



UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Master Degree in Physics

Final Dissertation

A Modified Fock Space Quantization of the Bosonic String

Thesis supervisor

Prof. Roberto Volpato

Thesis co-supervisor

Prof. Dr. Johannes Walcher

Candidate

Fabio Taccani

Academic Year 2025/2026

Acknowledgements

I would like to thank Professor Johannes Walcher for welcoming me to Heidelberg, for introducing me to the formal aspects of string theory, and for his guidance throughout the development of this work. I am particularly grateful for the time and attention he has continued to devote to discussing its results and their further development beyond the thesis itself.

I am also deeply grateful to Professor Roberto Volpato, who introduced me to conformal field theory, vertex operator algebras, and Moonshine, and whose guidance has played an important role in shaping my understanding of mathematical physics. I am grateful to both of them for their supervision, for the many discussions that shaped this work, and for their continued encouragement throughout its development.

I would also like to thank my twin brother, Andrea, for the many valuable conversations we had while I was working on this thesis, and for the inspiration he has given me and continues to give me. A special thank you goes to my parents, Giulietta and Claudio, for their unwavering support and love, and to my brother Christian for always being there.

Finally, I would like to thank the friends who accompanied me through these years in Padua, in particular Giacomo, Margherita, and the Perugini; and those I met in Heidelberg, especially Gabriele, Aglaé, Flavio, Alicia, Anna, and Vicky.

Contents

Acknowledgements	iii
1 Introduction	1
2 The Bosonic String	7
2.1 String theory as a fundamental theory	7
2.2 The classical relativistic string	8
2.2.1 The Nambu–Goto action	8
2.2.2 The Polyakov action	8
2.3 Classical solutions	9
2.3.1 Gauge fixing	9
2.3.2 Equations of motion and boundary conditions	9
2.3.3 Mode expansion	10
2.3.4 The classical Virasoro constraints	10
2.4 Canonical quantization	11
2.4.1 Canonical commutation relations	11
2.4.2 The standard Fock vacuum	11
2.4.3 Negative norm states	12
2.4.4 Quantum Virasoro generators	12
2.5 Old covariant quantization	13
2.5.1 Physical state conditions	13
2.5.2 The ground state	13
2.5.3 The first excited level	14
2.5.4 Null states	14
2.5.5 The no-ghost theorem	15
2.6 Light cone quantization	15
2.6.1 Target space light cone coordinates	15
2.6.2 Solving the constraints	15
2.6.3 Lorentz invariance	16
2.7 Gauge fixing and Faddeev–Popov ghosts	16
2.7.1 The path integral	16
2.7.2 Matter and ghost central charges	17
2.7.3 Two different meanings of ghost	17
2.8 BRST quantization	18
3 The Modified Fock Space Representation	19
3.1 Modified vacuum conditions	19
3.2 The algebraic Fock space	20
3.3 Level operators and gradings	20
3.4 Lorentz transformations and unitary implementability	22
3.4.1 Spatial rotations and boosts	22
3.4.2 The domain of the boost generator	23

3.4.3	Finite boosts as Bogoliubov transformations	24
4	Positivity of the Modified Kinematical Space	27
4.1	Induction parameter	27
4.2	Positivity of the basis states	27
4.3	Orthogonality	29
4.4	Positive definite inner product	29
5	Virasoro Algebra and Ghost Sector	31
5.1	Modified normal ordering and vacuum energy shift	31
5.2	Matter Virasoro algebra	32
5.3	Virasoro and Lorentz generators	35
5.4	Standard ghost sector and total anomaly	36
6	The Physical State Space	39
6.1	Matter physical state conditions and positivity	39
6.2	Structure of the physical state constraints	39
6.2.1	Absence of finite K support states	41
6.2.2	The stronger algebraic obstruction	44
6.2.3	Passage to infinite K support	44
7	The Completed Virasoro Constraint Problem	45
7.1	Spatial and timelike factorisation	45
7.2	Hilbert–Schmidt formulation	46
7.3	Triviality of the completed matter physical space	47
8	Conclusions	51
A	General First Order Ghost Systems	53
B	A Formal Infinite Support State	55
C	The General Algebraic Virasoro Obstruction	57
C.1	The timelike Virasoro generator	57
C.2	Extension to the complete algebraic Fock space	59

Chapter 1

Introduction

One of the central difficulties in the covariant quantisation of relativistic systems is the appearance of states with negative norm. Such states originate from the indefinite signature of the Minkowski metric and cannot be interpreted as physical states in an ordinary probabilistic quantum theory. A consistent quantisation must therefore remove them from the physical spectrum or identify them as unphysical gauge degrees of freedom.

A familiar example is the covariant quantisation of the electromagnetic field. All four components of the gauge potential are retained in order to preserve manifest Lorentz covariance, but the temporal and longitudinal components produce states with zero or negative norm. The Gupta–Bleuler prescription deals with this problem by imposing a subsidiary condition and subsequently quotienting by the remaining null states [1, 2].

The same tension appears in the covariant quantisation of the bosonic string. The embedding coordinates $X^\mu(\tau, \sigma)$ are expanded in oscillator modes satisfying

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n}.$$

For the spatial components, this is the usual oscillator algebra with positive sign. In the timelike direction, instead,

$$[\alpha_m^0, \alpha_n^0] = -m\delta_{m+n}.$$

If all positive modes are taken to annihilate the vacuum, as in the standard representation, the timelike creation operators generate negative norm states.

In old covariant quantisation, this indefiniteness is accepted at the kinematical level. One constructs an indefinite Fock space and then imposes

$$L_n |\psi\rangle = 0, \quad n > 0,$$

together with

$$(L_0 + a) |\psi\rangle = 0.$$

At the critical dimension and intercept, the no-ghost theorem shows that the physical subspace has nonnegative norm and becomes positive definite after quotienting by null states [3, 4]. Positivity is therefore recovered only after the Virasoro constraints and the associated gauge identifications have been taken into account.

In this thesis we study a different approach to the same problem. Instead of accepting an indefinite kinematical space, one may ask whether a different representation of the same oscillator algebra can already have a positive definite inner product.

This possibility was considered by Bars in his analysis of the relativistic harmonic oscillator [5]. The Lorentzian oscillator algebra admits different Fock sectors, depending on which operators are chosen to annihilate the vacuum. In one of these sectors, the usual spatial polarisation is retained, while the

roles of the timelike creation and annihilation operators are reversed. The resulting Fock space has positive norm, although the chosen vacuum is not invariant under Lorentz boosts.

Bars also observed that the same choice can be made independently for each nonzero string mode. The corresponding string representation is kinematically positive, but the Virasoro constraints become nonstandard and their solution was left open [5]. This thesis studies that string representation more explicitly.

For every $n > 0$, the spatial vacuum conditions are kept unchanged

$$\alpha_n^i |0, p\rangle = 0, \quad i = 1, \dots, D-1.$$

In the timelike sector, instead, we impose

$$\alpha_{-n}^0 |0, p\rangle = 0.$$

The spatial creation operators are therefore α_{-n}^i , while the timelike creation operators are the positive modes α_n^0 . The oscillator algebra, the classical string equations and the zero mode sector are not modified. The only change is the choice of polarisation.

The reversed timelike polarisation compensates the negative sign in the commutator. For instance,

$$\|\alpha_n^0 |0, p\rangle\|^2 = \langle 0, p | \alpha_{-n}^0 \alpha_n^0 |0, p\rangle = n.$$

The same conclusion holds for arbitrary finite oscillator states. The occupation number states are mutually orthogonal and have strictly positive norm. Thus, the modified algebraic Fock space contains neither negative norm states nor nonzero null vectors, and its completion defines a positive definite kinematical Hilbert space.

Even if the resulting space is positive definite, one must still check whether the other structures of the theory are preserved. The first difficulty appears already for target space Lorentz transformations. Spatial rotations preserve the distinction between creation and annihilation operators, whereas boosts mix spatial and timelike oscillators and therefore do not preserve the chosen polarisation.

The oscillator contribution to a boost acts on the vacuum as

$$J_{\text{osc}}^{0i} |0, p\rangle = i \sum_{n>0} \frac{1}{n} \alpha_{-n}^i \alpha_n^0 |0, p\rangle.$$

After normalising the oscillators, the terms in this sum form an orthonormal family. The norm of the truncation to the first M modes is proportional to M , so the sequence does not converge in the kinematical Hilbert space. In particular, the infinitesimal boost generator is not defined on the vacuum with the natural oscillator domain prescription.

The same obstruction can be seen directly for finite boosts. On each spatial timelike oscillator pair, a boost acts as a bosonic Bogoliubov transformation. Its off diagonal part has coefficient $\sinh(\chi)$ at every oscillator mode, where χ is the rapidity. Consequently, its Hilbert–Schmidt norm contains the divergent sum

$$\sum_{n>0} |\sinh \chi|^2.$$

By the Shale–Stinespring criterion, a nontrivial boost is not unitarily implementable on the modified oscillator Fock space [6, 7]. The obstruction is therefore not only the non invariance of the vacuum. It is caused by the repetition of the same creation annihilation mixing over infinitely many independent modes.

The modified polarisation also changes the normal ordering prescription. In the timelike sector, negative modes are moved to the right rather than positive modes. The matter Virasoro generators must therefore be reconstructed using normal ordering with respect to the modified vacuum.

On the algebraic Fock space, the resulting generators satisfy

$$[L_m^X, L_n^X] = (m - n)L_{m+n}^X + \frac{D - 2}{12} (m^3 - m) \delta_{m+n}.$$

The formal matter central extension is thus

$$c_m = D - 2,$$

and the corresponding matter vacuum energy shift is

$$a^X = -\frac{D - 2}{24}.$$

The $D - 1$ spatial oscillator families contribute with the standard sign, while the reversed timelike oscillator contributes with the opposite sign.

The value $c_m = D - 2$ is obtained from the formal oscillator mode algebra in a representation that is not of positive energy. It has not been shown that it is also the Weyl anomaly coefficient of a local worldsheet conformal field theory. In the usual Polyakov path integral, D scalar fields contribute $c_m = D$ regardless of the Lorentzian signature of the target space metric. A local worldsheet or path integral realisation of the modified representation would therefore have to be established separately.

If the standard anticommuting Faddeev–Popov bc system is retained as an additional assumption, its central charge remains

$$c_{gh} = -26.$$

The total formal Virasoro anomaly then cancels for

$$D = 28, \quad a = -1.$$

These values only express cancellation of the formal mode anomaly. They do not by themselves prove Weyl invariance or the existence of a consistent BRST quantisation.

The main difficulty of the construction lies in the matter physical state conditions. The signed oscillator level can be written as

$$N = N^{\text{sp}} + N^0.$$

Since timelike creation operators lower its eigenvalue, N is not bounded from below. At fixed signed level, there are therefore infinitely many oscillator states, and the usual finite level analysis of the Virasoro constraints cannot be applied.

It is useful to introduce a second operator,

$$K = N^{\text{sp}} - N^0,$$

which counts the total weighted number of spatial and timelike creation operators. Its spectrum is nonnegative, and every fixed K eigenspace is finite dimensional.

The spatial and timelike parts of a positive Virasoro generator act in opposite directions on this grading

$$[K, L_m^{\text{sp}}] = -mL_m^{\text{sp}}, \quad [K, L_m^0] = mL_m^0.$$

Writing a fixed N state as

$$|\psi_\nu\rangle = \sum_{k \geq 0} |\psi_{\nu,k}\rangle,$$

the equation $L_m^X |\psi_\nu\rangle = 0$ becomes

$$L_m^{\text{sp}} |\psi_{\nu,k+m}\rangle + L_m^0 |\psi_{\nu,k-m}\rangle = 0, \quad k \geq 0.$$

The constraints are therefore an infinite system of relations between different K components.

The finite support case can be excluded. If k_M is the largest value of K occurring in a nonzero state, the constraint at output grading $k_M + m$ gives

$$L_m^0 |\psi_{\nu, k_M}\rangle = 0.$$

For $m = 2$, this equation can be analysed directly by separating the spatial and timelike oscillator content of the top component. In the timelike sector, the term

$$-\frac{1}{2} (\alpha_1^0)^2$$

produces a contribution with two additional timelike creation operators. No other term has the same polynomial degree, and hence no nonzero top component can satisfy the equation.

The same reasoning extends to every $m \geq 2$. The relevant highest degree term is generated by

$$-\frac{1}{2} \sum_{r=1}^{m-1} \alpha_{m-r}^0 \alpha_r^0.$$

This is multiplication by a nonzero polynomial on the algebraic timelike Fock space and is therefore injective. One obtains

$$\ker \left(L_m^X |_{\mathcal{F}_{\text{alg}}(p)} \right) = \{0\}, \quad m \geq 2.$$

In particular, no nonzero algebraic vector satisfies all the positive mode constraints.

Any possible physical state in the Hilbert space completion must therefore contain infinitely many K components. Such a formal sum defines a Hilbert space vector only when

$$\sum_{k \geq 0} \|\psi_{\nu, k}\|^2 < \infty.$$

Normalisability is no longer automatic at fixed signed level.

An explicit infinite support solution is constructed in Appendix B. It satisfies the positive Virasoro constraints as a formal oscillator identity, but its norm diverges. The example shows that solving the recursive equations algebraically is not sufficient to obtain a state in the kinematical Hilbert space.

The structure of the problem becomes clearer after separating the spatial and timelike oscillator sectors

$$\mathcal{H}_{\text{kin}}(p) \simeq \mathcal{H}_{\text{sp}}(p^{\text{sp}}) \otimes \mathcal{H}^{(0)}(p^0).$$

The timelike modes can be written as

$$\beta_n = \alpha_{-n}^0.$$

They satisfy the standard one boson oscillator algebra and define ordinary Virasoro generators $\ell_m^{(0)}$. The modified timelike contribution is related to them by

$$L_m^0 = -\ell_{-m}^{(0)}.$$

Hence, on the algebraic tensor product core,

$$L_m^X = L_m^{\text{sp}} \otimes I - I \otimes \ell_{-m}^{(0)}.$$

A normalisable vector in this tensor product can be identified with a Hilbert–Schmidt operator

$$T_\psi : \mathcal{H}^{(0)} \longrightarrow \mathcal{H}_{\text{sp}},$$

with

$$\|\psi\rangle\| = \|T_\psi\|_{\text{HS}}.$$

Under this correspondence, the positive mode constraints become

$$L_m^{\text{sp}} T_\psi v = T_\psi \ell_m^{(0)} v, \quad v \in \mathcal{F}_{\text{alg}}^{(0)}(p^0), \quad m > 0.$$

Thus, the operator associated with a physical state must intertwine the positive Virasoro modes of the timelike and spatial Fock representations.

The zero mode condition fixes the difference between the two oscillator levels. If

$$\nu(p) = -a^X - \alpha' p^2,$$

then

$$N^{\text{sp}} T_\psi - T_\psi N_0^+ = \nu(p) T_\psi,$$

where $N_0^+ = -N^0$ is the nonnegative timelike level operator. The operator must therefore have a fixed degree with respect to the spatial and timelike level decompositions.

The completed problem can now be solved. The zero mode condition fixes the degree of a possible intertwiner with respect to the spatial and timelike level decompositions. Suppose that a nonzero Hilbert–Schmidt intertwiner exists and consider the lowest timelike oscillator level on which it does not vanish. Its image contains a nonzero vector annihilated by all positive spatial Virasoro modes.

Acting on this vector with negative Virasoro modes produces an orthonormal sequence of timelike descendants whose images under the intertwiner remain bounded away from zero in norm. This is incompatible with compactness, since every Hilbert–Schmidt operator is compact and must map an orthonormal sequence to a sequence whose norm tends to zero. Consequently, no nonzero Hilbert–Schmidt operator can satisfy simultaneously the positive mode intertwining equations and the zero mode condition. It follows that, for every real momentum and every finite dimension $D \geq 2$,

$$\mathcal{H}_{\text{phys}}^X(p) = \{0\}.$$

Thus, the obstruction found on the algebraic Fock space persists after Hilbert space completion: allowing infinite K support does not produce any nonzero normalisable matter physical state. The formal solution constructed in Appendix B is consistent with this result, since it lies outside the kinematical Hilbert space and corresponds to a noncompact rather than Hilbert–Schmidt intertwiner.

The construction studied here is not proposed as an alternative consistent quantisation of the bosonic string. It is instead a way of testing how the usual consistency conditions react when positivity is imposed before the Virasoro constraints. The modified representation has a positive kinematical Hilbert space, but boosts are not unitarily implementable and the formal Virasoro central extension is changed. Moreover, the obstruction to the Virasoro constraints is not confined to the algebraic Fock space, the completed matter physical state space is trivial. Thus, imposing positivity at the kinematical level does not lead to a nontrivial physical matter spectrum in this representation.

