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p-ADIC QUANTUM UNIQUE ERGODICITY ON $\mathrm{GL}_2(\mathbb{Q}_P)$

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Chapter I

Introduction

The study of the concentration properties of automorphic forms is a beautiful subject at the crossroad of number theory, ergodic theory, geometry and representation theory. Among the many open lines of research, the one related to quantum unique ergodicity has received increasing attention, thanks to striking partial results.

The quantum unique ergodicity conjecture (QUE) of Rudnick-Sarnak was inspired by the quantum ergodicity (QE) result of Colin de Verdière, Snirelman, Zelditch and has its roots in the expected properties of physical quantum systems in the semiclassical limit [RS94]. In particular, it deals with the concentration properties of eigenfunctions of the Laplacian on negatively curved compact Riemannian spaces (X, g), in the high frequency limit. It states the following.

Conjecture I.0.1 (QUE, Rudnick Sarnak)**.** *Let (X, g) be a compact Riemannian manifold with negative curvature, with probability measure μ_g induced by the Riemannian metric. Let $\varphi_j \in C^\infty(X)$ be a sequence of L^2-normalized eigenfunctions of the laplacian Δ_g, with eigenvalue λ_j. Denote by $\mu_{\varphi_j}(E) = \int_E |\varphi_j|^2 d\mu_g$, the L^2-mass probability measure on X induced by φ_j. If $\lambda_j \to +\infty$ then μ_{φ_j} converge weak-$*$ to μ_g.*

The problem in this form is far from being solved and, moreover, doesn't involve any arithmetic argument. However, one of the most important partial results was obtained by Lindenstrauss in [Lin04], by restricting to suitable arithmetic functions on an arithmetic surface, proving the celebrated arithmetic quantum unique ergodicity result (AQUE). We briefly describe the setting.

Let $X = \Gamma \setminus \mathbb{H}$ be a compact arithmetic hyperbolic surface, with constant curvature -1, for $\Gamma = R^\times \subset \mathrm{SL}_2(\mathbb{R})$, the units of an Eichler order R of an indefinite quaternion algebra. The space X naturally comes with a commuting family of Hecke normal operators T_p acting on $C^\infty(X)$, for p prime, that commute with the laplacian Δ_X. In this context, the *Hecke-Mass eigenforms* $\varphi \in C^\infty(X)$ are eigenfunctions of Δ_X and of all the Hecke operators T_p. Then the result states the following.

Theorem I.0.2 (AQUE, Lindenstrauss)**.** *For any sequence of L^2-normalized Hecke-Maass eigenforms $(\varphi_j)_{j\in\mathbb{N}}$, with eigenvalue $\lambda_j \to \infty$, the sequence μ_{φ_j} converges weak-$*$ to $d\mu_g$.*

This result gave the input to other studies of quantum unique ergodicity of arithmetic functions, also on non-compact arithmetic quotients [HS10]. In this setting, another prominent research line deals with the properties of the automorphic forms in the limit in which the volume of the arithmetic hyperbolic surface X tends to infinity. In this context, we record the following theorem.

Theorem I.0.3 (AQUE, Nelson, Nelson-Pitale-Saha, Hu, [Nel11], [NPS13], [Hu18])**.** *Fix a prime p and a non-negative integer n_0. Let n traverse a sequence of natural numbers tending to ∞. Let ϕ be an L^2-normalized holomorphic Hecke newform on $\Gamma_0(p^n)$. Then the pushforward to $\Gamma_0(p^{n_0}) \setminus H$ of the L^2-mass of ϕ equidistributes.*

In this essay, we will study a p-adic analogue of this theorem, following Nelson's work [Nel18]. Now, we will present the setting of this problem in a special case.

Fix a prime number p and let $G = \mathrm{GL}(\mathbb{Q}_p)$. We consider a definite quaternion algebra B over $\mathbb{Q}$ that splits at p and define the compact space $X = \Gamma \setminus \mathrm{GL}_2(\mathbb{Q}_p)$, for $\Gamma = R[1/p]^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_p)$, for a maximal order R of B.

The space X is modeled on the cotangent bundle of a real arithmetic hyperbolic surface, associated to an indefinite quaternion algebra. As in the real case, it comes with Hecke correspondences T_l, for primes $l \neq p$, as defined in section VI.1. Assume now, that

$$\mathrm{Tors}\Gamma \subset \{\pm 1\}. \tag{I.1}$$

As in the real case, we may consider the minimal quotient $Y = X/K$, for the maximal open compact subgroup $K = \mathrm{GL}_2(\mathbb{Z}_p)$, and study the space of functions $\mathcal{F}(Y)$ which correspond to global automorphic forms. However, since X is compact and K is an open compact, Y is finite and discrete. Thus, $\mathcal{F}(Y)$ is not an interesting space for the quantum unique ergodicity problem in Lindenstrauss' formulation. So, we shall look at X with a finer resolution. First, notice that

$$Y \cong R[1/p]^\times[1/p]^\times \setminus G/\mathbb{Z}_p^\times \mathrm{GL}_2(\mathbb{Z}_p) \cong \overline{\Gamma} \setminus \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p),$$

for $\overline{\Gamma}$ the image of Γ in $\mathrm{PGL}_2(\mathbb{Q}_p)$. Under the assumption I.1, Y can be regarded as a $p+1$-regular undirected graph, quotient of the regular tree of $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ (see the discussion of section IV.3). Then, the analogues of $\Gamma_0(p^N)$ (see section VI.1), correspond to the *path spaces* of the graph associated to Y.

Definition I.0.4 (Path Space)**.** Let $m, m' \in \mathbb{Z}$, $m \geq m'$ and consider the set $m' \dots m = \{m', m+1, \dots, m\}$. The *path space* $Y_{m' \dots m}$ of level $N = m' - m$ is

$$Y_{m' \dots m} = \{\text{non-backtracking paths } x = (x_{m'} \to x_{m'+1} \to \dots \to x_m) \text{ of } Y\}.$$

For $m' \dots m \supseteq n' \dots n$, the projection $p : Y_{m \dots m'} \to Y_{n' \dots n}$ is the map obtained by forgetting the part of the path with indices not contained in $n' \dots n$. This map induces a projection $p^* : L^2(Y_{m' \dots m}) \to L^2(Y_{n' \dots n})$, for the probability counting measure on these finite spaces, by

$$f \mapsto \tilde{f} : x \mapsto \sum_{y \in p^{-1}(x)} f(y).$$

We are then interested in studying the asymptotic behavior of the eigenfunctions for $L^2(Y_{m' \dots m})$, defined as an analogue of the classical newforms attached to real modular surfaces.

Definition I.0.5 (Newvectors)**.** Let $\mathcal{F}_{m' \dots m} \subseteq L^2(Y_{m' \dots m})$ be an orthonormal basis for the space of functions $\phi : Y_{m' \dots m} \to \mathbb{C}$ satisfying the following conditions:

1. the pullback of ϕ to $X = \Gamma \setminus G$ generates an irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ under the right translation action.

2. ϕ is an eigenfunction of the Hecke operators T_l, for any prime $l \neq p$.

3. ϕ is orthogonal to pullbacks from $Y_{n\dots n'}$, whenever $n' \dots n \subsetneq m' \dots m$.

Focus now, for simplicity, on the symmetric intervals $-N \dots N$. Fix $n \in \mathbb{N}$ and consider $N \geq n$ as a parameter tending to $+\infty$. We can associate to any $\phi \in \mathcal{F}_{-N\dots N}$, a probability measure μ_ϕ on $Y_{-n\dots n}$ by

$$\mu_\phi(E) = \frac{1}{|Y_{-N\dots N}|} \sum_{\substack{x \in Y_{-N\dots N}: \\ p(x) \in E}} |\phi|^2(x),$$

which is the pushforward of the measure μ_ϕ, as in Theorem I.0.2, under the covering $p : Y_{-N\dots N} \to Y_{-n\dots n}$, as in Theorem I.0.2. In this essay, following [Nel18], we will prove an analogue of the arithmetic quantum unique ergodicity result for *principal series* for the path spaces of Y, that can be expressed as follows.

Theorem I.0.6 (Nelson). *Fix $n \in \mathbb{N}$. Let $N \geq n$ traverse a sequence of positive integers tending to ∞. For any sequence of newvectors $\phi_N \in \mathcal{F}_{-N\dots N}$, such that ϕ_N generates an irreducible principal series representation of G, any weak-* limit of the measures μ_ϕ converge to the uniform probability measure μ_n on $Y_{-n\dots n}$.*

The proof of this result will ultimately rely on a measure classification theorem used for the proof of Lindenstrauss' AQUE. In particular, we will need to introduce a p-adic version of a representation theoretic construction called *microlocal lift*, defined in section VI.4, for which a more general setting is needed. Finally, we stress that the statement clearly resembles Theorem I.0.3, adapted to a discrete graph theoretic version, and so opens a possible link with quantum unique ergodicity on large graphs [AM15].

We give a summary of the contents of the various chapters.

In the second chapter we present the classical and adelic theory of quaternion algebras following [Vig80]. The main goal is to give the necessary background to appreciate the construction of the p-adic space X as an adelic quotient and to understand the naturality of the definition of the p-adic eigenfunctions in this context.

The third chapter deals with the local representation theory over $\mathrm{GL}_2(\mathbb{Q}_p)$ as presented in [Bum97] and some lecture notes [Che] and [Tai]. In this chapter, we try to lay the representation theoretic foundations for the proof of the main results of the last chapter.

The fourth chapter presents the link between the regular graphs and trees and the p-adic spaces that are quotients of $\mathrm{PGL}_2(\mathbb{Q}_p)$ as in [Ser80]. In particular, we introduce the Bruhat-Tits Tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ following [Ser80]. In the last section, we deal with analytic properties of the eigenfunctions of the tree that play an important role for the AQUE problem.

The fifth chapter is an introduction to some topics of ergodic theory that are useful for the understanding of the measure classication result of Theorem VI.1.13. The discussion is based mainly on [VO15], with the last topics drawn from [Lin04] and [Lin06].

The last chapter will present the representation theoretic setting that generalizes the path space construction we presented before. In this context, we will present the main proof, following closely [Nel18]. In particular, we will try to clarify and give a sense to some of the constructions and some details, also drawing the intuition and explanations from the previous chapters.

Chapter II

Quaternion Algebras

In this chapter, we will present the theory of quaternion algebras, with a particular emphasis on the rational ones. We won't prove the majority of the results, since it would take too much time, but the exposition will be based on the classical work of [Vig80]. We also acknowledge an intersection with [Dal] in the first chapters, due to the common source.

In the first section, we will present the basic definitions and results, developing the noncommutative analogue of the theory of commutative Dedekind domains. In particular, we will define the concept of orders, ideals and the class number of a quaternion algebra. Finally, we will introduce the different and discriminant.

In the second section, we will focus on the quaternion algebras over a local field. After proving a classification, we will study their orders and the local discriminant.

The third section is dedicated to the quaternion algebras over a global field K, specializing the discussion to $K = \mathbb{Q}$. We will present a classification and study the *Eichler* orders. We will define the *Brandt matrices* and the *Hecke operators* associated to maximal orders. Then, we will pass from the global setting to the adelic setting. We will define the automorphic forms on this setting and state the associated strong multiplicity one result.

II.1 Quaternion Algebras

Let K be a field of characteristics different from 2.

Definition II.1.1 (Quaternion Algebra)**.** A quaternion algebra B over K, is a central simple algebra of dimension 4 over K . Equivalently, a quaternion algebra B is a K-algebra of dimension 4, such that there exists a K-basis $1, i, j, ij$ that satisfies:

$$i^2 = a, \quad j^2 = b, \quad ij = -ji \quad a, b \in K^\times.$$

In this case, B will also be denoted by $B(a, b)$ or $\left(\frac{a,b}{K}\right)$.

Any quaternion algebra comes with an involutive anti-automorphism.

Definition II.1.2 (conjugation)**.** Let B be a quaternion algebra over K. The conjugation is the K-endomorphism : $h \mapsto \bar{h}$, defined on the basis $1, i, j, ij$ by $a + bi + cj + dij \mapsto a - bi - cj - dij$.

Definition II.1.3. Let B be a quaternion algebra over K. The *reduced trace* of $h \in B$ is $t(h) = h + \bar{h}$. The *reduced norm* of $h \in B$ is $n(h) = h\bar{h}$. The minimal polynomial of $h \notin K$ is

$$(X - h)(X - \bar{h}) = X^2 - t(h)X + n(h).$$

Proposition II.1.4. *Let B a quaternion algebra over K. The following hold:*

1. *the norm is a multiplicative homomorphism $n : B \to F$;*
2. *the trace is an additive homomorphism $t : B \to F$;*
3. *an element $h \in B$ is invertible if and only if $n(B) \neq 0$.*

Proof. It follows from the definition of the norm and the trace by expressing their action on the basis $1, i, j, ij$. See [Vig80, Lemme 1.1]. □

Example II.1.5. A classical and fundamental example of a quaternion algebra over K is given by the matrix algebra $M_2(K)$. It has dimension 4 over K and it is simple since any principal bilateral ideal contains the identity. In this setting, the trace is the classical trace of a matrix and the norm is the classical norm of a matrix. The conjugation is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Proposition II.1.6. *Let K be a field and let $a, b, c \in K^\times$. Then the following hold:*

1. $\left(\frac{a,b}{K}\right) \cong \left(\frac{b,a}{K}\right)$;
2. $\left(\frac{\mu^2 a, \nu^2 b}{K}\right) \cong \left(\frac{a,b}{K}\right)$, *for any* $\mu, \nu \in K^\times$;
3. $\left(\frac{1,c}{K}\right) \cong \left(\frac{b,a}{K}\right)$;

Proof. 1) The isomorphism comes from the K-algebra endomorphism $i \mapsto j$, $j \mapsto i$.

2) In this case, the isomorphism is induced by $i \mapsto \frac{i}{\mu}$, $j \mapsto \frac{j}{\nu^2}$.

3) The isomorphism is induced by

$$1 \mapsto I_2, \quad i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

□

Proposition II.1.7. *For an algebraically closed field K, the only quaternion algebra, up to isomorphism, is $M_2(K)$.*

Proof. Let $B = \left(\frac{a,b}{K}\right)$ be a quaternion algebra over K. Fix μ, such that $\mu^2 = a$. By Proposition II.1.6 we have that:

$$\left(\frac{a,b}{K}\right) = \left(\frac{\mu^2,b}{K}\right) \cong \left(\frac{1,b}{K}\right) \cong M_2(K).$$

□

For a given field extension $L \hookrightarrow F$ and a quaternion algebra B over F, the tensor product $B' = B \otimes_F K$ is a L-algebra of dimension 4 that satisfies the equivalent condition of Definition II.1.1, hence is a quaternion algebra over L. As a consequence of the previous proposition, there always exists an intermediate field $F \subseteq L \subseteq K$, for the algebraic closure K of F, such that $B \otimes_F L \cong M_2(L)$. This argument motivates the following definition.

Definition II.1.8. Let B be a quaternion algebra over a field F. Consider a field extension $F \hookrightarrow L$. We say that B *splits* over L if $B \otimes_F L \cong M_2(L)$.

Proposition II.1.9. *Let B be a quaternion algebra over K. Either B is split or it is a division algebra.*

Proof. By the Artin-Wedderburn Theorem, since B is a central simple algebra of finite dimension over the semisimple artinian ring F, we have that $B \cong M_n(D)$, for a division algebra D. Moreover, $\dim_F(B) = 4$, so there are only two choices: either $n = 1$ and $D = B$ is a division algebra or $n = 2$ and $\dim_F(D) = 1$, i.e. $D = F$. □

II.2 Ideals and Orders

Fix a Dedekind ring R. Denote with K its field of fractions and let B/K be a quaternion algebra over K.

Definition II.2.1. A R-lattice L of a K-vector space V is a finitely generated R-module on V. The R-lattice L is *complete*if $L \otimes_R K \cong V$.

Definition II.2.2. An element $x \in B$ is integral over K if $R[x]$ is a R-lattice of B.

As in the commutative case of number field extensions, there is a relation between integral elements and lattices.

Proposition II.2.3. *An element $x \in B$ is integral if and only if its reduced trace and its reduced norm are elements of $R \subset K$.*

In contrast with the commutative case, the integral elements do not form a ring in general. As a consequence, we sometimes need to restrict to subsets of the integral elements over R which form a ring.

Definition II.2.4 (Ideal, Order)**.** An ideal of B is a complete R-lattice. An order O of B is an ideal which is a ring. Equivalently, an order O of B is a subring of integral elements such that:

1. $R \subseteq O$;
2. $KO = H$.

A maximal order is an order which is not contained properly in another order. An Eichler order is the intersection of maximal orders.

To any given ideal I of B we can associate two orders:

$$O_l(I) = \{h \in B \mid hI \subset I\},$$
$$O_r(I) = \{h \in B \mid Ih \subset I\},$$

respectively the left and right orders associated to I. See [Vig80, §1.4].

Proposition II.2.5. *The two definitions of order are equivalent. Moreover, there exists at least an order. Any order is contained in a maximal one.*

Proof. See [Vig80, Proposition 4.2]. □

Definition II.2.6. An ideal I is a left ideal of $O_l(I)$, a right ideal of $O_r(I)$. It is *bilateral* if $O_l(I) = O_r(I)$, *normal* if $O_l(I)$ and $O_r(I)$ are minimal, *integral* if $I \subseteq O_r(I) \cap O_l(I)$, *principal* if $I = O_g h = hO_d$.

There are also natural operations associated to the ideals.

Definition II.2.7. The inverse of an ideal I is $I^{-1} = \{h \in H, IhI \subset I\}$. The product IJ of two ideals IJ is the ideal generated by hk for $h \in I$, $k \in J$.

Proposition II.2.8. *The product of ideals is associative. Moreover, an ideal I is integral if and only if it is contained in one of its associated orders.*

Proof. See [Vig80, Lemme 4.3]. □

Remark II.2.9. Let $I = O_l(I)b$ be a principal ideal. Then

1. $I^{-1} = h^{-1}O_l(I) = O_r(I)h^{-1}$;
2. $II^{-1} = O_l$, $I^{-1}I = O_r$,

where $O_r = b^{-1}O_l b$ is the right order of I.

As in the commutative case, we can define equivalence relations between ideals.

Definition II.2.10. Let I, J be ideals of B. Then I and J are right equivalent if and only if $I = Jh$, for $h \in B$. The right equivalence classes which are left ideals of the order O are called the left classes of O. In the same way, we can define the left equivalence between two ideals and the right classes of O.

The two similar notions are related as expected.

Lemma II.2.11. *The inverse map $I \mapsto I^{-1}$ induces a bijection between the left classes and right classes of an order O.*

Consider an ideal J. The map $I \mapsto JI$ induces a bijection between the left classes of $O_l(I)$ and the left classes of $O_l(J)$

Definition II.2.12 (Class number)**.** Fix an order O of B. The class number $\mathrm{cl}(O)$ is the number (finite or infinite) of equivalence classes of left (or right) ideals of O. The class number of B, $\mathrm{cl}(B)$ is the class number of a maximal order of B.

II.3 Different and Discriminant

Fix a quaternion algebra B over K and a Dedekind ring R, with field of fraction equal to K. All the orders and ideals will be relative to R.

Definition II.3.1 (Norm)**.** The *norm* of an ideal I, is the fractional ideal of R given by

$$n(I) = \langle n(h) \rangle_{h \in I}.$$

Definition II.3.2. The different $O^{*-1}$of an order O of B is the inverse ideal of the dual of O, respect to the bilinear form induced by the reduced trace:

$$O^* = \{x \in H \mid t(xO) \subseteq R\}.$$

Its reduced norm $n(O^{*-1})$ is the reduced discriminant of O, denoted by $d(O)$.

Lemma II.3.3. *The following is true:*

1. *if I is an ideal, then $I^* = \{x \in H \mid t(xI) \subseteq R\}$ is a bilateral ideal;*
2. *if O is an order, then O^{*-1} is an integral bilateral ideal;*
3. *If O is a free R-module with basis (u_j) and a principal ideal, then*

$$n(O^{*-1})^2 = R(det(t(u_i u_j))).$$

Proof. 3) The proof is similar to the commutative case, see [Vig80, Lemme 4.7]. □

Corollary II.3.4. *Let O, O' be orders of B. If $O' \subseteq O$, we have that $d(O') \subseteq d(O')$. Moreover, $d(O) = d(O')$ if and only if $O = O'$.*

Remark II.3.5. This corollary is useful for establishing conditions on orders to be maximal. For example, if an order O of B has the property that $d(O) = R$, then automatically it must be maximal. That's the case for the order $M_2(R)$ of the quaternion algebra $M_2(K)$, for K the field of fraction of a Dedekind ring R.

II.4 Quaternion Algebras over Local Fields

Let K be a local field. Moreover, if K is non-archimedean, let $\mathfrak{o}$ be its ring of integers, fix a uniformizer ϖ and let $\mathfrak{p} = (\varpi)$ be the unique maximal ideal of $\mathfrak{o}$.

Theorem II.4.1 (Classification)**.** *On a local field $K \neq \mathbb{C}$ there exists a unique quaternion division algebra B. In particular, for K non-archimedean of characteristic different from 2, $B \cong \left(\frac{\varpi, u}{K}\right)$, where u is a root of unity in K which is not a square.*

Proof. See [Vig80, §II.1 Théorème 1.1] and [Vig80, §II.1 Théorème 1.3]. □

Definition II.4.2. Let K be a field of characteristics different from 2. The *Hasse invariant* of $a, b \in K^\times$ is defined by

$$\epsilon_K(a,b) = \begin{cases} 1 & \text{if } \left(\frac{a,b}{K}\right) \text{ splits;} \\ -1 & \text{if } \left(\frac{a,b}{K}\right) \text{ doesn't split.} \end{cases}$$

Remark II.4.3. Since any quaternion algebra over K can be expressed as $\left(\frac{a,b}{K}\right)$, the Hasse invariant is actually a map $\epsilon_K : \mathrm{Quat}(K) \to \{\pm 1\}$.

We introduce now a related number theoretical invariant.

Definition II.4.4 (Hilbert symbol)**.** Let $a, b \in K^\times$. Then the *Hilbert symbol* is defined as

$$(a,b)_K = \begin{cases} 1 & \text{if } ax^3 + by^2 - z^2 = 0 \text{ admits a nontrivial solution for } x, y, z \in K; \\ -1 & \text{otherwise.} \end{cases}$$

The two objects, in the case of a local field, are related in the following way.

Theorem II.4.5. *Let K be a local field of characteristic different from 2. Then:*

$$\epsilon_K(a,b) = (a,b)_K.$$

Proof. See [Vig80, §2.1]. □

Now, we determine the discriminant of a maximal order of a non-split quaternion algebra over a non-archimedean local field K.

Proposition II.4.6. *Let B be a non-split quaternion algebra over a non-archimedean local field of characteristic different from 2. Then there is a unique maximal order O_M. Its reduced discriminant is $d(O_M) = \mathfrak{p}$.*

Definition II.4.7. Let B be a quaternion algebra over a local field. Then the *discriminant* of B, denoted by $d(B)$, is the discriminant of a maximal order.

Remark II.4.8. As a consequence of Remark II.3.5, for any non-archimedean local field K, the split quaternion algebra $M_2(K)$ has a maximal order given by $O_M = M_2(\mathfrak{o})$. Its reduced discriminant is then $d(O_M) = \mathfrak{o}$. As a result, it's possible to determine if a quaternion algebra over K is non-split or split, by looking at its discriminant. Thus, in light of the previous proposition, if $B \ncong M_2(K)$ we also say that B is ramified.

Remark II.4.9. Even though there is no unique maximal order or ideal in the case of a split local quaternion algebra B over a local field K, it's possible to find a standard one in the non-archimedean case. Let v be the valuation associated to the local field K. Then the set $X_p = \{h \in B \mid v(n(h)) \geq 0\}$ is a maximal ideal and order in B, see [Vig80, §2 Lemme 1.4].

II.5 Quaternion Algebras over Global Fields

Fix a global field K.

We present the classification of the quaternion algebras over K.

Definition II.5.1. Let B be a quaternion algebra over K and fix a place v. We say that B ramifies at v if $B_v = B \otimes_K K_v$ is ramified. In particular if $K = \mathbb{Q}$, and B ramifies at infinity, we say that B is definite.

Proposition II.5.2. *Let B be a quaternion algebra over K. Then there are only finitely many ramified places.*

Proof. Write B as $\left(\frac{a,b}{K}\right)$, for $a, b \in K^\times$. Then for almost any non-archimedean place, we have that $a, b \in \mathfrak{o}_v^\times$ and the order associated to the lattice by $O = R_v + R_v i + R_v j + R_v ij$ has discriminant $d(O) \not\subseteq \mathfrak{p}_v$. Hence by Proposition II.4.6, B is split. □

The classification depends on local information, through the *discriminant* of B.

Definition II.5.3. Let B be a quaternion algebra over a number field K. The reduced discriminant of B, denoted by $d(B)$, is the integral ideal $d(B)$ given by the product of the finite ramified places.

Theorem II.5.4. *Classification Let B be a quaternion algebra over a global field K. Let* $\mathrm{Ram}(B)$ *be the set of ramified places. Then:*

1. $\mathrm{Ram}(B)$ *is a finite set of even cardinality;*
2. *for any finite set S of places of even cardinality there exists one and only one quaternion algebra B', up to isomorphism, such that* $\mathrm{Ram}(B) = S$.

Proof. See [Vig80, §3.3 Théorème 3.1]. □

Remark II.5.5. In general, 1) follows by the *Hilbert reciprocity law*

$$\prod_{v \text{ place}} (a,b)_{K_v} = 1 = \prod_{v \text{ place}} \epsilon(a,b)_{K_v}.$$

Actually, it's also possible to deduce this theorem as a corollary of the classification. In the case $K = \mathbb{Q}$ Hilbert reciprocity becomes just quadratic reciprocity.

Remark II.5.6. In the case of $K = \mathbb{Q}$, the previous theorem specializes nicely. In fact, for any there exists one and only one quaternion algebra B over $\mathbb{Q}$ for a given reduced discriminant $d(B)$, i.e. there is a bijection between the squarefree integers (1 included) and the quaternion algebras. The reason behind this nice form is the existence of only one infinite place, whose splitting behaviour is then determined by the cardinality of $\mathrm{Ram}_{fin}(B)$, the set of finite ramified places. In fact, by the previous theorem $\mathrm{Ram}(B)$ has to be of even cardinality.

From these results, we noticed that the properties of a global quaternion algebra B over K are strictly related to the ones of the associated local quaternion algebras B_v over K_v, by localization. This connection is formalized by the notions of adèles and idèles associated to B.

First, notice that for any place v, the quaternion algebra B_v has a natural topology, given by considering the underlying topological vector space of dimension 4 over K_v.

Fix a finite set $S \supseteq S_\infty$ of places of K, containing the archimedean ones. Denote with $\mathcal{V}_K$ the set of places of K.

Let A_K be the ring of adèles of K and let $A_K^\times$ be the group of idèles, with their respective topologies.

Definition II.5.7. The ring of adèles of B is the topological group $B_A = B \otimes_K A_K$. Equivalently, it is the restricted direct product of the topological groups B_v, for $v \in \mathcal{V}_K$, with respect to the open compact subgroups $O_v = O \otimes R_v$, for a maximal order O of B, for $v \notin S$.

Definition II.5.8. The group of idèles $B_A^\times$ is the restricted direct product of the topological groups $B_v^\times$, for $v \in \mathcal{V}_K$, with respect to the open compact subgroups $O_v^\times = (O \otimes R_v)^\times$, for a maximal order O of B, for $v \notin S$.

Definition II.5.9. The unit idèles B_A^1 is the subgroup of the idèles $B_A^\times$, given by the kernel of the map induced on the idèles by the reduced norm $n : B_A^\times \to A_K$:

$$B_A^1 = \ker((x_v)_v \mapsto (n(x_v))_{v \in V}).$$

Remark II.5.10. With these definitions, it's possible to realize $B = B_K$ as a subgroup of B_A and $B_K^\times$ as a subgroup of $B_K^\times$, by diagonal embedding.

The same holds for B_v and $B_v^\times$, for a place v.

We report the following structural result in this setting.

Theorem II.5.11. *Let B be a quaternion algebra over a number field K. The following hold:*

1. *B is discrete and cocompact in B_A, i.e. B_A/B is compact;*
2. *For any place v, $B_v + B_k$ is dense on B_A.*
3. *$B^\times$ is discrete on $B_A^\times$;*
4. *if B is a division algebra, then B_A^1/B^1 is compact.*

Proof. See [Vig80, §3.1 Théoreme 1.4]. □

We end the section with the following result.

Theorem II.5.12 (Strong Approximation)**.** *Let K be a number field and B a quaternion algebra over K. Let $S \supseteq S_\infty$ be a finite set of places containing the infinite ones and a non ramified one. Set $B_S^1 = \prod_{s \in S} B_s^1$. Then $B^1 B_S^1$ is dense on B_A^1.*

Corollary II.5.13. *Let B a quaternion algebra over a field K of class number* 1*. Let $H \supset B_{A,fin}^\times$ be an open compact subgroup, such that $n(H) = \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^\times$, the maximal open compact subgroup of $A_{K,fin}$.*

1. *If B is indefinite, then $B^\times \prod_{v|\infty} B_v^\times H = B_A^\times$;*
2. *If B is definite and $\mathfrak{p}$ is a split finite place, then $B^\times B_{\mathfrak{p}}^\times H = B_{A,fin}^\times$.*

Proof. 1) By strong approximation, $B^1 \prod_{v|\infty} B_v^1 H \supset B_A^1$. So, we need to prove that $n(B^\times \prod_{v|\infty} B_v^\times H) = A_K^\times$. By multiplicativity of the norm

$$n(B^\times \prod_{v|\infty} B_v^\times H) = K^\times \prod_{v|\infty} K_v^\times \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^\times.$$

Thus, we need to prove that

$$|A_K^\times / K^\times \prod_{v|\infty} K_v^\times \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^\times| = 1.$$

This is true since the left hand side is the class number of K.

2) Again, as a consequence of strong approximation, $B^\times B_{\mathfrak{p}}^\times H \supseteq B_{A,fin}^1$. Arguing as before, we need to prove that $|A_{K,fin}^\times / K^\times \prod_{\mathfrak{p}} \mathfrak{o}^\times| = 1$. But this is just the class number of K. □

Remark II.5.14. Any Eichler order R of a quaternion algebra B over a number field K satisfies the property that $H_R = \prod_{\mathfrak{p}} (R \otimes \mathfrak{o}_{\mathfrak{p}})^\times$ is an open compact subgroup of $B_{A,fin}^\times$ such that $n(H_R) = \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}$.

II.6 Ideals and Brandt Matrices

The strong approximation theorem gives a way to translate global questions into adelic ones, that can be attacked through local results.

Let K be a number field, with ring of integers R. Let B be a quaternion algebra over K and fix a finite set S of places containing a non-ramified one. Denote by R_S the Dedekind domain given by

$$R_S = \{x \in R \mid x \in R_v,\ \forall v \notin S\},$$

for the ring of integers R_v of the completion K_v of K at the place v. Denote by Y an R_S-ideal of B and let $Y_v = Y \otimes_{R_S} R_{S,v}$ be the completion at the place v.

There is a correspondence between R_S-ideals of B and local ideals, given by the following proposition.

Proposition II.6.1. *1) There exists a bijection between the R_S-ideals Y of B and the set of local $R_{S,v}$-ideals Z_v of B_v, such that $Z_v = X_v$ for almost every split non-archimedean place $v \notin S$. The bijection is given in particular by:*

$$Y \mapsto (Y_v)_{v\in\mathcal{V}_k} \qquad (Z_v)_{v\in\mathcal{V}_k} \mapsto Y = \{h \in B \mid h \in Y_v,\ \forall v \in \mathcal{V}_K\}.$$

2) The norm and discriminant of an ideal are local:

$$n(Y)_p = n(Y_p) \qquad d(Y)_p = d(Y_p).$$

3) An order O is maximal if and only if

$$d(O) = \prod_{\mathfrak{p}\ fin} d(B_{\mathfrak{p}}),$$

where $(d(B_{\mathfrak{p}}))$ is the R_S ideal generated by $d(B_{\mathfrak{p}}) \cap R_S$.

Proof. 1) See [Vig80, §3.5A Proposition 5.1], [Vig80, §3.5A Proposition 5.1], [Vig80, §3.5A Corollaire 5.2], [Vig80, §3.5A Corollaire 5.2]. □

After this proposition, it seems then convenient to associate to any order and ideal Y of B, its adelic counterpart, denoted by $Y_A = \prod_v Y_v \subset B_A$. In particular, to any Eichler order O, there is the associated adelic order O_A, and to the group of invertible elements $O^\times$, we associate the group

$$O_A^\times = \prod_v (O_v)^\times \subset B_A^\times.$$

Then the following correspondence holds.

