Università degli Studi di Padova

Dipartimento di Fisica e Astronomia Galileo Galilei

Corso di Laurea Magistrale in Fisica

The role of asymmetric gain and loss perception in Minority Games

Il ruolo della percezione asimmetrica del guadagno e della perdita nei Minority Games

Candidato
Federico Bettin

Relatore
Prof. Attilio Stella

Correlatori
Prof. Anirban Chakraborti*
Prof. Damien Challet**

* School of Computational and Integrative Sciences, Jawaharlal Nehru University, 110067 New Delhi - INDIA

** Laboratoire de Mathématiques Appliquées aux Systèmes, École Centrale Paris, 92290 Châtenay-Malabry - FRANCE

Preface

The aim of this thesis is to show an application to economics of some usual statistical instruments very common in physics, or more generally a scientific method.

There is obviously no presumption of originality in this approach, as physics has a long tradition of moving to fields apparently far from their subject. The cycle of lectures by Erwin Schoredinger (1887-1961), which have been reported in the book *What is life?* [1] by the physicist himself in 1944, is an important example. This book is probably the first attempt to explain what happens inside a body of a living being starting from physics and chemistry concepts. It did not remain a futureless attempt, because it inspired other physicists, such as Max Delbruck and Francis Crick, and it was an important contribution to the birth of molecular biology.

As it emerges partially from this book, when a physicist tries to analyze situations far from his main field of application, his approach is his most important ally, rather than physical laws he already knows. As proof of that we can stress that physics use to test continuously their results through experiments, and the empirical results are usually the basis for future theories. This proper characteristic of physics makes it an useful *forma mentis* for studying complex systems, for which a pure mathematical axiomatic approach can be too rigid to appreciate the complexity involved.

In this thesis we are not interested in listing all contributions of physicists to other sciences, but we want to focus on a way to study systems with many entities interacting with each other. This general problem is very common in physics, both in large scale problems, such as many bodies gravitational systems, and in the small scale ones, such as the Ising Model. As underlined in [2], Adam Smith (1723-1790) himself *made the first attempt to formulate economical science, by modeling the society as a truly many-body system of selfish agents, each having no idea of benevolence or charity towards its fellow neighbors, or having no foresight, can indeed reach an equilibrium where the economy as a whole is most efficient, leading to the best acceptable price for each commodity.* This research of the most efficient situation, in the meaning of the most efficient to all participating agents, is known as the *invisible hand* determining the society dynamics, and it predates the self-organization mechanism in physics of many-body systems.

Smith models the society as sum of interacting entities, acting to get a maximisation of a quantity, i.e., the efficiency, and at the end an equilibrium is reached. All this concepts are very common in Game Theory, which is properly an attempt to model complex systems as a set of elements interacting with one another, following some rules, in order to maximize some function, i.e. *utility function.*

Basic concepts of Game Theory are given in chapter 1, but we are now interested in highlighting that this theory is used in very different fields, not only economics. As reported by *The Economist* in 2011 [3], there have been many successful Game Theory attempts to describe the behaviour of socio-political scenarios, e.g., the Egypt's President Mubarak fall from power and the name of the successor of the Ayatollah Khomeini five years before his death, both predictions given by Mr Bueno de Mesquit, who after other right predictions was hired by foreign governments, America's State Department, Pentagon and intelligence agencies. Again in biology, as argued in [4] and [5], we can regard to the distribution of genetic alleles as the results of the competition among phenotypes, which generates a selection pressure for optimal genotypes. This biological process can be considered, more generally speaking, as the Darwin evolutionary description, which asserts that the process of natural selection leads to the optimization of the reproductive process.

Chapter 1 is devoted to give some basic concepts of Game Theory, in chapters 2 and 3 we get back to economic applications analyzing a model called *Minority Game*. We have underlined some good results of Game Theory, now it is time to bring to light difficulties which we encounter when we try to apply it: as written in [6], economics is still a science in his infancy, so that not only we have not a universal theory, but we are very far from it. As a consequence, only some aspects of economics can be treated successfully through theory, in the mathematical sense, and all other ones are better treated in a qualitative way. In addition, when we decide to face an economic problem using a pure mathematical approach, we always have to remember that the initial problem is never clearly formulated, and this is why, to minimize this probelm a previous qualitative analysis is strictly necessary. To continue the comparison, when one builds a physical theory, then usually one has the possibility to test it by experiments. In economics this check is more difficult to perform, because the empirical background is frequently inadequate.

The model we analyze in depth in this thesis is the so-called *Minority Game*. This model, which is introduced in chapter 2, is a game, in the proper sense given in chapter 1, in which a (odd) number of players can choose between two options, let us say buy and sell, and each player will win if his choice is the minority one. Since losses are about twice as painful as gains [7] in chapter 3 we make a little variation in order to modify the learning scheme of agents in a simple version of the minority game [8] in order to differentiate their behaviour depending on what occurred at the previous time step, i.e. gain or loss. In chapter 4 conclusions about the analyzed variation of chapter 3 are summarized.

Contents

List of Figures

List of Tables

Chapter 1

Introduction to Game Theory

Game Theory can be defined as the study of mathematical models of conflict and co-operation between rational decision-makers, who make decisions that will influence one another's welfare. Basically these decision-makers are people forming a community, from the social science perspective.

From the historical point of view, Zermelo, Borel, von Neumann and Morgenstern can be considered as the founding fathers of the modern Game Theory. It is interesting to notice that early works on Game Theory were done during the World War II in Princeton, in the same community where many leaders of the theoretical physicists were working. This fact is not a pure casual event, indeed the structure of these social problems is very close to many statistical mechanics situations involving spatially extended systems. As stressed in [10] *interesting parallels between nonequilibrium phase transitions and spatial evolutionary game theory have added another dimension to the concept of universality classes.*

The term *game* identifies a situation of competition between two or more individuals acting according to some rules. This individuals are called *players* in this contest. When we write *rational* we mean that every player makes decisions with the only aim of maximizing his own payoff, which is measured in some utility scale.

In order to have a rigorous approach to Game Theory we start from the Bayesian Decision Theory, then we will generalize this theory to the case of two or more decision makers. We will follow mainly the perspective by Myerson [11].

Very basic Decision Theory concepts are given, then we will proceed with a general and deeper discussion on Game Theory itself.

1.1 Decision Theory

As we have already said Game Theory can be considered an extension of Decision Theory to two or more players. From these considerations, without going into deep details, we are now interested in getting main results of this theory in order to use them in the remainder of this work.

At first, given a finite set $\mathbf{Z}$, we define the set of probability distributions $\Delta(\mathbf{Z})$ as

$$\mathbf{\Delta}(\mathbf{Z}) = \{q : \mathbf{Z} \longrightarrow \mathbb{R} | \sum_{y \in \mathbf{Z}} q(y) = 1 \text{ and } q(z) \geq 0, \forall z \in \mathbf{Z}\} \tag{1.1}$$

Given this definition, let we define $\mathbf{X}$ as the set of available prizes that the player can get and $\mathbf{\Omega}$ the set of possible states, one of which is the true state of the world. For simplicity's sake we assume both $\mathbf{X}$ and $\mathbf{\Omega}$ to be finite.

Given those sets we can define a lottery as any real and non-negative function from every prize x in $\mathbf{X}$ and every (fixed) state t in $\mathbf{\Omega}$, such that $\sum_{x \in \mathbf{X}} f(x|t) = 1$. From these definitions we see that $f(x|t)$ is to be interpreted as the objective conditional probability of getting prize x in lottery f if t is the state of the world. We denote $\mathbf{L} = \{f : \mathbf{\Omega} \rightarrow \mathbf{\Delta}(\mathbf{X})\}$ as the set of all lotteries.

If we call event a non-empty subset of $\mathbf{\Omega}$, given two lotteries f and g in $\mathbf{L}$, we write $f \gtrsim_S g$ iff, in the opinion of the player, f would be at least as desirable as g if the event S occurred. Given this relation we define

$$f \sim g \text{ iff } f \gtrsim_S g \text{ and } g \gtrsim_S f$$
$$f \succ_S g \text{ iff } f \gtrsim_S g \text{ and } g \not\sim_S f$$

It is not difficult to see that given f and g in $\mathbf{L}$, $\alpha f + (1-\alpha)g$ is still a lottery that selects f or g with probability α and $1-\alpha$ respectively. We denote $\alpha[x] + (1-\alpha)[y]$, where $[x](y|t) \equiv \delta_{x,y}$, the lottery that gives either prize x or y, with probabilities α and $(1-\alpha)$ respectively. This lottery is a *random lottery*.

1.1.1 Axioms

All properties of a theory follow a list of axioms. Here these axioms are:

1. **completeness**: $f \gtrsim_S g$ or $g \gtrsim_S f$
2. **transitivity**: if $f \gtrsim_S g$ and $g \gtrsim_S h$, then $f \gtrsim_S h$
3. **relevance**: if $f(\cdot|t) = g(\cdot|t)\ \forall t \in S$, then $f \backsim_S g$
4. **monotonicity**: if $f \succ_S h$ and $0 \leqslant \beta < \alpha \leqslant 1$, then $\alpha f + (1-\alpha)h \succ_S \beta f + (1-\beta)h$
5. **continuity**: if $f \gtrsim_S g$ and $g \gtrsim_S h$, then $\exists \gamma \in \mathbb{R}$ such that $0 \leqslant \gamma \leqslant 1$ and $g \sim_S \gamma f + (1-\gamma)h$
6. **objective substitution**: if $e \gtrsim_S f$ and $g \gtrsim_S h$ and $0 \leqslant \alpha \leqslant 1$, then $\alpha e + (1-\alpha)g \gtrsim_S \beta f + (1-\beta)h$
7. **strict objective substitution**: if $e \succ_S f$ and $g \gtrsim_S h$ and $0 < \alpha \leqslant 1$, then $\alpha e + (1-\alpha)g \succ_S \beta f + (1-\beta)h$
8. **subjective substitution**: if $f \gtrsim_S g$ and $g \gtrsim_T h$ and $S \cap T \neq \emptyset$, then $f \gtrsim_{S \cup T} g$

9. **strict subjective substitution**: if $f \succ_S g$ and $f \succ_T g$ and $S \cap T \neq \emptyset$, then $f \gtrsim_{S \succ T} g$

10. **interest**: $\forall t \in \mathbf{\Omega}$, there exist prizes y and z in $\mathbf{X}$ such that $[y] \succ_t [z]$

11. **state neutrality**: for any two states r and t in $\mathbf{\Omega}$, if $f(\cdot|r) = f(\cdot|t)$, $g(\cdot|r) = g(\cdot|t)$ and $f \gtrsim_r g$, then $f \gtrsim_t g$

Axioms (1) and (2) assert that preferences form a complete transitive order over the set of lotteries.

Axiom (3) assures that only possible states are relevant to a decision-maker.

Axiom (4) asserts that a higher probability of getting a better lottery is always better, and the following (5), built on the previous one, implies that any lottery ranked between f and h is just as good as some randomization between f and h.

It is not difficult to notice that if substitution axioms (6)÷(9) hold, then an intuitive concept follows: if a decision-maker must choose between two alternatives and if there are two mutually exclusive events, one of which must occur, such that in each event he would prefer the first alternative, then he must prefer the first alternative before he learns which event occurs. Axioms (6) and (7) events are objective randomization in a random lottery process, on the other hand axioms (8) and (9) are for events which are subjective unknowns. The difference from objective and subjective unknowns is that the former implies that prizes depends on events that have obvious objective probabilities, the latter is the case when an event has not obvious probability (e.g., result of a soccer match or future course of the stock market).

Axiom (10) is just a regularity condition, which guarantees that the decision maker is never indifferent between all prizes.

Axiom (11) is optional and it asserts the decision-maker has the same preference ordering over objective gambles in all states of the world.

1.1.2 Main theorems in Decision Theory

In this subsection we are only interested in giving some important results which follow axioms we have written above, without any interest in writing demonstrations. These theorems will be very useful in the following parts of the thesis, when we introduce the Game Theory.

Let Ξ be the set of all events and $p : \Xi \longrightarrow \mathbf{\Delta}(\mathbf{\Omega})$ the conditional probability for every state $t \in \mathbf{\Omega}$ and every event S. Now we define an utility function $u : \mathbf{X} \times \mathbf{\Omega} \longrightarrow \mathbb{R}$, which we denote with U iff it satisfies the condition of state independency $U(x) = u(x,t)\, \forall x, t$.

Given any utility function u and any conditional probability p, we denote $E_p(u(f)|S)$ as the expected value of the prize determined by f

$$E_p(u(f)|S) = \sum_{t \in S} p(t|S) \sum_{x \in \mathbf{X}} u(x,t) f(x,t).$$

Now, given p and u, a theorem gives us a quantitative interpretation of the hierarchy of the lotteries in terms of the expected utility:

Theorem 1.1.1 (Expected-Utility Maximization Theorem) *Axioms from 1 to 10 are jointly satisfied iff there exist a utility function* $u: \mathbf{X} \times \mathbf{\Omega} \to \mathbb{R}$ *and a conditional probability* $p: \mathbf{\Xi} \to \mathbf{\Delta}(\mathbf{\Omega})$ *such that:*

1. $\max_{x \in \mathbf{X}} u(x,t) = 1$ *and* $\min_{x \in \mathbf{X}} u(x,t) = 0 \; \forall t \in \mathbf{\Omega}$
2. $p(R|T) = p(R|S)p(S|T) \, \forall R, S, T$ *such that* $R \subseteq S \subseteq T \subseteq \mathbf{\Omega}$ *and* $S \neq \emptyset$
3. $f \gtrsim_S g$ *iff* $E_p(u(f)|S) \geqslant E_p(u(g)|S) \, \forall f, g \in \mathbf{L}$, $\forall S \in \mathbf{\Xi}$

where $p(R|S) \equiv \sum_{r \in R} p(r|S)$.

Axiom (11) is also satisfied iff all the results of this theorem holds for a state-independent utility function.

The last point of the theorem is the most important and interesting, because it assures that the decision-makers always prefers lotteries with the high expected utilities. In addition we notice that, with $\mathbf{X}$ and $\mathbf{\Omega}$ finite, there are only finite many utility and probability numbers. Thus, the decision-maker's preferences over all the infinite many lotteries in $\mathbf{L}$ can be completely characterized by finite many numbers.

Since more than one utility function can satisfy the results of the theorem (1.1.1), we write a theorem which makes us able to recognize such equivalent representations:

Theorem 1.1.2 (Equivalent Representation Theorem) *Let* S *in* $\mathbf{\Xi}$ *be any given subjective event and both the conditional-probability function* p *and the utility function* u *satisfying the results of the theorem (1.1.1). Suppose that the decision-maker's preferences satisfy axioms 1* $\div$ *10. Then* v *and* q *represent the preference ordering* $\geq_S$ *iff* $\exists$ *a number* $A \in \mathbb{R}^+$ *and a function* $B: S \to \mathbb{R}$ *such that*

$$q(t|S)v(x|t) = Ap(t|S)u(x|t) + B(t) \, \forall t \in S, \, \forall x \in \mathbf{X}$$

If we assume axiom (11) to hold and we require utility functions to be state-independent, we eliminate the ambiguity on the probability function:

Theorem 1.1.3 *Under the same assumptions of the theorem (1.1.2), if we add the axiom (11) and the condition of state-independency on the utility function* v*, then*

$$\begin{aligned} q(t|S) &= p(t|S) \, \forall t \in S \\ v(x) &= Au(x) + C \, \forall x \in \mathbf{X} \end{aligned}$$

where $A, C \in \mathbb{R}$ *and* $A > 0$.

