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Consistent truncations with reduced supersymmetry of D=11 supergravity through exceptional field theory

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Abstract

In this thesis we will examine some aspects of the theory of supergravity, which is the low energy limit of the most widely accepted candidate to solve the tension between quantum mechanics and general relativity, namely string theory. Supergravity is the rewriting of general relativity coupled to matter fields in a supersymmetric invariant way, where supersymmetry acts as local transformations exchanging, broadly speaking, bosons and fermions, among the field content of the theory. Supergravity inherits from string theory the gauge/gravity holographic correspondence, which is the existence of a dictionary between elements of quantum gravity, such as fields and their masses, and quantities of a conformal field theory in one less dimension, such as operators and scaling dimensions. The thesis will be concerned with the particular instance of $\mathrm{AdS}_4 \times S^7 \simeq \mathrm{ABJM}$, which is the correspondence between solutions of the spacetime metric in a supergravity theory, tending asymptotically to a four dimensional anti de Sitter spacetime times a seven dimensional sphere, and a conformal field theory with Chern-Simons terms and matter couplings. Studying new solutions on the l.h.s. will give new insights on the conformal field theory, even on its non-perturbative dynamics, as the correspondence associates a weakly coupled theory to a strongly coupled one. In order to make the l.h.s. more manageable, we will study consistent truncations of the supergravity theory on S^7, which means selecting a subset of fields such that their dynamics is embedded consistently in the original theory's. In order to perform such consistent truncation we will employ the toolkit of exceptional field theory, which allows to study the metric and the other fields of the supergravity theory in a unified way. Furthermore, we will study one particular instance of supergravity theory in AdS_4 which is known to exist, with a peculiar choice of gauge group (being a subgroup of the isometry group of the theory's scalar manifold) given by $\mathrm{SO}(6) \times \mathrm{SO}(2)$ and coupling constants $g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)} \equiv \rho \neq 1$, and research whether it can be reached by consistent truncation of 11d supergravity on an aptly chosen internal space. The choice of such gauge group purposefully aims to exploit the local isomorphism between S^7 and the product of S^1 with the complex projective space of complex dimension 3, namely $S^7 \simeq \mathbb{CP}^3 \times S^1$, as to associate ρ to a deformation of the radius of S^1.

We will provide original results regarding the structures that allow to truncate from the 11d supergravity to 4d supergravity, with supersymmetry $\mathcal{N} = 2$ and gauge group $\mathrm{SO}(6) \times \mathrm{SO}(2)$, as well as elaborate on some further steps one could take to study the relation between deformation of the S^1 radius and upliftability of a 4d gauged supergravity theory, which in general will be different from the one we started with.

Contents

Chapter 1

Introduction

Quantum field theories are a central concept in modern high energy physics. Indeed, the standard model of particles is formulated in terms of quantum fields and has yielded spectacularly precise experimental results. However, there exists a tension between quantum mechanics and general relativity, the first one describing particles at a microscopic level, the latter studying spacetime and its curvature at large scales. General relativity itself is an effective theory, meaning the low energy limit of a more fundamental theory, in the same way as newtonian gravity is an effective theory of general relativity itself. Such more fundamental theory is called the ultraviolet completion of general relativity, meaning the completion at higher (or ideally infinite) energy. The search for a formulation of such ultraviolet completion, formulated in terms of quantised fields, has attracted a huge effort in the last decades but is still an open question. Nevertheless, the more plausible candidate has emerged in the form of string theory. String theory proposes that at the fundamental level there exist open and closed strings, whose vibrational modes give rise to particles and their features. In order to ensure mathematical consistency of the theory, these strings inhabit a higher dimensional space. The maximum number of dimensions in which one can formulate supersymmetric string theory is 10d, compatibly with mathematical consistency. In 11d dimensions, the existence of a quantum gravity theory of membranes, called M-theory, is widely suspected although it has not been formulated yet. Supersymmetry is a global symmetry group that a consistent quantum field theory may admit, as we are going to explain further in the next chapter of the thesis, being the maximal extension of the Poincaré group (up to internal symmetries). Supersymmetry allows for a greater control over the theory, as indeed only some couplings are allowed, making the theory more manageable. We will see, in the next chapter, that the more amount of supersymmetry is present in the theory, the fewer the arbitrariness.

The requirement of 11 dimensions of M theory may seem in contradiction with the ordinary four dimensions; the remaining seven dimensions are hidden from our experiments as they might be curled up at still undetectably small scales. Technically, the remaining dimensions are compactified on a internal space, whose geometrical features will leave trace on the external, lower dimensional theory. Through compactification, one is able to formulate string theory in dimensions lower than 11 as well.

Among the many areas studied by string theory, there exists a very active area of research known originally as AdS/CFT correspondence, meaning the correspondence between string theory solutions which tend asymptotically to an Anti de Sitter spacetime, and a conformal field theory, where typically the correspondence exchanges strong and weak coupling, broadly speaking. Anti de Sitter spacetime is the maximally symmetric solution of the Einstein's equation, describing the dynamics of the spacetime metric, with a negative cosmological constant. A de Sitter spacetime has on the other hand a positive cosmological constant, like our own universe. The conformal field theory on the right hand side of the correspondence is, broadly, a theory without a metric and with a symmetry group containing the Lorentz group. Conformal field theories are not a novel concept in modern physics, as they arise in statistical mechanics too for instance, and intuitively can be seen as theories without any dimensionful parameter (like classical electrodynamics).

The AdS/CFT correspondence has been extended to a large set of examples, and constitutes a framework, also known as gauge/gravity holographic correspondence, which has allowed to discover many features of both string theory and gauge theories. Indeed it provides a dictionary between elements, such as fields, of the quantum gravity theory to elements, such as operators, of the CFT. Furthermore, given the exchange of weak/strong coupling, the gauge/gravity correspondence allows to study non perturbative features of one side by means of a perturbative treatment of the other. The correspondence is called holographic because the conformal field theory lives in one less dimension than the quantum gravity theory; the standard references for the AdS/CFT correspondence are [1], [2]. For string theory in general we refer to [3], while for CFT's [4].

In the course of this thesis, we will not deal with string theory itself, but with its low energy limit, supergravity. Supergravity is the supersymmetric formulation of general relativity or, equivalently, the theory of gauged supersymmetry. In summary, a supergravity theory comprises of the metric and other fields of various spin, as we are going to explain in the next chapter, whose couplings to one another are built in order for the local, i.e. space-dependent, supersymmetric transformations to leave the action invariant.

In particular, we will study solutions of supergravity in 11 dimensions which are asymptotically equivalent to the product of AdS in four dimensions times a seven-sphere S^7. In the gauge/gravity holographic correspondence, such solutions are dual [5] to a 3d topological gauge theory called the ABJM model, being a supersymmetric conformal field theory, with gauge group $\mathrm{SU}(n)$ and coupled to matter fields. Features of the supergravity solutions will translate into features of the gauge theory; being 11d supergravity hard to manage computationally, one often performs consistent truncations thereof. A consistent truncation of a theory consists in the selection of a subset of fields of the original theory such the solution of their equations of motions are solutions of the equations of motion of the original theory as well. Hence in particular we can perform a consistent truncation on S^7, the seven dimensional sphere, which leaves us with a manageable theory, whose dynamics can be embedded in the higher dimensional theory's.

Consistent truncations of 11d supergravity on internal manifolds however may lead to gauged supergravity. Indeed, in any dimension $2 \leq d \leq 9$, supergravity with the maximal number of supersymmetry has a global symmetry group G, such that the scalar fields of the theory parameterise G/H, with H being the maximal compact subgroup of G. We will explain in the next chapter what we mean by maximal amount of supersymmetry and how it is related to the spacetime dimensions. A gauged supergravity is a supergravity theory with a gauged subgroup $G_0 \subset G$; thus the theory will need to be invariant with respect to local G_0 transformations. Gauged supergravity has desirable features, which the ungauged kind lacks. Indeed gauged supergravity admits a potential for the scalar fields, while ungauged supergravity (with $\mathcal{N} > 1$) does not. A scalar potential may lead to non trivial vacuum expectation values for the scalar fields; this leads to the scalar potential behaving as an effective cosmological constant. Such possibility would be interesting from the point of view of cosmology and inflation; furthermore, non trivial vevs of the scalars lead to scalar and fermions mass terms, from their coupling in the lagrangian. Moreover, a cosmological constant leads to the possibility of spontaneous supersymmetry breaking. We will indeed mention that a non vanishing energy of the ground state of a supersymmetric theory leads to spontaneous breaking of rigid supersymmetry. Spontaneous supersymmetry breaking may take place in ungauged supergravity too, however in that case there would not be an effective cosmological constant and the ground state would just describe Minkowski spacetime.

Spontaneous supersymmetry breaking is crucial to recover the known low energy particle spectrum; indeed we do not observe, below the current energy thresholds, the spectrum predicted by supersymmetry, an example of which is the supersymmetric bosonic partner of the electron, degenerate in mass. Hence, if supersymmetry and supergravity are to describe the universe, albeit in an effective way, they must be broken[1] at an energy scale above the scales we can probe today (namely 14 TeV at CERN, as of June 2023). We will not, however, deal with supersymmetry breaking, nor with phenomenological issues, as we are going to focus on aspects of the gauge/gravity correspondence that need supersymmetry in order to be defined.

[1]Supersymmetry breaking is not the only condition to ensure consistency with the detected particle spectrum, as the latter also comprises chiral interactions.

The gauging of the lower dimensional theory, after consistent truncation, depends on the geometry of the internal manifold and the fluxes of the higher dimensional fields along the internal manifold, where by fluxes we mean constant integrals of the fields strengths along non trivial cycles of the internal manifold. The primary focus of the thesis is to study whether a 4d gauged supergravity, with gauge group $\mathrm{SO}(6) \times \mathrm{SO}(2)$ and coupling constants $g_{\mathrm{SO}(6)} \neq g_{\mathrm{SO}(2)}$, with an amount of supersymmetry $\mathcal{N} = 2$, can be reached by consistent truncation of 11d supergravity on S^7. $\mathcal{N}$ quantifies the amount of supersymmetry of the theory, and varies from 1 to 8 in 4d. In particular we want to study whether the deformation parameters $\rho = g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)} \neq 1$ can be attributed to the geometry of the internal manifold.
In order to study the possibility of such consistent truncation, we will use the framework of exceptional field theory. Exceptional field theory describes 11d supergravity and 10d supergravity in a unified way, by requiring covariance with respect to G, where G is the global symmetry group of maximally supersymmetric supergravity in d dimensions. In order to ensure such covariance, exceptional field theory rests upon exceptional generalised geometry, in the same way general relativity is based upon the mathematics of Riemannian geometry. Exceptional generalised geometry is however a generalisation of Riemannian geometry, as it describes the spacetime metric and matter content in a unified way. In the course of this thesis we are going to explain in which sense 11d supergravity and its local symmetries are reproduced, by introducing concepts such as generalised diffeomorphisms and a generalised tangent bundle.
In particular, exceptional generalised geometry is purpusefully designed to make consistent truncations to d dimensions straightforward; the variety of ways we can perform such truncations has been expanded over time. Indeed, before exceptional field theory, consistent truncation were performed by using Riemannian geometry and labour intensive techniques, for example of 11d supergravity on S^7, leading to $\mathcal{N} = 8$ SO(8) gauged supergravity. However, by employing exceptional field theory, we will be able to consider a wide set of examples of consistent truncations of increasing generality. The first instance will be that of generalised group manifolds, i.e. internal manifolds that behave as a group manifolds in a generalised sense, as we are going to explain. In particular, we will explain that any S^n allows for a generalised consistent truncation thereupon, as long as there is a top-dimensional constant flux on the sphere. The lower dimensional theory turns out to be a maximally supersymmetric $\mathrm{SO}(n+1)$ gauged supergravity.
As we said, the aim of the thesis is to understand whether we can consistently truncate from 11d supergravity to 4d $\mathcal{N} = 2$ $\mathrm{SO}(6) \times \mathrm{SO}(2)$, with coupling constants $g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)} \neq 1$. Indeed, we

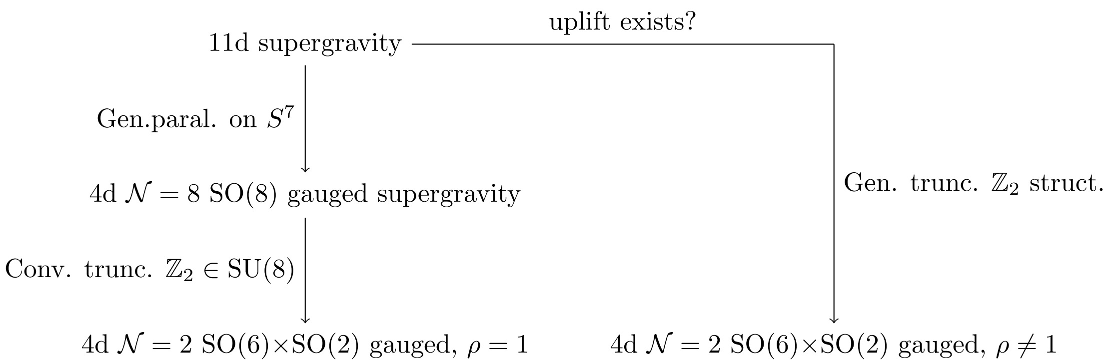

Figure 1.1: Outline of the thesis; $\rho = g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)}$.

can reach 4d $\mathcal{N} = 8$ SO(8) gauged supergravity by means of a generalised consistent truncation on S^7, to be precise a generalised parallelisation. Furthermore, we can reach from such theory a 4d $\mathcal{N} = 2$ theory, with gauge group $\mathrm{SO}(6) \times \mathrm{SO}(2)$, with $\rho \equiv g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)} = 1$. It is a known result that the same theory with $\rho \neq 1$ does exist, but it is unknown whether it admits an uplift to 11d supergravity. In this thesis, in particular in chapter 6, we will present the necessary ingredients to perform a generalised consistent truncation from 11d supergravity to 4d $\mathcal{N} = 2$ $\mathrm{SO}(6) \times \mathrm{SO}(2)$

and ask ourselves to which extent we can deform ρ, while keeping those necessary ingredients well defined. Namely, these ingredients will be a so called HV structure (hypermultiplet-vector multiplet structure), which we will build out of (an instance of) the generalised parallelising frame on S^7, which leads to 4d SO(8) $\mathcal{N} = 8$ supergravity.

What we found, is that the HV structure is well defined for a discrete subset of values of ρ. In practice, this defines an orbifold:

$$\frac{S^7}{\mathbb{Z}_4} \simeq \mathbb{CP}^3 \times \frac{S^1}{\mathbb{Z}_4} , \tag{1.1}$$

however we cannot conclude that the truncation of 11d supergravity on such orbifold leads to the $\rho \neq 1$ theories. Indeed, we could not conclusively attribute the deformation ρ to a deformation of the 11d metric. Our starting hypothesis was that such deformation parameter could be associated to the radius of S^1 in 1.1, i.e. to the vacuum expectation value of a modulus that gets truncated away from 11d supergravity to 4d. However, as we will explain in the last chapter, S^1 is non trivially patched along S^7, whence the deformation parameter ρ does not translate naively into a rescaling of the S^1 radius. In the end of the thesis, we suggest some next steps that might provide a conclusive answer to whether the $\rho \neq 1$ theories admit an uplift to 11d supergravity.

The structure of the thesis is quite straightforward: in the first three chapters we provide the necessary ingredients to the comprehension of the problem. We introduce the basic ideas of supergravity, its description in terms of exceptional field theory and the geometry underlying such exceptional generalised construction. In the fifth chapter, we deal with consistent truncations, which will be explained in order of increasing generality. In the last chapter, we will use all the previous elements to provide some answer to the aforementioned questions.

Chapter 2

Fundamentals of Supergravity

In this chapter we will lay out the basic elements of supersymmetry and supergravity. We will explain how supersymmetry fits among the symmetries of quantum field theories and we will introduce the super-Poincaré algebra. After mentioning the spinorial representation of the Poincaré algebra, and in particular the gravitino, we will use the so called Noether procedure to build the simplest supergravity action. Furthermore we will explain how different dimensions support supersymmetric and supergravity theories, and ultimately we will describe their basic features, such as p-forms duality and scalar geometry. This is not a pedagogical introduction, but aims to provide a logically consistent framework for the following chapters of this thesis. There are many valuable books on the subject, among which the standard references: [6], [7] and, more focused on supergravity, [8] and [9].

Supersymmetric algebra A famous theorem by Sidney Coleman and Jeffrey Mandula [10] states that the maximal Lie algebra of symmetries of the S matrix of a local, interacting, relativistic quantum field theory in dimension greater than 1+1, with an energy gap, is the direct sum of the Poincarè algebra and a compact Lie algebra of internal symmetries[1], generated by Lorentz invariant charges. There are two loopholes to the previous no-go theorem, either to consider theories without an energy gap, which leads to conformal symmetry, or to consider *graded* Lie algebras, leading to supersymmetry. Indeed a theorem by Haag, Lopuskanski and Sohnius [11] builds on the previous result and states that the maximal graded Lie algebra of symmetries of the S matrix of the abovementioned theory, is the supersymmetric extension of the Poincaré algebra (thus called super-Poincaré algebra) with some possible additional Lorentz invariant generators of internal symmetries. The super-Poincaré algebra is a $\mathbb{Z}_2$-graded Lie algebra, as it comprises two kind of generators: bosonic and fermionic, algebraically related to one another through commutation or anticommutation relations.
The bosonic generators are[2]: P^μ generator of spacetime translations and $M^{\mu\nu}$ generator of the Lorentz transformations. Their algebraic relations are:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \ , \\ [M^{\mu\nu}, P^\sigma] &= i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}) \ , \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma}) \ . \end{aligned}$$

The fermionic generators, constituting the abovementioned maximal extension of the Poincarè algebra, are a number $\mathcal{N}$ of Weyl spinors Q^I_α and their hermitian conjugates $\bar{Q}^{\dot{\alpha}\, I}$, where $I = 1 \dots \mathcal{N}$. These supercharges generate supersymmetric transformations of fields, transforming bosons into fermsions and viceversa; their spinorial representation depends on the dimension of spacetime, as we are going to explain in the following paragraph. They do not need to be always Weyl fermions and indeed, for odd spacetime dimensions, chirality is not defined in the first place. In 3+1 dimensions, supercharges can be equivalently represented as Majorana spinors, which is somewhat more common

[1] The full statement and proof can be found in [7].

[2] We define both of them to be Hermitian. The anti-hermitian definition just differs by the multiplication with $-i$.

in supergravity books.
The algebraic relations between the supercharges, in the Weyl representation, is:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu \ . \tag{2.1}$$

It is sometimes said that this relation can be seen as the supersymmetric transformations being the square root of translations. As a simple application, from 2.1 it can be proved that the energy of any supersymmmetric state is always non-negative, and that the energy of the ground state is positive if and only if supersymmetry is broken.
Moreover, 2.1 suggests how supergravity comes about. Indeed supergravity can be seen equivalently as the supersymmetric extension of general relativity or as the gauge theory of supersymmetry, which necessarily includes gauged diffeomorphisms. The latter description can be understood by taking the supercharges to be space dependent, which implies in 2.1 that the momentum is space dependent as well, thus generating local coordinate transformations.
There are two more aspects of the supersymmetric algebra to explain. First of all, the supercharges have the following algebraic relations as well:

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ}$$
$$\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}}Z^{IJ} \ .$$

The charges Z^{IJ}, satisfying $Z^{IJ} = -Z^{JI}$, are called central charges and commute with all generators of the super-Poincaré algebra. As shown in [6], the central charges are related to the internal, scalar, compact symmetries allowed by the Coleman Mandula theorem.
The second aspect to point out is that, embedded in this algebra of internal symmetries, there are also generators that do not commute with the supercharges:

$$[T_A, Q^I_\alpha] = (U_A)^I{}_J Q^J_\alpha \ ,$$

while commuting with all other generators. These generators are said to form the R-symmetry, which is the largest subgroup of the group of automorphisms of the super-Poincaré algebra, commuting with the Poincaré algebra. In four dimensions and with $\mathcal{N}$ supercharges, the R-symmetry group is U($\mathcal{N}$); as we will mention in the following paragraph, the explicit R-symmetry group depends on the representations of supercharges which in turn depends on the spacetime dimensions.
The algebraic relations between the bosonic and fermionic generators are (considering only the nontrivial case):

$$[P^\mu, Q^I_\alpha] = [P^\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \ ,$$
$$[M^{\mu\nu}, Q^I_\alpha] = (\sigma^{\mu\nu})_\alpha{}^\beta Q^I_\beta \ ,$$
$$[M^{\mu\nu}, \bar{Q}^{\dot{\alpha}\ I}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}\ I} \ .$$

The first equation tells us that, in any representation of the supersymmetric algebra on the Hilbert space, all of its states are degenerate in energy. The second and third equations just tell us how the supercharges transform under Lorentz transformations, with $\sigma^{\mu\nu}$ and its hermitian conjugate $\bar{\sigma}^{\mu\nu}$ being spinorial representations of the Lorentz generators. They are defined as:

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \ .$$

Since the super-Poincaré algebra is a Lie algebra, it is also endowned with super-Jacobi identities, which we will not deal with here but are treated in appendix B of [9].
There are a few more comments in order. We wrote the super-Poincaré algebra in 3+1 spacetime dimensions, in a chiral representation of the supercharges. Of course, a different spacetime dimension would imply a different representation of the algebra generators. Secondly, the Lorentz group is the isometry group of Minkowski spacetime; a different spacetime, such as de Sitter or Anti de Sitter,

has a different isometry group. In particular, the algebra of symmetries of (A)dS differs with respect to the Poincaré algebra only in one relation:

$$[P_\mu, P_\nu] = \pm\frac{1}{l^2}M_{\mu\nu} \ ,$$

where the upper sign is for AdS, the lower for dS, and l is their radius of curvature, as indeed in the limit $l \to \infty$, we recover the Poincaré case. It is interesting to notice that the De Sitter spacetime, that is the maximally symmetric spacetime with positive cosmological constant, does not admit a consistent supersymmetric extension of the isometry group, as computed in [8]. In other words, a positive cosmological constant breaks supersymmetry; on the other hand, AdS does admit a supersymmetric completion.

Spinorial representations of the Lorentz group The Lorentz group SO(1, 3) does not admit spinorial representations, however its universal covering group Spin(1, 3), being simply connected, does. Its representations are indexed by a pair of real integer or half integer numbers (j_1, j_2). These two numbers classify the representations of the two $\mathfrak{su}(2)$ algebras isomorphic to the universal covering group:

$$\mathfrak{spin}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}^*(2) \ ,$$

where the complex conjugation is a consequence of the construction of $\mathfrak{su}(2)$ by complexification of the generators of $\mathfrak{so}(1, 3)$, as explained in [12]. In practice, it implies that the complex conjugate of the representation (j_1, j_2) is the same representation with exchanged indices:

$$(j_1, j_2)^* = (j_2, j_1) \ .$$

The spin associated to the particle corresponding to a certain Spin(1, 3) representation is given by $s = j_1 + j_2$; the dimension of the representation is $(2j_1 + 1)(2j_2 + 1)$. The trivial representation is the scalar field (0, 0), then the right chiral fermion (1/2, 0), the left chiral fermion (0, 1/2), the vector field (1/2, 1/2) and so on. Particularly important in the following will be the $s = 3/2$ representation, commonly known as Rarita Schwinger field, or gravitino.
Spinorial representation of the (universal covering group of the) Lorentz group can be found explicitly by looking at the irreducible representations of the Clifford algebra; whose elements satisfy:

$$\{\gamma_a, \gamma_b\} = \eta_{ab} \ , \tag{2.2}$$

with $\eta_{ab} = \mathrm{diag}(-1, +1, \cdots + 1)$, in D dimensions. To highlight the connection, we can rewrite the elements of the Lorentz algebra $M_{\mu\nu}$, in terms of the elements of Clifford's as:

$$M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \ ,$$

and it can be easily showns that 2.2 implies the Lorentz algebra. However, irreducible representations of the Clifford algebra may not be irreducible for Lorentz: indeed, already in four dimensions, the Dirac representations of the Clifford algebra can be decomposed in terms of Weyl fermions.
A priori, in D dimensions, a Clifford irrep has $2^{\mathrm{Int}[D/2]}$ components. The symbol Int stands for the integer part of its argument; hence in $D = 2m$ or $D = 2m + 1$ a Dirac spinor has the same number of components, however there is a most important caveat. Indeed a Dirac spinor can be decomposed in different ways, according to the spacetime dimensions. There are four different possibilities:

- In even dimensions, it can be shown that all non-trivial representations of the Clifford algebra are equivalent, that is, are mapped to one another through a similiarity transformation. In even dimensions, from a Dirac spinor we can define two Weyl spinors, each enconding half of the original degrees of freedom. Indeed we can build, from the elements of the Clifford algebra (also called gamma matrices) an element Γ^* commuting with all the other matrices, and squaring to 1. Such Γ^* allows to define left and right chiral projectors:

$$P_{\mathrm{L/R}} = \frac{1}{2}(1 \pm \Gamma*) \ ,$$

which single out the chirality of the spinor. Hence chirality can be defined in even dimensions only, and it is a Lorentz covariant condition since $[M_{\mu\nu}, \Gamma^*] = 0$.

- In odd dimensions, there exist two classes of inequivalent representations of the algebra. To build spinorial irreps of the Lorentz group, it is sometimes possible to define Majorana spinors, through a reality condition:
$$\psi^* \overset{!}{=} B\psi \ ,$$
or equivalently, using the charge conjugation matrix $\mathcal{C}$:
$$\psi^c = \mathcal{C}\bar{\psi}^T \overset{!}{=} \psi \ ,$$
where the matrix B must square to the identity, for consistency, and must allow Lorentz covariance of the condition. The latter requirement entails that not all odd dimensions allow for Majorana spinors. Furthermore, a Majorana spinor has only half of the degrees of freedom of a complex Dirac spinor.
In some cases, a reality condition can only be imposed using pairs of Dirac spinors and a real matrix Ω such that $\Omega^2 = -\mathbb{1}$. When this possibility arises, one speaks of symplectic Majorana spinors, encoding the same number of degrees of freedom of a single Dirac spinor.
In the rest of the thesis, unless stated otherwise, when considering spinors we will always consider Majorana spinors.

- One last possibility arises for some even dimensions, when the Majorana condition and the Weyl condition can be simultaneously imposed, producing a Majorana Weyl spinor, which encodes one quarter of the degrees of freedom of a Dirac spinor. This representations can be found in dimensions $D = 2$ mod 8. Furthermore, in dimensions $D = 6$ mod 8, one can define symplectic Majorana Weyl spinors.

We will not dwell on the details of each condition, which can be found for instance in [8], as our goal is to show the minimal amount of supercharges that each dimension can support. This equals the number of components of the minimal spinorial representation of the Lorentz group in such dimensions. In table 2.1, we list, for every D, the type of spinor, the number of components and the R-symmetry enjoyed by $\mathcal{N}$ extended supersymmetry. We will only consider $4 \leq D \leq 11$, as in 3 dimensions supergravity does not have any propagating degrees of freedom, while for $D \geq 11$, fields with helicity greater than 2 are necessarily involved, implying a non interacting theory in Minkowski spacetime. It should be noted, moreover, that the abovementioned conditions only hold in Lorentzian signature; in a different signature, such as the Euclidean, different conditions will define the spinorial irreps of the Lorentz algebra.
The number of degrees of freedom can be found by:
$$k\ 2^{[D/2]-1} \ ,$$
with $k = 2$ for Dirac spinors, $k = 1$ for Majorana and Weyl, $k = 1/2$ for Majorana Weyl spinors.

Gravitino and the Noether procedure As we mentioned before, supergravity is the gauge theory of supersymmetry. In this paragraph, we will give provide some evidence to substantiate this claim. In order to do so, we need to introduce the spin 3/2 representation of the Poincaré algebra, which is the gravitino. This field will act as the gauge connection of local supersymmetry, analogally to the photon potential A_μ being the connection of electrodynamics and the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ being the connection for general relativity.
Although the irreducible representation of spin 3/2 would be the $(1, 1/2) \oplus (1/2, 1)$, it was shown by Fierz and Pauli, in [13], that no local, Lorentz covariant action can be written for such representations. However, a consistent action can be written in terms of a vector spinor $\psi_{\mu\alpha}$, which is a reducible representation of the Lorentz group that decomposes into a spin 3/2 representation and a spin 1/2 representation in the following way:
$$(1/2, 1/2) \otimes [(1/2, 0) + (0, 1/2)] = (1, 1/2) \oplus (1/2, 1) \oplus (1/2, 0) \oplus (0, 1/2) \ .$$

D	Minimal spinor	No. real components	R.symmetry group
4	M or W	4	$\mathrm{U}(\mathcal{N})$
5	SM	8	$\mathrm{Usp}(2\mathcal{N})$
6	SMW	8	$\mathrm{Usp}(2\mathcal{N}_L)\times\mathrm{Usp}(2\mathcal{N}_R)$
7	SM	16	$\mathrm{Usp}(2\mathcal{N})$
8	M or W	16	$\mathrm{U}(\mathcal{N})$
9	M	16	$\mathrm{SO}(\mathcal{N})$
10	MW	16	$\mathrm{SO}(\mathcal{N}_L)\times\mathrm{SO}(\mathcal{N}_R)$
11	M	32	$\mathrm{SO}(\mathcal{N})$

Table 2.1: Spinors in various dimensions

The consistency of the associated action (with suppressed spinor indices):

$$\mathrm{S} = \int d^4x \left(-\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \frac{1}{2}m\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \right)$$

was shown by W. Rarita e J. Schwinger in [14]. Such representation of the Poincarè algebra carries two on shell degrees of freedom when massless, while four in the massive case. In the massless case, it can be proved that the action is invariant under $\delta\psi_\mu = \partial\Lambda$ with Λ spinor with adequate mass dimensions. This transformation rule is typical of infinitesimal transformations of gauge connections in Yang Mills gauge theories; we will now show, through the so called Noether procedure, that in a very simple theory with a scalar and a majorana fermion, making the supersymmetric transformations space dependent will lead to the necessary introduction of a metric and a gravitino, with the latter being supersymmetric partner of the metric and acting as gauge connection. This procedure closely follows [8]; let us consider the lagrangian density of a complex scalar ϕ and a Majorana spinor χ:

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi^* - (\bar{\chi}_R\not{\partial}\chi_L + \bar{\chi}_L\not{\partial}\chi_R)\ ,$$

and the supersymmetric transformations of the fields:

$$\delta\phi = \bar{\epsilon}_L\chi_L$$
$$\delta\chi_L = \frac{1}{2}\not{\partial}\phi\epsilon_R$$

and their complex conjugates. The Lagrangian variation is:

$$\begin{aligned}\delta\mathcal{L} &= -\partial_\mu(\delta\phi)\partial^\mu\phi^* - \delta\bar{\chi}_R\not{\partial}\chi_L + \partial_\mu(\delta\bar{\chi}_R)\gamma^\mu\chi_L + \text{h.c.}\\ &= -\partial_\mu(\bar{\epsilon}_L)\not{\partial}\phi^*\gamma^\mu\chi_L + \partial_\mu\mathcal{K}^\mu + \text{h.c.}\\ &= (\partial_\mu\bar{\epsilon}_L)j^\mu_L + (\partial_\mu\bar{\epsilon}_R)j^\mu_R + \partial_\mu\mathcal{K}^\mu + \text{h.c.}\ ,\end{aligned}$$

where, in the intermediate passages, an integration by parts has been performed and $\mathcal{K}^\mu$ has been defined as:

$$\mathcal{K}^\mu = -\delta\phi\partial^\mu\phi^* - \delta\bar{\chi}_R\gamma^\mu\chi_L\ ,$$

and the currents:

$$j^\mu_L \equiv -\not{\partial}\phi^*\gamma^\mu\chi_L \qquad j^\mu_R \equiv -\not{\partial}\phi^*\gamma^\mu\chi_R\ .$$

We can see that, for constant transformation parameter ϵ, the lagrangian density is invariant up to total derivatives, as expected. The current $j^\mu = j^\mu_L + j^\mu_R$ is conserved upon using the equation of motion of ψ and χ; taking $\epsilon = \epsilon(x)$, requires the introduction of a new field whose supersymmetric transformation cancels out $\delta_\epsilon\mathcal{L}$. The corresponding additional lagrangian is:

$$\mathcal{L}' = -\frac{1}{M_p}(\bar{\psi}_{\mu L}j^\mu_L + \bar{\psi}_{\mu R}j^\mu_R)\ ,$$

with:

$$\delta_\epsilon \psi_{\mu\ L,R} = M_p \partial_\mu \epsilon_{L,R} \ , \tag{2.3}$$

where the Planck mass M_p enforces the correct mass dimension of the gravitino, which is 3/2 in natural units, and ensures that the Lagrangian density has mass dimension 4. The additional $\mathcal{L}'$ contributes however to a new piece in $\delta_\epsilon \mathcal{L}$, namely the piece proportional to the variation of the currents. It can be shown that such variations can be split into two parts: a part bilinear in ϕ and another bilinear in χ. Both of these quantities can be rewritten as:

$$\delta \mathcal{L}' \supset \frac{1}{M_p} \bar{\epsilon} \gamma_\nu \psi_\mu T^{\mu\nu} + \dots \ ,$$

where $T^{\mu\nu}$ is the energy momentum tensor of ϕ and χ. To compensate for this piece, one needs to introduce a new term in the lagrangian:

$$\mathcal{L}'' \sim -g_{\mu\nu} T^{\mu\nu} \ ,$$

where $g_{\mu\nu}$ is the spacetime metric, which must transforms as:

$$\delta_\epsilon g_{\mu\nu} \sim \frac{1}{M_p} \epsilon \gamma_{(\mu} \psi_{\nu)} \ .$$

The latter transformation rule supports the idea that the gravitino is the supersymmetric partner of the graviton, while 2.3 shows ψ_μ must transform as the gauge connection, with the derivative of the transformation parameter, to ensure invariance of the action. The full action can found by considering kinetic terms for $g_{\mu\nu}$ and ψ_μ as well.
Although such action was found starting from what is called a Wess Zumino model, the action of pure supergravity should exist independently of the matter content, and thus involves only the kinetic terms of gravity and the gravitino (with appropriate spacetime covariantization) and possibly interactions among $g_{\mu\nu}$ and ψ_μ.

Representation of super-Poincaré algebra Unlike Poincaré's, the representations of the super-Poincaré algebra are not labeled by mass and spin. Indeed one cannot simply use the Casimir operator of the Poincaré group $W^\mu W_\mu$, built from the Pauli-Lubański vector, because $[W_\mu, Q_\alpha] \neq 0$ as a consequence of $[M_{\mu\nu}, Q_\alpha] \neq 0$. In practice this means that state belonging to the same representation will have different helicity (or spin); in other words representations of the super-Poincaré algebra are built in terms of representations (scalars, Weyl fermions, vectors...) of the Poincarè algebra. This is not a surprise, since the former is an extension of the latter.
Each supermultiplet has the same number of on shell bosonic and fermionic states, unless supersymmetry happens to be broken. Supersymmetry breaking is however a topic we will not touch upon.
We will only consider massless representations in $3+1$ dimensions, however we will not build such representations in each case, but only mention the results. The idea behind their construction is that, starting from a maximum helicity state $h_{\max}$, we can use some of the supercharges as creation and annihilation operators, allowing to find states of lower helicity. The amount of supercharges one can use depends on whether the representations are massive or massless, whether supersymmetry is extended and whether central charges are present.
For $\mathcal{N} = 1$ the supermultiplets (including only fields with helicity $|h| \leq 2$) are:

- $|h_{\max}| \leq 1/2$; Chiral multiplet (complex scalar, Weyl fermion);
- $|h_{\max}| \leq 1$; Vector multiplet (vector field, Weyl fermion);
- $|h_{\max}| \leq 3/2$; Gravitino multiplet (gravitino, vector field);
- $|h_{\max}| \leq 2$; Supergravity multiplet (graviton, gravitino);

Each supermultiplet is made of four states, that is two helicity states for each field; in general, in extended supersymmetry, massless multiplets are made of $2^{\mathcal{N}+1}$ states.
Theories with rigid extended supersymmetry are special cases of theories with rigid $\mathcal{N} = 1$, that is,

the $\mathcal{N} \geq 1$ supermultiplets can be recast in terms of those with $\mathcal{N} = 1$. Thus rigid $\mathcal{N} = 1$ constitutes a more general case, while extended supersymmetry theories have couplings and parameters that sit in the subspaces of parameter space that, while satisfying $\mathcal{N} = 1$, also allow for further supercharges. The same cannot be said for local supersymmetry, or supergravity, as in this case non linear and non renormalizable couplings are introduced, as it will be explained in further paragraphs. Therefore, in general, the parameters of local $\mathcal{N} \geq 1$ satisfy deformed requirements with respect to those of $\mathcal{N} = 1$.
Let us now consider massless $\mathcal{N} = 2$ supermultiplets:

- Hypermultiplet (made of two $\mathcal{N} = 1$ chiral multiplets: 2 complex scalars, two Weyl fermions);
- Vector multiplet (made of $\mathcal{N} = 1$ vector and chiral multiplets; one vector field, two Weyl fermions, one complex scalar);
- Supergravity multiplet (made of $\mathcal{N} = 1$ supergravity and gravitino multiplets: one graviton, two gravitino, one vector field);

Increasing the number of supercharges leads to a decrease in the number of possible representations one can build respecting $|h| \leq 2$. For $\mathcal{N} = 4$ we only have:

- A matter multiplet, built out of two $\mathcal{N} = 2$ vector and hypermultiplets, made of: one vector field, four Weyl fermions, three complex scalars;
- A gravity multiplet, built out of two $\mathcal{N} = 2$ supergravity and vector multiplets, made of: one graviton, two gravitinos, two vector fields, two Weyl fermions, one complex scalar.

For $\mathcal{N} > 4$ one cannot have "matter multiplets", as any multiplet will also include the graviton and gravitino. In $\mathcal{N} = 8$ case, we only have one possible representation, made of 1 graviton, 8 gravitinos, 28 vector fields, 56 Weyl fermions and 70 complex scalars.
The reason why we did not consider, for example, $\mathcal{N} = 3$ is that it can be proved that in rigid supersymmetry such theory has the same matter content as $\mathcal{N} = 4$, by CPT conjugation, as explained in [12]. On the other hand, local $\mathcal{N} = 3$ does exist in its own right, as it has a different supergravity multiplet than $\mathcal{N} = 4$'s. Similarly, $\mathcal{N} = 7$ can be proved to have the same matter content, by CPT conjugation, as $\mathcal{N} = 8$.

