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Schauder Space Counterparts of Some Theorems on The Neumann-Poincaré Operator of D. Khavinson, M. Putinar and H.S. Shapiro

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UNIVERSITÀ DEGLI STUDI DI PADOVA

Abstract

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by Khai An TRAN

Layer potentials entered early into the main stream of mathematics as mathematical tools. Their role in the development of the main problems of Mathematical Physics of the XIX-th Century is well known. The dawn of modern spectral analysis is also rooted in the attempts to exploit layer potentials as integral transforms with a singular kernel.

The aim of Dissertation is studying spectral analysis properties of the double layer potential operator, also known as Neumann-Poincaré operator. More specifically, we study Poincaré variational problems examined by D. Khavinson, M. Putinar and H.S. Shapiro in Sobolev spaces in the frame of Schauder spaces.

The Dissertation is organized as follows. Chapter 1 collects the preliminaries and notation of Functional Analysis and Potential Theory. In Chapter 2, we present some properties of single and double layer potentials for the Laplace equation and some basic results of potential theory in Schauder Spaces. In Chapter 3 we consider boundary integral operator for the Laplace equation. After that, we collect some results of spectral theory for the Laplace operator. Chapter 4 exploits the material of Chapter 3 in order to prove the spectral properties of boundary integral operators and a characterization of the ball. At the end of the Dissertation, we have enclosed some Appendices with some results that we have exploited.

***Keyword**: Schauder Space, Spectral Analysis, Potential Theory*

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Contents

This is for my family. Thank you for always being there for me.

Introduction

This Dissertation explains how the balance of inner-outer harmonic field energies led Poincaré to an early discovery of a discrete spectrum of real critical values for the Neumann-Poincaré operator.

The Neumann–Poincaré operator is a boundary integral operator which appears naturally when solving classical boundary value problems using layer potentials. We mention V.G. Mazya [Maz91]; S. Agmon, A. Douglis, L. Nirenbert [Nir59]; P. Ebenfelt [Sha01; Sha02]; J.L. Lions; E. Magences [Mag72] as an example of their works. Its study (for the Laplace operator) goes back to C. Neumann [Neu87] and H. Poincaré [Poi97; Poi99] as the name of the operator suggests, and appeared in the first time in T. Carleman's Dissertation [Car16]. C. Neumann's series method was successfully applied it towards the solvability of Dirichlet's problem on convex domains. More general method to solve the Dirichlet problem was discovered by Poincaré and called by him "balayage". However, the convergence of Neumann series and in general the invertibility of the double layer potential operator on the boundary were computationally more accessible techniques.

In a long and technical memoir of 1897, Poincaré attacked the convergence of the Neumann series on non-convex boundaries [Poi97]. Poincaré concludes his remarks by the following words: *"After having established these results [concerning convergence of the Neumann series] rigorously, I felt obliged in the two final chapters to give an idea of the insights which initially led me to foresee these results. I thought that, despite their lack of rigor, these could be useful as tools for research insofar as I had already used them successfully once."* Poincaré's variational principle was formulated in modern terms in a recent D. Khavinson, M. Putinar and H.S. Shapiro's article [Sha07] which serves as a strarting point. The study of spectral properties of the Neumann-Poincaré operator was initiated by S. Zaremba [Zar04]. In D. Khavinson, M. Putinar and H.S. Shapiro paper [Sha07], the Neumann-Poincaré operator is not self-adjoint. Generally, in L^2, Neumann-Poincaré operator can be realized as a self-adjoint operator in the $H^{-1/2}$ Sobolev space, provided a new inner product is introduced there.

One of the aims of this Dissertation is establishing the counterpart of some results Poincaré variational problem and related problems which were obtained by D. Khavinson, M. Putinar and H.S. Shapiro in Sobolev Spaces in the frame of Schauder spaces.

This dissertation consists of four chapters. We now describe in details the content of each chapter.

Chapter 1: In this chapter we introduce some preliminary classical knowledge in real analytic functions, Hölder continuous functions and the Schauder spaces that we use thoughout the Dissertation. Terminology and basic facts of Newtonian Potential Theory can be found in N.S. Landkof [Lan72] or D. Gilbarg; N.S. Trudinger [Tru01]. We can also consult the monographs G.B. Folland's book [Fol99] L.C. Evan's book [Eva10] or V.I. Burenkov's book [Bur13].

Chapter 2: For the properties of harmonic functions, we refer to the classical textbooks of Evans [Eva10] and Folland [Fol95]. This content is not included in the dissertation. We warn that there is no consensus in the vast literature on the subject of signs and constants in the definitions of potentials. We hope this will not be a cause of any confusion.

Chapter two is devoted to the Green Identities and to the investigation of some properties of the layer potentials corresponding to the fundamental solution of the Laplace operator in Schauder spaces. This results are applied to the special case of Laplace equations.

In Section 2.1, we introduce the Green Identites. The Green Identities play an important role in the representation of the solutions for boundary value problems. For more details of the proofs, we refer to Folland's book [Fol95] or to M. Dalla Riva, M. Lanza de Cristoforis and P. Musolino [CPM19] or the monograph of M. Lanza de Cristoforis [Criad].

In Section 2.2, we introduce the Newtonian kernel, a fundamental solution for the Laplace operator,

$$S_n(x-y) = \begin{cases} \frac{1}{s_n} \log|x-y| & (n=2) \\ \frac{1}{(2-n)s_n}|x-y|^{2-n} & (n \geq 3) \end{cases}$$

for all $x, y \in \mathbb{R}^n, x \neq y$, where s_n is the surface area of the unit sphere in $\mathbb{R}^n$. Then, we introduce

$$v_\Omega[\phi](x) \equiv \int_{\partial\Omega} S_n(x-y)\phi(y)d\sigma_y \qquad \forall x \in \mathbb{R}^n$$

as single (or simple) layer potential, and if $\psi \in C^0(\partial\Omega)$, we denote

$$w_\Omega[\psi](x) \equiv \int_{\partial\Omega} \psi(y)\frac{\partial}{\partial\nu_\Omega(y)} S_n(x-y)d\sigma_y \qquad \forall x \in \mathbb{R}^n$$

as double layer potential. We introduce classical Schauder Regularity results and the Jump Formulas of the potentials for the normal derivative of a single layer potential and for the limit value of the double layer potential on the boundary.

In section 2.3, we shall consider the basic boundary value problems for the Laplace operator in an open, bounded set of Ω class $C^{1,\alpha}$ for some $\alpha \in]0,1]$ and in its exterior Ω^-.

In the last section, we consider the Dirichlet problem for Laplace equation a bounded domain. In particular, by the third Green Identity, we can reformulate the Laplace equation by integral equations, the so called Fredholm equation of second kind. Using the Fredholm Alternative Theorem, such integral equation has a unique solution and accordingly the second layer potential satisfies Dirichlet problem in Ω.

Chapter 3: In this chapter, we demonstrate certain properties of boundary integral operators related to the single layer potential v_Ω and double layer potential w_Ω.

In Section 3.1, we will review certain results of Fredholm operators in dual systems. We also recall some classical theorems of Functional Analysis on Banach spaces by following Chapter 5 of M. Dalla Riva, M. Lanza de Cristoforis and P. Musolino [CPM19]. We do not include any proof.

In Section 3.2, we study the boundary integral operators associated to the single and double Layer potentials. We define the Neumann-Poincaré operator W_Ω from $C^{1,\alpha}(\partial\Omega)$ to itself and its transpose operator W_Ω^t from $C^{0,\alpha}(\partial\Omega)$ to itself defined by setting

$$W_\Omega f(x) = -\int_{\partial\Omega} f(y)\nu_\Omega(y)\nabla S_n(x-y)d\sigma_y,\ \forall f \in C^{1,\alpha}(\partial\Omega),$$
$$W_\Omega^t f(x) = \int_{\partial\Omega} f(y)\nu_\Omega(x)\nabla S_n(x-y)d\sigma_y,\ \forall f \in C^{0,\alpha}(\partial\Omega).$$

Then we rewrite the jump formulas in terms of the boundary operators W_Ω and W_Ω^t.

In Section 3.3, we shall be concerned with the relation between the spectral analysis of the Neumann-Poincaré operator and some extremal problems from comparing the energies in Ω, Ω^- of the single layer potentials. For that purpose, we will use the space of harmonic function in Shauder Space,

$$\begin{aligned}\mathfrak{H} =& \{h^+ \in C^{1,\alpha}(\overline{\Omega}) | \Delta h^+ = 0 \text{ in } \Omega\} \\ &\times \{h^- \in C^{1,\alpha}_{loc}(\overline{\Omega^-}) | \Delta h^- = 0 \text{ in } \Omega^-, h^- \text{ harmonic at infinity}\}.\end{aligned}$$

Then we consider the finite energy semi-norm

$$\|h\|^2_{\mathfrak{H}} = \int_\Omega |\nabla h^+|^2 dx + \int_{\Omega^-} |\nabla h^-|^2 dx.$$

Then we have

$$\mathfrak{H} = \mathfrak{S} \oplus \mathfrak{D}$$

where $\mathfrak{S}, \mathfrak{D}$ are subspace of single and double layer potentials, respectively (c.f. D. Khavinson, M. Putinar, H.S. Shapiro [Sha07]). Plemelj's symmetrization principle and a non-trivial observation of general nature lead to the following theorem, whose main points were foreseen by Poincaré [Poi97].

Chapter 4: This chapter presents results on Sobolev spaces of D. Khavinson, M. Putinar, H.S. Shapiro's paper [Sha07]. Then, we shall construct similar results on Schauder spaces by using potential theory.

In Section 4.1, we study Poincaré's variational problem: as study of an angle operator between decompositions of the subspace of single and double layer potentials. We have two Hermitian forms on $\mathfrak{H}$

$$J^+[f] = \int_\Omega |\nabla V_\Omega[f]|^2 dx, \qquad J^-[f] = \int_{\Omega^-} |\nabla V_\Omega[f]|^2 dx$$

Poincaré proposes to analyze the characteristic value of the Reyleigh quotient $\dfrac{J^-[f] - J^+[f]}{J^-[f] + J^+[f]}$. Poincaré's variational principle was formulated in modern terms in a recent article [Sha07]. Now, we extend there results in Schauder spaces.

In Section 4.2 and Section 4.3, we consider the case in unit ball and another characteristic property of the ball. Poincaré has conjectured (by analogy with the sphere) that the spectrum of W_Ω is always non-negative in $\mathbb{R}^n, n \geq 3$ and moreover that W_Ω is injective. These assertions are true in the case the boundary is a sphere. W_Ω is never self-adjoint unless Ω is a ball. However, using the so-called symmetrization procedure one can show that W_Ω has indeed a real spectrum and its eigenfunctions do span the Schauder space.

The Appendix: The Appendix contains some results and proofs that support the results in the main text. It includes Spectral Theory in Functional Analysis, c.f. B. Helffer [Hel13]; E.B. Davies [Dav95], and Spherical Harmonics, c.f. Folland [Fol95]; R.P. Feynman, R.B. Leighton, M. Sands [San], N.N. Lebedev [Leb72] which are useful to solve the Laplace equation on the ball.

Chapter 1

Preliminaries and Notation

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, the set of integer numbers, the set of real numbers, the set of complex numbers, respectively. We denote K be either field $\mathbb{R}$ or $\mathbb{C}$. Let a real number $\alpha \in]0,1[$ and Ω be open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We denote by $\Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega}$ is a exterior of Ω. We also remark that, when no misunderstanding is possible, we will use the symbol I instead of $I_{C^{0,\alpha}(\partial\Omega)}$ or $I_{C^{1,\alpha}(\partial\Omega)}$ for the identity operator from $C^{0,\alpha}(\partial\Omega)$ to itself and from $C^{1,\alpha}(\partial\Omega)$ to itself, respectively.
Let X be a set. We denote by $D_X = \{(x_1, x_2 \in X \times X : x_1 = x_2)\}$ the diagonal of $X \times X$. If Y is also set, we denote by Y^X the set of maps from X to Y.
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces on $\mathbb{K}$. Let U and V be subsets of X and Y, respectively. We denote by $\overline{U}$ be the closure of U, by ∂U the boundary of U, and by $diam(U) \equiv \sup\{\|x-y\|_X : x, y \in U\}$ the diameter of U. We denote either by U^+ or $\dot{U}$ the interior of U and U^- by exterior of U. We set $U^+ \equiv \dot{U}$ and $U^- \equiv X \setminus \overline{U}$. We endow the space $X \times Y$ with the norm defined by $\|(x,y)\|_{X\times Y} \equiv \|x\|_X + \|y\|_Y$ for all $(x,y) \in X \times Y$ while we use the Euclidean norm for $\mathbb{R}^n$.

The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$ or in $\mathbb{C}$. For all $R \in]0,+\infty[$, $x \in \mathbb{R}^n$, x_j denotes the $j-th$ coordinate of x, and $\mathbb{B}_n(x,R)$ denotes the ball

$$\mathbb{B}_n(x,R) \equiv \{y \in \mathbb{R}^n : |x-y| < R\},$$

and $\mathbb{B}_n$ be the unit ball $\{y \in \mathbb{R}^n : |y| < 1\}$.

Let Ω be an bounded subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued (resp. complex-valued) functions on Ω is denoted by $C^m(\Omega, \mathbb{K})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in C^m(\Omega)^r$. The $s-th$ component of f is denoted f_s and the gradient matrix of f is denoted Df. Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ and $|\beta| = \beta_1 + \ldots + \beta_n$, $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$. Then $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}$.
The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\beta f$ of order $|\beta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega}, \mathbb{K})$ or more simply $C^m(\bar{\Omega})$. We consider the space

$$C_b^m(\overline{\Omega}) \equiv \{f \in C^m(\overline{\Omega}) : D^\beta f \in B(\overline{\Omega}), \forall \beta \in \mathbb{N}^n, |\beta| \leq m\}$$

endowed with the norm

$$\|f\|_{C_b^m(\overline{\Omega})} \equiv \sum_{|\beta| \leq m} \sup_{x \in \overline{\Omega}} |D^\beta f(x)|$$

is a Banach space.