Let us focus now on the second result of the theorem (1.1.1), that is the Bayes' formula. Contemporary we define $\mathbf{\Delta}^*(\mathbf{\Omega})$ the set of all Bayesian conditional probability systems on $\mathbf{\Omega}$, and $\mathbf{\Delta}^0(\mathbf{Z})$ the set of all probability distributions on Z that assigns positive probability to every element in Z. Any element in the set $\mathbf{\Delta}^0(\mathbf{Z})$ generates an element of $\mathbf{\Delta}^*(\mathbf{\Omega})$ through the relation

$$p(t|S) = \begin{cases} \frac{\hat{p}^k(t)}{\sum_{r \in \mathbf{S}} \hat{p}(r)} & \text{if } t \in S \\ 0 & \text{if } t \notin S \end{cases} \tag{1.2}$$

We stress that through (1.2) we are sure that, starting from an element of $\mathbf{\Delta}^0(\mathbf{Z})$ we get an element of $\mathbf{\Delta}^*(\mathbf{Z})$, but this does not implies that we generate all $\mathbf{\Delta}^*(\mathbf{Z})$. Fortunately, a theorem tells us how to do:

Theorem 1.1.4 *The probability function p is Bayesian probability-system in $\mathbf{\Delta}^*(\mathbf{Z})$ iff $\exists$ a sequence of probability distribution $\{\hat{p}^k\}_{k=1}^{\infty}$ such that, $\forall$ event S in $\mathbf{\Omega}$ and $\forall t$ in $\mathbf{\Omega}$*

$$p(t|S) = \begin{cases} \lim_{k \to \infty} \frac{\hat{p}^k(t)}{\sum_{r \in \mathbf{S}} \hat{p}(r)} & \textit{if } t \in S \\ 0 & \textit{if } t \notin S \end{cases}$$

A question may arise at this point: can we say that, independently from the present state of the system, some decision-options are no optimal for sure? The answer that sometimes we can say it, and in these cases we say that a decision-option is strongly dominated.

As intuition suggests, we define a strategy y to be optimal if, given a state-dependent utility function $u : \mathbf{X} \times \mathbf{\Omega} \longrightarrow \mathbb{R}$ where we interpret $\mathbf{X}$ as the set of decisions available to the decision-maker, and given $p(t) = p(t|\mathbf{\Omega})\, \forall t \in \mathbf{\Omega}$ as the subjective probability of each state $t \in \mathbf{\Omega}$, then

$$\sum_{t \in \mathbf{\Omega}} p(t) u(y,t) \geq \sum_{t \in \mathbf{\Omega}} p(t) u(x,t)\, \forall x \in \mathbf{X}. \tag{1.3}$$

For the p of the relation (1.3) a geometrical property holds:

Theorem 1.1.5 *Given $u : \mathbf{X} \times \mathbf{\Omega} \longrightarrow \mathbb{R}$ and $y \in \mathbf{X}$, the set of all $p \in \mathbf{\Delta}(\mathbf{\Omega})$ such that y is optimal is convex.*

Let us now finally give an example of a strongly dominated strategy: suppose $\mathbf{X} = \{\alpha, \beta, \gamma\}$, $\mathbf{\Omega} = \{\theta_1, \theta_2\}$ and the utility function to be as in table 1.1.

Decision	θ_1	θ_2
α	8	1
β	5	3
γ	4	7

Table 1.1: The expected utility payoff of decision-options $\{\alpha, \beta, \gamma\}$ depending on the states $\{\theta_1, \theta_2\}$

It is easy to verify that the optimality condition (1.3) is satisfied for option α against β and γ if $p(\theta_1) \leq 0.6$, and for γ if $p(\theta_1) \geq 0.6$, whereas it is never satisfied for β. For this reason we say that β is strongly dominated.

If we define a *randomized strategy* σ any probability distribution over the set of decision options $\mathbf{X}$, it is possible to find in literature an alternative definition of a strongly dominated strategy which uses a randomized strategy $\sigma = (\sigma(x))_{x\in\mathbf{X}} \in \mathbf{\Delta}(\mathbf{X})$ and the usual utility function. According to this definition we say that an option $y \in \mathbf{X}$ is strongly dominated by a random strategy γ iff

$$\sum_{x\in\mathbf{X}} \sigma(x)u(x,t) > u(y,t)\,\forall t \in \mathbf{\Omega} \tag{1.4}$$

A theorem states that the two definitions are equivalent.

We can relax the condition (1.4), getting the definition of a weakly dominated option $y \in \mathbf{X}$

$$\sum_{x\in\mathbf{X}} \sigma(x)u(x,t) \geq u(y,t)\,\forall t \in \mathbf{\Omega}$$

and there exist at least one $s \in \mathbf{\Omega}$ such that

$$\sum_{x\in\mathbf{X}} \sigma(x)u(x,s) > u(y,s)$$

A theorem asserts that $y \in \mathbf{X}$ is weakly dominated by the random strategy σ if there does not exist a distribution $p \in \mathbf{\Delta}^0(\mathbf{\Omega})$ such as y is optimal.

1.2 Game Theory

In this section we extend concepts of Decision Theory to a situation with multiple decision-makers, i.e. players, somehow interacting with each other. We recall that we are considering a system of rational and limited players, in the meaning of what we wrote at the beginning of chapter 1.

1.2.1 The extensive and the strategic form

In representing schematically a game we have two possibilities:

- The extensive form Γ^e is a tree with notes at every step of the game. In correspondence of every note two situations can occur, either it is a player's turn or the choice is determined by a chance event. Each node is labelled by the name of the player and the available information to him. The ending point of the tree is the end of the game, and here a label with a utility function for every player u_i is written. This $u_i \in \mathbb{R}$ is also called *payoff*.

- The strategic form is a trio $\Gamma = N, (C_i)_{i\in N}, (u_i)_{i\in N}$, where N is the set of players. For a fixed player i, C_i is the set of (*pure*) strategies, and u_i is real utility function acting on the space $\mathbf{C} = \times_{j\in N} C_j$. In the following examples we will indiscriminately talk about strategic and normal form. Precisely speaking this is not correct but, if given a strategy profile $c = (c_j)_{j\in N}$, we define the utility function for the strategic form as the expected utility payoff

$$u_i(c) = \sum_{x \in \boldsymbol{\Omega}^*} P(x|c) w_i(x)$$

where $\boldsymbol{\Omega}^*$ is the set of all terminal node of Γ^e, then the equality between the strategic and normal description holds. $P(x|c)$ is the probability that the game goes through the node x, given the cumulative strategy c and starting from the initial point of Γ^e.

Let us show some examples and how they can be schematised.

- the first example is a card game: at the beginning players 1 and 2 put one dollar in the pot, then player 1 picks a card from a deck with half red and half black card. After picking player 1 can looks at his card privately and decide to raise or fold. Player 1 wins the money in the pot if the card is red, player 2 wins otherwise. If player 1 decides to raise putting another dollar in the pot, then the game continues and it is player 2's turn. Options for player 2 are two: he can pass, so game ends and player 1 takes the money in the pot, or meet putting a dollar in the pot. We can describe this scenario in two different but (obviously) equivalent ways, i.e. the extensive (picture 1.1) and the strategic form (or normal representation) (table 1.2).

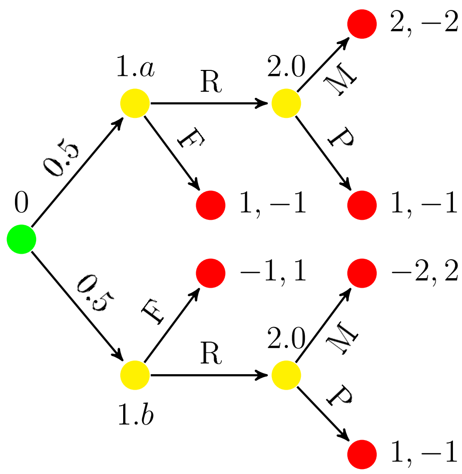


Figure 1.1: The card game in the extensive form. Each colored point denotes a node in this tree representation. The starting point is green and it is the *root*, then two *branches* follow leading to two nodes corresponding to player 1's turn. All nodes corresponding to a player's turn, both 1 and 2, are yellow, and each yellow point is labelled by a number, which indicates the player acting at that node, followed by a letter denoting the knowledge of the player about past events. We notice that player 1 knows the color of the first card, whose player 2 knows nothing, so the latter cannot distinguish whether he is on the upper or lower branch. Finally *terminal nodes* are red and correspond to the ways the game can end. Each available sequence of events is known as *path of play*. Above arrows the decision made (or card picked in the initial node) is indicated.

- let us consider now a model in which player 2 can distinguish the past event and the first choice is not a chance one in order to introduce some variation to the card game.

$\mathbf{C}_1$	$\mathbf{C}_2$: M	P
Rr	$0, 0$	$1, 1$
Rf	$0.5, -0.5$	$0, 0$
Fr	$-0.5, 0.5$	$1, -1$
Ff	$0, 0$	$0, 0$

Table 1.2: The card game in the strategic form. Strategies for player 1 are in the set $C_1 = \{Rr, Rf, Fr, Ff\}$, where the first capitol letter is the choice in the red case and the second lower one in the black case, on the other hand for player 2 only two strategies $C_2 = \{M, P\}$ are available. Given a couple strategy the payoff for player 1 and 2 is reported, separated by a comma.

The extensive form of this model is shown in picture 1.2, whereas the strategic form (or normal representation) in table 1.3.

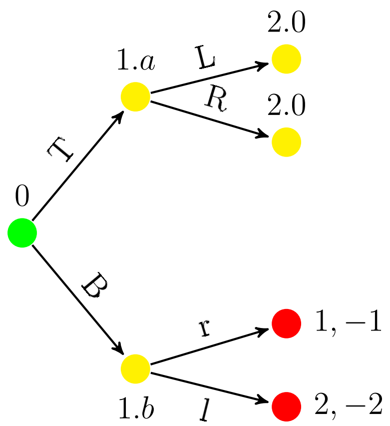


Figure 1.2: Variation of the card game: no chance nodes are present and player 2 knows the choice of player 1

$\mathbf{C}_1$	$\mathbf{C}_2$: Ll	Lr	Rl	Rr
T	$2, 2$	$2, 2$	$4, 0$	$4, 0$
B	$1, 0$	$3, 1$	$4, 1$	$3, 1$

Table 1.3: The card game in the strategic form. Strategies for player 2 are in the set $C_2 = \{Ll, Lr, Rl, Rr\}$, where the first capitol letter is the choice in the T case and the second lower one in the B case, on the other hand for player 1 the only two available strategies are in the set $C_1 = \{T, B\}$

An obvious question might now arise: why have we defined this two forms? Probably the extensive form is the easiest to write, once we have the description of the game. However, the strategic form can be considered as the static version of the extensive form, which is the dynamic description. It follows that, if we are not interested in time, we can delete a dimension getting a simpler game. This rationing has been confirmed by Von Neumann and Morngenstern, who assert that, in a very general sense, the normal representation is all what we need to analyze the game. This conclusion arises from the assumption of rational players.

Generalizing what we have described in section 1.1, we can write that two strategic forms Γ and $\hat{\Gamma}$ are *fully equivalent* iff $\forall$ player i and $\forall$ probability distribution $\mu = \mu(c)_{c\in\mathbf{C}}$ (the probability of choosing the strategy profile c) in $\mathbf{\Delta}(\mathbf{C})$, and $\forall\lambda \in \mathbf{\Delta}(\mathbf{C})$, a player i would prefer μ over λ in both games. Other definitions of equivalence exist, but we are not interested in listing them.

1.2.2 Simplifying a game

Let us consider a more complicated game, as shown in the extensive form in picture 1.3. Our hope is that, once written in the strategic form (table 1.4), we will be able to ease it in some way.

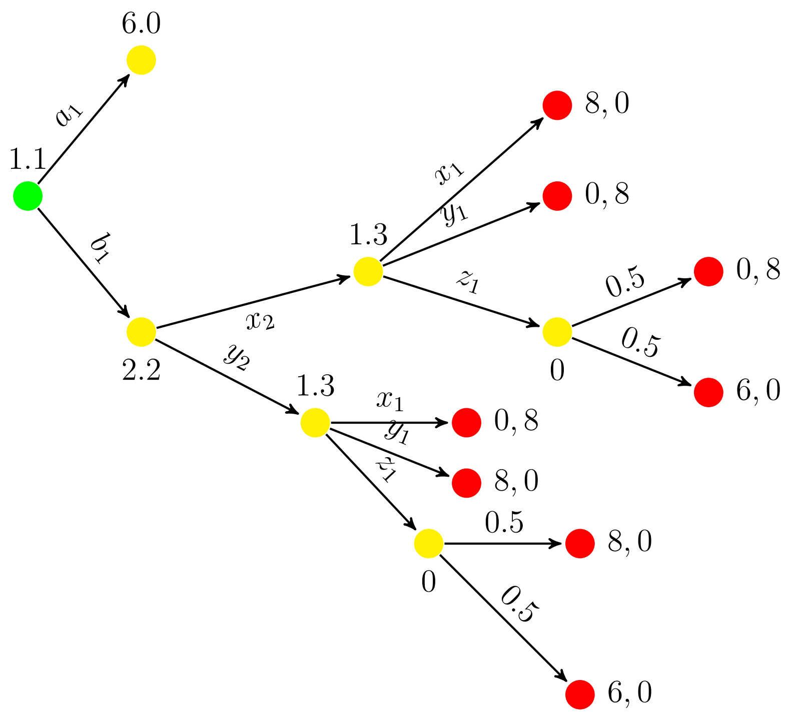


Figure 1.3: The game in the extensive form.

After having written the game in the strategic form we notice immediately that the subset of C_1, given by $\{a_1x_1, a_1y_1, a_1z_1\}$, leads to identical payoffs, so that we can replace $\mathbf{C}_1$ by its quotient getting the *purely reduced normal representation* as in table 1.5

After having deleted equivalent strategies we are now interested in checking if some strategy is redundant, in the meaning that can be generated by other ones. In this sense we can neglect the element $\{b_1z_1\}$ because it can be obtained by a randomized strategy $0.5[a_1\cdot] + 0.5[b_1y_1]$, giving the payoff

$\mathbf{C}_1$	$\mathbf{C}_2$ x_2	y_2
a_1x_1	$6, 0$	$6, 0$
a_1y_1	$6, 0$	$6, 0$
a_1z_1	$6, 0$	$6, 0$
b_1x_1	$8, 0$	$0, 8$
b_1y_1	$0, 8$	$8, 0$
b_1z_1	$3, 4$	$7, 0$

Table 1.4: The game in the strategic form.