Supersymmetry and dimensions In this paragraph we will summarise which kind of supergravity or supersymmetry theories one can have in various dimensions, depending on the minimal spinor representation and the matter content. The entire paragraph is summarised in table 2.3, which is borrowed from many supergravity reviews (for example [9]).
An important distinction is that with 16 or less (real) supercharges, matter multiplets, i.e. multiplets without graviton or gravitino, exist on their own and can be used to formulate a rigid supersymmetric theory[3]. However, this does not imply that supergravity theory cannot exist with 16 or less supercharges, as one can couple to the matter multiplets the supergravity multiplet. Furthermore, for 8 or less supercharges, there exist chiral multiplets as well, i.e. multiplets that only contain complex scalars and spin 1/2 fermions.
As mentioned above, a supermultiplet should have equal numbers of bosonic and fermionic degrees of freedom. However, by simply looking at the degrees of freedom of, say, the graviton and the gravitino as a function of spacetime dimensions, we can see that such balance requires some adjustements. Indeed, the massless irreps of the Lorentz group in D dimensions are classified by SO(D-2), which is simply the extension of what is called *little group* in 3+1 dimensions. A graviton in D dimension has:

$$\frac{(D-2)(D-2+1)}{2} - 1$$

degrees of freedom, hence 2 in four dimensions and 5 in five dimensions. On the other hand, the gravitino has 8 degrees of freedom in five dimensions, as in general the gravitino has:

$$k\, 2^{[\mathrm{D}/2]-1}(D-2-1)$$

[3]with the exception of $\mathcal{N} = 3$, which only exists separately from $\mathcal{N} = 4$ for supergravity theories, as mentioned above.

degrees of freedom, coming from its spinor index and its vector index[4]. Therefore, some additional bosonic degrees of freedom are needed: this is the reason why p-forms are introduced, i.e. antisymmetric tensor fields. Indeed, a rank n p-form has:

$$\frac{(D-2)!}{(D-2-n)!n!}$$

degrees of freedom. In particular, in five dimension, a massless vector field A_μ, sometimes called the graviphoton, is needed to close the supergravity supermultiplet.
The highest dimensional theory, 11 supergravity, has the simple matter content of the metric, with vielbein e^a_μ (44 degrees of freedom), a gravitino ψ_μ (128) and a 3-form $C_{\mu\nu\rho}$ (86), enjoying an Abelian gauge symmetry:

$$\delta C_{\mu\nu\rho} = 3\partial_{[\mu}\Lambda_{\nu\rho]} \ .$$

Upon dimensional reduction, we obtain three different kinds of chiral supergravity theory, as one could expect in even dimensions. One can consider a theory with only one supercharge, obtaining type I supergravity, or with both supercharges. In the latter case, we can distinguish between two supercharges of opposite chirality (type IIA) or same (IIB). The first theory can be obtained by dimensional reduction from 11d supergravity, while the latter cannot. They share a sector of their field content: which is consistent on its own if one restricts to a single supercharge. However, their

{	Φ ,	λ^- ,	$B_{\mu\nu}$,	ψ^+_μ ,	$g_{\mu\nu}$	} Common sector
	1	8	28	56	35	$= 64_F + 64_B$
	{	λ^+ ,	$C_{\mu\nu\rho}$,	ψ^+_μ ,	A_μ	} Type IIA RR sector
		8	56	56	8	$= 64_F + 64_B$
{	C_0 ,	$\lambda^{-(2)}$,	$C_{\mu\nu}$,	$\psi^{+(2)}_\mu$,	$C_{\mu\nu\rho\sigma}$	} Type IIB RR sector
	1	8	28	56	35	$= 64_F + 64_B$

Table 2.2: Field content of type II supergravity

field content is completed by different Ramond-Ramond (RR) sectors.

[4]The factor $(D-2-1)$ also accounts for the auxiliary spin 1/2 spinor in the action of the gravitino, as explained in [8] and mentioned above

D	spinor	32	24	20	16	12	8	4
11	M	M						
10	MW	IIA / IIB			I			
9	M	$\mathcal{N}=2$			$\mathcal{N}=1$			
8	M or W	$\mathcal{N}=2$			$\mathcal{N}=1$			
7	SM	$\mathcal{N}=4$			$\mathcal{N}=2$			
6	SMW	(2,2) [or (3,1), (4,0)]	(2,1) [or (3,0)]		(1,1) [or (2,0)]		(1,0)	
5	SM	$\mathcal{N}=8$	$\mathcal{N}=6$		$\mathcal{N}=4$		$\mathcal{N}=2$	
4	M or W	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=5$	$\mathcal{N}=4$	$\mathcal{N}=3$	$\mathcal{N}=2$	$\mathcal{N}=1$
		sugra	sugra	sugra	sugra/susy	sugra	sugra/susy	sugra/susy

Table 2.3: Allowed supergravity and rigid supersymmetry theories in various dimensions and with various amounts of supercharges. The number on the top row are the real supercharges, while D stands for spacetime dimensions, with Lorentzian signature. It should be noted that the theories (4,0), (3,1), (3,0) are not based on a dynamical metric $g_{\mu\nu}$, but on a more complicated representation.

Hodge duality and electric magnetic duality As mentioned above, a massless p-form sits in a representation of the $\mathrm{SO}(D-2)$ subgroup of the Lorentz group, and therefore contains:

$$\binom{D-2}{p} \tag{2.4}$$

degrees of freedom. This can also be seen as a consequence of the Abelian gauge freedom of a p-form:

$$A_{(p)} \to A_{(p)} + \mathrm{d}\Lambda_{(p-1)} \ ,$$

which removes from the a priori $\dfrac{D!}{(D-n)!n!}$ components, the components of the gauge parameters. However, from the latter, only the effective gauge parameters must be considered; indeed parameters of the form $\Lambda_{(p-1)} = \mathrm{d}\Lambda_{(p-2)}$ are not fungible. Hence it can be proved that:

$$\binom{D}{n} - \binom{D}{n-1} + \binom{D}{n-2} - \cdots = \binom{D-1}{n} \ ,$$

which becomes 2.4 upon using the equations of motion of massless p-forms.
The counting of the on shell degrees of freedom of a p-form leads however to a natural consequence: a $(D-2-p)$-form, which we will call the *dual* form, encodes the same number of degrees of freedom, in D dimension, of a p-form. It turns out that the dual form actually encodes the *same* degrees of freedom of the p-form, in a theory with Abelian vector fields not minimally coupled to matter. Indeed, let us consider the equation of motion of the p-form:

$$d * F_{(p+1)} \quad \text{with} \quad F_{(p+1)} = \mathrm{d}A_{(p)}$$

and its Bianchi identity:

$$dF_{(p+1)} = 0 \ .$$

We could equivalently consider the $(D-p-1)$-form $G_{(D-p-1)}$ defined as the Hodge dual of $F_{(p+1)}$:

$$G_{(D-p-1)} \equiv *F_{(p+1)} \ ,$$

which therefore obeys a dual Bianchi identity:

$$\mathrm{d}G_{(D-p-1)} = 0 \ ,$$

implying that it can be locally expressed as:

$$G_{(D-p-1)} = \mathrm{d}A_{(D-p-2)} \ .$$

The equation of motion also holds:

$$\mathrm{d} * G_{(D-p-1)} = \mathrm{d} * *F_{(p+1)} = \ (-1)^c \,\mathrm{d}F_{(p+1)} = 0 \ ,$$

in absence of sources for the dual $A_{(D-p-2)}$. Therefore p-form $A_{(p)}$ and its dual $A_{(D-p-2)}$ encode the same degrees of freedom, as the equation of motions and Bianchi identity of one form automatically imply the equation of motion and Bianchi identity of the other.
Let us point out that the transformation that maps $A_{(p)}$ to its dual, while obeying e.o.m. and Bianchi identity, is non local. Indeed the following equivalences hold:

$$*\mathrm{d}A_{(p)} = *F = G = \mathrm{d}A_{(D-p-2)} \ ,$$

and by integration:

$$A_{(D-p-2)} = \int *\mathrm{d}A_{(p)}$$

which is manifestly a non local trasformation.
The previous argument can be extended to theories in 3+1 dimensions, with n_v Abelian vectors A^I

and some interacting fields ϕ (fermionic and/or bosonic), which may interact with the field strength $F^I_{\mu\nu}$, with $I = 1 \dots n_v$, but are not minimally coupled to A^I. The results which we would like to prove, which first appeared in [15], is that the group of duality symmetries that preserve the equation of motions and Bianchi identities is $\mathrm{Sp}(2n_v, \mathbb{R})$.
Let us consider the action of such theories:

$$e^{-1}\mathcal{L}(F^I, \phi, \partial_\mu\phi) = \frac{1}{4}\mathcal{F}_{IJ}F^I_\mu F^{J\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J\mu\nu} + + \frac{1}{2}\mathcal{O}^{\mu\nu}_I F^I_{\mu\nu} + e^{-1}\mathcal{L}_{\mathrm{rest}}(\phi, \partial_\mu\phi) \ , \tag{2.5}$$

where $\mathcal{O}^I_{\mu\nu}$ are operators built out of the ϕ and $\partial_\mu\phi$, while $\mathcal{F}_{IJ}$ is definite negative to ensure unitarity, $\mathcal{F}_{IJ}$ and $\mathcal{R}_{IJ}$ are symmetric and depend on the scalars only.
We defined the field strength as:

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \quad \text{or} \quad F^I = \mathrm{d}A^I \ ,$$

naturally obeying:

$$\mathrm{d}F^I = 0 \ .$$

We can define, as before:

$$\tilde{G}^I_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{I\rho\sigma} \equiv 2\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}_I} \ , \tag{2.6}$$

naturally obeying:

$$\mathrm{d}G_I = 0 \ .$$

It could appear that the system of equations of motion and Bianchi identity are invariant under general linear transformations like:

$$\mathbb{F}' = \mathrm{S}\mathbb{F} \quad \text{with} \quad \mathbb{F} = \begin{pmatrix} F^I \\ G_J \end{pmatrix} \quad \text{with} \quad \mathrm{S} \in \mathrm{GL}(2n_v, \mathbb{R}) \ ,$$

however requiring invariance of the definition 2.6 restricts S to a subgroup of $\mathrm{GL}(2n_v, \mathbb{R})$, as we are going to see.
First of all we construct the dual Lagrangian, that is the Legendre transform of Lagrangian 2.5 written in terms of the dual curvature:

$$e^{-1}\mathcal{L}_D = \left[e^{-1}\mathcal{L} - \frac{1}{2}F^I_{\mu\nu}G^{\mu\nu}_J\right]_{F=F(\tilde{G})} \ ,$$

where the second term is a total derivative[5]. The curvature F^I can be found by:

$$F^I_{\mu\nu} = -2\frac{\partial\mathcal{L}_D}{\partial\tilde{G}^{\mu\nu}_I} \ , \tag{2.7}$$

and we may even write $\mathcal{L}_D$ in terms of $\tilde{G}$, in the same form of 2.5 by linearity of the Legendre transformations:

$$\begin{aligned} e^{-1}\mathcal{L}_D = \frac{1}{4}\mathcal{A}^{IJ}\tilde{G}_{I\mu\nu}\tilde{G}^{\mu\nu}_J + \frac{1}{4}\mathcal{B}_{IJ}\tilde{G}^I_{\mu\nu}G^{J\mu\nu} - \\ - \frac{1}{2}\mathcal{O}^{I\mu\nu}\tilde{G}_{I\mu\nu} + e^{-1}\mathcal{L}'_{\mathrm{rest}}(\phi, \partial_\mu\phi) \ , \end{aligned}$$

where $\mathcal{A}_{IJ}, \mathcal{B}_{IJ}$ are symmetric matrices. Now that we have both the Lagrangian and its dual, we can require the consistency of the definitions 2.7 and 2.6, which will impose nontrivial constrains

[5] This terms is added to the original action as it allows, if we vary with respect to the dual potential, to find the Bianchi identity $\mathrm{d}F_I = 0$ and, if we vary with respect to the potential, to recover $G_I = \mathrm{d}A_I$. Hence this terms allows to write $\mathcal{L}_D$ with respect to the dual potentials A_I.

among the matrices that act as coefficients in the Lagrangians. Indeed we can compute $G^I_{\mu\nu}$, $\tilde{G}^I_{\mu\nu}$, $F^I_{\mu\nu}$, $\tilde{F}^I_{\mu\nu}$ using the definitions and we would find:

$$\begin{aligned}
\tilde{G}^I_{\mu\nu} &= \mathcal{F}^{IJ} F_{J\mu\nu} + \mathcal{R}^{IJ} \tilde{F}_{J\mu\nu} + \mathcal{O}^I_{\mu\nu} \ , \\
G^I_{\mu\nu} &= -\mathcal{F}^{IJ} \tilde{F}_{J\mu\nu} + \mathcal{R}^{IJ} F_{J\mu\nu} - \tilde{\mathcal{O}}^I_{\mu\nu} \ , \\
F^I_{\mu\nu} &= -\mathcal{A}^{IJ} \tilde{G}_{J\mu\nu} - \mathcal{B}^{IJ} G_{J\mu\nu} - \tilde{\mathcal{O}}^I_{\mu\nu} \ , \\
\tilde{F}^I_{\mu\nu} &= \mathcal{A}^{IJ} \tilde{G}_{J\mu\nu} + \mathcal{B}^{IJ} G_{J\mu\nu} - \tilde{\mathcal{O}}^I_{\mu\nu} \ .
\end{aligned}$$

Requiring consistency implies:

$$\begin{aligned}
\mathcal{A} &= -\left(\mathcal{F} + \mathcal{R}\mathcal{F}^{-1}\mathcal{R}\right)^{-1} \ , \\
\mathcal{B} &= \mathcal{A}\mathcal{R}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{R}\mathcal{A} \ , \\
\mathcal{O}^I &= \mathcal{A}^{IJ}\mathcal{O}_J - \mathcal{B}^{IJ}\tilde{\mathcal{O}}_J \ , \\
\mathcal{L}'_{\text{rest}} &= \mathcal{L}_{\text{rest}} + \frac{1}{4}\mathcal{O}^I\mathcal{O}_I \ .
\end{aligned}$$

We then perform a general linear transformation of the curvatures:

$$\begin{aligned}
\delta F^I_{\mu\nu} &= A^I{}_J F^J + B^{IJ} G_J \ , \\
\delta G_I &= C_{IJ} F^J + D_I{}^J G_J \ ,
\end{aligned} \tag{2.8}$$

with A, B, C, D real square matrices. Applying these transformations on the above relations among the curvatures, in particular to the first line, leads to:

$$\delta\tilde{G}_{I\mu\nu} = -\delta\mathcal{F}_{IJ} F^J_{\mu\nu} + \delta\mathcal{R}_{IJ}\tilde{F}^J_{\mu\nu} - \mathcal{F}_{IJ}\delta F^J_{\mu\nu} + \mathcal{R}_{IJ}\delta\tilde{F}^J_{\mu\nu} + \delta\mathcal{O}_{I\mu\nu} \ .$$

Substituting 2.8 and expressing G_I in terms of F_I or viceversa leads to transformation rules for the coefficients $\mathcal{F}_{IJ}, \mathcal{R}_{IJ}, \mathcal{O}_I$, in terms of A, B, C, D. Requiring that:

$$\begin{aligned}
\delta\mathcal{F}^T &= \delta\mathcal{F} \ , \\
\delta\mathcal{R}^T &= \delta\mathcal{R} \ ,
\end{aligned}$$

implies:

$$C = C^T \quad B = B^T \quad A = -D^T \ ,$$

or in other words:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(2n_v, \mathbb{R}) \ .$$

This goes to prove that the transformation:

$$\mathrm{S} = \exp\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n_v, \mathbb{R}) \ .$$

A few more comments are in order: first of all, given this symplectic duality transformations acting on field strenghts, we can define a symplectic invariant as:

$$\langle Y, Z\rangle := Y^T \Omega Z \ ,$$

where Y, Z are two vectors belonging two the $2n_v$ dimensional representation of $\mathrm{Sp}(2n_v, \mathbb{R})$, or more commonly, symplectic vectors.
Let us moreover point out that the duality transformations do not necessarily leave invariant the Lagrangian. Indeed it can be proved the the Lagrangian is preserved, up to total derivatives, only if the block $B = 0$ in S, which corresponds to perturbative transformations. The latter statement can be checked, as does [8], by further studying how the duality transformation affect the scalar matrices and operators.

2.1 Geometry of supergravity theories

Any supergravity theory will contain a supergravity multiplet, in some spacetime dimensions and with a certain number of supercharges, possibly interacting with some matter multiplets. In the course of this section we will try to describe the principal features of supergravity theories, as well as its matter couplings.

Freedom in the building of the theory Generally, the higher the number of supercharges, the less arbitrariness one has in building the supergravity (or supersymmetry) theory. Indeed, supergravity is uniquely defined in 11 dimensions, together with the aforementioned boundary on the helicity of fields $|h_{\text{max}}| \leq 2$. Lowering the number of supercharges and dimensions, one can have some freedom in the following aspects:

- Matter couplings. For $\mathcal{N} \leq 4$, matter multiplets exist by themselves and can be coupled to the theory; the choice of matter coupling does not affect the kinetic terms for $\mathcal{N} \leq 2$, but it does for $\mathcal{N} = 4$. For $\mathcal{N} > 4$, no matter couplings exist.
- Choice of kinetic terms. For $\mathcal{N} = 1$, scalar and gauge kinetic terms can be chosen independently, while for $\mathcal{N} = 2$ they are related to each other by supersymmetric transformations. For $\mathcal{N} \geq 4$, kinetic terms cannot be chosen, but depend univocally on the number of supercharges. For $\mathcal{N} = 1$, one can add a superpotential to the action, being a holomorphic function of the scalar fields.

We will now examine how the scalar geometry is affected by the number of supercharges.

Scalar geometry Scalar fields in supergravity theories appear in the action through their own kinetic term, which in general will be non minimal:

$$e^{-1}\mathcal{L}_{\text{sugra}} \supset \frac{1}{2} g_{ij}(\phi) D_\mu \phi^i D^\mu \phi^j ,$$

where i, j count the number of scalars of the theory, e is the vielbein's determinant. The scalars and the other fields of the theory will transform in a representation of a global symmetry of the theory, at least for $D \leq 9$, whose (semi)simple group will be denoted with G. The origin of this group is explained in section 2.2. If some subgroup of G is gauged, i.e. the transformations' parameters are made space dependent, then the derivatives D_μ will need to be covariant with respect to it.
Scalars field also appear in the kinetic term of Abelian vector fields:

$$e^{-1}\mathcal{L}_{\text{sugra}} \supset -\frac{1}{4}\text{Im}\big(\mathcal{N}(\phi)_{IJ}\big) F^I_{\mu\nu} F^{\mu\nu\, J} - \frac{1}{8}\text{Re}\big(\mathcal{N}_{IJ}\big) e^{-1} \epsilon^{\mu\nu\rho\sigma} F^I_{\mu\nu} F^J_{\rho\sigma} ,$$

such coupling requires as well interactions between fermions and vector fields due to the necessary supersymmetric invariance of the action. Scalar fields may also be present in the superpotential, which only exists for $\mathcal{N} = 1$ as mentioned before.
The set of the scalar fields' possible values draws out a manifold $\mathcal{M}$, aptly called scalar manifold. Some of the geometric features of this manifold are a direct consequence of the amount of supercharges; for $\mathcal{N} = 1$, $\mathcal{M}$ may be any Kähler (for rigid supersymmetry) or Kähler-Hodge (for supergravity) manifold, whose definition we are going to give briefly. The case $\mathcal{N} = 2$ is also special, as we will see. On the other hand, if more than eight real supercharges are present, then $\mathcal{M}$ is necessarily the symmetric Kähler space given by the coset space G/H, where G is the abovementioned Lie group and H is its maximal compact subgroup. In any case, the group of isometries G of the scalar manifold will be affected by the R-symmetry group acting on fermions and, for eight or more supercharges, G will need to be a subset of the duality group of the theory.
Let us examine in detail how the scalar manifold changes as a consequence of the supercharges:

- Global $\mathcal{N} = 1$. In this case $\mathcal{M}$ is a Kähler manifold. The idea behind such peculiar geometry is that such a manifold should be complex, allow for scalar reparameterizations and require

that the chirality of the supersymmetric partners of the scalars should not mix under scalar redefinitions, being a property of the underlying spacetime.
Mathematically, a Kähler manifold is a complex manifold with some additional structure, which we are going to specify. First of all a complex manifold can be seen as a regular differentiable manifold with biholomorphic transition functions on the intersection of the coordinate patches. Such manifold is naturally endowed with a (almost)[6] complex structure, given by a tensor field $\mathcal{F}$ of type $(1,1)$, defined by:

$$\mathcal{F} = i\, \mathrm{d}\phi^m \otimes \frac{\partial}{\partial \phi^m} - i\, \mathrm{d}\phi^{\bar{m}} \otimes \frac{\partial}{\partial \phi^{\bar{m}}} \ ,$$

where ϕ^m and $\phi^{\bar{m}}$ are the complex coordinates on the manifold. Such complex structure satisfies:

$$\mathcal{F}^2 = -\mathrm{id} \ .$$

If it is also compatible with the metric (written in terms of real coordinates[7]):

$$\mathcal{F}_i^{\ j} \mathcal{F}_k^{\ l} g_{jl} = g_{ik} \ ,$$

then the complex manifold is said to be hermitian. A Kähler manifold is an Hermitian manifold with a closed fundamental two form, also called a Kähler two form, defined in terms of $\mathcal{F}$ as:

$$J = i\, \mathrm{d}\phi^m \wedge \mathrm{d}\phi^{\bar{m}} g_{m\bar{n}}(\phi, \bar{\phi}) \quad \text{such that} \quad \mathrm{d}J = 0 \ .$$

A Kähler manifold is thus automatically endowed with a real function $K(\phi, \bar{\phi})$ called Kähler potential, in terms of which we can write the metric and the Kähler two form by means of:

$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K(\phi, \bar{\phi}) \ .$$

We can thus see that the metric is invariant under Kähler transformations:

$$K(\phi, \bar{\phi}) \to K(\phi, \bar{\phi}) + \phi + \bar{\phi} \ .$$

This goes to prove that a global $\mathcal{N} = 1$ theory is determined by the Kähler potential (through the metric) and the superpotential. A possible deformation of this geometry is performed by gauging a subgroup of the group G of isometries of $\mathcal{M}$, which would lead to derivatives covariant under such gauge symmetry. Indeed the scalar manifold mya enjoy some internal symmetries, that are transformations of the scalar fields leaving the metric invariant:

$$g_{ij} \stackrel{!}{=} g'_{kl} \equiv g_{ij} \frac{\partial \phi^i}{\partial \phi'_k} \frac{\partial \phi^j}{\partial \phi'_l} \ .$$

The isometric transformations of the scalar fields can be seen as generated, at the infinitesimal level, by Killing vectors ξ^i_J, with J running over the dimension of the isometry group, as in:

$$\phi^i \to \phi^i + \alpha^I \xi^i_I(\phi) \ .$$

The coefficients α^I can be made space dependent, which then leads to a covariantization of the action with respect to such local symmetry as well. Notice moreover that the presence of isometries on a Kähler manifold is not devoid of consequences. Indeed the isometries must respect the chiral separation of the scalar's partner, as well as leaving the Kähler potential invariant up to Kähler transformations. This naturally leads to the definion of Killing prepotentials $\mathcal{P}_I$ obeying:

$$i_{\xi_I} J = \mathrm{d}\mathcal{P}_I \ ,$$

which appear in the Lagrangian also within the scalar potential (in addition to a possible superpotential term).

[6]It can be shown that the tensor field we are about to define needs to satisfy an additional identity in order to be fully considered a complex structure, as explained in [16].

[7]Given n complex coordinates, we can give an equivalent description of $\mathcal{M}$ in terms of $2n$ real coordinates, for example the real and imaginary parts of their complex counterparts.

- Local $\mathcal{N} = 1$. In this case it can be proved that $\mathcal{M}$ is a Kähler-Hodge manifold, see [8], however this would require a rather technical explanation which goes beyond the purpose of this thesis.

- Global $\mathcal{N} = 2$. In this case, the scalars of the theory belong either to hypermultiplets or to vector multiplets; in each case, scalars do not mix, but transform inside their multiplets, therefore the scalar manifold is the product of two scalar manifolds: the hypermultiplet's and the vector multiplet's:
$$\mathcal{M} = \mathcal{M}_{\text{vec}} \times \mathcal{M}_{\text{hyper}} \ .$$
The geometry of these manifolds is affected by two additional constraints: the duality symmetry acting on the vector fields, forming in 3+1 dimension the group $\text{Sp}(2n_v, \mathbb{R})$, and the non trivial R-symmetry group acting on the fermions, being U(2) for $\mathcal{N} = 2$. As we will see, this implies that $\mathcal{M}_{\text{vec}}$ is a rigid, or affine, special Kähler manifold. On the other hand, $\mathcal{M}_{\text{hyper}}$ is called a hyper-Kähler manifold.
Let us start with $\mathcal{M}_{\text{vec}}$; in this case the form of the action is constrained to respect the above-mentioned additional symmetries. Imposing the R-symmetry on both the gravitino and spin 1/2 fermion kinetic terms leads to the relation:
$$2i\, \partial_K \big(\text{Im}\mathcal{N}_{IJ}\big) = \Gamma_{KI\bar{J}}$$
between the vector's kinetic term and the connection of the symmetry of local scalar reparameterization, with $I, J = 1 \dots \dim_{\mathbb{C}}\mathcal{M}$. Knowing the relation between the scalar's metric and the Kähler potential:
$$g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K \ ,$$
we may conclude that:
$$K = i\,\left(X^I \bar{F}_I + \bar{X}^I F_I\right) \ ,$$
where X^I are the scalars, i.e. what we called ϕ^i previously, while F^I are holomorphic functions thereof. Such Kähler potential is invariant under symplectic transformations, as can be seen by writing K as the symplectic product:
$$K = i\mathcal{V}^T \Omega \bar{\mathcal{V}} \ ,$$
with
$$\mathcal{V} = \begin{pmatrix} X^I \\ F_J \end{pmatrix} \ ,$$
and the symplectic transformations of $\text{Sp}(2n_v, \mathbb{R})$ acting like:
$$\mathcal{V}' = S\mathcal{V} \ .$$
By symmetry properties it can be proved that $F_I = \partial_I F$, with F called holomorphic prepotential, in terms of which we can write all kinetic terms, related to each other by:
$$g_{I\bar{J}} = \mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ} \ .$$
As described in [8], there is a slight problem with the previous construction, as the scalars X^I admit non-linear reparameterizations, however they belong to a symplectic vector that only allows for linear transformations. The conundrum is solved by patching the symplectic structure along all the manifold; in other words a rigid special Kähler manifold is a Kähler manifold with a tensor bundle given by the product of a vector bundle with a symplectic structure and a flat U(1) bundle. This means that a section of such tensor bundle is given by a symplectic vector $\mathcal{V}$ with transitions functions, in the intersection of patches, given by:
$$\mathcal{V}_A = e^{i\phi_{AB}} S_{AB} \mathcal{V}_B + c_{AB} \ ,$$
where A, B indicate the two patches and we suppressed the dependence of S on the coordinates on the intersection. The U(1) transformation and the shift with respect to c are both constant

transformation that, while acting on $\mathcal{V}$, do not affect the action.
For what concerns $\mathcal{M}_{\text{hyper}}$ instead, the definition of hyper-Kähler manifolds, which are a restricted type of regular Kähler manifold, is a bit more complex. Let us mention that the manifold is strongly constrained by the R symmetry group acting among the scalars' partners in the hypermultiplets. As explained in [8], this constraint leaves the possibility of a transformation A acting on the fields, with:

$$A \in \mathrm{U}(2n_h) \cap \mathrm{Sp}(2n_h, \mathbb{C}) = \mathrm{Usp}(2n_h) = \mathrm{Sp}(n_h) \ .$$

There is moreover a residual SU(2) invariance acting on the fields, hence the ultimate symmetry group of the manifold is $\mathrm{Sp}(n_h)\times\mathrm{SU}(2)$. Nevertheless, it can be proved that in global $\mathcal{N} = 2$ the curvature of SU(2) should be null, which makes $\mathcal{M}_{\text{hyper}}$ a hyper-Kähler manifold, which is a special subcase of a Kähler manifold characterized by the presence of three independent complex structures satisfying the algebra of quaternions.

- Local $\mathcal{N} = 2$. Let us first examine $\mathcal{M}_{\text{vector}}$. In the local case, we have to take into account also the supergravity multiplet, with the vierbein, two gravitini and a vector field. This additional vector field produces a mismatch between the number of vector fields and scalar fields. Nevertheless, as in the global case, the scalar manifold is equipped with a tensor bundle given by the product of a flat holomorphic vector bundle with a symplectic structure and a holomorphic line bundle. A section of this tensor bundle is given by:

$$\mathcal{V} := \begin{pmatrix} X^I \\ F_J \end{pmatrix} \ ,$$

with transition functions:

$$\mathcal{V}_A = e^{-h_{AB}} S_{AB} \mathcal{V}_B \ ,$$

with $S_{AB} \in \mathrm{Sp}(2n+2, \mathbb{R})$ constant matrix and h_{AB} holomorphic function. The non trivial transition function $e^{-h_{AB}}$ is required by consistency of the Kähler transformation properties of fermions.
The functions F_J need not satisfy $F_J = \partial_J F$, as in the rigid case, but for a right choice of $S \in \mathrm{Sp}(2n_v + 2, \mathbb{R})$, they do. In other words, for a correct choice of symplectic frame, $\tilde{F}_J = SF_J = \partial F(X)/\partial X^J$. Notice also that the holomorphic prepotential F is restricted to be a homogeneous function of degrees two, meaning:

$$X^I \to kX^I \quad \Longrightarrow \quad F(kX) = k^2 F(X) \ ,$$

to ensure the correct transfromation properties of $\mathcal{N}_{IJ}$. The Kähler potential is still symplectically invariant but slightly differently defined:

$$K = -i \log\big(i \, \langle \mathcal{V}, \bar{\mathcal{V}} \rangle\big) \ .$$

For what concerns instead $\mathcal{M}_{\text{hyper}}$, the scalar manifold can be proved to be a quaternionic Kähler manifold, characterized by a non trivial curvature of the SU(2) part of the isometry group. This implies that $\mathcal{M}_{\text{hyper}}$ is not a particular case of the rigid case, and actually it cannot be considered a Kähler manifold in the first place.

- Local $\mathcal{N} > 2$. For $\mathcal{N} > 2$, the constraints on the scalar manifold require it to be a symmetric coset space G/H, with G given by the group of isometries of the manifold and H is the isotropy subgroup. Furthermore, the R-symmetry group is either the full H, for pure supergravities, or a factor thereof.
Let us recall[8] that a coset space G/K is given by the elements of G equivalent up to right multiplication by K; a coset space is a homogeneous space i.e. it admits the transitive action of the group G. By transitive action, we mean that any point of the manifold can reach any

[8]These and the notions of coset geometry of the following chapters are taken from [17].

other point by G transformations. With regards to the quotient G/K, the Lie algebra of G can be split into:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{c} \ ,$$

where $\mathfrak{k}$ is the algebra of generators of H and $\mathfrak{c}$ is the algebra of residual generators, thus called generators of the coset. A coset space is said to be symmetric if:

$$[\mathfrak{c}, \mathfrak{c}] = \mathfrak{k} \ ,$$

that is, if the commutator of the coset generators closes into the isotropy algebra.

For $\mathcal{N} \geq 3, 4$, the scalar manifold is determined once the number of vector multiplets is given. For more than 16 supercharges, on the other hand, $\mathcal{M}$ is determined regardless.

Duality and Lagrangian description In a previous paragraph we explained the Hodge duality of forms and the duality of interacting vector fields. An important point, in that derivation, was noticing that a p-form and a $(D-p-2)$-form are *on shell* duals, but not generally off shell. Hence there might be different Lagrangian descriptions of some field content which may be on shell dual to one another, but off shell different. Furthermore, their off shell formulation implies different isometry groups G, which is a crucial point to consider when it comes to gauging subgroups of G.
In 4d, since the duality group of equations of motion and Bianchi identities is $\mathrm{Sp}(2n_v, \mathbb{R})$, one might say that Lagrangians that are on shell equivalent are mapped to one another by transformations S in such symplectic group. However we must quotient by two isotropy factors. Firstly, some S can be reabsorbed by linear combination of the vector fields, meaning by $\Lambda \in \mathrm{GL}(n_v, \mathbb{R})$, which leaves the Lagrangian invariant. Secondly, some S transformations can be reabsorbed by G transformations on the fields, which is anyway a symmetry of the theory. Therefore, Lagrangians that are on shell equivalent are said to parameterise the coset space:

$$\mathrm{G} \setminus \mathrm{Sp}(2n_v, \mathbb{R}) \ / \ \mathrm{GL}(n_v, \mathbb{R}) \ .$$

Choosing different Lagrangian representations which are on shell dual, is called choosing a symplectic frame.
In four dimensions, vector fields are dual to vector fields, hence they are self dual; in a supersymmetric theory, 2.6 will translate in:

$$F^P_{\mu\nu} = -\frac{e}{2}\epsilon_{\mu\nu\rho\sigma}\Omega^{PQ}\mathcal{N}_{QR}F^{\rho\sigma R} \ ,$$

which is sometimes called twisted self duality equation. Self duality is a general property of $(m-1)$-forms in $D = 2m$ dimensions. In general the twisted self duality will be (with $m - 1 = k$):

$$F^P_{\nu_1\ldots\nu_k} = -\frac{e}{K!}\epsilon_{\nu_1\ldots\nu_k\mu_1\ldots\mu_k}\Omega^{PQ}\mathcal{M}_{QR}F^{\mu_1\ldots\mu_k\ R} \ , \tag{2.9}$$

with $\mathcal{M}_{IJ}$ the scalar matrix of the k-form kinetic terms, and:

$$\Omega_{IJ} = \begin{pmatrix} 0 & \mathbb{1}_m \\ \epsilon\mathbb{1}_m & 0 \end{pmatrix} \ ,$$

with $\epsilon = (-1)^{K+1}$. By consistency of the self duality equation, the factor $\Omega^{IJ}\mathcal{M}_{JP}$ must square to ϵ (such that the right hand side of 2.9 is just F^P). Such condition leads to:

$$\mathcal{M}_{IJ}\Omega^{JK}\mathcal{M}_{KN} = \Omega_{IN} \ .$$

This means that $\mathcal{M}_{IJ}$ must be symplectic or orthogonal for, respectively, K even or odd. Hence, the global symmetry group G must embed into $\mathrm{Sp}(m, m)$ or $\mathrm{SO}(m, m)$ for even or odd m.

Returning to four dimensions, only half of the vector fields propagate degres of freedom, while the other half is just on shell dual. Therefore, one might choose a symplectic frame by splitting in two the vector fields: half electric and half magnetic. In the action, only the electric fields should appear and the Lagrangian would enjoy only a subgroup of the isometry group G. However, instead of choosing a symplectic frame, i.e. an explicit split in electric and magnetic components, one might want to keep the entire G group as the manifest group of isometries of $\mathcal{M}$. This is the spirit of the "democratic formulation" of supergravity, wherein one writes a pseudo-action for all vector fields (or generally all self dual p-forms). Then, the equations of motion of the vector fields are found by varying the pseudo action and imposing, only after, the twisted self duality constraint. For example, in D=4 maximal supergravity, the pseudo action would be:

$$S = \int d^4x\sqrt{-g}\big(R + \frac{1}{48}g^{\mu\nu}\partial_\mu\mathcal{M}^{MN}\partial_\nu\mathcal{M}_{MN} - \frac{1}{4}\mathcal{M}_{MN}F^M_{\mu\nu}F^{\mu\nu M}\big) \ ,$$

equipped with the twisted self duality:

$$*F^M = \Omega^{MP}\mathcal{M}_{PN}F^N \ ,$$

where the matrix $\mathcal{M}$ encodes the scalar fields of the theory, as we are going to explain at the beginning of the next chapter.

