UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea in Fisica

Tesi di Laurea

Bosonic and fermionic quantum gases in D dimensions

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Anno Accademico 2019/2020

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Chapter 1

Introduction

Systems composed of quantum gases which follow the Bose-Einstein statistics (introduced in 1924) or the Fermi-Dirac statistics (introduced in 1926) have been thoroughly studied in the last century, and are of fundamental importance in modern physics of matter. Quantum statistical mechanics has been essential in successfully describing a variety of phenomena ranging from the superfluidity of helium-4 to the heat capacity of metals.

The study of quantum gases has been object of a renewed interest in the last decades due to the experimental results obtained thanks to the development of new techniques in laser cooling and evaporative cooling. With these techniques it was possible to bring dilute alkali-metal atoms to very low temperatures (of the order of $100\,\mathrm{nK}$) and achieve Bose-Einstein condensation (BEC) in 1995 [1] and the Fermi quantum degeneracy in 1999 [2].

In this thesis we will firstly rederive the formulas for the finite temperature spatial distributions of fermionic and bosonic quantum gases in a generic external potential, along with the Fermi temperature and the Bose transition temperature for the rigid box case, but generalizing those formulas for D-dimensional spaces. We will show that these predictions are consistent with recent experiments with ultracold gases.

Then we will study the case of the isotropic power-law potential in a D-dimensional space, deriving the momentum distribution and eventually the Fermi temperature, Bose transition temperature and the formula for the condensed fraction of a Bose gas, deducing the condition that allows a bosonic gas to show the BEC. The power-law potential covers many common potentials studied in the theory and used in recent experiments, the most important probably being the harmonic potential.

Throughout this thesis we will follow the approach introduced in [3] for studying quantum ideal gases by solving integrals in a $2D$-dimensional phase space.

The study of these systems in D-dimensional spaces is not just a mathematical abstraction but it is motivated by recent experiments with reduced dimension.

Chapter 2

External potential and rigid box in D dimension

2.1 Quantum gases and the semiclassical limit

In this chapter we will consider a confined quantum gas of noninteracting identical fermions or bosons in a D-dimensional space. We will call $|\alpha\rangle$ the single-particle eigenstate of the Hamiltonian H, and the corresponding energy eigenvalue ϵ_α.
We recall that, due to the different behavior of the wavefunction of fermions and bosons with respect to exchange which leads to the Pauli exclusion Principle, calling N_α the number of particles in the state $|\alpha\rangle$ we have

$$N_\alpha = \begin{cases} 0, 1, 2, 3\ldots & \text{for bosons} \\ 0, 1 & \text{for fermions} \end{cases} \tag{2.1}$$

Working in the grand canonical ensemble of equilibrium mechanics [4] we can write the average number N_α of particles in the state $|\alpha\rangle$ as

$$N_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} \pm 1} \tag{2.2}$$

where the sign $+$ is for fermions and the sign $-$ is for bosons, while following the standard notation μ is the chemical potential and $\beta = \frac{1}{kT}$, with k the Boltzmann constant and T the absolute temperature.
Summing over all the values of ϵ_α we can obtain the average total number of particles N

$$N = \sum_\alpha N_\alpha \tag{2.3}$$

This condition fixes the chemical potential, which becomes a function of β and N.
In the case of fermions μ has no limitations, at zero temperature it is called the **Fermi energy** and denoted by E_F. From the Fermi energy we immediately find the **Fermi temperature**

$$T_F = \frac{E_F}{k} \tag{2.4}$$

Below this temperature the particles begin to fill the lowest available single-particle states, with only one particle being allowed per state in accordance with the Pauli exclusion Principle shown in equation 2.1. This phenomenon is called the **Fermi quantum degeneracy**.
In the case of bosons on the other hand we have that $\mu < \epsilon_0$, where ϵ_0 is the lowest single-particle energy level. When μ approaches ϵ_0 the function N_0 diverges. The physical meaning of this is that the lowest single-particle level becomes macroscopically occupied: this is the **Bose-Einstein condensation** (BEC). One can calculate the condensed fraction $\frac{N_0}{N}$ and the **Bose transition temperature** T_B by studying the system at $\mu = \epsilon_0$.
One can see the interpretation of the difference between bosons and fermions at low temperature in figure 2.1.

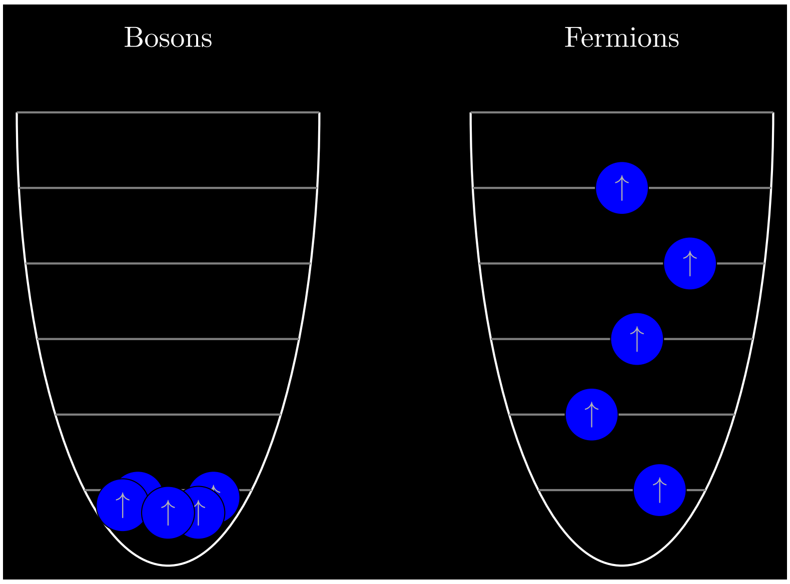

Figure 2.1: The different behavior of bosons and fermions approaching $T = 0\,\mathrm{K}$