The thesis is organised as follows. Chapter 2 reviews the classical bosonic string and its standard covariant, light cone and BRST quantisations, with particular attention to negative norm states. Chapter 3 introduces the modified vacuum, the algebraic Fock space and the two level operators N and K . It also discusses the failure of unitary implementability of Lorentz boosts. Chapter 4 proves positivity and orthogonality of the modified occupation number basis and constructs the kinematical Hilbert space. Chapter 5 defines the modified normal ordering, computes the formal matter Virasoro algebra and discusses the standard ghost sector. Chapter 6 studies the constraints on the algebraic Fock space and proves that the algebraic physical state space is trivial. Chapter 7 reformulates the completed constraint problem in terms of Hilbert–Schmidt operators and Virasoro intertwining equations, and proves that the completed matter physical state space is trivial. The final Chapter 8 summarises the results and discusses the remaining questions concerning BRST quantisation, the worldsheet interpretation and target space Lorentz symmetry.

Chapter 2

The Bosonic String

In this chapter we will introduce the necessary background regarding the bosonic string theory following the references [3, 8–13]. We will also emphasise the emergence of negative norm states. We will mainly consider an open bosonic string with Neumann boundary conditions in every target space direction. This is the setting that will be used in the subsequent construction.

2.1 String theory as a fundamental theory

String theory tries to solve the problem of reconciling quantum theory with general relativity. If the Einstein–Hilbert action is quantized perturbatively using the standard methods of quantum field theory, ultraviolet divergences arise that cannot be absorbed into a finite number of coupling constants. General relativity can nevertheless be treated consistently as an effective quantum field theory at energies below the Planck scale. In this regime, quantum corrections can be computed systematically.

String theory changes the basic degrees of freedom of the theory. Instead of point particles, it describes one dimensional extended objects propagating in a D dimensional target spacetime. As a string evolves, it sweeps out a two dimensional surface called the worldsheet. The characteristic length scale is

$$\ell_s = \sqrt{\alpha'}.$$

At distances much larger than ℓ_s , a string appears point like, whereas at shorter distances its extended structure becomes relevant.

The different particle species arise as different vibrational states of the string. In particular, the spectrum of the closed bosonic string necessarily contains a massless spin two state. Its low energy interactions reproduce those expected for the graviton, and the corresponding low energy effective action contains the Einstein–Hilbert term. Gravity is therefore not added to string theory as an independent ingredient, but emerges from the closed string spectrum.

The bosonic string is not a realistic description of nature. Its spectrum contains a tachyonic state with negative mass squared, and it does not contain spacetime fermions. These problems are addressed in superstring theory by enlarging the worldsheet theory to include fermionic degrees of freedom and supersymmetry.

Nevertheless, the bosonic string remains a useful theoretical model. It combines target space Lorentz symmetry, worldsheet gauge invariance, an infinite dimensional oscillator algebra and a system of Virasoro constraints. For these reasons it is a natural model in which to investigate the relation between positivity of Fock space, Lorentz symmetry, gauge constraints and the completion of Fock space into Hilbert space.

2.2 The classical relativistic string

Consider a string propagating in a D dimensional target spacetime with Minkowski metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1), \quad \mu, \nu = 0, \dots, D-1.$$

Its embedding is described by

$$X^\mu = X^\mu(\tau, \sigma), \quad \sigma^a = (\tau, \sigma), \quad a = 0, 1,$$

where τ is a timelike worldsheet coordinate and σ parametrizes the string. For an open string, we take

$$0 \leq \sigma \leq \pi.$$

At fixed τ , the map $X^\mu(\tau, \sigma)$ gives the spatial configuration of the string.

The target space metric induces the worldsheet metric

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = \partial_a X \cdot \partial_b X.$$

2.2.1 The Nambu–Goto action

The natural relativistic action is proportional to the invariant area of the worldsheet

$$S_{\text{NG}} = -T \int d\tau d\sigma \sqrt{-\det(h_{ab})},$$

where T is the string tension. Introducing

$$\dot{X}^\mu = \partial_\tau X^\mu, \quad X'^\mu = \partial_\sigma X^\mu,$$

one finds

$$\det(h_{ab}) = \dot{X}^2 X'^2 - (\dot{X} \cdot X')^2,$$

and therefore

$$S_{\text{NG}} = -T \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}.$$

The action is invariant under target space Poincaré transformations,

$$X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu + a^\mu,$$

and under worldsheet reparametrizations,

$$\sigma^a \longrightarrow \sigma'^a(\sigma).$$

Even if this action is geometrically natural, the square root in the Nambu–Goto action make quantization harder. It is useful to introduce the classically equivalent Polyakov formulation.

2.2.2 The Polyakov action

Let γ_{ab} be an independent Lorentzian metric on the worldsheet. The Polyakov action is

$$S_{\text{P}} = -\frac{T}{2} \int d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu, \quad \text{with } \gamma = \det \gamma_{ab}.$$

Varying with respect to the inverse metric gives

$$\delta S_{\text{P}} = -\frac{T}{2} \int d^2\sigma \sqrt{-\gamma} T_{ab} \delta \gamma^{ab},$$

where

$$T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X \cdot \partial_d X,$$

is the worldsheet energy momentum tensor. The equation of motion for the metric is

$$T_{ab} = 0.$$

The Polyakov action is invariant under Poincaré transformations of the target space, worldsheet diffeomorphisms and Weyl transformations,

$$\gamma_{ab}(\sigma) \longrightarrow e^{2\omega(\sigma)} \gamma_{ab}(\sigma).$$

In two dimensions, the Weyl factors in $\sqrt{-\gamma}$ and γ^{ab} cancel. Thus, the worldsheet metric may be fixed locally by an appropriate gauge choice.

2.3 Classical solutions

2.3.1 Gauge fixing

A two dimensional symmetric metric has three independent components. Worldsheet diffeomorphisms remove two of them, while Weyl invariance removes the remaining conformal factor. Locally, one may therefore choose

$$\gamma_{ab} = e^{2\omega(\tau, \sigma)} \eta_{ab},$$

and then use Weyl invariance to set

$$\gamma_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the conformal gauge, the Polyakov action becomes

$$S_P = \frac{T}{2} \int d\tau d\sigma \left(\dot{X}^2 - X'^2 \right).$$

It describes D free massless scalar fields on the worldsheet, with the timelike field X^0 carrying the opposite kinetic sign because of the Lorentzian target space metric.

Gauge fixing does not eliminate the equations obtained by varying the worldsheet metric. The constraints

$$T_{ab} = 0$$

must therefore be imposed in addition to the equations of motion derived from the gauge fixed action.

2.3.2 Equations of motion and boundary conditions

Varying the coordinates gives

$$\delta S_P = -T \int d\tau d\sigma \delta X_\mu \left(\partial_\tau^2 - \partial_\sigma^2 \right) X^\mu - T \int d\tau X'_\mu \delta X^\mu \Big|_{\sigma=0}^{\sigma=\pi}.$$

The bulk equation of motion is therefore

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0.$$

For an open string, the boundary term vanishes under either Neumann conditions,

$$X'^\mu(\tau, 0) = X'^\mu(\tau, \pi) = 0,$$

or Dirichlet conditions,

$$\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, \pi) = 0.$$

In the following, we impose Neumann boundary conditions in every target space direction.

Introducing the worldsheet light cone coordinates

$$\sigma^\pm = \tau \pm \sigma, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma),$$

the equation of motion becomes

$$\partial_+ \partial_- X^\mu = 0.$$

Its general solution is

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-).$$

For an open string, the boundary conditions relate the left and right moving sectors, leaving a single independent set of oscillator modes.

2.3.3 Mode expansion

The solution satisfying Neumann boundary conditions is

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos(n\sigma).$$

Here x^μ and p^μ are the centre of mass position and momentum, while α_n^μ are the oscillator modes. Reality of X^μ requires

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu.$$

Defining

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$$

allows the zero and nonzero modes to be treated uniformly.

The canonical momentum density and the total target space momentum are

$$P^\mu(\tau, \sigma) = T \dot{X}^\mu = \frac{1}{2\pi\alpha'} \dot{X}^\mu, \quad p^\mu = \int_0^\pi d\sigma P^\mu(\tau, \sigma).$$

2.3.4 The classical Virasoro constraints

In conformal gauge, the independent components of the worldsheet energy momentum tensor are

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}(\dot{X}^2 + X'^2), \quad T_{\tau\sigma} = \dot{X} \cdot X'.$$

Thus, $T_{ab} = 0$ is equivalent to

$$\dot{X}^2 + X'^2 = 0, \quad \dot{X} \cdot X' = 0.$$

In worldsheet light cone coordinates, these constraints become

$$T_{++} = \partial_+ X \cdot \partial_+ X = 0, \quad T_{--} = \partial_- X \cdot \partial_- X = 0.$$

For the open string, the Fourier modes of the energy momentum tensor are

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{n-m} \cdot \alpha_m,$$

and the classical constraints are equivalent to

$$L_n = 0, \quad n \in \mathbb{Z}.$$

With the standard oscillator Poisson bracket conventions, these generators satisfy

$$\{L_m, L_n\}_{\text{PB}} = -i(m-n)L_{m+n}.$$

They therefore generate the classical Witt algebra. The central extension of the corresponding Virasoro algebra arises only after quantization and normal ordering.

2.4 Canonical quantization

2.4.1 Canonical commutation relations

The canonical quantization is done by promoting X^μ and P^μ to operators and imposing the equal time commutation relations

$$[X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = i\eta^{\mu\nu}\delta(\sigma - \sigma'), \quad [X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = 0, \quad [P^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = 0.$$

Substituting the mode expansion into these relations gives

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n}.$$

Hermiticity of X^μ requires

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu.$$

For a spatial component $i = 1, \dots, D-1$, the algebra becomes

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n}.$$

For the timelike component, instead,

$$[\alpha_m^0, \alpha_n^0] = -m\delta_{m+n}.$$

This relative minus sign, that comes from the metric, is the origin of the indefinite inner product in covariant quantization.

2.4.2 The standard Fock vacuum

In the standard representation, the Fock vacuum is defined by

$$\alpha_n^\mu |0, p\rangle = 0, \quad n > 0,$$

and

$$\hat{p}^\mu |0, p\rangle = p^\mu |0, p\rangle.$$

The operators α_n^μ with $n > 0$ are interpreted as annihilation operators, while α_{-n}^μ with $n > 0$ are interpreted as creation operators. A generic state is obtained by applying a finite number of creation operators to the vacuum

$$\alpha_{-n_1}^{\mu_1} \alpha_{-n_2}^{\mu_2} \cdots \alpha_{-n_r}^{\mu_r} |0, p\rangle.$$

The oscillator level is defined by

$$N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n.$$

It satisfies

$$[N, \alpha_{-m}^\mu] = m\alpha_{-m}^\mu$$

for $m > 0$. Therefore, the state above has level

$$n_1 + n_2 + \cdots + n_r.$$

2.4.3 Negative norm states

The oscillator algebra contains the target space metric. Consequently, the norm of a one oscillator state depends on the spacetime index of the oscillator.

For a spatial excitation,

$$|\psi_i\rangle = \alpha_{-1}^i |0, p\rangle,$$

we obtain

$$\langle\psi_i|\psi_i\rangle = \langle 0, p | \alpha_1^i \alpha_{-1}^i | 0, p \rangle = \langle 0, p | [\alpha_1^i, \alpha_{-1}^i] | 0, p \rangle = 1.$$

For the corresponding timelike excitation,

$$|\psi_0\rangle = \alpha_{-1}^0 |0, p\rangle,$$

the norm is

$$\langle\psi_0|\psi_0\rangle = \langle 0, p | \alpha_1^0 \alpha_{-1}^0 | 0, p \rangle = \langle 0, p | [\alpha_1^0, \alpha_{-1}^0] | 0, p \rangle = \eta^{00} = -1.$$

The conventional Fock space is therefore not a Hilbert space with a positive definite inner product. It is an indefinite inner product space containing positive norm, zero norm and negative norm vectors.

The negative norm is not caused by an inconsistent choice of oscillator algebra. The algebra follows directly from canonical quantization and Lorentz covariance. Rather, the problem arises from combining the indefinite Minkowski metric with the conventional choice of vacuum, according to which all positive modes annihilate the vacuum.

The standard solution is not to modify this representation, but to identify a physical subspace through the Virasoro constraints. The no-ghost theorem then guarantees that negative norm states do not survive in the physical quotient space.

2.4.4 Quantum Virasoro generators

At the quantum level, the Virasoro generators are defined by normal ordering the oscillator expression,

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : \alpha_{n-m} \cdot \alpha_m :.$$

In the conventional normal ordering prescription, annihilation operators are placed to the right of creation operators.

For $n \neq 0$, the normal ordering prescription does not generate an independent constant. For L_0 , one obtains

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n.$$

Using $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$, this becomes

$$L_0 = \alpha' p^2 + N.$$

The normal ordering ambiguity is conventionally represented by an intercept a , and the physical mass shell condition is written as

$$(L_0 + a) |\psi\rangle = 0.$$

Since

$$p^2 = -M^2,$$

the mass formula is

$$\alpha' M^2 = N + a.$$

The quantum Virasoro generators satisfy

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{12} (m^3 - m) \delta_{m+n}.$$

The coefficient D is the matter central charge, since each free scalar field X^μ contributes one unit

$$c_m = D.$$

2.5 Old covariant quantization

2.5.1 Physical state conditions

The old covariant quantization preserves manifest target space Lorentz covariance. All components of X^μ are quantized on the same basis. The price to pay for this quantization is the presence of negative norm states in the kinematical Fock space.

Physical states are selected by imposing the quantum Virasoro constraints

$$L_n |\psi\rangle = 0, \quad n > 0,$$

together with

$$(L_0 + a) |\psi\rangle = 0.$$

The condition involving L_0 determines the mass of the state. The positive mode constraints remove unphysical polarizations and relate states containing timelike and longitudinal oscillators.

It is not possible to impose

$$L_n |\psi\rangle = 0$$

for every integer n as a strong operator condition. Indeed, the commutator of positive and negative Virasoro generators contains both L_0 and the central term. The constraints are imposed asymmetrically, in a manner analogous to the Gupta–Bleuler quantization of the electromagnetic field.

The Hermitian conjugate of the physical state condition implies

$$\langle\psi| L_{-n} = 0, \quad n > 0.$$

As a consequence, the expectation value of the negative mode constraints vanishes between physical states even though L_{-n} does not annihilate an individual physical ket.

2.5.2 The ground state

At level $N = 0$, the only oscillator state is

$$|0, p\rangle.$$

The positive mode Virasoro constraints are automatically satisfied, while the L_0 condition gives

$$\alpha' M^2 = a.$$

For the critical open bosonic string, $a = -1$, and therefore

$$M^2 = -\frac{1}{\alpha'}.$$

The ground state is tachyonic. This instability is one of the reasons why the bosonic string is not a realistic fundamental theory, but it does not affect the structural discussion of the oscillator quantization.

2.5.3 The first excited level

At level $N = 1$, the general state is

$$|\psi\rangle = \zeta_\mu \alpha_{-1}^\mu |0, p\rangle,$$

where ζ_μ is a polarization vector.

The mass shell condition gives

$$\alpha' M^2 = 1 + a.$$

For $a = -1$, the state is massless.

The only nontrivial positive Virasoro constraint is $L_1 |\psi\rangle = 0$. Using

$$[L_1, \alpha_{-1}^\mu] = \alpha_0^\mu,$$

we find

$$L_1 |\psi\rangle = \zeta_\mu [L_1, \alpha_{-1}^\mu] |0, p\rangle = \zeta_\mu \alpha_0^\mu |0, p\rangle = \sqrt{2\alpha'} (\zeta \cdot p) |0, p\rangle.$$

Therefore,

$$\zeta \cdot p = 0.$$

This condition removes one polarization. A further reduction follows from the presence of null states. Consider

$$L_{-1} |0, p\rangle = \alpha_{-1} \cdot \alpha_0 |0, p\rangle = \sqrt{2\alpha'} p_\mu \alpha_{-1}^\mu |0, p\rangle.$$

For a massless momentum, this state has zero norm and is orthogonal to all physical states. So, polarizations related by

$$\zeta^\mu \sim \zeta^\mu + \lambda p^\mu$$

describe the same physical state.

After imposing transversality and quotienting by the null direction, the massless vector has $D - 2$ physical polarizations. This agrees with the result obtained directly in light cone quantization.

The level one example illustrates the basic mechanism of old covariant quantization. The original Fock space contains timelike and longitudinal excitations, some of which have negative norm. The Virasoro constraints and the null state equivalence remove these unphysical components from the physical spectrum.