For some fixed $\alpha \in]0,1]$, then f is α-Hölder continuous provided that

$$|f : \mathbb{D}|_\alpha \equiv \sup\left\{\frac{|f(x) - f(y)|}{|x-y|^\alpha} : x, y \in \mathbb{D}, x \neq y\right\} < \infty.$$

If $\alpha = 1$, we say that f is Lipschitz continuous provided that f is 1 - Hölder continuous and set $Lip(f) \equiv |f : \mathbb{D}|_1$.

Let $m \in \mathbb{N}, \alpha \in]0,1]$. The Schauder space of exponents m, α is defined as

$$C^{m,\alpha}(\overline{\Omega}) \equiv \{f \in C^m(\overline{\Omega}) : D^\eta f \in C^{0,\alpha}(\overline{\Omega}), \forall \eta \in \mathbb{N}^n, |\eta| = m\}.$$

We define the norm in $C^{m,\alpha}(\overline{\Omega})$ as follows

$$\|f\|_{C^{m,\alpha}(\overline{\Omega})} = \sup \|f\|_{C^m(\overline{\Omega})} + \sum_{|\eta|=m} |D^\eta f|_\alpha, \forall f \in C^{k,\alpha}(\overline{\Omega}).$$

We define the spaces $C^{m,\alpha}(\partial\Omega)$ for $k \in \{0, \dots, m\}$ by exploiting the local parametrizations (cf. Gilbarg and Trudinger [Tru01]). The trace operator from $C^{k,\alpha}(\overline{\Omega})$ to $C^{m,\alpha}(\partial\Omega)$ is linear and continuous. For standard properties of functions in Schauder spaces, we refer to Gilbarg and Trudinger [Tru01].

Chapter 2

Layer Potentials in Schauder Spaces

2.1 Green Identities

We remind the first Green formulae in interior and exterior domain

Theorem 2.1 (First Green's Identity in interior domains).
Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $u, v \in C^1(\overline{\Omega}), v \in C^2(\Omega)$ and $u\Delta v \in L^1(\Omega)$, then

$$\int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} u \Delta v dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} d\sigma.$$

Proof. See for example in [Fol95], page 69. □

Theorem 2.2 (Second Green's Identity in interior domains).
Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $u, v \in C^1(\overline{\Omega}) \cap \in C^2(\Omega)$ and $u\Delta v, v\Delta u \in L^1(\Omega)$, then

$$\int_{\Omega} \Delta u v - u \Delta v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} d\sigma.$$

Proof. See for example in [Fol95], page 69. □

Corollary 2.1 (First Green's Identity in exterior domains).
Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $u, v \in C^1(\overline{\Omega^-})$ be harmonic in Ω^- and harmonic in infinty. Then

$$\lim_{R \to +\infty} \int_{\Omega^- \cap \mathbb{B}_n(0,R)} \nabla u \nabla v dx = - \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} d\sigma.$$

Proof. See for example in [Fol95], page 69. □

Corollary 2.2 (Second Green's Identity in exterior domains).
Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $u, v \in C^1(\overline{\Omega^-})$ be harmonic in Ω^- and harmonic in infinty. Then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} d\sigma = 0.$$

Proof. See for example in [Fol95], page 69. □

Theorem 2.3. *(Third Green's Identity)*

Let $\alpha \in]0,1]$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $g \in C^{1,\alpha}(\partial\Omega)$, and Δu is integrable in Ω, then the following formula holds.

$$u(x) = \int_{\Omega} \Delta u(y) S_n(x-y)dy + \int_{\partial\Omega} u(y) \frac{\partial}{\partial \nu_{\Omega}(y)} S_n(x-y) - \frac{\partial u}{\partial \nu_{\Omega}}(y) S_n(x-y) d\sigma_y, \tag{2.1.1}$$

for all $x \in \Omega$, where ν_{Ω} denotes the outward unit normal to $\partial\Omega$ and

$$\frac{\partial}{\partial \nu_{\Omega}(y)} S_n(x-y) \equiv -\nu_{\Omega} \cdot \nabla S_n(x-y), \forall (x,y) \in \mathbb{R}^n \times \partial\Omega, x \neq y.$$

Proof. See for example [CPM19], Chapter 4. □

2.2 Single and Double Layer Potential

We denote $S_n(x-y)$ be the fundamental solution of Laplace equation,

$$S_n(x-y) = \begin{cases} \frac{1}{s_n} \log|x-y| & (n=2) \\ \frac{1}{(2-n)s_n} |x-y|^{2-n} & (n \geq 3) \end{cases}$$

for all $x, y \in \mathbb{R}^n, x \neq y$, where s_n is the surface area of the unit sphere in $\mathbb{R}^n$. The signs were chosen so that $\Delta S_n(x-y) = \delta$ (Dirac's delta function).

If $\phi \in C^0(\partial\Omega)$, we denote

$$v_{\Omega}[\phi](x) \equiv \int_{\partial\Omega} S_n(x-y)\phi(y) d\sigma_y \qquad \forall x \in \mathbb{R}^n$$

as single (or simple) layer potential of support $\partial\Omega$ and moment ϕ associated to the fundamental solution S_n, and if $\psi \in C^0(\partial\Omega)$, we denote

$$w_{\Omega}[\psi](x) \equiv \int_{\partial\Omega} \psi(y) \frac{\partial}{\partial \nu_{\Omega}(y)} S_n(x-y) d\sigma_y \qquad \forall x \in \mathbb{R}^n$$

where

$$\frac{\partial}{\partial \nu_{\Omega}(y)} S_n(x-y) \equiv -\nu_{\Omega}(y) \nabla S_n(x-y) \qquad \forall (x,y) \in \mathbb{R}^n \times \partial\Omega, x \neq y.$$

as double layer potential of support $\partial\Omega$ and moment ψ associated to the fundamental solution S_n.

We set

$$v^+_{\Omega}[\phi] = v_{\Omega}[\phi]|_{\overline{\Omega}} \qquad v^-_{\Omega}[\phi] = v_{\Omega}[\phi]|_{\overline{\Omega^-}}$$

We have the following theorems, which clarifies the behavior of $v^-_{\Omega}[\phi]$ and $w^-_{\Omega}[\phi]$ at infinity.

Theorem 2.4. *Let $\alpha \in]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi \in C^0(\partial\Omega)$, the following statments hold*

1. *For $n \geq 3$, $\lim_{x\to\infty} v^-_{\Omega}[\phi](x) = 0$ and accordingly $v^-_{\Omega}[\phi]$ is harmonic at infinity.*
2. *For $n = 2$, If $\int_{\partial\Omega} \phi = 0$, then $\lim_{x\to\infty} v^-_{\Omega}[\phi](x) = 0$ and accordingly $v^-_{\Omega}[\phi]$ is harmonic at infinity.*

Proof. see for example [CPM19], Chapter 4. □

Theorem 2.5. *Let $\alpha \in]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\psi \in C^0(\partial\Omega)$, the following statments hold*

1. $w_\Omega[\psi]$ *is harmonic in* $\mathbb{R}^n \setminus \partial\Omega$.

2. $\lim_{x\to\infty} w^-_\Omega[\psi](x) = 0$ *and accordingly* $w_\Omega[\psi]$ *is harmonic at infinity.*

Proof. see for example [CPM19], Chapter 4. □

We have the following classical Schauder Regularity results for the restriction of the single layer potential to Ω and Ω^-. For a proof we refer to Miranda [Mir65], page 307.

Theorem 2.6. *Let $\alpha \in]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

1. *If $\phi \in C^{0,\alpha}(\partial\Omega)$, then $v^+_\Omega[\phi]$ belongs to $C^{1,\alpha}(\overline{\Omega})$. Moreover, the map $v^+_\Omega[\cdot]$ is linear and continuous from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega})$.*

2. *If $\phi \in C^{0,\alpha}(\partial\Omega)$, then then $v^-_\Omega[\phi]$ belongs to $C^{1,\alpha}_{loc}(\overline{\Omega})$. Moreover, if $r \in]0,\infty[$ and $\overline{\Omega} \subset \mathbb{B}_n(0,r)$, then the map $v^-_\Omega[\cdot]|_{\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ is linear and continuous from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$.*

In the next theorem, we introduce a know formula for the first order partial derivatives of a single layer potential

Theorem 2.7. *Let $\alpha \in]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi \in C^{0,\alpha}(\partial\Omega)$. Let $j \in \{1,\dots,n\}$. Let $\tilde{x} \in \partial\Omega$.*

1. *The principal value*

$$p.v. \int_{\partial\Omega} \phi(y)\frac{\partial}{\partial x_j} S_n(\tilde{x}-y)d\sigma_y = \lim_{\epsilon\to 0}\int_{\partial\Omega\setminus\mathbb{B}_n(\tilde{x},\epsilon)} \phi(y)\frac{\partial}{\partial x_j} S_n(\tilde{x}-y)d\sigma_y$$

exists finite and we have the jump formula for the first order partial derivatives of a single layer potential

$$\lim_{t\to 0^\mp} \frac{\partial}{\partial x_j} v_\Omega[\phi](\tilde{x}+t\nu_\Omega(\tilde{x})) = \mp\frac{1}{2}\phi(\tilde{x})(\nu_\Omega)_j(\tilde{x}) + p.v.\int_{\partial\Omega}\phi(y)\frac{\partial}{\partial x_j}S_n(\tilde{x}-y)d\sigma_y.$$

2. *The function $\nu_\Omega\nabla S_n(x-y)\phi(y)$ is integrable in $y \in \partial\Omega$ and we have the following jump formula for the normal derivative of a single layer potential*

$$\frac{\partial}{\partial\nu_\Omega} v^\pm_\Omega[\phi](\tilde{x}) = \mp\frac{1}{2}\phi(\tilde{x}) + \int_{\partial\Omega}\nu_\Omega(\tilde{x})\nabla S_n(\tilde{x}-y)\phi(y)d\omega_y.$$

Proof. see for example [CPM19], Chapter 4. □

If Ω is a bounded open subset of $\mathbb{R}^n$ of calss $C^{1,\alpha}$ for some $\alpha \in]0,1]$, then the boundary $\partial\Omega$ is a compact local graph of $\mathbb{R}^n$ of class $C^{1,\alpha}$, and we can consider the double layer potential

$$\int_{\partial\Omega}\psi(y)\frac{\partial}{\partial\nu_\Omega(y)}S_n(x-y)d\sigma_y = -\sum_{l=1}^{n}\int_{\partial\Omega}\frac{\partial}{\partial x_l}S_n(x-y)(\nu_\Omega)_l(y)\psi(y)$$

corresponding to Hölder continuous function ψ defined on $\partial\Omega$. Next we introduce the following known jump formula for the limiting value of the double layer potential on the boundary.

Theorem 2.8. *Let $\alpha \in \,]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\psi \in C^{0,\alpha}(\partial\Omega)$ and $\tilde{x} \in \partial\Omega$. Then we have the following jump formula for the double layer potential*

$$\lim_{t\to 0^{\mp}} w_\Omega[\psi](\tilde{x} + t\nu_\Omega(\tilde{x})) = \pm\frac{1}{2}\psi(\tilde{x}) + w_\Omega[\psi](\tilde{x}).$$

Proof. see for example [CPM19], Chapter 4. □

The following theorem statement Schauder Regularity of double layer potential to Ω and Ω^-.

Theorem 2.9. *Let $\alpha \in \,]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

1. *If $\psi \in C^{1,\alpha}(\partial\Omega)$ then the restriction $w_\Omega[\psi]|_\Omega$ extends to function $w^+_\Omega[\psi]$ of class $C^{1,\alpha}(\overline{\Omega})$. Moreover, the map $w^+_\Omega[\cdot]$ is linear and continuous from $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega})$.*
2. *If $\psi \in C^{1,\alpha}(\partial\Omega)$ then the restriction $w_\Omega[\psi]|_\Omega$ extends to function $w^-_\Omega[\psi]$ of class $C^{1,\alpha}_{loc}(\overline{\Omega^-})$. Moreover, if $r \in \,]0,\infty[$ and $\overline{\Omega} \subset \mathbb{B}_n(0,r)$, then the map $w^-_\Omega[\cdot]|_{\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ is linear and continuous from $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$.*
3. *If $\psi \in C^{1,\alpha}(\partial\Omega)$ and $\tilde{x} \in \partial\Omega$, then $\nu_\Omega(\tilde{x})\nabla w^+_\Omega[\psi](\tilde{x}) = \nu_\Omega(\tilde{x})\nabla w^-_\Omega[\psi]()$.*

Proof. see for example [CPM19], Chapter 4. □

2.3 Statement of the Fundamental Boundary Value Problems for Laplace Operator

We now introduce the following boundary value problems for the Laplace operator in the bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in \,]0,1]$ and on the exterior Ω^- which are important in the applications.

The interior Dirichlet boundary value problem
Given $g \in C^{1,\alpha}(\partial\Omega)$, find $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{2.3.1}$$

The exterior Dirichlet boundary value problem
Given $g \in C^{1,\alpha}(\partial\Omega)$, find $u \in C^{1,\alpha}(\overline{\Omega^-}) \cap C^2(\Omega^-)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^-, \\ u = g \text{ on } \partial\Omega^- = \partial\Omega, \\ u \text{ harmonic at infinity.} \end{cases} \tag{2.3.2}$$

The interior Neumannn boundary value problem
Given $g \in C^{0,\alpha}(\partial\Omega)$, find $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu_\Omega} = g \text{ on } \partial\Omega. \end{cases} \tag{2.3.3}$$

The exterior Neumannn boundary value problem
Given $g \in C^{0,\alpha}(\partial\Omega)$, find $u \in C^{1,\alpha}(\overline{\Omega^-}) \cap C^2(\Omega^-)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^-, \\ \frac{\partial u}{\partial \nu_\Omega} = g \text{ on } \partial\Omega^- = \partial\Omega, \\ u \text{ harmonic at infinity.} \end{cases} \tag{2.3.4}$$

Then, we mention the following known third Green Identity, which enables us to write a function on $\overline{\Omega}$ in terms of an integral operator applied to Δu in Ω and of integral operator applied to $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial \nu_\Omega}$ on $\partial\Omega$ and which turns out to useful in the solution of the above interior boundary value problems.