If the number of particles is large and the energy level spacing is smaller than kT we can work in the semiclassical limit. In this approximation the D-dimensional system is described by a continuum of states, therefore instead of ϵ_α we can use the classical single-particle phase space energy $\epsilon(\vec{r},\vec{p})$, where

$$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_D \end{pmatrix} \qquad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_D \end{pmatrix} \tag{2.5}$$

are respectively the position vector and the linear momentum vector.
In this context equation 2.2 becomes

$$n(\vec{r},\vec{p}) = \frac{1}{e^{\beta(\epsilon(\vec{r},\vec{p})-\mu)} \pm 1} \tag{2.6}$$

We will follow the usual procedure [5] and set the quantum elementary volume of the $2D$-dimensional phase space cell to $h^D = (2\pi\hbar)^D$ where h and $\hbar$ are respectively the Planck constant and the reduced Planck constant. We can then write the average number N of particles in the D-dimensional space as

$$N = \int \frac{d^D r d^D p}{(2\pi\hbar)^D} n(\vec{r},\vec{p}) \tag{2.7}$$

If we introduce the spatial distribution

$$n(\vec{r}) = \int \frac{d^D p}{(2\pi\hbar)^D} n(\vec{r},\vec{p}) \tag{2.8}$$

and the momentum distribution

$$n(\vec{p}) = \int \frac{d^D r}{(2\pi\hbar)^D} n(\vec{r},\vec{p}) \tag{2.9}$$

it follows that N can also be written in the following ways:

$$N = \int d^D r n(\vec{r}) = \int d^D p n(\vec{p}) \tag{2.10}$$

The semiclassical formula for the density of states reads

$$\rho(\epsilon) = \int \frac{d^D r d^D p}{(2\pi\hbar)^D} \delta(\epsilon - \epsilon(\vec{r}, \vec{p})) \tag{2.11}$$

where $\delta(x)$ is the Dirac delta function. With this we can easily see that

$$\begin{aligned} N &= \int \frac{d^D r d^D p}{(2\pi\hbar)^D} n(\vec{r}, \vec{p}) = \\ &= \int \frac{d^D r d^D p}{(2\pi\hbar)^D} \frac{1}{e^{\beta(\epsilon(\vec{r},\vec{p})-\mu)} \pm 1} = \\ &= \int_0^\infty d\epsilon \left(\int \frac{d^D r d^D p}{(2\pi\hbar)^D} \delta(\epsilon - \epsilon(\vec{r}, \vec{p})) \right) \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} \end{aligned}$$

It follows that we can write

$$N = \int_0^\infty d\epsilon \rho(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} \tag{2.12}$$

It is important to notice that for fermions in the limit $\beta \to \infty$ (so at zero temperature) where $\mu \to E_F$ the phase space distribution becomes

$$n(\vec{r}, \vec{p}) = \Theta(E_F - \epsilon(\vec{r}, \vec{p})) \tag{2.13}$$

where $\Theta(x)$ is the Heaviside step function, as can be seen in figure 2.2.

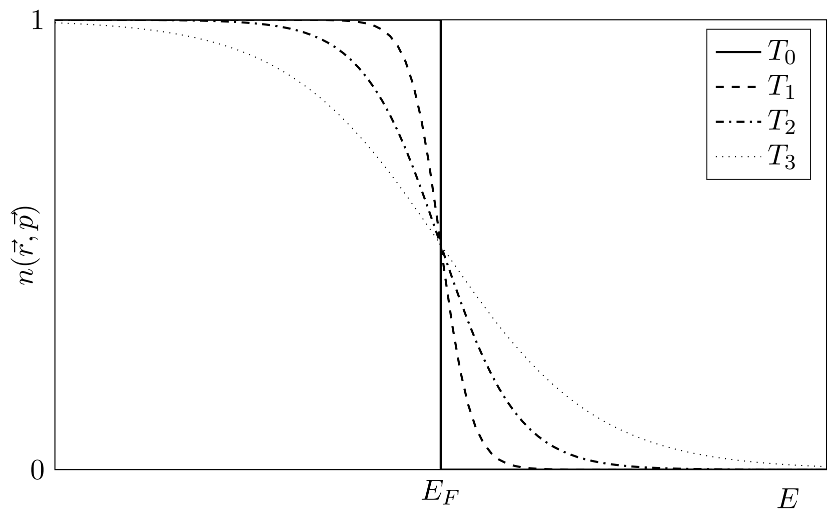

Figure 2.2: Fermi-Dirac distribution for different temperatures.

In the case of bosons equation 2.6 only describes the noncondensed thermal cloud. In this thesis we will avoid studying the density profile of the condensed fraction, but the most important properties of the Bose-Einstein gas at low temperatures (i.e. the BEC transition temperature T_B and the condensed fraction $\frac{N_0}{N}$) can be obtained by studying the noncondensed fraction (thermal cloud).