Theorem II.6.2 (Global-Adelic correspondence)**.** *The following hold:*

1. *(Ideals) The right ideals of the Eichler order O are in bijection with the set $B_A^\times/O_A^\times$. The bijection preserves the norm.*

2. *(Ideal classes) The classes of right ideals are in bijection with $B^\times \setminus B_A^\times/O_A^\times$.*

Proof. See [Vig80, pp.87-88]. □

Corollary II.6.3. *The class number $cl(O)$ of an Eichler order O is finite.*

Proof. See [Vig80, Théorème 5.4-5.5]. □

In the following discussion, we restrict to the field of rational numbers and thus consider only rational quaternion algebras.

In the adelic setting, there is a family of operators acting on the smooth functions $\phi : B_A^\times \to \mathbb{C}$, i.e. smooth at the finite places as in Definition III.2.1 and smooth at the infinite place as functions of a real variable. In fact, let $\mathfrak{p}$ be a finite place of $\mathbb{Q}$ and $B_v^\times$ the localization of a quaternion algebra B. Then ϕ is a smooth vector for the representation given by right translation of $B_v^\times$ on the space of smooth functions $S(B_A^\times)$. Then, the Hecke algebra $\mathcal{H}_{B_v^\times}$ acts on $S(B_A^\times)$, as defined in III.3.3. In particular, the spherical Hecke operators $T(p)$ are defined, for any p prime.

In the light of Theorem II.6.2, the spherical Hecke operators can be interpreted as correspondences on the space of ideals of an Eichler order O. Let p be a finite prime that splits a quaternion algebra B over $\mathbb{Q}$, such that $O_p = M_2(\mathfrak{o}_p)$. The adelic Hecke operator $T(p^n)$, as defined in III.6.5, acts on spherical functions $\phi : B_p^\times / O_p^\times \cong \mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathfrak{o}) \to \mathbb{C}$ by:

$$\phi(x) \to \sum_i \phi(xg_i) \quad \text{for } g_i \in M_{p^n}/\mathrm{GL}_2(\mathfrak{o}),$$

where $M_{p^n} = \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid |g|_p = p^n\}$. Hence, this operator defines a correspondence that to any $(\cdot, x_p, \cdot) \in B^{\times_A}/O_A^\times$ associates the formal sum $\sum_i (\cdot, x_p g_i, \cdot)$ for $g_i \in M_{p^n}/\mathrm{GL}_2(\mathfrak{o})$. By Theorem II.6.2 and the local property of the norm, this correspondence associates to any ideal I of O the formal sum $\sum_i J_i$, where $n(J_i) = p^n n(I)$. By composing the Hecke operators for distinct primes p, we can define the Hecke operator $T(n)$, for $n \in N^*$ and see it is well-defined. This argument explains the introduction of the following object.

Definition II.6.4 (Brandt Matrix)**.** Let O be an Eichler order of a quaternion algebra B over $\mathbb{Q}$, with $cl(O) = h$. Let $[I_j]$, $1 \leq j \leq h$, be representatives of the classes of right ideals of O. Fix a positive natural number $N \in \mathbb{N}^*$. The n-Brandt Matrix $T_B(n)$ is the $h \times h$ matrix defined by:

$$T_B(N)_{i,j} = \#\{J \subset I_j \mid n(J) = Nn(I_j) \text{ and } [I_i] = [J]\}.$$

In the case of definite quaternion algebras, this object has proved to be fundamental since its introduction in [Eic55]. In this exposition, we are however interested in a particular property of the n-Brandt Matrix.

Proposition II.6.5. *Let $T(N) : B_A^\times/O_A^\times \to \sum B_A^\times/O_A^\times$ be the N-Hecke correspondence. Then there is an associated map $T'(N) : \sum B^\times \setminus B_A^\times/O_A^\times \to \sum B^\times \setminus B_A^\times/O_A^\times$. If we fix representatives $[I_j]$ for the classes of right ideals of O, i.e. a basis of $\sum B^\times \setminus B_A^\times/O_A^\times$, then $T_B(N) = T'(N)$.*

Proof. The Hecke correspondence acts on an element of $B_A^\times/O_A^\times$ by right translation, so it is still possible to quotient at the left by $B^\times$ and the resulting map $T(N) : B^\times \setminus B_A^\times/O_A^\times \to \sum B^\times \setminus B_A^\times/O_A^\times$ is well-defined. Then it is possible to extend it $\mathbb{Z}$-linearly to a map $T'(N) : \sum B^\times \setminus B_A^\times/O_A^\times \to \sum B^\times \setminus B_A^\times/O_A^\times$. Once we fix representatives $[I_j]$ for the classes of right ideals of O, we have a matrix expression associated to $T'(N)$, that matches with the definition of $T_B(N)$, the N-Brandt Matrix. □

Remark II.6.6. For a prime p that splits the quaternion algebra, the Hecke operator $T(p)$ acts on the space of spherical automorphic functions $\phi : B^\times \setminus B_A^\times / O_A^\times \to \mathbb{C}$ by the inverse transpose matrix $T_B^{-T}(N)$, by interpreting ϕ as an element of the dual $(\mathbb{C} \otimes_{\mathbb{Z}} \sum B^\times \setminus B_A^\times / O_A^\times)^*$. The eigenvectors of the Hecke operator coincide with the eigenvectors of this matrix and have an interpretation as analogues of the classical automorphic functions for definite quaternion algebras.

II.7 Automorphic Representations on Quaternion Algebras

Consider a quaternion algebra B over $\mathbb{Q}$.

Definition II.7.1. An automophic form on $B^\times$ is a function $\phi : B_A^\times \to \mathbb{C}$ that satisfies the following properties:

1. ϕ is *smooth*: ϕ is continuous, ϕ is smooth at the real place and locally constant at the finite places;
2. $\phi(\gamma x) = \phi(x)$, for any $\gamma \in B^\times$, i.e. $\phi : B^\times \setminus B_A^\times \to \mathbb{C}$;
3. $\phi(xz) = \psi(z)\phi(x)$, for a continuous character $\psi(z)$, for any $z \in Z_A^\times = Z(B_A^\times)$;
4. $\phi(xg) = \phi(x)$ for any $g \in U_f$, for an open compact subgroup $U_f \leq B_{A,fin}^\times$;
5. ϕ is right K_∞-finite, for a choice of a maximal compact subgroup of $B_\mathbb{R}^\times$;
6. ϕ is $Z(\mathcal{U}(\mathfrak{b}))$-finite, for the center of the complex universal enveloping algebra of the Lie group $\mathfrak{b}$ of $B_\infty^\times$;
7. ϕ is of moderate growth;

Denote the space of the automorphic functions by $\mathcal{A}(B^\times \setminus B_A^\times)$. The subspace of the automorphic functions with fixed central character ψ is denoted by $\mathcal{A}(B^\times \setminus B_A^\times, \psi)$. More details will be given in the following section.

Remark II.7.2. Since in this essay we will treat in particular automorphic functions on definite quaternion algebras, with trivial infinite part, we will not explain the meaning of point 6) in general, for this see [Bum97, §3.2-3.3]. By II.5.11 the only quaternion algebra for which actually point 7) is necessary is the split one. In the non-split case, the quotient $Z_A^\times B^\times \setminus B_A^\times / U_f K_\infty$ is always compact.

Remark II.7.3. The space $B_A^\times$ is locally compact and is unimodular, with Haar measure μ, and the subgroup $Z_A^\times B^\times$ is closed. As a consequence, the quotient $Z_A^\times B^\times \setminus B_A^\times$ admits a right $B_A^\times$-invariant Radon measure μ^*, induced by the quotient map.

Definition II.7.4. Let $B^\times$ be a quaternion algebra over $\mathbb{Q}$. The space of square-integrable automorphic forms $L^2(B^\times \setminus B_A^\times, \psi)$ is the Hilbert space obtained by completing the subspace of automorphic forms ϕ of $B^\times$ such that

$$\int_{Z_A^\times B^\times \setminus B_A^\times} |\phi|^2 \, d\mu^* < +\infty.$$

In the case of a split quaternion algebra B, $L_0^2(B^\times \setminus B_A^\times)$ is the subspace of cuspidal square-integrable automorphic forms.

II.7.1 Global Models

As a consequence of the definition, any automorphic form can be realised as a function $\phi : B^\times \setminus B_A^\times / U_f \to \mathbb{C}$, for some open compact subgroup $U_f \leq B_{A,fin}^\times$.

Consider the subspace of automorphic forms fixed by K_∞. It has a classical interpretation in the following way.

In the split case, let $B = M_2(\mathbb{Q})$, $B^\times = GL_2(\mathbb{Q})$, $K_\infty = O_2(\mathbb{R})$. An open compact subgroup $U_f \leq B_{A,fin}^\times$ is associated to the local units of an Eichler order. Moreover, choosing an Eichler order R of B we have that, up to conjugation, $\mathrm{SL}_2(\mathbb{Z}) \supseteq R^\times \supseteq \mathrm{I}(N)$, for some $N \in \mathbb{N}^*$ and $\mathrm{I}(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z}))$. By Corollary II.5.13 with respect to the infinite place, an automorphic form ϕ of $B^\times$ can be then realised as a complex function on an arithmetic quotient

$$B^\times \setminus B^\times \mathrm{GL}_2(\mathbb{R}) U_f / U_f O_2(\mathbb{R}) \cong \Gamma \setminus \mathrm{GL}_2(\mathbb{R}) / O_2(\mathbb{R}),$$

where $\Gamma = B^\times \cap U_f \mathrm{GL}_2(\mathbb{R})$ and the isomorphism is given explicitly by the map $\pi : B_{\mathbb{Q}}^\times \setminus B_A^\times \to \Gamma \setminus \mathrm{GL}_2(\mathbb{R})$,

$$\pi : B_{\mathbb{Q}}^\times g \mapsto \pi_{B_A^\times \to B_\infty^\times}(B_{\mathbb{Q}}^\times g \cap [\mathrm{GL}_2(\mathbb{R}) \times U_f]),$$

for the standard projection $\pi_{B_A^\times \to B_\sigma^\times}$ of the idèle group to its local $B_\infty^\times$ component. In the case of trivial central character at infinity $\chi_\phi|_{B_\mathbb{R}} = 1$, we have the isomorphism

$$R \setminus \mathrm{GL}_2(\mathbb{R}) / \mathbb{R}^\times \mathrm{O}_2(\mathbb{R}) \cong R \setminus \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}) \cong R \setminus \mathcal{H},$$

after the Iwasawa decomposition

$$\mathrm{SL}_2(\mathbb{R}) \cong \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \mathrm{SO}_2(\mathbb{R}),$$

for the upper-half complex plane $\mathcal{H}$, and where the action of $\gamma \in R$ is given by the classical

$$\gamma(z) = \frac{az+b}{cz+d} \qquad \gamma \in R,\ z = x + iy \in \mathcal{H}.$$

So, we recover the classical automorphic functions on arithmetic quotients of $\mathcal{H}$. In this setting, the moderate growth condition coincides with the moderate growth at the cusps and the *cuspidal* forms are the subspace of the automorphic forms $\mathcal{A}_0(B^\times \setminus B_A^\times, \psi)$ that vanish at the cusps of $R \setminus \mathcal{H}$.

In the case of an indefinite quaternion division algebra B, we still get a classical automorphic function with respect to a lattice $\Gamma \leq \mathrm{SL}_2(\mathbb{Q})$. However, by theorem II.5.11, $\Gamma \setminus \mathcal{H}$ is compact and then there is no need to require the moderate condition at the cusps in Definition II.7.1.

In both cases where the quaternion algebra is split at infinity, the action of the center of the universal enveloping algebra of the lie algebra $\mathfrak{b}$ coincides with the action of the Laplacian in the global model.

In the case of a definite quaternion algebra B, we have that $B_{\mathbb{R}}^\times \cong \mathbb{H}^\times$. Moreover, $\mathbb{H}^1(\mathbb{Q})$ is the finite quaternion group, so we can't hope in the strong approximation with respect to an infinite place. However, it's still possible to express an automorphic form ϕ as a function of a quotient of a local quaternion algebra, in the case of trivial infinity type, i.e. $\phi = \phi|_{B_{A,fin}}$, in the following way. Let p be a prime such that B_p is split. Again by Corollary II.5.13, we obtain that

$$B_{A,fin}^\times = B^\times B_p^\times U_f,$$

where $U_f = \prod_{v \neq p} R^\times_{\mathbb{Q}_v}$, for an Eichler order R. Then ϕ can be realized as a function on

$$B^\times \setminus B^\times B_p^\times U_f/U_f \cong R[1/p]^\times \setminus B_p^\times = R[1/p]^\times \setminus \mathrm{GL}_2(\mathbb{Q}_p),$$

where the bijection is realized by $B^\times \cap B_p^\times U_f = R[1/p]^\times$.

II.7.2 Strong Multiplicity One

As we have seen, the relation with the classical theory of modular functions on quotients of the upper-half plane is present only in the case of indefinite quaternion algebras, while the definite case, in which we are interested in this essay, really benefits from a representation theoretic viewpoint. In fact, from the definition of automorphic forms, there is a well-defined action of the lie algebra $\mathfrak{b}$ and of K_∞ on the infinite component of $B_A^\times$ and an action of $B^\times_{A,fin}$ by right translation on the finite components of $B_A^\times$. These actions commute and make $\mathcal{A}(B^\times \setminus B_A^\times)$ a $(\mathfrak{b}, K_\infty)$-module, see [Bum97, §3.3.], and a $B^\times_{A,fin}$-representation (see $\S III.1$ for the relation between representations and left modules) that we will call a $B_A^\times$ representation.

Definition II.7.5. Let $K = K_\infty \times H_R$, for a choice of a maximal order R of $B^\times$ and $H_R = \prod_{v \text{ fin}} (R \otimes \mathfrak{o}_v)^\times$. A representation (π, V), with $V \subseteq \mathcal{A}(B^\times \setminus B_A^\times)$, of $B_A^\times$ is *admissible* if:

1. every $v \in V$ is K-finite;
2. for any finite dimensional irreducible representation ρ of K, the ρ-isotypic component of V, denoted by $V(\rho)$, is finite dimensional.

The importance of the admissible representations comes from the following result.

Theorem II.7.6 (Tensor Product Theorem)**.** *Let (V, π) be an irreducible admissible representation of $B_A^\times$. Then there exists an irreducible admissible $(\mathfrak{b}, K_\infty)$-module (π_∞, V_∞) and for each non-archimedean place v there exists an irreducible admissible representation (π_v, V_v) of $B_v^\times$ such that for almost all v, V_v contains a non-zero K_v-fixed vector ξ_v^o such that π is the restricted product tensor product of the representations π_v:*

$$(\pi, V) \cong \bigotimes_v (\pi_v, V_v),$$

with respect to ξ_v^o.

Remark II.7.7. The notion of restricted tensor product is given in [Bum97, §3.3-3.4], taken from [Fla79].

Definition II.7.8. Let (π, V) be a smooth representation of $B_A^\times$. We say that (π, V) has strong multiplicity one, if, for any irreducible admissible representations $(\rho_1, W_1), (\rho_2, W_2) \subseteq (\pi, V)$ such that

$$(\rho_{1,v}, W_{1,v}) \cong (\rho_{2,v}, W_{2,v}) \qquad \text{for almost all place } v,$$

we have that $(\rho_1, W_1) = (\rho_2, W_2) \subset (\pi, V)$.

We report the two crucial results, whose proof can be found in [Bum97, §3.2-3.-4] and [Gel75].

Theorem II.7.9. *Let $B = M_2(\mathbb{Q})$ be the split algebra. Then the L^2-completion of the space of cuspidal automorphic forms $L_0^2(B^\times \setminus B_A^\times, \psi)$, with fixed central character ψ, decomposes as a direct sum of irreducible invariant subspaces under $B_A^\times$. The space of K-finite vectors of any irreducible component generates an irreducible automorphic representation in $\mathcal{A}(B^\times \setminus B_A^\times, \psi)$.*

Theorem II.7.10 (Strong Multiplicity One)**.** *The space $\mathcal{A}_0(B^\times \setminus B_A^\times, \psi)$ has strong multiplicity one.*

Theorem II.7.11 (Strong Multiplicity One)**.** *Let $B^\times$ be a quaternion division algebra. Then the space $L^2(B^\times \setminus B_A^\times, \psi)$ decomposes into a direct sum of irreducible admissible representations of $B_A^\times$. The space of K-finite vectors of any irreducible component generates an irreducible automorphic representation in $\mathcal{A}(B^\times \setminus B_A^\times, \psi)$. Moreover, strong multiplicity one holds.*

Remark II.7.12. All these theorems are deep and were crucial steps in the development of the theory of automorphic forms. The theorems in the split case rely on the presence of cusps in the global model we built using strong approximation at the infinite place, so they are more direct to prove. While strong multiplicity one for non-split quaternion algebras is proved indirectly, as a corollary of the celebrated Jacquet-Langlands correspondence, see [Gel75]. This correspondence relates bijectively subsets of the irreducible components in the decomposition of the space of cuspidal forms for the split quaternion algebra, with the irreducible components of the decomposition of the space of automorphic forms of non-split quaternion algebras. Thus, if strong multiplicity one holds in the split case, we obtain as a corollary of the correspondence a strong multiplicity one in the non-split case.

Chapter III

Representation Theory of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this chapter we will focus on the representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$.

After recalling the basic definitions, we will restrict to the category of *smooth* representations over a locally compact group G, privileging the *admissible* ones. The first basic results will highlight the fact that this category is better behaved than the broader algebraic one. We will then introduce a family of averaging operators, the *Hecke Algebra*. Even though defined analytically, it will embed into the endomorphisms of the vector space underlying the smooth representation, thanks to the topology of G. Additionally, we will prove that a smooth admissible representation is determined by the action of this algebra.

Starting from the second section, we will restrict to $G = \mathrm{GL}_2(\mathbb{Q}_p)$, even though many results generalize to $\mathrm{GL}_n(\mathbb{Q}_p)$ or arbitrary reductive groups over $\mathbb{Q}_p$. The second part will deal with the introduction of models, i.e. concrete realizations of a representation, which are crucial for extrapolating information on certain families of irreducible representations. These will be the *Whittaker Model* and the *Kirillov Model*. The main result will be the characterization of irreducible representations of G that admit such models, called *generic* representations, through the introduction of the *Jacquet module*. As a consequence, we will obtain the existence of the *local newvector* and of the *conductor* of an irreducible generic representation, central notions in this thesis.

In the third section, we will study the *induced representations* and present a classification of the irreducible representations in three classes: the *principal series*, the *Steinberg* representation and the *supercuspidal* representations. In addition, we will define an intertwining operator from the induced model to the Whittaker model of a principal series representation.

In the fourth section, we will study the irreducible admissible generic representations by their conductor. In particular, we will study the zero conductor case given by the *spherical* representations, classified thanks to the action of the Hecke Algebra. Then, thanks to this classification and to a detailed study of the action of the Hecke algebra, we will prove the *uniqueness* of the local newvector. We will, in particular, give explicit expressions for the local newvectors of principal series representations in the Whittaker and Kirillov Model.

Finally, we will present the result of Prasad on the uniqueness of trilinear invariant forms, essential for relating computations involving local newvectors expressed in different models.

III.1 Representations

Let G be a group and V a vector space over a field F.

Definition III.1.1 (Representation)**.** A representation (π, V) of a group G is the datum of a vector space V over a field F together with a group homomorphism $\pi : G \to \mathrm{GL}(V)$.

Given two representations of G over the same field F, it's possible to define the concept of morphism between them.

Definition III.1.2 (Intertwining Operator)**.** Let (π_1, V_1), (π_2, V_2) be representations of G, then a morphism of representations or *intertwining operator* is a linear map $L : V_1 \to V_2$ such that:

$$L(\pi_1(g)v) = \pi_2(g)L(v) \quad \forall v \in V_1, g \in G.$$

Remark III.1.3. The representations of G over F, together with the intertwining operators, form a category denoted by $\mathrm{Rep}_{\mathrm{F}}(G)$.

Given a representation (π, V), it's possible to associate a subrepresentation to any given π-invariant linear subspace $W \leq V$, by $(\pi|_W, W)$. A subrepresentation is *proper* if the associated vector space W is a proper subspace of V. This leads naturally to a crucial definition.

Definition III.1.4 (Irreducible Representation)**.** A representation (π, V) of G is *irreducible* if it has no proper subrepresentations. Otherwise, it is called *reducible.*

On the other hand, there is an alternative definition of representation in terms of modules of a group algebra related to G.

Definition III.1.5 (Group Algebra)**.** Let G be a group and let F be a field. The *group algebra* $F[G]$ is the algebra whose elements lie in the free module over F generated by the set G, with multiplication defined by:

$$\left(\sum_{i=1}^{n} a_i g_i\right)\left(\sum_{j=1}^{m} b_i h_j\right) = \sum_{i,j}(a_i b_j)(g_i h_j) \quad \forall g_i, h_j \in G, a_i, b_j \in F,$$

with identity given by 1_g.

In this context, the natural category we shall be interested to is the category $LMod(F[G])$ of left $F[G]$-modules, by the following proposition.

Proposition III.1.6. *There is an equivalence of categories* $\mathrm{Rep}_F(G) \cong \mathrm{LMod}(F[G])$.

Sketch. Any representation (π, V) induces a left $F[G]$-module structure on V, by $(f \cdot g)v = f\pi(g)v$ for all $g \in G, f \in F, v \in V$. On the other hand, by the definition of group algebra, a left-module over $F[G]$ is also an F-vector space V, on which G has a left action by F-linear automorphisms, $g \mapsto \pi(g)$. It's then straightforward, by looking at the definitions, to see that these two maps induce two functors that give an equivalence of categories. See [Lan02, §XVIII.1]. □

Under this equivalence, the F-vector space V underlying a left $F[G]$-module corresponds to the vector space underlying the associated representation (π, V) and vice versa. As a consequence, classical operations and properties of $LMod(F[G])$ induce equivalent ones on $\mathrm{Rep}_F(G)$. For example, to irreducible representations correspond simple modules. While the semisimple modules are associated to direct sums of irreducible representations.

Proposition III.1.7. *Let (π, V), (π', V') be representations of G over F. Suppose that (π, V) is semisimple. Then $\mathrm{Hom}_{\mathrm{Rep}_F(G)}((\pi, V), (\pi', V')) \neq 0$ if and only if there exists a non-zero subrepresentation of (π, V) contained in (π', V').*

Proof. Consider V, V' with the $F[G]$-module structure induced respectively by π and π'. If V is semisimple, any quotient is isomorphic to $F[G]$-submodule of V. So that, for any non-trivial map $\rho : V \to V'$, we have that $V \supseteq V/\ker(V) \cong \mathrm{Im}(V) \neq 0 \leq V'$.

On the other hand, any submodule W of V is a quotient. So that, if $W \leq V'$ is non-zero, then $\rho = p \circ i : V \to V'$ is a non-trivial map of $F[G]$-modules, where $p : V \to W$ is the projection into the quotient W and $i : W \to V'$ is the inclusion. □

Corollary III.1.8. *Let (π, V), (π', V') be irreducible representations of G over F. Then $\mathrm{Hom}_{\mathrm{Rep}_F(G)}((\pi, V), (\pi', V')) \neq 0$ if and only $(\pi, V) \cong (\pi', V')$.*

Proof. The only nontrivial submodule of V is V. The same holds for W. As a consequence of the previous proposition $V \leq W$, hence $V = W$. □

One of the main goals in representation theory is the complete classification and description of irreducible representations. In general, a complete description of $\mathrm{Rep}_F(G)$ doesn't come only from the knowledge of the irreducible representations, since there could exist non-semisimple ones. However, in some cases much stronger results can be obtained.

Theorem III.1.9 (Maschke). *Let G a finite group and F a field. Then every left $F[G]$-module is semisimple if and only if* $\mathrm{char}(F) \nmid |G|$.

Proof. See [Lan02, §XVIII.1 Theorem 1.2]. □

Finally, we introduce representations compatible with the analytic structure of the associated vector space.

Definition III.1.10 (Unitary Representation)**.** Let V be vector space over $\mathbb{R}$ or over $\mathbb{C}$, with respectively a positive definite or a hermitian inner product. A representation (π, V) of a group G is *unitary* if

$$\langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle \quad \forall v, w \in V, g \in G.$$

Remark III.1.11. Any unitary representation (π, V) extends to a unitary representation of $(\pi, \bar{V})$, where $\bar{V}$ is the Hilbert space obtained by metric completion of V.

We will say (π, V) is *unitarizable* if it can be endowed with a $\pi(G)$-invariant hermitian or positive definite inner product.

Remark III.1.12. In general, unitary representations have nicer properties, since we imposed stronger constraints on their structure. However, for this reason, it's usually not convenient to study directly irreducible unitary representations, since the classification is easier in broader categories. As a consequence, it's better to understand first the irreducible representations in a different subcategory of $\mathrm{Rep}(G)$. After that, one can try to characterize the unitarizable ones inside the other subcategory. In the case of a locally compact totally disconnected group, as $\mathrm{GL}_2(\mathbb{Q}_p)$ is, the good category is the one of *smooth* or *admissible* representations.

III.2 Smooth and Admissible Representations

Fix F a non-archimedean local field, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ its maximal ideal and ϖ a generator of $\mathfrak{p}$.

Let G be totally disconnected locally compact group. We will consider only complex representation of G.

The category $\mathrm{Rep}_{\mathbb{C}}(G)$ doesn't account for the topology of G and contains objects which are not meaningful in the theory of automorphic forms. As a consequence, we will restrict to a smaller and better-behaved subcategory.

Definition III.2.1 (Smooth and Admissible Representation)**.** A representation (π, V) is *admissible* if the following two conditions are satisfied:

1. *smoothness*: for any $v \in V$ there exists an open subgroup $U \leq G$, such that $\pi(u)v = v$ for any $u \in U$;
2. *local finiteness*: for any open subgroup $U \leq G$, the space V^U of vectors stabilized by U is finite dimensional.

A representation satisfying (1) is called *smooth*.

Remark III.2.2. Since G is a locally compact totally disconnected group, any open subgroup U contains a compact open subgroup $K \leq U$. As a consequence, the previous definitions don't change if we restrict to compact open subgroups in the smoothness and local finiteness conditions.

Remark III.2.3. For any representation (π, V), the space of smooth vectors V^S defines a subrepresentation. It's easy to verify then that the restriction to V^S defines a functor $\mathrm{Rep}_{\mathbb{C}}(G) \rightsquigarrow \mathrm{SmRep}_{\mathbb{C}}(G)$.

The restriction to smooth representations has different advantages. First, let K be a compact open subgroup of G. Denote with $\hat{K}$ the set of equivalence classes of irreducible representations of K fixed by an open subgroup. Any such representation will be finite dimensional, since it restricts to an irreducible representation of the compact and discrete, hence finite, group K/K_0, for a normal open subgroup K_0 of K. For any $\rho \in \hat{K}$, define the *ρ-isotypic component* $V(\rho)$ of the representation (π, V) of G, as the sum of all the $\mathbb{C}[K]$-submodules of V isomorphic to ρ. Then V decomposes in the following way.

Proposition III.2.4. *Let (π, V) be a smooth representation of G. Consider a compact subgroup K. Then*

$$V = \bigoplus_{\rho \in \hat{K}} V(\rho).$$

The representation π is admissible if and only if each of the ρ-isotypic components is finite dimensional.

Proof. Any vector $v \in V$ is fixed by a compact open subgroup K_0, that can be chosen to be normal. Hence, the $\mathbb{C}[K]$-submodule generated by V is a complex representation of the finite group $K/K \cap K_0$, so by Maschke Theorem it is semisimple. As a result, $v \in \sum_{\rho \in \hat{K}} V(\rho)$.

We have to show that $\sum_{\rho \in \hat{K}} V(\rho)$ is a direct sum. Consider a non-trivial relation $\sum_{i=1}^{n} v_{\rho_i} = 0$, for $v_{\rho_i} \in V(\rho_i)$. Then there exists a compact open subgroup K' that fixes all the ρ_i, by taking the intersection of the stabilizers of the representations. As a consequence, $V' = \sum_{i=1}^{n} V(\rho_i)$ is a semisimple representation of the finite group K/K' and factors as $\bigoplus_{i=1}^{n} V'(\rho_i) = \bigoplus_{i=1}^{n} V(\rho_i)$.

For the last statement, suppose that (π, V) is admissible. If a ρ-isotypic component was infinite dimensional, considering its stabilizer K', we would obtain that $V^{K'}$ is infinite dimensional. By the assumption that K' contains an open subgroup, we obtain a contradiction with the admissibility. On the other hand, if (π, V) is not admissible, we can choose a normal open subgroup K' such that $V^{K'}$ is infinite dimensional. But then $V^{K'}$ is an infinite dimensional representation of the finite group K/K' and as a consequence has only finitely many equivalence classes of irreducible representation. This imply that at least one of the ρ-isotypic components of $V^{K'}$, and hence of V, is infinite dimensional. □

Given a representation (π, V), the vector space of $\mathbb{C}$-linear functionals $\hat{v} : V \to \mathbb{C}$ has a natural structure of left $\mathbb{C}[G]$-module defined by $g \cdot \hat{v}(w) = \hat{v}(g^{-1} \cdot w)$, for any $w \in V$. It is called the *dual* or *algebraic dual representation*. In the context of smooth representations, its smooth part is called the *contragredient representation* and denoted by $(\hat{\pi}, \hat{V})$.

Proposition III.2.5. *Let (π, V) be an admissible representation, then there is an isomorphism of representations $(\hat{\hat{\pi}}, \hat{\hat{V}}) \cong (\pi, V)$.*

Proof. There is a natural injective morphism of representations given by the evaluation $ev : (\pi, V) \to (\hat{\hat{\pi}}, \hat{\hat{V}})$. We only need to verify that it is surjective.

Fix an open compact subgroup K and consider the decomposition of V in its ρ-isotypic components. A smooth functional $\hat{v}$ is fixed by an open normal subgroup K^0, which implies that it can be realized as an element of the subspace $(\bigoplus_{\rho \in \widehat{K/K^0}} V(\rho))^* = \bigoplus_{\rho \in \widehat{K/K^0}} (V(\rho))^*$, since it is a finite dimensional space by the admissibility of (π, V). This means that $V^* \subseteq \bigoplus_{\rho \in \hat{K}} V(\rho)^*$ and $V^{**} \subseteq \bigoplus_{\rho \in \hat{K}} V(\rho)^{**}$. Then, the surjectivity follows by the fact that the evaluation factors into the ρ-isotypic components $ev|_\rho : V(\rho) \to V(\rho)^{**}$. □

We see that admissible representation inherit properties of finite dimensional representations.

Proposition III.2.6 (Schur's lemma)**.** *Let (π, V) be an irreducible admissible representation of G. Let $T : V \to T$ be an intertwining operator for π. Then there exists a complex number c such that $T(v) = cv$, for all $v \in V$.*

Proof. If $V = 0$ it is trivial, with $c = 0$. If $V \neq 0$, there exists a compact open subgroup K such that V^K is non-trivial and is finite dimensional. Then $T(V^K) \subseteq$

V^k, so that $T|_{V^K}$ has a non-trivial eigenvector with eigenvalue c. Then, $\ker(T - cId)$ is a nontrivial subrepresentation of V, which means that $\ker(T - cId) = V$, since (π, V) is simple. □

Corollary III.2.7 (Central character)**.** *Let Z be the center of G. Consider an irreducible admissible representation (π, V) of G. Then there exists a smooth character $\chi_\pi : Z \to \mathbb{C}$ such that $\pi(z)v = \chi_\pi(z)v$, for all $z \in Z$.*

Proof. For any $z \in Z$ the map $\pi(z) : v \mapsto \pi(z)v$ is an invertible intertwining operator $V \to V$. So it acts by multiplication by a non-zero scalar $\chi_\pi(z)$. By the fact that $\pi(z)\pi(z') = \pi(zz')$ we conclude that χ_π is a character. □

We conclude with a strong result on admissible unitary representations.

Proposition III.2.8. *Any admissible unitary representation (π, V) of G is semisimple.*

Proof. It's enough to prove that any submodule W has a complement W' such that $W \oplus W' = V$. Take $W' = W^\perp$. With this choice W' is again a submodule since the hermitian product is G-invariant and W is a submodule. Moreover $W \cap W' = \langle 0 \rangle$ is trivial since the hermitian form is positive definite. We need to verify that $W \oplus W' = V$. For a compact open subgroup K we have that $V = \bigoplus_{\rho \in \hat{K}} V(\rho)$ for finite dimensional subspaces $V(\rho)$. Since the ρ-isotypic components are mutually orthogonal under a K-invariant hermitian product, by the theory of characters of finite groups, we obtain that $(W \cap V(\rho)) \oplus (W^\perp \cap V(\rho)) = V(\rho)$, so that $W \oplus W' = V$. □

III.3 Hecke algebra and Distributions

Another reason for preferring smooth representation to algebraic ones is the existence of a family of averaging operators acting as endomorphisms of the smooth representations.

Definition III.3.1 (Smooth Functions)**.** The space $C^\infty(G)$ of smooth functions is the space of complex locally constant functions $f : G \to \mathbb{C}$.