Then one can prove a corresponding Green Identity for harmonic functions defined on the exterior domain Ω^-. Both the third Green identities in Ω and Ω^- show that we can write a large class of functions u in Ω and in Ω^-, in term of integrals of the forms

$$\int_{\partial\Omega} \mu(y) S_n(x-y) d\sigma_y,$$

i.e., as single (or simple) layer potential of support $\partial\Omega$ and moment μ associated to the fundamental solution S_n, and in the form

$$\int_{\partial\Omega} \omega(y) \frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y) d\sigma_y,$$

i.e., as double layer potential of support $\partial\Omega$ and moment ω associated to the fundamental solution S_n and in terms of integrals of the form

$$\int_\Omega S_n(x-y)\tau(y)dy$$

which are volume potentials of support Ω associated to the fundamental solution S_n and density τ and which are useful in solving the above boundary value problems and in understanding the regularity of solutions which as u above can be written as sums of the above layer potentials.

2.4 Dirichlet Problem in Ω

Let $\alpha \in \,]0,1]$ and let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. By the third Green Identity, if $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ solves the Dirichlet problem , then

$$u(x) = w_\Omega^+[u](x) - v_\Omega^+\left[\frac{\partial u}{\partial \nu_\Omega}\right](x), \forall x \in \Omega,$$

where

$$v_\Omega^+\left[\frac{\partial u}{\partial \nu_\Omega}\right](x) \equiv \int_{\partial\Omega} S_n(x-y)\frac{\partial u}{\partial \nu_\Omega}(y) d\sigma_y, \forall x \in \overline{\Omega}.$$

Since the singlularity of the kernel $S_n(x-y)$ is weak, one could exploit the Vitali Convergence Theorem [1] and prove that v_Ω^+ is continuous in $\overline{\Omega}$. Then by the continuity of both the left and the right hand side of the equality in Ω, we obtain

$$u(x) = w_\Omega^+[u](x) - v_\Omega^+\left[\frac{\partial u}{\partial \nu_\Omega}\right](x), \forall x \in \overline{\Omega}.$$

[1] This is a generalization of the better-known dominated convergence theorem.

Now such a formula contains the unknown $\frac{\partial u}{\partial \nu_\Omega}$ in the right hand side. Thus, if we hope to exploit such a formula to solve the Dirichlet problem, we need to find $\frac{\partial u}{\partial \nu_\Omega}$, which is not a part of data of the Dirichlet Problem. A way to get rid of $\frac{\partial u}{\partial \nu_\Omega}$ is to introduce the Green function for Dirichlet Problem, but to obtain the Green function, we need to solve the explicitly a Dirichlet Problem, which we can solve only in a limited number of cases.

We now to try to find an integral equation for $\frac{\partial u}{\partial \nu_\omega}$. We exploit the jump properties of $w_\Omega[u]$ and the continuity of $v_\Omega^+\left[\frac{\partial u}{\partial \nu_\Omega}\right]$, and equality $u = g$ on $\partial\Omega$, and we obtain

$$g(x) = \frac{1}{2}g(x) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y)g(y)d\sigma_y + \int_{\partial\Omega} \frac{\partial u}{\partial \nu_\Omega}(y)S_n(x-y)\sigma_y,$$

for all $x \in \partial\Omega$, which we can rewrite as

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu_\Omega}(y)S_n(x-y)\sigma_y = \frac{1}{2}g(x) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y)g(y)d\sigma_y$$

for all $x \in \partial\Omega$. Then, we may wonder we can solve such an integral equation on the unknown $\frac{\partial u}{\partial \nu_\Omega}$, which is a Fredholm equation of the first kind. This certainly a way to treat problem, the so-called direct method. However first kind Fredholm integral equations are somewhat complicated.

Instead, here we follow the method of Fredholm, which consists on searching for solution in the form

$$u(x) = w_\Omega^+[\psi](x) + v_\Omega^+[\phi](x), \forall x \in \overline{\Omega},$$

with ψ and ϕ unknown functions. Then by exploiting the jump properties of the double layer potential, and the continuity of the single layer potential, we obtain

$$g(x) = \frac{1}{2}\psi(x) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega} S_n(x-y)\psi(y)d\sigma_y + \int_{\partial\Omega} \phi(y)S_n(x-y)\sigma_y, \forall x \in \partial\Omega.$$

Then we observe that if the second integral in the right hand side were to be absent, then we would deal with a Fredholm equation of the second kind. Namely, with

$$g(x) = -\frac{1}{2}\psi(x) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega} S_n(x-y)\psi(y)d\sigma_y, \forall x \in \partial\Omega. \tag{2.4.1}$$

The method of Fredholm proposes to analyze such equation first. Since we have deliberately deleted the second integral in the right hand size, we can imagine that there may be some data g for which the Dirichlet Problem is solvable, but for which such modified equation is not solvable. However, the Fredholm theory allows to characterize such data g.

We consider ase a case which the equation (2.4.1) can be solved for all data $g \in C^{1,\alpha}(\partial\Omega)$. Namely the case in which both Ω amd Ω^- are connected. Now, by Theorem of Schauder implies that the integral operator W from $C^{1,\alpha}(\partial\Omega)$ to itself defined by $W_\Omega[\psi] \equiv w_\Omega[\psi]|_{\partial\Omega}, \forall \psi \in C^{1,\alpha}(\partial\Omega)$ is compact. Since we have assumed that Ω^- is connected, one could prove that $\frac{1}{2}I + W_\Omega$ is injective. Since W_Ω is compact and $\frac{1}{2}I + W_\Omega$ is injective, the Fredholm Alternative Theorem implies that the compact perturbation of the indentiy $\frac{1}{2}I + W_\Omega$ is an isomorphism of $C^{1,\alpha}$ onto itself. Then for each datum $g \in C^{1,\alpha}(\partial\Omega)$, the integral equation (2.4.1) has one and only one solution $\psi \in C^{1,\alpha}(\partial\Omega)$. Since ψ satisfies equation (2.4.1), then the function $w_\Omega^+[\psi]$

satisfies the equatlity

$$w_\Omega^+[\psi]|_{\partial\Omega} = \frac{1}{2}\psi + W_\Omega[\psi] = g \text{ on } \partial\Omega$$

and accordingly the function $u \equiv w_\Omega^+[\psi]$ satisfies the boundary condition $u = g$ on $\partial\Omega$ of the Dirichlet problem. On the other hand. the classical theorem of differentiability fo integrals depending on a parameter implies that $w_\Omega^+[\psi]|_{\partial\Omega} \in C^2(\Omega)$ and that $\Delta w_\Omega^+[\psi](x) = 0, \forall x \in \Omega$. Indeed, if $y \in \partial\Omega$, then the function $\frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y)$ of variable $x \in \Omega$ is of class C^∞ and is harmonic in Ω. Then $w_\Omega^+[\psi] \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$. One the other hand the Maximum Principle implies that the Dirichlet Problem has at most a solution and thus $w_\Omega^+[\psi]$ is the only solution of Dirichlet Problem in $C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$.

Chapter 3

Boundary Integral Equations in Schauder Spaces

3.1 Fredholm Alternative in Dual System

We introduce in Fredholm operators in dual systems, c.f. Wendland [Wen67; Wen70]. We do not include proofs, however, as these can be found in monographs such as those of Kress's book in [Kre14], as well as in the above mentioned papers of Wendland.

We tacitly assume that all linear spaces under consideration are complex linear spaces; the case of real linear spaces can be treated analogously.

Definition 3.1. *Let X, Y be linear spaces. A mapping*

$$\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C}$$

is called a bilinear form if

$$\begin{aligned} \langle \alpha_1\varphi_1 + \alpha_2\varphi_2, \psi \rangle &= \alpha_1 \langle \varphi_1, \psi \rangle + \alpha_2 \langle \varphi_2, \psi \rangle, \\ \langle \varphi, \beta_1\psi_1 + \beta_2\psi_2 \rangle &= \beta_1 \langle \varphi, \psi_1 \rangle + \beta_2 \langle \varphi, \psi_2 \rangle \end{aligned}$$

for all $\varphi_1, \varphi_2, \varphi \in X, \psi, \psi_1, \psi_2 \in Y, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. The bilinear form is called non-degenerate if for every $\varphi \in X$ with $\varphi \neq 0$ there exists $\psi \in Y$ such that $\langle \varphi, \psi \rangle \neq 0$; and for every $\psi \in Y$ with $\psi \neq 0$ there exists $\varphi \in X$ such that $\langle \varphi, \psi \rangle \neq 0$.

Definition 3.2. *Two normed spaces X and Y equipped with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C}$ are called a dual system and denoted by $\langle X, Y \rangle$.*

Definition 3.3. *Let $\langle X_1, Y_1 \rangle$ and $\langle X_2, Y_2 \rangle$ be two dual systems. Then two operators $A : X_1 \to X_2, B : Y_2 \to Y_1$ are called adjoint (with respect to these dual systems) if*

$$\langle A\varphi, \psi \rangle = \langle \varphi, B\psi \rangle$$

for all $\varphi \in X_1, \psi \in Y_2$.

Theorem 3.1 (Fredholm Alternative in a dual system)**.**

Let $\langle X_1, X_2 \rangle$ be dual system. Let K_1 be a linear compact operator form X_1 onto inself. Let K_2 be a linear compact operator form X_2 onto inself. Assume that K_1 is adjoint operator with K_2. Then one of the following statements holds:

- *The operator $I_{X_1} + K_1$ is an isomorphism from X_1 to itself and the operator $I_{X_2} + K_2$ is an isomorphism from X_2 to itself.*

- *The null space* $\ker(I_{X_1} + K_1)$ *and* $\ker(I_{X_2} + K_2)$ *have the same nonzero finite dimension and*

$$\mathrm{Im}(I_{X_1} + K_1) = \{\varphi \in X_1 : \langle \varphi, \psi \rangle = 0 \text{ for all } \psi \in \ker(I_{X_2} + K_2)\},$$
$$\mathrm{Im}(I_{X_2} + K_2) = \{\psi \in X_2 : \langle \varphi, \psi \rangle = 0 \text{ for all } \varphi \in \ker(I_{X_1} + K_1)\}.$$

Proof. See for example in [Kre14], Chapter 4, page 45. □

3.2 Boundary Integral Operators

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. We consider the integral kernels

$$K(x,y) := \frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y); \qquad K^*(x,y) := \frac{\partial}{\partial \nu_\Omega(x)} S_n(x-y)$$

for all $x, y \in \partial\Omega \setminus \{x, y \in \partial\Omega : x = y\}$.

For $n \geq 2$, since Ω is of class $C^{1,\alpha}$ a classical argument of [Fol95], page 128, implies the existence of $C_{\Omega,\alpha} > 0$ such that

$$|(x-y)\nu_\Omega(x)| \leq C_{\Omega,\alpha}|x-y|^{1+\alpha}, x, y \in \partial\Omega, x \neq y.$$

Hence,

$$\left| \frac{|(x-y)|\nu_\Omega(x)\nabla S_n(x-y)}{|x-y|^n} \right| \leq \frac{C_{\Omega,\alpha}}{|x-y|^{n-1-\alpha}}.$$

This inequality note that K^* is weakly singular and compact (see for instance [Tri92] p. 128).

We denote the operator W_Ω from $C^{1,\alpha}(\partial\Omega)$ onto itself and W^t_Ω are from $C^{0,\alpha}(\partial\Omega)$ onto itself are defined by setting

$$\begin{aligned} W_\Omega f(x) &= -\int_{\partial\Omega} f(y)\nu_\Omega(y)\nabla S_n(x-y)d\sigma_y, \ \forall f \in C^{1,\alpha}(\partial\Omega) \\ W^t_\Omega f(x) &= \int_{\partial\Omega} f(y)\nu_\Omega(x)\nabla S_n(x-y)d\sigma_y \ \forall f \in C^{0,\alpha}(\partial\Omega). \end{aligned} \tag{3.2.1}$$

for all $x \in \partial\Omega$.
As W_Ω and W^t_Ω are also an integral operator with weakly singular kernel and then it is also compact in $\partial\Omega$. Note that W_Ω and W^t_Ω are adjoint with respect to the dual system $\langle \cdot, \cdot \rangle$ defined by

$$\langle \phi, \psi \rangle_{\partial\Omega} = \int \phi\psi d\sigma, \qquad \phi \in C^{0,\alpha}(\partial\Omega), \psi \in C^{1,\alpha}(\partial\Omega). \tag{3.2.2}$$

Theorem 3.2. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Let* $\phi, \psi \in C^{0,\alpha}(\partial\Omega)$. *Then* W^t_Ω *is adjoint operator of* W_Ω *with respect to bilinear form* (3.2.2).

Proof. By Fubini-Tonelli Theorem, we have

$$\begin{aligned}
&\langle W_\Omega[\phi], \psi\rangle_{\partial\Omega} \\
&= \int_{\partial\Omega} \psi(z) \left(\int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y)\phi(y) d\sigma_y \right) d\sigma_z \\
&= \int_{\partial\Omega} \phi(z) \left(\int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} S_n(x-y)\psi(y) d\sigma_y \right) d\sigma_z \\
&= \langle \phi, W^t_\Omega[\psi] \rangle_{\partial\Omega}.
\end{aligned}$$

□

By the compactness of the imbeddings of $C^{1,\alpha}(\partial\Omega)$ into $C^{1,\beta}(\partial\Omega)$ and of $C^{0,\alpha}(\partial\Omega)$ into $C^{0,\beta}(\partial\Omega)$ for all $\beta \in]0, \alpha[$, we can deduce that W is compact from $C^{1,\alpha}(\partial\Omega)$ into itself and W^t_Ω is compact from $C^{0,\alpha}(\partial\Omega)$ into itself.