2.2 Higher dimensional origin of global symmetries

In the previous paragraphs we showed that in four dimensions, with varying amounts of supergravity, the scalar geometry is either strongly or completely constrained. We will now consider the opposite point of view, by fixing the number of supercharges to 32 and varying the spacetime dimensions. This will yield an interesting point of view on the origin of the global symmetry group G.
We will vary dimension by Kaluza Klein reductions on a n dimensional spacelike torus[9]. Broadly defined, Kaluza Klein reductions decompose the spacetime coordinates in internal and external:

$$X^\mu = (x^\mu, y^m) \quad \text{with} \quad \mu = 0\dots D, \quad m = 1\dots n \ ,$$

such that $D+n=11$ (or $D+n=10$ if we were considering reductions from IIB supergravity). Furthermore, the higher dimensional theory is reduced in the sense that all dependence on the internal coordinates is discarded.
We will now see how the metric is truncated; the truncation ansatz for the 11d vielbein is:

$$e_{\hat{\mu}}^{\ \hat{a}} = \begin{pmatrix} \lambda e_\mu^{\ \underline{\mu}} & K_\mu^{\ m}e_m^{\ \underline{m}} \\ 0 & e_m^{\ \underline{m}} \end{pmatrix} \tag{2.10}$$

where hatted indices are 11 dimensional, underlined ones are flat. The form 2.10 is allowed by the torus being spacelike and the SO(1,10) Lorentz invariance, as indeed $e_{\hat{\mu}}^{\ \hat{a}}$ parametrises the coset space:

$$\frac{\text{GL}(11,\mathbb{R})}{\text{SO}(1,10)} \ .$$

The ansatz 2.10 preserves a $\text{SO}(1,D-1)\times\text{SO}(n)$ subgroup of $\text{SO}(1,10)$; the first factor corresponds to the local Lorentz symmetry in the truncated theory, while the latter tells us that the lower right corner of the ansatz is $\text{SO}(n)$ symmetric. Such corner corresponds to the $\frac{1}{2}n(n+1)$ scalar fields of the truncated theory, and corresponding to geometrical features of the n dimensional torus. Indeed, if we were simply considering the Kaluza Klein reduction on a circle, a unique scalar field would correspond to the radius of the circle.
The reduction also produces n Kaluza Klein vectors, corresponding to $K_\mu^{\ m}$. There is a factor λ in 2.10, where:

$$\lambda = \big(\det(e_m^{\ \underline{m}})\big)^{\left(\frac{1}{n-(D-2)}\right)} \ ,$$

[9] In this paragraph we will only be concerned with reductions of fields, without considering their equations of motion.

engineered such that the truncated, D dimensional theory is in its Einstein frame. In the Einstein frame, there are no fields multiplying the Ricci scalar in the Einstein Hilbert action. The exact power can be found by requiring the reduction of the 11d metric determinant to be the D dimensional metric determinant.
The original theory enjoyed invariance under supersymmetry and local coordinate transformations; after Kaluza Klein truncation, the latter symmetry is still retained by the D dimensional theory (in form of invariance under local x^μ transformations). However, the invariance under internal diffeomorphisms is broken, and only its subgroup defined by:

$$y^m \to y^m + y^n J_n{}^m$$

with constant $J_n{}^m \in \mathfrak{gl}(n, \mathbb{R})$, is retained by the truncated theory, in form of a a multiplication of the fields by J, which is compatible with the KK ansatz as it does not introduce dependence on y^m. Thus the truncated theory enjoys a rigid $\mathrm{GL}(n, \mathbb{R})$ symmetry, or equivalently a $\mathrm{GL}(1)\times\mathrm{SL}(n)$, acting as:

$$\delta\, e_{\underline{m}}^{\underline{m}} = \Lambda^n{}_m e_{\underline{n}}^{\underline{m}} \qquad \delta\, K_\mu{}^m = \Lambda^m{}_n B^n{}_\mu\ ,$$

wherein GL(1) is embedded as a diagonal matrix.
If these were the only symmetries at play, we could immediately conclude that the scalars parameterize a coset space given by:

$$\frac{\mathrm{GL}(n)}{\mathrm{SO}(n)}\ ,$$

but this is not the case. Indeed the higher dimensional theory will in general include a certain number of p-forms, which are themselves subjected to Kaluza Klein truncation and therefore originate a certain number of lower dimensional forms, scalars included. To be precise, a p-form in $D+n$ dimensions produces $\binom{n}{p}$ scalars in D dimensions if $n \geq p$. The original Abelian gauge symmetries of the p-forms affect the global symmetries of the resulting scalars, as indeed in the transformations:

$$\delta C_{m_1\dots m_p} = p\ \partial_{[m_1}\Lambda_{m_2\dots m_p]}\ ,$$

one can consider Λ linear in the internal coordinates, in order not to introduce dependence on them. While this constitutes a gauge symmetry of the t-forms, with $t < p$, in the lower dimensional theory, it only acts as a global symmetry on the lower dimensional scalars. The symmetry group can be shown to be Abelian if only one p-form contributes to the lower dimensional scalars, otherwise it is non Abelian. Nevertheless, the algebra $\mathcal{P}$ of local scalar shifts can be proved to be solvable[10].
An additional factor that goes to affect the scalar geometry is the presence of scalars in the higher-dimensional theory, which parameterize on their own the coset space:

$$\frac{\mathcal{G}_0}{K(\mathcal{G}_0)}$$

where $\mathcal{G}_0$ is the global symmetry group of the higher dimensional theory, $K(\mathcal{G}_0)$ its maximal compact subgroup. To be precise, the higher dimensional theory also has a global $\mathbb{R}^+$ symmetry, called *trombone* symmetry[11], acting on fields, for examples on the 11d's:

$$g_{\mu\nu}\ \to\ \lambda^2\, g_{\mu\nu}\ , \qquad C_{\mu\nu\rho}\ \to\ \lambda^3\, C_{\mu\nu\rho}$$

and leaves the equations of motion invariant, while modifies the action by a factor λ^{D-2}.
Hence, the scalars of the truncated theory generally parameterize:

$$\frac{(\mathrm{GL}(n)\times\mathcal{G}_0)\ltimes\mathcal{P}}{\mathrm{SO}(n)\times K(\mathcal{G}_0)}\ , \tag{2.11}$$

[10] A solvable algebra has the property that nested commutators ultimately vanish.
[11] Note that the rescaling action of the trombone symmetry and the aforementioned GL(1) will combine into a unique rescaling transformation, with charges given by linear combinations of the two sets of charges, fixed by the truncated theory being in the Einstein frame.

where we did not include the trombone symmetry, as there are no scalars associated to it. This is not the end of the story however, since there is an addition group of symmetries (called *hidden* symmetries) whose effect is to mix the scalars coming from the different factors. The combined effects of 2.11 and the hidden symmetries is to form a semisimple global group G, with a maximal compact subgroup K(G), reported for maximal supergravity in table 2.4, such that the lower dimensional scalars parameterise:

$$\frac{\mathrm{G}}{\mathrm{K}(\mathrm{G})} \ .$$

D	9	8	7	6	5	4	3
G	$SL(2) \times \mathbb{R}^+$	$SL(3)\times SL(2)$	$SL(5)$	$SO(5,5)$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$
K	$SO(2)$	$SO(3)\times SO(2)$	$SO(5)$	$SO(5)\times SO(5)$	$Usp(8)$	$SU(8)$	$Spin(16)$
scalars	$\mathbf{1}^0 + \mathbf{3}^0 - 1$	$(\mathbf{3} - 1, \mathbf{1}) + (\mathbf{1}, \mathbf{8} - 3)$	$\mathbf{24} - 10$	$\mathbf{45} - 20$	$\mathbf{78} - 36$	$\mathbf{133} - 63$	$\mathbf{248} - 120$
1-forms	$\mathbf{1}^{-4} + \mathbf{2}^{+3}$	$(\mathbf{2}, \mathbf{3}')$	$\mathbf{10}'$	$\mathbf{16_c}$	$\mathbf{27}'$	$\mathbf{56}$	
2-forms	$\mathbf{2}^{-1}$	$(\mathbf{1}, \mathbf{3})$	$\mathbf{5}$	$\mathbf{10}$			
3-forms	$\mathbf{1}^{+2}$	$(\mathbf{2}, \mathbf{1})$					

Table 2.4: Structure of lower dimensional maximal supergravity theories with varying D (external dimension). The number of scalar equals the result of each expression, which is just the difference between the dimension of G and K, and the scalar themselves transform in the representation in bold, which is the adjoint of G. The grading is with respect to the $GL(1) \subset GL(n)$, commuting with $SL(n) \subset GL(n)$. Discrete subgroup have been ignored in the isotropy group; the field content has been dualized to lowest possible level.

2.3 Gauged supergravity

A dimensional reduction of 11d supergravity on a torus technically yields a gauged supergravity too, with abelian gauge group given by $\mathrm{U}(1)^{n_v}$ and the field content of the lower dimensional theory, besides the vector fields, invariant with respect to it. The vector fields instead transform as:

$$\delta A_\mu{}^M \;=\; \partial_\mu \Lambda^M \;.$$

Let us examine more in general the gauging of the global symmetry group G of a maximally supersymmetric supergravity theory. Considering G, we might choose a subgroup thereof G_0 and make its transformations space-dependent. As known, this requires the introduction of G_0-covariant derivatives:

$$\partial_\mu \;\rightarrow\; D_\mu = \partial_\mu - g A_\mu^M X_M \;,$$

where g is the gauge coupling constant, and $A_\mu^M X_M$ are the gauge vector fields, among the whole set of vector fields of the theory, acting as connections of the gauge symmetry. These are selected by X_M, which can be written in turn as:

$$X_M \;\equiv\; \Theta_M^{\;\alpha} t_\alpha \;\in \mathfrak{g} \;,$$

where $\Theta_M^{\;\alpha}$ is a constant rectangular matrix, describing which among the $t_\alpha \in \mathfrak{g}$ generators generate the gauge algebra, with M index in the representation of G which we will call[12] $\mathbf{R}_1^*$, and α in the adjoint of G. The transformation of the scalars and the vector fields under gauged G_0 is then:

$$\begin{aligned}\delta\mathcal{V} \;&=\; g\Lambda^M X_M \mathcal{V} \;,\\ \delta A_\mu{}^M \;&=\; \partial\Lambda^M + g A_\mu{}^N X_{NP}{}^M \Lambda^P \;=\; D_\mu \Lambda^M \;,\end{aligned}$$

where we have written:

$$X_{MN}^{\;\;\;\;P} \;= \Theta_M^{\;\alpha} (t_\alpha)_N^{\;\;P} \;,$$

using the representation $(t_\alpha)_N^{\;\;P}$ of the generators.
We require that $\mathfrak{g}_0$ forms a subalgebra of $\mathfrak{g}$, and that $\Theta_M^{\;\alpha}$ itself is invariant under G_0, which implies that:

$$\begin{aligned}0 = \delta_P \Theta_M^{\;\alpha} \;&=\; \Theta_P^{\;\beta} \delta_\beta \Theta_M^{\;\alpha} = \\ &=\; \Theta_P^{\;\beta} (t_\beta)_M^{\;\;N} \Theta_N^{\;\alpha} + \Theta_P^{\;\beta} f_{\beta\gamma}^{\;\;\;\alpha} \Theta_M^{\;\gamma} \;.\end{aligned}$$

In the previous passage we have used that the generators of the $\mathfrak{g}_0$ in the adjoint representation are the structure constants of $\mathfrak{g}_0$ itself (possibly up to a normalisation). Now contracting with the $\mathfrak{g}$ generators we obtain the so called quadratic constraint:

$$[X_M, X_N] \;=\; -X_{MN}^{\;\;\;\;P} X_P \;,$$

which not only imposes closure of the algebra, but also tell us that $X_{(MN)}^{\;\;\;\;\;\;P} X_P = 0$.
Before outlining the other constraint on the embedding tensor, let us point out that this formalism, using a G-covariant tensor $X_{MN}^{\;\;\;\;P}$ allows to keep G intact until we explicitely give an expression for $\Theta_M^{\;\alpha}$, treating $X_{MN}^{\;\;\;\;P}$ as a spurionic object which transforms according to its indices. If a form of the embedding tensor is specified, G is broken to its gauged subgroup.
The other constraint on the embedding tensor comes from compatibility with supersymmetry, and depends on the number of supercharges and spacetime dimensions. We can give an example of such constraint in 4d, where maximal supergravity has $G = \mathrm{E}_{7(7)}$ and the representation of the embedding tensor can be written as:

$$\Theta_M^{\;\alpha} : \qquad \mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480} \;,$$

[12] Such representation is the conjugate one of the $\mathbf{R}_1$ of G wherein vector fields transform.

where one can show that the supersymmetry constraint imposes that the only non vanishing component of the embedding tensor is **912**. If the **56** component is non-vanishing as well, then it corresponds to a gauging of the trombone symmetry of the theory, which does not allow to write explicitly an action invariant under such gauge symmetries. Indeed, we recall that the trombone is a symmetry of the equations of motion, and not of the action. In going from the action to the equations of motion, an integration by parts has to be performed; the boundary term does not vanish if the trombone symmetry is gauged.

The promotion of $G_0 \subset G$ to a non abelian gauge symmetry leads to a modification of the vector fields' strength. Indeed, the subset of vector fields that gauge the symmetry will have a field strength (up to a term that we will discuss shortly) proportional to the commutator of covariant derivatives. Hence, it will have the form:

$$F^M_{\mu\nu} = 2\partial_{[\mu} A^M_{\nu]} + 2g X_{NP}{}^M A_{[\mu}{}^N A_{\nu]}{}^N \ ,$$

however we will now see that such field strength is not sufficient by itself to ensure covariance of the vector fields' kinetic terms in the action under gauged G_0, but needs a modification. Namely, such modification will be a Stueckelberg-type coupling between the vector fields and a two-form, as we will see.
The problem is that the embedding tensor $X_{MN}^{\ \ \ P}$ fails to satisfy the Jacobi identity up to:

$$X_{[MN]}^{\ \ \ \ \ P} X_{[QP]}^{\ \ \ \ \ R} + X_{[QM]}^{\ \ \ \ \ P} X_{[NP]}^{\ \ \ \ \ R} + X_{[NQ]}^{\ \ \ \ \ P} X_{[MP]}^{\ \ \ \ \ R} = -Z^R_{\ P[Q} X_{MN]}^{\ \ \ \ \ P} \ ,$$

where the coefficients $Z^P{}_{MN}$ satisfy:

$$Z^P{}_{MN} X_P = 0 \ .$$

The variation of the above defined field strength can be shown to be:

$$\delta F^M_{\mu\nu} \ = \ -g\Lambda^P X_{PN}{}^M F^N_{\mu\nu} - 2g Z^M{}_{PQ} \left(\Lambda^P F^Q_{\mu\nu} - A^P_{[\mu} \delta A^Q_{\nu]} \right) \tag{2.12}$$

which does not allow to write covariant kinetic terms for the vector fields. The solution is to redefine those field strengths as:

$$\mathcal{F}^M_{\mu\nu} = F^M_{\mu\nu} + g Z^M{}_{PQ} B^{PQ}_{\mu\nu} \ , \tag{2.13}$$

with $B^{MN}_{\mu\nu} = B^{(MN)}_{\mu\nu}$. The non covariant terms in 2.12 can be reabsorbed by the transformation of $B^{MN}_{\mu\nu}$ themselves. Furthermore, the introduction of these new two forms leads to a new gauge symmetry acting as:

$$\begin{aligned} \delta A^M_\mu &= D_\mu \Lambda^M - g Z^M{}_{PQ} \Xi^{PQ}_\mu \ , \\ \delta B^{MN}_{\mu\nu} &= 2 D_{[\mu} \Xi^{MN}_{\nu]} - 2\Lambda^{(M} \mathcal{F}^{N)}_{\mu\nu} + 2 A^{(M}_{[\mu} \delta A^{N)}_{\nu]} \ , \end{aligned}$$

with Ξ^{MN}_μ being the transformation parameter. The new 2.13 introduces, as we mentioned before, a Stueckelberg-type coupling among the vector fields and the two forms, with the latter being part of the field content of the theory. It can be shown that a consistent gauging requires a series of coupling among p-forms and $(p+1)$-forms, until saturation of forms. This is called a tensor hierarchy and we will encounter it again in the next chapter.

In the course of the thesis we will deal with the gauging of $\mathcal{N} = 8$ supergravity in four dimensions. As we mentioned before, in 4d vector fields are self-dual. Consequently, the lagrangian will have an off-shell global symmetry group G_e, and an on-shell global symmetry group G. The latter will act on the whole set of vector fields, independently of the symplectic frame chosen to describe the independent degrees of freedom. On the other hand, the off-shell global symmetry G_e depends on the symplectic frame; choosing for instance the splitting in half electric and half magnetic degrees of freedom, leads to $G_e = \mathrm{SL}(8)$, called the electric global symmetry group, with the electric vector

fields transforming in its **28** representation. The gauge group must be a subset of electric symmetry group; by considering the representations in which the embedding tensor lives and branching them with respect to G_e, one can find the possible gauging of the theory purely from group-theoretic deductions.
As reported in [18], since the only nonvanishing component of a maximally supersymmetric gauged action is **912**, we can understand which gauge groups are allowed. Indeed we can branch with respect to the symplectic frame SL(8):

$$\begin{aligned} \mathbf{56} &\to \mathbf{28} \oplus \bar{\mathbf{28}} \ , \\ \mathbf{133} &\to \mathbf{63} \oplus \mathbf{70} \ , \\ \mathbf{912} &\to \mathbf{36} \oplus \mathbf{420} \oplus \bar{\mathbf{36}} \oplus \bar{\mathbf{420}} \ . \end{aligned}$$

Furthermore, we branch the products of **56** and **133** as in:

	28	$\bar{\mathbf{28}}$
63	$\mathbf{36}+\mathbf{420}$	$\bar{\mathbf{36}}+\bar{\mathbf{420}}$
70	$\bar{\mathbf{420}}$	**420**

Table 2.5: Branching of product representations of $E_{7(7)}$ with respect to $SL(8,\mathbb{R})$

Since the **912** has only one **420**, the two **420** in 2.5 need to coincide. However this would imply a gauging of the electric vector fields to the **63** and at the same time a gauging of the dual magnetic vector fields to **70**, which is impossible. The same can be said for $\bar{\mathbf{420}}$. Therefore the only nonvanishing sub-representations in the branching with respect to SL(8) must be **36**. Such representation can be seen as a symmetric 8 by 8 matrix θ_{AB}, with eigenvalues ± 1 or 0. In particular, if there are p eigenvalues $+1$, q eigenvalues -1 and r eigenvalues 0, the gauge group that leaves such matrix invariant is $CSO(p,q,r)$, with $p+q+r=8$. If there are only positive eigenvalues, the gauge group will be SO(8), which we will encounter again in a few chapters.
In that case, however, the SO(8) gauging is not the result of a direct gauging of some generators of $\mathfrak{e}_{7(7)}$, but will instead arise from dimensional reduction on an internal manifold with fluxes, in a generalised geometry framework.

Let us mention that gauged supergravity has remarkable features, which raise much interest from a phenomenological point of view. Indeed the gauged supergravity lagrangian, in order to maintain invariance under supersymmetry, will need to be deformed; namely the supersymmetric variation of fermions needs to be changed, or more precisely shifted, by constant terms. These constant terms imply the introduction of mass terms for the fermions and their supersymmetric partners; in particular a scalar potential arises, proportional to the sum of the squares of the fermion shifts, which would otherwise have been prohibited in maximal supergravity. A scalar potential may lead, in turn, to non-vanishing vacuum expectation values, which allow to interpret the scalar potential as an effective cosmological constant. Such effective cosmological constant, already relevant from an inflationary point of view, is all the more interesting since it leads to spontaneous rigid supersymmetry breaking. Indeed, as we mentioned in the initial paragraphs, rigid supersymmetry is broken if the energy of the ground state is non-zero. Supergravity is broken instead by gravitino or gaugino condensation; nonetheless, local supersymmetry breaking below a certain energy scale is a necessary requirement for supergravity to be consistent with the experimental observations. We do not observe, indeed, supersymmetric particle partners below the current energy thresholds of particle colliders, thus, if supergravity is to describe the universe, local supersymmetry must be broken below such thresholds.

Chapter 3

Exceptional Field theory

In the previous chapter we mentioned that ungauged maximal supergravity in D dimensions can be found by Kaluza Klein dimensional reduction of 11d (or IIB) on a torus and enjoys a global G symmetry group. The aim of exceptional field theory is to reproduce such D+n dimensional maximal (or half-maximal) supergravity, its field content, gauge symmetries and dynamics, in a G-covariant way. Therefore, the field content is organized in G representations and its symmetries along the internal manifold, being gauge and diffeomorphism symmetries, are woven into a unique symmetry principle, the invariance under generalised diffeomorphisms. Technically, one speaks of exceptional field theory only for D$= 5, 4, 3, 2$, that is where G=$\mathrm{E}_{n(n)}$. A formulation with G=O$(10, 10)$ is also available, called double field theory, which historically preceded the discovery of exceptional field theories, although we will not expand upon it.
The main references for this chapter are: [19], [20], [21].

3.1 G-covariant description of the field content

In this section we will explain in which sense the field content of D dimensional supergravity is organized in terms of representations of G. Let us point out that this description holds for $2 \leq \mathrm{D} \leq 9$, maximal supergravity. The identification of the field content in scalars, fermions and p-forms reflects their transformation properties with respect to general coordinate transformations of the external space; besides this, they also sit in G representations, as we are going to see[1].

Scalar sector As we mentioned in the previous chapter, scalars parameterise a symmetric coset space G/K, with G a non compact real (semi)simple Lie group and K its maximal compact subgroup, G enforcing a global symmetry of the theory and K a local invariance, to avoid overcounting of the scalars.
Indeed, let us choose a coset representative $\mathcal{V}(y)$, with y^m coordinates on the scalar manifold. Under a $g \in$ G transformation, such a choice will not be respected in general and needs a compensating $h \in$ K transformation, depending on g and y:

$$g\ \mathcal{V}(y) = \mathcal{V}(y')h(g, y')\ ,$$

or equivalently, under infinitesimal transformations:

$$\delta\mathcal{V} = \Lambda\mathcal{V} - \mathcal{V}k_\Lambda \qquad g \approx 1 - \Lambda, \quad h \approx 1 + k\ . \tag{3.1}$$

[1] Notation convention: consistently with the other chapters, indices from the greek alphabet denote representations of the $\mathrm{SO}(1, \mathrm{D}-1)$ Lorentz group in the external spacetime, while lowercase indices from the english alphabet, like $m, n \ldots$ index representations of the internal $\mathrm{GL}(n)$ group of reparametrizations of internal coordinates, uppercase letters from the english alphabet denote representations of the exceptional group or its maximal compact subgroup. Hatted greek indices denote representations of the $\mathrm{SO}(1, \mathrm{D}+n-1)$ Lorentz group of the whole D+n dimensional spacetime.

Using this coset representative, we can build a G left-invariant one form (assuming the group acts from the left), namely the Maurer Cartan one form:

$$\sigma = \sigma_\mu \mathrm{d}x^\mu = \sigma^A t_A = \mathcal{V}^{-1}\mathrm{d}\mathcal{V} \qquad A = 1, \dots \dim(\mathfrak{g}) \ ,$$

which transforms under G as:

$$\sigma^A t_A \to h^{-1}\sigma h + h^{-1}\mathrm{d}h \ ,$$

hence only local K transformations appear. Using the decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{c}$, with $\mathfrak{k}$ algebra of K, we can split:

$$\sigma_\mu = Q_\mu + P_\mu \qquad Q_\mu = Q^i_\mu T_i \in \mathfrak{k}, \ \ P_\mu = e^{\underline{m}}_\mu T_{\underline{m}} \in \mathfrak{c} \ ,$$

from which we see that, under infinitesimal g and h:

$$\begin{aligned} \delta Q_\mu &= +\partial_\mu k + [k, Q_\mu] \ , \\ \delta P_\mu &= [k, P_\mu] \ . \end{aligned}$$

Q^i is called K-connection and it behaves as a gauge connection under G transformations, while $e^{\underline{m}} = e^{\underline{m}}{}_\mu \mathrm{d}x^\mu$ is the vielbein one-form on the coset space. We can define K-covariant derivatives as:

$$D_\mu \mathcal{V} = \partial_\mu \mathcal{V} - \mathcal{V} Q_\mu \ ,$$

thus satisfying: $\mathcal{V}^{-1} D_\mu \mathcal{V} = P_\mu$. These covariant derivatives can be used to write the scalar action as:

$$\mathcal{L}_{\text{scalar}} \propto \frac{1}{2}\mathrm{Tr}\big[D_\mu \mathcal{V}^{-1} D^\mu \mathcal{V}\big] = -\frac{1}{2}\,\mathrm{Tr}\big[P_\mu P^\mu\big] \ , \tag{3.2}$$

describing the K inequivalent degrees of freedom, being G and K invariant. Equivalently, if a symmetric positive-definite bilinear Δ_{AB} exists, invariant under K, then we can write:

$$\mathcal{M}_{MN} = \mathcal{V}_M^A \mathcal{V}_N^B \Delta_{AB} \ ,$$

which is a symmetric, K invariant expression, transforming under infinitesimal $g \in$ G, with $g \approx 1+\Lambda$, as:

$$\delta\mathcal{M} = \Lambda\mathcal{M} + \mathcal{M}\Lambda^T \ .$$

Therefore, the scalar kinetic term of the Lagrangian density in the external spacetime can be written as:

$$\mathcal{L}_{\text{scalar}} = \frac{1}{8} e \ \mathrm{Tr}\big(\partial_\mu \mathcal{M}\ \partial^\mu \mathcal{M}^{-1}\big) \ , \tag{3.3}$$

which is manifestly K-invariant, with ∂_μ derivatives along the external coordinates and e determinant of the external space vielbein, and $\mathcal{M}$ depending on internal and external coordinates. Notice that the coset representative $\mathcal{V}$ has two indices, one in a G representation and the other in a K representation, the same way the vielbein e^a_μ has two indices, one curved transforming under GL(d) and the other, flat, transforming under local $\mathrm{SO}(1, \mathrm{d}-1)$. Indeed $\mathcal{V}$ can be seen as the generalised vielbein on the coset space, with $\mathcal{M}$ being the generalised metric. This provides the link between the two actions 3.2 and 3.3.

By convention, indices in a K fundamental representation will always be denoted by the initial letters of the english alphabet: $A, B, C \dots$, while G representation by letters in the middle: M, N, P, Q etc. The invariance under local K transformations, evident in 3.3, can be eliminated by fixing a special form of the coset representative, i.e. by gauge fixing. Although there are many choices of gauge fixing, we will mention two:

- Unitary gauge, in which:

  $$\mathcal{V} = \exp\left(\phi^a Y_a\right) \ ,$$

  where Y_a are the generators of $\mathfrak{k}$, which are non compact. The scalars ϕ^a transform under a linear representation of K, as can be seen from 3.1, hence a global K symmetry remains manifest.

- Triangular gauge, in which:

$$\mathcal{V} = \exp\left(\phi^m N_m\right) \exp(\phi^\lambda h_\lambda\right) \ ,$$

where h_λ are the generators of the Cartan subalgebra of $\mathfrak{g}$, while N_m are $\mathfrak{g}$ generators such that the set $\{h_\lambda, N_m\}$ constitutes the Borel subalgebra[2] of $\mathfrak{g}$. This gauge allows for the higher dimensional origin of the scalar fields to be more straightforward, as a consequence of their grading under rescaling of the volume of the internal manifold (the rescaling being itself an element of the Cartan subalgebra).

As mentioned in 2.2, scalars of the lower dimensional theory represent geometrical features of the internal manifold and higher dimensional p-forms. For instance, if a compactification was performed on a circle, the scalar resulting from the higher dimensional metric would represent the radius of the circle. Therefore, scalars are the *moduli* of the internal manifold, as they describe how the internal manifold changes along the space of the lower dimensional theory.

Spinor sector In the rest of the thesis, we will deal with spinors only when we will have to truncate to an arbitrary amount of supersymmetries. Let us mention that, as we said in the previous chapter, spinors transform under scalar reparameterizations of $\mathcal{M}$ as tangent vectors. Similarly, spinors in spacetime transform under local $\mathrm{SO}(1, D-1)$ Lorentz transformations[3], thus it is only natural to state that spinors in exceptional field theory are organised in representation of the local K symmetry. To write their interaction with the other pieces of the Lagrangian, objects with a flat K index are required, such as the generalised vielbein itself.

***p*-forms** p-forms generally transform in representations of G, which we listed in table 2.4; in particular the G representation of vector fields is denoted with $\mathbf{R}_1$ and might not be, in general, the fundamental representation of G. We mentioned in a previous paragraph the difficulty in writing a G-invariant action in even dimension, due to self duality; indeed also in ExFT, the G invariant Lagrangian will be a pseudo-Lagrangian, where all components of the self-dual vector fields will enter, endowed with the self duality constraint.

3.2 Generalised diffeomorphisms

In the introduction of this chapter we claimed that exceptional field theories aim to reproduce, among the other things, the internal symmetries of D+n dimensional maximal supergravity, in particular 11d and IIB. Its fields transform under local coordinate transformations as:

$$\begin{aligned} L_\xi g_{\hat{\mu}\hat{\nu}} &= \xi^{\hat{\rho}} \partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} + 2\partial_{(\hat{\mu}} \xi^{\hat{\rho}} g_{\hat{\nu})\hat{\rho}} \ , \\ L_\xi B_{\hat{\nu_1}\dots\hat{\nu_k}} &= \xi^{\hat{\rho}} \partial_{\hat{\rho}} B_{\hat{\nu_1}\dots\hat{\nu_k}} + 2\partial_{[\hat{\nu_1}} \xi^{\hat{\rho}} B_{\hat{\nu_2}\dots\hat{\nu_k}]\hat{\rho}} \ , \end{aligned}$$

and under gauge transformations as:

$$\delta B_{\hat{\nu_1}\dots\hat{\nu_k}} = k\ \partial_{[\hat{\nu_1}} \Lambda_{\hat{\nu_2}\dots\hat{\nu_k}]} \ .$$

ExFT is able to reproduce these symmetries along the internal space only, i.e. the associated transformations parameters have exclusively internal legs. The idea behind reproducing these symmetries in a unified way is the following. First of all, we perform split into external and internal coordinates of the fields of the higher dimensional theory without performing a full Kaluza Klein truncations of the field content. The internal coordinates are then extended to the full $\mathbf{R}_1$ representation of G, whence the coordinate dependence of such fields becomes:

$$(x_\mu, y_m) \longrightarrow (x_\mu, Y_M) \qquad m = 1\dots n, \quad M = 1\dots \mathrm{dim}\mathbf{R}_1$$

[2] Let us recall that the Borel subalgebra is the maximal solvable algebra.

[3] Since the spinorial representation is not acted upon like a change of coordinates by Lorentz transformations, as the bosonic representations are instead.

which is equal to enlarging the internal manifold's tangent space to dim$\mathbf{R}_1$ dimensions, with a natural action of G upon any tensor defined upon it (in the same way conventional tensors are naturally acted upon by Lorentz transformations). In the next chapter we will dwell further on the exceptional generalised geometry that describes this enlargement of the tangent space, but for the moment we will focus on its practical purpose.
We can define a generalised Lie derivative with respect to a generalised vector Λ^M (a section of the tangent bundle of the internal manifold, as we will see) which by construction has enough number of entries to encode all gauge transformations' parameters:

$$\Lambda^M \;=\; \left(\xi^m, \lambda_{[mn]}, \xi_{[mnpqr]}, \dots\right) , \tag{3.4}$$

where the vector and the p-forms are the result of the branching of $\mathbf{R}_1$ of G with respect to SL(n), with n dimension of the internal manifold[4].

3.2.1 Construction of the generalised Lie derivative

In analogy with the conventional Lie derivative, the generalised one should have a transport term, a term projecting onto the algebra of $\mathrm{E}_{n(n)}$ and a density term, for tensor densities.
Given a basis $\{t_{\alpha M}{}^N\}$ of the algebra $\mathfrak{e}_{\mathfrak{n}(\mathfrak{n})}$, we can build a projector onto it as:

$$\mathbb{P}^{M}{}_{N}{}^{P}{}_{Q} = t_{\alpha N}{}^{M}\eta^{\alpha\beta}t_{\beta Q}{}^{P} \ ,$$

with $\alpha = 1 \dots \mathrm{dim}\mathfrak{e}_{\mathfrak{n}(\mathfrak{n})}$ in the adjoint representation, $\eta^{\alpha\beta}$ proportional to the inverse of the Cartan Killing metric and the projector normalised as:

$$\mathbb{P}^{M}{}_{N}{}^{P}{}_{Q}\mathbb{P}^{P}{}_{Q}{}^{R}{}_{S} = \mathbb{P}^{M}{}_{N}{}^{R}{}_{S} \ .$$

The transport term is the least one would expect from a Lie derivative, as it is the only component when acting upon scalars. Moreover let us recall that conventional tensor densities transform under coordinate transformations as:

$$V^{\mu}(x') = \left|\frac{\partial x'}{\partial x}\right|^{\beta}\frac{\partial x^{\mu'}}{\partial x^{\nu}}V^{\nu}(x) \ ,$$

where β is thus the weight of the vector.
The generalised Lie derivative should be the linear combination of these three terms:

$$\mathcal{L}_U V^M = U^N\partial_N V^M - \alpha\mathbb{P}^{M}{}_{N}{}^{P}{}_{Q}\partial_P\Lambda^Q V^N + \beta\partial_P\Lambda^P V^M \ ,$$

with α, β real coefficients (in particular the latter is the weight under generalised diffeomorphisms).
We can rewrite the previous expression by defining a constant, $\mathrm{E}_{n(n)} \times \mathbb{R}^+$-invariant tensor:

$$Y^{MN}{}_{PQ} = \alpha\mathbb{P}^{M}{}_{N}{}^{P}{}_{Q} + \delta^M_Q\delta^P_N + \beta\delta^M_N\delta^P_Q \ ,$$

in terms of which:

$$\mathcal{L}_U V^M = U^N\partial_N V^M - V^N\partial_N U^M + Y^{MN}{}_{PQ}\partial_N U^Q V^N \ .$$

The generalised Lie derivative represents a tensor's variation under generalised coordinate transformation; we expect that the repeated action of two such transformations is equivalent to another such transformation. More formally, we expect the generalised Lie derivative to form an algebra:

$$[\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_2}]\Lambda_3^M = \mathcal{L}_{\mathcal{L}_{\Lambda_1}\Lambda_2}\Lambda_3^M \ .$$

The previous equation is called *Leibniz identity*; the algebra is called a Leibniz algebra, which only differs from a Lie algebra in that its brackets are not skew-symmetric.

[4]If one is concerned with IIB and its truncation, the group with respect to which we branch is going to be different, as we are going to explain shortly.

If the above expression of the generalised Lie derivative is inserted into the Leibniz identity, it can be found that the identity is not obeyed unless the partial derivatives satisfy a constraint. Such a constraint is called *section constraint* and is the natural counterpart of the aforementioned tangent space enlargement: the section constraint tells us that the theory depends only on the original n coordinates of the internal manifold, while enlarging was a useful mathematical trick, without physical manifestations. The section constrain can be formulated in terms of $Y^{MN}{}_{PQ}$ as:

$$Y^{MN}{}_{PQ}\, \partial_M \otimes \partial_N = 0\ ,$$

Considering the theory on a solution of the section constraint, which we will consider in the next paragraphs, is called considering it *on section*. Imposing the Leibniz identity on section allows to find the correct α, β coefficients in the previous formulae.
Another, equivalent way to find those coefficients is the following. As we know, generalised coordinate transformations encode the gauge and conventional diffeomorphisms transformations; hence in particular a component of the generalised vector Λ^M with respect to which we are deriving is going to encode the ξ^m parameter of general coordinate transformations. Recalling, from paragraph 2.2, that $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ contains a $\mathrm{GL}(n)$ subgroup related to internal coordinate transformation; by branching the $\mathbf{R_1}$ with respect to $\mathrm{GL}(n)$ one finds that ξ^m belongs to a vector representation of $\mathrm{GL}(n)$.
Furthermore, we can decompose ∂_M to an internal ∂_m component and require that all other components vanish by section constraint. If then we put all Λ^M components in the $\mathrm{GL}(n)$ branching to zero, other than ξ^m, then the generalised Lie derivative will simply recover the conventional L_ξ.
For completeness, the values of β are:

$$\beta = \frac{1}{9-n}\ ,$$

while $\alpha = \{3, 4, 6, 12\}$ for respectively $n = \{4, 5, 6, 7\}$.

Maximal solutions of the section constraint There exist two inequivalent orbits of maximal solutions of the section constraint, both consisting in reducing the $\mathbf{R}_1$ of $\mathrm{E}_{n(n)}$ under (different) subgroups thereof. These decompositions are:

$$\begin{aligned} &\mathrm{E}_{n(n)} \to \mathrm{SL}(n) && \text{11D orbit of solutions}\ ,\\ &\mathrm{E}_{n(n)} \to \mathrm{SL}(n-1)\times\mathrm{SL}(2) && \text{tybe IIB orbit of solutions}\ . \end{aligned}$$

The name of the different orbits is due to the fact that, on section, exceptional field theory will describe the D-dimensional truncation of 11D or IIB maximal supergravity.
A solution of the section constraint can be written equivalently as:

$$\partial_M \;=\; \epsilon_M{}^m \partial_m\ ,$$

with M in the $\mathbf{R}_1^*$ representation of $\mathrm{E}_{n(n)}$ and m in the fundamental of $\mathrm{SL}(n)$, $\epsilon_M{}^m$ being a constant rectangular matrix of maximal rank. Let us notice that the subgroups in the previous decompositions correspond to the group of coordinate transformations on the internal manifold, up to the $\mathrm{GL}(1) \subset \mathrm{GL}(n)$ rescaling factor.

Trivial parameters Being that $\mathcal{L}_\Lambda$ satisfies the Bianchi identity, we might ask if it also satisfies a Jacobi identity. As can be computed, a Jacobi identity is satisfies up to *trivial parameters*:

$$[[\mathcal{L}_{U_1}, \mathcal{L}_{U_2}], \mathcal{L}_{U_3}] + \text{cyclic permutations} = \text{trivial parameters}\ ,$$

where trivial parameters are generalised vectors φ^N such that:

$$\mathcal{L}_\varphi\,(\Phi) = 0\ ,$$

where Φ is any generalised tensor. The very existence of trivial parameters will have consequences on the gauging of generalised diffeomorphisms, as we will in the next paragraph.

3.3 Exceptional field theories

In the previous section we introduced generalised diffeomorphisms. Exceptional field theory is the gauge theory of such generalised diffeomorphisms, i.e. the vector Λ^M with respect to which we are Lie-deriving will depend on internal and external coordinates. Exceptional field theory comprises (at least) the field content of D-dimensional maximal supergravity: the vielbein of the external metric, the internal metric $\mathcal{M}_{MN}$ representing the scalars, p-forms in their representations. Starting from $n \geq 7$, ExFT might have more p-forms than D dimensional maximal supergravity, as a consequence of the trivial parameters, as we are going to see shortly. Noticeably, one is able to write a bosonic action of exceptional field theory as a consequence of the mere bosonic symmetries (gauge symmetries, external and generalised diffeomorphisms). Indeed, as mentioned previously, we will not deal with fermions in ExFT.
Let us point out, that while the field content is that of D dimensional maximal supergravity, these fields depend on the whole D+dim$\mathbf{R_1}$ coordinates, both external and enlarged internal; on section ExFT describes the entire 11 dimensional (or IIB) supergravity[5].

