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Pointwise dispersive estimates in general relativity

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Introduction

The Dirac equation represents a milestone of modern physics, being able to describe the behaviour of spin-1/2 particles in a relativistic setting. The aim of this Master's thesis is to study the *dispersive properties of the Dirac operator,* in order to apply them to the context of general relativity.

With the term *dispersion* we mean, roughly speaking, the property of each component of a wave packet to travel with different speeds. Due to this phenomenon, certain physical quantities, such as energy, have the peculiar trait of decaying locally while being conserved globally.

In Chapter 1, we present the *construction of the Dirac equation,* which is defined to be invariant under Lorentz transformations. Hence, it is not difficult to imagine that there is a deep connection between the differential operator and the Lorentz group. Therefore, a brief overview of this group and its spin representations is given to better understand the derivation of the Dirac equation. Denoting with $\eta := \mathrm{diag}(1,-1,-1,-1)$ the Minkoswki metric, in *flat spacetime* $(\mathbb{R}^{1+3},\eta)$, the equation writes

$$i\gamma^\mu\partial_\mu\psi = m\psi \ ,$$

where ψ is the unknown, $m \geq 0$ is the mass of the particle and the *Dirac matrices* γ^μ satisfy the anticommutation property $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ and are given by

$$\gamma^0 := \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_2 \end{pmatrix} , \qquad \gamma^j := \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} , \qquad \text{for } j = 1,2,3 \ ,$$

where σ^j denote the Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

A crucial feature of the Dirac equation is that it squares to the Klein–Gordon equation

$$\partial_t^2\psi - \Delta_x\psi + m^2\psi = 0 \ ,$$

for which many dispersive properties are known. Therefore, it is straightforward to infer that the Dirac flow shows the same behaviour.

However, things become much more delicate when you generalize the equation to a spacetime manifold $(\mathcal{M}, g)$. Indeed, many tools of differential geometry and a new notion of covariant spinorial derivative are needed to preserve the Lorentz invariance in this latter case. In particular, the definition of a matrix bundle $e^a{}_\mu$, called *vierbein*, is a core step to locally link the curved and the flat spacetimes as follows

$$g_{\mu\nu} = e^a{}_\mu \, \eta_{ab} \, e^b{}_\nu \ .$$

Thanks to the power of this formalism, the equation still maintains an extremely elegant and compact form

$$i\underline{\gamma}^\mu D_\mu \psi = m\psi \ ,$$

where $\underline{\gamma}^\mu := e^\mu{}_a \gamma^a$ and D_μ is the new *covariant derivative for Dirac spinors*

$$D_\mu := \partial_\mu + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b] \ ,$$

where $[\cdot\, , \cdot]$ denotes the commutator and ω is the spin connection, which writes in terms of the affine connection Γ associated to g as $\omega_\mu^{ab} := e_\nu{}^a \partial_\mu e^{\nu b} + e_\nu{}^a \Gamma^\nu_{\mu\sigma} e^{\sigma b}$.

In *curved background*, the squaring trick mentioned above is no longer as effective due to the spinorial structure and hence proving dispersive inequalities is definitely a much harder task. However, it is quite reasonable to expect that when the underlying metric is not that far from the flat one, also the solution behaves well. In Chapter 2, we confirm this intuition by presenting some *Strichartz estimates*, proven in [CdSM23], when the manifold $(\mathcal{M}, g)$ is assumed to *decouple space and time* and to be *static* and *asymptotically flat*. Mathematically speaking, this means

$$g(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & -h(x) \end{pmatrix} \ ,$$

where h is positive and with smooth entries satisfying for any multi-index $|\alpha| \leq 3$ and all x,

$$|\partial^\alpha (h_{jk}(x) - \delta_{jk})| \leq C_h \langle x \rangle^{-|\alpha|-1-\sigma} \ , \qquad \text{for } j, k = 1, 2, 3 \ ,$$

for some constants $C_h \ll 1$ and $\sigma \in (0, 1)$. The crucial fact that is heavily used in the proof of the theorem is the squaring property of the Dirac equation, which now yields a *spinorial Klein–Gordon equation*,

$$\partial_t^2 u - \Delta_h u - \frac{1}{4} R_h u + m^2 u = 0 \ ,$$

where R_h is the scalar curvature associated to h and $\Delta_h u = D^j D_j = h^{jk} D_k D_j$ is a "spinorial" Laplacian. The strategy consists in rewriting the modified Laplacian in terms of the standard Laplace–Beltrami operator. In this way, one can take advantage of the geometric assumptions to deduce proper decay and local smoothing estimates and thus control effectively the perturbative spinorial terms. Thanks to these arguments, one finally obtains the following Strichartz estimates,

$$\text{massless case } (m=0): \quad \|e^{it\mathcal{D}} u_0\|_{L^q_t \dot{H}^{1-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{\dot{H}^1(\mathcal{M}_h)} ,$$

$$\text{massive case } (m>0): \quad \|e^{it\mathcal{D}_m} u_0\|_{L^q_t H^{1/2-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{H^1(\mathcal{M}_h)} ,$$

where $\mathcal{D}_m$ denotes the Dirac operator (with $\mathcal{D} = \mathcal{D}_0$) and the exponents (p, s, q) must satisfy specific admissibility conditions associated to the wave and the Klein–Gordon flows.

These results become extremely relevant in the context of general relativity, where the dynamics of the particles described by the Dirac equation is influenced by gravity, whose law is governed by the Einstein equations. Coupling these two aspects, one obtains the so-called Einstein–Dirac system.
To approach this delicate topic, Chapter 3 begins with its derivation as the Euler–Lagrange equations of the following action

$$\mathcal{S}[\psi, e^\mu{}_a] = \int R_g \, e \, d^4x + \int \left[\frac{i}{2} \left(\overline{\psi} \underline{\gamma}^\mu D_\mu \psi - D_\mu \overline{\psi} \underline{\gamma}^\mu \psi \right) - m \overline{\psi} \psi \right] e \, d^4x ,$$

where $e = \sqrt{|\det g|}$. Computing the variations with respect to the vierbein and the mass field, one obtains the *Einstein–Dirac system*

$$G_{\mu\nu} + T_{\mu\nu} = 0 ,$$
$$i \underline{\gamma}^\mu D_\mu \psi = m \psi ,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_g$ is the *Einstein tensor* and

$$T_{\mu\nu} = \frac{i}{4} \left(\overline{\psi} \underline{\gamma_\mu} D_\nu \psi - D_\nu \overline{\psi} \underline{\gamma_\mu} \psi \right) + \frac{i}{4} \left(\overline{\psi} \underline{\gamma_\nu} D_\mu \psi - D_\mu \overline{\psi} \underline{\gamma_\nu} \psi \right)$$

is the *energy-momentum tensor*.
Moreover, by taking advantage of the first spherical harmonics, we present an explicit form of the Einstein–Dirac system in the *spherically symmetric case*.
Eventually, we present some open questions related to this model that will be addressed in future doctoral studies, such as the *well-posedness issue* and *stability of solutions*.

Finally, Appendix A is devoted to the detailed study of a *Dirac local smoothing estimate* [CdS19b], crucial in the second chapter. Indeed, the proof is quite instructive since it relies on techniques that are widely used in the analysis of dispersive PDEs. In particular, we retrace the arguments to establish a *virial identity* and we carefully bound its perturbative terms to deduce the result.

Chapter 1

The Dirac equation

Introducing the Dirac operator in a self-contained way and covering all the underlying ideas needed for its construction is not an easy task. In this chapter, we give the essential concepts of Lie groups and Lie algebras, in order to better understand the Lorentz group and its representations. This set of transformations plays indeed a key role in relativity theory and in the definition of the Dirac operator. Finally, we see how it generalizes to the curved case, through a new notion of covariant derivative.

1.1 Lie groups, Lie algebras and representations

We start this first section by briefly presenting the main ingredients that are useful to understand the inner structure of the Lorentz group. To this end, we follow the approach given in [Woi17].

Definition 1.1 (Lie group)**.** A *Lie group* is a smooth manifold G where the maps multiplication and inverse

$$m : (g_1, g_2) \in G \times G \mapsto g_1 g_2 \in G \quad \text{and} \quad i : g \in G \mapsto g^{-1} \in G$$

are smooth with respect to the differentiable structure.

In our context, a Lie group is seen as a "transformation group", that is a group of elements acting as geometric transformations. In particular, we especially treat *matrix Lie groups*, which are hence contained in $GL(n, \mathbb{C})$.

Definition 1.2 (Complex representation)**.** Let G a matrix Lie group. Then a *complex representation* of G is a continuous group homomorphism

$$\pi : g \in G \to \pi(g) \in GL(n, \mathbb{C}) \ .$$

The representation π is called *irreducible* if it has no subrepresentations, meaning non-zero proper subspaces $W \subset GL(n, \mathbb{C})$ such that $(\pi|_W, W)$ is a representation. Furthermore, (π, V) is said to be *unitary* if $\pi(g)$ is unitary for any $g \in G$.

We prefer to work with complex representations, since we can rely on many important results and properties, for instance Schur's lemma and diagonalization of operators.

The differential structure of a Lie group G gives rise to another important object, the Lie algebra $\mathfrak{g}$, which is the tangent space at the identity of G. Since we restricted to matrix Lie groups, we can define the Lie algebra in a more concrete way.

Definition 1.3 (Lie algebra)**.** Let G be a matrix Lie group. The *Lie algebra* $\mathfrak{g}$ of G is the set of all matrices $X \in M(n, \mathbb{C})$ such that $e^{itX} \in G$ for all $t \in \mathbb{R}$, where e^A denotes the matrix exponential.

Remark 1.1 (Lie bracket and matrix commutator). In the case of a matrix Lie group, the Lie algebra is naturally endowed with a Lie bracket given by the matrix commutator $[A, B] := AB - BA \in \mathfrak{g}$ for $A, B \in \mathfrak{g}$.

Remark 1.2 (Physicists' convention). Note that we adopted the physicists' convention of multiplying by i before exponentiating to be consistent with the notations in Parker and Toms' book [PT09]. Below, we will refer to the elements of the Lie algebra as *infinitesimal group elements*.

Another important fact is that representations of Lie groups also induce representations at the level of their Lie algebras. This is motivated by the fact that the homomorphism property causes the map π to be largely determined by its behaviour infinitesimally near the identity, and thus by the derivative π'. The next theorem shows a way to define the derivative of such a map in terms of velocity vectors of paths.

Theorem 1.1 (Lie algebra representation)**.** *Let $\pi : G \to GL(n, \mathbb{C})$ be a group representation of a matrix Lie group G. Then*

$$\pi' : X \in \mathfrak{g} \to \pi'(X) := \frac{d}{dt}\Big|_{t=0} \pi(e^{itX}) \in \mathfrak{gl}(n, \mathbb{C}) = M(n, \mathbb{C}) ,$$

is such that

1. $\pi(e^{itX}) = e^{it\pi'(X)}$ *, for all* $t \in \mathbb{R}$, $X \in \mathfrak{g}$ *;*

2. $\pi'(gXg^{-1}) = \pi(g)\pi'(X)(\pi(g))^{-1}$ *, for all* $g \in G$, $X \in \mathfrak{g}$ *;*

3. $\pi'([X,Y]) = [\pi'(X), \pi'(Y)]$ *, for all* $X, Y \in \mathfrak{g}$.

A linear map satisfying the last property is called a Lie algebra representation.

Proof. See [Woi17, Section 5.4]. □

This theorem shows that every representation of a matrix Lie group gives rise to a representation of the associated Lie algebra. For our purposes, a crucial aspect is understanding the reverse process. That is, what circumstances are sufficient to guarantee that, given a Lie algebra representation, we have an associated representation of the Lie group. An answer to this problem is given by the connectedness and simple-connectedness properties.

Theorem 1.2 (One-to-one correspondence)**.** *Let G be a connected and simply connected matrix Lie group. If $\widetilde{\pi}$ is a representation of $\mathfrak{g}$, then there exists a representation π of G such that $\widetilde{\pi} = \pi'$.*

Proof. See [Hal00, Corollary 5.35]. □

Remark 1.3*.* Below, we will notice that the Lorentz group is not simply connected. Therefore, to recover this crucial property, we will consider its double covering $SL(2,\mathbb{C})$ and study the representations of $\mathfrak{sl}(2,\mathbb{C})$.

Remark 1.4 (Complexification)*.* It is important to stress that Lie algebras are real vector spaces (even if they can be made of complex matrices). This property makes the Lie algebra representation π' to be real, even if π is a complex Lie group representation. To obtain a complex Lie algebra representation from a real one, we define the complexified Lie algebra as

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g} \,.$$

Consequently, we can extend π' to a representation of $\mathfrak{g}_{\mathbb{C}}$ by complex linearity

$$\tilde{\pi}'(X + iY) := \pi'(X) + i\pi'(Y) \,.$$

We now give the definition of another algebraic structure, the Clifford algebra. To simplify its statement, we will directly restrict to the case of our interest.

Definition 1.4 (Clifford algebra $\mathrm{Cliff}(1,3,\mathbb{C})$)**.** The *Clifford algebra* $\mathrm{Cliff}(1,3,\mathbb{C}) \subset M(4,\mathbb{C})$ is the algebra generated by $1, \gamma^\mu$ for $\mu = 0,1,2,3$ satisfying the relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathrm{Id}_4 \,, \tag{1.1}$$

where $\{a, b\} := ab + ba$ denotes the anticommutator and $\eta := \mathrm{diag}(1, -1, -1, -1)$. These elements can be chosen so that γ^0 is hermitian while the other γ^j are anti-hermitian.

Remark 1.5 (Sign convention). In this work, we follow the sign convention for the Minkowski metric η given, for instance, in [BD64].

Remark 1.6. If $(\gamma^\mu)_\mu$ and $(\tilde{\gamma}^\mu)_\mu$ both satisfy (1.1), then there exists $U \in GL(4, \mathbb{C})$ such that $\tilde{\gamma}^\mu = U^{-1}\gamma^\mu U$. Hence, all families of generators are equivalent.

The choice we will opt for along this work is the following.

Definition 1.5 (Dirac matrices)**.**

$$\gamma^0 := \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_2 \end{pmatrix} , \qquad \gamma^j := \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} , \qquad \text{for } j = 1, 2, 3 ,$$

where σ^j denote the Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We will refer to these matrices as *Dirac* or *Gamma matrices*.

In the next section, we will see how to express a representation of the Lorentz algebra in terms of the generators of the Clifford algebra.

1.2 Lorentz group and spin representation

The fundamental principles of relativity theory are that space and time must be treated together and that every law must be invariant under admissible changes of frame of reference. Mathematically speaking, this leads to the definition of a four dimensional spacetime and of the corresponding group of transformations preserving this structure.

Definition 1.6 (Minkowski spacetime)**.** The *Minkowski spacetime* is the vector space $\mathbb{R}^{1+3}$ equipped with the inner product defined by

$$(x, y) := x^T \eta \, y = \sum_{\mu,\nu=0}^{3} x_\mu \eta_{\mu\nu} y_\nu$$

where (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) are the coordinates of respectively $x, y \in \mathbb{R}^{1+3}$. The metric $\eta = \mathrm{diag}(1, -1, -1, -1)$ is called the Minkowski metric.

Definition 1.7 (Lorentz group and restricted Lorentz group)**.** The *Lorentz group* $O(1,3)$ is the group of linear transformations preserving the Minkowski space inner product. In other words, a Lorentz transformation Λ is real square matrix of size $1+3$ satisfying the following condition

$$\Lambda \in O(1,3) \iff \Lambda^T \eta \, \Lambda = \eta \ .$$

The *restricted Lorentz group* $SO^+(1,3)$ is the connected component of the identity and it is given by proper orthochronous Lorentz transformations (i.e. Lorentz transformations which have determinant +1 and preserve the time orientation). Furthermore, we note that $SO^+(1,3)$ is not simply connected (cf. [Hal00, Section 2.5]).

$SO^+(1,3)$ is a Lie group of dimension 6 and is generated by the rotations around the three spatial axes and the boosts in the three spatial directions: for instance, the rotation around the x-axis and the first boost are respectively

$$R_x(\theta) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} , \qquad B_x(\phi) := \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

For a basis of its Lie algebra, one can consider

$$l_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \quad l_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} , \quad l_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$k_1 := \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad k_2 := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad k_3 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} ,$$

which satisfy the following commutation relations

$$[l_i, l_j] = i\epsilon_{ijk} l_k \ , \quad [k_i, k_j] = -i\epsilon_{ijk} l_k \ , \quad [l_i, k_j] = i\epsilon_{ijk} k_k \ ,$$

where ϵ_{ijk} is the Levi-Civita symbol. These elements are respectively called *infinitesimal rotations and boosts*. Indeed, according to Theorem 1.1, exponentiating for instance by l_1 and k_1, one recovers respectively the rotation $R_x(\theta)$ and the boost $B_x(\sigma)$.

Even if these elements give us the geometric idea of a Lorentz transformation, for our purposes we need to define other ways to represent $SO^+(1,3)$. To this end, we give a brief argument showing that $SL(2,\mathbb{C})$ is the double cover of $SO^+(1,3)$, meaning that there exists a two-to-one mapping

$$\widetilde{\Phi} : SL(2,\mathbb{C}) \to SO^+(1,3) \ .$$

Idea (Construction of $\widetilde{\Phi}$). We start by identifying $\mathbb{R}^{1+3}$ with the space of 2 by 2 complex self-adjoint matrices by

$$(x_0, x_1, x_2, x_3) \leftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

and observe that

$$\det \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0^2 - x_1^2 - x_2^2 - x_3^2 \ .$$

Hence, the Minkowski space can be seen as the space of complex self-adjoint matrices with norm-squared the determinant of the matrix.

For $\Omega \in SL(2,\mathbb{C})$, let us define the linear transformation given by conjugation

$$\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \mapsto \Omega \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \Omega^* \ ,$$

where $\cdot^*$ denotes the conjugate transpose. One can see that it preserves the determinant, i.e. the inner product, and maps self-adjoint matrices to self-adjoints matrices, i.e. it maps $\mathbb{R}^4$ to $\mathbb{R}^4$. Furthermore, we notice that both Ω and $-\Omega$ induce the same linear transformation above and it can be proven that all elements of $SO^+(1,3)$ arise from a conjugation map with an appropriate Ω. This gives indeed the double covering map $\widetilde{\Phi} : SL(2,\mathbb{C}) \to SO^+(1,3)$ we were looking for.

Remark 1.7. In this digression, we avoided to present the deep motivation explaining this 2-fold property, which involves the theory of spin groups. Indeed, we took advantage of the isomorphism $SL(2,\mathbb{C}) = \mathrm{Spin}(1,3)$ to simplify the argument. For more details, we refer to [Woi17, Chapter 40].

This doubling property emerges also at the level of the Lie algebras. To see it, we complexify $\mathfrak{so}^+(1,3)$, by defining the following combinations

$$A_j := \frac{1}{2}(l_j + ik_j) \ , \quad B_j := \frac{1}{2}(l_j - ik_j) \ ,$$

which now satisfy the relations

$$[A_i, A_j] = i\epsilon_{ijk}A_k \ , \quad [B_i, B_j] = i\epsilon_{ijk}B_k \ , \quad [A_i, B_j] = 0 \ ,$$

that are the laws defining the algebra $\mathfrak{so}(3)_{\mathbb{C}}$. This means that the Lie algebra $\mathfrak{so}^+(1,3)_{\mathbb{C}}$ splits into a sum of two copies of $\mathfrak{so}(3)_{\mathbb{C}}$. Finally, using the fact that $\mathfrak{so}(3)_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ (see [Woi17, Section 8.1.2]), we conclude that

$$\mathfrak{so}^+(1,3)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \ .$$

Doing this allows us to classify the finite dimensional irreducible representations of $\mathfrak{so}^+(1,3)_{\mathbb{C}}$ by studying $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ representations.

Remark 1.8 (Projective representations). We will take advantage of this "doubled" version of the restricted Lorentz group to define below the so-called *spin representation*. However, we stress that the 2-fold covering map $\widetilde{\Phi}$ gives rise to a sign ambiguity, preventing us from defining a true representation of $SO^+(1,3)$. Examples like this are known as "projective representations". Nevertheless, this aspect will not affect our analysis and thus we shall overlook this technicality. In this way, thanks to Theorem 1.2 and since $SL(2,\mathbb{C})$ is connected and simply connected, we can construct its group representations by looking at the Lie algebra representations of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$.

Theorem 1.3 (Classification of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ representations)**.**
The finite dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ are labeled by (s_1, s_2) for $s_j = 0, \frac{1}{2}, 1, \dots$. These representations are given by the tensor product representations

$$(\pi_{s_1} \otimes \overline{\pi}_{s_2}, V^{s_1} \otimes V^{s_2}) \ ,$$

where (π_s, V^s) is the $\mathfrak{sl}(2,\mathbb{C})$ irreducible representation of dimension $2s+1$ and $(\overline{\pi}_s, V^s)$ its complex conjugate. Such representations have dimension $(2s_1+1)(2s_2+1)$.

Proof. See [Woi17, Section 41.1]. □

The representations of particular interests are the two half-spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, which give rise to respectively to left-handed and right-handed Weyl spinors. Since Dirac fermions are supposed to satisfy a chiral symmetry, we also define the bi-spinor (or Dirac) representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, that is a four-dimensional and reducible complex representation. This choice allows us to describe charged, massive, $\frac{1}{2}$-spin particles using Dirac spinors, that are expressed in terms of chiral eigenstates as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} ,$$

where ψ_L and ψ_R are two-components wavefunctions, of left and right chirality respectively.

Given a family of generators Σ_{ab} providing a matrix representation of the Lie algebra $\mathfrak{so}^+(1,3)$, we can represent a Lorentz transformation Λ as

$$\pi(\Lambda) = \exp\left(i\varepsilon^{ab}\Sigma_{ab}\right) , \tag{1.2}$$

where ε^{ab} are the parameters characterizing the Lorentz transformation. In this way, we can write an infinitesimal Lorentz transformation as

$$\Lambda^a{}_b = \delta^a{}_b + \varepsilon^a{}_b(x) , \qquad \pi(1+\varepsilon) = 1 + i\varepsilon^{ab}\Sigma_{ab} . \tag{1.3}$$

Note that, from the characterization of a Lorentz transformation $\Lambda^a{}_c\eta_{ab}\Lambda^b{}_d = \eta_{cd}$, one finds that the infinitesimal parameters satisfy $\varepsilon_{ab} = -\varepsilon_{ba}$. Therefore, we can assume that the generators satisfy $\Sigma_{ab} = -\Sigma_{ba}$.
Furthermore, we can deduce the commutation rules that any matrices that are to represent the Lorentz algebra must obey.

Lemma 1.4 (Commutation rules of the Lorentz algebra $\mathfrak{so}^+(1,3)$)**.**
Let $\Sigma_{ab} = -\Sigma_{ba}$ be the generators of a matrix representation of the Lie algebra $\mathfrak{so}^+(1,3)$. Then, they satisfy

$$[\Sigma_{ab}, \Sigma_{cd}] = \frac{i}{2}\Big(\eta_{ac}\Sigma_{bd} - \eta_{ad}\Sigma_{bc} - \eta_{bc}\Sigma_{ad} + \eta_{bd}\Sigma_{ac}\Big) . \tag{1.4}$$

Proof. Since π is a representation, given Λ and Λ' two Lorentz transformations, we have that

$$\pi(\Lambda)\pi(\Lambda')\pi(\Lambda^{-1}) = \pi(\Lambda\Lambda'\Lambda^{-1}) .$$

Linearizing first in Λ' and $\pi(\Lambda')$, using (1.3), we obtain

$$\pi(\Lambda)(1 + i\varepsilon^{ab}\Sigma_{ab})\pi(\Lambda^{-1}) = \pi(\Lambda(1+\varepsilon)\Lambda^{-1}) ,$$

and by linearity of π,

$$\pi(\Lambda)\Sigma_{ab}\pi(\Lambda^{-1})\,\varepsilon^{ab} = \varepsilon^{ab}(\Lambda^c{}_a(\Lambda^{-1})_b{}^d)\Sigma_{cd} .$$

From the antisymmetry of ε^{ab}, it follows

$$\pi(\Lambda)\Sigma_{ab}\pi(\Lambda^{-1}) = \frac{1}{2}\Big(\Lambda^c{}_a(\Lambda^{-1})_b{}^d - \Lambda^c{}_b(\Lambda^{-1})_a{}^d\Big)\Sigma_{cd} \; .$$

Finally, linearizing in Λ and $\pi(\Lambda)$, using (1.3), the last identity rewrites

$$\begin{aligned} i\varepsilon^{cd}\Sigma_{cd}\Sigma_{ab} - i\Sigma_{ab}\varepsilon^{cd}\Sigma_{cd} &= \frac{1}{2}\Big(\varepsilon^c{}_a\delta_b{}^d - \delta^c{}_a\varepsilon_b{}^d - \varepsilon^c{}_b\delta_a{}^d + \delta^c{}_b\varepsilon_a{}^d\Big)\Sigma_{cd} \\ -i\varepsilon^{cd}[\Sigma_{ab}, \Sigma_{cd}] &= \frac{1}{2}\Big(\eta_{ad}\varepsilon^{cd}\Sigma_{cb} - \eta_{bc}\varepsilon^{cd}\Sigma_{ad} - \eta_{bd}\epsilon^{cd}\Sigma_{ca} + \eta_{ac}\epsilon^{cd}\Sigma_{bd}\Big) \; , \end{aligned}$$

from which we obtain, using $\Sigma_{ab} = -\Sigma_{ba}$,

$$[\Sigma_{ab}, \Sigma_{cd}] = \frac{i}{2}\Big(\eta_{ac}\Sigma_{bd} - \eta_{ad}\Sigma_{bc} - \eta_{bc}\Sigma_{ad} + \eta_{bd}\Sigma_{ac}\Big) \; .$$

□

The Dirac representation is related, as the name suggests, to the Dirac matrices of Definition 1.5, that generate the Clifford algebra.

Definition 1.8 (Dirac representation)**.** The *Dirac representation* is given by the elements

$$\Sigma^{ab} := -\frac{i}{8}[\gamma^a, \gamma^b] \; , \tag{1.5}$$

which generate the Lorentz algebra $\mathfrak{so}^+(1,3)$ and indeed satisfy the commutation relations (1.4). In particular, in this representation, the infinitesimal rotations and boosts are respectively given by

$$\Sigma^{jk} = -\frac{i}{8}[\gamma^j, \gamma^k] = \frac{1}{4}\epsilon^{jkl}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix} \; , \qquad \text{for } j < k \text{ and } j, k = 1, 2, 3,$$

$$\Sigma^{0j} = -\frac{i}{8}[\gamma^0, \gamma^j] = -\frac{i}{4}\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} \; , \qquad \text{for } j = 1, 2, 3.$$

Remark 1.9 (Non-unitary representation). Note that this representation is not unitary due to the presence of the boost generators Σ^{0j}. For instance, if we compute

$$\begin{aligned} \pi(B_x(\phi)) = e^{i\phi\Sigma^{01}} &= \sum_k \frac{1}{(2k)!}\Big(\frac{\phi}{4}\Big)^{2k}\mathrm{Id}_4 + \sum_k \frac{1}{(2k+1)!}\Big(\frac{\phi}{4}\Big)^{2k+1}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\ &= \cosh\frac{\phi}{4}\,\mathrm{Id}_4 + \sinh\frac{\phi}{4}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \; , \end{aligned}$$

we see that it is clearly not unitary. More generally, this is a consequence of the fact that connected simple non-compact Lie groups cannot have any nontrivial unitary finite-dimensional representations, see [RB86, Section 8.1.B]. The non-compactness of the Lorentz group is indeed caused by the presence of boosts: intuitively, unlike rotations, iterating a boost makes you go to infinity never bringing back to the starting point and this is clearly preventing compactness.