We now set $V_\Omega[\phi] = v[\phi]|_{\partial\Omega}$. Let $\phi \in C^{0,\alpha}(\partial\Omega), \psi \in C^{1,\alpha}(\partial\Omega)$. The well known the jump formulae become

$$\begin{aligned}
v^+_\Omega[\phi]|_{\partial\Omega} &= v^-_\Omega[\phi]|_{\partial\Omega}, \\
\nu_\Omega \nabla v^+_\Omega[\phi] &= -\frac{1}{2}\phi + W^t_\Omega \phi, \\
\nu_\Omega \nabla v^-_\Omega[\phi] &= \frac{1}{2}\phi + W^t_\Omega \phi, \\
w^+_\Omega[\psi]|_{\partial\Omega} &= \frac{1}{2}\psi + W_\Omega \psi, \\
w^-_\Omega[\psi]|_{\partial\Omega} &= -\frac{1}{2}\psi + W_\Omega \psi, \\
\nu_\Omega \nabla w^+_\Omega[\psi]|_{\partial\Omega} &= \nu_\Omega \nabla w^-_\Omega[\psi]|_{\partial\Omega},
\end{aligned} \tag{3.2.3}$$

on $\partial\Omega$.

Lemma 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi, \psi \in C^{0,\alpha}(\partial\Omega)$. Then V_Ω is self-adjoint operator with respect to bilinear form defined in formula* (3.2.2).

Proof. We have

$$\begin{aligned}
\langle V_\Omega \phi, \psi\rangle_{\partial\Omega} &= \int_{\partial\Omega} V_\Omega \phi \psi d\sigma \\
&= \int_{\partial\Omega} \int_{\partial\Omega} S_n(x-y)\phi(y)\psi(z) d\sigma_y d\sigma_z \\
&= \int_{\partial\Omega} \phi(z) \left(\int_{\partial\Omega} S_n(x-y)\phi(y) d\sigma_y \right) d\sigma_z \\
&= \langle \phi, V_\Omega \psi \rangle_{\partial\Omega}.
\end{aligned}$$

□

Now, we study the linear map $\frac{1}{2}I + W_\Omega : C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ and the linear map $\frac{1}{2}I + W^t_\Omega : C^{0,\alpha}(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$. Similarly, we study the linear the map $-\frac{1}{2}I + W_\Omega : C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ and the linear map $-\frac{1}{2}I + W^t_\Omega : C^{0,\alpha}(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$.

Lemma 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $f \in C^{0,\alpha}(\partial\Omega)$. Then*

$$\int_{\partial\Omega}\left(-\frac{1}{2}I + W_\Omega^t\right) f d\sigma = \int_{\partial\Omega} f d\sigma.$$

Proof. See in example in [CPM19], Chapter 6. □

Theorem 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi, \psi \in C^{0,\alpha}(\partial\Omega)$. If $f \in \ker\left(-\frac{1}{2}I + W_\Omega^t\right)$, then $Vf \in \ker\left(-\frac{1}{2}I + W_\Omega\right)$.*

Proof. See in example in [CPM19], Chapter 6. □

Then we prove that V_Ω is an isomorphism from $\ker\left(\frac{1}{2}I + W_\Omega^t\right)$ to $\ker\left(\frac{1}{2}I + W_\Omega\right)$.

Theorem 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi, \psi \in C^{0,\alpha}(\partial\Omega)$. The map $\ker\left(-\frac{1}{2}I + W_\Omega^t\right)$ to $\ker\left(-\frac{1}{2}I + W_\Omega\right)$ taken f by Vf is isomorphism.*

Proof. See in example in [CPM19], Chapter 6. □

Hence, if Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$, then the map $\frac{1}{2}I + W_\Omega$ takes from $C^{1,\alpha}(\partial\Omega)$ to itself and the map $\frac{1}{2}I + W_\Omega^t$ takes from $C^{0,\alpha}(\partial\Omega)$ to itself are isomorphism. Similar, The map $-\frac{1}{2}I + W_\Omega$ takes from $C^{1,\alpha}(\partial\Omega)$ to itself and the map $-\frac{1}{2}I + W_\Omega^t$ takes $C^{0,\alpha}(\partial\Omega)$ to itself are isomorphism.
By Schauder regularity, V_Ω is bounded from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$.
In general, for all dimensions $n \geq 2$ we have the following

Theorem 3.5. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. The map from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega)$ which takes (ϕ, ρ) to $V_\Omega[\phi] + \rho$ is an isomorphism, where*

$$C^{0,\alpha}(\partial\Omega)_0 = \{f \in C^{0,\alpha}(\partial\Omega) : \int_{\partial\Omega} f d\sigma = 0\}.$$

Proof. See in example in [CPM19], Chapter 6. □

For $n \geq 3$ we can show that V_Ω is an isomorphism.

Theorem 3.6. *If $n \geq 3$, then V_Ω is an isomorphism from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$.*

Proof. See in example in [CPM19], Chapter 6. □

We show that harmonic functions in a bounded open set and in the exterior of a bounded open set can be written as a sum of a single layer and of a constant function. The proof can be deduced by Theorems 3.5 and 3.6 and by the uniqueness of the solution of the interior and exterior Dirichlet problems.

Proposition 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

1. *The map that takes a pair (μ, c) to $v_\Omega^+[\mu] + c$ is a linear bijection from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to the space of the functions of $C^{1,\alpha}(\overline{\Omega})$ that are harmonic in Ω.*
2. *If $n \geq 3$, then the map that takes a μ to $v_\Omega^-[\mu]$ is a linear bijection from $C^{0,\alpha}(\partial\Omega)$ to the space of the functions of $C^{1,\alpha}_{loc}(\overline{\Omega^-})$ that are harmonic in Ω^- and harmonic at infinity.*

3. *If $n = 2$, the map that takes a pair (μ, c) to $v_\Omega^-[\mu] + c$ is a linear bijective from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to the space of the functions of $C^{1,\alpha}_{loc}(\overline{\Omega^-})$ that are harmonic in Ω^- and harmonic at infinity.*

The statements are an immediate consequence of the third Green Identities in Ω and Ω^- and of the jump formulas:

Theorem 3.7. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $u \in C^{1,\alpha}(\overline{\Omega})$ is harmonic in Ω, then*

$$w_\Omega[u|_{\partial\Omega}] - v_\Omega[\nu_\Omega \nabla u|_{\partial\Omega}] = \begin{cases} u(x) \text{ in } \Omega, \\ \dfrac{1}{2}u(x) \text{ on } \partial\Omega, \\ 0 \text{ in } \Omega^-. \end{cases}$$

Theorem 3.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $u \in C^{1,\alpha}_{loc}(\overline{\Omega^-})$ is harmonic in Ω^- and harmonic at infinity, then*

$$-w_\Omega[u|_{\partial\Omega}] + v_\Omega[\nu_\Omega \nabla u|_{\partial\Omega}] + \lim_{x\to\infty} u(x) = \begin{cases} u(x) \text{ in } \Omega^-, \\ \dfrac{1}{2}u(x) \text{ on } \partial\Omega, \\ 0 \text{ in } \Omega. \end{cases}$$

We define the hypersingular operator $T[\psi] = \frac{\partial}{\partial\nu_\Omega(x)} W_\Omega[\psi]$.

Lemma 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi, \psi \in C^{0,\alpha}(\partial\Omega)$. Then T is self-adjoint operator via bilinear the form* (3.2.2).

Proof. Let w_1, w_2 be two double layer with density ϕ, ψ.
By Green Identities, and by Jump formulae, we have

$$\begin{aligned}
\langle T\phi, \psi\rangle &= \int_{\partial\Omega} T\phi\psi d\sigma \\
&= \int_{\partial\Omega} \frac{\partial w_1}{\partial\nu_\Omega(x)} \left(\int_{\partial\Omega} \frac{\partial}{\partial_{\nu_\Omega}(y)} S_n(y-z)\phi(z) d\sigma_z \right) d\sigma_x \\
&= \frac{1}{2} \int_{\partial\Omega} \frac{\partial w_1}{\partial\nu_\Omega(x)} \left(w_2^+[\phi] + w_2^-[\phi] \right)(z) d\sigma_x \\
&= \frac{1}{2} \int_{\partial\Omega} \left(w_1^+[\psi] + w_1^-[\psi] \right)(x) \frac{\partial w_2}{\partial\nu_\Omega(x)} d\sigma_x \\
&= \int_{\partial\Omega} \frac{\partial w_2}{\partial\nu_\Omega(x)} \left(\int_{\partial\Omega} \frac{\partial}{\partial_{\nu_\Omega}(y)} S_n(y-z)\psi(z) d\sigma_z \right) d\sigma_x \\
&= \langle \phi, T\psi\rangle .
\end{aligned}$$

□

Definition 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We denote by C_Ω^+ the operator from $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$ defined by*

$$C_\Omega^+[\phi, \psi] \equiv \begin{pmatrix} -\frac{1}{2}I + W_\Omega & V_\Omega \\ T & -\frac{1}{2}I + W_\Omega^t \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad \forall(\phi, \psi) \in C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega).$$

The operator C^+_Ω is said to be the Calderón projection in Ω.
Similarly, we denote by $C^-_{\Omega^-}$ the operator from $C^{1,\alpha}\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$

$$C^-_\Omega[\phi,\psi] \equiv \begin{pmatrix} \frac{1}{2}I + W_\Omega & V_\Omega \\ T & \frac{1}{2}I + W^t_\Omega \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad \forall(\phi,\psi) \in C^{0,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\Omega).$$

The operator C^-_Ω is said to be the Calderón projection in exterior domain Ω^-.

Theorem 3.9. *Let Ω be bounded open subset of $C^{1,\alpha}$. Let u be harmonic in Ω and $u \in C^{1,\alpha}(\partial\Omega)$. The boundary value and the normal derivative satisfy*

$$\begin{pmatrix} u \\ \frac{\partial u}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + W_\Omega & V_\Omega \\ T & -\frac{1}{2}I + W^t_\Omega \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial v} \end{pmatrix}$$

Moreover, the operator $C_\Omega : C^{0,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$ takes (ϕ,ψ) to $C_\Omega[\phi,\psi]$ such that $C^-_\Omega - C^+_\Omega = I$.

Proof. This follows from the third Green Identity using the jump relations.
We have

$$\begin{aligned} &C^-_\Omega[\phi,\psi] - C^+_\Omega[\phi,\psi] \\ &= \begin{pmatrix} \frac{1}{2}I + W_\Omega & V_\Omega \\ T & \frac{1}{2}I + W^t_\Omega \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} - \begin{pmatrix} -\frac{1}{2}I + W_\Omega & V_\Omega \\ T & -\frac{1}{2}I + W^t_\Omega \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \end{aligned}$$

for all $(\phi,\psi) \in C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$. □

3.3 Spectral Analysis of Neumann - Poincaré Operator

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We set $h = (h^+, h^-)$, where $h^+ = h|_\Omega, h^- = h|_{\Omega^-}$.
We denote the space of harmonic function

$$\begin{aligned} \mathfrak{H} = &\{h^+ \in C^{1,\alpha}(\overline{\Omega}) | \Delta h^+ = 0 \text{ in } \Omega\} \\ &\times \{h^- \in C^{1,\alpha}_{loc}(\overline{\Omega^-}) | \Delta h^- = 0 \text{ in } \Omega^-, h^- \text{ harmonic at infinity}\} \end{aligned}$$

could be defined as

$$\begin{aligned} &\{h^+ \in W^{1,2}(\Omega) | \Delta h^+ = 0 \text{ in } \Omega\} \\ &\times \{h^- \in W^{1,2}(\Omega^-) | \Delta h^- = 0 \text{ in } \Omega^-, h^- \text{ harmonic at infinity}\} \end{aligned}$$

by Schauder Regularity. Indeed, if $\Delta h^+ = 0, h^+ \in W^{1,2}(\Omega)$ and Ω is of class $C^{1,\alpha}$, then $h^+ \in C^{1,\alpha}(\overline{\Omega})$. Similarly, if $\Delta h^- = 0, h^- \in W^{1,2}(\Omega^-)$, h^- harmonic at infinity and Ω is of class $C^{1,\alpha}$, then $h^- \in C^{1,\alpha}_{loc}(\overline{\Omega^-}), \Delta h = 0$ in Ω^-, h^- harmonic at infinity.
We introducethe positive Hermitian form on $\mathfrak{H}$

$$\langle h, g \rangle_{\mathfrak{H}} = \int_\Omega \nabla h^+ \nabla g^+ dx + \int_{\Omega^-} \nabla h^- \nabla g^- dx \tag{3.3.1}$$

where $h = (h^+, h^-) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}_{loc}(\overline{\Omega^-}), g = (g^+, g^-) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}_{loc}(\overline{\Omega^-})$.
The corresponding finite energy semi-norm

$$\|h\|^2_{\mathfrak{H}} = \int_\Omega |\nabla h^+|^2 dx + \int_{\Omega^-} |\nabla h^-|^2 dx.$$

The next aim is to identify a closed subspace of $\mathfrak{H}$.
Consider the linear and continuous trace operator

$$\text{Tr} \ : C^{0,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\partial\Omega).$$

The trace operator of $\partial\Omega$ is linear, surjective and it has a continuous right inverse, see for instance Gilbarg and Trudinger [Tru01]. We will set $h|_{\partial\Omega} = \text{Tr}\, h$.

By the Jump formula (3.2.3), the operator $\frac{1}{2}I + W_\Omega$, $-\frac{1}{2}I + W_\Omega$ and and their transpose operators $-\frac{1}{2}I + W^t_\Omega$, $-\frac{1}{2}I + W^t_\Omega$ to the boundary conditions of the interior and exterior Dirichlet and Neumann problems. By Lemma 3.1, by Theorem 3.6, we have proved that V_Ω is self-adjoint, injective, W_Ω and W^t_Ω are compact operators. By the Fredholm Alternative Theorem with in the dual systems $\langle C^{0,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega)\rangle$, we deduce existence results for the boundary value problems.