$\mathbf{C}_1$	$\mathbf{C}_2$ x_2	y_2
$a_1\cdot$	$6, 0$	$6, 0$
b_1x_1	$8, 0$	$0, 8$
b_1y_1	$0, 8$	$8, 0$
b_1z_1	$3, 4$	$7, 0$

Table 1.5: The game in the strategic form.

$$\begin{cases} 0.5(6,0) + 0.5(0,8) = (3,4) & \text{against } x_2 \\ 0.5(6,0) + 0.5(8,0) = (7,0) & \text{against } y_2 \end{cases}$$

which is effectively equal to the payoff given by $\{b_1z_1\}$.

After having seen some strategies can be neglected in the case they are repeated, also from the payoff point of view we are now interested in knowing if it is possible to simplify further the game using the hypothesis of rational player. Indeed, intuitively if some strategy is always the worst we are led to think that it will be never chosen by players, so let us say it is negligible too. This intuition is right and it is again a generalization of concepts given in the section 1.1, even if it is necessary to distinguish from two kind of dominated strategies.

Given a game $\Gamma = \{N, (C_i)_{i \in N}, (u_i)_{i \in N}\}$, we say that a strategy d_i is strongly dominated iff there exists some randomized strategy $\sigma_i \in \mathbf{\Delta}(C_i)$ such that

$$u_i(c_i, d_i) < \sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \, \forall c_{-i} \in C_{-i}$$

where $\mathbf{C}_{-i}$ is the product of all each single player's strategies set $\mathbf{C}_j$ with $j \neq i$. Weakening this requirement we get the definition of a weakly dominated strategy

$$u_i(c_i, d_i) < \sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \, \forall c_{-i} \in C_{-i}.$$

After eliminating a strongly dominated strategy other new strongly dominated ones may arise, so that other eliminations may be possible. Due to our initial assumption of rationality this iterative elimination of strong dominated strategies does not affect the game. An obvious question is now if the elimination of weakly dominated strategy let the game unchanged. The answer is negative. In addition the resulting game from the elimination of the weakly dominated strategies depends on the order of elimination. Probably the most important theorem about the difference in eliminating strongly or weakly dominated strategies deals with equilibria, and for this reason it will be discussed later.

1.2.3 Nash Equilibrium

From a intuitive point of you we can imagine that at a certain moment the game stabilizes and each player finds a strategy he will never change in the future. This concept is better explained and defined in the following rows.

Given a game with rational players, we can assume that players' strategies are independent random variables. We call σ a randomized strategy profile and we denote with (σ_{-i}, τ_i) as the randomized-strategy profile in which the i-component is τ_i and all other components are as in σ. In other words $\sigma_i(c_i)$ represents the probability that player i chooses c_i the following equality follows

$$u_i(\sigma_{-i}, \tau_i) = \sum_{c \in \mathbf{C}} \left(\prod_{j \in N-i} \sigma_j(c_j) \right) \tau_i(c_i) u_i(c).$$

We can define a randomized strategy σ an (*Nash*) *equilibrium* iff no player could increase his expected playoff by unilaterally deviating from the prediction of the randomized strategy profile:

$$u_i(\sigma) \geq u_i(\sigma_{-i}, \tau_i) \, \forall i \in N, \, \forall \tau_i \in \mathbf{\Delta}(C_i) \tag{1.5}$$

We stress that in this definition we have used the word *unilaterally* and, as we will see in the final example in subsection 1.2.5, this word is very important.

Nash equilibria can be (*weakly*) *Pareto efficient*, iff there is no other outcome that would make all players better off. In addition more equilibria may be present.

After defining the concept of the Nash equilibrium we are interested in finding it, or simpler in knowing whether it can be found or not. A theorem which deals with this problem was written by Nash:

Theorem 1.2.1 *Given any finite game* Γ *in strategic form, there exists at least one equilibrium in* $\times_{i \in N} \Delta(C_i)$.

We stress that we are not considering only pure but also randomized strategies, in effect considering only pure strategies may be insufficient.

As in physics, we can distinguish between stable and unstable equilibria: let us see what these concepts mean in Game Theory. At fist we start defining the usual game $\Gamma = (N, (\mathbf{C_i})_{i \in N}, (u_i)_{i \in N})$ in strategic form. From Γ we can define a perturbation of this game as $\hat{\Gamma} = (N, (\mathbf{C_i})_{i \in N}, ((u_i(\delta_{\vec{e},\lambda}([\cdot])))_{i \in N}))$ where $\delta_{\vec{e},\lambda} : \times_{i \in N} \mathbf{\Delta}(C_i) \to \times_{i \in N} \mathbf{\Delta}^0(C_i)$ is defined by $\delta_{\vec{e},\lambda} = \hat{\sigma}_i$. Given a vector $\vec{e}$ with every component satisfying the condition

$0 < e_i \leq 1$, in the perturbed game every player i has an independent possibility e_i of implementing the randomized strategy $\lambda \in \times_{i \in N}$:

$$\hat{\sigma} = (1 - e_i)\sigma_i + e_i \lambda_i. \tag{1.6}$$

Before defining a stable set of equilibria we need to define the so-called *prestable set*: a closed subset of equilibria Θ of Γ is a prestable set iff it is arbitrarily close to an equilibrium of every perturbed game $\hat{\Gamma}_{\vec{e},\lambda}$, $\forall \vec{e}$ with sufficient small components and $\forall \lambda$. From this definition if we require the prestable set to be minimal we obtain a stable set.

We report an important theorem, which assures us the existence of at least one stable subset:

Theorem 1.2.2 *For any strategic-form game, there is at least one connected set of equilibria that contains a stable subset.*

In the previous subsection we have seen what happens generally to the game if we delete weakly dominated strategies. Now we want to know what happens from the equilibria point of view when we delete a weakly dominated strategy. The answer is that equilibria of the simplified game are still equilibria of the original game. However, eliminating a weakly dominated strategy, we may delete some equilibria too.

1.2.4 Classification of Games

We can classify different games according to their characteristics: a game with N players is called a *N-person game*. If the sum of all gains is equal to the total amount of losses we are dealing with a *zero sum game*, on the contrary it is a *non-zero sum game*. If participants can communicate and they use this ability in order to cooperate to increase their payoffs they are playing a *collaborative game*, otherwise it is a *non-collaborative game*.

An important feature in games is the memory, which can be defined as the basis on which a player decides what option to play. Players can learn working on the memory, in the sense that one changes his strategy over time if the previous one is working bad.

1.2.5 The Prisoner's Dilemma

In order to see some practical applications we present a first example of a basic and classic game: the *Prisoners' dilemma*. The original version has been written by Luce and Raiffa in 1957, and after that many different versions have been written.

The situation can be described as follows: there are two players who are arrested because they are suspected of committing a crime. They are immediately separated, so that they cannot communicate. Each player can confess or remain silent, if one confesses and the other remains silent the confessor is free and the silent one goes to jail for 6 years, if both confess, then both go to jail for 5 years, and finally if both stay silent, they go to jail for only 1 year.

The strategic form of the game, where the payoff is the number of years the player can spend free with respect to the maximum punishment of 6 years, is written in table 1.6.

	$\mathbf{C}_2$	
$\mathbf{C}_1$	s_2	c_2
s_1	$5,5$	$0,6$
c_1	$6,0$	$1,1$

Table 1.6: The Prisoners' dilemma in strategic form. Every player can confess (c_i) or remain silent (s_i). The payoff is given by the number of years each player can spend in freedom, and it spreads from 0, corresponding to 6 years in jail, to 6, corresponding to 0 years in jail.

We immediately notice the silence is always worse than the confession, irrespective of what the other player does. Thus, we can say that s_i is always strongly dominated by c_i. This means that the only (Nash) equilibrium of the game is given by $([c_1],[c_2])$. On the other hand we notice that the cooperative choice, i.e. $([s_1],[s_2])$ would lead to a better payoff for both players, indeed the equilibrium of this game is not Pareto efficient. From this remark we argue that, generally speaking, the elimination of strongly dominated strategies does not lead to the best payoff for each player.

Chapter 2

The general Minority Game model

This chapter is devoted to introduce a financial Game Theory model, i.e., the Minority Game. At first we present the *El Farol Bar Problem*, of which the Minority Game grew out. In the following section 2.2 we define the Minority Game, both the basic model and a specific version that we will modify in chapter 3 in order to take into account some psychological aspects that influence players' behavior. The interpretation of the Minority Game as a financial market description is explained in subsection 2.2.2.

2.1 El Farol Bar Problem

We have already stressed that assuming players to be perfectly rational is too strong a hypothesis for describing real interactions between people, specifically if we are dealing with economic systems, in which players are influenced by emotions and the environment cannot be rigorously defined. In 1994, W. B. Arthur formulated a game [9] characterized by a inductive reasoning, instead of a perfect deductive one, to formulate an efficient description of some complex economic systems.

He started considering some psychological studies which shew that people are not so able to use a deductive reasoning: to face some complex situation, one prefers to shape his decision scheme starting from some hypotheses that uses to verify continuously over time. If the strategy one is using ceases to perform, one will replace it with a new one. This behavior is perfect to describe an evolutionary context and it is a inductive reasoning.

The situation described below takes place in the bar El Farol in Santa Fe, which really offered Irish music on Thursday nights, and it was originally formulated as follows:

> *N people decide independently each week whether to go to a bar that offers entertainment on a certain night. For concreteness, let us set N at 100. Space is limited, and the evening is enjoyable if things are not too crowded-specifically, if fewer than 60% of the possible 100 are present. There is no way to tell the numbers coming for sure in advance, therefore a person or agent: goes - deems it worth going - if he expects fewer than 60 to show up, or stays home if he expects more than 60 to go.*

In the same article he stressed there are neither communication nor cooperation between players, and they all follow a pure inductive behavior, since *the only information available is the numbers who came in past weeks.*

As reported in [2] the only way to face the problem is the inductive method. Indeed, if a deductive solution existed, then each player would choose the same method. However, if all players make the same decision, then all of them will fail. This argumentation shows that no deductive solution exists.

The study of this problem through numerical simulations leads to a surprising conclusion, i.e., the mean attendance converges to the capacity of the bar (figure 2.1). In other words, the predictors self-organize into an equilibrium pattern.

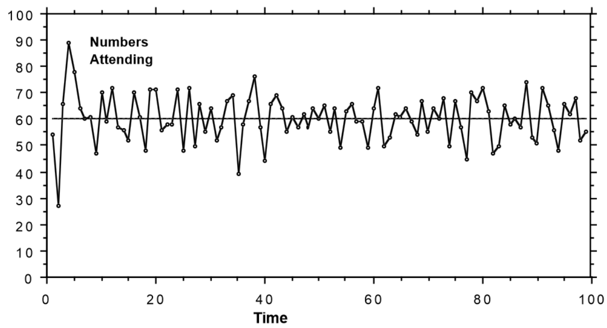


Figure 2.1: Bar attendance in the first 100 weeks [9].

2.2 Minority Game

As Arthur wrote in the preface of the book written by Challet, Zhang and Marsili [12]

> *The Minority Game grew out from my El Farol Bar Problem [...] To me, El Farol was not a problem of how to arrive at a coordinated solution (although the Minority Game very much is). I saw it as a conundrum for economics: How do you proceed analytically when there is no deductive, rational solution? [...] The physics community took it up, and in the hands of Challet, Marsili and Zhang, it inspired something different than I expected the Minority Game. El Farol emphasized (for me) the difficulties of formulating economic behavior in ill-defined problems. The Minority Game emphasizes something different: the efficiency of the solution. This is as it should be. The investigation reveals explicitly how strategies co-adapt and how efficiency is related to information. This opens an important door to understanding financial markets.*

The first formulation of game [8] defines a Minority Game as a play in which N (odd) players participate. Time after time, each player i can choose between two options, i.e., buy or sell assets, denoted by 0 or 1 respectively. We store the number of player choosing the option $a_i(t) = +1$ in the function $A_1(t)$. One player wins if he is in the minority, so the two extreme situations are when $N-1$ players loss and only one wins, and when $(N+1)/2$ loss and $(N-1)/2$ win.

The quantity we are interested in studying is $A_1(t)$, and more precisely fluctuations of this variable which are the measure of the system's total utility. We say that the more efficient the system is the smaller fluctuations are.

The bounded rationality is realized by limiting information every player can access to, basically the memory string is fixed to be M bits long. If we identify by I the possible histories each player can distinguish, it is easy to find that $I = 2^M$. Thus, it follows immediately that the number of strategies between a player can choose is $G = 2^P$ (table 2.1)

I	G															
00	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
01	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 2.1: In the first column all histories I accessible for each player ($M = 2$). In all other ones all the stragies G based on the two history-bits.

The Nash equilibrium is generally hard to find, so we now briefly discuss the simplest version of the game, that is M=0 and $p[a_i(t) = +1] = p[a_i(t) = -1] = 0.5$. As written in [13], any state with $|A| = 1$ is a Nash equilibrium. Indeed, winners would decrease their payoff switching to the minority side, whereas agents in the majority would stay in majority if they decide to change their decisions, because the aggregated decision would change from A to $-A$ as well.

Thus, the number Ω_{NE} of Nash equilibria is

$$\Omega_{NE} = \binom{N}{\frac{N-1}{2}} + \binom{N}{\frac{N+1}{2}} \tag{2.1}$$

Evolutionary Minority Game

We can build a more complex game, in which at the beginning every agent randomly picks l strategies and at each time steps opts for a strategy randomly selected from its pool of l options. One agent values each of his l strategies by assigning one virtual point to those which would have won, so the best strategy is the one with the highest number of virtual points.

If we measure the performance of each player by the number of times he wins, then we can extend the basic Minority Game to include the Darwinist selection [8]: we can consider a model in which after some time steps we worst player is substituted by a clone of the best one, and the new player's performance is reset to zero. To keep a certain difference, we introduce a possible mutation by allowing one of the strategy of the new player to be randomly replaced by a new one. What arises from the analysis is that *fluctuations are reduced and saturated, this implies the average gain for everybody is improved but never reaches the ideal limit*. The learning process is evident if we plot the temporal attendance of $A(t)$ as in figure (2.2).

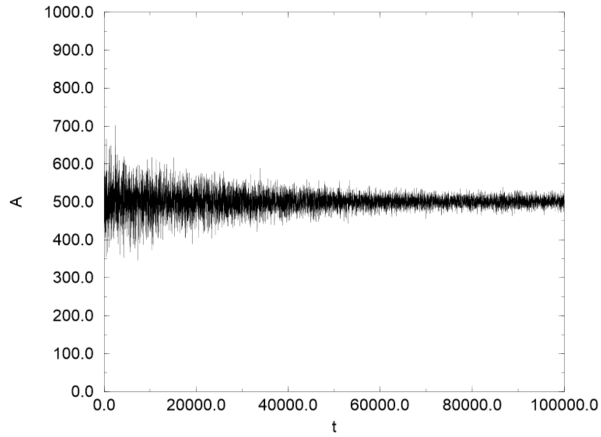


Figure 2.2: We see that the distance from equilibrium of A decreases with time as a result of the learning process [8].

Adaptive Minority Game

Sysi-Aho, Chakraborty and Kaski [14–17] introduced a new modification: the learning process is not realized by replacing the worst player with a new one, but players modify their strategies periodically depending on their performances after time interval τ.

When we define the game we fix a parameter f, that is a fraction of the total number of players. If after the time interval τ a player is one of the worst f players, then he adapts himself modifying his strategy. The adaptation mechanism is inspired by biology: let us consider two parents, i.e., strategies s_i and s_j. The adaptation consists of choosing a random breaking-point and, trough this one-point genetic crossover, the children s_k and s_l are produced (figure 2.3).