Gauging $\mathcal{L}$ Considering (x^μ, Y^M) dependent vectors in the Lie derivative is like considering conventional coordinate transformations of the type: $y^m \to y^m + \xi^m(x,y)$, or (x,y)-dependent gauge parameters. As in Yang Mills theory, the gauging of $\mathcal{L}$ implies substitution of partial derivatives with covariant derivatives:

$$D_\mu = \partial_\mu - \mathcal{L}_{A_\mu} \ ,$$

where $A_\mu^M(x,Y)$ is the gauge connection of the symmetry, and are none other than the vector fields of ExFT, transforming in the $\mathbf{R}_1$ representation. A generic field is covariant under gauge generalised diffeomorphisms if $\delta_\Lambda \Phi = \mathcal{L}_\Lambda \Phi$; by imposing covariance of the covariant derivative, one deduces the required transformation rule for the connection:

$$\begin{aligned}
\delta_\Lambda(D_\mu\Phi) &\stackrel{!}{=} \mathcal{L}_\Lambda D_\mu \Phi \ , \\
\delta_\Lambda(D_\mu\Phi) - \mathcal{L}_\Lambda D_\mu\Phi &= \big(\mathcal{L}_{\partial_\mu\Lambda}\Phi + \mathcal{L}_\Lambda\partial_\mu\Phi - \mathcal{L}_{\delta_\Lambda A_\mu}\Phi - \mathcal{L}_{A_\mu}\mathcal{L}_\Lambda\Phi\big) - \\
&\quad - \big(\mathcal{L}_\Lambda\partial_\mu\Phi - \mathcal{L}_\Lambda\mathcal{L}_{A_\mu}\Phi\big) \ , \\
\Longrightarrow \quad \delta_\Lambda A_\mu^M &= \partial_\mu\Lambda^M - \mathcal{L}_{A_\mu}\Lambda^M + \text{trivial parameters} = \\
&= D_\mu\Lambda^M + \text{trivial parameters} \ .
\end{aligned}$$

The presence of trivial parameters in the above transformation rule, which is exactly like the infinitesimal transformation rule of connections in Yang Mills theories, should not surprise us, since A_μ^M is the vector field with respect to which we Lie-derive. By convention, however, we choose:

$$\delta_\Lambda A_\mu^M = D_\mu \Lambda^M$$

without trivial parameters. In analogy with Yang Mills theories, we can build A_μ^M's field strength tensor by the commutator of covariant derivatives:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\Phi = -\mathcal{L}_{F_{\mu\nu}}\Phi = 2\big(\partial_{[\mu} - \mathcal{L}_{A_{[\mu}}\big)\Big(\big(\partial_{\nu]} - \mathcal{L}_{A_{\nu]}}\big)\Phi\Big) - 2\mathcal{A}_{\partial_{[\mu}A_{\nu]}}\Phi + [\mathcal{L}_{A_\mu}, \mathcal{L}_{A_\nu}]\Phi \ ,$$

whence:

$$F_{\mu\nu}^M = 2\partial_{[\mu}A_{\nu]}^M - [A_\mu, A_\nu]_E^M \ , \tag{3.5}$$

where the last expression is called E-bracket, defined as:

$$[\Lambda_1, \Lambda_2]_E = \frac{1}{2}\big(\mathcal{L}_{\Lambda_1}\Lambda_2 - \mathcal{L}_{\Lambda_2}\Lambda_1\big) \ .$$

[5] This is sometimes called "generalised oxidation", while the opposite, i.e. considering the KK truncation of the theory, is actually a reduction, with a jargon somewhat nodding to chemistry.

The so defined field strength is covariant under generalised diffeomorphisms up to trivial parameters; indeed, one cannot write a kinetic term for A_μ^M which is completely invariant under generalised diffeomorphisms, as trivial parameters will spoil invariance. Therefore we will need to add trivial parameters to the field strength definition, thus yielding a Stueckelberg-type coupling for the gauge connection.
Restricting to $n \leq 6$ internal dimensions for simplicity, the transformation rule of 3.5 is:

$$\delta_\Lambda F_{\mu\nu}^M = \mathcal{L}_\Lambda F_{\mu\nu}^M + Y^{MN}{}_{PQ}\partial_N\big(- \Lambda^P F_{\mu\nu}^Q + A_{[\mu}^P \delta_\Lambda A_{\nu]}^Q \big) \ .$$

It can be proved that the right $F_{\mu\nu}^M$ redefinition is:

$$\mathcal{F}_{\mu\nu}^M = F_{\mu\nu}^M + Y^{MN}{}_{PQ}\partial_N B_{\mu\nu}^{PQ} \ ,$$

where the two-forms $B_{\mu\nu}^{MN}$ satisfy:

$$B_{\mu\nu}^{MN} = \frac{1}{2(n-1)} Y^{MN}{}_{PQ} B_{\mu\nu}^{PQ} \ .$$

The transformation rule of $B_{\mu\nu}^{MN}$ required to have strictly covariant $F_{\mu\nu}^M$ is:

$$\delta_\Lambda B_{\mu\nu}^{MN} = -\frac{1}{2(n-1)} Y^{MN}{}_{PQ}\Big(- \Lambda^P F_{\mu\nu}^Q + A_{[\mu}^P \delta_\Lambda A_{\nu]}^Q \Big) \ .$$

Therefore we have found that a covariant definition of field strength for 1-forms require the intervention of 2-forms; actually the full calculation would show that the 2-forms' field strength requires 3-forms and so on.
Let us mention moreover that the resulting theory enjoys more than gauged diffeomorphisms: there is also a gauge transformation acting upon the 2-forms and the vector fields as:

$$\begin{aligned} \delta_\Xi B_{\mu\nu}^{MN} &= 2\mathcal{D}_{[\mu]}\Xi_{\nu]}^{MN} + \frac{1}{4(n-1)^2} Y^{MN}{}_{PQ} A_{[\mu}^P \partial_R \Xi_{\nu]}^{QR} \ , \\ \delta_\Xi A_\mu^{MN} &= -Y^{MN}{}_{PQ}\partial_N \Xi_\mu^{PQ} \ , \end{aligned}$$

with gauge parameters $\Xi_\mu^{MN} = \Xi_\mu^{(MN)}$. This additional gauge symmetry justifies our previous claim about the convention on A_μ^M's transformation: indeed any trivial parameter could have been shifted away by Ξ_μ^{MN} gauge transformations.
Let us notice that this systematic covariant definition of field strengths, called tensor hierarchy, might require fields that do not appear originally in D dimensional maximal supergravity, as we will see in the next section for the specific $E_{7(7)}$ case. Furthermore, tensor hierarchy is is not a new concept in supergravity, as systematic redefinitions are needed as well in gauged supergravity.

Action of ExFT An action for ExFT is build in terms of the covariant derivatives, the covariantised field strengths such that each term of the action is a scalar density with weight 1, meaning that the Lagrangian is invariant under generalised diffeomorphisms up to total derivatives.
The weight of fields is fixed by their kinetic term having weight 1. The same can be said about the metric, which needs to transform under generalised diffeomorphisms as a scalar density of weight $\beta = 2/(9-n)$. This is obtained by noticing that the scalar matrix $\mathcal{M}_{MN}$, having unit determinant, has weight 0.
As we will see for the specific $E_{7(7)}$ case, the ExFT action also presents a topological term and a scalar potential, unlike maximal supergravity, the latter associated to the curvature of the internal manifold and leading to the possibility of non trivial vacuum expectation values, thus cosmological constant, scalar and fermion masses and supersymmetry breaking.
The coefficients of the various terms are fixed by invariance under external general coordinate transformations and, as mentioned previously, one is able to determine the full bosonic action just by imposing the bosonic symmetries.

3.3.1 The $E_{7(7)}$ case

We will now examine exceptional field theory with group $E_{7(7)}$. The exceptional Lie groups $E_{n(n)}$ are called the *split real forms* of the more general $E_{n(K)}$, with K being the difference between noncompact and compact generators. In particular $E_{7(7)}$ is a non compact simple Lie group, with dimension 133, equal of course to the dimension of the adjoint representation; the generators of the algebra will be denoted as t_α with $\alpha, \beta, \ldots$ in the adjoint. Indices in the adjoint can be raised or lowered by means of the Cartan-Killing metric $k_{\alpha\beta} = \left(t_\alpha\right)_M{}^N \left(t_\beta\right)_N{}^M$, which, for $E_{7(7)}$, can be diagonalised to diag$(+1, \ldots, -1 \ldots)$.
The fundamental representation is 56 dimensional and will be denoted by capital $M, N, P, \ldots$; the group is embedded in the Sp$(56, \mathbb{R})$, being the duality group for D=4. This embedding leads to the existence of the symplectic invariant Ω_{MN} (such that $\Omega_{MN} = \Omega^{MN}$, $\Omega^{MP}\Omega_{NP} = \delta_N^M$) which can be used to lower and raise fundamental indices[6], thus making **56** self-conjugate.
Under its GL(7) subgroup, corresponding to the group of reparameterizations of the internal manifold, the **56** of $E_{7(7)}$ decomposes as[7]:

$$\mathbf{56} \overset{\text{GL}(7)}{\rightarrow} \mathbf{7_{-3}}' + \mathbf{21_{-1}} + \mathbf{21_{+1}}' + \mathbf{7_{+3}} \ ,$$

or equivalently:

$$\Lambda^M \rightarrow (\xi^m, \lambda_{[mn]}, \lambda_{[mnpqr]}, \zeta_m) \qquad m, n = 1, \ldots 7 \ . \tag{3.6}$$

Where we recognize that the components of the generalised vector are: the vector ξ^m responsible for diffeomorphisms transformations, the 3-form gauge parameter, the 11d dual 6-form gauge parameter, some vectors ζ^m which, for reasons we will say in section 3.3.2, do not appear in the action. Notice that, thanks to the existence of the completely antisymmetric $\epsilon_{mnpqrst}$, we can trade, for instance, $\lambda_{[mnpqr]}$ for $\lambda^{[st]}$.
Let us also branch the **133** representation with respect to GL(7):

$$E_{7(7)} \overset{\text{GL}(7)}{\longrightarrow} \begin{array}{c} \mathbf{7'}_{+4} \\ \mathbf{35}_{+2} \\ \hline \mathbf{1}_0 + \mathbf{48}_0 \\ \hline \mathbf{35'}_{-2} \\ \mathbf{7'}_{-4} \end{array} \tag{3.7}$$

where $\mathbf{1}_0 + \mathbf{48}_0$ is the adjoint of GL(7), the $\mathbf{35}_{+2}$ corresponds to the solvable algebra of shifts of the 35 scalars coming from the 11d 3-form, while the $\mathbf{7'}_{+4}$ comes from the shifts of scalars dual in 4d to the seven 2-forms coming from the 11d 3-form.

For future convenience, let us also mention the branching under SL(8). This is useful because, as we said, in four dimensions vector fields are self-dual and form a vector in the **56**. There are different way to choose a symplectic frame[8], i.e. a splitting in electric and magnetic vectors, one of them consists in taking a SL(8) basis. In other words, we branch $E_{7(7)}$ with respect to its SL(8) and split the vectors in:

$$\mathbf{56} \overset{\text{SL}(8)}{\rightarrow} \mathbf{28} + \mathbf{28'} \ ,$$

or equivalently:

$$V^M \rightarrow (V_{AB}, V^{AB}) \qquad A, B = 1 \ldots 8 \ .$$

This[9] branching also helps us in getting some insight on the generators of $\mathfrak{e}_{7(7)}$, as indeed **133** splits in the following way:

$$\mathbf{133} \overset{\text{SL}(8)}{\rightarrow} \mathbf{63} + \mathbf{70} \ ,$$

[6]Since Ω is antisymmetric, we need conventions when contracting its indices; these are the NW-SE conventions: $V_M = V^N\Omega_{NM}$, $W^N = \Omega^{MN}W_N$.
[7]The following charges correspond to th GL(1)⊂GL(n) grading, not the trombone's.
[8]Different bases are described in [18].
[9]We are using uppercase english letters A, B for SL(8) representations as well, in addition to our previous conventions.

where **63** corresponds to the adjoint of SL(8), given by traceless matrices $\Lambda_A{}^B$, with $A, B = 1, \dots 8$. On the other hand, the **70** correspond to antisymmetric $\Sigma_{ABCD} = \Sigma_{[ABCD]}$. We can thus decompose the explicit $(t_\alpha)_M{}^N$ representations of the generators as:

$$t_M{}^N = \begin{pmatrix} t_{AB}{}^{CD} & t_{ABCD} \\ t^{ABCD} & t^{AB}{}_{CD} \end{pmatrix} = \begin{pmatrix} 2\delta_{[A}{}^{[C}\Lambda_{B]}{}^{D]} & \Sigma_{ABCD} \\ \frac{1}{24}\epsilon^{ABCDEFGH}\Sigma_{EFGH} & -2\delta_{[C}{}^{[A}\Lambda_{D]}{}^{B]} \end{pmatrix} . \tag{3.8}$$

This representation of the algebra generators are obviously not duality covariant, and we shall not employ these expressions.

Let us point out that, besides the symplectic invariant, the $E_{7(7)}$ also has a symmetric quartic invariant d_{MNPQ} given by[10]:

$$d_{MNPQ} \propto k^{\alpha\beta} t_{\alpha(MN} t_{|\beta|PQ)} .$$

We can write the aforementioned invariant tensor $Y^{MN}{}_{PQ}$ as a combination of the two previous invariants:

$$Y^{MN}{}_{PQ} = -12 k^{\alpha\beta} t_\alpha^{MN} t_{\beta PQ} - \frac{1}{2}\Omega^{MN}\Omega_{PQ} .$$

We can write the section constraint condition of $E_{7(7)}$ exceptional field theory in terms of $Y^{MN}{}_{PQ}$:

$$Y^{MN}{}_{PQ}\partial_M \otimes \partial_N = 0 ,$$

or equivalently as:

$$\begin{aligned} (t_\alpha)^{MN}\partial_M \otimes \partial_N &= 0 \\ \Omega^{MN}\partial_M \otimes \partial_N &= 0 . \end{aligned}$$

These section constraints can be also written as:

$$\left(\mathbb{P}_{\mathbf{1+133}}\right)^{MN}\partial_M \otimes \partial_N = 0 ,$$

given that the generators project on the adjoint rep., while the symplectic invariant is indeed a singlet.

Generalised diffeomorphisms of $E_{7(7)}$ Generalised Lie derivative admits two classes of trivial parameters:

$$\begin{aligned} \Lambda^M &= (t^\alpha)^{MN}\partial_M \chi_\alpha && \chi_\alpha \text{ arbitrary} \\ \Lambda^M &= \Omega^{MN}\partial_M \chi && \chi \text{ arbitrary} , \end{aligned}$$

however $E_{7(7)}$ ExFT admits a more general class of trivial parameters of the form:

$$\Lambda^M = \Omega^{MN}\chi_M \qquad \text{with } \left(\mathbb{P}_{\mathbf{1+133}}\right)^{MN}\chi_M\partial_N = 0 \quad \text{and} \quad \left(\mathbb{P}_{\mathbf{1+133}}\right)^{MN}\chi_M\chi_N = 0 .$$

The latter constrained imply, in other words, that the field χ_M is *covariantly constrained.* The very presence of these trivial parameters imply that the Jacobi identity is not exactly satisfied. This can be proved in terms of the E-bracket, by defining the so-called Jacobiator[11] as:

$$J(V_1, V_2, V_3) = 3[[V_{[1}, V_2]_E, V_{3]}]_E ,$$

which measures the lack of closure of the Jacobi identity. It turns out that the Jacobiator equals:

$$\begin{aligned} J^M(V_1, V_2, V_3) = &-\frac{1}{2}(t_\alpha)^{MK}\partial_K\big((t^\alpha)_{PL}(V_1^P[V_2, V_3]_E^L + \text{cycl.})\big) + \\ &+ \frac{1}{12}\Omega^{MK}\Omega_{NL}\big(V_1^N\partial_K[V_2, V_3]_E^L + [V_1, V_2]_E^N\partial_K V_3^L + \text{cycl.}\big) , \end{aligned}$$

hence equals trivial parameters.

[10] The generators of the algebra with both index lowered are symmetric in these indices.
[11] We refer to [19] for the full calculation.

Gauging As explained before, gauging generalised diffeomorphisms implies covariantizing the derivatives as:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathcal{L}_{A_\mu} \ ,$$

with suitable δA_μ^M to ensure covariance of the covariant derivative. The associated field strength:

$$F_{\mu\nu}^M = 2\partial_{[\mu} A_{\nu]}^M - [A_\mu, A_\nu]_E^M \ ,$$

does not transform covariantly:

$$\begin{aligned}\delta F_{\mu\nu}^M = &\ 2D_{[\mu}\delta A_{\nu]}^M - \partial_K A_{[\mu}^K \delta A_{\nu]}^M - 12(t_\alpha)^M K(t_\alpha)_{NL}\partial_K(A_{[\mu}^N \delta A_{\nu]}^L + \\ &- \frac{1}{2}\Omega^{MK}\Omega_{LN}(A_{[\mu}^N\partial_K\delta A_{\nu]}^L - \partial_K A_\mu^N \delta A_{\nu]}^L \ .\end{aligned}$$

To restore strict covariance, we need to introduce two corrections to the $F_{\mu\nu}^M$, describing a Stueckelberg type coupling. The first corrections employs a 2-form in the adjoint representation, $B_{\mu\nu}^\alpha$, of $E_{7(7)}$ which was already present in the field content of 4-dimensional supergravity (after suitable redefinitions, which we will not describe here). On the other hand, the second correction requires the introduction of a new two form in the fundamental representation, $B_{\mu\nu M}$, having no counterpart in the field content of 4-dimensional supergravity. The resulting field strength is:

$$\mathcal{F}_{\mu\nu}^M \equiv F_{\mu\nu}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MK}B_{\mu\nu K} \ , \tag{3.9}$$

where this compensator field $B_{\mu\nu M}$ needs to be covariantly constrained:

$$\left(\mathbb{P}_{\mathbf{1+133}}\right)^{MN} B_M \partial_N = 0 \qquad \left(\mathbb{P}_{\mathbf{1+133}}\right)^{MN} B_M B_N = 0 \ .$$

This allows for a covariant transformation of the field strength:

$$\delta\mathcal{F}_{\mu\nu}^M = 2D_{[\mu}\delta A_{\nu]}^M - 12(t^\alpha)^{MN}\partial_N \Delta B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MK}\Delta B_{\mu\nu K} \ ,$$

with:

$$\begin{aligned}\Delta B_{\mu\nu\alpha} &\equiv \delta B_{\mu\nu\alpha} + (t_\alpha)_{KL} A_{[\mu}^L \delta A_{\nu]}^L \ , \\ \Delta B_{\mu\nu K} &\equiv \delta B_{\mu\nu K} + \Omega_{LN}(A_{[\mu}^N \partial_K \delta A_{\nu]}^L - \partial_K A_{[\nu}^N \delta A_{\nu]}^L) \ .\end{aligned}$$

Notice that the corrected field strength $\mathcal{F}_{\mu\nu}^M$ and $F_{\mu\nu}^M$ differ by trivial paramereters, hence in particular:

$$\mathcal{L}_{\mathcal{F}_{\mu\nu}^M} = \mathcal{L}_{F_{\mu\nu}^M} \ .$$

Furthermore, one could worry that a transformation on $B_{\mu\nu M}$ of the kind written above might lead the 2-form to not being covariantly constrained anymore. This does not happen, i.e. the covariant constraint and the transformation property are compatible.
Let us finally mention that, as anticipated, the deformed tensor algebra enjoys new gauge symmetries acting as:

$$\begin{aligned}\delta A_\mu^M &= 12(t^\alpha)^{MN}\partial_N \Xi_{\mu\alpha} + \frac{1}{2}\Omega^{MN}\Xi_{\mu N} \ , \\ \Delta B_{\mu\nu\alpha} &= 2D_{[\mu}\Xi_{\nu]\alpha} \ , \\ \Delta B_{\mu\nu M} &= 2D_{[\mu}\Xi_{\nu]M} + 48(t^\alpha)_L^{\ K}(\partial_K\partial_M A_{[\mu}^L)\Xi_{\nu]\alpha} \ ,\end{aligned}$$

which leave the action invariant (with the correct weights for the gauge parameters).

Action In order to write a G invariant action, due to self-duality of the vector fields we need to write a pseudo action with all vector fields and their duals, endowed with a twisted self duality equation, the latter being:

$$*\mathcal{F}^M = \Omega^{MP}\mathcal{M}_{PN}\mathcal{F}^N \ , \tag{3.10}$$

and the action being the following:

$$S = \int d^4x \int d^{56}Y \sqrt{-g}\Big(\hat{R} + \frac{1}{48}g^{\mu\nu}D_\mu\mathcal{M}^{MN}D_\nu\mathcal{M}_{MN} - \frac{1}{4}\mathcal{M}_{MN}\mathcal{F}^M_{\mu\nu}\mathcal{F}^{\mu\nu N} - V_{\mathrm{E}_{7(7)}}\Big) + \\ + S_{\mathrm{top}} \ ,$$

based on the field content of D dimensional supergravity (with additional forms) and reproducing on section 11d or IIB maximal supergravity. Notice that the twisted self duality equation can be recovered in part by varying the action with respect to the two forms; varying with respect to $B_{\mu\nu\alpha}$ reproduces 3.10 only with internal derivatives; while varying with respect to $B_{\mu\nu M}$ yields 3.10, with the caveat that $B_{\mu\nu M}$ is constrained as well.
The weights under generalised diffeomorphisms are:

	e^a_μ	$\mathcal{M}_{MN}$	A^M_μ	$B_{\mu\nu\alpha}$	$B_{\mu\nu M}$
λ	1/2	0	1/2	1	1/2

Table 3.1: Weights under generalised diffeomorphisms

The Einstein Hilbert term is covariantized with respect to generalised diffeomorphisms

$$\hat{R}_{\mu\nu}{}^{ab} \equiv R_{\mu\nu}{}^{ab}[\omega] + \mathcal{F}_{\mu\nu}{}^{M} e^{a\rho}\partial_M e_\rho{}^b \ ,$$

with $R_{\mu\nu}{}^{ab}$ spin connection curvature, and derivatives covariantised as in:

$$D_\mu e^a_\nu \equiv \partial_\mu e_\nu{}^a - A^M_\mu \partial_M e_\nu{}^a - \frac{1}{2}\partial_M A^M_\mu e_\nu{}^a \ .$$

The topological term is written as a surface term in 4+1 external dimensions as:

$$S_{\mathrm{top}} = \frac{1}{6}\int d^{56}Y \int \mathcal{F}^M \wedge D\mathcal{F}^N \Omega_{MN} \ ,$$

and there exist an explicit expression for the scalar potential as well:

$$V_{\mathrm{E}_{7(7)}} = -\frac{1}{48}\mathcal{M}^{MN}\partial_M\mathcal{P}\mathcal{Q}\partial_N\mathcal{P}\mathcal{Q} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_P\mathcal{M}_{NQ} + \\ -\frac{1}{2}\frac{1}{g}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\frac{1}{g}\mathcal{M}^{MN}\partial_M g\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu}\partial_N g_{\mu\nu} \ .$$

The relative coefficients in the action, which are fixed in maximal supergravity by supersymmetry requirements, are instead fixed in ExFT by invariance of the action with respect to conventional diffeomorphisms in the external coordinates (depending on the Y internal coordinates as well).

3.3.2 Embedding 11d supergravity

We will now outline how one can recover the dynamics, gauge symmetries and fields of 11d supegravity from $\mathrm{E}_{7(7)}$ ExFT.
First of all, by evaluating the equations of motion of the $\mathrm{E}_{7(7)}$ ExFT on a solution of the section constraint, one recovers the dynamics of 11 dimensional supergravity, with coordinates in a 4+7 split. Recalling the branching 3.6, a solution of the section constraint is given by considering only the ∂_m component, and putting to zero $\partial^{mn} \to 0$, $\partial_{mn} \to 0$ and $\partial^m \to 0$.

For what concerns the symmetries of 11d supergravity, they are found by imposing the section constraint onto the generalised Lie derivative and by acting with a component of the branching of the generalised vector Λ^M, namely the vector with respect to SL(7), as in 3.6 for 11d orbit of solutions, upon the various components of the generalised tensors. For what concerns the fields of the theory, fields of the low dimensional 4d supergravity, also depending on the internal coordinates, denoted by lowercase english letters, are recovered upon branching the $E_{7(7)}$ representations with respect to SL(7). So for example the gauge fields become:

$$A_\mu^M \to \{A_\mu^m, A_{\mu\ mn}, A_\mu^{mn}, A_{\mu\ m}\} \ .$$

Where A_μ^m correspond to the Kaluza Klein vectors upon KK truncation, $A_{\mu\ mn}$ come from the 11d 3-form and A_μ^{mn} from the 6-form. The remaining $A_{\mu\ m}$ descend from the 11d dual graviton, however they do not enter the action. The dual graviton can be defined by considering one index of the metric (at linearised level) as a form index and computing its dual, obtaining:

$$h_{\mu_1,\ldots\mu_8;\nu} \ ,$$

where $\mu_1, \ldots, \mu_8$ are antisymmetric. The dual graviton too will have its gauge symmetries $\Lambda_{\mu_1,\ldots\mu_7;\nu}$. However, in general gauge vectors enter the action through the covariant derivative, which is subjected to the section constraint. Hence the only surviving components of the covariant derivative are:

$$D(A_\mu^M) \to D(A_\mu^m, \partial_{[k}A_{mn]}, \partial_k A^{km}) \ ,$$

with no trace of $A_{\mu\ m}$. Counting the number of components, we see that:

$$A_\mu^m : 7 \qquad \partial_{[k}A_{mn]} : 15 \qquad \partial_k A^{km} : 6 \ ,$$

for a total of 28 vector fields entering the dynamics of the theory, in compliance with vector fields' self-duality in 3+1 dimensions. The counting of the vector fields exploits the gauge symmetry $A_{mn} \to A_{mn} + \partial_{[m}a_{n]}$, which can be used to put 6 components to zero.

We can similarly check that the 2-forms $B_{\mu\nu}^\alpha$, $B_{\mu\nu}^M$, entering the field strenght definition, yield the correct number of components, i.e. 28. This is actually the case, since $B_{\mu\nu}^M$ is covariantly constrained and only 7 of its components survive, while $B_{\mu\nu}^\alpha$ enters the field strength definition with a derivative. It follows that there are only a few remaining components, discounting the gauge equivalent components:

$$\partial_{[m}B_{n]} : 6 \quad \partial_K B^{kmn} : 15 \ ,$$

yielding the correct number of components (28) entering the field strength $F_{\mu\nu}^M$.

For what concerns the scalars instead, they can be recovered upon choosing a specific gauge for the coset representative, for instance the triangular gauge, wherein the the higher dimensional origin of the scalar fields is more transparent. We can parameterise:

$$\mathcal{V} = \exp[\phi t_{(0)}]\ \mathcal{V}_7\ \exp[c_{kmn}t_{+2}^{kmn}]\exp[\epsilon^{klmnpqr}c_{klmnpq}t_{(+4)\ r}] \ ,$$

where $t_{(0)}$ is associated to the $GL(1) \subset GL(n)$ grading, and $t_{(+n)}$ are the other generators of $E_{7(7)}$ of positive grading, as described in 3.7. $\mathcal{V}_7$ is an element of the SL(7) subgroup; the scalars c_{kmn} and c_{klmnpq} descend from the internal components of the 3-form and 6-form.

Chapter 4

Generalised geometry

We will now examine the generalised geometry lying underneath exceptional field theory. We will define the generalised tangent bundle and structures thereupon, which will be crucial in the next chapters to perform consistent truncations. The references for this chapter are: [22], [23], [24], [25].

4.1 $\mathbf{E}_{n(n)} \times \mathbb{R}^+$ structure

Before delving into the topic, let us refresh some concepts from Riemannian geometry. A differential n-dimensional manifold has a GL(n) structure group, because tensors in different coordinate patches admit GL(n)-valued transition functions. A Riemannian manifold in particular admits a metric g_{mn} which can written at a certain $x \in \mathcal{M}$ as $g_{mn}(x) = e^a_m(x) e^b_n(x)\delta_{ab}$, where the (inverse) vielbein e^m_a parameterises the coset space GL(n)/SO(d). This means that the set of vectors $\{e_a(x)\}$ or any local SO(d) transformations thereof are bases of the tangent space at that point $x \in \mathcal{M}$. Hence, at any point we can dress tensors by factors of the vielbein $V^m = V^a e^m_a$.
Exceptional generalised geometry shares many similarities with Riemannian geometry (like the Lie derivative, the covariant derivatives, the torsion, curvature) but presents some differences too, such as the generalised Lie derivative forming a Leibniz algebroid and the existence of a whole family of torsionless connections compatible with the generalised metric.

Starting from a n dimensional Riemannian manifold, in order to describe 11d supergravity in terms of exceptional generalised geometry we define the generalised tangent bundle, locally isomorphic to:

$$\mathrm{E} \simeq \mathrm{TM} \oplus \wedge^2\mathrm{T}^*\mathrm{M} \oplus \wedge^5\mathrm{T}^*\mathrm{M} \oplus \left(\mathrm{TM} \otimes \wedge^7\mathrm{T}^*\mathrm{M}\right) \ , \tag{4.1}$$

where obviously if $n < 7$, some terms will be vanishing. A section $V^M \in \Gamma(\mathrm{E})$ is a generalised vector with components:

$$V^M = \begin{pmatrix} v^m \\ \lambda_{[m_1 m_2]} \\ \sigma_{[m_1 \dots m_5]} \\ \tau_{m[m_1 \dots m_7]} \end{pmatrix} \ , \tag{4.2}$$

or in an equivalent notation, we ca write the formal sum $V = v+\lambda+\sigma+\tau$. The p-forms λ, σ, τ are the parameters of gauge transformations of the p-forms of eleven dimensional supergravity, in particular λ is associated to the gauge symmetry of $C_{(3)}$, σ to the dual $C_{(6)}$'s, τ to the dual graviton's; τ only has seven of its indices antisymmetrized.
The isomorphisms 4.1 is allowed by a choice of the gauge connections of 11d supergravity. Indeed sections of E are made isomorphic to sections of the r.h.s. of 4.1 by the gauge connections satisfying (restricting to $n \leq 5$ for simplicity):

$$V \ = \ v \ + \lambda \ + \ i_v A \ = \ e^A \tilde{V} \ ,$$

where i_vA is the 2-form given by the interior product of v and A. Several A may allow the isomorphism, which is thus not unique; the exponential of the 3-form A, if developed, yields the l.h.s. formal sum.
Since A is patched in different coordinate patches (i) and (j) by gauge transformations:

$$A_{(i)} \ \to \ A_{(j)} \ + \ \mathrm{d}\Lambda_{(ij)} \qquad \text{on the overlap of } (i),(j) \ ,$$

this gives an explicit patching of sections of E along the manifold:

$$V_{(i)} \ = e^{\mathrm{d}\Lambda_{(ij)}} V_{(j)} \ , \qquad \text{on the overlap of } (i),(j) \ . \tag{4.3}$$

The expression $e^{\mathrm{d}\Lambda_{(ij)}}$, where $\Lambda_{(ij)}$ is a 2-form, is the exponentiation of the formal sum, yielding a number of tensors and p-forms of maximal n rank. It translates on v and λ as:

$$\begin{aligned} v_{(i)} \ &= \ v_{(j)} \ , \\ \lambda_{(i)} \ &= \ \lambda_{(j)} \ + \ i_{v_{(ij)}} \mathrm{d}\Lambda_{(ij)} \ . \end{aligned} \tag{4.4}$$

The generalised tangent bundle E encodes the conventional tangent bundle TM, as well as the twisting and fluxes of the form field potentials. We will see in the next chapter what we mean by twisting and fluxes and the role they will have in consistent truncations.

The fiber E_x, upon $x \in \mathcal{M}$, of the generalised tangent bundle is by construction a representation space of the $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ global symmetry group of maximal supergravity in $D = \{5,4,3,2\}$, as described in the previous chapters. In other words, sections of the bundle are naturally acted upon by transformations belonging to $\mathrm{E}_{n(n)} \times \mathbb{R}^+$. Such action can be explicitly seen by branching the $\mathbf{R}_1$ representation of the generalised vectors with respect to the $\mathrm{GL}(n)$ subgroup, as in 4.2, that acts in a conventional way on each vector and p-form component.
One might ask if the action of $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ is compatible with the patching, i.e. if the transformed section is still a section in some other coordinate charts (on the patches overlap). This is indeed the case; let us consider frames for the conventional tangent and cotangent space: $\hat{e}_a$ and e^a. We can extend these to frames of the fibers of wedge product of bundles, defining a specific choice of generalised frame of the tangent bundle:

$$\{E_A\} = \{\hat{e}_a\} \cup \{e^{ab}\} \cup \{e^{a_1 \dots a_5}\} \cup \{e^{a,[a_1 \dots a_7]}\} \ , \tag{4.5}$$

with $e^{a_1 \dots a_p} = e^{a_1} \wedge \dots \wedge e^{a_p}$, and $e^{a,[a_1 \dots a_7]} = e^a \ e^{a_1} \wedge \dots \wedge e^{a_7}$. The previous equation denotes a formal union, on the same line of the aforementioned formal sum; it is actually a tensor product of frames. A section V^M of E can thus be written as:

$$V = V^A E_A = v^a \hat{e}_a + \frac{1}{2}\lambda_{ab} e^{ab} + \frac{1}{5!}\sigma_{a_1 \dots a_5} e^{a_1 \dots a_5} + \frac{1}{7!}\tau_{a,a_1 \dots \dots a_7} e^{a,a_1 \dots a_7} \ .$$

Let us point out that $\{E_A\}$ is a set of generalised vectors, with $E_A \ = \ E_A{}^M \partial_M$; $\{E_A\}$ is just an example of a basis of E_x, coordinate or non-coordinate; choosing a coordinate chart on a patch $U \subset \mathcal{M}$ would mean a particular choice of 4.5, where $\{E_A\} = \{\partial/\partial x^m\} \cup \{\mathrm{d}x^m \wedge \mathrm{d}x^n\} \cup \dots$ and so forth.
Any frame $\{E_A\}$ is a collection of linearly independent vectors, with dual frame E^A satisfying: $E_B{}^M E_N{}^A = \delta_N{}^M$ and $E_M{}^B E_A{}^M = \delta_A{}^B$. For future convenience, let us mention that $\{\hat{E}_A\}$ can be split into:

$$E_A{}^M(y) = \rho(y)\, U_A{}^M(y) \quad \text{with } \rho \in \Gamma\big(\det(\mathrm{T}^*\mathrm{M})^{1/(9-n)}\big), \ \det U_A{}^M = 1 \ ,$$

while $\det E_A{}^M = r^{\dim \mathbf{R_1}}$, with $r = e^{1/(9-n)}$ (e being the conventional frame determinant), r representing the weight of $E_A{}^M$ under $\mathbb{R}^+$.
Technically the set $\{E_A\}$ is section of the frame bundle associated to E_x, the frame bundle F being:

$$F = \coprod_{x \in \mathcal{M}} \left\{ \big(x, \{E_A\}\big) \middle| \ E_A \text{ is a frame of } E_x \right\} \ .$$

which is a $\mathrm{G}(\dim\mathbf{R}_1,\mathbb{R})$ principal bundle, as any fiber is given by the group itself $\mathrm{G}(\dim\mathbf{R}_1,\mathbb{R})$. Since any linear combination of these vectors is itself a basis of E_x, there is a natural action of $\mathrm{GL}(\dim\mathbf{R}_1,\mathbb{R})$ onto $\{E_A\}$, i.e. F is a $\mathrm{GL}(\dim\mathbf{R}_1,\mathbb{R})$ principal bundle and $\mathrm{GL}(\dim\mathbf{R}_1,\mathbb{R})$ is the structure group of the bundle.
We can now define $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ bases as those related to any other by constant $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ transformations:

$$E_A \to E'_A = E_B(M^{-1})^B{}_A \qquad V^A \to V'^A = M^A{}_B V^B \ ,$$

with $M\in\mathrm{E}_{n(n)}\times\mathbb{R}^+$ The frame bundle $\tilde{F}$, or *structure bundle*, defined as:

$$\tilde{F} = \coprod_{x\in\mathcal{M}} \left\{ \big(x,\{E_A\}\big)\big|\ E_A \text{ is a } \mathrm{E}_{n(n)}\times\mathbb{R}^+\text{-equivalent frame of } E_x \right\}$$

is a sub-bundle of the frame bundle F, as its fibers are only subgroups of $\mathrm{G}(\dim\mathbf{R}_1,\mathbb{R})$, and is by construction a $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ principal bundle; we therefore say that the structure group of E is $\mathrm{E}_{n(n)}\times\mathbb{R}^+$. The action of $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ can be branched with respect to $\mathrm{GL}(n)$ on each component of E^M_A, thus on $\hat{e}_a$, e^a in the explicit example 4.5, which transform in a conventional way through $\mathrm{GL}(n)$.
The statement that $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ is the structure group of E is also supported by the existence of invariant tensors, such as the quartic invariant and the symplectic invariant in the $\mathrm{E}_{7(7)}$ case. Actually, one can define a restricted $\mathrm{GL}(n)\ltimes\mathcal{P}$ structure group on E, as can be seen by the patching 4.3, where $\mathrm{GL}(n)$ is as usual the geometric subgroup of $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ and $\mathcal{P}$ is associated to shifts of the p-forms of the higher dimensional theory (as described in the section 2.2). Indeed one can define a set of frames equivalent under $\mathrm{GL}(d)\ltimes\mathcal{P}$, called conformal split frames, which thus in general restrict to such semisimple group the structure group of E.
Let us mention that, in general, if the generalised tangent bundle has a structure group G, all subgroup of $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ containing G are also structure groups of the bundle.

The fact that the patching and the $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ action on E are compatible, translates in:

$$V^A_{(i)} = C^{\ A}_{B(ij)}V^B_{(j)} \quad \text{on the overlap } U_i\cap U_j \ ,$$

with $C^{\ A}_{B(ij)}\in\mathrm{E}_{n(n)}\times\mathbb{R}^+$, i.e. a change of coordinate can be written off as a $\mathrm{E}_{n(n)}\times\mathbb{R}^+$ transformation (on the overlap of patches).