If $n = 2$, the single layer potential with moment f is harmonic at infinity if and only $\int_{\partial\Omega} f = 0$, in which case the potential vanishes at infinity.

As a first application we note an important isometric identification, see [Lan72].

Lemma 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $f \in C^{0,\alpha}(\partial\Omega)$. Then $\langle V_\Omega f, f\rangle_{\partial\Omega} = \|v_\Omega[f]\|^2_{\mathfrak{H}}$.*

Proof. Since $f \in C^{0,\alpha}(\partial\Omega)$, by Schauder regularity Theorem 2.6 of single layer potential, we have $v^+[f] \in C^{1,\alpha}(\overline{\Omega})$ and $v^-[f] \in C^{1,\alpha}_{loc}(\overline{\Omega^-})$. Since $v^+[f]$ is harmonic in Ω, by the first Green's formulae in interior domain Ω for $v^+[f]$ and first Green's formulae in exterior domain Ω^- for $v^-[f]$, we have

$$\begin{aligned}\|v_\Omega[f]\|^2_{\mathfrak{H}} &= \int_\Omega |\nabla v^+_\Omega[f]|^2 dx + \int_{\Omega^-} |\nabla v^-_\Omega[f]|^2 dx \\ &= \int_{\partial\Omega} v^+[f]|_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} v^+[f]|_{\partial\Omega} - \int_{\partial\Omega} v^-[f]|_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} v^-[f]|_{\partial\Omega},\end{aligned}$$

and by jump formula

$$\begin{aligned}\nu_\Omega \nabla v^+_\Omega[f]|_{\partial\Omega} &= \frac{1}{2} f + W^t_\Omega f, \\ \nu_\Omega \nabla v^-_\Omega[f]|_{\partial\Omega} &= -\frac{1}{2} f + W^t_\Omega f, \\ v^+_\Omega[f]|_{\partial\Omega} &= v^-_\Omega[f]|_{\partial\Omega}\end{aligned}$$

on $\partial\Omega$.
Then

$$\|v_\Omega[f]\|^2_{\mathfrak{H}} = \int_{\partial\Omega} \left(\frac{1}{2} f + W^t_\Omega f + \frac{1}{2} f - W^t_\Omega f\right) V_\Omega[f] = \int_{\partial\Omega} f V_\Omega[f].$$

On the other hand, since $f \in C^{0,\alpha}(\partial\Omega)$ and $V_\Omega[f] \in C^{1,\alpha}(\partial\Omega)$, by the dual system defined in (3.2.2), we have $\langle f, V_\Omega f\rangle_{\partial\Omega} = \int_{\partial\Omega} f v_\Omega[f]$. Moreover, by Lemma 3.1, we conclude that $\langle f, V_\Omega f\rangle_{\partial\Omega} = \langle V_\Omega f, f\rangle_{\partial\Omega} = \|v_\Omega[f]\|^2_{\mathfrak{H}}$ □

By Lemma 3.4 and Lemma 3.1, V_Ω is non - negative and is self-adjoint operator via the bi-linear form defined in (3.2.2). Moreover, by $V_\Omega f = 0$ implies that $\nabla v^+_\Omega[f] = 0$ in Ω and $\nabla v^-_\Omega[f] = 0$ in Ω^-, whence $v_\Omega[f]$ is constant on in Ω and Ω^-. However, since $v^+_\Omega[f]$ is harmonic at infinity, so $v^+_\Omega[f] = 0$ on Ω. Therefore, by Jump formulae, $f = \nu_\Omega \nabla v^-_\Omega[f] - \nu_\Omega \nabla v^+_\Omega[f] = 0$. This prove that V_Ω is strictly positive operator .

Proposition 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Assume that $n \geq 3$ and let $h = (h^+, h^-) \in \mathfrak{H}$. Then $h^+|_{\partial\Omega} = h^-|_{\partial\Omega}$ if and only if there exists $\rho \in C^{0,\alpha}(\partial\Omega)$ such that $h = V_\Omega \rho$.*

Proof. Assume that $h^+|_{\partial\Omega} = h^-|_{\partial\Omega} = f$. The third Green Identity implies that

$$\frac{1}{2}f = W_\Omega[f] - V_\Omega[\nu_\Omega \nabla h^+|_{\partial\Omega}] \text{ on } \partial\Omega.$$

Similarly, the third Green Identity implies that

$$\frac{1}{2}f = -W_\Omega[f] + V_\Omega[\nu_\Omega \nabla h^-|_{\partial\Omega}] + b,$$

where $b = \lim_{x\to\infty} h^-(x) \equiv 0$. Now, taking the sum of two equations above and set $\rho = \nu_\Omega \nabla h^-|_{\partial\Omega} - \nu_\Omega \nabla h^+|_{\partial\Omega}$, we verify that $V_\Omega[\rho] = f$.

Conversely, by the solvability for the Dirichlet problem and Jump Formulas, we have $v^+_\Omega[\rho]|_{\partial\Omega} = v^-_\Omega[\rho]|_{\partial\Omega}$ and $h^+|_{\partial\Omega} = h^-|_{\partial\Omega}$. □

We define the *space of single layer potentials* by

$$\mathfrak{S} = \{h \in \mathfrak{H} : h^+|_{\partial\Omega} = h^-|_{\partial\Omega}\}.$$

The orthogonal complement in $\mathfrak{H}$ will be denoted $\mathfrak{D} = \{g \in \mathfrak{H} | \langle h, g\rangle_{\mathfrak{H}} = 0, \forall h \in \mathfrak{H}\}$ where $\langle h, g\rangle$ is defined in (3.3.1) and we will identify this with the *space of double layer potentials* belonging to $\mathfrak{H}$.

Proposition 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. The normal derivaties of the entries of a pair $(h^+, h^-) \in \mathfrak{H}$ are $\frac{\partial h^+}{\partial \nu_\Omega}$, $\frac{\partial h^-}{\partial \nu_\Omega}$, belong in $C^{0,\alpha}(\partial\Omega)$ and satisfy*

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial h^+}{\partial \nu_\Omega} g^+ d\sigma &= \int_{\partial\Omega} h^+ \frac{\partial g^+}{\partial \nu_\Omega} d\sigma = \int_\Omega \nabla h^+ \nabla g^+ dx, \\ \int_{\partial\Omega} \frac{\partial h^-}{\partial \nu_\Omega} g^- d\sigma &= \int_{\partial\Omega} h^- \frac{\partial g^-}{\partial \nu_\Omega} d\sigma = -\int_{\Omega^-} \nabla h^- \nabla g^- dx. \end{aligned} \tag{3.3.2}$$

for every $g = (g^+, g^-) \in \mathfrak{H}$.

Proof. Since h^+, g^+ are harmonic functions in Ω, by the first Green's formulae, we have

$$\int_{\partial\Omega} \frac{\partial h^+}{\partial \nu_\Omega} g^+ d\sigma = \int_{\partial\Omega} h^+ \frac{\partial g^+}{\partial \nu_\Omega} d\sigma = \int_\Omega \nabla h^+ \nabla g^+ dx.$$

Similarly, we have $h^-, g^- \in C^{1,\alpha}(\overline{\Omega})$ and $h^-, g^- \in C^{1,\alpha}_{loc}(\overline{\Omega^-})$. Since h^-, g^- are harmonic functions in Ω, by first Green's formulae in exterior domain Ω^-, we have

$$\int_{\partial\Omega} \frac{\partial h^-}{\partial \nu_\Omega} g^- d\sigma = \int_{\partial\Omega^-} h^- \frac{\partial g^-}{\partial \nu_\Omega} d\sigma = -\int_\Omega \nabla h^- \nabla g^- dx.$$

□

Corollary 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $h = (h^+, h^-) \in \mathfrak{H}$, the following statements are equivalent*

1. $h \in \mathfrak{D}$
2. $\dfrac{\partial h^+}{\partial \nu_\Omega} = \dfrac{\partial h^-}{\partial \nu_\Omega} \in C^{0,\alpha}(\partial\Omega)$

3. *There exists f in $C^{1,\alpha}(\partial\Omega)$ such that $h = w_\Omega[f]$, where $h^+ = w^+_\Omega[f], h^- = w^-_\Omega[f]$. In this case, $f = h^+ - h^-$.*

Proof. (1) implies (2)
Assume that $h \in \mathfrak{D}$, by Proposition above, we have

$$\int_{\partial\Omega} \left(\frac{\partial h^+}{\partial \nu_\Omega} g^+ - \frac{\partial h^-}{\partial \nu_\Omega} g^- \right) d\sigma = 0.$$

Any element $f \in C^{1,\alpha}(\partial\Omega)$ can be realized as $f = g^+ = g^-$ for a proper choice of g, and hence $\frac{\partial h^+}{\partial \nu_\Omega} = \frac{\partial h^-}{\partial \nu_\Omega} \in C^{0,\alpha}(\partial\Omega)$.
(2) implies (1)
Assume that $\frac{\partial h^+}{\partial \nu_\Omega} = \frac{\partial h^-}{\partial \nu_\Omega} \in C^{0,\alpha}(\partial\Omega)$ Then the same identity implies (1).
(2) implies (3)
Assume that $\frac{\partial h^+}{\partial \nu_\Omega} = \frac{\partial h^-}{\partial \nu_\Omega} \in C^{0,\alpha}(\partial\Omega)$. Let $f = h^+ - h^-$. Thus

$$w_\Omega[f](x) = \int_{\partial\Omega} \frac{\partial S_n(x-y)}{\partial \nu_\Omega(y)} f d\sigma < \infty$$

for all $x \in \partial\Omega$ and by Jump formulae,

$$w^+_\Omega[f]|_{\partial\Omega} - w^-_\Omega[f]|_{\partial\Omega} = \frac{1}{2} f + W_\Omega f - \left(-\frac{1}{2} f + W_\Omega f \right) = f,$$
$$\frac{\partial}{\partial \nu_\Omega} w^+_\Omega[f]|_{\partial\Omega} = \frac{\partial}{\partial \nu_\Omega} w^-_\Omega[f]|_{\partial\Omega}.$$

Hence

$$h(x) = w_\Omega[f](x) = \int_{\partial\Omega} S_n(x-y) f(y) dy,$$

h harmonic at infinity,

for all $x \in \partial\Omega$.
(3) implies (1)
We will only prove that, if $h = W^+_\Omega f$ then

$$\int_{\partial\Omega^-} \nu_\Omega \nabla h d\sigma = 0.$$

Indeed, since $\nu_\Omega \nabla w^+_\Omega[f]|_{\partial\Omega} = \nu_\Omega \nabla w^-_\Omega[f]|_{\partial\Omega}$,

$$\int_{\partial\Omega} \nu_\Omega \nabla w^+_\Omega[f] d\sigma = - \int_{\partial\Omega^-} \nu_\Omega \nabla w^+_{\Omega^-}[f|_{\partial\Omega}] d\sigma.$$

The last integral can be seen to be equal to zero by a standard argument based on the first Green Identity in Ω^-. □

The next result is so called *Plemelj's symmetrization princible*, see [Kor99], [Kor13], [Ple11].

Theorem 3.10. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. The operator V_Ω, W^t_Ω from $C^{0,\alpha}(\partial\Omega)$ into itself satisfies $W_\Omega V_\Omega = V_\Omega W^t_\Omega$.*

Proof. Let $f \in C^{1,\alpha}(\partial\Omega)$, for $x \in \Omega$, we have

$$\begin{aligned}
w_\Omega^+[V_\Omega f]|_{\partial\Omega}(x) &= \int_{\partial\Omega} \frac{\partial}{\partial_{\nu_\Omega}(y)} S_n(x-y) \left(\int_{\partial\Omega} S_n(y-z) f(z) d\sigma_z \right) \\
&= \int_{\partial\Omega} f(z) \left(\int_{\partial\Omega} \frac{\partial}{\partial_{\nu_\Omega}(y)} S_n(x-y) S_n(z-y) d\sigma_y \right) d\sigma_z \\
&= \int_{\partial\Omega} f(z) \left(\int_{\partial\Omega} S_n(x-y) \frac{\partial}{\partial_{\nu_\Omega}(y)} S_n(z-y) d\sigma_y \right) d\sigma_z \\
&= \int_{\partial\Omega} S_n(x-y) \left(\frac{\partial}{\partial_{\nu_\Omega}(y)} \int_{\partial\Omega} S_n(z-y) f(z) d\sigma_z \right) \\
&= v_\Omega^+[\frac{\partial}{\partial\nu_\Omega} v_\Omega^+ f]|_{\partial\Omega}(x)
\end{aligned}$$

Passing from x to a point y in boundary, by the Jump formula, we obtain

$$-\frac{1}{2} V_\Omega f(y) + W_\Omega V_\Omega f(y) = -\frac{1}{2} V_\Omega f(y) + V_\Omega W_\Omega^t f(y).$$

Hence $W_\Omega V_\Omega(y) = V_\Omega W_\Omega^t(y)$ for $y \in \partial\Omega$. □

Chapter 4

Poincaré's Variational Problem

4.1 Poincaré 's Variational Problem

Definition 4.1. *Let $m \in \mathbb{N} \setminus \{0\}, \alpha \in \,]0,1[$. Let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\lambda \in \mathbb{C}$. We say that λ is Dirichlet eigenvalue of $-\Delta$ in Ω if there exists $u \in C(\bar{\Omega}, \mathbb{C}) \cap C^2(\Omega, \mathbb{C}) \setminus \{0\}$ such that*

$$\begin{cases} \Delta u + \lambda u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

We recall $\mathfrak{H}$ be the space of pairs of harmonic functions (h^+, h^-) defined on Ω, respectively Ω^-, $h^-(\infty) = 0$, and have finite energy.