Remark III.3.2. Since G is a totally disconnected locally compact group, $f \in C^\infty(\mathbb{C})$ if and only if there exists an open compact subgroup $K \leq G$ such that $f(gK) = f(g)$ for any $g \in G$.

Suppose now that G is unimodular, endowed with a Haar measure.

The space $C_c^\infty(G)$ of smooth functions with compact support has an algebra structure, where the product is defined by convolution:

$$\phi_1 * \phi_2 = \int_G \phi_1(gh^{-1})\phi_2(h)dh \quad \forall \phi_1, \phi_2 \in \mathbb{C}_c^\infty(G).$$

Definition III.3.3 (Hecke Algebra)**.** The *Hecke Algebra* $\mathcal{H}$ is the algebra $(C_c^\infty(G), *)$.

For any compact open subgroup K_0, the subalgebra of bi-K_0-invariant functions in the Hecke algebra is denoted by $\mathcal{H}_{K_0}$.

Let (π, V) be a smooth representation of G. Then, there is a left action of $\mathcal{H}$ on V. For any $\phi \in \mathcal{H}$ define

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v\, dg.$$

This action is actually algebraic and well-defined since, choosing a compact open K, small enough to fix both v and ϕ, and a compact open $K' \supseteq \mathrm{Supp}(\phi)$, the previous action can be written as:

$$\mathrm{vol}(K) \sum_{g \in K'/K} \phi(g)\pi(g)v,$$

which is a finite sum. Additionally, it's easy to see that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2).$$

As a consequence, a smooth representation of G has a structure of left $\mathcal{H}$-module. Actually, the mapping defines a functor

$$F : \mathrm{SmLMod}(\mathbb{C}[G]) \rightsquigarrow \mathrm{LMod}(\mathcal{H}).$$

Moreover, since $\mathcal{H}_K$ is a subalgebra for any compact open subgroup K, and for $K \supseteq K'$ there is an inclusion $\mathcal{H}_K \subseteq \mathcal{H}_{K'}$, we have an associated algebra $\hat{\mathcal{H}} = \varinjlim_{(K,\supseteq)} \mathcal{H}_K$, by inductive limit. This fact leads to the definition of a functor

$$\hat{F} : \mathrm{SmLMod}(\mathbb{C}[G]) \rightsquigarrow \mathrm{LMod}(\hat{\mathcal{H}})$$

that maps a module $V \mapsto \hat{V} = \varinjlim_{K \subseteq G} V^K$, with the inclusion $V^K \subseteq V^{K'}$ for $K \supseteq K'$, and the natural action of $\mathcal{H}_K$ on V^K.

Theorem III.3.4. *The functor* $F : \mathrm{SmLMod}(\mathbb{C}[\mathbf{G}]) \rightsquigarrow \mathrm{LMod}(\mathcal{H})$ *is fully faithful. The functor* $\hat{F} : \mathrm{SmLMod}(\mathbb{C}[\mathbf{G}]) \rightsquigarrow \mathrm{LMod}(\hat{\mathcal{H}})$ *is fully faithful. Moreover, the functors induce a bijection between the submodules of an object and the submodules of its image.*

Proof. For any vector $v \in V$, of a module $V \in \mathrm{Im}(F)$, and any $g \in G$ we have that there exists an open compact subgroup $K_{v,g}$ such that $\pi(hgj)v = \pi(g)v$, for any $h, j \in K_{v,g}$. For example, we could take $K_{v,g}$ as an open compact in the intersection of the stabilizer of v with the stabilizer of $\pi(g)v$. With this choice, we get that

$$e_{K_{v,g}gK_{v,g}} = \frac{1}{\mathrm{vol}(K_{v,g}gK_{v,g})} 1_{K_{v,g}gK_{v,g}}$$

acts as $\pi(g)$ on v. Then, for any morphism of left $\mathcal{H}$-modules $\rho : F(V) \to F(W)$, we have that, taking $w = \rho(v)$ and $K'_{v,g} = K_{v,g} \cap K_{w,g}$,

$$\rho(e_{K'_{v,g}gK'_{v,g}}v) = e_{K'_{v,g}gK'_{v,g}}\rho(v) \Rightarrow \rho(\pi(g)v) = \pi'(g)\rho(v).$$

It's straightforward to conclude the first part from this, by checking this is the inverse map on the set of morphisms.

Notice that we only used the fact that both v and $\rho(v)$ were smooth, so that we could consider also mappings $i : W \hookrightarrow F(V)$, for a $\mathcal{H}$-submodule W of $F(V)$.

By the previous reasoning, we conclude that these subspaces are also closed under $\pi(G)$.

For the second functor, notice first that $V \cong \varinjlim_{K\subseteq G} V^K$ as $\mathbb{C}$-vector spaces, since V is smooth. So, any morphism $\rho : \hat{V} \to \hat{W}$ of $\hat{\mathcal{H}}$-modules induces a $\mathbb{C}$-linear map on $\rho' : V \to W$. A careful inspection of the first part of the proof, together with the previous observation, already provides the key result. For any $\hat{v} \in \hat{V}$, let $\hat{w} = \rho(v)$, v, w the corresponding elements in V, W. Then, for any $g \in G$ there exists an open compact subgroup $K'_{v,g}$ and an element $w \in \mathcal{H}_{K'_{v,g}}$, such that

$$\rho(e_{K'_{v,g}gK'_{v,g}}\hat{v}) = e_{K'_{v,g}gK'_{v,g}}\rho(\hat{v}) \Rightarrow \rho(\pi(g)v) = \pi'(g)\rho(v).$$

This implies, after a quick check, the surjectivity of $\hat{F}$ on the set of homomorphisms. The injectivity comes directly by the first observation that $\hat{F}(\rho) = \rho$ as $\mathbb{C}$-linear maps, after the identification $V \cong \varinjlim_{K\subseteq G} V^K$.

Notice again, that we only used the fact that both v and $\rho(v)$ were smooth, so that we could consider also mappings $i : W \hookrightarrow F(V)$, for a $\mathcal{H}$-submodule W of $F(V)$. By the previous reasoning, we conclude that these subspaces are also closed under the action of $\pi(G)$ $\square$

Remark III.3.5. Actually, there is an equivalence of categories $\mathrm{LMod}(\mathcal{H}) \cong \mathrm{LMod}(\hat{\mathcal{H}})$, since for our choice of G we have that $\mathcal{H} \cong \hat{\mathcal{H}}$ as algebras.

The previous result has already interesting consequences.

Corollary III.3.6. *Let (π, V) be a smooth representation of G. Assume that V is non-zero. Then the following are equivalent:*

1. *The representation (π, V) is irreducible;*
2. *V is simple as a $\mathcal{H}$-module;*
3. *V^K is simple as a $\mathcal{H}_K$-module, for any compact open subgroup $K \leq G$.*

If (π, V) is moreover irreducible and admissible, and (π', V') is smooth, then $(\pi, V) \cong (\pi', V')$ if and only if $V^K \cong V'^K$ as $\mathcal{H}_K$-modules, for any compact open subgroup $K \leq G$.

Proof. The first part follows easily by Theorem III.3.4, since the functors F and $\hat{F}$ introduced before preserve the submodule structure.

For the last statement, the direct implication is immediate. For the converse, since $V^K \equiv V'^K$, for any open compact K, both representations are irreducible. Moreover, by Schur's Lemma, since V^K is finite dimensional, any such isomorphism $V^K \equiv V'^K$ lies in a 1-dimensional $\mathbb{C}$-vector space. As a consequence, by adjusting the coefficients, we get a compatible system of morphisms $\rho^K : V^K \to V'^K$, hence a morphism $\hat{\rho} : \hat{V} \to \hat{V}'$. Since the morphism is non-trivial and both modules are simple, this implies that the corresponding representations (π, V) and (π', V') are isomorphic. $\square$

A reason for looking actually at an action of $\mathbb{C}_c^\infty(G)$ on an admissible representation (π, V) is the theory of characters. Any element of the Hecke algebra $\mathcal{H}$ acts as an endomorphism of V with finite image, since it lies in $\mathcal{H}_K$ for some small enough open compact subgroup $K \leq G$. As a consequence, we can define a linear functional $\chi : \mathbb{C}_c^\infty(G) \to \mathbb{C}$, called the character of π, by $\chi(f) = \mathrm{tr}(f)$. The following result explains the introduction of this object.

Proposition III.3.7. *Let (π_1, V_1), (π_2, V_2) be irreducible admissible representations of G. If the characters of π_1 and π_2 are equal, then the two representations are isomorphic.*

Proof. This result holds for finite dimensional simple R-modules N, M, for an algebra R over a field of characteristic 0, by [Lan02, §XVII.3 Corollary 3.3]. Then, by applying that theorem to $R = \mathcal{H}_K$ and $N = V_1^K$, $M = V_2^K$, we get that if the characters of π_1 and π_2 agree on any $f \in \mathcal{H}^K$, for any compact open subgroup $K \leq G$, then $V_1^K \equiv V_2^K$ for any compact open subgroup $K \leq G$. This is enough to conclude that $(\pi_1, V_1) \equiv (\pi_2, V_2)$ by the previous result. □

Remark III.3.8. Looking at the proof in [Lan02, §XVII.3 Corollary 3.3], in this setting, the previous proposition could be extended to semisimple admissible representations, as the unitary admissible ones.

This result implies that some questions on simple admissible representations, can be translated in terms of problems on *distributions* $\mathfrak{D}(X)$ on some totally disconnected locally compact space X, where a distribution D is just a $\mathbb{C}$-linear map $D : \mathbb{C}_c^\infty(X) \to \mathbb{C}$. In fact, taking $X = G$, we see that any character of an irreducible admissible representation (π, V) is a distribution, and it is zero on $\mathcal{H}_K$ if and only if V^K is zero, for some open compact subgroup $K \leq G$. This point of view can be developed in a sheaf theoretic setting, by reinterpreting distributions as duals of a class of $\mathbb{C}_C^\infty$-sheaves on a totally disconnected locally compact space X. See [Bum97, §4.3] and [BZ76] for more details.

III.4 Whittaker and Kirillov Model

From now on, we will restrict to $G = GL_2(F)$, for a local non-archimedean field F, even though many results hold also on $G = GL_n(F)$. Let ψ be a non-trivial additive character of F. Denote with $N(F)$ the group of upper triangular unipotent matrices of G. Under the isomorphism $N(F) \cong F$, such that $n(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto a$, we see that the character ψ defines a character ψ_N of $N(F)$ by $\psi_N(n(a)) = \psi(a)$.

Let (π, V) be a smooth representation of G. A *Whittaker functional* is a linear functional $\lambda : V \to \mathbb{C}$ such that $\lambda(\pi(u)x) = \psi_N(u)\lambda(x)$, for all $u \in N(F)$, $x \in V$. We assume the following theorem.

Theorem III.4.1 (Uniqueness of the Whittaker functional)**.** *Let (π, V) be an irreducible admissible representation of $GL_2(F)$. Then the dimension of the space of Whittaker functionals on V is at most one.*

The proof of the theorem is out of the scope of this work and can be deduced by properties of the distributions on G. See [Bum97, §4.3] for a proof.

Now we will determine which irreducible smooth representations actually admit such a functional. In order to accomplish this, we introduce a fundamental object called *Jacquet module* and the associated *Jacquet functor*. Let (π, V) be a smooth representation of the Borel subgroup $B(F)$ of upper triangular matrices of G. Let $V_N \subseteq V$ be the linear subspace generated by the elements:

$$\pi(u)v - v \quad u \in N(F), v \in V.$$

Let $T(F)$ be the diagonal torus of G. Since $T(F)$ is in the normalizer of $N(F)$, we see that $\pi(t)V_N = V_N$, for any $t \in T(F)$. Thus, the quotient $J(V) = V/V_N$ has the structure of a left $\mathbb{C}[T(F)]$-module, so it defines a representation $(\pi_N, J(V))$ of $T(F)$, that is still smooth. Then $J(V)$ is called the *Jacquet module* of V. The *Jacquet functor* $J : \mathrm{SmRep}(B) \rightsquigarrow \mathrm{SmRep}(T)$ maps then $V \mapsto J(V)$ and associates to any morphism $\rho : V \to W$ of smooth left $\mathbb{C}[B]$-modules V, W the induced quotient morphism of T-modules, that is well-defined since $\rho(J(V)) \subseteq J(W)$ and $T(F)$ is normalized by $B(F)$.

Proposition III.4.2. *Let (π, V) be a smooth representation of $B(F)$. Then the following hold:*

1. *$v \in V_N$ if and only if*
$$\int_{\mathfrak{p}^{-n}} \pi(n(x))v dx = 0$$
for sufficiently large $n \in \mathbb{N}$;

2. *The Jacquet functor is exact.*

Proof. 1) First, we stress that the integral is well-defined since $\mathfrak{p}^{-n}$ is an open compact in $T(F)$, totally discrete locally compact group. So that $1_{\mathfrak{p}^{-n}} \in \mathcal{H}$, the Hecke algebra of $T(F)$. Since $T(F)$ is moreover a closed subgroup of $B(F)$ it's not difficult to see that its Hecke algebra is obtained by restriction of the Hecke algebra of $B(F)$, see [Bum97, §4.3]. For the first inclusion, notice that, since any $v \in V_N$ is a finite sum of elements of the form $\pi(n(a))w - w$, for $w \in V$, $a \in F$, without loss of generality, we can restrict to a generic $v = \pi(n(a))w - w$, for some $a \in \mathfrak{p}^{-m}$, $m \in \mathbb{N}$. Then, we have that:

$$\int_{\mathfrak{p}^{-m}} \pi(n(x))\pi(n(a))w dx = \int_{\mathfrak{p}^{-m}} \pi(n(x+a))w dx = \int_{\mathfrak{p}^{-m}} \pi(n(x))w dx,$$

since $a \in \mathfrak{p}^{-m}$. By subtracting the first and last term of the equation, we get that $\int_{\mathfrak{p}^{-n}} \pi(n(x))v dx = 0$, which is the desired result. Conversely, if $\int_{\mathfrak{p}^{-n}} \pi(n(x))v dx = 0$, let $m > -n$, such that $v \in V^{\mathfrak{p}^m}$. Then the integral breaks as a finite sum:

$$0 = \int_{\mathfrak{p}^{-n}} \pi(n(x))v dx = \sum_{a \in \mathfrak{p}^{-n}/\mathfrak{p}^m} \pi(n(a))v.$$

By weighting this term by q^{-n-m} and adding v on both sides, we obtain:

$$v = \sum_{a \in \mathfrak{p}^{-n}/\mathfrak{p}^m} q^{-n-m} v - q^{-n-m}\pi(n(a))v,$$

which implies that $v \in V_N$.

2) Notice that $J(V) = \mathrm{coker}(V_N \hookrightarrow V)$. Then, by an application of the Snake Lemma, we just need to show that if

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{p} V'' \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow V'_N \xrightarrow{i_N} V_N \xrightarrow{p_N} V''_N \longrightarrow 0$$

is exact. The injectivity of the first map and the surjectivity of the last map are straightforward. We only need to prove that $\mathrm{im}(i_N) = \ker(p_N)$. One inclusion is trivial. The other inclusion comes from the previous point. In fact, let $v \in V_N$ such that $p_N(v) = 0$, or equivalently $p(v) = 0$. Then $v \in \mathrm{Im}(V') \cap V_N$. Thus we just need to show that, for a given submodule $M \leq V$, $M_N = M \cap V_N$. The condition $\int_{\mathfrak{p}^{-n}} \pi(n(x))v dx = 0$, for $v \in M$ and some $n \in \mathbb{N}$, clearly implies that. □

We can define a twisted version of the Jacquet functor. Recall that ψ is a non-trivial additive character of F, that defined a character ψ_N of $N(F)$. Let $Z(F)$ be the center of G. If we denote with $V_{N,\psi}$ the $Z(F)$-submodule generated by $\pi(u)v - \psi_N(u)v$, for $u \in N(F)$, $v \in V$, the *twisted Jacquet module* is the smooth $Z(F)$-module $V/V_{N,\psi}$. As before, we have the associated *twisted Jacquet functor* $J_\psi : \mathrm{SmRep}(B) \rightsquigarrow \mathrm{SmRep}(Z)$ that maps $V \mapsto V_{N,\psi}$.

Proposition III.4.3. *Let (π, V) be a smooth representation of $B(F)$. Then the following hold:*

1. *$v \in V_{N,\psi}$ if and only if*

$$\int_{\mathfrak{p}^{-n}} \overline{\psi(x)}\pi(n(x))v dx = 0$$

for sufficiently large $n \in \mathbb{N}$;

2. *The twisted Jacquet functor J_ψ is exact.*

Proof. The proof is identical to the one of the non-twisted version. □

Remark III.4.4. Consider the forgetful functor $\mathrm{Res}_B : \mathrm{SmRep}(G) \rightsquigarrow \mathrm{SmRep}(B)$. The composition $J_\psi \circ \mathrm{Res}_B$ is again called a (twisted) Jacquet functor which has the same properties stated in the previous propositions.

Proposition III.4.5. *Let (π, V) be an irreducible representation of G and let ψ be a non-trivial character of F. Then* $\mathrm{dim} J_\psi(V) \leq 1$.

Proof. A linear functional $\tilde{v}$ on V is a Whittaker functional if and only if $\tilde{v}(V_{N,\psi}) = 0$. As a consequence, the linear space of Whittaker functionals can be identified with the algebraic dual of $J_\psi(V)$. Then, by Theorem III.4.1, it follows that $\mathrm{dim} J_\psi(V) \leq 1$. □

Now, we are going to construct a sheaf on the totally disconnected locally compact group F, whose properties will be used to determine the existence of the Whittaker model for a large class of irreducible admissible representation.

Let $\phi \in C_c^\infty(F)$. Define the Fourier transform of ϕ by:

$$\hat{\phi}(x) = \int_F \phi(x)\psi(xy) dy.$$

Some formal properties can be derived, see [Tat67] for the proofs:

1. for a given non-trivial character ψ, there exists a unique Haar measure dy on F, called self-dual, such that the Fourier Inversion formula $\hat{\hat{\phi}}(x) = \phi(-x)$ holds for any $\phi \in C_c^\infty(F)$;

2. the Fourier transform is an isomorphism of the algebra $(C_C^\infty(F), *)$ into $(C_C^\infty(F), \cdot)$, endowed with the pointwise product;

3. if $\mathfrak{p}^n$ is the conductor of ψ, then the characteristic function $1_{\mathfrak{p}^{-k}}$ of the ideal $\mathfrak{p}^{-k}$ has Fourier transform equal to $\mathrm{Vol}(\mathfrak{p}^{-\mathrm{k}})1_{\mathfrak{p}^{n+k}}$.

Remark III.4.6. The last point, together with the Fourier Inversion formula implies that $\mathrm{Vol}(\mathfrak{p}^{-k})\mathrm{Vol}(\mathfrak{p}^{n+k}) = 1$.

If we consider a smooth $B(F)$-module (π, V) and we restrict it to a smooth representation (π_1', V) of F by

$$\pi_1(x)v = \pi(n(-x))v \quad \forall x \in F,$$

we obtain an action of the Hecke algebra $\mathcal{H}$ of F, which coincides with $(C_c^\infty(F), *)$, choosing as Haar measure the self-dual measure associated to the character ψ.

By Fourier Inversion, we obtain also a $(C_c^\infty(F), \cdot)$-module. Just define:

$$\phi \cdot (v) = \rho(\hat{\phi})v = \int_F \hat{\phi}(x)\pi(n(-x))v dx.$$

Any vector v is fixed by some open subgroup $\mathfrak{p}^{-k}$. By the fact that F is locally compact, we obtain an associated presheaf on F of $C_c^\infty(F)$ modules. Its sheafification, in the sense of [Har77], is denoted by $\mathcal{S}(V)$.

Proposition III.4.7. *Let V be a smooth $B(F)$-module, and let $a \in F$. Then the stalk*

$$\mathcal{S}(V) \cong \begin{cases} J(V) & \text{if } a = 0; \\ J_{\psi_a}(V) \cong J_\psi & \text{if } a \neq 0. \end{cases}$$

More specifically, the projection map $V \to \mathcal{S}(V)_a$ is surjective, and its kernel is V_N if $a = 0$, and V_{N,ψ_a} if $a \neq 0$.

Proof. For any given a, we first determine the elements $v \in V$ such that there exists an open subgroup $\mathfrak{p}^k$, such that $1_{a+\mathfrak{p}^k} \cdot v = 0$. We have that $\hat{1}_{a+\mathfrak{p}^k} = \psi(ax)\mathrm{Vol}(\mathfrak{p}^k)\mathfrak{p}^{n-k}(x)$. Thus, by definition:

$$0 = 1_{a+\mathfrak{p}^k} \cdot v \iff \int_{\mathfrak{p}^{n-k}} \psi(ax)\pi(n(-x))v dx = \int_{\mathfrak{p}^{n-k}} \overline{\psi(ax)}\pi(n(x))v dx = 0.$$

For $a = 0$, by proposition III.4.2 the condition is equivalent to $v \in V_N$. While for $a \neq 0$ we recognize the condition for $v \in V_{N,\psi_a}$. The isomorphism $J_{\psi_a}(V) \cong J_\psi$ is realized by applying $\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$ to J_ψ. □

We are ready to prove the main result.

Theorem III.4.8. *Let (π, V) be a smooth representation of $\mathrm{GL}_2(F)$ that has no Whittaker functional. Then π factors through the determinant map $\det : \mathrm{GL}_2(F) \to F^\times$. In particular, if π is irreducible and admissible, it is one dimensional.*

Proof. Recall that a linear functional $\Lambda : V \to \mathbb{C}$ is a Whittaker functional if and only if $\Lambda(J_\psi(V)) = 0$. Then, the twisted Jacquet module J_ψ must be trivial, if not, since it has at most dimension 1, we would have a non-trivial Whittaker functional. This implies, by proposition III.4.7, that the sheaf $\mathcal{S}(V)$ is supported in 0. So it is a skyscraper sheaf generated by $J(V)$ at 0. The restriction $V \to V_0$ to the stalk at 0 is an isomorphism, hence $V \cong J(V)$ and $V_N = 0$. As a result, any vector is fixed by the action of $N(F)$, and by the action of its conjugates. Since the conjugates of $N(F)$ generate $\mathrm{SL}_2(F)$, we obtain that π factors through the determinant and is one dimensional, by the fact it is a smooth irreducible representation of $F^\times$. □

Definition III.4.9. A smooth representation (π, V) of G is called *generic* if it admits a non-trivial Whittaker functional.

Corollary III.4.10. *Let (π, V) be a irreducible admissible generic representation of G. Then $V^{N(F)} = \langle 0 \rangle$.*

Proof. See [Bum97, Proposition 4.4.6] □

A direct consequence of this theory is the following crucial property of generic representations.

Proposition III.4.11. *Let (π, V) be an irreducible admissible generic representation of G, with a Whittaker functional $\Lambda : V \to \mathbb{C}$. For any non-zero vector v, there exists $a \in F^\times$ such that*

$$\Lambda\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) \neq 0.$$

Proof. Suppose by contradiction that there exists $v \neq 0 \in V$ such that

$$v \in \bigcap_{a \in F^\times} \ker\left(v \mapsto \Lambda\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right)\right).$$

Consider the sheaf $\mathcal{S}(V)$ constructed previously. The above condition is equivalent to $v \in V_{N,\psi_a}$, for any $a \in F^\times$ by Proposition III.4.7. Then v, seen as a section of $\mathcal{S}(V)$, is supported only in 0. If we introduce then

$$v' = v - \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v,$$

for any $x \in F$, it follows that v' has again all the stalks $v'_a = 0$, for $a \in F^\times$ and moreover, by construction, $a \in V_N$, so that $v'_0 = 0$. Thus $v' = 0$. As a result, $v \in V^{N(F)}$, which is a contradiction to Corollary III.4.10. □

Consider an irreducible admissible generic representation (π, V), with a Whittaker functional $\Lambda : V \to \mathbb{C}$ associated to a non-trivial character ψ of F. To any vector $v \in V$ we can associate a function $W_v : G \to \mathbb{C}$ by:

$$W_v(g) = \Lambda(\pi(g)v) \quad \forall g \in G.$$

By construction, any such function W_v is smooth, has the property that $W_v(n(x)g) = \psi(x)W_v(g)$ and is endowed with a left action of G by right translations

$$h \cdot W_v(g) = W_v(gh) \quad \forall h, g \in G.$$

Let $\mathcal{W}(\pi,\psi)$ be the $\mathbb{C}$-vector space of these functions. Then the map $\rho : v \mapsto W_v$ is linear, surjective, injective by Proposition III.4.11 and G-equivariant, so it is a bijective intertwining operator $\rho : (\pi, V) \to \mathcal{W}(\pi,\psi)$, which implies that $(\pi, V) \cong \mathcal{W}(\pi,\psi)$. The space $\mathcal{W}(\pi,\psi)$ endowed with the action of G by right translations is called the *Whittaker model*.

Remark III.4.12. Let W be a $\mathbb{C}[G]$-submodule of the space of the complex functions $f : G \to \mathbb{C}$ endowed with the right translation. Suppose that there exists a subset $D \subset G$, such that the restriction $W|_D = \{f|_D \mid f \in W\}$ is injective. Then $W|_D$ has a natural structure of $\mathbb{C}[G]$-submodule given by $g \cdot f|_D = (g \cdot f)|_D$, well-defined since the restriction is injective. With this action $W|_D \cong W$ as $\mathbb{C}[G]$-modules, with intertwining operator given by the restriction. In fact, this is just a consequence of the first isomorphism theorem for $\mathbb{C}[G]$-modules.

A second model, related to the Whittaker model $\mathcal{W}(\pi,\psi)$, is obtained by the restriction to the subgroup $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, for $a \in F^\times$. In fact, by Proposition III.4.11, the restriction $\mathcal{W}(\pi,\psi) \to \mathcal{W}(\pi,\psi)|_A$ is injective. By the previous remark there is a natural structure of $\mathbb{C}[G]$-module induced by the restriction, such that $\mathcal{W}(\pi,\psi)|_A \cong (\pi, V)$. This model is called the *Kirillov model* associated to (π, V) and is denoted by $\mathcal{K}(\pi,\psi)$. The underlying vector space is a subspace of $C^\infty(F^\times)$. Even though the action of G is not as easy to describe as in the Whittaker model, a finer study of its structure can be achieved. For example, since it is a restriction to the diagonal of the Whittaker model, it's easy to see that:

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \phi(ax) \quad \pi\left(\begin{pmatrix} 0 & 1 \\ b & 1 \end{pmatrix}\right)\phi(x) = \psi(bx)\phi(x). \tag{III.1}$$

For our scope, the next result will be sufficient.

Proposition III.4.13. *Let (π, V) be an irreducible admissible generic representation of G. Let $\mathcal{K}(\pi,\psi)$ be the Kirillov model associated to the non-trivial character ψ of F. Then $C_c^\infty(F^\times) \subseteq \mathcal{K}(\pi,\psi)$ and corresponds to V_N in this model.*

Proof. We need the following lemma.

Lemma III.4.14. *Let $\phi \in \mathcal{K}(\pi,\psi)$, Kirillov model of the admissible irreducible generic representation (π, V). Then ϕ is locally constant and there exists a constant $C > 0$ such that $\phi(y) = 0$ if $|y| > C$. If moreover $\phi \in V_N$, then there exists an $\epsilon > 0$ such that $\phi(y) = 0$ when $|y| < \epsilon$.*

Proof. The smoothness is immediate, by Eq. III.1.

Since π is smooth, there exists a $\mathfrak{p}^k$ such that $\pi(N(\mathfrak{p}^k))\phi = \phi$, which is equivalent to

$$\phi(y) = \psi(xy)\phi(y) \quad \forall x \in \mathfrak{p}^k, y \in A.$$

This is possible only if $\phi(y) = 0$ for $|y| > C$, for some $C > 0$, since ψ is a nontrivial character. While if $\phi \in V_N$, it is a finite sum of elements of the form $\phi' - \pi(n(x))\phi'$, for $x \in F$. So we just need to prove the result for $\phi = \phi' - \pi(n(x))\phi'$. Again by Eq. III.1 we see that

$$\phi(y) = (\psi(xy) - 1)\phi'(y),$$

for a fixed $x \in F$. This implies that for $|y|$ small enough $\phi(y) = 0$, since ψ is a smooth character of F. $\square$

From the above lemma we obtain that $V_N \subseteq C_c^\infty(F^\times)$ in the Kirillov model. Moreover, by Corollary III.4.10, we get that there exists $v \neq 0 \in V_N$, since (π, V) is generic.

We just need to prove that $C_c^\infty(F^\times)$, with the action of the group

$$PB(F) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad a \in F^\times, b \in F,$$

described by Eq. III.1, is irreducible. So that $\langle v \rangle = C_c^\infty(F^\times)$. In particular, let $\phi \neq 0 \in C_c^\infty(F^\times)$. We show that $\langle \phi \rangle = C_c^\infty(F^\times)$, by proving that $1_{a(1+\mathfrak{p}^n)} \in \langle \phi \rangle$, for any $a \in A$ and sufficiently large $n \in N$.

Suppose $\phi(b) \neq 0$, for some $b \in A$. After a translation by $\pi(A(b/a))$, we may assume $\phi(a) \neq 0$. Consider a non-zero function $f \in C_c^\infty(F)$. Since π is smooth, $f \in \mathcal{H}(F)$ and acts on $C_c^\infty(F^\times)$ by integration. Then, for $f \in C_c^\infty(F)$, we have that

$$\phi_1 = \pi(f)\phi = \int_F f(x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\phi\, dx,$$

for dx the self-dual Haar measure on F. We may rewrite the previous expression by Eq. III.1 as

$$\phi_1(y) = \int_F f(x)\psi(xy)\phi(y)dx = \hat{f}(y)\phi(y).$$

Finally, since the Fourier transform is a bijective map $\mathcal{F} : C_c^\infty(F) \to C_c^\infty(F)$, we may choose f such that $\hat{f} = \frac{1_{a(1+\mathfrak{p}^n)}}{\phi(a)}$, for any $n \in \mathbb{N}$ large enough, so that $\phi|_{a(1+\mathfrak{p}^n)}$ is constant. As a result, $\hat{f}(y)\phi(y) = 1_{a(1+\mathfrak{p}^n)}$. This proves the proposition. □

This result, it's crucial in the theory of automorphic forms of $GL_2(F)$. In fact, let's introduce the congruence subgroups

$$K_0(\mathfrak{p}^n) = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{pmatrix} \cap \mathrm{GL}_2(\mathfrak{o}) \quad K_1(\mathfrak{p}^n) = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{pmatrix} \cap \mathrm{GL}_2(\mathfrak{o}.$$

For a smooth irreducible representation (π, V), any vector $v \in K_1(\mathfrak{p}^n)$ is a local analogue of an automorphic form for the congruence subgroup $\Gamma_0(p^n)$. It would be useful to understand which irreducible admissible representations admit vectors in $V^{K_1(\mathfrak{p}^n)}$.

Proposition III.4.15. *Let (π, V) be an irreducible admissible generic representation of G. Then there exists a non-zero vector $v \in V^{K_1(\mathfrak{p}^n)}$, for some $n \in \mathbb{N}$ large enough.*

Proof. Let $\mathcal{K}(\pi, \psi)$ be the Kirillov model of (π, V), for nontrivial character ψ of F which is unramified, i.e. $\psi(\mathfrak{o}) = 1$. Then $C_c^\infty(F^\times) \subseteq \mathcal{K}(\pi, \psi)$. Consider the function $\phi = 1_{\mathfrak{o}^\times}$. Then, since π is smooth there exists a compact normal subgroup $I(\mathfrak{p}^n) = \begin{pmatrix} 1+\mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{pmatrix}$ that fixes ϕ, for $n \in \mathbb{N}$ big enough. Moreover, by Eq. III.1, we have that $\pi(A(a))1_{\mathfrak{o}^\times} = 1_{\mathfrak{o}^\times}$, for any $a \in \mathfrak{o}^\times$, and $\pi(n(x))\phi(y) = \psi(xy)\phi(y) = \phi(y)$, for $y \in \mathfrak{o}^\times$ and $x \in \mathfrak{o}$. Hence ϕ is fixed by $PB(\mathfrak{o}) \cdot I(\mathfrak{p}^n) = K_1(\mathfrak{p}^n)$. □

Definition III.4.16. Let (π, V) be an irreducible admissible generic representation of G. The conductor $C(\pi)$ of the (π, V) is the smallest non-negative integer such that $V^{K_1(\mathfrak{p}^n)} \neq \langle 0 \rangle$. Any non-zero vector $v \in V^{K_1(\mathfrak{p}^n)}$ is called a *local newvector*.

Actually, we will prove later that a local newvector is unique up to constants.

