down the axion EFT, we require the PQ current to be orthogonal to the hypercharge current $J_\mu^Y\big|_a = \sum_i Y_i v_i \partial_\mu a_i$, which, if the two Higgs doublets carry opposite hypercharge, implies

$$\frac{X_u}{X_d} = \frac{v_d^2}{v_u^2}. \tag{3.20}$$

Combining these results, we can introduce an angle β such that

$$\sin\beta = \frac{v_u}{v}, \quad \cos\beta = \frac{v_d}{v} \implies \tan^2\beta = \frac{v_u^2}{v_d^2} = \frac{X_d}{X_u}. \tag{3.21}$$

One can then write down the order parameter for PQ symmetry breaking in terms of v as

$$v_a^2 = v^2 \sin^2(2\beta). \tag{3.22}$$

We therefore observe that in the PQWW model $v_a \sim v$. This possibility has been ruled out experimentally [44–47], and therefore the PQWW model has been abandoned in favour of the so-called *invisible axion models*, where the scale v_a is decoupled from v.

3.4 DFSZ axion model

The *Dine-Fishler-Srednicki-Zhitnitzky* (DFSZ) model is a generalisation of the PQWW one, in which the scalar sector is further extended by the introduction of a complex scalar singlet $\Phi \sim (1,1,0)_{\rm SM}$ of PQ charge X_Φ. The structure of the scalar potential in the model is

$$V(H_u, H_d, \Phi) = \tilde{V}_{\rm moduli}(|H_u|, |H_d|, |\Phi|, |H_u H_d|) + \lambda H_u H_d (\Phi^\dagger)^2 + \text{h.c.}, \tag{3.23}$$

where the last term represents a source of explicit breaking of the global $U(1)^3$ symmetry of the moduli terms into the product of two $U(1)$ subgroups to be identified with those associated to PQ and hypercharge symmetry:

$$U(1)_\Phi \times U(1)_{H_u} \times U(1)_{H_d} \to U(1)_{\rm PQ} \times U(1)_Y. \tag{3.24}$$

The PQ charge assignment of fermions is again assumed generation-independent, and it is related to that of the scalars through the Yukawa sector. There are two possible choices for the Yukawa terms in the lepton sector, defining two different DFSZ models:

- The DFSZ-I model corresponds to the choice

$$\mathcal{L}^{\rm Y}_{\text{DFSZ-I}} = -\bar{q}_L Y_U H_u u_R - \bar{q}_L Y_D H_d d_R - \bar{\ell}_L Y_E H_d e_R + \text{h.c.} \tag{3.25}$$

- The DFSZ-II model, instead, corresponds to the choice

$$\mathcal{L}^{\rm Y}_{\text{DFSZ-II}} = -\bar{q}_L Y_U H_u u_R - \bar{q}_L Y_D H_d d_R - \bar{\ell}_L Y_E \tilde{H}_u e_R + \text{h.c.} \tag{3.26}$$

By choosing appropriately the scalar potential, one can impose that the three scalars all acquire a nontrivial VEV

$$H_u \supset \frac{v_u}{\sqrt{2}} e^{i\frac{a_u}{v_u}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H_d \supset \frac{v_d}{\sqrt{2}} e^{i\frac{a_d}{v_d}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi \supset \frac{v_\Phi}{\sqrt{2}} e^{i\frac{a_\Phi}{v_\Phi}}, \tag{3.27}$$

where we assume $v_\Phi \gg v_{u,d}$, which by the identification $v^2 = v_u^2 + v_d^2$ implies $v_\Phi \gg v$, and neglected radial modes and modes charged under QED, which do not contain the axion.

As we did for the PQWW model, to identify the axion we consider the contribution of these scalars to the $U(1)_{\rm PQ}$ Noether current

$$J_\mu^{\rm PQ} \supset -X_\Phi \Phi^\dagger i\overleftrightarrow{\partial}_\mu \Phi - X_{H_u} H_u^\dagger i\overleftrightarrow{\partial}_\mu H_u - X_{H_d} H_d^\dagger i\overleftrightarrow{\partial}_\mu H_d \supset \sum_{i=\Phi,u,d} X_i v_i \partial_\mu a_i. \tag{3.28}$$

We define the axion field as

$$a = \frac{1}{v_a} \sum_{i=\Phi,u,d} X_i v_i a_i, \quad v_a^2 = \sum_{i=\Phi,u,d} X_i^2 v_i^2, \tag{3.29}$$

so that $J_\mu^{\rm PQ}\big|_a = v_a \partial_\mu a$ implying $\langle 0 | J_\mu^{\rm PQ}\big|_a | a \rangle = i v_a p_\mu$. Under a PQ transformation $a_i \to a_i + \kappa X_i v_i$ the axion thus defined transforms as expected $a \to a + \kappa v_a$.

We again require the orthogonality between $J_\mu^{\rm PQ}$ and the hypercharge current $J_\mu^Y\big|_a = \sum_i Y_i v_i \partial_\mu a_i$ to avoid axion mixing with the physical Z boson, which yields

$$\sum_{i=u,d} X_i v_i^2 = -X_u v_u^2 + X_d v_d^2 = 0. \tag{3.30}$$

From the PQ invariance of the term $H_u H_d (\Phi^\dagger)^2$ in the scalar potential one finds

$$X_u + X_d - 2X_\Phi = 0 \tag{3.31}$$

This latter result fixes the PQ charges of the scalars up to a normalisation. We choose to normalise them by requiring $X_\Phi = 1$; with this choice eq. (3.31) becomes $X_u + X_d = 2$, allowing us to write

$$X_u = 2\cos^2\beta, \quad X_d = 2\sin^2\beta \implies \tan^2\beta = \frac{X_u}{X_d}. \tag{3.32}$$

Combining this result with eq. (3.30) we find

$$\tan^2\beta = \frac{X_d}{X_u} = \frac{v_u^2}{v_d^2}. \tag{3.33}$$

We are now ready to understand the effect of the introduction of the scalar singlet Φ. From the definition of v_a, we find

$$v_a = v_\Phi + v^2 \sin^2(2\beta) \approx v_\Phi, \tag{3.34}$$

where we exploited the fact that $v_\Phi \gg v$. The presence of Φ therefore drives the value of v_a far from v, implementing the invisibility of the model.

We are now interested in finding the bare effective axion couplings. To do that we start by expressing $a_{u,d}$ in terms of a in the DFSZ Lagrangian, which after the selection of the axion-dependent terms, gives for the DFSZ-I model

$$\mathcal{L}_{\text{DFSZ-I}} \supset -m_U \bar{u}_L u_R e^{iX_u \frac{a}{v_a}} - m_D \bar{d}_L d_R e^{iX_d \frac{a}{v_a}} - m_E \bar{e}_L e_R e^{iX_d \frac{a}{v_a}} + \text{h.c.} \tag{3.35}$$

Then, if one performs the field redefinitions

$$u \to e^{-i\gamma_5 X_u \frac{a}{2v_a}} u, \quad d \to e^{-i\gamma_5 X_d \frac{a}{2v_a}} d, \quad e \to e^{-i\gamma_5 X_d \frac{a}{2v_a}} e, \tag{3.36}$$

the anomaly generates the $aG\tilde{G}$ and $aF\tilde{F}$ terms, with the anomaly coefficients which, from the general expressions of eq. (3.4), can be shown to take the values

$$N = n_g \left(\frac{X_u}{2} + \frac{X_d}{2} \right) = 3, \tag{3.37a}$$

$$E = n_g \left(3\left(\frac{2}{3}\right)^2 X_u + 3\left(-\frac{1}{3}\right)^2 X_d + (-1)^2 X_d \right) = 8. \tag{3.37b}$$

In particular, we observe that for this model $f_a = v_a/2N = v_a/6$.

The couplings with fermions, instead, come from the noninvariance of the kinetic terms under the transformation in eq. (3.36)

$$\delta(\bar{u}i\partial\!\!\!/ u) = X_u \frac{\partial_\mu a}{2v_a}\bar{u}\gamma^\mu\gamma_5 u = \left(\frac{1}{3}\cos^2\beta\right)\frac{\partial_\mu a}{2f_a}\bar{u}\gamma^\mu\gamma_5 u, \tag{3.38a}$$

$$\delta(\bar{d}i\partial\!\!\!/ d) = X_d \frac{\partial_\mu a}{2v_a}\bar{d}\gamma^\mu\gamma_5 d = \left(\frac{1}{3}\sin^2\beta\right)\frac{\partial_\mu a}{2f_a}\bar{d}\gamma^\mu\gamma_5 d, \tag{3.38b}$$

$$\delta(\bar{e}i\partial\!\!\!/ e) = X_d \frac{\partial_\mu a}{2v_a}\bar{e}\gamma^\mu\gamma_5 e = \left(\frac{1}{3}\sin^2\beta\right)\frac{\partial_\mu a}{2f_a}\bar{e}\gamma^\mu\gamma_5 e, \tag{3.38c}$$

yielding

$$c^0_{u_i} = \frac{1}{3}\cos^2\beta, \quad c^0_{d_i} = \frac{1}{3}\sin^2\beta, \quad c^0_{e_i} = \frac{1}{3}\sin^2\beta. \tag{3.39}$$

where $i = 1, 2, 3$ denotes the fermion generation. Of course, the discussion can be extended without any difficulty to the DFSZ-II model, where one can find that $E = 2$, $c^0_{e_i} = -\frac{1}{3}\cos^2\beta$.

By imposing a tree-level perturbative unitarity bound on Yukawa-mediated $2 \to 2$ fermion scattering amplitudes at $\sqrt{s} \gg m_{H_{u,d}}$, *i.e.* by imposing $|\mathrm{Re}\, a_{J=0}| < 1/2$, where $a_{J=0}$ is the $J = 0$ partial wave, one can find the allowed range for $\tan\beta$. Ignoring renormalisation group effect, and taking into account group theoretical factors [48, 49], we get [50] $y^{\mathrm{DFSZ}}_{t,b} < \sqrt{16\pi/3}$ from $Q_L\bar{u}_R \to Q_L\bar{u}_R$, with the initial state prepared into a $SU(3)_c$ singlet. The label DFSZ reminds us that these are not the Yukawa couplings of the SM, which in turn can be obtained as

$$y_t = y^{\mathrm{DFSZ}}_t \sin\beta, \quad y_b = y^{\mathrm{DFSZ}}_b \cos\beta. \tag{3.40}$$

This bound can be converted on a bound on $\tan\beta$:

$$\tan\beta \in [0.25, 170]. \tag{3.41}$$

This bound holds both for DFSZ-I and DFSZ-II models, since the τ Yukawa plays a subleading role for perturbativity.

3.5 KSVZ axion model

The *Kim-Shifman-Vainshtein-Zakharov* (KSVZ) model is the other benchmark invisible axion model. In this model, differently from the previously discussed ones, the PQ anomaly is generated by a Beyond the SM (BSM) vector-like fermion $\mathcal{Q} = \mathcal{Q}_L + \mathcal{Q}_R$ with zero bare mass, transforming as $\mathcal{Q} \sim (3, 1, 0)_{\mathrm{SM}}$ under $SU(3)_C \times SU(2)_L \times U(1)_Y$. The scalar sector of the SM is extended by the introduction of a complex scalar singlet $\Phi \sim (1, 1, 0)_{\mathrm{SM}}$, which, as we are going to see, is needed to make the model invisible.

The Lagrangian of the model reads

$$\mathcal{L}_{\mathrm{KSVZ}} = |\partial_\mu\Phi|^2 + \bar{\mathcal{Q}}iD\!\!\!\!/\,\mathcal{Q} - \left(\bar{\mathcal{Q}}_L Y_{\mathcal{Q}}\Phi\mathcal{Q}_R + \mathrm{h.c.}\right) - V(\Phi). \tag{3.42}$$

It features a symmetry under PQ transformations of the form $\Phi \to e^{i\alpha}\Phi$, $\mathcal{Q} \to e^{i\alpha\gamma_5}\mathcal{Q}$. The potential is defined in such a way to give rise to the SSB of the $U(1)_{\mathrm{PQ}}$ symmetry

$$V(\Phi) = \lambda_\Phi\left(|\Phi|^2 - \frac{v_a^2}{2}\right)^2, \tag{3.43}$$

with order parameter v_a which we assume much larger than the electroweak one v. We write the scalar field Φ as

$$\Phi = \frac{v_a + \rho_a}{\sqrt{2}} e^{i \frac{a}{v_a}} . \tag{3.44}$$

The Goldstone mode is identified with the axion field, while both the radial mode and the fermion $\mathcal{Q}$ acquire a mass: $m_{\rho_a} = \sqrt{2\lambda_\Phi} v_a$, $m_{\mathcal{Q}} = Y_{\mathcal{Q}} v_a / \sqrt{2}$.

In the broken phase, the Lagrangian contains a term

$$\mathcal{L}_{\text{KSVZ}} \supset -m_{\mathcal{Q}} \bar{\mathcal{Q}}_L \mathcal{Q}_R e^{i \frac{a}{v_a}} + \text{h.c.} \tag{3.45}$$

Performing the field redefinition

$$\mathcal{Q} \to e^{-i\gamma_5 \frac{a}{2v_a}} \mathcal{Q}, \tag{3.46}$$

the heavy fermion gets decoupled from the axion, and can be integrated out. Moreover, the anomaly of the transformation in eq. (3.46) generates the $aG\tilde{G}$ term in the axion effective Lagrangian

$$\delta\mathcal{L}_{\text{KSVZ}} = \frac{g_s^2}{32\pi^2} \frac{a}{f_a} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu}, \tag{3.47}$$

where we used the fact that, since $\mathcal{Q}$ is in the fundamental of $SU(3)_C$ and $X_{\mathcal{Q}} = 1$, we have $N = 1/2$ and therefore $f_a = v_a$.

The effective coupling of the axion to photons can be deduced from the same anomaly matching procedure described in the previous section. In particular since $E \propto \text{Tr}\, q_{\mathcal{Q}}^2 = 0$, being the fermion $\mathcal{Q}$ uncharged under $SU(2)_L \times U(1)_Y$, the axion-photon coupling vanishes. For what concerns the bare couplings to SM fermions, these too all vanish in the KSVZ model, because the PQ Noether current just depends of the BSM fermion $\mathcal{Q}$.

We mention that the KSVZ model can be generalised by considering BSM fermions $\mathcal{Q}$ transforming under a more general representation of the SM gauge group $\sum_{\mathcal{Q}} (\mathcal{C}_{\mathcal{Q}}, \mathcal{I}_{\mathcal{Q}}, \mathcal{Y}_{\mathcal{Q}})$, as long as at least one of the $\mathcal{C}_{\mathcal{Q}}$ is nontrivial, to allow for $N \neq 0$, see eq. (3.4a). It is also possible to define a set of phenomenological criteria to determine a preferred range of values for the ratio E/N in realistic models, the so-called *axion band*, see [51, 52].

Chapter 4

Axion astrophysics

As we have seen from the previous chapters, the experimental observation of the QCD axion would give us an important test of the PQ solution of the strong CP problem. Moreover, being the axion a very light and feebly interacting particle, it would certainly contribute to the observed dark matter (DM) relic density, thus solving, at least partially, two outstanding problems in present day high energy physics.

We have also seen how the model-independent physics of the axion is essentially determined just by one parameter, the axion decay constant f_a, which fixes the axion mass and its coupling constants to SM particles. It is then clear that, in order to design experiments aiming to the direct observation of the QCD axion, it is necessary to set bounds on the value of f_a, or, equivalently, on the values of the axion couplings to SM particles.

Among the various bounds on the axion parameter space, the most stringent ones are those stemming from astrophysical systems [38, 53–55]. The rationale behind these bounds is that, even though a direct measurement of the axion flux produced by these systems cannot be carried out due to the extremely feeble interactions of the axion with SM particles, the emission of axions would still have an effect on the evolution of the system, in the form, *e.g.*, of an increase in its cooling rate.

In this chapter we will review the main astrophysical bounds on the axion parameter space, with a particular focus on the supernova (SN) bound on the axion-nucleon couplings, which, as we will see, represents the main source of interest towards the role of finite density effects on the axion dynamics.

4.1 Stellar evolution

In this section, we aim to give a brief introduction to the topic of stellar evolution. This can be studied by looking at the so-called Hertzsprung-Russel (HR) or colour-magnitude (CM) diagram, in which the absolute magnitude of the star is represented as a function of the temperature of the star.

In the initial stage of their evolution, stars belong to the so-called *Main Sequence* (MS). MS stars are burning hydrogen in their core. The existence of this phase is largely independent from the initial mass, provided that this is above a minimal limit of roughly $0.1M_\odot$, where $M_\odot \simeq 2 \times 10^{30}\,\text{kg}$ is the solar mass. On the other hand, the qualitative evolution during the following stages is heavily determined by the initial mass of the star.

- If the mass is close to that of our Sun, when the hydrogen in the core starts to run out, the star passes first through a *sub-giant phase*, in which hydrogen is burnt in a thick shell, to move to the *Red Giant Branch* (RGB), where hydrogen fusion happens in a thin shell around an inert helium core. When the temperature becomes high enough to ignite helium burning in the stellar core (*He-flash*), the star moves to the *Horizontal Branch* (HB), and finally ends its life cycle by becoming a *White Dwarf* (WD), made up primarily of carbon and oxygen, since the temperature needed to ignite fusion of heavier nuclei is never reached. Stars with a mass a few times that of the Sun do not show a He-flash and ignite helium burning right after their MS stage, still leaving a WD as a remnant.

- For stars with mass $M \gtrsim 8M_\odot$, the gravitational attraction between the constituent particles is strong enough to cause a core collapse, followed by a recoil shock triggering an explosive event known as *type-II supernova* (SN). After the supernova, the stellar remnant can either be a *neutron star* (NS) or a *black hole* (BH), depending on the initial mass of the star.

4.2 Main astrophysical axion bounds

4.2.1 White Dwarf bound

The White Dwarf Luminosity Function (WDLF) gives the number density of WDs in terms of their brightness, expressed in terms of absolute bolometric magnitudes $M_{\rm bol}$. If the WD birth rate is assumed to be constant, the slope of this curve gives the cooling rate. In the early stages, while the WD is still very hot, the main cooling channel is through neutrinos from the plasmon decay $\gamma \to \bar{\nu}\nu$. However, while the star cools down, the neutrino emission is suppressed, as the efficiency of the neutrino cooling depends steeply on the temperature. Eventually, the photon cooling becomes predominant. If axions exist, they would also contribute to the cooling. Interestingly, the axion effect would be predominant in the time between the neutrino and the photon cooling. In [56] an analytic expression for the WDLF was obtained

$$\frac{\mathrm{d}N}{\mathrm{d}M_{\rm bol}} = B_3\, 2.2\times 10^{-4}\,\mathrm{pc}^{-3}\,\mathrm{mag}^{-1} \frac{10^{-4M_{\rm bol}/35}L_\odot}{78.7L_\odot 10^{-2M_{\rm bol}/5} + L_\nu + L_a}\left(\frac{M}{M_\odot}\right)^{5/7}\sum_j \frac{X_j}{A_j}, \tag{4.1}$$

where X_j is the mass fraction of element j with atomic mass A_j, B_3 is the birth rate normalised to $10^{-3}\,\mathrm{pc}^{-3}\mathrm{Gyr}^{-1}$, L_ν the neutrino luminosity, and L_a the axion luminosity.

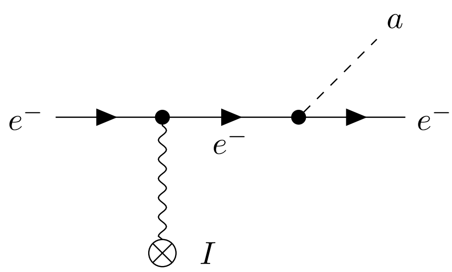


Figure 4.1: Axion production by electron bremsstrahlung

Production of axions through electron bremsstrahlung $e^- I \to I e^- a$, see Fig 4.1 would change the WDLF both in shape and in amplitude. We can therefore exploit this fact to set a bound on c_e. The current bound was found in [57] and reads

$$g_{aee} < 2.8\times 10^{-13} \quad (99\%\ \text{C.L.}), \tag{4.2}$$

where $g_{aee} = c_e m_e / f_a$.

4.2.2 Red Giant bound

As we described in Sec. 4.1, after a star runs out of hydrogen in its core, it moves from the MS to the RGB until it becomes hot enough to trigger helium fusion, reaching the HB. The tip of the RGB (TRGB), *i.e.* the point of the HR diagram at which helium burning is ignited, heavily depends on the density and the temperature of the star. So, if an additional cooling channel is introduced, like axion production by electron bremsstrahlung, the star will grow brighter and more massive before the He-flash.