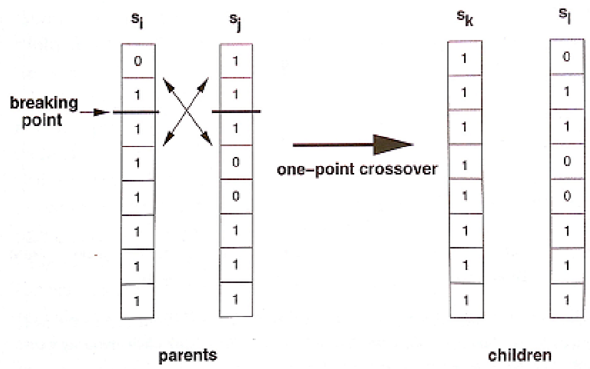


Figure 2.3: We see that the distance from equilibrium for A decreases with time as a result of the learning process [2].

The measure of the total utility of the system can be expressed as:

$$u(x_t) = [1 - H(x_t - x_M)]x_t + H(x_t - x_M)(N - x_t) \qquad (2.2)$$

where $x_M = (N - 1)/2$, x_t is either equal to $A_1(t)$ or $A_0(t)$ and so $x_t \in \{0, 1, 2, ..., N\}$.

The utility function of the system is maximum as the highest number of players wins.

As a result of the analysis, fluctuations disappear totally and the system stabilizes to a state with the maximum of the utility function. In addition a dependence on τ arises, that is, both very frequent adaptation and very slow adaptation lead to bad performances.

2.2.1 A Basic Minority Game

Other formulations of Minority Game are possible, so let us consider a model [18] with N (odd) agents, each agent i can choose between two options $a_i(t) = \pm 1$. The total amount of these choices is stored in the aggregated decision variable $A(t) = \sum_{i=1}^{N} a_i(t)$. The payoff $\Delta_i(t)$ each player experiences is given by

$$\Delta_i(t+1) = \Delta_i(t) - \frac{A(t)}{N} \tag{2.3}$$

and his choice is influenced by the probability function

$$p[a_i(t) = 1] = \frac{1 + \tanh[\Gamma \Delta_i(t)]}{2} \equiv p(t) \tag{2.4}$$

To convince ourselves this model is a Minority Game let us assume that at a certain time step t the probability $p(t)$ is $1/2$ and $A(t) > 0$. Thanks to equation (2.3) the score of every player decreases ($\Delta_i(t+1) < \Delta_i(t)$), so that the probability of choosing $a_i = +1$ decreases as well. Since $a_i(t) = +1$ was the choice of the majority, this is actually a Minority Game. We stress the game is a minority one thanks to two conditions, i.e., $p(t)$ is a monotonic increasing function with respect to $\Delta_i(t)$ and the aggregated decision $A(t)$ updates the score of a single player by being multiplied by a negative number.

This model can describe the behavior of a financial market, indeed if we assume that the log-price $\mathrm{LV}(t)$ evolves according to

$$\mathrm{LV}(t+1) = \mathrm{LV}(t) + \frac{A(t)}{N} \tag{2.5}$$

Rewriting $\Delta_i(t) = -\sum_{t'=0}^{t} A(t)/N + \Delta_i(0)$, we see immediately that

$$\Delta_i(t) = -\,\mathrm{LV}(t) + \Delta_i(0) \tag{2.6}$$

From this point of view $\Delta_i(0)$ is the asset value of agent i and this equation describes a model with N investors having each a value in mind for the price, and acting following their suppositions.

Further financial interpretation to the model will be given in the following subsection 2.2.2, and now we focus on listing main results obtained by the analysis of the model.

If all initial condition are identical and zero, that is $\Delta_i(t) = 0$, we can neglect the index i and the conditional expected value of the score is

$$\begin{aligned} E[\Delta(t+1)|\Delta(t)] &= \Delta(t) - \frac{1}{N} E[A(t)|\Delta(t)] \\ &= \Delta(t) - \tanh[\Gamma \Delta(t)] \end{aligned} \tag{2.7}$$

Thus, equation (2.3) becomes

$$\Delta(t+1) = \Delta(t) - \tanh[\Gamma\Delta(t)] + n(t) \tag{2.8}$$

and $n(t)$ is a white noise term, that is $E[n(t)] = 0$ and $E[n(t)n(t')] = \delta_{t,t'}\,(1 - \tanh[\Gamma\Delta(t)]^2/N)$; it vanishes is the limit $N \to \infty$.

Once we have found a fixed point, i.e., Δ^* such that $E[\Delta(t+1)|\Delta^*] = \Delta^*$, , for verifying the stability we expand the dynamic near Δ^*

$$\Delta(t+1) \approx \Delta^* + \left.\frac{\partial\Delta(t+1)}{\partial\Delta(t)}\right|_{\Delta^*}[\Delta(t) - \Delta^*] \tag{2.9}$$

so

$$\frac{\Delta(t+1) - \Delta^*}{\Delta(t) - \Delta^*} \approx \left.\frac{\partial\Delta(t+1)}{\partial\Delta(t)}\right|_{\Delta^*} \tag{2.10}$$

It is evident that the derivative measures the rate at which the successive iterates approach the fixed point or diverge from it. Thus, the fixed point is stable iff the derivative is included in the interval between -1 and $+1$. If its value is one of extremes of the interval a further investigation needs.

Our model has a fixed point, i.e., $\Delta^* = 0$. It can be demonstrated that it is stable whether $\Gamma < \Gamma_c = 2$ and unstable otherwise. The critical value Γ_c has the property to separate the region in which the time-average of fluctuations $\langle A^2\rangle$ is proportional to N, that is $\Gamma < \Gamma_c$, from the region where $\langle A^2\rangle \propto N^2$, that is $\Gamma > \Gamma_c$. This separation role of Γ_c is evident if we simulate fluctuations versus Γ (figure 2.4).

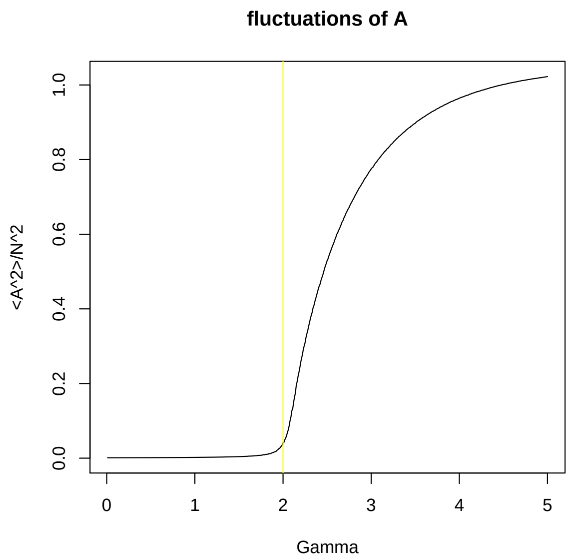


Figure 2.4: The black line is the simulation for fluctuations divided by N^2. The yellow line is the value of Γ_c, which is calculated analytically.

2.2.2 Minority Game as financial model

We have already given a connection between Minority Game and market in the previous subsection (2.2.1), let us now discuss it a bit deeper [18].

One of the most usual and important assumption we use to make for a financial market is that the arbitrage opportunity does not exist. This feature is implemented in Minority Game with the time-average of A to be zero.

The most difficult question is if financial markets are really Minority Games. Naive common sense suggests that if everybody is going to buy, the price will raise and therefore, buying is convenient. From this point of view markets should be rather similar to majority games. On the other hand, one may argue that only the minority of agents who buy first win whereas the others lose. The problem with this approach is the calculation of the payoff of a single transaction. Thus, we change point of view and we try to think how an agent makes a prediction about the future price, given the recent evolution of an asset. Depending on expectations of agents we can distinguish between *fundamentalists* (or *contrarians*), who perceive the market as a Minority Game, and *trend followers*, who perceive it as a majority one. Let us now see why these names for classifying them.

We divide the trading process into three time steps:

- *time* $t-\epsilon$: players know the value of the portfolio reached at the previous time step, that is $V(t-1)$, and submit their choices on the basis of their experiences up to time $t-1$. We stress that they have not any information about what decisions other agents are making.
- *time* t: market aggregates orders $a_i(t)$ getting $A(t)$ and gives the new price $V(t)$ as outcome.
- *time* $t+\epsilon$: agents discover if their choices were successful and they learn updating their experiences. They are ready for the next decision process that will start at time $t+1-\epsilon$.

Options available for any player are two and they have a precise meaning if we consider them actions on a speculative market:

- $a_i > 0$ means that agent i contributes with $a_i(t)$\$ to the the demand for the asset;
- $a_i(t) < 0$ means that i sells $-a_i(t)/V(p-1)$ units of asset.

Thus, the demand is given by $D(t) = N + A(t)/2$ and the supply by $S(t) = N - A(t)/[2V(t-1)]$. As usual the price is defined

$$\begin{aligned} V(t) &= \frac{D(t)}{S(t)} \\ &= V(t-1)\frac{N+1}{N-A(t)} \end{aligned} \tag{2.11}$$

We have written each player evaluates his decision once the price has been updated. The way they evaluate it is the payoff $u_i(t)$:

- *the agent plays* $a_i(t) = +1$: he spends 1$ to buy $1/V(t)$ units of asset. If the price of the asset increases, his choice is the right one. The payoff can be defined the difference between the money he would gain selling tomorrow and what he has spent today to buy the same asset amount, i.e.,

$$u_i(t) = \frac{V(t+1)}{V(t)} - 1 \tag{2.12}$$

- *the agent plays* $a_i(t) = -1$: he decides to sell $1/V(t-1)$ units of an asset gaining $V(t)/V(t-1)$. If the price of the asset decreases, his choice is successful. Thus, the payoff is the difference between what he has gained selling a certain amount of an asset today and how much he would spend to buy the same amount tomorrow, i.e.,

$$u_i(t) = \frac{V(t)}{V(t-1)} - \frac{V(t+1)}{V(t-1)} \tag{2.13}$$

If we now assume the expectation of each agent i to be

$$E_i[V(t+1)] = (1-\psi_i)V(t) + \psi_i V(t-1) \tag{2.14}$$

we see $\psi_i > 0$ implies that agents believe market prices fluctuate around a fixed value, so that the future price is an average of past prices. These players are called *fundamentalists.*

From equation 2.14, it is easy to find that the expectation for the payoff is

$$\begin{cases} E_i[u_i(t)|a_i(t) = +1] = -2\psi_i \dfrac{A(t)}{N + A(t)} & (2.15) \\ E_i[u_i(t)|a_i(t) = -1] = 2\psi_i \dfrac{A(t)}{N - A(t)} & (2.16) \end{cases}$$

Thus, we can generally write

$$E_i[u_i(t)] = -2\psi_i a_i(t) \frac{A(t)}{N + a_i A(t)} \tag{2.17}$$

Notice that if $\psi_i > 0$, then agents taking the majority action will receive a negative expected-payoff, whereas agents in the minority expect to receive a positive one. Hence, if $\psi_i > 0$, then equation 2.17 reduces to the usual payoff for a Minority Game. On the other hand, $\psi_i < 0$ leads to a payoff for Majority Game.

Let $\Delta V(t+1)$ be the price difference $V(t+1) - V(t)$ and insert it in equation 2.14 obtaining

$$E_i[\Delta V(t+1)] = \psi_i \Delta V(t) \tag{2.18}$$

From this viewpoint, if $\psi_i > 0$ agent i may be called *contrarians* since they believe that the future price increment is negatively correlated with the last one. On the other hand, $\psi_i < 0$ is a characteristic of *trend followers*, in the meaning that they think the price is following a monotonic trend and they prefer to stay in majority.

Since all players consider the same price history, expectations should converge. For this reason we require all players to play either a Minority Game or a Majority one. So, albeit with totally different expectations and outcomes, both models provide a description of market dynamics.

We briefly outline that describing a generic Minority Game as physical system is possible: the Hamiltonian of the system is described by the time average of the fluctuations $< E[A^2(t)] >$. If we consider adaptive agents following an exponential learning (as in 2.2.1) this dynamics admits for a Lyapunov function, in the continuous limit. This result is very important, since it turns the probability of studying the stationary state of a stochastic dynamical system into that of characterizing the local minima of a function. Since this function is an Hamiltonian, tools of statistical mechanics are available to treat this problem.

Chapter 3

The asymmetric gains/losses Minority Game model

In the real world, as we have stressed in the preface, a pure mathematical attempt to foresee the future may lead to wrong conclusions. In other words, we can say that *the derived value (utility) function of an individual does not always reflect pure attitudes to money, since it could be affected by additional consequences associated with specific amounts* [19]. Therefore, even if two events have the same probability to occur, generally losses loom larger than gains. The reason is that the aggravation one feels losing a certain amount uses to be greater than the pleasure he would have experienced if he had won the same amount. Moreover, *the aversiveness of symmetric fair bets generally increases with the size of the stake* [7], so that the utility function, defined with respect to the starting point, is concave for gains and convex for losses and it is not symmetric because it is steeper for losses than for gains.

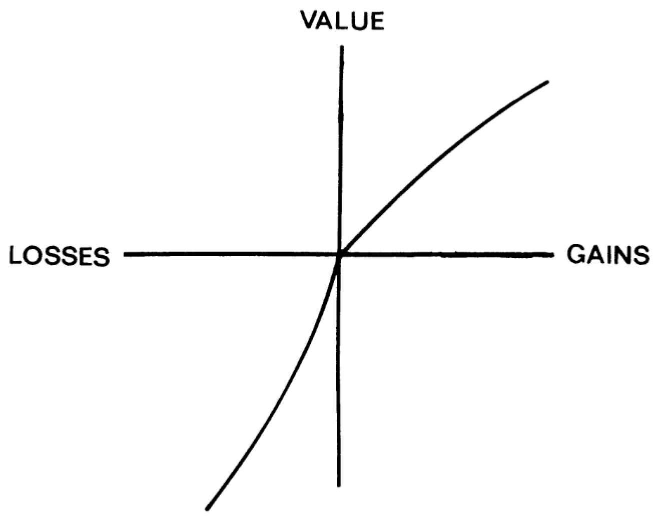


Figure 3.1: The utility function for a generic agent playing a game.

The aim of the variation we are introducing now to the model is to take into account this different attitude to sell or buy an asset, depending on if the decision one has made at the previous time step was successful or not. We want to stress that this approach has never been implemented in the minority game models before.

The simplest way to include this asymmetry in minority game's payoffs is realized simply by adding an asymmetry factor k, which puts more (or less) weight to losses, to the standard model in section 2.2.1:

$$\Delta_i(t+1) = \begin{cases} \Delta_i(t) - \dfrac{A(t)}{N} & \text{winners} \quad (3.1) \\ \Delta_i(t) - (k+1)\dfrac{A(t)}{N} & \text{losers} \quad (3.2) \end{cases}$$

$$p[a_i(t) = 1] = \frac{1 + \tanh[\Gamma \Delta_i(t)]}{2} \equiv p(t) \tag{3.3}$$

The agents play a Minority Game as long as $k + 1 > 0$, otherwise it is a Majority Game. In addition, losses have more weight for losers than winners if $k > 0$.