The frame bundle $\tilde{F}$ can be used to build any kind of generalised tensors on $\mathcal{M}$, which are just sections of the vector bundle constructed from $\tilde{F}$ with different representations of $\mathrm{E}_{n(n)}\times\mathbb{R}^+$.
For example, the generalised cotangent bundle E^*:

$$\mathrm{E}^* \simeq \mathrm{T^*M}\oplus\wedge^2\mathrm{TM}\oplus\wedge^5\mathrm{TM}\oplus\big(\mathrm{T^*M}\otimes\wedge^7\mathrm{TM}\big)\ ,$$

is built from frames in the representation dual to the fundamental. In the $\mathrm{E}_{7(7)}$ case, the representation of the generalised frame of fibers of the generalised tangent bundle is $\mathbf{56_1}$, with the subscript $\mathbf{1}$ being the $\mathbb{R}^+$ charge, with $\mathbf{1_1}$ being $(\det\mathrm{T}^*M)^{1/2}$ (or in general $(\det\mathrm{T}^*M)^{1/(9-n)}$). On the other hand, the representation of the generalised frame of fibers of the generalised cotangent bundle is $\mathbf{56_{-1}}$. Notice that in the $\mathrm{E}_{7(7)}$ case, $\mathbf{56}$ is self conjugate because of the symplectic embedding, as we mentioned previously, but in general it may not be the case.
We can also define an adjoint bundle $\mathrm{ad}\tilde{F}$ given by:

$$\mathrm{ad}\tilde{F}\simeq\mathbb{R}\oplus(\mathrm{TM}\otimes\mathrm{T^*M})\oplus\wedge^3\mathrm{T^*M}\oplus\wedge^6\mathrm{T^*M}\oplus\wedge^3\mathrm{TM}\oplus\wedge^6\mathrm{TM}\ ,$$

whose fibers have a generalised frame in the adjoint representation. Let us also mention the vector bundle N given by the symmetric product of two copies of the tangent bundle; a section of N is given (up to a $\det(E_A)$ factor) by the generalised metric $\mathcal{M}_{MN}$, which we will mention later.

We can define generalised connections, which are first order linear differential operators D_M, whose action on $W^A E_A$ is:

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B \ ,$$

with $\Omega_M \in \Gamma(E^*)$, taking values in $\mathrm{ad}\tilde{F}$ (whence the A,B indices). The action of the connection straightforwardly extends to generalised tensors. For future convenience, we can define a specific choice of connection, Weitzenböck connection, as $W_{AB}{}^C = E_A{}^M E_B{}^N \partial_M E_N{}^C$.
Let us also define the generalised torsion; given a generalised connection D and the generalised Lie derivative, we define:

$$L^D_V(\alpha) - L_V(\alpha) = T(\alpha) \ ,$$

where L^D is the generalised Lie derivative with D instead of ∂, α is any generalised tensor such that the torsion T acts on it in the adjoint representation:

$$T : \Gamma(E) \to \Gamma(\mathrm{ad}\tilde{F}) \ . \tag{4.6}$$

We can even write an explicit expression of the generalised torsion acting upon a vector V^M as:

$$T(V) = V^C \Big[\Omega_C{}^A{}_B - \Omega_B{}^A{}_C - E^A (L_{\hat{E}_C} \hat{E}_B) \Big] \ \hat{E}_A \times_{\mathrm{ad}} E^B \ , \tag{4.7}$$

where $\times_{\mathrm{ad}}$ represents the projection onto the adjoint representation. We can give an explicit equation in coordinates, in terms of the Weitzenböck connection, as:

$$T_{AB}{}^C = 2W_{[AB]}^{\ C} + Y^{CD}_{\ \ EB} W^E_{DA} \ .$$

which is usually non constant.
Although the generalised torsion belongs a priori to the whole $E^* \otimes \mathrm{ad}\tilde{F}$, actually it is constrained to belong only to some sub-representations. Hence, while for the $\mathrm{E}_{7(7)}$ case, the full array of sub representations is:

$$\mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480} \ , \tag{4.8}$$

T has components only in the $\mathbf{912}_{-1}$ and $\mathbf{56}_1$ of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$.
As discussed in the section of gauged supergravity, the embedding tensor Θ^α_M too belongs to $E^* \otimes \mathrm{ad}\tilde{F}$ and occupies only some of the sub-representations (those allowed by the constraints) which are the same of the generalised torsion. In the following section we will see the tight connection between the torsion of the parallelising frames and the embedding tensor of the gauge group of the lower dimensional theory: in that case, they are identified. Indeed in general the embedding tensor encodes the torsion of the frame on the internal manifold (i.e. *geometric flux*) as well as the possible fluxes of p-forms, and other components with a less straightforward geometrical interpretation.

4.1.1 H_d structure

A special case of structure group is H_d, the maximal compact subgroup of $\mathrm{E}_{n(n)} \times \mathbb{R}^+$, which for $n = 7$ is $\mathrm{SU}(8)/\mathbb{Z}_2$. The existence of such structure group is equivalent, as we will see, to the existence of a generalised metric on the manifold, which encodes the bosonic higher dimensional fields' components along the internal space.
A H_d structure means that there is exist a set of frames of E_x, point by point, mapped to itself by H_d transformations, i.e. a H_d principal sub-bundle P_d of $\tilde{F}$:

$$P_d \subset \tilde{F} \text{ with fiber } H_d \ .$$

Equivalently there exists a H_d invariant tensor, the generalised unimodular metric $\mathcal{M}_{MN}$, which point by point parameterises the coset space $\mathrm{E}_{n(n)} \times \mathbb{R}^+/H_d$; if moreover a H_d invariant, symmetric, positive definite Δ_{AB} exists, then we can write:

$$\mathcal{M}_{MN} = U_M{}^A U_N{}^B \Delta_{AB}$$

as mentioned in the last chapter, with $U \in \mathrm{E}_{n(n)}/H_d$, in compliance with det$\mathcal{M}$=1.
It can be proven that, given a H_d structure $P_d \subset \tilde{F}$, there always exists a torsion-free, compatible generalised connection D, but it is not unique (unlike Riemannian geometry). Notice that the H_d structure of generalised tangent bundle is the direct counterpart of the SO(d) structure of the conventional tangent bundle of the internal surface.
In the literature, there may be some ambiguities in the name generalised metric. Indeed the above defined generalised metric is not, strictly, a metric, as its image is not scalars by scalar densities. This is due to its unimodularity; if instead we take the proper generalised metric, with non fixed determinant, then from two generalised vectors we would obtain a scalar, as in the conventional case. We will denote this last non-unimodular generalised metric by $\mathcal{G}$.

4.1.2 More general structure bundles

We have seen that the generalised tangent bundle has a $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ frame bundle, or equivalently, structure group. One may be interested in more general structure group; we define a generalised G structure group P as a $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ principal sub-bundle of $\tilde{F}$:

$$P \subset \tilde{F} \quad \text{with fiber } \; G \; .$$

This is just the statement that the transitions functions $C_M{}^N$:

$$V^N_{(i)} = C^N{}_M V^M_{(j)} \quad \text{on the overlap } U_i \cap U_j$$

between coordinate charts (i, j), belong to the group G. Given the existence of a G-equivalent frame, we can write any V^M as:

$$V^M = V^{\underline{M}} E_{\underline{M}}{}^M \; , \tag{4.9}$$

with $E_{\underline{M}}^M$ belonging to the coset space $\mathrm{E}_{n(n)} \times \mathbb{R}^+/G$, instead of $\mathrm{E}_{n(n)} \times \mathbb{R}^+/H_d$.
Typically a structure group can be identified by a set of tensors $\{K_a\}$ invariant under the action of G. By definition, at each point $x \in \mathcal{M}$ these G-invariant tensors define a G-equivalent set of frames, comprising the frames that leave $\{K_a\}$ invariant. These frame thus parameterise the coset space:

$$\frac{\mathrm{E}_{n(n)} \times \mathbb{R}^+}{G} \; .$$

A generalised connection D is called compatible with the structure group if it annihilates all the invariant tensors defining the structure groups itself:

$$D\{K_a\} = 0 \; .$$

Previously we defined the generalised torsion of a connection. We can now define the intrinsic torsion; we recall the formula for the torsion:

$$(L^D - L)(\alpha) \; = \; T(\alpha) \; , \tag{4.10}$$

with D any generalised connection compatible with the structure group. By branching the representation wherein $T(\alpha)$ lives with respect to G, the sub-representations that are independent of the choice of D form the *generalised intrinsic torsion.* Such concept will be useful when we will deal with consistent truncations.

4.2 Identity structure and generalised parallelisations

Identity structures are of particular interest in exceptional generalised geometry as they allow truncations of 11d (or IIB) supergravity, to gauged supergravity theories with the same number of supersymmetries; hence they allow truncations to maximal supergravity as well. In analogy to the

general structure groups described above, the identity structure implies that there exists a set of vectors $\{E_A(x)\}$ constituting a frame for the fiber E_x for every $x \in \mathcal{M}$, i.e. they do not need to be H_d-transformed (or generically G-transformed) from point to point, as they are globally defined.
The existence of these frames is non trivial, we will see their role in the case group manifolds, and implies that the generalised tangent bundle E is actually globally trivial:

$$E \overset{\text{glob}}{=} \mathcal{M} \times \mathbb{R}^{\dim \mathbf{R}_1} .$$

In turn, the global triviality of E implies that the spinor bundle[1] is trivial as well. We will see in the next chapter what the triviality of the spinor bundle implies at the level of supersymmetry, i.e. how many supersymmetries survive a generalised parallelisation or a consistent truncation in general.
Let us mention that, while there are cases of manifolds whose conventional and generalised tangent bundles are both parallelisable, such as tori, this is not generally the case. We will now turn to the case where the existence of globally defined $\{\hat{E}_A\}$ is guaranteed: group manifolds.

Group manifolds in conventional and generalised geometry As mentioned in the previous chapter, conventional local group manifolds admit the transitive action of a Lie group G and admit an algebra-valued one form, invariant under G:

$$g^{-1}dg = e^a t_a ,$$

where $g \in G$, and t_a are the $\mathfrak{g}$ generators. $e^a(y)$ is the vielbein 1-form upon the manifold itself, which, belonging to $\mathfrak{g}$, is G invariant (on the left, by convention). The structure of the algebra is then encoded in the G transformation of e^a from the right:

$$[e_a, e_b] = f_{ab}{}^c e_c \quad \text{or} \quad \mathcal{L}_{e_a} e_b = f_{ab}{}^c e_c ,$$

where e_a is e^a's inverse, and $f_{ab}{}^c$ are $\mathfrak{g}$'s structure constants. We thus found a set $\{e_a\}$ of globally defined and linearly independent vectors: elements of the algebra can be left (or right) translated all over the manifold and, if they were not linearly independent, they would not form an algebra in the first place.
In the generalised case, the analogy is straightforward: a generalised group manifolds admit a set $\{E_A\}$ of vectors, globally defined and linearly independent, forming a frame of the fibers of the generalised tangent bundle and satisfying the *Leibniz parallelisation condition*:

$$L_{E_A} E_B = -X_{AB}{}^C E_C \quad \text{with } X_{AB}{}^C \text{ constant} , \tag{4.11}$$

$X_{AB}{}^C$ being the generalised torsion of the generalised frame.

4.3 The generalised parallelisation of S^n

Famously, only S^1, S^3 and S^7 are conventionally parallelisable, while generalised geometry allows to generalised parallelise all spheres S^n, under the conditions we are about to describe, as first reported in [23].
The only requirement for S^n to be generalised parallelisable is the existence of a top-dimensional field strength $F = \mathrm{d}A$, such that the equation of motion of the metric g_{mn} is:

$$R_{mn} = \frac{1}{n-1} F^2 g_{mn} \qquad \text{with} \qquad F = \frac{n-1}{R} \mathrm{vol}_g ,$$

where we are using coordinates on the sphere x^m with $m = 1 \dots n$, and the sphere has radius R.
The field strength is needed to be proportional to the volume form in order to consistently factorise in the equations of motion, as we are going to explain at length in the next chapter.

[1] The spinor bundle is just a vector bundle built from the $\mathrm{E}_{n(n)}$ frame bundle in a spinorial representation

Let us review a few aspects of the conventional geometrical descriptions of spheres. The sphere S^n can be embedded in a $d+1$ dimensional space as the locus of y^i, with $i = 1 \ldots n+1$, satisfying $y^i y^j \delta_{ij} = R^2$. In these coordinates, the line element on the sphere is:

$$\mathrm{d}s^2 = R^2 \delta_{ij} \mathrm{d}y^i \mathrm{d}y^j \ ,$$

while the metric in embedding coordinates is simply found by:

$$g_{mn} = R^2 \partial_m y^i \partial_n y^j \delta_{ij} \ .$$

The sphere has rotational Killing vectors v_{ij}^m whose expression is:

$$v_{ij}^m = y_{[i} \partial_n y_{j]} g^{mn} \ ,$$

forming a SO$(n+1)$ algebra:

$$[v_{ij}, v_{kl}] = R^{-1} \, 4 \, \delta_{[i|[k} v_{l]|j]} \ ,$$

and by definition $\mathcal{L}_v g_{mn} = 0$.
The generalised geometry description of the S^n extends its tangent bundle to:

$$E \simeq \mathrm{TM} \oplus \wedge^{n-2} \mathrm{T}^* \mathrm{M} \ ,$$

where the $(n-2)$-form encodes the parameter of gauge transformations of the aforementioned A $(n-1)$-form.
Sections of E have $n(n+1)/2$ entries, which suggests to locate a GL$(n+1)$ structure on E, i.e. to express indices of sections of E as a pair of antisymmetrized fundamental GL$(n+1)$ indices. This allows to branch:

$$V^M = V^{ab} = (V^{m0}, V^{mp}) \ ,$$

with $a, b = 1 \ldots n+1$ GL$(n+1)$ indices, $m, p = 1 \ldots n$ denoting the chosen coordinates on the sphere. We can identify:

$$\begin{aligned} V^{m0} &= v^m \in \Gamma(\mathrm{TM}) \ , \\ V^{mp} &= \lambda^{mp} \in \Gamma(\wedge^2 \mathrm{TM} \otimes \det \mathrm{T}^* \mathrm{M}) \ , \end{aligned}$$

where we used the isomorphisms $\wedge^2 \mathrm{TM} \otimes \det \mathrm{T}^*\mathrm{M} \simeq \wedge^{d-2} \mathrm{T}^* \mathrm{M}$.
The generalised structure bundle E has a H_d structure: indeed in all cases the scalars of the lower dimensional theory parameterise a coset space G/H_d through a generalised metric $\mathcal{M}_{MN}$. The generalised metric is of course invariant under H_d, but is also SO$(n+1)$ invariant, as the structure group GL$(n+1)$ we built on E has SO$(d+1)$ as maximal compact subgroup.
The generalised frame are thus labeled by a pair of antisymmetrized SO$(d+1)$ indices $\{\hat{E}_{ij}\}$ and obey the orthogonality condition:

$$\mathcal{G}(\hat{E}_{ij}, \hat{E}_{kl}) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \ ,$$

and, by SO$(n+1)$ invariance, any local rotation of the frame:

$$\hat{E}'_{ij} = U_i{}^k U_j{}^l \hat{E}_{kl}$$

with $U \in$ SO$(n+1)$, is an orthonormal generalised frame as well. Actually, since U and $-U$ generate the same transformation on E, the structure group we are considering on E is SO$(n+1)/\mathbb{Z}_2$, for n odd.
The result by [23], is that there exists a generalised orthonormal frame on any sphere S^n, globally defined and linearly independent, that yields a constant torsion. Such frame is:

$$\hat{E}_{ij} = v_{ij} + \sigma_{ij} + i_{v_{ij}} A \ , \tag{4.12}$$

thus the first component $v_{ij} = v_{ij}^m \partial_m$ of the frame are the rotational Killing vectors, while the second is the formal sum is σ_{ij}, with:

$$\sigma_{ij} = *\big(R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j\big) = \frac{R^{n-2}}{(n-2)!}\epsilon_{ijk_1\dots k_{n-1}} y^{k_1} \mathrm{d}y^{k_2} \wedge \dots \wedge \mathrm{d}y^{k_{d-1}} \ ,$$

and $i_{v_{ij}}A$ is just the interior product between the rotational killing vectors and the $(n-1)$ form A, thus the last two elements of the formal sum are both $(n-2)$-forms. We remind that the meaning of this formal sum is that the first element constitutes the first entry of the vector ${E_{ij}}^M$, while the last two belong to the second entry.
The frame is globally defined because the two components never vanish at the same time: the Killing vectors vanish only at $y^i = y^j = 0$, while the wedge product $\mathrm{d}y^i \wedge \mathrm{d}y^j$ vanishes on the surface $y_i^2 + y_j^2 = R^2$, i.e. when $\delta_{ij} y^i y^j = R^2$ holds.
The so defined frame can be shown to be orthonormal:

$$\mathcal{G}(\hat{E}_{ij}, \hat{E}_{kl}) = v_{ij} \cdot v_{kl} + \sigma_{ij} \cdot \sigma_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \ .$$

Moreover, we can compute the generalised Lie derivative:

$$\begin{aligned} L_{\hat{E}_{ij}} \hat{E}_{kl} &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}(\sigma_{kl} + i_{v_{kl}}A) - i_{v_{kl}} \mathrm{d}(\sigma_{ij} + i_{v_{ij}}A) = \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + i_{[v_{ij}, v_{kl}]}A - i_{v_{kl}}(\mathrm{d}\sigma_{ij} - i_{v_{ij}}F) = \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + i_{[v_{ij}, v_{kl}]}A \ . \end{aligned}$$

We will not dwell on the identities needed to go from one passage to the other, which can be found indeed in [23] and its appendices. At the end:

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = 4R^{-1} \delta_{[i|[l} \hat{E}_{k]|j]} \ , \tag{4.13}$$

which is an $\mathfrak{so}(\mathfrak{n}+\mathfrak{1})$ algebra obeyed by the frames and being, as we will see, the gauge group of lower dimensional gauged supergravity with the same number of supercharges as the higher dimensional theory.

4.4 Case study: ${X_{AB}}^C$ of S^4 parallelisation

We are now going to study the generalised parallelisation of S^4, which is already a known fact, however we will examine it in a form which, as far as we know, has not appeared yet in the literature. We are going to give an ansatz of the parallelising frame, in terms of the y^i coordinates and a 4-form, and actually prove that it leads to the Leibniz parallelisation condition 4.11 under some assumption on the frame's components.
In the generalised geometric description of S^4, we consider the GL(5) structure group on the generalised tangent bundle, given by GL(5) = SL(5) $\times \mathbb{R}^+$. Its frame ${E_A}^M$ can be written as:

$${E_A}^M = {E_{ij}}^{\tilde{i}\tilde{j}} = {E_{[ij]}}^{[\tilde{i}\tilde{j}]} = {E_{[i}}^{[\tilde{i}}{E_{j]}}^{\tilde{j}]} \ ,$$

with M curved index in the $\mathbf{R_1} = \mathbf{10}$ representation of SL(5), A flat index in the $\mathbf{10}$ of SO(5), $i, j = 1 \dots 5$ in the fundamental of SL(5) and $\tilde{i}, \tilde{j} = 1 \dots 5$ in the fundamental of SO(5). The last passage is allowed since the $\mathbf{10}$ is given by the antisymmetrization of two $\mathbf{5}$.
Since $\mathbf{10}$ branches according to SL(4) $\subset$ SL(5) as:

$${E_A}^M \overset{\mathrm{SL}(4)}{\to} ({E_A}^{mn}, {E_A}^{m5}) \ ,$$

where $m = 1 \dots 4$, fundamental representation of SL(4), then the 11-d orbit of solution of the section constraint leads to:

$$\hat{E}_A{}^M \partial_M = \hat{E}_A{}^{m5} \partial_m \ .$$

Let us now give the ansatz for $E_i{}^{\tilde{j}}(y)$:

$$E_i{}^{\tilde{j}} = \big(A(x)\big(g^{mm}\partial_n y_i + c^m y_i\big) \quad | \quad B(x) y_i\big) \ ,$$

and its inverse:

$$E_{\tilde{i}}{}^{i} = \begin{pmatrix} B(x)\partial_m y^i \\ A(x)(y^i - c^n\partial_n y^i) \end{pmatrix}$$

satisfying orthogonality:

$$E_i{}^{\tilde{j}} E_{\tilde{j}}{}^{k} = \delta_i^k \qquad E_{\tilde{j}}{}^{i} E_i{}^{\tilde{k}} = \delta_{\tilde{j}}^{\tilde{k}}$$

if $B(y) = A^{-1}(y)$. The function c^m can be rewritten as:

$$c^m \ \epsilon_{mn_1\dots n_{d-1}} = c_{n_1\dots n_{d-1}}$$

a $(d-1)$-form, being related to the potential A mentioned in section 4.3. The function A(x) represents the contribution of the trombone symmetry (it has nothing to do with the potential).
The goal of this section is to compute the torsion of such generalised frame and verify that, under conditions upon A and c^m, it is constant. In order to do so, we will employ the Weitzenböck connection (in GL(5) indices):

$$W_{ij}{}_{kl}^{pq} = E_{ij}^{\tilde{i}\tilde{j}} E_{kl}^{\tilde{k}\tilde{l}} \partial_{\tilde{i}\tilde{j}} E_{\tilde{k}\tilde{l}}^{\tilde{p}\tilde{q}} \ ,$$

and the formula of the torsion in terms of the connection:

$$-X_{AB}{}^{C} = W_{AB}{}^{C} - W_{BA}{}^{C} + Y^{CD}{}_{EB} W_{DA}^{E} \ ,$$

where:

$$Y^{AB}{}_{CD} = Y^{pq}_{st}{}^{df}_{kl} = 6\ \delta^{pqdf}_{stkl} \ .$$

Before delving into the calculation, let us rewrite explicitely the above expression:

$$\begin{aligned} Y^{pq}_{st}{}^{df}_{kl} &= \delta_s^{[p}\delta_t^{q]}\delta_k^{[d}\delta_l^{f]} + \delta_s^{[p}\delta_t^{d]}\delta_k^{[f}\delta_l^{q]} + \delta_s^{[p}\delta_t^{f]}\delta_k^{[q}\delta_l^{d]} + \\ &+ \delta_s^{[d}\delta_t^{q]}\delta_k^{[f}\delta_l^{p]} + \delta_s^{[f}\delta_t^{q]}\delta_k^{[p}\delta_l^{d]} + \delta_s^{[d}\delta_t^{f]}\delta_k^{[p}\delta_l^{q]} \ . \end{aligned}$$

Whence:

$$Y^{pq}_{st}{}^{df}_{kl} W_{df}{}^{st}_{ij} = W_{kl}{}^{pq}_{ij} + W_{df}{}^{df}_{ij}\delta^{pq}_{kl} - 4\delta^{[q}_{[l} W_{k]d}{}^{p]d}_{ij} \ ,$$

whence:

$$X_{ij}{}^{pq}_{kl} = -W_{ij}{}^{pq}_{kl} - W_{df}{}^{df}_{ij}\delta^{pq}_{kl} + 4\delta^{[q}_{[l} W^{p]d}_{k]d\,ij} \ . \tag{4.14}$$

Let us examine the first piece:

$$\begin{aligned} -W^{pq}_{ij\,kl} &= E_{ij}^{\tilde{i}\tilde{j}} E_{kl}^{\tilde{k}\tilde{l}} \partial_{\tilde{i}\tilde{j}} E_{\tilde{k}\tilde{l}}^{\tilde{p}\tilde{q}} = -E_{ij}{}^{m5} E_{kl}^{\tilde{k}\tilde{l}} \partial_{m5} E_{\tilde{k}\tilde{l}}{}^{pq} = \\ &= v^m_{ij} E_{kl}^{\tilde{k}\tilde{l}} \partial_m E_{\tilde{k}\tilde{l}}{}^{pq} = 2 v^m_{ij} E_{[k}{}^{\tilde{k}} \delta^{[q}_{l]} \partial_m E_{\tilde{k}}{}^{p]} \ . \end{aligned}$$

In the above passages we employed the solution of the section constraint and the definition of v^m_{ij}.
In the following we will insert the explicit ansatz for $U_i{}^{\tilde{j}}$ and its inverse; the calculations are made easier by employing the following results:

$$\begin{aligned} v^m_{d[j} y_{i]} &= \frac{1}{2} y_d v^m_{ij} \ , \\ y_d \partial^n y^d &= 0 \ , \\ y_d \partial_m \partial_n y^d &= -g_{mn} \ , \\ \partial^m y_i \partial_m y_j &= y_i y_j - \delta_{ij} \ , \end{aligned}$$

and by assumming that $A = \det(g)^\beta$, from which:

$$\mathcal{L}_v A = 0 \;\rightarrow\; v^m \partial_m \ln(A) + 2\beta \partial_m v^m_{ij} = 0 \;,$$
$$\partial_m \ln A = \beta \partial_m \ln(g) = \beta \mathrm{tr}\big[g^{-1}\partial g\big] = 2\beta \partial_m \partial_a y_i \partial^a y^i \;.$$

Let us examine the first piece in eq. 4.14:

$$\begin{aligned}
-W_{ij}{}^{pq}_{kl} &= 2v^m_{ij}\big[U^n_{[k}\delta^{[q}_{l]}\partial_m U^{p]}_n\big] + E^5_{[k}\delta^{[q}_{l]}\partial_m E^{p]}_5\big] = \\
&= 2v^m_{ij}\delta^{[q}_{[l}\partial_m \ln(A)\big[-\delta^{p]}_{k]} - 2c^n\partial_n y^{p]}y_{k]} + 2y_{k]}y^{p]}\big] + \\
&+ 2v^m_{ij}\delta^{[q}_{[l}\partial^n y_{k]}\partial_m\partial_n y^{p]} - 2v^m_{ij}\delta^{[q}_{[l}y_{k]}\partial_m c^n \partial_n y^{p]} + \\
&+ 2y_{[i}\delta^{[p}_{j]}y_{[k}\delta^{q]}_{l]} \;.
\end{aligned}$$

The middle piece (without the delta) in 4.14 is:

$$\begin{aligned}
-W_{df}{}^{df}_{ij} &= 2v^m_{df}\delta^{[f}_{[j}\partial_m \ln(A)\big[-\delta^{d]}_{i]} - 2c^n\partial_n y^{d]}y_{i]} + 2y_{i]}y^{d]}\big] + \\
&+ 2v^m_{df}\delta^{[f}_{[j}\partial^n y_{i]}\partial_m\partial_n y^{d]} - 2v^m_{df}\delta^{[f}_{[j}y_{i]}\partial_m c^n\partial_n y^{d]} + \\
&+ 2y_{[d}\delta^{[d}_{f]}y_{i]}\delta^{f]}_{j]} = \\
&= -2v^m_{ij}\partial_m \ln(A) + 2v^m_{ij}\partial_m \ln(A) + 2v^n_{ij}\partial^m y_d \partial_m \partial_n y^d = \\
&= \frac{1}{2\beta} v^m_{ij}\partial_m \ln(A) \;.
\end{aligned}$$

The third and last piece is a bit trickier:

$$\begin{aligned}
W_{kd}{}^{pd}_{ij} &= -2v^m_{kd}\delta^{[d}_{[j}\partial_m \ln(A)\big[-\delta^{p]}_{i]} - 2c^n\partial_n y^{p]}y_{i]} + 2y_{i]}y^{p]}\big] + \\
&- 2v^m_{kd}\delta^{[d}_{[j}\partial^n y_{i]}\partial_m\partial_n y^{p]} + 2v^m_{kd}\delta^{[d}_{[j}y_{i]}\partial_m c^n\partial_n y^{y]} + \\
&- 2y_{[k}\delta^{[p}_{d]}y_{[i}\delta^{d]}_{j]} \;,
\end{aligned}$$

which, after a few lengthy manipulations, can be written as:

$$\begin{aligned}
W_{kd}{}^{pd}_{ij} &= y_k\partial^m y_{[j}\delta^{[p}_{i]}\partial_m \ln(A) - y_{[j}\delta^{p}_{i]}\partial^m y_k \partial_m \ln(A) + \frac{1}{2}y_k v^m_{ij}\partial_m \ln(A)\big[2c^n\partial_n y^p - 2y^p\big] + \\
&- y_{[i}\delta^{[p}_{j]}\partial^m y_k\partial_m \ln(A) - \frac{1}{2}v^m_{ij}\partial^n y_k \partial_m\partial_n y^p + + \frac{1}{2}y_k\partial^n y_{[i}\delta^p_{j]}\frac{1}{2\beta}\partial_n \ln(A) + \\
&+ \frac{1}{2}\delta_{k[i}\delta^p_{j]} - \frac{1}{2}y_k y_{[i}\delta^p_{j]}\big[1 + \partial_m c^m - D + 2\big] + + \frac{1}{2}y_k v^m_{ij}\partial_m c^n \partial_n y^p + \\
&- c^m\partial_m \ln(A) y_{[i}\delta^p_{j]}y_k \;.
\end{aligned}$$

which in turn can be simplified to:

$$\begin{aligned}
W_{kd}{}^{pd}_{ij} &= \frac{1}{2}y_k\partial^n y_{[i}\delta^p_{j]}\partial_n \ln(A)\left(\frac{1}{2\beta} - 2\right) + \frac{1}{2}y_k v^m_{ij}\partial_m \ln(A)\Big[2c^n\partial_n y^p - 2y^p\Big] - \frac{1}{2}v^m_{ij}\partial^n y_k\partial_m\partial_n y^p + \\
&+ \frac{1}{2}\delta_{k[i}\delta^p_{j]} - \frac{1}{2}y_k y_{[i}\delta^p_{j]}\Big[3 - D + \partial_m c^m + 2c^m \ln(A)\Big] + \frac{1}{2}y_k v^m_{ij}\partial_m c^n\partial_n y^p \;.
\end{aligned}$$

Now we put all three pieces together as in equation 4.14; the pieces underlined with the same number will cancel out:

$$X_{ij}{}^{pq}_{kl} = 2v^m_{ij}\delta^{[q}_{[l}\partial_m \ln(A)\Big[-\delta^{p]}_{k]} - \underbrace{2c^n\partial_n y^{p]}y_{k]} + 2y_{k]}y^{p]}}_{(1)}\Big] + \underbrace{2v^m_{ij}\delta^{[q}_{[l}\partial^n y_{k]}\partial_m\partial_n y^{p]}}_{(2)} +$$

$$-\underbrace{2v^m_{ij}\delta^{[q}_{[l}y_{k]}\partial_m c^n\partial_n y^{p]}}_{(3)} + 2y_{[i}\delta^{[p}_{j]}y_{[k}\delta^{q]}_{l]} + \frac{1}{2\beta}v^m_{ij}\partial_m\ln(A)\delta^{pq}_{kl} +$$

$$+\underbrace{2y_{[k}v^m_{ij}\partial_m\ln(A)\Big[2c^n\partial_n y^{[p} - 2y^{[p}\Big]\delta^{p]}_{l]}}_{(1)} - 2\underbrace{v^m_{ij}\partial^n y_{[k}\partial_m\partial_n y^{[p}}_{(2)}\delta^{q]}_{l]} + 2y_{[k}\partial^n y_{[i}\delta^{[p}_{j]}\partial_n\ln(A)\delta^{q]}_{l]}\Big[\frac{1}{2\beta} - 2\Big] +$$

$$+ 2\delta^{[q}_{[l}\delta_{k][i}\delta^{p]}_{j]} + \underbrace{2y_{[k}v^m_{ij}\partial_m c^n\partial_n y^{[p}\delta^{q]}_{l]}}_{(3)} - 2y_{[k}\delta^{[q}_{l]}y_{[i}\delta^{p]}_{j]}\Big[3 - D + \partial_m c^m + 2c^m\partial_m\ln(A)\Big] \ .$$

We are now required to make the assumption $\beta = \frac{1}{4}$, in order for the torsion to be constant. We will also use the equation:

$$\nabla_m c^m = \frac{1}{\sqrt{|g|}}\partial_m(\sqrt{|g|}c^m)$$

where g is the determinant of the internal manifold. We thus arrive to:

$$X_{ij}{}^{pq}_{kl} = 2\delta^{[q}_{[l}\delta_{k][i}\delta^{p]}_{j]} - 2y_k y_i\delta^{[p}_{j]}\delta^{q]}_{l]}\Big[2 - D + \nabla_m c^m\Big] \ .$$

Assuming now that:

$$\nabla_m c^m = D - 2$$

where D=5 (the dimensions of the embedding coordinates) which is an assumption on the flux of the 4-form on the sphere, we finally obtain:

$$X_{ij}{}^{pq}_{kl} = 2\delta^{[q}_{[l}\delta_{k][i}\delta^{p]}_{j]}$$

describing the embedding tensor of 7d $\mathcal{N} = 8$ SO(5) gauged supergravity, which, as we are going to describe in the next chapter, is the lower dimensional theory obtained by consistent truncation of 11d on the S^4.

4.5 S^7 generalised parallelisation

In order to build the S^7 parallelisation we will consider the GL($d+1$) structure on the generalised tangent bundle; recalling that 4d maximal supergravity has a $E_{7(7)} \times \mathbb{R}^+$ global symmetry group, we can branch the **56** representation with respect to SL(8) $\subset$ GL(8) in:

$$\mathbf{56} \overset{\mathrm{SL}(8)}{\to} 28 + \bar{28} \ ,$$

which is equivalent to branching the generalised tangent bundle into:

$$E \simeq E^{(0)} \oplus E^{(1)} \ ,$$

with:

$$E^{(0)} \simeq \mathrm{TM} \oplus \wedge^5\mathrm{T}^*\mathrm{M} \qquad E^{(1)} \simeq \wedge^2\mathrm{T}^*\mathrm{M} \oplus \big(\mathrm{TM} \otimes \wedge^7\mathrm{T}^*\mathrm{M}\big) \ .$$

We can give ansatzes for the generalised parallelising frames for both parts of the bundle:

$$\hat{E}_A = \begin{cases} \hat{E}_{ij} = v_{ij} + \sigma_{ij} + i_{v_{ij}}A & \text{for } E^{(0)} \\ \hat{E}^{ij} = \omega_{ij} + \tau_{ij} - jA \wedge \omega_{ij} & \text{for } E^{(1)} \ . \end{cases} \tag{4.15}$$

In the above definition, $\tau_{ij} = R\big(y_i \mathrm{d}y_i - y_j \mathrm{d}y_j\big) \otimes \mathrm{vol}_g$, and j is defined to be:

$$(jA \wedge \omega)_{i,j_1 \ldots j_7} = \frac{7!}{5!\ 2!} A_{i[j_1 \ldots j_5} \omega_{j_6 j_7]} \ .$$

Analogously to the general case, ω_{ij} vanishes on the hypersurface $y_i^2 + y_j^2 = R^2$, while τ_{ij} only at the origin $y_i = y_j = 0$. We can compute the generalised Lie derivative of each component of the frame with respect to the other, finding:

$$\begin{aligned} L_{\hat{E}_{ij}} \hat{E}_{kl} &= 4R^{-1} \left(\delta^n_{[j} \delta_{i][l} \delta^m_{k]} \hat{E}_{mn} \right) , \\ L_{\hat{E}_{ij}} \hat{E}^{kl} &= 4R^{-1} \left(\delta^{[l}_{[i} \delta_{j]p} \hat{E}^{k]p} \right) , \\ L_{\hat{E}^{ij}} \hat{E}_{kl} &= 0 \ , \\ L_{\hat{E}^{ij}} \hat{E}^{kl} &= 0 \ . \end{aligned}$$

The constant torsion proves that the frame 4.15 provides a generalised parallisation of S^7. As we will explain in the following chapter, the 4d theory obtained by consistent truncation on S^7 with such frame, from 11d supergravity, is maximal and has a gauge group SO(8). In the next paragraph, we will indeed verify that the embedding tensor of SO(8) in 4d $\mathcal{N} = 8$ supergravity corresponds to the above expression of the generalised torsion.

SO(8) gauging of 4d $\mathcal{N} = 8$ supergravity The embedding tensor is constrained to transform in a sub-representation of the tensor product $\mathbf{R}_{\mathrm{adj}} \otimes \mathbf{R_1}^*$. In the 4d case, such tensor product produces:

$$\mathbf{133} \otimes \mathbf{56} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480} \ , \tag{4.16}$$

however, as we mentioned in the section of gauged supergravity, the only nonvanishing representation of $\Theta_A{}^\alpha$ is **912**. Branching such representation with respect to the SL(8) symplectic frame, we obtain:

$$\mathbf{912} \overset{\mathrm{SL}(8)}{\to} \mathbf{36} + \mathbf{420} + \bar{\mathbf{36}} + \bar{\mathbf{420}}$$

It can be proved, as does [18], that for SO(8) gaugings only the **36** representation is nonvanishing. Such representation is given by the symmetrization of two fundamental representations of SL(8) and of course can be further branched in $\mathbf{36} = \mathbf{1} + \mathbf{35}$, the trace and the traceless part. The other $\bar{\mathbf{36}}$ vanishes by choice of the symplectic frame: branching with respect to SL(8) implies splitting in half the 56 vector fields in 28 electric and 28 magnetic; only half of them, being physically independent, ought to be gauged.
Recalling that $X_{AB}{}^C = \Theta_A{}^\alpha (t_\alpha)^C_B$, and recalling the SL(8) representation of the $\mathfrak{e}_{7(7)}$ generators as in 3.8, we can write the embedding tensor of SO(8) as:

$$X_{AB}{}^C = \begin{pmatrix} X_{ii'\ jj'}{}^{kk'} & 0 \\ 0 & X_{ii'}{}^{kk'}{}_{jj'} \end{pmatrix} ,$$

with:

$$X_{ii'\ jj'}{}^{kk'} = -X_{ii'}{}^{kk'}{}_{jj'} = 4 X_{k[j} \delta^{k'k}_{j']l} \delta^{kl}_{ii'}$$

with $X_{ij} = R^{-1} \delta_{ij}$, in agreement with the embedding tensor found by generalised parallelisation.