2.2 Confining external potential

We will now consider an ideal Fermi or Bose gas in an external confining potential $U(\vec{r})$, again in a D-dimensional space. We know that the classical single-particle energy can be written as

$$\epsilon(\vec{r}, \vec{p}) = \frac{\vec{p}^2}{2m} + U(\vec{r}) \tag{2.14}$$

where $\frac{\vec{p}^2}{2m}$ is the kinetic energy and m is the mass of the particle.
We can now begin our study by showing a preliminary result:

Theorem 2.2.1. *The semiclassical density of states can be approximated as*

$$\rho(\epsilon) = \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} \int d^D r(\epsilon - U(\vec{r}))^{\frac{D-2}{2}} \tag{2.15}$$

where $\Gamma(x)$ *is the Euler gamma function.*

Proof. From equations 2.11 and 2.14 we have

$$\begin{aligned}\rho(\epsilon) &= \int \frac{d^D r d^D p}{(2\pi\hbar)^D} \delta(\epsilon - \epsilon(\vec{r}, \vec{p})) = \\ &= \int \frac{d^D r d^D p}{(2\pi\hbar)^D} \delta\left(\epsilon - \left(\frac{\vec{p}^2}{2m} + U(\vec{r})\right)\right)\end{aligned}$$

To solve this we will recall a formula which will be used frequently throughout this thesis [6]. Suppose we have a function f such that $f(\vec{x}) = f(|\vec{x}|)$ (f only depends upon the distance from the origin) and λ_n is the Lebesgue measure in $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} f(\vec{x}) d\lambda_n(\vec{x}) = s_{n-1} \int_0^{+\infty} r^{n-1} f(r) dr \tag{2.16}$$

where $s_{n-1} = \lambda_{\mathbb{S}^{n-1}}(\mathbb{S}^{n-1})$ is the measure of the $(n-1)$-dimensional unit sphere, which reads

$$s_{n-1} = n\lambda_n(B_n) = n\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{2.17}$$

B_n being the n-dimensional unit ball.
With this we can write

$$\rho(\epsilon) = \frac{1}{(2\pi\hbar)^D} \int d^D r D \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} dp p^{D-1} \delta\left(\epsilon - \left(\frac{p^2}{2m} + U(\vec{r})\right)\right)$$

We now need to exploit the following property of the Dirac delta function:

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{2.18}$$

where x_i are the points such that

$$g(x_i) = 0$$

so that we can substitute[1]

$$\begin{aligned}\delta\left(\epsilon - \left(\frac{p^2}{2m} + U(\vec{r})\right)\right) &= \delta\left(\epsilon - \frac{p^2}{2m} - U(\vec{r})\right) = \\ &= \frac{\delta\left(p - \sqrt{2m(\epsilon - U(\vec{r}))}\right)}{\sqrt{\frac{2(\epsilon - U(\vec{r}))}{m}}}\end{aligned}$$

We then get

$$\rho(\epsilon) = \frac{D\pi^{\frac{D}{2}}}{(2\pi\hbar)^D} \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} \int d^D r \sqrt{\frac{m}{2(\epsilon - U(\vec{r}))}} (2m)^{\frac{D-1}{2}} (\epsilon - U(\vec{r}))^{\frac{D-1}{2}}$$

which eventually leads to equation 2.15, after a few calculations and remembering that for $n \in \mathbb{N}$

$$\Gamma(n) = (n-1)!$$

□

[1] Remembering that we are integrating in $[0, +\infty]$.

It is essential now to introduce the Fermi function and the Bose function [7] [8].
The Fermi function is given by

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{z e^{-y} y^{n-1}}{1 + z e^{-y}} \tag{2.19}$$

while the Bose function is

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{z e^{-y} y^{n-1}}{1 - z e^{-y}} \tag{2.20}$$

These two functions are connected by the property

$$f_n(z) = -g_n(-z)$$

For $|z| < 1$ they can be rewritten as

$$f_n(z) = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{z^\ell}{\ell^n} \tag{2.21}$$

and

$$g_n(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^n} \tag{2.22}$$

Another property of the g_n function which we will need later on is that $g_n(1) = \zeta(n)$, where ζ is the Riemann ζ-function.
We can now state two theorems about ideal Fermi and Bose gases in an external potential.

Theorem 2.2.2. *For an ideal Fermi gas in an external potential $U(\vec{r})$ and D-dimensional space the finite temperature spatial distribution is*

$$n(\vec{r}) = \frac{1}{\lambda^D} f_{\frac{D}{2}} \left(e^{\beta(\mu - U(\vec{r}))} \right) \tag{2.23}$$

where

$$\lambda = \left(\frac{2\pi\hbar^2\beta}{m} \right)^{\frac{1}{2}}$$

is the thermal length and μ the chemical potential. The zero temperature spatial distribution is

$$n(\vec{r}) = \left(\frac{m}{2\pi\hbar^2} \right) \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} (E_F - U(\vec{r}))^{\frac{D}{2}} \Theta(E_F - U(\vec{r})) \tag{2.24}$$

where E_F is the Fermi energy.

Proof. To find the spatial distribution defined in equation 2.8 we need to integrate 2.6 over the momenta with the sign $+$. Using again the formula 2.16 we have

$$\begin{aligned} n(\vec{r}) &= \int d^D p \frac{1}{(2\pi\hbar)^D} \frac{1}{e^{\beta\left(\frac{\vec{p}^2}{2m} + U(\vec{r}) - \mu\right)} + 1} = \\ &= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} dp \frac{p^{D-1}}{e^{\beta\left(\frac{p^2}{2m} + U(\vec{r}) - \mu\right)} + 1} = \\ &= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} dp \frac{p^{D-1} e^{\beta(\mu - U(\vec{r}))} e^{-\beta\frac{p^2}{2m}}}{1 + e^{\beta(\mu - U(\vec{r}))} e^{-\beta\frac{p^2}{2m}}} \end{aligned}$$

If we now define $z = e^{\beta(\mu - U(\vec{r}))}$ and $y = \beta\frac{p^2}{2m}$ we can rewrite the integral as follows

$$n(\vec{r}) = \frac{\pi^{\frac{D}{2}}}{(2\pi\hbar)^D} \left(\frac{2m}{\beta} \right)^{\frac{D}{2}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} \int_0^{+\infty} \frac{z e^{-y} y^{\frac{D}{2}-1}}{1 + z e^{-y}}$$