Below we will require the solution of the Dirac equation to be invariant under the action given by the Dirac representation of the Lorentz group.

1.3 The Dirac equation in flat spacetime

In this section, we briefly retrace the original approach that Paul Dirac followed almost a century ago, in 1928 [Dir28], when he stated his famous equation. For further details regarding the historical derivation of the Dirac equation, refer to [BD64] or [Pes18].

Dirac's goal was to describe the motion of spin fermions in $\mathbb{R}^3$ with the addition of relativistic corrections. Mathematically speaking, this translates in two core properties:

- *quantum mechanics evolution*: we are looking for an equation that can be written in the form of a Schrödinger equation $i\partial_t u = Hu$, where H is a Hamiltonian function to be determined. This condition comes from quantum formalism, whereby H can be interpreted as a self-adjoint differential operator representing the "physical observable" energy;
- *relativistic covariance*: if a solution satisfies an equation of the form $\mathcal{D}\psi = 0$, for some differential operator $\mathcal{D}$, we expect that if we perform a Lorentz transformation (i.e. basically, a rotation or a boost) the transformed solution in the new frame satisfies the same equation. This condition comes from the relativistic principle that laws of physics must be independent by the choice of the frame of reference.

The transition from classical to quantum mechanics, at least at a formal level, can be achieved by replacing the classical quantities with suitable differential operators. In this way, the state of the system which is given by a pair $(x(t), p(t))$ of position and momentum, is replacecd by a wavefunction $\psi(t)$ representing the density of probability associated to the particle. In particular, the "observables" energy E and the momentum p of a particle are generalized as follows

$$E \to i\hbar\partial_t \,, \qquad p \to -i\hbar\nabla_x \,,$$

where $\hbar$ is the Planck's constant. With this substitution, the relativistic energy–momentum relation

$$E = \sqrt{c^2p^2 + m^2c^4} \,, \tag{1.6}$$

where c is the speed of light and m the mass of the particle, gives the square-root of the Klein–Gordon equation

$$i\hbar\,\partial_t\psi(t,x) = \sqrt{-c^2\hbar^2\Delta_x + m^2c^4}\,\psi(t,x) \,, \qquad \text{for } (t,x) \in \mathbb{R}^{1+3} \,,$$

where $\Delta_x = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}$ is the standard Laplace operator. However, the asymmetry of space and time derivatives was not compatible with a relativistic invariant description in presence of external fields. Also, squaring the equation was not appealing since, as said above, we are looking for a first order in time evolution. Hence, Dirac restarted again from the energy-momentum relation and linearized it before performing the formal transition to quantum mechanics. Thus, we have

$$E = c\sum_{j=1}^{3}\alpha_j p_j + \beta mc^2 = c\alpha\cdot p + \beta mc^2 \,, \tag{1.7}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are matrices determined by (1.6). Indeed, squaring both (1.6) and (1.7), the following system must be satisfied

$$\begin{aligned} &\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{ik}\mathrm{Id}_n \,, && \text{for } j,k = 1,2,3 \,, \\ &\alpha_j\beta + \beta\alpha_j = 0 \,, && \text{for } j = 1,2,3 \,, \\ &\beta^2 = \mathrm{Id}_n \,, \end{aligned}$$

where the size n of the matrices is not clear yet and δ_{jk} denotes the Kronecker symbol. At this point, performing $E \to i\hbar\partial_t$, $p \to -i\hbar\nabla_x$ we obtain the Dirac equation written in the form

$$i\hbar\partial_t\psi = \mathcal{D}_m\psi \,, \qquad \mathcal{D}_m := -i\hbar c\alpha\cdot\nabla + \beta mc^2 \,. \tag{1.8}$$

To gather some information on these matrices and to discuss covariance, we prefer to rewrite the Dirac equation in a four-dimensional notation. Therefore, we multiply (1.8) by β and we define the matrices

$$\gamma^0 := \beta \,, \qquad \gamma^j := \beta\alpha^j \,,$$

to finally obtain

$$i\gamma^\mu\partial_\mu\psi = m\psi \,, \tag{1.9}$$

where we set for convenience $\hbar = c = 1$. In this form, we notice that the new $(\gamma^\mu)_\mu$ now satisfy a more elegant and compact anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathrm{Id}_n ,$$

with γ^0 hermitian and the other γ^j antihermitian. These anticommutation relations are exactly defining the Clifford algebra structure Cliff$(1, 3, \mathbb{C})$. This tells us $\gamma^\mu \in M_4(\mathbb{C})$, and thus the solution $\psi \in C(\mathbb{R}; L^2(\mathbb{R}^3; \mathbb{C}^4))$.

On the other hand, as noticed in Remark 1.6, the choice of the matrices γ^μ is, mathematically speaking, arbitrary since any $(\gamma^\mu)_\mu$ and $(\tilde{\gamma}^\mu)_\mu$, both satisfying (1.1), are equivalent up to an invertible transformation U. Furthermore, if ψ solves $i\gamma^\mu\partial_\mu\psi = m\psi$, then $\tilde{\psi} := U^{-1}\psi$ solves $i\tilde{\gamma}^\mu\partial_\mu\tilde{\psi} = m\tilde{\psi}$.
(However, we stress that, from a physical point of view, some choices may be preferable to better highlight some behaviours and quantities of the observed particle.)

Going back to the covariance requirement, we impose that our solution is invariant under Lorentz transformations, which means that if ψ solves (1.9), then $\psi' := \psi \circ \Lambda$ must solve $i(\gamma')^\mu\partial_\mu\psi' = m\psi'$. This is equivalent to require that the Dirac matrices must transform under Lorentz transformations as

$$(\gamma')^\mu = \gamma^\nu(\Lambda^{-1})^\mu{}_\nu .$$

Note that this condition is compatible with the anticommutation relations found before, indeed

$$\{(\gamma')^\mu, (\gamma')^\nu\} = \{\gamma^\alpha, \gamma^\beta\}(\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\nu{}_\beta = 2\eta^{\alpha\beta}(\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\nu{}_\beta = 2\eta^{\mu\nu} .$$

Putting together the two aspects, in the Minkowski spacetime $(\mathbb{R}^{1+3}, \eta)$ one obtains a structure that assures that if ψ is a solution of the Dirac equation, then $\psi' := U^{-1}(\Lambda)\psi \circ \Lambda$ is again a solution of the equation with corresponding matrices $(\gamma')^\mu := U^{-1}(\Lambda)\gamma^\nu U(\Lambda)(\Lambda^{-1})^\mu{}_\nu$. Indeed,

$$\begin{aligned}
i(\gamma')^\mu\partial_\mu\psi'(x) &= i\, U^{-1}(\Lambda)\gamma^\nu U(\Lambda)(\Lambda^{-1})^\mu{}_\nu\, \partial_\mu(U^{-1}(\Lambda)\psi(\Lambda x)) \\
&= i\, U^{-1}(\Lambda)\gamma^\nu(\Lambda^{-1})^\mu{}_\nu\, \partial_\sigma\psi(\Lambda x)\Lambda^\sigma{}_\mu \\
&= U^{-1}(\Lambda)(i\gamma^\nu\partial_\nu\psi(\Lambda x)) = U^{-1}(\Lambda)(m\psi(\Lambda x)) \\
&= m\psi'(x) .
\end{aligned}$$

Note that the mapping $\Lambda \in SO^+(1, 3) \mapsto U(\Lambda) \in GL(4, \mathbb{C})$ defines a group representation (which is however not unitary, as already remarked). At this point, writing explicitly the Dirac equation, is a matter of choosing a family of matrices $(\gamma^\mu)_\mu$ with the desired anticommutation relations and covariance laws. We recall that the convention we fixed in Definition 1.5 is given by

$$\gamma^0 := \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_2 \end{pmatrix} , \qquad \gamma^j := \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} , \qquad \text{for } j = 1, 2, 3 ,$$

where σ^j denote the Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

To conclude this section, we stress two important properties of the Dirac equation $i\gamma^\mu \partial_\mu \psi = m\psi$, that were indeed the starting points for our derivation:

- if we multiply by γ^0 the Dirac equation, we indeed obtain a Schrödinger-type equation

$$i\partial_t \psi = \mathcal{D}_m \psi , \qquad \mathcal{D}_m := -i\alpha \cdot \nabla + \beta m , \tag{1.10}$$

  where $\alpha^j := \gamma^0 \gamma^j$, $\beta := \gamma^0$ and $\mathcal{D}_m$ is the Dirac operator, that is now fully determined by our choice of the Gamma matrices;

- by construction, when we square the Dirac equation, we recover the Klein–Gordon equation (or the wave equation in the massless case),

$$(\partial_t^2 - \Delta_x + m^2)\psi = 0 .$$

  This feature becomes extremely useful (as we will see later) in the study of the behaviour of Dirac solutions, since many properties can be thus deduced from the Klein–Gordon flow.

Finally, we observe one can see that $\mathcal{D}_m$ is indeed self-adjoint but not bounded from below. This mathematical fact appeared paradoxical and absurd from a physical point of view, since it would imply that negative energy states were admissible. Dirac interpreted these states as positive energy states associated to some *antiparticles*, which indeed have been discovered experimentally years later.

1.4 Preliminaries of differential geometry

In order to generalize the Dirac operator to curved spacetime, we briefly recall the main definitions and formulas of Riemannian and Lorentzian geometry.

We start with the geometrical object that generalizes the flat Minkowski space.

Definition 1.9 (Four-dim Lorentzian manifold)**.** A *four-dimensional Lorentzian manifold* $(\mathcal{M}, g)$ is a smooth manifold $\mathcal{M}$ equipped with Lorentzian metric g, i.e. an everywhere non-degenerate, smooth, symmetric metric tensor g with signature (1,3) in every point $p \in \mathcal{M}$.

When we deal with non-flat manifolds, the presence of curvature requires some corrections in the definition of the derivative. We recall the core object that permits a first generalization of covariant derivative.

Definition 1.10 (Levi-Civita connection and Christoffel symbols)**.** The *Levi-Civita connection* ∇ is the unique affine connection that preserves the Lorentzian metric g and is torsion-free, i.e.

$$\nabla g = 0 \quad \text{and} \quad \nabla_X Y - \nabla_Y X = [X, Y]\,, \text{for any } X, Y \text{ vector fields.}$$

Fixed a coordinate basis, the *Christoffel symbols* are the coefficients of the Levi-Civita connection and they write

$$\Gamma^{\sigma}_{\mu\nu} := \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})\ .$$

In coordinates, the compatibility with the metric and the absence of torsion rewrite

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^{\sigma}_{\alpha\mu} g_{\sigma\nu} - \Gamma^{\sigma}_{\alpha\nu} g_{\mu\sigma} = 0\ ,$$
$$T^{\mu}_{\alpha\beta} = \frac{1}{2}(\Gamma^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\beta\alpha}) = 0\ .$$

Thus, the Levi-Civita connection differentiates tensors as follows.

Definition 1.11 (Covariant derivative for a tensor field in coordinates)**.** Let T be a $(s, r)-$tensor. The *covariant derivative in coordinates* is given by the formula

$$\begin{aligned}\nabla_\mu T^{\alpha_1\alpha_2\ldots\alpha_s}_{\beta_1\beta_2\ldots\beta_r} :&= \partial_\mu T^{\alpha_1\alpha_2\ldots\alpha_s}_{\beta_1\beta_2\ldots\beta_r} + \Gamma^{\alpha_1}_{\mu\nu} T^{\nu\alpha_2\ldots\alpha_s}_{\beta_1\beta_2\ldots\beta_r} + \ldots + \Gamma^{\alpha_s}_{\mu\nu} T^{\alpha_1\alpha_2\ldots\nu}_{\beta_1\beta_2\ldots\beta_r} \\ &- \Gamma^{\nu}_{\mu\beta_1} T^{\alpha_1\alpha_2\ldots\alpha_s}_{\nu\beta_2\ldots\beta_r} - \ldots - \Gamma^{\nu}_{\mu\beta_r} T^{\alpha_1\alpha_2\ldots\alpha_s}_{\beta_1\beta_2\ldots\nu}\ .\end{aligned}$$

We conclude this section by introducing the principal curvature tensors and their main properties.

Definition 1.12 (Riemann curvature tensor)**.** The *Riemann curvature tensor* R is a $(1, 3)$-tensor defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and in coordinates write

$$R^{\alpha}{}_{\beta\gamma\mu} := \partial_\mu \Gamma^{\alpha}_{\gamma\beta} - \partial_\gamma \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\nu}_{\gamma\beta} - \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\nu}_{\mu\beta}\ .$$

Proposition 1.5 (Symmetries of the curvature tensor)**.** *The Riemann curvature tensor R satisfies the following properties:*

$$\begin{aligned} R_{\alpha\beta\gamma\mu} &= -R_{\alpha\beta\mu\gamma} \ , \\ R_{\alpha\beta\gamma\mu} &= -R_{\beta\alpha\gamma\mu} \ , \\ R_{\alpha\beta\gamma\mu} &= R_{\gamma\mu\alpha\beta} \ , \\ R_{\alpha\beta\gamma\mu} + R_{\alpha\mu\beta\gamma} + R_{\alpha\gamma\mu\beta} &= 0 \ , \\ \nabla_\nu R_{\alpha\beta\gamma\mu} + \nabla_\beta R_{\nu\alpha\gamma\mu} + \nabla_\alpha R_{\beta\nu\gamma\mu} &= 0 \ , \end{aligned}$$

where the last two equations are respectively the first and the second Bianchi identities.

Definition 1.13 (Ricci curvature tensor and scalar curvature)**.** The *Ricci curvature tensor* $R_{\beta\mu}$ and the *scalar curvature* R_g are defined by contracting the Riemann tensor

$$R_{\beta\mu} := R^{\alpha}{}_{\beta\alpha\mu} \ , \qquad R_g := g^{\beta\mu} R_{\beta\mu} \ .$$

1.5 Vierbein formalism and spin connection

This section will be devoted to the construction of another, and in some sense more general, connection that will be crucial in the definition of the Dirac operator in curved spacetime. A more precise discussion regarding this topic can be found in [PT09, Section 5.6].

To achieve this goal, we begin with a naive introduction of the vierbein formalism.

Idea (Vierbein). Given a Lorentzian manifold $(\mathcal{M}, g)$, if we consider some local spacetime coordinates $\{x^\mu\}_\mu$, we can associate a coordinate basis $\{dx^\mu\}_\mu$ in the cotangent space. Hence, the line element is given in local coordinates by

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

On the other hand, one may pass to a local orthonormal frame $\{e^a(x)\}$, whereby the line element writes

$$ds^2 = \eta_{ab}\, e^a(x) e^b(x) \ ,$$

where η is the Minkowski metric. Since $\{dx^\mu\}_\mu$ and $\{e^a(x)\}_a$ both span the cotangent space, there exist some (spacetime dependent) coefficients such that

$$e^a(x) = e^a{}_\mu(x) dx^\mu \ . \tag{1.11}$$

We will call this object $e^a{}_\mu$ *vierbein* (or *tetrad*). From the relations above, we deduce the characterizing property of a vierbein,

$$g_{\mu\nu}(x) = e^a{}_\mu(x)\,\eta_{ab}\,e^b{}_\nu(x)\ , \tag{1.12}$$

which could be interpreted, in some sense, as the "square root" of the metric.

Remark 1.10 (Global vierbein). In principle, vierbeins are defined only locally, since in the above construction we considered local orthonormal frames. In practice, dealing with small perturbations of flat spacetime will allow us to consider a globally defined vierbein. We conclude by mentioning that the presence of a global vierbein is a topological requirement characterizing an important class of manifolds, called *spin manifolds* (see [GLJ80, Section 1]).

Defining the *dual vierbein* $e_a{}^\mu := g^{\mu\nu}\eta_{ab}e^b{}_\nu$, one has

$$e_a{}^\mu(x)e^a{}_\nu(x) = \delta^\mu_\nu\ , \qquad e_a{}^\mu(x)e^b{}_\mu(x) = \delta^a_b\ ,$$

which allows to reverse the previous formulas

$$dx^\mu = e_a{}^\mu(x)\,e^a(x)\ , \qquad \eta_{ab} = e_a{}^\mu(x)\,g_{\mu\nu}(x)\,e_b{}^\nu(x)\ .$$

Hence, from (1.11), $e^a{}_\mu$ can be seen as a matrix transforming the coordinate basis dx^μ of the cotangent space to an orthonormal basis, and similarly the dual $e_a{}^\mu$ behaving in the same way on the tangent space of $\mathcal{M}$,

$$e_a(x) := e_a{}^\mu(x)\partial_\mu\ .$$

Notation (Flat and curved indices). From now on, Greek and Latin indices will be fundamental to distiguish between the coordinate system representation and the orthonormal frame. In particular, we will also refer to the former as flat indices and to the latter as curved ones, and they will be raised or lowered respectively by the metric g and η. The power of this formalism lies in the fact that the vierbein allows us to switch between Greek and Latin bases: indeed, for example, given a $(1,1)$-tensor, we have

$$T^a_b = e^a{}_\mu\,e_b{}^\nu\,T^\mu_\nu\ , \qquad T^\mu_\nu = e_a{}^\mu\,e^b{}_\nu\,T^a_b\ .$$

Idea (Spin connection). At this point, we notice that the standard Levi-Civita connection ∇ only interacts with curved indices. Thus, we need to extend it to a new covariant derivative, being able to act also on flat indices, i.e.

$$D_\mu X^a = \partial_\mu X^a + \omega_\mu{}^a{}_b X^b\ ,$$

where $\omega_\mu{}^a{}_b$ replace the standard Christoffel symbols and define the components of a new connection to be determined: this is indeed the *spin connection* we want to construct.

To gather some information about it, we start by imposing the Leibniz rule for the covariant differentiation, i.e.

$$D_\mu X^a = D_\mu(e^a{}_\nu X^\nu) = (D_\mu e^a{}_\nu)X^\nu + e^a{}_\nu(\nabla_\mu X^\nu) \ .$$

From Definition 1.11, $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\sigma} X^\sigma$, and thus it follows

$$D_\mu \, e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\sigma_{\nu\mu} e^a{}_\sigma + \omega_\mu{}^a{}_b \, e^b{}_\nu \ , \tag{1.13}$$

which relates the covariant derivative of the vierbein to the spin connection.

To completely determine $\omega_\mu{}^a{}_b$, we introduce Cartan's formalism which, roughly speaking, approaches differential geometry through the language of differential forms rather than tensors. For further details regarding similarities and differences between Cartan's and Riemann's approaches, we refer to [EGH80].

Now, let us define the spin one-form

$$\omega^a{}_b := \omega_\mu{}^a{}_b \, dx^\mu \ . \tag{1.14}$$

From (1.11), the exterior derivative of the orthonormal frame is given by

$$de^a = (\partial_\mu e^a{}_\nu - \Gamma^\sigma_{\mu\nu} e^a{}_\sigma) \, dx^\mu \wedge dx^\nu \ . \tag{1.15}$$

Following Cartan's approach, we state the two Cartan's structure equations, by defining the torsion 1-form $\mathcal{T}^a$ and the curvature 2-form $\mathcal{R}^a{}_b$,

$$\begin{aligned} \mathcal{T}^a &: = de^a + \omega^a{}_b \wedge e^b \ , \\ \mathcal{R}^a{}_b &:= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \end{aligned}$$

These two objects should be thought of, as their names suggests, as the equivalents of the torsion and curvature tensors in Riemannian geometry. In Definition 1.10, we recalled the two conditions uniquely determining the Levi-Civita connection; similarly, using Cartan's formalism, the spin connection is now defined imposing the two following conditions

$$\begin{aligned} \omega_{ab} &= -\omega_{ba} \ , \\ \mathcal{T}^a &= de^a + \omega^a{}_b \wedge e^b = 0 \ . \end{aligned}$$

The first compatibility condition makes the spin connection coefficients antisymmetric in the flat indices, $\omega_{\mu ab} = -\omega_{\mu ba}$. On the other hand, the absence of torsion, together with (1.15), determines their explicit formula

$$\omega_{\mu}{}^{a}{}_{b} = -e_{b}{}^{\nu}(\partial_{\mu}e^{a}{}_{\nu} - \Gamma^{\sigma}_{\mu\nu}e^{a}{}_{\sigma}) \ . \tag{1.16}$$

As a consequence, (1.13) and (1.16) imply the so-called *tetrad postulate*

$$D_{\mu}e^{a}{}_{\nu} = 0 \ . \tag{1.17}$$

This property shows that covariant differentiation commutes with the conversion of tensors to and from the local orthonormal frame, since the Leibniz rule written before simply reduces to

$$D_{\mu}X^{a} = D_{\mu}(e^{a}{}_{\nu}X^{\nu}) = e^{a}{}_{\nu}(\nabla_{\mu}X^{\nu}) \ .$$

Finally, we derived a new generalized covariant derivative that acts both on curved and flat indices, through the Levi-Civita and the spin connections respectively: for instance, we have

$$D_{\mu}T^{a}{}_{\nu} = \partial_{\mu}T^{a}{}_{\nu} - \Gamma^{\sigma}_{\mu\nu}T^{a}{}_{\sigma} + \omega_{\mu}{}^{a}{}_{b}T^{b}{}_{\nu} \ .$$

As a last remark, we give the formula relating the Cartan curvature form $\mathcal{R}^{a}{}_{b}$ and the Riemann curvature tensor $R^{\lambda}_{\sigma\mu\nu}$. Indeed, from the definition of $\mathcal{R}^{a}_{b}$ and explicitly computing the exterior derivative of the spin form, it follows

$$\begin{aligned}\mathcal{R}^{a}{}_{b} &= \frac{1}{2}(\partial_{\mu}\omega_{\nu}{}^{a}{}_{b} - \partial_{\nu}\omega_{\mu}{}^{a}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\,\omega_{\nu}{}^{c}{}_{b} - \omega_{\nu}{}^{a}{}_{c}\,\omega_{\mu}{}^{c}{}_{b})\, dx^{\mu} \wedge dx^{\nu} \\ &=: -\frac{1}{2}R_{\mu\nu}{}^{a}{}_{b}\, dx^{\mu} \wedge dx^{\nu} \ .\end{aligned}$$

Using (1.16) and Definition 1.12, one obtains

$$R_{\mu\nu}{}^{a}{}_{b} = e^{a}{}_{\lambda}e_{b}{}^{\sigma}R^{\lambda}_{\sigma\mu\nu} \ . \tag{1.18}$$

1.6 The Dirac equation in curved spacetime

At this point, we are ready to present the construction of the Dirac operator in curved background. As mentioned before, the Dirac equation represents a milestone in the framework of Quantum Field Theory, describing charged massive particles with relativistic corrections. The extension to the curved setting is fundamental to study this equation in the context of general relativity, where, as we will see later, the spacetime metric is governed by the Einstein Field Equations.

To this end, we start by pointing out a fact, quite obvious from the properties of the frame, but definitely crucial.

Remark 1.11 (The vierbein is not uniquely determined). Let $e^a{}_\mu$ be a vierbein and Λ a matrix representing a local Lorentz transformation (i.e. the parameters representing the Lorentz transformation are functions of spacetime coordinates). Then

$$(e')^a{}_\mu(x) := \Lambda^a{}_b(x) e^b{}_\mu(x) \ , \tag{1.19}$$

is again a vierbein, indeed

$$(e')^a{}_\mu \eta_{ab} (e')^b{}_\nu = \Lambda^a{}_c \, e^c{}_\mu \, \eta_{ab} \, \Lambda^b{}_d \, e^d{}_\nu = e^c{}_\mu \, \eta_{cd} \; e^d{}_\nu = g_{\mu\nu} \ .$$

Hence, we deduce that the metric structure characterizes a vierbein up to Lorentz transformations. Since the spacetime metric is not affected by the different choice of the vierbein, we expect that the same property of Lorentz invariance should hold also for any field equation.

This principle translates into a specific law of covariance for the spin connection. Consider, for instance, a vector field $X^\mu(x)$, whose flat components are $X^a = e^a{}_\mu X^\mu$. Now, from what we said, X^a must transform, under local Lorentz transformations Λ, like

$$X'^a(x') = \Lambda^a{}_b X^b(x) \ ,$$
$$D'_\mu X'^a(x') = \frac{\partial x^\nu}{\partial x'^\mu} \Lambda^a{}_b(x) D_\nu X^b(x) \ ,$$

where $D_\mu X^a = \partial_\mu X^a + \omega_\mu{}^a{}_b X^b$ is the covariant derivative found above. These covariance laws are compatible if the spin connection satisfies

$$\omega'_\mu{}^a{}_b = \frac{\partial x^\nu}{\partial x'^\mu} \Big(\Lambda^a{}_c \, \omega_\nu{}^c{}_d (\Lambda^{-1})^d{}_b - (\partial_\nu \Lambda^a{}_c)(\Lambda^{-1})^c{}_b \Big) \ . \tag{1.20}$$

So far, we defined the covariant behaviour of vector fields. We are now interested in the construction of a suitable covariant derivative being able to to preserve the invariance of the Dirac equation. Therefore, we now focus on Dirac spinors.

Idea (Dirac spinor). A four-component field ψ is called a *Dirac spinor* if it transforms under local Lorentz transformations Λ as

$$\psi'(x') = \pi(\Lambda(x))\psi(x) \ ,$$
$$D'_\mu \psi'(x') = \frac{\partial (x')^\rho}{\partial x^\mu} \pi(\Lambda(x)) D_\rho \psi(x) \ ,$$

where π is the spin representation of $SO^+(1,3)$, induced by the Dirac representation of Definition 1.8. (For simplicity, from now on we suppress the component indices of ψ and we will use a matrix notation.)