2.5.4 Null states

A null state is a state with vanishing norm that is orthogonal to every physical state. Such states are physically indistinguishable from the zero vector and must be quotiented out.

States of the form

$$L_{-n} |\chi\rangle, \quad n > 0,$$

provide an important class of null states when $|\chi\rangle$ satisfies suitable conditions. If $|\psi\rangle$ is physical, then

$$\langle\psi| L_{-n} |\chi\rangle = 0.$$

The physical state space is therefore not simply the kernel of the positive Virasoro generators. One must also identify states that differ by null vectors. Schematically, the physical Hilbert space is constructed as

$$\mathcal{H}_{\text{phys}} = \frac{\{|\psi\rangle : L_n |\psi\rangle = 0 \quad n > 0, \quad (L_0 + a) |\psi\rangle = 0\}}{\{\text{null physical states}\}}.$$

The quotient is essential. Before taking it, the physical subspace may have a positive semidefinite inner product rather than a strictly positive definite one.

2.5.5 The no-ghost theorem

The absence of negative norm physical states is established by the no-ghost theorem. In the critical bosonic string, with

$$D = 26$$

and

$$a = -1,$$

the space of physical states has nonnegative norm. After quotienting by its null subspace, the resulting physical Hilbert space is positive definite and isomorphic to the state space obtained in light cone quantization.

2.6 Light cone quantization

2.6.1 Target space light cone coordinates

An alternative approach is to solve the constraints before quantization. We introduce target space light cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1}).$$

The Minkowski scalar product becomes

$$A \cdot B = -A^+ B^- - A^- B^+ + A^i B^i,$$

where

$$i = 1, \dots, D-2$$

labels the transverse directions.

The conformal gauge still leaves residual worldsheet transformations. These can be used to impose the light cone gauge

$$X^+ = x^+ + 2\alpha' p^+ \tau.$$

Equivalently, the nonzero oscillator modes satisfy

$$\alpha_n^+ = 0, \quad n \neq 0.$$

The gauge condition requires

$$p^+ \neq 0.$$

It identifies the target space coordinate X^+ with the worldsheet time and removes the corresponding oscillator excitations.

2.6.2 Solving the constraints

The Virasoro constraints can now be solved for the remaining longitudinal coordinate X^- . Define the transverse Virasoro generators

$$L_n^\perp = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{n-m}^i \alpha_m^i.$$

For $n \neq 0$, the constraint $L_n = 0$ gives

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha' p^+}} L_n^\perp.$$

The longitudinal oscillators are not independent degrees of freedom. They are completely determined by the transverse ones.

The zero mode constraint determines p^- in terms of the transverse momentum and oscillator excitations. It is equivalent to the mass shell condition

$$\alpha' M^2 = N^\perp + a,$$

where

$$N^\perp = \sum_{n>0} \alpha_{-n}^i \alpha_n^i.$$

Only the $D - 2$ transverse oscillators remain independent. Their algebra is

$$[\alpha_m^i, \alpha_n^j] = m \delta^{ij} \delta_{m+n}.$$

The corresponding Fock space has a manifestly positive definite inner product.

There are no independent timelike or longitudinal creation operators. Negative norm states are therefore absent from the beginning.

2.6.3 Lorentz invariance

The advantage of light cone quantization is that it directly isolates the physical degrees of freedom. Positivity and unitarity are manifest. But the problem is that target space Lorentz invariance is no longer manifest, since the choice of X^+ distinguishes a preferred light cone direction.

The Lorentz generators are

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n>0} \frac{\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu}{n}.$$

The commutator

$$[J^{-i}, J^{-j}]$$

contains anomalous contributions. The Lorentz algebra closes only if

$$D = 26,$$

and

$$a = -1.$$

Lorentz invariance therefore requires the critical dimension $D = 26$. In the covariant formulation, the same condition arises from the Virasoro anomaly and from the consistency of the ghost sector.

2.7 Gauge fixing and Faddeev–Popov ghosts

2.7.1 The path integral

After a Wick rotation to a Euclidean worldsheet, the gauge fixed Polyakov path integral takes the form

$$Z = \int \frac{\mathcal{D}X \mathcal{D}\gamma}{\text{Vol}(\text{Diff} \times \text{Weyl})} e^{-S_{\text{P}}^{\text{E}}[X, \gamma]}.$$

The denominator indicates that metrics related by worldsheet diffeomorphisms and Weyl transformations represent the same physical configuration.

To define the path integral, one imposes a gauge condition such as the conformal gauge. The gauge fixing procedure introduces a functional determinant, the Faddeev–Popov determinant. This determinant can be represented as a path integral over anticommuting fields denoted by b and c .

In complex worldsheet coordinates z and $\bar{z}$, the ghost action can be written as

$$S_{\text{gh}} = \frac{1}{2\pi} \int d^2z (b_{zz} \bar{\partial} c^z + b_{\bar{z}\bar{z}} \partial c^{\bar{z}}).$$

The holomorphic fields $b(z)$ and $c(z)$ have conformal weights

$$h_b = 2, \quad h_c = -1.$$

The corresponding mode expansions are

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-2},$$

and

$$c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n+1}.$$

Their nonvanishing anticommutation relations are

$$\{b_m, c_n\} = \delta_{m+n}.$$

The ghost energy momentum tensor is

$$T_{\text{gh}}(z) = -2 : b \partial c : - : (\partial b) c : .$$

Its Virasoro algebra has central charge

$$c_{\text{gh}} = -26.$$

2.7.2 Matter and ghost central charges

In the standard bosonic string, the matter sector consists of D free scalar fields X^μ . Its central charge is

$$c_{\text{m}} = D.$$

The total central charge is therefore

$$c_{\text{tot}} = c_{\text{m}} + c_{\text{gh}} = D - 26.$$

Quantum Weyl invariance requires

$$c_{\text{tot}} = 0.$$

Consequently,

$$D = 26.$$

This is the critical dimension of the bosonic string. From the path integral perspective, it ensures the cancellation of the Weyl anomaly. From the light cone perspective, it ensures the closure of the target space Lorentz algebra. From the covariant operator perspective, it is required for the consistency of the Virasoro constraints and the no-ghost theorem.

2.7.3 Two different meanings of ghost

It is important to distinguish the Faddeev–Popov bc ghosts from negative norm states.

Negative norm states arise in the matter Fock space because the target space metric is indefinite. For example

$$\alpha_{-1}^0 |0, p\rangle$$

has negative norm in the standard covariant representation.

The b and c ghosts, instead, are anticommuting worldsheet fields introduced by the gauge fixing procedure. Their name reflects their statistics and state space structure, but they are not the timelike negative norm oscillators of the matter sector.

2.8 BRST quantization

The BRST formalism provides a systematic treatment of gauge symmetries at the quantum level [11]. It combines the matter Virasoro generators with the ghost degrees of freedom into a nilpotent operator.

The BRST charge can be written schematically as

$$Q_B = \sum_{n \in \mathbb{Z}} c_{-n} (L_n^m + a\delta_n) - \frac{1}{2} \sum_{m, n \in \mathbb{Z}} (m - n) : c_{-m} c_{-n} b_{m+n} : .$$

The precise form depends on the normal ordering conventions, but its structural role is universal.

The BRST charge is required to be nilpotent

$$Q_B^2 = 0.$$

For the bosonic string, nilpotency requires

$$D = 26, \quad a = -1.$$

A state is BRST closed if

$$Q_B |\psi\rangle = 0.$$

States of the form

$$|\psi\rangle = Q_B |\chi\rangle$$

are BRST exact. Since $Q_B^2 = 0$, every exact state is automatically closed.

Physical states are identified with elements of the BRST cohomology,

$$\mathcal{H}_{\text{phys}}^{\text{BRST}} = \frac{\ker Q_B}{\text{Im } Q_B}.$$

This construction is the cohomological counterpart of imposing the Virasoro constraints and quotienting by null states in old covariant quantization.

Chapter 3

The Modified Fock Space Representation

In the present chapter we will define the modified algebraic Fock space and describe its immediate structural consequences. The positivity of its inner product will be proved in Chapter 4. The corresponding normal ordering prescription and Virasoro algebra will instead be studied in Chapter 5.

Throughout the oscillator analysis, the momentum p^μ will be kept fixed. We therefore suppress the continuous delta function normalization associated with the zero mode sector and normalize the oscillator vacuum as

$$\langle 0, p | 0, p \rangle = 1.$$

3.1 Modified vacuum conditions

The oscillator algebra remains

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m+n},$$

together with the Hermiticity condition

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu.$$

For the spatial components, we retain the conventional vacuum condition

$$\alpha_n^i |0, p\rangle = 0, \quad n > 0, \quad i = 1, \dots, D-1.$$

Thus, as in the standard representation, α_n^i is an annihilation operator and α_{-n}^i is a creation operator for $n > 0$.

In the timelike sector, instead, we reverse the standard assignment and impose

$$\alpha_{-n}^0 |0, p\rangle = 0, \quad n > 0.$$

Consequently, the negative timelike modes α_{-n}^0 act as annihilation operators, while the positive timelike modes α_n^0 act as creation operators.

The creation and annihilation operators of the modified representation are therefore

	creation operators	annihilation operators	
spatial sector	α_{-n}^i	α_n^i	$n > 0.$
timelike sector	α_n^0	α_{-n}^0	

This modification compensates the negative sign in the timelike oscillator algebra. For example, consider the one oscillator state

$$|\psi_n^0\rangle = \alpha_n^0 |0, p\rangle, \quad n > 0.$$

Its norm is

$$\langle \psi_n^0 | \psi_n^0 \rangle = \langle 0, p | \alpha_{-n}^0 \alpha_n^0 | 0, p \rangle = \langle 0, p | [\alpha_{-n}^0, \alpha_n^0] | 0, p \rangle = n.$$

Indeed,

$$[\alpha_{-n}^0, \alpha_n^0] = (-n)\eta^{00} = n.$$

The complete proof will be given in the next chapter.

3.2 The algebraic Fock space

Let

$$\mathbf{r} = \{r_{n,i}\}_{n>0, i=1,\dots,D-1}, \quad \mathbf{s} = \{s_n\}_{n>0},$$

denote the spatial and timelike occupation numbers. We assume that only finitely many entries of $\mathbf{r}$ and $\mathbf{s}$ are nonzero.

The corresponding occupation number states are

$$|\mathbf{r}, \mathbf{s}; p\rangle = \prod_{n>0} \prod_{i=1}^{D-1} (\alpha_{-n}^i)^{r_{n,i}} \prod_{m>0} (\alpha_m^0)^{s_m} |0, p\rangle. \quad (3.1)$$

The algebraic Fock space at fixed momentum is defined as the finite linear span

$$\mathcal{F}_{\text{alg}}(p) = \text{span} \{|\mathbf{r}, \mathbf{s}; p\rangle\}.$$

Thus, a generic vector in $\mathcal{F}_{\text{alg}}(p)$ is a finite linear combination

$$|\psi\rangle = \sum_{\mathbf{r}, \mathbf{s}} c_{\mathbf{r}, \mathbf{s}} |\mathbf{r}, \mathbf{s}; p\rangle,$$

where only finitely many coefficients are nonzero.

At this stage, $\mathcal{F}_{\text{alg}}(p)$ is only an algebraic vector space equipped with the inner product inherited from the adjoint relation

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu.$$

Only after proving that this inner product is positive definite will it be appropriate to consider the Hilbert space completion of $\mathcal{F}_{\text{alg}}(p)$.

3.3 Level operators and gradings

The spatial contribution to the oscillator level operator is defined in the usual way

$$N^{sp} = \sum_{n>0} \sum_{i=1}^{D-1} \alpha_{-n}^i \alpha_n^i.$$

In the timelike sector, the corresponding normal ordered contribution is

$$N^0 = - \sum_{n>0} \alpha_n^0 \alpha_{-n}^0.$$

The total signed level operator is therefore

$$N = N^{sp} + N^0.$$

The minus sign in N^0 is inherited from the target space metric. It implies that a timelike creation operator lowers the eigenvalue of N . Indeed,

$$[N, \alpha_{-m}^i] = m\alpha_{-m}^i, \quad [N, \alpha_m^0] = -m\alpha_m^0, \quad m > 0.$$

On an occupation number state, the three operators act as

$$N^{sp} |\mathbf{r}, \mathbf{s}; p\rangle = \left(\sum_{n>0} \sum_{i=1}^{D-1} nr_{n,i} \right) |\mathbf{r}, \mathbf{s}; p\rangle,$$

$$N^0 |\mathbf{r}, \mathbf{s}; p\rangle = - \left(\sum_{m>0} ms_m \right) |\mathbf{r}, \mathbf{s}; p\rangle,$$

and hence

$$N |\mathbf{r}, \mathbf{s}; p\rangle = \left(\sum_{n>0} \sum_{i=1}^{D-1} nr_{n,i} - \sum_{m>0} ms_m \right) |\mathbf{r}, \mathbf{s}; p\rangle.$$

Unlike the standard string level operator, N is not bounded from below. It is therefore not a non negative grading of the modified Fock space. Moreover, a fixed eigenvalue of N generally has infinite degeneracy. For instance, the operator

$$\alpha_{-n}^i \alpha_n^0$$

changes neither the eigenvalue of N nor the momentum, while it introduces two additional oscillator excitations.

It is useful to introduce a second operator that counts the total weighted number of creation operators,

$$K = N^{sp} - N^0 = \sum_{n>0} \sum_{i=1}^{D-1} \alpha_{-n}^i \alpha_n^i + \sum_{n>0} \alpha_n^0 \alpha_{-n}^0. \quad (3.2)$$

Its commutators with the creation operators are

$$[K, \alpha_{-m}^i] = m\alpha_{-m}^i, \quad [K, \alpha_m^0] = m\alpha_m^0.$$

Therefore,

$$K |\mathbf{r}, \mathbf{s}; p\rangle = \left(\sum_{n>0} \sum_{i=1}^{D-1} nr_{n,i} + \sum_{m>0} ms_m \right) |\mathbf{r}, \mathbf{s}; p\rangle.$$

The operator K has nonnegative integer eigenvalues and provides a genuine grading of the algebraic Fock space

$$\mathcal{F}_{\text{alg}}(p) = \bigoplus_{k \geq 0} \mathcal{F}_k(p),$$

where $\mathcal{F}_k(p)$ denotes the eigenspace with eigenvalue k .

At fixed k , the number of oscillator partitions is finite. By contrast, a fixed N sector contains infinitely many different K components. This distinction will become essential when the physical state conditions are studied in Chapter 6.

The operators N , K and p^μ mutually commute

$$[N, K] = 0, \quad [N, p^\mu] = 0, \quad [K, p^\mu] = 0.$$

The basis states may therefore be organized simultaneously according to momentum, signed level and K grading.

3.4 Lorentz transformations and unitary implementability

The target space Lorentz generators retain their standard formal expression,

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n>0} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

It is useful to separate the zero mode and oscillator contributions

$$J^{\mu\nu} = J_0^{\mu\nu} + J_{\text{osc}}^{\mu\nu},$$

where

$$J_0^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

and

$$J_{\text{osc}}^{\mu\nu} = -i \sum_{n>0} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

The zero mode representation has not been modified. The orbital generators $J_0^{\mu\nu}$ act on the momentum variables. A Lorentz transformation therefore maps a state with momentum p to a state with momentum Λp and does not preserve an individual fixed momentum space $\mathcal{H}_{\text{kin}}(p)$. The discussion below is about the nonzero mode oscillator representation, which is the same in every momentum fibre. The obstruction to Lorentz symmetry is already visible in this sector.

3.4.1 Spatial rotations and boosts

Consider first a spatial rotation. Both terms in J_{osc}^{ij} contain a spatial annihilation operator when they act on the vacuum. Consequently,

$$J_{\text{osc}}^{ij} |0, p\rangle = 0.$$

More generally, spatial rotations mix spatial annihilation operators only with spatial annihilation operators, and spatial creation operators only with spatial creation operators. They preserve the spatial oscillator polarisation.

The situation is different for a boost mixing the timelike direction with a spatial direction i . Acting formally on the vacuum gives

$$\begin{aligned} J_{\text{osc}}^{0i} |0, p\rangle &= -i \sum_{n>0} \frac{1}{n} (\alpha_{-n}^0 \alpha_n^i - \alpha_{-n}^i \alpha_n^0) |0, p\rangle \\ &= i \sum_{n>0} \frac{1}{n} \alpha_{-n}^i \alpha_n^0 |0, p\rangle. \end{aligned} \tag{3.3}$$

The first term vanishes because α_n^i is a spatial annihilation operator. The second term does not vanish, since both α_{-n}^i and α_n^0 are creation operators in the modified representation.

Thus, spatial rotations preserve the vacuum polarisation, whereas boosts do not. This already shows that the modified Fock representation is not manifestly Lorentz invariant.

