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Black Hole Microstates in String Theory

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Introduction

The aim of this thesis is the derivation of black hole microstates through the calculation of string amplitudes.

Black holes have always been fascinating and interesting from many points of view, for both astronomical and theoretical reasons. The idea of black holes appears when one studies simple solutions of Einstein's equations of motion in general relativity. One can easily predict the existence of solutions with an *event horizon*. The simplicity of such solutions and the properties related to the event horizon make the study of black holes a fundamental way to inspect the key features of a theory of gravitation.

The analysis of black holes in general relativity reveals interesting –and sometimes surprising– properties; first of all, a number of uniqueness theorems shows that there are not many different possibilities for a black hole geometry. It turns out that a black hole is completely determined by a finite set of quantities, called *charges*; by this name we refer to quantities like mass, angular momentum, electric and magnetic charge. If we think at a black hole as coming from the gravitational collapse of a sufficient quantity of matter, we can say that the creation of an event horizon has the effect of soften all the particular characteristics of the starting situation, and to keep track only of some general features.

On the other hand, there are other interesting properties: black holes in general relativity satisfy a number of laws that resembles the laws of thermodynamics. In particular it seems possible to associate to black holes a temperature and an entropy, depending only on the event horizon and not on what lies in its interior. The entropy of a black hole is proportional to the area of the event horizon; the analogous of the second law of thermodynamics states that the area of the event horizon of an isolated black hole cannot decrease through physical processes. The thermodynamic interpretation is not restricted to an analogy in the three laws; the interpretation of the area as entropy, for example, has a more fundamental origin. In fact, it has been shown that black holes must be interpreted as real thermodynamic objects. A key process is the radiation emission: it has been shown by Hawking ([1]) that this radiation is completely thermalised; it depends only on the temperature of the black hole, which is a local property at the horizon.

Uniqueness theorems and thermodynamic properties are somehow conflictual, and produce a number of puzzles and paradoxes whose solution is completely non-trivial. The most famous is, with no doubt, the black hole information paradox, which is closely related to the thermalisation of the emitted radiation. The paradox emerges when considering the formation and the following evaporation of a black hole. The formation of the event horizon keeps track only of few charges; on the other hand the evaporation produces a radiation which is dependent only on the temperature, which is a quantity related only to the horizon. It seems that the information about the particular initial state is lost after the creation and evaporation of the black hole ([2]). A related problem is the microscopical interpretation of the entropy; we could ask if it is possible to relate it to the number of microstates of a black hole. However, the existence of uniqueness theorems implies that there are not enough possible geometries.

It is natural to think that these problems will find a solution only in the framework of a quantum gravity theory. It is clear that the classical description of black holes given by general relativity is incomplete, and that all the paradoxes will find a solution, once a complete quantum description will be found.

There are essentially two ways in which we can think to solve the information paradox. One may think that the Hawking's calculation for the spectrum of the radiation emitted from a black hole would receive quantum corrections: we expect these corrections to be of order $\frac{l_P}{r_H}$, where l_P is the Planck length and r_H is the radius of the event horizon. Alternatively, it is possible that even the classical description of black holes in general relativity has to be changed; the actual microstate geometries would differ from the classical solution up to the horizon scale, invalidating the assumptions of the Hawking's calculation.

Unfortunately we do not know what the right theory of quantum gravity is; anyway string theory is the only complete and consistent attempt in this direction. A black hole in string theory is a bound state of strings and multidimensional objects, called *branes*. The number of possible vibrational modes of strings and branes is responsible for the existence of a large number of states corresponding to the same classical solution (which is a supergravity solution). Strominger and Vafa have shown ([3]) that there is a perfect matching between the microscopic count of states and the entropy calculated looking only at the properties of the event horizon of the classical solution.

This is with no doubt a great success of string theory, but leads to more questions: how do the geometries of these microstates look like? How do these microstates help to solve the information paradox? Many attempts have been made in order to find an answer to these important questions, but they are still open problems.

The explicit form of some microstates have been calculated, starting from simple cases which, to be precise, are not black holes. A real black hole, in fact, must be characterized by at least three charges. Solutions with only one or two different charges, instead, have a vanishing horizon area. Anyway from these systems one can learn features that are valid also in the black hole case. It has been shown by Mathur ([4]), in the two charges case, that the microstates can be different from the corresponding classical solution at scales comparable with the event horizon. If this is true the classical picture of black holes is replaced by a description in which quantum gravity effects are not confined around the singularity, but extend up to the horizon. The information of the system is not constrained to lie at the center of the hole, but it is spread in all the interior of the horizon. The region between the singularity and the horizon is no more made up by empty space, but has a more complicated description; we say that it is a *fuzzball*. The role of the classical solution is to represent a statistical average of all the actual quantum microstates. If this is true also for black hole microstates (i.e. in the three-charge case), we can see that the information paradox can vanish. the derivation of the thermalisation properties of the emitted radiation, in fact, was based on the assumption that the information and the quantum effects are located in a small size region around the singularity.

The derivation of microstates, or more precisely of their large distance behaviour, is possible in string theory through the calculation of the amplitude of the emission of a graviton from the bound state of branes corresponding to the hole. This procedure has been applied to many systems, but a complete characterization and classification of the microstates of black holes is still lacking. The goal of this thesis is to summarize which progresses have been made in this direction, and to make an explicit derivation of a new type of microstate. In the two-charge case, this method will not reveal anything new: a complete characterization of the microstates of this system is already known. Anyway, we will understand exactly which is the microscopical interpretation of each microstate. On the other hand, the three-charge black hole is a much more complicated system; there is not a systematic way to characterize its microstates. Therefore the calculation of string amplitudes is one of the few methods to obtain some of the microstates (at least their large distance behaviour).

In chapter 1 we recall how black holes arise in general relativity; then we describe the uniqueness theorems and the thermodynamic laws. In chapter 2 we see how black holes are described in supergravity theories; for this reason we introduce briefly the key concept of supersymmetry and supergravity, and make the explicit derivation of a black hole solution in five dimension, the Strominger–Vafa black hole. In chapter 3 we introduce string theory, and we see why it is a consistent theory of quantum gravity. In chapter 4 we explain what is the microscopical description of black holes and sketch the explicit count of the microstates of the Strominger–Vafa black hole. In chapter 5 we explain how we can find the explicit form of a microstate through the calculation of a string amplitude: we review the progresses made in this direction in the last 15 years. In the end, in chapter 6 we do the explicit derivation of a new type of microstate of the Strominger–Vafa black hole, and compare it to a known supergravity solution.

Chapter 1

Black holes in general relativity

1.1 Black holes in four dimensions

1.1.1 The Schwarzschild solution

The action that describes Einstein's general relativity is the so-called Einstein–Hilbert action. It involves the Ricci scalar R, which is derived from the Rienmann tensor through $R = g^{\mu\nu}R_{\mu\nu}$. The action should be a Lorentz scalar invariant under diffeomorphisms, involving second derivatives of the metric. In fact it is

$$I_E = I_M + \frac{1}{16\pi G}\int d^4x\,\sqrt{g(x)}R\,, \tag{1.1}$$

where I_M represents the action of all fields different from the metric. The corresponding equation of motion are the Einstein field equation; they are a set of 10 non-linear, coupled differential equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{1.2}$$

where the energy-momentum tensor $T_{\mu\nu}$ is defined by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{g(x)}}\frac{\delta I_M}{\delta g^{\mu\nu}(x)}\,. \tag{1.3}$$

The purpose of solving Einstein equation seems to be quite difficult. The easiest particular solution to such a set of equations is known since 1916, when Karl Schwarzschild discovered it. It describes a spherically symmetric vacuum spacetime; in vacuum ($T_{\mu\nu} = 0$) the 10 equations become quite simpler:

$$R_{\mu\nu} = 0\,. \tag{1.4}$$

It turns out that an isotropic metric in vacuum, must also be static (this result is known as Birkhoff's theorem). The resulting metric is therefore unique (it depends only on the parameter M) and reads (see for example [5]):

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{1.5}$$

We will see later that M can be interpreted as a mass. This solution is a good approximation for the spacetime outside a massive object, such as a star. We see that the metric becomes divergent at $r = 0$ and $r = r_s := 2GM$ (Schwarzschild radius): normally this radius is quite small with respect to the dimension of the object generating the Schwarzschild solution, and so it lies in the interior of the object. But the Schwarzschild solution make sense only in vacuum:

this means that the divergence does not bother us, because the Schwarzschild geometry does not represent the interior of a massive object.

A very massive and compact object can have a Schwarzschild radius bigger than its dimension. Such an object is called a *black hole*. In this case the locus of points lying at $r = r_s$ is called *event horizon*. The divergence of the metric at the event horizon is not physical, but it is only due to the particular choice of coordinates we have done. Rather than looking at the metric components, we could detect a physical singularity when the curvature becomes infinite. What this exactly means is hard to say, but we can check that the scalar quantity

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12G^2M^2}{r^6} \tag{1.6}$$

diverges only at the point $r = 0$ ([6]). One can explicitly see that the event horizon does not represent a true singularity, looking at the metric in the so-called Eddington–Finkelstein coordinates:

$$v = t + r* = t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right| . \tag{1.7}$$

The metric (1.5) becomes:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{1.8}$$

The event horizon is again at $r = 2GM$, but now the metric coefficient becomes divergent only at $r = 0$.

1.1.2 Event horizons, Killing horizons and surface gravity

There is a precise way to characterise symmetries of a metric: they are given by the existence of Killing vectors. The Schwarzschild metric possesses three (spacelike) Killing vectors corresponding to the spherical symmetry, and a (timelike) Killing vector corresponding to the invariance under time translation. A metric is stationary if it has a timelike Killing vector at infinity; furthermore a metric is static, if it posses a timelike Killing vector (at infinity) which is orthogonal to a family of hypersurfaces. In the case of the Schwarzschild metric the timelike Killing vector field $\zeta = \frac{\partial}{\partial t}$ is orthogonal to all the hypersurfaces $t =$ const: thus the metric (1.5) is indeed static, not only stationary.