4.6 Systematics of generalised parallelisations

In this paragraph we will outline the results of [25], which, from a bottom-up approach, explains the conditions that the embedding tensor of a lower dimensional gauged supergravity theory must satisfy in order to admit an uplift to a higher dimensional theory with the same number of supersymmetries. In so doing, the features of the internal manifold $\mathcal{M}$, the parallelising frame $\hat{E}_A$ and possible

deformations of the generalised Lie derivative are also fixed.
First of all, the condition of generalised parallelisation can be written locally (on a certain coordinate patch) as:

$$L_{\hat{E}_A} E_B^M - E_A{}^P E_B{}^Q F_{PQ}{}^M = -X_{AB}{}^C E_C{}^M \ , \tag{4.17}$$

where we considered deformations $F_{MN}{}^P$ of the generalised Lie derivative, encoding deformations of the gauge symmetries of the higher dimensional theory (such as gauging of massiveness), $X_{AB}{}^C$ the torsion of the generalised frame. Defining a modified generalised Lie derivative $\tilde{L}_{\hat{E}_A} E_B^M$ equal to the r.h.s of 4.17, and requiring closure (Leibniz identity) and Jacobi identity:

$$[\tilde{L}_\Gamma, \tilde{L}_\Lambda]\Sigma^M = \tilde{L}_{\tilde{L}_\Gamma \Lambda}\Sigma^M \qquad \tilde{L}_{\{\Gamma,\Lambda\}}\Sigma^M = 0 \ ,$$

puts three constraints on the deformation $F_{MN}{}^P$, the simplest being:

$$F_{MN}{}^P \epsilon_P^m = 0 \ ,$$

where ϵ_M^m is the rectangular constant matrix of maximal rank defining the maximal solution of the section constraint:

$$\partial_M = \epsilon_M^m \partial_m \ .$$

Most components of the generalised flux $F_{MN}{}^P$ can be reabsorbed in the generalised frame by writing:

$$\hat{E}_A{}^M(y) = E_A{}^N(y) C_N^M(y) \ , \tag{4.18}$$

where $C_N{}^M$ satisfies:

$$C_N{}^M \epsilon_M^m = \epsilon_N^m \ ,$$

and descends from the gauge potentials of the higher dimensional theory associated to the fluxes $F_{MN}{}^P$, thus is not defined globally. Through $C_M{}^N$, the Leibniz parallelisation condition can be written alternatively as:

$$L_{\hat{E}_A} \hat{E}_B{}^M - \hat{E}_A{}^P \hat{E}_B{}^Q F_{PQ}{}^M = -X_{AB}{}^C \ ,$$

or

$$\tilde{L}_{E_A} E_B = -X_{AB}{}^C E_C \ . \tag{4.19}$$

Assuming to have a suitable constant $X_{AB}{}^C$ satisfying the quadratic and the representation constraints, in order to uplift such gauged supergravity to higher dimension, one has to exhibit a generalised frame $\{\hat{E}_A\}$, globally defined and linearly independent obeying 5.5.
The conventional vectors $K_A^m = \epsilon_M^m \hat{E}_A{}^M$, satisfy by construction:

$$[K_A, K_B] = -X_{AB}{}^C K_C \ .$$

We can define $\Theta_A{}^a$, with A $\mathrm{E}_{n(n)}$ index denoting the gauged generators, a in the adjoint of G, such that:

$$[K_A, K_B] = \Theta_A{}^a \Theta_B{}^b f_{ab}{}^c K_c \ ,$$

with $f_{ab}{}^c$ structure constants of $\mathfrak{g}$. In the previous passage we ignored the possibility of central charges in the G-algebra, as we will in the rest of this section, since the extension to central charge of the following results is relatively straightforward.
We thus have $K_a{}^m$ non vanishing components of the generalised frame, which point by point generate the action of the gauge group G, which is transitive by the global definition of $\{E_A\}$. The internal manifold $\mathcal{M}$ is thus the coset space G/H, with H being the isotropy subgroup of G. As we know, we can build the Maurer Cartan one form using the coset representative $L(y)$:

$$L^{-1}\partial_m L = e_m{}^{\underline{m}} t_{\underline{m}} + h_m{}^i t_i \ ,$$

with $L(y)$ transforming under $g \in G$ as:

$$gL(y) = L(y')h(y', g) \ , \tag{4.20}$$

which infinitesimally reads, after contraction on the left with $\Theta_A{}^a$, as:

$$\Theta_A{}^a K_a{}^m = \left(LX_A L^{-1}\right)|^{\underline{m}} e_{\underline{m}}{}^m \tag{4.21}$$

where $|^{\underline{m}}$ denotes the projection onto the coset generators. The previous equation can be found by recalling that the action of G on the manifold is generated by k_A^m(y), thus infinitesimally 4.20 becomes:

$$k_A^m \partial_m L = T_A L + L h_A \ ,$$

where T_A are the $\mathfrak{g}$ generators, such that the infinitesimal $g \in G$ transformation can be written as $g \simeq 1 + g_A T^A$, and h_A is the compensating transformation. By contracting with L^{-1} from the left, projecting on the coset generators and contracting with $\Theta_A{}^a$, one recovers eq. 4.21.
Employing the invariance of the embedding tensor with respect to G transformations, i.e. the quadratic constraint, we can rewrite 4.21 as:

$$\Theta_A{}^a K_a{}^m = L_A^{-1}\ {}^B \Theta_B^{\ \underline{m}} e_{\underline{m}}{}^m \ ,$$

we thus conclude:

$$\hat{E}_A{}^M \Theta_M{}^m e_m{}^{\underline{m}} = L_A^{-1}\ {}^B \Theta_B^{\ \underline{m}} \ .$$

The l.h.s of the previous equation satisfies the section constraint, since $Y^{AB}{}_{CD}$ is G-invariant and by definition:

$$Y^{mn}{}_{CD} = 0 \ .$$

Hence in particular $\Theta_B^{\ \underline{m}}$ satisfies the section constraint as well:

$$Y^{AB}{}_{CD} \Theta_A^{\ \underline{m}} \Theta_B^{\ \underline{m}} = 0$$

Since equivalent solutions of the section constraint belong to the same orbit under $\mathcal{G} \times \mathbb{R}^+$ transformations acting on the R_1^* index, $\Theta_B^{\ \underline{m}}$ and $\epsilon_B^{\ \underline{m}}$ cannot differ in more than one such transformations. In particular, one can use $\Theta_B{}^m$ and $\epsilon_B{}^m$ interchangeably, with $\epsilon_M{}^m = \delta_M{}^A \delta_{\underline{m}}{}^m \Theta_A^{\ \underline{m}}$.
Defining an embedding of $e_{\underline{m}}^m \in \mathrm{GL}(n)$ and its inverse into $\mathcal{G} \times \mathbb{R}^+$, via:

$$e_M{}^A \Theta_A^{\ \underline{m}} = \Theta_M{}^m e_m{}^{\underline{m}} \qquad e_A{}^M \Theta_M{}^m = \Theta_A^{\ \underline{m}} e_{\underline{m}}{}^m \ ,$$

we can give an ansatz of the generalised frame yielding an uplift of the lower dimensional gauge supergravity to a higher dimension supergravity, possibly deformed through generalised fluxes, with the same number of supersymmetries, as:

$$\hat{E}_A{}^M = L_A^{-1}\ {}^B\ e_B{}^N\ C_N{}^M \ . \tag{4.22}$$

Let us comment the global definiteness of the generalised parallelising frame written above. The function $C_N{}^M$ is patched through gauge transformations, as we know. The vielbein $e_B{}^N$ transforms under local H transformations, through similarity transformations that can be seen as acting as a rotation of the flat B index. The coset represenative $L_A{}^B$, by construction is acted upon too by local H transformations. In formulae:

$$\begin{aligned}(L^{-1})_A{}^B &\rightarrow (L^{-1})_A{}^C (h^{-1})_C{}^B \ , \\ e_B{}^N &\rightarrow h_B{}^D e_D{}^N \ ,\end{aligned}$$

whence we can see that in the generalised parallelising frame, the H-patching transformations cancel out. Thus, up to gauge transformations on C, the generalised parallelising frame $\hat{E}_A{}^M$ is globally defined along the internal manifold.

Chapter 5

Consistent truncations

In this chapter, we will explain what consistent truncations are and how to perform them. Although we will present examples from conventional geometry too, the focus will especially be on the purposeful employment of exceptional field theory, which makes performing consistent truncations particularly straightforward. The references for this chapter are: [23], [26], [27], [28].

5.1 The idea of consistent truncations

Given a theory with global symmetry group G and a set of fields of various spin $\{\varphi_i\}$, the goal of a consistent truncation thereof is to find a subset of $\{\varphi_i\}$ such that the solutions of their equations of motion are also solutions of the equations of motion of the fields of the original theory, i.e. the final theory's dynamics can be embedded in the original's in a consistent way.
Let us make a simple example: consider a theory with two massless scalars ϕ and Φ with some interaction among them (take for instance $\phi\Phi\Phi$) such that the equations of motion are:

$$\begin{aligned} \Box\phi &= \Phi\Phi \;, \\ \Box\Phi &= 2\phi\Phi \;. \end{aligned} \tag{5.1}$$

A consistent truncation of such theory consists in putting Φ to zero, which clearly does not imply in the first equation that ϕ should be zero as well. The truncation is consistent because, in the second equation, both members go identically to zero, so there is no risk for Φ to be dynamically switched on. On the other hand, if we had a different theory with equations of motion:

$$\begin{aligned} \Box\phi &= \ldots \;, \\ \Box\Phi &= \phi \;, \end{aligned} \tag{5.2}$$

putting Φ to zero is not consistent with the second equation, since Φ is switched on by ϕ.
Consistent truncation often involve a dimensional reduction on an internal manifold $\mathcal{M}$. The aim of a consistent truncation of a theory living a in space $\mathcal{M}_{\text{ext}} \times \mathcal{M}$ is to find a lower dimensional theory, with support on $\mathcal{M}_{\text{ext}}$, such that all dependence on internal manifold is consistently factored out of the equations of motion. Consistency is ensured by that fact that no modes on the internal manifold can be dynamically switched on by the lower dimensional theory.
How does one make sure that the truncation is consistent? All instances of consistent truncations that we will examine are based on symmetry principles. To be precise: a truncation is consistent if the lower dimensional theory comprises only the singlets of G or of a subgroup thereof.
Let us see this principle in practice for the previous example of the scalar theory. Given a global symmetry group G of the lagrangian, every term thereof must transform as a G-singlet. If the matter content is formed of two fields, a singlet ϕ and a non-singlet Φ, interactions terms with ϕ cannot have one single Φ, but we need at least another Φ in the conjugate representation. Whence we would not get equations of motion like 5.2, but we could have 5.1. Whence, truncating with respect to the singlets of G would lead to a consistent theory, since a singlet representation can never source non

singlet representations.

5.2 Consistent truncations on group manifolds

Before studying consistent truncatins on generalised group manifolds, let us examine consistent truncations on conventional ones. As we have seen in the last chapter, an n-dimensional group manifold $\mathcal{M}$, with group G, admits by definition a globally defined, linearly independent set of vectors $\{e_a(y)\}$, with $e_a = e_a{}^m\partial_m$, satisfying the algebra $\mathfrak{g}$ associated to G:

$$[e_a, e_b] = f_{ab}{}^c e_c \ , \tag{5.3}$$

with $f_{ab}{}^c$ structure constants of the algebra, and the brackets being the conventional Lie derivative, such that:

$$L_{e_a} e_b{}^m = e_a{}^n\partial_n e_b{}^m - e_b{}^n\partial_n e_a{}^m \ .$$

A consistent truncation of a pure metric higher dimensional theory living on $\mathcal{M}_{\text{ext}} \times \mathcal{M}$ is given by the so-called Scherk-Schwarz ansatz:

$$e_{\underline{a}}(x,y) = S_{\underline{a}}{}^b(x)e_b(y) \quad \text{or equivalently} \quad g_{mn}(x,y) = e_m{}^k(y)e_n{}^l(y)g_{kl}(x) \ . \tag{5.4}$$

Along with the previous condition 5.3, in order to allow a consistent truncation upon a conventional group manifolds, we need the unimodular requirement as well:

$$f_{ab}{}^b = 0 \ ,$$

which ensures that the truncation is consistent at the level of the action as well. Indeed, if the unimodular condition was not satisfied, the lower dimensional theory would undergo a trombone symmetry gauging as well, which makes it impossible to perform the integration by parts required to derive the equations of motion from the action. The same condition will appear in the generalised case.
In equation 5.4, underlined indices transform according to SO(n), all other with respect GL(n), $\hat{e}_{\underline{a}}(x,y)$ is the vielbein of the higher dimensional theory along the internal manifold, $S_{\underline{a}}{}^b(x)$ is the matrix of the scalar fields of the lower dimensional theory, parameterising the coset space GL(n)/SO(n), x are the coordinates on the external manifold and y on the internal. Similarly, $g_{mn}(x,y)$ is the metric of the higher dimensional theory along the internal manifold, and $g_{kl}(x)$ is an equivalent parameterization of the lower dimensional scalars, which can be written as $g_{kl}(x) = S_k{}^{\underline{a}}(x)S_l{}^{\underline{b}}(x)\delta_{\underline{ab}}$, as usual. The truncation with 5.4 in ensured to be consistent because all dependence on the coordinates on the internal manifold is entrusted in the parallelising frame $\hat{e}_a(y)$, which is also called *twist matrix*, and is invariant under G.
If the higher dimensional theory presented p-forms as well, then the ansatz is:

$$\begin{aligned} C_{m_1\ldots m_p}(x,y) = e_{m_1}^{a_1}(y)\cdots e_{m_p}^{a_p}(y)C_{a_1\ldots a_p}(x) + c_{m_1\ldots m_p}(y) \ , \\ \text{with } \partial_m c_{m_1\ldots m_p} \ = \ F_{mm_1\ldots m_p} \quad \text{and} \quad F_{a_1\ldots a_p} \ = \ \text{const} \ . \end{aligned}$$

The ansatzes for the metric and the p-forms, while necessary for a consistent truncation, are not sufficient by themselves. Indeed they do allow for a factorization of the y-dependence in the equations of motion of the higher dimensional theory, however derivatives with respect to y will act upon the twists $e_m{}^a$ and $c_{m_1\ldots m_p}$ yielding non trivial expressions, called fluxes, encoding geometrical information on the internal manifold.
Indeed, while geometrical fluxes are related to the features of the frame adopted on a certain manifold, fluxes of $(p+1)$-forms are defined as constant non-vanishing integral of the type:

$$\int_\Sigma F_{(p+1)} \ = \ \xi \ ,$$

with Σ a non-trivial cycle along the manifold; in turn, non-vanishing fluxes encode the cohomology class the $(p+1)$-form sits in, as such differential form cannot be exact. Indeed, if such differential form was exact, the integral would vanish by Stokes theorem.
The generalised Scherk-Schwarz ansatzes for the vielbein and the p-forms hold in the following instances of group manifolds with fluxes:

- In a Kaluza Klein truncation, the dependence on the internal coordinates is completely discarded, i.e. $e_m{}^a = \delta_m{}^a$, and $c_{m_1\ldots m_p} = 0$;
- We can consider instead a truncation on a n dimensional torus with $e_m^a(y) = \delta_m^a$ and constant flux $h_{m_1\ldots m_p} = \delta_{m_1}^{a_1}\cdots\delta_{m_n}^{a_n}\partial_{[a_1}c_{a_1\ldots a_n]}$, being top-dimensional.
- In group manifolds, with group G, the left-invariant Maurer-Cartan one form:

  $$\sigma \;=\; g^{-1}\mathrm{d}g \;,$$

  with $g \in \mathrm{G}$, and $\sigma \in \mathfrak{g}$, can be expanded in terms of the algebra generators:

  $$\sigma \;=\; e^a T_a \;,$$

  yielding left-invariant one forms $e^a = e^a_{\ m}\mathrm{d}y^m$, which are the vielbein on the manifold. These vielbein satisfy by definition the Maurer-Cartan equation:

  $$\mathrm{d}e^a = \frac{1}{2} f_{ab}{}^{c} e^b \wedge e^c \;,$$

  whence they would generate a constant so-called geometric flux, being proportional to the torsion of the such parallelising frame and to the structure constants of the algebra. If the flux associated to p-forms is constant as well, in truncating the higher dimensional theory on the internal manifold, these fluxes have the effect of gauging a subgroup of the global symmetry group of the higher dimensional theory. Indeed, the Scherk-Schwarz ansatz for the Kaluza Klein vectors is:

  $$A^m_\mu(x,y) = e^m_{\ a}(y) A^a_\mu(x) \;,$$

  which means that $A_\mu{}^a$ become non Abelian. Indeed, by $\Lambda^m(x,y) = e^m_{\ a}\Lambda^a(x)$, we see:

  $$\delta A_\mu{}^a = \partial_\mu \Lambda^a + f_{bc}{}^a A_\mu{}^b \Lambda^c \;.$$

  Furthermore, such constant geometric fluxes lead to a scalar potential for the lower dimensional theory, which is phenomenologically interesting as it may lead to supersymmetry breaking, a cosmological constant, mass terms, as we explained in the section of gauged supergravity.

In supergravity, consistent truncations upon group manifolds in conventional geometry have been performed, as well as consistent truncations on some spheres, such as S^7, S^4 for 11d supergravity, S^5 and S^3 for sectors of type II supergravity.
However, consistent truncations in exceptional generalised geometry allow for a more systematic understanding, for instance by allowing consistent truncations on spheres of all dimensions S^d, as long as the matter content comprises a $d-1$ dimensional form, such that its field strength is proportional to the volume form. In the next chapter we will see the general conditions that the twist matrices (both their unimodular part and their trombone-related part) have to obey in order to generate a consistent truncations. However, in the rest of this section, we will consider consistent truncations on generalised group manifolds, i.e. manifolds forming a group manifold in a generalised sense.
In complete analogy to the conventional case, we can perform a consistent truncation on a manifold, in generalised geometry, by finding a set of generalised vectors $\{E_A\}$ globally defined and linearly independent, satisfying the aforementioned (generalised) Leibniz parallelisation condition:

$$L_{E_A} E_B = -X_{AB}{}^C E_C \quad \text{with} \quad X_{AB}{}^C \text{ constant} \;, \tag{5.5}$$

where $X_{AB}{}^{C}$ are the structure constants of the Leibniz algebra[1] formed by the frames themselves, and A, B, C indices in the $\mathbf{R}_1^*$ of $\mathrm{E}_{n(n)}$.
The generalised Scherk-Schwarz ansatz for the unimodular part of generalised frame of the higher dimensional theory along the internal space is:

$$U_{\underline{P}}(x,y) = S_{\underline{P}}{}^{B}(x) U_B(y) \ , \tag{5.6}$$

with B in the $\mathbf{R}_1$ of $\mathrm{E}_{n(n)}$, $\underline{P}$ in the representation of H_d with same dimension of $\mathbf{R}_1$, and $S_{\underline{P}}{}^{B}$ the scalar fields of the lower dimensional theory, which for generalised parallelisation parameterise the coset $\mathrm{E}_{n(n)}/H_d$. Notice that we are being consistent with our notation, according to which:

$$E_B(y) = \rho(y)\; U_B(y) \ .$$

Condition 5.6, which translates for the unimodular generalised metric of the higher dimensional theory along the internal space as:

$$\mathcal{M}_{MN}(x,y) = U_M{}^{P}(y) U_N{}^{Q}(y) \mathcal{M}_{PQ}(x)$$

is consistent with the generalised metric being itself unimodular. We will not give the ansatzes for the other fields of Exceptional field theory here, but we will go through them in the following section.
Let us summarise what we described so far: given a generalised parallelisation of a manifold, there exists a consistent truncation upon such manifold, with scalars of the lower dimensional theory given by $S_{\underline{P}}{}^{B}$ in eq. 5.6, and gauge algebra of the lower dimensional theory being the algebra formed by the twist matrices, encoded in the structure constants $X_{AB}{}^{C}$.
Spheres, in particular, are generalised parallelisable (as long as there exists a top dimensional constant flux on it); as described in the previous chapter, there exists a choice of frame such that the algebra formed by the frames themselves is $\mathfrak{so}(n+1)$, which by construction is also gauge algebra of the lower dimensional theory.
As mentioned in the previous chapter, a generalised parallelisation yields a consistent truncation to the same number of supercharges as the higher dimensional theory's, since the spinorial bundle is globally trivial. Thus sections of the spinorial bundle, which are the parameters of supersymmetric transformations, are trivially patched along the manifold and are invariant under the global symmetry group of the higher dimensional theory. A thorough explanation of these concepts will be given in section 5.4, when we will examine consistent truncations to reduced supersymmetry.
Let us mention that different choices of the parallelising frame will lead to different gaugins, as highlighted in [23]. In such paper, indeed two different frames for S^3 are presented, one leading to $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \times \mathfrak{so}(3)$ gauging, the other leading to $\mathfrak{so}(3)$ gauging, with the embedding tensor of the latter case comprising both the torsion of the generalised frame and the constant 3-form flux.

In particular, a consistent truncation of 11d supergravity on S^7 leads to maximal 4d $\mathcal{N} = 8$ gauged supergravity with SO(8) gauge group. Indeed, given the unimodular part $\{U_{ij}\}$ of the complete frame $\{\hat{E}_{ij}\}$ of eq. 4.15 we can write a generalised Scherk-Schwarz ansatz:

$$U_{\underline{mn}}(x,y) = S_{\underline{m}}{}^{i}(x) S_{\underline{n}}{}^{j}(x) U_{ij}(y) \ ,$$

with $S_{\underline{n}}{}^{i}$ paramerising the $\mathrm{GL}(n+1)/\mathrm{O}(n+1)$, encoding the scalar fields of the lower dimensional theory. If a parameterisation of the generalised metric $\mathcal{M}(x,y)$ is given in terms of the internal components of the higher dimensional metric and p-forms, one can deduce which scalar fields descend from which part of the higher dimensional theory.

5.3 Systematics of maximally supersymmetric consistent truncations through Exceptional field theory

We have seen consistent truncation on generalised group manifolds; we will now give generalised Scherk-Schwarz ansatzes for all fields of $\mathrm{E}_{7(7)}$ exceptional field theory, allowing for consistent truncations on hyperboloids (and spheres), inducing 4d $\mathcal{N} = 8$ gauged supergravities with gauge groups

[1] We recall that a Leibniz algebra differs from a Lie algebra for its brackets are non-antisymmetric.

SO(p,q) or, more generally, CSO(p,q,r). We thus see that this systematic approach, pioneered in [26], provides a generalisation with respect to the previous section, although it still deals with generalised parallelisations.
In order to allow a consistent truncation, the unimodular twist matrices $\{U_A\}$ must satisfy two conditions, equivalent to the generalised parallelisation condition 5.5:

$$\begin{aligned} \left[(U^{-1})_M{}^P (U^{-1})_N{}^Q \partial_P U_Q{}^K\right]_{(\mathbf{912})} &= \frac{1}{7}\rho\, \Theta_M{}^\alpha (t_\alpha)_N{}^K \qquad (5.7)\\ \partial_N (U^{-1})_M{}^N - 3\rho^{-1}\partial_N \rho (U^{-1})_M{}^N &= 2\rho\, \theta_M\ . \qquad (5.8) \end{aligned}$$

In this equation, (**912**) represents the projection onto the **912** sub-representation, of eq. 4.16, which we know to be of the only two nonvanishing representations, along with **56**, due to the constraints on the embedding tensor. $\Theta_M{}^\alpha$ is indeed the embedding tensor of the lower dimensional theory and must be constant, as must θ_M, which is instead associated to a gauging of the trombone symmetry (ρ(y) is related to θ_A by $\mathcal{L}_{E_A}\rho = \theta_A\, \rho$). The previous two conditions are exactly the same as the Leibniz generalised parallelisation condition 5.5, where the embedding tensor $X_{AB}{}^C$ has been mapped to $\Theta_M{}^\alpha$ and θ_M. The embedding tensor automatically satisfy the quadratic constraint, enforcing the closure of the algebra, once the previous conditions 5.7, 5.8 and the section constraint are satisfied.

Such conditions can be swiftly generalised to different dimensions as:

$$\begin{aligned} \left[(U^{-1})_M{}^P (U^{-1})_N{}^L \partial_P U_L{}^K\right]_{(\mathbb{P})} &= \rho\, \Theta_M{}^\alpha (t_\alpha)_N{}^K\ ,\\ \partial_N (U^{-1})_M{}^N - (D-1)\rho^{-1}\partial_N \rho (U^{-1})_M{}^P &= \rho\, (D-2)\theta_M\ , \end{aligned}$$

where as before $(\mathbb{P})$ denotes a projection, and some numerical factors might be absorbed into the embedding tensors.

An equivalent way to write these equations is:

$$\begin{aligned} [\chi_M^\alpha]_{(\mathbb{P})} &= \rho\, \Theta_M^\alpha\ ,\\ \chi_{KM}{}^K &= (1-D)\rho^{-1}\partial_N\rho (U^{-1})_M{}^N + \rho(2-D)\theta_M\ , \end{aligned}$$

with:

$$\chi_{MN}{}^K \;=\; (U^{-1})_M{}^P (U^{-1})_N{}^L \partial_P U_L{}^K \;=\; X_M^\alpha (t_\alpha)_N{}^K$$

where we recognize in $\chi_{MN}{}^K$ the Weitzenböck connection (with $U_M{}^N$ instead of the full frame). Recalling the expression of the frame torsion in terms of the Weitzenböck connection, we see that the previous conditions can be seen as enforcing constant torsion.
The general form of the embedding tensors in terms of $U_A{}^M$ and ρ is:

$$\begin{aligned} \theta_A &= \frac{1}{n-1}\rho^{-1}\bigg((n-1)(U^{-1})_A{}^N \partial_N \ln\rho^{-1} + \partial_N U_A{}^N\bigg)\ ,\\ \chi_{AB}{}^C &= \Theta_{AB}{}^C + \frac{n-2}{n-1}(2\delta^C_{[A}\theta_{B]} - Y^{CD}{}_{AB}\theta_D)\ ,\\ \text{with }\ \Theta_{AB}{}^C &= \rho\bigg(2U_M{}^C (U^{-1})_{[B}{}^N \partial_{|N|} (U^{-1})_{A]}{}^M - Y^{MN}{}_{PQ} U_M{}^C \partial_N (U^{-1})_A{}^P (U^{-1})_B{}^Q +\\ &\quad - \frac{1}{n-1}(2\delta^C_{[A}\partial_{|N|}(U^{-1})_{B]}{}^N - Y^{CD}{}_{AB}\partial_N (U^{-1})_D{}^N)\bigg)\ . \end{aligned}$$

For a generic parallelisation, the embedding tensors may of course not be constant. If they are instead constant, then the frame $E_A{}^M$ provides a *Leibniz* generalised parallelisation of the background, which as we know leads to a lower dimensional gauged supergravity with the same number of supercharges. As we mentioned in the case of conventional group manifolds, a nonzero trombone gauging leads to the impossibility of performing the consistent truncations at the level of the action too. Indeed, in going from the action to the equations of motion, an integration by parts has to be made.

The resulting boundary term will be proportional to X_{AB}^{B}, and in general will not vanish unless such components of the embedding tensor vanish themselves. Such condition, called unimodularity condition, is the same as restricting the representation of the 4d $\mathcal{N}=8$ embedding tensor to **912** only.
We will now give Scherk-Schwarz ansatzes for all the fields of Exceptional Field theory:

$$\begin{aligned}
\mathcal{M}_{MN}(x,y) &= U_M{}^P(y)U_N{}^Q(y)\mathcal{M}_{PQ}(x) , && (5.9)\\
e_\mu{}^a(x,y) &= \rho^{-1}(y)e_\mu{}^a(x) , && (5.10)\\
A_\mu{}^M(x,y) &= A_\mu{}^N(x)(U^{-1})_N{}^M(y)\rho^{-1}(y) , && (5.11)\\
B_{\mu\nu\ \alpha}(x,y) &= \rho^{-2}(y)U_\alpha{}^\beta(y)B_{\mu\nu\ \beta}(x) , && (5.12)\\
B_{\mu\nu\ M} &= -2\rho^{-2}(U^{-1})_S{}^P\partial_M U_P{}^R(t^\alpha)_R{}^S B_{\mu\nu\ \alpha}(x) . && (5.13)
\end{aligned}$$

Before examining each ansatz, let us notice that except for 5.13, in the other ansatzes the twist matrices simply act as a rotation of the indices of the lower dimensional fields, each according to their own representation (i.e. U_α^β is evaluated in the adjoint representation). Moreover, recalling the table 3.1 of weights λ under generalised diffeomorphisms, we see that each field presents a factor of $\rho^{-2\lambda}$ in their ansatz.
Let us consider the ansatz for a generalised vector Λ^M:

$$\Lambda^M(x,y) = \rho^{-1}(U^{-1})_P{}^M(y)\Lambda^P(x) ,$$

we can ask what is the action of generalised diffeomorphisms on $\mathcal{M}_{MN}(x)$ with parameters Λ^P; we would find:

$$\delta_\Lambda\mathcal{M}_{MN}(x) = 2\Lambda^L(x)\,(\Theta_L{}^\alpha + 8\theta_R(t^\alpha)_L{}^R)(t_\alpha)_{(M}^{\ P}\mathcal{M}_{N)P}(x) .$$

The variation of $\mathcal{M}_{MN}(x)$ is precisely the one generated by the gauged generators; θ represents instead the gauging of the trombone symmetry, and, through (t_α), gauges the SO(8) $\in$ $E_{7(7)}$ as well.
Let us consider the variation of the the external vielbein with respect to generalised diffeomorphisms

$$\begin{aligned}
\delta_\Lambda e_\mu{}^a(x,Y) &= \Lambda^M(x)\,(\partial_M(\rho^{-1}) + \lambda\partial_M(\rho^{-1}(U^{-1})_N{}^M))\,e_\mu{}^a(x) =\\
&= \Lambda^M(x)\lambda\rho^{-1}(\partial_M(U^{-1})_N{}^M - (1+\lambda^{-1})\rho^{-1}\partial_M\rho)e_\mu{}^a(x) =\\
&= \Lambda^M(x)\,\theta_M\,e_\mu{}^a(x) .
\end{aligned}$$

Thus we see that the action of generalised diffeomorphisms on the external vielbein is the just the action of the trombone symmetry.
Generalised covariant derivatives $D_\mu = \partial_\mu - L_{A_\mu}$ also factorize consistently. When acting on the external vielbein:

$$\mathcal{D}_\mu e_\nu{}^a(x,y) = \rho^{-1}(y)(\partial_\mu - A_\mu{}^N\theta_N)\,e_\nu{}^a .$$

When acting on the generalised metric instead:

$$\mathcal{D}_\mu\mathcal{M}_{MN}(x,y) = U_M{}^P(y)U_N{}^Q(y)\left(\partial_\mu\mathcal{M}_{PQ} - 2A_\mu^L\big(\Theta_L{}^\alpha + 8\,\theta_R\,(t^\alpha)_L{}^R\big)(t_\alpha)_{(M}^{\ P}\mathcal{M}_{N)P}\right) .$$

For what concerns the field strength tensor 3.9, the y-dependence factorises as:

$$\begin{aligned}
\mathcal{F}_{\mu\nu}^M(x,y) &= \rho^{-1}(U^{-1})_N{}^M\big\{2\partial_{[\mu}A_{\nu]}^N + \Theta_K{}^\alpha(t_\alpha)_L{}^N A_{[\mu}{}^K A_{\nu]}{}^L - \frac{1}{3}(\Omega^{NP}\Omega_{KL} + 4\delta_{KL}^{NP})\,\theta_P A_{[\mu}{}^K A_{\nu]}{}^L +\\
&\quad + (\Theta^{N\ \alpha} - 16\,(t^\alpha)^{NK}\theta_K)B_{\mu\nu\ \alpha}\big\} =\\
&\quad = \rho^{-1}(U^{-1})_N{}^M\mathcal{F}_{\mu\nu}{}^N(x) ,
\end{aligned}$$

with $\mathcal{F}_{\mu\nu}^N$ reproducing correctly the field strength tensor of the lower dimensional gauged supergravity, as we argued in gauged supergravity section.

5.3.1 Explicit construction of the twist matrices

Twist matrices solving the conditions 5.7 and 5.8, would describe consistent truncations to spheres, warped hyperboloids with possibly some factors of warped tori as well, which lead to lower dimensional theories gauged by CSO(p, q, r), which is the group of square real matrices of unit determinant that leave the metric:

$$\eta_{AB} = (\underbrace{+1, \ldots, +1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{0, \ldots 0}_{r}) \tag{5.14}$$

invariant, describing for generic (p, q, r) a non compact, non semisimple group. The computation that leads to this result, which we are going to outline, employs the fact that the trombone symmetry is ungauged and that the only non vanishing component of the embedding tensor is:

$$\Theta_{ABC}{}^{D} = \delta^{D}_{[A}\eta_{B]C} \ ,$$

with A, B, C, D indices in the fundamental representation of an apt SL(n) embedded in the global symmetry group of the D dimensional maximal supegravity, which for 4d would be[2] SL$(8) \subset$ E$_{7(7)}$. The components of the twist matrices satisfying conditions 5.7 and 5.8 are found by solving these differential equations while evaluating derivatives on a maximal solution of the section constraint. In the case of $r = 0$, solutions to those conditions exist only for three pairs of (D, n), external dimensions and n of the embedded SL(n):

$$(D, n) = (7, 5), \qquad (D, n) = (5, 6), \qquad (D, n) = (4, 8) \ , \tag{5.15}$$

while the case $r \neq 0$ admits more general solutions.
Although we will not use them explicitly, let us write the components of the twist matrix $U_A{}^B$ solving the section constraint. Let us first focus on the case $r = 0$; the ansatz for the matrix components are:

$$\begin{aligned}
U_0{}^0 &= (1-v)^{(n-1)/n} \ , \\
U_0{}^i &= \eta_{ij} y^j (1-v)^{(n-2)/(2n)} K(u, v) \ , \\
U_i{}^0 &= \eta_{ij} y^j (1-v)^{(n-2)/(2n)} \ , \\
U_i{}^j &= (1-v)^{-1/n} \Big(\delta^{ij} + \eta_{ik}\eta_{jl} y^k y^l K(u, v) \Big) \ ,
\end{aligned}$$

with i, j representing the original $n-1$-dimensions of the internal manifold, on section. Moreover $u = y^i y^i$ and $v = y^i \eta_{ij} y^j$ with η_{ij} being η_{AB} restricted to the last $n-1$ values, i.e. the diagonal matrix:

$$\eta_{ij} = (\underbrace{+1, \cdots +1}_{p-1}, \underbrace{-1, \cdots -1}_{q}) \ ,$$

with $p + q = n$. Furthermore, the function $K(u, v)$ satisfies:

$$2(1-v)(u\partial_v K + v\partial_u K) = ((1+q-p)(1-v) - u)K - 1 \ .$$

The ansatz for the trombone factor is:

$$\rho = \rho_{p,q} = (1-v)^{(n-4)/(2n)} \ .$$

It can be proven that such ansatzes do satisfy the abovementioned conditions. In the case of $r \neq 0$, there is a simple generalisation of the ansatz for the twist matrix, with:

$$U_A{}^B = \begin{pmatrix} \beta^{-r} U_{(p,q)} & 0 \\ 0 & \beta^{p+q} \mathbb{I}_s \end{pmatrix} \ ,$$

[2]For $n = 6$ the correct choice is instead SL(6), however the fundamental representation of E$_{6(6)}$ cannot be expressed in terms of the antisimmetrisation of two fundamental SL(6) representations.

where $\beta = (1-v)^{-1/(p+q)(p+q+r)}$ for the aforementioned cases 5.15, although in general one could consider more general truncations for $r \neq 0$, as we said before. The matrix $U_{(p,q)}$ is the same as the $r = 0$ case, thus we see that in this case too the twist matrix depends only the coordinates $y = i \dots p+q-1$, instead of all $n = p+q+r-1$.
In the $n = 7$ case, given the known decomposition:

$$\mathbf{56} \overset{\mathrm{SL}(8)}{\rightarrow} \mathbf{28} + \bar{\mathbf{28}} \ ,$$

the twist matrix $U_M{}^N$ branches into:

$$U_M{}^N = \begin{pmatrix} U_{[AB]}{}^{[CD]} & 0 \\ 0 & U^{[AB]}{}_{[CD]} \end{pmatrix} \ ,$$

with $U_{[AB]}{}^{[CD]} = U_{[A}{}^{[C} U_{B]}{}^{D]}$ and $U_A{}^B$ as above (where obviously $A, B, C, D = 1, \dots 8$ are SL(8) indices).

Internal metric from the twist matrices Since the internal components of the metric sit in the generalised metric, we could deduce the geometry of the internal space from the twist matrices. To be precise, if we branch $\mathcal{M}_{MN}$ with respect to SL(n) into blocks, the higher dimensional metric along the internal space appears in:

$$\mathcal{M}^{(i0),(j0)} = g^{(4-n)} g^{ij}$$

with g determinant of g^{ij}. Recalling the ansatz for the generalised metric, we can consider it where all scalars of the lower dimensional theory vanish (i.e. at the origin of the truncation), such that:

$$\mathcal{M}_{MN}(x,y) = U_M{}^P U_N{}^P \ ,$$

with P summed upon. Therefore, the relevant block of $\mathcal{M}_{MN}$ under SL(n) branching reads:

$$\mathcal{M}_{(i0),(j0)} = \frac{1}{2}\big(m_{ij} m_{00} - m_{i0} m_{j0}\big) \ ,$$

with $m_{AB} = U_A{}^C U_B{}^C$. Given the explicit form of the twist matrices, we can find:

$$\begin{aligned} \mathrm{d}s^2 &= g_{ij} \mathrm{d}y^i \mathrm{d}y^j \\ &= (1+u+v)^{-2/(p+q+r-2)} \left(\mathrm{d}y^z \mathrm{d}y^z + \mathrm{d}y^a \mathrm{d}y^b \big(\delta_{ab} + \frac{\eta_{ac}\eta_{bd} y^d y^c}{1-v}\big) \right) \ , \end{aligned}$$

where we split coordinates in $y^i = \{y^a, y^z\}$, with $a = 1, \dots p+q-1$ and $z = p+q \dots r$. Such space is conformally related to:

$$H^{(p,q)} \times \mathbb{R}^r \ , \tag{5.16}$$

where $H^{(p,q)}$ is the hyperboloid described by the embedding surface:

$$y^A \eta_{ab} y^B = 1$$

with η as in 6.9, and A restricted to $A = 1 \dots p+q-1$, within a $(p+q)$ dimensional Euclidean space. Such 5.16 reduces to the round sphere for $r = q = 0$ (since the prefactors also reduces to a constant).
Although we computed the internal metric at a point of the lower dimensional theory scalar manifold where all scalar vanish, this choice is not unique. Furthermore, the geometry does not need to be a part of a solution of the higher dimensional theory, as indeed the lower dimensional theory might not have an extremum of the scalar potential at the origin of the scalar manifold.