It is clear that using the definition of the Fermi function and of thermal length this becomes

$$\begin{aligned} n(\vec{r}) &= \left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{D}{2}} f_{\frac{D}{2}}(z) = \\ &= \frac{1}{\lambda^D} f_{\frac{D}{2}}(z) \end{aligned}$$

which is the result we were looking for.
In the limit $T \to 0$ we need to integrate the distribution 2.13

$$\begin{aligned} n(\vec{r}) &= \frac{1}{(2\pi\hbar)^D} \int d^D p \Theta\left(E_F - \frac{\vec{p}^2}{2m} - U(\vec{r})\right) = \\ &= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} dp p^{D-1} \Theta\left(E_F - \frac{p^2}{2m} - U(\vec{r})\right) \end{aligned}$$

As a function of p the Θ is different from 0 only in $\left[0, \sqrt{2m(E_F - U(\vec{r}))}\right]$, we can then restrict the domain and compute the integral

$$\begin{aligned} n(\vec{r}) &= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \Theta(E_F - U(\vec{r})) \int_0^{\sqrt{2m(E_F - U(\vec{r}))}} dp p^{D-1} = \\ &= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \Theta(E_F - U(\vec{r})) \frac{1}{D} (2m(E_F - U(\vec{r})))^{\frac{D}{2}} \end{aligned}$$

which leads to the desired result. □

The second theorem is analogous to the one we just discussed but this time we are considering an ideal Bose gas.
Theorem 2.2.3. *For an ideal Bose gas in an external potential $U(\vec{r})$ and D-dimensional space the finite temperature spatial distribution is*

$$n(\vec{r}) = \frac{1}{\lambda^D} g_{\frac{D}{2}}\left(e^{\beta(\mu - U(\vec{r}))}\right) \tag{2.25}$$

where λ is the thermal length and μ the chemical potential.

Proof. The proof is identical to the one already provided for fermions, one only need to substitute the sign $+$ with a $-$ in equation 2.6, then again using the substitution $z = e^{\beta(\mu - U(\vec{r}))}$ one gets the finite temperature spatial distribution. □

These theorems are a generalization of the classical results for ideal homogeneous Fermi and Bose gases, they allow us to study the system in the 3-dimensional space or in a space with reduced dimension and in a generic confining external potential.

2.3 The D-dimensional rigid box

We can now show how these last two theorems allow us to study the Fermi temperature and Bose transition temperature in the rigid box case, and comparing the results we will obtain with the classical ones will assess the validity of our methods.
Setting $U(\vec{r}) = 0$ (which translates to studying the box of volume V) and imposing the normalization condition 2.10 to the zero temperature spatial distribution for a Fermi gas 2.24 we have

$$\begin{aligned} \int d^D r n(\vec{r}) &= \int d^D r \left(\frac{m}{2\pi\hbar^2}\right) \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} E_F^{\frac{D}{2}} \Theta(E_F) = \\ &= V \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} E_F^{\frac{D}{2}} = \\ &= N \end{aligned}$$

By solving for $E_F = kT_F$ we can easily get

$$kT_F = \left(\frac{2\pi\hbar^2}{m}\right)\left[\Gamma\left(\frac{D}{2}+1\right)n\right]^{\frac{2}{D}} \tag{2.26}$$

where $n = \frac{N}{V}$ is the homogeneous density of particles.
For the Bose transition temperature we must remember that at low temperatures we have $\mu = \epsilon_0$ and that in the case of a box of volume $V \to \infty$ we have $\epsilon_0 \to 0$. This is due to the wavenumber of the lowest single-particle energy state being inversely proportional to the volume of the box, which causes the momentum to approach 0. It can be easily seen for a particle in a 1-dimensional box but it holds in general.
Keeping this in mind along with the fact that $g_n(1) = \zeta(n)$ we can write

$$\begin{aligned}\int d^D r n(\vec{r}) &= \int d^D r \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} g_{\frac{D}{2}}(e^{\beta\epsilon_0}) = \\ &= V\left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \zeta\left(\frac{D}{2}\right) = \\ &= N\end{aligned}$$

In the end we have

$$kT_B = \frac{2\pi\hbar^2}{m}\left(\frac{n}{\zeta\left(\frac{D}{2}\right)}\right)^{\frac{2}{D}} \tag{2.27}$$

We can now compare these formulas with the known ones for the $D = 3$ case. Setting $v = \frac{1}{n}$ equations 2.23 and 2.25 become

$$\frac{\lambda^3}{v} = \begin{cases} f_{\frac{3}{2}}(z) & \text{for fermions} \\ g_{\frac{3}{2}}(z) & \text{for bosons}\end{cases} \tag{2.28}$$

remembering that the second equation only describes the noncondensed cloud. These are the well known formulas for ideal quantum gases in a 3-dimensional box.
We can now compute the Fermi temperature and Bose temperature for $D = 3$, using the fact that $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$$kT_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{\frac{2}{3}} \tag{2.29}$$

$$kT_B = \frac{2\pi\hbar^2}{m\left(v\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}} \tag{2.30}$$

These results are the same one can find in [4], showing the effectiveness of our approach.
Another thing we can notice is that for $D = 2$ the Riemann ζ function diverges, which shows us that Bose-Einstein condensation can not happen in 2-dimensional spaces (in the box potential case, we will see that this does not hold for different potentials later on). This is another known fact and a confirmation of the validity of equation 2.27.