According to the definition of minority game a player wins if his choice is minority ($A(t)a_i(t) < 0$), otherwise he loses ($A(t)a_i(t) > 0$). For precision's sake no draw possibilities are available, because N is an odd number. The splitting between winners and losers can be easily realized through the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

so that equations (3.1) and (3.2) can be rewritten in a more compact form

$$\Delta_i(t+1) = \Delta_i(t) - \frac{A(t)}{N}\big(1 + kH[A(t)a_i(t)]\big) \tag{3.4}$$

From this last equation we realize that the special case $k = 0$ is the standard symmetric model we have already discussed.

In the following sections both numerical and analytical analyses will be given. After getting familiar with the new model we will find some general properties of the system, in other words how the aggregate decision $A(t)$ behaves with respect to the two parameters (k, Γ) of the model and how they influence the stability of the fixed point.

Before working analytically on the equation which describes the dynamic of the score we get acquainted with the model by simulating the evolution of the score of a single agent, focusing on the role of k and Γ (section 3.1). After that we will move the spotlight to the *time to equilibrium* (section 3.2), that is the time the system needs to stabilize. This value will be very important when we will look for general properties of the system (section 3.3) using numerical simulations, indeed we will calculate them after this transition time. For all the simulations the software R will be used.

3.1 General overview

If we focus on the single agent payoff plotting the output of the code

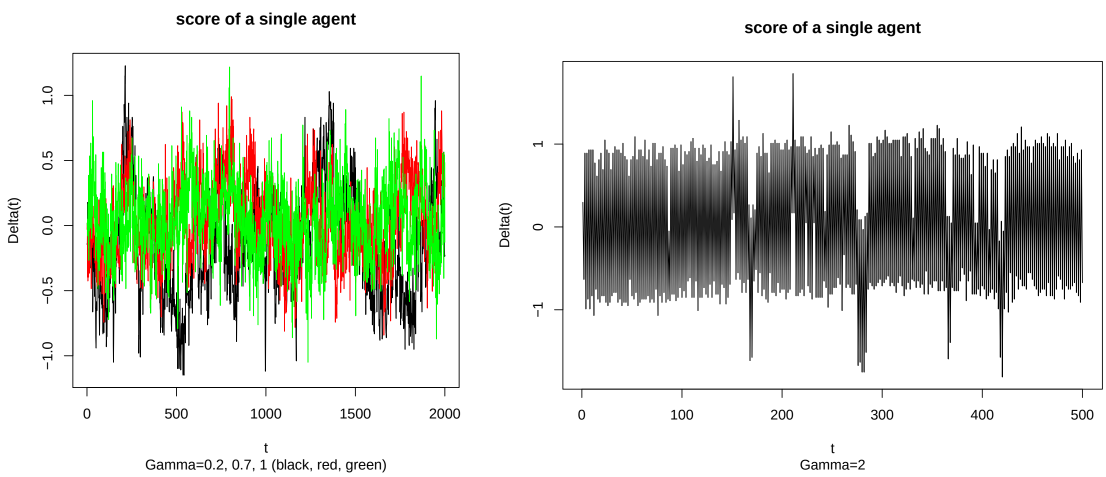
```
model<-function(N=101, k=1, Gamma=1, NIT=1000, agent2observe=1) {

  x=rep(0,length=NIT)
  Deltak=rep(0, length=N)

  for(it in 1:NIT) {
    p=(1+tanh(Gamma*(Deltak)))/2
    r=runif(N)
    a=(-1+2*(r<p))
    A=sum(a)
    theta = ( sign(A)==sign(a) )
    Deltak=Deltak-A*(1+k*theta)/N
    x[it]=Deltak[agent2observe]
  }
  return(x)
}
```

one can distinguish different kinds of path, depending on Γ and k.

The role of the learning factor Γ is to quantify how the probability of choosing $a_i(t) = +1$ depends on the score $\Delta_i(t)$, indeed we see that if Γ is very large, then the value of $\tanh[\Gamma\Delta_i(t)]$ will take almost only two values, ± 1. This means that the payoff jumps between two values at every time step (figure 3.2b).

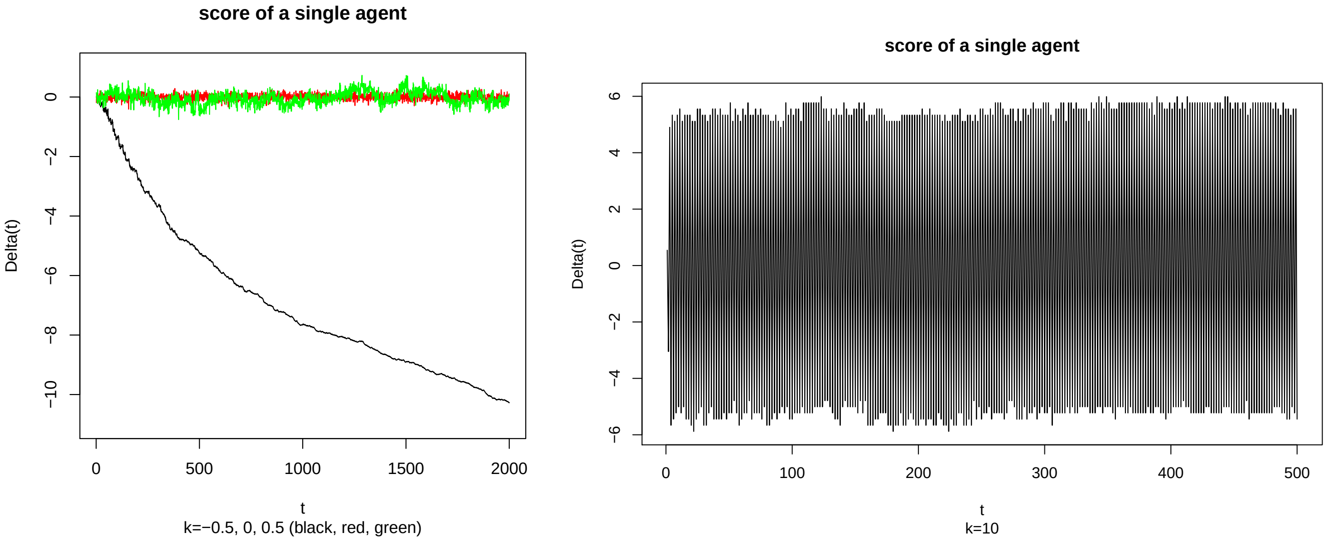


(a) **(b)**

Figure 3.2: The score $\Delta_i(t)$ of a generic agent with different values of Γ. $k = 1$ is fixed.

On the other hand, the role of k is to differentiate payoffs, depending on whether one wins or loses. Varying this parameter we obtain the figure 3.3

One sees that, if $k < 0$, a strong drift characterizes the dynamics (figure 3.3). On the other hand, if $k > 0$ the dynamics is rather random (figure 3.3a), but if k exceeds some large value, then the payoff jumps again between only two values (figure 3.3b).



(a) **(b)**

Figure 3.3: The score $\Delta_i(t)$ of a generic agent with different values of k. $\Gamma = 1$ is fixed.

3.2 Time to equilibrium

As we have already written the aim of this thesis is to study the global properties of (3.4). In order to get convincing results, all statistics have to be computed after the system stabilizes.

The quantity we consider to find this transition time is the exponential moving average

$$\mathrm{EMA}(t) = (1 - \lambda)\,\mathrm{EMA}(t-1) + \lambda f(t) \tag{3.5}$$

where

- $f(t)$ is the value of the variable we are analysing;
- EMA is the exponential moving average;
- λ $(0 < \lambda < 1)$ is the constant which modulates the EMA update.

Generally we will say the *time to equilibrium* is reached when the value of EMA pegs. The evolution of EMA $[A^2(t)]$ is given by the code

```
teq1_0<-function(N=101, k=1, Gamma=1, NIT=1000, lambda=0.05) {
 Deltak=rep(0, length=N)
 ema_A2=rep(0, length=NIT)
 for(it in 1:NIT) {
   p=(1+tanh(Gamma*(Deltak)))/2
   r=runif(N)
   a=(-1+2*(r<p))
   A=sum(a)
```

```
    theta = ( sign(A)==sign(a) )
    Deltak=Deltak-A*(1+(k-1)*theta)/N
    if(it==1) ema_A2[it] = A*A
    if(it!=1) ema_A2[it]=(1-lambda)*ema_A2[it-1]+lambda*A*A
  }
return(ema_A2)
}
```

where all parameters of the function can be varied.

At first we investigate the role of k in the behavior of $\mathrm{EMA}[A^2(t)]$ with $\Gamma = 1$ (figure 3.4) and we see (figure 3.4a) that the shape of the line describing the evolution of $A(t)$ varies, depending on whether $k > 0$ or $-1 < k < 0$, so that

- $-1 < k < 0$: TEQ increases considerably with k, e.g., $\mathrm{TEQ}(k = -0.8) \sim 500$ and $\mathrm{TEQ}(k = -0.1) > 2000$ (figure 3.4b);
- $k \geq 0$: TEQ does not depend on k (figures 3.4b and 3.4c).

Now, if we fix the value of k and we vary Γ (figure 3.5) we see that

- $-1 < k < 0$, TEQ increases when Γ decreases, e.g., $\mathrm{TEQ}(\Gamma = 0.2) > 2000$ and $\mathrm{TEQ}(\Gamma = 0.8) \sim 800$ (figures 3.5c and 3.5d);
- $k \geq 0$, TEQ is Γ-independent (figures 3.5a, 3.5b, 3.5e and 3.5f).

3.3 Analytical analysis

At first we want to get an approximation of the dynamic of the score $\Delta_i(t)$.

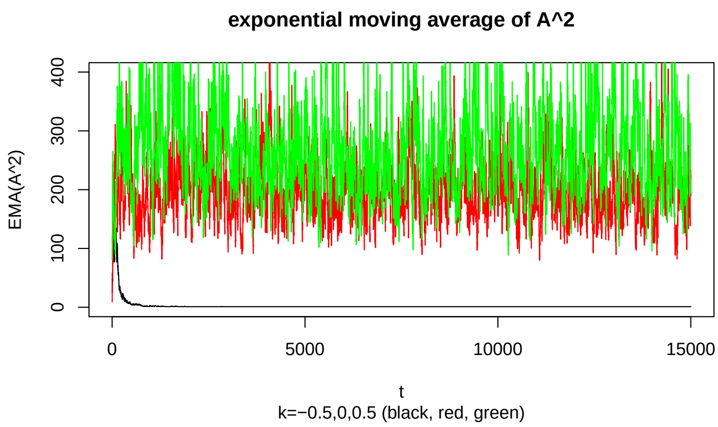
In order to get this, we have got to calculate the expected aggregated decision

$$\begin{aligned} E\big(A(t)\big|\,\{\Delta_j(t)\}\,\big) &= \sum_{j=1}^{N} \tanh[\Gamma\Delta_j(t)] \\ &= \frac{\tanh[\Gamma\Delta_i(t)]}{N} + \frac{\sum_{j\neq i}\tanh[\Gamma\Delta_j(t)]}{N} \\ &= \frac{\tanh[\Gamma\Delta_i(t)]}{N} + \frac{\mu_{-i}}{N} \end{aligned} \tag{3.6}$$

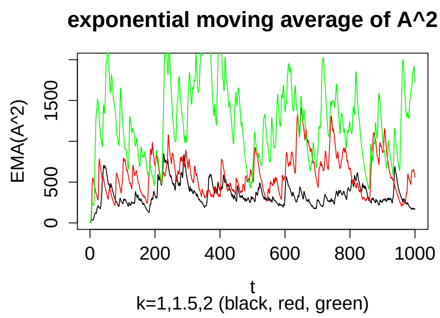
where we have introduced the notation $\mu_{-i} = \sum_{j\neq i}\tanh(\Gamma\Delta_j(t))$.

However, the main difficulty is the calculation of the average of the conditional term

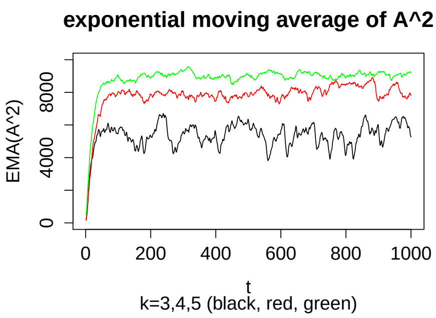
$$\begin{aligned} E\big(A(t)H[A(t)a_i(t)]\big|\,\{\Delta_j(t)\}\,\big) &= E\big([A_{-i}(t)+a_i(t)]H\left([A_{-i}(t)+a_i(t)]a_i(t)\right)\big|\,\{\Delta_j(t)\}\,\big) \\ &= p(t)E\big(A_{-i}(t)+1\big|\,\{\Delta_j(t)\}\,, A_{-i}(t)+1>0\big)+ \\ &\quad + [1-p(t)]E\big(A_{-i}(t)-1\big|\,\{\Delta_j(t)\}\,, -A_{-i}(t)+1>0\big) \end{aligned} \tag{3.7}$$



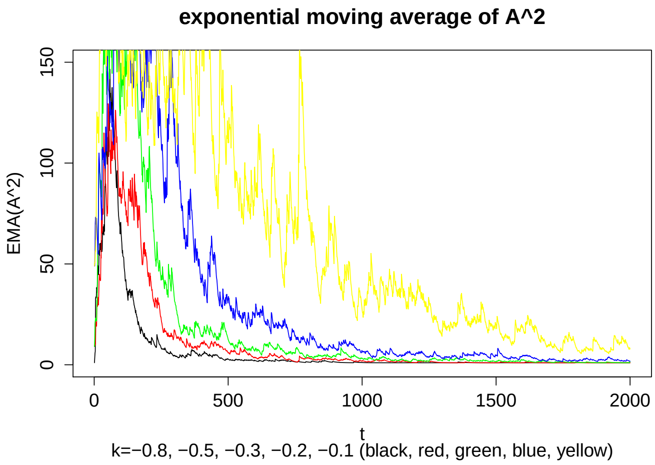
(a)



(b)



(c)



(d)

Figure 3.4: The behavior of the EMA$[A^2(t)]$ with different k. Γ is set to 1 and all initial conditions for Δ_i are 0.