In practice, we are looking for a new covariant derivative

$$D_\mu\psi := \partial_\mu\psi + B_\mu(x)\psi \ ,$$

where B_μ is a field, to be determined, which lies in the Lie algebra representation of $\mathfrak{so}^+(1,3)$. This requirement comes from the fact that the covariant differentiation must be defined on the tangent space and thus on the Lie algebra.
Therefore, we impose that the following map

$$F(\Lambda) : \psi \mapsto D'_\mu\psi' - \frac{\partial(x')^\rho}{\partial x^\mu}\pi(\Lambda)D_\rho\psi \tag{1.21}$$

is identically zero. Since $SO^+(1,3)$ is connected, it is sufficient to require that both F and its differential are vanishing at the identity. Hence, we linearize near the identity and choose B_μ such that F is zero up to the first order. To this end, we recall from (1.3), that infinitesimal variations of the identity write

$$\Lambda^a{}_b = \delta^a{}_b + \varepsilon^a{}_b(x) \ , \qquad \pi(1+\varepsilon) = 1 + i\varepsilon^{ab}\Sigma_{ab} \ ,$$

where the coefficients satisfies $\varepsilon_{ab} = -\varepsilon_{ba}$ and the generators of the representation of $\mathfrak{so}^+(1,3)$ are such that $\Sigma_{ab} = -\Sigma_{ba}$. Hence, we have

$$\begin{aligned} D'_\mu\psi' &= \partial_\mu\psi' + B'_\mu\psi' = \partial_\mu(\pi(\Lambda)\psi) + B'_\mu(\pi(\Lambda)\psi) \\ &= (\partial_\mu\pi(\Lambda))\psi + \pi(\Lambda)\partial_\mu\psi + B'_\mu(\pi(\Lambda)\psi) \end{aligned}$$

and, on the other hand,

$$D'_\mu\psi' = \pi(\Lambda)D_\mu\psi = \pi(\Lambda)(\partial_\mu + B_\mu)\psi \ ,$$

where the prefactor $\partial(x')^\rho/\partial x^\mu$ has been replaced with δ^ν_μ, since we are restricting to infinitesimal local Lorentz transformations. From these two last relations, we deduce that the field B_μ must satisfy

$$B'_\mu = \pi(\Lambda)B_\mu\pi^{-1}(\Lambda) - (\partial_\mu\pi(\Lambda))\pi^{-1}(\Lambda)$$

and thus using the infinitesimal form of a Lorentz transformation (1.3),

$$B'_\mu = B_\mu + i\varepsilon^{ab}\Sigma_{ab}B_\mu - i\,B_\mu\varepsilon^{ab}\Sigma_{ab} - i\partial_\mu\varepsilon^{ab}\Sigma_{ab} = B_\mu + i\varepsilon^{ab}[\Sigma_{ab}, B_\mu] - i\partial_\mu\varepsilon^{ab}\Sigma_{ab} \ .$$

Since B_μ takes values on the Lie algebra, we can express it in terms of the generators, i.e. $B_\mu(x) = B_\mu^{ab}(x)\Sigma_{ab}$. Therefore, after some manipulations of indices and commutators,

$$B'_\mu = B_\mu - i\varepsilon^{cd} B_\mu^{ab}[\Sigma_{ab}, \Sigma_{cd}] - i\partial_\mu \varepsilon^{ab}\Sigma_{ab} \ .$$

By Lemma 1.4, we deduce the infinitesimal law of covariance

$$B'_\mu{}^{ab} = B_\mu{}^{ab} + B_\mu{}^b{}_c\,\varepsilon^{ca} - B_\mu{}^a{}_c\,\varepsilon^{cb} - i\partial_\mu \varepsilon^{ab} \ .$$

If we compare it with the infinitesimal form of the spin connection (1.20),

$$\omega'_\mu{}^{ab} = \omega_\mu{}^{ab} + \omega_\mu{}^b{}_c\,\varepsilon^{ca} - \omega_\mu{}^a{}_c\,\varepsilon^{cb} - \partial_\mu \varepsilon^{ab} \ ,$$

we finally conclude that

$$B_\mu{}^{ab} = i\omega_\mu{}^{ab} \ .$$

Hence the covariant derivative of ψ is given by

$$D_\mu \psi = \partial_\mu \psi + i\omega_\mu{}^{ab}\Sigma_{ab}\psi \ . \tag{1.22}$$

In this formula, we see that the covariant derivative is determined by two main factors: the spin connection, which is purely given by the geometry of the manifold (and the choice of the vierbein), and the algebraic spin representation, that is related to the nature of the particle observed.

Before stating the final form of the Dirac equation, we give an important technical lemma regarding the new covariant derivatives. Indeed, in order to preserve Lorentz invariance, we sacrificed the commutativity of derivatives.

Lemma 1.6 (Commutator of covariant derivatives)**.** *Let D_μ the covariant derivative defined as in* (1.22)*. Then*

$$[D_\mu, D_\nu] = -iR_{\mu\nu}{}^{ab}\,\Sigma_{ab} \ , \tag{1.23}$$

where $R_{\mu\nu}{}^{ab}$ is the "mixed" curvature tensor given by (1.18)*.*

Proof. The assertion follows from direct computations. Indeed,

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu + i\omega_\mu{}^{ab}\Sigma_{ab}\,, \partial_\nu + i\omega_\nu{}^{cd}\Sigma_{cd}] \\ &= i\partial_\mu\omega_\nu{}^{cd}\Sigma_{cd} - i\partial_\nu\omega_\mu{}^{ab}\Sigma_{ab} - \omega_\mu{}^{ab}\omega_\nu{}^{cd}[\Sigma_{ab}, \Sigma_{cd}] \ .\end{aligned}$$

Using Lemma 1.4, the last summand rewrites

$$\begin{aligned}\omega_\mu{}^{ab}\omega_\nu{}^{cd}[\Sigma_{ab}, \Sigma_{cd}] &= \frac{i}{2}\Big(- \omega_\mu{}^b{}_a\omega_\nu{}^{cd}\Sigma_{bd} - \omega_\mu{}^{ab}\omega_\nu{}^c{}_a\Sigma_{bc} - \omega_\mu{}^a{}_c\omega_\nu{}^{cd}\Sigma_{ad} + \omega_\mu{}^{ab}\omega_\nu{}^c{}_b\Sigma_{ac}\Big) \\ &= i(\omega_\mu{}^a{}_c\omega_\nu{}^{cb} - \omega_\mu{}^{cb}\omega_\nu{}^a{}_c)\Sigma_{ab} \ ,\end{aligned}$$

where in the last equality we used $\omega_{\mu ab} = -\omega_{\mu ba}$. Finally, from (1.18), we conclude

$$\begin{aligned}[D_\mu, D_\nu] &= -i(\partial_\nu \omega_\mu{}^{ab} - \partial_\mu \omega_\nu{}^{ab} - \omega_\mu{}^a{}_c \omega_\nu{}^{cb} + \omega_\mu{}^{cb} \omega_\nu{}^a{}_c)\Sigma_{ab} \\ &= -iR_{\mu\nu}{}^{ab}\,\Sigma_{ab}\ .\end{aligned}$$

□

We are ready to generalize the Dirac equation. We recall that in flat spacetime, it writes $i\gamma^\mu \partial_\mu \psi = m\psi$. We perform the following substitution

$$\partial_\mu \leftarrow e_a{}^\mu D_\mu\ ,$$

where D_μ is the covariant derivative is defined as in (1.22). Thus, we find

$$i\gamma^a e_a{}^\mu D_\mu \psi = m\psi\ .$$

We denote $\underline{\gamma}^\mu(x) := e_a{}^\mu(x)\gamma^a$, and we call them *spacetime-dependent Dirac matrices*, which satisfy the following anticommutation rules

$$\{\underline{\gamma}^\mu(x), \underline{\gamma}^\nu(x)\} = 2g^{\mu\nu}(x)\mathrm{Id}_4\ . \tag{1.24}$$

Furthermore, we note that the new matrices $\underline{\gamma}^\mu$ are covariantly constant: this fact can be easily deduced from the tetrad postulate (1.17).

Theorem 1.7. *The Dirac matrices* $\underline{\gamma}^\mu := e_a{}^\mu \gamma^a$ *are covariantly constant, i.e.* $D_\nu \underline{\gamma}^\mu = 0$.

Proof. Using the Leibniz rule and the tetrad postulate $D_\nu e_a{}^\mu = 0$, we have

$$D_\nu \underline{\gamma}^\mu = D_\nu(e_a{}^\mu \gamma^a) = e_a{}^\mu (D_\nu \gamma^a) = e_a{}^\mu(\partial_\nu \gamma^a + \frac{1}{8}\omega_\nu{}^{bc}[\gamma_b, \gamma_c]\gamma^a) = \frac{1}{8} e_a{}^\mu\, \omega_\nu{}^{bc}[\gamma_b, \gamma_c]\gamma^a\ .$$

Recalling that $\omega_\nu{}^{cb} = -\omega_\nu{}^{bc}$ and $\{\gamma^b, \gamma^c\} = 2\eta^{bc}$, we have

$$D_\nu \underline{\gamma}^\mu = \frac{1}{8} e_a{}^\mu\, \omega_\nu{}^{bc}(\gamma_b\gamma_c - \gamma_c\gamma_b)\gamma^a = \frac{1}{8} e_a{}^\mu\, \omega_\nu{}^{bc}\{\gamma_b, \gamma_c\}\gamma^a = \frac{1}{4} e_a{}^\mu\, \omega_\nu{}^b{}_b \gamma^a = 0\ .$$

□

Finally, recalling the spin representation of Definition 1.8, $\Sigma^{ab} = -\frac{i}{8}[\gamma^a, \gamma^b]$, we fully determine the spinorial covariant derivative

$$D_\mu \psi = \partial_\mu \psi + B_\mu \psi := \partial_\mu \psi + \frac{1}{8}\omega_\mu{}^{ab}[\gamma_a, \gamma_b]\psi\ , \tag{1.25}$$

which satisfies, by Lemma 1.6, the following

$$[D_\mu, D_\nu]\psi = -\frac{1}{8} R_{\mu\nu}{}^{ab}[\gamma_a, \gamma_b]\psi\ . \tag{1.26}$$

At the end of Section 1.3, we noticed that the flat Dirac equation has the remarkable property of squaring to the Klein–Gordon equation. We will see, with the help of the next lemma, that the same happens even in the curved (more delicate) case.

Lemma 1.8 (Schrödinger-Lichnerowicz identity, [Sch32]). *Let D_μ the spinorial covariant derivative defined as in* (1.25). *Then*

$$(\underline{\gamma}^\mu D_\mu)^2 = \Box_g + \frac{1}{4}R_g \ ,$$

where $\Box_g := D^\mu D_\mu = g^{\mu\nu} D_\nu D_\mu$ is the d'Alembertian operator and R_g is the scalar curvature associated to g.

Remark 1.12 (Spinorial d'Alembertian operator). Before presenting the proof, let us briefly show how $\Box_g$ acts in coordinates. Given a spinor ψ, we have

$$\begin{aligned}\Box_g\psi = g^{\mu\nu} D_\nu D_\mu \psi &= g^{\mu\nu}\Big((\partial_\nu + B_\nu) D_\mu\psi - \Gamma^\sigma_{\nu\mu} D_\sigma\psi\Big) \\ &= g^{\mu\nu}(\partial_\nu\partial_\mu\psi - \Gamma^\sigma_{\nu\mu}\partial_\sigma\psi) + g^{\mu\nu}(B_\nu\partial_\mu + B_\nu B_\mu + \partial_\nu B_\mu - \Gamma^\sigma_{\nu\mu}B_\sigma)\psi \\ &= \widetilde{\Box}_g\psi + B^\mu\partial_\mu\psi + B^\mu B_\mu\psi + \widetilde{D}^\mu B_\mu\psi \ ,\end{aligned}$$

where $\widetilde{\Box}_g$ denotes the wave operator for scalar fields and $\widetilde{D}^\nu X_\mu := \partial^\nu X_\mu - \Gamma^\sigma{}_\mu{}^\nu X_\sigma$.

Proof. We start by multiplying by $\underline{\gamma}^\mu\underline{\gamma}^\nu$ both sides of (1.26), obtaining

$$\underline{\gamma}^\mu\underline{\gamma}^\nu[D_\mu, D_\nu] = -\frac{1}{8}\underline{\gamma}^\mu\underline{\gamma}^\nu R_{\mu\nu}{}^{ab}[\gamma_a, \gamma_b] \ .$$

We focus first on the right-hand side: recalling that $R_{\mu\nu}{}^a{}_b = e^a{}_\lambda e_b{}^\sigma R^\lambda{}_{\sigma\mu\nu}$, we have

$$\begin{aligned}-\frac{1}{8}\underline{\gamma}^\mu\underline{\gamma}^\nu R_{\mu\nu}{}^{ab}[\gamma_a, \gamma_b] &= -\frac{1}{8}\underline{\gamma}^\mu\underline{\gamma}^\nu e^a{}_\lambda e^b{}_\sigma R^{\lambda\sigma}{}_{\mu\nu}[\gamma_a, \gamma_b] = -\frac{1}{8}\underline{\gamma}^\mu\underline{\gamma}^\nu R^{\lambda\sigma}{}_{\mu\nu}[\underline{\gamma_\lambda}, \underline{\gamma_\sigma}] \\ &= -\frac{1}{4}\underline{\gamma}^\mu\underline{\gamma}^\nu\underline{\gamma}^\lambda\underline{\gamma}^\sigma R_{\mu\nu\lambda\sigma} \ ,\end{aligned}$$

where, in the last equality, we used the symmetries of the Riemann curvature tensor, stated in Proposition 1.5. Due to the first Bianchi identity, and using repeatedly the anticommutation properties of $\underline{\gamma}^\mu$, we obtain

$$\underline{\gamma}^\mu\underline{\gamma}^\nu\underline{\gamma}^\lambda\underline{\gamma}^\sigma R_{\mu\nu\lambda\sigma} = -2\underline{\gamma}^\mu\underline{\gamma}^\lambda g^{\sigma\nu} R_{\mu\nu\lambda\sigma} \ .$$

Using again $\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$, one has

$$\underline{\gamma}^\mu\underline{\gamma}^\lambda R_{\mu\nu\lambda\sigma} = 2g^{\mu\lambda} R_{\mu\nu\lambda\sigma} - \underline{\gamma}^\lambda\underline{\gamma}^\mu R_{\mu\nu\lambda\sigma} = 2R^\lambda_{\nu\lambda\sigma} - \underline{\gamma}^\lambda\underline{\gamma}^\mu R_{\lambda\sigma\mu\nu} \ ,$$

which gives, relabeling the indices

$$\underline{\gamma}^\mu\underline{\gamma}^\lambda R_{\mu\nu\lambda\sigma} = R^\lambda_{\nu\lambda\sigma} = R_{\nu\sigma} \ .$$

Summarizing, the right-hand side gives

$$-\frac{1}{8}\underline{\gamma}^\mu\underline{\gamma}^\nu R_{\mu\nu}{}^{ab}[\gamma_a,\gamma_b] = -\frac{1}{4}\underline{\gamma}^\mu\underline{\gamma}^\nu\underline{\gamma}^\lambda\underline{\gamma}^\sigma R_{\mu\nu\lambda\sigma} = \frac{1}{2}g^{\sigma\nu}\underline{\gamma}^\mu\underline{\gamma}^\lambda R_{\mu\nu\lambda\sigma} = \frac{1}{2}g^{\sigma\nu}R_{\nu\sigma} = \frac{1}{2}R_g\ .$$

On the other hand, by Theorem 1.7, we have that $(\underline{\gamma}^\mu D_\mu)^2 = \underline{\gamma}^\mu\underline{\gamma}^\nu D_\mu D_\nu$. Thus, the left-hand side simply rewrites

$$\underline{\gamma}^\mu\underline{\gamma}^\nu(D_\mu D_\nu - D_\nu D_\mu) = (\underline{\gamma}^\mu D_\mu)^2 - (2g^{\mu\nu} - \underline{\gamma}^\nu\underline{\gamma}^\mu)D_\nu D_\mu = 2(\underline{\gamma}^\mu D_\mu)^2 - 2\Box_g\ .$$

Hence, the identity follows

$$(\underline{\gamma}^\mu D_\mu)^2 = \Box_g + \frac{1}{4}R_g\ .$$

□

Therefore, it is straightforward to see that squaring the Dirac equation, we recover a curved Klein–Gordon equation

$$\Big(\Box_g + \frac{1}{4}R_g + m^2\Big)\psi = 0\ .$$

However, we stress once again that the wave operator $\Box_g = D^\mu D_\mu$ is "spinorial", meaning that is defined by spinorial covariant derivatives. This feature, in general, severely complicates the understanding of the curved Dirac operator. In the following chapter, we will analyze in detail this aspect.

For later use, we conclude the paragraph by briefly introducing the adjoint spinor, without entering in further details.

Definition 1.14 (Adjoint spinor and main properties)**.** Denoting with $\cdot^*$ the hermitian adjoint, we define the *adjoint spinor* $\overline{\psi}$ of ψ as

$$\overline{\psi} := \psi^*\gamma^0\ ,$$

which transforms under local Lorentz transformations as $\overline{\psi}' = \overline{\psi}\pi^{-1}(\Lambda)$. The action of the spinorial covariant derivative writes

$$D_\mu\overline{\psi} := \partial_\mu\overline{\psi} - \overline{\psi}B_\mu = \partial_\mu\overline{\psi} - \frac{1}{8}\overline{\psi}\,\omega_\mu{}^{ab}[\gamma_a,\gamma_b]\ .$$

Thanks to this construction, we can deduce the adjoint equation of the Dirac equation.

Theorem 1.9 (Adjoint Dirac equation)**.** *If ψ is a solution of the Dirac equation $i\underline{\gamma}^\mu D_\mu\psi = m\psi$, then the adjoint spinor $\overline{\psi}$ solves*

$$iD_\mu\overline{\psi}\underline{\gamma}^\mu = -m\overline{\psi}\ . \tag{1.27}$$

Proof. To prove the statement, we start with two properties of the Dirac matrices, that can be easily checked recalling that γ^0 is hermitian while the $\gamma^j, j = 1, 2, 3$ are antihermitian:

$$\gamma^0[\gamma_a, \gamma_b]^*\gamma^0 = -[\gamma_a, \gamma_b] \quad \text{and} \quad \gamma^0(\gamma^a)^*\gamma^0 = \gamma^a \ .$$

Therefore, we deduce $\overline{D_\mu\psi} = D_\mu\overline{\psi}$, indeed

$$\begin{aligned}\overline{D_\mu\psi} &= (\partial_\mu\psi + B_\mu\psi)^*\gamma^0 = \partial_\mu\psi^*\gamma^0 + \frac{1}{8}(\omega_\mu{}^{ab}[\gamma_a, \gamma_b]\psi)^*\gamma^0 \\ &= \partial_\mu\overline{\psi} + \frac{1}{8}\psi^*\omega_\mu{}^{ab}[\gamma_a, \gamma_b]^*\gamma^0 = \partial_\mu\overline{\psi} - \frac{1}{8}\psi^*\omega_\mu{}^{ab}\gamma^0[\gamma_a, \gamma_b] = D_\mu\overline{\psi} \ .\end{aligned}$$

Finally, recalling that $(\gamma^0)^2 = \mathrm{Id}_4$, we have

$$\overline{\underline{\gamma}^\mu D_\mu\psi} = (D_\mu\psi)^*(\underline{\gamma}^\mu)^*\gamma^0 = (D_\mu\psi)^*\gamma^0\gamma^0(\underline{\gamma}^\mu)^*\gamma^0 = \overline{D_\mu\psi}\,\underline{\gamma}^\mu = D_\mu\overline{\psi}\,\underline{\gamma}^\mu \ ,$$

and thus we conclude

$$im\overline{\psi} = \overline{\underline{\gamma}^\mu D_\mu\psi} = D_\mu\overline{\psi}\,\underline{\gamma}^\mu \ .$$

□

Chapter 2

Dispersive estimates for the Dirac equation

Now that the Dirac operator in curved background has been constructed, in the second chapter we finally study some of its properties. In particular, we observe how things become more difficult in the curved case and show how to recover, under suitable assumptions, a ”dispersive” behaviour.

The main result of the chapter concerns some Strichartz estimates on asymptotically flat manifolds, that have been proven in the recent years by Cacciafesta, de Suzzoni and Meng, see [CdS19b] and [CdSM23].

2.1 Dispersion in flat spacetime

Before looking at the analytical properties of the Dirac operator in curved spacetime, let us take a step back to the flat case. In this simplified setting, we briefly introduce the concept of dispersion, which may help understanding the behaviour of the solution.

By *dispersion* we mean, roughly speaking, the property of each component of a wave packet to travel with different speeds. This peculiar feature often yields, on the one hand, some physical quantities (the energy, for instance) to be globally conserved, but on the other, on every compact region a decay is observed. The most important equations of quantum mechanics share this trait and we have encountered them from time to time throughout this work. We are referring to the Schrödinger, the Klein–Gordon and the wave equations,

$$\text{Schrödinger :} \begin{cases} i\partial_t\psi + \Delta_x\psi = 0 \ , \\ \psi(0,x) = \psi_0(x) \ ; \end{cases}$$

$$\text{Klein–Gordon :} \begin{cases} \partial_t^2\psi - \Delta_x\psi + m^2\psi = 0 \ , \\ \psi(0,x) = \psi_0(x) \ , \ \partial_t\psi(0,x) = \psi_1(x) \ ; \end{cases}$$

$$\text{Wave :} \begin{cases} \partial_t^2\psi - \Delta_x\psi = 0 \ , \\ \psi(0,x) = \psi_0(x) \ , \ \partial_t\psi(0,x) = \psi_1(x) \ ; \end{cases}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$. In the flat case, the Dirac equation is also clearly dispersive. Indeed, we recall from Section 1.3, that each component of a Dirac solution solves a Klein–Gordon (or wave) equation. Hence, exploiting this *"squaring trick"*, many estimates for the Dirac flow can be easily deduced from the well-known results for these latter equations.

For instance, let us look at the Strichartz estimates, which write respectively

$$\begin{aligned} \text{Klein–Gordon :} \quad & \|e^{it\sqrt{m^2-\Delta_x}}f\|_{L^p_t H^s_q} \lesssim \|f\|_{L^2} \ , \\ \text{Wave :} \quad & \|e^{it\sqrt{-\Delta_x}}f\|_{L^p_t \dot{H}^s_q} \lesssim \|f\|_{L^2} \ , \end{aligned}$$

where by $A \lesssim B$ we mean that $A \leq CB$, with C a constant independent of the parameters. At this stage we omit the details regarding the exponents (p,s,q) and the Sobolev spaces appearing in the mixed norms, since these objects will be discussed extensively in the following sections.
As anticipated, thanks to the squaring property, one deduces immediately the same Strichartz estimates for the Dirac flow

$$\begin{aligned} \text{massive case :} \quad & \|e^{it\mathcal{D}_m}f\|_{L^p_t H^s_q} \lesssim \|f\|_{L^2} \ , \\ \text{massless case :} \quad & \|e^{it\mathcal{D}}f\|_{L^p_t \dot{H}^s_q} \lesssim \|f\|_{L^2} \ , \end{aligned}$$

where $\mathcal{D}_m := -i\alpha \cdot \nabla + \beta m$ is the Dirac operator (with $\mathcal{D} = \mathcal{D}_0$).

These estimates formalize the idea of dispersion mentioned above. Indeed, locally in time they describe a sort of smoothing effect, through a gain of integrability in a L^p time-averaged sense. On the other hand, globally in time, they prescribe a decay effect of the spatial norm, again in some L^p time-averaged sense.

However, this approach becomes extremely complicated when we pass to curved spacetime. As we already noticed in the end of Chapter 1, the curved Dirac equation squares to a spinorial Klein–Gordon equation, where all the components are not decoupled anymore. Below, we will see how to deal with this new difficulty, starting with a digression regarding the geometric hypotheses on the manifold.

The reason why we are interested in this family estimates is related to the analysis of nonlinear problems. Strichartz estimates represent indeed a fundamental tool for many techniques used in the study of well-posedness for dispersive equations. For some (non exhaustive) references regarding these topics, see for instance [Tao06] and [Str77].

2.2 Asymptotically flat manifolds

Going back to the curved context, the spinorial covariant derivative D_μ is defined in terms of the spin connection, whose behaviour is deeply related to the geometrical structure of the manifold. Hence, it is quite reasonable to expect a well-behaved solution when the underlying manifold is nice: for instance, when it is not that far from the standard flat Minkowski metric. In this section, we formalize this intuition specifying the geometric hypotheses on the Lorentzian manifold and we study the consequent behaviour of the Dirac operator.

We start with the following assumption, that considerably simplifies the spin structure.

Definition 2.1 (Static and decoupling metric)**.** Given a Lorentzian manifold $(\mathcal{M}, g)$, we say that the metric g is *static in time and decoupling*, if it writes

$$g_{\mu\nu}(x) = \begin{cases} 1 & \text{if } \mu = \nu = 0; , \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu \ , \\ -h_{\mu\nu}(x) & \text{otherwise} \ , \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ is the space variable and $h \in C^\infty(\mathbb{R}^3)$ is a Riemannian metric.

Notation (Index notation). We recall that the Latin indices $a, b, ...$ denote the flat components, while the Greek letters $\mu, \nu, ...$ denote the spacetime curved ones. In addition, to refer specifically to the curved spatial components, we use $j, k, ...$.

The first important property of a metric decoupling space and time, is that the vierbein $e^a{}_\mu$ reduces to a *"dreibein"* $f^a{}_j$, satisfying

$$h_{jk}(x) = f^a{}_j(x)\, \delta_{ab}\, f^b{}_k(x) \ ,$$

where all the indices belong to $\{1,2,3\}$ and δ_{ab} is the Kronecker symbol.

In particular, for a static decoupling metric, we can build a vierbein given a dreibein in a very natural way.

Lemma 2.1. *Let h be a Riemannian metric and $f^a{}_j$ be a spatial dreibein associated to h, then*

$$e^a{}_\mu(x) := \begin{cases} 1 & \text{if } \mu = a = 0; , \\ 0 & \text{if } \mu a = 0 \text{ and } \mu \neq a \ , \\ f^a{}_\mu(x) & \text{otherwise} \ . \end{cases}$$

defines a spacetime vierbein associated to g as in Definition 2.1.

Proof. We omit the proof since it is only a matter of evaluating explicitly the three possible cases. For more details, see [CdS19b, Proposition 2.1]. □

Furthermore, under the standing assumptions on g, we also have a clear relation at the level of the affine connections.

Lemma 2.2. *Let g be a static and decoupling metric, and let Φ^i_{jk} and $\Gamma^\sigma_{\mu\nu}$ be the affine connection coefficients associated respectively to h and g. Then*

$$\Gamma^\sigma_{\mu\nu} = \begin{cases} \Phi^\sigma_{\mu\nu} & \text{if } \sigma\mu\nu \neq 0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof. As with the previous lemma, we omit this proof for the same reasons. For more details, see [CdS19b, Proposition 2.2] □

Remark 2.1 (Scalar curvature). Thanks to the last lemma, we immediately deduce that $R_g = -R_h$, the scalar curvature associated to h.

Therefore, due to these results, it is not difficult to imagine what the behaviour of the spin connection is.