An *event horizon* is, roughly speaking, a surface that divides a spacetime into an exterior part which can not be affected by events happening in the interior part. For a spherically symmetric spacetime it coincide with the surface where r switches from being a spacelike to a timelike coordinate; we take this as a definition for event horizon. For the Schwarzschild metric the event horizon coincide with the surface where $g^{rr} = 0$. In fact:

$$\left(\frac{\partial}{\partial r}\right)^{\mu} = (0, 1, 0, 0) \quad \Leftrightarrow \quad \left|\frac{\partial}{\partial r}\right|^2 = -g_{rr} = -(g^{rr})^{-1} . \tag{1.9}$$

We thus see that the surface $r = r_s$ is indeed an event horizon.

There is another important concept of horizon in general relativity: the *Killing horizon*. A Killing horizon is a null hypersurface, to which a Killing vector field is normal. For the Schwarzschild metric, the Killing horizon corresponding to the vanishing of the norm of $\frac{\partial}{\partial t}$ is again the surface $r = r_s$. In fact:

$$\zeta^{\mu} = \left(\frac{\partial}{\partial t}\right)^{\mu} = (1, 0, 0, 0) \quad \Leftrightarrow \quad |\zeta|^2 = -g_{tt} = \left(1 - \frac{2MG}{r}\right) . \tag{1.10}$$

There is an important quantity associated to the Killing horizon of a stationary black hole, the *surface gravity*. It is a measure of the acceleration experienced by a test particle close to the horizon. For a stationary black hole there exist a Killing vector field χ^μ normal to the horizon. A possible definition for the surface gravity κ is ([8])

$$\kappa^2 = D^\mu|\chi|D_\mu|\chi| = \partial^\mu|\chi|\partial\mu|\chi| \,, \tag{1.11}$$

where $\partial^\mu|\chi|$ is calculated at the horizon. In the case of the Schwarzschild metric, we can take $\chi^\mu = \zeta^\mu = (1,0,0,0)$. Thus:

$$|\chi| = \sqrt{-\chi^\mu\chi_\mu} = \sqrt{-g_{tt}} = \sqrt{1-\frac{2GM}{r}} \,. \tag{1.12}$$

We can now calculate:

$$\partial^\mu|\chi|\partial\mu|\chi| = g^{rr}(\partial_r|\chi|)^2 = \left(1-\frac{2GM}{r}\right)\left(\frac{1}{2|\chi|}\frac{2GM}{r^2}\right)^2 = \frac{G^2M^2}{r^4} \,. \tag{1.13}$$

Calculating this quantity at the Killing horizon $r = 2GM$ we obtain:

$$\kappa = \frac{GM}{4G^2M^2} = \frac{1}{4GM} \,. \tag{1.14}$$

We will later see that this quantity can be related to the temperature of the black hole.

1.1.3 The Reissner–Nordstrøm metric

The Schwarzschild solution is the simplest solution of the Einstein equations. We now turn to the analysis of another solution, corresponding to a charged black hole. Such a solution should exist, because in principle nothing prevents us from adding some charge to an object generating a Schwarzschild metric. Anyway we expect the resulting solution to be stationary and static. The key difference is that a charge produce a non-zero energy-momentum tensor. This means that the solution we are looking for is not a vacuum solution. We know that

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \,. \tag{1.15}$$

We restrict to the case of only electric charge (a magnetic contribution gives rise to similar results), which means that the only non-zero contribution to $F_{\mu\nu}$ is $F_{tr} = -F_{rt} = -\frac{Q}{r^2}$. We should solve Einstein's equations (1.2), which are now coupled to Maxwell's equations:

$$g^{\mu\nu}D_\mu F_{\nu\rho} = 0 \,, \qquad D_{[\mu}F_{\nu\rho]} = 0 \,. \tag{1.16}$$

We state here only the final result, the so called Reissner–Nordstrøm metric ([6]):

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \quad \text{where} \quad \Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \,. \tag{1.17}$$

We see that the solution is quite similar to the Schwarzschild one, and reduces to it in the limit $Q \to 0$: the only difference is an additional term in the expression of g_{tt} (and g^{rr}). There are two surfaces where the metric coefficient g_{rr} becomes singular, corresponding to the vanishing of Δ:

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2} \,. \tag{1.18}$$

In the following we will ignore the case $GM^2 < Q^2$: this geometry would have the singularity at $r = 0$ not hidden by an event horizon. This would be in contrast with the cosmic censorship conjecture, which states that we do not observe in nature any naked singularity. Anyway this situation ($GM^2 < Q^2$) sounds unreasonable, because it would roughly correspond to an object with the contribution to the energy coming from the charge bigger than the total energy. Let us look at the case $GM^2 > Q^2$. Now we have two interesting concentric surfaces, which are both event horizons. A particle coming from infinity, and passing through r_+, must continue decreasing its distance from the center; in fact r is a timelike coordinate in the region between the two horizons. At $r < r_-$ a particle could start going away from the origin, because r is again a spacelike coordinate. When it arrives again in the region between the two horizons, this particle must keep increasing r, until it reaches r_+. The particle must then exit the outer event horizon, emerging in an outside region which is actually different from the starting region; from this point of view of the Reissner–Nordstrøm hole behave as a white hole. In the limit $Q \to 0$ we have that the inner event horizon reduces to a point ($r_- \to 0$) and $r_+ \to 2GM$; we thus recover the structure of the Schwarzschild black hole.

There is a particular case, when $GM^2 = Q^2$. In this situation there is just one event horizon (r_+ coincide with r_-), and the coordinate r is never timelike. The resulting black hole is called *extremal*. Thanks to their simplicity, extremal black holes are useful for thought experiments (even if they seem to be quite non-physical): we will use them in the following.

We observe also that $r = r_\pm$ are both Killing horizons, where the norm of the Killing vector $\zeta = \frac{\partial}{\partial t}$ vanishes. We are mostly interested in the properties of the external event horizon $r = r_+$ because it is the only "accessible" from the exterior. The surface gravity is given by:

$$\begin{aligned}
\chi^\mu &= \left(\frac{\partial}{\partial t}\right)^\mu = (1,0,0,0) \quad \Leftrightarrow \quad |\chi|^2 = -g_{tt} = \Delta\,, \\
\partial_r|\chi| &= \partial_r\sqrt{\Delta} = \frac{1}{2\sqrt{\Delta}}\cdot 2\left(\frac{GM}{r^2} - \frac{GQ^2}{r^3}\right)\,, \\
\kappa^2 &= g^{rr}(\partial_r|\chi|)^2 = \left(\frac{GM}{r_+^2} - \frac{GQ^2}{r_+^3}\right) = \frac{G^2M^2 - GQ^2}{r_+^4}\,.
\end{aligned} \tag{1.19}$$

We see that the surface gravity can be written as $\kappa = \frac{r_+ - r_-}{2r_+^2}$; from this expression it is straightforward to verify that it reduces to the surface gravity of a Schwarzschild black hole in the limit $r_+ \to r_s = 2GM,\ r_- \to 0$. We also observe that the extremal black hole is characterised by $\kappa = 0$.

In the extremal case, it is useful to express the metric using a change of coordinates ([7]): let r_0 be the position of the two coinciding event horizon, and let us define $r' = r - r_0$. In this new coordinates the event horizon is at $r' = 0$ and the metric reads:

$$ds^2_{\text{ext}} = -H(r')^{-2}dt^2 + H(r')^2(dr'^2 + r'^2 d\Omega^2) \quad \text{where} \quad H(r') = \left(1 + \frac{r_0}{r'}\right)\,. \tag{1.20}$$

These new coordinates (t, r', θ, ϕ) are called isotropic: the reason is that in these coordinates the metric shows explicitly an $SO(3)$ symmetry. We will see later, when we will deal with black holes in supergravity, that the presence of this symmetry in the extremal case can be seen as the presence of a supersymmetry.

1.1.4 The Kerr metric

It is much more difficult to derive a solution corresponding to a rotating black hole, even if we restrict to the vacuum case. We look for a stationary solution, but not static. The resulting metric was discovered by Kerr only in 1963, and reads ([6]):

$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta d\phi - dt)^2\,, \qquad (1.21)$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \quad \text{and} \quad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta\,. \qquad (1.22)$$

The parameter a is a measure of the rotation of the black hole; in the limit $a \to 0$ the metric (1.21) reduces to the Schwarzschild metric (1.5). We note also that in the limit $M \to 0$, (1.21) reduces to flat spacetime, even if not in ordinary polar coordinates, but in ellipsoidal coordinates. This means that surfaces at $r = \text{const}$ are not spheres, but ellipsoids. The curvature singularity corresponds now to $\rho = 0$, and not $r = 0$. We remember that $r = 0$ is not a point in space, but a disk. The condition $\rho = 0$ is equivalent to $r = 0$, $\theta = \frac{\pi}{2}$, which corresponds to a ring at the edge of the disk: the curvature singularity is not confined to a single point, but spread over a ring.

The metric coefficients are independent from both t and ϕ: thus we have two manifest Killing vectors, that we call $\zeta^\mu = \frac{\partial}{\partial t} = (1,0,0,0)$ and $\eta^\mu = \frac{\partial}{\partial\phi} = (0,0,0,1)$. ζ^μ expresses the stationarity of the solution; the fact that $g_{t\phi} \neq 0$ implies that ζ^μ is not orthogonal to the hypersurfaces $t = \text{const}$. It turns out that there is not any hypersurface orthogonal to ζ^μ, which means that the Kerr metric is non-static. The fact that η^μ is a Killing vector expresses the axial symmetry of the metric, where the axis of symmetry is the axis of rotation of the black hole.

As for the Reissner–Nordstrøm metric, the Kerr metric possesses two event horizons, given by the vanishing of Δ:

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}\,. \qquad (1.23)$$

As before we exclude the case which would lead to a naked singularity ($G^2M^2 < a^2$), and concentrate on the case $G^2M^2 > a^2$; the extremal black hole, given by $G^2M^2 = a^2$, is just a particular case of the latter. The two event horizons behave exactly like the event horizons of the Reissner–Nordstrøm black hole, and a test particle would follow similar geodesics.