The same distinction can be expressed in terms of the signed timelike level operator. For $n > 0$,

$$[N^0, \alpha_n^0] = -n \alpha_n^0, \quad [N^0, \alpha_{-n}^0] = n \alpha_{-n}^0, \quad [N^0, \alpha_n^i] = 0.$$

It follows that

$$[N^0, J_{\text{osc}}^{ij}] = 0,$$

while

$$[N^0, J_{\text{osc}}^{0i}] = -i \sum_{n>0} (\alpha_{-n}^0 \alpha_n^i + \alpha_{-n}^i \alpha_n^0) \neq 0.$$

The timelike occupation structure is therefore invariant under spatial rotations but not under boosts.

This statement concerns the chosen representation, rather than the formal oscillator algebra itself. The oscillator commutation relations remain Lorentz covariant, but the distinction between creation and annihilation operators selected by the modified vacuum is not preserved by boosts.

3.4.2 The domain of the boost generator

There is an additional problem in the infinite sum in equation (3.3). It is convenient to introduce normalised oscillators.

For every $n > 0$, define

$$a_n^i = \frac{\alpha_n^i}{\sqrt{n}}, \quad a_n^{i\dagger} = \frac{\alpha_{-n}^i}{\sqrt{n}},$$

in the spatial sector, and

$$b_n = \frac{\alpha_{-n}^0}{\sqrt{n}}, \quad b_n^\dagger = \frac{\alpha_n^0}{\sqrt{n}},$$

in the timelike sector. They satisfy

$$[a_n^i, a_m^{i\dagger}] = \delta_{nm}, \quad [b_n, b_m^\dagger] = \delta_{nm},$$

with all mixed commutators vanishing. The modified vacuum conditions take the standard form

$$a_n^i |0, p\rangle = 0, \quad b_n |0, p\rangle = 0.$$

In terms of these operators, the oscillator contribution to the boost generator becomes formally

$$J_{\text{osc}}^{0i} = i \sum_{n>0} \left(a_n^{i\dagger} b_n^\dagger - a_n^i b_n \right). \quad (3.4)$$

The boost therefore contains both pair creation and pair annihilation terms.

To investigate the convergence of the mode sum, consider the finite mode truncation

$$J_{\text{osc},M}^{0i} = i \sum_{n=1}^M \left(a_n^{i\dagger} b_n^\dagger - a_n^i b_n \right).$$

Its action on the vacuum is

$$J_{\text{osc},M}^{0i} |0, p\rangle = i \sum_{n=1}^M a_n^{i\dagger} b_n^\dagger |0, p\rangle.$$

The states

$$a_n^{i\dagger} b_n^\dagger |0, p\rangle, \quad n > 0,$$

are mutually orthonormal. Therefore, for $M > N$,

$$\| (J_{\text{osc},M}^{0i} - J_{\text{osc},N}^{0i}) |0, p\rangle \|^2 = \left\| \sum_{n=N+1}^M a_n^{i\dagger} b_n^\dagger |0, p\rangle \right\|^2 = M - N.$$

The sequence

$$J_{\text{osc},M}^{0i} |0, p\rangle$$

is therefore not Cauchy and has no norm limit in $\mathcal{H}_{\text{kin}}(p)$.

As a consequence the formal operator J_{osc}^{0i} is not defined on the vacuum, it cannot be defined on a domain containing the complete algebraic Fock space $\mathcal{F}_{\text{alg}}(p)$, since the vacuum belongs to this space.

This is stronger than the statement that the vacuum is not invariant under boosts. It shows that the infinitesimal boost generator does not map the vacuum to a vector in the Hilbert space completion.

Nevertheless, this conclusion alone does not exclude the existence of finite unitary boost transformations. In particular, the fact that the vacuum does not belong to the domain of the infinitesimal generator does not imply that a finite unitary transformation cannot act on it.

For this reason, the finite boost transformation must be analysed directly.

3.4.3 Finite boosts as Bogoliubov transformations

The corresponding unitary transformation is

$$U_M(\chi) = \exp(-i\chi J_{\text{osc},M}^{0i}),$$

where χ denotes the boost rapidity.

For $n \leq M$, the commutation relations give

$$\begin{aligned} U_M(\chi) a_n^i U_M(\chi)^{-1} &= \cosh \chi a_n^i - \sinh \chi b_n^\dagger, \\ U_M(\chi) b_n U_M(\chi)^{-1} &= \cosh \chi b_n - \sinh \chi a_n^{i\dagger}. \end{aligned}$$

The overall signs depend on the convention chosen for the boost parameter. The relevant point is that the coefficient multiplying the creation operators has modulus $|\sinh \chi|$.

The finite boost mixes annihilation and creation operators. Such a linear transformation preserving the canonical commutation relations is called a bosonic Bogoliubov transformation. For general background on Bogoliubov transformations and their implementation on bosonic Fock space see [14].

Let

$$\mathcal{H}_{0i}^{op} = \overline{\text{span}} \left\{ a_n^{i\dagger} |0, p\rangle, b_n^\dagger |0, p\rangle : n > 0 \right\}$$

be the one particle Hilbert space associated with the spatial direction i and the timelike oscillator. It is isomorphic to

$$\mathcal{H}_{0i}^{op} \simeq \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}).$$

Regarding this decomposition, the part of the Bogoliubov transformation that mixes annihilation and creation operators is

$$V_\chi = -\sinh \chi \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (3.5)$$

where I is the identity operator on $\ell^2(\mathbb{N})$.

The full oscillator one particle space, with also the other spatial directions, can be decomposed as

$$\mathcal{H}^{op} = \mathcal{H}_{0i}^{op} \oplus \mathcal{H}_\perp^{op},$$

where $\mathcal{H}_\perp^{op}$ contains all oscillator directions that are not mixed by the chosen boost. The transformation acts trivially on $\mathcal{H}_\perp^{op}$, so the complete off diagonal part is $V_\chi \oplus 0$.

A bosonic Bogoliubov transformation is unitarily implementable in a Fock representation if and only if its off diagonal part is a Hilbert–Schmidt operator. This is the Shale–Stinespring criterion [6, 7].

In the present case,

$$\|V_\chi\|_{\text{HS}}^2 = 2 \sum_{n=1}^{\infty} |\sinh \chi|^2.$$

For every nonzero rapidity χ , the quantity $|\sinh \chi|^2$ is a strictly positive constant, independent of the mode number n . The sum therefore diverges

$$\|V_\chi\|_{\text{HS}}^2 = \infty, \quad \chi \neq 0.$$

Thus, the off diagonal part of the boost transformation is not Hilbert–Schmidt. By the Shale–Stinespring criterion, no unitary operator on the modified oscillator Fock space can implement a nontrivial boost.

The failure of unitary implementability is an infinite mode effect. For every finite cutoff M , the truncated transformation is unitarily implementable because it acts nontrivially on only finitely many oscillator pairs. The obstruction appears when the cutoff is removed, the same nonzero mixing is repeated independently for every mode, producing a divergent Hilbert–Schmidt norm.

The conclusion concerns the nonzero mode oscillator representation. A complete Lorentz boost also transforms the zero modes and maps the momentum fibre $\mathcal{H}_{\text{kin}}(p)$ to $\mathcal{H}_{\text{kin}}(\Lambda_\chi p)$. A representation of the complete Lorentz transformation would therefore require a direct integral Hilbert space over the momentum variables. This construction is not needed for the present conclusion, since the failure of unitary implementability already occurs in the oscillator sector common to all momentum fibres.

Chapter 4

Positivity of the Modified Kinematical Space

The positivity of one oscillator states was already verified in Chapter 3. In the present chapter, we prove the corresponding statement for arbitrary finite oscillator states. More precisely, we show that the inner product on $\mathcal{F}_{\text{alg}}(p)$ is positive definite.

The proof proceeds in two steps. We first prove, by induction, that every occupation number basis state has strictly positive norm. We then show that basis states with different occupation numbers are mutually orthogonal. Together, these two results imply positivity for every nonzero finite linear combination of basis states.

4.1 Induction parameter

We use the occupation number basis introduced in equation (3.1),

$$|\mathbf{r}, \mathbf{s}; p\rangle = \prod_{n>0} \prod_{i=1}^{D-1} (\alpha_{-n}^i)^{r_{n,i}} \prod_{m>0} (\alpha_m^0)^{s_m} |0, p\rangle.$$

Only finitely many occupation numbers $r_{n,i}$ and s_m are nonzero. For the induction argument, we introduce the total unweighted number of creation operators,

$$\ell(\mathbf{r}, \mathbf{s}) = \ell_{\text{sp}}(\mathbf{r}) + \ell_{\text{time}}(\mathbf{s}),$$

where

$$\ell_{\text{sp}}(\mathbf{r}) = \sum_{n>0} \sum_{i=1}^{D-1} r_{n,i}, \quad \ell_{\text{time}}(\mathbf{s}) = \sum_{m>0} s_m.$$

The quantity ℓ counts the number of creation operators appearing in a basis state, without weighting them by their mode number. It should therefore not be confused with either the signed level N or the nonnegative grading K introduced in Chapter 3.

4.2 Positivity of the basis states

We prove by strong induction on ℓ that

$$\langle \mathbf{r}, \mathbf{s}; p | \mathbf{r}, \mathbf{s}; p \rangle > 0$$

for every occupation number basis state. For $\ell = 0$, the only state is the vacuum. By the normalization chosen in the previous chapter,

$$\langle 0, p | 0, p \rangle = 1 > 0.$$

Assume now that every basis state containing at most ℓ creation operators has positive norm. Let $|I'\rangle$ be a basis state containing $\ell + 1$ creation operators. We may remove one of its creation operators and write

$$|I'\rangle = A^\dagger |I\rangle,$$

where $|I\rangle$ contains ℓ creation operators. The pair $(A, A^\dagger)$ can be of one of the following two types

	$A^\dagger$	A	$[A, A^\dagger]$	
spatial	α_{-n}^i	α_n^i	n	$n > 0.$
timelike	α_n^0	α_{-n}^0	n	

For the timelike sector, the positive sign follows from

$$[\alpha_{-n}^0, \alpha_n^0] = (-n)\eta^{00} = n.$$

The norm of $|I'\rangle$ is therefore

$$\langle I'|I'\rangle = \langle I|AA^\dagger|I\rangle = \langle I|([A, A^\dagger] + A^\dagger A)|I\rangle = n\langle I|I\rangle + \langle AI|AI\rangle. \quad (4.1)$$

By the induction hypothesis,

$$\langle I|I\rangle > 0.$$

It remains to analyze the second term in equation (4.1). Since $|I\rangle$ is an occupation number basis state, the annihilation operator A either annihilates it or removes one corresponding creation operator. Thus,

$$A|I\rangle = 0,$$

or

$$A|I\rangle = c|K\rangle,$$

where c is a nonzero numerical coefficient and $|K\rangle$ is a basis state containing $\ell - 1$ creation operators. More explicitly, if the corresponding occupation number is q , then

$$\alpha_n^i (\alpha_{-n}^i)^q |\chi\rangle = qn (\alpha_{-n}^i)^{q-1} |\chi\rangle,$$

provided that $\alpha_n^i |\chi\rangle = 0$, while in the timelike sector

$$\alpha_{-n}^0 (\alpha_n^0)^q |\chi\rangle = qn (\alpha_n^0)^{q-1} |\chi\rangle,$$

provided that $\alpha_{-n}^0 |\chi\rangle = 0$. In the first case,

$$\langle AI|AI\rangle = 0.$$

In the second case, the strong induction hypothesis gives

$$\langle AI|AI\rangle = |c|^2 \langle K|K\rangle > 0.$$

Consequently,

$$\langle AI|AI\rangle \geq 0.$$

Returning to equation (4.1), we find

$$\langle I'|I'\rangle = n\langle I|I\rangle + \langle AI|AI\rangle > 0.$$

This completes the induction and proves that every occupation number basis state has strictly positive norm.

4.3 Orthogonality

We now show that two occupation number basis states are orthogonal unless all their occupation numbers coincide. Consider first a single spatial oscillator pair

$$A^\dagger = \alpha_{-n}^i, \quad A = \alpha_n^i,$$

satisfying

$$[A, A^\dagger] = n.$$

Using the vacuum condition $A|0, p\rangle = 0$, repeated application of the commutation relation gives

$$\langle 0, p | A^r \left(A^\dagger \right)^{\tilde{r}} | 0, p \rangle = \delta_{r, \tilde{r}} n^r r!.$$

The same argument applies to a timelike oscillator pair,

$$B^\dagger = \alpha_n^0, \quad B = \alpha_{-n}^0,$$

for which

$$[B, B^\dagger] = n.$$

Therefore,

$$\langle 0, p | B^s \left(B^\dagger \right)^{\tilde{s}} | 0, p \rangle = \delta_{s, \tilde{s}} n^s s!.$$

Oscillators with different mode numbers or different spatial indices commute. The complete inner product therefore factorizes into independent single mode contributions

$$\langle \mathbf{r}, \mathbf{s}; p | \tilde{\mathbf{r}}, \tilde{\mathbf{s}}; p \rangle = \left[\prod_{n>0} \prod_{i=1}^{D-1} \delta_{r_{n,i}, \tilde{r}_{n,i}} n^{r_{n,i}} r_{n,i}! \right] \left[\prod_{m>0} \delta_{s_m, \tilde{s}_m} m^{s_m} s_m! \right]. \quad (4.2)$$

All products are effectively finite because the occupation sequences have finite support. It follows immediately that

$$\langle \mathbf{r}, \mathbf{s}; p | \tilde{\mathbf{r}}, \tilde{\mathbf{s}}; p \rangle = 0$$

while

$$(\mathbf{r}, \mathbf{s}) \neq (\tilde{\mathbf{r}}, \tilde{\mathbf{s}}).$$

For equal occupation numbers, equation (4.2) gives the explicit norm

$$\| |\mathbf{r}, \mathbf{s}; p\rangle \|^2 = \prod_{n>0} \prod_{i=1}^{D-1} n^{r_{n,i}} r_{n,i}! \times \prod_{m>0} m^{s_m} s_m! > 0.$$

This formula provides an explicit version of the positivity result established by induction.

4.4 Positive definite inner product

Let

$$|\psi\rangle = \sum_{\mathbf{r}, \mathbf{s}} c_{\mathbf{r}, \mathbf{s}} |\mathbf{r}, \mathbf{s}; p\rangle$$

be a nonzero vector in $\mathcal{F}_{\text{alg}}(p)$. By definition, only finitely many coefficients are nonzero. Using the orthogonality of the occupation number basis, we obtain

$$\langle \psi | \psi \rangle = \sum_{\mathbf{r}, \mathbf{s}} |c_{\mathbf{r}, \mathbf{s}}|^2 \| |\mathbf{r}, \mathbf{s}; p\rangle \|^2.$$

Every term in this sum is nonnegative. As a result $|\psi\rangle \neq 0$, at least one coefficient is nonzero, and the corresponding basis state norm is strictly positive. Hence

$$\langle \psi | \psi \rangle > 0 \quad \text{for every } |\psi\rangle \neq 0.$$

We have proved that the inner product on $\mathcal{F}_{\text{alg}}(p)$ is positive definite. In particular, the modified matter Fock space contains no nonzero null vectors and no negative norm oscillator states at the kinematical level. It is useful to introduce normalized occupation number states,

$$|\mathbf{r}, \mathbf{s}; p\rangle_{\text{N}} = \left[\prod_{n>0} \prod_{i=1}^{D-1} \frac{1}{\sqrt{n^{r_{n,i}} r_{n,i}!}} \right] \times \left[\prod_{m>0} \frac{1}{\sqrt{m^{s_m} s_m!}} \right] |\mathbf{r}, \mathbf{s}; p\rangle.$$

They satisfy

$${}_{\text{N}} \langle \mathbf{r}, \mathbf{s}; p | \tilde{\mathbf{r}}, \tilde{\mathbf{s}}; p \rangle_{\text{N}} = \delta_{\mathbf{r}, \tilde{\mathbf{r}}} \delta_{\mathbf{s}, \tilde{\mathbf{s}}}.$$

The positive definite norm allows us to define the kinematical Hilbert space as the completion

$$\mathcal{H}_{\text{kin}}(p) = \overline{\mathcal{F}_{\text{alg}}(p)}^{\|\cdot\|}.$$

A vector in $\mathcal{H}_{\text{kin}}(p)$ may therefore contain an infinite superposition of normalized occupation number states, provided that its coefficients are square summable. The K grading introduced in equation (3.2) becomes an orthogonal Hilbert space decomposition,

$$\mathcal{H}_{\text{kin}}(p) = \widehat{\bigoplus_{k \geq 0} \mathcal{H}_k(p)},$$

where $\mathcal{H}_k(p)$ is the finite dimensional subspace with K eigenvalue k . So, for a decomposition

$$|\psi\rangle = \sum_{k \geq 0} |\psi_k\rangle, \quad |\psi_k\rangle \in \mathcal{H}_k(p),$$

the norm satisfies

$$\| |\psi\rangle \|^2 = \sum_{k \geq 0} \| |\psi_k\rangle \|^2.$$

This identity will be important when studying physical states with infinite support in the K grading. The positivity of the kinematical state space is thus established. It remains to determine how the modified vacuum affects the Virasoro generators, their central extension and the physical state conditions.