III.5 Parabolically Induced Representations

We still don't have concrete examples of irreducible generic representations. A simple way of constructing them is to *induce* them from representations of smaller subgroups, which are easier to handle. In the case of $G = GL_2(\mathbb{Q}_p)$:

$$G = BK_0,$$

for the closed Borel subgroup $B(F) \leq G$ and the maximal compact subgroup $K_0 = \mathrm{GL}_2(\mathfrak{o})$. Actually, by the Iwasawa decomposition, $B(F)$ can be further decomposed as

$$B = NT,$$

for the diagonal split torus T and the unipotent group $N(F) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ we already met. We will consider only smooth representations of $B(F)$ invariant under $N(F)$, hence which come from characters of T, and obtain a smooth representation of G. Before doing that, we should account for an important detail: $B(F)$ is not a unimodular locally compact group.

Define the *modular character* $\delta_B : B \to \mathbb{C}$ as follows:

$$\delta_B \left(\begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \right) = \left| \frac{x}{y} \right|.$$

The modular character arises as the unique character with the property that $\int_B f(ba)db = \delta_B(a) \int_B f(b)db$ for any smooth function $f : B \to \mathbb{C}$.

Lemma III.5.1. *Let db_L be a* **left** *Haar measure on B, then $\delta_B db_L$ is a right invariant measure.*

Proof. See [Tai, §2.3]. □

Definition III.5.2. Let χ be the smooth character of $T(F)$. Extend χ to a character of $B = NT$ by imposing that $\chi|_N = 1$. Denote by $\mathrm{Ind}_B^G \chi$ the space of complex functions $f : G \to \mathbb{C}$ such that:

1. $f(bg) = \delta_B(b)^{1/2} \chi(b) f(g)$ for any $b \in B(F)$, $g \in G$;

2. $f(gk) = f(g)$ for any $g \in G$ and $k \in K$, for some open compact subgroup K.

The space $\mathrm{Ind}_B^G(\chi)$ endowed with an action of G by right translation is called a *parabolically induced* representation of G. If we write $\chi(a_1, a_2) = \chi_1(a_1)\chi_2(a_2)$, we denote by $\mathcal{B}(\chi_1, \chi_2)$ the associated induced representation.

Remark III.5.3. The reason behind the presence of the modular character δ_B in the definition of the parabolic induction is to preserve unitarity. If χ is a unitary character, i.e. $|\chi| = 1$, the normalization imposed makes the representation $\mathrm{Ind}_B^G(\chi)$ unitarizable, with a G-invariant hermitian product given by

$$\langle f, g \rangle = \int_{K_0} f(k)\bar{g}(k)dk.$$

By the Iwasawa decomposition, Definition III.5.2.1 and the compactness of K_0 it is well-defined and non-degenerate. See [Tai, §2.3] for a more careful explanation and for the proof.

Remark III.5.4 (Line Model)**.** By the previous remark and Definition III.5.2.1, any $v \in \pi$, for a parabolically induced representation $\pi \cong \mathcal{B}(\chi_1, \chi_2)$, is defined by its restriction to K_0. By the decomposition

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \det(g)/d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix},$$

we get that any function vector v, can be then expressed as follows:

$$v(g) = f(c/d) \left| \frac{\det(g)}{d^2} \right|^{1/2} \chi_1(\det(g)/d)\chi_2(d) \quad \forall g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0.$$

The function $f_1 \in C^\infty(\mathbb{Q}_p)$, defines $f \in C^\infty(G)$ and so the linear map $f \mapsto f_1$ is injective and we get a new model for π, called the *line model*. For f in the line model, we denote by v_f the corresponding function in $\mathcal{B}(\chi_1, \chi_2)$. Moreover, if χ_1, χ_2 are unitary, it follows by Remark III.5.3 that $\|v\|^2 = \int_{\mathbb{Q}_p} |f|^2(x)dx$.

A smooth representation of G which injects into a parabolically induced one admits then an *induced model*, given by Definition III.5.2. In the previous section, we introduced the Jacquet modules and functors, with particular emphasis on the twisted ones, that resulted very helpful in establishing the existence of the Whittaker Model. For the parabolically induced representations the main actor is, however, $J(V)$, the non-twisted Jacquet module, for the following classical result.

Proposition III.5.5 (Frobenius Reciprocity)**.** *The Jacquet functor* $J : \mathrm{Rep}(G) \to \mathrm{Rep}(T)$ *is left adjoint to* $\mathrm{Ind}_B^G : \mathrm{Rep}(T) \to \mathrm{Rep}(G)$.

Proof. Let (π, V), be a smooth representation of G and (π', W') be a smooth representation of T. For any $v \in V$, denote with $\bar{v}$ its image on the quotient module $J(V)$. The first explicit morphism $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G W) \to \mathrm{Hom}_T(J(V), W)$ is given by $\alpha \mapsto (\bar{v} \mapsto \alpha(v)(1))$. First $v \mapsto \alpha(v)(1)$ vanishes on $V(N)$, since any $f \in \mathrm{Ind}_B^G W$ has the property that $f(n(x)1) = \delta_B(1)^{1/2} f(1)$. As a consequence, the morphism we presented is well-defined. The fact it is T-equivariant is obtained from the fact α is G-equivariant and from the construction of $J(V)$ and $\mathrm{Ind}_B^G W$.

The second morphism $\mathrm{Hom}_T(J(V), W) \to \mathrm{Hom}_G(V, \mathrm{Ind}_B^G W)$ is given by $\beta \mapsto (v \mapsto (x \mapsto \beta(\overline{\pi(x)v})))$. The G-invariance follows directly by definition.

The fact that the two maps are inverses one of the other is a formal consequence of their explicit construction. □

Corollary III.5.6. *Let* (π, V) *be an irreducible smooth representation of* G. *Assume that* $J(V) \neq 0$. *Then* V *embeds in a representation induced from a smooth character of* T.

Proof. The smooth characters of T coincide with its smooth irreducible representations. By Frobenius reciprocity, it's enough to prove that $J(V)$ admits a map of left $\mathbb{C}[T]$-modules towards a simple left $C[T]$-module W. This is equivalent to proving that $J(V)$ admits a simple quotient. To infer this, let's show that $J(V)$ is finitely generated. Choose a non-zero vector $v \in V$, such that $\bar{v} \neq 0$, and let K be an open compact subgroup of G fixing v. As a consequence of the Iwasawa decomposition, there exists a finite set $R \subset G$ such that $G = BRK$. Then, V is generated by linear combinations of $\pi(g)v = \pi(brk)v = \pi(br)v$, for some $b \in B$, $r \in R$, $k \in K$, since it is irreducible. If we write $b = nt$, for some $n \in N$, $t \in T$, then $J(V)$ is generated by $\pi(tr)\bar{v}$, hence as a left $\mathbb{C}[T]$-module it is generated by the finitely many $\pi(r)\bar{v}$.

By a simple application of Zorn's lemma, it's possible to prove that any finitely generated module M admits a maximal submodule M'. Then M/M' is a simple quotient. □

The previous results suggest a classification of the smooth irreducible representations (π, V) of G.

Definition III.5.7. Let (π, V) be a smooth irreducible representation of G. Suppose it is not one-dimensional, so that it is generic. Then the following happens:

1. $J(V) \neq 0$ and $(\pi, V) \cong \mathrm{Ind}_B^G \chi$, for a smooth character $\chi(a_1, a_2) = \chi_1(a_1)\chi_2(a_2)$ of the diagonal torus T. In this case (π, V) is called an irreducible *principal series.*

2. $J(V) \neq 0$ and $(\pi, V) \subsetneq \mathrm{Ind}_B^G \chi$ for a smooth character $\chi(a_1, a_2) = \chi_1(a_1)\chi_2(a_2)$ of the diagonal torus T.

3. $J(V) = 0$, so that (π, V) is not a subrepresentation of a parabolically induced representation. In this case (π, V) is called a *cuspidal* representation.

4. $J(V) = 0$ and (π, V) is not a subquotient of a parabolically induced representation. In this case (π, V) is called a *supercuspidal* representation.

Remark III.5.8. Case 2 gives rise to the *Steinberg* representation, see [Bum97, §4.5] for more details.

Actually, for $\mathrm{GL}_2(\mathbb{Q}_p)$ any cuspidal representation is supercuspidal. This can be seen by studying with more care the induced representations $\mathrm{Ind}_B^G(\chi)$. It turns out, by the following theorem, that characterizes for which characters $B(\chi_1, \chi_2)$ is irreducible.

Theorem III.5.9. *Let χ_1 and χ_2 be smooth quasicharacters of $F^\times$. Then $\mathcal{B}(\chi_1, \chi_2)$ is irreducible except in the following two cases:*

1. *if $(\chi_1\chi_2^{-1})(y) = |y|^{-1}$ for all $y \in F^\times$, then $\mathcal{B}(\chi_1, \chi_2)$ has a one dimensional invariant subspace and the quotient representation is irreducible;*

2. *if $(\chi_1\chi_2^{-1})(y) = |y|$ for all $y \in F^\times$, then $\mathcal{B}(\chi_1, \chi_2)$ has an irreducible invariant subspace of codimension 1.*

Moreover, $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\mu_1, \mu_2)$ if and only if $\mu_1 = \chi_2$ and $\mu_2 = \chi_1$ or $\mu_1 = \chi_1$ and $\mu_2 = \chi_2$. Finally, $\chi \otimes \mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi_1\chi, \chi_2\chi)$, for any character χ of $GL_2(\mathbb{Q}_p)$.

Proof. Both results are crucial and not immediate to prove, so we just give an idea, referring to [Bum97, §4.5] for the proof. The first one can be tackled by proving that $\mathcal{B}(\chi_1, \chi_2)$ always admits at most one Whittaker functional, up to scalars, see [Bum97, §4.5 Proposition 4.5.4], so that, it's possible to argue with the twisted Jacquet modules. If $0 \subsetneq V' \subsetneq V = \mathcal{B}(\chi_1, \chi_2)$ is a proper subrepresentation of $\mathcal{B}(\chi_1, \chi_2)$, then there is an exact sequence

$$0 \longrightarrow V' \xrightarrow{\ i\ } V \xrightarrow{\ p\ } V'' \longrightarrow 0\ ,$$

which induces an exact sequence of twisted Jacquet modules, since the twisted Jacquet functor is exact,

$$0 \longrightarrow J_\psi(V') \xrightarrow{\;i\;} J_\psi(V) \xrightarrow{\;p\;} J_\psi(V'') \longrightarrow 0 \ .$$

By the uniqueness of the Whittaker functional, $J_\psi(V)$ has dimension 1, we are left with the case $J_\psi(V') = 0$ or $J_\psi(V'') = 0$. The first case can be treated by showing it can happen only as stated in point 1) of this theorem, the second case can be treated by dualizing and considering point 2) of this theorem.

The last assertions are proved in [Bum97, §4.5 Theorem 4.5.2]. □

The last part of this section is devoted to the construction of an intertwining operator from the induced model to the Whittaker model of a principal series representation $\mathcal{B}(\chi_1, \chi_2)$, for $\chi_1\chi_2^{-1}(y) \neq |y|^{\pm 1}$. In fact, each of these concrete realizations helps the study of the structure of these representations, for example in this way we have two distinct realizations of a local newvector, and it would be helpful to transport this information from one model to the other in a canonical way, by the way unique by Schur's Lemma.

Proposition III.5.10. *Let $(\pi, V) \cong \mathcal{B}(\chi_1, \chi_2)$ be a principal series representation. Suppose that $\chi_1\chi_2^{-1}$ is ramified, then an intertwining operator $\rho : \mathcal{B}(\chi_1, \chi_2) \to \mathcal{W}(\pi, \psi)$, for an unramified non-trivial character ψ of F, is given by*

$$\rho(f) = f' : g \mapsto \lim_{n \to +\infty} \int_{\mathfrak{p}^{-n}} f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x)dx, \tag{III.2}$$

for $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. If $\Lambda(f) = \rho(f)(1)$ converges, then it is a Whittaker functional. Since

$$\begin{aligned} \Lambda(\pi(n(y))f) &= \lim_{n \to +\infty} \int_{\mathfrak{p}^{-n}} f\left(w \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}\right) \psi(-x)dx \\ &= \lim_{n \to +\infty} \int_{\mathfrak{p}^{-n}} f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \psi(-x+y)dx. \\ &= \psi(y)\Lambda(f). \end{aligned}$$

To prove it is a non-zero functional, by direct computation we obtain that $\rho(f) \neq 0$ for local newvector $f \in \mathcal{B}(\chi_1, \chi_2)$ as in III.5.12.

Now, we have to prove that the integral converges, in particular we have to study the behaviour of the integral for $|x| \to +\infty$. First, notice that

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Then by the definition of the induced model

$$f\left(w \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right) = |x|^{-1}\chi_1^{-1}(x)\chi_2(x) f\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right).$$

By the smoothness of f, there exists an integer $n > 0$, such that $f\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right) = f(1)$, for $x \in F \setminus \mathfrak{p}^{-n}$. Then, for any $m > n$, we have that:

$$\begin{aligned}\int_{\mathfrak{p}^{-m}\setminus\mathfrak{p}^{-m+1}} f\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)dx &= p^{-m}f(1)\int_{v(x)=-m}\chi_1^{-1}(x)\chi_2(x)\psi(-x)dx \\ &= \chi_1^{-1}(\varpi^{-m})\chi^2(\varpi^m)f(1)\int_{\mathfrak{o}^\times}\chi_1^{-1}(x)\chi_2(x)\psi(-\varpi^{-m}x)dx \\ &= \chi_1^{-1}(\varpi^{-m})\chi^2(\varpi^m)f(1)\mathcal{G}((\chi_1^{-1}\chi_2)|_{\mathfrak{o}^\times}, \bar{\psi}_{\varpi^{-m}}),\end{aligned}$$

where $\mathcal{G}((\chi_1^{-1}\chi_2)|_{\mathfrak{o}^\times}, \bar{\psi}_{\varpi^{-m}})$ is a Gauss sum. By the basic theory of Gauss sums, reported in [Kob14, §2.2], $\mathcal{G}((\chi_1^{-1}(x)\chi_2|_{\mathfrak{o}^\times}), \bar{\psi}_{\varpi^{-m}}) \neq 0$ only for finitely many $m \in \mathbb{N}$, depending on the conductor of the character $\chi_1^{-1}(x)\chi_2|_{\mathfrak{o}^\times}$, i.e. the smallest n such that the character is trivial on the subgroup $1 + \mathfrak{p}^n$. So for any $m'' > m' > n$, for some m', we have that

$$\int_{\mathfrak{p}^{-m}\setminus\mathfrak{p}^{-m+1}} f\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi(-x)dx = 0.$$

This implies that the limit exists. □

Remark III.5.11. The intertwining operator can also be defined for $\chi_1^{-1}\chi_2$ unramified, even though one has to pay attention to convergence issues. The idea is to consider in block the whole family of unramified characters, parametrized by the complex exponents (s_1, s_2), and treat the Whittaker functional as a holomorphic function of $s_1 - s_2$, for $\Re(s_1 - s_2) > 0$, where the integral expression provided in III.2 is absolutely converging, hence well-defined. Then, by analytic continuation, it's possible to extend the definition for all $s_1 - s_2 \neq 1$.

We end the section giving an explicit expression for local newvectors associated to principal series representations in the induced model.

Proposition III.5.12. *Let (π, V) be an irreducible principal series representation. Consider its induced model $\mathcal{B}(\chi_1, \chi_2)$, for smooth characters χ_1, χ_2 of $F^\times$. Set $n_1 = c(\chi_1)$ and $n_2 = c(\chi_2)$. Then the conductor $c(\pi) = n_1 + n_2$, with a local newvector given by*

$$f(g) = \begin{cases} \chi_1(a)\chi_1(d)|ad^{-1}|^{1/2} & \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(F)\gamma_{n_2}K_1(\mathfrak{p}^{n_2}), \\ 0 & \text{if } g \notin B(F)\gamma_{n_2}K_1(\mathfrak{p}^{n_2}), \end{cases}$$

where $\gamma_{n_2} = \begin{pmatrix} 1 & 0 \\ \varpi^{n_2} & 1 \end{pmatrix}$. In particular:

1. *if $\chi_1\chi_2$ are ramified:*

$$f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = \chi_1(x)^{-1}1_{\varpi^{n_2}\mathfrak{o}^\times}(x);$$

2. *if χ_1 is unramified and χ_2 is ramified:*

$$f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = 1_{\mathfrak{p}^{n_2}}(x);$$

3. *if χ_1 is ramified and χ_2 is unramified:*

$$f\left(\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\right)=\begin{cases}\chi_1^{-1}(x)\chi_2(x)|x|^{-1} & \text{if } v(x)<0;\\ 0 & \text{if } v(x)\geq 0;\end{cases}$$

4. *if χ_1, χ_2 are unramified:*

$$f\left(\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\right)=\begin{cases}\chi_1^{-1}(x)\chi_2(x)|x|^{-1} & \text{if } v(x)<0;\\ 1 & \text{if } v(x)\geq 0.\end{cases}$$

Proof. For the details and other explicit expression for local newvectors, see [Sch02]. □

III.6 Spherical Representations

Denote by K_0 the maximal compact subgroup $GL_2(\mathfrak{o})$ of $G = GL_2(F)$. Fix the Haar measure of G that assigns volume 1 to K_0.

Definition III.6.1. An irreducible admissible representation (π, V) is called *spherical* if it contains a non-zero K_0-fixed vector. Any non-zero vector $v \in V^{K_0}$ is then called *spherical*.

Remark III.6.2. Consider an irreducible admissible representation (π, V). If it is spherical then, by definition III.4.16, any non-zero vector in V^{K_0} is a local newvector. Moreover, as a consequence of the definitions, we get that (π, V) is spherical if and only if the conductor $c(\pi)$ is zero.

Consider the Hecke algebra $\mathcal{H} = (C_c^\infty(G), *)$ associated to (π, V). We recall then that V has a natural structure of $\mathcal{H}$-module. It makes then sense to study the properties of the spherical vectors in V^{K_0} through the subalgebra $\mathcal{H}_{K_0}$, called the spherical Hecke algebra.

Theorem III.6.3. *The spherical Hecke algebra $\mathcal{H}$ is commutative.*

Proof. The idea is to use Gelfand's involution method, or Gelfand's trick. First, we have to find an involution on $\mathcal{H}_{K_0}$. Define $i : \mathbb{C}_c^\infty(G) \to \mathbb{C}_c^\infty(G)$, such that $i(\phi)(g) = \phi(g^T)$, where g^T is the transpose matrix of G. Actually, $i(\phi)$ is an antiinvolution, i.e. an involution and an anti-automorphism:

$$\imath(\phi_1*\phi_2)(g) = \int_G \phi_1(g^T h^{-1})\phi_2(h)dh = \int_G \phi_1(g^T h^{-T})\phi_2(h^T)dh = \int_G \phi_1(h^T)\phi_2((gh^{-1})^T)dh.$$

Moreover, by the p-adic Cartan decomposition of G, $e_{\varpi_{n_1},\varpi_{n_2}} = 1_{K_0 T(\varpi^{n_1},\varpi^{n_2})K_0}$, for $T(\varpi^{n_1},\varpi^{n_2}) = \begin{pmatrix}\varpi^{n_1} & 0\\ 0 & \varpi^{n_2}\end{pmatrix}$, $n_1, n_2 \in \mathbb{Z}$, is a basis of $\mathcal{H}_K$ as a $\mathbb{C}$-vector space. By symmetry under transposition of $K_0T(\varpi^{n_1},\varpi^{n_2})K_0$, we obtain that $i(e_{\varpi_{n_1},\varpi_{n_2}}) = e_{\varpi_{n_1},\varpi_{n_2}}$. By linearity, then $i : \mathcal{H}_K \to \mathcal{H}_K$ is the identity. So we have that

$$\phi_1 * \phi_2 = i(\phi_1 * \phi_2) = i(\phi_2) * i(\phi_1) = \phi_2 * \phi_1.$$

□

Corollary III.6.4 (Uniqueness spherical local newvector)**.** *Let (π, V) be an irreducible admissible spherical representation. Then $\dim_{\mathbb{C}}(V^{K_0}) = 1$. Hence, there exists a unique spherical local newvector up to constants.*

Proof. By Corollary III.3.6, V^{K_0} is a simple $\mathcal{H}_{K_0}$-module and, since (π, V) is admissible, it is finite dimensional. Any element of $\phi \in \mathcal{H}_{K_0}$ acts as an endomorphism of V^{K_0} that commutes with every element of $\mathcal{H}_{K_0}$. Then by Schur's Lemma, ϕ acts as the multiplication by a constant $c \neq 0 \in \mathbb{C}$. This implies that any line is a $\mathcal{H}_{K_0}$-submodule. The only possibility is that V^{K_0} is one dimensional. □

Next, it is possible to determine completely the action of $\mathcal{H}_{K_0}$ on V^{K_0} by looking at the generators of this algebra. So, the next step is studying the action of $e_{\varpi^{n_1},\varpi^{n_2}} = 1_{K_0 T(\varpi^{n_1},\varpi^{n_2})K_0}$. Let $e_{K_0} = 1_{K_0}$. Consider an irreducible admissible spherical representation (π, V) of G. Then we have the identity

$$\pi(e_{\varpi^{n_1},\varpi^{n_2}})v = \pi(e_{K_0})\pi(T(\varpi^{n_1}, \varpi^{n_2}))\pi(e_{K_0})v \quad \forall v \in V,$$

which specializes to

$$\pi(e_{\varpi^{n_1},\varpi^{n_2}})v = \pi(e_{K_0})\pi(T(\varpi^{n_1}, \varpi^{n_2})) \quad \forall v \in V^{K_0}. \tag{III.3}$$

Then, if we denote by $R(\mathfrak{p})$ the operator $\pi(e_{\varpi,\varpi})$ on V, we get by Eq. III.3 that:

$$R(\mathfrak{p})v = \chi_\pi(\varpi)v \quad \forall v \in V^{K_0},$$

for the central character χ_π of (π, V). This implies in particular that $e_{\varpi^{n_1},\varpi^{n_2}} * R(\mathfrak{p}) = e_{\varpi^{n_1+1},\varpi^{n_2+1}}$. As a consequence, now we only need to determine the behaviour of the operators $e_{\varpi^n,1}$. To achieve this we introduce the operators $T(\mathfrak{p}^k) \in \mathcal{H}_{K_0}$.

Definition III.6.5. The operator $T(\mathfrak{p}^k)$ or $T(p^k)$ is the Heke operator defined by

$$T(\mathfrak{p}^k) = 1_{M_{p^k}},$$

where $M_n = \{g \in G \mid g \in \det^{-1}(\mathfrak{p}^k \setminus \mathfrak{p}^{k+1}) \cap \mathrm{Mat}_2(\mathfrak{o})\}$.

The $T_{\mathfrak{p}^k}$ operators satisfy the following relations.

Proposition III.6.6. *For $n \geq 1$, we have the following relation*

$$T(\mathfrak{p})T(\mathfrak{p}^k) = T(\mathfrak{p}^{k+1}) + qR(\mathfrak{p})T(\mathfrak{p}^{k-1}),$$

where q is the cardinality of the residue field of the local field F.

Proof. Notice that $T(\mathfrak{p}) * T(\mathfrak{p}^k)$, $T(\mathfrak{p}^{k+1})$, and $R(\mathfrak{p}) * T(p^{k-1})$ are supported on the double K_0-cosets, inside $\mathrm{Mat}_2(\mathfrak{o})$, with elements of determinant in $\mathfrak{p}^{k+1} \setminus \mathfrak{p}^{k+2}$, i.e. have support of the form $\cup_{i\in I} K_0 T(\varpi^{n_1},, \varpi^{n_2})K_0$, for $n_1 + n_2 = k + 1$, $n_1, n_2 \geq 0$, thanks to the Cartan decomposition. As a consequence, to verify the result, it's enough to test the inequality over elements of the form $T(\varpi^{k+1-r}, \varpi^r)$, for $0 \leq r \leq k+1$. First, let's compute the product $T(\mathfrak{p}) * T(\mathfrak{p}^k)$:

$$\begin{aligned}
(T(\mathfrak{p}) * T(\mathfrak{p}^k))(g) &= \int_G T(\mathfrak{p})(gh^{-1})T(\mathfrak{p}^k)(h)dh \\
&= \int_G T(\mathfrak{p})(h)T(\mathfrak{p}^k)(h^{-1}g)dh \\
&= \int_{K_0T(\varpi,1)K_0} T(\mathfrak{p}^k)(h^{-1}g)dh \\
&= T(\mathfrak{p})^k(T(1,\varpi)^{-1}g) + \sum_{b \mod \mathfrak{p}} T(\mathfrak{p}^k)\left(\begin{pmatrix} \varpi & b \\ 0 & 1 \end{pmatrix}^{-1} g\right),
\end{aligned}$$

since $T(\mathfrak{p}) = T(1,\varpi)K_0 \cup \bigcup_{b \mod \mathfrak{p}} \begin{pmatrix} \varpi & b \\ 0 & 1 \end{pmatrix} K_0$. From this last expression, we get that

$$(T(\mathfrak{p}) * T(\mathfrak{p}^k))(T(\varpi^{k+1-r}, \varpi^r)) = \begin{cases} 1 & \text{if } r = 0 \text{ or } r = k+1; \\ q+1 & \text{if } 1 \leq r \leq k+1. \end{cases}.$$

it is easily verified that these values coincide with the ones obtained by evaluating the other side of the equation. □

Remark III.6.7. From the proof of the previous proposition, it's possible to obtain the identity $e_{\varpi^{n_1},1} = T(\mathfrak{p}^{n_1}) - R(p)T(\mathfrak{p}^{n_1-1})$. This fact implies the following result.

Proposition III.6.8. *The spherical Hecke algebra $\mathcal{H}_{K_0}$ is generated by $T(\mathfrak{p})$, $R(\mathfrak{p})$ and $R(\mathfrak{p})^{-1}$.*

Proof. By proposition III.6.6, together with a simple induction, we get that $T(\mathfrak{p}^k)$, is generated by $T(\mathfrak{p})$ and $R(\mathfrak{p})$. This implies that also $e_{\varpi,0}$ is generated by $T(\mathfrak{p}^k)$, $R(\mathfrak{p})$. Finally, by the Cartan decomposition $\mathcal{H}_{K_0}$ is generated by $e_{\varpi^{n_1},\varpi^{n_2}}$, for $n_2 \geq n_1$, hence by $\mathbb{R}(\mathfrak{p})^n$, for $n \in \mathbb{Z}$, and $e_{\varpi^m,1}$, for $m > 0$. □

Thanks to these results, we can prove the following.

Theorem III.6.9 (Structure Theorem). *Let (π, V) be an irreducible admissible spherical representation. Denote by $v_0 \in V^{K_0}$ the unique, up to constants, local newvector.*

1. *The evaluation $\langle \pi(\phi)v_0, v_0 \rangle$, $\phi \in \mathcal{H}_{K_0}$, defines a $\mathbb{C}$-algebra morphism $\chi : \mathcal{H}_{K_0} \to \mathbb{C}$, called the spherical character.*
2. *Write $\chi(R(\mathfrak{p})) = \alpha_1\alpha_2$, $\chi(T(\mathfrak{p})) = q^{1/2}(\alpha_1 + \alpha_2)$. Then*

$$\chi(T(\mathfrak{p}^k)) = q^{k/2}\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta},$$

   for any $k \geq 1$.

3. *Consider the unique unramified characters χ_1, χ_2 of $F^\times$, with the property that $\chi_1(\varpi) = \alpha_1$, $\chi_2(\varpi) = \alpha_2$. Then $(\pi, V) \cong B(\chi_1, \chi_2)$, the principal series representation generated by χ_1, χ_2.*

Proof. 1) For any $\phi \in \mathcal{H}_{K_0}$, necessarily $\pi(\phi)v_0 = \chi(\phi)v_0$, for some $\chi(\phi) \in \mathbb{C}$. The function $\chi : \mathcal{H}_{K_0} \to \mathbb{C}$, such that $\phi \mapsto \chi(\phi)$ is obviously multiplicative and $\mathbb{C}$-linear. Moreover we have that $\chi(\phi) = \langle v_0, v_0 \rangle$, so that it coincides with the spherical function and is independent of the choice of $v_0 \neq 0 \in V^{K_0}$.

2) By proposition III.6.6, we know that

$$\chi(T(\mathfrak{p}))\chi(T(\mathfrak{p}^k)) = \chi(T(\mathfrak{p}^{k+1})) + q\chi(R(\mathfrak{p}))\chi(T(\mathfrak{p}^{k-1})).$$

Let $a_k = \chi(T(\mathfrak{p}^k))$, $k \geq 1$. The previous expression is equivalent to

$$a_{k+1} = a_1 a_k - q\chi(R(\mathfrak{p}))a_{k-1}.$$

This is a linear recurrence. By a simple linear algebra argument, we conclude that $a_k = A_1(q^{1/2}\alpha_1)^k + B_2(q^{1/2}\alpha_2)^k$, for $q^{1/2}\alpha_1, q^{1/2}\alpha_2$ solutions of $X^2 = a_1 X - q\chi(R(\mathfrak{p}))$. Hence $a_1 = q^{1/2}(\alpha_1 + \alpha_2)$, $R(\mathfrak{p}) = \alpha_1\alpha_2$. Finally the constants are determined by the conditions $a_1 = q^{1/2}(\alpha_1 + \alpha_2)$ and $a_2 = a_1^2 - q\alpha_1\alpha_2 = q(\alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2)$. Hence $a_k = q^{k/2}\frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2}$, for $\alpha_1 \neq \alpha_2$. While for $\alpha_1 = \alpha_2$ we obtain $a_k = (k+1)q^{k/2}\alpha_1^k$.
3) See [Bum97, Theorem 4.6.4]. □

Finally, it is natural to ask what happens for irreducible admissible generic representations of conductor greater than 0. The following result is similar to proposition III.6.3 and the proof is due to Novodvorskii, [Nov71].

Theorem III.6.10. *Let $H_{K_0(\mathfrak{p}^n)}$, be the Hecke algebra associated to the congruence subgroup $K_0(\mathfrak{p}^n)$, for $n \geq 1$. The subalgebra $C_{k+1} = (1 - e_k) * A_{k+1} * (1 - e_k)$, for $A_k = \mathcal{H}_{K_1(\mathfrak{p}^n)}$, $k \geq 0$, $e_k = \frac{1}{\mathrm{vol}(K_1(\mathfrak{p}^n))} 1_{K_1(\mathfrak{p}^n)}$ is commutative.*

The following result presented also in [Cas73a], follows as a corollary.

Corollary III.6.11 (Uniqueness local newvector)**.** *Let (π, V) be an irreducible admissible generic representation of G and let $c(\pi)$ be its conductor. Then* $\dim V^{K_1(\mathfrak{p}^{c(\pi)})} =$ *1.*

Proof. For $c(\pi) = 0$, it is the content of Corollary III.6.4. For $c(\pi) \geq 1$, by Theorem III.6.10, the space $V^{K_1(\mathfrak{p}^{c(\pi)})}$ is a simple $\mathcal{H}_{K_1(\mathfrak{p}^{c(\pi)})}$-module. Moreover, by the definition of conductor, we have that $\pi(\mathcal{H}_{K_1(\mathfrak{p}^{c(\pi)-1})})V^{K_1(\mathfrak{p}^{c(\pi)})} = 0$. Hence, the space is also a C_{k+1}-simple module. But C_{k+1} is abelian and $V^{K_1(\mathfrak{p}^{c(\pi)})}$ is finite dimensional, so, by Schur's lemma, we conclude that $V^{K_1(\mathfrak{p}^{c(\pi)})}$ has dimension 1. □

III.7 Trilinear invariant forms

Many of the results we presented on the previous sections could be extended, with little modification, to more general settings, like $\mathrm{GL}_n(\mathbb{Q}_p)$ or reductive algebraic groups over $\mathbb{Q}_p$. While the space of *trilinear invariant forms* exhibits a crucial property for $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

Definition III.7.1. Let π_1, π_2, π_3 be representations of G. Then denote by $\mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3)$ the space of trilinear forms $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ that are G-invariant, i.e.

$$\ell(\pi(g)v_1, \pi(g)v_2, \pi(g)v_3) = \ell(v_1, v_2, v_3) \quad \forall v_i \in \pi_i,\ g \in G.$$

Example III.7.2 (Integration form)**.** Consider the topological space $X = H \setminus G$, for a closed subgroup H of G. Then, by [Bum97, §4.3], there exists a right invariant Haar measure on X, that we will denote by μ_H. Suppose that $\mu_H(X) < +\infty$. Let $\mathcal{A}(X)$ be the space of smooth functions on X, i.e. the functions that are smooth vectors for the action of G by right translation. Let $\pi_1, \pi_2, \pi_3 \subset \mathcal{A}(X)$ be irreducible representations. There is a natural trilinear invariant form $\ell \in \mathrm{Hom}(\pi_1 \otimes \pi_2 \otimes \pi_3)$ given by integration:

$$\ell(\phi_1, \phi_2, \phi_3) = \int_X \phi_1(x)\phi_2(x)\phi_3(x) d\mu_H(x).$$

However, we need to stress that for some choices of π_i, the integration trilinear form could be trivially zero, i.e. $\ell = 0$.

Then, from Prasad's result in [Pra90, Theorem 1.1-1.3], we obtain the following theorem.