The key feature that characterises a Kerr black hole is the fact that there is a Killing horizon which do not coincide with any of the two event horizon. This is given by the vanishing of the norm of the Killing vector ζ^μ:

$$\zeta^\mu\zeta_\mu = g_{tt} = -1 + \frac{2GMr}{\rho^2} = -\frac{1}{\rho^2}(\Delta - a^2\sin^2\theta)\,. \qquad (1.24)$$

We see that this norm does not vanish at the event horizons (where $\Delta = 0$). The norm actually vanishes at two surfaces, given by

$$r_{1,2} = GM \pm \sqrt{G^2M^2 - a^2\cos^2\theta}\,. \qquad (1.25)$$

The surface $r = r_2$ lies entirely inside the inner event horizon: we do not consider it. The surface $r = r_1$, on the other end, lies outside the outer event horizon (and touches it at the poles, where $\theta = 0, \pi$). A consequence of this fact is that there exist a region between r_1 and r_+, called *ergosphere*, where a test particle would inevitably keep moving in the direction of the rotation of the black hole. The situation is shown in picture 1.1.

This feature of the Kerr black hole has important consequences: the fact that ζ^μ is spacelike in the ergosphere means that the energy of a particle $E = -\zeta_\mu p^\mu$ can be negative. We can then imagine a process by which we can extract energy from the hole; this is known as *Penrose process*. We start from a composite object outside the Killing horizon with momentum given by

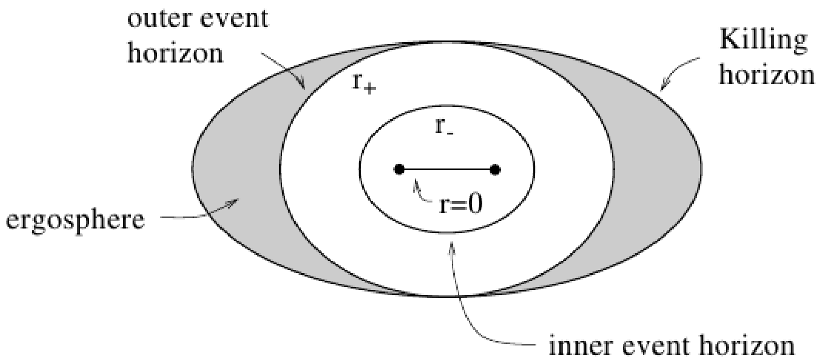

Figure 1.1: Horizons of a Kerr black hole.

$p_0^\mu = p_1^\mu + p_2^\mu$, and energy $E_0 = E_1 + E_2$. Let us suppose that this object enters the ergosphere, and here separates into its two parts. It is possible that the part (2) has energy $E_2 < 0$ and move toward the black hole finally entering its event horizon, and that the part (1) (with positive energy $E_1 > E_0$) follows a geodesic going again outside the ergosphere. Obviously this process does not violate energy conservation: the object (2) must move in a direction opposite to the rotation of the hole; the gain in mass due to the presence of the object (2) is opposed by a reduction of the angular momentum of the hole. The nett result is a decrease of the total energy of the black hole.
It is worth noting that even the outer horizon is a Killing horizon. This can be seen defining a Killing vector χ^μ in the following way:

$$\chi^\mu = \zeta^\mu + \Omega_H \eta^\mu \,. \tag{1.26}$$

This vector is null at the outer horizon, provided that the value of Ω_H is $\Omega_H = \frac{a}{r_+^2 + a^2}$. This value can be interpreted as the angular velocity of the black hole evaluated at the horizon.

We could calculate the the surface gravity at the horizon of the Kerr black hole, in the same way we have done for the Schwarzschild or Reissner–Nordstrøm black holes. The calculation would be much more complicated: we only state that the result is the following:

$$\kappa = \frac{\sqrt{G^2 M^2 - a^2}}{2GM(GM + \sqrt{G^2 M^2 - a^2})} \,. \tag{1.27}$$

We see that in the limit $a \to 0$ we recover the surface gravity of the Schwarzschild black hole, while in the extremal limit $G^2 M^2 = a^2$ the surface gravity vanishes. This is a common feature with the Reissner–Norstrøm extremal black hole.

1.2 Charges of a black hole

The goal of this section is to find practical rules in order to derive interesting quantities characterizing a black hole, starting from its geometry. We take into account black holes in a spacetime of generic dimension ($D \geq 4$). This is because in the following we will be interested also in higher dimensional black holes.

1.2.1 Mass and angular momentum

Starting from the generalisation of the Einstein–Hilbert action (1.1) for a generic dimension D,

$$I_E = \frac{1}{16\pi G} \int \mathrm{d}^D x \, \sqrt{g(x)} R \,, \tag{1.28}$$

we derive the Einstein field equations, which look the same for any dimension D. In order to define quantities like mass and angular momentum, we suppose that our solution can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{1.29}$$

where $h_{\mu\nu}$ is the perturbation from flat spacetime. The metric is normally chosen to be in the so-called harmonic gauge, i.e.:

$$\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h^\alpha{}_\alpha = 0 \,, \tag{1.30}$$

where the indices are raised and lowered with the flat metric η. With this assumption, Einstein equations become, at the first linear order,

$$\nabla^2 h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{D-2} T^\alpha{}_\alpha \right) =: -16\pi G \overline{T}_{\mu\nu} \,, \tag{1.31}$$

where ∇^2 is the D-dimensional Laplacian. The solution to such equation is known to be

$$h_{\mu\nu}(x) = \frac{16\pi G}{(D-3) A_{D-2}} \int \frac{\overline{T}_{\mu\nu}(y)}{|x-y|^{D-3}} d^D y \,. \tag{1.32}$$

Here A_{D-2} represents the area of a unit sphere of dimension $D-2$. The trace of the energy-momentum tensor is dominated by the energy density, so we can approximate: $T^\alpha{}_\alpha \simeq -T_{00}$. We now recall that the mass (or total energy) of the system is defined by

$$M = \int d^D x \, T_{00}(x) \,. \tag{1.33}$$

We always assume that we are considering the system in its rest frame, i.e.

$$\int d^D x \, T_{0i}(x) = 0 \,. \tag{1.34}$$

In order to define the angular momentum, we suppose that the origin of the coordinates coincide with the center of mass of the system; thus the angular momentum is given by

$$J_{\mu\nu} = \int d^D x \, (x_\mu T_{\nu 0} - x_\nu T_{\mu 0}) \,. \tag{1.35}$$

Using also the conservation of the energy-momentum tensor we can derive the following expressions:

$$\begin{aligned} J_{0i} &= 0 \,, \\ J_{ij} &= 2 \int d^D x \, (x_i T_{j0} - x_\nu T_{\mu 0}) \,. \end{aligned} \tag{1.36}$$

One should now expand the solution (1.32) in the region $r = |x| \gg |y|$, and using the definitions of mass and angular momentum one can compare this quantities with the metric coefficients. We write here the final expressions ([9]):

$$
\begin{aligned}
h_{00} &\simeq \frac{16\pi G}{(D-2)A_{D-2}} \frac{M}{r^{D-3}}\,, \\
h_{ij} &\simeq \frac{16\pi G}{(D-2)(D-3)A_{D-2}} \frac{M}{r^{D-3}} \delta_{ij}\,, \\
h_{0i} &\simeq -\frac{8\pi G}{A_{D-2}} \frac{x^k J_{ki}}{r^{D-1}}\,.
\end{aligned}
\tag{1.37}
$$

The Schwarzschild and Reissner–Nordstrøm solutions (in dimension $D = 4$) do not satisfy the harmonic gauge condition (1.30). Anyway, we can express the two solutions in harmonic gauge by means of a shift of the coordinates r; this implies that we can safely use (1.37) in order to derive the mass and the angular momentum. In both cases the parameter M appearing in the solution is indeed the mass, and the angular momentum vanishes. In the case of the Kerr metric, we have again that the mass coincide with M, and the angular momentum is $J = |J_{xy}| = Ma$.

1.2.2 Electric Charge

The action for a Maxwell field in D dimensions reads:

$$
I = -\frac{1}{4} \int d^D x \, \sqrt{g(x)} F^{\mu\nu} F_{\mu\nu}\,. \tag{1.38}
$$

The electric charge Q is completely defined by $F_{\mu\nu}$ through:

$$
F_{0i} = -Q \frac{x^i}{r^{D-1}}\,. \tag{1.39}
$$

This is the generalisation to a generic dimension of the formula we have used to derive the Reissner–Nordstrøm metric. The presence of an electric charge Q gives a non-trivial contribution to the energy-momentum tensor $T_{\mu\nu}$, and thus influences the metric. It is a general feature that contributions coming from the charge are of higher order in r^{-1} than those coming from the mass; this can be seen, for example, from the metric (1.17).

1.2.3 Uniqueness theorems

In section (1.1) we have seen that every isotropic solution of the Einstein equations in vacuum must be static, and thus it must be the Schwarzschild metric. This means that every isotropic vacuum solution is completely determined by a single parameter M. This is an example of a family of uniqueness theorems, that state that there is only a very limited family of stationary, asymptotically flat black hole solution to the Einstein equations ([10]).

The Schwarzschild metric is the only static vacuum solution, while the Kerr metric is the only stationary vacuum solution. If we add an electric charge Q, the Reissner–Nordstrøm metric is the unique static solution, while a suitable generalisation of the Kerr metric (the Kerr-Newman metric) is the unique stationary solution. We can also consider the presence of magnetic charges; the resulting metrics are completely similar to the ones with electric charge.

Summing up, we can say that a stationary black hole can be completely determined by a finite set of quantities (mass, angular momentum, electric and magnetic charges): in the following we refer to these quantities with the common name *charges*.

1.3 Black hole thermodynamics

During the study of the Kerr geometry we have learned that energy can not only be absorbed by a black hole; it is possible to have processes where a black hole acts as an intermediary, exchanging energy with other objects. From this trivial observation we may be led to think whether it is possible to consider a black hole as a thermodynamic object. In this section we will examine if we can define a temperature for a black hole, and if there exist some laws which may be considered as the analogous of the thermodynamic laws.