Chapter 5

Virasoro Algebra and Ghost Sector

Throughout the matter analysis, we work first on the algebraic core $\mathcal{F}_{\text{alg}}(p)$ introduced in Chapter 3. For every fixed Virasoro mode, the corresponding operator is well defined on this domain. Indeed, on a finite oscillator state only finitely many terms containing annihilation operators act nontrivially, while the part containing only creation operators is finite for fixed mode number.

We denote the matter Virasoro generators by L_n^X . The superscript will be retained in this chapter in order to distinguish the matter, ghost and total generators.

5.1 Modified normal ordering and vacuum energy shift

The matter Virasoro generators are defined by

$$L_n^X = \frac{1}{2} \sum_{m \in \mathbb{Z}} : \eta_{\mu\nu} \alpha_{n-m}^\mu \alpha_m^\nu :_*, \quad (5.1)$$

where $: - :_*$ denotes normal ordering with respect to the modified vacuum.

For $n > 0$, the spatial operators α_{-n}^i are creation operators and α_n^i are annihilation operators, while in the timelike sector α_n^0 are creation operators and α_{-n}^0 are annihilation operators. In particular,

$$: \alpha_{-n} \cdot \alpha_n :_* = \sum_{i=1}^{D-1} \alpha_{-n}^i \alpha_n^i - \alpha_n^0 \alpha_{-n}^0, \quad n > 0.$$

With this convention,

$$L_0^X = \frac{1}{2} \alpha_0^2 + N,$$

where N is the signed level operator introduced in Chapter 3. Since

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu,$$

we may also write

$$L_0^X = \alpha' p^2 + N.$$

On the algebraic core, the adjoint relation of the oscillators implies

$$(L_n^X)^\dagger = L_{-n}^X.$$

We now determine the c number generated by rewriting the symmetric, unordered expression in modified normal order. Consider

$$L_{0,\text{unord}}^X = \frac{1}{2} \alpha_0^2 + \frac{1}{2} \sum_{n>0} (\alpha_{-n} \cdot \alpha_n + \alpha_n \cdot \alpha_{-n}).$$

For each spatial oscillator,

$$\frac{1}{2} (\alpha_{-n}^i \alpha_n^i + \alpha_n^i \alpha_{-n}^i) = \alpha_{-n}^i \alpha_n^i + \frac{n}{2}.$$

There are $D - 1$ spatial components.

For the timelike oscillator, instead,

$$-\frac{1}{2} (\alpha_{-n}^0 \alpha_n^0 + \alpha_n^0 \alpha_{-n}^0) = -\frac{1}{2} (2\alpha_n^0 \alpha_{-n}^0 + [\alpha_{-n}^0, \alpha_n^0]) = -\alpha_n^0 \alpha_{-n}^0 - \frac{n}{2}.$$

The timelike oscillator therefore contributes with the opposite sign to the vacuum energy. Combining the spatial and timelike sectors,

$$L_{0,\text{unord}}^X = L_0^X + \frac{D-2}{2} \sum_{n>0} n.$$

Using zeta function regularization,

$$\sum_{n>0} n = \zeta(-1) = -\frac{1}{12},$$

we obtain

$$L_{0,\text{unord}}^X = L_0^X + a^X,$$

with

$$a^X = -\frac{D-2}{24}.$$

The quantity a^X is the matter vacuum energy shift associated with the modified normal ordering. It should be distinguished from the total intercept a , which will also contain the contribution of the ghost sector.

5.2 Matter Virasoro algebra

Before computing the Virasoro algebra, it is useful to determine the action of L_n^X on a single oscillator. Using equation (5.1), we find

$$\begin{aligned} [L_n^X, \alpha_m^\rho] &= \frac{1}{2} \sum_{t \in \mathbb{Z}} \eta_{\mu\nu} (\alpha_{n-t}^\mu [\alpha_t^\nu, \alpha_m^\rho] + [\alpha_{n-t}^\mu, \alpha_m^\rho] \alpha_t^\nu) \\ &= \frac{1}{2} \sum_{t \in \mathbb{Z}} \eta_{\mu\nu} (t \delta_{t+m} \eta^{\nu\rho} \alpha_{n-t}^\mu + (n-t) \delta_{n-t+m} \eta^{\mu\rho} \alpha_t^\nu) = -m \alpha_{n+m}^\rho. \end{aligned} \quad (5.2)$$

Thus,

$$[L_n^X, \alpha_m^\rho] = -m \alpha_{n+m}^\rho.$$

This commutator has the same form as in the conventional representation. The effect of the modified vacuum appears instead in the normal ordering anomaly of the Virasoro algebra.

Using equation (5.2), we obtain

$$\begin{aligned} [L_n^X, L_m^X] &= \frac{1}{2} \sum_{v \in \mathbb{Z}} \eta_{\mu\nu} ([L_n^X, \alpha_{m-v}^\mu] \alpha_v^\nu + \alpha_{m-v}^\mu [L_n^X, \alpha_v^\nu]) \\ &= \frac{1}{2} \sum_{v \in \mathbb{Z}} \eta_{\mu\nu} ((v-m) \alpha_{n+m-v}^\mu \alpha_v^\nu - v \alpha_{m-v}^\mu \alpha_{n+v}^\nu). \end{aligned} \quad (5.3)$$

If $n + m \neq 0$, we may set $r = n + v$ in the second term. After restoring normal order, no c number is generated, and

$$[L_n^X, L_m^X] = \frac{1}{2} \sum_{r \in \mathbb{Z}} \eta_{\mu\nu} (n-r) \alpha_{n+m-r}^\mu \alpha_r^\nu + \frac{1}{2} \sum_{r \in \mathbb{Z}} \eta_{\mu\nu} (r-m) \alpha_{n+m-r}^\mu \alpha_r^\nu = (n-m) L_{n+m}^X.$$

The anomalous contribution can therefore arise only when the mode numbers sum to zero.

Let $q > 0$ and consider

$$[L_q^X, L_{-q}^X].$$

From equation (5.3),

$$[L_q^X, L_{-q}^X] = \frac{1}{2} \sum_{v \in \mathbb{Z}} \eta_{\mu\nu} \left(-v \alpha_{-q-v}^\mu \alpha_{q+v}^\nu + (v+q) \alpha_{-v}^\mu \alpha_v^\nu \right).$$

We split the sum into three regions

$$v < -q, \quad -q \leq v \leq 0, \quad v > 0.$$

Accordingly,

$$[L_q^X, L_{-q}^X] = \frac{1}{2} (S_< + S_0 + S_>).$$

It is convenient to define

$$B_p = \sum_{i=1}^{D-1} \alpha_{-p}^i \alpha_p^i - \alpha_p^0 \alpha_{-p}^0, \quad p > 0.$$

Then

$$L_0^X = \frac{1}{2} \alpha_0^2 + \sum_{p>0} B_p.$$

For the region $v < -q$, write $v = -q - p$ with $p > 0$. We obtain

$$S_< = \sum_{p>0} \eta_{\mu\nu} \left((q+p) \alpha_p^\mu \alpha_{-p}^\nu - p \alpha_{q+p}^\mu \alpha_{-q-p}^\nu \right).$$

After applying the modified normal ordering, the spatial c number terms cancel between the two contributions, while the timelike terms are put in the reversed order. Thus,

$$S_< = \sum_{p>0} (q+p) B_p + \sum_{p>q} (q-p) B_p. \quad (5.4)$$

For the region $v > 0$, write $v = p$ with $p > 0$. Then

$$S_> = \sum_{p>0} \eta_{\mu\nu} \left(-p \alpha_{-q-p}^\mu \alpha_{q+p}^\nu + (q+p) \alpha_{-p}^\mu \alpha_p^\nu \right).$$

The same cancellation of the external c number terms gives

$$S_> = \sum_{p>0} (q+p) B_p + \sum_{p>q} (q-p) B_p. \quad (5.5)$$

The finite region $-q \leq v \leq 0$ is responsible for the anomaly. Setting $v = -p$, with $0 \leq p \leq q$, gives

$$S_0 = \sum_{p=0}^q \eta_{\mu\nu} \left(p \alpha_{p-q}^\mu \alpha_{q-p}^\nu + (q-p) \alpha_p^\mu \alpha_{-p}^\nu \right).$$

The operator part is

$$S_0^{\text{op}} = 2q \alpha_0^2 + 2 \sum_{p=1}^{q-1} (q-p) B_p. \quad (5.6)$$

Normal ordering also produces a finite c number. The $D - 1$ spatial oscillators contribute

$$(D - 1) \sum_{p=0}^q p(q - p),$$

whereas the reversed timelike oscillator contributes

$$- \sum_{p=0}^q p(q - p).$$

Therefore,

$$S_0^{\text{anom}} = (D - 2) \sum_{p=0}^q p(q - p).$$

Combining equations (5.4), (5.5) and (5.6), the operator terms satisfy

$$S_{<}^{\text{op}} + S_0^{\text{op}} + S_{>}^{\text{op}} = 2q \alpha_0^2 + 4q \sum_{p>0} B_p.$$

After restoring the overall factor $1/2$, this becomes

$$\frac{1}{2} (S_{<}^{\text{op}} + S_0^{\text{op}} + S_{>}^{\text{op}}) = 2q L_0^X.$$

For the anomalous contribution, we use

$$\sum_{p=0}^q p(q - p) = \frac{q^3 - q}{6}.$$

Hence,

$$\frac{1}{2} S_0^{\text{anom}} = \frac{D - 2}{12} (q^3 - q).$$

We conclude that

$$[L_q^X, L_{-q}^X] = 2q L_0^X + \frac{D - 2}{12} (q^3 - q).$$

Combining the anomalous and non anomalous cases, the matter Virasoro algebra is

$$[L_m^X, L_n^X] = (m - n) L_{m+n}^X + \frac{D - 2}{12} (m^3 - m) \delta_{m+n}. \quad (5.7)$$

The matter central charge is therefore

$$c_m = D - 2.$$

The $D - 1$ spatial oscillators each contribute $+1$, while the reversed timelike oscillator contributes -1 .

This result does not contradict the positivity proved in Chapter 4. The modified representation is not a positive energy representation of the Virasoro algebra, the signed level operator N is not bounded from below, and the modified vacuum is not a Virasoro highest weight state. Positivity of the inner product therefore does not imply that every oscillator sector contributes positively to the Virasoro central extension.

5.3 Virasoro and Lorentz generators

We now study the commutator between the matter Virasoro generators and the target space Lorentz generators. The latter were decomposed in Chapter 3 as

$$J^{\mu\nu} = J_0^{\mu\nu} + J_{\text{osc}}^{\mu\nu},$$

where

$$J_0^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

and

$$J_{\text{osc}}^{\mu\nu} = -i \sum_{t>0} \frac{1}{t} (\alpha_{-t}^\mu \alpha_t^\nu - \alpha_{-t}^\nu \alpha_t^\mu).$$

The calculation in this section is a formal identity of the oscillator mode algebra. As shown in Chapter 3, the boost generators are not automatically well defined operators on the completed kinematical Hilbert space. Consequently, the vanishing of the formal commutator below does not establish a unitary Lorentz action on that space.

For $n > 0$, equation (5.2) gives

$$\begin{aligned} [L_n^X, J_{\text{osc}}^{\mu\nu}] &= -i \sum_{t>0} \frac{1}{t} \left([L_n^X, \alpha_{-t}^\mu] \alpha_t^\nu + \alpha_{-t}^\mu [L_n^X, \alpha_t^\nu] - [L_n^X, \alpha_{-t}^\nu] \alpha_t^\mu - \alpha_{-t}^\nu [L_n^X, \alpha_t^\mu] \right) \\ &= -i \sum_{t>0} \left(\alpha_{n-t}^\mu \alpha_t^\nu - \alpha_{-t}^\mu \alpha_{n+t}^\nu - \alpha_{n-t}^\nu \alpha_t^\mu + \alpha_{-t}^\nu \alpha_{n+t}^\mu \right). \end{aligned}$$

Reindexing the second and fourth terms by setting $s = n + t$, we obtain

$$\begin{aligned} [L_n^X, J_{\text{osc}}^{\mu\nu}] &= -i \left[\sum_{t>0} (\alpha_{n-t}^\mu \alpha_t^\nu - \alpha_{n-t}^\nu \alpha_t^\mu) - \sum_{s>n} (\alpha_{n-s}^\mu \alpha_s^\nu - \alpha_{n-s}^\nu \alpha_s^\mu) \right] \\ &= -i \sum_{t=1}^n (\alpha_{n-t}^\mu \alpha_t^\nu - \alpha_{n-t}^\nu \alpha_t^\mu). \end{aligned}$$

For $1 \leq t \leq n-1$, the terms cancel pairwise under $t \leftrightarrow n-t$. Only the boundary contribution $t = n$ remains

$$[L_n^X, J_{\text{osc}}^{\mu\nu}] = -i (\alpha_0^\mu \alpha_n^\nu - \alpha_0^\nu \alpha_n^\mu).$$

We now compute the zero mode contribution. Since

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu,$$

and

$$[x^\mu, p^\nu] = i\eta^{\mu\nu},$$

we find

$$[L_n^X, x^\mu] = -i\sqrt{2\alpha'} \alpha_n^\mu, \quad [L_n^X, p^\mu] = 0.$$

Therefore,

$$[L_n^X, J_0^{\mu\nu}] = [L_n^X, x^\mu] p^\nu - [L_n^X, x^\nu] p^\mu = -i\sqrt{2\alpha'} (\alpha_n^\mu p^\nu - \alpha_n^\nu p^\mu) = i (\alpha_0^\mu \alpha_n^\nu - \alpha_0^\nu \alpha_n^\mu).$$

The oscillator and zero mode contributions cancel

$$[L_n^X, J^{\mu\nu}] = 0, \quad n > 0.$$

The result for $n < 0$ follows formally by taking the adjoint, while the $n = 0$ case follows directly because $J_{\text{osc}}^{\mu\nu}$ has vanishing total signed level and $J_0^{\mu\nu}$ commutes with α_0^2 . Hence,

$$[L_n^X, J^{\mu\nu}] = 0, \quad n \in \mathbb{Z},$$

as a formal mode algebra identity.

This result means that the Virasoro generators remain Lorentz scalars at the algebraic level. It does not imply that the modified representation is Lorentz invariant. The obstruction found in Chapter 3 concerns the implementation of boosts as operators on the chosen Fock representation and its Hilbert completion.

5.4 Standard ghost sector and total anomaly

We now assume that the worldsheet gauge fixing is accompanied by the standard anticommuting Faddeev–Popov bc system reviewed in Chapter 2. This is an additional structural assumption, the modified matter representation does not by itself prove that the complete path integral and BRST constructions remain unchanged.

The nonvanishing anticommutation relation is

$$\{b_m, c_n\} = \delta_{m+n},$$

and the ghost Virasoro generators are

$$L_m^{\text{gh}} = \sum_{n \in \mathbb{Z}} (m - n) : b_{m+n} c_{-n} :_{\text{gh}}.$$

With the standard nonzero mode normal ordering, their algebra is

$$[L_m^{\text{gh}}, L_n^{\text{gh}}] = (m - n) L_{m+n}^{\text{gh}} + \frac{1}{12} (-26m^3 + 2m) \delta_{m+n}. \quad (5.8)$$

The coefficient of m^3 corresponds to the standard ghost central charge

$$c_{\text{gh}} = -26.$$

The coefficient of the term linear in m depends on the additive normal ordering convention for L_0^{gh} .

Define the total shifted generators by

$$\mathcal{L}_m = L_m^X + L_m^{\text{gh}} + a\delta_m.$$

Using equations (5.7) and (5.8), we obtain

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n) \mathcal{L}_{m+n} + A(m) \delta_{m+n},$$

where

$$\begin{aligned} A(m) &= \frac{D-2}{12} (m^3 - m) + \frac{1}{12} (-26m^3 + 2m) - 2am \\ &= \frac{D-28}{12} m^3 + \frac{4-D-24a}{12} m. \end{aligned}$$

The vanishing of the cubic term requires

$$D = 28.$$

For this value of D , the vanishing of the linear term requires

$$a = -1.$$

Thus, under the assumption that the standard bc ghost system is retained, cancellation of the total Virasoro anomaly occurs for

$$D = 28, \quad a = -1.$$

We may verify the value of the intercept directly from the normal ordering constants. For the nonzero ghost modes, take

$$b_n |0\rangle_{\text{gh}} = c_n |0\rangle_{\text{gh}} = 0, \quad n > 0.$$

The treatment of the ghost zero modes does not affect the vacuum energy calculation below.