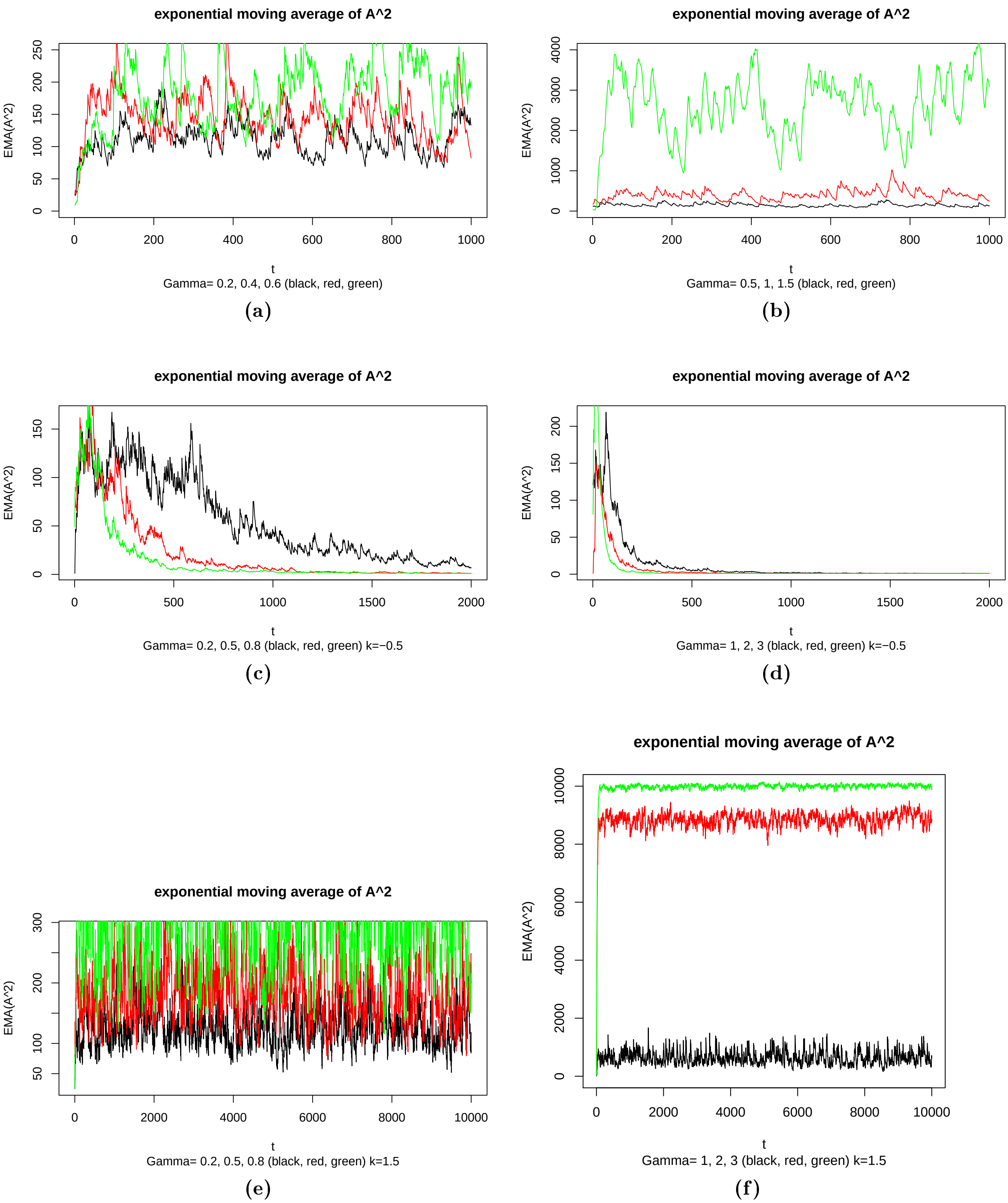


Figure 3.5: The behavior of the EMA$[A^2(t)]$ with different Γ. If not specified k is set to 1 and all initial conditions for Δ_i are 0.

where $A_i(t) = \sum_{j\neq i} a_j(t)$. Without any additional assumption we cannot go further, indeed generally $\Delta_l(t) \neq \Delta_m(t)\ \forall l, m$ and this leads to

- $E[\Delta_l(t+1)|\{\Delta_j(t)\}] \neq E[\Delta_m(t+1)|\{\Delta_j(t)\}]$
- strictly speaking we cannot use the *Central Limit Theorem* (CLT) for the distribution of $A_{-i}(t)$ because $E[a_l(t)|\{\Delta_j(t)\}] \neq E[a_k(t)|\{\Delta_j(t)\}]$

Through equations (3.6) and (3.7) , neglecting fluctuations, one approximates $\Delta_i(t+1)$ (3.4) by its expectation, i.e.,

$$\begin{aligned}\Delta_i(t+1) \simeq \Delta_i(t) - \frac{\tanh[\Gamma\Delta_i(t)]}{N} - \frac{\mu_{-i}}{N} + \\ -\frac{k}{N}\left[p(t)E\big(A_{-i}(t)+1\big|\left\{\Delta_j(t)\right\}, A_{-i}(t)+1>0\big)+\right. \\ \left.+[1-p(t)]E\big(A_{-i}(t)-1\big|\left\{\Delta_j(t)\right\}, -A_{-i}(t)+1>0\big)\right]\end{aligned} \tag{3.8}$$

In order to get a deeper analysis of the dynamic of the model we have to add to the model the condition that initial conditions are the same for every player.

3.3.1 Identical initial conditions

The basic case $k = 0$ is much easier to solve when all agents have the same initial conditions, i.e., $\Delta_i(0) = \Delta(0)\ \forall i$ [18]. This is also the case here, indeed under this assumption

$$\begin{aligned}E\big(\Delta_i(1)\big|\left\{\Delta_j(0)=\Delta(0)\right\}\big) = \Delta_i(0) - \frac{\tanh[\Gamma\Delta_i(0)]}{N} - \frac{\mu_{-i}}{N} + \\ -\frac{k}{N}\left[p(0)E\big(A_{-i}(0)+1\big|\left\{\Delta_j(t)\right\}, A_{-i}(0)+1>0\big)\right. \\ \left.+[1-p(0)]E\big(A_{-i}(0)-1\big|\left\{\Delta_j(t)\right\}, -A_{-i}(0)+1>0\big)\right]\end{aligned} \tag{3.9}$$

where $\mu_{-i}\left[\{\Delta_{j\neq i}(t=0)\}\right] = (N-1)\tanh\left[\Gamma\Delta(0)\right]$.

As we wrote we are not allowed to use the CLT because generally $\mu_i \neq \mu_j$ if $i \neq j$. Anyway, if we plot the coefficient

$$d_1(t) = \frac{\sum_{j\neq 1}|\mu_1(t)-\mu_j(t)|}{N-1} = \frac{\sum_{j\neq 1}|\tanh[\Gamma\Delta_1(t)]-\tanh[\Gamma\Delta_j(t)]|}{N-1} \tag{3.10}$$

in order to understand how large the spread is between the different μ_is we get the figure 3.6 and its time average (after $t = \text{TEQ}$) is 0.3437831. Hence, we can apply the CLT to $A(t)$ considering $E[a_l(t)|\{\Delta_j(t)\}] = \langle a_k(t)|\{\Delta_j(t)\}\rangle\ \forall l, k$ and we write always $\tanh[\Gamma\Delta_l(t)]$.

Under this approximation we obtain

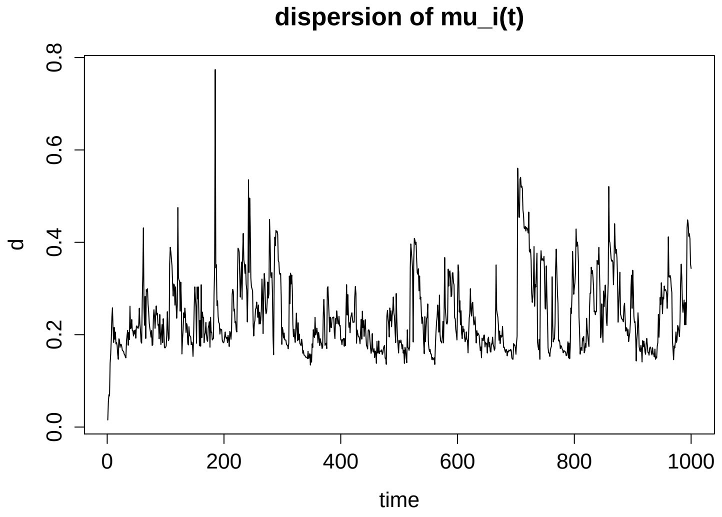


Figure 3.6: the dispersion coefficient $d(t)$.

$$E\big(1 + A_{-i}(t)\big|\left\{\Delta_j(t)\right\}, A_{-i}(t) > -1\big) \simeq \int_{-1}^{+\infty} (1+x)\mathcal{N}[x; (N-1)\mu_{\Delta_j}(t), (N-1)\sigma^2_{\Delta_j}(t)]\, dx \tag{3.11}$$

$$E\big(-1 + A_{-i}(t)\big|\left\{\Delta_j(t)\right\}, A_{-i}(t) < 1\big) \simeq \int_{-\infty}^{1} (-1+x)\mathcal{N}[x; (N-1)\mu_{\Delta_j}(t), (N-1)\sigma^2_{\Delta_j}(t)]\, dx \tag{3.12}$$

where

$$\mathcal{N}[x; (N-1)\mu_j(t), (N-1)\sigma_j^2(t)] = C \exp\left(-\frac{(x-(N-1)\mu_j(t))^2}{2(N-1)\sigma_j^2(t)}\right)$$

$$\begin{cases} C = \frac{1}{\sqrt{2\pi(N-1)\sigma_j^2(t)}} \\ \mu[a_j(t)(\{\Delta_k(t)\})] = \mu_j(t) \equiv \tanh[\Gamma\Delta_j(t)] \\ \sigma^2[a_j(t)(\{\Delta_k(t)\})] = \sigma_j^2(t) \equiv 1 - \tanh^2[\Gamma\Delta_j(t)] \end{cases}$$

If now define the function $\Phi(x)$ as

$$\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt \overset{u=\frac{t-\mu}{\sigma}}{=} \int_{-\infty}^{\frac{z-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \equiv \Phi\left(\frac{z-\mu}{\sigma}\right) \tag{3.13}$$

we can write the equality

$$\begin{aligned}
\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\sigma} t e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt &\overset{u=\frac{t-\mu}{\sigma}}{=} \int_{-\infty}^{\frac{z-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} (\sigma u + \mu) e^{-\frac{u^2}{2}}\, du \\
&= \frac{\sigma}{\sqrt{2\pi}} \left(-e^{-\frac{1}{2}u^2} \Big|_{-\infty}^{\frac{z-\mu}{\sigma}} + \mu \Phi\left(\frac{z-\mu}{\sigma}\right) \right) \\
&= -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(z-\mu)^2}{\sigma^2}} + \frac{1}{2}\mu + \mathrm{erf}\left(\frac{z-\mu}{\sqrt{2}\sigma}\right)
\end{aligned} \tag{3.14}$$

Finally we can get back to (3.11) and (3.12) and through (3.14) we can rewrite them obtaining the relation

$$\begin{aligned}
E\big(1 + A_{-i}(t)\big|\,\{\Delta_j(t)\}, A_{-i}(t) > -1\big) &= 1 - \Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) + \\
&+ E\big(A_{-i}(t)\big|\,\{\Delta_j(t)\}, A_{-i}(t) > -1\big) \\
&= 1 - \Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) + E\big(A_{-i}(t)\big|\,\{\Delta_j(t)\}\big) + \\
&- E\big(A_{-i}(t)\big|\,\{\Delta_j(t)\}, A_{-i}(t) < -1\big) \\
&= 1 - \Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) + \mu_{-i} + \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\mu_{-i})^2}{\sigma_{-i}^2}} + \\
&- \mu_{-i}\Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right)
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
E\big(-1 + A_{-i}(t)\big|\,\{\Delta_j(t)\}, A_{-i}(t) < 1\big) &= -\Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right) + E\big(A_{-i}(t)\big|\,\{\Delta_j(t)\}, A_{-i}(t) < 1\big) \\
&= -\Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right) - \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\mu-i)^2}{\sigma_{-i}^2}} + \\
&+ \mu_{-i}\Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right)
\end{aligned} \tag{3.16}$$

where

$$\begin{cases}
\mu_{-i}[A_{-i}(t)(\{\Delta_{j\neq i}(t)\})] \equiv \mu_{-i} = \sum_{j\neq i} \tanh[\Gamma\Delta_j(t)] & (3.17) \\
\sigma^2_{-i}[A_{-i}(t)(\{\Delta_{j\neq i}(t)\})] \equiv \sigma^2_{-i} = \sum_{j\neq i} (1 - \tanh^2[\Gamma\Delta_j(t)]) & (3.18)
\end{cases}$$

Hence, under the assumption that all initial conditions are the same we have found that we can approximate the score after the first time step (3.9) with

$$\begin{aligned}
\Delta_i(1) \simeq \Delta_i(0) - \frac{\tanh(\Gamma\Delta_i(0))}{N} - \frac{\mu_{-i}}{N} + \\
- \frac{k}{N}\frac{\tanh(\Gamma\Delta_i(0))}{2} \cdot \\
\cdot \left[1 - \Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) + \Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right) + \mu_{-i} + \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\mu_{-i})^2}{\sigma_{-i}^2}} + \right. \\
\left. -\mu_{-i}\Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) + \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\mu_{-i})^2}{\sigma_{-i}^2}} - \mu_{-i}\Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right)\right] + \\
- \frac{k}{2N} \cdot \\
\cdot \left[1 - \Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) - \Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right) + +\mu_{-i} + \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\mu_{-i})^2}{\sigma_{-i}^2}} + \right. \\
\left. -\mu_{-i}\Phi\left(\frac{-1-\mu_{-i}}{\sigma_{-i}}\right) - \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\mu_{-i})^2}{\sigma_{-i}^2}} + \mu_{-i}\Phi\left(\frac{1-\mu_{-i}}{\sigma_{-i}}\right)\right] \equiv \hat{\Delta}_i(1)
\end{aligned} \tag{3.19}$$

Consequently, as $\hat{\Delta}_i(1)$ depends only on initial conditions and they are all equivalent, we have that all $\hat{\Delta}_i(1)$ are the same, so we can find $\hat{\Delta}_i(2)$ in the same way and so on at all time steps.

Now, let us consider the new process, that is merely an approximation of the dynamic of the model itself:

$$\begin{aligned}
\hat{\Delta}_i(t+1) = \hat{\Delta}_i(t) - \frac{\tanh(\Gamma\hat{\Delta}_i(t))}{N} - \frac{\hat{\mu}_{-i}}{N} + \\
- \frac{k}{N}\frac{\tanh(\Gamma\hat{\Delta}_i(t))}{2} \cdot \\
\cdot \left[1 - \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \hat{\mu}_{-i} + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}_{-i}^2}} + \right. \\
\left. -\hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}_{-i}^2}} - \hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right] + \\
- \frac{k}{2N} \cdot \\
\cdot \left[1 - \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) - \Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \hat{\mu}_{-i} + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}_{-i}^2}} + \right. \\
\left. -\hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) - \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}_{-i}^2}} + \hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right]
\end{aligned} \tag{3.20}$$

We will now study some properties of the dynamics of the approximated score $\hat{\Delta}_i(t)$.