The main cooling channel near the TRGB in standard RG models is given by neutrino emission by plasmon decay $\gamma \to \bar{\nu}\nu$. A bound on g_{aee} was found in [58] by studying the TRGB of stars in a sample of 22 Globular Clusters and reads

$$g_{aee} < 1.48 \times 10^{-13} \quad (95\% \text{ C.L.}). \tag{4.3}$$

4.2.3 Horizontal Branch bound

Stars in the HB have an inner core in which helium undergoes fusion, with production of carbon and oxygen, and an outer shell in which hydrogen is still burning. The effect of additional energy loss due to Primakoff emission of axions $\gamma\, Ze \to Ze\, a$, see Fig. 4.2 would be that of decreasing the lifetime of the HB star, due to an acceleration in helium consumption.

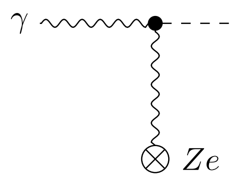


Figure 4.2: Axion production by Primakoff process

One can introduce a parameter R defined as the ratio between the number of stars in the HB and the number of stars in the RGB that are brighter than those in the HB, which is equal to the ratio of the respective lifetimes

$$R = \frac{N_{\text{HB}}}{N_{\text{RGB}}} = \frac{t_{\text{HB}}}{t_{\text{RGB}}}. \tag{4.4}$$

The parameter R depends both on $g_{a\gamma}$ and g_{aee}, since g_{aee} conditions the evolution in the RGB, while $g_{a\gamma}$ influences the lifetime in the HB. An analytic expression of R can be found in [59]. In a model with no tree-level couplings to electrons, as in KSVZ models, one can then translate any bound on R on a bound on $g_{a\gamma}$. In the DFSZ model, in which axions couple to electrons at tree level, a bound on R just yields a contraint on a combination of $g_{a\gamma}$ and g_{aee}.

The current bound, stemming from the analysis of a sample of 39 Globular Clusters, was obtained in [60, 61]:

$$g_{a\gamma} < 0.65 \times 10^{-10}\,\text{GeV}^{-1}, \quad (95\% \text{ C.L.}). \tag{4.5}$$

For KSVZ axion this is equivalent to $f_a > 3.4 \times 10^7\,\text{GeV}$.

4.2.4 Neutron Star bound

The cooling rate of NSs can be used to constrain the axion-nucleon bound. In particular, according to the current models, NSs cool down by neutrino emission in the first stages of their life, while photon emission dominates later on. The introduction of an additional nucleon bremsstrahlung $NN \to NNa$ cooling channel should change the cooling rate of the NS, allowing to set a bound on the relevant axion-neutron coupling c_n.

The most stringent bound comes from from the Magnificent Seven NSs together with PSR J0659, and, in the case of the DFSZ model, reads [62]

$$m_a < \left(33 - 17\sin^2\beta - 1.8\sin^4\beta\right)\mathrm{meV}, \tag{4.6}$$

and varies between 33 and 14 meV for $0 \leq \sin^2\beta \leq 1$.

4.3 Supernova bound

4.3.1 Core-collapse supernova

As anticipated in Section 4.1, when a particular fuel goes off in a star, the lack of the pressure which was generated by the nuclear reactions breaks the hydrostatic equilibrium, so that the star undergoes gravitational collapse. Usually, this collapse heats the stellar core enough to ignite the fusion of a new element. However, successive burning stages require hotter and hotter temperatures to overcome the larger and larger Coulomb barriers between heavier elements. For stars with mass $M < 8M_\odot$, the energy provided by the gravitational collapse never reaches the value needed to ignite carbon fusion, and the star ends up becoming a carbon-oxygen WD sustained by electron degeneracy pressure.

However, if the mass of the original star was higher, $M \gtrsim 8M_\odot$, the fusion of heavier elements can be initiated, until a Fe core is generated. Nuclear fusion of iron is not energetically favoured, so that this Fe core never ignites. Hence, the gravitational collapse of the iron core will eventually lead it to reach the Chandrasekhar mass, at which electron degeneracy pressure is not enough to overcome the gravitational pressure, and to become a proto-neutron star (PNS).

However, this collapse gets halted very rapidly. This is due, on the one hand, to the fact that the equation of state of the PNS becomes stiffer at higher density, due to the repulsive nature of short-range strong interactions between nuclei, and on the other hand to the pressure coming from the neutrons becoming degenerate. This generates a recoil shock wave in the inner region of the PNS, which is rapidly transferred to the outer core, losing energy through dissociation of iron nuclei in their constituent nucleons. Hence, the shock wave loses intensity and is counterbalanced by the pressure stemming from the accretion of the PNS by gravitational attraction of the mass surrounding it.

However, the high temperature and density yield the production of a large number of neutrinos, which then scatter and thermalise with the nuclear medium. The neutrino heating, together with other effects, allows to break the balance and trigger an explosion, see [63] for more details. This explosive event is known as type-II supernova or *core-collapse supernova*. The neutrinos are then emitted from a thin layer, known as the *neutrinosphere*, beyond which they do not undergo further scattering. This blackbody-like neutrino signature will be fundamental in Section 4.3.2. In the meantime, the density of the PNS core reaches the *nuclear saturation density* $n_0 = 0.16\,\mathrm{fm}^{-3}$, *i.e.* the baryon number density found in atomic nuclei. [1] The extremely high density of the PNS makes its inner core completely opaque to all species of neutrinos, making it the only astrophysical system, except for the early universe, in which neutrinos are trapped.

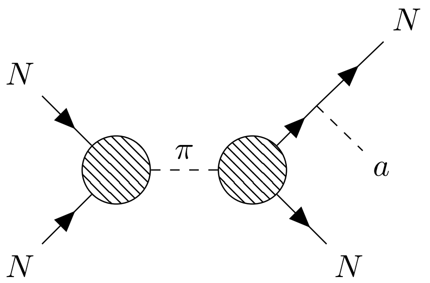
The mass of the PNS core then determines the fate of the stellar remnant. If this is small, then we can model the matter within the PNS as a mixture of a nonrelativistic Fermi gas of nucleons and a ultrarelativistic Fermi gas of electrons. Since the electrons which could be emitted by β-decay of the nonrelativistic neutrons would have a small momentum, falling well within the Fermi sphere, this decay is suppressed. Then, electron capture $e^- p \to n\nu_e$ is favoured, increasing the neutron fraction in the system. The gravitational collapse is then counterbalanced by the neutron degeneracy pressure, giving rise to a stable system, a neutron star.

If the PNS has a gravitational mass larger than $2.1M_\odot - 2.4M_\odot$ then the degeneracy pressure is insufficient to balance the gravitational pull, and the collapsing matter shrinks to a radius smaller than the Schwarzschild radius, generating a black hole.

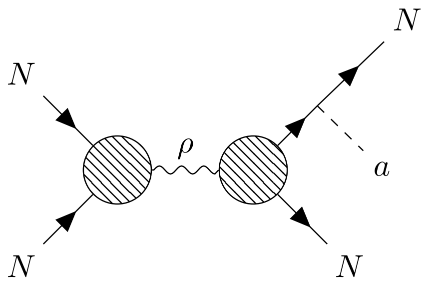
4.3.2 The SN1987A axion bound

On the 23rd of February of 1987 a flux of neutrinos associated to the SN1987A core-collapse supernova event were observed in the IBM, Kamiokande-II and BUST experiments [12–17]. The dominant detection channel was inverse β-decay $\bar{\nu}_e p \to ne^+$. The observed flux is in good agreement with theoretical models [54], and can be used to set a bound on the axion parameter space.

As we have seen from the description of the type-II supernova in the previous section, the main energy loss channel for the SN is given by the emission of neutrinos. The existence of another light and feebly intercating particle, like the axion, which can be ejected from the PNS core, would provide an additional cooling channel for the SN, which would affect the observed flux of neutrinos.



(a) OPE contribution



(b) ρ exchange contribution

Figure 4.3: Main contributions to axion production by nucleon bremsstrahlung

One of the main production channels for the axion within the SN is given by *nucleon bremsstrahlung* $NN \to NNa$, see Fig 4.3, and the energy loss due to axion emission is therefore determined by the value of the axion-nucleon couplings $c_{p,n}$. An upper bound on the energy loss per unit mass due to

[1] The number density of an atomic nucleus is given by

$$n = \frac{A}{\frac{4\pi}{3}R^3},$$

where A is the mass number and R is the nuclear radius. This, for typical nuclei, satisfies $R \approx r_0 A^{1/3}$, where $r_0 \simeq 1.25\,\mathrm{fm}$. It follows that, for typical nuclei

$$n = \frac{3}{4\pi r_0^3} \equiv n_0^{\mathrm{theory}} \simeq 0.122\,\mathrm{fm}^{-3}.$$

The observed value of n_0 is $n_0^{\mathrm{exp.}} \simeq 0.16\,\mathrm{fm}^{-3}$.

axion production was obtained in [64] assuming a free-streaming regime for axion propagation within the SN. This reads

$$\varepsilon_a \lesssim 1 \times 10^{19}\,\mathrm{erg\,g^{-1}\,s^{-1}}. \tag{4.7}$$

The current SN bound stemming from nucleon bremsstrahlung was obtained by *ab initio* methods in [18] where the nucleon-nucleon interactions were described by taking into account both a one pion exchange (OPE) contribution and a two pion exchange one, modelled as a ρ exchange, see Fig. 4.3. It can be written as

$$m_a < \frac{7.0\,\mathrm{meV}}{\sqrt{c_p^2 + 1.64 c_n^2 + 0.87 c_p c_n}}. \tag{4.8}$$

For KSVZ axions, this is equivalent to

$$f_a > 4 \times 10^8\,\mathrm{GeV}. \tag{4.9}$$

In the last years another axion production process has been taken into account for the supernova bound [19, 65] (see also [66–68]); this is *pion-nucleon scattering* $\pi^- p \to na$, see Fig. 4.4. Even though the number density of one species of thermal pions in the SN is expected to be $n_\pi = 3 \times 10^{-15}\,\mathrm{fm}^{-3} \ll n_0$, the pions contribute to the thermodynamic equilibrium between nucleons and the chemical potential of charged pions is $\mu_\pi = \mu_n - \mu_p$, favouring negative pions over positive ones. This generates an enhancement in the abundance of pions approximately given by $e^{\beta\mu_\pi}$, yielding a sizeable rate for $\pi^- p \to na$.

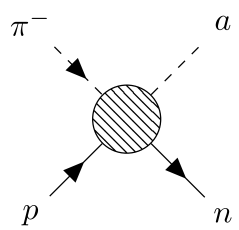


Figure 4.4: Pion-nucleon scattering axion production channel

The bound on the axion-nucleon couplings due to pion-nucleon scattering can be expressed as follows [69]

$$\bar{g}_{aNN} < 0.6 \times 10^{-9}, \tag{4.10}$$

where

$$\bar{g}_{aNN} = \frac{1}{2}\big(g_{app}^2 + g_{ann}^2\big)\big(1 + g_A^{-4}\big) + \frac{1}{3} g_{app} g_{ann}\big(1 - 3g_A^{-4}\big), \qquad g_{aNN} = \frac{c_N m_N}{f_a}. \tag{4.11}$$

We finally mention that the *electric dipole portal* term in the axion effective Lagrangian

$$\mathcal{L}_a \supset -\frac{i}{2} c_{aN\gamma}\, \mu_N \frac{a}{f_a} \bar{N} \gamma_5 \sigma^{\mu\nu} N F_{\mu\nu}, \tag{4.12}$$

where $\mu_N = e/2m_p$ is the nuclear magneton, introduces an emission channel $\gamma N \to Na$ for the axion, see [70], which becomes dominant in the SN only if the couplings between axions and nucleons are somehow suppressed. Axion models in which this is the case will be described in Chapter 6.

4.3.3 Relevance of finite density effects for the SN axion bound

As we have already discussed, the SN is an extremely dense object, whose baryonic number density is of the order of the nuclear saturation density n_0. The production of axions inside a SN will therefore be influenced in a non-negligible way by the dense nuclear medium. It is therefore clear that, in order to obtain a reliable bound from the cooling argument, it is necessary to determine how finite density corrections affect axion-nucleon interactions.

To find the bounds presented in the previous section, the presence of nonvanishing chemical potentials for proton and neutron numbers, μ_p and μ_n, was considered to find the expression of axion emissivity, *i.e.* the energy emitted through the axion cooling channel per unit time and volume. In particular, it was necessary to properly take into account Pauli blocking, through factors depending on the Fermi-Dirac distribution function [18, 19, 71]

$$f_N = \frac{1}{\exp[\beta(E(p) - \mu_N)] + 1}, \tag{4.13}$$

where, working in the nonrelativistic limit for nucleons

$$E(p) = \sqrt{p^2 + m_N^{*2}} \approx m_N + \frac{\vec{p}^2}{2m_N} + U, \tag{4.14}$$

where

$$m_N^* = m_N + \Sigma_S, \quad U = \Sigma_S + \Sigma_V, \tag{4.15}$$

Σ_S and Σ_V being respectively the scalar and vector contribution to the nucleon self-energy.

However, this calculation did not take into account the modifications to the axion-nucleon couplings which are introduced when the interactions happen inside a highly dense medium, and which can sizeably modify the SN bound. Taking into account such corrections will be the main focus of the rest of this work, following the steps outlined in Ref. [20]. In particular, in the next chapter we will develop the formalism needed to describe the in-medium corrections to the axion-nucleon couplings, and in Chapter 6 we will apply this formalism to study the behaviour at finite density of a particularly interesting class of axion models.

Chapter 5

Finite density effects in axion physics

In this chapter we want to analyse how the behaviour of the axion is modified in systems at finite baryonic density n, following Ref. [20]. Our main focus will be on densities in the proximity of the *nuclear saturation density* $n_0 = 0.16\,\mathrm{fm}^{-3}$, *i.e.* the baryonic density of standard nuclear matter, which is the expected density in astrophysical systems of interest in axion physics like supernovae (SNe). At this density, quarks are still confined in hadrons, so that a description of strong interactions in terms of hadronic degrees of freedom is still possible.

5.1 Quark condensates at finite density

A finite baryonic density affects axion physics in two main ways: it modifies the axion potential and it changes the axion-nucleon couplings. To understand how these effects appear, we start from the fact that, as explained in detail in Appendix C, the $SU(3)_L \times SU(3)_R$-breaking quark condensate $\langle \bar{q}_R q_L \rangle$ can be parametrised as as

$$\begin{gathered} \left\langle \bar{q}_R^i q_L^j \right\rangle \equiv \langle \bar{q}q \rangle_0 \, (\Sigma_0)^{ij} \\ \Sigma_0 = \cos\vartheta \mathbb{1}_3 + i \sin\vartheta \vec{n} \cdot \vec{\lambda}, \quad |\vec{n}| = 1, \end{gathered} \tag{5.1}$$

where $\langle \bar{q}q \rangle_0$ is a constant number, $\vartheta \in [-\pi/2, \pi/2)$, $\vec{\lambda} = (\lambda^1, \ldots, \lambda^8)$ are the Gell-Mann matrices, $i, j = u, d, s$ are flavour indices, and Σ_0 transforms as $\Sigma_0 \to L \Sigma_0 R^\dagger$. The matrix Σ_0 characterises the *orientation of the ensemble average*, which for the usual QCD vacuum is trivial, $\Sigma_0 = \mathbb{1}_3$.

The change in the axion potential is then due to two effects: on the one hand, as we are going to see, the presence of baryons gives corrections to $\langle \bar{q}q \rangle_0$, whose value acquires a dependence on density; on the other hand, at high enough density, *kaon condensation* can occur, see Appendix C. The effect of kaon condensation is that of giving a nontrivial orientation to the QCD vacuum, $\Sigma_0 \neq \mathbb{1}_3$, so that the axion potential acquires a dependence on the orientation of the ground state. Since these effects are expected to appear at densities higher than those we are interested in, as explained in [20], we will neglect the effect of kaon condensation and set $\Sigma_0 = \mathbb{1}_3$.

The explicit density dependence of the chiral condensates can be obtained, following [20], by exploiting the Hellmann-Feynman theorem [72]

$$\zeta_{\bar{q}q}(n) \equiv \frac{\langle \bar{q}q \rangle_n}{\langle \bar{q}q \rangle_0} = 1 + \frac{1}{\langle \bar{q}q \rangle_0} \frac{\partial \Delta E(n)}{\partial m_q}, \quad q = u, d, s. \tag{5.2}$$

In this formula, $\Delta E(n)$ represents the shift in the QCD ground state energy due to the presence of the nucleonic background. This receives two contributions: a first one from the free baryonic Fermi gas E^{free} and a second from nuclear interactions E^{int}.

We take the so-called *linear approximation* for the chiral condensates, which is obtained by neglecting both interactions and relativistic corrections. Next, we discuss the validity of this approximation. In this limit, in fact, $\Delta E = \sum_{x=n,p} m_x n_x$, so that

$$\zeta_{\bar{q}q}(n) = 1 + \frac{1}{\langle \bar{q}q \rangle_0} \sum_x n_x \frac{\partial m_x}{\partial m_q}, \quad q = u, d, s. \tag{5.3}$$

We observe that the change in the quark condensates is entirely determined by the derivatives $\partial m_x / \partial m_q$. Even though there are six of these derivatives, being $x = n, p$ and $q = u, d, s$, only three of them turn out to be independent. These can be described in the isospin basis for the quark masses

$$\bar{m} \equiv \frac{1}{2}(m_u + m_d), \quad \Delta m \equiv \frac{1}{2}(m_u - m_d), \tag{5.4}$$

by introducing the sigma terms

$$\sigma_{\pi N} \equiv \bar{m}\left(\frac{\partial m_p}{\partial \bar{m}}\right) = \bar{m}\left(\frac{\partial m_n}{\partial \bar{m}}\right), \tag{5.5a}$$

$$\tilde{\sigma}_{\pi N} \equiv \Delta m\left(\frac{\partial m_n}{\partial \Delta m}\right) = -\Delta m\left(\frac{\partial m_p}{\partial \Delta m}\right), \tag{5.5b}$$

$$\sigma_s \equiv m_s\left(\frac{\partial m_p}{\partial m_s}\right) = m_s\left(\frac{\partial m_n}{\partial m_s}\right), \tag{5.5c}$$

such that

$$m_n = m_N + \sigma_{\pi N} + \tilde{\sigma}_{\pi N} + \sigma_s, \tag{5.6a}$$

$$m_p = m_N + \sigma_{\pi N} - \tilde{\sigma}_{\pi N} + \sigma_s, \tag{5.6b}$$

where m_N is the nucleon mass in the chiral limit $m_q \to 0$, $q = u, d, s$.

By using the relation $\langle \bar{q}q \rangle_0 (m_u + m_d) = -m_\pi^2 f_\pi^2$, we can then rewrite the $\zeta_{\bar{q}q}(n)$ as

$$\zeta_{\bar{u}u}(n) = 1 - b_1 \frac{n}{n_0} + b_2\left[2\frac{n_p}{n} - 1\right]\frac{n}{n_0}, \tag{5.7a}$$

$$\zeta_{\bar{d}d}(n) = 1 - b_1 \frac{n}{n_0} - b_2\left[2\frac{n_p}{n} - 1\right]\frac{n}{n_0}, \tag{5.7b}$$

$$\zeta_{\bar{s}s}(n) = 1 - b_3 \frac{n}{n_0}, \tag{5.7c}$$

where $n = n_p + n_n$ and we defined the b-terms

$$b_1 \equiv \frac{\sigma_{\pi N} n_0}{m_\pi^2 f_\pi^2} = 3.5 \times 10^{-1}\left(\frac{\sigma_{\pi N}}{45\,\mathrm{MeV}}\right), \tag{5.8a}$$

$$b_2 \equiv \frac{\tilde{\sigma}_{\pi N} n_0}{m_\pi^2 f_\pi^2}\frac{\bar{m}}{\Delta m} = -2.2 \times 10^{-2}\left(\frac{\tilde{\sigma}_{\pi N}}{1\,\mathrm{MeV}}\right), \tag{5.8b}$$

$$b_3 \equiv \frac{\tilde{\sigma}_s n_0}{m_\pi^2 f_\pi^2}\frac{2\bar{m}}{m_s} = 1.7 \times 10^{-2}\left(\frac{\sigma_s}{30\,\mathrm{MeV}}\right). \tag{5.8c}$$

We observe that the $\langle \bar{s}s \rangle_n$ condensate is only weakly affected by the nucleonic background. Moreover,

$\langle \bar{u}u \rangle_n \approx \langle \bar{d}d \rangle_n$ up to a small isospin correction [73].