Starting from

$$L_{0,\text{unord}}^{\text{gh}} = \sum_{n \in \mathbb{Z}} (-n) b_n c_{-n},$$

we split the positive and negative modes

$$\begin{aligned} L_{0,\text{unord}}^{\text{gh}} &= \sum_{n>0} (-n b_n c_{-n} + n b_{-n} c_n) = \sum_{n>0} [-n(1 - c_{-n} b_n) + n b_{-n} c_n] \\ &= - \sum_{n>0} n + \sum_{n>0} n (c_{-n} b_n + b_{-n} c_n). \end{aligned}$$

Using

$$- \sum_{n>0} n = \frac{1}{12},$$

we find

$$L_{0,\text{unord}}^{\text{gh}} = L_0^{\text{gh}} + a^{\text{gh}},$$

with

$$a^{\text{gh}} = \frac{1}{12}.$$

The total vacuum energy shift is therefore

$$\begin{aligned} a &= a^X + a^{\text{gh}} \\ &= -\frac{D-2}{24} + \frac{1}{12} \\ &= \frac{4-D}{24}. \end{aligned}$$

For $D = 28$,

$$a = -1,$$

in agreement with the direct cancellation of the Virasoro anomaly.

At the level of the formal mode algebra, the same conditions cancel the anomalous term entering the square of the conventional BRST charge. They therefore provide the natural candidate values for a nilpotent BRST construction

$$D = 28, \quad a = -1.$$

This does not constitute a complete proof of BRST consistency. One must still define the BRST charge on an appropriate matter ghost domain, analyze its closure and determine its cohomology. In particular, anomaly cancellation alone does not establish the nontriviality of the physical state space or the implementation of target space Lorentz symmetry.

A more general first order system with arbitrary conformal weights is discussed in Appendix A. Such systems are not alternative normal ordering prescriptions for the standard Faddeev–Popov ghosts. Changing their conformal weights changes the worldsheet conformal field theory itself.

In the following chapter, unless explicitly stated otherwise, L_n will denote the matter generator L_n^X . The total generators including the ghost sector will continue to be denoted by $\mathcal{L}_n$.

Chapter 6

The Physical State Space

We now investigate the states selected by the matter Virasoro constraints.

As announced at the end of Chapter 5, throughout this chapter we write

$$L_n \equiv L_n^X$$

for the matter Virasoro generators. The total matter ghost generators introduced in the previous chapter will instead be denoted by $\mathcal{L}_n$.

6.1 Matter physical state conditions and positivity

On the algebraic Fock space, the matter physical state conditions are well defined without any further operator domain assumptions. We therefore introduce the algebraic constrained space

$$\mathcal{F}_{\text{phys,alg}}^X(p) := \left\{ |\psi\rangle \in \mathcal{F}_{\text{alg}}(p) \mid L_m |\psi\rangle = 0 \text{ for every } m > 0, (L_0 + a^X) |\psi\rangle = 0 \right\}. \quad (6.1)$$

Since $\mathcal{F}_{\text{alg}}(p)$ is contained in the positive definite kinematical Hilbert space,

$$\mathcal{F}_{\text{alg}}(p) \subset \mathcal{H}_{\text{kin}}(p),$$

every nonzero vector satisfying equation (6.1) would have strictly positive norm

$$\langle \psi | \psi \rangle > 0 \quad \text{for } |\psi\rangle \in \mathcal{F}_{\text{phys,alg}}^X(p), \quad |\psi\rangle \neq 0.$$

Thus, unlike the standard covariant matter representation, the modified algebraic matter space contains neither negative norm states nor nonzero null vectors. Positivity alone, however, does not imply that the Virasoro constraints admit any nonzero solution.

The statement above concerns only the matter inner product. A full BRST construction would also contain the ghost state space and would define physical states cohomologically rather than simply as a constrained subspace.

6.2 Structure of the physical state constraints

Let $|\psi_\nu\rangle$ be an eigenstate of the signed oscillator level

$$N |\psi_\nu\rangle = \nu |\psi_\nu\rangle.$$

The zero mode condition in equation (6.1) becomes

$$(\alpha' p^2 + \nu + a^X) |\psi_\nu\rangle = 0.$$

Equivalently,

$$\alpha' M^2 = \nu + a^X.$$

Consider first the oscillator vacuum. Since $N|0, p\rangle = 0$, its matter mass shell condition would give

$$\alpha' M^2 = a^X = -\frac{D-2}{24}.$$

For $D > 2$, this is a tachyonic mass. The mass shell condition alone, however, does not imply that the vacuum is physical. One must also verify all positive mode Virasoro constraints.

In the standard Fock representation, the relation

$$[N, L_m] = -mL_m$$

implies

$$N(L_m|\psi_\nu\rangle) = (\nu - m)L_m|\psi_\nu\rangle.$$

Since the conventional level operator is nonnegative, one immediately finds

$$L_m|\psi_\nu\rangle = 0 \quad \text{for } m > \nu.$$

The same argument cannot be applied in the modified representation. As shown in Chapter 3, the signed level operator N is not bounded from below. Consequently, $L_m|\psi_\nu\rangle$ may be a nonzero state of level $\nu - m$, even when $\nu - m < 0$.

To analyse the constraints, we instead use the nonnegative K grading introduced in Chapter 3. The first few K graded subspaces contain states of the form

$$|\psi_0\rangle = \beta|0, p\rangle,$$

$$|\psi_1\rangle = (\beta_i \alpha_{-1}^i + \lambda \alpha_1^0)|0, p\rangle,$$

and

$$|\psi_2\rangle = \left(\tau_i \alpha_{-2}^i + \varepsilon_{ij} \alpha_{-1}^i \alpha_{-1}^j + \chi \alpha_2^0 + \epsilon (\alpha_1^0)^2 + \rho_i \alpha_{-1}^i \alpha_1^0 \right) |0, p\rangle,$$

where ε_{ij} may be taken symmetric.

We decompose the matter Virasoro generator into spatial and timelike contributions

$$L_m = L_m^{\text{sp}} + L_m^0,$$

where

$$L_m^{\text{sp}} = \frac{1}{2} \sum_{t \in \mathbb{Z}} \sum_{i=1}^{D-1} : \alpha_{m-t}^i \alpha_t^i :,$$

and

$$L_m^0 = -\frac{1}{2} \sum_{t \in \mathbb{Z}} : \alpha_{m-t}^0 \alpha_t^0 :.$$

Their commutators with K are

$$[K, L_m^{\text{sp}}] = -mL_m^{\text{sp}}, \quad [K, L_m^0] = mL_m^0.$$

Therefore, L_m^{sp} lowers the K grading by m , while L_m^0 raises it by m .

Let

$$|\psi_\nu\rangle = \sum_{k \geq 0} |\psi_{\nu, k}\rangle,$$

where

$$N |\psi_{\nu,k}\rangle = \nu |\psi_{\nu,k}\rangle, \quad K |\psi_{\nu,k}\rangle = k |\psi_{\nu,k}\rangle.$$

We set

$$|\psi_{\nu,k}\rangle = 0 \quad \text{for } k < 0.$$

The condition

$$L_m |\psi_{\nu}\rangle = 0$$

must hold after summing over all K components. Extracting the component with K eigenvalue k gives

$$L_m^{\text{sp}} |\psi_{\nu,k+m}\rangle + L_m^0 |\psi_{\nu,k-m}\rangle = 0, \quad k \geq 0, \quad m > 0. \quad (6.2)$$

Thus, the physical state conditions form a coupled recursive system relating different K components. This is the principal difference from the conventional positive energy representation.

6.2.1 Absence of finite K support states

Suppose that $|\psi_{\nu}\rangle$ has finite support in the K grading

$$|\psi_{\nu}\rangle = \sum_{k=0}^{k_M} |\psi_{\nu,k}\rangle,$$

where

$$|\psi_{\nu,k_M}\rangle \neq 0.$$

Consider equation (6.2) at output grading $k = k_M + m$. We obtain

$$L_m^{\text{sp}} |\psi_{\nu,k_M+2m}\rangle + L_m^0 |\psi_{\nu,k_M}\rangle = 0.$$

The first term vanishes because k_M is the maximal grading appearing in the state. Therefore,

$$L_m^0 |\psi_{\nu,k_M}\rangle = 0, \quad m > 0. \quad (6.3)$$

We now show directly, using the mode $m = 2$, that equation (6.3) has no nonzero solution in a fixed finite K subspace.

Purely spatial component

Consider first a top component containing only spatial creation operators

$$|\psi_{\nu,k_M}\rangle = P^{\text{sp}} |0, p\rangle,$$

where P^{sp} is a finite linear combination of spatial oscillator monomials of total K grading k_M .

Since L_m^0 commutes with every spatial oscillator,

$$L_m^0 P^{\text{sp}} |0, p\rangle = P^{\text{sp}} L_m^0 |0, p\rangle.$$

For $m = 2$, the timelike Virasoro generator acts on the vacuum as

$$L_2^0 |0, p\rangle = - \left(\alpha_0^0 \alpha_2^0 + \frac{1}{2} (\alpha_1^0)^2 \right) |0, p\rangle.$$

Here

$$\alpha_0^0 = \sqrt{2\alpha'} p^0.$$

The two temporal states

$$\alpha_2^0 |0, p\rangle, \quad (\alpha_1^0)^2 |0, p\rangle$$

are linearly independent. In particular, the second term cannot be removed by any choice of p^0 . Hence

$$L_2^0 |0, p\rangle \neq 0.$$

Equation (6.3) therefore implies

$$P^{\text{sp}} = 0.$$

No nonzero purely spatial top component is allowed.

Purely timelike component

Consider now a purely timelike top component

$$|\psi_{\nu, k_M}\rangle = P^0 |0, p\rangle,$$

where P^0 is a polynomial in the timelike creation operators α_n^0 , with $n > 0$.

For $m > 0$, we split the timelike Virasoro generator as

$$L_m^0 = L_m^{0,+} + L_m^{0,-},$$

where the part containing only creation operators and zero modes is

$$L_m^{0,+} = -\frac{1}{2} \sum_{t=0}^m \alpha_{m-t}^0 \alpha_t^0,$$

while the part containing one timelike annihilation operator is

$$L_m^{0,-} = -\frac{1}{2} \left(\sum_{t<0} \alpha_{m-t}^0 \alpha_t^0 + \sum_{t>m} \alpha_t^0 \alpha_{m-t}^0 \right).$$

Equivalently,

$$L_m^{0,-} = -\sum_{r>0} \alpha_{m+r}^0 \alpha_{-r}^0.$$

Since

$$L_m^{0,-} |0, p\rangle = 0,$$

we obtain

$$L_m^0 P^0 |0, p\rangle = ([L_m^{0,-}, P^0] + P^0 L_m^{0,+}) |0, p\rangle.$$

Decompose P^0 according to the unweighted number of timelike creation operators

$$P^0 = P_0^0 + P_1^0 + \cdots + P_d^0, \quad P_d^0 \neq 0, \quad (6.4)$$

where P_ℓ^0 is homogeneous of degree ℓ in the operators α_n^0 , with $n > 0$.

The commutator

$$[L_m^{0,-}, P_d^0] |0, p\rangle$$

contains d timelike creation operators, the annihilation operator in $L_m^{0,-}$ removes one creator from P_d^0 , while the remaining positive mode introduces one creator.

The zero mode part of

$$P_d^0 L_m^{0,+} |0, p\rangle$$

contains $d + 1$ timelike creation operators. The strictly positive part of $L_m^{0,+}$ contains two creation operators and therefore produces terms of degree $d + 2$.

For $m = 2$,

$$L_2^{0,+} = -\alpha_0^0 \alpha_2^0 - \frac{1}{2} (\alpha_1^0)^2.$$

The unique contribution containing $d + 2$ timelike creation operators is therefore

$$-\frac{1}{2} (\alpha_1^0)^2 P_d^0 |0, p\rangle.$$

No other term in $L_2^0 P^0 |0, p\rangle$ has the same degree. So this component must vanish independently

$$(\alpha_1^0)^2 P_d^0 |0, p\rangle = 0.$$

Multiplication by $(\alpha_1^0)^2$ is injective on the algebraic timelike Fock space. Consequently,

$$P_d^0 = 0,$$

contradicting equation (6.4). Thus, no nonzero purely timelike top component satisfies equation (6.3).

Mixed component

A general mixed top component can be written as

$$|\psi_{\nu, k_M}\rangle = \sum_A S_A^{\text{sp}} P_A^0 |0, p\rangle,$$

where the S_A^{sp} are chosen to be linearly independent spatial oscillator monomials and the P_A^0 are timelike polynomials.

Since L_m^0 commutes with all spatial oscillators,

$$L_m^0 |\psi_{\nu, k_M}\rangle = \sum_A S_A^{\text{sp}} L_m^0 P_A^0 |0, p\rangle.$$

The linear independence of the spatial monomials implies that

$$L_m^0 P_A^0 |0, p\rangle = 0$$

for every A . The preceding timelike argument then gives

$$P_A^0 = 0$$

for every A . Therefore,

$$|\psi_{\nu, k_M}\rangle = 0,$$

contradicting the assumption that k_M was the maximal nonvanishing K component.

The argument has used only the single constraint

$$L_2 |\psi_{\nu}\rangle = 0.$$

Since the algebraic Fock space is the algebraic direct sum of its N eigenspaces and L_2 lowers the signed level by two, the same statement holds on the complete algebraic core

$$\ker \left(L_2|_{\mathcal{F}_{\text{alg}}(p)} \right) = \{0\}. \quad (6.5)$$

In particular, no nonzero state satisfying all positive mode Virasoro constraints can have finite support in the K grading

$$L_m |\psi\rangle = 0 \quad \text{for all } m > 0, \quad |\psi\rangle \neq 0 \quad \implies \quad |\psi\rangle \text{ has infinite } K \text{ support}.$$

Since every fixed K subspace is finite dimensional, this excludes in particular every nonzero finite linear combination of occupation number basis states.

6.2.2 The stronger algebraic obstruction

The role played by L_2 in the preceding argument is not special. For every integer $m \geq 2$, the individual positive mode generator has trivial kernel on the algebraic Fock space

$$\ker \left(L_m |_{\mathcal{F}_{\text{alg}}(p)} \right) = \{0\}, \quad m \geq 2. \quad (6.6)$$

The proof is given in Appendix C. It follows the same general idea as the explicit $m = 2$ argument but replaces the factor $(\alpha_1^0)^2$ by the nonzero quadratic polynomial

$$\frac{1}{2} \sum_{r=1}^{m-1} \alpha_{m-r}^0 \alpha_r^0.$$

This quadratic creation term is sufficient to prove that the timelike part L_m^0 is injective on the algebraic timelike Fock space for every $m \geq 2$.

Equation (6.6) is stronger than what is required to exclude algebraic physical states. It states that an algebraic vector cannot be annihilated even by a single generator L_m with $m \geq 2$.

Consequently,

$$\mathcal{F}_{\text{phys,alg}}^X(p) = \{0\}.$$

6.2.3 Passage to infinite K support

The algebraic obstruction does not exclude states in the Hilbert space completion. A nonzero constrained vector, if it exists, must have the form

$$|\psi_\nu\rangle = \sum_{k \geq 0} |\psi_{\nu,k}\rangle,$$

with infinitely many nonzero K components.

The positivity result of Chapter 4 shows that the K decomposition is orthogonal. Consequently,

$$\| |\psi_\nu\rangle \|^2 = \sum_{k \geq 0} \| |\psi_{\nu,k}\rangle \|^2. \quad (6.7)$$

A formal infinite support solution defines a genuine state in the kinematical Hilbert space only if

$$\sum_{k \geq 0} \| |\psi_{\nu,k}\rangle \|^2 < \infty.$$

This requirement is not specific to the modified quantization, it is part of the definition of a Hilbert space state. What is new is that, at fixed signed level $N = \nu$, infinitely many K components are available. Normalisability is therefore no longer automatic.

Appendix B gives an explicit formal state with fixed N and infinite K support that satisfies all positive mode Virasoro constraints but violates equation (6.7). This demonstrates that the constraints can be solved algebraically without producing a normalisable Hilbert space state.

The completed constraint problem will be studied in the next chapter.

Chapter 7

The Completed Virasoro Constraint Problem

In the previous chapter, we proved that no nonzero vector in the algebraic Fock space satisfies the positive mode Virasoro constraints. Any possible physical state in the Hilbert space completion must therefore have infinite support in the nonnegative K grading.