Fixed point

Due to the definition, the fixed point of our series satisfies the condition

$$\hat{\Delta}_i(t+1) \stackrel{!}{=} \hat{\Delta}_i(t) \tag{3.21}$$

Thus, if we force this condition in our model we find that it implies

$$\begin{aligned}
0 \stackrel{!}{=} & -\frac{\tanh(\Gamma\hat{\Delta}_i(t))}{N} - \frac{\hat{\mu}_{-i}}{N} + \\
& -\frac{k}{N}\frac{\tanh(\Gamma\hat{\Delta}_i(t))}{2} \cdot \\
& \cdot \left[1 - \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \hat{\mu}_{-i} + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} + \right. \\
& \left. -\hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} - \hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right] + \\
& -\frac{k}{2N} \cdot \\
& \cdot \left[1 - \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) - \Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \hat{\mu}_{-i} + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} + \right. \\
& \left. \cdot -\hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) - \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} + \hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right]
\end{aligned} \tag{3.22}$$

If we calculate how the score in equation (3.20) evolves starting from $\Delta_i(0) = 0\ \forall i$ we find that

$$\begin{aligned}
\hat{\Delta}_i(t+1)|_{\hat{\Delta}_i(t)=\hat{\Delta}_j(t)=0} &= -\frac{k}{2N}\left[1 - \Phi\left(\frac{-1}{\sqrt{N-1}}\right) - \Phi\left(\frac{1}{\sqrt{N-1}}\right) + \right. \\
& \left. \frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}} - \frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}}\right] \\
&= -\frac{k}{2N}\left[1 - \Phi\left(\frac{-1}{\sqrt{N-1}}\right) - \Phi\left(\frac{1}{\sqrt{N-1}}\right)\right] = 0
\end{aligned} \tag{3.23}$$

Therefore, $\Delta^* = 0$ satisfy the condition is a fixed point of our model.

Generalizing what we have written in the previous chapter to the multidimensional case, a fixed point Δ_i^* is stable if all eigenvalues of the Jacobian are real or complex numbers with absolute value strictly less than 1.

At first to simplify the reading we highlight the derivatives

$$\begin{cases} \dfrac{\partial}{\partial \hat{\Delta}_{j\neq i}(t)} \hat{\mu}_{-i} = \dfrac{\Gamma}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} & (3.24) \\ \dfrac{\partial}{\partial \hat{\Delta}_{j\neq i}(t)} \hat{\sigma}_{-i} = \dfrac{1}{2} \dfrac{-\frac{2\Gamma \tanh[\Gamma \hat{\Delta}_j(t)]}{\cosh^2[\Gamma \hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}} \Rightarrow \dfrac{\partial}{\partial \hat{\Delta}_{j\neq i}(t)} \hat{\sigma}^2_{-i} = -\dfrac{2\Gamma \tanh[\Gamma \hat{\Delta}_j(t)]}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} & (3.25) \end{cases}$$

Finally, to compute the derivative of the score at the time $t+1$ with respect to the score at the previous time we calculate

$$\begin{aligned} \frac{\partial}{\partial \hat{\Delta}_{j\neq i(t)}} \left(\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} \right) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \frac{-\frac{2\Gamma \tanh[\Gamma \hat{\Delta}_j(t)]}{\cosh^2[\Gamma \hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} + \\ &\quad - \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} \frac{1}{2} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} \frac{\partial}{\partial \hat{\Delta}_{j\neq i(t)}} \left[\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}} \right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \frac{-\frac{2\Gamma \tanh[\Gamma \hat{\Delta}_j(t)]}{\cosh^2[\Gamma \hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} - \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} \frac{1}{2} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} \cdot \\ &\quad \cdot \frac{-2(-1-\hat{\mu}_{-i}) \frac{\Gamma}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} \hat{\sigma}^2_{-i} + (-1-\hat{\mu}_{-i})^2 \frac{2\Gamma \tanh[\Gamma \hat{\Delta}_j(t)]}{\cosh^2[\Gamma \hat{\Delta}_j(t)]}}{\hat{\sigma}^4_{-i}} \end{aligned} \tag{3.26}$$

$$\begin{aligned} \frac{\partial}{\partial \hat{\Delta}_{j\neq i(t)}} \left(\hat{\mu}_{-i} \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}} \right) \right) &= \frac{\Gamma}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}} \right) + \\ &\quad + \hat{\mu}_{-i} \frac{\partial}{\partial \hat{\Delta}_{j\neq i(t)}} \left(\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{-1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}} \right) \right) \\ &= \frac{\Gamma}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}} \right) + \\ &\quad + \hat{\mu}_{-i} \frac{1}{2} \frac{2}{\sqrt{\pi}} e^{-\left(\frac{-1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}}\right)^2} \frac{\partial}{\partial \hat{\Delta}_j(t)} \left(\frac{-1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}} \right) \\ &= \frac{\Gamma}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}} \right) + \hat{\mu}_{-i} \frac{1}{2} \frac{2}{\sqrt{\pi}} e^{-\left(\frac{-1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}}\right)^2} \cdot \\ &\quad \cdot \frac{-\frac{\Gamma}{\cosh^2[\Gamma \hat{\Delta}_j(t)]} \sqrt{2}\hat{\sigma}_{-i} - (-1-\hat{\mu}_{-i}) \sqrt{2} \frac{1}{2} \frac{-\frac{2\Gamma \tanh[\Gamma \hat{\Delta}_j(t)]}{\cosh^2[\Gamma \hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}}}{2\hat{\sigma}^2_{-i}} \end{aligned} \tag{3.27}$$

$$\frac{\partial}{\partial\hat{\Delta}_{j\neq i(t)}}\left(\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\right) = \frac{1}{\sqrt{2\pi}}\frac{1}{2}\frac{-\frac{2\Gamma\tanh[\Gamma\hat{\Delta}_j(t)]}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}}e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}+$$
$$-\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}}\frac{1}{2}e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\frac{\partial}{\partial\hat{\Delta}_{j\neq i(t)}}\left[\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}\right]$$
$$=\frac{1}{\sqrt{2\pi}}\frac{1}{2}\frac{-\frac{2\Gamma\tanh[\Gamma\hat{\Delta}_j(t)]}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}}e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}-\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}}\frac{1}{2}e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\cdot$$
$$\cdot\frac{-2(1-\hat{\mu}_{-i})\frac{\Gamma}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}\hat{\sigma}^2_{-i}+(1-\hat{\mu}_{-i})^2\frac{2\Gamma\tanh[\Gamma\hat{\Delta}_j(t)]}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}}{\hat{\sigma}^4_{-i}} \tag{3.28}$$

$$\frac{\partial}{\partial\hat{\Delta}_{j\neq i(t)}}\left(\hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right) = \frac{\Gamma}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)+$$
$$+\hat{\mu}_{-i}\frac{\partial}{\partial\hat{\Delta}_{j\neq i(t)}}\left(\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(\frac{1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}}\right)\right)$$
$$=\frac{\Gamma}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)+$$
$$\hat{\mu}_{-i}\frac{1}{2}\frac{2}{\sqrt{\pi}}e^{-\left(\frac{1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}}\right)^2}\frac{\partial}{\partial\hat{\Delta}_j(t)}\left(\frac{1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}}\right)$$
$$=\frac{\Gamma}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)+\hat{\mu}_{-i}\frac{1}{2}\frac{2}{\sqrt{\pi}}e^{-\left(\frac{1-\hat{\mu}_{-i}}{\sqrt{2}\hat{\sigma}_{-i}}\right)^2}\cdot$$
$$\cdot\frac{-\frac{\Gamma}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}\sqrt{2}\hat{\sigma}_{-i}-(1-\hat{\mu}_{-i})\sqrt{2}\frac{1}{2}\frac{-\frac{2\Gamma\tanh]\Gamma\hat{\Delta}_j(t)]}{\cosh^2[\Gamma\hat{\Delta}_j(t)]}}{\hat{\sigma}_{-i}}}{2\hat{\sigma}^2_{-i}} \tag{3.29}$$

It is easy to notice now that the our Jacobian is symmetric

$$\begin{pmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ B & B & \ddots & \cdots & B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & B & B & \cdots & A \end{pmatrix} \tag{3.30}$$

where

$$
\begin{aligned}
A \equiv \frac{\partial \hat{\Delta}_i(t+1)}{\partial \hat{\Delta}_i(t)} &= 1 - \frac{\Gamma}{N\cosh^2[\Gamma\hat{\Delta}_i(t)]} \cdot \\
&\cdot \left[1 + \frac{k}{2}\left(1 - \Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \right.\right. \\
&+ \hat{\mu}_{-i} + \frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} - \hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \\
&\left.\left. + \frac{\sigma_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}} - \hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right)\right]
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
B \equiv \frac{\partial \hat{\Delta}_i(t+1)}{\partial \hat{\Delta}_{j\neq i}(t)} &= -\frac{\Gamma}{N\cosh^2[\Gamma\hat{\Delta}_j(t)]} + \\
&- \frac{k}{2}\frac{\tanh[\Gamma\hat{\Delta}_i(t)]}{N}\left[-\frac{\partial}{\partial\hat{\Delta}_j(t)}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \frac{\partial}{\partial\hat{\Delta}_j(t)}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \right. \\
&+ \frac{\partial}{\partial\hat{\Delta}_j(t)}\hat{\mu}_{-i} + \\
&+ \frac{\partial}{\partial\hat{\Delta}_j(t)}\left(\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\right) - \frac{\partial}{\partial\hat{\Delta}_{j\neq i}(t)}\left(\hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right) + \\
&\left. + \frac{\partial}{\partial\hat{\Delta}_{j\neq i}(t)}\left(\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\right) - \frac{\partial}{\partial\hat{\Delta}_{j\neq i}(t)}\left(\hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right)\right] + \\
&- \frac{k}{2N}\left[-\frac{\partial}{\partial\hat{\Delta}_j(t)}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) - \frac{\partial}{\partial\hat{\Delta}_j(t)}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right) + \right. \\
&+ \frac{\partial}{\partial\hat{\Delta}_j(t)}\hat{\mu}_{-i} + \\
&+ \frac{\partial}{\partial\hat{\Delta}_j(t)}\left(\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(-1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\right) - \frac{\partial}{\partial\hat{\Delta}_{j\neq i}(t)}\left(\hat{\mu}_{-i}\Phi\left(\frac{-1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right) + \\
&\left. - \frac{\partial}{\partial\hat{\Delta}_j(t)}\left(\frac{\hat{\sigma}_{-i}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(1-\hat{\mu}_{-i})^2}{\hat{\sigma}^2_{-i}}}\right) + \frac{\partial}{\partial\hat{\Delta}_{j\neq i}(t)}\left(\hat{\mu}_{-i}\Phi\left(\frac{1-\hat{\mu}_{-i}}{\hat{\sigma}_{-i}}\right)\right)\right]
\end{aligned}
\tag{3.32}
$$

Before going further we introduce the *circulant matrix* as a $n \times n$ matrix satisfying the property

$$\begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \vdots & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & \vdots & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix} \tag{3.33}$$

The meaning of circulant is easy explained realizing that this matrix is completely specified by the first column, in effect all $n-1$ remaining ones are given by a cyclic permutation of the first one. This special matrix has the useful property to have all eigenvalues given by the relation

$$\chi_j = A + B\left(\omega_j + \omega_j^2 + \cdots + \omega_j^{(N-1)}\right) \tag{3.34}$$

where $\omega_j = \exp\left(\imath \frac{2\pi j}{n}\right)$,with the index $j = 0, \ldots, n-1$, are the n-th roots of unity and $\imath$ is the imaginary unit.

After introducing this special type of matrix we notice that our Jacobian is not only symmetric but circulant as well. According to the notation we have used to define the circulant matrix, A is the element c_0 and B al the remaining element of the first column.

Thus, we can easy calculate that

$$\chi_0 = A - B\left(\sum_{k=1}^{N-1} \omega_0^k\right) = A + (N-1)B \tag{3.35}$$

$$\chi_{j\neq 0} = A - B\left(\sum_{k=1}^{N-1} \omega_j^k\right) = A + B\left(\frac{\omega_j - \omega_j^N}{1-\omega_j}\right) = A - B \tag{3.36}$$

Since $\chi_{j\neq 0}$ does not depend on j this has multiplicity $n-1$ and we rename it as χ.

To check the stability of the fixed point $\Delta^* = 0$, we need to calculate the Jacobian (3.30) in it. This calculation leads to

$$
\begin{aligned}
A &= 1 - \frac{\Gamma}{N}\left[1 + \frac{k}{2}\left(1 - \Phi\left(\frac{-1}{\sqrt{N-1}}\right) + \Phi\left(\frac{1}{\sqrt{N-1}}\right) + \right.\right. \\
&\left.\left. + \frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}} + \frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}}\right)\right] \\
&= 1 - \frac{\Gamma}{N}\left[1 + k\left(\Phi\left(\frac{1}{\sqrt{N-1}}\right) + \frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}}\right)\right] \\
&= 1 + \Gamma\alpha(k, N)
\end{aligned} \tag{3.37}
$$

$$
\begin{aligned}
B &= -\frac{\Gamma}{N} - \frac{k}{2N}\left[-2\frac{1}{\sqrt{\pi}} e^{\frac{1}{2(N-1)}} \frac{-\Gamma\sqrt{2}\sqrt{N-1}}{2(N-1)} + \Gamma + \right. \\
& - \frac{\sqrt{N-1}}{\sqrt{2\pi}} \frac{1}{2} e^{-\frac{1}{2}\frac{1}{N-1}} \frac{2(N-1)\Gamma}{(N-1)^2} - \Gamma\Phi\left(\frac{-1}{\sqrt{N-1}}\right) + \\
&\left. + \frac{\sqrt{N-1}}{\sqrt{2\pi}} \frac{1}{2} e^{-\frac{1}{2}\frac{1}{N-1}} \frac{-2(N-1)\Gamma}{(N-1)^2} + \Gamma\Phi\left(\frac{1}{\sqrt{N-1}}\right)\right] \\
&= -\frac{\Gamma}{N} - \frac{k\Gamma}{2N}\left[\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2(N-1)}} \frac{\sqrt{2}\sqrt{N-1}}{(N-1)} - 2\frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}} \frac{1}{(N-1)} + \right. \\
&\left. + 2\Phi\left(\frac{1}{\sqrt{N-1}}\right)\right] \\
&= \Gamma\beta(k, N)
\end{aligned} \tag{3.38}
$$

where we have introduced functions

$$
\begin{aligned}
\alpha(k, N) &= -\frac{1}{N} - \frac{k}{N}\left[\Phi\left(\frac{1}{\sqrt{N-1}}\right) + \frac{\sqrt{N-1}}{\sqrt{2\pi}}\right] \\
&= -\frac{1}{N} + f(N)k
\end{aligned} \tag{3.39}
$$

$$
\begin{aligned}
\beta(k, N) &= -\frac{1}{N} - \frac{k}{2N}\left[\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2(N-1)}} \frac{\sqrt{2}\sqrt{N-1}}{(N-1)} - 2\frac{\sqrt{N-1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{N-1}} \frac{1}{(N-1)} + \right. \\
&\left. + 2\Phi\left(\frac{1}{\sqrt{N-1}}\right)\right] \\
&= -\frac{1}{N} + g(N)k
\end{aligned} \tag{3.40}
$$

to ease the notation.
The specific condition to satisfy for the stability is

$$
\begin{cases}
-1 < 1 + \Gamma(\alpha(k, N) + (N-1)\beta(k, N)) < 1 & \text{for } \chi_0 \\
-1 < 1 + \Gamma(\alpha(k, N) - \beta(k, N)) < 1 & \text{for } \chi
\end{cases} \tag{3.41}
$$

From the condition on χ_0 we get:

$$-1 < 1 + \Gamma\left[-\frac{1}{N} - \frac{N-1}{N} + \left(f(N) + (N-1)g(N)\right)k\right] < 1 \tag{3.42}$$

and from the one on χ

$$-1 < 1 + \Gamma\left(f(N) - g(N)\right)k < 1 \tag{3.43}$$

Since our approximations hold in the limit $N \to \infty$ we can rewrite equations (3.42) and (3.43) obtaining

$$\begin{cases} -\frac{2}{\Gamma} < -\left(1 + \frac{k}{2}\right) < 0 & \text{for } \chi_0 \\ \lambda = 1 & \text{for } \chi \end{cases} \tag{3.44}$$

From the first one we get

$$\Gamma\left(1 + \frac{k}{2}\right) < 2 \Rightarrow \Gamma_c = \frac{2}{1 + \frac{k}{2}} \tag{3.45}$$

which, setting $k = 0$, recovers the case of the model without splitting [18]. From the equality which defines the value of Γ_c it arises that k destabilizes the dynamics, in the sense that in the limit of $k \to \infty$ all values of Γ make Δ^* unstable (figure 3.7).