Theorem III.7.3 (Uniqueness of the trilinear invariant forms)**.** *Let π_1, π_2, π_3 be irreducible admissible representations of G such that the product of their central characters is trivial. Then $\dim(\mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3)) \leq 1$. In particular, if at least two among π_1, π_2, π_3 are irreducible principal series, then there exists a non-trivial trilinear invariant form.*

Example III.7.4 (Local-Rankin Selberg integral)**.** Let π_1 be an irreducible generic unitary representation of $\mathrm{GL}_2(k)$ with trivial central character and let $\pi_2 = \mathcal{B}(\chi_1, \chi_2)$ be an irreducible unitary principal series representation of G, realized in its induced model. The hypothesis of the previous theorem apply for $\pi_1, \bar{\pi}_2, \pi_2$. For this, define the local Ranking-Selberg integral $l_{RS} \in \mathrm{Hom}_G(\pi_1 \otimes \bar{\pi}_2 \otimes \pi_2)$ is defined by

$$\ell_{RS}(W_1, \bar{W}_2, v_3) = \int_{ZN/G} W_1 \bar{W}_2 v_3 = \int_{y \in \mathbb{Q}_p^\times} \int_{x \in k} W_1 \bar{W}_2 v_3(a(y)n'(x)) \frac{d^\times(y)}{|y|} dx,$$

for $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, Z the center of G, $N'(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, for $v_3 \in \pi$, $W_1 \in \mathcal{W}(\sigma, \psi)$, $W_2 \in \mathcal{W}(\pi, \psi)$, for a non-trivial unramified character ψ. Where we used the Iwasawa decomposition to reexpress the integral. This form is well-defined since by the analogue $III.1$ for the Whittaker model, multiplication by $n(x)$ is equivalent to multiplication by $\psi\bar{\psi}$ and the product of the central characters is trivial. By non-trivial results (see [MV10, §3]) the local Rankin-Selberg integral ℓ_{RS} is a non-trivial trilinear invariant form.

As a consequence of the previous examples, for a choice of the representations π_1, π_2, π_3 as in Example III.7.4, we have that there always exists a constant L such that

$$\ell = L\ell_{RS}.$$

Chapter IV

Trees

In this chapter, we present the theory of regular trees and the associated quotients, as found in [Ser80] and [Cas].

After recalling the basic definitions, we will focus on the connection between the quotient $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ and the $p+1$-regular tree, as in [Cas]. We will then show some useful consequences of this simple geometric realization.

In the second section, we will focus on geometrical constructions associated to regular graphs and trees, namely the quotients and the space of paths. We will relate them to local and global quaternion algebras.

In the third section, we will review the spectral properties of the regular trees and prove a delocalization estimate for their eigenfunctions.

IV.1 Graphs and Trees

Definition IV.1.1. An (oriented) graph X consists of sets $V = V(X)$, $E = E(X)$ and a map

$$\gamma : E \to V \times V, \quad \gamma : e \mapsto (o(e), t(e)).$$

An element $v \in V$ is called a *vertex* of X, an element $e \in E$ is called an (*oriented*) *edge*. For any $e \in E$, the vertices $o(e), t(e)$ are called the *endpoints* of e, in particular, the vertex $o(e)$ is called the *origin* of e, while $t(e)$ is the *terminus* of e.

A vertex v_2 is a neighbour of $v_1 \in V$, $v_1 \to v_2$, if $(v_1, v_2) \in \mathrm{Im}(\gamma)$. We say that v_1 and v_2 are related, $v_1 \sim v_2$, if $v_1 \to v_2$ and $v_2 \to v_1$.

Remark IV.1.2. We can also define an undirected graph, by composing the map $E \to V \times V$, given in the previous definition, with the map $V \times V \to (V \times V)/\sim$, where $(x_1, w_1) \sim (x_2, w_2)$ if and only if $x_1 = x_2$ and $w_1 = w_2$ or $x_1 = w_2$ and $w_1 = x_2$. Thus, to any oriented graph X, we can associate an undirected graph $\tilde{X}$.

Remark IV.1.3. Given a graph, it's possible to define a *subgraph* by restricting the set of vertices and the set of edges, in a (obvious) compactible way, to create a new graph.

Definition IV.1.4 (morphism)**.** Let G_1, G_2 be two graphs. A morphism of graphs $g : G_1 \to G_2$ consists of two maps $g_V : V(G_1) \to V(G_2)$ and $g_E : E(G_1) \to E(G_2)$, with the property that

1. if $v_1 \to v_2$ then $g_V(v_1) \to g_V(v_2)$;

2. $g_V \circ o(e) = o \circ g_E(e)$ and $g_V \circ t(e) = t \circ g_E(e)$.

We denote by $Im(f)$ the subgraph of G determined by the image of f.

Consider an oriented graph G. The set of *neighbours* of $v \in V(X)$ is the subset of vertices $N(v) \subset V(X)$, such that $n \in N(X)$ if and only if there exists an edge $e \in E$ whose endpoints are n and v. The set $O(v) \subset E(X)$ of a vertex $v \in V(X)$ consists of all the edges $e \in E(X)$ such that $o(e) = v$. Similarly, The set $T(v) \subset E(X)$ of a vertex $v \in V(X)$ consists of all the edges $e \in E(X)$ such that $t(e) = v$. We can trivially extend the definitions to an undirected graph. The *degree* of a vertex $v \in V$, denoted by $d(v)$, is the cardinality of $T(V) \cup O(v)$.

Definition IV.1.5. A *regular graph* X of degree d is a graph with the property that $d(v_1) = d(v_2) = d$, for any $v_1, v_2 \in V(X)$.

To any graph G is associated a geometric realization as a topological space. Consider the disjoint union $M = V(G) \cup E(G) \times [0, 1]$, for the unit segment $[0, 1] \subset \mathbb{R}$ with the Euclidean topology. Then $T(G) = M/R$, for the finest equivalence relation on M for which $(e, 0) \sim o(e)$, $(e, 1) \sim t(e)$, for any $e \in E(G)$. The topological space is homotopic to a *bouquet*.

Remark IV.1.6. Starting from the geometric realization of an undirected graph G, together with the vertices V, it's possible to determine uniquely G. The same doesn't hold for direct graphs, where the orientation of each edge has to be specified.

Remark IV.1.7. The geometric realization is actually a functor F : Graphs $\to$ Top$_p$, where Top$_p$ is the category of pointed topological spaces, i.e. topological spaces with a choice of a subset of points, the vertices.

Example IV.1.8 (Path graph)**.** Let $n \geq 0$ be an integer. The oriented graph P_n with vertices $V(P_n) = \{0, \dots, n\}$, with edges $E(P_n) = \{0, \dots, n-1\}$, for $n > 0$, otherwise $E = \emptyset$, such that $\gamma(i) = (i, i+1)$, is called a *path* graph of length n. Its geometric realization is actually a path of "length n".

Definition IV.1.9. A path of length n in a graph G is a morphism $g : P_n \to G$. The endpoints of a path are the image of $0, n+1 \in V(P_n)$; the former is called the origin and the latter the terminus.

Definition IV.1.10. Let G be a graph. Two vertices v_1, v_2 are connected if there exists a path whose origin is v_1 and whose endpoint is v_2 and a path of finite length whose origin is v_2 and whose endpoint is v_1. Equivalently, two vertices in an undirected graph G are connected if they are path-connected in the geometric realization of G.

A graph G is connected if every couple of vertices $(v_1, v_2) \in V \times V$ is connected.

Definition IV.1.11 (Distance)**.** Let G be a connected graph. Then we define the *distance* of two vertices $v, w \in V(G)$ as

$$d(v, w) = \inf_{P \in P(v,w)} \text{length}(P),$$

where $P(v, w)$ is the set of all paths starting from v and ending in w.

Remark IV.1.12. If G is an undirected graph, the distance is actually a real metric on $V(G)$.

Example IV.1.13 (Cyclic Graph)**.** The cyclic graph C_n of length $n \geq 1$, is the oriented graph with vertices $V(P_n) = \mathbb{Z}/n\mathbb{Z}$, with edges $E(P_n) = \mathbb{Z}/n\mathbb{Z}$, such that $\gamma(i) = (i, i+1)$, where the sum is taken in $\mathbb{Z}/n\mathbb{Z}$.

Definition IV.1.14 (Cycle)**.** A cycle (of length n) in an oriented graph is any subgraph isomorphic to C_n. A cycle (of length n) in an undirected graph is any subgraph isomorphic to $\tilde{C}_n$, the undirected cyclic graph of length n.
A cycle of length 1 is also called a *loop*.
A graph is *acyclic* if it has no cycles.

Remark IV.1.15. Fix a vertex v in an undirected graph G. This choice also fixes a point v in the geometric realization $T(G)$. Suppose that the graph is connected, then $T(G)$ is path connected and we can consider its fundamental group $\pi_1(T(G), v)$. The group is non-trivial if and only if there exists a loop which is not homotopically equivalent to the trivial one $p([0, 1]) = v$. This happens if and only if there is a non-trivial loop in $T(G)$, hence if and only if G has a cycle.

IV.2 Trees

Definition IV.2.1 (Tree)**.** A tree is an undirected connected non-empty graph without cycles. A d-regular tree, is a tree with vertices of constant degree d, i.e. $d(v) = d$ for any $v \in V$.

This definition can be restated in the following way.

Proposition IV.2.2. *A graph G is a tree if and only if it has no loops and for each couple of distinct vertices $v_1 \neq v_2 \in V(G)$, there exists a unique path connecting them.*

Proof. The existence is necessary for the connectedness of G. The absence of loops leaves only the possibility for cycles of length greater than 1. Suppose by contradiction that there were two vertices v_1, v_2 connected by two non-isomorphic paths $f_1 : P_{n_1} \to G$, $f_2 : P_{n_2} \to G$, starting at v_1. We can choose v_1, v_2 such that the paths have minimal length, by possibly admitting $v_1 = v_2$. Then, necessarily $f_{1,V}(v) = f_{2,V}(w)$ only for $v = w = 0$ and $v = n_1$, $w = n_2$. Thus, we get that $Im(f)$ is isomorphic to the cycle graph with length $n_1 + n_2$. □

Corollary IV.2.3. *Let G be a tree. Then there exists a unique path of minimal length from two vertices, i.e. a unique* geodesic.

Proposition IV.2.4. *Let G be a tree. Then $T(G)$ is simply connected.*

Proof. First, $T(G)$ is path connected, since G is connected. Moreover, since G is acyclic, we have that all the loops are homotopically equivalent to the trivial one. Hence $\pi_1(T(G)) = \pi_1(T(G), v) = \{1\}$, for the choice of a vertex $v \in V(G)$. This implies that $T(G)$ is simply connected. □

Let G be a finite connected regular graph of degree d. We show now that it can be realized as the quotient of a n-regular tree, under the action of a subgroup of the tree automorphisms.

Definition IV.2.5. Let G be a connected undirected graph with finite degree. A graph covering consists of a connected graph G' and a surjective map $p : G' \to G$, such that p_E induces a bijection $O(v) \cong O(f(v))$, for any $v \in V$.

A direct consequence of the definition is the following result.

Proposition IV.2.6. *Let G be a connected undirected graph with finite degree. There is a bijection between the set of topological coverings and the graph coverings, induced by the functor that maps each graph to its geometrical realization.*

Proof. The part that is not clear from the definition is that any topological covering can be realized as a graph covering. So, consider a topological covering $P : M \to T(G)$, for a graph G. We claim that M it isomorphic to $T(G')$ for a graph G', with a graph covering $p : G' \to G$, such that $\tilde{p} = P$. Consider the set of vertices $V(G) \subset T(G)$, we define the set of vertices of G' as $V(G') = P^{-1}(V(G))$. The set of edges $E(G')$ is the set of connected components of $M \setminus V(G')$. The endpoint of an edge $e \in E(G')$ are defined by the points of the set $\overline{e} \setminus e \subset M$. Then it's easy to check that the map P induces a graph covering $p : G' \to G$. □

Theorem IV.2.7. *Let G be an undirected finite connected d-regular graph. Then, for $d > 1$, the d-regular tree T_d is the universal cover of G, i.e. its geometric realization is the universal cover of $T(G)$.*

Proof. We already proved that the geometric realization of T_d is simply connected. We just need to find a graph covering map $p : T_d \to G$. If $d > 1$, then fix a vertex $v \in V(G)$ and define $P_{n,G}$ the set of distinct non-backtracking paths on G starting from v of length n, for $n \geq 0$. Set $V(G') = P_{n,G}$. Since G is undirected, any path $f : P_k \to G$ can be identified with the image $(f_V(i))_{0 \leq i \leq n} \in V(G)^{n+1}$. We connect two vertices $p_1 = (v_1, \dots, v_j), p_2 = (w_1, \dots, w_i)$ by an edge if and only if $i = j+1$ and $v_k = w_k$, for $0 \leq k \leq j$. Then the undirected graph G' constructed in this way has constant degree d and it is connected since any vertex is connected to $P_0 = v$. The only fact we need to prove is that G is acyclic. First notice that for any $w = (v, \dots, v_j)$ there exists only one non-backtracking path that starts at P_0 and ends at w, the one which passes through $(v, \dots, v_k)$, for $1 \leq k \leq j$. This defines a distance $d_0(w)$ of a vertex w from P_0. So, suppose by contradiction that there exists a path $\hat{P}$ of length $n \geq 1$ whose image is a non-trivial cycle $C \subset G'$. Necessarily $P_0 \notin C$ for the previous observation. Then, let $w = \hat{P}(0)$ be a vertex in C. By definition of the map $d_0(\hat{P}(i))$ is stricly monotonic when i increases, but $\hat{P}(0) = \hat{P}(n)$, a contradiction. Hence, any cycle reduces to a single point and G' is also acyclic. Thus $G' \cong T(G)$.

The covering map $p : G' \to G$ is induced just by the projection $w = (v, \dots, v_n) \mapsto v_n$. □

Corollary IV.2.8. *Let X be an undirected finite connected d-regular graph, $d > 1$. Then X can be realized as the quotient $H \backslash T_d$ for a subgroup H of the automorphisms of T_n, i.e. the graph associated to the topological quotient $G \setminus T_d$.*

Corollary IV.2.9. *Let H be a subgroup of the automorphisms of the regular tree T_d whose elements act without torsion elements and such that $|\mathrm{Stab}(\Gamma, \mathrm{v})| < +\infty$, for any $v \in V(T_d)$. Then $X = H \setminus T_d$ is a connected d-regular graph.*

Proof. The absence of torsion is necessary to ensure that the projection $T_d \to H \backslash T_d$ sends edges in edges, so that the set of vertices is the image of the set of vertices of T_d. While the bound on the elements of Γ that fix a vertex is equivalent to a properly discontinuous action of H on the topological realization of T_d. □

Remark IV.2.10. If $d = 1$, then G consists of a vertex and a loop and $p : T_2 \cong \mathbb{R} \to \mathbb{S}^1 \cong T(G)$ induces the required covering of graphs.

IV.3 The tree of $\mathrm{PGL}_2(F)$

Let F be a local field, with residue field k with cardinality q, and let ϖ be a uniformizer. We denote by $\mathfrak{o}$ the ring of integers and $\mathfrak{p} = (\varpi)$ the unique maximal ideal. In this section, we will see how the $q+1$-regular tree T arises from the lattices of F^2 and, as a consequence, we will obtain an action of $\mathrm{PGL}_2(F)$ on T. This construction is a simple but important case of a more general geometric construction, called *affine Bruhat-Tits Building* associated to a reductive algebraic group, which can be interpreted as a generalization of the real symmetric spaces to arbitrary local fields. The geometry of these objects encodes much of the group structure, playing a signifcant role in the representation theory of the local reductive groups. In the case of $\mathrm{PGL}_2(F)$ the building coincide with the tree T, which is then called the Bruhat-Tits tree.

Let T' be the undirected graph such that

1. its vertices $V(T')$ are the equivalence classes of $\mathfrak{o}$-lattices on F^2, where two lattices L_1, L_2 are similar if and only if $L_1 = cL_2$, for some $c \in F^\times$;

2. two vertices $[L], [M]$ are connected by an edge if and only if there exist representatives L', M' of the equivalence classes such that

$$\varpi L' \subset M' \subset L'.$$

Remark IV.3.1. If $\varpi L' \subset M' \subset L'$, then $\varpi M' \subset \varpi L' \subset M'$ and then $[L] \to [M]$ implies also $[M] \to [L]$.

Moreover, the neighbours of a vertex $[L]$, are in bijection with the lines in $L/\varpi L \cong F^2$. As a consequence, they correspond to the $q+1$ points of $P^1(k)$.

Remark IV.3.2. Any lattice L, can be defined by the choice of a basis (v_1, v_2) that defines a matrix $(v_1\ v_2) \in \mathrm{GL}_2(F)$ and vice versa any element of $GL_2(F)$ spans a lattice. At the same way, to any element of $\mathrm{PGL}_2(F)$ corresponds an equivalence class of lattices, and the map is surjective. Under this notation, there is a well-defined action of any $g \in \mathrm{GL}_2(F)$ on the vertices of T, defined by left multiplication, $g \times [(v_1\ v_2)] = [(g(v_1)\ g(v_2))]$, i.e. $L \mapsto g(L)$. The neighbours of L are so generated by left multiplication by matrices with determinant of norm q. For this reason, T' is also called the Hecke tree.

Fix the lattice $L_0 = (e_1, e_2)$, for the standard basis e_1, e_2 of F^2. The action of $\mathrm{PGL}_2(F)$ is transitive, and the stabilizer of L_0 is $\mathrm{PGL}_2(\mathfrak{o})$. This implies, by the orbit-stabilizer theorem, that $\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathfrak{o})$ is isomorphic as a $\mathrm{PGL}_2(F)$-space to $V(T')$ with the action of $\mathrm{PGL}_2(F)$. So we can endow $\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathfrak{o})$ with the graph structure of T' and identify the two objects.

Definition IV.3.3. A chain in T' is a, possibly infinite, path in the graph T'. A chain is called *simple* if the path is non-backtracking. The infinite simple chain defined by the vertices $((e_0, e_1), (e_0, \varpi e_1), \ldots, (e_0, \varpi^n e_1), \ldots$ is called the *standard chain.*

Theorem IV.3.4. *Any chain may be transformed to the standard one by an element of* $\mathrm{GL}_2(F)$.

Proof. See [Cas, Proposition 2.1]. □

Corollary IV.3.5. $\mathrm{GL}_2(F)$ *acts transitively on the space of infinite chains.*

Corollary IV.3.6. *The graph T' is a tree.*

Proof. By the Iwasawa decomposition, any element in $\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathfrak{o})$ is represented by (the equivalence class of) a matrix of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varpi^n \end{pmatrix}$. So, by multiplying by matrices of determinant q it's possible to construct a path from L_0 to any other vertex.

Finally, if T' had a non-trivial cycle, it would also have a non-trivial chain that can't be transformed into a subchain of the standard one. This would be a contradiction to Theorem IV.3.4. □

Now that we have established that T' is actually a tree we can denote it directly T or T_{q+1}.

Now we can study the action of $\mathrm{PGL}_2(F)$ on T. We will see how stabilizers of subsets of T with natural geometric interpretation correspond to subgroups of $\mathrm{PGL}_2(F)$ already met in the local representation theory.

1. The stabilizer of a vertex is a maximal compact subgroup of the form gK_0g^{-1}, for $K_0 = \mathrm{PGL}_2(\mathfrak{o})$, for some $g \in \mathrm{PGL}_2(F)$, since $T \cong_{\mathrm{PGL}_2(F)} \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathfrak{o})$.

2. The stabilizer of an edge is called an *Iwahori subgroup*, and it coincides with a conjugate of the standard Iwahori congruence subgroup
$$K_0(\mathfrak{p}) = K_0 \cap \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K_0 \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

3. In general, the stabilizer of a finite non-backtracking path of length $n \geq 0$, is a conjugate of the standard congruence subgroup $K_0(\mathfrak{p}^n)$, which stabilizes the subchain of length n starting at L_0 of the standard simple chain. Moreover, every such stabilizer coincides with the intersection of the stabilizers of the endpoints of the finite path $a^{-1}K_0a \cap b^{-1}K_0b$, since the geodesics in a tree are unique.

4. The stabilizer of a simple chain is a conjugate of the Borel subroup $B(\mathfrak{o})$ of $\mathrm{PGL}_2(\mathfrak{o})$, the stabilizer of the standard chain, by the previous point.

There is still a geometric construction we have not defined: the geodesic lines.

Definition IV.3.7. An *apartment* on T is a doubly infinite geodesic path, i.e. an infinite bidirectional non-backtracking path.

Proposition IV.3.8. $\mathrm{PGL}_2(F)$ *acts transitively on the apartments of* T.

Proof. See [Cas, Proposition 4.1]. □

As for the chains, we can define the standard apartment by considering the bidirectional infinite path $((e_1, \varpi^m e_2))_{m\in\mathbb{Z}}$. Then its stabilizer is the intersection of the upper and lower Borel subgroup of PGL_2 over $\mathfrak{o}$. Thus, we get that the stabilizer of every point of an apartment is a split torus in $\mathrm{PGL}_2(\mathfrak{o})$, a conjugate of the diagonal subgroup, which is the stabilizer of the standard apartment. On the other hand, the split tori over F preserve apartments translating however their points.

Remark IV.3.9 (The tree of a local quaternion algebra)**.** The group $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponds to the group of units of a local split quaternion algebra B_p. Its standard maximal order is $O = M_2(\mathbb{Z}_p)$ and the Eichler orders correspond to intersections of two conjugates of O. Since the conjugation by an element of the center is a trivial action, the group of units of the Eichler orders correspond then to the stabilizers of the finite paths in the Bruhat-Tits tree and so the Eichler orders contained in O are in bijection with the vertices of T. It's then possible to associate a degree to an Eichler order O', given by the distance of O' from O in T.

Through the orbit-stabilizer theorem and the previous observations, it's possible to state the following.

Proposition IV.3.10. *Let T be the Bruhat-Tits tree, $G = \mathrm{PGL}_2(F)$. Then:*

1. *the space of paths of length n, as a G-space, is isomorphic to $G/K_0(\mathfrak{p}^n)$;*
2. *the space of infinite simple chains is isomorphic to $G/B(\mathfrak{o})$;*
3. *the space of the apartments is isomorphic to $G/\mathrm{diag}(\mathfrak{o})$, for the split diagonal torus* diag *of G.*

Remark IV.3.11. The result extends trivially to quotients $\Gamma \setminus T$ of the regular tree, provided that $T \to \Gamma \setminus T$ is a covering map.

Corollary IV.3.12. *Let B be a global quaternion algebra over Q that splits at a finite place p. Let R be an Eichler order, whose local component at p, defined as in proposition II.6.1, is a maximal order. Then the symmetric space*

$$A^\times B_{\mathbb{Q}}^\times \setminus B_A^\times / H_R$$

has a canonical structure of finite quotient of the Bruhat-Tits Tree, where H_R is defined as the invertible elements of the global R_A as in II.6.1.

Proof. Recall by section §II.7.1, that

$$A^\times B_{\mathbb{Q}}^\times \setminus B_A^\times / H_R \cong R[1/p]^\times \setminus \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times \mathrm{GL}_2(\mathbb{Z}_p).$$

This last term is isomorphic to the quotient $\overline{R[1/p]^\times} \setminus \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$, for the image of $R[1/p]^\times$ in $\mathrm{PGL}_2(\mathbb{Q}_p)$. The finiteness is given by teh corollary of Theorem II.6.2. □

Remark IV.3.13. Recall by remark IV.3.2 that the neighbours of a lattice in L correspond to the lattices L' such that $[L' : L] = q$. From this, we obtain that the graph structure on $A^\times B_{\mathbb{Q}}^\times \setminus B_A^\times / H_R$ is also the one induced by the Brandt matrix. In this way, we get a concrete expression of the graph $\overline{R[1/p]^\times} \setminus \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$.

Remark IV.3.14. As a consequence of the construction, unbounded subsets of the tree are in bijection with unbounded $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$.

IV.4 Spectral theory

In this section we are going to prove a delocalization property for the eigenfunctions of the regular tree T of degree d.

Definition IV.4.1 (Adjacency matrix)**.** Let G be an undirected graph of finite degree, i.e. each vertex has finite degree. We denote by $\mathbb{C}(G)$ the set of complex functions $f : V(G) \to \mathbb{C}$. The *adjacency operator* $A_G : \mathbb{C}(G) \to \mathbb{C}(G)$ is the linear operator defined by :

$$A_G f(v) = \sum_{w \sim v} f(w).$$

The eigenfunctions of A_G are called the eigenfunctions of the graph G.

We shall be interested now in the eigenfunctions of a regular tree T_p of degree p. Define the spherical operator $S_{p^k} : \mathbb{C}(G) \to \mathbb{C}(G)$ by:

$$S_{p^k}(f) = \sum_{d_{T_p}(v,w)=n} f(w),$$

for the distance in d_{T_p} in T_p. Hence $A_{T_p} = S_p$.

Theorem IV.4.2. *Assume that $S_p f = \lambda f$ for some $\lambda \in \mathbb{R}$. Then there exists a constant c, independent of λ and p such that for all $n \geq 0$,*

$$\sum_{0\leq i\leq n} \sum_{d_{T_p}(y,v)=i} |f(y)|^2 \geq cn|f(v)|^2 \quad \forall v \in V(G).$$

Proof. For brevity, we shall rewrite $\sum_{0\leq i\leq n} \sum_{d_{T_p}(y,e)=i} |f(y)|^2$ as $\sum_{B_n} |f(y)|^2$, for B_n the closed ball of radius n, centered at e.

By the graph structure of tree, and since $S_p = A_{T_p}$ we have that the following holds:

$$S_{p^k} \circ S_p = S_{p^{k+1}} + pS_{p^{k-1}} \quad \forall k \geq 2,$$

and $S_p \circ S_p = S_{p^2} + (p+1)Id$. Let λ_{p^k} be the eigenvalue of f relative to S_{p^k}. Then, by the recurrence relations, as done in greater generality in the proof Proposition III.6.6, we have that $\lambda_{p_k} = a_1 r_1^k + a_2 r_2^k$, for the roots in $\mathbb{C}$ of the polynomial $x^2 - \lambda x + p$. We distinguish two cases: $|lambda| > 2\sqrt{p}$, the untempered case, and $|lambda| \leq 2\sqrt{p}$.

Consider first the untempered case. Define $\alpha \in \mathbb{R}$ such that $\cosh(\alpha) = |\lambda/2\sqrt{p}$. Then $r_1 = \text{sgn}(\lambda)\sqrt{p}e^{\alpha}$ and $r_2 = \text{sgn}(\lambda)e^{-\alpha}$. By requiring that $\lambda_p = \lambda$ and that $\lambda_{p^2} = -p - 1 + \lambda^2$, we determine a_1 and a_2. After the computation we obtain:

$$\lambda_{p^k} = \left[\frac{1 - pe^{2\alpha}}{p(1 - e^{2\alpha})} e^{k\alpha} + \frac{1 - pe^{-2\alpha}}{p(1 - e^{-2\alpha})e^{-k\alpha}}\right] \text{sgn}(\alpha) p^{k/2}.$$

With the convention $\lambda_{p^0} = 1$ we obtain

$$\sum_{k=0}^{n} \lambda_{p^{2k}} = p^n \frac{\sinh{(2n+1)\alpha}}{\sinh \alpha} \geq (2n+1)p^n.$$

Thus

$$\left| \sum_{d_{T_p}(e,y)\in\{0,2,\ldots,2n\}} f(y) \right| \geq (2n+1)p^n f(e).$$

By Cauchy-Schwarz

$$\sum_{d_{T_p}(e,y)\in\{0,2,\ldots,2n\}} |f(y)|^2 \geq n^2 |f(e)|^2.$$

Now, assume that $|\lambda| \leq 2\sqrt(p)$ and let $\theta = \lambda/(2\sqrt{p})$. Again by computing λ_{p^k}, we get that

$$\sum_{k=0} \lambda_{p^{2k}} = p^n \frac{\sin(2n+1)\theta}{\sin \theta}. \tag{IV.1}$$

By Cauchy-Schwarz again:

$$\sum_{d_{T_p}(e,y)=2k} |f(y)|^2 \geq \frac{\left|\sum_{d(e,y)=2k} f(y)\right|^2}{(p+1)p^{2k-1}} = \frac{|\lambda_{p^{2k}}|^2|f(e)|^2}{(p+1)p^{2k-1}}.$$

Subtracting IV.1 with $n = k-1$ from IV.1 with $n = k$ we get that

$$\lambda_{p^{2k}} = p^k \left[\frac{\sin(2k+1)\theta}{\sin\theta} - \frac{\sin(2k-1)\theta}{p\sin\theta}\right].$$

Thus

$$\sum_{d_{T_p}(x,y)=2k} |f(y)|^2 \geq c|f(e)|^2$$

if $(2k+1)\theta \in [2\pi/5, 3\pi/5] \mod \pi$.

Now if n is such that $n \gg 1/\theta$, we can assume that

$$|\{k \mid 1 \leq k \leq n,\ (2k+1)\theta \in [2\pi/5, 3\pi/5] \mod \pi\}| > c_2 n,$$

so that the claim of the theorem is satisfied for some constant c. On the other hand, if n is sufficiently small ($n \ll 1/\theta$), we have that $\frac{\sin(2n+1)\theta}{\sin\theta} \geq n$,so that by IV.1 we get

$$\left|\sum_{k=0}^{n} \lambda_{p^{2k}}\right| \geq np^n.$$

By again Cauchy Schwartz we obtain

$$\sum_{y\in B_{2n}} |f(y)|^2 \geq cn^2|f(e)|^2 \geq cn|f(e)|^2.$$

By collecting the three cases, we can choose a constant c such that the claim holds. □

Remark IV.4.3. A similar inequality holds if the operators S_{l^k} have a slightly different definition, allowing some type of twist.

Chapter V

Ergodic Theory

In this chapter, we present a theorem of Einsiedler-Lindenstrauss [EL08] on the rigidity of measures on Lie Groups satisfying diagonal invariance. This measure classification result was the key input for establishing long-sought properties on the limit distribution of the quantum limits of arithmetic eigenfunctions on arithmetic quotients of the hyperbolic plane in [Lin06] and [Lin04], and more generally on locally symmetric spaces by [SV07]. In particular, we will try to give all the basic theory necessary to appreciate the result, the proof being out of the scope of this essay.

In the first part, we will recall some basic definitions on dynamical systems and measures and the ergodic theory. After that, we will focus on two key concepts: the recurrence and the entropy of a measure of a dynamical system.

In the second part, following [Lin04], we will specialize to measures invariant by the action of a local split torus in the adelic setting. We will present the result of Einsiedler-Lindenstrauss and explain how the theorem applies to measures associated to the mass of Hecke eigenfunctions.

V.1 Dynamical Systems

In this section, we recall some basic definitions.

Definition V.1.1. Let X be a set, $\mathcal{B}$ a sigma algebra and $\mu : \mathcal{B} \to \mathbb{R}_{\geq 0}$ a positive measure. The couple $(X, \mathcal{B})$ is called a *measurable space*. The triple $(X, \mathcal{B}, \mu)$ is called a *measure space*.

In the context of dynamics systems, we usually have a measurable transformation $f : X \to X$ which governs the evolution of the system, and we are interested in the *invariant measures*.

Definition V.1.2. Let $(X, \mathcal{B})$ be a measurable space and consider a measurable function $f : X \to X$. We say that a measure μ on $(X, \mathcal{B})$ is f-invariant, or that f preserves μ, if

$$\mu(E) = \mu(f^{-1}(E)) \quad \text{for any } E \in \mathcal{B}.$$

Example V.1.3. Consider the measure space $(\mathbb{S}^1, \mathcal{L}, \mu)$, with the Lebesgue probability measure induced by the standard projection $p : [0, 1] \to \mathbb{S}^1$. Then any rotation preserves mu.

Often the dynamics is not governed by a unique transformation, but by a family depending on a continuous parameter $t \in \mathbb{R}$.

Definition V.1.4 (Flow)**.** Let $(X, \mathcal{B})$ be a measurable space. A *flow* is a family of measurable transformations $f^t : M \to M$, with $t \in R$, satisfying the following conditions:

$$f^0 = id, \qquad f^{s+t} = f^s \circ f^t \quad \forall s, t \in \mathbb{R}.$$

We say that a measure μ on $(X, \mathcal{B})$ is invariant under the flow, if μ is invariant under f^t, for any $t \in \mathbb{R}$.

Example V.1.5. Classically, flows are associated to the solutions of differential equations over a "time" variable $t \in \mathbb{R}$. Let $M = \mathbb{R}^n$, or more generally a smooth manifold, and take a section of the tangent bundle $X \in TX$. Under suitable conditions on X, there is a well-defined flow $f^t : X \to X$, defined locally, both on $x \in M$ and on $t \in \mathbb{R}$, by the exponential map

$$f^t(x) = e^{tX}x.$$

A dynamical system associated to a flow is also called a time-continuous dynamical system, while the one associated to a transformation T is also called a discrete-time dynamical system, since the map $n \mapsto T^n$ can be interpreted as a discrete time flow.