We now want to discuss the validity of the linear approximation we have used, in order to give an estimate of the densities up to which our results can be trusted, as done in [20]. First of all, we should include relativistic corrections. In the fully relativistic limit, the free part of the energy of a fermion x is given by

$$E_x^{\text{free}} = 2\int^{k_F^x} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sqrt{k^2+m_x^2} = m_x n_x F(k_F^x/m_x), \tag{5.9a}$$

$$F(q) = \frac{3q\sqrt{q^2+1}\left(2q^2+1\right) - 3\sinh^{-1}(q)}{8q^3} = 1 + \frac{3q^2}{10} + \mathcal{O}(q^4), \tag{5.9b}$$

where k_F^x is the Fermi momentum, $k_F^x = \sqrt{m_x^2 - \mu_x^2}$, which fixes the number density

$$n_x = 2\int^{k_F^x} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} = \frac{(k_F^x)^3}{3\pi^2}. \tag{5.10}$$

We note that relativistic corrections become important at large densities, where also the corrections from nuclear interactions become important. These are mainly due to one pion exchange, but they also come from four-nucleon contact interactions. They clearly become sizeable when $n_x/\Lambda_\chi f_\pi^2$ becomes of order one, where $\Lambda_\chi \simeq 700\,\mathrm{MeV}$ is the energy scale at which ChPT breaks down. This density corresponds to that at which ChPT is beyond control, $k_F^x \sim \Lambda_\chi$. In addition, since $\Lambda_\chi \approx m_p \approx m_n$, relativistic corrections are approximately controlled by the same expansion parameter

$$\frac{k_F^2}{\Lambda_\chi^2} \approx \frac{(3\pi^2 n/2)^{2/3}}{\Lambda_\chi^2} \approx (15\%)\left(\frac{n}{n_0}\right)^{2/3}\left(\frac{700\,\mathrm{MeV}}{\Lambda_\chi}\right)^2, \tag{5.11}$$

where we took $k_F = k_F^p \sim k_F^n$. It follows that the best way to assess the validity of our approximation is to compute the relevant NLO corrections. This has been done in [74–78]. These authors found $\mathcal{O}(1)$ deviations from the linear approximation for densities slightly above n_0.

5.2 Axion-nucleon couplings at finite density

In Sec. 2.2.3 we introduced the axion-nucleon interactions by matching the axion-dressed HBChPT Lagrangian

$$\mathcal{L}_{aN} = \bar{N}\left(v\cdot\partial + 2g_A c_- \frac{\partial_\mu a}{2f_a} S^\mu \tau^3 + 2g_0^{ud} c_+ \frac{\partial_\mu a}{2f_a} S^\mu\right) N \tag{5.12}$$

to the effective axion-quark Lagrangian. The result of this matching were the expressions of the axion-nucleon couplings

$$(c_p)_0 = (g_A)_0 (c_-)_0 + (g_0^{ud})_0 (c_+)_0, \tag{5.13a}$$

$$(c_n)_0 = -(g_A)_0 (c_-)_0 + (g_0^{ud})_0 (c_+)_0, \tag{5.13b}$$

where we introduced the index "0" to underline that these expressions hold at zero density, and where

$$(g_A)_0 = (\Delta u)_0 - (\Delta d)_0, \quad (g_0^{ud})_0 = (\Delta u)_0 + (\Delta d)_0, \quad s^\mu \Delta q = \langle p|\bar{q}\gamma^\mu\gamma_5 q|p\rangle\,, \tag{5.14}$$

with values $(g_A)_0 = 1.2723(23)$ and $(g_0^{ud})_0 = 0.521(53)$.

In a three flavour formalism, the same approach gives us expressions taking into account also a correction stemming from the contribution of the strange quark

$$(c_p)_0 = (g_A)_0(c_-)_0 + (g_0^{ud})_0(c_+)_0 + (\Delta s)_0(c_s)_0, \tag{5.15a}$$

$$(c_n)_0 = -(g_A)_0(c_-)_0 + (g_0^{ud})_0(c_+)_0 + (\Delta s)_0(c_s)_0, \tag{5.15b}$$

where $(\Delta s)_0 = -0.026$. In these expressions the c_q are the axion-quark couplings, which, as we found in Sec. 2.2 have the form $c_q = c_q^0 - [Q_a]_q$ where c_q^0 is the model-dependent axion-quark coupling of eq. (3.10b) and Q_a is the transformation matrix introduced in eq. (2.39). The specific form of Q_a is chosen so as to remove axion-pion mass mixing (see Sec. 2.2.1), and reads

$$(Q_a^*)_0 = \frac{\mathrm{diag}\,(1, z, zw)}{1 + z + zw}, \tag{5.16}$$

where $z = m_u/m_d = 0.47^{+0.06}_{-0.07}$ and $w = m_d/m_s = (17-22)^{-1}$.

Since we are working in an effective theory at energies way lower than the PQ-breaking scale f_a at which these axion-quark couplings are defined, we need to take into account renormalisation group (RG) effects. Following [79], we employ the approximate formula, keeping only the leading top quark contribution,

$$c_q^{\mathrm{IR}} \approx c_q + r_q^t(m_{\mathrm{BSM}})c_t, \tag{5.17}$$

where c_q^{IR} is the axion-quark coupling at the IR scale, which we take to be $2\,\mathrm{GeV}$, c_t is the axion coupling to the top quark at the scale f_a, and m_{BSM} is the heavy mass scale associated to the Higgs doublets of our axion model. Assuming $m_{\mathrm{BSM}} \simeq f_a$ and $f_a \sim 10^{12}\,\mathrm{GeV}$, the function $r_q^t(m_{\mathrm{BSM}})$ takes the values

$$r_u^t(f_a) = -0.291859, \quad r_d^t(f_a) = 0.294052. \tag{5.18}$$

where u and d stand for up-type and down-type quarks.

Taking into account these RG effects, the axion-nucleon couplings take the form

$$(c_p^{\mathrm{IR}})_0 = (g_A)_0(c_-^{\mathrm{IR}})_0 + (g_0^{ud})_0(c_+^{\mathrm{IR}})_0 + (\Delta s)_0(c_s^{\mathrm{IR}})_0, \tag{5.19a}$$

$$(c_n^{\mathrm{IR}})_0 = -(g_A)_0(c_-^{\mathrm{IR}})_0 + (g_0^{ud})_0(c_+^{\mathrm{IR}})_0 + (\Delta s)_0(c_s^{\mathrm{IR}})_0. \tag{5.19b}$$

Now we are interested in analysing how these couplings change in presence of nonzero baryon number and electromagnetic charge densities. The effects of finite density are twofold, in fact we now want to show that both the axion-quark couplings and the hadronic matrix elements acquire a density dependence.

5.2.1 In-medium mixing angles

The change in the chiral condensates at finite density has an important consequence: it modifies the scalar potential correcting the quark masses as $m_q \to (\langle \bar{q}q \rangle_n / \langle \bar{q}q \rangle_0)m_q$. This implies that, at finite density, the form of the matrix Q_a which diagonalises the axion-pion mixing matrix changes, becoming

$$(Q_a^*)_n = \frac{\mathrm{diag}\,(1, zZ, zZwW)}{1 + zZ + zZwW}. \tag{5.20}$$

Here,

$$Z = \frac{\langle \bar{u}u \rangle_n}{\langle \bar{d}d \rangle_n} = 1 - 2b_2 \frac{n - 2n_p}{n_0}, \tag{5.21a}$$

$$W = \frac{\langle \bar{d}d \rangle_n}{\langle \bar{s}s \rangle_n} = 1 - \left[b_1 - b_2 \left(1 - \frac{2n_p}{n} \right) - b_3 \right] \frac{n}{n_0}, \tag{5.21b}$$

where we made use of eq. (5.7).

This, of course, yields a change in the in-medium values of the axion-quark couplings, which now read

$$(c_q^{\mathrm{IR}})_n \approx (c_q)_n + r_q^t(m_{\mathrm{BSM}})c_t = \left(c_q^0 - [(Q_a^*)_n]_q \right) + r_q^t(m_{\mathrm{BSM}})c_t, \tag{5.22}$$

where we neglected the density dependence of c_t.

5.2.2 In-medium matrix elements

We are now interested in describing the in-medium corrections to the hadronic matrix elements of eq. (5.14). Let us start by observing that Δs is already subleading at zero density with respect to g_A and g_0^{ud}, and since finite density corrections are expected to be even smaller we can safely neglect them and work with the zero density value $(\Delta s)_0$. We therefore need to find a suitable formalism to obtain the in-medium corrections to g_A and g_0^{ud}. As shown in sec. 2.2.3, they can be identified with the low energy constants (LECs) appearing in the Baryon ChPT Lagrangian describing the coupling of the nucleonic isotriplet and isoscalar axial currents to external fields, which in our case, as shown in eq. (5.12), describe the axion.

We first of all outline the programme to obtain the in medium corrections to g_A: we start by considering the matrix elements of the spatial components[1] $J_5^{i,a}$ of the isotriplet axial current in Baryon ChPT in the presence of the corresponding external field and in the nonrelativistic limit, taken between the initial and final one-nucleon states $|\alpha\rangle$ and $|\beta\rangle$; this provides us the very definition of g_A in vacuum. Following [80], we then include the effects of the nuclear medium by exploiting an *independent particle approximation*, *i.e.* by using as initial and final states the simple products of $|\alpha\rangle$ and $|\beta\rangle$ with a Fock state $|F\rangle$ describing the Fermi sea

$$|F\rangle = \prod_{h \in F} a_h^\dagger |0\rangle , \tag{5.23}$$

where $|0\rangle$ is the vacuum state, and h denotes a one-nucleon state in the Fermi sea. We then end up with matrix elements between two-nucleon states, which need to be antisymmetrised to impose the Fermi statistics:

[1]We consider only the spatial components since, in the nonrelativistic limit, the time component turns out to be higher order in m_N^{-1}. In fact, from the expansion in the Dirac basis

$$u_r(p) = \frac{\not{p} + m_N}{\sqrt{2m_N(E_{\vec{p}} + m_N)}} \begin{pmatrix} \varphi_r \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2m_N(E_{\vec{p}} + m_N)}} \begin{pmatrix} (E_{\vec{p}} + m_N)\varphi_r \\ \vec{p} \cdot \vec{\sigma} \varphi_r \end{pmatrix}$$

where $r = 1, 2$, $\varphi_1 = (1, 0)^T$, $\varphi_2 = (0, 1)^T$ one finds

$$\bar{u}_{r'}(p')\gamma^0\gamma_5 u_r(p) = \frac{1}{2m_N} \varphi_{r'}^T \left(\vec{p}^{\,'} \cdot \vec{\sigma} + \vec{p} \cdot \vec{\sigma} \right) \varphi_r = \frac{\vec{P} \cdot \vec{\sigma}}{m_N}$$

where $\vec{P} = (\vec{p} + \vec{p}^{\,'})/2$.

$$\langle\beta;F|J_5^{i,a}|\alpha;F\rangle = \sum_{h\in F}\Big[\langle\beta,h|J_5^{i,a}|\alpha,h\rangle - \langle\beta,h|J_5^{i,a}|h,\alpha\rangle\Big]. \tag{5.24}$$

Here the first term is known as the *Hartree term*, while the second one is called the *Fock term.* Working in the limit in which the momentum carried by the external field goes to zero, the momenta of the one-nucleon states $|\alpha\rangle\,,|\beta\rangle$ must lie on the surface of the Fermi sphere, so that

$$\vec{p}_\alpha = \vec{p}_\beta \equiv \vec{p}, \quad |\vec{p}| = k_F, \tag{5.25}$$

where k_F if the Fermi momentum. Then, by averaging over the direction of $\vec{p}$ we obtain the desired correction

$$\int \frac{\mathrm{d}\Omega_{\hat{p}}}{4\pi}\,\langle\beta;F|J_5^{i,a}|\alpha;F\rangle = \delta g_A \tau^{\pm}_{\beta\alpha}\sigma^i_{\beta\alpha}, \quad \delta g_A = \delta^H g_A + \delta^F g_A, \tag{5.26}$$

where τ^a, with $a=1,2,3$, are the isospin Pauli matrices, $\tau^{\pm}=(\tau^1\pm i\tau^2)/2$, σ^i, with $i=1,2,3$, are the spin Pauli matrices, $\tau^{\pm}_{\beta\alpha} = \langle\beta|\tau^{\pm}|\alpha\rangle$, $\sigma^i_{\beta\alpha} = \langle\beta|\sigma^i|\alpha\rangle$, and we split the correction δg_A into its Hartree and Fock contributions. This method was exploited in [81] to explain the observed quenching of β-decay rates.

To carry on this calculation explicitly, we first of all need to write down the relevant terms of the Baryon ChPT Lagrangian [2], see Appendix A

$$\mathcal{L}_{\pi N} = \mathcal{L}_0 + \mathcal{L}_1 \tag{5.27a}$$

$$\mathcal{L}_0 \supset \bar{B}[iv\cdot D + 2i(g_A)_0 S\cdot\Delta]B \tag{5.27b}$$

$$\begin{aligned}\mathcal{L}_1 =& \bar{B}\left[\frac{v^\mu v^\nu - \eta^{\mu\nu}}{2m_N}D_\mu D_\nu + 4c_3 i\Delta\cdot i\Delta + \left(2c_4 + \frac{1}{2m_N}\right)[S^\mu,S^\nu][i\Delta_\mu, i\Delta_\nu]\right]B \\ &- 4id_1\bar{B}S\cdot\Delta B\bar{B}B + 2id_2\varepsilon^{abc}\varepsilon_{\mu\nu\lambda\kappa}v^\mu\Delta^{\nu,a}\bar{B}S^\lambda\tau^b B\bar{B}S^\kappa\tau^c B + \cdots\,.\end{aligned} \tag{5.27c}$$

Here, v^μ is the four-velocity of the nucleon field B, S^μ is the spin operator, and we work in the rest frame of the nucleon, where $v^\mu = (1,\vec{0})$ and $S^\mu = (0,\vec{\sigma}/2)$. Moreover,

$$\xi = \exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{2f_\pi}\right), \tag{5.28a}$$

$$D_\mu B = (\partial_\mu + \Gamma_\mu)B, \tag{5.28b}$$

$$\Gamma_\mu = \frac{1}{2}\Big[\xi^\dagger,\partial_\mu\xi\Big] - \frac{i}{2}\xi^\dagger\frac{\tau^a}{2}a^a_\mu\xi + \frac{i}{2}\xi\frac{\tau^a}{2}a^a_\mu\xi^\dagger, \tag{5.28c}$$

$$\Delta_\mu = \frac{1}{2}\Big\{\xi^\dagger,\partial_\mu\xi\Big\} - \frac{i}{2}\xi^\dagger\frac{\tau^a}{2}a^a_\mu\xi - \frac{i}{2}\xi\frac{\tau^a}{2}a^a_\mu\xi^\dagger, \quad \Delta_\mu = \frac{\tau^a}{2}\Delta^a_\mu, \tag{5.28d}$$

where a^a_μ is the isotriplet axial vector external field. The index n in $\mathcal{L}_n$ denotes the order in the following power counting scheme. A Feynman diagram for an A-nucleon process scales as Q^ν, where Q is a small momentum, and

$$\nu = 4 - A - 2C + 2L + \sum_i V_i\Delta_i, \qquad \Delta_i = d_i + \frac{n_i}{2} - 2, \tag{5.29}$$

where C is the number of separately connected subdiagrams, L is the number of loops, V_i is the

[2] From now on we will denote three-vectors in Minkowski space by an arrow, $\vec{v}$, while we will denote vectors in isospin space by a bold letter, $\mathbf{v}$.

number of vertices of type i, d_i is the number of derivatives or powers of m_π in the vertex of type i, and n_i is the number of nucleon lines attached to a vertex of type i. The Lagrangian $\mathcal{L}_n$ has only vertices with $\Delta_i = n$.

As a last observation we note that, in the nonrelativistic limit, the d_1 term and the d_2 term in eq. (5.27c) turn out to be equivalent, as shown in [82], so that we can substitute these terms with a single one

$$\mathcal{L}_1 \supset -i\frac{c_D}{f_\pi^2\Lambda_\chi}\bar{B}S\cdot\Delta B\bar{B}B, \tag{5.30}$$

where $c_D = 4f_\pi^2\Lambda_\chi(d_1+2d_2)$.

The leading order contributions to the axial current coupling to a_μ^a come from the $\Delta_i = -1$ and $\Delta_i = 0$ two-body currents. The spatial components of the former are actually vanishing in the rest frame of the nucleon, since they turn out to be proportional to $v^\mu = (1,\vec{0})$. So, the first nontrivial in-medium corrections to g_A come from the $\Delta_i = 0$ two-body currents, which receive contributions from one pion exchange (OPE) and four-nucleon contact graphs, see fig. 5.1.

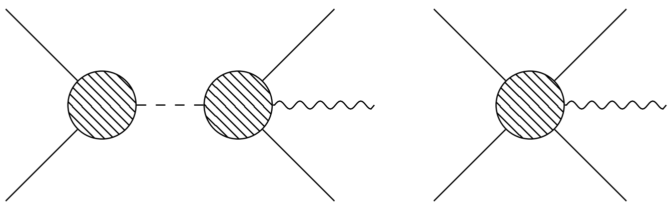


Figure 5.1: Leading order contributions to the in-medium corrections to g_A: OPE graphs and four-nucleon contact graphs in presence of the external field a_μ^a. Figure inspired by fig. 1 of [80]

By expanding ξ up to second order in f_π^{-1}

$$\begin{aligned}\xi &= \exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{2f_\pi}\right) \approx \mathbb{1} + i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{2f_\pi} - \frac{(\boldsymbol{\tau}\cdot\boldsymbol{\pi})^2}{8f_\pi^2},\\ \xi^\dagger &= \exp\left(-i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{2f_\pi}\right) \approx \mathbb{1} - i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{2f_\pi} - \frac{(\boldsymbol{\tau}\cdot\boldsymbol{\pi})^2}{8f_\pi^2}\end{aligned} \tag{5.31}$$

we can then obtain an expansion of Γ_μ and Δ_μ:

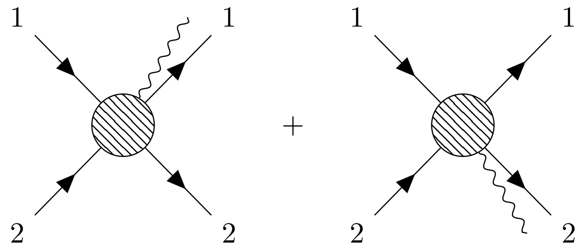
$$\Gamma_\mu \approx \frac{i}{2f_\pi}(\boldsymbol{\pi}\times\boldsymbol{\tau})\cdot\boldsymbol{a}_\mu + \frac{i}{4f_\pi^2}(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}, \tag{5.32a}$$

$$\Delta_\mu \approx -\frac{i}{2}(\boldsymbol{\tau}\cdot\boldsymbol{a}_\mu) + i\frac{\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}}{2f_\pi} - \frac{1}{4f_\pi^2}\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}. \tag{5.32b}$$

By substituting these expansion in the Lagrangian of eq. (5.27) we obtain the relevant interaction terms for our calculation. In particular, the four-nucleon contact term is given by

$$\mathcal{L}_1 \supset -\frac{c_D}{4f_\pi^2\Lambda_\chi}\bar{B}\sigma^i\tau^a a_i^a B\bar{B}B \tag{5.33}$$

The relevant diagrams are



, (5.34)

corresponding to the amplitude

$$i\mathcal{M} = -i\frac{c_D}{4f_\pi^2\Lambda_\chi}\left[\bar{u}_1\sigma_1^i\tau_1^a a_i^a u_1\bar{u}_2 u_2 + \bar{u}_2\sigma_2^i\tau_2^a a_i^a u_2\bar{u}_1 u_1\right]. \tag{5.35}$$

In the nonrelativistic limit $\bar{u}u \to \mathbb{1}$, so that

$$i\mathcal{M} = -i\frac{c_D}{4f_\pi^2\Lambda_\chi}\left(\sigma_1^i\tau_1^a + \sigma_2^i\tau_2^a\right)a_i^a. \tag{5.36}$$