Lemma 2.3. *Let g be a static and decoupling metric, and let $\alpha_j{}^{ab}$ and $\omega_\mu{}^{ab}$ be the spin connection coefficients associated respectively to h and g. Then*

$$\omega_\mu{}^{ab} = \begin{cases} \alpha_\mu{}^{ab} & \text{if } \mu ab \neq 0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof. From (1.16), we recall that $\omega_\mu{}^{ab} = e^a{}_\nu(\partial_\mu e^{b\nu} + \Gamma^\nu_{\mu\sigma} e^{b\sigma})$. We proceed by cases: if $\mu = 0$,

$$\omega_0{}^{ab} = e^a{}_\nu(\partial_t e^{b\nu} + \Gamma^\nu_{0\sigma} e^{b\sigma}) = e^a{}_\nu \partial_t e^{b\nu} = 0 \ ,$$

since $\Gamma^{\nu}_{0\sigma} = 0$ and $e^{a}{}_{\mu}$ does not depend on t, by Lemmas 2.2 and 2.1. Hence

$$\omega_0{}^{ab} = 0 \quad \text{for any } a, b\ .$$

If $a = 0$, and assuming $\mu \neq 0$,

$$\omega_\mu{}^{0b} = e^{0}{}_{\nu}(\partial_\mu e^{b\nu} + \Gamma^{\nu}_{\mu\sigma} e^{b\sigma}) = \partial_\mu e^{b0} + \Gamma^{0}_{\mu\sigma} e^{b\sigma} = 0\ ,$$

where in the first equality the sum over ν gives only $\nu = 0$ and thus one concludes using again Lemmas 2.2 and 2.1. Hence

$$\omega_\mu{}^{0b} = 0 \quad \text{for any } \mu, b\ .$$

Since $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$, we immediately deduce the case $b = 0$

$$\omega_\mu{}^{a0} = 0 \quad \text{for any } \mu, a\ .$$

If $\mu a b \neq 0$, which means $\mu = j$,

$$\omega_j{}^{ab} = e^{a}{}_{\nu}(\partial_j e^{b\nu} + \Gamma^{\nu}_{j\sigma} e^{b\sigma}) = e^{a}{}_{k}(\partial_j e^{bk} + \Gamma^{k}_{jl} e^{bl}) = f^{a}{}_{k}(\partial_j f^{bk} + \Phi^{k}_{jl} f^{bl}) = \alpha_j{}^{ab}\ ,$$

since the sums over ν and σ give only the spatial indices. □

We are now ready to formalize the idea of dealing with a "controlled and decaying perturbation" of the Minkowski flat metric.

Definition 2.2 (Asymptotically flat manifold)**.** Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian manifold with g static and decoupling and geodesically complete (i.e. maximal geodesics are defined on the whole $\mathbb{R}$).
$(\mathcal{M}, g)$ is *asymptotically flat* if there exist constants C_h and $\sigma \in (0,1)$ such that for any multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq 3$ and all x,

$$|\partial^\alpha (h_{jk}(x) - \delta_{jk})| \leq C_h \langle x\rangle^{-|\alpha|-1-\sigma}\ , \qquad \text{for } j,k = 1,2,3\ , \tag{2.2}$$

where $\partial^\alpha := \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$ and $\langle x\rangle := (1 + |x|^2)^{1/2}$.

Remark 2.2 (Non-trapping condition). The constant C_h plays a fundamental role and it is strongly related to the so-called *non-trapping condition* on the metric g. Indeed, to observe dispersive dynamics on non-flat manifolds, it is necessary that the underlying metric does not confine geodesic flows in some compact and localized regions. In our context, this is guaranteed if C_h is small enough and indeed this will be an implicit requirement throughout this chapter.

Remark 2.3 (Uniform bounds on inverse and derivatives). The asymptotically flat condition (2.2) implies uniform bounds on the inverse $h^{jk}(x)$. For instance, supposing C_h small enough, we can establish

$$|h^{-1}(x)| \leq 2 \;, \qquad \text{for all } x \;,$$

where $|\cdot|$ denotes the matrix norm. Furthermore, regarding the bounds on the derivatives of h, we can rewrite (2.2) in a more compact way as

$$|h'(x)| \leq C_h \langle x \rangle^{-2-\sigma} \;, \quad |h''(x)| \leq C_h \langle x \rangle^{-3-\sigma} \;, \quad |h'''(x)| \leq C_h \langle x \rangle^{-4-\sigma} \;,$$

where $|h'(x)| := \sum_{|\alpha|=1} |\partial^\alpha h(x)| \;,\; |h''(x)| := \sum_{|\alpha|=2} |\partial^\alpha h(x)| \;,\; |h'''(x)| := \sum_{|\alpha|=3} |\partial^\alpha h(x)| \;.$

The next lemma shows how the assumption on the metric affects both the geometric structure and the Dirac operator.

Lemma 2.4. *Let $(\mathcal{M}, g)$ be an asymptotically flat manifold with constant $C_h \ll 1$. Then there exist some geometric constants C_R and C_Γ such that for all x,*

$$|R_h(x)| \leq C_R C_h \langle x \rangle^{-3-\sigma} \;, \quad |\Gamma(x)| \leq C_\Gamma C_h \langle x \rangle^{-2-\sigma} \;.$$

Furthermore, the dreibein e exists globally and can be chosen such that there exist constants C_B and C_B'

$$|B(x)| \leq C_B C_h \langle x \rangle^{-2-\sigma} \;, \quad |\partial B(x)| \leq C_B' C_h \langle x \rangle^{-3-\sigma} \qquad \text{for all } x.$$

Proof. We start by estimating the the geometric quantities, not depending on the choice of the dreibein. Recalling that the scalar curvature is given by

$$R_h = h^{jk}(\partial_k \Gamma^i_{ij} - \partial_i \Gamma^i_{jk} + \Gamma^i_{kl}\Gamma^l_{ij} - \Gamma^i_{il}\Gamma^l_{jk}) \;,$$

we obtain, for C_h small enough,

$$|R_h(x)| \leq C_R'(|\partial\Gamma(x)| + |\Gamma(x)|^2) \;,$$

where we used the uniform bound on the inverse given in Remark 2.3. Hence, recalling that $\Gamma^i_{jk} = \frac{1}{2} h^{il}(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk})$,

$$|\Gamma(x)| \leq C_\Gamma |h'(x)| \leq C_\Gamma C_h \langle x \rangle^{-2-\sigma} \;.$$

On the other hand, writing $(h^{-1})'(x) = -h^{-1}(x) h'(x) h^{-1}(x)$, we obtain

$$|\partial\Gamma(x)| \leq C_\Gamma'(|h'(x)|^2 + |h''(x)|) \leq C_\Gamma' C_h \langle x \rangle^{-3-\sigma} \;.$$

Hence, we get the following estimate on the curvature

$$|R_h(x)| \leq C_R C_h \langle x\rangle^{-3-\sigma} .$$

Let us now focus on the choice of a suitable dreibein, which by definition must satisfy $h_{jk}(x) = e^a{}_j(x)\,\delta_{ab}\,e^b{}_k(x)$. If we impose $(e_{ja})_{1\leq j,a\leq 3}$ to be symmetric, the relation gives

$$h = e^2 .$$

Hence, we can interpret the choice of the dreibein given the spatial metric h as the inverse map of $e \in \mathrm{Sym}(3,\mathbb{R}) \mapsto e^2 \in \mathrm{Sym}(3,\mathbb{R})$, which sends symmetric matrices to their square. Such a map is smooth, $\mathrm{Id}_3^2 = \mathrm{Id}_3$ and its differential at the identity is twice the identity. Thus, by inverse function theorem, it can be reversed, in a neighborhood of Id_3, into a smooth map F. For C_h suitably small, for any x, $h(x)$ takes values in the domain of definition K of F, which means that the dreibein exists globally. In particular, we define $e(x) := F(h(x))$. Using the increment theorem, we find

$$|e(x) - \mathrm{Id}_3| \leq \sup_K |DF||h(x) - \mathrm{Id}_3| .$$

Furthermore, we can also obtain estimates on its derivatives

$$\begin{aligned}
|e'(x)| &\leq \sup_K |DF||h'(x)| , \\
|e''(x)| &\leq \sup_K |D^2F||h'(x)|^2 + \sup_K |DF||h''(x)| , \\
|e'''(x)| &\leq \sup_K |D^3F||h'(x)|^3 + 3\sup_K |D^2F||h'(x)||h''(x)| + \sup_K |DF||h'''(x)| .
\end{aligned}$$

Therefore, by the asymptotically flat assumption and Remark 2.3,

$$\begin{aligned}
|e(x) - \mathrm{Id}_3| &\leq C_e C_h \langle x\rangle^{-1-\sigma} , & |e'(x)| &\leq C_e C_h \langle x\rangle^{-2-\sigma} , \\
|e''(x)| &\leq C_e C_h \langle x\rangle^{-3-\sigma} , & |e'''(x)| &\leq C_e C_h \langle x\rangle^{-4-\sigma} .
\end{aligned}$$

Finally, recalling that $\omega_\mu{}^{ab} = e^a{}_\nu(\partial_\mu e^{b\nu} + \Gamma^\nu_{\mu\sigma} e^{b\sigma})$ and $B_\mu = \frac{1}{8}\omega_\mu{}^{ab}[\gamma_a, \gamma_b]$, the remaining estimates follow immediately

$$|\omega(x)| \leq C_\omega C_h \langle x\rangle^{-2-\sigma} , \quad |B(x)| \leq C_B C_h \langle x\rangle^{-2-\sigma} , \quad |\partial B(x)| \leq C'_B C_h \langle x\rangle^{-3-\sigma} .$$

□

2.3 Strichartz estimates for the Dirac equation

Thanks to the previous section, we can easily deduce the explicit formulas for the spinorial covariant derivatives,

$$D_0 = \partial_t , \qquad D_j = \partial_j + B_j = \partial_j + \frac{1}{8}\alpha_j{}^{ab}[\gamma_a, \gamma_b] ,$$

where $\alpha_j{}^{ab}$ are the coefficients of the spin connection associated to h. Therefore, the Dirac operator rewrites

$$i\underline{\gamma}^\mu D_\mu = i\gamma^a e_a{}^\mu D_\mu = i\gamma^0\partial_t + i\gamma^a f_a{}^j D_j \ ,$$

where $f_a{}^j$ is the spatial dreibein linking h to the flat space metric. Finally, multiplying by γ^0, the Dirac equation rewrites

$$i\partial_t u = (-i\gamma^0\gamma^a f_a{}^j D_j + \gamma^0 m)u =: \mathcal{D}_m u \ .$$

Coupling it with an initial datum u_0, we obtain the following Cauchy problem

$$\begin{cases} i\partial_t u - \mathcal{D}_m u = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \ , \\ u(0,x) = u_0(x) \ , & \text{for } x \in \mathbb{R}^3 \ . \end{cases} \tag{2.3}$$

To state the main result of this section, we recall the definition of admissible exponents. The following conditions are, in some sense, related to the scaling properties of the equations they are related to.

Definition 2.3 (Admissible Strichartz exponents)**.** In dimension $d = 3$, a triple of exponents (s,q,r) is called *wave admissible* if

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{r} \ , \quad 2 \le q, r \le \infty \ , \quad r \ne \infty \ , \quad s = \frac{1}{2} - \frac{1}{r} + \frac{1}{q} \ .$$

The triple (s,q,r) is called *Klein–Gordon (or Schrödinger) admissible* if

$$\frac{2}{q} = \frac{3}{2} - \frac{3}{r} \ , \quad 2 \le q \le \infty \ , \quad 2 \le r \le 6 \ , \quad s = \frac{1}{2} - \frac{1}{r} + \frac{1}{q} \ .$$

We are are finally ready to state the theorem regarding Strichartz estimate for the Dirac flow.

Theorem 2.5 (Strichartz estimates for Dirac, [CdSM23])**.** *Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian and asymptotically flat manifold. Then the following estimates for the Dirac flow hold:*

- *massless Strichartz estimate, i.e. $m = 0$,*

$$\|e^{it\mathcal{D}} u_0\|_{L^q_t \dot{H}^{1-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{\dot{H}^1(\mathcal{M}_h)} \ ,$$

for any wave admissible triple (s,q,r) ;

- *massive Strichartz estimate, i.e. $m > 0$,*

$$\|e^{it\mathcal{D}_m} u_0\|_{L^q_t H^{1/2-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{H^1(\mathcal{M}_h)} \ ,$$

for any Klein–Gordon admissible triple (s,q,r) with $q > 2$.

Notation (Lebesgue and Sobolev norms). On $(\mathcal{M}, g)$ Lorentzian manifold, with g static and decoupling, we define the space Lebesgue norm as

$$\|f\|^p_{L^p(\mathcal{M}_h)} := \int_{\mathbb{R}^3} |f(x)|^p \sqrt{\det h(x)}\, d^3x \ .$$

The homogeneous and inhomogeneous Sobolev spaces are defined via Fourier multipliers by

$$\|f\|_{\dot{H}^s_p(\mathcal{M}_h)} := \|(-\widetilde{\Delta}_h)^{s/2} f\|_{L^p(\mathcal{M}_h)} \ , \quad \|f\|_{H^s_p(\mathcal{M}_h)} := \|(1-\widetilde{\Delta}_h)^{s/2} f\|_{L^p(\mathcal{M}_h)} \ ,$$

where $\widetilde{\Delta}_h$ is the standard Laplace–Beltrami operator. We recall that these spaces are defined also for negative s by duality and for fractional s by interpolation, see for instance [DNPV12] and [Tar07]. Finally, the mixed spacetime norms are given by

$$\|f\|_{L^q_t X(\mathcal{M}_h)} := \Big(\int_{\mathbb{R}^+} \|f(t,\cdot)\|^q_{X(\mathcal{M}_h)}\, dt\Big)^{1/q} \ , \quad \text{with } X = \dot{H}^s_p \ , \ H^s_p \ .$$

The proof of the theorem, as we will see later, is not difficult, but relies on many important and advanced results of local smoothing estimates and Strichartz estimates on manifolds.

At the beginning of the chapter, we emphasised how effective the squaring trick was in deducing properties about the Dirac flow in the flat case. In general, however, this argument no longer works due to the presence of spinorial derivatives. Hence, the strategy is to take advantage of the geometric assumptions on the manifold to control and bound such new terms using Lemma 2.4. With the help of some additional local smoothing estimates, we will be able to prove the desired results using the standard Strichartz estimates for the Klein–Gordon and wave equations.

The next section will be devoted to presenting all the ingredients necessary for the proof of Theorem 2.5.

2.4 Preliminary estimates

We start by showing the relation between the Dirac equation (2.3) and its square, that is a "spinorial" Klein–Gordon (or wave) equation. The result is a straightforward consequence of Lemma 1.8 and Remark 1.12.

Theorem 2.6 (Squared Dirac equation)**.** *If u is a (smooth enough) solution to* (2.3)*, then it also solves*

$$\begin{cases} \partial_t^2 u - \Delta_h u - \dfrac{1}{4} R_h u + m^2 u = 0 \ , & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \ , \\ u(0,x) = u_0(x) \ , \quad \partial_t u(0,x) = -i\mathcal{D}_m u_0(x) \ , & \text{for } x \in \mathbb{R}^3 \ , \end{cases} \tag{2.4}$$

where $\Delta_h = D^j D_j = h^{jk} D_k D_j$. Furthermore,

$$\Delta_h v = \widetilde{\Delta}_h v + B^j \partial_j v + \widetilde{D}^j B_j v + B^j B_j v \ , \tag{2.5}$$

where $\widetilde{\Delta}_h$ is the standard Laplace–Beltrami operator, $\widetilde{D}^j X_k = \partial^j X_k - \Gamma^l{}_k{}^j X_l$ and $B^j = h^{jk} B_k$.

Remark 2.4. We stress that the opposite sign of the scalar curvature term (see [CdSM23, Theorem 2.2]) is due to the different convention of the Riemann tensor (cf. Definition 1.12 and [CdSM23, Formula (10)]).

We now compare the problem above with the Klein–Gordon and wave equations defined via the standard Laplace–Beltrami operator $\widetilde{\Delta}_h$, for which we have the following Strichartz estimates.

Theorem 2.7 (Strichartz estimates for wave/Klein–Gordon)**.** *Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian and asymptotically flat manifold and u be a solution to the Cauchy problem*

$$\begin{cases} \partial_t^2 u - \widetilde{\Delta}_h u + m^2 u = 0 \ , & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \ , \\ u(0,x) = u_0(x) \ , \quad \partial_t u(0,x) = u_1(x) \ , & \text{for } x \in \mathbb{R}^3 \ . \end{cases}$$

Then the following estimates hold:

- *massless Strichartz estimate, i.e. $m = 0$,*

  $$\|u\|_{L^q_t \dot{H}^{1-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{\dot{H}^1(\mathcal{M}_h)} + \|u_1\|_{L^2(\mathcal{M}_h)} \ ,$$

  for any wave admissible triple (s, q, r) ;

- *massive Strichartz estimate, i.e. $m > 0$,*

  $$\|u\|_{L^q_t H^{1/2-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{H^{1/2}(\mathcal{M}_h)} + \|u_1\|_{H^{-1/2}(\mathcal{M}_h)} \ ,$$

  for any Klein–Gordon admissible triple (s, q, r) .

Proof. We omit the details of the proof.

For the case $m = 0$, one can look at [SW10, Theorem 1.4].

Regarding the case $m > 0$, we notice that an asymptotically flat manifold in the sense of our definition, satisfies also the decay $|\partial^\alpha(h-\delta)| \lesssim \langle x\rangle^{-|\alpha|-1}$. Hence, it is also asymptotically conic (see [HTW05, Definition 1.1 and Remark 1.2]). Therefore, our statement is a specific case of the more general global-in-time Strichartz estimate on non-trapping conic manifolds, [ZZ19, Theorem 1.1 with $F = 0$]. □

Therefore, from Theorems 2.6 and 2.7, we see that a solution of the Cauchy problem (2.3) solves a perturbed Klein–Gordon (or wave) equation

$$\begin{cases} \partial_t^2 u - \widetilde{\Delta}_h u + m^2 u = \frac{1}{4}R_h u + B^j\partial_j u + \widetilde{D}^j B_j u + B^j B_j u \ , \\ u(0,x) = u_0(x) \ , \quad \partial_t u(0,x) = -i\mathcal{D}_m u_0(x) \ . \end{cases}$$

Now, using the Duhamel formula, we can write the solution as follows

$$u(t,x) := e^{it\mathcal{D}_m}u_0 = \dot{W}_m(t)u_0 - iW_m(t)\mathcal{D}_m u_0 + \int_0^t W_m(t-s)\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \ , \tag{2.6}$$

where

$$W_m(t) := \frac{\sin\left(t\sqrt{m^2-\widetilde{\Delta}_h}\right)}{\sqrt{m^2-\widetilde{\Delta}_h}} \ , \quad \dot{W}_m(t) := \frac{d}{dt}W_m(t) \ ,$$

and

$$\Omega_1(u) := 2B^j\partial_j u \ , \quad \Omega_2 := \partial^j B_j + B^j B_j - \Gamma^j{}_k{}^k B_j + \frac{1}{4}R_h \ . \tag{2.7}$$

Hence, proving Theorem 2.5 is clearly a matter of controlling the perturbative terms (2.7) appearing in the Duhamel formulation.

The following local smoothing estimates will in fact do this work.

Theorem 2.8 (Local smoothing estimate for Dirac – I)**.** *Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian and asymptotically flat manifold and u be a solution to the Cauchy problem* (2.3)*. Then, for $m \geq 0$, the following local smoothing estimate holds*

$$\|\langle x\rangle^{-3/2-}u\|_{L_t^2L^2(\mathcal{M}_h)} + \|\langle x\rangle^{-1/2-}\widetilde{\nabla}u\|_{L_t^2L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} \ ,$$

where $\widetilde{\nabla}$ denotes the scalar gradient and $\langle x\rangle^{\alpha\pm} := \langle x\rangle^{\alpha\pm\eta}$ for $\eta > 0$.

Proof. The theorem can be easily deduced from the following result proven in [CdS19b] (see Appendix A for an idea of the proof),

$$\|\langle x\rangle^{-3/2-}u\|_{L^2_tL^2(\mathcal{M}_h)} + \|\langle x\rangle^{-1/2-}\nabla u\|_{L^2_tL^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} \ . \tag{A.1}$$

This estimate, together with Lemma 2.4, gives

$$\|\langle x\rangle^{-1/2-}Bu\|_{L^2_tL^2(\mathcal{M}_h)} \leq C_B C_h \|\langle x\rangle^{-3/2-}u\|_{L^2_tL^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} \ .$$

Therefore, recalling that the spinorial gradient $\nabla = \widetilde{\nabla} + B$, we conclude by triangular inequality

$$\begin{aligned}\|\langle x\rangle^{-3/2-}u\|_{L^2_tL^2(\mathcal{M}_h)} &+ \|\langle x\rangle^{-1/2-}\widetilde{\nabla}u\|_{L^2_tL^2(\mathcal{M}_h)} \\ &\leq \|\langle x\rangle^{-3/2-}u\|_{L^2_tL^2(\mathcal{M}_h)} + \|\langle x\rangle^{-1/2-}\nabla u\|_{L^2_tL^2(\mathcal{M}_h)} + \|\langle x\rangle^{-1/2-}Bu\|_{L^2_tL^2(\mathcal{M}_h)} \\ &\lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} \ .\end{aligned}$$

□

Theorem 2.9 (Local smoothing estimates for wave/Klein–Gordon)**.** *Let* $(\mathcal{M}, g)$ *be a four-dimensional Lorentzian and asymptotically flat manifold. Then, the following local smoothing estimates hold*

$$\|\langle x\rangle^{-1/2-}e^{it\sqrt{-\widetilde{\Delta}_h}}f\|_{L^2_tL^2(\mathcal{M}_h)} \lesssim \|f\|_{L^2(\mathcal{M}_h)} \ ,$$

for any $f \in L^2(\mathcal{M}_h)$ *, and*

$$\|\langle x\rangle^{-1}\, e^{it\sqrt{m^2-\widetilde{\Delta}_h}}f\|_{L^2_tL^2(\mathcal{M}_h)} \lesssim \|(1-\widetilde{\Delta}_h)^{1/4}f\|_{L^2(\mathcal{M}_h)} \ ,$$

for any f *such that* $(1-\widetilde{\Delta}_h)^{1/4}f \in L^2(\mathcal{M}_h)$ *.*

Proof. We start with the massless case $m = 0$, briefly retracing the argument presented in [BH09]. Let us define the unitary transform

$$\mathcal{V} : u \in L^2(\mathcal{M}_h) = L^2(\mathbb{R}^3, \sqrt{\det h(x)}dx) \longmapsto (\det h(x))^{1/4}u \in L^2(\mathbb{R}^3, dx) \ ,$$

which maps $-\widetilde{\Delta}_h$ to

$$P := -\mathcal{V}\widetilde{\Delta}_h\mathcal{V}^{-1} = -(\det h(x))^{1/4}\widetilde{\Delta}_h(\det h(x))^{-1/4} \ .$$

Therefore, if we consider $u := e^{it\sqrt{-\widetilde{\Delta}_h}}f \in L^2(\mathcal{M}, dg)$ and $v := \mathcal{V}u$, we obtain the following equivalence

$$\partial_t^2 u - \widetilde{\Delta}_h u = 0 \Longleftrightarrow \partial_t^2 v + Pv = 0 \ .$$

Hence, from [BH09, Theorem 1.3 with $G = 0$], we obtain the desired estimate

$$\|\langle x\rangle^{-1/2-}u\|_{L^2_tL^2(\mathcal{M}_h)} = \|\langle x\rangle^{-1/2-}v\|_{L^2(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim \|v(0)\|_{L^2(\mathbb{R}^3)} = \|f\|_{L^2(\mathcal{M}_h)} \,.$$

Regarding the Klein–Gordon estimate, we need two more results. The first one is [ZZ17, Formula (3.5) with $V = 0$],

$$\|\langle x\rangle^{-1}e^{-it\widetilde{\Delta}_h}f\|_{L^2_tL^2(\mathcal{M}_h)} \lesssim \|f\|_{L^2(\mathcal{M}_h)} \,,$$

which means that $\langle x\rangle^{-1}$ is $-\widetilde{\Delta}_h$- smooth.
On the other hand, [D'A15, Theorems 2.2 and 2.4] together imply that $\langle x\rangle^{-1}(m^2 - \widetilde{\Delta}_h)^{-1/4}$ is $\sqrt{m^2-\widetilde{\Delta}_h}$- smooth. Hence, we exactly obtain the second statement

$$\|\langle x\rangle^{-1}e^{it\sqrt{m^2-\widetilde{\Delta}_h}}f\|_{L^2_tL^2(\mathcal{M}_h)} \lesssim \|(m^2-\widetilde{\Delta}_h)^{1/4}\|_{L^2(\mathcal{M}_h)} \,.$$

□

Remark 2.5 (Dual local smoothing estimates). Once the two local smoothing estimates above are proven, one can use standard TT^* arguments to deduce the dual estimates (that will actually be the ones we will use in the proof of Theorem 2.5):

$$\left\|\int_0^T e^{-is\sqrt{-\widetilde{\Delta}_h}}f(s,\cdot)\,ds\right\|_{L^2(\mathcal{M}_h)} \lesssim \|\langle x\rangle^{1/2+}f\|_{L^2_tL^2(\mathcal{M}_h)} \,,$$
$$\left\|\int_0^T e^{-is\sqrt{m^2-\widetilde{\Delta}_h}}f(s,\cdot)\,ds\right\|_{H^{-1/2}(\mathcal{M}_h)} \lesssim \|\langle x\rangle f\|_{L^2_tL^2(\mathcal{M}_h)} \,.$$

The last tool needed is the following lemma, which states an equivalence between the Sobolev norm and the one that is induced by the Dirac operator. The proof is a bit technical, but it relies only on the geometric estimates of Lemma 2.4 and on the self-adjointness of $-\widetilde{\Delta}_h$ and $\mathcal{D}_m$, proven in [Che73].