1.3.1 Black hole temperature

In order to define a temperature for black holes, we now schematically review the key features of a finite temperature field theory. Let us suppose to have a theory with Lagrangian $\mathcal{L}$ in a 4 dimensional spacetime (the extension to higher dimensions is straightforward). If $|\phi_a\rangle$ is a basis for the Hilbert space of our field theory, the partition function Z is given by

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int d\phi_a \, \langle\phi_a|e^{-\beta H}|\phi_a\rangle \, . \tag{1.40}$$

Here H is the Hamiltonian of our theory, and β is related to the temperature by $\beta = \frac{1}{k_B T}$. Let us remember that the following Green's function can be expressed via a path integral:

$$\langle\phi_a|e^{-\frac{iHt}{\hbar}}|\phi_b\rangle = \int_{\phi(0,x)=\phi_a(x)}^{\phi(t,x)=\phi_b(x)} [d\phi] \, e^{-\frac{i}{\hbar}\int_0^t dt' \int d^3x \, \mathcal{L}} \, . \tag{1.41}$$

Defining now $\tau = it$ we have:

$$\langle\phi_a|e^{-\frac{iHt}{\hbar}}|\phi_b\rangle = \langle\phi_a|e^{-\frac{\tau H}{\hbar}}|\phi_b\rangle = \int_{\phi(0,x)=\phi_a(x)}^{\phi(-i\tau,x)=\phi_b(x)} [d\phi] \, e^{-\frac{1}{\hbar}\int_0^\tau d\tau' \int d^3x \, \mathcal{L}} \, . \tag{1.42}$$

We can now rewrite the partition function as

$$\begin{aligned} Z &= \int d\phi_a \, \langle\phi_a|e^{-\beta H}|\phi_a\rangle = \int d\phi_a \int_{\phi(0,x)=\phi_a(x)}^{\phi(-i\beta\hbar,x)=\phi_a(x)} [d\phi] \, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x \, \mathcal{L}} \\ &= \int_{\phi(0,x)=\phi(-i\beta\hbar,x)} [d\phi] \, e^{-\int_0^\beta d\tau \int d^3x \, \mathcal{L}} = \int_{\phi(0,x)=\phi(-i\beta\hbar,x)} [d\phi] \, e^{-i\int_0^{-i\beta} dt \int d^3x \, \mathcal{L}} \, . \end{aligned} \tag{1.43}$$

We see that the partition function at temperature T is simply the path integral over configurations with euclidean time $\tau = it$ periodic with period $\beta\hbar$. We can thus argue that we should associate a temperature $T = \frac{1}{k_B \beta}$ to a solution with a periodic imaginary time with period $\beta\hbar$.

We now want to inspect whether we can assign a certain temperature to the simple black hole solutions we know. Let us start with the Schwarzschild geometry (1.5), and perform the change of variable $\tau = it$. The resulting metric is

$$ds^2 = \left(1 - \frac{2MG}{r}\right) d\tau^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \, . \tag{1.44}$$

From this expression one can not easily say if this solution has some periodicity in τ. In order to make it manifest, we expand around $r = r_s$, introducing the parameter $\epsilon = r - 2GM$. We make this because we want to inspect the geometry barely outside the black hole horizon, which should be the region that characterises all black hole properties. Assuming ϵ small and positive, and taking only leading order terms, we obtain

$$ds^2 \simeq \frac{\epsilon}{2GM}d\tau^2 + \frac{2GM}{\epsilon}dr^2 + r_s^2 d\Omega^2 = \frac{\epsilon}{2GM}d\tau^2 + \frac{2GM}{\epsilon}d\epsilon^2 + r_s^2 d\Omega^2 \,, \tag{1.45}$$

where we have expressed the trivial angular part as $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We now perform a further change of variable, defining $\rho = \sqrt{2\epsilon}$. Thus:

$$ds^2 \simeq \frac{\rho^2}{4GM}d\tau^2 + 4GMd\rho^2 + r^2 d\Omega^2 = 4GM\left(d\rho^2 + \frac{\rho^2}{16G^2M^2}d\tau^2\right) + r^2 d\Omega^2 \,. \tag{1.46}$$

Remembering that a euclidean space in 2 dimensions can be expressed in polar coordinates by the metric $ds^2 = dr^2 + r^2 d\phi^2$ with the period ϕ being 2π, we easily see that we must assign to τ the periodicity $2\pi \cdot 4GM = 8\pi GM$ in order to make the space smooth at $r = r_s$. We then state that to a Schwarzschild black hole with mass M can be associated a temperature

$$T = \frac{\hbar}{8\pi k_B GM} \,. \tag{1.47}$$

We now turn to the Reissner–Nordstrøm black hole. Using the euclidean time, and remembering the expression for $r_\pm$, we can write the metric (1.17) as

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r^2}d\tau^2 + \frac{r^2}{(r - r_+)(r - r_-)}dr^2 + r^2 d\Omega^2 \,. \tag{1.48}$$

We consider the geometry outside the outer event horizon, and expand it using the parameter $\epsilon = r - r_+$. Keeping only leading order terms we get

$$ds^2 \simeq \frac{(r_+ - r_-)\epsilon}{r_+^2}d\tau^2 + \frac{r_+^2}{(r_+ - r_-)\epsilon}d\epsilon^2 + r_+^2 d\Omega^2 \,. \tag{1.49}$$

We see that the expression has the same structure as the Schwarzschild metric, with the substitution $\frac{1}{2GM} \to \frac{r_- - r_-}{r_+^2}$: we can then conclude that τ has periodicity $4\pi \cdot \frac{r_+^2}{r_+ - r_-}$. Finally we find that to a Reissner–Nordstrøm black hole is associated a temperature

$$T = \frac{\hbar(r_+ - r_-)}{4\pi k_B G r_+^2} = \frac{2\hbar\sqrt{M^2 - Q^2}}{4\pi k_B G r_+^2} \,. \tag{1.50}$$

We can make two simple observation: firstly, as expected, the temperature reduces to the Schwarzschild temperature in the limit $Q \to 0$. Furthermore we can observe the temperature of the extremal black hole vanishes.

We can now observe an interesting fact: it seems that there is a proportionality between temperature and surface gravity, i.e.: $T = \frac{\hbar\kappa}{2\pi k_B}$. It turns out that this is true even for the Kerr black hole. This correspondence leads to the following properties (which we will not prove, but sound reasonable thanks to what we know):

- although κ is defined locally on the Killing horizon, it is constant (this is true for static and axisymmetric black holes). This means that the temperature of a black hole is constant over the horizon;
- the surface gravity (and hence the temperature) of a black hole is always non-negative, and vanishes only in the extremal limit.

The first of these two properties sounds similar to the zeroth law of thermodynamics, which states that the temperature is uniform in a system in thermal equilibrium. This is a good confirmation of the validity of our definition of temperature for a black hole.

It may seem a bit unclear what is the physical meaning of this derivation. Anyway there is another way to calculate the temperature of a black hole, and it takes into account quantum effects near the horizon. One can argue that the creation of pairs of particle in the proximity of the horizon gives rise to an emission of radiation from the hole; Hawking ([1]) has shown that this radiation is completely thermalised. In the simple cases we have taken into account, this Hawking temperature coincides exactly with the one we have derived. We thus see that the temperature of a black hole, which is the temperature of the emitted radiation, is a local property on the horizon; this observation will be important in the following.

1.3.2 First law of black hole thermodynamics

In this section we will derive the form of the so-called first law of black hole thermodynamics. We will see that there is an evident analogy with the first law of thermodynamics. Furthermore, using our previous definition of temperature of a black hole, we will be able to identify a quantity playing the role of entropy. We will follow the procedure in [11] and [13].

We are going to derive the first law in a very general framework. Let us simply assume that we have a gravity theory in D dimensions, deriving from a diffeomorphisms invariant Lagrangian $\mathcal{L}$. For practical reasons we do not deal directly with the scalar density $\mathcal{L}$, but with its Hodge-dual D-form $\mathbf{L} = *\mathcal{L}$; in the same way other quantities will appear in their dualised version, as differential forms. Let us suppose that the Lagrangian locally depends on the metric $g_{\mu\nu}$ and possibly on other dynamical fields; we denote all the fields, including the metric, with ϕ. Performing a first order variation of the fields ($\phi \to \phi + \delta\phi$) we get

$$\delta\mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}(\phi, \delta\phi)\,, \tag{1.51}$$

where summation over the fields and indices contractions are understood. Integration over the spacetime yields to the equation of motion $\mathbf{E} = 0$, while the $(D-1)$-form $\mathbf{\Theta}$ is the boundary term, which is locally constructed from the fields and their first order variation. It should be noted that, thanks to the diffeomorphisms invariance, $\mathbf{\Theta}$ is determined only up to the addition of a closed form.

We can now perform a further variation of the fields, and then define the symplectic current $\mathbf{\Omega}$ (which is a $(D-1)$-form) in the following way:

$$\mathbf{\Omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1[\mathbf{\Theta}(\phi, \delta_2\phi)] - \delta_2[\mathbf{\Theta}(\phi, \delta_1\phi)]\,. \tag{1.52}$$

Diffeomorphisms invariance implies that also $\mathbf{\Omega}$ is defined up to a closed $(D-1)$-form. Let now ξ^μ be an arbitrary vector field, from which we can construct the field variations $\hat{\delta}\phi = \mathcal{L}_\xi\phi$. We remember that the Lie derivative of a generic form $\mathbf{\Lambda}$ can be written as $\hat{\delta}\mathbf{\Lambda} = \xi \cdot d\mathbf{\Lambda} + d(\xi \cdot \mathbf{\Lambda})$, where the product $\xi\cdot$ denotes the contraction with the first index of the form. We can apply this formula and calculate the variation of the Lagrangian form:

$$\hat{\delta}\mathbf{L} = \mathcal{L}_\xi\mathbf{L} = d(\xi \cdot \mathbf{L})\,. \tag{1.53}$$

We see that ξ defines an infinitesimal diffeomorphism that leaves invariant $\mathbf{L}$ up to a closed form: thus the action remains invariant, and we can say that this diffeomorphism is a local symmetry. We can then define a Noether current $(D-1)$-form, associated to this local symmetry, as

$$\mathbf{j} = \mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi \cdot \mathbf{L}\,. \tag{1.54}$$

This is indeed a Noether current, in the sense that it is conserved whenever the equation ($\mathbf{E} = 0$) of motion are satisfied. In fact:

$$d\mathbf{j} = d\mathbf{\Theta}(\phi, \mathcal{L}_\xi \phi) - d(\xi \cdot \mathbf{L}) = \mathcal{L}_\xi \mathbf{L} - \mathbf{E}\mathcal{L}_\xi \phi - \mathcal{L}_\xi \mathbf{L} = -\mathbf{E}\mathcal{L}_\xi \phi \,. \tag{1.55}$$