The OPE contributions instead come from the terms

$$\begin{aligned}\mathcal{L}_{\pi N} \supset\, & 2i(g_A)_0\bar{B}S\cdot\Delta B + \bar{B}\bigg[\frac{v^\mu v^\nu - \eta^{\mu\nu}}{2m_N}D_\mu D_\nu + 4c_3 i\Delta\cdot i\Delta \\ & + \left(2c_4 + \frac{1}{2m_N}\right)[S^\mu, S^\nu][i\Delta_\mu, i\Delta_\nu]\bigg]B\end{aligned} \tag{5.37}$$

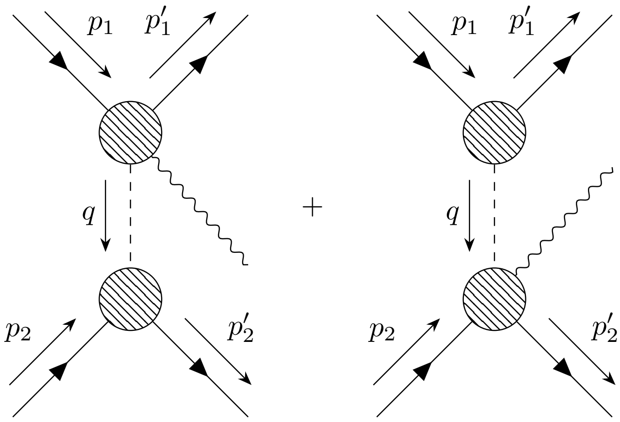
Upon substitution of the expansion of Δ_μ, the first term gives us the πNN vertex without external field instertions:

$$\mathcal{L}_{\pi N} \supset -\frac{(g_A)_0}{2f_\pi}\bar{B}\sigma^i\tau^a\partial_i\pi^a B \tag{5.38}$$

The πNN vertex with one external field insertion can instead be obtained from the other terms:

$$\begin{aligned}\mathcal{L}_{\pi N} \supset\, & \frac{i}{4m_N f_\pi}\left[\bar{B}(\boldsymbol{\pi}\times\boldsymbol{\tau})^a a_i^a\partial^i B - \partial^i\bar{B}(\boldsymbol{\pi}\times\boldsymbol{\tau})^a a_i^a B\right] \\ & - \frac{2c_3}{f_\pi}\bar{B}a_\mu^a\partial^\mu\pi^a B + \frac{1}{f_\pi}\left(c_4 + \frac{1}{4m_N}\right)\bar{B}\varepsilon^{ijk}\varepsilon^{abc}a_i^a\partial_j\pi^b\tau^c\sigma^k B\end{aligned} \tag{5.39}$$

The relevant diagrams are



, (5.40)

with corresponding amplitude in the nonrelativistic limit

$$\begin{aligned} i\mathcal{M} = &\left\{ -\frac{i^2}{2}\frac{(g_A)_0}{2m_N f_\pi^2}\frac{1}{\vec{q}^2+m_\pi^2}(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2)^a\left(P_1^i\vec{\sigma}_2\cdot\vec{q}+P_2^i\vec{\sigma}_1\cdot\vec{q}\right)\right. \\ &- i\frac{(g_A)_0}{f_\pi^2}c_3\frac{1}{\vec{q}^2+m_\pi^2}\left(q^i\vec{\sigma}_2\cdot\vec{q}\tau_2^a+q^i\vec{\sigma}_1\cdot\vec{q}\tau_1^a\right) \\ &\left. - i\frac{(g_A)_0}{2f_\pi^2}\left(c_4+\frac{1}{4m_N}\right)\frac{1}{\vec{q}^2+m_\pi^2}(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2)^a\left[(\vec{\sigma}_1\times\vec{q})^i\vec{\sigma}_2\cdot\vec{q}+(\vec{\sigma}_2\times\vec{q})^i\vec{\sigma}_1\cdot\vec{q}\right]\right\}a_i^a, \end{aligned} \tag{5.41}$$

where

$$\begin{aligned} \vec{q} &= \vec{p}_2^{\,'}-\vec{p}_2=\vec{p}_1-\vec{p}_1^{\,'}, \\ \vec{P}_{1,2} &= \frac{\vec{p}_{1,2}+\vec{p}_{1,2}^{\,'}}{2}. \end{aligned} \tag{5.42}$$

By combining these results, we finally obtain the expression of the axial current

$$\begin{aligned} \vec{J}_5^{\pm}(\vec{q}) = &-\frac{(g_A)_0}{m_N f_\pi^2}\frac{1}{\vec{q}^2+m_\pi^2}\left\{\frac{i}{2}(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2)^{\pm}\left(\vec{P}_1\vec{\sigma}_2\cdot\vec{q}+\vec{P}_2\vec{\sigma}_1\cdot\vec{q}\right)+2\bar{c}_3\vec{q}\left(\tau_1^{\pm}\vec{\sigma}_1\cdot\vec{q}+\tau_2^{\pm}\vec{\sigma}_2\cdot\vec{q}\right)\right. \\ &\left.+\left(\bar{c}_4+\frac{1}{4}\right)(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2)^{\pm}[(\vec{\sigma}_1\times\vec{q})(\vec{\sigma}_2\cdot\vec{q})-(\vec{\sigma}_2\times\vec{q})(\vec{\sigma}_1\cdot\vec{q})]\right\} \\ &-\frac{2(g_A)_0}{m_N f_\pi^2}\bar{c}_D\left(\tau_1^{\pm}\vec{\sigma}_1+\tau_2^{\pm}\vec{\sigma}_2\right), \end{aligned} \tag{5.43}$$

where we defined

$$\bar{c}_{3,4}=m_N c_{3,4}, \quad \bar{c}_D=\frac{m_N}{4(g_A)_0\Lambda_\chi}c_D. \tag{5.44}$$

We now consider the limit $\vec{P}_1=\vec{P}_2=\vec{0}$. In this limit, the Hartree contribution to the matrix element vanishes, since the condition $2\vec{P}_1=\vec{p}_\alpha+\vec{p}_\beta=\vec{0}$ is incompatible with eq. (5.25). So, we just need to consider the Fock contribution. Given a state h in the Fermi sea with momentum $\vec{k}$, observing that, due to rotational invariance each spatial component of the current must give the same contribution to $\delta^F g_A$, and using the identities $\sum_i q^i(\vec{\sigma}\cdot\vec{q})=-\sum_i\sigma^i\vec{q}^2$ and $(\vec{\sigma}_1\times\vec{q})^i\vec{\sigma}_2\cdot\vec{q}-(\vec{\sigma}_2\times\vec{q})^i\vec{\sigma}_1\cdot\vec{q}=-4iq^i(\vec{\sigma}\cdot\vec{q})+4i\vec{q}^2\sigma^i$, one finds that

$$\langle\beta,h|J_5^{i,\pm}|h,\alpha\rangle=-\frac{4(g_A)_0}{m_N f_\pi^2}\tau_{\beta\alpha}^{\pm}\sigma_{\beta\alpha}^i\left[-\frac{1}{3}\left(\bar{c}_3-2\bar{c}_4-\frac{1}{2}\right)\frac{\vec{q}^2}{\vec{q}^2+m_\pi^2}+\bar{c}_D\right], \tag{5.45}$$

so that, by summing over all the $h\in F$, or, equivalently, by integrating over the Fermi sea, one finds

$$\begin{aligned} \frac{\delta g_A}{(g_A)_0} &= \frac{4}{m_N f_\pi^2}\int\frac{\mathrm{d}\Omega_{\hat{p}}}{4\pi}\int^{k_F}\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left[-\frac{1}{3}\left(\bar{c}_3-2\bar{c}_4-\frac{1}{2}\right)\frac{\vec{q}^2}{\vec{q}^2+m_\pi^2}+\bar{c}_D\right] \\ &= -\frac{4}{m_N f_\pi^2}\int^{k_F}\frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\left[-\frac{1}{3}\left(\bar{c}_3-2\bar{c}_4-\frac{1}{2}\right)\frac{\vec{q}^2}{\vec{q}^2+m_\pi^2}+\bar{c}_D\right], \end{aligned} \tag{5.46}$$

where we used the fact that $\vec{q}=\vec{k}-\vec{p}$. Now, the integral can be solved as

$$\begin{aligned}\int^{k_F}\frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\frac{\vec{q}^2}{\vec{q}^2+m_\pi^2}&=\int^{k_F}\frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}-\int^{k_F}\frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\frac{m_\pi^2}{\vec{q}^2+m_\pi^2}\\&=\frac{k_F^3}{6\pi^2}-\frac{m_\pi^3}{2\pi^2}\int_0^{\frac{k_F}{m_\pi}}\mathrm{d}w\,\frac{w^2}{1+w^2}\\&=\frac{k_F^3}{6\pi^2}-\frac{m_\pi^2 k_F}{2\pi^2}+\frac{m_\pi^3}{2\pi^2}\arctan\left(\frac{k_F}{m_\pi}\right),\end{aligned}\tag{5.47}$$

where $w = q/m_\pi$. Hence,

$$\frac{\delta g_A}{(g_A)_0}=\frac{n}{m_N f_\pi^2}\left[\bar{c}_D-\frac{I(m_\pi/k_F)}{3}\left(2\bar{c}_4-\bar{c}_3+\frac{1}{2}\right)\right],\tag{5.48}$$

where n is the density of the Fermi gas $n = 2k_F^3/3\pi^2$, and

$$I(x)=1-3x^2+3x^3\arctan\left(\frac{1}{x}\right).\tag{5.49}$$

Finally, one can write down the expression of g_A as a function of n

$$\frac{(g_A)_n}{(g_A)_0}=1+\frac{n}{f_\pi^2\Lambda_\chi}\left[\frac{c_D}{4(g_A)_0}-\frac{I(m_\pi/k_F)}{3}\left(2\hat{c}_4-\hat{c}_3+\frac{\Lambda_\chi}{2m_N}\right)\right],\tag{5.50}$$

where we defined $\hat{c}_{3,4} = c_{3,4}\Lambda_\chi$.

The values of the LECs can be obtained by a fit of the amplitudes for πN scattering obtained through HBChPT on experimental data: the values reported in [81] are

$$c_D=-0.85\pm 2.15,\quad (2\hat{c}_4-\hat{c}_3)=9.1\pm 1.4.\tag{5.51}$$

Using these values, in [20] it was found that

$$\frac{(g_A)_n}{(g_A)_0}\approx 1-(30\pm 20)\%\frac{n}{n_0}.\tag{5.52}$$

A similar calculation could be performed to obtain, from the operators

$$\mathcal{L}_{\pi N}\supset-\frac{4\tilde{c}_3}{\Lambda_\chi}\bar{B}\Delta^\mu\tilde{\Delta}_\mu B-\frac{4\tilde{c}_4}{\Lambda_\chi}\bar{B}[S^\mu,S^\nu]\Delta_\mu\tilde{\Delta}_\nu B-i\frac{\tilde{c}_D}{f_\pi^2\Lambda_\chi}\bar{B}S\cdot\tilde{\Delta}B\bar{B}B,\tag{5.53}$$

the expression of g_0^{ud} as a function of density. However, the LECs appearing in the Lagrangian are currently not known, so that we can at most parametrise our ignorance, as in [20], by introducing an unknown parameter κ such that

$$\frac{(g_0^{ud})_n}{(g_0^{ud})_0}\approx 1+\kappa\frac{n}{n_0}.\tag{5.54}$$

Using the behaviour of g_A as a reference, we will set $\kappa = \pm 0.3$ as the benchmark values of κ.

Finally, one can write down the density-dependent axion-nucleon couplings as

$$(c_p)_n\simeq(g_A)_n(c_-^{\mathrm{IR}})_n+(g_0^{ud})_n(c_+^{\mathrm{IR}})_n+(\Delta s)_0(c_s^{\mathrm{IR}})_n,\tag{5.55a}$$

$$(c_n)_n\simeq-(g_A)_n(c_-^{\mathrm{IR}})_n+(g_0^{ud})_n(c_+^{\mathrm{IR}})_n+(\Delta s)_0(c_s^{\mathrm{IR}})_n,\tag{5.55b}$$

Chapter 6

Finite density effects in nucleophobic axion models

The aim of this chapter is to study finite density effects for the so-called *nucleophobic axion models*, introduced for the first time in [21]. These are UV completions of the axion EFT characterised by a suppression of the axion-nucleon couplings g_{aN}, a condition known as *nucleophobia*. We will show that nucleophobia can only be implemented in *nonuniversal* axion models, *i.e.* models with fermion generation-dependent PQ charge assignment. Nucleophobic axion model turn out to be very interesting due to their phenomenological consequences. In particular, the nucleophobia condition allows to relax the SN and NS bounds on the axion parameter space, see Chapter 4

An interesting subclass of nucleophobic models is given by the *astrophobic axion models*, in which also the axion-electron coupling is suppressed (*electrophobia*), allowing to relax also the WD and RG bounds and thus yielding an axion in the heavy mass window $m_a \sim \mathcal{O}(0.1\,\text{eV})$, about one order of magnitude heavier than in standard axion models.

It was shown in [21] that the nucleophobia conditions can be realised in a consisten way in vacuum; however, since the main feature of such models is to relax the astrophysical bounds stemming from SNe and NSs, it is of the utmost importance to test whether these conditions survive finite density effects.

The first section of the chapter will be devoted to a formal introduction to these models, whilst in the second section we will consider the effects of finite density corrections on the nucleophobia condition.

6.1 Nucleophobic axion models

6.1.1 Nucleophobia and nonuniversality

Our starting point is the effective axion Lagrangian for the two-flavours case

$$\begin{aligned}\mathcal{L} \supset\, & \frac{\alpha_S}{8\pi}\frac{a}{f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + \frac{\alpha}{8\pi}\frac{E}{N}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} \\ & + \frac{\partial_\mu a}{2f_a}\sum_{Q=U,D}\left[\bar{Q}_L\frac{X_{Q_L}}{N}\gamma^\mu Q_L + \bar{Q}_R\frac{X_{Q_R}}{N}\gamma^\mu Q_R\right],\end{aligned} \tag{6.1}$$

where N and E are respectively the QCD and QED anomaly coefficients, and f_a is the axion decay constant $f_a = v_a/2N$, where $v_a = \sqrt{2}\,\langle\Phi\rangle$ and Φ is $U(1)_{\text{PQ}}$-breaking singlet field, see Chapter 3.

$Q_{L,R} = U_{L,R}, D_{L,R}$ are vectors containing respectively the left-handed and right-handed quarks of the three generations and $X_{Q_{L,R}}$ are the corresponding PQ charges. In the KSVZ model these are vanishing, while in the DFSZ model they are nonzero but universal, *i.e.* independent of the quark generation.

We now want to show that nucleophobia cannot be realised in universal models. We start, as usual by removing the axion-gluon term by the field redefinition of the first generation quarks

$$q_{L,R} \to e^{\mp i \frac{a}{2f_a} f_q} q_{L,R}, \quad q = u, d, \tag{6.2}$$

where $f_q = [Q_a]_q$ and we choose Q_a in such a way to remove axion-pion mixing. Then, by defining $z = m_u/m_d$, one has

$$f_u = \frac{1}{1+z} \approx \frac{2}{3}, \quad f_d = \frac{z}{1+z} \approx \frac{1}{3}, \quad \mathrm{Tr}[Q_a] = f_u + f_d = 1. \tag{6.3}$$

After the rotation (6.2), the coefficient of the QED term changes as $E/N \to E/N + f_\gamma$, with $f_\gamma \simeq 1.92$, while the axion coupling to the first generation quarks becomes

$$\begin{aligned}\mathcal{L}_{aq} &= \frac{\partial_\mu a}{2f_a} \sum_{q=u,d} \left[\bar{q}\gamma^\mu\gamma_5 \left(\frac{X_{q_R} - X_{q_L}}{2N} - f_q \right) q \right] \\ &\equiv \frac{\partial_\mu a}{2f_a} \sum_{q=u,d} \left[\bar{q}\gamma^\mu\gamma_5 \left(c_q^0 - f_q \right) q \right],\end{aligned} \tag{6.4}$$

where the vector coupling vanishes due to the equations of motion and we introduced the axion-quark couplings as outlined in Section 2.2. The axion-nucleon couplings can then be obtained by matching the nonrelativistic axion-nucleon Lagrangian with (6.4) over a single nucleon matrix element, as seen in Section 2.2.3. The resulting interaction terms have the form

$$\mathcal{L}_{aN} = \frac{\partial_\mu a}{2f_a} c_N \bar{N}\gamma^\mu\gamma_5 N, \tag{6.5}$$

with $N = n, p$. The matching gives

$$c_p + c_n = 0.50(5)\left(c_u^0 + c_d^0 - 1\right) - 2\delta_s, \tag{6.6a}$$

$$c_p - c_n = 1.273(2)\left(c_u^0 - c_d^0 - f_{ud}\right), \tag{6.6b}$$

with $f_{ud} = f_u - f_d \approx 1/3$, and the 1 in eq. (6.6a) is just $f_u + f_d$. δ_s is a correction dominated by the strange quark contribution.

The conditions for nucleophobia are

$$c_p + c_n \approx 0, \tag{6.7a}$$

$$c_p - c_n \approx 0, \tag{6.7b}$$

which, due to eq. (6.6), can be rewritten as

$$c_u^0 + c_d^0 \approx 1, \tag{6.8a}$$

$$c_u^0 - c_d^0 \approx f_{ud}. \tag{6.8b}$$

To understand why these conditions require nonuniversal PQ charge assignment we have to look at the QCD anomaly coefficient. Recalling eq. (3.4a), this is given by

$$N = \sum_{i=1}^{3} \sum_{f_i} X_{f_i} d(\mathcal{I}_{f_i}) T(3), \tag{6.9}$$

where $d(\mathcal{I}_{f_i})$ is the dimension of the weak isospin representation for the quark f_i, while $T(3) = 1/2$ is the colour Dynkin index for the fundamental representation. We then find that N can be expressed as the sum of contributions coming from each irreducible representation of the SM gauge group (all taken to be left-handed, so that $X_{f_L} = -X_{f_R}$)

$$2N = \sum_{i=1}^{3} (X_{u_i} + X_{d_i} - 2X_{q_i}), \tag{6.10}$$

where u_i, d_i are the right-handed chiral fields, and q_i the left-handed ones. The contribution from first generation quarks is

$$2N_\ell = X_{u_1} + X_{d_1} - 2X_{q_1}. \tag{6.11}$$

The condition of eq. (6.8a) can then be written as

$$c_u^0 + c_d^0 = \frac{N_\ell}{N} \approx 1. \tag{6.12}$$

One then observes that

- In the KSVZ model $c_u^0 = c_d^0 = 0$, so

$$c_p + c_n = -0.50 - 2\delta_s \neq 0, \tag{6.13a}$$

$$c_p - c_n = -1.273 \times \frac{1}{3} \neq 0. \tag{6.13b}$$

- In the DFSZ model, $c_u^0, c_d^0 \neq 0$, but their value is independent of the fermion generation, so that eq. (6.12) implies

$$c_u^0 + c_d^0 = \frac{N_\ell}{N} = \frac{1}{3} \neq 1. \tag{6.14}$$

So, we understand that nucleophobia requires the model to be nonuniversal.

6.1.2 Two Higgs doublet models

Nucleophobia in 2HDMs

We can now proceed and build explicitly a nucleophobic axion model, following the steps outlined in [21]. The defining properties of a nucleophobic model are just the conditions (6.7), so we are left with a great amount of freedom in the choice of the specific model. We will make two simplifying assumptions:

i) The scalar sector is composed of the PQ-breaking Higgs singlet Φ with PQ charge X_Φ and two Higgs doublets $H_{1,2}$ with PQ charges $X_{1,2}$ and hypercharge $Y = -1/2$. The doublets interact with the SM quarks via Yukawa operators, so that we can relate their PQ charges to those of the quarks by imposing PQ invariance of the Yukawa terms;

ii) The PQ charges are assigned in such a way to allow all SM Yukawa operators.

The first assumption defines what are known as *two Higgs doublets models* (2HDMs).

The SM quarks are described as usual, with the left-handed components collected into $SU(2)_L$ doublets q_i and the right-handed components u_i, d_i that transform as $SU(2)_L$ singlets. One then has

$$c^0_{u_i} = \frac{X_{u_i} - X_{q_i}}{2N}, \quad c^0_{d_i} = \frac{X_{d_i} - X_{q_i}}{2N}. \tag{6.15}$$

We now have to assign PQ charges to the quarks. To do that we start by observing that a generic PQ charge matrix for a left-handed or right-handed quark generation Q can be written as

$$X_Q = X^0_q \mathbb{1}_3 + X^8_q \lambda_8 + X^3_q \lambda_3, \tag{6.16}$$

where $\mathbb{1}_3 = \mathrm{diag}(1,1,1)$, $\lambda_8 = \mathrm{diag}(1,1,-2)$, $\lambda_3 = \mathrm{diag}(1,-1,0)$. Under the assumptions above one can show that two generations must have the same charges, so that we can drop the term proportional to λ_3. Hence, in general

$$X_Q = X^0_q \mathbb{1}_3 + X^8_q \lambda_8. \tag{6.17}$$

This matrix satisfies an $SU(2)$ symmetry on the generation indices $\{1,2\}$ and we therefore refer to this structure as *2+1*.