2.4 Experimental results for a 3-dimensional box

Despite being the simplest and most studied system, the low temperature quantum gas confined in a box potential is extremely difficult to achieve, as it is usually much easier to create a harmonic trap. It wasn't until these last years that state-of-the-art experiments with ultracold gases managed to create quasi-uniform box potentials.
We will now briefly describe a recent experiment [9] which achieved Bose-Einstein condensation in a 3-dimensional box. Without entering into the detail, a sample of ^{87}Rb atoms has been pre-cooled in a harmonic trap and then loaded in a box trap. Figure 2.3 shows a scheme of the apparatus: in a)

we can see that the box is composed of three 532 nm laser beams (one "tube" beam and two "sheet" beams) confining the gas in a cylindrical region. These green beams are created reflecting a single Gaussian beam on a phase-imprinting spatial light modulator, as shown in b). The gravitational force is also cancelled by a magnetic field gradient.
c) shows the loading of the atoms in the box after being cooled in a harmonic trap, while in d) we can see the images of the cloud just before and after being loaded in the box, along with the corresponding density profiles. The colors represent the optical density (OD), which is the standard way of measuring the atomic density in these experiments (it is obtained recording the absorption of a probe beam and the resulting intensity is related to the atomic density).
The blue and green dashed lines are the predicted density profiles of a respectively harmonic potential and box potential, and we can see that the green line perfectly fits the profile of the box trap.

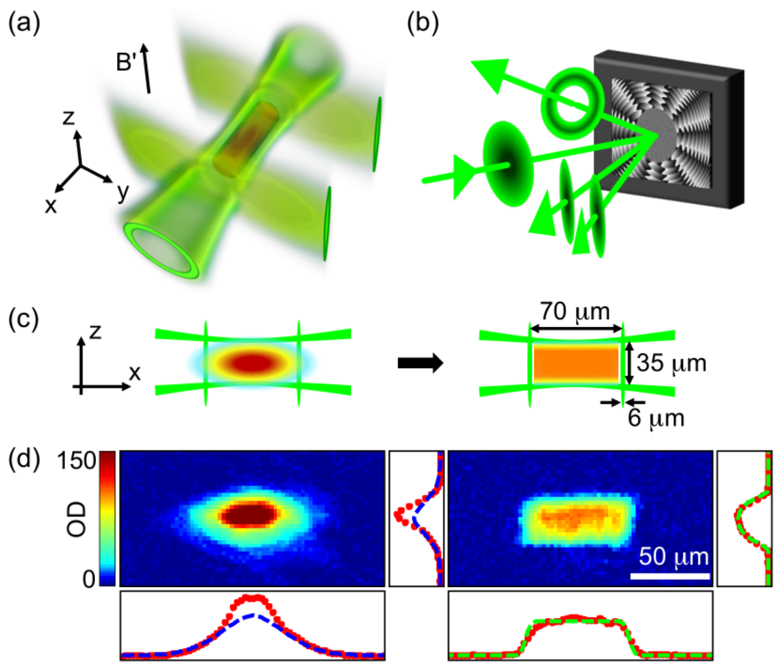

Figure 2.3: A scheme of the experimental apparatus and the images of the gas before and after being loaded into the box. Adapted from [9].

Evaporative cooling is then used to bring the $N \approx 6 \times 10^5$ atoms (which after being confined in the box have temperature of $T \approx 140\,\mathrm{nK}$) below the condensation temperature. Evaporative cooling roughly consists in selecting the hottest atoms of the sample and removing them from the trap, then waiting for the system to re-thermalize at a lower average temperature and then iterate the procedure until the desired temperature is reached.
The results are shown in figure 2.4: P and P_0 are respectively the trapping power and the total laser power, we can see that as the evaporative cooling brings down the temperature there is no dramatic effect on the atomic distribution. This confirms that the system is approximately a uniform box potential, as with harmonic trapping the condensation is both in the position space and in the momentum space.
We can however study the momentum distribution by turning off the trap and letting the gas expand for a time $\tau = 50\,\mathrm{ms}$ and then taking an image of the result. This way one can study the momentum profile of the gas. We can see (again in image 2.4) that as the temperature gets lower eventually a peak in the momentum distribution starts to appear, which indicates the presence of Bose-Einstein condensation.
Fitting the experimental data one can find a Bose temperature $T_B = (92 \pm 3)\mathrm{nK}$, which is consistent with the theoretical prediction for a uniform gas in a rigid box $T_B^0 = (98 \pm 10)\mathrm{nK}$.

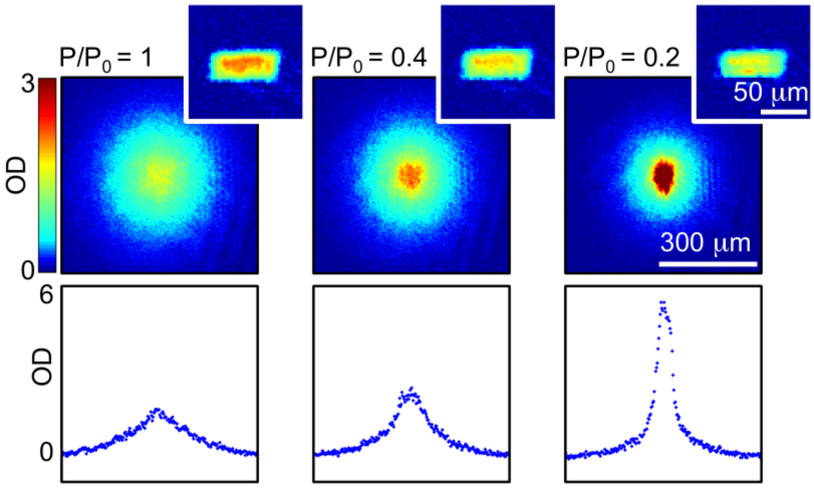

Figure 2.4: Evaporation and Bose-Einstein condensation in a 3-dimensional box. Adapted from [9].

We have thus seen that the approach we have followed leads us to results which are consistent both with the milestone equations for ideal Fermi and Bose quantum gases confined in a box and with the experimental results achieved in recent studies.