More generally, notice that the dynamical systems we introduced were essentially related to the action of the abelian group $(\mathbb{R}, +)$, or the semigroup $(+, \mathbb{N})$, on a measure space $(X, \mathcal{B})$, by measurable functions. This suggests the following definition.

Definition V.1.6. Let G be a group. A measurable space $(X, \mathcal{B})$ is a right (left) G-space if there is a right (left) action of G on X, such that

$$f^g : x \mapsto x \cdot g \qquad \text{is measurable } \forall g \in G.$$

We say that a measure μ on $(X, \mathcal{B})$ is right (left) G-invariant if it is invariant under the transformation f^g, for any $g \in G$.

Remark V.1.7. Similarly, it's possible to define a G-space $(X, \mathcal{B})$ for a semigroup G.

V.2 Ergodicity

Fix a measurable space $(X, \mathcal{B})$ and let $f : X \to X$ be a measurable transformation. When necessary, we will identify the space X with the dynamical system $(X, \mathcal{B}, f)$. We will denote by μ a positive measure on X invariant by f.

Among the class of invariant measures μ on X there are some which are more compatible with the dynamics on X.

Definition V.2.1. A dynamical system (X, f, μ) is *ergodic* if

$$\lim_{n\to+\infty} \sum_{j=0}^{n} 1_E(f^j(x)) = \mu(E) \quad \forall E \in \mathcal{B}, \text{ for a.e } x \in X,$$

where 1_E is the indicator function of the measurable set E.

Remark V.2.2. The function $\tau(E, x) = \lim_{n\to+\infty} \sum_{j=0}^{n} 1_E(x)$ is called the mean sojourn time. While for an integrable function $\phi \in L^1(X)$ we define

$$\tilde{\phi}(x) = \lim_{n\to+\infty} \sum_{j=0}^{n} \phi(f^j(x)),$$

the time average of ϕ.

Remark V.2.3. It's not obvious from the definition that the mean sojourn time and the time average of an integrable function even exist. This is guaranteed by the celebrated Birkhoff pointwise ergodic theorem, see [VO15, §3.2] for the proof and the background.

There are different equivalent definitions.

Proposition V.2.4. *Let (X, f, μ) be a dynamical system. Then the following are equivalent:*

1. *(X, f, μ) is ergodic;*
2. *for every measurable set $E \in \mathcal{B}$ the function $\tau(B, \cdot)$ is constant μ-a.e.;*
3. *for every $\phi \in L^1(\mu)$ one has $\tilde{\phi} = \int_X \phi d\mu$ for μ-a.e. $x \in X$;*
4. *for every $\phi \in L^1(\mu)$ the time average $\tilde{\phi}$ is constant for μ-a.e. $x \in X$;*
5. *for every invariant $\psi \in L^1(X)$ one has $\psi(X) = \int \psi d\mu$ for μ-a.e. $x \in X$;*
6. *every invariant $\psi \in L^1(X)$ is constant for μ-a.e. $x \in X$;*
7. *for every invariant measurable subset $A \in \mathcal{B}$ we have either $\mu(A) = 0$ or $\mu(A) = 1$;*

Proof. See [VO15, Proposition 4.1.3]. □

Remark V.2.5. Through the last equivalent condition, the definition of ergodic measure can be extended in the case of flows and more generally G-spaces. However, for the theory to be non-trivial, one has to prove first that the space of G-invariant probability measures on X is not empty. This is not usually true. A counterexample is given by the spaces $X = G$, where G is a *non*-amenable group, like $\mathrm{SL}_2(\mathbb{Z})$ with the discrete topology.

Hopefully, there is still a large class of G-spaces that admits G-invariant measures: G a locally compact Hausdorff unimodular topological group, and $X = H\backslash G$, for a closed subgroup H. In this case, the pushforward measure on X is right invariant. Moreover, if the space has finite measure, we can find also a G-invariant probability measure on X.

The ergodic systems then coincide with the ones that can't be decomposed into dynamically stable subspaces of non-trivial measure. This suggests that, among the invariant measures, they could be the indecomposable elements while the non-ergodic ones may be decomposed.

First, we restrict to the space of invariant probability measures $\mathcal{M}_1(f)$, which is convex since $(1-t)\mu_1 + t\mu_2$ is still a probability measure, for any $t \in [0,1]$. Actually, by interpreting positive measures as positive functionals in $\mu : L^1(X) \to \mathbb{R}$, it is actually a compact subset of $L^1(X)^*$ with the weak-* topology by the Banach-Alaoglu Theorem. We record some results in this context.

Proposition V.2.6. *1) If μ and ν are invariant probability measures such that μ is ergodic and ν is absolutely continuous with respect to μ, ($\nu \ll \mu$), then $\mu = \nu$.*

2) An invariant probability measure μ is ergodic if and only if it is not possible to write it as $\mu = (1-t)\mu_1 + t\mu_2$ with $t \in (0,1)$ and $\mu_1, \mu_2 \in \mathcal{M}_1(f)$, with $\mu_1 \neq \mu_2$.

Proof. See [VO15, Proposition 4.3.1-2]. □

The previous result states that the ergodic measures coincide with the *extremal points* of the convex set $\mathcal{M}_1(f)$. What we would like to achieve is then a decomposition of any other invariant measure ν in terms of ergodic measures. However, only stating the correct theorem requires some care and additional definitions. In fact, in the most naive form we would want that $\nu = \sum_{i\in I} a_i \nu_i$, with $a_i \geq 0$ and $\sum_{i \in I} a_i \mu_i$. This is true in the case in which $\mathcal{M}_1(f)$ lives inside a finite dimensional space $\mathbb{R}^n$, but this is in general not verified. We need a more advanced way of decomposing a measure, which is called a *disintegration.*

Let ν a probability measure on X. Let $\mathcal{P}$ a partition of X into measurable subsets. Denote by $\pi : X \to \mathcal{P}$ the projection that assigns to each point $x \in M$ the element $\mathcal{P}(x)$ of the partition that contains it, also called *atom* and denoted $[x]_{\mathcal{P}}$. Then $\mathcal{P}$ has an induced structure of probability space given by the pushforward of the measured space structure on $\mathcal{P}$ by the quotient map π. We denote by $\hat{\mu}$ this pushforward measure.

Definition V.2.7 (Disintegration of a measure)**.** Let ν be a measure on X. A *disintegration* of ν with respect to a partition $\mathcal{P}$ of X into measurable subsets is a family $\{\nu_P \mid P \in \mathcal{P}\}$ of probability measures on X such that, for every measurable set $E \in \mathcal{B}$:

1. $\mu_P(P) = 1$ for $\hat{\nu}$-a.e. $P \in \mathcal{P}$;

2. the map defined by $P \mapsto \nu_P(E)$ is measurable;

3. $\mu(E) = \int \nu_P(E) d\hat{\nu}(P)$.

The measures μ_P are called the *conditional proababilities* of μ with respect to $\mathcal{P}$.

Without imposing conditions on the space X and on the partition $\mathcal{P}$ it's not possible to always achieve a disintegration of a given probability measure. However, there is a large class of partitions that is actually well-behaved.

Consider a sequence partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, \ldots$. For any $m \in N^* \cup \{\infty\}$, denote by $\bigvee_{n=1}^m \mathcal{P}_n$ the partition whose elements are the non-trivial intersections of the form $\cap_{n=1}^m P_n$, for $P_n \in \mathcal{P}_n$. We say that the sequence is increasing if $\mathcal{P}_i \leq \mathcal{P}_{i+1}$, for all $i \in \mathbb{N}^*$, where the order relation is defined as follows:

$$\mathcal{P}_i \leq \mathcal{P}_{i+1} \iff \forall P_i \in \mathcal{P}_i \; \exists \; P_{i+1} \in \mathcal{P}_{i+1} \text{ such that } P_i \subset P_{i+1,i}.$$

$\mathcal{P}_i$ is called a coarser partition than $\mathcal{P}_{i+1}$ and the latter is a finer partition.

Definition V.2.8 (Measurable Partition)**.** Let (X, μ) be a measured space. We say that $\mathcal{P}$ is a *measurable partition* if there exists some measurable set $X_0 \subset X$ with full measure, such that, restricted to X_0,

$$\mathcal{P} = \bigvee_{n=1}^{\infty} \mathcal{P}_n$$

for a sequence of increasing countable partitions $\mathcal{P}_1 \leq \mathcal{P}_2 \leq \cdots \leq \mathcal{P}_n \leq \ldots$.

Remark V.2.9. If the sequence of partitions is not increasing, we can consider the standard increasing sequence $\bigvee_{i=1}^1 \mathcal{P}_i, \ldots, \bigvee_{i=1}^n \mathcal{P}_i, \ldots$. If the partitions $\mathcal{P}_i$ are countable, then also $\bigvee_{i=1}^n \mathcal{P}_i$ are countable.

Then the *Rokhlin disintegration theorem* states the following.

Theorem V.2.10 (Rokhlin disintegration theorem)**.** *Assume that X is a complete separable metric space, endowed with the Borel σ-algebra $\mathcal{B}$, and $\mathcal{P}$ is a measurable partition. Then a probability measure ν admits some disintegration with respect to $\mathcal{P}$.*

Proof. See [VO15, §5.2]. □

Remark V.2.11. Actually, a stronger result holds in the case in which the σ-algebra is countably generated: the disintegration of ν is essentially unique, i.e. $\nu_P = \nu'_P$ for $\hat{\nu}$-a.e. $P \in \mathcal{P}$.

Now, we have all the necessary results in order to appreciate the following theorem.

Theorem V.2.12. *Let X be a complete separable metric space, endowed with the Borel σ-algebra $\mathcal{B}$, let $f : M \to M$ be a measurable transformation and let μ be an invariant probability measure. Then there exists a measurable set $X_0 \subseteq X$ of full measure, a partition $\mathcal{P}$ of X_0 into measurable subsets and a disintegration $\{\mu_P \mid P \in \mathcal{P}\}$ of μ such that μ_P is invariant and ergodic for $\hat{\mu}$-almost every $P \in \mathcal{P}$.*

Definition V.2.13 (Ergodic components)**.** With the notation of Theorem V.2.12, the ergodic measures μ_P are called the *ergodic components* of the dynamical system (X, f, μ).

V.3 Entropy

The notion of *entropy* arose first in the context of thermodynamics, whose first role was a quantifier of the degree of "disorder" in a thermodynamical system in equilibrium. Actually, through the entropy formalism, it became the main actor, able to fully describe a system in equilibrium.

A distinct but related notion of entropy was introduced later by Claude Shannon, in the domain of information theory. For a given finite set $\mathcal{A}$, called alphabet, one associates a probability measure μ on $\mathcal{P}(A)$, by assigning a frequency $p_a = \mu(\{a\})$ to the letters $a \in \mathcal{A}$. The mean information associated to an alphabet $\mathcal{A}$ is given by

$$I(\mathcal{A}) = \sum_{a\in\mathcal{A}} -p_a \log p_a,$$

where by convention $0 \log 0 = 0$.

In general, it's possible to associate a sequence μ_n of probability measures on the set $\mathcal{A}^n$, with the compatibility requirement that for $\pi_{i+1} : \mathcal{A}^{n+1} \to \mathcal{A}^n$, given by $(a_1, \dots, a_{n+1}) \mapsto (a_1, \dots, a_n)$, we have that $\pi^*_{n+1}\mu_{n+1} = \mu_n$. Then the entropy of a communication channel is defined by:

$$I = \lim_{n\to+\infty} \frac{1}{n} I(\mathcal{A}^n).$$

Also in the context of dynamical systems, it's possible to introduce the notion of entropy of a system $((X, \mathcal{B}), f, \mu)$. As in the thermodynamical case, we define this quantity only under the "equilibrium" condition that μ is f-invariant.

Fix a dynamical system $(((X, \mathcal{B}), f, \mu))$, for a measurable transformation $f : X \to X$ and an f-invariant probability measure μ.

Definition V.3.1 (Entropy of a partition)**.** Let $\mathcal{P}$ be a countable partition of $X_0 \subset X$, measurable subset of full measure of X. Let $I_{\mathcal{P}} : X \to \mathbb{R}$ be the information function defined by

$$I_{\mathcal{P}}(x) = -\log \mu(\mathcal{P}(x)).$$

The the *entropy* of the partition $\mathcal{P}$ is:

$$H_\mu(\mathcal{P}) = \int I_{\mathcal{P}} d\mu = \sum_{P\in\mathcal{P}} -\mu(P) \log \mu(P).$$

Remark V.3.2. The function $\phi : (0, \infty) \to \mathbb{R}$ given by $\phi(x) = -x \log x$ is strictly concave. As a result,

$$\phi\left(\int_X f d\mu\right) \geq \int_X \phi \circ f d\mu,$$

for any measurable function $f : X \to \mathbb{R}_{>0}$, by Jensen's inequality.

Remark V.3.3. Since f is μ-invariant, we have the equality $H_\mu(\mathcal{P}) = H_\mu(f^{-j}(\mathcal{P}))$, for any $n \geq 0$.

Proposition V.3.4. *Let $\mathcal{P}, \mathcal{Q}$ be countable partitions of $X_0 \subset X$, measurable subset of full measure of X. Then*

1. *if $\mathcal{P} \leq \mathcal{Q}$ then $H_\mu(\mathcal{P}) \leq H_\mu(\mathcal{Q})$;*

2. $H_\mu(\mathcal{P} \vee \mathcal{Q}) \leq H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q})$.

Proof. See [VO15, Lemma 9.1.5]. □

Let $\mathcal{P}$ be a countable partition of $X_0 \subset X$, measurable subset of full measure of X, with finite entropy. Denote

$$\mathcal{P}^n = \bigvee_{i=0}^{n-1} f^{-1}(\mathcal{P}), \text{ for any } n \geq 1.$$

The sequence $(\mathcal{P}^n)_{n\in\mathbb{N}^*}$ is thus non-decreasing. Moreover, since $\mathcal{P}^{n+m} = \mathcal{P}^m \vee f^{-m}\mathcal{P}^n$, we have that the sequence of entropies $(H_\mu(\mathcal{P}^n))_{n\in\mathbb{N}^*}$ satisfies, by proposition V.3.4,

$$H_\mu(\mathcal{P}^{n+m}) \leq H_\mu(\mathcal{P}^m) + H_\mu(f^{-m}\mathcal{P}^n) = H_\mu(\mathcal{P}^m) + H_\mu(\mathcal{P}^n),$$

hence is subadditive. As a consequence of Fakete's Lemma, we get that the quantity:

$$h_\mu(f, \mathcal{P}) = \lim_{n\to+\infty} \frac{1}{n} H_\mu(\mathcal{P}^n)$$

is well-defined and equals $\inf_{n\to+\infty} \frac{1}{n} H_\mu(\mathcal{P}^n)$. Then $h_\mu(f, \mathcal{P})$ is the entropy of f with respect to the partition $\mathcal{P}$.

Definition V.3.5. The *entropy* of the dynamical system $((X, \mathcal{B}), f, \mu)$ is defined by

$$h_\mu(f) = \sup_{\mathcal{P}} h_\mu(f, \mathcal{P}),$$

where the supremum is taken over all the partitions with finite entropy.

The computation of the supremum is potentially a major problem in the determination of the entropy of a dynamical system. Thus, the following theorem of Kolmogorov-Sinai is fundamental.

Theorem V.3.6 (Kolmogorov-Sinai)**.** *Let $\mathcal{P}_1 \leq \cdots \leq \mathcal{P}_n \leq \ldots$ be a non-decreasing sequence of countable partitions with finite entropy, such that $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ generated the σ-algebra of measurable sets, up to measure zero. Then,*

$$h_\mu(f) = \lim_{n\to+\infty} h_\mu(f, \mathcal{P}_n)$$

Proof. See [VO15, Theorem 9.2.1]. □

Corollary V.3.7. *Let $\mathcal{P}$ be a partition with finite entropy such that the union of the iterates $\mathcal{P}^n = \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})$, $n \geq 1$ generates the σ-algebra of measurable sets, up to measure zero. Then, $h_\mu(f) = h_\mu(f, \mathcal{P})$.*

Proof. By the previous theorem, it's enough to compute $\lim_{n\to+\infty} h_\mu(f, \mathcal{P}^n) =$. But by the definition of

$$h_\mu(f, \mathcal{P}) = \lim_{n\to+\infty} H_\mu(\mathcal{P}^n),$$

we get that $h_\mu(f, \mathcal{P}^n) = h_\mu(f, \mathcal{P})$, for any $n \in N^*$. □

Suppose now that X is a metric space and μ is a Borel probability measure.

Corollary V.3.8. *Let $\mathcal{P}_1 \leq \cdots \leq \mathcal{P}_n \leq \ldots$ be a non-decreasing sequence of countable partitions with finite entropy such that* $\mathrm{diam}\mathcal{P}_n(x) \to 0$ *for μ-a.e. every $x \in M$. Then,*

$$h_\mu(f) = \lim_{n\to+\infty} h_\mu(f, \mathcal{P}_n).$$

Corollary V.3.9. *Let $\mathcal{P}$ be a partition with finite entropy such that we have that* $\mathrm{diam}\mathcal{P}^n \to 0$ *for μ-a.e. $x \in M$. Then $h_\mu(f) = h_\mu(f, \mathcal{P})$.*

Proof. See [VO15, Corollary 9.2.9-10]. □

The following crucial result, defines a local analogue of the entropy and relates it to the entropy of the dynamical system.

Theorem V.3.10 (Shannon-McMillan-Breiman)**.** *Given any partition $\mathcal{P}$ with finite entropy, the limit*

$$h_\mu(f, \mathcal{P}, x) = \lim_{n\to+\infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(x)) \tag{V.1}$$

exists at μ-a.e. point.

The function $x \mapsto h_\mu(f, \mathcal{P}, x)$ is μ-integrable, and the limit in V.1 also holds in $L^1(\mu)$. Moreover,

$$\int h_\mu(f, \mathcal{P}, x) d\mu(x) = h_\mu(f, \mathcal{P}).$$

If $(f, \mu)$is ergodic then $h_\mu(f, \mathcal{P}, x) = h_\mu(f, \mathcal{P})$ at μ-almost every point.

Proof. See [VO15, Theorem 9.3.1]. □

What is left to discuss is the relation between the entropy associated to an f-invariant probability measure μ and the ones of its ergodic decomposition, in the case in which $(X, \mathcal{B})$ is a complete separable metric space with the Borel σ-algebra $\mathcal{B}$.

Theorem V.3.11 (Jacobs)**.** *Suppose that $(X, \mathcal{B})$ is a complete separable metric space with the Borel σ-algebra $\mathcal{B}$. Given any invariant probability measure μ, let $\{\mu_p \mid P \in \mathcal{P}\}$, for a measurable partition $\mathcal{P}$, be its ergodic decomposition. Then, $h_\mu(f) = \int h_{\mu_P}(f) d\hat{\mu}(P)$.*

Proof. See [VO15, Theorem 9.6.2]. □

Corollary V.3.12. *Suppose that $(X, \mathcal{B})$ is a complete separable metric space with the Borel σ-algebra $\mathcal{B}$. Let μ be an f-invariant probability measure, let $\{\mu_p \mid P \in \mathcal{P}\}$, for a measurable partition $\mathcal{P}$, be its ergodic decomposition. Let $\mathcal{Q}$ be a partition with finite μ-entropy. Then $h_{\mu_P}(f, \mathcal{Q}, x) = h_\mu(f, \mathcal{Q}, x)$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$ and for μ-a.e. $x \in P$.*

V.4 Recurrence

Fix a locally compact Hausdorff group G, with a right invariant metric d_G, and a locally compact metric space X, with Borel σ-algebra $\mathcal{B}$. We assume that $(X, \mathcal{B})$ with the structure of a G-space, with a Radon measure μ, where the map $(x, g) \mapsto x \cdot g$ is assumed to be continuous, for any $x \in X, g \in G$. We assume moreover that the map $t_x : g \mapsto x \cdot g$ is injective for μ-a.e. $x \in X$.

Definition V.4.1. We say that a Radon measure μ on a G-space X is *recurrent* if for every measurable $B \in \mathcal{B}$, with $\mu(B) > 0$, for almost every $x \in B$ it holds that $t_x^{-1}(B)$ is unbounded, i.e. has a non-compact closure.

The opposite of recurrence is *transience*.

Remark V.4.2. If $G = (\mathbb{Z}, +)$ with the discrete topology, the G-action is generated by an invertible transformation $f : X \to X$, defined by $x \mapsto x \cdot 1$. Then μ is transient if and only if $\# t_x^{-1}(B) = \infty$, for every measurable set $B \in \mathcal{B}$, with $\mu(B) > 0$.

The right action of G induces a *foliation* on X, whose *leaves* are the G-orbits. We would like to achieve a disintegration of μ with respect to the partition given by the orbits xG. However, this is unlikely a measurable partition, and we can't apply directly Theorem V.2.10. Nonetheless, through a more delicate limit process, it's possible to obtain conditional measures on the orbits. The key idea is to obtain a disintegration with respect to measurable partitions $\mathcal{P}$ whose atoms are partial orbits and "patch" them to obtain a conditional measure on the full orbits. More formally:

Definition V.4.3. A set $A \subset X$ is an open G-plaque if for any $x \in A$:

1. $A \subset t_x(G)$;
2. $t_x^{-1}(A)$ is open in G.

Remark V.4.4. A particular example is given by measurable partitions $\mathcal{P}_r$ whose atoms $[x]_{\mathcal{P}_r} = xB_r(G)$ for μ-a.e. $x \in X$, where $B_r(G)$ is the open ball of radius r centered at the identity of G . The associated conditional measures $\mu_{x,r}$ on $[x]_{\mathcal{P}_r}$ can be normalized so that $\mu_{x,r}(xB_1(G)) = 1$, for μ-a.e. $x \in X$. Then, for some particular choices of X, for μ-a.e. fixed $x \in X$, the sequence $(\mu_{x,r})$ forms a direct limit of measures on $xG = G$, that we denote by $\mu_{x,G}$. We stress that, with this choice, there is no direct way of obtaining a disintegration of μ with respect to $\mu_{x,G}$.

The following holds.

Theorem V.4.5. *Let X be a G-space, and μ a probability measure on X with t_x injective a.s. Then there is a Borel measurable map ${}_{x,G} : X \to M^\infty(G)$ on G , where $M^\infty(G)$ is the space of Radon measures with the weak-* topology, which is uniquely determined (up to -measure 0) by the following two conditions:*

1. *For -a.e. x, we have that $\mu_{x,G}(B_1(G)) = 1$;*
2. *For any measurable partition $\mathcal{A}$ of E, measurable subset of full measure of X, if for every $x \in E$ the atom $[x]_\mathcal{A}$ is an open G-plaque, then for μ-a.e. $x \in E$,*

$$t_x^{-1}{}_* \mu_x^{\mathcal{A}} \propto \mu_{x,G}\big|_{t_x^{-1}[x]_\mathcal{A}}.$$

In addition, $\mu_{x,G}$ has the following property:

(a) There is a set $X_0 \subseteq X$ of full measure so that every $x, y \in X_0$, with $y \in xG$, and for any $g \in G$ satisfying $t_y \circ g = t_x$, we have that

$$g_* \mu_{x,G} \propto \mu_{y,G}.$$

Thus, for any disintegration with atoms given by open G-plaques, we obtain information on $\mu_{x,G}$. In particular, if, for a sequence of disintegrations whose atoms are larger and larger open G-plaques, we have that the conditional probabilities at the atom of x give to $xB_1(G)$ a volume which tends to 0, for μ-a.e. x, then $\mu_{x,G}(G) = \infty$ for μ-a.e. x. This is the idea behind the following crucial criterion.

Theorem V.4.6. *Let μ be a probability measure on a G-space X. Then μ is G-recurrent if and only if $\mu_{x,G}$ is an infinite measure μ-a.s.*

Proof. See [Lin04, Theorem 4.2]. □

V.5 Unique Ergodicity

In section V.2 we saw that, in the class of dynamical systems $((X, \mathcal{B}), f, \mu)$, for an invariant probability measure μ, the ergodic systems had a special role, acting as building blocks. In this context, a much sought property is the *unique ergodicity.*

Definition V.5.1. A measurable transformation $f : X \to X$ is *uniquely ergodic* if it admits exactly one invariant probability measure.

Remark V.5.2. By using proposition V.2.6, the unique invariant probability measure must be ergodic, since it must be an extremal point of the convex set $\mathcal{M}_1(f)$, which has only one element. From this observation follows the name of these special dynamical systems.

We can extend the previous definition to arbitrary G-spaces in the trivial way. A property related to unique ergodicity is the *minimality.*

Definition V.5.3. Let $(X, \mathcal{B})$ be a G-space. Let $Y \subset X$ be a closed measurable set invariant under G. We say that Y is *minimal* if $Y = \overline{xG}$, for any $x \in Y$.

Proposition V.5.4. *If the G-space X is uniquely ergodic, then the support of the unique invariant probability measure μ is a minimal set.*

Proof. See [VO15, Proposition 6.2.1]. □

Remark V.5.5. The converse is not true: there could be distinct measures supported on distinct minimal sets on a single G-space. Take for example an indefinite rational quaternion algebra B with Eichler order R. Define $H_R = \prod_{v\neq\infty}(R \otimes \mathbb{Z}_v)^\times$. Then the space $X = A^\times B_\mathbb{Q}^\times \setminus B_A^\times / H_R$ is isomorphic to $\mathbb{R}^\times R^\times \setminus \mathrm{GL}_2(\mathbb{R})$. Now let's restrict to the right $O_2(\mathbb{R})$-invariant measures. Then $X' = X/O_2(\mathbb{R}) \cong \mathbb{R}^\times \mathcal{H}$, for the upper-half plane $\mathcal{H}$ with the standard action of $\mathrm{SL}_2(\mathbb{R})$ on it. Consider the diagonal flow $f^t(x) = x \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$, for $t \in \mathbb{R}$, associated to the diagonal split torus A_1 of $\mathrm{PGL}_2(\mathbb{R})$. This flow coincides with the *geodesic flow* on X' endowed with the standard Riemannian metric that has constant sectional curvature equal to -1. As a consequence, any invariant probability measure μ on X, right-invariant by O_2, supported on a minimal set, corresponds to a measure $\tilde{\mu}$ on μ which is supported in the closure of a geodesic. Since there are non-trivial compact closed geodesics, it's possible to find probability measures which are supported on a minimal set distinct from the whole space X'.

The most interesting cases where some kind of unique ergodicity, or similar strong classifications of the set of invariant measures, is verified are related to algebraic constructions, in particular to algebraic groups. In this setting, assumptions on the entropy and recurrence of the dynamical system reveal to be essential. Thanks to these additional hypotheses, Einsiedler and Lindenstrauss in [EL08] were able to prove the following classification result for invariant measures on algebraic groups.

Theorem V.5.6 (Einsiedler-Lindenstrauss). *Let $G = G_1 \times G_2$ where G_1 is a semisimple linear algebraic group over a characteristic zero local field K_σ with K_σ rank 1 and G_2 is a zero characteristic S-algebraic group. Let $\Delta \subset G$ be a discrete group. Let A_1 be a K_σ split torus of G_1 and let χ be a non-trivial K_σ-character of A_1 that can be extended to $C_G(A_1)$. Let $M_1 = \{h \in C_G(A_1) \mid \chi(h) = 1\}$. Let μ be an A_1-invariant, G_2-recurrent probability measure on $\Delta \setminus G$ such that*

1. *almost every A_1-ergodic component of μ has positive ergodic theoretic entropy with respect to some $a \in A_1$ with $|\chi(a)|_\sigma \neq 1$ and*

2. *for μ-a.e. x the group*
$$\{h \in M_1 \times G \mid x \cdot h = x\}$$
is finite.

Then μ is a convex combination of homogeneous measures. Each of these homogeneous measures is supported on an orbit of a subgroup H which, after restriction of scalars to a local subfield F_σ, contains a finite index subgroup of a semisimple algebraic subgroup of G_1 of F_σ-rank one.

Remark V.5.7. We shall explain some of the terms in the previous theorem.

First, a measure on $\Gamma \setminus G$ is called *homogeneous* if it is the unique H-invariant probability measure on a periodic H-orbit, i.e. an orbit that supports a finite H-invariant measure.

Second, an S-algebraic group, for a finite set S, is a product of K_i-algebraic groups, for local fields K_i, $i \in S$.

Finally, the rank of a semisimple algebraic group is the (algebraic) dimension of a Cartan subgroup, which is just the centralizer of a maximal torus.

Remark V.5.8. One of the most important assumptions is the presence of two commuting actions on $\Delta \setminus G$. This is crucial for making the dynamical system more rigid, see [Ber16, §9.4] for an example of this phenomenon in a simpler setting.

In this thesis we are only interested in its application in number theory, in particular for the concentration properties of automorphic forms in global quaternion algebras. We will find out that, even though in its general form this classification doesn't prove a unique ergodicity result, by restricting to particular arithmetic measures we will obtain a p-Adic version of the celebrated *Arithmetic Quantum Unique Ergodicity* [Lin06]. In the next section, we will see how this broad theorem can be specialized and made more effective in the arithmetic setting.

V.6 Dynamics on Quaternion Algebras

Let B be a quaternion algebra over $\mathbb{Q}$. Let σ, l be two places of $\mathbb{Q}$, for a finite prime l that splits the quaternion algebra and for another split place σ, possibly infinite. Consider an Eichler order R of B and let $H_R = \prod_{v \neq \sigma} R_v^\times$, where $R_v = R \otimes \mathbb{Z}_v$, coming from Proposition II.6.1. Then, as in section II.7.1, there is a natural identification (and homeomorphism):

$$B_{\mathbb{Q}}^\times \setminus B_{\mathbb{A}}^\times / H_R \cong \Gamma \setminus B_\sigma^\times,$$

where $\Gamma = B^\times \cap H_R$ and is discrete by II.5.11 (in case B is a division algebra it is also cocompact). The isomorphism is given explicitly by the map $\pi : B_{\mathbb{Q}}^\times \setminus B_{\mathbb{A}}^\times \to \Gamma \setminus B_\sigma^\times$,

$$\pi : B_{\mathbb{Q}}^\times g \mapsto \pi_{B_{\mathbb{A}}^\times \to B_\sigma^\times}(B_{\mathbb{Q}}^\times g \cap [G_\sigma \times H_{\ R}]),$$

for the standard projection $\pi_{B_{\mathbb{A}}^\times \to B_\sigma^\times}$ of the idèle group to its local $B_\sigma^\times$ component.

Let $X = \Gamma \setminus B_\sigma^\times$. With an analogue argument, we can identify:

$$X \cong \Gamma' \setminus B_\sigma^\times B_l^\times / R_l^\times,$$

where $\Gamma' = B^\times \cap \prod_{v \neq \sigma, l} R_v^\times$. We can set $X' = \Gamma' \setminus B_\sigma^\times B_l^\times$. It has the structure of a locally compact metric space, with an action of both groups $B_\sigma^\times$, $B_l^\times$ by right translation. In this setting, Theorem V.5.6 takes the following form.

Theorem V.6.1. *Let μ be a $Z(B_\sigma^\times)$-invariant probability Radon measure on X such that:*

1. *μ is A-invariant, for the diagonal subgroup A of $B_\sigma^\times$;*
2. *μ is $B_l^\times / Z(B_l^\times)$-recurrent, as a measure of $X'/Z(B_\sigma^\times B_l^\times)$;*
3. *almost every A-ergodic component of μ has positive entropy with respect to $\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$, for some $|s|_\sigma \neq 1$.*

Then μ is the $B_\sigma^\times$-right invariant probability Radon measure on X, obtained by pushforward of a Haar measure on $B_\sigma^\times$.

Proof. First, we can identify $B_\sigma^\times \cong \mathrm{GL}_2(\mathbb{Q}_\sigma)$ and $B_l^\times \cong \mathrm{GL}_2(\mathbb{Q}_l)$, since they are split local quaternion algebras. As a result, the $Z(B_\sigma^\times)$-invariant measure on X lives, by pushforward, on the space:

$$Y = \Gamma \setminus \mathrm{GL}_2(\mathbb{Q}_\sigma) / \mathbb{Q}_\sigma^\times \cong \bar{\Gamma} \setminus \mathrm{PGL}_2(\mathbb{Q}_\sigma),$$

where $\bar{\Gamma} = (\Gamma \mathbb{Q}_\sigma^\times)/\mathbb{Q}_\sigma^\times$ is the group of σ-projective units of the Eichler order R, and where we identified $Z(B_\sigma^\times) \cong \mathbb{Q}_\sigma^\times$, since B_σ is a central simple algebra.

Again by strong approximation,

$$Y \cong \Delta \setminus \mathrm{PGL}_2(\mathbb{Q}_\sigma)\mathrm{PGL}_2(\mathbb{Q}_l)/\overline{R_l^\times},$$

for $\Delta = \Gamma' \mathbb{Q}_\sigma^\times \mathbb{Q}_l^\times / \mathbb{Q}_\sigma^\times \mathbb{Q}_l^\times$, the image of Γ' in $\mathrm{PGL}_2(\mathbb{Q}_\sigma^\times)\mathrm{PGL}_2(\mathbb{Q}_l^\times)$, and $\overline{R_l^\times} \cong \mathrm{PGL}_2(\mathbb{Z}_l)$ the image of $R_l^\times$ in $B_l^\times / \mathbb{Q}_l^\times$.