Let us call $\mathbf{Q}$ the Noether charge $(D-2)$-form which can be locally constructed out of the fields ϕ and the vector ξ and such that, when the equation of motion are satisfied, obeys

$$d\mathbf{Q} = \mathbf{j} \,. \tag{1.56}$$

Again, $\mathbf{Q}$ is uniquely defined up to a local closed $(D-2)$-form. What is uniquely determined by the relation (1.56), by the way, is the integral of $\mathbf{Q}$ over a closed surface Σ of dimension $D-2$. Let us now consider the first order variation of equation (1.54):

$$\delta\mathbf{j} = \delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \delta\mathbf{L} = \delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot [\mathbf{E}\delta\phi] - \xi \cdot d\mathbf{\Theta}(\phi, \delta\phi) \,. \tag{1.57}$$

We now evaluate this expression on the equations of motion, and use again the definition of Lie derivative; thus we get

$$\delta\mathbf{j} = \delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi \phi) - \mathcal{L}_\xi \mathbf{\Theta}(\phi, \delta\phi) + d[\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)] \,. \tag{1.58}$$

Looking at the definition of the symplectic current, we can recognise that

$$\delta\mathbf{j} = \mathbf{\Omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi) + d[\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)] \,. \tag{1.59}$$

The next step is to integrate this last equation over a Cauchy surface $\mathcal{C}$. The key observation is that the symplectic current is related to the variation of the Hamiltonian H corresponding to the evolution generated by ξ. In particular, Hamilton's equations of motion are equivalent to (see [12])

$$\delta H_\xi = \int_\mathcal{C} \mathbf{\Omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi) \,, \tag{1.60}$$

which can now be written as

$$\delta H_\xi = \delta \int_\mathcal{C} \mathbf{j} - \int_\mathcal{C} d(\xi \cdot \mathbf{\Theta}) \,. \tag{1.61}$$

If we now assume that the equations of motion are satisfied, we can express the Noether current in terms of the Noether charge:

$$\delta H_\xi = \int_\mathcal{C} d(\delta\mathbf{Q}[\xi] - \xi \cdot \mathbf{\Theta}) \,. \tag{1.62}$$

What is the meaning of the Hamiltonian H_ξ? It turns out that in an asymptotically flat space-time it is natural to associate the surface contribution to the Hamiltonian from infinity with the conserved quantity associated with the vector field ξ. For example, the canonical energy $\mathcal{E}$ and the canonical momentum $\mathcal{J}$ are associated respectively with the time translation vector ζ and a rotation vector η. These definitions of energy and angular momentum are quite standard in general relativity; even if it is not obvious how to prove that they coincide with the definition we have made in section (1.2.1), one can at least verify that they give the same result in some particular cases. This is indeed true from the black hole solutions we have encountered up to now.

Let us specialise equation (1.62) to the two cases $\xi = \zeta$ and $\xi = \eta$:

$$\begin{aligned} \delta\mathcal{E} &= \int_\infty (\delta\mathbf{Q}[\zeta] - \zeta \cdot \mathbf{\Theta}) \,, \\ \delta\mathcal{J} &= -\int_\infty \delta\mathbf{Q}[\eta] \,. \end{aligned} \tag{1.63}$$

The minus sign in the second expression is just a matter of convention: it depends on the definition of the sign of the angular momentum. In the same equation we have not written any term like $\eta \cdot \mathbf{\Theta}$, because the vector η is assumed to be tangent to the sphere at infinity; this means that the contribution to the integral of such a term would vanish. Furthermore we suppose that it exists a $(D-1)$-form $\mathbf{B}$ such that $\delta \int_\infty t \cdot \mathbf{B} = \int_\infty t \cdot \mathbf{\Theta}$. We can thus write

$$\begin{aligned} \mathcal{E} &= \int_\infty \left(\mathbf{Q}[\zeta] - \zeta \cdot \mathbf{B} \right) , \\ \mathcal{J} &= - \int_\infty \mathbf{Q}[\eta] \, . \end{aligned} \tag{1.64}$$

We now want to specialise to a Killing vector field, and in particular a Killing vector field with vanishing norm on a Killing horizon Σ. We have in mind the three particular black holes we have studied, and so we define ξ as

$$\xi^\mu = \zeta^\mu + \Omega_H \eta^\mu \, , \tag{1.65}$$

where Ω is the angular velocity of the horizon. In principle a black hole in higher dimensions could rotate not only in one direction, and so in the product $\Omega_H \eta^\mu$ should be understood the presence of a sum over all the possible rotational directions.

Being ξ a Killing vector field, we have that $\mathcal{L}_\xi \phi = 0$ for every field ϕ. This means that the symplectic current vanishes, and then

$$\delta \mathbf{j} = d[\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)] \, , \tag{1.66}$$

or, when evaluated on the equations of motion,

$$d(\delta \mathbf{Q}) - d[\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)] = 0 \, . \tag{1.67}$$

Let us now integrate this latter expression over an asymptotically flat Cauchy surface with interior boundary on the Killing horizon Σ; equation (1.67) becomes

$$\int_\Sigma \delta \mathbf{Q}[\xi] - \xi \cdot \mathbf{\Theta} = \int_\infty \delta \mathbf{Q}[\xi] - \xi \cdot \mathbf{\Theta} \, . \tag{1.68}$$

The right hand side can be expressed in terms of energy and angular momentum. In the left hand side, the fact that ξ has vanishing norm on the horizon Σ means that the term $\xi \cdot \mathbf{\Theta}$ does not give any contribution. The final result is:

$$\delta \int_\Sigma \mathbf{Q}[\xi] = \delta \mathcal{E} - \Omega_H \delta \mathcal{J} \tag{1.69}$$

This is the so-called *first law of black hole thermodynamics*; anyway, we can go further and extract some more informations. An important fact is that, at least if we consider general relativity with the Einstein–Hilbert Lagrangian, $\mathbf{Q}[\xi]$ depends only on ξ^μ and its antisymmetrised first covariant derivatives $D^{[\mu}\xi^{\nu]}$; furthermore the dependence on these derivatives is linear. These derivatives can be expressed in terms of the bi-normal tensor $\epsilon^{\mu\nu}$ to the surface Σ, because the definition of surface gravity (1.11) is equivalent to

$$D^{[\mu}\xi^{\nu]} = \kappa \epsilon^{\mu\nu} \, . \tag{1.70}$$

However $\epsilon^{\mu\nu}$ does not depend on ξ: it is an object inherently related only to the surface Σ. Hence, defining $\widetilde{\mathbf{Q}}$ as $\mathbf{Q} = \kappa \widetilde{\mathbf{Q}}$, we have that it is completely independent on ξ when evaluated on the surface Σ. This because the Killing vector is null on the horizon, and the dependence on the covariant derivatives reduces to a dependence only on the bi-normal tensor. With this definition, the first law of black hole thermodynamics becomes

$$\kappa\,\delta\int_\Sigma \widetilde{\mathbf{Q}} = \delta\mathcal{E} - \Omega_H\delta\mathcal{J}\,. \tag{1.71}$$

It is worth noting that if we allow the electric charge Q to be variable, the first law has another term, i.e.

$$\kappa\,\delta\int_\Sigma \widetilde{\mathbf{Q}} = \delta\mathcal{E} - \Omega_H\delta\mathcal{J} - \Phi_e\delta Q\,, \tag{1.72}$$

where Φ_e can be interpreted as electrostatic potential.

Recalling the definition of temperature given above, we can write

$$T\,\frac{2\pi k_B}{\hbar}\delta\int_\Sigma \widetilde{\mathbf{Q}} = T\delta S = \delta\mathcal{E} - \Omega_H\delta\mathcal{J} - \Phi_e\delta Q\,. \tag{1.73}$$

Here we have defined what should be interpreted as the entropy of the black hole. A key observation is that this entropy depends only on local quantities defined on the horizon of the black hole.

1.3.3 Entropy and second law of black hole thermodynamics

Our derivation of the first law of black hole thermodynamics naturally led to a candidate for the entropy of a black hole. We have seen that it depends only on the horizon Σ. Let us now see what this entropy exactly is in the case of the Einstein–Hilbert Lagrangian in four dimensions. The Lagrangian (in its dual form) is

$$\mathbf{L}_{\mu\nu\rho\sigma} = \frac{1}{16\pi G}\epsilon_{\mu\nu\rho\sigma}R\,. \tag{1.74}$$

One can derive the boundary term $\boldsymbol{\Theta}$ with a quite tedious calculation; here we only state the result:

$$\boldsymbol{\Theta}_{\beta\gamma\delta}(g,\delta g) = \frac{1}{16\pi G}\epsilon_{\mu\beta\gamma\delta}g^{\mu\sigma}g^{\alpha\nu}(D_\nu\delta g_{\alpha\sigma} - D_\sigma\delta g_{\alpha\nu})\,. \tag{1.75}$$

Let us now recall that, given a vector field ξ, the Lie derivative of the metric is given by

$$\mathcal{L}_\xi g_{\mu\nu} = D_\mu\xi_\nu + D_\nu\xi_\mu\,. \tag{1.76}$$

One can now calculate the Noether current associated with this vector field, which is

$$\mathbf{j}_{\beta\gamma\delta} = \frac{1}{8\pi G}\epsilon_{\mu\beta\gamma\delta}\left[D_\alpha(D^{[\alpha}\xi^{\mu]}) + \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\xi_\nu\right]\,. \tag{1.77}$$

When calculated over the equations of motion (that are the Einstein field equations) the second term vanishes. Hence it is straightforward to recognise that we can take as Noether charge the following expression:

$$\mathbf{Q}_{\beta\gamma} = -\frac{1}{16\pi G}\epsilon_{\beta\gamma\mu\nu}D^\mu\xi^\nu\,. \tag{1.78}$$

We see that $\mathbf{Q}$ indeed depends only linearly on the first derivatives of the vector field ξ. We can now calculate what our candidate entropy is:

$$S = \frac{2\pi k_B}{\hbar\kappa}\int_\Sigma \mathbf{Q} = -\frac{k_B}{8\hbar G}\int_\Sigma \epsilon_{\beta\gamma\mu\nu}\epsilon^{\mu\nu} = \frac{Ak_B}{4\hbar G}\,, \tag{1.79}$$

where A is the area of the event horizon Σ. This entropy is known as *Bekenstein–Hawking entropy*; it is just, up to numerical factors, the area of the event horizon of the black hole.