We now note that the nucleophobia condition of eq. (6.12) can be satisfied if either

i) $N_2 = -N_3, \quad N = N_1,$

ii) $N_1 = N_2 = 0, \quad N = N_3,$

where we used two different orderings for the fermion generations in the two cases, in particular having the light quarks correspond to the first generation in i) and to the third generation in ii).

To study which Yukawa structures can enforce either of these conditions we exploit the $SU(2)$ symmetry in the quark PQ charge matrices and therefore consider just one generation in *2* together with the generation in *1* carrying the $\{3\}$ index ad write

$$\begin{aligned} &\bar{q}_2 u_2 H_1, \quad \bar{q}_3 u_3 H_a, \quad \bar{q}_2 u_3 H_b, \qquad \bar{q}_3 u_2 H_{1+a-b}, \\ &\bar{q}_2 d_2 \tilde{H}_c, \quad \bar{q}_3 d_3 \tilde{H}_d, \quad \bar{q}_2 d_3 \tilde{H}_{d+a-b}, \quad \bar{q}_3 d_2 \tilde{H}_{c-a+b}, \end{aligned} \tag{6.18}$$

where $\tilde{H} = i\sigma^2 H^*$, and we assigned H_1 to the first term without loss of generality, while the other Higgs indices must take values in $\{1, 2\}$. One can see that in each line the charges of the first three quark bilinears determine the fourth one, while the third term of the second line is obtained by equating $X_{q_3} - X_{q_2}$ as extracted from the second and third terms of both lines.

Denoting the Higgs ordering in the two lines by their indices, we have

i) For the case $N_1 = N_2 = -N_3$ there are two models:

M1: $(1, 2, 1, 2)_u\,(2, 1, 2, 1)_d$,

M2: $(1, 2, 2, 1)_u\,(2, 1, 1, 2)_d$;

ii) For the case $N_1 = N_2 = 0$, $N = N_3$ there are two models:

M3: $(1, 1, 1, 1)_u\,(1, 2, 2, 1)_d$,

M4: $(1, 2, 2, 1)_u\,(1, 1, 1, 1)_d$.

Let us now discuss the second nucleophobia condition (6.8b). We set $\tan\beta = v_2/v_1$, with $v_{1,2} = \sqrt{2}\,\langle H_{1,2}\rangle$, and we introduce the shorthand notation $s_\beta = \sin\beta$, $c_\beta = \cos\beta$. Imposing the orthogonality condition between the $U(1)_{\rm PQ}$ and $U(1)_Y$ Noether currents, we find

$$X_1 v_1^2 + X_2 v_2^2 = 0, \tag{6.19}$$

yielding

$$\frac{X_1}{X_2} = -\tan^2\beta. \tag{6.20}$$

The charge normalisation is given in terms of the light quark anomaly

$$X_2 - X_1 = \pm 2N_\ell, \tag{6.21}$$

where the upper signs correspond to models M1, M2, M3 and the lower sign to model M4. In all cases

$$c_u^0 - c_d^0 = -\frac{X_1 - X_2}{2N} = \pm\big(s_\beta^2 - c_\beta^2\big). \tag{6.22}$$

Then condition (6.8b) is realised for $s_\beta^2 = 2/3$ in models M1, M2, M3 and for $s_\beta^2 = 1/3$ in model M4.

We make an important observation: even under our very restrictive assumptions we were able to find four different ways to realise nucleophobia. Even more possibilities could become viable by relaxing some of the assumptions, *e.g.* by allowing for PQ charges to forbid some SM Yukawa operator [83]. We also observe that condition (6.7a) is imposed just by PQ charge assignment, while the second condition needs a specific value of $\tan\beta \approx 2^{\pm 1/2}$.

Finally, since we need our model to describe an invisible axion, it is necessary to couple $H_{1,2}$ to Φ via a nonhermitian operator, so that the PQ symmetry gets spontaneously broken at the scale $v_a \gg v_{1,2}$, suppressing all axion couplings. There are two possibilities, *i.e.* the operator $H_2^\dagger H_1 \Phi$, for which $|X_\Phi| = 2N$ and the number of domain walls is $N_{\rm DW} = 1$, and the operator $H_2^\dagger H_1 \Phi^2$ for which $|X_\Phi| = N_\ell = N$ and $N_{\rm DW} = 2$.

Electrophobia in 2HDMs

In 2HDMs electrophobia can be induced at the cost of tuning a cancellation between c_e^0 and a correction coming from flavour mixing in the lepton sector [21]. This requires large lepton mixings and one fine tuning. Since the lepton sector is indeed characterised by large mixings, the first requirement is not unplausible.

We can assign the electron either to the doublet or the singlet in *2+1*. We denote the former case by E1 and the latter by E2, and in both cases we can consider $(1, 2, \dots)_\ell$ or $(2, 1, \dots)_\ell$ structures and combine these with the four quark cases. Moreover, for $(a, b, a, b)_\ell$-type of structures, electrophobia is enforced by a cancellation from LH mixing, while for $(a, b, b, a)_\ell$ the cancellation comes from RH mixing. We therefore have a total of $2 \times 2 \times 4 \times 2 = 32$ physically different astrophobic 2HDMs. However, these correspond to just four values of the ratio E/N, *i.e.* of the axion-photon coupling $g_{a\gamma}^0$ (see Ref. [21] for details).

6.1.3 Three Higgs doublet models

We saw that in a 2HDM, the lepton sector is necessarily charged under $U(1)_{\rm PQ}$, and electrophobia requires an extra fine tuning. There is, however, a more elegant way to impose electrophobia within the so-called *three Higgs doublet models* (3HDMs) [50]. In this kind of models, we introduce a third Higgs doublet H_3 with PQ charge X_3 coupled only to the lepton sector, and in order to impose electrophobia it is enough for the condition $X_3 \approx 0$ to be consistent with nucleophobia. In this case, in fact, the lepton sector turns out to be approximately uncharged under $U(1)_{\rm PQ}$, and in particular the axion decouples from the electron.

Let us consider such a model, in which $H_{1,2}$ couple only to the quarks, with the same couplings as in the previous case, while H_3 couples to the leptons. Five conditions must be satisfied in order for the model to be well defined:

i) The two conditions for nucleophobia, eq. (6.7);

ii) The orthogonality between $U(1)_{PQ}$ and $U(1)_Y$ Noether currents;

iii) The two conditions stemming from the requirement that the $U(1)^4$ rephasing symmetry of the kinetic terms of the scalars $H_{1,2,3}$, Φ is broken down to $U(1)_Y \times U(1)_{\rm PQ}$. These conditions can be implemented either by coupling H_3 to both $H_{1,2}$ or by coupling one between $H_{1,2}$ to one of the other doublets

$$H_3^\dagger H_1 \Phi^m + H_3^\dagger H_2 \Phi^n \quad \text{or} \quad H_3^\dagger H_{1,2} \Phi^m + H_2^\dagger H_1 \Phi^n. \tag{6.23}$$

For renormalisable operators one has, without loss of generality, $m = 1, 2$ and $n = \pm 1, \pm 2$ where negative values mean hermitian conjugation: $\Phi^n = \left(\Phi^\dagger\right)^{|n|}$.

Assuming $c_p + c_n \approx 1$ to be satisfied, and choosing the first type of operator in eq. (6.23), our conditions can be written as

1) $\dfrac{X_1}{X_2} = -\dfrac{m_d}{m_u}$,

2) $X_1 v_1^2 + X_2 v_2^2 + X_3 v_3^2 = 0$,

3) $-X_3 + X_1 + m = 0$,

4) $-X_3 + X_2 + n = 0$.

Let us now set $X_3 \to 0$, so as to check whether there is a consistent charge assignment compatible with both electrophobia and nucleophobia. In this limit, 2) reduces to 1), while 3), 4) imply $X_1/X_2 = m/n$, which, together with 1) yields

$$\frac{m_d}{m_u} = -\frac{m}{n}. \tag{6.24}$$

Hence, electrophobia can be consistently implemented for the following values of the light quark mass ratio: $m_d/m_u = 2, 1, 1/2$. Fortunately, the measured value is $m_u/m_d = 0.48(3)$ [84], which is compatible with the first possibility.

By contrast, if we had chosen the second operator in eq. (6.23), electrophobia would have required $m_d/m_u = 1, \infty$ choosing H_1 in the first term, and $m_d/m_u = 1, 0$ choosing H_2, resulting incompatible with nucleophobia.

Implementation of astrophobic 3HDMs

We now want to study the constraints on the axion parameter space arising in astrophobic 3HDMs. We start by assuming that the scalar potential contains the following terms

$$H_3^\dagger H_1 \Phi^2 + H_3^\dagger H_2 \Phi^\dagger, \tag{6.25}$$

which correspond to the choice $m = 2$, $n = -1$. For the quark sector we instead assume a *2+1* structure with equal PQ charges for the first two generations. We therefore need to list only the Yukawa operators involving the second and third generations. For the M1 model

$$\begin{gathered} \bar{q}_2 u_2 H_1, \quad \bar{q}_3 u_3 H_2, \quad \bar{q}_2 u_3 H_1, \quad \bar{q}_3 u_2 H_2, \\ \bar{q}_2 d_2 \tilde{H}_2, \quad \bar{q}_3 d_3 \tilde{H}_1, \quad \bar{q}_2 d_3 \tilde{H}_2, \quad \bar{q}_3 d_2 \tilde{H}_1. \end{gathered} \tag{6.26}$$

We now assume that the lepton sector couples to a third Higgs doublet with universal PQ charge assignment

$$\bar{\ell}_i e_j \tilde{H}_3. \tag{6.27}$$

The conditions $-X_3 + X_1 + m = 0$ and $-X_3 + X_2 + n = 0$ imply, for our choice of m and n,

$$X_1 = X_3 - 2, \quad X_2 = X_3 + 1. \tag{6.28}$$

Neglecting mixing effects, the diagonal axion couplings read

$$
\begin{aligned}
c^0_{u,c} &= \frac{2}{3} - \frac{X_3}{3}, \quad c^0_t = -\frac{1}{3} - \frac{X_3}{3}, \\
c^0_{d,s} &= \frac{1}{3} + \frac{X_3}{3}, \quad c^0_b = -\frac{2}{3} + \frac{X_3}{3}, \\
c^0_{e,\mu,\tau} &= \frac{X_3}{3}.
\end{aligned}
\tag{6.29}
$$

Inserting the expressions of $c^0_{u,d}$ in eq. (6.8b) one gets

$$
X_3 = \frac{1}{2} - \frac{3}{2} f_{ud} \approx -0.03, \tag{6.30}
$$

which confirms that the suppression of the axion-electron couplings is indeed compatible with nucleophobia.

We can now parametrise the vacuum expectation values of the Higgs doublets as

$$
v_1 = v c_1 c_2, \quad v_2 = v s_1 c_2, \quad v_3 = v s_2, \tag{6.31}
$$

where $s_i = \sin\beta_i$, $c_i = \cos\beta_i$, and $\tan\beta_1 = v_2/v_1$, $\tan\beta_2 = \sqrt{v_3^2/(v_1^2+v_2^2)}$. In this parametrisation, the orthogonality condition reads

$$
X_3 = \left(3c_1^2 - 1\right) c_2^2. \tag{6.32}
$$

Requiring the Yukawa couplings to be perturbative restricts the allowed regions in the (β_1, β_2) space. In particular, we can obtain a conservative limit by imposing the tree-level partial wave unitarity bound on the $2 \to 2$ fermion scattering, $|\mathrm{Re}\, a_{J=0}| < 1/2$, in the 3HDM involving Yukawas as $\sqrt{s} \gg M_{H_{1,2,3}}$. Ignoring RG effect, and taking into account group theoretical factors [48, 49], we get

- $y^{\rm 3HDM}_{t,b} < \sqrt{16\pi/3}$ from $Q_L \bar{u}_R \to Q_L \bar{u}_R$, with the initial state prepared into a $SU(3)_c$ singlet;
- $y^{\rm 3HDM}_{\tau} < \sqrt{4\sqrt{2}\pi}$ from $Q_L \bar{Q}_L \to u_R \bar{u}_R$, with the initial state prepared into a $SU(2)_L$ singlet.

The label 3HDM reminds us that these are not the Yukawa couplings of the SM, which in turn can be obtained as

$$
y_t = y^{\rm 3HDM}_t s_1 c_2, \quad y_b = y^{\rm 3HDM}_b c_1 c_2, \quad y_\tau = y^{\rm 3HDM}_\tau s_2. \tag{6.33}
$$

These unitarity bounds can then be translated into perturbativity bounds in the (β_1, β_2) plane, and they correspond to the black hatched region of Fig. 6.1.

6.2 Finite density effects in nucleophobic axion models

In this section, we want to study how finite density corrections affect nucleophobic axion models. This is an important analysis, since one of the main advantages of nucleophobic axion models is the possibility to relax the SN and NS bounds, and, as we have seen, both these astrophysical systems are characterised by extremely high baryonic densities. A similar analysis was performed in [86] on RG corrections, showing that the nucleophobia condition survives running effects, although with a shift in the cancellation point of $c_p - c_n$. We will consider for simplicity a specific nucleophobic model, the

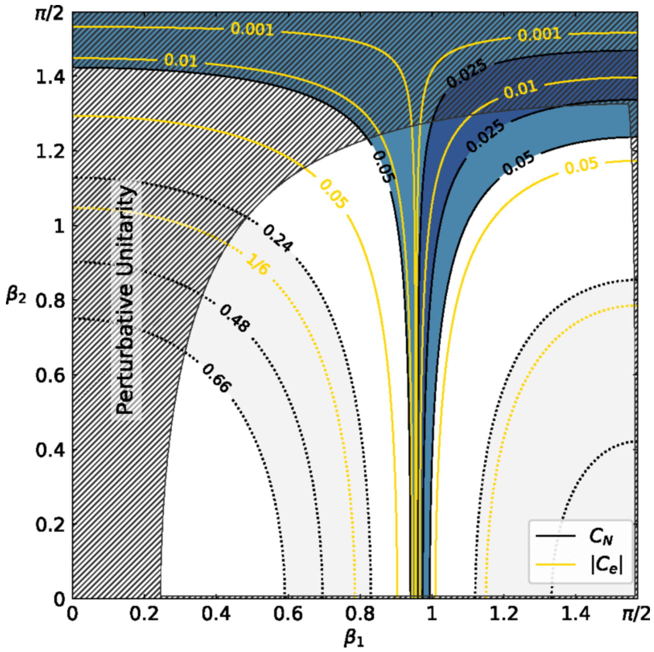


Figure 6.1: $c_N = \sqrt{c_p^2 + c_n^2}$ (black) and $|c_e|$ (yellow) on the (β_1, β_2) plane for the astrophobic 3HDM. For reference, we show the contour lines corresponding to the values of c_N and c_e for the KSVZ ($c_N = 0.48$ dotted line) and the DFSZ axion models (grey region between $c_N = 0.24$ and $c_N = 0.66$ and $c_e = 1/6$ yellow dotted line) [85]. The light (dark) blue shaded area represents the region where the SN bound is relaxed with respect to the KSVZ case by a factor of 10 (20). The figure is reproduced from [50].

M1 model with the choice $H_2^\dagger H_1 \Phi$ for the nonhermitian operator in the scalar potential; however, our analysis can be easily generalised to the other models with analogous results.

Our starting point is the determination of the density-dependent axion-quark couplings in the M1 model. The Yukawa structure of the model is $(1,2,1,2)_u(2,1,2,1)_d$. The Yukawa matrices then read

$$\begin{aligned} Y_1^u &= \begin{pmatrix} y_{11}^u & 0 & y_{13}^u \\ 0 & y_{22}^u & y_{23}^u \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{31}^u & y_{32}^u & y_{33}^u \end{pmatrix}, \\ Y_1^d &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{31}^d & y_{32}^d & y_{33}^d \end{pmatrix}, \quad Y_2^d = \begin{pmatrix} y_{11}^d & 0 & y_{13}^d \\ 0 & y_{22}^d & y_{23}^d \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{6.34}$$

Imposing $U(1)_{\rm PQ}$ invariance of the Yukawa sector yields, see *e.g.* [87],

$$-X_q Y_{1,2}^u + Y_{1,2}^u X_u + X_{1,2} Y_{1,2}^u = 0, \tag{6.35a}$$

$$-X_q Y_{1,2}^u + Y_{1,2}^u X_u + X_{1,2} Y_{1,2}^u = 0, \tag{6.35b}$$

so that

$$X_{u_1} - X_{q_1} = -X_1, \tag{6.36a}$$

$$X_{d_1} - X_{q_1} = X_2, \tag{6.36b}$$

$$X_{d_2} - X_{q_2} = X_2. \tag{6.36c}$$

Then, by using eq. (6.15) we find

$$c_u^0 = -\frac{X_1}{2N}, \quad c_{d,s}^0 = \frac{X_2}{2N}. \tag{6.37}$$

With the normalisation $X_\Phi = 1$ we have

$$X_2 - X_1 = 1 \implies X_1 = -\sin^2\beta, \quad X_2 = \cos^2\beta. \tag{6.38}$$

Moreover, the first nucleophobia condition implies

$$c_u^0 + c_d^0 = \frac{X_2 - X_1}{2N} = \frac{1}{2N} = 1 \implies 2N = 1. \tag{6.39}$$

Hence, the model-dependent contribution to the axion-quark couplings reads

$$c_u^0 = \sin^2\beta, \tag{6.40a}$$
$$c_{d,s}^0 = \cos^2\beta. \tag{6.40b}$$

The full density-dependent axion-quark couplings, taking into account also RG effects through eq. (5.17), are then given by

$$(c_q^{\text{IR}})_n \simeq c_q^0 - [(Q_a^*)_n]_q + r_q^t(f_a)(c_t)_0, \tag{6.41}$$

where $(Q_a^*)_n$ is given by eq. (5.20), we chose f_a as the UV scale, and we are neglecting the density dependence of c_t. From this expression of the axion-quark couplings, we immediately obtain the axion-nucleon couplings by means of eq. (5.55).

We observe that there are three free parameters in our result, *i.e.* $\tan\beta$, κ (the unknown parameter of eq. (5.54)), and obviously the density n. Since our goal is to test the nucleophobia condition against finite density corrections, we first of all need to determine if there exists a value of $\tan\beta$ which realises the second nucleophobia condition. Moreover, we need to check wheter the first nucleophobia condition is spoiled by in-medium corrections. Finally, due to our ignorance about the actual value of κ, we also need to analyse the behaviour of our couplings under changes in its value.

The second nucleophobia condition, $c_p - c_n = 0$, is realised at zero density for $\tan\beta = \sqrt{2}$ in model M1. We now want to show that this cancellation point for $c_p - c_n$ is left invariant to a good approximation by finite density effects.

Let us start from the finite density expressions of c_p and c_n, neglecting for the moment running effects:

$$(c_p)_n = (g_A)_n(c_-)_n + (g_0^{ud})_n(c_+)_n + (\Delta s)_0(c_s)_n, \tag{6.42a}$$
$$(c_n)_n = -(g_A)_n(c_-)_n + (g_0^{ud})_n(c_+)_n + (\Delta s)_0(c_s)_n. \tag{6.42b}$$

Subtracting the latter from the former, we get

$$\begin{aligned}(c_p)_n - (c_n)_n = 2(g_A)_n(c_-)_n &= (g_A)_n\big(c_u^0 - c_d^0 - (f_{ud})_n\big)\\ &= (g_A)_n\left(c_u^0 - c_d^0 - \frac{1 - zZ}{1 + zZ + zZwW}\right).\end{aligned} \tag{6.43}$$

The cancellation point corresponds to the value of $\tan\beta$ such that $c_u^0 - c_d^0 = f_{ud}$. We are considering the case of symmetric nuclear matter, $n_p = n/2$, so that from eq. (5.21) we immediately get

$$Z = 1, \quad W = 1 - (b_1 - b_3)\frac{n}{n_0}. \tag{6.44}$$

Hence,

$$(f_{ud})_n = \frac{1-z}{1+z+zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]}, \tag{6.45}$$

to be compared with the value at zero density

$$(f_{ud})_0 = \frac{1-z}{1+z+zw}. \tag{6.46}$$

In order to make this comparison, we observe that

$$\frac{(f_{ud})_n}{(f_{ud})_0} = \frac{1}{1-\frac{zw(b_1-b_3)}{1+z+zw}\frac{n}{n_0}}, \tag{6.47}$$

which yields the relative difference

$$\frac{(f_{ud})_n-(f_{ud})_0}{(f_{ud})_0} = \frac{\frac{zw(b_1-b_3)}{1+z+zw}\frac{n}{n_0}}{1-\frac{zw(b_1-b_3)}{1+z+zw}\frac{n}{n_0}}. \tag{6.48}$$

This is an increasing function of n/n_0, and in our interval of values of n it is always small:

$$\frac{(f_{ud})_n-(f_{ud})_0}{(f_{ud})_0} < 0.02 \quad \forall n \in [0, 2n_0]. \tag{6.49}$$

Thus, to a good approximation, the value of f_{ud} is unaffected by finite density corrections, and the cancellation point of $c_p - c_n$ stays the same, as shown in Fig. 6.2.