The space $\mathfrak{H}$ possesses two natural direct sum decompositions

$$\mathfrak{H} = \mathfrak{S} \oplus \mathfrak{D}$$

where $\mathfrak{S}, \mathfrak{D}$ are space of single and double layer potentials. Note that the direct sum may not orthogonal.

Let C^+_Ω, C^-_Ω be corresponding Calderón projections in interior domain Ω and exterior domain Ω^- as Definition 3.4. We define inner, outer enegry functionals

$$J^+[f] = \int_\Omega |\nabla v^+_\Omega[f]|^2 dx, \qquad J^-[f] = \int_{\Omega^-} |\nabla v^-_\Omega[f]|^2 dx$$

Poincaré proposes to analyze the characteristic value of the Reyleigh quotient as a form $\dfrac{J^-[f] - J^+[f]}{J^-[f] + J^+[f]}$ and predicts that they fill a discrete spectrum. Hence, we guss in the morderm terms that some relative compactness is responsible for this phenomenon.

Lemma 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $g \in C^{0,\alpha}(\partial\Omega)$. Then*

$$\frac{\left\langle (C^-_\Omega - C^+_\Omega) v_\Omega[g], V_\Omega[g] \right\rangle_{\mathfrak{H}}}{\|v_\Omega[g]\|^2_{\mathfrak{H}}} = \frac{\langle W_\Omega V_\Omega g, g \rangle_{2,\partial\Omega}}{\langle V_\Omega g, g \rangle^2_{2,\partial\Omega}}.$$

Proof. Since $g \in C^{0,\alpha}(\partial\Omega)$, then $v^+_\Omega[g] \in C^{1,\alpha}(\overline{\Omega})$ and $v^-_\Omega[g] \in C^{1,\alpha}_{loc}(\overline{\Omega^-})$ and $v_\Omega[g]$ is a harmonic function in $\mathbb{R}^n \setminus \partial\Omega$.
By the Jump formula, and by Green Formula, and by the opposite orientation of $\partial\Omega$ with respect to the exterior domain, we have that

$$\int_{\Omega^-} |\nabla v^-_\Omega[g]|^2 dx = -\int_{\partial\Omega} V_\Omega[g] \frac{\partial}{\partial \nu_\Omega} v^-_\Omega[g] d\sigma = \left\langle V_\Omega g, \frac{1}{2} g + W^t_\Omega g \right\rangle_{\partial\Omega}$$

and

$$\int_{\Omega} |\nabla v_{\Omega}^{+}[g]|^2 dx = \int_{\partial\Omega} V_{\Omega}[g] \frac{\partial}{\partial \nu_{\Omega}} v_{\Omega}^{+} d\sigma = \left\langle V_{\Omega} g, -\frac{1}{2} g + W_{\Omega}^{t} g \right\rangle_{\partial\Omega}.$$

Since

$$\|v_{\Omega}[f]\|_{\mathfrak{H}}^2 = \int_{\Omega} |\nabla v_{\Omega}^{+}[f]|^2 dx + \int_{\Omega^-} |\nabla v_{\Omega}^{-}[f]|^2 dx$$

and V_{Ω} is self adjoint by Lemma 3.1 and $C_{\Omega}^{-} - C_{\Omega}^{+} = I$ by Theorem 3.9, we obtain

$$\left\langle (C_{\Omega}^{-} - C_{\Omega}^{+}) v_{\Omega}[g], v_{\Omega}[g] \right\rangle_{\mathfrak{H}} = \|v_{\Omega}[f]\|_{\mathfrak{H}}^2 = \langle V_{\Omega} g, W_{\Omega} g \rangle_{\partial\Omega} = \langle W_{\Omega} V_{\Omega} g, g \rangle_{\partial\Omega}$$

and

$$\|v_{\Omega}[g]\|_{\mathfrak{H}}^2 = \langle V_{\Omega} g, g \rangle_{\partial\Omega}.$$

□

Let us start with eigenfunction $f \in C^{0,\alpha}(\partial\Omega)$ of operator W_{Ω}^{t}. Then

$$W_{\Omega}^{t} f = \lambda f \Rightarrow W_{\Omega} V_{\Omega} f = V_{\Omega} W_{\Omega}^{t} f = \lambda V_{\Omega} f$$

and by jump formula

$$\nu_{\Omega} \nabla v^{+}[f]|_{\partial\Omega} = \left(-\frac{1}{2} + \lambda\right) f, \qquad \nu_{\Omega} \nabla v^{-}[f]|_{\partial\Omega} = \left(\frac{1}{2} + \lambda\right) f$$

on $\partial\Omega$. Hence, by (3.3.2), we obtain

$$J^{+}[f] = \int_{\Omega} |\nabla v_{\Omega}^{+}[f]|^2 dx = \left(-\frac{1}{2} + \lambda\right) \langle V_{\Omega} f, f \rangle_{\partial\Omega}$$

and

$$J^{-}[f] = \int_{\Omega^-} |\nabla v_{\Omega}^{-}[f]|^2 dx = \left(-\frac{1}{2} - \lambda\right) \langle V_{\Omega} f, f \rangle_{\partial\Omega}.$$

Therefore

$$\frac{J^{+}[f] - J^{-}[f]}{J^{+}[f] + J^{-}[f]} = \frac{\left(-\frac{1}{2} + \lambda + \frac{1}{2} + \lambda\right) \langle V_{\Omega} f, f \rangle_{\partial\Omega}}{\left(-\frac{1}{2} + \lambda - \frac{1}{2} - \lambda\right) \langle V_{\Omega} f, f \rangle_{\partial\Omega}} = 2\lambda. \tag{4.1.1}$$

Proposition 4.1. *(Extension of Theorem in D. Khavinson, M.P. Putinar, H.S Shapiro [Sha07])*
Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$ of class of $C^{1,\alpha}$ and let $\Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega}$. Let $v_{\Omega}[\rho]$ denote the single layer potential of a distribution $\rho \in C^{0,\alpha}(\partial\Omega)$.
The energy quotients are defined by

$$\lambda_k^{+} = \max_{\rho \perp \{\rho_0^{+}, \cdots \rho_{k-1}^{+}\}} \frac{\|\nabla v_{\Omega}[\rho]\|_{L^2(\Omega^-)}^2 - \|\nabla v_{\Omega}[\rho]\|_{L^2(\Omega)}^2}{\|\nabla v_{\Omega}[\rho]\|_2^2} \qquad \forall k \in \{1, \ldots n\}$$

The maximum is attained at a smooth distribution $\rho^{+} \in C^{0,\alpha}(\partial\Omega)$.
Similarly,

$$\lambda_k^{-} = \min_{\rho \perp \{\rho_0^{-}, \cdots \rho_{k-1}^{-}\}} \frac{\|\nabla v_{\Omega}[\rho]\|_{L^2(\Omega^-)}^2 - \|\nabla v_{\Omega}[\rho]\|_{L^2(\Omega)}^2}{\|\nabla v_{\Omega}[\rho]\|_2^2} \qquad \forall k \in \{1, \ldots n\}$$

The minmimum is attained at a smooth distribution $\rho^{-} \in C^{0,\alpha}(\partial\Omega)$.

Proof. Firsly, by Plemelj's symmetrization principle, we have $W_\Omega V_\Omega = V_\Omega W_\Omega^t$. Secondly, by the Jump formula and Lemma 4.1, we have

$$\frac{J^+[f] - J^-[f]}{J^-[f] + J^+[f]} = \frac{\langle W_\Omega V_\Omega \rho, \rho\rangle_{\partial\Omega}}{\langle V_\Omega \rho, \rho\rangle^2_{\partial\Omega}} = \frac{\left\langle (C^-_\Omega - C^+_\Omega) v_\Omega[\rho], v_\Omega[\rho]\right\rangle_{\mathfrak{H}}}{\|v_\Omega[\rho]\|^2_{\mathfrak{H}}} = \lambda.$$

Finally, let $\lambda_1^- \le \lambda_1^- \le \ldots \le 0 \le \ldots \le \lambda_2^+ \le \lambda_1^+$ be eigenvalue of W_Ω repeated according to their multiplicity and $\rho_k^+, \rho_k^- \in C^{0,\alpha}(\partial\Omega)$ be the corresponding eigenvalues. By the Courant-Fischer minimax principle, we have

$$\lambda_k^+ = \max_{\rho\perp\{\rho_0^+,\cdots\rho_{k-1}^+\}} \frac{\langle W_\Omega V_\Omega \rho, \rho\rangle_{\partial\Omega}}{\langle V_\Omega \rho, \rho\rangle^2_{\partial\Omega}} = \max_{\rho\perp\{\rho_0^+,\cdots\rho_{k-1}^+\}} \frac{\|\nabla v_\Omega[\rho]\|^2_{L^2(\Omega^-)} - \|\nabla v_\Omega[\rho]\|^2_{L^2(\Omega)}}{\|\nabla v_\Omega[\rho]\|^2_2}$$

and

$$\lambda_k^- = \min_{\rho\perp\{\rho_0^-,\cdots\rho_{k-1}^-\}} \frac{\langle W_\Omega V_\Omega \rho, \rho\rangle_{\partial\Omega}}{\langle V_\Omega \rho, \rho\rangle^2_{\partial\Omega}} = \min_{\rho\perp\{\rho_0^-,\cdots\rho_{k-1}^-\}} \frac{\|\nabla v_\Omega[\rho]\|^2_{L^2(\Omega^-)} - \|\nabla v_\Omega[\rho]\|^2_{L^2(\Omega)}}{\|\nabla v_\Omega[\rho]\|^2_2}.$$

Thanks for Schauder regularity Theorem B.1, the spectrum of operator W not only in L^2-theory but also in $C^{1,\alpha}$. Any the corresponding eigenfunctions must belong $C^{1,\alpha}$ □

Corollary 4.1. *Let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Assume that $\lambda \in \mathbb{R}$, $u \in W^{1,2}(\Omega)$ and $\Delta u + \lambda u = 0$ in Ω. Then $u \in C^{1,\alpha}(\overline{\Omega})$.*

This eigenvalue variational problem is correlated to the eigenvalue problem of the operator W_Ω. Precisely, we have

Corollary 4.2. *The spectrum of the operator W_Ω, multiplicities included, coincides with the set of values $\{\lambda_k^\pm\}_{k\in\mathbb{N}}$ of the Poincarié variational problem, together with possively the point zero. The extremal distributions for the Poincarié problem are exactly the eigenfunctions of W_Ω.*

4.2 Poincaré Problem in the Ball

We denote that $\mathbb{B}_3 = \mathbb{B}_3(0,1) = \{x \in \mathbb{R}^3 : |x| < 1\}$. Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{B}_3, \\ u = g \text{ on } \partial\mathbb{B}_3. \end{cases}$$

Thus, to solve Dirichlet's problem, it is sufficient to solve the integral equation

$$-\frac{1}{2} f + W_\Omega f = g \text{ on } \partial\mathbb{B}_3$$

where $f \in C^{1,\alpha}(\partial\Omega)$. Then $u = W^+_\Omega[f]$ would be harmonic function in $\mathbb{B}_3$. Carl Neumann [Neu87] remarked that the infinite sum, known today as Neumann series, $-g + Wg + W^2 g + \cdots$ if convergent, provides the density f. The complete solution of Dirichlet problem in the unit ball we can refer to [Fol95].

The complete solution of Poincaré's variational problem for unit ball in $\mathbb{R}^3$ is obtained in [Poi97; Poi99]. Now, we consider the extremal Poincaré variation problem in the ball. It is enough to consider the case of dimension $n = 3$, the case of $n > 3$ is similar. Instead, case $n = 2$ is somewhat different.

For $n \geq 3$, let H_k be the set of homogeneous polynomials of degree k satisfy the Laplace equation. By Corollary C.2 with $n = 3$, we have $\dim H_k = 2k + 1$. For each F in H_k, we can write $F(x) = r^k f(y)$, where $r = |x|, y = \frac{x}{r}$ is a point of $\partial\Omega$ and the function$f$ on $\partial\Omega$ is so called the spherical harmonic of order k. Since $\dfrac{F(x)}{|x|^{k+1}}$ is harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{B}}$, the pair $u^+ = r^k f(y)$ and $u^- = r^{-k-1} f(y)$ fit together continuously across $\partial\Omega$ to form the single layer potential of a moment g on $\partial\Omega$. We have

$$\frac{\partial u^+(y)}{\partial \nu_\Omega} = k f(y)$$
$$\frac{\partial u^-(y)}{\partial \nu_\Omega} = -(k+1) f(y)$$

for $y \in \partial\Omega$. Hence $g = (2k+1) f(y)$.
We have

$$J^+[g] = \int_{\partial\Omega} (\partial_\nu u^+)(y) g = k(2k+1) \int_{\partial\Omega} f^2 d\sigma$$
$$J^-[g] = \int_{\partial\Omega} (\partial_\nu u^-)(y) g = (k+1)(2k+1) \int_{\partial\Omega} f^2 d\sigma$$

By (4.1.1), we have

$$\frac{J^+[g] - J^-[g]}{J^-[g] + J^+[g]} = 2\lambda = \frac{(2k+1)(k+1-k)}{(2k+1)^2} = \frac{1}{2k+1}.$$

This assocoated Neumann-Poincaré eigenvalue. To summarize

The set of eigenvalues of Neumann- Poincaré operator coincides with the set $\{1/2; 1/6; 1/10; \ldots\}$ *and the eigenspace corresponding to the eigenvalues* $\dfrac{1}{2(2k+1)}$ *has dimension* $2k+1$ *coincides of all spherical harmonics of order* k, $k \in \mathbb{N}$.

The Neumann -Poincaré integral operators is injective in this case; i.e. the spectral point 0 of the Neumann - Poincarié operator is not an eigenvalue.