A natural question arises: is there an analogous statement to the second law of thermodynamics? It should say that the area of the horizon of a black hole can not decrease during physical processes. This statement (called *area theorem*) was not proven in absolute generality; under certain conditions, by the way, it was proven that the cosmic censorship conjecture implies that the area of an event horizon cannot be decreasing ([10]).

The interpretation of the area of a black hole as entropy is not only restricted to a formal analogy between the laws of thermodynamics; one can derive the entropy of a black hole in analogy with the entropy of a thermodynamic system. We can calculate the partition function Z of the theory via a path integral, and then derive the entropy from Z; this method has the advantage to have a more clear physical meaning. In the path integral one usually considers only the contribution of the classical solution, which is the classical black hole metric of a given system. Using the partition function, as we have done in section (1.3.1), one can derive the thermodynamic potentials. Once these are known, one can derive the entropy of a black hole: the result agrees exactly with (1.79) (see [14]).

We can finally say that the laws of black hole thermodynamics are not only identities formally similar to the law of standard thermodynamics; the fact that quantities like temperature and entropy can be derived from the partition function of the quantum gravity theory, means that thermodynamic properties can be associated to a black hole. The following step is to investigate whether or not a microscopic interpretation for these thermodynamic laws is possible. This is a completely non trivial purpose, and it will drive us throughout this work.

1.4 The black hole information paradox

There are some serious problems which compromise our interpretation of black holes as true thermodynamic objects. A first observation is that the Bekenstein–Hawking entropy is proportional to the area of the black hole; we would have expected that the entropy was an extensive quantity, then proportional to a volume, rather than an area. This fact opens interesting perspectives involving holography.

Another serious puzzle arises from Hawking's calculation of the radiation coming out from a black hole ([1]): he has shown that this radiation is completely thermalised, i.e. its spectrum depends only on the temperature (and possibly other charges) associated to the black hole. A direct consequence is that information is lost during the process of formation and evaporation of a black hole. This is simply because there are many possible configuration which can lead to a black hole with the same charges: but the evaporation of the black hole is sensible only to these charges, and does not keep track of all the information related to the matter which has formed the black hole. This apparent loss of information is in contrast with the unitary evolution of quantum mechanics; a true quantum theory of gravity should solve this puzzle, known as *black hole information paradox*.

The information loss should be just an trickery due to the semi-classical nature of the calculation of the radiation spectrum; many attempts have been done, trying to identify where semi-classical arguments fail. Different solutions have been proposed, each one with its own virtues and vices; let us now briefly see which are the ideas:

- information is stored in a Planck-sized remnant;
- information is stored in a baby universe;

- information suddenly escapes from the black hole at the end of the evaporation;
- information actually flows slowly out of the black hole during evaporation, due to small correction to the semi-classical calculation;
- the classical picture of black holes is drastically modified at the horizon scale, resulting in the disappearance of the information paradox.

Let us concentrate in particular in the two last possibilities. It could be possible that the solution to the information paradox lies in the quantum corrections to the Hawking's calculation: in this case we would expect these corrections to be of order l_P/r_H, where l_P is the Planck scale and r_H is the radius of the horizon of the black hole (this means that these corrections are typically very small). An alternative is to think that even the classical description of black holes must be modified up to the horizon scale, resulting in correction to the Hawking's calculation of order 1. We will investigate this possibility in the following. We refer to [2] for an extended discussion on the information paradox.

Another problem, related to the information paradox, is the following: the interpretation of the area of a black hole as entropy naturally leads to the problem of identifying what are the microstates responsible for this entropy. It seems that there is not a satisfying answer to this problem, because of uniqueness theorems. We know that all different black holes geometry are parametrised by a finite family of charges. It seems that we have no way to have such a number of different microstates that could generate a finite entropy. Also this problem should be solved by a quantum theory of gravity; in the following we will learn how it is possible to find suitable black hole microstates in the framework of what is the better candidate to be a theory of quantum gravity: string theory. First of all, however, it is necessary to study supergravity, which is the (classical) low energy limit of string theory.

Chapter 2

Black holes in supergravity

There are at least three reasons to deal with supergravity theories. The first reason is that supergravity is an extension of general relativity (in four or more dimensions) that naturally implements a theory which is invariant under diffeomorphisms. The second reason is that if supersymmetry is indeed realised in nature, it would be very obvious to think that it would also apply in the context of gravity. Third, but not less important, supergravity theories turn out to be the low energy limit of superstring theories. This gives to supergravity a more fundamental interpretation; if string theory is indeed the real quantum theory of gravity, then the correct gravity theory describing our world will be a supergravity theory, if supersymmetry is not broken at high energy scales. In this chapter we will briefly sketch the key features of supergravity in four and more dimensions, and we will try to understand which are the generalisations of black holes in supergravity.

The study of black holes in supergravity is a key passage toward the analysis of black holes in string theory. This is because we will always need to compare a string solution with its classical limit, which will be a supergravity solution. In this chapter we will also derive the explicit form of a supergravity black hole; in the following we will also try to understand which are the microstates of this hole.

2.1 Basic features of supergravity

An extended review of supergravity is far beyond the purposes of this work. Here we derive only some basic properties. We refer to a complete reference such as [16] for details.

Supergravity arises from making a global supersymmetry local. A global supersymmetry transformation relates bosons and fermions to each other; we can schematically write the transformation rules, following [15], as

$$\delta_\epsilon B = \bar{\epsilon} F\,, \quad \delta_\epsilon F = \epsilon \gamma^\mu \partial_\mu B\,, \tag{2.1}$$

where ϵ is the infinitesimal parameter characterizing the supersymmetry transformation carrying a spinor index and γ^μ are matrices satisfying the Clifford algebra. The commutator of two of such transformations for a bosonic field is given by

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] B \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu B\,. \tag{2.2}$$

Making this symmetry local, means to promote the parameter ϵ to depend on the spacetime coordinate x; thus the commutator of two local supersymmetry transformations reads:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] B \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2)(x) \partial_\mu B\,. \tag{2.3}$$

The fact that the result has this form (i.e. the derivative of the field B times a function of x) means that it can be interpreted as an infinitesimal diffeomorphism; hence we have found that a theory which is invariant under local supersymmetry, must also be invariant under diffeomorphisms (which means that the spacetime metric must be considered as a dynamical object). But this is exactly what characterises general relativity; in fact the Einstein theory of gravity is the simplest diffeomorphisms invariant theory, even if other higher order corrections to the Einstein–Hilbert Lagrangian are possible.

The simplest supergravity theory in four dimension ($N = 1$, $D = 4$ supergravity) predicts the presence of a doublet formed by the graviton $g_{\mu\nu}$ and its fermionic super-partner, the gravitino ψ_μ^α (here and in the following we will denote a spacetime index with μ, ν, etc... and a spinor index with α, β, etc...).

The necessity of the presence of the gravitino can be related to the general way one constructs a local theory, starting from the global one. In the case of an ordinary gauge theory, one must define a covariant derivative with the use of vector fields in order to make the Lagrangian invariant under the local transformation. For example, a scalar theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(x) \partial^\mu \bar{\phi}(x) \tag{2.4}$$

has an $U(1)$ global symmetry corresponding to the transformation

$$\phi(x) \to e^{i\Lambda}\phi(x) \, . \tag{2.5}$$

If we make the symmetry local, promoting the parameter Λ to depend on the spacetime coordinate x, the Lagrangian (2.4) would no more be invariant. If we want to construct a Lagrangian invariant under this new local symmetry we must substitute the ordinary derivative with the covariant derivative $D_\mu = \partial_\mu - iA_\mu(x)$. Here we have introduced a vector field A_μ transforming as

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \, . \tag{2.6}$$

Obviously, the Lagrangian should also be completed with a kinetic term for the new vector field.

In the case of a supersymmetry transformation, the role of the parameter Λ is played by the spinor parameter ϵ^α. We then expect the new field appearing in the covariant derivative to be a spin-$\frac{3}{2}$ field ψ_μ^α, the gravitino.

For practical reasons, when dealing with spinors in curved spacetime it is useful to reformulate the theory in the *vielbein formalism*. Let us define a coordinate system which is inertial at the point x_0 ($y^A(x_0; x)$); the vielbein is defined by

$$e_\mu{}^A(x_0) := \left. \frac{\partial y^A(x_0; x)}{\partial x^\mu} \right|_{x=x_0} . \tag{2.7}$$

The choice of the locally inertial frame is unique up to a Lorentz transformation, under which the vielbein transforms as

$$e_\mu{}^A = e_\mu{}^B \Lambda_B^A \, . \tag{2.8}$$

With this object we can write the metric of the curved spacetime $g_{\mu\nu}$ in terms of the flat metric η_{ab}, through

$$g_{\mu\nu}(x) = e_\mu{}^A(x) e_\nu{}^B(x) \eta_{AB} \, . \tag{2.9}$$

We want to define a suitable covariant derivative, which should be the generalisation of both the covariant derivative of gauge theories and the one of general relativity. Its relevant part is

$$\widetilde{D}_\mu = \partial_\mu + \frac{1}{2}\omega_\mu{}^{AB} M_{AB} \, , \tag{2.10}$$

where M_{AB} are the antisymmetric generators of the Lorentz transformations. The new term is needed for invariance under local Lorentz transformations, and involves the so-called spin connection ${\omega_\mu}^{AB} = {\omega_\mu}^{[AB]}$. If one wants to derive the expression of the total covariant derivative D_μ when applied to a tensor, one must add to the expression $\widetilde{D}_\mu$ all the necessary terms involving the affine connection, in the very same way one does in general relativity. In order to determine the spin connection one has to impose the *vielbein postulate*; it is the analogous of the statement that in general relativity the metric is covariantly constant, i.e.

$$D_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} \,. \tag{2.11}$$

The vielbein postulate requires that the vielbein is covariantly constant, i.e.

$$0 = D_\mu {e_\nu}^A = \widetilde{D}_\mu {e_\nu}^A - \Gamma^\lambda_{\mu\nu} {e_\lambda}^A = \partial_\mu {e_\nu}^A + {\omega_\mu}^A{}_B {e_\nu}^B - \Gamma^\lambda_{\mu\nu} {e_\lambda}^A \,. \tag{2.12}$$

Antisymmetrizing this equation, and writing it with the use of forms, we get

$$de^A + \omega^A{}_B \wedge e^B = 0 \,. \tag{2.13}$$

2.2 Supergravity in various dimensions

There are essentially two parameters characterizing a supersymmetry theory: the dimension of the spacetime D, and the number of supersymmetries N. One usually divides the fields of the theory into multiplets, which are formed by fields related each other by supersymmetry transformations. We are obviously interested in the supergravity multiplet containing a spin-2 particle, which we will interpret as the graviton. We notice that if a multiplet contains a particle with spin greater than 2, we should consider it non-physical. Given this constraint, one can see that the possible supersymmetry (and supergravity) theories are not many. In $D = 4$, for example, the most common theories have $N = 1, 2, 4, 8$. The field content of the graviton multiplet in all these examples is represented in table 2.1 (see for example [15]).