5.4 Systematics of consistent truncations through Exceptional field theory, with reduced supersymmetry

The previous section provided a broader explanation of generalised Leibniz parallelisations, which allow to consistently truncate to a thoery with the same number of supersymmetries. Indeed the Scherk-Schwarz ansatz for the parameters of the supersymmetric transformations are (keeping the spinor index manifest):

$$\epsilon^{\hat{\alpha}}(x,y) \;=\epsilon^{\alpha I}(x,y) \;=\; \epsilon^{\alpha \underline{I}}(x)\eta_{\underline{I}}{}^{I}(y) \;,$$

where $\hat{\alpha}$ is the spinor index in the higher dimensional theory, i.e. in $\mathrm{SO}(1,10)$. However, truncating to 4d breaks such higher dimensional Lorentz group into $\mathrm{SO}(1,3)$, represented by α, and the internal $\mathrm{SO}(7)$, represented by I. Actually, as reported in [29], I can be promoted to a representation of the double cover of the maximal compact subgroup of the global symmetry group of the lower dimensional theory, in the 4d case $\mathrm{SU}(8) \subset \mathrm{E}_{7(7)}$. The index $\underline{I}$ instead transforms according to the structure group on the generalised tangent bundle; for a generalised parallelisation, such structure group is the identity, thus all the $\epsilon^{\alpha \underline{I}}(x)$ of the lower dimensional theory are singlets with respect to $\mathrm{SU}(8)$, leading to a lower dimensional theory with the same number of supersymmetries. If we start from maximal supergravity, we will reach again by generalised prallelisation lower dimensional maximal supergravity, usually gauged, as we will see.

In this section we will focus on a general approach allowing consistent truncations to an arbitrary reduced number of supersymmetries, while employing the concepts of generalised geometry. We will follow closely [27] and references therein.

For clarity, we will first focus on the conventional geometric case. Given a n-dimensional manifold $\mathcal{M}$ in Riemannian geometry, with its natural $\mathrm{GL}(n,\mathbb{R})$ frame bundle, a consistent truncation upon $\mathcal{M}$ requires two conditions: firstly the existence of a restricted structure group $\mathrm{G}_S \subset \mathrm{O}(n)$ on $\mathcal{M}$, specified by a set $\{\Xi_i\}$ of G_S invariant tensors. Secondly, we need the intrinsic torsion (defined in eq.4.10) associated to the structure G_S to be constant and G_S invariant.

The reason behind these requirements is the following: to perform a consistent truncation we need to decompose all higher dimensional fields in G_S representations, by means of the $\{\Xi_i\}$ tensors, and keep only the singlet representations. In this way the lower dimensional theory will not display any dependence on the internal coordinates, since, again, singlet representations cannot source non singlet representations. The condition on the intrinsic torsion instead is needed for the covariant internal derivative of these $\{\Xi_i\}$. Indeed, since the internal manifold has a $\mathrm{O}(n)$ structure as well, there exists a metric. We can thus built a metric compatible, torsionless connection, i.e. the Levi-Civita connection, and compute it on the G_S-invariant tensors:

$$\begin{aligned}\nabla_m \Xi_i^{n_1\dots n_r}{}_{p_1\dots p_r} \;&=\; K_m{}^{n_1}{}_{p_1}\Xi_i^{q\dots n_r}{}_{p_1\dots p_r} + \dots + K_m{}^{n_r}{}_q \Xi_i^{n_1\dots q}{}_{P_1\dots p_r} + \\ &\quad - K_m{}^{q}{}_{p_1}\Xi_i^{n_1\dots n_r}{}_{q\dots p_s} + \dots - K_m{}^{q}{}_{p_s}\Xi_i^{n_1\dots n_r}{}_{p_1\dots q} \;,\end{aligned}$$

where $K_m{}^n{}_p$ is a section of $\mathrm{T^*M} \otimes \mathfrak{g}^{\perp}$, m being the index in the cotangent bundle and n,p those in $\mathfrak{g}^{\perp}$, which is the subalgebra of $\mathfrak{so}(\mathfrak{n})$ orthogonal to $\mathfrak{g}$, the latter being the G_S algebra. Indeed, a fiber of $\wedge^2\mathrm{T^*M}$ can be seen as the algebra $\mathfrak{so}(n)$, which can be decomposed with respect to the Cartan-Killing metric in $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, with $\mathfrak{g}^{\perp}$ the generators of $\mathfrak{so}(\mathfrak{n})$ orthogonal to $\mathfrak{g}$. The part $\mathrm{T^*M} \otimes \mathfrak{g}$ is absent in $K_m{}^n{}_p$, as $\{\Xi_i\}$ are G_S invariant.

The tensors $K_m{}^n{}_p$ define the intrinsic torsion of the G_S structure as:

$$(T_{\mathrm{int}})_{mn}{}^{p} = K_n{}^p{}_m - K_m{}^p{}_n \;,$$

thus if the intrinsic torsion if G_S singlet, so is $K_m{}^n{}_p$, allowing for a consistent y-factorisation.

This systematic approach to consistent truncations allow to determine the features of the lower dimensional theory as a consequence of the structure group that has been employed. We can determine:

- The scalars of the lower dimensional theory. As we recall from the previous section, scalars of the lower dimensional theory in generalised parallelisations can be found via the Scherk-Schwarz ansatz as deformations of the internal metric. Indeed they are the moduli of the internal metric, describing how the internal geometry changes along the lower dimensional space. Their moduli space is the coset $\mathrm{GL}(n)/\mathrm{SO}(n)$. In this systematic approach, scalars are deformations of the metric that cannot be reabsorbed by the structure group, that is:

$$\text{metric scalars} \;=\; H \in \frac{\mathrm{C}_{\mathrm{GL}(n,\mathbb{R})}(\mathrm{G}_S)}{\mathrm{C}_{\mathrm{O}(n,\mathbb{R})}(\mathrm{G}_S)} \ ,$$

where $C_K(A)$ is the commutant of A inside K.
We can see that in the sub-case of conventional parallelisation, where the structure group is trivial $\mathrm{G}_S = \mathbb{I}$, the metric scalars are all possible deformations of the metric:

$$\text{metric scalars in conv. parallelisations} \;=\; H \in \frac{\mathrm{GL}(n,\mathbb{R})}{\mathrm{O}(n,\mathbb{R})}$$

- Gauge fields. Among the invariant tensors $\{\Xi_i\}$, there are a number of G_S-invariant vectors $\hat{\eta}_a$, yielding:

  $$\text{metric gauge fields} \;=\; \mathcal{A}^a_\mu \hat{\eta}_a \ ,$$

  in the lower dimensional theory. The intrinsic torsion, if G_S singlet and constant, is determined by:

  $$\mathcal{L}_{\hat{\eta}_a} \Xi_i = f_{ai}{}^{j} \Xi_j \ .$$

  Indeed, we recall the definition of the torsion of a connection D:

  $$(L^D - L)(\alpha) \;=\; T(\alpha) \ .$$

  The intrinsic torsion of a structure group G_S is the restriction of T to its sub-representations under branching with respect to G_S, independent of the choice of connection D compatible with the G_S invariant tensors $\{\Xi_i\}$. The gauging of the lower dimensional theory is given by:

  $$[\hat{\eta}_a, \hat{\eta}_b] = f_{ab}{}^{c} \hat{\eta}_c \ .$$

  Indeed, in the specific case of of conventional parallelisations, we have a set of $\{e_a\}$ of G-invariant one-forms, defining n gauge vector fields. The ansatz for the metric truncation is:

  $$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h_{ab}(e^a + \mathcal{A}^a)(e^b + \mathcal{A}^b) \ ,$$

  with $\mathcal{A}^a_\mu$ being indeed the gauge vector fields.

Having analyzed the conventional case, let us now pass to the exceptional generalised geometric instance, which will be analogous in many ways.

Consider the generalised tangent bundle, in exceptional generalised geometry, with structure group given by H_d, maximal compact subgroup of the global symmetry group $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ of exceptional field theory. A consistent truncation of such ExFT on $\mathcal{M}$ can be performed if there exists a suitable structure group $\mathrm{G}_S \subset H_d$, determined by a set of G_S-invariant generalised tensors $\{Q_i\}$, and such that the intrinsic generalised torsion associated to G_S is constant and G_S-singlet[3].

The reasons behind these requirements are analogous to the ones of the conventional case: the invariant tensors allow to expand fields of exceptional field theory in G_S representations and entrust the dependence on the internal coordinates y in G_S singlets only; the constant and singlet generalised torsion is needed for the covariant internal derivatives of these $\{Q_i\}$. Indeed, given the existence of the generalised metric, we can define metric-compatible, torsionless connections D_M, such that:

$$D_M Q_i = \Sigma_M \cdot Q_i$$

[3] If fermions were considered too, we should consider the double covers of G_S and H_d

with $\Sigma_M \in \Gamma(E^* \otimes \mathrm{ad}\tilde{F})$ (E^* being the generalised cotangent bundle, $\mathrm{ad}\tilde{F}$ the adjoint bundle) completely determined by the generalised intrinsic torsion. The dot in the above formula denotes the adjoint action of Σ_M on Q_i, wherein Σ_M is expanded in terms of the generators $\Sigma_M{}^\alpha t_\alpha$, which then act on Q_i according to their representations.
As before, given the reduced structure group, we can determine the features of the lower dimensional theory. The scalars indeed will parameterise:

$$\mathcal{M}_{\text{scalar}} = \frac{C_{\mathrm{E}_{n(n)}}(\mathrm{G}_S)}{C_{H_d}(\mathrm{G}_S)} := \frac{\mathcal{G}}{\mathcal{H}} ,$$

on the other hand, among the Q_i there will be a subset of invariant vectors parameterizing a vector space $\mathcal{V} \subset \Gamma(E)$, with a basis given by $\{K_A\}$. The lower dimensional gauge vector fields, in the space X, will be:

$$\mathcal{A}^A_\mu K_A \quad \in \quad \Gamma(\mathrm{T}^*\mathrm{X}) \otimes \mathcal{V} ,$$

and similarly we could deduce the two forms of the lower dimensional theory. Recalling the definition of the intrinsic torsion, in eq. 4.6, with a connection $\tilde{D}$ compatible with the structure group, i.e. $\tilde{D}Q_i = 0$, then:

$$L_{K_A} Q_i = -T_{\text{int}}(K_A) \cdot Q_i ,$$

with L being the generalised Lie derivative, the dot denoting the adjoint action, for instance on vectors $K^M \in \{Q_i\}$:

$$-T_{\text{int}}(K_A) \cdot K^M \; = \; -T_{\text{int}}{}^\alpha(K_A)(t_\alpha)^M{}_N K^N .$$

By hypothesis, T_{int} is a G_S-singlet, and acts on tensors in the adjoint representation. However, singlets of $\mathfrak{g}_s$ are exactly the elements of the Lie algebra of the commutant of the structure group in $\mathrm{E}_{n(n)}$ group. Therefore, T_{int} defines the gauge group of the lower dimensional theory; in particular we can define an embedding tensor $\Theta : \mathcal{V} \to \mathrm{Lie}\ \mathcal{G}$, such that:

$$L_{K_A} K_B = -\Theta_A \cdot K_B = \Theta_A{}^{\hat{\alpha}}(t_{\hat{\alpha}})_B{}^C K_C := -X_{AB}{}^C K_C ,$$

with $\hat{\alpha}$ running over the algebra of the gauge subgroup generated by the action of $\{K_A\}$ on themselves, and with $X_{AB}{}^C$ satisfying the quadratic constraint due to the Leibniz identity of the generalised Lie derivative. In general, the gauge group defined by Θ_A as above will only be a subgroup of the complete gauge group, as we will see.
We thus see that the gauge group the lower dimensional theory is not given by the whole $\mathcal{G}$, but instead by a subgroup thereof, given by the action of the subset $\{K_A\}$ on the G_S-invariant generalised tensors.

5.5 Consistent truncation of 11d supergravity to 4d $\mathcal{N} = 2$

As we set out in the introduction, the goal of the thesis is to study a particular instance of consistent truncation of 11d supergravity to 4d $\mathcal{N} = 2$, through exceptional field theory. In the course of the next sections, we will provide all the elements that will allow us to study how we can reach, from 11d, a 4d $\mathcal{N} = 2$ gauged supergravity, with $\mathrm{SO}(6) \times \mathrm{SO}(2)$ gauge group. Before dealing with the gauge group, we will concern ourselves with reaching $\mathcal{N} = 2$.
The requirement of truncating from $\mathcal{N} = 8$ to $\mathcal{N} = 2$ fixes the maximal structure group G_S, however there are in general several subsets of G_S that fulfill the same goal, as we are going to see shortly.
Fermions in exceptional field theory transform, as mentioned in the previous chapters, in representations of the maximal compact subgroup H_d, as the R-symmetry group of the lower dimensional theory is in general a subgroup of H_d. Maximal supergravity in 4d has $H_d = \mathrm{SU}(8)/\mathbb{Z}_2$, hence its double cover is $\tilde{H}_d = \mathrm{SU}(8)$. The parameters of supersymmetry transformations in maximally supersymmetric 4d supergravity thus transform in the $\mathbf{8}$ of SU(8). In order to have a consistent truncation to 4d, with a number $\mathcal{N}$ of supersymmetries, we need to employ the G_S subgroup of H_d given by: $\mathrm{G}_S = \mathrm{SU}(8-\mathcal{N})$; indeed, branching $\mathbf{8}$ with respect to $\mathrm{SU}(8-\mathcal{N})$ will yield some direct sum of representations, some of which will be G_S singlets (and in general non singlets of the commutant

of G_S in H_d). The amount of these G_S singlets gives the desired number of supersymmetries in the truncated theory. Necessarily, among the commutant of G_s in $E_{7(7)}$, there will be the R-symmetry group of $\mathcal{N}$ supegravity in 4d.
Let us make list the relevant groups for variable $\mathcal{N}$, in 4 external dimensions:

$\mathcal{N}$	maximal G_S	$C_{E_{7(7)}}(\mathrm{G}_S)$	$C_{\mathrm{SU}(8)}(\mathrm{G}_S)$
1	SU(7)	U(1)	U(1)
2	SU(6)	SU(2) × U(1)	SU(2) × U(1)
3	SU(5)	SU(3) × U(1)	SU(3) × U(1)
4	SU(4)	SU(4) × SL(2, $\mathbb{R}$)	SU(4) × U(1)
5	SU(3)	SU(5, 1)	SU(5) × SU(1)
6	SU(2)	SO*(12)	SU(6) × U(1)

Table 5.1: Features of 4d consistent truncations to variable $\mathcal{N}$.

In particular, considering the truncation to $\mathcal{N}=2$, the $\mathbf{8}$ of SU(8) branches with respect to SU(6) × SU(2) × U(1), where SU(6) is the maximal structure group, as:

$$\mathbf{8} \overset{\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\longrightarrow} (\mathbf{6},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{2})_{-3} \ . \tag{5.17}$$

Let us briefly explain the branching. Among the special unitary eight by eight matrices we consider the sub-matrices:

$$\mathrm{SU}(8) \ni \begin{pmatrix} A_{6\times 6} & 0 \\ 0 & \mathbb{I}_{2\times 2} \end{pmatrix} \ A \in \mathrm{SU}(6) \ , \qquad \mathrm{SU}(8) \ni \begin{pmatrix} \mathbb{I}_{6\times 6} & 0 \\ 0 & B_{2\times 2} \end{pmatrix} \ B \in \mathrm{SU}(2) \ ,$$

while the $\mathrm{U}(1) \subset \mathrm{SU}(8)$ is given by:

$$\mathrm{SU}(8) \ni \begin{pmatrix} e^{i\alpha}\mathbb{I}_6 & 0 \\ 0 & e^{-3i\alpha}\mathbb{I}_2 \end{pmatrix} \ ,$$

which also explains the U(1) charge assignments. These are of course conventional, as we could have assigned charge $-1/3$ to the SU(6) block and 1 to the SU(2) block.
Furthermore, as mentioned at the beginning of this section, we can choose a smaller $\mathrm{G}_S \subset \mathrm{SU}(6)$, as long as the branching of $\mathbf{8}$ with respect to it only contains two G_S-singlets. Since in 5.17, SU(2) is exactly the R-symmmetry group of $\mathcal{N}=2$ supergravity in 4d, then necessarily only SU(6) can change.

Recall that a structure group is almost always identified by a set of singlet tensors. In order to find these tensors, with $\mathrm{G}_S = \mathrm{SU}(6)$, let us branch the $\mathbf{56}$ and $\mathbf{133}$ of $\mathrm{E}_{7(7)}$ with respect to G_S. The number and type of singlet tensors will determine so-called structures, which in turn have consequences on the field content and gauging of the lower dimensional theory, as we explained before.
The $\mathbf{56}$ representation branches as:

$$\mathbf{56} \overset{\mathrm{SU}(8)}{\to} \mathbf{28} + \bar{\mathbf{28}} \ , \tag{5.18}$$

$$\mathbf{28} \overset{\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\longrightarrow} (\mathbf{15},\mathbf{1})_2 \oplus (\mathbf{6},\mathbf{2})_{-2} \oplus (\mathbf{1},\mathbf{1})_{-6} \ , \tag{5.19}$$

or more explicitly:

$$\begin{aligned} V_M &= (V_{[mn]}, V^{[mn]}) = (V_{[ab]}, V_{[ai]}, V_{[ij]}, V^{[ab]}, V^{[ai]}, V^{[ij]}) \ , \\ &\text{with } m,n = 1,\dots 8, \quad a,b = 1,\dots 6, \quad i,j = 1,2 \ , \end{aligned}$$

and the U(1) charges in 5.19 are given by the sum of the underlying representations'. We can see in 5.19 that there is one single component which is a singlet of SU(6), given by the last sub-representation. A generalised vector K$\in \Gamma(E)$ transforming as $(\mathbf{1},\mathbf{1})_{-6}$ under $\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ corresponds to a *vector multiplet structure*, with K$\in \{Q_i\}$. A K structure is crucial in determining the gauging of the lower dimensional theory; furthermore, it has to satisfy a normalization constraint:

$$d(K,K,K,K) > 0 \ ,$$

where d is the quadratic invariant of $\mathrm{E}_{7(7)}$. This latter condition leads to the fact that there exists a subgroup of $\mathrm{E}_{7(7)}$, namely $\mathrm{E}_{6(2)}$, such that transformations thereof leave K invariant. In other words, the existence of K defines a $\mathrm{E}_{6(2)}$ structure group on E. Other than being a G_S singlet and having positive norm with respect to the quartic invariant, K^M is a rather generic tensor.
Let us now examine the branching of **133**:

$$\mathbf{133} \stackrel{\mathrm{SU}(8)}{\rightarrow} \mathbf{63}\oplus\mathbf{70} \ ,$$

where as we know, **63** corresponds to the adjoint matrices $\Lambda_A{}^B$ of SU(8), which are traceless and skew-symmetric. On the other hand, **70**, corresponds to generators $\Sigma_{[ABCD]}$. With respect to $\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, these branch as:

$$\begin{aligned}
\mathbf{63} &\stackrel{\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\longrightarrow} (\mathbf{35},\mathbf{1})_0\oplus(\mathbf{6},\mathbf{2})_{-2}\oplus(\bar{\mathbf{6}},\mathbf{2})_2\oplus(\mathbf{1},\mathbf{3})_0 \ ,\\
\mathbf{70} &\stackrel{\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\longrightarrow} (\mathbf{15},\mathbf{1})_4\oplus(\mathbf{10},\mathbf{2})_0\oplus(\mathbf{15},\mathbf{1})_{-4}
\end{aligned}$$

or[4] equivalently

$$\begin{aligned}
\Lambda_A{}^B &\rightarrow (\Lambda_a{}^b,\Lambda_a{}^i,\Lambda_i{}^a,\Lambda_i{}^j) \ ,\\
\Sigma_{[ABCD]} &\rightarrow (\Sigma_{[abcd]},\Sigma_{[abc]i},\Sigma_{[ab][ij]]}) \ .
\end{aligned}$$

Among these subrepresentations, the triplet $(\mathbf{1},\mathbf{3})_0$ of $\{J_\alpha\}$, with $\alpha=1,2,3$ and $J_\alpha\in\Gamma(\mathrm{ad}\tilde{F})$, the latter being the adjoint bundle; if globally defined and singlet under (at least) SU(6), forms a so-called *hypermultiplet structure*[5]. These currents J_α generate the highest root SU(2) subalgebra of $\mathfrak{e}_{7(7)}$, hence:

$$[J_\alpha,J_\beta] = 2\epsilon_{\alpha\beta\gamma}J_\gamma \ ,$$

and are normalised through:

$$\mathrm{Tr}[J_\alpha J_\beta] = -\delta_{\alpha\beta} \ .$$

It can be shown that the subgroup of $\mathrm{E}_{7(7)}$ leaving J_α invariant is actually[6] $\mathrm{Spin}^*(12)$; thus the structure group, i.e. the sub-bundle of the $\mathrm{E}_{7(7)}$ structure bundle, is given by $\mathrm{Spin}^*(12)\cap E_{6(2)}\simeq \mathrm{SU}(6)$. In order to form such structure group, the hypermultiplet and vector multiplet structures, or H and V structures for short, have to be compatible:

$$J_\alpha\cdot K = 0 \ ,$$

where $\cdot$ denotes the adjoint action. It can be shown that the existence of such a structure leads to a $\mathcal{N}=2$ 4d supergravity with the supergravity multiplet only, i.e. without vector or hypermultiplets in the matter content. On the other hand, if the structure group was smaller than SU(6), such that $\mathrm{SU}(8)\supset \mathrm{G}_S\times\mathrm{SU}_R(2)$, we would obtain a larger number of G_S singlets in the **56** and in the **133**

[4] The fundamental and dual representations of SU(2) are self conjugate due to the existence of ϵ_{ij}.

[5] Notice that in the literature, J_α are equivalently defined as belonging to $\Gamma(\mathrm{ad}\tilde{F}\otimes\det(\mathrm{T^*M})^{1/2}$, which in turn changes the following algebra and normalisation condition by a factor $\mathbf{k^2}$, with $\mathbf{k}\in\Gamma(\det(\mathrm{T^*M})^{1/2})$.

[6] $\mathrm{Spin}^*(12)$ is the double cover of $\mathrm{SO}^*(12)$, the real subgroup of $\mathrm{SO}(12;\mathbb{C})$. The latter is the group of unit determinant complex matrices such that $\Omega A^*\Omega = A$, with Ω real, skew-symmetric and squaring to minus the identity (being the symplectic invariant).

decomposition. We would denote these additional vectors and triplets as K_I, with $I = 1, \ldots n_v$, and J_A, wih $A = 1 \ldots \dim\mathcal{H}$. The Lie algebra $\mathfrak{h}$ related to the Lie group $\mathcal{H}$, is generated by J_A, as in:

$$[J_A, J_B] = f_{AB}{}^C J_C \ ,$$

with $f_{AB}{}^C$ being the $\mathfrak{h}$ structure constants, H being the global symmetry group acting on the hypermultiplet scalar manifold. Compatibility of the two structure is imposed by:

$$J_A \cdot K_I = 0 \quad \forall\, I, A$$

and lead to a restricted structure group $\mathrm{G}_S \subset \mathrm{SU}(6)$, as well as a 4d $\mathcal{N} = 2$ with n_v vector multiplets and $\dim\mathcal{H}$ hypermultiplets.

5.6 HV structure in terms of the generalised parallelising frame

In order to find the generalised tensors K^M and $J_\alpha{}^a$ which define the SU(6) structure, an obvious choice would be to build them from the generalised parallelising frame $E_A{}^M$ of S^7. Indeed, $E_A{}^M$ is globally defined too; furthermore, under $\mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, the flat index, i.e. the $\mathrm{E}_{7(7)}$ index A, branches as:

$$E_A{}^M \overset{\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\longrightarrow} (E^M_{[ab]}, E^M_{ai}, E^M_{[ij]}, E^{[ab]M}, E^{ai\,M}, E^{[ij]M}) \ ,$$

where $a, b = 1, \ldots 6$, $i, j = 1, 2$. In particular, one can build combinations of these components such that the final generalised tensor is SU(6) invariant. Let us construct them:

- Vector structure. To build the SU(6) invariant vector K^M, a natural choice would be to use either $E^M_{[ij]}$ or $E^{[ij]M}$ or a linear combination thereof.
  Hence any linear combination of the kind:

  $$K^M \;\propto\; E_{[ij]}^{\;M} + \beta E^{[ij]M} \qquad \beta \in \mathbb{R} \ , \tag{5.20}$$

  is a consistent choice. One should check that the norm of the vectors with respect to the $\mathrm{E}_{7(7)}$ quartic invariant is positive definite; this is case the, although we will not prove it here, for $\beta \neq 0$.

- Hypermultiplet structure. An ansatz for the currents defining the hypermultiplet structure is:

  $$(J_\alpha)_N{}^M \quad \propto \quad (\sigma_\alpha)_i{}^j E^M_{bj} E_N{}^{bi} \ ,$$

  with σ the Pauli matrices. The normalization is given by:

  $$\begin{aligned} \mathrm{tr}(J_\alpha J_\beta) &\overset{!}{=} -\delta_{\alpha\beta} \\ (J_\alpha)_M{}^N (J_\beta)_N{}^M &= (\sigma_\alpha)_i{}^j E^M_{aj} E^{ai}_N (\sigma_\beta)_k{}^l E_{bl}{}^N E_M{}^{bk} = \\ = \frac{1}{4}\delta_a{}^a (\sigma_\alpha)_i{}^j (\sigma_\beta)_j{}^i &= \; 3\delta_{\alpha\beta} \end{aligned}$$

  thus setting:

  $$J_\alpha \;=\; \frac{i}{\sqrt{3}} (\sigma_\alpha)_i{}^j E^M_{bj} E_N{}^{bi}$$

Indeed this ansatz satisfies $[J_\alpha, J_\beta] = -\sqrt{3}\,\epsilon_{\alpha\beta\gamma} J_\gamma$:

$$
\begin{aligned}
-\frac{1}{3}(\sigma_\alpha)_i{}^j E_{bj}{}^M E_N{}^{bi} (\sigma_\beta)_k{}^l E_{al}{}^P E_M{}^{ak} - (\alpha \leftrightarrow \beta) =\\
= -\frac{1}{3}(\sigma_\alpha)_i{}^j (\sigma_\beta)_k{}^l \delta^{ak}_{bj} E_N{}^{bi} E_{al}{}^P - (\alpha \leftrightarrow \beta) =\\
= -\frac{1}{6}(\sigma_\alpha)_i{}^j (\sigma_\beta)_k{}^l \delta^a_b \delta^k_j E_N{}^{bi} E_{al}{}^P - (\alpha \leftrightarrow \beta) =\\
= -\frac{1}{6}(\sigma_\alpha)_i{}^j (\sigma_\beta)_j{}^l E_N{}^{ai} E_{al}{}^P - (\alpha \leftrightarrow \beta) =\\
= -\frac{1}{3} i\epsilon_{\alpha\beta\gamma} (\sigma_\gamma)_i{}^l E_N{}^{ai} E_{al}{}^P \stackrel{!}{=} -\sqrt{3}\,\epsilon_{\alpha\beta\gamma} (J_\gamma)_M{}^P \ ,
\end{aligned}
$$

where we used the algebra of Pauli matrices $[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$ (and we recall that the multiplication by a constant of the structure constants of the algebra does not change the algebra itself).
Although the given ansatz for the currents seems the correct one, we need to make sure that it actually is a section of the adjoint bundle. In order to do so, we need to find, among the generators of $\mathfrak{e}_{7(7)}$, three of them such that the currents J_α can be written in terms of them. In order to do so, let us consider the representation $(t_{\rm a})_M{}^N$, with a $\in \mathbf{133}$; among such values of a, we consider three of them, indexed by α. We then consider the three $\mathfrak{e}_{7(7)}$ generators given by:

$$
(t_\alpha)_A{}^B = \begin{pmatrix}
\mathbb{O}_{15} & & & & & \\
& (t_\alpha)_{ak}{}^{bl} & & & & \\
& & \mathbb{O}_1 & & & \\
& & & \mathbb{O}_{15} & & \\
& & & & (t_\alpha)^{ak}{}_{bl} & \\
& & & & & \mathbb{O}_1
\end{pmatrix} ,
$$

where we keep only the components of the generators in the representations $(\mathbf{6}, \mathbf{2})_{-2}$ and its conjugate, in the branching of $\mathbf{56}$ with respect to SU(6) $\times$ SU(2) $\times$ U(1). We thus require:

$$
(t_\alpha)_A{}^B (t_\beta)_B{}^C - (\beta \leftrightarrow \alpha) \stackrel{!}{=} 2\epsilon_{\alpha\beta\gamma} (t_\gamma)_A{}^C \ ,
$$

which translates in:

$$
\begin{aligned}
[(t_\alpha)_{ak}{}^{bl}, (t_\beta)_{bl}{}^{cs}] &= 2\epsilon_{\alpha\beta\gamma} (t_\gamma)_{ak}{}^{cs} \ ,\\
[(t_\alpha)^{ak}{}_{bl}, (t_\beta)^{bl}{}_{cs}] &= 2\epsilon_{\alpha\beta\gamma} (t_\gamma)^{ak}{}_{cs} \ .
\end{aligned}
$$

We see that $(t_\alpha)_{ak}{}^{bs} = \delta^{[b}_{[a} (\sigma_\alpha)^{l]}_{k]}$ does indeed satisfy the above requirements. Thus we see that the currents are just proporional to these generators dressed by factors of the vielbein, i.e. they actually are sections of ad$\tilde{F}$:

$$
\begin{aligned}
(J_\alpha)_M{}^N &\propto (t_\alpha)_A{}^B E_B{}^N E_M{}^A =\\
&\propto (\sigma_\alpha)_k{}^l E_{al}{}^N E_M{}^{bk} \delta^b_a \ .
\end{aligned}
$$

Let us verify the compatibility of these structures:

$$
J_\alpha \cdot K = J_\alpha{}^a (t_a)_M{}^N K^M = 0 \ ,
$$

by orthogonality of the frames:

$$
E_A{}^M E_M{}^B = \delta_A{}^B \qquad \Longrightarrow \ E_{ai}{}^M E_M^{[ij]} = 0 \ .
$$

In order to determine the gauge group of the lower dimensional theory, let us recall from last chapter's section 4.5, the only non vanishing generalised Lie derivative between the components of the frame (with flat index decomposed under SU(8)):

$$L_{E_{mn}} E_{pq} = 4R^{-1}\left(\delta^{[k}_{[n}\delta_{m][p}\delta^{l]}_{q]} E_{kl}{}^{M}\right) ,$$
$$L_{E_{mn}} E^{pq} = 4R^{-1}\left(\delta^{[p}_{[m}\delta_{n]f} E^{q]f}\right) ,$$

whence we can deduce the action of the generalised Lie derivative of the SU(6) × SU(2) × U(1) components of the frame[7], with respect to $E_{ij} = E_{78}$:

$$\begin{aligned} &L_{E_{78}} E^{M}_{ab} = 0 , && L_{E_{78}} E^{ab\ M} = 0 , \\ &L_{E_{78}} E^{M}_{78} = 0 , && L_{E_{78}} E^{78\ M} = 0 , \\ &L_{E_{78}} E^{M}_{ai} = 2\delta_{i[7} E^{M}_{|a|8]} , && L_{E_{78}} E^{ai\ M} = 2\delta^{i}_{[7}\delta_{8]m} E^{am\ M} . \end{aligned}$$

Since E_{78} correspond to the SU(2) component in the **56** decomposition 5.19, understandably it has no effect on the E^{M}_{ab} part or its inverse. On the mixed E^{M}_{ai} (and inverse) component, it acts as an infinitesimal rotation of SO(2). Indeed, a generic rotation in the fundamental representation reduces in the infinitesimal form to:

$$\begin{pmatrix} E_{a7} \\ E_{a8} \end{pmatrix}' = \begin{pmatrix} \sin(x) & \cos(x) \\ -\cos(x) & \sin(x) \end{pmatrix} \cdot \begin{pmatrix} E_{a7} \\ E_{a8} \end{pmatrix} \overset{\text{inf.lly}}{\Longrightarrow} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} E_{a7} \\ E_{a8} \end{pmatrix} .$$

On itself, $E_{78}{}^{M}$ yields null generalised Lie derivative. Since, in general:

$$L_{K_A} K_B = -X_{AB}{}^{C} K_C ,$$

we conclude that the gauge group under which the vector multiplet is charged is trivial, obviously, being there no vector multiplets in the first place. On the other hand, by computing the action of the generalised Lie derivative with respect to $E^{M}_{[78]}$ on the currents J_α we will discover the gauge group of the hypermultiplets. Notice that, according to equation 5.20, we should compute the generalised Lie derivative both with respect to $E_{[ij]}$ and $E^{[ij]}$, however the latter component, by choice of the symplectic frame, satisfies:

$$L_{E^{[ij]}} E^{M}_{[mn]} = L_{E^{[ij]}} E^{[mn]M} = 0 \qquad m, n = 1, \ldots 8 ,$$

thus, the generalised Lie derivative of J_α with respect to K^M is equal to:

$$\begin{aligned} L_{E_{78}} J_\alpha &\propto (\sigma_\alpha)_i{}^j \left((L_{E_{78}} E_{aj}{}^{M}) E_N{}^{ai} + E_{aj}{}^{M} (L_{E_{78}} E_N{}^{ai}) \right) = \\ &= 2(\sigma_\alpha)_i{}^j \left(\delta_{j[7} E^{M}_{|a|8]} E_N{}^{ai} + E_{aj}{}^{M} \delta_{[7}{}^{i}\delta_{8]m} E_N{}^{am} \right) = \\ &= 2(\sigma_\alpha)_i{}^j \left(\delta_{j[7}\delta^{k}_{8]}\delta_l{}^{i} + \delta_j{}^{k}\delta_{[7}{}^{i}\delta_{8]l} \right) E_{ak}{}^{M} E_N{}^{al} = \\ &= [\sigma_\alpha, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}]_l{}^k \, E_{ak}{}^{M} E_N{}^{al} , \end{aligned}$$

where in the last line, the matrix:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{5.21}$$

can be considered a representation of the generator of the transformations whose finite form is given by $E_{78}{}^{M}$. We can see that 5.21 is proportional to the second Pauli matrix, thus we expect that the

[7] We are going to put the radius of the sphere to $R = 1$.

action of ${E_{78}}^M$ on the currents is that of a rotation among them, with respect to one of the three α "directions", to be precise, $\alpha = 2$. Indeed:

$$\begin{aligned} L_{E_{78}} J_1 &= -\frac{1}{2} J_3 \ , \\ L_{E_{78}} J_2 &= 0 \ , \\ L_{E_{78}} J_3 &= +\frac{1}{2} J_1 \ , \end{aligned}$$

which is just an SO(2) rotation of J_1 and J_3 among themselves. We thus conclude that the gauge group of the lower dimensional theory is SO(2), isomorphic to the U(1) subgroup of the $\mathrm{SU}(2)_R$ R-symmetry group of the lower dimensional theory.

Chapter 6

Deformation of $\mathcal{N} = 2$ theory and uplift to 11d

In the previous chapter, we explained how to perform consistent truncations through exceptional field theory. In particular, we described the consistent truncation of 11d supergravity on S^7, yielding 4d $\mathcal{N} = 8$ SO(8) gauged supergravity. Furthermore, we used a more systematic approach to consistent truncations through exceptional field theory with reduced supersymmetry, to truncate consistently 11d supergravity to 4d $\mathcal{N} = 2$ SO(2) gauged supergravity. In the course of this chapter, we will first perform a consistent truncation of 4d $\mathcal{N} = 8$ SO(8) gauged supergravity to 4d $\mathcal{N} = 2$ SO(6)×SO(2) gauged supergravity, with coupling constant obeying $g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)} \equiv \rho = 1$. We will perform such truncation without involving the full machinery of exceptional generalised geometry, just by truncating to the singlets of a subset of the global SO(8) symmetry, namely $\mathbb{Z}_2 \in \mathrm{SO}(8)$. Figure 6.1 may be helpful for the reader to get his bearings. Since it is a known result that such theories can be deformed to $\rho \neq 1$, as reported in [30], we will ask ourselves whether such deformed theories may be obtained by consistent truncation of 11d supergravity, on a certain HV structure which will be described. In other words, the aim of this chapter, and ultimately of the whole thesis, is to study whether 4d $\mathcal{N} = 2$ SO(6) × SO(2), with $\rho \neq 1$, admits an uplift to 11d and to understand the role of the deformation parameter ρ.