Chapter 3

Power-law potential in D dimensions

3.1 Power-law potential and the density of states

As we have already seen it is very difficult to create a uniform potential in experiments, so it is important to study other types of confining potentials. The power-law potential has the form

$$U(\vec{r}) = Ar^n \tag{3.1}$$

where $r = |\vec{r}| = \left(\sum_{\ell=1}^{D} x_\ell^2\right)^{\frac{1}{2}}$ and it is important since it includes the harmonic potential, which is the most widely used one in modern applications. Power-law potentials are also important for studying the effects of adiabatic changes in the trap.

We introduce now a mathematical result that will be useful to us in the solution of the next integrals [8]. Let $\mu, \nu \in \mathbb{C}, \lambda \in \mathbb{R}$ with $\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0$ and $\lambda > 0$. Then

$$\int_0^1 x^{\mu-1}(1-x^\lambda)^{\nu-1}dx = \frac{1}{\lambda}B\left(\frac{\mu}{\lambda},\nu\right) \tag{3.2}$$

where $B(x,y)$ is the Euler beta function, which can be written as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

We can now prove the following result:
Theorem 3.1.1. *The density of states of a quantum gas in a power-law potential can be written as*

$$\rho(\epsilon) = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}}\left(\frac{1}{A}\right)^{\frac{D}{m}}\frac{\Gamma\left(\frac{D}{m}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D}{2}+\frac{D}{m}\right)}\epsilon^{\frac{D}{2}+\frac{D}{m}-1} \tag{3.3}$$

Proof. We need to substitute the potential 3.1 in equation 2.15. Since we are working in the semiclassical approximation, the integral is defined for $\epsilon > U(\vec{r})$. We can then write

$$\begin{aligned}\rho(\epsilon) &= \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}}\frac{1}{\Gamma\left(\frac{D}{2}\right)}\int_{\epsilon>U(\vec{r})} d^D r(\epsilon - Ar^n)^{\frac{D-2}{2}} = \\ &= \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}}\frac{1}{\Gamma\left(\frac{D}{2}\right)}\frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)}\int_0^{\left(\frac{\epsilon}{A}\right)^{\frac{1}{n}}} dr r^{D-1}(\epsilon - Ar^n)^{\frac{D-2}{2}}\end{aligned}$$

where we have followed the usual procedure of solving D-dimensional integrals since the potential is a function of $|\vec{r}|$.

We will now make use of the substitution $x = \left(\frac{A}{\epsilon}\right)^{\frac{1}{n}} r$ which allows us to write the density of states in the following form

$$\rho(\epsilon) = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \frac{D}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{A}\right)^{\frac{D}{n}} \epsilon^{\frac{D}{2}+\frac{D}{n}-1} \int_0^1 dx x^{D-1}(1-x^n)^{\frac{D-2}{2}}$$

We can now use the formula 3.2 and write

$$\rho(\epsilon) = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \frac{D}{n} \left(\frac{1}{A}\right)^{\frac{D}{n}} \epsilon^{\frac{D}{2}+\frac{D}{n}-1} \frac{\Gamma\left(\frac{D}{n}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D}{2}+\frac{D}{n}\right)} =$$

$$= \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \left(\frac{1}{A}\right)^{\frac{D}{m}} \frac{\Gamma\left(\frac{D}{m}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D}{2}+\frac{D}{m}\right)} \epsilon^{\frac{D}{2}+\frac{D}{m}-1}$$

which is what we wanted to prove. □

3.2 Fermi temperature and condensate properties

After finding the density of states we now have to find a formula for the momentum distribution of both bosons and fermions, then we will be able to calculate the information we need about our quantum gases, i.e. Fermi temperature, Bose transition temperature and the condensate fraction for bosons.

We will begin with the Fermi gas:

Theorem 3.2.1. *Let us consider an ideal gas of fermions in a power-law isotropic potential. The finite temperature momentum distribution is*

$$n(\vec{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} f_{\frac{D}{n}}\left(e^{\beta(\mu-\frac{\vec{p}^2}{2m})}\right) \tag{3.4}$$

The zero temperature momentum distribution is

$$n(\vec{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{A}\right)^{\frac{D}{n}} \left(E_F - \frac{\vec{p}^2}{2m}\right)^{\frac{D}{n}} \Theta\left(E_F - \frac{\vec{p}^2}{2m}\right) \tag{3.5}$$

The Fermi energy E_F and the Fermi temperature T_F are given by

$$E_F = kT_F = \left[\left(\frac{2\hbar^2}{m}\right)^{\frac{D}{2}} A^{\frac{D}{n}} \frac{\Gamma\left(\frac{D}{2}+1\right)}{\Gamma\left(\frac{D}{n}+1\right)} \Gamma\left(\frac{D}{2}+\frac{D}{n}+1\right) N\right]^{\frac{1}{\frac{D}{2}+\frac{D}{n}}} \tag{3.6}$$

Proof. To find the momentum distribution we need to use formula 2.9 integrating the function 2.6 with the sign +, with $\epsilon(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2m} + Ar^n$

$$n(\vec{p}) = \int d^D r \frac{1}{(2\pi\hbar)^D} \frac{1}{e^{\beta\left(\frac{\vec{p}^2}{2m}+Ar^n-\mu\right)}+1} =$$

$$= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} dr r^{D-1} \frac{e^{-\beta Ar^n} e^{\beta\left(\mu-\frac{\vec{p}^2}{2m}\right)}}{1+e^{-\beta Ar^n} e^{\beta\left(\mu-\frac{\vec{p}^2}{2m}\right)}}$$

with the substitutions $z = e^{\beta\left(\mu-\frac{\vec{p}^2}{2m}\right)}$ and $y = \beta Ar^n$ and using the definition of Fermi function we get