In this chapter, we complete the analysis of the matter physical state space. We first reformulate the Virasoro constraints in terms of Hilbert–Schmidt operators and then show that no nonzero normalisable solution can satisfy both the positive mode constraints and the zero mode condition.

7.1 Spatial and timelike factorisation

At fixed momentum, the spatial and timelike oscillator sectors are independent, and the kinematical Hilbert space factorises as

$$\mathcal{H}_{\text{kin}}(p) \simeq \mathcal{H}_{\text{sp}}(p^{\text{sp}}) \otimes \mathcal{H}^{(0)}(p^0).$$

The spatial factor is the usual bosonic Fock space generated by the spatial oscillators. The timelike factor is also positive, but its creation operators are the positive modes α_n^0 .

It is convenient to rewrite the timelike oscillator algebra by defining

$$\beta_n = \alpha_{-n}^0.$$

The β_n satisfy the standard oscillator algebra

$$[\beta_m, \beta_n] = m\delta_{m+n},$$

and the vacuum satisfies $\beta_n |0, p\rangle = 0$ for $n > 0$.

We also denote the nonnegative timelike level by

$$N_0^+ = -N^0 = \sum_{n>0} \alpha_n^0 \alpha_{-n}^0.$$

Thus

$$N = N^{\text{sp}} - N_0^+.$$

Let $\ell_m^{(0)}$ denote the standard one boson Virasoro generators constructed from the β oscillators. The timelike part of the modified Virasoro generator can then be written as

$$L_m^0 = -\ell_{-m}^{(0)}.$$

Hence, on the algebraic tensor product core, the complete matter generator takes the form

$$L_m = L_m^{\text{sp}} \otimes I - I \otimes \ell_{-m}^{(0)}.$$

This form separates the two positive Fock representations and makes the completed constraint problem particularly transparent.

To discuss the constraints in the Hilbert space completion, we use the natural maximal closed realisation of the positive Virasoro modes. Writing

$$|\psi\rangle = \sum_{k \geq 0} |\psi_k\rangle, \quad |\psi_k\rangle \in \mathcal{H}_k(p),$$

we define the domain of L_m , for $m > 0$, by the condition

$$\sum_{k \geq 0} |L_m^{\text{sp}} |\psi_{k+m}\rangle + L_m^0 |\psi_{k-m}\rangle|^2 < \infty,$$

where $|\psi_k\rangle = 0$ for $k < 0$. On this domain,

$$L_m |\psi\rangle = \sum_{k \geq 0} (L_m^{\text{sp}} |\psi_{k+m}\rangle + L_m^0 |\psi_{k-m}\rangle).$$

This extends the algebraic Virasoro generator introduced in the previous chapters and allows the infinite K support states to be treated as genuine Hilbert space vectors.

7.2 Hilbert–Schmidt formulation

Choose normalised real occupation number bases in $\mathcal{H}_{\text{sp}}$ and $\mathcal{H}^{(0)}$. With this choice, a vector in

$$\mathcal{H}_{\text{sp}} \otimes \mathcal{H}^{(0)}$$

can be identified with an operator

$$T : \mathcal{H}^{(0)} \longrightarrow \mathcal{H}_{\text{sp}}.$$

Normalisability of the vector is equivalent to the requirement that T be Hilbert–Schmidt. In particular,

$$||\psi\rangle|^2 = |T|_{\text{HS}}^2.$$

Under this correspondence, the positive mode Virasoro constraint

$$L_m |\psi\rangle = 0$$

becomes the intertwining equation

$$L_m^{\text{sp}} T v = T \ell_m^{(0)} v, \quad m > 0,$$

for vectors v in the algebraic timelike Fock space. In particular, Tv must belong to the domain of the corresponding spatial Virasoro generator.

The operator T must therefore intertwine the positive Virasoro actions of the timelike and spatial oscillator sectors.

The zero mode condition imposes in addition

$$N^{\text{sp}} T - T N_0^+ = \nu(p) T, \quad \nu(p) = -a^X - \alpha' p^2.$$

Since both N^{sp} and N_0^+ have nonnegative integer spectra, a nonzero solution can exist only if $\nu(p)$ is an integer. When this is the case, the zero mode condition implies

$$T \mathcal{H}_q^{(0)} \subset \mathcal{H}_{q+\nu(p)}^{\text{sp}},$$

where a spatial level with negative index is understood to be zero.

Thus, the zero mode condition fixes the difference between the spatial and timelike oscillator levels. The completed physical state problem is reduced to the existence of a Hilbert–Schmidt operator of fixed degree satisfying all positive mode intertwining equations.

We define the completed matter physical state space by

$$\mathcal{H}_{\text{phys}}^X(p) = \ker(L_0 + a^X) \cap \bigcap_{m \geq 1} \ker L_m. \quad (7.1)$$

We now show that this space is trivial.

7.3 Triviality of the completed matter physical space

Suppose that a nonzero state

$$|\psi\rangle \in \mathcal{H}_{\text{phys}}^X(p)$$

exists, and let T be the corresponding Hilbert–Schmidt operator. Since every Hilbert–Schmidt operator is compact, T is in particular a compact operator.

The zero mode condition implies that T has fixed degree. If $\nu(p)$ is not an integer, this already forces $T = 0$. We may therefore assume

$$\nu(p) \in \mathbb{Z}.$$

Since T is assumed to be nonzero, there is a smallest timelike oscillator level q_0 on which it does not vanish,

$$q_0 = \min \left\{ q \geq 0 : T|_{\mathcal{H}_q^{(0)}} \neq 0 \right\}.$$

Choose a unit vector

$$v \in \mathcal{H}_{q_0}^{(0)}$$

such that

$$w = Tv \neq 0.$$

For every $m > 0$, the intertwining equation gives

$$L_m^{\text{sp}} w = T \ell_m^{(0)} v.$$

The vector $\ell_m^{(0)} v$ belongs to the timelike level $q_0 - m$. By the minimality of q_0 , T vanishes on every lower level. Consequently,

$$L_m^{\text{sp}} w = 0, \quad m > 0.$$

Thus w is annihilated by every positive spatial Virasoro mode.

Since v and w belong to fixed oscillator levels, they are eigenvectors of the corresponding zero modes. We write

$$\ell_0^{(0)} v = h_0 v, \quad L_0^{\text{sp}} w = h_{\text{sp}} w.$$

Consider now an integer

$$n > \max\{q_0, 1\}.$$

Since $\ell_n^{(0)}$ lowers the timelike oscillator level by n ,

$$\ell_n^{(0)} v = 0.$$

The timelike β oscillators form a standard one boson Virasoro representation with central charge one. Therefore,

$$\left| \ell_{-n}^{(0)} v \right|^2 = 2nh_0 + \frac{n^3 - n}{12}.$$

For convenience, define

$$\lambda_n = 2nh_0 + \frac{n^3 - n}{12}.$$

For sufficiently large n , one has $\lambda_n > 0$.

The spatial sector contains $D - 1$ standard bosonic oscillator families and therefore has central charge $D - 1$. Since $L_n^{\text{sp}} w = 0$, the same Virasoro commutation relation gives

$$|L_{-n}^{\text{sp}} w|^2 = \left(2nh_{\text{sp}} + \frac{D-1}{12}(n^3 - n) \right) |w|^2.$$

We define

$$\mu_n = 2nh_{\text{sp}} + \frac{D-1}{12}(n^3 - n).$$

We now apply the positive mode intertwining equation to the algebraic vector $\ell_{-n}^{(0)} v$. Since $\ell_n^{(0)} v = 0$,

$$L_n^{\text{sp}} T \ell_{-n}^{(0)} v = T \ell_n^{(0)} \ell_{-n}^{(0)} v = T [\ell_n^{(0)}, \ell_{-n}^{(0)}] v = \lambda_n w.$$

Taking the inner product with w and using the adjoint relation gives

$$\lambda_n |w|^2 = \langle L_{-n}^{\text{sp}} w, T \ell_{-n}^{(0)} v \rangle.$$

The Cauchy–Schwarz inequality therefore implies

$$\lambda_n |w|^2 \leq \sqrt{\mu_n} |w| \left| T \ell_{-n}^{(0)} v \right|.$$

Define the normalised vectors

$$e_n = \frac{1}{\sqrt{\lambda_n}} \ell_{-n}^{(0)} v.$$

They belong to the mutually orthogonal timelike levels $\mathcal{H}_{q_0+n}^{(0)}$ and satisfy

$$|e_n| = 1.$$

Hence they form an orthonormal sequence.

The preceding inequality gives

$$|Te_n| \geq \sqrt{\frac{\lambda_n}{\mu_n}} |w|.$$

For large n , the cubic terms dominate both λ_n and μ_n , and therefore

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\mu_n} = \frac{1}{D-1}.$$

It follows that

$$\liminf_{n \rightarrow \infty} |Te_n| \geq \frac{|w|}{\sqrt{D-1}} > 0.$$

This contradicts compactness. Indeed, a compact operator maps an orthonormal sequence to a sequence whose norm tends to zero. Since every Hilbert–Schmidt operator is compact, the contradiction shows that no nonzero Hilbert–Schmidt operator can satisfy simultaneously the positive mode intertwining equations and the zero mode condition.

We have therefore proved

$$\boxed{\mathcal{H}_{\text{phys}}^X(p) = \{0\}} \quad D \geq 2. \quad (7.2)$$

The result holds for every real momentum p . The zero mode condition determines the degree that a possible intertwiner would have, but no Hilbert–Schmidt intertwiner of such fixed degree can satisfy all the positive mode Virasoro constraints.

The formal infinite support state discussed in Appendix B does not contradict this result. It solves the positive mode constraints only as a formal infinite sum and fails to belong to the kinematical Hilbert space. In the operator formulation, it corresponds to a noncompact intertwiner rather than to a Hilbert–Schmidt operator.

The obstruction found in Chapter 6 therefore persists after Hilbert space completion. The absence of algebraic physical states is not replaced by a normalisable infinite support solution: the completed matter physical state space itself is trivial.

Chapter 8

Conclusions

This thesis studied a modified Fock representation of the bosonic string obtained by reversing the usual choice of creation and annihilation operators in the timelike sector. The oscillator algebra and the classical theory were left unchanged. For $n > 0$, the vacuum conditions were taken to be

$$\alpha_n^i |0, p\rangle = 0, \quad \alpha_{-n}^0 |0, p\rangle = 0.$$

The timelike creation operators are therefore the positive modes α_n^0 .

With this choice, the negative sign in the timelike commutator no longer produces negative norm excitations. The occupation number states are mutually orthogonal and have strictly positive norm. Consequently, the algebraic matter Fock space is positive definite and admits a genuine Hilbert space completion. In particular, there are no negative norm states or nonzero null vectors in the kinematical matter sector.

The modified polarisation is preserved by spatial rotations but not by Lorentz boosts. The formal boost generator produces an infinite sum of mutually orthogonal states when acting on the vacuum, and the corresponding finite mode truncations do not converge in norm. Moreover, finite boosts act as Bogoliubov transformations whose off diagonal part is not Hilbert–Schmidt. The Shale–Stinespring criterion therefore shows that nontrivial boosts are not unitarily implementable in this representation.

The change of polarisation also modifies the normal ordering of the timelike oscillators. Recomputing the matter Virasoro algebra gives

$$[L_m^X, L_n^X] = (m - n)L_{m+n}^X + \frac{D - 2}{12} (m^3 - m) \delta_{m+n},$$

with

$$c_m = D - 2, \quad a^X = -\frac{D - 2}{24}.$$

The reversed timelike oscillator contributes with the opposite sign to the central extension. If the standard Faddeev–Popov ghost system is retained, formal cancellation of the total anomaly occurs for

$$D = 28, \quad a = -1.$$

These values concern the formal mode algebra. They do not establish the existence of a local worldsheet conformal field theory or a complete BRST construction.

The main obstruction appears in the Virasoro constraints. The signed level operator is not bounded from below, because timelike creation operators lower its eigenvalue. At fixed signed level there are therefore infinitely many oscillator configurations. To organise them, the nonnegative grading

$$K = N^{\text{sp}} - N^0$$

was introduced. Its eigenspaces are finite dimensional, while the spatial and timelike parts of L_m^X shift K in opposite directions.

This grading makes it possible to exclude all algebraic physical states. If a state has finite K support, its highest component must be annihilated by the timelike operator L_m^0 . For $m = 2$, the highest polynomial degree term shows directly that this cannot happen for a nonzero vector. The argument extends to every $m \geq 2$ and gives

$$\ker \left(L_m^X|_{\mathcal{F}_{\text{alg}}(p)} \right) = \{0\}, \quad m \geq 2.$$

Thus, no nonzero finite linear combination of occupation number states satisfies the positive mode Virasoro constraints.

A possible solution in the Hilbert space completion would necessarily have infinitely many nonzero K components. The formal state constructed in Appendix B shows that the positive mode constraints can indeed be solved outside the Hilbert space: it satisfies them as a formal oscillator identity, but its norm diverges. This example isolates the remaining analytic question, namely whether an infinite support solution can also satisfy the normalisability condition. Chapter 7 shows that this is impossible.

After separating the spatial and timelike oscillator sectors,

$$\mathcal{H}_{\text{kin}}(p) \simeq \mathcal{H}_{\text{sp}}(p^{\text{sp}}) \otimes \mathcal{H}^{(0)}(p^0),$$

a normalisable state can be identified with a Hilbert–Schmidt operator

$$T : \mathcal{H}^{(0)}(p^0) \longrightarrow \mathcal{H}_{\text{sp}}(p^{\text{sp}}).$$

Under this correspondence, the positive Virasoro constraints become the intertwining equations

$$L_m^{\text{sp}} T v = T \ell_m^{(0)} v, \quad m > 0,$$

while the zero mode condition fixes the level shift of T .

The zero mode condition fixes the degree of a possible intertwiner with respect to the spatial and timelike oscillator levels. Suppose that a nonzero Hilbert–Schmidt intertwiner exists and consider the lowest timelike level on which it does not vanish. Its image contains a nonzero vector annihilated by all positive spatial Virasoro modes.

The Virasoro descendants of this vector can then be used to construct an orthonormal sequence in the timelike Fock space whose images under T remain bounded away from zero in norm. This is incompatible with compactness, since every Hilbert–Schmidt operator is compact. Consequently, no nonzero normalisable state satisfies all the matter Virasoro constraints together with the zero mode condition.

Thus, for every real momentum p and every finite dimension $D \geq 2$,

$$\mathcal{H}_{\text{phys}}^X(p) = \{0\}.$$

The obstruction found on the algebraic Fock space therefore persists after Hilbert space completion. Allowing infinite K support does not produce a nonzero normalisable matter physical state.

The formal infinite support state of Appendix B does not contradict this conclusion. In the Hilbert–Schmidt formulation, it corresponds to a noncompact formal intertwiner and therefore lies precisely outside the class of normalisable states excluded by the argument above.

The modified representation is therefore positive at the kinematical level, but this positivity comes with two decisive obstructions. Lorentz boosts are not unitarily implementable, and the matter Virasoro constraints admit no nonzero normalisable physical state. In addition, the formal Virasoro central extension differs from that of the standard string representation.

The construction therefore shows that positivity of the kinematical Fock space, by itself, is not sufficient to obtain a consistent string quantisation. In the representation studied here, removing negative norm states at the kinematical level comes at the price of losing unitary implementability of target space boosts and of obtaining a trivial matter physical state space.

Appendix A

General First Order Ghost Systems

In this appendix, we briefly consider a general first order conformal system. The purpose is to clarify whether a ghost central charge different from -26 can be obtained by changing the conformal weights of the fields.

This construction should not be confused with a different normal ordering prescription for the Faddeev–Popov ghosts of the Polyakov string. The conformal weights $(2, -1)$ of the standard bc system are fixed by the gauge fixing problem. Changing these weights defines a different worldsheet conformal field theory [15] [13] [16].