Theorem 4.1. *If* Ω *is a disk in* $\mathbb{R}^2$*, then the double layer potential* W_Ω *has rank one.*

Proof. For $f \in C^{1,\alpha}(\partial\Omega)$, we have

$$W_\Omega f(x) = -\int_{\partial\Omega} f(y) \nu_\Omega(y) \nabla S_n(x-y) d\sigma_y = \int_{\partial\Omega} \frac{\langle x-y, \nu(y)\rangle}{|y-x|^2} f(y) d\sigma_y.$$

If Ω is a disk, then there exists $x_0 \in \mathbb{R}^2, R > 0$ such that

$$\mathbb{D}(x_0, R) = \{x \in \mathbb{R}^2 : |x - x_0| < R\}).$$

Let $x, y \in \partial\Omega$, $\nu(x)$ and let $\nu(y)$ denotes the unit outer normal to $\partial\Omega$ at x and y. By change the polar coordinate

$$x = x_0 + r\cos\theta \qquad\qquad y = x_0 + r\sin\theta$$

where $0 < r < R, 0 < \theta < 2\pi$, we have

$$\frac{\langle x-y, \nu(y)\rangle}{|y-x|^2} = \frac{1}{2R}.$$

Hence we have

$$W_\Omega f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta.$$

Let f equal $\cos m\theta, \sin m\theta, m \in \mathbb{N}$, we have $W_\Omega[1] = 1$ and $W_\Omega[\cos m\theta] = W_\Omega[\sin\theta] = 0$. Thus $\lambda = 1$ is an eigenvalue of W_Ω with corresponding eigenfunction 1, and $\lambda = 0$ is also an eigenvalue of W_Ω with corresponding eigenfunctions

$$\{\cos m\theta, \sin m\theta, m \in \mathbb{N}\}.$$

From the completeness of the orthogonal set of eigenfunctions

$$\{1, \cos m\theta, \sin m\theta, m \in \mathbb{N}\}$$

in $L^2[0, 2\pi]$. By Schauder Regularity Theorem B.1, these eigenfunctions are of class $C^{1,\alpha}$. That is, $\lambda = 1$and $\lambda = 0$ are the only eigenvalues of W_Ω for the case $\partial\Omega$ is a circle. Furthermore, since $\cos m\theta, \sin m\theta, m \in \mathbb{N}$ are all eigenfunctions of W_Ω corresponding to $\lambda = 0$, it follows that $\dim N(W_\Omega) = \infty$. □

Conversely, it is well known in H.S. Shapiro book, c.f [Sha92], Chapter 7, disk is the only domain for which the Neumann - Poincaré operator has finite rank.

4.3 Symmetric of Boundary Integral Operator on the Ball

Theorem 4.2. *Let $\alpha \in]0,1[$. Let Ω be bounded, open, connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\partial\Omega$ be connected. Let $K(x,y) = \dfrac{\langle x-y, \nu(y)\rangle}{|x-y|^n}, \forall x, y \in \partial\Omega, x \neq y$ be Neumann - Poincaré kernel. Then K is symmetric if and ony if Ω is a ball.*

Proof. Let Ω be the ball $\mathbb{B}(x_0, R)$ in $\mathbb{R}^n, n \geq 3$. Let $x, y \in \partial\Omega$, let $\nu(x)$ and $\nu(y)$ denote the unit outer normal to $\partial\Omega$ at x and y. We have

$$\langle x-y, \nu(y)\rangle = \frac{(x-y)(y-x_0)}{R} = \frac{(y-x)(x-x_0)}{R} = \langle y-x, \nu(x)\rangle$$

for all $x, y \in \partial\Omega$. Hence W_Ω is selft - adjoint. For $n = 2$, since $K(x,y) = \frac{1}{2r}$, then $W_\Omega f(x) = 0, x \in \partial\Omega$.

Now we study converse. Assume that for all $x, y \in \partial\Omega$, $\nu(x)$ and $\nu(y)$ denotes the unit outer normal to $\partial\Omega$ at x and y then we have

$$\langle x-y, \nu(y)\rangle = \langle y-x, \nu(x)\rangle. \tag{4.3.1}$$

Take $x \in \partial\Omega$, set a point y of $\partial\Omega$ at minimal distance from x. The line joing y to x is defined by $L(y,x) = \{x + t\nu(x) : t < 0\}$. Since there is a normal vector at x, there exists a point z in $(L(x) \cap \partial\Omega) \setminus \{x\}$ for which the line segment joining x and z is contained in $\overline{\Omega}$. Since $\nu(x)$ is the outward normal vector at x, we have $\nu(x) = \dfrac{x-z}{|x-z|}$. By (4.3.1), we have

$$\langle x-z, \nu(y)\rangle = \left\langle z-x, \frac{x-z}{|x-z|}\right\rangle.$$

Hence

$$|x - z| = \langle z - x, \nu(y) \rangle .$$

Since $|\nu(z)| = 1$, we conclude that

$$\nu(z) = \frac{z - x}{|z - x|} = -\nu(x)$$

We have

$$\nu(y)x - \nu(y)y - \nu(x)y + \nu(x)x = 0.$$
$$\nu(y)z - \nu(y)x - \nu(x)z + \nu(x)x = 0.$$

It follows thats

$$2\nu(x)x - \nu(x)(y + z) - \nu(y)(y - z) = 0.$$

Since the last term is independent of y, we get

$$2\nu(x) - \nu(x)(y + z) - c = 0. \tag{4.3.2}$$

On the other hand, we have

$$v_\Omega[1](x) = \begin{cases} \dfrac{1}{(2-n)s_n} \displaystyle\int_{\partial\Omega} \frac{1}{|y - x|^{n-2}} d\sigma & n \geq 3 & (4.3.3a) \\ \dfrac{1}{2\pi} \displaystyle\int_{\partial\Omega} \log|x - y| d\sigma & n = 2 & (4.3.3b) \end{cases}$$

and satisfies the Jump formula,

$$\nu_\Omega \nabla v_\Omega^+[1] = -\frac{1}{2} + W_\Omega^t[1]$$

and

$$\nu_\Omega \nabla v_\Omega^-[1] = \frac{1}{2} + W_\Omega^t[1].$$

Moreover, $W_\Omega^t[1] = \frac{1}{2}$, c.f. [Fol95] page 125. Hence $V_\Omega 1$ is constant in the interior of Ω, and its exterior boundary gradient is equal to the unit normal vector $v(x)$. Let $u(x) = v_\Omega^-[1](x), x \in \Omega^-$. By (4.3.2), we have

$$2\nabla u(x)x - \nabla u(x)(y + z) - c = 0.$$

Hence, for each y, $\nabla u(x)(y + z)$ is harmonic at ∞ with the same boundary values. The maximum principle, implies that this function is harmonic. From this we conclude that the function $y + z$ is independent of y. Indeed, if $y + z$ were dependent of y, then u would have a directional derivative is constant and equal to zero, which is impossible because u is decaying at ∞. Since $|y - z| = c$, it follows that Ω is a ball.

□

Appendix A

Functional analysis

We also recall some classical theorems of Functional Analysis in Banach spaces, c.f. Kress [Kre14], Folland [Fol99] and Spectral Theory in Functional Analysis, c.f. B. Helffer [Hel13]; E.B. Davies [Dav95].

A.1 Fredholm Alternative

Let X be a normed space on the filed $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We denote by X' the topological dual space $\mathcal{L}(X, \mathbb{K})$ of X, so that X' consists of the linear continuous operator from X to $\mathbb{C}$.

Definition A.1. *Let X, Y be a normed spaces. If L is a linear map from X to Y, then the transpose L^t is the linear map from Y' to X' that takes a functional $\psi \in Y'$ to the functional $L^t[\psi] \in X'$ defined by*

$$(L^t[\psi])[\phi] = \psi[L[\phi]],\ \forall \phi \in X.$$

Definition A.2 (Fredholm operator)**.**
We say that a bounded linear operator L from a Banach space X to a Banach space Y is Fredholm if and only if both the following conditions hold.

1. *The null space* $\ker L$ *has finite dimension.*
2. *the cokernel* $Y / \operatorname{Im} L$ *has finite dimension.*

If L is a Fredhohm operator, then the index of L is the integer defined by

$$\textit{Index } L \equiv \dim \ker L - \dim(Y / \operatorname{Im} L).$$

Theorem A.1 (Compact perturbation of Fredholm operator)**.**
Let X and Y are Banach spaces. If L is a Fredholm operator from X to Y and K is a compact operator from X to Y, then $L + k$ is a Fredholm operator from X to Y and Index $(L + K) =$ Index L.

Theorem A.2 (Fredholm Alternative)**.**
Let X and Y are Banach spaces. Let L is a Fredholm operator of index 0 from X to Y and K is a compact operator from X to Y Then one of the following statements holds:

- *The operator L is an isomorphism from X to Y and the operator L^+ is an isomorphism from Y to X.*
- *The null space* $\ker L$ *and* $\ker L^t$ *have the same nonzero finite dimension and*

$$\operatorname{Im} L = \{\phi \in X : \psi[\phi] = 0 \textit{ for all } \psi \in \ker L^t\}$$
$$\operatorname{Im} L^t = \{\psi \in X' : \psi[\phi] = 0 \textit{ for all } \varphi \in \ker L\}$$

A.2 The spectral theory

For more detail and proof of spectral theory, we could refer Helffer's book [Hel13].

Definition A.3. *Let X be Banach spaces. Let A be a closed operator on X. The resolvent set of A is given by*

$$\rho(A) = \{\lambda \in \mathbb{C} | \lambda I - A : Dom(A) \to X \text{ is bijectiove,}\}$$

and its sepctrum by

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

We further define the point spectrum of A by

$$\sigma_\rho(A) = \{\lambda \in \mathbb{C} | \exists v \in Dom(A) \setminus \{0\} \text{ with } \lambda v = Av\} \subseteq \sigma(A),$$

where we call $\lambda \in \sigma_\rho(A)$ an eigenvalue of A and the corresponding v an eigenvector or eigenfunction of A. For $\lambda \in \rho(A)$ the operator

$$R(\lambda, A) \equiv (\lambda I - A)^{-1} : X \to X$$

and the set $\{R(\lambda, A) | \lambda \rho(A)\}$ are called the resolvent.

Theorem A.3. *Let A be a closed operator on X and let $\lambda \in \rho(A)$. Then the following assertions hold.*

1. *$AR(\lambda, A) = \lambda R(\lambda, A) - I, AR(\lambda, A)x = R(\lambda, A)Ax$ for all $x \in Dom(A)$ and*

$$\frac{1}{\mu - \lambda}(R(\lambda, A) - R(\mu, A)) = R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A)$$

   if $\mu \in \rho(A) \setminus \{\lambda\}$. The formula in display is called the resolvent equation.

2. *The spectrum $\sigma(A)$ is closed.*

3. *The function $\rho(A) \to \mathcal{B}(X, [Dom(A)]); \lambda \mapsto R(\lambda, A)$ is infinitely differentiable with*

$$\left(\frac{d}{d\lambda}\right)^n R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \text{ for every } n\mathbb{N}.$$

4. $\|R(\lambda, A)\| \geq \frac{1}{d(\lambda, \sigma(A))}$.

Theorem A.4 (Spectral theory for compact operators)**.**
Let $A \in \mathcal{K}(E)$ where E is an infinite dimensional Banach space. Then

1. $0 \in \sigma(A)$.

2. $\sigma(A) \setminus \{0\} = \sigma_\rho(A) \setminus \{0\}$.

3. *We are in one and only one the following cases*

   - *either $\sigma(A) = \{0\}$,*
   - *either $\sigma(A) \setminus \{0\}$ is finite,*
   - *or $\sigma(A) \setminus \{0\}$ can be dscribed as a sequence of distincts points tending to* 0.

4. *Each $\lambda \sigma_\rho \setminus \{0\}$ is isolated and* $\dim N(A - \lambda I) < +\infty$.

As $A = A^*$, the spectrum is real. Indeed, if $\operatorname{Im} \lambda \neq 0$, one immediately verifies that

$$|\operatorname{Im} \lambda| \|u\|^2 \leq |\operatorname{Im} \langle (A-\lambda)u, u\rangle| \leq \|(A-\lambda)u\| \|u\|.$$

This shows that the map $(A - \lambda)$ is injective and with close range. But the orthogonal of the range of $(A - \lambda)$ is the kernel of $(A - \bar{\lambda})$ which is reduced to 0. So $(A - \lambda)$ is bijective.

Theorem A.5 (Spectral theory for selfadjoint operators)**.**
Let A be linear selfadjoint operator. Then the spectrum of A is contatined in $[m, M]$ with $m = \dfrac{\inf \langle Tu, u\rangle}{\|u\|^2}$ and $m = \dfrac{\sup \langle Tu, u\rangle}{\|u\|^2}$. Moreover m and M belong to spectrum of A.

Corollary A.1. *Let A be linear selfadjoint operator such that $\sigma(A) = \{0\}$. Then $A = 0$.*

Proposition A.1. *If A is positive and selfadjoint then $\|A\| = M$.*

Theorem A.6 (Spectral theory for compact self adjoint operators)**.**
Let H be a spearable Hilbert space and A is a compact self adjoint operator. Then H admits an Hilbertian basis consisting of eigenfunctions of T.

Appendix B

Results of Classical Potential Theory on Layer Potentials

Theorem B.1. *[Schauder regularity result for Fredholm integral equations]*
Let $\alpha \in]0,1[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.

1. *Let $k \in \{1, \ldots, m\}$, let $g \in C^{k-1,\alpha}(\partial\Omega)$, $\mu \in L^2(\partial\Omega)$ and*

$$g(x) = -\frac{1}{2}\mu(x) + \int_{\partial\Omega} \frac{\partial}{\partial\nu(x)} S_n(x-y)\mu(y)d\sigma_y \text{ for allmost all } x \in \partial\Omega,$$

   then $\mu \in C^{k-1,\alpha}(\partial\Omega)$.

2. *Let $k \in \{0, \ldots, m\}$, let $g \in C^{k,\alpha}(\partial\Omega)$, $\mu \in L^2(\partial\Omega)$ and*

$$g(x) = -\frac{1}{2}\mu(x) + \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} S_n(x-y)\mu(y)d\sigma_y \text{ for allmost all } x \in \partial\Omega,$$

   then $\mu \in C^{k,\alpha}(\partial\Omega)$.