Lemma 2.10 (Norm equivalence)**.** *Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian and asymptotically flat manifold. For $m \geq 0$, it holds*

$$\|(m^2-\widetilde{\Delta}_h)^{1/2}u\|_{L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u\|_{L^2(\mathcal{M}_h)} \lesssim \|(m^2-\widetilde{\Delta}_h)^{1/2}u\|_{L^2(\mathcal{M}_h)} \,.$$

Proof. Since $-\widetilde{\Delta}_h$ is self-adjoint on $L^2(\mathcal{M}_h)$, we have for $m \geq 0$

$$\|(m^2-\widetilde{\Delta}_h)^{1/2}u\|^2_{L^2(\mathcal{M}_h)} = \langle (m^2-\widetilde{\Delta}_h)u, u\rangle_{L^2(\mathcal{M}_h)} \,.$$

Using that

$$-\langle \widetilde{\Delta}_h u, u\rangle_{L^2(\mathcal{M}_h)} = h^{ij}\langle \partial_i u, \partial_j u\rangle_{L^2(\mathcal{M}_h)} \,,$$

we can rewrite

$$-\langle\widetilde{\Delta}_h u, u\rangle_{L^2(\mathcal{M}_h)} = \delta^{ij}\langle\partial_i u, \partial_j u\rangle_{L^2(\mathcal{M}_h)} + (h^{ij}-\delta^{ij})\langle\partial_i u, \partial_j u\rangle_{L^2(\mathcal{M}_h)} .$$

From the asymptotically flatness condition, $|h^{ij}-\delta^{ij}| \ll 1$ and hence by Cauchy-Schwarz inequality

$$\delta^{ij}\langle\partial_i u, \partial_j u\rangle_{L^2(\mathcal{M}_h)} \lesssim -\langle\widetilde{\Delta}_h u, u\rangle_{L^2(\mathcal{M}_h)} \lesssim \delta^{ij}\langle\partial_i u, \partial_j u\rangle_{L^2(\mathcal{M}_h)} .$$

We also have, using the self-adjointness of $\mathcal{D}_m$,

$$\|\mathcal{D}_m u\|^2_{L^2(\mathcal{M}_h)} = \langle(m^2-\widetilde{\Delta}_h)u, u\rangle_{L^2(\mathcal{M}_h)} - \langle\Omega_1(u)+\Omega_2 u, u\rangle_{L^2(\mathcal{M}_h)} ,$$

where $\mathcal{D}_m = m^2 - \widetilde{\Delta}_h - \Omega_1 - \Omega_2$, with Ω_1, Ω_2 as in (2.7). Since B_i are skew-symmetric, we have

$$\langle B^i\partial_i u, u\rangle_{L^2(\mathcal{M}_h)} = -\langle\partial_i u, B^i u\rangle_{L^2(\mathcal{M}_h)} , \qquad \langle B_i\partial^i u, u\rangle_{L^2(\mathcal{M}_h)} = -\langle\partial^i u, B_i u\rangle_{L^2(\mathcal{M}_h)} .$$

Thus, by Cauchy-Schwarz inequality, we obtain

$$|\langle\Omega_1(u), u\rangle_{L^2(\mathcal{M}_h)}| \leq \sum_i \|\partial_i u\|_{L^2(\mathcal{M}_h)}\|\langle x\rangle B\|_{L^\infty}\|\langle x\rangle^{-1}u\|_{L^2(\mathcal{M}_h)} .$$

Thanks to the assumption on h, the norms $L^2(\mathcal{M}_h)$ and $L^2(\mathbb{R}^3)$ are equivalent and, using Hardy inequality, we find

$$|\langle\Omega_1(u), u\rangle_{L^2(\mathcal{M}_h)}| \lesssim \|\langle x\rangle B\|_{L^\infty}\|(m^2-\widetilde{\Delta}_h)^{1/2}u\|^2_{L^2(\mathcal{M}_h)} .$$

Let us now consider the term in Ω_2, which gives

$$|\langle\Omega_2 u, u\rangle_{L^2(\mathcal{M}_h)}| \lesssim \|\langle x\rangle^2\Omega_2\|_{L^\infty}\|(m^2-\widetilde{\Delta}_h)^{1/2}u\|^2_{L^2(\mathcal{M}_h)} .$$

According to Lemma 2.4, it finally follows

$$\begin{aligned}&\|\langle x\rangle B\|_{L^\infty} \ll 1 ,\\ &\|\langle x\rangle^2\Omega_2\|_{L^\infty} \leq \|\langle x\rangle^2(\partial^i B_i + B^i B_i - \Gamma^j{}_i{}^i B_j + \frac{1}{4}R_h)\|_{L^\infty} \ll 1\end{aligned}$$

and hence we conclude

$$\|(m^2-\widetilde{\Delta}_h)^{1/2}u\|^2_{L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u\|^2_{L^2(\mathcal{M}_h)} \lesssim \|(m^2-\widetilde{\Delta}_h)^{1/2}u\|^2_{L^2(\mathcal{M}_h)} .$$

□

2.5 Strichartz estimates for the Dirac equation – Proof

We are finally ready to prove the Strichartz estimates of Theorem 2.5.

Proof. We recall the Duhamel formulation (2.6) of the solution

$$u(t,x) = \dot{W}_m(t)u_0 - iW_m(t)\mathcal{D}_m u_0 + \int_0^t W_m(t-s)\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \ ,$$

where

$$W_m(t) = \frac{\sin\left(t\sqrt{m^2 - \widetilde{\Delta}_h}\right)}{\sqrt{m^2 - \widetilde{\Delta}_h}} \ , \quad \dot{W}_m(t) = \frac{d}{dt}W_m(t) \ ,$$

and

$$\Omega_1(u) = 2B^j\partial_j u \ , \quad \Omega_2 = \partial^j B_j + B^j B_j - \Gamma^j{}_k{}^k B_j + \frac{1}{4}R_h \ .$$

We start with the massless case. Thanks to Theorem 2.7 and Christ–Kiselev Lemma [CK01], we only need to study the inhomogeneous term. Hence,

$$\begin{aligned}
&\left\| \int_0^T W_m(t-s)\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{L^q_t \dot{H}^{1-s}_r(\mathcal{M}_h)} \\
&\qquad \lesssim \left\| \frac{e^{it\sqrt{-\widetilde{\Delta}_h}}}{\sqrt{-\widetilde{\Delta}_h}} \int_0^T e^{-is\sqrt{-\widetilde{\Delta}_h}}\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{L^q_t \dot{H}^{1-s}_r(\mathcal{M}_h)} \\
&\qquad \lesssim \left\| \int_0^T e^{-is\sqrt{-\widetilde{\Delta}_h}}\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{L^2(\mathcal{M}_h)} .
\end{aligned}$$

Using the dual form of the wave local smoothing estimate in Theorem 2.9,

$$\left\| \int_0^T e^{-is\sqrt{-\widetilde{\Delta}_h}}\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{L^2(\mathcal{M}_h)} \lesssim \|\langle x\rangle^{1/2+}(\Omega_1(u) + \Omega_2 u)\|_{L^2_t L^2(\mathcal{M}_h)} \ .$$

Combining the local smoothing estimate for the Dirac equation of Theorem 2.8 and Lemma 2.4, we control both Ω_1 and Ω_2 :

$$\begin{aligned}
\|\langle x\rangle^{1/2+}\Omega_1(u)\|_{L^2_t L^2(\mathcal{M}_h)} &\lesssim \|\langle x\rangle^{1/2+} B^j \partial_j u\|_{L^2_t L^2(\mathcal{M}_h)} \\
&\lesssim \|\langle x\rangle^{1+} B\|_{L^\infty_x} \|\langle x\rangle^{-1/2-}\widetilde{\nabla} u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}u_0\|_{L^2(\mathcal{M}_h)}
\end{aligned}$$

and

$$\|\langle x\rangle^{1/2+}\Omega_2 u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\langle x\rangle^{2+}\Omega_2\|_{L^\infty_x} \|\langle x\rangle^{-3/2-} u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}u_0\|_{L^2(\mathcal{M}_h)} \ .$$

Finally, using Lemma 2.10, the first case is proven

$$\|u\|_{L^q_t \dot{H}^{1-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{\dot{H}^1(\mathcal{M}_h)} + \|\mathcal{D}u_0\|_{L^2(\mathcal{M}_h)} \lesssim \|u_0\|_{\dot{H}^1(\mathcal{M}_h)} \ .$$

Let us focus now on the massive case. As before, thanks to Theorem 2.7, we restrict to the inhomogeneous term and we can apply Christ–Kiselev Lemma [CK01] since $q > 2$. Hence,

$$\begin{aligned}\left\| \int_0^T W_m(t-s)\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{L^q_t H^{1/2-s}_r(\mathcal{M}_h)} \\ \lesssim \left\| \frac{e^{it\sqrt{m^2-\widetilde{\Delta}_h}}}{\sqrt{m^2-\widetilde{\Delta}_h}} \int_0^T e^{-is\sqrt{m^2-\widetilde{\Delta}_h}}\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{L^q_t H^{1/2-s}_r(\mathcal{M}_h)} \\ \lesssim \left\| \int_0^T e^{-is\sqrt{m^2-\widetilde{\Delta}_h}}\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{H^{-1/2}(\mathcal{M}_h)} .\end{aligned}$$

Using the dual form of the local smoothing estimate in Theorem 2.9,

$$\left\| \int_0^T e^{-is\sqrt{m^2-\widetilde{\Delta}_h}}\Big(\Omega_1(u)(s) + \Omega_2 u(s)\Big)\, ds \right\|_{H^{-1/2}(\mathcal{M}_h)} \lesssim \|\langle x\rangle(\Omega_1(u) + \Omega_2 u)\|_{L^2_t L^2(\mathcal{M}_h)} .$$

Combining the local smoothing estimate for the Dirac equation of Theorem 2.8 and Lemma 2.4, we control both $\Omega_1(u)$ and Ω_2:

$$\begin{aligned}\|\langle x\rangle \Omega_1(u)\|_{L^2_t L^2(\mathcal{M}_h)} &\lesssim \|\langle x\rangle B^j \partial_j u\|_{L^2_t L^2(\mathcal{M}_h)} \\ &\lesssim \|\langle x\rangle^{3/2+} B\|_{L^\infty_x} \|\langle x\rangle^{-1/2-} \widetilde{\nabla} u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)}\end{aligned}$$

and

$$\|\langle x\rangle \Omega_2 u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\langle x\rangle^{5/2+} \Omega_2\|_{L^\infty_x} \|\langle x\rangle^{-3/2-} u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} .$$

Finally, using Lemma 2.10 and Sobolev embeddings, also the second case follows

$$\|u\|_{L^q_t H^{1/2-s}_r(\mathcal{M}_h)} \lesssim \|u_0\|_{H^{1/2}(\mathcal{M}_h)} + \|\mathcal{D}_m u_0\|_{H^{-1/2}(\mathcal{M}_h)} + \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} \lesssim \|u_0\|_{H^1(\mathcal{M}_h)} .$$

□

Chapter 3

Einstein–Dirac system with spherical symmetry

The following chapter is devoted to the derivation of the Einstein–Dirac system, which links Curved Quantum Field Theory and General Relativity. In particular, we present an explicit model describing the interaction between gravity and Dirac particles in the spherically symmetric case. Part of the calculations presented below were inspired by [FSY99].

The purpose of this chapter is to define the starting point for the next doctoral studies. In fact, we intend to present a series of open problems, to be addressed in the future, concerning the Einstein–Dirac system under this geometric assumption, such as the well-posedness issue and stability of solutions.

3.1 Derivation of the Einstein–Dirac system

So far, when dealing with the curved Dirac equation, we have treated the underlying manifold $(\mathcal{M}, g)$ as something given a priori. However, we know thanks to Einstein's works, between 1905 [Ein05] and 1915 [Ein15], that matter affects space and time. Hence, gravity is no longer treated as a force, but as inducing a deformation of the spacetime manifold, whose curved geometry determines the geodesic flows followed by particles.

In standard General Relativity, the coupled system describing this phenomenon is given by the Euler–Lagrange equations associated to the Einstein–Hilbert action $\mathcal{S}[\psi, g]$, where g is the metric and ψ the mass field. However, due to the spin structure, if we want to preserve local Lorentz invariance, we need to simultaneously vary the vierbein (and the related spin connection), in place of g. This correction yields the following action (cf. [Che22] for the massless case),

$$\begin{aligned}\mathcal{S}[\psi, e^\mu{}_a] :&= \mathcal{S}_G[e^\mu{}_a] + \mathcal{S}_M[\psi, e^\mu{}_a] \\ &= \int R_g \, e \, d^4x + \int \Big[\frac{i}{2}\big(\overline{\psi}\underline{\gamma}^\mu D_\mu\psi - D_\mu\overline{\psi}\underline{\gamma}^\mu\psi\big) - m\overline{\psi}\psi\Big] \, e \, d^4x \,,\end{aligned}$$

where R_g is the scalar curvature associated to g, $\overline{\psi} = \psi^*\gamma^0$ denotes the adjoint spinor (recall Definition 1.14) and $e := |\det(e^a{}_\mu)|$. Notice that, using $g_{\mu\nu} = e^a{}_\mu \, \eta_{ab} \, e^b{}_\nu$, we recover $e = \sqrt{|\det g|}$. (All the physical constants have been normalized for readability.)

The term $\mathcal{S}_G[e^\mu{}_a]$ corresponds to the action of the gravitional field, while $\mathcal{S}_M[\psi, e^\mu{}_a]$ determines the action of the mass Dirac field. Now, to obtain the Einstein–Dirac system, we need to compute the Euler–Lagrange equations

$$\begin{aligned}\delta_e\mathcal{S}_G + \delta_e\mathcal{S}_M &= 0 \,, \\ \delta_\psi\mathcal{S}_M &= 0 \,,\end{aligned}$$

where δ_e and δ_ψ denote respectively the variation with respect to the vierbein and the Dirac field. We briefly outline the main computations leading to the final system: for more details, see for instance [Yep11].

Let us begin with the variation with respect to the field ψ, which yields the so-called *equation of motion*. Recall that, due to the skew-symmetry of D_μ , to $\overline{D_\mu\psi} = D_\mu\overline{\psi}$ and to the fact that the $\underline{\gamma}^\mu$ are covariantly constant (see Theorem 1.7), we have

$$\int_{\mathcal{M}_g} \overline{\psi_1}\underline{\gamma}^\mu D_\mu\psi_2 = -\int_{\mathcal{M}_g} D_\mu\overline{\psi_1}\underline{\gamma}^\mu\psi_2 \,, \text{ for any spinors } \psi_1, \psi_2 \,.$$

Thus, we deduce

$$0 = \delta_\psi\mathcal{S}_M = \int \Big[\delta\overline{\psi}(i\underline{\gamma}^\mu D_\mu\psi - m\psi) - (iD_\mu\overline{\psi}\underline{\gamma}^\mu + m\overline{\psi})\delta\psi\Big] \, e \, d^4x \,.$$

Using $\delta\overline{\psi} = \overline{\delta\psi}$, we note that the second addend is nothing but the adjoint of the first one and thus we recover, as expected, the Dirac equation

$$i\underline{\gamma}^\mu D_\mu\psi - m\psi = 0 \,. \tag{3.1}$$

On the other hand, varying the vierbein, we obtain the Einstein equations. If we considered $\delta_g\mathcal{S}_G$, after standard computations we would find

$$\delta_g\mathcal{S}_G = -\int \Big[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_g\Big]\delta g_{\mu\nu} \sqrt{|\det g|} \, d^4x \,.$$

Now, from

$$\delta(e^a{}_\mu\, e_a{}^\nu) = \delta(e_{b\mu}\,\eta^{ab}\, e_{a\lambda}\, g^{\lambda\nu}) = \delta(g_{\mu\lambda} g^{\lambda\nu}) = \delta(\delta^\nu_\mu) = 0\ ,$$

we deduce

$$\delta(e^a{}_\mu)\, e_a{}^\nu = -e^a{}_\mu\, \delta(e_a{}^\nu)\ .$$

Hence, we can express the variation δg in terms of the vierbein as

$$\begin{aligned}\delta g_{\mu\nu} &= \delta(e^a{}_\mu\, \eta_{ab}\, e^b{}_\nu) = \delta(e^a{}_\mu)\, e_{a\nu} + e_{b\mu}\, \delta(e^b{}_\nu) = -e^a{}_\mu\, \delta e_{a\nu} - \delta e_{a\mu} e^a{}_\nu \\ &= -(g_{\nu\lambda} e^a{}_\mu + g_{\mu\lambda} e^a{}_\nu)\delta e_a{}^\lambda\ ,\end{aligned}$$

and consequently

$$\delta_e \mathcal{S}_G = \int \Big[\Big(R^\mu_\lambda - \frac{1}{2}\delta^\mu_\lambda R_g\Big) e^a{}_\mu\Big] \delta e_a{}^\lambda \sqrt{|\det g|}\, d^4x\ .$$

Multiplying the quantity in round brackets by $g^{\lambda\nu}$, we obtain the *Einstein tensor*

$$G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_g\ .$$

Regarding $\delta_e\mathcal{S}_M$, assuming that ψ satisfies the *on shell condition* (that is, it solves its own equation of motion), the variation reduces to

$$\delta_e \mathcal{S}_M = \int \frac{i}{2}\big(\overline{\psi}\gamma^a D_\mu\psi - D_\mu\overline{\psi}\gamma^a\psi\big)\, \delta e_a{}^\mu \sqrt{|\det g|}\ d^4x\ .$$

We define the *Belifante–Rosenfeld energy-momentum tensor* as

$$T^a{}_\mu := \frac{i}{2}\big(\overline{\psi}\gamma^a D_\mu\psi - D_\mu\overline{\psi}\gamma^a\psi\big)\ ,$$

which reads in curved indices as

$$\begin{aligned}T_{\mu\nu} :&= \frac{1}{2}\big(e^a{}_\mu\, \eta_{ab}\, T^b{}_\nu + e^a{}_\nu\, \eta_{ab}\, T^b{}_\mu\big) \\ &= \frac{i}{4}\big(\overline{\psi}\underline{\gamma_\mu} D_\nu\psi - D_\nu\overline{\psi}\underline{\gamma_\mu}\psi\big) + \frac{i}{4}\big(\overline{\psi}\underline{\gamma_\nu} D_\mu\psi - D_\mu\overline{\psi}\underline{\gamma_\nu}\psi\big)\ .\end{aligned}$$

Hence, we have found

$$\delta_e\mathcal{S}[\psi, e^\mu{}_a] = \int \Big[\Big(R^\mu_\lambda - \frac{1}{2}\delta^\mu_\lambda R_g\Big) e^a{}_\mu + T^a{}_\lambda\Big] \delta e_a{}^\lambda \sqrt{|\det g|}\, d^4x = 0\ .$$

Since the variation of the vierbein field does not vanish in general, me must impose that the quantity in square brackets is zero. Multiplying it by $g^{\lambda\nu}$ and noting that $e^a{}_\mu \neq 0$, we deduce the *Einstein equations under the Dirac action*

$$G^{\mu\nu} + T^{\mu\nu} = 0\ . \tag{3.2}$$

Finally, coupling (3.1) and (3.2), we obtain the *Einstein–Dirac system*

$$\begin{aligned} G^{\mu\nu} + T^{\mu\nu} &= 0\ , \\ i\underline{\gamma}^\mu D_\mu\psi &= m\psi\ . \end{aligned}$$

Remark 3.1. If ψ solves the Dirac equation (and consequently $\overline{\psi}$ satisfies the adjoint equation $iD_\mu \overline{\psi}\underline{\gamma}^\mu = -m\overline{\psi}$), the trace of T can be easily computed. Indeed,

$$\mathrm{tr}_g T = g^{\mu\nu} T_{\mu\nu} = \frac{i}{2}\Big(\overline{\psi}\underline{\gamma}^\mu D_\mu \psi - D_\mu \overline{\psi}\underline{\gamma}^\mu \psi\Big) = m\overline{\psi}\psi \,.$$

Furthermore, if the Einstein equations also hold, then it follows

$$\mathrm{tr}_g G = g^{\mu\nu} G_{\mu\nu} = -R_g = -\mathrm{tr}_g T \,,$$

which means $R_g = m\overline{\psi}\psi$. Note that when $m = 0$, the energy–momentum tensor is hence traceless, and consequently $R_g = 0$.

Remark 3.2 (Einstein Vacuum Equations). In general, one can couple the Einstein equations with other matter fields, obtaining different systems, such as the Einstein–Maxwell or the Einstein–Klein–Gordon systems.
On the other hand, in absence of an external field, the energy-momentum tensor is zero and thus one obtains the *Einstein Vacuum Equations (EVE),*

$$R^{\mu\nu} = 0 \,. \tag{3.3}$$

We stress that the zero scalar curvature constraint, found in the previous remark, is not sufficient to ensure that the entire curvature tensor is zero and thus it does not imply the Einstein Vacuum Equations.
Furthermore, we mention that, although this system of equations may seem quite simple, it is actually very rich. Indeed, in addition to the trivial Minkowski metric η, also spacetime metrics modelizing black holes solve (3.3). For brevity, we limit ourselves to stating the *Schwarzschild metric*

$$g_M(t,r,\theta,\phi) := \Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,.$$

3.2 Spherically symmetric manifolds

In this section, we present the geometric hypotheses we make on the manifold and we compute the quantities needed to explicitly write the Einstein–Dirac system.

In particular, we assume that the Lorentzian manifold $(\mathcal{M}, g)$ is endowed with a time-dependent, decoupling and spherically symmetric metric, that is (cf. [FSY99, Section 8])

$$g(t,r,\theta,\phi) = F^2(t,r)dt^2 - G^2(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,, \tag{3.4}$$

where $F, G \in C^\infty(\mathbb{R}^2)$. Hence, in coordinates,

$$g_{\mu\nu} = \mathrm{diag}\big(F^2(t,r), -G^2(t,r), -r^2, -r^2\sin^2\theta\big) \;,$$
$$g^{\mu\nu} = \mathrm{diag}\Big(\frac{1}{F^2(t,r)}, -\frac{1}{G^2(t,r)}, -\frac{1}{r^2}, -\frac{1}{r^2\sin^2\theta}\Big) \;.$$

We will see that, thanks to the spherical symmetry, raising an index simplifies the angular dependence. Therefore, we are interested in writing the Einstein–Dirac system in the following form

$$T^\mu_\nu = -G^\mu_\nu \;,$$
$$i\underline{\gamma}^\mu D_\mu \psi = m\psi \;.$$

We begin by giving the formulas of the Christoffel symbols.

Lemma 3.1. *Let g be a metric of the form* (3.4) *and $\Gamma^\sigma_{\mu\nu}$ be the associated affine connection coefficients. Then, the non-vanishing Christoffel symbols are given by*

$$\Gamma^0_{00} = \frac{\partial_t F}{F} \;, \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{\partial_r F}{F} \;, \quad \Gamma^0_{11} = \frac{G\,\partial_t G}{F^2} \;,$$
$$\Gamma^1_{00} = \frac{F\,\partial_r F}{G^2} \;, \qquad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\partial_t G}{G} \;, \quad \Gamma^1_{11} = \frac{\partial_r G}{G} \;, \qquad \Gamma^1_{22} = -\frac{r}{G^2} \;, \Gamma^1_{33} = -\frac{r\sin^2\theta}{G^2} \;,$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \;, \quad \Gamma^2_{33} = -\sin\theta\cos\theta \;,$$
$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \;, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \;.$$

Proof. If follows from direct computations. □

Therefore, we can now compute the Ricci curvature tensor and the scalar curvature.

Lemma 3.2. *Let g be a metric of the form* (3.4) *and $R_{\beta\mu}$ be the Ricci curvature tensor. Then, the non-vanishing components are given by*

$$R_{00} = -\frac{F\partial_r^2 F - G\partial_t^2 G}{G^2} + \frac{F(\partial_r F)(\partial_r G)}{G^3} - \frac{(\partial_t F)(\partial_t G)}{FG} - 2\frac{F(\partial_r F)}{rG^2} \;,$$
$$R_{11} = -\frac{G\partial_t^2 G - F\partial_r^2 F}{F^2} + \frac{G(\partial_t G)(\partial_t F)}{F^3} - \frac{(\partial_r F)(\partial_r G)}{FG} - 2\frac{\partial_r G}{rG} \;,$$
$$R_{22} = -\frac{r\partial_r G}{G^3} + \frac{r\partial_r F}{FG^2} + \frac{1}{G^2} - 1 \;,$$
$$R_{33} = \sin^2\theta\, R_{22} \;,$$
$$R_{01} = -2\frac{\partial_t G}{rG} \;.$$

As a consequence, the scalar curvature $R_g := g^{\alpha\beta}R_{\alpha\beta}$ writes

$$R_g = \frac{2}{FG}\Big(\frac{\partial_t^2 G}{F} - \frac{\partial_r^2 F}{G} + \frac{(\partial_r F)(\partial_r G)}{G^2} - \frac{(\partial_t F)(\partial_t G)}{F^2}\Big) + \frac{4}{rG^2}\Big(\frac{\partial_r G}{G} - \frac{\partial_r F}{F}\Big) + \frac{2}{r^2}\Big(1 - \frac{1}{G^2}\Big) \;.$$

Proof. Recalling that the Ricci curvature tensor is given by

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} := \partial_\nu \Gamma^{\alpha}_{\alpha\mu} - \partial_\alpha \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\alpha\mu} - \Gamma^{\alpha}_{\alpha\sigma}\Gamma^{\sigma}_{\mu\nu} \ ,$$

we proceed by cases. Starting from R_{00}, we have

$$\begin{aligned}
R_{00} &= \partial_t(\Gamma^0_{00} + \Gamma^1_{01}) - \partial_t \Gamma^0_{00} - \partial_r \Gamma^1_{00} + \Gamma^0_{00}\Gamma^0_{00} + 2\Gamma^1_{00}\Gamma^0_{01} + \Gamma^1_{01}\Gamma^1_{01} \\
&\quad - \Gamma^0_{00}(\Gamma^0_{00} + \Gamma^1_{01}) - \Gamma^1_{00}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) \\
&= \partial_t \Gamma^1_{01} - \partial_r \Gamma^1_{00} + \Gamma^1_{00}\Gamma^0_{01} + \Gamma^1_{01}\Gamma^1_{01} - \Gamma^0_{00}\Gamma^1_{01} - \Gamma^1_{00}(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) \\
&= \frac{G\partial_t^2 G - (\partial_t G)^2}{G^2} - \frac{((\partial_r F)^2 + F\partial_r^2 F)G^2 - 2FG(\partial_r F)(\partial_r G)}{G^4} \\
&\quad + \frac{(\partial_r F)^2}{G^2} + \frac{(\partial_r G)^2}{F^2} - \frac{(\partial_t F)(\partial_t G)}{FG} - \frac{F\partial_r F}{G^2}\Big(\frac{\partial_r G}{G} + \frac{2}{r}\Big) \\
&= \frac{G\partial_t^2 G - F\partial_r^2 F}{G^2} + \frac{F(\partial_r F)(\partial_r G)}{G^3} - \frac{(\partial_t F)(\partial_t G)}{FG} - 2\frac{F(\partial_r F)}{rG^2} \ .
\end{aligned}$$

Then, passing to R_{11},

$$\begin{aligned}
R_{11} &= \partial_r(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) - \partial_t \Gamma^0_{11} - \partial_r \Gamma^1_{11} + \Gamma^0_{01}\Gamma^0_{01} + \Gamma^1_{01}\Gamma^0_{11} + \Gamma^0_{11}\Gamma^1_{01} \\
&\quad + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13} - \Gamma^0_{11}(\Gamma^0_{00} + \Gamma^1_{10}) - \Gamma^1_{11}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) \\
&= \partial_r(\Gamma^0_{01} + \Gamma^2_{12} + \Gamma^3_{13}) - \partial_t \Gamma^0_{11} + \Gamma^0_{01}\Gamma^0_{01} + \Gamma^0_{11}\Gamma^1_{01} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13} \\
&\quad - \Gamma^0_{11}\Gamma^0_{00} - \Gamma^1_{11}(\Gamma^0_{01} + \Gamma^2_{12} + \Gamma^3_{13}) \\
&= \frac{F\partial_r^2 F - (\partial_r F)^2}{F^2} - \frac{2}{r^2} - \frac{((\partial_t G)^2 + G\partial_t^2 G)F^2 - 2FG(\partial_t F)(\partial_t G)}{F^4} \\
&\quad + \frac{(\partial_r F)^2}{F^2} + \frac{(\partial_t G)^2}{F^2} + \frac{2}{r^2} - \frac{G(\partial_t F)(\partial_t G)}{F^3} - \frac{\partial_r G}{G}\Big(\frac{\partial_r F}{F} + \frac{2}{r}\Big) \\
&= \frac{F\partial_r^2 F - G\partial_t^2 G}{F^2} + \frac{G(\partial_t G)(\partial_t F)}{F^3} - \frac{(\partial_r F)(\partial_r G)}{FG} - 2\frac{\partial_r G}{rG} \ .
\end{aligned}$$