Elicity:	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$N=1$	1	1							
								1	1
$N=2$	1	2	1						
							1	2	1
$N=4$	1	4	6	4	1				
					1	4	6	4	1
$N=8$	1	8	28	56	70	56	28	8	1

Table 2.1: Graviton multiplet field content in $D = 4$.

We see that the number of particles is always 2^N, eventually multiplied by 2 because of CPT invariance. $N = 1$ is called minimal supersymmetry; it is the consistent theory (in $D = 4$) with the smallest possible number of supersymmetries. On the other end, $N = 8$ is called maximal supersymmetry; in fact $N > 8$ would imply the presence of particles with spin greater than 2, and so it would not be a physical theory.

We now want to extend the analysis of supersymmetry and supergravity theories to higher dimensional spacetime. There are several motivation that drive us to consider supergravity in

$D > 4$. On the one hand supergravity in $D = 4$ naturally arises as dimensional reduction of higher dimensional theories; on the other hand supergravity in $D = 10$ can be seen as the low energy limit of superstring theory. The extension is trivial if one considers only bosonic fields; in order to deal also with fermions, one should first study spinor representations in dimension greater than 4. What really matters to us here, is just what is the dimension of the irreducible spinor representation in D dimension. The first step is to identify the representation of the Clifford algebra in D dimension, i.e. the generalisation of the γ matrices satisfying

$$\{\gamma_A, \gamma_B\} = 2\eta_{AB} \,. \tag{2.14}$$

Let us parametrise the dimension D as

$$\begin{aligned} D &= 2k + 2 \quad \text{if } D \text{ is even}\,, \\ D &= 2k + 3 \quad \text{if } D \text{ is odd}\,. \end{aligned} \tag{2.15}$$

It turns out that there is always a representation for the γ matrices; the corresponding spinors transform under a representation with (complex) dimension 2^{k+1}, and are called *Dirac spinors*. This representation, however, is not always irreducible: in even dimension it is possible to define the matrix $\bar{\gamma} = i^k \gamma_0 \gamma_1 \ldots \gamma_{D-1}$, which generalise the γ_5 in 4 dimensions. The Clifford algebra implies that $\bar{\gamma}$ satisfies

$$\{\bar{\gamma}, \gamma_A\} = 0\,, \quad \bar{\gamma}^2 = -\mathbb{1}, . \tag{2.16}$$

Thanks to these two properties, the Dirac spinor representation can be decomposed into two irreducible parts by virtue of the projectors

$$\mathbb{P}_\pm = \frac{1}{2}(\mathbb{1} \pm i\bar{\gamma}) \,. \tag{2.17}$$

The two resulting representations have (complex) dimension 2^k, and the corresponding elements are called *Weyl spinor*. Up to now we have considered only complex spinors: one may ask whether it exists a way to define real spinors. There are two ways to impose reality conditions, and the resulting spinors are called respectively *Majorana* and *pseudo-Majorana spinors*. We do not see explicitly how this can be done, but only state that it is not always possible to impose these conditions: it depends on the dimension D of the spacetime (it is possible only if $D \equiv 0, 1, 2, 3, 4 \mod 8$). This fact allows to have a representation with half the (real) dimension. Finally one may further ask if, in even dimension, these reality condition are compatible with the split into Weyl spinors: if this is so, the resulting irreducible representation has real dimension 2^k, and its elements are called *Majorana–Weyl spinors*.
In table 2.2 we summarise all the possible spinor representation in a D dimensional spacetime, and write the real dimension of the irreducible representation d_R (see for example [17]).

D	2	3	4	5	6	7	8	9	10	11
k	0	0	1	1	2	2	3	3	4	4
Weyl	✓		✓		✓		✓		✓	
(pseudo-)Majorana	✓	✓	✓				✓	✓	✓	✓
Majorana–Weyl	✓								✓	
d_R	1	2	4	8	8	16	16	16	16	32

Table 2.2: Possible types of spinors in D dimensions.

We have stopped our table at $D = 11$, and this is not by chance. The reason is that one may view supergravity theories in $D = 4$ as the dimensional reduction of supergravity theories in greater dimension; but this process does not change the total (real) degrees of freedom of the theory. We have seen that the maximal supergravity theory in $D = 4$ corresponds to $N = 8$; hence the total number of supercharges is $4 \cdot 8 = 32$. This theory can be viewed as the dimensional reduction of a supergravity theory in a spacetime of dimension at most 11, because theories in $D > 11$ have d_R greater than 32. Thus the supergravity in 11 dimensions plays a significant role: it can be seen as one of the most fundamental theories, since many other theories in lower dimensions can be derive from this one.

The massless field content of the $11D$ supergravity can be derived easily. We know that we have a total number of 256 degrees of freedom, just like $N = 8$ supergravity in 4 dimensions: due to supersymmetry, these are divided into 128 fermions and 128 bosons. Our goal is to identify representations of the little group of massless particles in 11 dimension (i.e. $SO(9)$) with the appropriate degrees of freedom. The bosonic 128 can be divided into the following two irreducible representations:

- a symmetric traceless field $g_{\mu\nu}$ ($d_g = \frac{9 \cdot 10}{2} = 45$);
- a completely antisymmetric 3-form $A_{\mu\nu\rho}$ ($d_A = \binom{9}{3} = 84$).

On the other hand, there is a fermionic irreducible 128 representation of $SO(9)$, and thus the (massless) fermionic content of the 11 dimensional supergravity is given only by

- a γ-traceless vector spinor ψ_μ^α ($d_\psi = 128$).

This theory is not only more fundamental, but also much simpler than other supergravity theories in lower dimension, because its field content is very simple. This is the reason why it is often useful to work with this theory and then, if necessary, to obtain physical results via dimensional reduction.

Another important role is played by the supergravity theories in 10 dimension, which correspond to the low energy limit of superstring theories. Differently from $D = 11$, there is more than one possibility in $D = 10$, because the real dimension of the irreducible spinor representation is 16. The minimal supergravity in $D = 10$ is called *Type I*. On the other hand, there are two possible maximal supergravity theories, which are called *Type IIA* and *Type IIB* and are slightly different in their field content. It turns out that only Type IIA can be obtained via dimensional reduction starting from the 11 dimensional supergravity.

2.3 Supergravity in 11 and 10 dimensions

When dealing with a supersymmetric theory, one often writes only the bosonic part of the Lagrangian, the fermionic one being completely fixed by supersymmetry. In the case of the $11D$ dimensional supergravity, we have to write a Lagrangian for the metric g and the 3-form A. The metric part should be the generalisation of the Einstein–Hilbert Lagrangian, while the kinetic term for the 3-form should be the generalisation of the $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ corresponding to a vector field. The correct action is (using the conventions of [17])

$$S_{11} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2}|F^{(4)}|^2\right) - \frac{1}{12\kappa^2}\int A^{(3)} \wedge F^{(4)} \wedge F^{(4)}\,, \tag{2.18}$$

where we have defined the 4-form field strength $F^{(4)} = dA^{(3)}$. The last term is called *Chern–Simons* term, and it is required by supersymmetry; in spite of its explicit appearance it is gauge-invariant.

2.3.1 Dimensional reduction

The fields and the action for Type IIA superstring theory can be derived via dimensional reduction; this is obtained compactifying the coordinate x^{10} over a circle. Let us call $y \equiv x^{10}$, while all the other coordinates will have an analogous significance also in 10 dimension; we then parametrise the metric and the 3-form as

$$\begin{aligned} ds_{11}^2 &= e^{2\sigma}(dy + C_\mu^{(1)} dx^\mu)^2 + ds_{10}^2\,, \\ A^{(3)} &= B^{(2)} \wedge dy + C^{(3)}\,. \end{aligned} \tag{2.19}$$

We then see that the bosonic field content of Type IIA supergravity is given by:

- the metric $g_{\mu\nu}$, with $\mu, \nu = 0, 1, \ldots, 9$;
- the scalar field σ or, equivalently, the *dilaton* $\Phi \equiv \frac{3}{2}\sigma$;
- the 2-form $B^{(2)}$;
- the forms $C^{(1)}$ and $C^{(3)}$.

The first three fields are called *NSNS* (Neveu–Schwarz Neveu–Schwarz) fields, while the forms $C^{(n)}$ (with $n = 1, 3$) are called *RR* (Ramond Ramond) fields. These names derive from the interpretation of this theory as the low energy limit of superstring theory. We can now write down the action, directly deriving it from (2.18):

$$\begin{aligned} S_{IIA} = &\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^\sigma R_{10} - \frac{1}{2} e^{3\sigma} |F^{(2)}|^2 \right) + \\ &- \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^{-\sigma} |H^{(3)}|^2 + e^\sigma |\hat{F}^{(4)}|^2 \right) - \frac{1}{4\kappa_{10}^2} \int B^{(2)} \wedge F^{(4)} \wedge F^{(4)}\,. \end{aligned} \tag{2.20}$$

Here we have defined the field strengths $F^{(2)} = dC^{(1)}$, $H^{(3)} = dB^{(2)}$, $F^{(4)} = dC^{(3)}$ and $\hat{F}^{(4)} = dC^{(3)} - C^{(1)} \wedge F^{(3)}$. The dilaton Φ is related to the string theory coupling constant by

$$g_s = e^{\Phi_\infty}\,, \tag{2.21}$$

where Φ_∞ is the value of the dilaton at spatial infinity. We can observe that in the strong coupling limit we have $\sigma \to \infty$, which means that we can describe Type IIA theory as a 11 dimensional theory, the *M theory*. We say that the strong coupling limit of Type IIA is M theory.