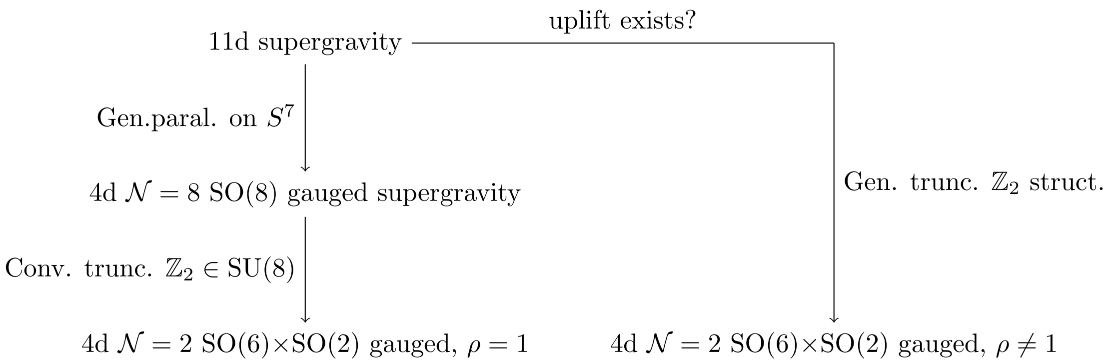

Figure 6.1: Outline of the chapter

6.1 Consistent truncation of 4d $\mathcal{N} = 8$ SO(8) gauged supergravity to $\mathcal{N} = 2$

In the course of this section, we will describe the consistent truncation of 4d $\mathcal{N} = 8$ SO(8) gauged supergravity, to $\mathcal{N} = 2$. In order to perform such truncation we will make use of the aforementioned symmetry principle, i.e. we will truncate to the singlets of the subgroup of the global symmetry group of the higher $\mathcal{N}$ theory.
Maximal supergravity in four dimensions has an R-symmetry group given by SU(8); we recall that the field content of maximal supergravity in 4d is uniquely fixed: 70 scalars, 42 spin 1/2 fermions, 28 vector fields, 8 gravitinos, 1 metric. These fields transform under the SU(8) according to the following representations:

$$\phi^{[IJKL]} \;, \quad \xi^{[IJK]} \;, \quad A_\mu^{[IJ]} \;, \quad \psi_\mu^{[I]} \;, \quad I, J, K, L = 1, \ldots 8$$

in order of increasing spin. Such representations of SU(8) can be deduced when constructing the supermultiplet through the supercharges, in the following way: starting from the highest spin field, acting with half of the supercharges (due to masslessness of the fields), one adds an index I for each iteration and decreases the spin by 1/2. The lowest spin one can reach is 0, the scalar's, which will have necessarily four SU(8) fundamental indices.
Since we want to truncate from $\mathcal{N} = 8$ to $\mathcal{N} = 2$, we need to consider a subset of the R-symmetry SU(8), whose singlets we will truncate to. Since we want the lower $\mathcal{N}$ theory to be supergravity as well, we need, among those singlets, to have one $\mathcal{N} = 2$ supergravity multiplet. We recall that the latter comprises the metric, two gravitinos and one vector field, sometimes called the *graviphoton*. Thus, the subgroup of SU(8) we need to consider is $\mathbb{Z}_2$, as in:

$$\mathbb{Z}_2 \;=\; \begin{pmatrix} -\mathbb{I}_6 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \;. \tag{6.1}$$

Indeed, from the original 8 gravitinos ψ_μ^I, by acting with $\mathbb{Z}_2$ we can see that only 2 are invariant: ψ_μ^i with $i = 7, 8$. The other six components get a minus sign which makes them not invariant; indeed, if we had considered a $\mathbb{Z}_2$ transformation with, on the diagonal, the six by six identity and minus the two by two identity, we would have truncated to a 4d $\mathcal{N} = 6$ supergravity theory. Let us find the field content of the lower $\mathcal{N}$ theory:

- Scalars; since the scalars have an even number of SU(8) indices, the surviving ones are: $\phi^{[abcd]}$ and $\phi^{[ab]78}$, with $a, b, c, d = 1, \ldots 6$. This makes for $30 = 15 + 15$ scalars. The latter ones can be written as $\phi^{[ab]78} = \phi^{[ab]}\epsilon^{78}$, with ϵ^{78} being the SU(2)-invariant completely antisymmetric tensor. The scalar manifold can be shown to be:

$$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)} \;,$$

- Spin 1/2 fermions; the only remaining spin 1/2 fermions are the following: $\xi^{[ab]7}$ and $\xi^{[ab]8}$, thus 30 spin 1/2 fermions.
- Vector fields; the only surviving vector fields are A_μ^{ab} and A_μ^{78}, the latter rewritable as $A_\mu^{78} = \epsilon^{78}\dot{A}_\mu$, where $\dot{A}_\mu$ can be seen as the graviphoton. We thus find 15+1 vector fields.
- Gravitino; the only surviving spin 3/2 field, as explained above, are two: ψ_μ^7 and ψ_μ^8.
- The metric is invariant, carrying no SU(8) indices.

This is however not the end of the story. Indeed the theory we are truncating from has a SO(8) gauge group; we might wonder which is the gauge group, if any, of the $\mathcal{N} = 2$ theory. The gauge group of the $\mathcal{N} = 2$ theory is the SO(8) subgroup that is "left invariant" by 6.1, i.e. the subgroup of SO(8) which commutes with $\mathbb{Z}_2$. Such residual gauge group can be shown to be SO(6) $\times$ SO(2). Indeed, branching **56** with respect to SO(6) $\times$ SO(2), we obtain:

$$\mathbf{56} \overset{\mathrm{SO(8)}}{\rightarrow} \mathbf{28} + \bar{\mathbf{28}} \overset{\mathrm{SO(6)\times SO(2)}}{\longrightarrow} (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{6},\mathbf{2}) \cdot 2 \ ,$$

where the **6** of SO(6) is self-conjugate due to the existence of δ_{ab}. We clearly see that, although $(\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$ is invariant under $\mathbb{Z}_2$, the last piece is not.
Notice that there are just enough vector fields in the $\mathcal{N}=2$ theory to gauge this whole group; in particular, the SO(6) and SO(2) singlet $\dot{A}_\mu$ is the gauge field of SO(2); such gauge group corresponds to the gauging of the U(1)$\subset$ SU(2) R-symmetry group of the 4d $\mathcal{N}=2$ theory, acting on the two gravitini in the supergravity multiplet. This is also called Fayet-Ilioupolus gauging.

The complete gauge group SO(6)$\times$SO(2), however, does not have different coupling constant for its two different factors. Both $g_{\mathrm{SO(6)}}$ and $g_{\mathrm{SO(2)}}$ are equal to the original $g_{\mathrm{SO(8)}}$ coupling constant, as it could not be otherwise. To be explicit, if the covariant derivative acting on the spin 1/2 fermions in the $\mathcal{N}=8$, SO(8) gauged, theory was:

$$D_\mu \xi^{[IJK]} = \partial_\mu \xi^{[IJK]} - 3\ g\ A_\mu^{LR} \delta^{[I}_{[L} \delta_{R]M} \xi^{JK]M} \ ,$$

in the $\mathcal{N}=2$, gauged SO(6) $\times$ SO(2) theory instead:

$$D_\mu \xi^{ij\ \Delta} = \partial_\mu \xi^{ij\ \Delta} - g 2 A_\mu^{kl} \delta^{[i}_{[k} \delta_{l]m} \xi^{j]k\ \Delta} - g \dot{A}_\mu \epsilon^{\Delta\Omega} \xi^{ij\ \Omega} \qquad \Delta,\Omega = 1,2 \ .$$

Nevertheless, one can conceive 4d $\mathcal{N}=2$ SO(6)$\times$ SO(2) supergravity theoris with $\rho \equiv g_{\mathrm{SO(6)}}/g_{\mathrm{SO(2)}} \neq 1$. These theories exist on their own, and ρ is a genuine deformation parameter interpolating them, as reported in [30]. However, one can show that such theories with $\rho \neq 1$ do not admit an uplift to 4d $\mathcal{N}=8$ SO(8) gauged supergravity; i.e. if $\rho \neq 1$ we exit the landscape of theories which are consistent truncations of the above mentioned $\mathcal{N}=8$ theory. On the other hand, we cannot exclude instead that theories with $\rho \neq 1$ do not admit an uplift to 11d supergravity. We know from the previous chapter that we can perform a consistent truncation of 11d supergravity to 4d $\mathcal{N}=2$ SO(2) gauged supergravity through an HV structure, which can be extended in some sense to yield a theory with a larger matter content and gauge group. In the rest of the chapter, we will study whether the HV structure leading to the SO(6) $\times$ SO(2) gauge theory can be deformed, and to which extent, in order to ensure $\rho \neq 1$. However, before studying such point, we will describe the Hopf fibration $S^7 \simeq \mathbb{CP}^3 \times S^1$, because, as we are going to explain, the deformation parameter ρ might be attributed to a rescaling of the radius of S^1.

6.2 The Hopf fibration $S^7 \overset{\mathrm{loc}}{\simeq} \mathbb{CP}^3 \times S^1$

Let us consider the Hopf fibration:

$$S^1 \hookrightarrow S^7 \hookrightarrow \mathbb{CP}^3 \ ,$$

or equivalently:

$$S^7 \overset{\mathrm{loc}}{\simeq} \mathbb{CP}^3 \times S^1 \ ,$$

and let us explain how this fibration comes about.

The complex projective space of complex dimension 3, $\mathbb{CP}^3$, is defined by $\vec{z}^T = (z_1, z_2, z_3, z_4)$, $z_i \in \mathbb{C}$, such that:

$$\vec{z} \simeq \lambda \vec{z} \qquad \text{with } \lambda \in \mathbb{C} \setminus \{0\} \ ,$$

where we can fix the norm of $\vec{z}$, i.e. $\vec{z}\bar{\vec{z}} = 1$, such that $\lambda = e^{i\alpha}$, α being a real phase.
The projective space $\mathbb{CP}^3$ can be seen as a homogeneous space; indeed it can be shown that any $\vec{z}$

can be transformed into a chosen reference point, take the "north pole" $z_N^T = (1,0,0,0)$, via a U(4) transformation. Indeed, taking a generic:

$$\vec{z} \;=\; \begin{pmatrix} \vdots \\ \rho_j e^{i\alpha_j} \\ \vdots \end{pmatrix} \;,$$

we can act with the U(4) transformation:

$$\mathrm{U}(4) \;\ni\; \mathrm{diag}\big(e^{-i\alpha_1}, \dots\, e^{-i\alpha_4}\big)$$

leading to a vector with all real entries. Subsequently we can perform a O(4) transformation, leading to the north pole. There is however a isotropy subgroup of U(4) that leaves the north pole invariant (up to a phase), namely the U(3) and U(1) transformations embedded as:

$$\begin{pmatrix} 1 & \\ & A \end{pmatrix} \;\text{ with } A \in \mathrm{U}(3) \;, \qquad \begin{pmatrix} e^{i\alpha} & \\ & \mathbb{I}_3 \end{pmatrix} \;,$$

thus:

$$\mathbb{CP}^3 \;\simeq\; \frac{\mathrm{U}(4)}{\mathrm{U}(3) \times U(1)} \;. \tag{6.2}$$

We can rewrite the coset space in a different way; we consider the coset representative L as a U(4) element with unit determinant. We know that U(3) = U(1) × SU(3), whence such U(1) factor and the U(1) factor in the original 6.2 combine into a unique U(1) transformation embedded in SU(4) as:

$$\begin{pmatrix} e^{i\alpha} & \\ & e^{-i\frac{\alpha}{3}}\,\mathbb{I}_3 \end{pmatrix} \;. \tag{6.3}$$

Thus, the coset space 6.2 can be rewritten as:

$$\mathbb{CP}^3 \;\simeq\; \frac{\mathrm{SU}(4)}{\mathrm{SU}(3) \times \mathrm{U}(1)} \;. \tag{6.4}$$

The seven sphere S^7 can be similarly identified with a coset space, namely:

$$S^7 \;\simeq\; \frac{\mathrm{U}(4)}{\mathrm{U}(3)} \;.$$

Indeed S^7 can be written in embedding coordinates in a 8d flat space, with coordinates y^i, as:

$$y^i y^i = 1$$

or introducing the complexified coordinates z^a, with $a = 1, \dots 4$:

$$z^a \bar{z}^a = 1$$

with $z^a = y_j + i y_{j+4}$. Let us specify that the isotropy U(3) is embedded as diag$(1, A)$, with $A \in \mathrm{U}(3)$. Hence in particular the $\mathrm{U}(1) \subset U(3)$ is embedded in U(4) as $\mathrm{diag}(1, \exp(i\alpha)\mathbb{I}_3) \in \mathrm{U}(4)$.
As before, we can choose in the coset space only the unit determinant representatives; in that case, the only part of the isotropy subgroup is SU(3), thus:

$$S^7 \;\simeq\; \frac{\mathrm{SU}(4)}{\mathrm{SU}(3)} \;.$$

Since U(1) has as group manifold S^1, we can conclude that:

$$S^7 \;\overset{\text{locally}}{\simeq}\; \mathbb{CP}^3 \times S^1 \;, \tag{6.5}$$

thus the seven sphere is locally isomorphic to $\mathbb{CP}^3$, with S^1 fibered upon it.

6.3 Truncation with $\mathbf{G}_S \subset \mathbf{SU}(6)$

As we set out in the introduction, we will now consider the truncation through exceptional field theory of 11d supergravity to 4d $\mathcal{N}=2$ SO(6)×SO(2) gauged supergravity, by means of a reduced structure group $\mathrm{G}_S \subset \mathrm{SU}(6)$. If we take as structure group $\mathrm{G}_S = \mathbb{Z}_2 \subset \mathrm{SU}(6)$, embedded into SU(8) as:

$$\mathrm{SU}(8) \ni \mathbb{Z}_2 \;=\; \begin{pmatrix} -\mathbb{I}_6 & \\ & \mathbb{I}_2 \end{pmatrix} \;,$$

we can clearly see that there are only two $\mathbb{Z}_2$-invariant elements in the branching of $\mathbf{8}$ with respect to $\mathbb{Z}_2$, corresponding to the two desired supercharges. For clarity, the branching is performed with respect to $\mathrm{SU}(8) \to \mathrm{SU}(6) \times \mathrm{SU}_R(2)$, while keeping the even/odd behaviour under $\mathbb{Z}_2$, respectively for invariant or non-invariant sub-representations; we can see that in the branching of $\mathbf{28}$ of SU(8), similarly, there are $\mathbf{15}+\mathbf{1}$ $\mathbb{Z}_2$ invariants:

$$\begin{aligned} \mathbf{8} &\overset{\mathrm{SU}(6)\times\mathrm{SU}(2)_R}{\longrightarrow} (\mathbf{6},\mathbf{1})_{\mathrm{odd}} \oplus (\mathbf{1},\mathbf{2})_{\mathrm{even}} \;, \\ \mathbf{28} &\overset{\mathrm{SU}(6)\times\mathrm{SU}(2)_R}{\longrightarrow} (\mathbf{15},\mathbf{1})_{\mathrm{even}} \oplus (\mathbf{6},\mathbf{2})_{\mathrm{odd}} \oplus (\mathbf{1},\mathbf{1})_{\mathrm{even}} \;, \end{aligned}$$

We can thus define $\mathbb{Z}_2$-invariant generalised vectors in terms of the generalised parallelising frame $E_A{}^M$. We can define K_0 as in the previous section:

$$K_0^M \;\propto\; E_{[ij]}{}^M + \beta\, E^{[ij]M} \qquad i,j = 7,8, \quad \beta \in \mathbb{R}$$

and 15 new vectors $K_{[ab]}^M$, given by:

$$K_{[ab]}^M \;\propto\; E_{[ab]}{}^M + \alpha\, E^{[ab]M} \qquad a,b = 1,\ldots 6, \quad \alpha \in \mathbb{R} \;.$$

We will denote the whole set of them as $K_I = \{K_0, K_{[ab]}\}$, with $I = 1 \ldots 16$. From the previous section, we can swiftly conclude that the lower dimensional theory will have a number (15) of vector multiplets, with a gauge group SO(6) generated by the K_I, as indeed:

$$L_{E_{ab}} E_{cd}{}^M \;=\; 4\, \delta^{[m}_{[a} \delta_{b][c} \delta^{n]}_{d]} E_{[mn]}{}^M \qquad a,b,c,d,m,n = 1,\ldots 6 \;,$$

as we recall from the algebra of the parallelising frame.
On the other hand, the branching of $\mathbf{133}$ with respect to SU(6) × SU(2) yields:

$$\begin{aligned} \mathbf{133} &\overset{\mathrm{SU}(8)}{\to} \mathbf{63} + \mathbf{70} \;, \\ \mathbf{63} &\overset{\mathrm{SU}(6)\times\mathrm{SU}(2)}{\longrightarrow} (\mathbf{35},\mathbf{1})_{\mathrm{even}} \oplus (\mathbf{6},\mathbf{2})_{\mathrm{odd}} \oplus (\mathbf{6},\mathbf{2})_{\mathrm{odd}} \oplus (\mathbf{1},\mathbf{3})_{\mathrm{even}} \;, \\ \mathbf{70} &\overset{\mathrm{SU}(6)\times\mathrm{SU}(2)}{\longrightarrow} (\mathbf{15},\mathbf{1})_{\mathrm{even}} \oplus (\mathbf{20},\mathbf{2})_{\mathrm{odd}} \oplus (\mathbf{15},\mathbf{1})_{\mathrm{even}} \;, \end{aligned}$$

however the only $\mathbb{Z}_2$-invariant, non trivial representation of SU(2) is $(\mathbf{1},\mathbf{3})$ in the $\mathbf{63}$ decomposition, therefore such restricted structure group has not yielded more currents than the previous case. We only have one J_α, with $\alpha = 1,2,3$, which we have built before. We have seen that the action of K_0 upon J_α yields an SO(2) rotation of the $\alpha = 1$ and $\alpha = 3$ directions; SO(2) is thus the gauge group acting on the supergravity multiplet in 4d $\mathcal{N}=2$. The newly defined $K_{[ab]}^M$ have clearly no effect on J_α, as we can see from the algebra of the parallelising frame 4.13, since

$$L_{E_{[ab]}} E_{78}{}^M \;=\; L_{E_{[ab]}} E^{78\;M} \;= 0 \;,$$

and also trivially, by choice of the symplectic frame:

$$L_{E^{[ab]}} E_{[mn]}^M \;=\; L_{E^{[ab]}} E^{[mn]\;M} \qquad a,b = 1,\ldots 6 \;,\;\; m,n = 1,\ldots 8 \;.$$

Thus the gauge group of the 4d $\mathcal{N}=2$, consistently truncated by means of the $\mathbb{Z}_2$ structure group, is SO(6)×SO(2).

6.4 Deformation of the $\mathcal{N}=2$ truncation ansatz

In the light of the two previous sections, we can interpret the SO(6)$\times$ SO(2) gauging of the 4d $\mathcal{N}=2$ theory in a different perspective. Indeed, let us recall the isomorphisms $\mathrm{SO}(6)\simeq \mathrm{SU}(4)/\mathbb{Z}_2$ and $\mathrm{SO}(2)\simeq \mathrm{U}(1)$. The theory we obtain in 4d has, as gauge group, the groups that admit a transitive action upon $\mathbb{CP}^3$ and S^1. Therefore, we hypothesize that the deformation parameter ρ has an interpretation in terms of $\mathbb{CP}^3$ and S^1. The hypothesis we will try to prove is that the deformation $\rho\neq 1$ corresponds to a rescaling of the radius of S^1, or equivalently, that is corresponds to a deformation of the 11d metric, thus showing that the $\rho\neq 1$ theories do admit an uplift to 11d. In order to prove whether a rescaling of the S^1 radius leads to $\rho\neq 1$, we will consider the angle $\theta\in[0,2\pi]$, $\theta\ \sim\theta+2\pi n$, $n\in\mathbb{Z}$, parameterising the circle S^1. Since we know that the SO(2) gauge group corresponds to S^1, and since the SO(2) action upon certain components of the generalised frame can be written as a simple 2 by 2 rotation matrix $R(\theta)$ (as we will see), we are going to prove whether changing the periodicity of θ in $R(\theta)$ leaves the HV structure invariant.
We will then check whether any such reduced periodicity can be attributed back to a rescaling or the radius of S^1, or equivalently to a deformation of the 11d metric; indeed, if the 11d metric only contains the coordinate θ along S^1 in the term $\mathrm{d}^{11}s^2\supset\ \mathrm{d}\theta^2$, with no off-diagonal term in θ, a change of periodicity of θ does correspond to a metric deformation.

Since we have written the HV structure in terms of the generalised parallelising frame, we first need to examine what is the dependence of $\{E_A\}$ upon θ. Let us recall from 4.6 that the generalised parallelising frame can be written in all generality as:

$$E_A{}^M=(L^{-1})_A{}^B e_B{}^N C_N{}^M\ ,$$

where L is a coset representative of the internal manifold, $e_B{}^N$ is the inverse vielbein on the internal manifold, embedded into $E_{n(n)}\times\mathbb{R}^+$. Instead $C_N{}^M$ encodes the gauge potentials of the higher dimensional theory.
Let us examine each of these pieces. Given L coset representative of $S^7\simeq \mathrm{U}(4)/\mathrm{U}(3)$, we can build the Maurer-Cartan left invariant one form on S^7:

$$L^{-1}\mathrm{d}L\ =\ e^{\underline{m}}t_{\underline{m}}\ +\ h^i t_i$$

with m index along the coset generators, i along the $\mathfrak{u}(3)$ generators, which are embedded in $\mathfrak{u}(4)$ as $\mathrm{diag}(0,A)$, with $A\in\mathfrak{u}(3)$. Given the S^7 Hopf fibration, we can factorise the coset representative as:

$$L(\theta,\phi_1,\ldots,\phi_6)\ =R(\theta)\ P(\phi_1,\ldots\phi_6)$$

with $R(\theta)$ being a representation of U(1), which is embedded in SU(4) as the traceless hermitian $\mathrm{diag}(\exp(i\alpha),\exp(-i\alpha/3)\mathbb{I}_3)$. Instead $P(\phi_1,\ldots\phi_6)$ is the unimodular coset representative of $\mathbb{CP}^3$. These two elements commute; in particular we can rewrite the Maurer-Cartan form as:

$$L^{-1}\mathrm{d}L\ =\ R^{-1}\mathrm{d}R\ +\ P^{-1}\,\mathrm{d}P$$

where in turn:

$$P^{-1}\mathrm{d}P\ =\ e^{\underline{n}}t_{\underline{n}}^{\mathrm{SU}(4)\smallsetminus\mathrm{SU}(3)\times\mathrm{U}(1)}\quad +h^i t_i^{\mathrm{SU}(3)\times\mathrm{U}(1)}\ .$$

Clearly no term of this last equation depends on θ. The only dependence comes from $R^{-1}\mathrm{d}R$. In particular, since L is a coset representative for U(4)/U(3), R will be the representation:

$$R(\theta)\ =\ \exp(i\alpha)\mathbb{I}_4$$

whence in particular:

$$R^{-1}\mathrm{d}R\ =\ i\,\mathrm{d}\theta,$$

which itself does not depend on θ. Hence in particular no element of $e_A{}^M$ depends on θ, whence its determinant will not depend on θ either.

Let us consider $C_M{}^N$, which corresponds to the gauge potential C_6 of 11d supergravity. In particular, we must have:

$$\mathrm{d}C \;\propto\; \big((\det(e)\big)\mathrm{d}\theta \wedge \mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_6 \ ,$$

or in other words the field strenght should be proportional to the S^7 volume form. However, the determinant of the vielbein does not depend on θ as we have seen; furthermore, there exists necessarily a C_6 such that $F_7 = \mathrm{d}C = \mathrm{d}C_6$. Such C_6 can be taken to be $C_6 = C_5 \wedge \mathrm{d}\theta$, where C_5 satisfies:

$$\mathrm{d}C_5 = \big(\det(e)\big)\mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_6$$

being proportional to the volume form on $\mathbb{CP}^3$. Thus we have that neither e nor C depend on θ. Indeed, the only θ-dependent term is enclosed in L through $R(\theta)$. Let us recall the branching of **56** with respect to SU(6) $\times$ SU(2) $\times$ U(1); the action of the generalised Lie derivatives of the various sub-representations of $E_A{}^M$ with respect to $E_{78}{}^M$, yields non vanishing result only for:

$$L_{E_{78}} E_{ai} \quad \text{and} \quad L_{E_{78}} E^{ai} \ ,$$

thus the only components of $E_A{}^M$ with a θ dependence are $E_{ai}{}^M$ and $E^{ai\,M}$. As we pointed out in the previous chapters, they are rotated by an SO(2) 2 by 2 matrix. All other components of $E_A{}^M$ have no dependence whatsoever on θ. We may ask what is the charge of $E_{ai}{}^M$ and its conjugate under the U(1) $\simeq SO(2)$. In order to do that, we need to decompose the **8** representation of SO(8) with respect to SU(4) $\times\, U(1)$, which is homeomorphic to SO(6)$\times$ SO(2). The branching is:

$$\mathbf{8} \;\overset{\mathrm{SU}(4)\times\mathrm{U}(1)}{\longrightarrow}\; \mathbf{4}_{1/2} \oplus \bar{\mathbf{4}}_{-1/2} \ .$$

The charge 1/2 is purely conventional; it has been given a non integer value because SU(4) is the double cover of SO(6), thus **4** can be seen as a spinorial representation. In turn, the **28** representation of SO(8) branches as:

$$\begin{aligned}\mathbf{28} \;&\overset{\mathrm{SU}(4)\times\mathrm{U}(1)}{\longrightarrow}\; \big(\mathbf{4}_{1/2} \oplus \bar{\mathbf{4}}_{-1/2}\big) \wedge \big(\mathbf{4}_{1/2} \oplus \bar{\mathbf{4}}_{-1/2}\big) \;= \\ &=\; \mathbf{6}_1 \oplus \bar{\mathbf{6}}_{-1} \oplus \mathbf{15}_0 \oplus \mathbf{1}_0 \ .\end{aligned}$$

In such branching, we see that $\mathbf{15} + \mathbf{1}$ have no charge with respect to U(1), corresponding to the $E_{[ab]}$ and E_{78} components of the generalised frame. On the other hand, the E_{ai} component has a charge $+1$ with respect to U(1), which is double with respect to the 1/2 charge of **4**. Thus, if the U(1) transformation acting on **4** is parameterised by θ with 2π periodicity, the transformation acting on E_{ai} instead will have half that period, i.e. π.
What are the implications on the HV structure? The $K_{[ab]}$ vectors defined previously have no dependence on θ, thus they are defined for every period of θ, and same for the graviphoton K_0. The currents instead, remain well defined for $\theta \sim \theta + \dfrac{\pi}{2}$. Indeed, the finite transformation acting on E_{ai} can be seen as:

$$R(\theta) \;=\; \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix} \ ,$$

whence in particular:

$$R(\theta + \pi/2) \;=\; -R(\theta) \ .$$

However, the currents J_α are formed by bilinears of E_{ai} and its conjugate, therefore they remain well defined under $\pi/2$ shifts:

$$J_\alpha(\theta + \pi/2) \;=\; J_\alpha(\theta) \ .$$

In particular, this means that changing the period of θ leaves the HV structure invariant.
Such reduced periodicity of θ define a so-called *orbifold*:

$$\frac{S^7}{\mathbb{Z}^4} \;\simeq\; \mathbb{CP}^3 \;\times\; \frac{S^1}{\mathbb{Z}_4} \ .$$

Nevertheless, to admit that the $\rho \neq 1$ theories descend from 11d supergravity truncated upon such orbifold, we should be able to attribute the reduced periodicity of θ to the rescaling of the S^1 radius, i.e. to a change of the 11d metric. In this way, we would trace a direct line between 11d supergravity and its lower dimensional deformed consistent truncations. However changing the periodicity of θ does not naively translates into a 11d metric deformation, for the reasons which we will outline in the next section.

6.5 Complications to liftability

The above mentioned theory with $\rho \neq 1$ would be liftable to 11d supegravity if the S^1 radius rescaling could be attributed to a rescaling of the θ term of the 11d metric, as in:

$$\mathrm{d}^{11}s^2 \;=\; \mathrm{d}^{10}s^2(\varphi_1, \dots \varphi_{10}) \;+\; \lambda\, \mathrm{d}\theta^2 \;.$$

This, however, is a nontrivial requirement; indeed the HV structure we built in terms of the S^7 generalised parallelising frame, invariant under $\theta \to \theta + \pi/2$, does not allow in general allow for a consistent truncation of 11d supergravity to 4d $\mathcal{N} = 2$, $\rho \neq 1$. In fact, let us consider again the Maurer-Cartan SU(8)-left invariant 1-form on the sphere, wherein we choose $L \in \mathrm{U}(4)/\mathrm{U}(3)$, and P to have unit determinant:

$$\begin{aligned} L^{-1}\mathrm{d}L &= (e^{-i\theta}\mathbb{I}_4 P^{-1})\,\mathrm{d}(e^{i\theta}\mathbb{I}_4 P) \\ &= i\mathrm{d}\theta \mathbb{I}_4 + P^{-1}\mathrm{d}P \;, \end{aligned}$$

while in general:

$$L^{-1}\mathrm{d}L = e^{\underline{\theta}} t_{\underline{\theta}} \;+\; e^{\underline{m}} t^{\mathbb{CP}^3}_{\underline{m}} \;+\; h^i t^{\mathbb{CP}^3}_i \;. \tag{6.6}$$

In particular, $e^{\underline{\theta}}$ is the einbein on S^1, with $\underline{\theta}$ flat index. The h-connection h^{CP^3}, proportional to the $\mathrm{SU}(3) \times \mathrm{U}(1)$ generators in SU(4), in turn splits in:

$$h^i t^{\mathbb{CP}^3}_i \;=\; ih^{\mathrm{U}(1)} \begin{pmatrix} 1 & \\ & -\mathbb{I}_3/3 \end{pmatrix} \;+\; ih^a t^{\mathrm{SU}(3)}_a$$

where the factors i enforce hermitianity of the generators and the form of the U(1) generator in SU(4) is, rightly, infinitesimal. On the other, hand, knowing that $L \in \mathrm{U}(4)/\mathrm{U}(3)$:

$$L^{-1}\mathrm{d}L = e^{\underline{\theta}} t_{\underline{\theta}} + e^{\underline{m}} t^{\mathbb{CP}^3}_{\underline{m}} + h^i t^{\mathrm{U}(3)}_i \;,$$

we can split:

$$h^i t^{\mathrm{U}(3)}_i = iq^{\mathrm{U}(1)} \begin{pmatrix} 0 & \\ & \mathbb{I}_3 \end{pmatrix} + iq^j t^{\mathrm{SU}(3)}_j \;,$$

whence we find the following equation:

$$ie^{\underline{\theta}} \begin{pmatrix} 1 & \\ & \mathbb{I}_3 \end{pmatrix} \;+\; iq^{\mathrm{U}(1)} \begin{pmatrix} 0 & \\ & \mathbb{I}_3 \end{pmatrix} \;\;=\;\; i\mathrm{d}\theta \begin{pmatrix} 1 & \\ & \mathbb{I}_3 \end{pmatrix} \;+\; ih^{\mathrm{U}(1)} \begin{pmatrix} 1 & \\ & -\dfrac{\mathbb{I}_3}{3} \end{pmatrix} \;,$$

where the last matrix is the generator of 6.3. We thus find that the einbein one form on the circle equals:

$$e^{\underline{\theta}} \;=\; \mathrm{d}\theta \;+\; h^{\mathrm{U}(1)} \;. \tag{6.7}$$

The previous equation tells us that the S^1 is non trivially patched along the S^7. Indeed, the $h^{\mathrm{U}(1)}$ connection is patched through gauge transformations:

$$h^{\mathrm{U}(1)'} \;\to\; h^{\mathrm{U}(1)} \;+\; \mathrm{d}\Lambda \;,$$

therefore, $\mathrm{d}\theta$ must too undergo gauge transformations in different coordinate patches, to ensure that $e^{\underline{\theta}}$ is invariant. In fact $e^{\underline{\theta}}$ is patched only through local H transformation, in particular by H-similarity transformation, where H is the isotropy subgroup of the internal manifold.
This implies that, in the 11d metric, a reduced periodicity of θ cannot be attributed to a simple rescaling $\mathrm{d}\theta^2 \to \lambda \mathrm{d}\theta^2$, because the metric contains terms in θ that depend non-trivially on θ, those arising from the mixing of the r.h.s. in eq. 6.7. In particular, we cannot prove upliftability through $\rho = g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)}$.

Outline of some next steps Although we did did not prove liftability of 4d $\mathcal{N} = 2$ SO(6)×SO(2), $\rho \neq 1$, we are going to outline, in this section, some steps that might instead succeed.
In the previous chapter, we mentioned that decomposing the generalised metric $\mathcal{M}$ with respect to SL(n), with n number of physical dimensions of the internal manifold, one of the sub-representations is proportional to the 11d metric up to a power of its determinant. Let us recall the generalised Scherk-Schwarz ansatz for the generalised metric:

$$\mathcal{M}_{MN}(x,y) \;=\; \left(U_M{}^P\right)^T(y) M_{PQ}(x) U_N{}^Q(y) \;, \tag{6.8}$$

where $M_{PQ}(x)$ are the scalar fields of the lower dimensional theory. The generalised parallelising unimodular frame $U_M{}^P$ is related to their non-unimodular part by:

$$E_A{}^M(y) \;=\; \rho(y)\, U_A{}^M \;,$$

where $E_A{}^M$ can be written in all generality as:

$$E_A{}^M \;=\; (L^{-1})_A{}^B(y)\, e_B{}^N\, C_N{}^M(y) \;.$$

In such expression, which we recall from 4.6, $L(y)$ is the coset representative of the internal manifold, which is always a coset space if it is generalised parallelisable. The other factors are: the (inverse) vielbein on the coset space $e_A{}^M$, embedded in $\mathrm{E}_{7(7)}$, while being an element of GL(7). The factor $C_N{}^M$ encodes instead the fluxes of the higher dimensional theory.
Furthermore we can consider the origin of the lower dimensional scalar manifold, where all moduli vanish, or in other words where $M_{PQ}(x) = \mathbb{I}$. In such case, we can write the equation:

$$\det(E)^2 \left(E_A{}^M\right)^T E^{A\,N} \epsilon_M^{\;m}\, \epsilon_N^{\;n} \;\propto g^{mn}(x,y) \;. \tag{6.9}$$

However, knowing the general form of the parallelising frame, we can perform some simplification on the l.h.s.. Indeed, in the (symbolic) contraction on the "flat index" of:

$$(E^T \cdot E) \;=\; C^T\, e^T\, \underbrace{L^{-T}\, L^{-1}}\; e\, C$$

the under-bracketed part equals the identity by orthogonality of L (being an element of U(4)). The subsequent vielbein contraction yields the internal metric $\hat{\mathrm{d}s}^2$. Such metric, if we think of the $S^7 \simeq \mathbb{CP}^3 \times S^1$, can be expanded into:

$$\hat{\mathrm{d}s}^2 \;=\; (e^{\underline{\theta}})^2 \;+\; (e^{\underline{i}})^2$$

where $e^{\underline{\theta}}$ is the vielbein on S^1, $e^{\underline{i}}$ the one along $\mathbb{CP}^3$. Indeed the vielbein along the internal S^7 can be expanded into:

$$e_m{}^{\underline{m}} \mathrm{d}y^m t_{\underline{m}} \;=\; e_m{}^{\underline{\theta}} \mathrm{d}y^m\, i\, \mathbb{I}_4 \;+ e_m{}^{\underline{i}} \mathrm{d}y^m\, t_{\underline{i}} \;,$$

where $\underline{i}$ runs along the generators of the $\mathbb{CP}^3$ coset. We can see that rescaling $e^{\underline{\theta}} \;\to\; \lambda\, e^{\underline{\theta}}$ would imply a rescaling of the internal metric $\hat{\mathrm{d}s}^2$, and due to eq. 6.9, a rescaling of some elements of the 11d metric as well. However, what are the consequences on the global definiteness generalised parallelising frame? As we mentioned above, the vielbein $\tilde{e} = (e^{\underline{\theta}},\, e^{\underline{i}})$ is patched around by means of local H-similarity transformations (on the flat index), while the curved index transforms with an element of GL(7), being the jacobian of the coordinate change. The coset representative L

too changes with a local H transformation, however in ${E_A}^M$ these local H transformations cancel out purpusefully. Rescaling $e^{\underline{\theta}}$ has no interference with the global definiteness of ${E_A}^M$, since $e^{\underline{\theta}}$ is proportional to $\mathbb{I}_4$, which certainly commutes with the H transformations (being the latter U(3) embedded in U(4)). Thus the rescaled vielbein still belongs to a generalised parallelising frame ${E_A}^M$; what we expect however is that such new parallelising generalised frame will not yield constant torsion. There is no reason to expect that such rescaling mantains the $\mathfrak{so}(8)$ algebra of the frame, while it is on the other hand foreseeable that the torsion will have both constant and nonconstant elements.

6.6 Conclusions

This thesis was set out with the proposal that, exploiting the local fibration $S^7 \simeq \mathbb{CP}^3 \times S^1$, the gauged supergravity theory in 4d, with $\mathcal{N} = 2$ and gauge group $\mathrm{SO}(6) \times \mathrm{SO}(2)$, with deformation parameter $\rho = g_{\mathrm{SO}(6)}/g_{\mathrm{SO}(2)} \neq 1$, admits an uplift to 11d supergravity because the deformation parameter corresponds to a deformation of the S^1 radius.
We studied, in order of increasing generality, generalised parallelisation of spheres, generalised parallelisations in general and truncations to a lower amount of supersymmetries. We were able to build explicit generalised tensors, invariant under the tructure group SU(6) of the generalised tangent bundle, in the case of $\mathrm{E}_{7(7)}$ exceptional field theory. We built them in terms of the S^7 generalised parallelising frame yielding an SO(8) gauge group of the lower dimensional theory. By identifying these SU(6)-invariant tensors, with constant SU(6) intrinsic torsion, we were able to truncate to 4d $\mathcal{N} = 2$ SO(2) gauged supergravity. We were able to generalise to a smaller structure group, with additional G_s-invariant tensors leading to a $\mathrm{SO}(6) \times \mathrm{SO}(2)$ gauge group, with $\mathrm{G}_S = \mathbb{Z}_2 \subset \mathrm{SU}(6)$. For $\rho = 1$, such theory can be obtained from 4d $\mathcal{N} = 8$ SO(8) gauged supergravity as well, by a simple truncations devoid of any exceptional generalised geometry machinery. We then asked ourselves to what extent the so-built HV structures depended on the S^1 radius, in the fibration $S^7 \simeq \mathbb{CP}^3 \times S^1$, reaching the conclusions that it is well defined for $\theta \sim \theta + \pi/2$, thus with a reduced periodicity. This defines a so called *orbifold*:

$$\frac{S^7}{\mathbb{Z}^4} \simeq \mathbb{CP}^3 \times \frac{S^1}{\mathbb{Z}_4} ,$$

however, we cannot conclusively say that truncating 11d supergravity on such orbifold yields the 4d $\mathcal{N} = 2$ $\rho \neq 1$ theories. Indeed, the circle is non trivially patched on S^7 due to the $\mathbb{CP}^3$ U(1)-connection, and the mixing of the terms in the r.h.s. of eq. 6.7 does not allow to interpret the θ reduced periodicity as a 11d metric deformation. The next step would be to consider a generalised parallelising frame on S^7 which explicitly encodes a rescaling of the 11d metric, in the way mentioned above. Nevertheless, in general one cannot prove upliftability of 4d $\mathcal{N} = 2$ $\rho \neq 1$ theories by means of an HV structure built in terms of the generalised parallelising frame $E_A{}^M$ obeying a $\mathfrak{so}(8)$ algebra. A different frame is in general going to be needed, one that may not obey $\mathfrak{so}(8)$.

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