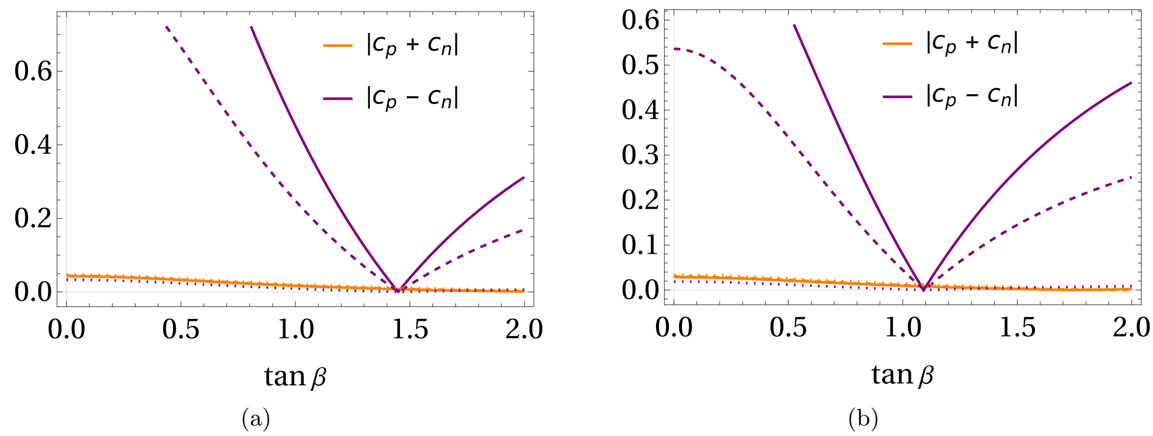


Figure 6.2: Dependence on $\tan\beta$ of the two combinations $|c_p + c_n|$ (orange) and $|c_p - c_n|$ (purple) for three values of density, $n/n_0 = 0$ (full lines), $n/n_0 = 1$ (dashed lines), and $n/n_0 = 2$ (dotted lines), and for $\kappa = 0.3$. The figure on the left does not take into account running effects, which are instead considered on the right. We observe that density does not affect the cancellation point of $|c_p - c_n|$, in agreement with our analytical argument, which is instead shifted to a lower value of $\tan\beta$ by RG effects.

A similar argument can be used to show that also $c_p + c_n$ is only slightly modified by finite density effects. In fact, by summing eq. (6.42a) and eq. (6.42b) we find

$$\begin{aligned}(c_p)_n + (c_n)_n &= 2(g_0^{ud})_n (c_+)_n + 2(\Delta s)_0 (c_s)_n = \\ &= (g_0^{ud})_n \left[c_u^0 + c_d^0 - \frac{1+z}{1+z+zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]} \right] \\ &+ 2(\Delta s)_0 \left[c_s^0 - \frac{zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]}{1+z+zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]} \right],\end{aligned} \tag{6.50}$$

which, since in nucleophobic models $c_u^0 + c_d^0 = 1$, yields

$$\begin{aligned}(c_p)_n + (c_n)_n &= (g_0^{ud})_n \left[1 - \frac{1+z}{1+z+zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]} \right] \\ &+ (\Delta s)_0 \left[c_s^0 - 1 + \frac{1+z}{1+z+zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]} \right].\end{aligned} \tag{6.51}$$

We again observe that

$$\begin{aligned}(f_{ud}^+)_n &\equiv \frac{1+z}{1+z+zw\left[1-(b_1-b_3)\frac{n}{n_0}\right]} \\ &= \frac{1+z}{1+z+zw} \frac{1}{1-\frac{zw(b_1-b_3)}{1+z+zw}\frac{n}{n_0}},\end{aligned} \tag{6.52}$$

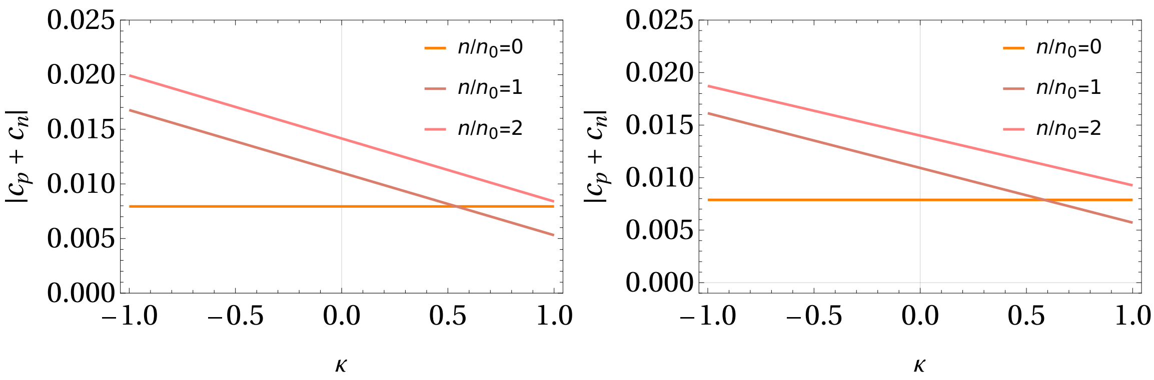
which can be compared to the value at zero density

$$(f_{ud}^+)_0 = \frac{1+z}{1+z+zw} \tag{6.53}$$

to give

$$\frac{(f_{ud}^+)_n - (f_{ud}^+)_0}{(f_{ud}^+)_0} = \frac{\frac{zw(b_1-b_3)}{1+z+zw}\frac{n}{n_0}}{1-\frac{zw(b_1-b_3)}{1+z+zw}\frac{n}{n_0}} < 0.02 \quad \forall n \in [0, 2n_0]. \tag{6.54}$$

Hence, both c_+ and c_s are essentially left unaffected by finite density corrections. Moreover, since $(c_+)_n \approx (c_+)_0 \approx 0 \ \forall n \in [0, 2n_0]$, it follows that the contribution of $(g_0^{ud})_n$ can be safely neglected, making our result independent of κ to a good approximation, as one can see in Fig. 6.3, where we fixed the value of $\tan\beta$ so as to impose the second nucleophobia condition.



(a) (b)

Figure 6.3: Dependence on κ of $|c_p + c_n|$, for $\tan\beta$ corresponding to the cancellation point of $|c_p - c_n|$ and three values of density, $n/n_0 = 0$ (orange), $n/n_0 = 1$ (dark pink), and $n/n_0 = 2$ (pink). RG effects are neglected on the left, and taken into account on the right.

We therefore showed that the nucleophobia conditions are left unspoiled by in-medium corrections for values of density within theoretical control. In Fig. 6.4 we show the density dependence of $|c_p + c_n|$ and $|c_p - c_n|$ when both nucleophobia conditions are satisfied and for $\kappa = 0.3$.

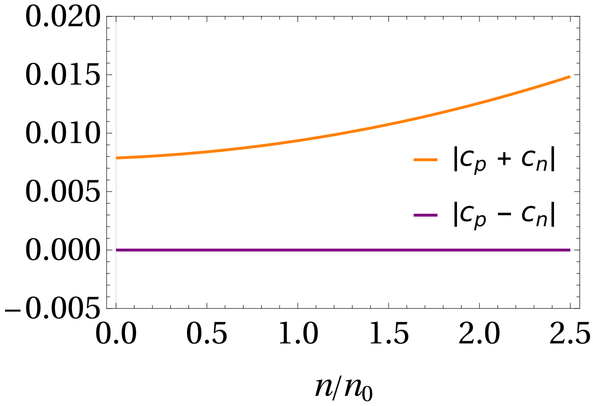
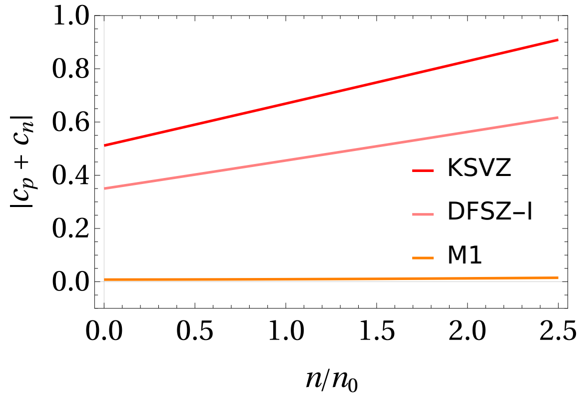
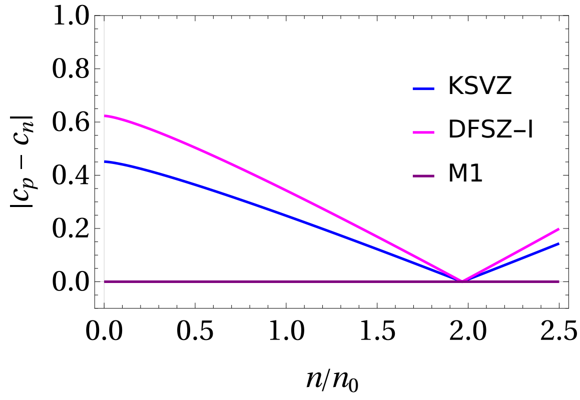


Figure 6.4: Density dependence of $|c_p + c_n|$ (orange) and $|c_p - c_n|$ (purple) for model M1, at the cancellation point for $c_p - c_n$ and for $\kappa = 0.3$. Running effects are taken into account.

To understand the entity of the suppression of the couplings, we plot in Fig. 6.5 the axion-nucleon couplings as functions of density for the model M1 and for the two benchmark axion models, the KSVZ model and the DFSZ model.



(a)



(b)

Figure 6.5: Depedence of $|c_p + c_n|$ (left) and $|c_p - c_n|$ (right) on density for the KSVZ and DFSZ-I benchmark axion models, as well as for the nucleophobic model M1, for $\tan\beta$ corresponding the cancellation point of $c_p - c_n$ of model M1, and $\kappa = 0.3$. RG effects are taken into account.

Chapter 7

Conclusions

The QCD axion is a theoretically well motivated beyond the Standard Model particle which, if observed, would allow to solve both the strong CP problem and the problem of the nature of DM. Moreover, it is, in principle, within reach of current and future experiments. An accurate determination of the bounds on the parameter space of the axion is therefore fundamental to determine the allowed region in which this particle could be observed.

The SN1987A bound on the axion-nucleon couplings finds its origin in a highly dense stellar system, in which the interactions of the axion with nucleons are heavily affected by in-medium corrections. In this work, we discussed in detail the problem of the determination of the finite density corrections to axion-nucleon couplings, first independently of the axion model and then for the specific case of nucleophobic axion models.

In Chapter 5, we analysed how the chiral quark condensates are modified by finite density, and how this affects the axion-quark couplings. Moreover, we computed the in-medium corrections to the isotriplet hadronic matrix element g_A by exploiting a Fermi gas approximation for the nuclear medium. By means of these calculations, we were able to find an expression of the couplings c_p and c_n as functions of baryon number density. This, although incomplete, due to the missing corrections to the isoscalar matrix element g_0^{ud}, was enough to show that the nucleophobia condition, *i.e.* the suppression of axion-nucleon couplings, is left unspoiled by finite density corrections. This was the main result of Chapter 6.

The lack of an explicit expression for g_0^{ud} as a function of density offers a possible direction for future work, with the aim of obtaining a complete description of the density dependence of axion-nucleon couplings in more general axion models. This, in turn, would allow us to update the value of the SN1987A bound.

Appendices

Appendix A

Baryon Chiral Perturbation Theory

A.1 Relativistic Baryon ChPT

Baryon Chiral Perturbation Theory, Baryon ChPT for short, is an EFT of QCD which allows a consistent description of the low energy dynamics of the light meson and baryon octets. To build such an effective theory we need to construct an effective Lagrangian written in terms of these hadronic degrees of freedom. We will follow a bottom-up approach, where we will build the Lagrangian based on symmetry arguments, and then match its predictions with those of QCD to obtain the values of the low energy constants.

We start from the following expression of the QCD Lagrangian for the three light quarks given by Gasser and Leutwyler in [88, 89]

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2}\,\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] + \bar{q}i\not{D}q + \bar{q}\gamma^\mu\left(v_\mu + \frac{1}{3}v_\mu^{(s)} + \gamma_5 a_\mu + \frac{1}{3}\gamma_5 a_\mu^{(s)}\right)q - \bar{q}(s - i\gamma_5 p)q. \tag{A.1}$$

Here, $q = (u, d, s)^T$ and we have introduced the external fields $v_\mu = v_\mu^a\lambda^a/2$, $v_\mu^{(s)}$, $a_\mu = a_\mu^a\lambda^a/2$, $a_\mu^{(s)}$, $s = s^a\lambda^a$, and $p = p^a\lambda^a$, where the λ^a are the Gell-Mann matrices in flavour space. We will denote this set of external fields as (v, a, s, p).

The matching will be carried over the connected transition amplitude between two one-baryon states in the presence of the external fields [90]

$$\mathscr{F}(\vec{p}, \vec{p}\,'; v, a, s, p) = \left\langle \vec{p}\,', \mathrm{out}\middle|\vec{p}, \mathrm{in}\right\rangle^c_{v,s,a,p}\,, \tag{A.2}$$

where $|\vec{p}, \mathrm{in}\rangle$ and $|\vec{p}\,', \mathrm{out}\rangle$ are respectively the asymptotic one-baryon in and out states. The usual three-flavour QCD Lagrangian can be obtained from eq. (A.1) by the substitutions $v_\mu, v_\mu^{(s)}, a_\mu, a_\mu^{(s)}, p \to 0$ and $s \to \mathcal{M}_q = \mathrm{diag}(m_u, m_d, m_s)$.

We now want to build our Lagrangian. To do that we first of all need a field operator to represent the baryons. We associate to each baryon in the light octet a Dirac fermion field which we arrange in a matrix in flavour space

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{6}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{6}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} \tag{A.3}$$

This field is required to transform as an octet under $SU(3)_V$ isospin transformations:

$$B \to B' = VBV^\dagger, \quad \forall\, V \in SU(3)_V. \tag{A.4}$$

The transformation law under more general chiral transformations $(L, R) \in SU(3)_L \times SU(3)_R$ can instead be chosen arbitrarily, since many possibilities agree with eq. (A.5) and, in the presence of the pseudo-Goldstone bosons associated to chiral spontaneous symmetry breaking, they can be shown to be connected by field redefinitions. We make a choice which is particularly advantageous to construct the Lagrangian. Let

$$\Sigma = \exp\left(i\frac{\pi^a \lambda^a}{f_\pi} \right) \tag{A.5}$$

be the meson field, transforming as $\Sigma \to L\Sigma R^\dagger$ under $SU(3)_L \times SU(3)_R$. We define a field ξ such that $\xi^2 = \Sigma$. Hence

$$\xi = \exp\left(i\frac{\pi^a \lambda^a}{2f_\pi} \right). \tag{A.6}$$

We now introduce a function $K = K(\Sigma, L, R)$ such that under a chiral transformation

$$\xi \to \xi' = \sqrt{L\Sigma R^\dagger} \equiv L\xi K^{-1}(\Sigma, L, R). \tag{A.7}$$

Hence,

$$K(\Sigma, L, R) = \xi'^{-1} L\xi = \left(\sqrt{L\Sigma R^\dagger} \right)^{-1} L\sqrt{\Sigma}. \tag{A.8}$$

We then choose as the transformation law for our baryon field B under chiral transformations

$$B \to B' = K(\Sigma, L, R) B K(\Sigma, L, R)^\dagger \tag{A.9}$$

We observe that this realisation of the $SU(3)_L \times SU(3)_R$ group is nonlinear and local, since the transformation matrix $K(\Sigma, L, R)$ depends on the spacetime coordinates through $\Sigma(x)$.

To build a Lagrangian invariant under chiral transformations, following [91, 92], we will then need covariant derivatives associated to the local transformation. We define it as

$$D_\mu B = \partial_\mu B + [\Gamma_\mu, B], \tag{A.10a}$$
$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i r_\mu \Sigma + i\Sigma \ell_\mu, \tag{A.10b}$$

where $r_\mu = v_\mu + a_\mu$ and $\ell_\mu = v_\mu - a_\mu$ and where we introduced the so-called *chiral connection*

$$\Gamma_\mu = \frac{1}{2}\Big[\xi^\dagger, \partial_\mu \xi\Big] - \frac{i}{2}\xi^\dagger r_\mu \xi - \frac{i}{2}\xi \ell_\mu \xi^\dagger, \tag{A.11}$$

The covariant derivatives thus defined correctly transform as $(D_\mu B) \to (D_\mu B)' = K(D_\mu B)K^\dagger$ and $(\nabla_\mu \Sigma) \to (\nabla_\mu \Sigma)' = L(\nabla_\mu)R^\dagger$ under chiral transformations.

We also define the *vielbeins*

$$\Delta_\mu = \frac{1}{2}\Big\{\xi^\dagger, \partial_\mu \xi\Big\} - \frac{i}{2}\xi^\dagger r_\mu \xi + \frac{i}{2}\xi \ell_\mu \xi^\dagger \tag{A.12a}$$
$$\tilde{\Delta}_\mu = -\frac{i}{2}\Big[\xi^\dagger a_\mu^{(s)} \xi + \xi a_\mu^{(s)} \xi^\dagger\Big] = -i a_\mu^{(s)}, \tag{A.12b}$$

which also transform as $\Delta_\mu \to \Delta'_\mu = K\Delta_\mu K^\dagger$ and $\tilde{\Delta}_\mu \to \tilde{\Delta}'_\mu = K\tilde{\Delta}_\mu K^\dagger$. These are the building blocks which can be combined to obtain the expression of the Lagrangian.

We are just missing an ingredient, *i.e.* a power counting scheme to organise the infinite terms of our effective Lagrangian. Regarding the meson sector of the theory, the power counting is the same as the one in regular Chiral Perturbation Theory. Let $q \ll \Lambda_\chi$ be a small momentum, where Λ_χ is the breaking scale of ChPT. Assuming the Σ field to count as $\mathcal{O}(q^0)$, the derivatives $\partial_\mu\Sigma$ count as $\mathcal{O}(q)$. Similarly, the external fields v_μ, a_μ appearing in the covariant derivative $\nabla_\mu\Sigma$ alongside $\partial_\mu\Sigma$ also count as $\mathcal{O}(q)$. The field s is instead proportional to the quark mass matrix, and, since meson masses squared are linear in quark masses, we must count s as $\mathcal{O}(q^2)$. Finally, since p appears always with s we count it too as $\mathcal{O}(q^2)$. In summary:

$$\Sigma \sim \mathcal{O}(q^0), \quad \partial_\mu\Sigma, v_\mu, a_\mu \sim \mathcal{O}(q), \quad s, p \sim \mathcal{O}(q^2). \tag{A.13}$$

The power counting in the baryonic sector is defined in a similar fashion, but with an important subtlety: the masses of the baryons are not small, so the action of the derivative does not produce a small quantity. To understand how to determine the correct power counting rule, we consider the positive-energy plane wave solution to the Dirac equation

$$\psi^{(+)}(t,\vec{x}) = e^{-ip\cdot x}\sqrt{E+m_B}\begin{pmatrix}\chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m_B}\chi\end{pmatrix}, \tag{A.14}$$

where χ is a Pauli spinor and $p = (E,\vec{p}) = \left(\sqrt{\vec{p}^2+m_B^2},\vec{p}\right)$. In the limit in which p is a small momentum, *i.e.* the nonrelativistic limit $p \ll m_B$, the small component is suppressed with respect to the large one. The action of the derivative on the plane wave in eq. (A.14) gives $p^\mu = (m_B,\vec{0}) + (E-m_B,\vec{p})$, where the second term is a small momentum of $\mathcal{O}(q)$. Therefore $(i\not{\partial} - m_B)$ acting on the baryon field gives a quantity of $\mathcal{O}(q)$. The Dirac bilinears can instead be studied by noticing that $\{\mathbb{1}, \gamma^0, \gamma_5\gamma^i, \sigma^{ij}\}$ mix large and large components, while $\{\gamma_5, \gamma_5\gamma^0, \gamma^i, \sigma^{i0}\}$ only mix large and small components. Finally, the action of the covariant derivative D_μ on the baryon field does not imply the lowering of a small momentum, so it counts as $\mathcal{O}(q^0)$. Hence,

$$\begin{aligned} B, D_\mu B \sim \mathcal{O}(q^0), \quad (i\not{D} - m_B)B \sim \mathcal{O}(q), \\ \mathbb{1}, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu} \sim \mathcal{O}(q^0), \quad \gamma_5 \sim \mathcal{O}(q), \end{aligned} \tag{A.15}$$

where we only give the minimal order.