So, set $G_1 = \mathrm{PGL}_2(Q_\sigma)$ and $G_2 = \mathrm{PGL}_2(\mathbb{Q}_l)$, and notice that Δ is a discrete subgroup of $G = G_1 \times G_2$. Both G_1 and G_2 are semisimple linear algebraic groups with rank one over, respectively, the zero characteristic local fields $\mathbb{Q}_\sigma$ and $\mathbb{Q}_l$. Under the previous isomorphisms, μ can be identified with a right $\mathrm{PGL}_2(\mathbb{Z}_l)$-invariant measure on $\Delta \setminus G$. We want to apply Theorem V.6.1 for this choice of G_1, G_2, Δ and for $A_1 = \bar{A}$, the image of the diagonal group of $B_\sigma^\times$ on $\mathrm{PGL}_2(\mathbb{Q}_\sigma)$, generated by the image of the matrices $\begin{pmatrix} \mathbb{Q}_\sigma^\times & 0 \\ 0 & 1 \end{pmatrix}$. For clarity, we point out that the hypothesis 2) of being $B_l^\times/Z(B_l^\times)$-recurrent, as a measure of $X'/Z(B_\sigma^\times B_l^\times)$ is equivalent to the G_2-recurrence of the measure on $\Delta \setminus G$.

We still need to define the character χ. We choose in particular the map $\chi(diag(y_1, y_2)) = \frac{y_1}{y_2}$, which is trivially well-defined on A_1. Moreover, $C_{G_1}(A_1) = A_1$ and by the choice of χ, we have that

$$M_1 = \{\mathrm{diag}(a_1, a_2) \mid a_1 = a_2\} = \{1\},$$

is trivial. Finally, the hypothesis 3) is equivalent to the hypothesis 1) on Theorem V.6.1.

Thus, μ is invariant by some finite index subgroup H_1 of some semisimple algebraic subgroup of G_1, that contains A_1. The smallest such H_1 is the image of $\mathrm{SL}_2(\mathbb{Q}_\sigma)$ in $\mathrm{PGL}_2(\mathbb{Q}_\sigma)$. As a result, μ, as a measure on X, is a Radon measure right invariant by the diagonal torus A and by $\mathrm{SL}_2(\mathbb{Q}_\sigma)$, which generate $\mathrm{GL}_2(\mathbb{Q}_\sigma)$. Hence, it is the unique probability Haar right $B_\sigma^\times$-invariant measure on X. □

We conclude the discussion by giving equivalent conditions for hypothesis 2) and 3) of the previous theorem, that are easier to check in practice. We will use the notation of the proof of the previous theorem.

For condition 2), the $B_l^\times/Z(B_l^\times)$ recurrence of μ as a measure on

$$Y' = X'/Z(B_\sigma^\times B_l^\times)$$

is just a $PGL_2(\mathbb{Q}_l)$-recurrence on the action by right multiplication in the second component of $\Delta \setminus G$. Since Δ is discrete, the action is μ-a.s. injective. So, we can use Theorem V.4.6 and try to prove that $\mu_{x,G_2}(G_2) = +\infty$, for μ-a.e. $x \in Y'$. Recall now that μ is right $\mathrm{PGL}_2(\mathbb{Z}_l)$-invariant, and that $B_1(G_2) = \mathrm{PGL}_2(\mathbb{Z}_l)$. Then, consider disintegrations with respect to a measurable partition $\mathcal{P}^S$ whose atoms are μ-a.e. open G-plaques A_x, of the form $A_x = xS\mathrm{PGL}_2(\mathbb{Z}_l)$, for a subset $S \subset T_l = \mathrm{PGL}_2(\mathbb{Q}_l)/\mathrm{PGL}_2(\mathbb{Z}_l)$. The conditional measures will be μ-a.s. right $\mathrm{PGL}_2(\mathbb{Z}_l)$-invariant and will thus live naturally as measures $\tilde{\mu}_{x,G_2}$ supported on the subset S of $T_{l+1} == \mathrm{PGL}_2(\mathbb{Q}_l)\mathrm{PGL}_2(\mathbb{Z}_l)$. We already met T_{l+1} as a $l+1$-regular tree in section IV.2 and we noticed the isometry between the quotient and the regular tree metric. So, as in the discussion after Theorem V.4.5, it's enough to prove that the conditional measures of the partitions $\mathcal{P}^S$, for $S\mathrm{PGL}_2(\mathbb{Z}_l) \supset B_1(G_2)$ larger and larger, assign volume $\mu_{x,G}(B_1(G_2))$ that tends to zero μ-a.s. Equivalently, it's enough to prove that for choices of S larger ad larger in the regular tree T_{l+1}, we have that $\tilde{\mu}_{x,G}(\{1\})$ tends to zero, μ-a.s. on $Y/\mathrm{PGL}_2(\mathbb{Z}_l) = X'$, or on X due to the $Z(B_\sigma^\times)$-invariance of μ. This is called a *T_l-recurrence*, from the operators T_l that generate the regular graph (see §IV.3). A reasonable choice of S_r is just $B_r(1)$, the ball of radius r in T_l, for $r \to +\infty$. This partially explains the following proposition.

Proposition V.6.2. *Let μ_φ be a $Z(B_\sigma^\times)$-invariant probability measure on X, induced by a L^2-normalized function $\varphi \in L^2(X)$, i.e. $\mu_\varphi(E) = \int_E |\varphi|^2 d\mu$, for the right*

Haar probability measure μ on X. Consider finite sets $\{1\} \subset S_i \subset \mathrm{GL}_2(\mathbb{Q}_l)/\mathrm{GL}_2(\mathbb{Z}_l)$, $i \in \mathbb{N}$, such that the projection $S_i \to S_i/Q_l^\times$ is injective. Denote by $S_i(x)$ the set xS_i, by identifying X with $\Gamma' \setminus \mathrm{GL}_2(\mathbb{Q}_\sigma)\mathrm{GL}_2(\mathbb{Q}_l)/\mathrm{GL}_2(\mathbb{Z}_l)$. If $A_i = \sum_{y\in S_i(x)} |\varphi(y)|^2$ has the property that

$$\lim_{n\to+\infty} \frac{|\varphi(x)|^2}{A_i} = 0,$$

then μ_φ is T_l-recurrent.

Proof. See [Lin04] for a detailed discussion. □

While for the entropy, we report a criterion of Lindenstrauss for our case of X coming from arithmetic quotients of quaternion algebras. To reduce the notation, we will present only the case of $\sigma = p$ a finite place of $\mathbb{Q}$, which is the relevant case for this work. The case $\sigma = \infty$ has an analogue definition.

Call $\epsilon > 0$ *admissible* if it belongs to the image of $|.| : \mathbb{Q}_p^\times \to \mathbb{R}_+^\times$. For a compact open subgroup C of $\mathbb{Q}_p^\times$ and admissible $\epsilon > 0$, set

$$B(C, \epsilon) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, d \in C,\ |b|, |c| \le \epsilon\},$$

the *dynamical ball* of radius ϵ and support C.

Theorem V.6.3. *Assume the notation of theorem V.6.2. Suppose that for each compact subset Ω of $B_p^\times$ there exists C as above and $C_1, c_2 > 0$ so that for all admissible $\epsilon \in (0, 1)$ and for all $x \in \Omega$, one has $\mu_\varphi(xB(C, \epsilon)) \le C_1\epsilon^{c_2}$. Then μ_φ has positive entropy on almost every A-ergodic component with respect to $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof. See [Lin04, §8]. □

Remark V.6.4. If B is a division algebra, so that X is compact, the result directly follows by an application of a topological version of the Shannon-McMillan-Breiman Theorem, proved by Brin-Katok in its general form in [BK83]. The point of Lindenstrauss' result is that it applies also to split quaternion algebras, so to non-compact X, which is another interesting but harder setting for the Arithmetic Quantum Unique Ergodicity problem.

Remark V.6.5. If we require that the conditions in Theorem V.6.3 and Proposition V.6.2 hold uniformly for a sequence of L^2-normalized functions $\varphi_i \in L^2(X)$, then the theorems hold also for any weak-* limit of the measures μ_{φ_i}, as a consequence of the upper semi-continuity of the entropy ([VO15, §9.2.2]) and because the decompositions over fixed open G-plaques leading to V.4.6 converge weak-* $\hat{u}$-a.s. See [Lin04] for the precise results.

Chapter VI

p-Adic Arithmetic Quantum Unique Ergodicity on $\mathrm{GL}_2(\mathbb{Q}_p)$

This chapter contains the main results on this essay, proving the p-adic arithmetic quantum unique ergodicity for a class of arithmetic quotients of $\mathrm{GL}_2(Q_p)$, coming from definite rational quaternion algebras.

After translating the setting presented in the section I in representation theoretical terms, we will prove the main results from Nelson's paper [Nel18].

VI.1 The setting

As in the introduction, fix a prime number p. Let $\mathfrak{o}^\times$ be the ring of integers of $\mathbb{Q}_p$, and denote by $\mathfrak{p} = (\varpi)$ its maximal ideal, generated by a uniformizer ϖ. Let $G = \mathrm{GL}(\mathbb{Q}_p)$.

Let B be a definite quaternion algebra B over $\mathbb{Q}$ that splits at p. Fix an Eichler order R and let $\Gamma = R[1/p]^\times$. Then, $B \hookrightarrow M_2(\mathbb{Q}_p)$ and the arithmetic quotient $X = \Gamma \setminus \mathrm{GL}_2(\mathbb{Q}_p)$ is compact, by II.5.11 and section II.7. In fact, the space X is a locally symmetric space isomorphic to

$$X \cong B_{\mathbb{Q}}^\times \setminus B_A^\times / H_R,$$

for $H_R = \prod_{v \neq p} R_v^\times$, for $R_\infty = B_\infty$.

The adelic quotient X comes with Hecke correspondences T_l, for primes $l \neq p$, as defined in Definition III.6.5 for split l. By composing them, we obtain the Hecke operators T_n, for $(n, p) = 1$. These assume the explicit form

$$T_l \varphi(x) = \sum_{s \in M_l/\Gamma} \varphi(s^{-1}x) \quad \forall x \in X, \tag{VI.1}$$

for $M_l = R[1/p]^\times \cap \mathrm{nr}^{-1}(l\mathbb{Z}[1/p])$.

More generally, with respect to section I, we allow non-trivial central characters for the newvectors. In this context, a direct geometric representation is less immediate and the problem benefits from a more abstract representation theoretic point of view.

Recall from section III.4 the congruence subgroups $K_0(\mathfrak{p}^n)$ for G. We will define now a suitable generalization.

Definition VI.1.1. Define the compact subgroup $K_{m' \dots m}$ of G as

$$K_{m' \dots m} = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-m'} \\ \mathfrak{p}^m & \mathfrak{o} \end{pmatrix} \cap \mathrm{GL}_2(\mathfrak{o}).$$

Thus, we shall be interested in functions which are invariant by open subgroups.

Definition VI.1.2. Denote by $\mathcal{A}(X)$ the space of smooth functions on X, as a G-module with the right translation action by G. An (Hecke) *eigenfunction* on X is an element $\phi \in \mathcal{A}(X)$ that is a T_l-eigenfunction for each prime $l \neq p$ and that generates an irreducible representation of G.

Remark VI.1.3. The link with the path space interpretation is clarified by section IV.3. In fact, there is a bijection:

$$X/K_{m' \dots m} = \Gamma \setminus G/K_{m' \dots m} \to Y_{m' \dots m},$$

for the path space $Y_{m' \dots m}$, since $K_{m' \dots m}$ is the stabilizer of the balanced path $m' \dots m$ in the standard apartment of the Bruhat-Tits Tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The simplifying requirement $\mathrm{Tors}(\Gamma) = \pm 1$ was then the necessary hypothesis to apply Remark IV.3.11.

Notation. On X there is a canonical right G-invariant Radon probability measure μ_H, given by pushforward of a Haar measure under $p : G \to \Gamma \setminus G$, for the unimodular group G, see [Bum97, §4.3] for the construction. The measure μ_H will be called the *uniform* measure. When equipped with the uniform measure, $\mathcal{A}(G)$ is a unitary (smooth) admissible representation of G, in the sense of III.2.1. So, an element $\phi \in \mathcal{A}(X)$ is L^2-normalized if $\int_X |\phi|^2 d\mu_H = 1$. In that case, the L^2-mass of ϕ is the probability measure μ_ϕ on X given by $\mu_\phi(\Psi) = \int_X \Psi|\phi|^2$. A sequence *equidistributes* if it converges to the uniform measure.

Definition VI.1.4. Let $\mathcal{H} \subseteq \mathrm{End}(\mathcal{A})(X)$ be the ring generated by the right translations of $g \in G$ and by the T_l operators. We denote by $\mathrm{A}(X)$ the set of irreducible $\mathcal{H}$-submodules of $\mathcal{A}(X)$, by $\mathrm{A}_0(X) \subseteq \mathrm{A}(X)$ the subset consisting of those that are not one-dimensional, and by $\mathcal{A}_0(X) \subseteq \mathcal{A}(X)$ the sum of the elements of $\mathrm{A}_0(X)$.

Remark VI.1.5. The 1-dimensional elements of $\mathrm{A}(X)$ have the form $\mathbb{C}(\chi \circ \det)$, for χ a character of $\mathbb{Q}_p^\times / \det(\Gamma)$.

As a consequence of the following proposition, any irreducible $\mathcal{H}$-module in $\mathcal{A}(X)$ can be safely regarded as a more simple irreducible G-module.

Proposition VI.1.6. *Let $\pi \in \mathrm{A}(X)$. Then π is also an irreducible representation of G. Moreover $\mathcal{A}(X) = \oplus_{\pi \in \mathrm{A}(X)} \pi$ and $\mathcal{A}_0(X) = \oplus_{\pi \in \mathrm{A}_0(X)} \pi$.*

Proof. Under the equivalence $X \cong B_{\mathbb{Q}}^\times \setminus B_A^\times / H_R$, the elements of $\mathrm{A}(X)$ corresponds to irreducible automorphic representations of $B^\times$, as defined in section II.7.2. Then, by the strong multiplicity one result, each $\pi \in \mathrm{A}(X)$ occurs in $\mathcal{A}(X)$ with multiplicity one. Moreover, the infinite component of the automorphic representation is trivial, and all the local components π_l are spherical for $l \neq p$. Then, by strong multiplicity one for $B^\times$, as seen in Theorem II.7.11, the restriction of π to a G-module yields an irreducible representation of G, since the T_l eigenvalues (Satake parameters), for $l \neq p$, uniquely determine the local component π_l of an automorphic representation of $B^\times$.

Finally $\mathcal{A}(X)$ is an admissible and unitary (unitarizable) G-module, so it is semisimple and the T_l operators are normal operators, with adjoint $T^* - l = z(l)T_l$ that commute with the translation by $g \in G$. □

We define now the class of functions, whose quantum limits will be the object of the p-adic quantum unique ergodicity.

Definition VI.1.7 (Generalized newvector)**.** Let $\pi \in \mathrm{A}_0(X)$. For integers m, m', a vector $\phi \in \pi$ will be called a *generalized newvector of support $m \ldots m'$* if $|m' - m| = c(\pi)$ and

$$\pi(g)\phi = \chi_\pi(d)\phi \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{m' \ldots m}.$$

Remark VI.1.8. Recall that the local newvectors were already defined in section III.4. The generalized ones are defined similarly, and are just conjugates of the classical ones. These vectors are the local components of the classical Maass or Hecke newforms in the classical real indefinite setting.

The main equidistribution result that we will prove is the following.

Theorem VI.1.9. *Let $\pi \in \mathrm{A}_0(X)$ traverse a sequence of irreducible principal series representations $\mathcal{B}(\chi_1, \chi_2)$ of G, with $c(\chi_1/\chi_2) \to \infty$. Let $\phi \in \pi$ be an L^2-normalized generalized newvector. Then μ_ϕ equidistributes.*

Remark VI.1.10. The condition $c(\chi_1/\chi_2) \to +\infty$ is a stronger condition than $c(\pi) \to +\infty$, since $c(\pi) = c(\chi_1) + c(\chi_2)$. However, it is a necessary restriction, since for any character χ of $\mathbb{Q}_p^\times$ the twist $\chi(\det) \otimes \pi \cong B(\chi_1\chi, \chi_2\chi)$, by Theorem III.5.9. Then we could have possibly a sequence of irreducible principal series with conductor tending to infinity, but such that the generalized newvectors have the form $\phi\chi_n$, for a fixed $\phi \in \mathcal{A}(X)$ and unitary character χ_n. Then the quantum limit $\mu = \mu_\phi$ trivially and there is no equidistribution.

As in the classical case, the equidistribution results for the newvectors passes through the introduction of a special class of measures, in these case smooth functions on $\mathcal{A}(X)$, called *microlocal lift*.

Fix for each positive integer N a partition $N = N_1 + N_2$ into nonnegative integers, with the property that $N_1, N_2 \to +\infty$ for $N \to +\infty$.

Definition VI.1.11 (microlocal lift)**.** Let π be an irreducible generic representation of G. Then $v \in \pi$ is a *microlocal lift* of support $-N_1 \dots N_2$ and level $N > 0$ if there exists characters ω_1, ω_2 of $\mathfrak{o}^\times$ so that $c(\chi_1/\chi_2) = N$ and

$$\pi(g)v = \omega_1(a)\omega_2(\det(g)/a) \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{-N_1 \dots N_2}.$$

In that case (ω_1, ω_2) is the *orientation* of v.

In particular, for the following applications, this choice is useful: we fix $N_1 = N_2$ or $N_1 = N_2 + 1$. These vectors are important since their L^2-mass achieve the diagonal invariance in the following sense.

Theorem VI.1.12. *Let N traverse a sequence of positive integers tending to ∞, and let $\varphi \in \pi \in \mathrm{A}_0(X)$ be an L^2-normalized microlocal lift of level N on X with L^2-mass μ_φ. Then any weak subsequential limit of the measures μ_φ.*

As in the classical case, the strategy is using Theorem V.5.6 on the weak-* limits of the masses of L^2-normalized microlocal lifts, in the form of Theorem V.6.1. With a more friendly notation, it can be stated as follows.

Theorem VI.1.13 (Measure Classification)**.** *Let μ be a probability measure on X, invariant by the center of G, with the properties:*

1. *μ is $a(\mathbb{Q}_p^\times)$-invariant;*

2. *μ is T_l-recurrent for some split $l \neq p$;*

3. *The entropy of almost every ergodic component of μ is positive for the $a(p)$-action.*

Then μ is the uniform measure.

Then, once proved this result, we will get the following.

Theorem VI.1.14. *Let N traverse a sequence of positive integers tending to ∞. Let $\varphi \in \mathcal{A}(X)$ be an L^2-normalized microlocal lift of level N on X. Then μ_φ equidistributes.*

The equidistribution of the local newvectors will be then a corollary of some local results.

VI.2 Recurrence

Fix a prime $l \neq p$ that splits the quaternion algebra B. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $X = \Gamma \setminus G$, for $\Gamma = R[1/p]$.

The aim of this section is to prove the T_l-recurrence of the weak-* limits of the L^2-masses μ_{ϕ_i} of L^2-normalized Hecke eigenfunctions $\phi_i \in \mathcal{A}_0(X)$.

Theorem VI.2.1. *Let μ be any sequence subsequential limit of a sequence of L^2-masses μ_ϕ of L^2-normalized automorphic forms $\phi \in \pi \in \mathrm{A}_0(X)$. Then μ is T_l-recurrent.*

Remark VI.2.2. The theorem holds also when we replace automorphic forms with simple L^2-normalized T_l-eigenfunctions. In fact, that's the only used property needed for the proof in the case of automorphic forms.

Recall from section V.4-6 that the T_l-recurrence is associated to an action of $\mathrm{PGL}_2(\mathbb{Q}_l)$ on a complete metric space invariant by $\mathrm{PGL}_2(\mathbb{Z}_l)$. In that case, using Proposition V.6.2 and the results on the eigenfunction of the T_{l+1} tree of section IV.4, we would get the T_l-recurrence. In this case, by strong approximation and section II.7.2, $X \cong R[1/pl]^\times \setminus \mathrm{GL}_2(\mathbb{Q}_p)\mathrm{GL}_2(\mathbb{Q}_l)/\mathrm{GL}_2(\mathbb{Z}_l)$ and there is no natural action of $\mathrm{PGL}_2(\mathbb{Q}_l)$. However, the action is recovered by the fact that the measures μ_ϕ are naturally Z-invariant. So the T_l-recurrence has to be interpreted as in section V.5. Then, the aforementioned method is still valid, as long as we find a representative of the Hecke Tree T_{l+1} in $\mathrm{GL}_2(\mathbb{Q}_l)/\mathrm{GL}_2(\mathbb{Z}_l)$ and we deal with the presence of non-trivial central characters. For this, we introduce the following operator.

Definition VI.2.3. Fix $n \in \mathbb{N}$ coprime to p. Let $M_n = R[1/p] \cap \mathrm{nr}^{-1}(n\mathbb{Z}[1/p]^\times)$, so that $M_1 = R[1/p]^\times$. Denote by M_n^{prim} is the set of all *primitive* elements of M_n, i.e. the ones that are not divisible inside $R[1/p]$ by any divisor $d > 1$ of n. Then $S_n(x)$, for $x \in X$, is the formal sum

$$S_n(x) = \sum_{s \in M^{prim}/\Gamma} s^{-1}x.$$

It induces an operator S_n on $\mathcal{A}(X)$ given by $S_n\phi(x) = \sum_{y \in S_n(x)} \phi(y)$.

Remark VI.2.4. As a consequence of the previous definition $T_n = \sum_{d^2|n} z(d^{-1})S_{n/d^2}(x)$, for $z(q)$ the diagonal matrix qI_2 that corresponds to the scalar $q \in B_p^\times \cong \mathrm{GL}_2(\mathbb{Q}_p)$.

By the previous remark, the image of $S_{l^n}(1)$ in $\mathrm{GL}_2(\mathbb{Q}_l)/\mathrm{GL}_2(\mathbb{Z}_l)$, is sent bijectively into the elements of distance n in the T_{l+1} Hecke Tree of $\mathrm{PGL}_2(\mathbb{Q}_l)/\mathrm{PGL}_2(\mathbb{Z}_l)$ by the standard projection. Thus, by Proposition V.6.2 and Remark V.6.5, the following definition is compatible withTheorem V.4.6.

Definition VI.2.5 (T_l-recurrence)**.** A finite Z-invariant measure μ on X is called T_l-recurrent if for each Borel subset $E \subseteq X$ and μ-a.e. $x \in E$, there exists infinitely many positive integers n for which $S_{l^n}(x) \cap E \not\emptyset\emptyset$.

Then, by proposition V.6.2, it's enough to prove the following result.

Lemma VI.2.6. *Let $l \neq p$ be a split prime. There exists $c_0 > 0$ such that for each automorphic form $\phi \in \pi \in \mathrm{A}_0(X)$ and $x \in X$, one has $\sum_{k \leq n} \sum_{y \in S_{l^n}(x)} |\phi(y)|^2 \geq c_0 n|\phi(x)|^2$.*

Proof. The proof is an analogue of the one of section IV.4, with the Satake parameters of π_l , the l-th component of π, playing the same role of the eigenvalue of the T_l-operator in the mentioned proof. We treat the *tempered* case, which, as in Theorem IV.4.2, is the most difficult and, in this unitary setting, the only one required. By hypothesis $\phi \in \pi \in \mathrm{A}_0(X)$ is an eigenfunction of the T_l operator and generates a spherical irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_l)$. Then, by III.6.9.2 we have that $T_{l^n}\phi = p^{n/2}\lambda_\pi(l^n)$, for $\lambda_\pi(l) = \alpha + \beta$, the Satake parameters of π_l. In particular, α, β are roots of the polynomial $X^2 - l^{1/2}\lambda_\pi(l)X + l\theta$, for the unit $\theta = \chi_\pi(l) \in C^{(1)}$. Then the tempered condition $|\lambda_{pi}(l)| \leq 2l^{1/2}$ implies $\alpha, \beta \in \mathbb{C}^{(1)}$, i.e. $|\alpha| = |\beta| = 1$. Moreover, always by III.6.9.2, recall that

$$\lambda_\pi(l^n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

By Remark VI.2.4 we have that $T_{l^n} = \sum_{k \leq n:\, k \equiv n \mod 2} z(l^{(k-n)/2})S_{l^k}$. Conversely, $S_{l^k} = T_{l^k} - 1_{k\geq 2} z(l^{-1})T_{l^{k-2}}$. Thus, denoting by $l^{k/2}\sigma_k \in C$ the scalar by which S_{l^k} acts on π, one obtains $\sigma_k = \lambda_\pi(l^k) - 1_{k\geq 2}\theta l^{-1}\lambda_\pi(l^{k-2})$, which expands as

$$\sigma_k = \frac{\gamma_1\alpha^k - \gamma_2\beta^k}{\alpha - \beta},$$

for $\gamma_1 = \alpha - \theta l^{-1}\alpha^{-1}$, $\gamma_2 = \beta - \theta l^{-1}\beta^{-1}$. With this expression, $|\gamma_1|, |\gamma_2| \geq 1/2$.

Now, for Cauchy Schwarz

$$l^m|\lambda_\pi(l^m)\phi(x)|^2 = |T_{l^m}\phi(x)|^2 \leq (1 + l^{-1})l^m \sum_{y \in T_{l^m}} |\phi(y)|^2,$$
$$l^k|\sigma_k\phi(x)|^2 = |S_{l^k}\phi(x)|^2 \leq (1 + l^{-1})l^k \sum_{y \in S_{l^k}(x)} |\phi(y)|^2.$$

Then, as a result of Remark VI.2.4, we have that $\sum_{k\leq n}\sum_{y\in S_{l^n}(x)} |\phi(y)|^2 \gg c_\pi(n)|\phi(x)|^2$, with

$$c_\pi(n) = \max_{m\leq n} |\lambda_\pi(l^m)| \sum_{k\leq n} |\sigma_k|^2.$$

We are left to prove that $c(\pi)_n \gg n$ uniformly in π. By contradiction, suppose the estimate fails, so that there is a sequence (π_j, n_j), for $j \in \mathbb{N}$, such that $n_j \to +\infty$ for $j \to +\infty$, with the property that $c_{\pi_j}(n) = o(n)$, for $n \to +\infty$. Starting from now, the asymptotic notation will be with respect to $j \to +\infty$. Up to passing to subsequences, we need to consider two cases:

1. $\frac{\frac{1}{n_j}}{|\alpha_j - \beta_j|} = o(1)$;
2. $|\alpha_j - \beta_j| \ll 1/n_j$.

In case 1) we have $\frac{\frac{1}{n_j}}{|1-\alpha_j\bar{\beta}_j|} = o(1)$, so expanding the square and summing we get

$$c_{\pi_j}(n_j) \geq \sum_{k\leq n_j} |\sigma_k|^2 = \frac{|\gamma_1|^2 n_j + |\gamma_2|^2 n_j + o(n_j)}{|\alpha_j - \beta_j|^2} \geq \frac{n/3}{||\alpha_j - \beta_j|^2} \gg n_j.$$

In case 2), one has $|\alpha_j - \beta_j|^{-1} \gg n_j$, so the largest positive integer $m_j \leq n_j$ for which $m_j|\alpha_j - \beta_j| < 1/10$ satisfies $m_j \gg n_j$ and so $c_{\pi_j}(n_j) \geq |\lambda_\pi(l^{m_j})|^2 \gg m_j^2 \gg n_j^2 \geq n_j$. So we get the desired contradiction. □

Theorem VI.2.7.

VI.3 Strongly Positive Entropy

In this section, we prove that any weak-* limit μ of L^2-normalized Hecke eigenfunctions $\phi_n \in \pi \in \mathrm{A}_0(X)$ has positive entropy on almost every ergodic component.

Recall from section V.6 that $\epsilon > 0$ is *admissible* if it belongs to the image of $|.| : \mathbb{Q}_p^\times \to \mathbb{R}_+^\times$. For a compact open subgroup C of $\mathbb{Q}_p^\times$ and admissible $\epsilon > 0$, we set

$$B(C,\epsilon) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, d \in C,\ |b|, |c| \leq \epsilon\},$$

the *dynamical ball* of radius ϵ and support C. Then, by Theorem V.6.3 and Remark V.6.5, it's enough to prove the following.

Theorem VI.3.1. *For each compact subset Ω of G there exists $C_1, c_2 > 0$ so that for all admissible $\epsilon \in (0,1)$, all L^2-normalized $\varphi \in \pi \in \mathrm{A}_0(X)$, and all $x \in \Omega$, one has $\mu_\varphi(xB(C,\epsilon)) \leq C_1 \epsilon^{c_2}$, for $B(C,\epsilon)$ as in Theorem V.6.3.*

Proof. For a given Ω we set C to be an open subgroup of $\mathfrak{o}^\times$ such that for small enough ϵ, one has

$$xB(C,\epsilon)x^{-1} \subseteq K_0 \qquad \forall x \in \Omega, \tag{VI.2}$$

$$gB(C,\epsilon)g^{-1} \cap \Gamma = \{1\} \qquad \forall g \in G, \tag{VI.3}$$

the latter being possible since B is non-split, so that Γ is discrete in G.

We need first the following lemmas.