$$n(\vec{p}) = \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \frac{1}{n} \left(\frac{1}{\beta A}\right)^{\frac{1}{n}} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}-\frac{1}{n}} \int_0^{+\infty} dy \frac{y^{\frac{D}{n}-1} e^{-y} z}{1+e^{-y} z} =$$

$$= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \frac{1}{n} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} \Gamma\left(\frac{D}{n}\right) f_{\frac{D}{n}}(z) =$$

$$= \frac{1}{(2\sqrt{\pi}\hbar)^D} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} f_{\frac{D}{n}}(z)$$

and so we have found the momentum distribution.
The zero temperature momentum distribution on the other hand can be found by integrating equation 2.13:

$$n(\vec{p}) = \int d^D r \frac{1}{(2\pi\hbar)^D} \Theta\left(E_F - \frac{\vec{p}^2}{2m} - Ar^n\right) =$$
$$= \frac{1}{(2\pi\hbar)^D} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} drr^{D-1} \Theta\left(E_F - \frac{\vec{P}^2}{2m} - Ar^n\right)$$

The Θ function is nonzero only for $Ar^n < E_F - \frac{\vec{p}^2}{2m}$, so we can write

$$n(\vec{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{D}{\Gamma\left(\frac{D}{2}+1\right)} \Theta\left(E_F - \frac{\vec{p}^2}{2m}\right) \int_0^{\left(\frac{1}{A}\right)^{\frac{1}{n}}\left(E_F - \frac{\vec{p}^2}{2m}\right)^{\frac{1}{n}}} drr^{D-1} =$$
$$= \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{A}\right)^{\frac{D}{n}} \left(E_F - \frac{\vec{p}^2}{2m}\right)^{\frac{D}{n}} \Theta\left(E_F - \frac{\vec{p}^2}{2m}\right)$$

Now we only need to find the Fermi temperature. To do this we need to use once again the normalization condition 2.10 with the zero temperature momentum distribution

$$N = \int d^D p n(\vec{p}) =$$
$$= \frac{1}{(2\sqrt{\pi}\hbar)^D} \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{A}\right)^{\frac{D}{n}} \frac{D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)} \int_0^{+\infty} dpp^{D-1} \left(E_F - \frac{p^2}{2m}\right)^{\frac{D}{n}} \Theta\left(E_F - \frac{p^2}{2m}\right) =$$
$$= \frac{1}{(2\hbar)^D} \frac{DE_F^{\frac{D}{n}}}{\left(\Gamma\left(\frac{D}{2}+1\right)\right)^2} \left(\frac{1}{A}\right)^{\frac{D}{n}} \int_0^{\sqrt{2mE_F}} dpp^{D-1} \left(1 - \frac{p^2}{2m}\right)^{\frac{D}{n}}$$

Setting now $x = \frac{p^2}{2mE_F}$ we get

$$N = \frac{1}{(2\hbar)^D} \frac{D}{\left(\Gamma\left(\frac{D}{2}+1\right)\right)^2} \left(\frac{1}{A}\right)^{\frac{D}{n}} m^{\frac{D}{2}} 2^{\frac{D}{2}-1} E_F^{\frac{D}{n}+\frac{D}{2}} \int_0^1 dx x^{\frac{D}{2}-1} (1-x)^{\frac{D}{n}}$$

and using equation 3.2 we can write

$$N = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \left(\frac{1}{A}\right)^{\frac{D}{n}} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D}{2}+\frac{D}{n}+1\right)} E_F^{\frac{D}{2}+\frac{D}{n}}$$

finally by inverting this we can find $E_F = kT_F$. □

The final part of our study will consist in finding the momentum distribution for bosons and the essential properties of the condensate.

Theorem 3.2.2. *Let us consider an ideal gas of bosons in a power-law isotropic potential. The finite temperature noncondensed momentum distribution is*

$$n(\vec{p}) = \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} g_{\frac{D}{n}}\left(e^{\beta\left(\mu - \frac{\vec{p}^2}{2m}\right)}\right) \tag{3.7}$$

The Bose transition temperature T_B is

$$kT_B = \left[\left(\frac{2\hbar^2}{m}\right)^{\frac{D}{2}} A^{\frac{D}{n}} \frac{\Gamma\left(\frac{D}{2}+1\right)}{\Gamma\left(\frac{D}{n}+1\right)} \frac{1}{\zeta\left(\frac{D}{2}+\frac{D}{n}\right)} N\right]^{\frac{1}{\frac{D}{2}+\frac{D}{n}}} \tag{3.8}$$

and the condensed fraction reads

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_B}\right)^{\frac{D}{2}+\frac{D}{n}} \tag{3.9}$$

where N is the number of bosons in the gas, N_0 is the number of condensed bosons.

Proof. To find the finite temperature momentum distribution one only has to follow the same procedure shown in the Fermi gas case, integrating equation 2.6 with the sign $-$.
Finding the Bose transition temperature is a little bit trickier. Using the semiclassical approximation we have that the chemical potential μ is zero since the potential $U(\vec{r})$ has a minimum for $r_{\text{min}} = 0$ in which $U(r_{\text{min}}) = 0$ and at the Bose temperature the chemical potential is the minimum of the energy. Writing the Bose function as a power series as shown in 2.22 we have

$$\begin{aligned} N &= \int d^D p \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} g_{\frac{D}{n}}\left(e^{-\beta\frac{\vec{p}^2}{2m}}\right) = \\ &= \frac{1}{(2\hbar\sqrt{\pi})^D} \frac{\Gamma\left(\frac{D}{n}+1\right) D\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} \int_0^{+\infty} dp p^{D-1} g_{\frac{D}{n}}\left(e^{-\beta\frac{p^2}{2m}}\right) = \\ &= \frac{1}{(2\hbar)^D} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)\Gamma\left(\frac{D}{2}+1\right)} D \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\frac{D}{n}}} \int_0^{+\infty} dp p^{D-1} e^{-\ell\beta\frac{p^2}{2m}} \end{aligned}$$