We can now write down the leading order Lagrangian for Baryon ChPT

$$\begin{aligned} \mathcal{L}^{\mathrm{LO}}_{\mathrm{BChPT}} &= \mathcal{L}^{(2)}_{\pi\pi} + \mathcal{L}^{(1)}_{\pi N} \\ &= \frac{f^2}{4}\operatorname{Tr}\Big[(\nabla_\mu\Sigma)^\dagger(\nabla^\mu\Sigma)\Big] + \frac{f^2}{4}\operatorname{Tr}\Big[\chi^\dagger\Sigma + \chi\Sigma^\dagger\Big] \\ &+ \operatorname{Tr}\big[\bar{B}(i\not{D} - m_B)B\big] - iD\operatorname{Tr}\big(\bar{B}\gamma^\mu\gamma_5\{\Delta_\mu, B\}\big) - iF\operatorname{Tr}\big(\bar{B}\gamma^\mu\gamma_5[\Delta_\mu, B]\big) \\ &- i\tilde{D}\operatorname{Tr}\Big(\bar{B}\gamma^\mu\gamma_5\tilde{\Delta}_\mu B\Big) \end{aligned} \tag{A.16}$$

where we introduced the field $\chi = 2B_0(s+ip)$ and where $f \approx f_\pi = 92.3\,\mathrm{MeV}$, B_0 is related to the value of the chiral condensates, $D = 0.80$, and $F = 0.50$.

One can similarly write down the Baryon ChPT Lagrangian for the two-flavour QCD case. In this case the chiral group becomes $SU(2)_L \times SU(2)_R$, the Goldstone bosons are just the pions, organised in the field

$$\Sigma = \exp\left(i\frac{\pi^a\tau^a}{f_\pi}\right), \tag{A.17}$$

where τ^a are the isospin Pauli matrices, and the relevant baryons are the nucleons, organised in an isospin doublet

$$N = \begin{pmatrix} p \\ n \end{pmatrix}. \tag{A.18}$$

The transformation law for the nucleon doublet under chiral transformations is

$$N \to N' = K(\Sigma, L, R)N, \quad K(\Sigma, L, R) = \left(\sqrt{L\Sigma R^\dagger}\right)^{-1} L\sqrt{\Sigma}. \tag{A.19}$$

The covariant derivatives in this case take the form

$$D_\mu N = (\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})N, \tag{A.20a}$$

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - ir_\mu \Sigma + i\Sigma \ell_\mu, \tag{A.20b}$$

and the power counting rules are totally analogous to the previous case. The leading order Lagrangian then reads

$$\begin{aligned} \mathcal{L}^{\text{LO}}_{\text{BChPT}} = &= \mathcal{L}^{(2)}_{\pi\pi} + \mathcal{L}^{(1)}_{\pi N} \\ &= \frac{f^2}{4}\operatorname{Tr}\left[(\nabla_\mu\Sigma)^\dagger(\nabla^\mu\Sigma)\right] + \frac{f^2}{4}\operatorname{Tr}\left[\chi^\dagger\Sigma + \chi\Sigma^\dagger\right] \\ &+ \bar{N}\left(i\not{D} - m_N + ig_A\gamma^\mu\gamma_5\Delta_\mu + ig_0^{ud}\gamma^\mu\gamma_5\tilde{\Delta}_\mu\right)N \end{aligned} \tag{A.21}$$

where m_N is the nucleon mass, $g_A \simeq 1.27$, and $g_0^{ud} \simeq 0.52$.

A.1.1 Chiral power counting

Let us consider the meson sector of our theory. We want to find a consistent scheme to determine the contribution of each renormalised diagram to a given amplitude, where by renormalised diagrams we mean the diagrams computed by taking into account also the counterterms.

To this end, let us perform a mathematical trick. Starting from a generic amplitude $\mathcal{M}(p_i, m_q)$, where p_i are the external momenta and m_q the quark masses, we perform a rescaling of both the momenta and the quark masses, $p_i \to tp_i$ and $m_q \to t^2 m_q$. here we exploited the fact that $m_q \sim \mathcal{O}(q^2)$. Under this rescaling, the amplitude can be shown to scale as [93]

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q), \tag{A.22}$$

where the *chiral dimension* D of the amplitude is given by

$$D = 2 + (n-2)N_L + \sum_{k=1}^{+\infty} 2(k-1)N_{2k} \geq 2 \quad (\text{for } n = 4), \tag{A.23}$$

N_L being the number of independent loops in the diagram, N_{2k} the number of vertices from $\mathcal{L}_{2k}$, and n the number of spacetime dimensions.

By choosing $0 \leq t \leq 1$ we go to smaller momenta e quark masses, and we observe that diagrams with higher chiral dimensions get suppressed with respect to the ones with lower chiral dimension. Moreover, loop diagrams are always suppressed due to the term $(n-2)N_L$.

This relation introduces a correspondence between the loop expansion and the chiral expansion, since, due to eq. (A.23), for a given chiral dimension, the number of loops is bounded from above.

A.2 Heavy Baryon ChPT

The formalism outlined in the previous section allows for a consistent description of baryons within the framework of ChPT. However, it has an important flaw which makes its direct application difficult. In the meson sector, the Lagrangian is organised in powers of a small momentum, that we can identify with the momentum of the initial and final mesons. This identification relies on the fact that if the meson masses are small, and even vanish in the chiral limit, so that at the low energies at which we can apply ChPT we can lightheartedly assume $q \ll \Lambda_\chi$. The same, however, cannot be done in the baryon sector; the baryon masses do not vanish in the chiral limit, and even the lightest baryons, the nucleons, have a mass $m_N \sim \Lambda_\chi$. Then, while, as we have seen, a correspondence between the chiral power counting and the loop expansion can be built in the meson sector, the same cannot be done in the baryon sector [94], so that an amplitude of a given chiral dimension receives contributions from diagrams at all loop orders. A solution to this problem is given by the *Heavy Baryon ChPT* formalism, proposed by Jenkins and Manohar [95], which we are going to discuss in detail in this section.

Let us consider for simplicity the two-flavour case. As we have seen the pion-nucleon Lagrangian reads

$$\mathcal{L}^{(1)}_{\pi N} = \bar{N}\Big(i\not{D} - m_N + ig_A\gamma^\mu\gamma_5\Delta_\mu + ig_0^{ud}\gamma^\mu\gamma_5\tilde{\Delta}_\mu\Big)N. \tag{A.24}$$

Let us write the nucleon four-momentum as

$$p^\mu = m_N v^\mu + k^\mu, \tag{A.25}$$

where v^μ is a four-vector satisfying

$$v^2 = 1, \quad v^0 \geq 1 \tag{A.26}$$

and k is a small residual momentum with $v \cdot k \ll m_N$. We observe that, if we choose as v^μ the four-velocity of the nucleon, the first term is just the nonrelativistic four-momentum of the nucleon, while k^μ gives a negligible correction. We are therefore considering the extreme nonrelativistic limit of our theory. We choose to work in the nucleon rest frame where $v^\mu = (1, 0, 0, 0)$.

The Dirac equation for the nucleon field gives $(\not{p} - m_N)N = 0$. In the nonrelativistic limit $k^\mu \approx 0$, this is just $(\mathbb{1} - \not{v})N \approx 0$. By introducing the projectors

$$P_{v\pm} = \frac{\mathbb{1} \pm \not{v}}{2}, \tag{A.27}$$

satisfying, for our choice of v^μ,

$$P_{v+} + P_{v-} = \mathbb{1}, \quad P_{v\pm}^2 = P_{v\pm}, \quad P_{v\pm}P_{v\mp} = 0, \tag{A.28}$$

we introduce the fields

$$\mathcal{N}_v(x) = e^{im_N v\cdot x} P_{v_+} N, \tag{A.29a}$$
$$\mathcal{H}_v(x) = e^{im_N v\cdot x} P_{v_-} N. \tag{A.29b}$$

We observe that in the nonrelativistic limit $\mathcal{H}_v \sim (\mathbb{1} - \not{v})N \approx 0$, so that by writing

$$N = e^{-im_N v\cdot x}(\mathcal{N}_v + \mathcal{H}_v), \tag{A.30}$$

we have $N \approx e^{-im_N v\cdot x}\mathcal{N}_v$ at such low energies.

We now consider the equation of motion for N stemming from the Lagrangian in eq. (A.24)

$$\Big(i\not{D} - m_N + ig_A\not{\Delta}\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}_\mu\gamma_5\Big)N = 0. \tag{A.31}$$

Let us substitute the decomposition eq. (A.30) in this expression to find

$$\Big(i\not{D} + ig_A\not{\Delta}\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}\gamma_5\Big)\mathcal{N}_v + \Big(i\not{D} - 2m_N + ig_A\not{\Delta}\gamma_5 + ig_0^{ud}\gamma^\mu\gamma_5\tilde{\not{\Delta}}_\mu\Big)\mathcal{H}_v = 0. \tag{A.32}$$

Exploiting the gamma matrix algebra relations, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and $\{\gamma^\mu, \gamma_5\} = 0$, we can rewrite eq. (A.32) as two independent equations

$$\Big(iv\cdot D + ig_A\not{\Delta}_\perp\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)\mathcal{N}_v + \Big(i\not{D}_\perp + ig_A v\cdot\Delta\gamma_5 + ig_0^{ud}v\cdot\tilde{\Delta}\gamma_5\Big)\mathcal{H}_v = 0, \tag{A.33a}$$
$$\Big(i\not{D}_\perp - ig_A v\cdot\Delta\gamma_5 - ig_0^{ud}v\cdot\tilde{\Delta}\gamma_5\Big)\mathcal{N}_v + \Big(-iv\cdot D - 2m_N + ig_A\not{\Delta}_\perp\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)\mathcal{H}_v = 0. \tag{A.33b}$$

where we introduced the notation $A^\mu = v\cdot A v^\mu + A_\perp^\mu$ for a generic four-vector A^μ.

We can then integrate out the field $\mathcal{H}_v$, which as we have seen is not excited in the nonrelativistic limit, by solving eq. (A.33b) in terms of it and substituting in eq. (A.33a). The resulting equation of motion for $\mathcal{N}_v$ is

$$\begin{aligned}&\Big(iv\cdot D + ig_A\not{\Delta}_\perp\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)\mathcal{N}_v\\ &+ \Big(i\not{D}_\perp + ig_A v\cdot\Delta\gamma_5 + ig_0^{ud}v\cdot\tilde{\Delta}\Big)\Big(2m_N + iv\cdot D - ig_A\not{\Delta}_\perp\gamma_5 - ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)^{-1}\\ &\times\Big(i\not{D}_\perp - ig_A v\cdot\Delta\gamma_5 - ig_0^{ud}v\cdot\tilde{\Delta}\Big)\mathcal{N}_v = 0\end{aligned} \tag{A.34}$$

which can be traced back to a Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{HBChPT}} =& \bar{\mathcal{N}}_v\Big(iv\cdot D + ig_A\not{\Delta}_\perp\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)\mathcal{N}_v\\ &+ \bar{\mathcal{N}}_v\Big(i\not{D}_\perp + ig_A v\cdot\Delta\gamma_5 + ig_0^{ud}v\cdot\tilde{\Delta}\gamma_5\Big)\Big(2m_N + iv\cdot D - ig_A\not{\Delta}_\perp\gamma_5 - ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)^{-1}\\ &\times\Big(i\not{D}_\perp - ig_A v\cdot\Delta\gamma_5 - ig_0^{ud}c\cdot\tilde{\Delta}\gamma_5\Big)\mathcal{N}_v\end{aligned} \tag{A.35}$$

We observe that the second line is suppressed by the m_N at the denominator, so, at leading order in m_N^{-1} our Lagrangian reads

$$\mathcal{L}_{\text{HBChPT}} = \bar{\mathcal{N}}_v\Big(iv\cdot D + ig_A\not{\Delta}_\perp\gamma_5 + ig_0^{ud}\tilde{\not{\Delta}}_\perp\gamma_5\Big)\mathcal{N}_v. \tag{A.36}$$

We can now introduce the *spin operator*

$$S^\mu = \frac{i}{2}\gamma_5\sigma^{\mu\nu}v_\nu = -\frac{1}{2}\gamma_5(\gamma^\mu \not{v} - v^\mu), \tag{A.37}$$

satisfying

$$(S^\mu)^\dagger = \gamma^0 S^\mu \gamma^0, \tag{A.38a}$$

$$v \cdot S^\mu = 0, \tag{A.38b}$$

$$\{S^\mu, S^\nu\} = \frac{v^\mu v^\nu - \eta^{\mu\nu}}{2}, \tag{A.38c}$$

$$[S^\mu, S^\nu] = i\varepsilon^{\mu\nu\rho\sigma} v_\rho S_\sigma. \tag{A.38d}$$

One can show that $\bar{\mathcal{N}}_v \gamma^\mu \gamma_5 \mathcal{N}_v = 2\bar{\mathcal{N}}_v S^\mu \mathcal{N}_v$, so that our Lagrangian finally takes the form

$$\mathcal{L}_{\text{HBChPT}} = \bar{\mathcal{N}}_v \Big(iv \cdot D + 2ig_A S \cdot \Delta + 2ig_0^{ud} S \cdot \tilde{\Delta} \Big) \mathcal{N}_v. \tag{A.39}$$

Let us comment on this result. The first thing we observe is that the linear dependence on m_N of eq. (A.24) has disappeared in the nonrelativistic limit, so that the leading order Lagrangian is now completely independent of the nucleon mass. The terms we neglected can instead be organised in a series in powers of m_N^{-1}, with each term suppressed with respect to previous one. Moreover, also the problem regarding the small momentum expansion has disappeared. The action of a derivative on $\mathcal{N}_v$ gives $\partial_\mu \mathcal{N}_v \sim \partial_\mu e^{im_N v\cdot x - ip\cdot x} \sim -ik_\mu$, where k^μ is now a small momentum. Hence, whenever the physics of the problem allows us to take the nonrelativistic limit, the Heavy Baryon ChPT formalism allows us to recover the lost correspondence between the chiral power counting and the loop expansion.

Appendix B

Basics of Thermal Field Theory

Thermal Field Theory (TFT) is the formalism which generalises the usual tools of statistical mechanics to the case in which the microscopic theory is a quantum field theory (QFT). In particular, it provides the techniques to compute the *thermal Green's functions*, which in turn allow the determination of the thermodynamic state functions associated to the system of interest, *e.g.* by means of linear response theory. Our aim in this section is to give a brief and self-contained introduction to TFT, mainly focusing on the problem of the description of systems at finite density. We will, in particular, find a prescription to include the effects of a nonzero chemical potential, and therefore of a nonvanishing value of the corresponding density, in the Lagrangian of a QFT.

B.1 The grand canonical ensemble

Let us consider a physical system $\mathcal{S}$ described by a Hamiltonian H, and characterised by a set of mutually commuting conserved charges Q_k. Let us assume that our system is put in contact with an external bath $\mathcal{B}$ through rigid walls which allow $\mathcal{S}$ and $\mathcal{B}$ to exchange energy and charges, which are then free to fluctuate around their average values $\langle E\rangle$ and $\langle Q_k\rangle$ determined by the equilibrium values of the temperature T and chemical potentials μ_k.

To connect the microscopic and macroscopic descriptions of the system, we consider an ensemble of copies of our system, each corresponding to a different microscopic state, but all compatible with the same macroscopic state characterised by the fixed values of T and μ_k and by the volume V of the system. The quantum state associated to the ensemble is described by the density operator ρ. To find the explicit form of the density operator, we maximise the entropy under the following three constraints:

$$\operatorname{Tr}\rho = 1 \tag{B.1a}$$

$$\operatorname{Tr}(\rho H) = \langle E\rangle \tag{B.1b}$$

$$\operatorname{Tr}(\rho\, Q_k) = \langle Q_k\rangle \tag{B.1c}$$

The von Neumann entropy associated to our ensemble is (in units of $k_B = 1$)

$$S = -\operatorname{Tr}(\rho\log\rho). \tag{B.2}$$

To impose the constraints of eq. (B.1), we introduce a set of Lagrange multipliers α_i, so that the maximisation condition becomes

$$\delta\left\{-\operatorname{Tr}\left(\rho\log\rho+\alpha_0\rho+\alpha_E\rho H+\sum_k\alpha_k\rho\,Q_k\right)\right\}=0, \tag{B.3}$$

which yields

$$\rho=\exp\left[(\alpha_0-1)+\alpha_E H+\sum_k\alpha_k Q_k\right]. \tag{B.4}$$

Taking the trace of this expression and imposing eq. (B.1a) we find

$$\mathcal{Z}(\alpha_E,\{\alpha_k\})\equiv\exp(1-\alpha_0)=\operatorname{Tr}\left[\exp\left(\alpha_E H+\sum_k\alpha_k Q_A\right)\right], \tag{B.5}$$

where we introduced the *grand canonical partition function* $\mathcal{Z}(\alpha_E,\{\alpha_k\})$.

From eq. (B.4) one easily finds, by imposing the remaining constraints of eq. (B.1),

$$-\log\mathcal{Z}(\alpha_E,\{\alpha_k\})+\alpha_E\langle E\rangle+\sum_k\alpha_k\langle Q_k\rangle+S=0, \tag{B.6}$$

which, by the identifications $\alpha_E=-1/T$ and $\alpha_k=\mu_k/T$, becomes

$$\Omega(T,\{\mu_k\})\equiv-\log\mathcal{Z}(T,\{\mu_k\})=\langle E\rangle-TS-\sum_k\mu_k\langle Q_k\rangle\,, \tag{B.7}$$

which is the thermodynamic definition of the *Landau free energy* or *grand potential.*

The density operator can therefore be written as

$$\rho=\frac{e^{-\beta(H-\sum_k\mu_kQ_k)}}{\operatorname{Tr}e^{-\beta(H-\sum_k\mu_kQ_k)}}=\frac{1}{\mathcal{Z}(T,(\mu_k))}e^{-\beta(H-\sum_k\mu_kQ_k)}, \tag{B.8}$$

where $\beta=1/T$, and where the grand canonical partition function now reads

$$\mathcal{Z}(T,\{\mu_A\})=\operatorname{Tr}e^{-\beta(H-\sum_k\mu_kQ_k)}. \tag{B.9}$$

B.2 Thermal Field Theory: real time formalism

We now want to study the case in which the microscopic dynamics of our system is described by a QFT, following [96]. For simplicity, let us consider the case of a single real multi-component scalar field $\Phi(x)$ evolving in the Heisenberg picture

$$\Phi(x)=e^{itH}\Phi(t,\vec{x})e^{-itH}. \tag{B.10}$$

We assume our time coordinate t to be a complex function of a real variable v, $t=t(v)$. Fixed a path C in the complex plane, we define the path-dependent functions

$$\vartheta_C(t-t')=\vartheta(v-v'), \tag{B.11a}$$

$$\delta_C(t-t')=\left(\frac{\partial t}{\partial v}\right)^{-1}\delta(v-v'). \tag{B.11b}$$

We are interested in computing the *thermal Green's functions*

$$G_C(x_1,\ldots,x_n) = \langle T_C \Phi(x_1)\cdots\Phi(x_n)\rangle_\beta\,, \tag{B.12}$$

where $\langle\cdot\rangle_\beta$ is the thermal average in the grand canonical ensemble

$$\langle\mathcal{O}\rangle_\beta = \frac{1}{\mathcal{Z}(\beta,\{\mu_k\})}\,\mathrm{Tr}\Big(e^{-\beta(H-\sum_k\mu_k Q_k)}\mathcal{O}\Big), \tag{B.13}$$

and T_C is the path-ordering defined as

$$T_C\Phi(t_1,\vec{x}_1)\cdots\Phi(t_n,\vec{x}_n) = \sum_k\left(\prod_{j=1}^{n-1}\vartheta_C(t_{k_j}-t_{k_{j+1}})\right)\Phi(t_{k_1},\vec{x}_{k_1})\cdots\Phi(t_{k_n},\vec{x}_{k_n}), \tag{B.14}$$

where k runs over all the possible permutations of the indices $\{1,\ldots,n\}$.