Proof. See [Tru01], Theorem 6.19. □

Appendix C

Remark of Spherical Harmonics

We summary the results of Spherical Harmonics in some monograph, as Folland [Fol95]; R.P. Feynman, R.B. Leighton, M. Sands [San], N.N. Lebedev [Leb72] or J. Gallier [Gal09].

Harmonic homogeneous polynomials and their restrictions to $\mathbb{S}^n$, where

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in R^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

turn out to play a crucial role in understanding the structure of the eigenspaces of the Laplacian on $\mathbb{S}^n$.

Definition C.1. *Let $\mathcal{P}_k(n+1)$ denote the space of homogeneous polynomial of degree k in $n+1$ variables with real coeffcients and let $\mathcal{P}_k(\mathbb{S}^n)$ denote the restrictions of homogeneous polynomials in $\mathcal{P}_k(n+1)$ to $\mathbb{S}^n$. Let $\mathcal{H}_k(n+1)$ denote the space of (real) harmonic polynomials, with*

$$\mathcal{H}_k(n+1) = \{P \in \mathcal{P}_k(n+1) | \Delta P = 0\}.$$

Let $\mathcal{H}_k(\mathbb{S}^n)$ denote the space of (real) spherical harmonics be the set of restrictions of harmonic polynomials in $\mathcal{H}_k(n+1)$ to $\mathbb{S}^n$.

The restriction map, $\rho : \mathcal{H}_k(n+1) \to \mathcal{H}_k(\mathbb{S}^n)$, is a surjective linear map. In fact, it is abijection. Indeed, if $P \in \mathcal{H}_k(n+1)$, observe that

$$P(x) = \|x\|^k P\left(\frac{x}{\|x\|}\right), \text{ with } \frac{x}{\|x\|} \in \mathbb{S}^n$$

for all $x \neq 0$. Consequently, if $P(\sigma) = Q(\sigma)$ for all $\sigma \in \mathbb{S}^n$, then

$$P(x) = \|x\|^k P\left(\frac{x}{\|x\|}\right) = \|x\|^k Q\left(\frac{x}{\|x\|}\right) = Q(x)$$

for all $x \neq 0$. which implies that $P = Q$. Therefore, we have a linear isomorphism between $\mathcal{H}_k(n+1)$ and $\mathcal{H}_k(\mathbb{S}^n)$.

Note that every homogeneous polynomial P, of degree k in the variables $x_1, \ldots, x_n$ can be written uniquely as

$$P = \sum_{|\alpha|=k} c_\alpha x^\alpha,$$

It is well known that $\mathcal{P}_k(n)$ is a (real) vector space of dimension

$$d = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$$

We can define an Hermitian inner product on $\mathcal{P}_k(\mathbb{S}^n)$ is an inner product by viewing a homogeneous polynomial as a differential operator as follows:

For any $P = \sum_{|\alpha|=k} c^{\alpha} x^{\alpha}$. Let

$$\partial(P) = c_{\alpha} \frac{\partial^k}{\partial x_1^{\alpha_n} \dots \partial x_n^{\alpha_n}}.$$

For any two polynomials $P, Q \in \mathcal{P}_k(n)$, let an inner product $\langle P, Q \rangle = \partial P Q$. Another useful property of our inner product is this:

$$\begin{aligned} \langle P, QR \rangle &= \langle QP, R \rangle = \partial(QR)P = \partial Q \partial(RP) \\ &= \partial R(P\partial Q) = \langle R, P\partial Q \rangle = \langle \partial Q P, R \rangle \end{aligned}$$

In particular $\left\langle (x_1^2 + \dots + x_n^2)P, Q \right\rangle = \langle P, \Delta Q \rangle$.

Theorem C.1. *The map $\Delta : \mathcal{P}_k(n) \to \mathcal{P}_{k-2}(n)$ is surjective for all $n, k \geq 2$. Furthermore, we have the following orthognal direct sum decompositions:*

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{H}_{k-2}(n) \oplus \dots \oplus \|x\|^{2j} \mathcal{H}_{k-2j}(n) \oplus \dots \oplus \|x\|^{2[k/2]} \mathcal{H}_{[k/2]}(n).$$

Proof. If the map $\Delta : \mathcal{P}_k(n) \to \mathcal{P}_{k-2}(n)$ is not surjective, then some nonzero polynomial $Q \in \mathcal{P}_{k-2}(n)$ is orthogonal to the image of Δ. In particular, Q must be orthogonal to ΔP with $P = \|x\|^2 Q \in \mathcal{P}_k(n)$. So

$$= 0 \langle Q, \Delta P \rangle = \left\langle \|x\|^2 Q, P \right\rangle = \langle P, P \rangle$$

which implies that $P = \|x\|^2 Q = 0$, and thus $Q = 0$, a contradiction.

We claim that we have an orthogonal direct sum decomposition,

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{P}_{k-2}(n).$$

When $k = 0, 1$, this case is trivial, Assume that $k \geq 2$, since $\ker \Delta = \mathcal{H}_k(n)$ and Δ is surjective, it is sufficient to prove that $\mathcal{H}_k(n)$ is orthogonal to $\|x\| \mathcal{P}_{k-2}(n)$. Now, if $H \in \mathcal{H}_k(n)$ and $P = \|x\|^2 Q \in \|x\|^2 \mathcal{P}_{k-2}(n)$, we have $\left\langle \|x\|^2 Q, H \right\rangle = \langle Q, \Delta H \rangle = 0$, so $\mathcal{H}_k(n)$ and $\|x\|^2 \mathcal{P}_{k-2}(n)$ are orthognal. Using induction, we immediately to get orthognal direct sum decomposition

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{H}_{k-2}(n) \oplus \dots \oplus \|x\|^{2j} \mathcal{H}_{k-2j}(n) \oplus \dots \oplus \|x\|^{2[k/2]} \mathcal{H}_{[k/2]}(n).$$

□

Since every polynomial in $n + 1$ variables is the sum of homogeneous polynomials, we get

Corollary C.1. *The restriction to $\mathbb{S}^n$ of every polynomial in $n + 1 \geq 2$ variables is a sum of trstriction to $\mathbb{S}^n$ of harmonic polynomials.*

We can also derive a fomula for the dimension of $\mathcal{H}_k(n)$

Corollary C.2. *The dimension $a_{k,n}$ of the space of harmonoc polynomials $\mathcal{H}_k(n)$ is given by the formula*

$$a_{k,n} = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} - \begin{pmatrix} n+k-3 \\ k-2 \end{pmatrix}$$

if $n, k \geq 2$, with $a_{0,n} = 1$ and $a_{1,n} = n$, and similarly for $\mathcal{H}_k(n)$. As $\mathcal{H}_k(n+1)$ is isomorphic to $\mathcal{H}_k(\mathbb{S}^n)$, we have

$$\dim \mathcal{H}_k(\mathbb{S}^n) = a_{k,n+1} = \begin{pmatrix} n+k \\ k \end{pmatrix} - \begin{pmatrix} n+k-2 \\ k-2 \end{pmatrix}$$

Let $L^2(\mathbb{S}^n)$ be the space of square-integrable functions of the sphere. We have an inner prodcut on $L^2(\mathbb{S}^n)$ given by

$$\langle f, g \rangle = \int_{\mathbb{S}^n} f g d\sigma_n$$

where $f, g \in L^2(\mathbb{S}^n)$ With this inner produce, $L^2(\mathbb{S}^n)$ is a complete normed vector space, and is a Hilbert space.

Proposition C.1. *The set of all finite linear combination of elements in*

$$\bigcup_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^n)$$

is

1. *dense in $C(\mathbb{S}^n)$ with respect to the L^∞ norm;*
2. *dense in $L^2(\mathbb{S}^n)$.*

Proof. 1. As $\mathbb{S}^n$ is compact, by the Stone-Weierstrass approximation theorem, g is cointinuous on $\mathbb{S}^n$, then then it can be approximated uniformly by polynomials P_j, restricted to $\mathbb{S}^n$. It is linear combination of elements in $\bigcup_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^n)$.

2. Given $f \in L^2(\mathbb{S}^n)$, for every $\epsilon > 0$, we can choose a continuous function g, so that $\|f - g\|_2 < \epsilon/2$. We can find a linear combination h of elements in $\bigcup_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^n)$ so that $\|g - h\|_\infty < \dfrac{\epsilon}{2\sqrt{vol(\mathbb{S}^n)}}$, where $m_{n-1}(\mathbb{S}^n)$ is a measure of $\mathbb{S}^n$. Thus, we get

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2 < \epsilon/2 + \sqrt{m_{n-1}(\mathbb{S}^n)} \|g - h\|_\infty < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Proposition C.2. *For every harmonic polynomial $P \in \mathcal{H}_k(n+1)$, the restriction $H \in \mathcal{H}_k(\mathbb{S}^n)$ of P to $\mathbb{S}^n$ is an eigenfunction of $\Delta_{\mathbb{S}^n}$ for the eigenvalue $-k(n+k-1)$.*

Proof. We have $P(r\sigma) = r^k H(\sigma), r > 0, \sigma \in \mathbb{S}^n$, and for any $f \in C^\infty(\mathbb{R}^{n+1})$, we have

$$\Delta f = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \Delta_{\mathbb{S}^n} f.$$

Consequently,

$$\begin{aligned}\Delta P = \Delta(r^k H) &= \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial(r^k H)}{\partial r}\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^n}(r^k H)\\ &= \frac{1}{r^n}\frac{\partial}{\partial r}\left(kr^{n+k-1}H\right) + r^{k-2}\Delta_{\mathbb{S}^n} H\\ &= \frac{1}{r^n}k(n+k-1)r^{n+k-2}H + r^{k-2}\Delta_{\mathbb{S}^n} H\\ &= r^{k-2}(k(n+k-1)H + \Delta_{\mathbb{S}^n} H).\end{aligned}$$

Thus $\Delta P = 0$ iff $\Delta_{\mathbb{S}^n} H = -k(n+k-1)H$. □

We conclude that the space $\mathcal{H}_k(\mathbb{S}^n)$ is a subspace of the eigenspace E_k associated with the eigenvalue $-k(n+k-1)$. We can deduce immediately is that $\mathcal{H}_k(\mathbb{S}^n)$ and $\mathcal{H}_l(\mathbb{S}^n)$ are pairwise orthogonal whenever $k \neq l$.

Theorem C.2. *The family of space $\mathcal{H}_k(\mathbb{S}^n)$ yields a Hilbert space direct sum decomposition*

$$L^2(\mathbb{S}^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^n)$$

which means that the summands are closed, pairwise orthogonal, and that every $f \in L^2(\mathbb{S}^n)$ is the sum of converging series

$$f = \sum_{k=0}^{\infty} f_k$$

in the L^2 norm, where $f_k \in \mathcal{H}_k(\mathbb{S}^n)$. are uniquely determined functions. Furthermore, given any orthonormal basis, $(U_k^1, \cdots, Y_k^{a_{k,n+1}})$ of $\mathcal{H}_k(\mathbb{S}^n)$ we have

$$f = \sum_{k=0}^{a_{k,n+1}} c_{k,m_k} Y_k^{m_k} \text{ with } c_{k,m_k} = \left\langle f, Y_k^{m_k} \right\rangle.$$

The coeffcients c_{k,m_k} are generalized Fourier coeffcients

Theorem C.3. *1. The eigenspaces of the Laplacian on $\mathbb{S}^n$ are the space of spherical harmonic $E_k = \mathcal{H}_k(\mathbb{S}^n)$ and E_k correesponds to the eigenvalue $-k(n+k-1)$.*

2. We have the Hilbert space direct sum decompositions

$$L^2(\mathbb{S}^n) = \bigoplus_{k=0}^{\infty} E_k.$$

Proof. 1. We have already known that $-k(n+k-1)$ are eigenvalue of $\Delta_{\mathbb{S}^n}$ and that $\mathcal{H}_k(\mathbb{S}^n) \subseteq E_k$. We will prove that $\Delta_{\mathbb{S}^n}$ has no other eigenvalue and eigenvectors.

Let λ be any eigenvalue of $\Delta_{\mathbb{S}^n}$ and let $f \in L^2(\mathbb{S}^n)$ be any eigenfunction associated with λ so that $\Delta f = \lambda f$. We have a unique series expansion

$$f = \sum_{k=0}^{\infty} \sum_{k=0}^{a_{k,n+1}} c_{k,m_k} Y_k^{m_k} \text{ with } c_{k,m_k} = \left\langle f, Y_k^{m_k} \right\rangle$$

Now Fourier coeffcients are given by

$$d_{k,m_k} = \left\langle \Delta f, Y_k^{m_k} \right\rangle = \left\langle f, \Delta Y_k^{m_k} \right\rangle = -k(n+k-1) \left\langle f, Y_k^{m+k} \right\rangle = -k(n+k-1)c_{k,m_k}.$$

On the other hand, the Fourier coefficients of Δf are given by $d_{k,m_k} = \lambda c_{k,m_k}$. By uniequness of the Fourier expasion, we have

$$\lambda c_{k,m_k} = -k(n+k-1)c_{k,m_k}$$

for all $k \geq 0$. Since $f \neq 0$ there some k such that $c_{k,m_k} \neq 0$ and we must have

$$\lambda = -k(n+k-1)$$

for any such k. However, the function $k \mapsto -k(n+k-1)$ reaches its maximum for $k = -\frac{n-1}{2}$ and as $n \geq 1$ it is strictly decreasing for $k \geq 0$, which implies that k is unique and that $c_{j,m_k} = 0$ for all $j \neq k$. Therefore $f \in \mathcal{H}_k(\mathbb{S}^n)$ and the eigenvalue of $\Delta_{\mathbb{S}^n}$ are exactly the integres $*k(n+k-1)$, so $E_k = \mathcal{H}_k(\mathbb{S}^n)$.

□

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