The components R_{22} and R_{33} are respectively given by

$$\begin{aligned}
R_{22} &= \partial_\theta \Gamma^3_{23} - \partial_r \Gamma^1_{22} + 2\Gamma^2_{12}\Gamma^1_{22} + \Gamma^3_{23}\Gamma^3_{23} - \Gamma^1_{22}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) \\
&= \partial_\theta \Gamma^3_{23} - \partial_r \Gamma^1_{22} + \Gamma^2_{12}\Gamma^1_{22} + \Gamma^3_{23}\Gamma^3_{23} - \Gamma^1_{22}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^3_{13}) \\
&= -\frac{1}{\sin^2\theta} - 2\frac{r\partial_r G}{G^3} + \frac{1}{G^2} - \frac{1}{G^2} + \cot^2\theta + \frac{r\partial_r F}{FG^2} + \frac{r\partial_r G}{G^3} + \frac{1}{G^2} \\
&= -\frac{r\partial_r G}{G^3} + \frac{r\partial_r F}{FG^2} + \frac{1}{G^2} - 1 \ ;
\end{aligned}$$

$$\begin{aligned}
R_{33} &= -\partial_r \Gamma^1_{33} - \partial_\theta \Gamma^2_{33} + 2\Gamma^3_{13}\Gamma^1_{33} + 2\Gamma^3_{23}\Gamma^2_{33} - \Gamma^1_{33}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) - \Gamma^3_{23}\Gamma^2_{33} \\
&= -\partial_r \Gamma^1_{33} - \partial_\theta \Gamma^2_{33} + \Gamma^3_{13}\Gamma^1_{33} + \Gamma^3_{23}\Gamma^2_{33} - \Gamma^1_{33}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{12}) \\
&= -(\sin^2\theta)\frac{2rG(\partial_r G) - G^2}{G^4} + (\cos^2\theta - \sin^2\theta) - \frac{\sin^2\theta}{G^2} - \cos^2\theta \\
&\quad + \frac{r\sin^2\theta}{G^2}\Big(\frac{\partial_r F}{F} + \frac{\partial_r G}{G} + \frac{1}{r}\Big) \\
&= \sin^2\theta\Big(-\frac{r\partial_r G}{G^3} + \frac{r\partial_r F}{FG^2} + \frac{1}{G^2} - 1\Big) = \sin^2\theta R_{22} \ .
\end{aligned}$$

The non-diagonal components write

$$
\begin{aligned}
R_{01} &= \partial_r(\Gamma^0_{00}+\Gamma^1_{01}) - \partial_t\Gamma^0_{01} - \partial_r\Gamma^1_{01} + +\Gamma^0_{01}\Gamma^0_{00} + \Gamma^1_{01}\Gamma^0_{01} + \Gamma^0_{11}\Gamma^1_{00} + \Gamma^1_{11}\Gamma^1_{01} \\
&\quad - \Gamma^0_{00}\Gamma^0_{01} - \Gamma^1_{01}\Gamma^0_{01} - \Gamma^1_{01}(\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13}) \\
&= \partial_r\Gamma^0_{00} - \partial_t\Gamma^0_{01} + \Gamma^0_{11}\Gamma^1_{00} - \Gamma^1_{01}\Gamma^0_{01} - \Gamma^1_{01}(\Gamma^2_{12}+\Gamma^3_{13}) \\
&= \frac{F\partial^2_{rt}F - (\partial_r F)(\partial_t F)}{F^2} - \frac{F\partial^2_{tr}F - (\partial_r F)(\partial_t F)}{F^2} + \frac{(\partial_t G)(\partial_r F)}{FG} \\
&\quad - \frac{(\partial_t G)(\partial_r F)}{FG} - 2\frac{\partial_t G}{rG} \\
&= -2\frac{\partial_t G}{rG}\ , \\
R_{\mu\nu} &= 0\ , \quad \text{for } \mu\neq\nu \text{ and } (\mu,\nu)\neq(0,1),(1,0)\ .
\end{aligned}
$$

From the above computations, it follows

$$
\begin{aligned}
R_g &= \frac{1}{F^2}R_{00} - \frac{1}{G^2}R_{11} - \frac{1}{r^2}R_{22} - \frac{1}{r^2\sin^2\theta}R_{33} \\
&= \frac{2}{FG}\Big(\frac{\partial^2_t G}{F} - \frac{\partial^2_r F}{G} + \frac{(\partial_r F)(\partial_r G)}{G^2} - \frac{(\partial_t F)(\partial_t G)}{F^2}\Big) + \frac{4}{rG^2}\Big(\frac{\partial_r G}{G} - \frac{\partial_r F}{F}\Big) + \frac{2}{r^2}\Big(1-\frac{1}{G^2}\Big)\ .
\end{aligned}
$$

□

Therefore, thanks to Lemma 3.2, we can evaluate the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R_g g_{\mu\nu}$,

$$
\begin{aligned}
G_{00} &= -2\frac{F^2\,\partial_r G}{rG^3} + \frac{F^2}{r^2}\Big(\frac{1}{G^2}-1\Big)\ , \\
G_{01} &= -2\frac{\partial_t G}{rG}\ , \\
G_{11} &= -2\frac{\partial_r F}{rF} + \frac{1}{r^2}\Big(G^2-1\Big)\ , \\
G_{22} &= \frac{r^2}{FG}\Big(\frac{\partial^2_t G}{F} - \frac{\partial^2_r F}{G} + \frac{(\partial_r F)(\partial_r G)}{G^2} - \frac{(\partial_t F)(\partial_t G)}{F^2}\Big) + \frac{r}{G^2}\Big(\frac{\partial_r G}{G} - \frac{\partial_r F}{F}\Big)\ , \\
G_{33} &= \sin^2\theta\, G_{22}\ .
\end{aligned}
$$

Raising up one index, the angular dependence in θ simplifies and we finally obtain the first ingredient of the Einstein–Dirac system

$$
\begin{aligned}
G^0_0 &= -2\frac{\partial_r G}{rG^3} + \frac{1}{r^2}\Big(\frac{1}{G^2}-1\Big)\ , \\
G^0_1 &= -2\frac{\partial_t G}{rF^2G}\ , \\
G^1_1 &= 2\frac{\partial_r F}{rFG^2} + \frac{1}{r^2}\Big(\frac{1}{G^2}-1\Big)\ , \\
G^2_2 = G^3_3 &= -\frac{1}{FG}\Big(\frac{\partial^2_t G}{F} - \frac{\partial^2_r F}{G} + \frac{(\partial_r F)(\partial_r G)}{G^2} - \frac{(\partial_t F)(\partial_t G)}{F^2}\Big) \\
&\quad - \frac{1}{rG^2}\Big(\frac{\partial_r G}{G} - \frac{\partial_r F}{F}\Big)\ .
\end{aligned}
\tag{$\star_1$}
$$

We conclude this section by presenting the choice of the vierbein and the consequent behaviour of the spin connection.
In order to have a vierbein $e^a{}_\mu$ satisfying $g_{\mu\nu} = e^a{}_\mu \, \eta_{ab} \, e^b{}_\nu$, we can choose

$$e^a{}_\mu(t, r, \theta, \phi) := \mathrm{diag}\big(F(t,r), G(t,r), r, r\sin\theta\big) \ .$$

Therefore, defining $e^a := e^a{}_\mu dx^\mu$, we obtain

$$e^0 = F\,dt \ , \quad e^1 = G\,dr \ , \quad e^2 = r\,d\theta \ , \quad e^3 = r\sin\theta\,d\phi \ .$$

Writing their exterior derivatives, we have

$$\begin{aligned}
de^0 &= \partial_r F\,dr \wedge dt = -\frac{\partial_r F}{FG} e^0 \wedge e^1 \ , \\
de^1 &= \partial_t G\,dt \wedge dr = \frac{\partial_t G}{FG} e^0 \wedge e^1 \\
de^2 &= dr \wedge d\theta = \frac{1}{rG} e^1 \wedge e^2 \ , \\
de^3 &= \sin\theta\,dr \wedge d\phi + r\cos\theta\,d\theta \wedge d\phi = \frac{1}{rG} e^1 \wedge e^3 + \frac{\cot\theta}{r} e^2 \wedge e^3 \ .
\end{aligned}$$

Letting $\omega^a{}_b := \omega_\mu{}^a{}_b dx^\mu$ be the spin-one form, we recall the the characterization of the spin connection, given in Section 1.5,

$$de^a + \omega^a{}_b \wedge e^b = 0 \ , \quad \text{for } a = 0, 1, 2, 3 \ .$$

Solving the linear system defined by this family of equations, one gets

$$\omega^0{}_1 = \frac{\partial_r F}{FG} e^0 + \frac{\partial_t G}{FG} e^1 \ , \quad \omega^1{}_2 = -\frac{1}{rG} e^2 \ , \quad \omega^1{}_3 = -\frac{1}{rG} e^3 \ , \quad \omega^2{}_3 = -\frac{\cot\theta}{r} e^3 \ .$$

Hence, passing from the orthonormal frame to the standard one and raising a flat index with η, we completely determine the spin connection.

Lemma 3.3. *Let g be a metric of the form* (3.4)*, $e^a{}_\mu$ the vierbein defined as above and $\omega_\mu{}^{ab}$ be the associated spin connection coefficients. Then, the non-vanishing components are given by*

$$\begin{aligned}
&\omega_0{}^{01} = -\frac{\partial_r F}{G} = -\omega_0{}^{10} \ , \qquad \omega_1{}^{01} = -\frac{\partial_t G}{F} = -\omega_1{}^{10} \ , \qquad \omega_2{}^{12} = \frac{1}{G} = -\omega_2{}^{21} \ , \\
&\omega_3{}^{13} = \frac{\sin\theta}{G} = -\omega_3{}^{31} \ , \qquad \omega_3{}^{23} = \cos\theta = -\omega_3{}^{32} \ .
\end{aligned}$$

3.3 Dirac operator with spherical symmetry

Now that the spin connection has been computed, we can focus on the Dirac equation. In this section we show how to exploit the radial symmetry of the spacetime to separate the variables and simplify the angular dependence. To this end, we introduce partial wave subspaces and spherical harmonics and we ”decompose” the Dirac operator in a sum of ”radial” operators. This approach is widely discussed in [Tha13, Section 4.6], [CH96] and [CdS19a].

To implement this strategy, from now on we will opt for a different choice for the Dirac matrices, which simply consists of a permutation of the γ^j , $j = 1, 2, 3$:

$$\gamma^t := \gamma^0 \ , \quad \gamma^r := \gamma^3 \ , \quad \gamma^\theta := \gamma^1 \ , \quad \gamma^\phi := \gamma^2 \ . \tag{3.5}$$

Remark 3.3 (Dirac operator on $\mathbb{S}^2$). We will see in a moment the importance of this change. Thanks to this choice, we will recover the Dirac operator on the sphere $-i\hat{\nabla}$, for which many diagonalization results are available.

Recalling the definition of the spinorial covariant derivative $D_\mu = \partial_\mu + \frac{1}{8}\omega_\mu{}^{ab}[\gamma_a, \gamma_b]$, we easily obtain

$$\begin{aligned}
D_0 &= \partial_t + \frac{1}{8}\omega_0{}^{ab}[\gamma_a, \gamma_b] = \partial_t - \frac{\partial_r F}{2G}\gamma_t\gamma_r \ , \\
D_1 &= \partial_r + \frac{1}{8}\omega_1{}^{ab}[\gamma_a, \gamma_b] = \partial_r - \frac{\partial_t G}{2F}\gamma_t\gamma_r \ , \\
D_2 &= \partial_\theta + \frac{1}{8}\omega_2{}^{ab}[\gamma_a, \gamma_b] = \partial_\theta + \frac{1}{2G}\gamma_r\gamma_\theta \ , \\
D_3 &= \partial_\phi + \frac{1}{8}\omega_3{}^{ab}[\gamma_a, \gamma_b] = \partial_\phi + \frac{1}{2}\Big(\frac{\sin\theta}{G}\gamma_r\gamma_\phi + \cos\theta\gamma_\theta\gamma_\phi\Big) \ ,
\end{aligned}$$

while the spacetime-dependent Dirac matrices $\underline{\gamma}^\mu = e_a{}^\mu\gamma^a$ are given by

$$\underline{\gamma}^t = \frac{1}{F}\gamma^t \ , \quad \underline{\gamma}^r = \frac{1}{G}\gamma^r \ , \quad \underline{\gamma}^\theta = \frac{1}{r}\gamma^\theta \ , \quad \underline{\gamma}^\phi = \frac{1}{r\sin\theta}\gamma^\phi \ .$$

Therefore, the Dirac equation $i\underline{\gamma}^\mu D_\mu\psi = m\psi$ with spherical symmetry writes

$$\begin{aligned}
i\Big[\frac{1}{F}\Big(\gamma^t\partial_t + \frac{\partial_r F}{2G}\gamma^r\Big) + \frac{1}{G}\Big(\gamma^r\partial_r + \frac{\partial_t G}{2F}\gamma^t\Big) + \frac{1}{r}\Big(\gamma^\theta\partial_\theta + \frac{1}{2G}\gamma^r\Big) \\
+ \frac{1}{r\sin\theta}\Big(\gamma^\phi\partial_\phi + \frac{1}{2}\Big(\frac{\sin\theta}{G}\gamma^r + \cos\theta\gamma^\theta\Big)\Big)\Big]\psi = m\psi \ ,
\end{aligned}$$

which yields, rearranging the terms,

$$i\Big[\frac{1}{F}\gamma^t\Big(\partial_t + \frac{\partial_t G}{2G}\Big) + \frac{1}{G}\gamma^r\Big(\partial_r + \frac{\partial_r F}{2F} + \frac{1}{r}\Big) + \frac{1}{r}\Big(\gamma^\theta\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) + \frac{1}{\sin\theta}\gamma^\phi\partial_\phi\Big)\Big]\psi = m\psi \, .$$

Multiplying the equation by $F\gamma^t$ and defining $\alpha^j := \gamma^t\gamma^j$, for $j = r, \theta, \phi$, we find the following Schrödinger form

$$i\partial_t\psi = \Big[-i\frac{\partial_t G}{2G} - i\alpha^r\frac{F}{G}\Big(\partial_r + \frac{\partial_r F}{2F} + \frac{1}{r}\Big) - i\frac{F}{r}\Big(\alpha^\theta\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) + \frac{1}{\sin\theta}\alpha^\phi\partial_\phi\Big) + \gamma^t Fm\Big]\psi \, .$$

If we define the angular part of the Dirac operator as

$$\mathcal{D}_{\mathbb{S}^2} := \alpha^\theta\Big(- i\partial_\theta - \frac{i\cot\theta}{2}\Big) - \alpha^\phi \frac{i}{\sin\theta}\partial_\phi \ ,$$

the Dirac equation rewrites

$$i\partial_t\psi = (\mathcal{D} + \gamma^t F m)\psi \ , \qquad \text{where } \mathcal{D} := -i\frac{\partial_t G}{2G} - i\alpha^r \frac{F}{G}\Big(\partial_r + \frac{\partial_r F}{2F} + \frac{1}{r}\Big) + \frac{F}{r}\mathcal{D}_{\mathbb{S}^2} \, .$$

Now, if we recall the Dirac operator on the sphere $\mathbb{S}^2$

$$-i\hat\nabla := -i\sigma^1\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) - i\frac{\sigma^2}{\sin\theta}\partial_\phi \, ,$$

and recalling our choice (3.5), we obtain as desired

$$\mathcal{D}_{\mathbb{S}^2} = -i\alpha^1\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) - i\frac{\alpha^2}{\sin\theta}\partial_\phi = \begin{pmatrix} & -i\hat\nabla \\ -i\hat\nabla & \end{pmatrix} \ .$$

Therefore, we have rewritten the Dirac equation as $i\partial_t\psi = H_{F,G}\psi$, where

$$H_{F,G} = \begin{pmatrix} -i\frac{\partial_t G}{2G} + Fm & -i\sigma^3\frac{F}{G}\Big(\partial_r + \frac{\partial_r F}{2F} + \frac{1}{r}\Big) + \frac{F}{r}(-i\hat\nabla) \\ -i\sigma^3\frac{F}{G}\Big(\partial_r + \frac{\partial_r F}{2F} + \frac{1}{r}\Big) + \frac{F}{r}(-i\hat\nabla) & -i\frac{\partial_t G}{2G} - Fm \end{pmatrix} \ .$$

In this way, we managed to isolate the radial and the angular components, which are now expressed in terms of the operator $-i\hat\nabla$. As anticipated, this allows us to take advantage of its diagonalization properties, which are stated in [AJ02].

Indeed, we have that $-i\hat\nabla$ diagonalizes into

$$-i\hat\nabla\Gamma^{\pm}_{j,m_j} = \pm\lambda_j\Gamma^{\pm}_{j,m_j} \ ,$$

where $j \in \frac{1}{2}+\mathbb{N}$, $m_j \in \frac{1}{2}+\mathbb{Z}$ with $-j \le m_j \le j$ and $\lambda_j = \frac{1}{2}+j$. The two-component eingenfuctions $\Gamma^{\pm}_{j,m_j}$ can be chosen such that

$$\Gamma^{\pm}_{j,m_j} = \pm i\sigma^3\Gamma^{\mp}_{j,m_j} \Leftrightarrow -i\sigma^3\Gamma^{\pm}_{j,m_j} = \pm\Gamma^{\mp}_{j,m_j}$$

and such that they form an orthonormal basis of $L^2(\mathbb{S}^2,\mathbb{C}^2)$,

$$\langle\Gamma^{\varepsilon_1}_{j_1,m_{j_1}}, \Gamma^{\varepsilon_2}_{j_2,m_{j_2}}\rangle = \delta^{\varepsilon_1,\varepsilon_2}\delta_{j_1,j_2}\delta_{m_{j_1},m_{j_2}} \ .$$

Defining $\mathcal{E}^{\pm}_{j,m_j} := \frac{1}{\sqrt{2}}(\Gamma^+_{j,m_j} \pm \Gamma^-_{j,m_j})$, we can construct an orthogonal basis of $L^2(\mathbb{S}^2,\mathbb{C}^4)$, which is given by

$$\mathcal{G}^+_{j,m_j} := \begin{pmatrix} \mathcal{E}^+_{j,m_j} \\ 0 \end{pmatrix} \ , \ \mathcal{G}^-_{j,m_j} := \begin{pmatrix} 0 \\ -\mathcal{E}^-_{j,m_j} \end{pmatrix} \ , \ \mathcal{F}^-_{j,m_j} := \begin{pmatrix} \mathcal{E}^-_{j,m_j} \\ 0 \end{pmatrix} \ , \ \mathcal{F}^+_{j,m_j} := \begin{pmatrix} 0 \\ \mathcal{E}^+_{j,m_j} \end{pmatrix} \ .$$

Hence, if we define the following spaces,

$$\widetilde{\mathcal{H}}^-_{j,m_j} := \mathrm{Vect}(\mathcal{G}^+_{j,m_j}, \mathcal{G}^-_{j,m_j}) , \quad \widetilde{\mathcal{H}}^+_{j,m_j} := \mathrm{Vect}(\mathcal{F}^-_{j,m_j}, \mathcal{F}^+_{j,m_j}) ,$$

we obtain the orthogonal decomposition

$$L^2(\mathbb{S}^2, \mathbb{C}^4) \cong \bigoplus_{j,m_j} (\widetilde{\mathcal{H}}^+_{j,m_j} \oplus \widetilde{\mathcal{H}}^-_{j,m_j}) .$$

As reported in [AJ02], the functions $\mathcal{E}^\pm_{j,m_j}$ (and consequently $\mathcal{F}^\pm_{j,m_j}, \mathcal{G}^\pm_{j,m_j}$) can be expressed in terms of the spherical harmonics Y^m_l, up to a local rotation $R_1 = e^{i\frac{\theta}{2}\sigma_2} e^{i\frac{\phi}{2}\sigma_3}$:

$$\mathcal{E}^+_{j,m_j} = R_1 \begin{pmatrix} \sqrt{\frac{j+m_j}{2j}} Y^{m_j^-}_{j^-} \\ \sqrt{\frac{j-m_j}{2j}} Y^{m_j^+}_{j^-} \end{pmatrix} , \quad \mathcal{E}^-_{j,m_j} = R_1 \begin{pmatrix} \sqrt{\frac{j-m_j+1}{2j+2}} Y^{m_j^-}_{j^+} \\ -\sqrt{\frac{j+m_j+1}{2j+2}} Y^{m_j^+}_{j^+} \end{pmatrix} , \tag{3.6}$$

where $j^\pm = j \pm \frac{1}{2}$ and $m^\pm_j = m_j \pm \frac{1}{2}$.

Remark 3.4 (Spherical harmonics Y^m_l). Similarly to $e^{in\theta}$ spanning the space of complex-valued functions on the circle, spherical harmonics $Y^m_l(\theta, \phi)$ define a complete orthonormal family in $L^2(\mathbb{S}^2, \mathbb{C}^2)$. They can be expressed in terms of Legendre polynomials

$$Y^m_l(\theta, \phi) := \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{im\phi} P^m_l(\cos\theta) , \text{ for } m \in \{-l, -l+1, ..., l-1, l\},$$

where $P^m_l(x) := \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2} \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^l$. One can use these functions to define the canonical *partial wave subspaces* $\mathcal{H}_{j,m_j,k_j}$, with $k_j = \pm\lambda_j$, in Thaller's book and orthogonally decompose $L^2(\mathbb{S}^2, \mathbb{C}^4)$. Thanks to (3.6), we infer that

$$\mathcal{H}_{j,m_j,k_j} = (R^*_1 \oplus R^*_1) \widetilde{\mathcal{H}}^\pm_{j,m_j}$$

and for this reason, we will also call $\widetilde{\mathcal{H}}^\pm_{j,m_j}$ partial wave subspaces. For more details regarding spherical harmonics and their application to the Dirac operator on the sphere, see for instance [Tha13, Section 4.6].

The above construction is extremely useful to simplify the angular dependence, but it hides a crucial subtlety, namely that in general the Dirac operator does not preserve radial spinors. Indeed, as proven in [Cac11, Section 5], the partial wave decomposition is preserved by the nonlinear dynamics provided the initial condition has an angular part belonging to one of the four $\widetilde{\mathcal{H}}^\pm_{1/2,m_{1/2}}$ spaces. For this reason, to construct our model, we restrict to $\widetilde{\mathcal{H}}^-_{1/2,1/2} = \mathrm{Vect}(\mathcal{G}^+_{1/2,1/2}, \mathcal{G}^-_{1/2,1/2})$.

We conclude this section by computing explicitly the Dirac equation in this latter case. Using $e^{i\theta\sigma_j} = \cos\theta I + i\sin\theta\sigma_j$, the rotation is given by

$$R_1 = \begin{pmatrix} e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ -e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}$$

Therefore,

$$\mathcal{E}^+_{1/2,1/2} = R_1 \begin{pmatrix} Y_0^0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{\pi}} R_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}} \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix},$$

$$\mathcal{E}^-_{1/2,1/2} = R_1 \begin{pmatrix} \frac{1}{\sqrt{3}} Y_1^0 \\ -\sqrt{\frac{2}{3}} Y_1^1 \end{pmatrix} = \frac{1}{2\sqrt{\pi}} R_1 \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} = \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

from which we deduce

$$\mathcal{G}^+_{1/2,1/2} = \begin{pmatrix} \mathcal{E}^+_{1/2,1/2} \\ 0 \end{pmatrix} = \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}} \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{G}^-_{1/2,1/2} = \begin{pmatrix} 0 \\ -\mathcal{E}^-_{1/2,1/2} \end{pmatrix} = \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}} \begin{pmatrix} 0 \\ 0 \\ -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix}.$$

Hence, the generic function $\psi \in L^2(F(t,r)G(t,r)r^2 dt\, dr) \otimes \widetilde{\mathcal{H}}^-_{1/2,1/2}$ can be written as

$$\psi(t,r,\theta,\phi) := u_+(t,r)\mathcal{G}^+_{1/2,1/2}(\theta,\phi) + u_-(t,r)\mathcal{G}^-_{1/2,1/2}(\theta,\phi) = \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}} \begin{pmatrix} u_+(t,r)\cos\frac{\theta}{2} \\ -u_+(t,r)\sin\frac{\theta}{2} \\ -u_-(t,r)\cos\frac{\theta}{2} \\ -u_-(t,r)\sin\frac{\theta}{2} \end{pmatrix}.$$

Now, let us compute the action of the Dirac operator on a spinor of this type. We start with the first addend

$$\begin{aligned} \underline{\gamma}^t D_0\psi &= \frac{1}{F}\Big(\gamma^t\partial_t + \frac{\partial_r F}{2G}\gamma^r\Big)\psi = \frac{1}{F}\Big(\gamma^0\partial_t + \frac{\partial_r F}{2G}\gamma^3\Big)\psi \\ &= \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}F}\left[\begin{pmatrix} \partial_t u_+\cos\frac{\theta}{2} \\ -\partial_t u_+\sin\frac{\theta}{2} \\ \partial_t u_-\cos\frac{\theta}{2} \\ \partial_t u_-\sin\frac{\theta}{2} \end{pmatrix} + \frac{\partial_r F}{2G}\begin{pmatrix} -u_-\cos\frac{\theta}{2} \\ u_-\sin\frac{\theta}{2} \\ -u_+\cos\frac{\theta}{2} \\ -u_+\sin\frac{\theta}{2} \end{pmatrix}\right]. \end{aligned}$$

The second term is then given by

$$\begin{aligned} \underline{\gamma}^r D_1\psi &= \frac{1}{G}\Big(\gamma^r\partial_r + \frac{\partial_t G}{2F}\gamma^t\Big)\psi = \frac{1}{G}\Big(\gamma^3\partial_r + \frac{\partial_t G}{2F}\gamma^0\Big)\psi \\ &= \frac{e^{i\frac{\phi}{2}}}{2\sqrt{\pi}G}\left[\begin{pmatrix} -\partial_r u_-\cos\frac{\theta}{2} \\ \partial_r u_-\sin\frac{\theta}{2} \\ -\partial_r u_+\cos\frac{\theta}{2} \\ -\partial_r u_+\sin\frac{\theta}{2} \end{pmatrix} + \frac{\partial_t G}{2F}\begin{pmatrix} u_+\cos\frac{\theta}{2} \\ -u_+\sin\frac{\theta}{2} \\ u_-\cos\frac{\theta}{2} \\ u_-\sin\frac{\theta}{2} \end{pmatrix}\right]. \end{aligned}$$

Passing to the two angular components, we find

$$\underline{\gamma}^\theta D_2\psi = \frac{1}{r}\Big(\gamma^\theta\partial_\theta + \frac{1}{2G}\gamma^r\Big)\psi = \frac{1}{r}\Big(\gamma^1\partial_\theta + \frac{1}{2G}\gamma^3\Big)\psi$$
$$= \frac{e^{i\frac{\phi}{2}}}{4\sqrt{\pi}r}\left[\begin{pmatrix} -u_-\cos\frac{\theta}{2} \\ u_-\sin\frac{\theta}{2} \\ u_+\cos\frac{\theta}{2} \\ u_+\sin\frac{\theta}{2}\end{pmatrix} + \frac{1}{G}\begin{pmatrix} -u_-\cos\frac{\theta}{2} \\ u_-\sin\frac{\theta}{2} \\ -u_+\cos\frac{\theta}{2} \\ -u_+\sin\frac{\theta}{2}\end{pmatrix}\right] ,$$

$$\underline{\gamma}^\phi D_3\psi = \frac{1}{r\sin\theta}\Big(\gamma^\phi\partial_\phi + \frac{\sin\theta}{2G}\gamma^r + \frac{\cos\theta}{2}\gamma^\theta\Big)\psi = \frac{1}{r\sin\theta}\Big(\gamma^2\partial_\phi + \frac{\sin\theta}{2G}\gamma^3 + \frac{\cos\theta}{2}\gamma^1\Big)\psi$$
$$= \frac{e^{i\frac{\phi}{2}}}{4\sqrt{\pi}r\sin\theta}\left[\begin{pmatrix} -u_-\sin\frac{\theta}{2} \\ u_-\cos\frac{\theta}{2} \\ u_+\sin\frac{\theta}{2} \\ u_+\cos\frac{\theta}{2}\end{pmatrix} + \frac{\sin\theta}{G}\begin{pmatrix} -u_-\cos\frac{\theta}{2} \\ u_-\sin\frac{\theta}{2} \\ -u_+\cos\frac{\theta}{2} \\ -u_+\sin\frac{\theta}{2}\end{pmatrix} + \cos\theta\begin{pmatrix} -u_-\sin\frac{\theta}{2} \\ -u_-\cos\frac{\theta}{2} \\ u_+\sin\frac{\theta}{2} \\ -u_+\cos\frac{\theta}{2}\end{pmatrix}\right] .$$

Finally, imposing ψ to solve the Dirac equation $\underline{\gamma}^\mu D_\mu\psi = -im\psi$, we obtain a system of two linearly independent PDEs with no angular dependence,

$$\begin{aligned} \frac{1}{F}\partial_t u_+ - \frac{1}{G}\partial_r u_- &= \Big(-\frac{\partial_t G}{2FG} - im\Big)u_+ + \Big(\frac{\partial_r F}{2FG} + \frac{1+G}{rG}\Big)u_- , \\ -\frac{1}{G}\partial_r u_+ + \frac{1}{F}\partial_t u_- &= \Big(\frac{\partial_r F}{2FG} + \frac{1-G}{rG}\Big)u_+ + \Big(-\frac{\partial_t G}{2FG} + im\Big)u_- . \end{aligned} \tag{$\star_2$}$$

Remark 3.5 (Comparison with the flat case). Notice that taking $F = 1, G = 1$ reduces $(\mathcal{M}, g)$ to the standard Minkowski space. Hence, if we define

$$\psi_+(t,r) := ru_+(t,r) , \qquad \psi_-(t,r) := -iru_-(t,r) ,$$

the Dirac equations on $L^2(r^2 dt\, dr) \otimes \widetilde{\mathcal{H}}^-_{1/2,1/2}$ become

$$i\partial_t\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = \begin{pmatrix} m & -\partial_r - \frac{1}{r} \\ \partial_r - \frac{1}{r} & -m\end{pmatrix}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} ,$$

which is exactly [Tha13, Formula (4.129) with $k_j = -1$ and null potential].