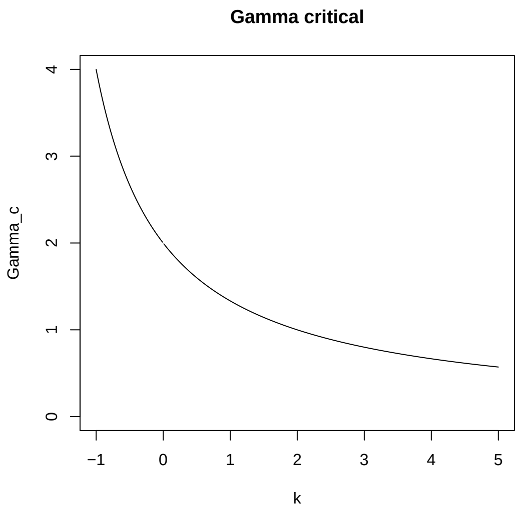


Figure 3.7: The dependence of Γ_c on k. $\Delta^* = 0$ is stable if $\Gamma < \Gamma_c$

If we simulate the case with $k = \Gamma = 1$ and $N = 1001$ we find that $\chi_0 = -0.53$ and $\chi = 0.99$. Hence, Δ^* is stable and it confirms the validity of the approximation we have made to find Γ_c analytically.

Fluctuations of $A(t)$

We now want to investigate how the aggregated decision $A(t)$ evolves in time.

Since the probability function is symmetric with respect to $\Delta_i(t) = 0$ we expect the average in time

$$\langle A \rangle = \lim_{t_0, T \to \infty} \frac{1}{T} \sum_{t_0}^{t_0+T} A(t) \tag{3.46}$$

to be 0. In order to verify this we plot $\langle A \rangle$ and how it depends on Γ and k (figure 3.8)

From figure (3.8) we have the confirmation that the approximation $\langle A \rangle \simeq 0$ hold generally for every value of the parameters of the game.

A second information about the average of $A(t)$ is now necessary, that is the magnitude of the fluctuations

$$\langle A^2 \rangle = \lim_{t_0, T \to \infty} \frac{1}{T} \sum_{t_0}^{t_0+T} A^2(t) \tag{3.47}$$

We want to check if the behavior reported in chapter 2 holds in this model as well. In order to estimate this, the average $\langle A^2 \rangle$ is plotted in figure (3.9)

For $k > 0$ the behavior of fluctuation is the same we have reported for the basic model, i.e. the transition between the the state with $\langle A^2 \rangle \propto N$ and $\langle A^2 \rangle \propto N^2$ occurs at the value $\Gamma = \Gamma_c$. On the other hand, for $k < 0$ the function is strictly decreasing.

At the end we can say that $k > 0$ destabilizes and $k < 0$ stabilizes the dynamics.

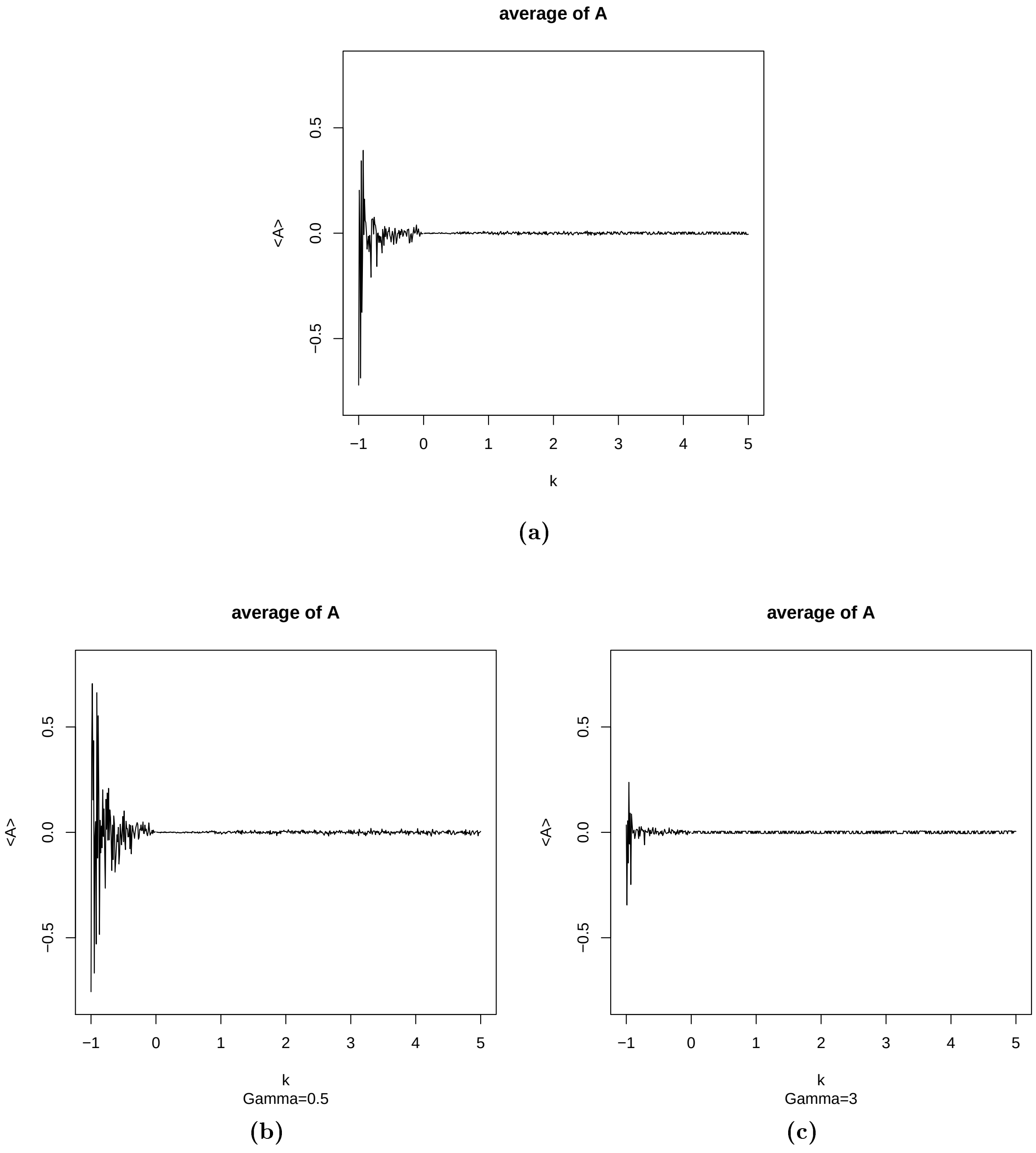


Figure 3.8: The dependence of $\langle A \rangle$ with respect to Γ and k. If not specified $\Gamma = 1$.

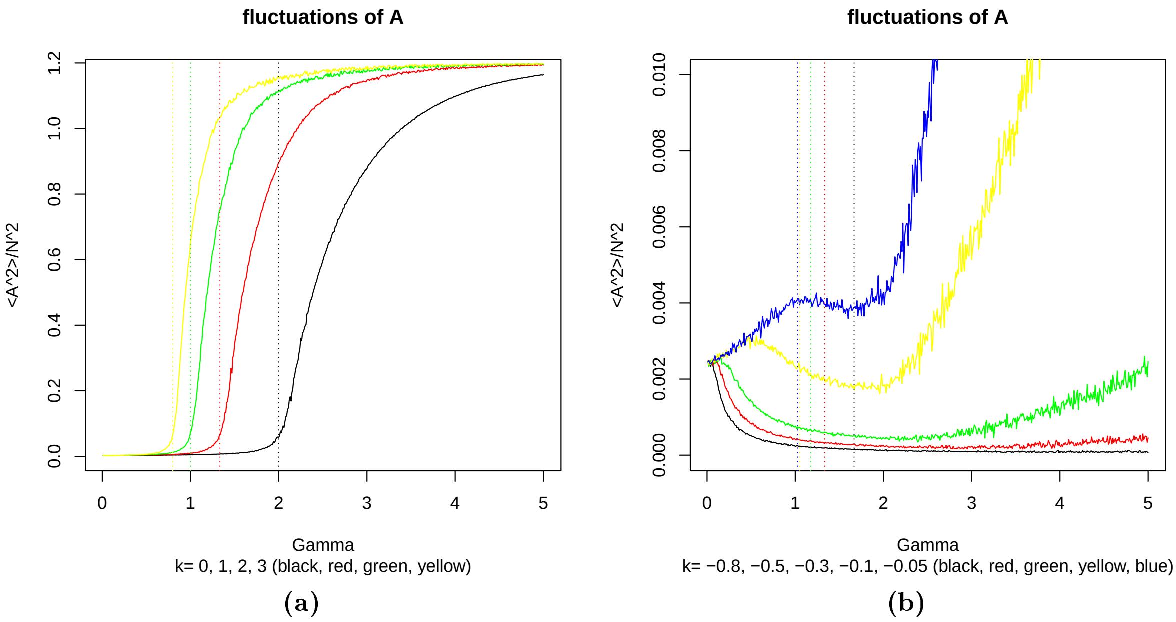


Figure 3.9: The dependence of the fluctuation $\langle A^2 \rangle / N^2$ with respect to Γ and k. Dotted lines identify the theoretical value of Γ_c as calculated in the limit $N \to \infty$.

Chapter 4

Conclusion and perspectives

Adding to the basic model the splitting between gains and losses we have moved the critical value of the learning rate. If we consider the case $k > 0$ no differences arise between this variation and the previous standard model. For the case $k < 0$ a different behaviour occurs. Anyway, as already written, this new case is not particularly revealing because it is a case in which the punishment for the winner is greater than the one for the loser. A second step may be done by considering different initial conditions, e.g., using $\Delta_i(0)$ normally distributed and focusing on how the variance affects the behavior of $A(t)$.

About further developments one can strengthen the role of the gains/losses splitting by differentiating the score through $\Delta_i^+(t)$ and $\Delta_i^-(t)$ and adding two more memory variables, i.e., ρ_+ and ρ_-, introducing the model

$$\begin{cases} \Delta_i^+(t)(1-\rho^+) - \dfrac{A(t)}{N}\dfrac{1-a_i(t)\,\mathrm{sgn}(A(t))}{2} & (4.1) \\ \Delta_i^-(t)(1-\rho^-) - k\dfrac{A(t)}{N}\dfrac{1+a_i(t)\,\mathrm{sgn}(A(t))}{2} & (4.2) \end{cases}$$

where the total score for each player is given by the sum of $\Delta_i^+(t)$ and $\Delta_i^-(t)$ and the probability distribution follows the usual law. ρ^+ and ρ^- determine how the new score is related to the previous one, so that we expect them to influence the approach to a possible equilibrium of the system.

This new model splits two cases depending on whether one wins

$$\begin{cases} \Delta_i^+(t)(1-\rho^+) - \dfrac{A(t)}{N} & (4.3) \\ \Delta_i^-(t)(1-\rho^-) & (4.4) \end{cases}$$

or loses

$$\begin{cases} \Delta_i^+(t)(1-\rho^+) & (4.5) \\ \Delta_i^-(t)(1-\rho^-) - k\dfrac{A(t)}{N} & (4.6) \end{cases}$$

The special case $\rho^+ = \rho^-$ recovers a model which is very similar to the one discussed in chapter 3.

Bibliography

[1] Erwin Scroedinger: *What is Life? The Physical Aspect of the Living Cell.* Cambridge University Press, 1946.

[2] Sitabhra Sinha, Arnab Chatterjee, Anirban Chakraborti, and Bikas Chakrabarti: *Econophysics. An introduction.* John Wiley & Sons, 2011.

[3] editorial staff: *Game theory in practice.* The Economist, Sept 2011.

[4] L. W. Buss: *The Evolution of Individuality.* Princeton University Press, 1987.

[5] W. Fontana and L.W. Buss: *The arrival of the fittest: Toward a theory of biological organization.* Bull. Math. Biol., 56, Sept 1994.

[6] John Von Neumann and Oskar Morgenstern: *Theory of Games and Economic Behaviour.* Princeton University Press, 1944.

[7] Daniel Kahneman and Amos Tversky: *Prospect theory: An analysis of decision under risk.* Econometrica, 47(2):263–292, 1979.

[8] Damien Challet and Y C Zhang: *Emergence of cooperation and organization in an evolutionary game.* Physica A: Statistical Mechanics and its Applications, 246(3):407–418, 1997.

[9] W.B. Arthur: *Bounded rationality and inductive behavior. the el farol problem.* American Economic Review, 84:406–411, 1994.

[10] Christoph Hauert and György Szabó: *Game theory and physics.* American Journal of Physics, 73(5):405–414, 2005.

[11] Roger B. Myerson: *Game Theory. Analysis of Conflict.* Harvard University Press, 1991.

[12] Damien Challet, , M. Marsili, and Y C Zhang: *Game Theory. Analysis of Conflict.* Oxford University Press, 2005.

[13] Matteo Marsili, Damien Challet, and Riccardo Zecchina: *Exact solution of a modified el farol's bar problem: Efficiency and the role of market impact.* Physica A: Statistical Mechanics and its Applications, 280(3):522–553, 2000.

[14] Marko Sysi-Aho, Anirban Chakraborti, and Kimmo Kaski: *Adaptation using hybridized genetic crossover strategies.* Physica A: Statistical Mechanics and its Applications, 322:701–709, 2003.

[15] Marko Sysi-Aho, Anirban Chakraborti, and Kimmo Kaski: *Biology helps you to win a game.* Physica Scripta, 2003(T106):32, 2003.

[16] Marko Sysi-Aho, Anirban Chakraborti, and Kimmo Kaski: *Intelligent minority game with genetic crossover strategies.* The European Physical Journal B-Condensed Matter and Complex Systems, 34(3):373–377, 2003.

[17] Marko Sysi-Aho, Anirban Chakraborti, and Kimmo Kaski: *Searching for good strategies in adaptive minority games.* Physical Review E, 69(3):036125, 2004.

[18] Matteo Marsili: *Market mechanism and expectations in minority and majority games.* Physica A, 299(1-2):93–103, 2001.

[19] Eugene Galanter and Patricia Pliner: *Cross-modality matching of money against other continua.* In HowardR. Moskowitz, Bertram Scharf, and JosephC. Stevens (editors): *Sensation and Measurement*, pages 65–76. Springer Netherlands, 1974, ISBN 978-94-010-2247-7. http://dx.doi.org/10.1007/978-94-010-2245-3_6.