We now need to set $x = \ell\beta\frac{p^2}{2m}$ and notice that

$$\int_0^{+\infty} dx x^{\frac{D}{2}-1} e^{-x} = \Gamma\left(\frac{D}{2}\right)$$

This way we can write

$$\begin{aligned} \frac{1}{\ell^{\frac{D}{n}}} \int_0^{+\infty} dp p^{D-1} e^{-\ell\beta\frac{p^2}{2m}} &= \frac{1}{\ell^{\frac{D}{n}+\frac{D}{2}}} \left(\frac{2m}{\beta}\right)^{\frac{D}{2}} \frac{1}{2} \int_0^{+\infty} dx d^{\frac{D}{2}-1} e^{-x} = \\ &= \frac{1}{\ell^{\frac{D}{n}+\frac{D}{2}}} \left(\frac{2m}{\beta}\right)^{\frac{D}{2}} \frac{1}{2} \Gamma\left(\frac{D}{2}\right) \end{aligned}$$

This way we can write N as

$$N = \left(\frac{m}{\beta\hbar^2}\right)^{\frac{D}{2}} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{\beta A}\right)^{\frac{D}{n}} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\frac{D}{n}+\frac{D}{2}}}$$

We must now use the definition of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad s \in \mathbb{C} \tag{3.10}$$

Now N becomes

$$N = \frac{1}{\beta^{\frac{D}{2}+\frac{D}{n}}} \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \frac{\Gamma\left(\frac{D}{n}+1\right)}{\Gamma\left(\frac{D}{2}+1\right)} \left(\frac{1}{A}\right)^{\frac{D}{n}} \zeta\left(\frac{D}{n}+\frac{D}{2}\right)$$

Finally by solving this for $\beta = \frac{1}{kT_B}$ one gets the Bose transition temperature.
Below T_B a macroscopic number N_0 of particles occupies the single-particle ground-state of the system. The previous equation as a function $N(T)$ gives the total number of particles for $T = T_B$, and the number $N - N_0$ of noncondensed particles for any other value of T. Therefore we can find the condensed fraction

$$\frac{N_0}{N} = \frac{T_B^{\frac{D}{2}+\frac{D}{n}} - T^{\frac{D}{2}+\frac{D}{n}}}{T_B^{\frac{D}{2}+\frac{D}{n}}} = 1 - \left(\frac{T}{T_B}\right)^{\frac{D}{2}+\frac{D}{n}}$$

□

We can make some important observations on these last theorems.
By setting $n = 2$ we have the case of the harmonic potential in a D-dimensional space, which has been

thoroughly studied in the last decades. We have $A = \frac{1}{2}m\omega^2$, where ω is the angular frequency of the trap. We can also write this potential as

$$U(\vec{r}) = \frac{1}{2}\frac{U_0}{R^2}r^2$$

where R is a range parameter for the potential, so

$$A = \frac{1}{2}\frac{U_0}{R^2}$$

Substituting this in 3.8 we get

$$T_B = \frac{\hbar}{k}\sqrt{\frac{U_0}{m}}\rho^{\frac{1}{D}}\zeta(D)^{-\frac{1}{D}}$$

where $\rho = \frac{N}{R^D}$.
This result is exactly the same one that can be found in the literature for a Bose gas in D dimensions [10], proving once again the correctness of this approach.

Finally we can see that in the formula of the BEC transition temperature one can find the function $\zeta\left(\frac{D}{2} + \frac{D}{n}\right)$. The Riemann zeta function $\zeta(x)$ diverges for $x \leq 1$, therefore in the case of an ideal Bose gas in a power-law potential Bose-Einstein condensation is possible if and only if

$$\frac{D}{2} + \frac{D}{n} > 1 \tag{3.11}$$

This inequality is very useful as it covers many different cases.
For $D = 3$ one can see from 3.11 that BEC is possible both for the rigid box ($\frac{D}{n} \to 0$) and for the harmonic potential ($n = 2$).
When $D = 2$ we have that condensation is impossible for the potential box case since the ζ function diverges, but it can happen for example for the harmonic trap.
Lastly, for $D = 1$ we have that BEC is possible for $n < 2$.

Chapter 4

Conclusions

In this thesis we have seen how using the grand canonical ensemble of equilibrium mechanics and the semiclassical approximation it is possible to compute the main properties of quantum bosonic and fermionic gases and predict the temperature at which phenomena such as the Fermi degeneracy and the Bose-Einstein condensation occur. This has been done in a generic D-dimensional space and with both a homogeneous and a power-law potential, which is useful for studying systems with reduced dimensions and with different kinds of magneto-optical traps.

The study of quantum gases is a very fertile field, as there are many interesting effects that can only be seen at extremely low temperatures. For example, in a Bose-Einstein condensate one can solve the Gross-Pitaevskii equation (the Schrödinger equation for bosons which are in the same quantum state) and find peculiar solutions such as the soliton or quantum vortices, which have been recently observed.
Another interesting phenomenon that has been studied and observed is the BCS-BEC crossover, where at very low temperatures weakly-correlated pairs of Fermions (Cooper pairs) can behave as bosons and thus form a condensate. Cooper pairs are important in the description of superconductivity, neutron stars and other quantum systems, so further studies on this subjects will probably lead to interesting results.

Quantum gases are a field of study that is fundamental in physics of matter, and future research in low temperature systems could lead to important applications hereafter. Of course a more thorough treatment of theoretical and experimental advances in this field would require a more advanced mathematical formalism and an in-depth examination of recent experimental methods, which are beyond the scope of this thesis.
We have nonetheless covered some extremely important aspects of fermionic and bosonic gases at very low temperatures, which can be the basis for further studies.

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