3.4 Energy-momentum tensor

The last remaining ingredient for writing the Einstein–Dirac system is the energy-momentum tensor $T_{\mu\nu}$. Therefore, we proceed with the computation of the non-vanishing components.

Let us start with the component $T_{00} = \frac{i}{2}\left(\overline{\psi}\underline{\gamma_t}D_0\psi - D_0\overline{\psi}\underline{\gamma_t}\psi\right)$: we treat the two addends separately

$$\begin{aligned}
\overline{\psi}\underline{\gamma_t}D_0\psi &= F\psi^*\Big(\partial_t + \frac{\partial_r F}{2G}\alpha^r\Big)\psi = F\psi^*\Big(\partial_t + \frac{\partial_r F}{2G}\alpha^3\Big)\psi \\
&= \frac{F}{4\pi}\Big(\overline{u_+}\partial_t u_+ + \overline{u_-}\partial_t u_-\Big) + \frac{F\partial_r F}{2G}\psi^*\alpha^3\psi \ , \\
D_0\overline{\psi}\underline{\gamma_t}\psi &= F\Big(\partial_t\psi^* + \frac{\partial_r F}{2G}\psi^*\alpha^r\Big)\psi = F\Big(\partial_t\psi^* + \frac{\partial_r F}{2G}\psi^*\alpha^3\Big)\psi \\
&= \frac{F}{4\pi}\Big(\overline{\partial_t u_+}u_+ + \overline{\partial_t u_-}u_-\Big) + \frac{F\partial_r F}{2G}\psi^*\alpha^3\psi \ .
\end{aligned}$$

Hence,

$$T_{00} = -\frac{F}{4\pi}\Im(\overline{u_-}\partial_t u_- + \overline{u_+}\partial_t u_+) \ .$$

Then, let us focus on $T_{01} = \frac{i}{4}\left(\overline{\psi}\underline{\gamma_t}D_1\psi - D_1\overline{\psi}\underline{\gamma_t}\psi\right) + \frac{i}{4}\left(\overline{\psi}\underline{\gamma_r}D_0\psi - D_0\overline{\psi}\underline{\gamma_r}\psi\right)$. In this case, computing preliminarly

$$\begin{aligned}
\overline{\psi}\underline{\gamma_t}D_1\psi &= F\psi^*\Big(\partial_r + \frac{\partial_t G}{2F}\alpha^r\Big)\psi = F\psi^*\partial_r\psi + \frac{\partial_t G}{2}\psi^*\alpha^3\psi \\
&= \frac{F}{4\pi}\Big(\overline{u_+}\partial_r u_+ + \overline{u_-}\partial_r u_-\Big) + \frac{\partial_t G}{2}\psi^*\alpha^3\psi \ , \\
D_1\overline{\psi}\underline{\gamma_t}\psi &= F\Big(\partial_r\psi^*\psi + \frac{\partial_t G}{2F}\psi^*\alpha^r\psi\Big) = F\partial_r\psi^*\psi + \frac{\partial_t G}{2}\psi^*\alpha^3\psi \\
&= \frac{F}{4\pi}\Big(\overline{\partial_r u_+}u_+ + \overline{\partial_r u_-}u_-\Big) + \frac{\partial_t G}{2}\psi^*\alpha^3\psi \ ,
\end{aligned}$$

$$\begin{aligned}
\overline{\psi}\underline{\gamma_r}D_0\psi &= -G\psi^*\Big(\alpha^r\partial_t + \frac{\partial_r F}{2G}\Big)\psi = -G\psi^*\alpha^3\partial_t\psi - \frac{\partial_r F}{2}\psi^*\psi \\
&= \frac{G}{4\pi}\Big(\overline{u_+}\partial_t u_- + \overline{u_-}\partial_t u_+\Big) - \frac{\partial_r F}{2}\psi^*\psi \ , \\
D_0\overline{\psi}\underline{\gamma_r}\psi &= -G\Big(\partial_t\psi^*\alpha^r\psi + \frac{\partial_r F}{2G}\psi^*\psi\Big) = -G\partial_t\psi^*\alpha^3\psi - \frac{\partial_r F}{2}\psi^*\psi \\
&= \frac{G}{4\pi}\Big(\overline{\partial_t u_+}u_- + \overline{\partial_t u_-}u_+\Big) - \frac{\partial_r F}{2}\psi^*\psi \ ,
\end{aligned}$$

we obtain

$$T_{01} = -\frac{F}{8\pi}\Im(\overline{u_+}\partial_r u_+ + \overline{u_-}\partial_r u_-) - \frac{G}{8\pi}\Im(\overline{u_+}\partial_t u_- + \overline{u_-}\partial_t u_+) \ .$$

Passing to $T_{11} = \frac{i}{2}\left(\overline{\psi}\underline{\gamma_r}D_1\psi - D_1\overline{\psi}\underline{\gamma_r}\psi\right)$, we have

$$\begin{aligned}
\overline{\psi}\underline{\gamma_r}D_1\psi &= -G\psi^*\Big(\alpha^r\partial_r + \frac{\partial_t G}{2F}\Big)\psi = -G\psi^*\Big(\alpha^3\partial_r + \frac{\partial_t G}{2F}\Big)\psi \\
&= \frac{G}{4\pi}\Big(\overline{u_+}\partial_r u_- + \overline{u_-}\partial_r u_+\Big) - \frac{G\partial_t G}{2F}\psi^*\psi \ , \\
D_1\overline{\psi}\underline{\gamma_r}\psi &= -G\Big(\partial_r\psi^*\alpha^r + \frac{\partial_t G}{2F}\psi^*\Big)\psi = -G\Big(\partial_r\psi^*\alpha^3 + \frac{\partial_t G}{2F}\psi^*\Big)\psi \\
&= \frac{G}{4\pi}\Big(\overline{\partial_r u_+}u_- + \overline{\partial_r u_-}u_+\Big) - \frac{G\partial_t G}{2F}\psi^*\psi \ ,
\end{aligned}$$

from which we deduce

$$T_{11} = -\frac{G}{4\pi}\Im(\overline{u_-}\partial_r u_+ + \overline{u_+}\partial_r u_-) \ .$$

Let us now compute $T_{22} = \frac{i}{2}\left(\overline{\psi}\underline{\gamma_\theta} D_2\psi - D_2\overline{\psi}\underline{\gamma_\theta}\psi\right)$:

$$\begin{aligned}
\overline{\psi}\underline{\gamma_\theta} D_2\psi &= -r\psi^*\Big(\alpha^\theta\partial_\theta + \frac{1}{2G}\alpha^r\Big)\psi = -r\psi^*\Big(\alpha^1\partial_\theta + \frac{1}{2G}\alpha^3\Big)\psi \\
&= -i\frac{r}{4\pi}\Im(\overline{u_-}u_+) - \frac{r}{2G}\psi^*\alpha^3\psi \ , \\
D_2\overline{\psi}\underline{\gamma_\theta}\psi &= -r\Big(\partial_\theta\psi^*\alpha^\theta + \frac{1}{2G}\psi^*\alpha^r\Big)\psi \\
&= -r\Big(\partial_\theta\psi^*\alpha^1 + \frac{1}{2G}\psi^*\alpha^3\Big)\psi = i\frac{r}{4\pi}\Im(\overline{u_-}u_+) - \frac{r}{2G}\psi^*\alpha^3\psi \ ,
\end{aligned}$$

hence,

$$T_{22} = \frac{r}{4\pi}\Im(\overline{u_-}u_+) \ .$$

Similarly, computing $T_{33} = \frac{i}{2}\left(\overline{\psi}\underline{\gamma_\phi} D_3\psi - D_3\overline{\psi}\underline{\gamma_\phi}\psi\right)$, we find

$$\begin{aligned}
\overline{\psi}\underline{\gamma_\phi} D_3\psi &= -r\sin\theta\psi^*\Big(\alpha^\phi\partial_\phi + \frac{\sin\theta}{2G}\alpha^r + \frac{\cos\theta}{2}\alpha^\theta\Big)\psi \\
&= -r\sin\theta\psi^*\Big(\alpha^2\partial_\phi + \frac{\sin\theta}{2G}\alpha^3 + \frac{\cos\theta}{2}\alpha^1\Big)\psi \\
&= -i\frac{r\sin^2\theta}{4\pi}\Im(\overline{u_-}u_+) - (r\sin\theta)\psi^*\Big(\frac{\sin\theta}{2G}\alpha^3 + \frac{\cos\theta}{2}\alpha^1\Big)\psi \ , \\
D_3\overline{\psi}\underline{\gamma_\phi}\psi &= -r\sin\theta\Big(\partial_\phi\psi^*\alpha^\phi + \frac{\sin\theta}{2G}\psi^*\alpha^r + \frac{\cos\theta}{2}\psi^*\alpha^\theta\Big)\psi \\
&= -r\sin\theta\Big(\partial_\phi\psi^*\alpha^2 + \frac{\sin\theta}{2G}\psi^*\alpha^3 + \frac{\cos\theta}{2}\psi^*\alpha^1\Big)\psi \\
&= i\frac{r\sin^2\theta}{4\pi}\Im(\overline{u_-}u_+) - (r\sin\theta)\psi^*\Big(\frac{\sin\theta}{2G}\alpha^3 + \frac{\cos\theta}{2}\alpha^1\Big)\psi \ ,
\end{aligned}$$

which yields

$$T_{33} = \frac{r\sin^2\theta}{4\pi}\Im(\overline{u_-}u_+) = \sin^2\theta\, T_{22} \ .$$

Finally, raising one index to simplify the angular dependence, we obtain

$$\begin{aligned}
T^0_0 &= -\frac{1}{4\pi F}\Im(\overline{u_-}\partial_t u_- + \overline{u_+}\partial_t u_+) \ , \\
T^0_1 &= -\frac{1}{8\pi F}\Im(\overline{u_+}\partial_r u_+ + \overline{u_-}\partial_r u_-) - \frac{G}{8\pi F^2}\Im(\overline{u_+}\partial_t u_- + \overline{u_-}\partial_t u_+) \ , \qquad (\star_3) \\
T^1_1 &= \frac{1}{4\pi G}\Im(\overline{u_-}\partial_r u_+ + \overline{u_+}\partial_r u_-) \ , \\
T^2_2 = T^3_3 &= -\frac{1}{4\pi r}\Im(\overline{u_-}u_+) \ .
\end{aligned}$$

Remark 3.6 (Trace of the energy-momentum tensor). We note that, if we impose the two Dirac PDEs ($\star_2$), the first component can be rewritten as

$$\begin{aligned} T_0^0 &= -\frac{1}{4\pi F}\Big(\frac{F}{G}\Im(\overline{u_-}\partial_r u_+ + \overline{u_+}\partial_r u_-) - \frac{2F}{r}\Im(\overline{u_-}u_+) + mF(|u_-|^2 - |u_+|^2)\Big) \\ &= -\frac{1}{4\pi G}\Im(\overline{u_-}\partial_r u_+ + \overline{u_+}\partial_r u_-) + \frac{1}{2\pi r}\Im(\overline{u_-}u_+) + \frac{m}{4\pi}(|u_+|^2 - |u_-|^2)\ . \end{aligned}$$

Hence, the trace of the energy-momentum is given by

$$\mathrm{tr}_g T = T_j^j = \frac{m}{4\pi}(|u_+|^2 - |u_-|^2) = m\overline{\psi}\psi\ ,$$

consistently with Remark 3.1.

3.5 The final system and open questions

Finally, putting together ($\star_1$), ($\star_2$) and ($\star_3$), we write the Einstein–Dirac system

$$\begin{aligned} -\frac{1}{4\pi F}\Im(\overline{u_-}\partial_t u_- + \overline{u_+}\partial_t u_+) &= 2\frac{\partial_r G}{rG^3} - \frac{1}{r^2}\Big(\frac{1}{G^2} - 1\Big)\ , \\ -\frac{1}{8\pi F}\Im(\overline{u_+}\partial_r u_+ + \overline{u_-}\partial_r u_-) - \frac{G}{8\pi F^2}\Im(\overline{u_+}\partial_t u_- + \overline{u_-}\partial_t u_+) &= 2\frac{\partial_t G}{rF^2 G}\ , \\ \frac{1}{4\pi G}\Im(\overline{u_-}\partial_r u_+ + \overline{u_+}\partial_r u_-) &= -2\frac{\partial_r F}{rFG^2} - \frac{1}{r^2}\Big(\frac{1}{G^2} - 1\Big)\ , \\ -\frac{1}{4\pi r}\Im(\overline{u_-}u_+) &= \frac{1}{FG}\Big(\frac{\partial_t^2 G}{F} - \frac{\partial_r^2 F}{G} + \frac{(\partial_r F)(\partial_r G)}{G^2} - \frac{(\partial_t F)(\partial_t G)}{F^2}\Big) \\ &\quad + \frac{1}{rG^2}\Big(\frac{\partial_r G}{G} - \frac{\partial_r F}{F}\Big)\ , \\ \frac{1}{F}\partial_t u_+ - \frac{1}{G}\partial_r u_- &= \Big(-\frac{\partial_t G}{2FG} - im\Big)u_+ + \Big(\frac{\partial_r F}{2FG} + \frac{1+G}{rG}\Big)u_-\ , \\ -\frac{1}{G}\partial_r u_+ + \frac{1}{F}\partial_t u_- &= \Big(\frac{\partial_r F}{2FG} + \frac{1-G}{rG}\Big)u_+ + \Big(-\frac{\partial_t G}{2FG} + im\Big)u_-\ . \end{aligned}$$

As already noticed in Remark 3.1, coupling the Einstein and the Dirac equations automatically guarantees an additional constraint on the trace of the energy-momentum tensor. Therefore, the fourth equation above, which corresponds to $T_2^2 = -G_2^2$, is linearly dependent on the others and thus it can be discarded.

Therefore, the Einstein–Dirac system with radial symmetry reduces to

$$\begin{aligned}
-\frac{1}{4\pi F}\Im(\overline{u_-}\partial_t u_- + \overline{u_+}\partial_t u_+) &= 2\frac{\partial_r G}{rG^3} - \frac{1}{r^2}\Big(\frac{1}{G^2} - 1\Big) \,, \\
-\frac{1}{8\pi F}\Im(\overline{u_+}\partial_r u_+ + \overline{u_-}\partial_r u_-) - \frac{G}{8\pi F^2}\Im(\overline{u_+}\partial_t u_- + \overline{u_-}\partial_t u_+) &= 2\frac{\partial_t G}{rF^2 G} \,, \\
\frac{1}{4\pi G}\Im(\overline{u_-}\partial_r u_+ + \overline{u_+}\partial_r u_-) &= -2\frac{\partial_r F}{rFG^2} - \frac{1}{r^2}\Big(\frac{1}{G^2} - 1\Big) \,, \\
\frac{1}{F}\partial_t u_+ - \frac{1}{G}\partial_r u_- &= \Big(-\frac{\partial_t G}{2FG} - im\Big)u_+ + \Big(\frac{\partial_r F}{2FG} + \frac{1+G}{rG}\Big)u_- \,, \\
-\frac{1}{G}\partial_r u_+ + \frac{1}{F}\partial_t u_- &= \Big(\frac{\partial_r F}{2FG} + \frac{1-G}{rG}\Big)u_+ + \Big(-\frac{\partial_t G}{2FG} + im\Big)u_- \,.
\end{aligned}$$

Finally, if we define the new components of ψ as

$$\psi_+(t,r) := \frac{r\sqrt{F(t,r)}}{8\pi}u_+(t,r) \,, \qquad \psi_-(t,r) := i\frac{r\sqrt{F(t,r)}}{8\pi}u_-(t,r) \,,$$

the Dirac spinor rewrites

$$\psi(t,r,\theta,\phi) := \psi_+(t,r)\mathcal{G}^+_{1/2,1/2}(\theta,\phi) + \psi_-(t,r)\mathcal{G}^-_{1/2,1/2}(\theta,\phi) = 4\sqrt{\pi}\,\frac{e^{i\frac{\phi}{2}}}{r\sqrt{F}}\begin{pmatrix} \psi_+(t,r)\cos\frac{\theta}{2} \\ -\psi_+(t,r)\sin\frac{\theta}{2} \\ i\psi_-(t,r)\cos\frac{\theta}{2} \\ i\psi_-(t,r)\sin\frac{\theta}{2} \end{pmatrix} .$$

Due to this change of variables, the Einstein–Dirac system with spherical symmetry slightly simplifies and it writes

$$\begin{aligned}
\frac{16\pi}{rF^2}\Im(\overline{\psi_+}\partial_t\psi_+ + \overline{\psi_-}\partial_t\psi_-) &= -2\frac{\partial_r G}{G^3} + \frac{1}{r}\Big(\frac{1}{G^2} - 1\Big) \,, \\
-\frac{8\pi}{rG}\Im(\overline{\psi_+}\partial_r\psi_+ + \overline{\psi_-}\partial_r\psi_-) - \frac{8\pi}{rF}\Re(\overline{\psi_-}\partial_t\psi_+ - \overline{\psi_+}\partial_t\psi_-) &= 2\frac{\partial_t G}{G^2} \,, \\
\frac{16\pi}{rFG}\Re(\overline{\psi_+}\partial_r\psi_- - \overline{\psi_-}\partial_r\psi_+) &= 2\frac{\partial_r F}{FG^2} + \frac{1}{r}\Big(\frac{1}{G^2} - 1\Big) \,, \\
\frac{1}{F}\partial_t\psi_+ + \frac{i}{G}\partial_r\psi_- &= \Big[\frac{1}{2F}\Big(\frac{\partial_t F}{F} - \frac{\partial_t G}{G}\Big) - im\Big]\psi_+ - \frac{i}{r}\psi_- \,, \\
\frac{1}{G}\partial_r\psi_+ + \frac{i}{F}\partial_t\psi_- &= \frac{1}{r}\psi_+ + i\Big[\frac{1}{2F}\Big(\frac{\partial_t F}{F} - \frac{\partial_t G}{G}\Big) + im\Big]\psi_- \,,
\end{aligned} \qquad (\star)$$

consistently with equations found in [FSY99, Section 8 with $T = F^{-1}, A = G^{-2}$]. In their paper, Finster, Smoller and Yau started from these computations to numerically construct linearly stable soliton-like solutions.

Throughout this work, we have tried to cover in the most self-contained way possible the arguments and motivations that led us to the statement of the Einstein–Dirac system with spherical symmetry. Our aim was not simply to compute the final equations, but also to start approaching some of the tools and techniques useful in future doctoral studies.

We conclude by providing an overview of the state of the art in the study of this system and by giving some ideas for the next research goals.
In fact, now that our object is determined, many questions arise and one of the first ones is related to the *local well-posedness* of the system and the understanding of the suitable regularity conditions for the initial data.
Furthermore, being the Minkowski metric and the null Dirac spinor $(g, \psi) = (\eta, 0)$ a solution to ($\star$), the question of its *stability* is natural.

For the Einstein Vacuum Equations, Lindblad and Rodnianski [LR10] showed the stability of the Minkowski metric with a proof based on specific commuting vector fields, pointwise estimates and bootstrap arguments. In [Che22] these techniques have been adapted to prove the stability of Minkowski spacetime for the Einstein–Dirac system with zero mass.
Regarding the *massive case*, we expect that additional pointwise estimates for the Dirac operator in asymptotically flat manifolds are needed. We aim to show them by establishing new virial identities, in the same spirit as what was done in [CdS19b].

Finally, one can turn to the study of other geometries. Indeed, as already mentioned, *Schwarzschild metrics* g_M also satisfy the EVE and thus the same issue about stability applies to the pairs $(g, \psi) = (g_M, 0)$. The focus on these manifolds is motivated by Birkhoff's theorem, which states that the only asymptotically flat spherically symmetric solutions to the EVE are precisely Schwarzschild black holes. Moreover, we notice that a metric g_M can be separated into two regions: far from the black hole the geometry is asymptotically flat, while it becomes *asymptotically hyperbolic* approaching the event horizon. Therefore, new dispersive estimates for the Dirac operator will be needed to deal with this latter behaviour of the metric.

Appendix A

Complements

The appendix is devoted to tracing the main ideas for the proof of a second local smoothing estimate for the Dirac equation. We recall that this result is fundamental to establish Theorem 2.8.

Notation. Throughout this last section, solutions to the Dirac equation are simply treated as $\mathbb{C}^4$-valued functions, ignoring the underlying spinorial properties. Hence, given u a $\mathbb{C}^4$-valued function, $\overline{u}$ will simply denote its hermitian adjoint, and not its spinorial adjoint. We prefer to keep the same notation, since it still holds $\overline{D_i u} = D_i \overline{u}$, where $D_i \overline{u} := \partial_i \overline{u} - \frac{1}{8}\omega_i{}^{ab}\overline{u}[\gamma_a, \gamma_b]$.

A.1 Local smoothing estimate for the Dirac equation

Theorem A.1 (Local smoothing estimate for Dirac – II, [CdS19b])**.** *Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian and asymptotically flat manifold and u be a solution to the Cauchy problem* (2.3)*. Then, for $m \geq 0$, the following local smoothing estimate holds*

$$\|\langle x\rangle^{-3/2-} u\|_{L^2_t L^2(\mathcal{M}_h)} + \|\langle x\rangle^{-1/2-} \nabla u\|_{L^2_t L^2(\mathcal{M}_h)} \lesssim \|\mathcal{D}_m u_0\|_{L^2(\mathcal{M}_h)} , \tag{A.1}$$

where ∇ denotes the spinorial gradient $\nabla = \widetilde{\nabla} + B$.

The result is implied by the following, and slightly more general, estimate that the authors proved in [CdS19b]

$$\|u\|^2_{XL^2_t} + \|\nabla u\|^2_{YL^2_t} \lesssim \|\mathcal{D}_m u_0\|^2_{L^2(\mathcal{M}_h)} , \tag{A.2}$$

where the Campanato-type norms are defined as

$$\|v\|_X^2 := \sup_{R>0} \frac{1}{\langle R\rangle^2} \int_{\mathcal{M}_h\cap S_R} |v|^2 = \sup_{R>0} \frac{1}{\langle R\rangle^2} \int_{\mathcal{M}_h\cap S_R} |v|^2 \sqrt{\det(h)}\, dS \ ,$$
$$\|v\|_Y^2 := \sup_{R>0} \frac{1}{\langle R\rangle} \int_{\mathcal{M}_h\cap B_R} |v|^2 = \sup_{R>0} \frac{1}{\langle R\rangle} \int_{B_R} |v|^2 \sqrt{\det(h)}\, dx \ ,$$

where dS denotes the surface measure, while S_R and B_R are respectively the surface and the interior of the sphere of radius R and centered at the origin. Furthermore, we note that $\|\cdot\|_Y$ is equivalent to the norm

$$\|u\|_{\tilde{Y}}^2 := \sup_{R\geq 1} \frac{1}{R} \int_{\mathcal{M}_h\cap B_R} |u|^2 \ . \tag{A.3}$$

Remark A.1 (Weaker geometrical assumption). Before outlining the proof of (A.2), we stress that the result holds with a weaker assumption on the decay of the asymptotically flat condition. Indeed, from now on, the power $-|\alpha|-1-\sigma$ is weakened to $-|\alpha|-\sigma$. Adapting Lemma 2.4 and assuming C_h small enough, one can still choose a global vierbein and deduce the following decays

$$|\Gamma(x)| \leq C_\Gamma C_h \langle x\rangle^{-1-\sigma},\ |\partial\Gamma(x)| \leq C'_\Gamma C_h \langle x\rangle^{-2-\sigma},\ |e(x)-\mathrm{Id}_3| \leq C_e C_h \langle x\rangle^{-\sigma} \ . \tag{A.4}$$

The strategy of the proof consists in using the properties of the Dirac operator and fixing a suitable *multiplier* function ψ to establish a *virial identity*. The last step, but still laborious, is to carefully estimate the terms appearing in both sides of the resulting equation and finally deduce the result.

We conclude this section by defining the preliminary function from which the virial identity will arise. For this purpose, recall from Theorem 2.6 that squaring the Dirac equation $i\partial_t u - \mathcal{D}_m u = 0$, we obtain

$$\partial_t^2 u + Lu = 0 \ ,$$

where $L := -\Delta_h - \frac{1}{4}R_h + m^2$ is self-adjoint by [Che73] for the inner product

$$\langle f, g\rangle_h := \int_{\mathcal{M}_h} \overline{f} g = \int \overline{f} g \sqrt{\det(h)}\, d^3x \, .$$

Then, given a a real-valued function of space ψ, we define

$$\Theta(t) := \langle \psi\partial_t u, \partial_t u\rangle_h + \Re\langle (2\psi L - L\psi)u, u\rangle_h \ ,$$

which satisfies the following properties.