Lemma VI.3.2 (Bounds for Hecke returns)**.** *For all admissible $\epsilon \in (0,1)$, all $n \in \mathbb{Z}_{\geq 1}$ coprime to p and satisfying $n < \sqrt{1/2}\epsilon^{-1}$, all $m \in \mathbb{Q}^\times$ with numerator and denominator coprime to p, and all $x \in \Omega$, the set $S = M_n \cap z(m)xB(C,\epsilon)x^{-1}$ has cardinality $\#S \leq 6\prod_{p^k||n}(k+1)$. In particular $\#S \leq 2^{13}$ if n has at most 10 prime divisors counted with multiplicity.*

Proof. Recall that $M_n = nr^{-1}(n\mathbb{Z}[1/p]^\times) \cap R[1/p]$. By VI.2, we have that $nr(M_n \cap z(m)xB(C,\epsilon)x^{-1}) \subseteq n\mathbb{Z}[1/p]^\times\mathbb{Q}_+ \times \mathbb{Z}_p = \{n\}$, since, by hypothesis, $z(m)$ is in K. Then $S \subseteq R(n) = \{\alpha \in R : nr(\alpha) = n\}$. We prove now that S is contained in an imaginary quadratic field $L \subset B_\mathbb{Q}$. Equivalently, since B is non-split and definite, it's enough to prove that the elements of S commute, or that any commutator $u = [s,t] = sts^{-1}t^{-1}$ is trivially equal to 1, for any $s, t \in S$. Notice first that $nr(u) = 1$ and $n^2u = st\bar{s}\bar{t} \in \mathbb{R}$, for the conjugates $\bar{s}, \bar{t}$ of s, t in $B_\mathbb{Q}$, so that $tr(u) \in n^{-2}\mathbb{Z}$. Now, any $s \in S$ is conjugate to an element of $B(C,\epsilon)$, which, after another conjugation by some $\begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix}$, is contained in $K_0(\mathfrak{q})$, for $\mathfrak{q} = \{x \in \mathfrak{o} : |x| \leq \epsilon^2\}$. As a consequence, any commutator u lies in the preimage of the upper unipotent in $\mathrm{GL}_2(\mathfrak{o}/\mathfrak{q})$, so that $|tr(u) - 2|_p \leq \epsilon^2$. Since B is definite, $tr(u)^2 - 4nr(u) \leq 0$ or $|tr(u)|_\infty \leq 2|nr(u)^{1/2}|_\infty = 2$. As a result, the integer $a = n^2tr(u) - 2n^2$ satisfies $|a|_\infty|a|_p \leq 2n^2\epsilon^2 < 1$, by hypothesis, and so must be zero, which implies $tr(u) = 2$. Then the reduced characteristic polynomial of u is $x^2 - 2x + 1 = (x-1)^2$ and since B is non-split we have that $u = 1$. So we can bound S by bounding $\mathcal{O}(n)$, the elements of norm n in the ring of integers $\mathcal{O}$ of L'. Thus, we have

$$\#S \leq \#\mathcal{O}(n) \leq \#\mathcal{O}^\times \cdot \#\{I \subseteq \mathcal{O} : nr(I) = n\} \leq 6\prod_{p^k||n}(k+1)$$

□

Lemma VI.3.3 (Geometric Amplification)**.** *Let $(c_l)_{l\in\mathbb{Z}_{\geq 1}}$ be a finitely-supported sequence of scalars. Set $T = \sum_l c_l T_l/\sqrt{l}$ and $T^a = \sum_l |c_l| T_l^*/\sqrt{l}$. let $\varphi \in \mathcal{A}(X)$, $\psi, \nu \in C_c^\infty(G)$. Define $\Psi \in \mathcal{A}(X)$ by $\Psi(g) = \sum_{\gamma\in\Gamma} |\Psi|(\gamma g)$ and $\psi * \nu \in C_c^\infty(G)$ by $\psi * \nu(x) = \int_{y\in G} \psi(xy)\nu(y)$. Then*

$$\|T\phi(\psi * \nu)\|_{L^2(G)} \leq \|\phi\|_{L^2(X)} \|T^a \Psi\|_{L^2(X)} \|\nu\|_{L^2(G)}.$$

Proof. Let $M = R[1/p]$. Then the operator T, evaluated at ϕ, can be rewritten as $T\phi(x) = \sum_{s\in M/\Gamma} h_s \phi(s^{-1}x)$, for some finitely supported coefficients h_s; as a consequence of the definition, $T^a\Psi(x) = \sum_{s\in M/\Gamma} |h_s| \Psi(sx)$. Let $I = \|T_\phi(\psi * \nu)\|_{L^2(G)}$. By the triangle inequality and the change of variables $x \mapsto sx$, we have

$$I \leq \sum_{s\in M/\Gamma} \left(\int_{x\in G} |\phi|^2(x) |\psi * \nu(sx)|^2 \right).$$

By another change of variables, we can express $\psi * \nu(x) = \int_{y\in G} \psi(sy)\nu_y^*(x)$, where $\nu_y^* = \nu(x^{-1}y)$. By triangle inequality, $I \leq \int_{y\in G} \sum_{s\in M/\Gamma} |h_s| |\psi(sy)| \|\phi\nu_y^*\|_{L^2(G)}$. By the unfolding $\int_{y\in G} \sum_{s\in M/\Gamma} = \sum_{y\in X} \sum_{s\in\Gamma\backslash M} \sum_{\gamma\in\Gamma}$, we get

$$I \leq \int_{y\in X} T^a \Psi(y) \|\phi \nu_y^*\|_{L^2(G)}.$$

We conclude by Cauchy-Schwart and the identity $\int_{y\in X} \|\phi\nu_y^*\|^2_{L^2(G)} = \|\nu\|^2_{L^2(G)} \|\phi\|^2_{L^2(X)}$. □

Since ϕ is a Hecke eigenfunction, we have that $T\phi = \lambda\phi$, for $\lambda = \sum_l c_l \lambda_\pi(l)$ and we recall that, for the chosen normalizations, $T_l\phi = \sqrt{l}\lambda_\pi(l)\phi$. We abbreviate the group $J = B(C,\epsilon)$. Let $x \in \Omega$. Take $\psi = 1_{xB(C,\epsilon)} \geq 0$ and $\nu = e_J = vol(J)^{-1} 1_J$, so that $1_{xB(C,\epsilon} = |\psi * \nu|^2$. By VI.3, we have that $\mu_{T_\phi}(|psi * \nu|^2) = \|T\phi(\psi * \nu)\|^2_{L^2(G)}$, and so by Geometric amplification we have $\mu_\phi(xB(C,\epsilon)) \leq |\lambda|^{-1} \|T^a\Psi\| L^2(X) \|\nu\|_{L^2}(G)$. The square $\|T^a\Psi\|^2_{L^2(X)}$ is a linear combination of terms $\langle T_l^*\Psi, T_{l'}^*\rangle = \langle T_{l'} T_l^* \Psi, \Psi\rangle$. Then by using the Hecke multiplicativity and unfolding the integrals and definition of ψ and ν, we obtain: for $m.n \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned}\langle z(m) T_n^* \Psi, \Psi\rangle \|\nu\|^2_{L^2(G)} &= \int_{g\in G} \sum_{s\in M_n} \psi(z(m)sg)\psi(g) vol(J)^{-1} \\ &= M_n \cap z(m^{-1})xJx^{-1}.\end{aligned}$$

So, by the bounds on Hecke returns, we thereby get

$$\mu_\phi(xB(C,\epsilon))^2 \leq 2^{13} |\lambda| \sum_{l,l'} |c_l c_l'| \sum_{d|(l,l')} d/\sqrt{ll'},$$

as long as C_l is supported on the integers $l \leq 2^{-1/4}\epsilon^{-1/2}$ having at most 5 primes counted with multiplicity. Now we have to choose c_l.Set $L = (1/\epsilon)^{0.1}$. Let $\mathcal{L}$ the set of numbers $l = q$ or $l = q^2$, for a prime $q \in [L, 2L]$. Notice that each such q splits B provided that ϵ is small enough. Set $c_l = 0$ unless $l \in \mathcal{L}$, where in that case $c_l = L^{-1} log(L) sgn(\lambda_\pi(l))^{-1}$. As a consequence of the density of primes, for L large enough, $\sum_l |c_l| \asymp 1$ and $|c_l| \leq L^{-1}log(L)$. By the Iwaniec's trick, $|\lambda_\pi(q)|^2 + |\lambda_\pi(q^2)| \geq 1$, we have that $\lambda \asymp 1$. Then, by Cauchy-Schwarz $\sum_{l,l'} |c_l c_{l'}| \ll L^{-1}\log L$, while $\sum_{d|(l,l')} d/\sqrt{ll'} \ll (\log L)^c$. Then $\mu_\phi(xB(C,\epsilon)) \ll L^{-1/2} \log(L)^{O(1)} \leq \epsilon^{0.01}$. □

VI.4 Local Newforms and Microlocal lifts

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $\mathfrak{o}$ the ring of integers of $\mathbb{Q}_p$, and let $\mathfrak{p}$ be its unique maximal ideal, generated by a uniformizer ϖ.

We will use the following notation:

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad n'(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$
$$\mathrm{diag}(y_1, y_2) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$a(x) = diag(y, 1), \quad z(y) = \mathrm{diag}(y, y),$$
$$K_{m'\dots m} = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{-m'} \\ \mathfrak{p}^{\mathfrak{m}} & \mathfrak{o} \end{pmatrix} \cap \mathrm{GL}_2(\mathfrak{o}), \quad K = K_0 = K_{0\dots 0} = \mathrm{GL}_2(\mathfrak{o}).$$

Here, we introduce a generalized notion of special invariant vectors, of which the generalized newform and microlocal lifts are a subspace.

Definition VI.4.1. Let π be an irreducible generic representation of G. Then $v \neq 0 \in \pi$ has *support* $m' \dots m$ and *orientation* (ω_1, ω_2), for characters ω_1, ω_2 of $\mathfrak{o}^\times$, if

$$\pi(\mathrm{diag}(a, d))v = \omega_1(a)\omega_2(\det(g)/a)v \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{m'\dots m}.$$

The number $|m' - m| = N$ is called the *level* of v. We say that such a vector v is maximal if the space of vectors with equal orientation and level $N - 1$ is trivial.

Example VI.4.2. A microlocal lift with support $m' \dots m$ and orientation (χ_1, χ_2) is a vector of support $m' \dots m$, for $m' \asymp m$, and orientation (ω_1, ω_2), for $c(\omega_1/\omega_2) = c(\pi) = |m' - m|$.

In the previous definition, if $c(\omega_2) \geq |m' - m|$, we recover a more natural definition of orientation: in this case $\det(g) \equiv ad \mod \mathfrak{p}^{|m'-m|}$, so that $\omega_2(\det(g)/a) = \omega_2(d)$. This happens, for example, when $c(\omega_1\omega_2) \geq |m' - m|$ and $c(\omega_1/\omega_2) \geq |m' - m|$. Also for this reason, we will find useful the following lemma, whose proof can be found in [Cas73b] or, more explicitly, in [Sch02].

Lemma VI.4.3. *Let π be an irreducible generic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with ramified central character χ_π. Then $c(\chi_\pi) \leq c(\pi)$ with equality precisely when π is isomorphic to an irreducible principal series representation $\mathcal{B}(\chi_1, \chi_2)$ for which at least one of the inducing characters χ_1, χ_2 is unramified.*

As a consequence of the previous lemma, a generalized newform $v \in \pi$ of support $m' \dots m$ is a maximal vector of support $m' \dots m$ and orientation $(1, \chi_\pi|_{\mathfrak{o}^\times})$, for the central character χ_π of π. This suggests a relation between maximal vectors of a given orientation and generalized newforms.

Remark VI.4.4. The existence of a vector of support $m \dots m'$ and orientation (χ_1, χ_2) implies the existence of a possibly infinite family of vectors with the same level and orientation. In fact, the conjugation by $\begin{pmatrix} \omega^i & 0 \\ 0 & 1 \end{pmatrix}$, for $i \in \mathbb{Z}$, fixes level and orientation but induces a bijection from the vectors with support $m' \dots m$ to the ones with support $m' + i \dots m + i$. As a consequence, when we will study the dimension of the space of vectors with a fixed orientation and level N, we will always refer to the dimension of the ones with support $m \dots m + N$, for some fixed $m \in \mathbb{Z}$.

Theorem VI.4.5. *Let π be an irreducible generic representation of G, with conductor $c = c(\pi)$. The space of maximal vectors of fixed orientation (χ_1, χ_2) and $c(\chi_1\chi_2^{-1}) \leq c$, is the inverse image of the space of generalized newforms under the linear isomorphism $\pi \to \pi \otimes \chi_1^{-1}$, given by $v \mapsto v \otimes \chi_1^{-1}$. The bijection preserves the support. In particular:*

1. *the space of these vectors has dimension 1;*
2. *the space of microlocal lifts of orientation (χ_1, χ_2) and level $N > 0$ is non-trivial if and only if $\pi \cong \mathcal{B}(\chi_1, \chi_2)$, the irreducible principal series determined by the characters χ_1, χ_2, extended trivially to $\mathbb{Q}_p^\times$. In this case, the space of all microlocal lifts π consists of two lines, the ones of orientation (χ_1, χ_2) and (χ_2, χ_1).*

Proof. The map defined in the hypothesis naturally preserves the support of the vectors. As a consequence, after conjugating, it's enough to prove the assertion for the maximal vectors of support $0 \ldots m$.

Let v be a local newform of the irreducible representation $\pi \otimes \chi_1^{-1}$. Let $m = c_{\chi_1^{-1}} = c(\pi \otimes \chi_1^{-1})$. By definition, we have that:

$$\begin{aligned}\pi \otimes \chi_1^{-1}(g)v &= \chi_\pi(d)\chi_1^{-2}(d)v \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0\ldots m} \\ &= \chi_2(d)\chi_1^{-1}(d)v \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0\ldots m}.\end{aligned}$$

Then, denote by v' the image of v under the inverse map $\pi \otimes \chi_1^{-1} \to \pi$, such that $v \mapsto v \otimes \chi_1$. We have that

$$\begin{aligned}\pi(g)v' &= \chi_2(d)\chi_1^{-1}(d)\chi_1(\det(g)) \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0\ldots m} \\ &= \chi_1^{-1}(d)\chi_2(d)\chi_2(\det(g))\chi_2^{-1}(\det(g))\chi_1(\det(g)) \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0\ldots m} \\ &= \chi_1^{-1}(d)\chi_2(d)\chi_2(\det(g))\chi_2^{-1}\chi_1(ad) \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0\ldots m} \\ &= \chi_1(a)\chi_2(a/\det(g)) \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0\ldots m},\end{aligned}$$

where we used the fact that, by Lemma VI.4.3, $m \geq c(\chi_1\chi_2^{-1})$, to rewrite $\chi_1\chi_2^{-1}(\det(g))$ as $\chi_1\chi_2^{-1}(ad)$.

Retracing backwards the previous proof we obtain the bijection between maximal vectors and generalized newvectors. In particular, as a corollary of Corollary III.6.11, we obtain that the space of maximal vectors of fixed orientation has dimension 1.

2) The generalized newvectors in $\pi\otimes\chi_1^{-1}$ must have level $c(\pi\otimes\chi_1^{-1}) = c(\chi_1/\chi_2) = c(\chi_{\pi\otimes\chi_1^{-1}})$. So, by Lemma VI.4.3, $\pi \otimes \chi_1^{-1} \cong \mathcal{B}(\nu_1, \nu_2)$, for ν_1 unramified. Then $\pi \cong \mathcal{B}(\nu_1, \nu_2)\otimes\chi_1 \cong \mathcal{B}(\nu_1\chi_1, \nu_2\chi_1)$, so that $\nu_1\otimes\chi_1|_{\mathfrak{o}^\times} = \chi_1$ and $\nu_2\otimes\chi_1|_{\mathfrak{o}^\times} = \chi_2$. Moreover, by Theorem III.5.9, we have that, for $\nu_1\chi_1/(\nu_2\chi_2)$ ramified, $\mathcal{B}(\nu_1\chi_1, \nu_2\chi_1) \cong \mathcal{B}(\nu_2\chi_1, \nu_1\chi_1)$, with no other additional isomorphisms with principal series induced by different characters. Thus, the space of microlocal lifts in this case consists of two lines, with orientation (χ_1, χ_2) and (χ_2, χ_1). ☐

As a direct result of the explicit expressions of the local newforms in proposition III.5.12 and the previous result, we report the explicit expression of the generalized newvectors and of the microlocal lifts.

Corollary VI.4.6. *Let $\pi \cong \mathcal{B}(\chi_1, \chi_2)$ be an irreducible principal series representation of G. Let $v \in \pi$ be a generalized newvector of support $m' \dots m$.*

1. *If χ_1 is ramified and χ_2 is ramified, then $v = v_f$ as in the line model of Remark III.5.4 for f a character multiple of the characteristic function of an $\mathfrak{o}^\times$-coset, thus $f = c\chi 1_{\varpi^n \mathfrak{o}^\times}$ for some cin$\mathbb{C}$, $\chi : \mathbb{Q}_p^\times \to \mathbb{C}$ and $n \in \mathbb{Z}$.*

2. *If χ_1 is unramified and χ_2 is ramified, then $v = v_f$ for $f = c1_{\mathfrak{a}}$ for some scalar c and fractional $\mathfrak{o}$-ideal $\mathfrak{a} \subset \mathbb{Q}_p$.*

Proof. This is just proposition III.5.12, for the case of support $0 \dots m'$. The general case is given by conjugation by $\begin{pmatrix} \varpi^{m'} & 0 \\ 0 & 1 \end{pmatrix}$ of the previous case and using the properties of Definition III.5.2 of the induced model. □

Corollary VI.4.7. *Let $\pi \cong \mathcal{B}(\chi_1, \chi_2)$ and $\omega_i = \chi_i|_{\mathfrak{o}^\times}$, with $N = c(\omega_1/\omega_2) \geq 1$ and $N_1 + N_2 = N$. Define $f_1, f_2 \in C^\infty(\mathbb{Q}_p)$ by*

$$f_1(x) = 1_{\mathfrak{p}^{N_2}}(x), \quad f_2(x) = 1_{\mathfrak{p}^{N_1}}(1/x)|1/x|\chi_1^{-1}\chi_2(x)$$

and $v_1, v_2 \in \pi$ in the induced model on $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G$ by

$$v_1(g) = v_{f_1}(g) = 1_{p^{N_2}}(c/d) \left| \frac{\det g}{d^2} \right|^{1/2} \chi_1(\det(g)/d)\chi_2(d), \tag{VI.4}$$

$$v_2(g) = v_{f_2}(g) = 1_{p^{N_1}}(d/c) \left| \frac{\det g}{c^2} \right|^{1/2} \chi_1(\det(g)/c)\chi_2(c) \tag{VI.5}$$

and $W_1, W_2 \in \pi$ in the Kirillov model $\mathcal{K}(\pi, \psi)$, for an unramified character ψ of $\mathbb{Q}_p$, by

$$W_1(y) = 1_{\mathfrak{p}^{-N_1}}(y)|y|^{1/2}\chi_1(y), \quad W_2(y) = 1_{\mathfrak{p}^{-N_1}}(y)|y|^{1/2}\chi_2(y). \tag{VI.6}$$

Then v_1, W_1 and v_2, W_2 are microlocal lifts of support $-N_1 \dots N_2$ and orientations (ω_1, ω_2) and (ω_2, ω_1), respectively.

Proof. By Theorem VI.4.5, the microlocal lifts are twists of generalized newvectors by characters. The explicit expressions in the induced model for the local newforms were shown in Proposition III.5.12. So, after conjugation, and multiplying by the necessary character, we get the result. See [Sch02], also for the computations of the Kirillov model. □

VI.5 Local Results

Fix $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and let $k = \mathbb{Q}_p$, $\mathfrak{o}^\times = \mathbb{Z}_p$ and $\mathfrak{p} = (\varpi)$, the maximal ideal of $\mathfrak{o}^\times$.

The aim of this section is to obtain bounds and explicit expression for the local Rankin-Selberg trilinear form, evaluated at generalized newvectors and microlocal lifts. These, thanks to the uniqueness of the trilinear invariant forms for $G =$

$\mathrm{GL}_2(\mathbb{Q}_p)$, will be the crucial technical step behind the proof of the equidistribution of the microlocal lifts and of the newvectors.

First, let σ be an irreducible generic unitary representation of $\mathrm{PGL}_2(k)$ and let $\pi = \mathcal{B}(\chi_1, \chi_2)$ be an irreducible unitary principal series representation of G, realized in its induced model. Recall that the local Ranking-Selberg integral $l_{RS} \in \mathrm{Hom}_G(\sigma \otimes \bar{\pi} \otimes \pi)$ is defined by

$$l_{RS}(W_1, \bar{W}_2, v_3) = \int_{ZN\backslash G} W_1 \bar{W}_2 v_3,$$

for $v_3 \in \pi$, $W_1 \in \mathcal{W}(\sigma, \psi)$, $W_2 \in \mathcal{W}(\pi, \psi)$, for a non-trivial unramified character ψ. By section III.7 or directly by Theorem VI.5.2 in the most interesting cases for this essay, this trilinear form is non-zero. This applies in particular when $W_2 = W_v$, the image of a vector v, for some $v \in \pi$ under the intertwiner shown in III.2.

We have the following lemma

Lemma VI.5.1. *Let $f \in C_c^\infty(\mathbb{Q}_p)$. Let U_1 be an open subgroup of $\mathfrak{o}^\times$ for which $\bar{f} \otimes f$ is U_1-invariant, in the sense that $\bar{f}(ux)f(uy) = \bar{f}(x)f(y)$, for all $u \in U_1,\ x, y \in \mathbb{Q}_p$. Let $W_1 \in \mathcal{W}(\sigma, \psi)$. Then*

$$l_{RS}(W_1, \overline{W_{v_f}}, v_f) = \int_{\substack{x \in k \\ y \in k^\times \\ t \in k}} f(x)\bar{f}(x + y/t)F(x, y, t; W_1, U_1)\frac{dt}{|t|}dx d^\times y,$$

where $F(x, y, t; W_1, U_1) = \mathbb{E}_{u \in U_1} W_1(a(y)n'(x/u))\chi_1\chi_2^{-1}(ut)\psi(ut)$ with $\mathbb{E}_{e \in U_1}$ denoting an integral with respect to the probability Haar measure on U_1.

Proof. This depends on the intertwiner given in III.2. See [Nel18, Lemma 46] for the proof. □

With the previous notations, we can state the technical result.

Theorem VI.5.2. *Let $v \in \pi$ be a microlocal lift of orientation $(\chi_1|_{\mathfrak{o}^\times}, \chi_2|_{\mathfrak{o}^\times})$ and let $v' \in \pi$ be a generalized newvector, both with balanced support. Let $W_1 \in \sigma$.*

1. *If the level $N = c(\chi_1/\chi_2)$ is large enough in terms of W_1, then*

$$l_{RS}(W_1, \overline{W_v}, v) = cp^{-N/2}\|v\|^2 \int_{y \in k^\times} W_1(y) d^\times y$$

where $c = q^{N/2} \int_{t \in k^\times} \chi_1\chi_2^{-1}(t)\psi(t)\frac{dt}{|t|}$, and $c \asymp 1$ is a complex scalar which is independent of W_1 and whose magnitude depends only upon k.

2. *One has $l_{RS}(W_1, \overline{W_{v'}}, v') \ll q^{-N/2}\|v'\|^2$ with the implied constant depending at most upon W_1.*

Proof. For the full proof, see [Nel18, Theorem 49]. We report here the first part. 1) Without loss of generality, let $v = v_f$ with $f(x) = 1_{p^{N_2}}(x)$, as in Corollary VI.4.6 and Remark III.5.4. Since N_2 is large enough in terms of W_1, we have whenever $f(x) \neq 0$ that $W_1(a(y)n'(x/u))$, for all $u \in \mathfrak{o}^\times$. By Lemma VI.5.1 and the previous argument, we get $f(x)\bar{f}(x+y/t) = 1_{\mathfrak{p}^{N_2}} 1_{\mathfrak{p}^{N_2}}(y/t)$ and $1_{\mathfrak{p}^{N_2}} F(x, y, t; W_1, \mathfrak{o}^\times) = 1_{\mathfrak{p}^{N_2}}(x) W_1(y) H(t)$

with $H(t) = \mathbb{E}u \in \mathfrak{o}^\times \chi_1\chi_2^{-1}(ut)\psi(ut)$, where $\mathbb{E}_{u\in\mathfrak{o}^\times}$ stands for the integral (finite Gauss sum) with the Haar probability measure on $o^\times$. Then

$$\ell_{RS}(W_1, \bar{W}_v, v) = \int_{y\in k^\times} W_1(y) \int_{x\in k} 1_{\mathfrak{p}^{N_2}}(x) \int_{t\in k} 1_{\mathfrak{p}^{N_2}}(y/t) H(t) \frac{dt}{|t|} dx d^\times y.$$

Then, by the properties of Gauss Sums, as in [Nel18, §5.7], $W_1(y)H(t) = 0$ unless $|t| \asymp p^N$ and $|y| \ll 1$. Since N_1 is eventually large enough in terms of W_1, the fact $1_{\mathfrak{p}^{N_2}}(y/t) = 1$ is thus redundant. Since $\int_{x\in k} 1_{\mathfrak{p}^{N_2}}(x)dx = \int_k |f|^2 = \|v\|^2$, we obtain the required identity.

2) As the previous point, the key insight is expressing $\ell_{RS}(W_1, \bar{W}_v, v)$ using VI.5.1. The result follows the by using the explicit expression of VI.4.6 and stationary phase analysis, thanks to the appearance of Gauss sums as before. However, it is rather technical, even tough not really long, so we refer to [Nel18, Theorem 49.2] for the proof. □

VI.6 Completion of the Proof

Recall that $X = \Gamma \setminus G$, for $G = \mathrm{GL}_2$.

In this section we will prove Theorem VI.1.9 and VI.1.9, i.e. the equidistribution of the microlocal lifts and of the local newvectors. We will start first with the equidistribution of the microlocal lifts.

Notation. Let $\varphi \in \pi \in \mathrm{A}_0(X)$ traverse a sequence of L^2-normalized microlocal lifts of level $N \to +\infty$, and balanced support $-N_1 \dots N_2$, for $N_1 + N_2 = N$ and $N_1, N_2 \to +\infty$. Both φ and π obviously depend on $N \in \mathbb{N}$, but to simplify the notation we will omit it. In place, we will say that an object is *fixed* if it is independent of N and *eventually* will mean "for large enough N". The asymptotic notation such as $o(1)$ refers to the $N \to +\infty$ limit. We say that φ equidistribute if any subsequential weak-* limit of the L^2-masses μ_φ, for $N \to \infty$ is equal to the right G-invariant Haar probability measure μ on X, obtained by psuhforward of a Haar measure on G, under the projection $p : G \to \Gamma \setminus G$. Equivalently, μ is the unique right G-invariant Radon probability measure on X.

First, by assumption $\pi \cong \mathcal{B}(\chi_1, \chi_2)$ for some unitary characters χ_1, χ_2 of $\mathbb{Q}_p^\times$, since π consists of smooth functions on a compact space X, with $c(\chi_1/\chi_2) = N$.

Second, recall that any $\pi' \in \mathrm{A}(X)$ is not only an irreducible G-module but is an irreducible module under the action of G and the T_l Hecke operators, for any $l \neq p$. In particular, this implies that the maximal vectors as in Definition VI.4.1 are eigenfunctions of all the T_l operators, being in a space of dimension 1 fixed by the T_l operators.

Recall that the L^2-mass μ_φ associated to φ is defined by

$$\mu_\varphi(\Psi) = \int_X \Psi |\varphi|^2 d\mu \qquad \Psi \in \mathcal{A}(X).$$

As a consequence of the definition and by the fact that φ has unitary central character, the probability measure μ_φ is Z-invariant, for the center Z of G. Moreover, the T_l operators for $l \mid \mathrm{disc}(B)$ are involutions and φ is an eigenfunction of them, thus they act by multiplication by ± 1 and μ_φ is also T_l-invariant, for $l \mid \mathrm{disc}(B)$. So we need to prove the invariance only for the space of *functions*

$$\mathcal{A}^+(X) = \left\{ \Psi \in \mathcal{A}(X) : \begin{cases} T_l\Psi = \Psi & \text{for } l \mid \mathrm{disc}(B) \\ \Psi(xz) = \Psi(z) & \text{for } z \in Z \end{cases} \right\}.$$

Observe that this space decomposes as $\mathcal{A}^+(X) = (\oplus_\chi \mathbb{C}(\chi \circ \det)) \oplus \mathcal{A}_0^+(X)$ where

1. χ traverses the set of quadratic characters of the compact group $\mathbb{Q}_p^\times/\mathbb{Z}[1/p]^\times$, i.e. $\det(\Gamma \setminus G)$, satisfying $\chi(l) = 1$ for $l \mid \mathrm{disc}(B)$, and
2. $\mathcal{A}_0^+ = \mathcal{A}^+(X) \cap \mathcal{A}_\prime(X)$, which decomposes further by definition as $\mathcal{A}_\prime^+ = \oplus_{\sigma \in \mathrm{A}_0^+(X)}\sigma$, where we recall that $\mathcal{A}_0$ is the space spanned by the irreducible automorphic representations $\sigma \in \mathrm{A}(X)$ of G that are infinite dimensional on G, which, by the previous argument, are the irreducible infinite dimensional representation of G in $\mathcal{A}(X)$.

Remark VI.6.1. We need to prove that for any fixed $\Psi \in \mathcal{A}(X)$, any weak-* subsequential limit ν of the μ_φ equidistributes. Equivalently, we just need to verify that $\nu(1) = \mu(1)$ and $\nu(\Psi) = \mu(\Psi) = 0$, for any $\Psi \in \sigma \in \mathrm{A}(X)$, $\sigma \neq \mathbb{C}(1)$, since in this case $\Psi \perp 1$.

Now let $\sigma \in \mathrm{A}^+(X)$ be fixed. Denote by $\ell : \sigma \otimes \bar{\pi} \otimes \pi$ the trilinear G-invariant form defined by integration on X, we will call it the *automorphic* trilinear form. First, if $\sigma = \mathbb{C}(\chi \circ \det)$, we have the following result.

Lemma VI.6.2. *Suppose that σ is one-dimensional and $\ell \neq 0$. Then σ is trivial eventually.*

Proof. The existence of a non-trivial G-invariant trilinear form $\ell : \sigma \otimes \bar{\pi} \otimes \pi \to \mathbb{C}$ is equivalent to the existence of a non-trivial intertwiner $\ell : \pi \to (\sigma \otimes \bar{\pi})^* \cong \sigma \otimes \pi$, since σ is quadratic. This is equivalent to $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi\chi_1, \chi\chi_2)$, by Theorem III.5.9. So, necessarily, again by Theorem III.5.9, $\chi_1 = \chi_1\chi$, in this case χ is trivial, or $\chi_2 = \chi_1\chi$. But by hypothesis $N = c(\pi) = c(\chi_1/\chi_2)$. So, in the second case $c(\chi) = N$, but $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ is finite, hence, eventually $\chi_2 \neq \chi_1\chi$. □

Since φ is L^2-normalized, the equidistribution is verified for $\Psi = 1$, and $\sigma = \mathbb{C}(1)$, the trivial character. By the previous lemma and Remark VI.6.1, then we have verified the equidistribution for $\sigma = \mathbb{C}(\chi \circ \det)$).

We are left with the more challenging case of $\sigma \in \mathrm{A}_0^+$. We want to use the measure classification result of Theorem VI.1.13. For this, we need to prove first the diagonal invariance, i.e. the A-invariance in the theorem. Since μ_φ is also invariant by $A(\mathfrak{o}^\times)$ by construction, since $\varphi \in \pi^{K_{m' \dots m}}$, it's enough to prove the $a(\varpi)$-invariance, or equivalently the $a(p)$-invariance.

In this case, by Theorem III.4.8, σ is generic. So, fix an unramified non-trivial character $\psi : \mathbb{Q}_p \to \mathbb{C}^(1)$, for $\psi(\omega) \neq 1$ and a G-equivariant isometric isomorphism $\sigma \cong \mathcal{W}(\sigma, \psi)$, the Whittaker model of σ. Fix also an isometric isomorphism $\pi \cong \mathcal{W}(\sigma, \psi)$ and $\pi \cong \mathcal{B}(\chi_1, \chi_2)$, the Whittaker and Induced models of π. Denote by $\ell_{RS} : \sigma \otimes \bar{\pi} \otimes \pi \to \mathbb{C}$ the trilinear form defined in section III.7. To simplify the notation, any $\phi \in \pi$, $\Psi \in \sigma$ will also denote their image in the other models.

Lemma VI.6.3. *Eventually, there exists a constant L, depending on N, such that*

$$\ell = L\ell_{RS}.$$

Proof. By the uniqueness of the trilinear invariant forms for $\sigma, \bar{\pi}, \pi$, as in Theorem III.7.3, it's enough to prove that ℓ_{RS} is eventually non-trivial. This follows by Theorem VI.5.2.1, choosing $\Psi \in \mathcal{W}(\sigma, \psi)$, such that $\Psi(a(y)) = 1_{\mathfrak{o}^\times}(y) \subset C_c^\infty(\mathbb{Q}_p^\times)$. This choice is possible by Proposition III.4.13, since $C_c^\infty(\mathbb{Q}_p)^\times \subseteq \mathcal{K}(\sigma, \psi)$, the Kirillov model of σ, with unramified character ψ. □

Remark VI.6.4. Actually, by the discussion of section III.7 or [MV10, §3], the trilinear invariant form ℓ_{RS} is never trivial, for our choice of σ, π.

As a consequence of Theorem VI.5.2.1, we have that eventually

$$\ell_{RS}(\sigma(a(p))\Psi, \bar{\varphi}, \varphi) = \ell_{RS}(\Psi, \bar{\varphi}, \varphi),$$

since $\int_{y \in k^\times} \Psi(a(y)) d^\times y = \int_{y \in k^\times} \Psi(a(yp)) d^\times y$. Then, from Lemma VI.6.3, we obtain eventually the $a(p)$-invariance for ℓ and any fixed $\psi \in \omega$.

Since any subsequential limit of the microlocal lifts φ is A-invariant and is every φ T_l eigenfunction, for any $l \neq p$ split prime, we can apply Theorem VI.3.1 and Lemma VI.2.6. As a consequence, the hypothesis of the measure rigidity result of Theorem V.5.6 are satisfied and any subsequential limit of the μ_φ equidistributes. So, μ_φ, for $N \to +\infty$, converge weak-* to the μ.

As for the equidistribution of the generalized newvectors ϕ of π, again by Lemma VI.6.2, we can restrict to proving the equidistribution for any fixed $\Psi \in \sigma \in \mathrm{A}_0^+$. By Remark VI.6.1, this is equivalent to proving that $\ell(\Psi, \bar{\phi}, \phi) = o(1)$. By Lemma [ref], this is equivalent to proving that eventually $L\ell_{RS}(\Psi, \bar{\phi}, \phi) = o(1)$. By Theorem VI.5.2.1, we know that $L\ell_{RS}(\Psi, \bar{\phi}, \phi) \ll Lp^{-N/2}$, so that we just need to prove that $L = o(p^{N/2})$. To estimate L, we can use the result we obtained with the equidistribution of the microlocal lifts: we already know that $L\ell_{RS}(\Psi', \bar{\varphi}, \varphi) = o(1)$, for any fixed $\Psi' \in \sigma$ and the microlocal lifts φ. Pick in particular Ψ' such that its corresponding function in the Kirillov Model $\mathcal{K}(\sigma, \psi)$ is $1_{\mathfrak{o}^\times} \in C_c^\infty(\mathbb{Q}_p^\times)$. By Theorem VI.5.2.1 we get that

$$L_{RS}(\Psi', \bar{\varphi}, \varphi) = cp^{-N/2} \|\varphi\|^2 \int_{y \in \mathbb{Q}_p^\times} 1_{\mathfrak{o}^\times}(y) d^\times y,$$

so that $Lcp^{-N/2} = o(1)$, since φ is L^2-normalized. This concludes the proof.

Remark VI.6.5. We see that the choice of microlocal lifts with balanced support, made the decay rate of VI.5.2.1 potentially slower respect to the unbalanced case. This is however crucial to establish the equidistribution of the balanced generalized newvectors. In fact, for the proof to work, the local decay rate needs to be faster than the one of the microlocal lifts.

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