Consider two fields $\tilde{b}(z)$ and $\tilde{c}(z)$ satisfying

$$\tilde{c}(z)\tilde{b}(w) \sim \frac{1}{z-w}, \quad \tilde{b}(z)\tilde{c}(w) \sim \frac{\varepsilon}{z-w},$$

where

$$\varepsilon = \begin{cases} +1, & \text{for an anticommuting system,} \\ -1, & \text{for a commuting system.} \end{cases}$$

Let the energy momentum tensor be

$$T(z) = (1 - \lambda) : (\partial \tilde{b}) \tilde{c} : - \lambda : \tilde{b} \partial \tilde{c} : .$$

Computing the OPE with the elementary fields gives

$$T(z)\tilde{b}(w) \sim \frac{\lambda \tilde{b}(w)}{(z-w)^2} + \frac{\partial \tilde{b}(w)}{z-w},$$

and

$$T(z)\tilde{c}(w) \sim \frac{(1 - \lambda) \tilde{c}(w)}{(z-w)^2} + \frac{\partial \tilde{c}(w)}{z-w}.$$

The conformal weights are therefore

$$h_{\tilde{b}} = \lambda, \quad h_{\tilde{c}} = 1 - \lambda.$$

To compute the central charge, define

$$A = : (\partial \tilde{b}) \tilde{c} :, \quad B = : \tilde{b} \partial \tilde{c} : .$$

The double contractions contributing to the fourth order pole are

$$A(z)A(w) \sim -\frac{\varepsilon}{(z-w)^4},$$

$$A(z)B(w) \sim -\frac{2\varepsilon}{(z-w)^4},$$

$$B(z)A(w) \sim -\frac{2\varepsilon}{(z-w)^4},$$

and

$$B(z)B(w) \sim -\frac{\varepsilon}{(z-w)^4}.$$

Since

$$T = (1 - \lambda)A - \lambda B,$$

the fourth order pole of $T(z)T(w)$ is

$$-\frac{\varepsilon [(1 - \lambda)^2 - 4\lambda(1 - \lambda) + \lambda^2]}{(z - w)^4}.$$

Equivalently,

$$T(z)T(w) \sim -\frac{\varepsilon (6\lambda^2 - 6\lambda + 1)}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w}.$$

Comparing with the standard Virasoro OPE,

$$T(z)T(w) \sim \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w},$$

we obtain

$$c = -2\varepsilon (6\lambda^2 - 6\lambda + 1).$$

For the standard anticommuting Faddeev–Popov system,

$$\varepsilon = 1, \quad \lambda = 2,$$

and therefore

$$c = -26.$$

One may formally ask which anticommuting first order systems have central charge -24 . Setting $\varepsilon = 1$ and imposing

$$-2 (6\lambda^2 - 6\lambda + 1) = -24,$$

gives

$$6\lambda^2 - 6\lambda - 11 = 0.$$

The two solutions are

$$\lambda_{\pm} = \frac{3 \pm 5\sqrt{3}}{6},$$

or numerically

$$\lambda_+ \simeq 1.943, \quad \lambda_- \simeq -0.943.$$

These systems have conformal weights

$$(h_{\tilde{b}}, h_{\tilde{c}}) = (\lambda_{\pm}, 1 - \lambda_{\pm}),$$

which differ from the Faddeev–Popov weights $(2, -1)$. Consequently, although a first order conformal system with central charge -24 can be defined algebraically, it cannot replace the standard Faddeev–Popov ghost sector without changing the gauge fixed worldsheet theory.

Appendix B

A Formal Infinite Support State

In this appendix, we give a simple explicit example of a infinite support solution of the positive Virasoro constraints. The purpose of the example is not to prove that the physical Hilbert space is nontrivial, but to illustrate why the obstruction is analytic rather than purely algebraic.

We restrict to the oscillator pair formed by one spatial direction, denoted by 1, and the timelike direction 0. The modified vacuum conventions are

$$\alpha_n^1 |0; p\rangle = 0 \quad (n > 0), \quad \alpha_n^0 |0; p\rangle = 0 \quad (n < 0).$$

We introduce normalized oscillators

$$a_n = \frac{\alpha_n^1}{\sqrt{n}}, \quad a_n^\dagger = \frac{\alpha_{-n}^1}{\sqrt{n}}, \quad n > 0,$$

for the spatial direction, and

$$b_n = \frac{\alpha_{-n}^0}{\sqrt{n}}, \quad b_n^\dagger = \frac{\alpha_n^0}{\sqrt{n}}, \quad n > 0,$$

for the timelike direction.

Consider the following formal state in the $N = 0$ sector

$$|B\rangle = \exp \left(\sum_{n>0} a_n^\dagger b_n^\dagger \right) |0; p\rangle = \exp \left(\sum_{n>0} \frac{1}{n} \alpha_{-n}^1 \alpha_n^0 \right) |0; p\rangle.$$

Each operator $\alpha_{-n}^1 \alpha_n^0$ increases the spatial and timelike occupation levels by the same amount. Therefore, it leaves N unchanged but increases K

$$\Delta N = n - n = 0, \quad \Delta K = n + n = 2n.$$

Thus, $|B\rangle$ has fixed $N = 0$ and infinite support in the K grading.

Let λ be a partition of M , and let $r_n(\lambda)$ be the multiplicity of the part n in the partition λ . Define the normalized state associated with λ by

$$|\lambda, \lambda\rangle = \prod_{n>0} \frac{(a_n^\dagger)^{r_n(\lambda)}}{\sqrt{r_n(\lambda)!}} \frac{(b_n^\dagger)^{r_n(\lambda)}}{\sqrt{r_n(\lambda)!}} |0; p\rangle.$$

Then

$$N |\lambda, \lambda\rangle = 0, \quad K |\lambda, \lambda\rangle = 2M |\lambda, \lambda\rangle.$$

The state $|B\rangle$ decomposes as

$$|B\rangle = \sum_{M \geq 0} |B_M\rangle, \quad |B_M\rangle = \sum_{\lambda \text{ partition of } M} |\lambda, \lambda\rangle.$$

We now check the positive Virasoro constraints. The state $|B\rangle$ satisfies

$$a_n |B\rangle = b_n^\dagger |B\rangle, \quad b_n |B\rangle = a_n^\dagger |B\rangle, \quad n > 0.$$

Equivalently, in terms of the original oscillators,

$$\alpha_n^1 |B\rangle = \alpha_n^0 |B\rangle, \quad \alpha_{-n}^1 |B\rangle = \alpha_{-n}^0 |B\rangle, \quad n > 0.$$

If we also choose the zero modes so that

$$p^1 = p^0,$$

then the same identification holds for $n = 0$. Therefore, the contribution of the pair $(0, 1)$ to the matter Virasoro generator cancels formally

$$\frac{1}{2} \sum_{t \in \mathbb{Z}} : (\alpha_{m-t}^1 \alpha_t^1 - \alpha_{m-t}^0 \alpha_t^0) : |B\rangle = 0, \quad m > 0.$$

Since the two oscillator indices in each term sum to $m > 0$, no contact commutator with total mode number zero is generated when the relations above are applied successively.

All other spatial directions are taken in their oscillator vacuum, so they are annihilated by the positive modes. Hence

$$L_m |B\rangle = 0, \quad m > 0,$$

as a formal identity.

The zero mode constraint can be imposed independently by choosing the momentum so that

$$\alpha' p^2 + a^X = 0.$$

This condition does not affect the oscillator norm computed below.

It remains to check whether $|B\rangle$ is a genuine Hilbert space state. Since the states $|\lambda, \lambda\rangle$ are orthonormal,

$$\| |B_M\rangle \|^2 = \sum_{\lambda \text{ partition of } M} 1 = p(M),$$

where $p(M)$ is the number of partitions of M . Therefore,

$$\| |B\rangle \|^2 = \sum_{M \geq 0} \| |B_M\rangle \|^2 = \sum_{M \geq 0} p(M) = \infty.$$

Thus, $|B\rangle$ is not an element of the Hilbert completion of the modified Fock space. It is only a infinite support solution of the physical state constraints.

This example shows that the Virasoro constraints can admit formal infinite support solutions which do not define states in the kinematical Hilbert space. Chapter 7 shows that no nonzero Hilbert—Schmidt solution exists. The present formal state therefore illustrates the noncompact, non Hilbert—Schmidt behaviour necessarily exhibited by a nonzero formal solution.

Appendix C

The General Algebraic Virasoro Obstruction

In Chapter 6, we gave a direct and detailed proof that the constraint

$$L_2 |\psi\rangle = 0$$

has no nonzero solution in the algebraic Fock space. The argument used the component with maximal K grading and the highest polynomial degree in the timelike creation operators.

The purpose of this appendix is to show that the same obstruction is present for every mode $m \geq 2$. More precisely, we prove

$$\ker \left(L_m |_{\mathcal{F}_{\text{alg}}(p)} \right) = \{0\}, \quad m \geq 2. \quad (\text{C.1})$$

The argument is entirely algebraic and concerns only $\mathcal{F}_{\text{alg}}(p)$.

C.1 The timelike Virasoro generator

Let $\mathcal{F}_{\text{alg}}^0(p)$ denote the algebraic timelike Fock space generated from the vacuum by the timelike creation operators

$$\alpha_n^0, \quad n > 0.$$

A generic vector in this space can be written as

$$|\chi\rangle = P^0 |0, p\rangle,$$

where P^0 is a polynomial in the operators $\alpha_1^0, \alpha_2^0, \dots$ involving only finitely many variables and finitely many monomials.

For $m > 0$, the timelike contribution to the Virasoro generator is

$$L_m^0 = -\frac{1}{2} \sum_{t \in \mathbb{Z}} : \alpha_{m-t}^0 \alpha_t^0 :.$$

For $m \geq 2$, it is useful to write this expression as

$$L_m^0 = -\alpha_0^0 \alpha_m^0 - \frac{1}{2} \sum_{r=1}^{m-1} \alpha_{m-r}^0 \alpha_r^0 - \sum_{r < 0} \alpha_{m-r}^0 \alpha_r^0, \quad (\text{C.2})$$

where

$$\alpha_0^0 = \sqrt{2\alpha'} p^0.$$

The three terms in equation (C.2) have different effects on the number of timelike creation operators. The first term contains one creation operator and therefore increases the polynomial degree by one. In the last term, α_r^0 with $r < 0$ is an annihilation operator, it removes one creation operator, while α_{m-r}^0 introduces another one. This term therefore preserves the polynomial degree.

The middle term is different. Define

$$Q_m := \frac{1}{2} \sum_{r=1}^{m-1} \alpha_{m-r}^0 \alpha_r^0. \quad (\text{C.3})$$

Every oscillator appearing in Q_m is a timelike creation operator. Consequently, Q_m increases the polynomial degree by two.

Now decompose P^0 according to its unweighted polynomial degree

$$P^0 = P_0^0 + P_1^0 + \cdots + P_d^0, \quad P_d^0 \neq 0, \quad (\text{C.4})$$

where P_ℓ^0 is homogeneous of degree ℓ .

When L_m^0 acts on $P^0 |0, p\rangle$, the only contribution with degree $d+2$ is

$$-Q_m P_d^0 |0, p\rangle. \quad (\text{C.5})$$

All other terms have degree at most $d+1$. Therefore, if $L_m^0 P^0 |0, p\rangle$ were zero, the vector in equation (C.5) would have to vanish independently.

We now show that this is impossible whenever $P_d^0 \neq 0$.

The linear independence of the timelike occupation number states allows us to identify the algebraic timelike Fock space with a polynomial algebra

$$\mathcal{F}_{\text{alg}}^0(p) \simeq \mathbb{C}[x_1, x_2, \dots].$$

Under this identification,

$$\alpha_n^0 \longleftrightarrow x_n,$$

and a polynomial P^0 corresponds to

$$P^0(\alpha_1^0, \alpha_2^0, \dots) |0, p\rangle.$$

The operator Q_m acts by multiplication by the polynomial

$$q_m(x) = \frac{1}{2} \sum_{r=1}^{m-1} x_{m-r} x_r. \quad (\text{C.6})$$

For every $m \geq 2$, this is a nonzero polynomial. Since a polynomial algebra has no zero divisors,

$$q_m(x) P_d^0(x) \neq 0$$

whenever

$$P_d^0(x) \neq 0.$$

Equivalently,

$$Q_m P_d^0 |0, p\rangle \neq 0.$$

It follows that

$$L_m^0 |\chi\rangle \neq 0 \quad \text{for every } |\chi\rangle \in \mathcal{F}_{\text{alg}}^0(p), \quad |\chi\rangle \neq 0, \quad m \geq 2. \quad (\text{C.7})$$

Thus, the timelike Virasoro generator L_m^0 is injective on the algebraic timelike Fock space for every $m \geq 2$, independently of the value of p^0 .

C.2 Extension to the complete algebraic Fock space

We now return to the complete matter Fock space. Suppose that

$$|\psi\rangle \in \mathcal{F}_{\text{alg}}(p), \quad |\psi\rangle \neq 0,$$

and assume, for contradiction, that

$$L_m |\psi\rangle = 0 \tag{C.8}$$

for some fixed $m \geq 2$.

Because $|\psi\rangle$ belongs to the algebraic Fock space, it has finite support in the K grading. We may therefore write

$$|\psi\rangle = \sum_{k=0}^{k_M} |\psi_k\rangle, \quad |\psi_{k_M}\rangle \neq 0.$$

Recall that

$$L_m = L_m^{\text{sp}} + L_m^0,$$

where L_m^{sp} lowers the K grading by m and L_m^0 raises it by m . The component of $L_m |\psi\rangle$ with output grading $k_M + m$ can therefore come only from the timelike action on the top component

$$L_m |\psi\rangle|_{K=k_M+m} = L_m^0 |\psi_{k_M}\rangle.$$

The spatial generator would have to act on a component with grading $k_M + 2m$, which is absent by the definition of k_M .

Equation (C.8) consequently implies

$$L_m^0 |\psi_{k_M}\rangle = 0. \tag{C.9}$$

Expand the top component using linearly independent spatial occupation number monomials

$$|\psi_{k_M}\rangle = \sum_A S_A^{\text{sp}} P_A^0 |0, p\rangle,$$

where the P_A^0 are timelike polynomials.

Since L_m^0 commutes with all spatial oscillators, equation (C.9) becomes

$$\sum_A S_A^{\text{sp}} L_m^0 P_A^0 |0, p\rangle = 0.$$

The spatial monomials S_A^{sp} are linearly independent. Therefore,

$$L_m^0 P_A^0 |0, p\rangle = 0$$

for every A .

The injectivity established in equation (C.7) then implies

$$P_A^0 = 0$$

for every A . Hence

$$|\psi_{k_M}\rangle = 0,$$

contradicting the choice of k_M .

We have therefore proved

$$L_m |\psi\rangle = 0, \quad |\psi\rangle \in \mathcal{F}_{\text{alg}}(p), \quad m \geq 2 \implies |\psi\rangle = 0.$$

Equivalently,

$$\ker \left(L_m|_{\mathcal{F}_{\text{alg}}(p)} \right) = \{0\}, \quad m \geq 2.$$

The detailed $m = 2$ argument in Chapter 6 makes the mechanism particularly transparent. The present proof shows that the obstruction is not specific to that mode, every generator L_m with $m \geq 2$ is already injective on the algebraic Fock space.

Bibliography

- [1] Suraj N. Gupta. Theory of Longitudinal Photons in Quantum Electrodynamics. *Proc. Phys. Soc. A*, 63:681–691, 1950.
- [2] Konrad Bleuler. Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen. *Helv. Phys. Acta*, 23:567–586, 1950.
- [3] P. Goddard and Charles B. Thorn. Compatibility of the Dual Pomeron with Unitarity and the Absence of Ghosts in the Dual Resonance Model. *Phys. Lett. B*, 40:235–238, 1972.
- [4] Richard C. Brower. Spectrum generating algebra and no ghost theorem for the dual model. *Phys. Rev. D*, 6:1655–1662, 1972.
- [5] Itzhak Bars. Relativistic Harmonic Oscillator Revisited. *Phys. Rev. D*, 79:045009, 2009.
- [6] David Shale. Linear symmetries of free boson fields. *Transactions of the American Mathematical Society*, 103(1):149–167, 1962.
- [7] S.N.M Ruijsenaars. On bogoliubov transformations. ii. the general case. *Annals of Physics*, 116(1):105–134, 1978.
- [8] Yoichiro Nambu. Duality and Hadrodynamics. In R. Chand, editor, *Proceedings of the International Conference on Symmetries and Quark Models*, pages 269–277, New York, 1970. Gordon and Breach.
- [9] Tetsuo Goto. Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model. *Prog. Theor. Phys.*, 46:1560–1569, 1971.
- [10] A.M. Polyakov. Quantum geometry of bosonic strings. *Physics Letters B*, 103(3):207–210, 1981.
- [11] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12 2007.
- [12] M. A. Virasoro. Subsidiary conditions and ghosts in dual resonance models. *Phys. Rev. D*, 1:2933–2936, 1970.
- [13] Daniel Friedan, Emil J. Martinec, and Stephen H. Shenker. Conformal invariance, supersymmetry and string theory. *Nucl. Phys. B*, 271:93–165, 1986.
- [14] Jan Dereziński and Christian Gérard. *Mathematics of Quantization and Quantum Fields*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2013.
- [15] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.
- [16] Mitsuhiro Kato and Kaku Ogawa. Covariant Quantization of String Based on BRS Invariance. *Nucl. Phys. B*, 212:443–460, 1983.