Proposition A.2. *Let u be a solution of the Dirac equation, then Θ satisfies*

$$\Theta'(t) = \Re\langle [L,\psi]u, \partial_t u\rangle_h \ ,$$
$$\Theta''(t) = -\Re\langle [L,[L,\psi]]u, u\rangle_h \ .$$

Proof. These identities are a straightforward consequence of the self-adjointness of L. We start by proving the first identity.

$$\Theta'(t) = \langle \psi\, \partial_t^2 u, \partial_t u\rangle_h + \langle \psi \partial_t u, \partial_t^2 u\rangle_h + \Re\langle (2\psi L - L\psi)\partial_t u, u\rangle_h + \Re\langle (2\psi L - L\psi)u, \partial_t u\rangle_h \ .$$

Since u is a solution of the Dirac equation, it also solves $\partial_t^2 u + Lu = 0$. Thus

$$\begin{aligned}\Theta'(t) &= -\langle \psi L u, \partial_t u\rangle_h - \langle \psi\partial_t u, Lu\rangle_h + \Re\langle (2\psi L - L\psi)\partial_t u, u\rangle_h + \Re\langle (2\psi L - L\psi)u, \partial_t u\rangle_h \\ &= -2\Re\langle \psi L u, \partial_t u\rangle_h + \Re\langle (2\psi L - L\psi)\partial_t u, u\rangle_h + \Re\langle (2\psi L - L\psi)u, \partial_t u\rangle_h \\ &= -\Re\langle \partial_t u, \psi L u\rangle_h + \Re\langle (2\psi L - L\psi)u, \partial_t u\rangle_h \\ &= -\Re\langle \psi L u, \partial_t u\rangle_h + \Re\langle L\psi u, \partial_t u\rangle_h = \Re\langle [L,\psi]u, \partial_t u\rangle_h \ .\end{aligned}$$

The second identity then follows by deriving again in time,

$$\begin{aligned}\Theta''(t) &= \partial_t \Re\langle [L,\psi]u, \partial_t u\rangle_h = \frac{1}{2}\partial_t\Big(\langle [L,\psi]u, \partial_t u\rangle_h + \langle \partial_t u, [L,\psi]u\rangle_h\Big) \\ &= \frac{1}{2}\Big(\langle [L,\psi]\partial_t u, \partial_t u\rangle_h + \langle [L,\psi]u, \partial_t^2 u\rangle_h + \langle \partial_t^2 u, [L,\psi]u\rangle_h + \langle \partial_t u, [L,\psi]\partial_t u\rangle_h\Big) \\ &= \langle [L,\psi]\partial_t u, \partial_t u\rangle_h - \langle [L,\psi]u, Lu\rangle_h = -\Re\langle [L,[L,\psi]]u, u\rangle_h \ .\end{aligned}$$

□

A.2 Virial identity

Finally, we can pass to the proof of the virial identity. From the last proposition above, we see that finding the formulas for the derivatives of Θ is a matter of computing explicitly the commutators.

Proposition A.3 (Virial identity). *Let u be a solution of the Dirac equation, then*

$$\begin{aligned}\Theta' &= -\Re\Big(\int_{\mathcal{M}_h} (\Delta_h\psi)\overline{u}\,\partial_t u + 2\int_{\mathcal{M}_h} \nabla_h\psi\cdot\overline{\nabla_h u}\,\partial_t u\Big) \ , \\ \Theta'' &= -\int_{\mathcal{M}_h}\Big(\Delta_h^2\psi - \frac{1}{2}\nabla_h\psi\cdot\nabla_h R_h\Big)|u|^2 + 4\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^2(\psi)^{ij} \\ &\quad -\frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\,D_i u \ ,\end{aligned}$$

where $D^2(\psi)^{ij} := h^{il}h^{kj}\partial_l\partial_k\psi - \Phi^{k,ij}\partial_k\psi$. Hence, we deduce the following virial identity

$$\begin{aligned}-\int_{\mathcal{M}_h}\Big(\Delta_h^2\psi - \frac{1}{2}\nabla_h\psi\cdot\nabla_h R_h\Big)|u|^2 + 4\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^2(\psi)^{ij} - \frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\,D_i u \\ = -\partial_t\Re\Big(\int_{\mathcal{M}_h}(\Delta_h\psi)\overline{u}\partial_t u + 2\int_{\mathcal{M}_h}\nabla_h\psi\cdot\overline{\nabla_h u}\,\partial_t u\Big) \ .\end{aligned}$$

Remark A.2 (Leibniz rule). Before proceeding with the computations, we recall that Δ_h and ∇_h denote respectively the spinorial Laplace–Beltrami operator and the spinorial gradient. Thanks to the properties of covariant derivative and connections, we have the Leibniz rule

$$D_j(\varphi u) = \partial_j \varphi \, u + \varphi D_j u \ ,$$

for any real scalar function φ and any spinor u .

Proof. We start by proving the first identity, which involves the commutator between $L = -\Delta_h - \frac{1}{4}R_h + m^2$ and ψ . Since ψ trivially commutes with R_h and m^2 , we have

$$[L, \psi] = [-\Delta_h, \psi] = -\Delta_h \psi - 2\nabla_h \psi \cdot \nabla_h \ , \tag{A.5}$$

where $\nabla_h f \cdot \nabla_h := h^{ij} \langle D_i f, D_j \cdot \rangle_{\mathbb{C}^4}$. Hence, the first identity follows

$$\begin{aligned} \partial_t \Theta &= \Re \langle [L, \psi] u, \partial_t u \rangle_h = \Re \langle (-\Delta_h \psi - 2\nabla_h \psi \cdot \nabla_h) u, \partial_t u \rangle_h \\ &= -\Re \Big(\int_{\mathcal{M}_h} (\Delta_h \psi) \overline{u} \, \partial_t u + 2 \int_{\mathcal{M}_h} \nabla_h \psi \cdot \overline{\nabla_h u} \, \partial_t u \Big) \ . \end{aligned} \tag{A.6}$$

Now, let us study the second commutator. Using (A.5), we obtain

$$[L, [L, \psi]] = [L, -\Delta_h \psi] + 2[\Delta_h, \nabla_h \psi \cdot \nabla_h] + \frac{1}{2}[R_h, \nabla_h \psi \cdot \nabla_h] =: M_1 + M_2 + M_3 \ .$$

The action of M_1 in the sense of distributions, is given by

$$M_1 = \Delta_h^2 \psi + 2\nabla_h (\Delta_h \psi) \cdot \nabla_h \ ,$$

hence,

$$\langle M_1 u, u \rangle_h = \int_{\mathcal{M}_h} \Delta_h^2 \psi |u|^2 + 2 \int_{\mathcal{M}_h} \nabla_h (\Delta_h \psi) \cdot \overline{\nabla_h u} \, u \ .$$

We focus our attention on the second addend. To understand its structure, we start by computing the symmetric of $\nabla_h \varphi \cdot \nabla_h$, for any φ regular enough. Testing the operator against v, w smooth, using the skew-symmetry of D_j and the Leibniz rule, we obtain

$$\begin{aligned} \langle \nabla_h \varphi \cdot \nabla_h v, w \rangle &= \int h^{ij} \partial_i \varphi \overline{D_j v} \, w \sqrt{\det h} \, d^3 x = - \int \overline{v} \, D_j \Big(\sqrt{\det h} \, h^{ij} \partial_i \varphi \, w \Big) \, d^3 x \\ &= - \int \partial_j \Big(\sqrt{\det h} \, h^{ij} \partial_i \varphi \Big) \overline{v} \, w \, d^3 x - \int \sqrt{\det h} \, h^{ij} \partial_i \varphi \, \overline{v} \, D_j w \, d^3 x \\ &= - \int_{\mathcal{M}_h} \overline{v} \Big((\Delta_h \varphi) \, w + \nabla_h \varphi \cdot \nabla_h w \Big) \ , \end{aligned}$$

since Δ_h on scalar functions coincides with the standard Laplace–Beltrami operator. Hence, for any φ, the symmetric operator is given by

$$(\nabla_h \varphi \cdot \nabla_h)^* = -\Delta_h \varphi - \nabla_h \varphi \cdot \nabla_h \ . \tag{A.7}$$

Choosing $\varphi = \Delta_h \psi$,

$$\langle \nabla_h(\Delta_h \psi) \cdot \nabla_h u, u \rangle = -\int_{\mathcal{M}_h} (\Delta_h^2 \psi)|u|^2 - \langle u, \nabla_h(\Delta_h \psi) \cdot \nabla_h u \rangle_h \ ,$$

from which we deduce

$$\Re \langle \nabla_h(\Delta_h \psi) \cdot \nabla_h u, u \rangle = -\frac{1}{2} \int_{\mathcal{M}_h} (\Delta_h^2 \psi)|u|^2 \ . \tag{A.8}$$

Hence, going back to the contribution of M_1,

$$\Re \langle M_1 u, u \rangle_h = 0 \ .$$

Passing to M_3, its action in the sense of distributions is

$$M_3 = -\frac{1}{2} \nabla_h \psi \cdot \nabla_h R_h \ ,$$

hence, we immediately deduce that

$$\Re \langle M_3 u, u \rangle_h = \langle M_3 u, u \rangle_h = -\frac{1}{2} \int_{\mathcal{M}_h} \nabla_h \psi \cdot \nabla_h R_h \, |u|^2 \ .$$

We now focus on $M_2 = 2[\Delta_h, \nabla_h \psi \cdot \nabla_h]$, which acts in terms of distributions as

$$M_2 = 2\Delta_h(\nabla_h \psi \cdot \nabla_h) - 2(\nabla_h \psi \cdot \nabla_h)\Delta_h \ .$$

Using (A.7) with $\varphi = \psi$ and the self-adjointness of Δ_h, we manipulate the second term to obtain

$$\langle M_2 u, u \rangle = 2\langle \Delta_h(\nabla_h \psi \cdot \nabla_h)u, u \rangle_h + 2\langle \Delta_h u, (\Delta_h \psi)u \rangle_h + 2\langle u, \Delta_h(\nabla_h \psi \cdot \nabla_h)u \rangle_h \ .$$

Taking the real part, we find

$$\Re \langle M_2 u, u \rangle_h = 4\Re \langle \Delta_h(\nabla_h \psi \cdot \nabla_h)u, u \rangle_h + 2\Re \langle \Delta_h u, (\Delta_h \psi)u \rangle_h \ . \tag{A.9}$$

From

$$\begin{aligned} \langle \Delta_h u, (\Delta_h \psi)u \rangle_h &= -\langle \nabla_h u, \nabla_h((\Delta_h \psi)u) \rangle_h \\ &= -\langle \nabla_h(\Delta_h \psi) \cdot \nabla_h u, u \rangle - \int_{\mathcal{M}_h} (\Delta_h \psi) \overline{\nabla_h u} \cdot \nabla_h u \ , \end{aligned}$$

and using (A.8), the second term of (A.9) rewrites

$$2\Re\langle \Delta_h u, (\Delta_h \psi) u\rangle_h = \int_{\mathcal{M}_h} (\Delta_h^2 \psi)|u|^2 - 2\int_{\mathcal{M}_h} (\Delta_h \psi)\overline{\nabla_h u}\cdot \nabla_h u\ .$$

To completely determine the contribution of M_2, it remains to compute the real part of

$$I := 4\langle \Delta_h(\nabla_h \psi \cdot \nabla_h) u, u\rangle_h\ .$$

We have

$$\begin{aligned} I &= -4\langle \nabla_h(\nabla_h\psi\cdot\nabla_h u), \nabla_h u\rangle_h \\ &= -4\int h^{ij}\overline{D_j\Big(h^{kl}\partial_k\psi\, D_l u\Big)}\, D_i u\sqrt{\det h}\, d^3x \\ &= -4\int h^{ij}\partial_j(h^{kl}\partial_k\psi)\overline{D_l u}\, D_i u\sqrt{\det h}\, d^3x - 4\int h^{ij}h^{kl}\partial_k\psi\,\overline{D_j D_l u}\, D_i u\sqrt{\det h}\, d^3x\ . \end{aligned}$$

Recalling that

$$[D_j, D_l] = -\frac{1}{8}R_{jl}{}^{ab}[\gamma_a,\gamma_b] = -\frac{1}{8}e^a{}_\mu e^b{}_\nu R_{jl}{}^{\mu\nu}[\gamma_a,\gamma_b] = -\frac{1}{8}R_{jl}{}^{\mu\nu}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\ ,$$

we can rewrite the last term as

$$\begin{aligned} -4\int & h^{ij}h^{kl}\partial_k\psi\,\overline{D_j D_l u}\, D_i u\sqrt{\det h}\, d^3x \\ &= -4\int h^{ij}h^{kl}\partial_k\psi\,\overline{D_l D_j u}\, D_i u\sqrt{\det h}\, d^3x + \frac{1}{2}\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\overline{[\underline{\gamma_\mu},\underline{\gamma_\nu}]u}\, D_i u \\ &= -4\int h^{ij}h^{kl}\partial_k\psi\,\overline{D_l D_j u}\, D_i u\sqrt{\det h}\, d^3x - \frac{1}{2}\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\, D_i u \\ &=: II + III\ , \end{aligned}$$

where we used that $\overline{[\gamma_a,\gamma_b]} = -[\gamma_a,\gamma_b]$, for $a, b\in\{1,2,3\}$. Hence, we find

$$\begin{aligned} \Re I = -4\Re\int h^{ij}\partial_j(h^{kl}\partial_k\psi)\overline{D_l u}\, D_i u\sqrt{\det h}\, d^3x - 4\Re\int h^{ij}h^{kl}\partial_k\psi\,\overline{D_l D_j u}\, D_i u\sqrt{\det h}\, d^3x \\ - \frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\, D_i u\ . \end{aligned}$$

We are now interested in estimating the term

$$II = -4\int h^{ij}h^{kl}\partial_k\psi\,\overline{D_l D_j u}\, D_i u\sqrt{\det h}\, d^3x\ .$$

Using again the skew-symmetry of the spinorial derivative,

$$\begin{aligned} II &= 4\int \overline{D_j u}\, D_l\Big(\sqrt{\det h}\, h^{ij}h^{kl}\partial_k\psi\, D_i u\Big)\, d^3x \\ &= 4\int \partial_l\Big(\sqrt{\det h}\, h^{ij}h^{kl}\partial_k\psi\Big)\overline{D_j u}\, D_i u\, d^3x - \overline{II}\ . \end{aligned}$$

Thus,

$$\Re II = 2\int \partial_l\Big(\sqrt{\det h}\, h^{ij}h^{kl}\partial_k\psi\Big)\overline{D_j u}\, D_i u\, d^3x \ ,$$

which yields

$$\Re I = -4\Re\int h^{ij}\partial_j(h^{kl}\partial_k\psi)\overline{D_l u}\, D_i u\,\sqrt{\det h}\, d^3x + 2\int \partial_l\Big(\sqrt{\det h} h^{ij}h^{kl}\partial_k\psi\Big)\overline{D_j u}\, D_i u\, d^3x$$
$$-\frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\, D_i u \ .$$

Relabeling the indices, we rewrite it as

$$\Re I = \Re\int_{\mathcal{M}_h}(\overline{D_j u}\, D_i u)D(\psi)^{ij} - \frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\, D_i u \ ,$$

where

$$D(\psi)^{ij} := \frac{2}{\sqrt{\det h}}\partial_l\Big(\sqrt{\det h}\, h^{ij}h^{kl}\partial_k\psi\Big) - 4h^{il}\partial_l(h^{kj}\partial_k\psi) \ .$$

Decomposing

$$\partial_l\Big(\sqrt{\det h}\, h^{ij}h^{kl}\partial_k\psi\Big) = h^{ij}\partial_l\Big(\sqrt{\det h}\, h^{kl}\partial_k\psi\Big) + (\partial_l h^{ij})\Big(\sqrt{\det h}\, h^{kl}\partial_k\psi\Big) \ ,$$

we have

$$D(\psi)^{ij} = 2(\Delta_h\psi)h^{ij} + 2h^{kl}\partial_k\psi\,\partial_l h^{ij} - 4h^{il}\partial_l(h^{kj}\partial_k\psi) \ .$$

Summarizing, recalling that $\Theta'' = -\Re\langle[L,[L,\psi]]u,u\rangle_h$, we have so far

$$\Theta'' = -\int_{\mathcal{M}_h}\Big(\Delta_h^2\psi - \frac{1}{2}\nabla_h\psi\cdot\nabla_h R_h\Big)|u|^2 + 2\int_{\mathcal{M}_h}(\Delta_h\psi)\overline{\nabla_h u}\cdot\nabla_h u$$
$$-\Re\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D(\psi)^{ij} - \frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\, D_i u \ .$$

Manipulating the two terms in the middle, we find

$$\Theta'' = -\int_{\mathcal{M}_h}\Big(\Delta_h^2\psi - \frac{1}{2}\nabla_h\psi\cdot\nabla_h R_h\Big)|u|^2 + \Re\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^1(\psi)^{ij}$$
$$-\frac{1}{2}\Re\int_{\mathcal{M}_h}\partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\, D_i u \ ,$$

where

$$\begin{aligned} D^1(\psi)^{ij} :&= 2(\Delta_h\psi)h^{ij} - D(\psi)^{ij} \\ &= -2h^{kl}\,\partial_k\psi\,\partial_l h^{ij} + 4h^{il}\partial_l(h^{kj}\partial_k\psi) \\ &= 2\partial_k\psi(-h^{kl}\,\partial_l h^{ij} + 2h^{il}\partial_l h^{kj}) + 4h^{il}\,h^{kj}\partial_l\partial_k\psi \ . \end{aligned}$$

Due to the real part in the second addend, we obtain a symmetry in i and j, that is

$$\Re\Big(\overline{D_j u}\, D_i u D^1(\psi)^{ij}\Big) = \Re\Big(\overline{D_i u}\, D_j u D^1(\psi)^{ij}\Big) \ .$$

Therefore, $D^1(\psi)^{ij} = \frac{1}{2}(D^1(\psi)^{ij} + D^1(\psi)^{ji})$ and

$$\Re \int_{\mathcal{M}_h} (\overline{D_j u}\, D_i u) D^1(\psi)^{ij} = 4 \int_{\mathcal{M}_h} (\overline{D_j u}\, D_i u) D^2(\psi)^{ij} \ ,$$

with

$$D^2(\psi)^{ij} := \frac{1}{2}\partial_k \psi(-h^{kl}\,\partial_l h^{ij} + h^{il}\partial_l h^{kj} + h^{jl}\partial_l h^{ki}) + h^{il}\, h^{kj}\partial_l\partial_k\psi \ .$$

Recognizing the Christoffel symbols associated to h,

$$\Phi^{k,ij} = h^{il}\, h^{jm}\Phi^k_{lm} = \frac{1}{2}(h^{kl}\,\partial_l h^{ij} - h^{il}\partial_l h^{kj} - h^{jl}\partial_l h^{ki}) \ ,$$

we rewrite

$$D^2(\psi)^{ij} = -\Phi^{k,ij}\partial_k\psi + h^{il}\, h^{kj}\partial_l\partial_k\psi \ .$$

Finally, relating the two expressions for Θ' and Θ'', we obtain the virial identity

$$\begin{aligned} -\int_{\mathcal{M}_h} \Big(\Delta_h^2\psi - \frac{1}{2}\nabla_h\psi\cdot\nabla_h R_h\Big)|u|^2 + 4\int_{\mathcal{M}_h} (\overline{D_j u} D_i u) D^2(\psi)^{ij} - \frac{1}{2}\Re \int_{\mathcal{M}_h} \partial_k\psi R^{ik\mu\nu}\, \overline{u}[\underline{\gamma_\mu}, \underline{\gamma_\nu}]\, D_i u \\ = -\partial_t \Re\Big(\int_{\mathcal{M}_h} (\Delta_h\psi)\overline{u}\partial_t u + 2\int_{\mathcal{M}_h} \nabla_h\psi\cdot\overline{\nabla_h u}\,\partial_t u\Big) \ . \end{aligned}$$

□

A.3 Choice of the multiplier and final arguments

Now that we have achieved the desired virial identity, we can give the explicit definition of the multiplier ψ. Finally, we outline the remaining arguments to obtain estimate (A.2).

Let us define,

$$\psi_0(x) := \int_0^r \psi_0'(s)\, ds \ , \quad \text{where } \psi_0'(r) := \begin{cases} \dfrac{r}{3} \ , & r \le 1 \ , \\ \dfrac{1}{2} - \dfrac{1}{6r^2} \ , & r > 1 \ , \end{cases}$$

where we committed a small abuse of notation, not distinguishing $\psi_(x)$ and $\psi(r) = \psi(|x|)$. Finally, the multiplier is given by the rescaled function $\psi := \psi_R$

$$\psi_R(r) := R\psi_0\Big(\frac{r}{R}\Big) \ , \quad \text{for } R > 0 \text{ fixed.}$$

Thanks to this choice, the multiplier and its derivatives enjoy many good properties. We limit ourselves to mentioning the one we will use most, that is

$$\psi_R'(r) \le \frac{1}{2} \ , \qquad \text{for any } r \ge 0 \ . \tag{A.10}$$

At this point, the strategy of the authors of [CdS19b] is to deal with the left (LHS) and the right (RHS) hand-sides of the virial identity of Theorem A.3 separately. In particular, one can recognize some terms as "leading" and others as "perturbative", depending on their contribution in the asymptotically flat case. For brevity, we omit the details of the proof, but the goal is to obtain a chain of inequalities of the following type

$$\|u\|_X^2 + \|\nabla_h u\|_Y^2 \lesssim \text{LHS} = \text{RHS} \lesssim \|\mathcal{D}_m u\|^2_{L^2(\mathcal{M}_h)} \ ,$$

which yields, integrating in time and exchanging the integrals, the estimate (A.2).

We conclude with a final argument proving the perturbative nature of the following term in the LHS,

$$III = -\frac{1}{2}\Re \int_{\mathcal{M}_h} \partial_k \psi R^{ik\mu\nu}\, \overline{u}[\underline{\gamma_\mu}, \underline{\gamma_\nu}]\, D_i u \ ,$$

since this analysis is missing in the original article. We stress that, due to the presence of the Riemann tensor, III becomes marginal as $C_h \ll 1$, therefore the Campanato-type estimate and thus Theorem A.1 still hold unaltered.

In particular, we bound the absolute value of III. From the definition of the spacetime-dependent Dirac matrices, we rewrite

$$|III| \leq \frac{1}{2}\int_{\mathcal{M}_h} |\partial_k \psi| |R^{ik}{}_{\mu\nu}\, e_a{}^\mu e_b{}^\nu| |\, \overline{u}[\gamma^a, \gamma^b]\, D_i u| \ .$$

We proceed by estimating separately the factors appearing in the integrand. Using (A.10), we immediately have

$$|\partial_k \psi| \leq |\psi'| \leq \frac{1}{2}$$

and thanks to the (weakened) asymptotically flat decays (A.4),

$$|R^{ik}{}_{\mu\nu} e_a{}^\mu e_b{}^\nu| \lesssim |R^{ik}{}_{\mu\nu}| = |h^{kj} R^i{}_{j\mu\nu}| \lesssim |h^{-1}||R^i{}_{j\mu\nu}| \lesssim |\partial\Gamma| + |\Gamma|^2 \lesssim C'_\Gamma C_h \langle x\rangle^{-2-\sigma} \ .$$

Therefore, noting that

$$|\, \overline{u}[\gamma^a, \gamma^b]\, D_i u| \lesssim |\nabla_h u||u| \ ,$$

if we split $\langle x\rangle^{-2-\sigma} = \langle x\rangle^{-1/2-\sigma/2}\langle x\rangle^{-3/2-\sigma/2}$ and use the Young inequality, we obtain

$$|III| \lesssim \frac{C'_\Gamma C_h}{2}\int_{\mathcal{M}_h} \Big(\langle x\rangle^{-1-}|\nabla_h u|^2 + \langle x\rangle^{-3-}|u|^2\Big) \ , \qquad \text{(A.11)}$$

where we omit σ to lighten the notation.

Let us deal with the two terms separately. Using [CdS19b, Estimate (4.6)], it follows

$$\int_{\mathcal{M}_h} \langle x\rangle^{-1-}|\nabla_h u|^2 \leq \int_{\mathcal{M}_h} \langle x\rangle^{-1-}|\nabla_h u|^2 \lesssim \|\nabla_h u\|_Y^2 \,.$$

On the other hand,

$$\int_{\mathcal{M}_h} \langle x\rangle^{-3-}|u|^2 = \int_{\mathcal{M}_h\cap B_1} \langle x\rangle^{-3-}|u|^2 + \int_{\mathcal{M}_h\cap B_1^c} \langle x\rangle^{-3-}|u|^2 \,.$$

Regarding the first summand, we have

$$\begin{aligned}\int_{\mathcal{M}_h\cap B_1} \langle x\rangle^{-3-}|u|^2 &\leq \int_{\mathcal{M}_h\cap B_1} \frac{1}{|x|^2\langle x\rangle^{1+}}|u|^2 \leq \int_{\mathcal{M}_h\cap B_1} \frac{|u|^2}{|x|^2} \leq 4\|\nabla_h u\|^2_{L^2(\mathcal{M}_h\cap B_1)} \\ &\lesssim \|\nabla_h u\|^2_{\widetilde{Y}} \approx \|\nabla_h u\|^2_Y \,,\end{aligned}$$

where we used respectively [CdS19b, Hardy-type inequality (4.1)] and the norm equivalence (A.3).
The other integral can be controlled using [CdS19b, Estimate (4.8)],

$$\int_{\mathcal{M}_h\cap B_1^c} \langle x\rangle^{-3-}|u|^2 \leq \int_{\mathcal{M}_h\cap B_1^c} \frac{1}{|x|^2\langle x\rangle^{1+}}|u|^2 \lesssim 2\|u\|_X^2 \,.$$

Finally, from (A.11), if we integrate in time, between 0 and T, and exchange the integrals, we find

$$\int_0^T \Big(-\frac{1}{2}\Re\int_{\mathcal{M}_h} \partial_k\psi R^{ik\mu\nu}\,\overline{u}[\underline{\gamma_\mu},\underline{\gamma_\nu}]\,D_i u\Big)dt \gtrsim -C_h\big(\|\nabla_h u\|^2_{YL^2_T} + \|u\|^2_{XL^2_T}\big)\,.$$

Therefore, adding this contribution to the other arguments shown in the article, one obtains an estimate of the form

$$\begin{aligned}(M_1 - C_{III}C_h)\|u\|^2_{XL^2_T} + (M_2 - C_{III}C_h)\|\nabla_h u\|^2_{YL^2_T}& \\ \lesssim \|\mathcal{D}_m u(T)\|^2_{L^2(\mathcal{M}_h)} + \|\mathcal{D}_m u(0)\|^2_{L^2(\mathcal{M}_h)}&\,,\end{aligned}$$

where $M_j := C_j - \widetilde{C}_j C_h$ are explicitly given in [CdS19b, Section 4.5]. Hence, for C_h small enough, the multiplicative constants are positive and the proof is concluded by letting T to infinity and using the conservation of the L^2-norm of $\mathcal{D}_m u$.

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