By defining a rule for functional differentiation for c-number-valued functions $f(x)$ whose time argument is defined on C

$$\frac{\delta f(x)}{\delta f(x')} = \delta_C(t-t')\delta^3(\vec{x}-\vec{x}\,'), \tag{B.15}$$

we can introduce a *generating functional* $\mathcal{Z}_C(\beta,\{\mu_k\},j)$ for the thermal Green's functions (B.12), such that

$$G_C(x_1,\ldots,x_n) = \frac{1}{\mathcal{Z}(\beta,\{\mu_k\})}\frac{\delta^n\mathcal{Z}(\beta,\{\mu_k\},j)}{i\delta j(x_1)\cdots i\delta j(x_n)}\bigg|_{j=0}. \tag{B.16}$$

It is easy to see that such a generating functional must have the form

$$\mathcal{Z}(\beta,\{\mu_k\},j) = \mathrm{Tr}\Big[e^{-\beta(H-\sum_k\mu_k Q_k)}T_C\exp\Big(i\int_C \mathrm{d}^4x\, j(x)\cdot\Phi(x)\Big)\Big], \tag{B.17}$$

where $\int_C \mathrm{d}^4x \equiv \int_C \mathrm{d}t\int\mathrm{d}^3\vec{x}$. We observe that by setting the sources $j(x)$ to zero, eq. (B.17) becomes the grand canonical partition function (B.9). This object has also a path integral representation

$$\mathcal{Z}(\beta,\{\mu_k\},j) = \int\mathcal{D}\Pi\,\mathcal{D}\Phi\,\exp\left[i\int_C\mathrm{d}^4x\left(\Pi(x)\cdot\partial_0\Phi(x)-\mathcal{H}(\Phi,\pi)+\sum_k\mu_k J_k^0(x)+j(x)\cdot\Phi(x)\right)\right] \tag{B.18}$$

where the integration is carried over configurations satisfying $\Phi(t,\vec{x}) = \Phi(t-i\beta,\vec{x})$, Π is the field canonically conjugated to Φ, and we used the relation between the conserved charges Q_k and the corresponding Noether currents J_k^μ

$$Q_k(t) = \int\mathrm{d}^3\vec{x}\,J_k^0(x). \tag{B.19}$$

Equation (B.19) has an important implication: in fact, starting from it, one can show that the net effect of the presence of the chemical potential terms in the generating functional is to introduce a shift in the time derivatives

$$\partial_0\to\partial_0+i\sum_k\mu_k T_k^{\mathcal{R}}, \tag{B.20}$$

where the $T_k^{\mathcal{R}}$ are the generators of the symmetry group associated to the charge Q_k in the appropriate representation $\mathcal{R}$.

We have still to choose a specific path C in order to obtain the explicit expressions of the Green's functions. In order to find a suitable path we need to study the analyticity properties of the *two-point function.*

$$D_C(x,x') = \vartheta_C(t-t')D_C^{>}(x,x') + \vartheta_C(t'-t)D_C^{<}(x,x'), \tag{B.21}$$

where we defined the *advanced and retarded propagators*

$$D_C^{>}(x,x') = \left\langle \Phi(x)\Phi(x')\right\rangle_\beta , \tag{B.22a}$$
$$D_C^{<}(x,x') = \left\langle \Phi(x')\Phi(x)\right\rangle_\beta = D_C^{>}(x',x). \tag{B.22b}$$

The two-point function satisfies the symmetry property

$$D_C(x,x') = D_C(x',x). \tag{B.23}$$

Given a set of common eigenstates of the Hamiltonian H and of the charges Q_k, $\{|n\rangle, \{q_k^n\}\}$, with $H|n,\{q_k^n\}\rangle = E_n|n,\{q_k^n\}\rangle$, we can write

$$\begin{aligned} D_C^{>}(x,x') = \left\langle \Phi(x)\Phi(x')\right\rangle_\beta &= \frac{1}{\mathcal{Z}(\beta,\{\mu_k\})}\,\mathrm{Tr}\Big[e^{-\beta\left(H-\sum_k \mu_k Q_k\right)}\Phi(x)\Phi(x')\Big] \\ &= \frac{1}{\mathcal{Z}(\beta,\{\mu_k\})}\sum_{n,m}\left\langle m,\{q_k^m\}\middle|\Phi(0,\vec{x})e^{i(t'-t)H}\middle|n,\{q_k^n\}\right\rangle \times \\ &\times \left\langle n,\{q_k^n\}\middle|\Phi(0,\vec{x}\,')e^{-\beta+it-it'}\left(\prod_k e^{\beta\mu_k q_k^m}\right)\middle|m,\{q_k^m\}\right\rangle, \end{aligned} \tag{B.24}$$

which yields

$$\begin{aligned} D_C^{>}(x,x') = \frac{1}{\mathcal{Z}(\beta,\{\mu_k\})}&\sum_{n,m} e^{\beta\sum_k \mu_k q_k^m}e^{(-\beta+it-it')E_m}e^{-i(t-t')E_n} \\ &\times \langle m,\{q_k^m\}|\Phi(0,\vec{x})|n,\{q_k^n\}\rangle \left\langle n,\{q_k^n\}\middle|\Phi(0,\vec{x}\,')\middle|m,\{q_k^m\}\right\rangle. \end{aligned} \tag{B.25}$$

We observe that the convergence of the sum implies that the advanced propagator is analytic for $-\beta < \mathrm{Im}(t-t') < 0$. A similar calculation shows that the retarded propagator is analytic in the interval $0 < \mathrm{Im}(t-t') < \beta$. Since the limit of an analytic function at the boundary of its analyticity domain is a distribution, we can consider also the extreme values in the previous intervals. This implies that the full two-point function is analytic in the interval $-\beta \le \mathrm{Im}(t-t') \le \beta$ if $\vartheta_C(t) = 0$ for $\mathrm{Im}(t) > 0$. In other words, looking at the definition of ϑ_C, the path C must be chosen in such a way that $\mathrm{Im}(t)$ is a nonincreasing function of v.

Another property of the path C can be found by looking at the generating functional. We have

$$\begin{aligned}\mathcal{Z}(\beta,\{\mu_k\},j) &= \mathrm{Tr}\left[e^{-\beta(H-\sum_k \mu_k Q_k)} T_C \exp\left(i\int_C \mathrm{d}^4x\, j(x)\cdot\Phi(x)\right)\right] \\ &= \int \mathcal{D}\Phi'\left\langle \Phi'(t_i,\vec{x})\middle| e^{-\beta(H-\sum_k \mu_k Q_k)} T_C \exp\left(i\int_C \mathrm{d}^4x\, j(x)\cdot\Phi(x)\right)\middle|\Phi'(t_i,\vec{x})\right\rangle \\ &= \int \mathcal{D}\Phi'\left\langle \Phi'(t_i - i\beta,\vec{x})\middle| e^{\sum_k \beta\mu_k Q_k} T_C \exp\left(i\int_C \mathrm{d}^4x\, j(x)\cdot\Phi(x)\right)\middle|\Phi'(t_i,\vec{x})\right\rangle,\end{aligned} \tag{B.26}$$

where $\{|\Phi'(t_i,\vec{x})\rangle\}$ is a set of instantaneous eigenstates of the field operator in the Heisenberg picture, evaluated at some initial time t_i. Equation (B.26) implies that if our path starts at time t_i it must end at time $t - i\beta$.

A path that satisfies these two constraints and that contains the real axis, as it should since we want to work with a real time variable, is the one depicted in Figure B.1.

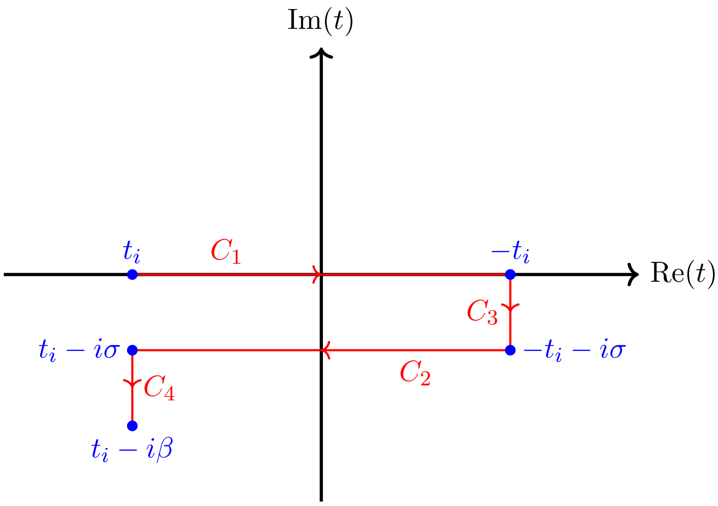


Figure B.1: Inspired by Fig. 3.6 of [96]

Out path starts at a time $t_i \to -\infty$ on the real axis, then moves through a segment C_1 to $-t_i$. A segment C_3 then follows, carrying the path to a point $-t_i - i\sigma$, where $\sigma \in [0,\beta]$ can be chosen arbitrarily. Then a segment C_2 moves towards $t_i - i\sigma$, followed by a final segment C_4 bringing the path to its end at $t_i - i\beta$.

One can prove that the contributions of C_3 and C_4 to the generating functional can be neglected, so that we end up with a doubled time axis composed of the oppositely directed infinite segments C_1 and C_2. We can then denote by an index $i = 1, 2$ our fields and sources depending on whether their time coordinate lies on C_1 or C_2; what we obtain is a doubling of our degrees of freedom, which must be taken into account in perturbation theory. Of course, external lines of correlation functions only depend on the "physical" degrees of freedom with time coordinate on C_1, but internal legs can be of either type.

Appendix C

Meson condensation

We want to give a brief introduction to meson condensation, an important effect of finite density in QCD, following closely [97]. Let us consider the case $N_f = 3$. Taking for the moment the assumption of massless quarks, the QCD Lagrangian has a global symmetry group $SU(3)_L \times SU(3)_R \times U(1)_V$. The chiral subgroup $SU(3)_L \times SU(3)_R$ is spontaneously broken to $SU(3)_V$ at low energies, when the chiral condensate $\langle \bar{q}_R q_L \rangle$ becomes nonzero. Moreover, due to the spontaneous breaking of the chiral symmetry, the condensate is not invariant under $SU(3)_L \times SU(3)_R$, but rather it transforms under the $(\mathbf{3}, \bar{\mathbf{3}})$ representation. We then have infinite degenerate vacuum configurations, connected by chiral transformations, each characterised by a specific value of the condensate.

In absence of chemical potential, the nonvanishing condensate can be parametrised as $\langle \bar{q}_R q_L \rangle = \langle \bar{q}q \rangle_0 \, \mathbb{1}_3$, with $\langle \bar{q}q \rangle_0$ a fixed value. The condensate for the other degenerate vacua can then be obtained by specifying a chiral transformation $(\tilde{L}, \tilde{R})$:

$$\begin{aligned} \langle \bar{q}_R q_L \rangle \to \tilde{L} \, \langle \bar{q}_R q_L \rangle \, \tilde{R}^\dagger = \tilde{L}(\langle \bar{q}q \rangle_0 \, \mathbb{1}_3)\tilde{R}^\dagger = \langle \bar{q}q \rangle_0 \, \tilde{L}\tilde{R}^\dagger \equiv \\ \equiv \langle \bar{q}q \rangle_0 \, \Sigma. \end{aligned} \tag{C.1}$$

As shown in [98], assuming $\Sigma = \Sigma(x)$, one can recognise in it the field describing the light meson octet in ChPT, $\Sigma(x) = e^{i \frac{\pi^a(x)}{f_\pi} \lambda^a}$.

Taking into account the explicit symmetry breaking due to the nonvanishing masses of the quarks, the degeneracy between the various vacua is lost, and one can find the homogeneous and time independent ground state configuration of the meson fields Σ_0 by maximising the static Lagrangian

$$\mathcal{L}_{\text{static}} = \frac{f_\pi^2 B_0}{2} \, \text{Tr}\Big[M(\Sigma_0 + \Sigma_0^\dagger) \Big], \tag{C.2}$$

corresponding to minus the scalar potential. In this case, the vacuum configuration in the trivial one $\Sigma_0 = \mathbb{1}$.

We can now exploit the main result of Appendix B, *i.e.* the prescription of eq. (B.20), to include the effects of nonvanishing chemical potentials for isospin, μ_I, and strangeness, μ_S, in the Lagrangian

$$\partial_0 \Sigma_0 \to \partial_0 \Sigma_0 + i[\mu, \Sigma_0], \quad \mu = \text{diag}\Big(\frac{\mu_I}{2}, -\frac{\mu_I}{2}, -\mu_S \Big), \tag{C.3}$$

It introduces another source of explicit symmetry breaking, with pattern $SU(3)_V \times U(1)_B \to U(1)_I \times U(1)_Y \times U(1)_B$, where the first two subgroups are generated by $\lambda^3/2$ and $\lambda^8/2$, thus modifying the ground state configuration, which can be obtained by maximisation of

$$\mathcal{L}_{\text{static}} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}\Big\{[\Sigma_0,\mu]\Big[\mu,\Sigma_0^\dagger\Big]\Big\} + \frac{f_\pi^2 B_0}{2}\,\mathrm{Tr}\Big[M(\Sigma_0+\Sigma_0^\dagger)\Big]. \tag{C.4}$$

To perform this maximisation, we parametrise the matrix Σ_0 as

$$\begin{aligned}\Sigma_0 &= e^{i\vartheta\vec{n}\cdot\vec{\lambda}} = \cos\vartheta\,\mathbb{1}_3 + i\sin\vartheta\,\vec{n}\cdot\vec{\lambda},\\ |\vec{n}| &= 1, \quad \vartheta\in[-\pi/2,\pi/2],\end{aligned} \tag{C.5}$$

where $\vec{\lambda} = (\lambda^1,\dots,\lambda^8)$ is a vector whose components are the Gell-Mann matrices. In principle, we can then use the angle ϑ and the eight components of $\vec{n}$ as variational parameters to find the ground state configuration Σ_0. This is, however, extremely difficult to do. What is done instead, as described in [97], is to use group theory arguments to impose constraints on $\vec{n}$ in the various cases of interest, and then perform the maximisation with a reduced number of variational parameters.

The presence of nonvanishing chemical potentials modifies also the dispersion relations of the light mesons, which now have masses

$$m_{\pi^0} = m_\pi, \tag{C.6a}$$

$$m_{\pi^\pm} = m_\pi \mp \mu_I, \tag{C.6b}$$

$$m_\eta = \sqrt{\frac{4m_K^2 - m_\pi^2}{3}}, \tag{C.6c}$$

$$m_{K^\pm} = m_K \mp \frac{1}{2}\mu_I \mp \mu_S, \tag{C.6d}$$

$$m_{K^0,\bar{K}^0} = m_K \pm \frac{1}{2}\mu_I \mp \mu_S, \tag{C.6e}$$

where m_π and m_K are the meson masses for $\mu_I = \mu_S = 0$. We observe that for given values of the chemical potentials some of the mesons can become massless. By group theory arguments, it can be shown that these massless modes correspond to Goldstone bosons for the spontaneous breaking of a subgroup of $U(1)_I\times U(1)_Y\times U(1)_B$, and that the corresponding fields acquire a nonvanishing vacuum expectation value. This regime is known as *meson condensation*.

The chiral condensate can be parametrised as

$$\langle \bar{q}_R q_L\rangle \equiv \langle\bar{q}q\rangle_0\, e^{i\alpha}\Sigma_0, \tag{C.7}$$

where Σ_0 characterises the *orientation* of the ground state, and where we added a phase $e^{i\alpha}$ associated to the anomalous $U(1)_A$ symmetry. Nonperturbative effects generate a potential for α, which is minimised for $\alpha = 0$. Meson-condensed phases correspond to nontrivial orientations of the QCD vacuum, $\Sigma_0 \neq \mathbb{1}_3$.

C.1 Kaon condensation

Let us explore more in detail the possibility of kaon condensation. As discussed in detail in [20], kaon condensation has interesting effects on axion physics. It modifies the axion potential, with an effective change in the axion mass. This, in turn, could lead, at even higher densities, to an *axion-condensed phase* in which CP symmetry is spontaneously broken in the neutral sector of the theory.

Our starting point is the modification of the dispersion relation of the charged kaons when a chemical potential for electric charge μ is introduced

$$\omega_{K^\pm}(\vec{k}) = \sqrt{(m^2_{K^\pm})_n + k^2} \pm \mu. \tag{C.8}$$

Here we defined the in-medium kaon mass

$$(m^2_{K^\pm})_n = \frac{1}{f_\pi^2}\left(-\frac{\langle \bar{u}u + \bar{s}s\rangle_n}{2} m_s - \frac{1}{2}(n + n_p)\mu\right), \tag{C.9}$$

where the first contribution is just the usual kaon mass term in which we took into account the change in the value of the quark condensate at finite density, and the second term just stems from the introduction of the chemical potential in the kinetic term of the baryon field. By using eq. (5.7), one finds

$$-\frac{\langle \bar{u}u + \bar{s}s\rangle_n}{2f_\pi^2} m_s = m_K^2\left\{1 - \frac{1}{2}\left[b_1 - b_2\left(\frac{2n_p}{n} - 1\right) + b_3\right]\frac{n}{n_0}\right\}, \tag{C.10}$$

where $m_K^2 = -m_s \langle \bar{q}q\rangle_0 / f_\pi^2$ is the neutral kaon mass in vacuum, neglecting $\mathcal{O}(m_{u,d}/m_s)$ terms. It follows that, since $b_1 \gg b_2$, the kaon mass decreases with density. This is different from the case of the in-medium charged pion mass

$$\left(m^2_{\pi^\pm}\right)_n = \frac{1}{f_\pi^2}\left(-\left\langle \bar{u}u + \bar{d}d\right\rangle_n \bar{m} + \frac{1}{2}(n - 2n_p)\mu\right), \tag{C.11}$$

where $\bar{m} = (m_u + m_d)/2$, which, in a neutron-rich background $n_p/n < 1/2$, becomes larger with increasing density. This, together with an increase in the in-medium pion mass due to higher order ChPT corrections [73], makes pion condensation in systems like NSs less likely than kaon condensation.

Condensation of the negative kaons occurs when $\omega_{K^-}(\vec{0}) = (\mu_{K^\pm})_n - \mu = 0$. This introduces a nontrivial orientation of the QCD vacuum

$$\Sigma_0 = \begin{pmatrix} \cos\vartheta & 0 & i\sin\vartheta \\ 0 & 1 & 0 \\ i\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}, \tag{C.12}$$

which gives a potential

$$V(\vartheta) = -\frac{1}{2}\mu^2 f_\pi^2 \sin^2\vartheta - f_\pi^2 (m_K^\pm)_n \cos\vartheta, \tag{C.13}$$

that, if minimised, gives

$$\cos\vartheta = \min\left[1, \frac{(m^2_{K^\pm})_n}{\mu^2}\right]. \tag{C.14}$$

Now, assuming electric charge neutrality in the medium $n_{\mathrm{EM}} = -\langle \partial\mathcal{L}/\partial\mu\rangle = 0$, we obtain the condition

$$-f_\pi^2 \mu \sin^2\vartheta + \cos\vartheta n_p - \sin^2\frac{\vartheta}{2} n_n - n_e(\mu) - n_\mu(\mu) = 0, \tag{C.15a}$$

$$n_\ell(\mu) = \vartheta(|\mu| - m_\ell)\,\mathrm{sgn}(\mu)\frac{(\mu^2 - m_\ell^2)^{3/2}}{3\pi^2}, \quad \ell = e, \mu, \tag{C.15b}$$

where n_p is the proton number density and $n_n = n - n_p$ is the neutron number density. Equations (C.14) and (C.15a) together yield the values of ϑ and μ as functions of the density n and the proton fraction n_p/n. One can see that, for symmetric nuclear matter, $n_p/n = 1/2$, kaon condensation happens for $n \simeq 2.5n_0$, so way above the density within control of our approximations in Chapter 5.

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