An important observation is that the metric part of the action has not the correct form of an higher dimensional Einstein–Hilbert action; this is due to the particular frame in which we have written the action. If we want to derive physical results, we have to write the action in the so-called *Einstein frame*, i.e the frame in which the part of the Lagrangian involving the metric takes the form $\sqrt{-g}R$. In order to do so, one has to mix the metric and the scalar field in the following way:

$$(g_E)_{\mu\nu} = e^{\frac{\sigma}{4}} (g_{10})_{\mu\nu} = e^{\frac{\Phi}{6}} (g_{10})_{\mu\nu}\,. \tag{2.22}$$

There is another useful frame, the *string frame*, which is the frame one uses when getting the action of Type IIA as low energy limit of superstring theory. It is related to the others by

$$(g_E)_{\mu\nu} = e^{-\frac{\Phi}{2}} (g_s)_{\mu\nu}\,. \tag{2.23}$$

The metric part of the action can be written in this three frames as

$$S_g = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_{10}}e^{\sigma}R_{10} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_E}R_E = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_s}e^{-2\Phi}R_s\,. \quad (2.24)$$

It is useful to write down explicitly how the action (2.20) reads in string frame:

$$S_{IIA} = S_{NSNS} + S_{RR} + S_{CS}\,, \quad (2.25)$$

where we have divided the action into the three parts corresponding to the NSNS and RR fields, and the last Chern–Simons term. The explicit expressions are

$$\begin{aligned} S_{NSNS} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_s}e^{-2\Phi}\left(R_s + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H^{(3)}|^2\right)\,,\\ S_{RR} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g_s}\left(|F^{(2)}|^2 + |\hat{F}^{(4)}|^2\right)\,,\\ S_{CS} &= -\frac{1}{4\kappa_{10}^2}\int B^{(2)}\wedge F^{(4)}\wedge F^{(4)}\,. \end{aligned} \quad (2.26)$$

2.3.2 T-duality

We have said that in 10 dimension there is another consistent supergravity theory (Type IIB) with 32 supercharges, which cannot be derived from an 11 dimensional theory via dimensional reduction. Anyway, this theory can be related to Type IIA supergravity, thanks to the presence of a duality between the fields of the two theories; this is the so-called T-duality. This duality can be derived by the properties of superstrings; in fact both theories are the low energy limit of respectively Type IIA and IIB superstring theory. The T-duality depends on a chosen compact direction y: starting from Type IIA one can derive the fields of the Type IIB theory parametrizing (in string frame):

$$\begin{aligned} ds^2 &= g_{yy}(dy + A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu\,,\\ B^{(2)} &= B_{\mu y}dx^\mu \wedge (dy + A_\nu dx^\nu) + \hat{B}^{(2)}\,,\\ C^{(p)} &= C_y^{(p-1)}\wedge(dy + A_\mu dx^\mu) + \hat{C}^{(p)}\,. \end{aligned} \quad (2.27)$$

The fields of the corresponding type IIB theory are the following ones:

$$\begin{aligned} ds'^2 &= g_{yy}^{-1}(dy + B_{\mu y}dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu\,,\\ e^{2\Phi'} &= g_{yy}^{-1}e^{2\Phi}\,,\\ B'^{(2)} &= A_\mu dx^\mu \wedge dy + \hat{B}^{(2)}\,,\\ C'^{(p)} &= \hat{C}^{(p-1)}\wedge(dy + B_{\mu y}dx^\mu) + C_y^{(p)}\,. \end{aligned} \quad (2.28)$$

We then see that the field content is very similar to that of Type IIA. The *NSNS* sector has the same type of fields (metric, dilaton and B fields). The *RR* sector is again made up by p-forms, but now p takes only even values (0, 2, 4), differently from Type IIA. One can also derive the action, starting from the Type IIA one in string frame (equations (2.25) and (2.26)); the result is ([17])

$$S_{IIB} = S_{NSNS} + S_R + S_{CS}\,, \quad (2.29)$$

where

$$
\begin{aligned}
S_{NSNS} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_s} e^{-2\Phi} \left(R_s + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H^{(3)}|^2 \right) , \\
S_{RR} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g_s} \left(|F^{(1)}|^2 + |\hat{F}^{(3)}|^2 + \frac{1}{2} |\hat{F}^{(5)}|^2 \right) , \\
S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int C^{(4)} \wedge H^{(3)} \wedge F^{(3)} .
\end{aligned}
\tag{2.30}
$$

Here we have defined the field strengths $F^{(p+1)} = dC^{(p)}$ (for $p = 0, 2, 4$), $H^{(3)} = dB^{(2)}$, $\hat{F}^{(3)} = F^{(3)} - C^{(0)} \wedge H^{(3)}$ and finally $\hat{F}^{(5)} = F^{(5)} - \frac{1}{2} C^{(2)} \wedge H^{(3)} + \frac{1}{2} B^{(2)} \wedge F^{(3)}$. The action is very similar to the corresponding one of Type IIA: the existence of this duality can be interpreted as the fact that the two theories are, in some sense, different points of view in the framework of the same theory.

2.3.3 S-duality

There is another important duality, that relates two different Type IIB theories. It is useful to use this duality in order to obtain another solution starting from a solution of the equation of motion of Type IIB supergravity, but it is also important for investigating the strong coupling limit of Type IIB. Roughly speaking, this duality interchanges the role of the two 2-forms $B^{(2)}$ and $C^{(2)}$, and changes the sign of the dilaton. The complete set of transformations is given by

$$
\begin{aligned}
\Phi' &= -\Phi , \\
g'_{\mu\nu} &= e^{-\Phi} g_{\mu\nu} , \\
B'^{(2)} &= C^{(2)} , \\
C'^{(2)} &= -B^{(2)} .
\end{aligned}
\tag{2.31}
$$

The other fields ($C^{(0)}$ and $C^{(4)}$) remain unchanged. Changing the sign of the dilaton has the effect of inverting the coupling constant:

$$
g'_s = \frac{1}{g_s} . \tag{2.32}
$$

From this relation we can relate two Type IIB theories, one with small coupling and one with big coupling. We thus see that the strong coupling limit of Type IIB is again a Type IIB theory.

2.4 Branes and charges

We have seen that supergravity theories in 10 dimensions (both Type IIA and IIB) are characterised by the presence of the 2-form B and other p-forms $C^{(p)}$; in Type IIA p is odd, while it is even in Type IIB. In this section we will see how these forms can be interpreted as the fields that give electric and magnetic charges to some multidimensional objects, called branes.

We start reviewing briefly how a vector field (or 1-form) $A^{(1)}$ can give electric and magnetic charge to a particle (which is a 0-dimensional object) in 4 dimensions. In standard electrodynamics the interaction term of the Lagrangian, connecting the field $A^{(1)}_\mu$ with a particle of charge q with trajectory $x^\mu(\lambda)$ is

$$
\mathcal{L}_{int} = q \int A^{(1)}_\mu \frac{dx^\mu}{d\lambda} d\lambda = q \int_\gamma A^{(1)} , \tag{2.33}
$$

where γ represents the world line of the particle, and the last integration in the above formula is the standard integration of an n-form over a n dimensional domain (in this case $n = 1$). The electric and magnetic charges can be calculated out of the field strength $F^{(2)} = dA^{(1)}$ and its hodge dual $\widetilde{F}^{(2)} = \star F^{(2)}$, through some integrals over a 2-sphere S_2, defined by t and r constant in polar coordinates. The exact definitions for the electric (Q_e) and magnetic (Q_m) charges are

$$Q_e = \int_{S_2} \star F^{(2)} = \int_{S_2} \widetilde{F}^{(2)} \,, \quad Q_m = \int_{S_2} F^{(2)} = \int_{S_2} \star \widetilde{F}^{(2)} \,. \tag{2.34}$$

The multidimensional generalisation of these charges derives from the generalisation of the interaction Lagrangian (2.33), which is (for a p-form field $C^{(p)}$)

$$\mathcal{L}_{int} = \mu_p \int_{\gamma_p} A^{(p)} \,, \tag{2.35}$$

where μ_p is called charge density. Here γ_p is a p dimensional domain, which we can interpret as the world volume of a $p-1$ dimensional object, i.e. the "trajectory" this object describes as time passes. This object is called *brane*, and it is the natural generalisation of the concept of particle (a particle can be seen as a 0-brane). Starting from $C^{(p)}$ we can construct the field strength $F^{(p+1)} = dC^{(p)}$ and its hodge dual $\widetilde{F}^{(D-p-1)} = \star F^{(p+1)}$. Thus each p-form couples electrically to a $(p-1)$-brane, while it couples magnetically to another type of brane, with dimension $D-p-3$, the values of the charges being:

$$\begin{aligned} Q_e &= \int_{S_{D-p-1}} \star F^{(D-p-1)} \qquad \text{(electric charge of a } (p-1)\text{-brane)}\,, \\ Q_m &= \int_{S_{p+1}} F^{(p+1)} \qquad \text{(magnetic charge of a } (D-p-3)\text{-brane)}\,. \end{aligned} \tag{2.36}$$

For example, in 11 dimensions the field $A^{(3)}$ gives electric charge to a 2-brane and magnetic charge to a 5-brane; they are called respectively $M2$ and $M5$ branes (here M stays for M-theory).

In table 2.3 we schematically write all the branes coupled to the fields we have encountered up to now.

$11D$ supergravity	Fields	$A^{(3)}$			
	Electric coupling	$M2$			
	Magnetic coupling	$M5$			
$10D$ Type IIA	Fields	$B^{(2)}$	$C^{(1)}$	$C^{(3)}$	
	Electric coupling	$F1$	$D0$	$D2$	
	Magnetic coupling	$NS5$	$D6$	$D4$	
$10D$ Type IIB	Fields	$B^{(2)}$	$C^{(0)}$	$C^{(2)}$	$C^{(4)}$
	Electric coupling	$F1$	✗	$D1$	$D3$
	Magnetic coupling	$NS5$	✗	$D5$	$D3$

Table 2.3: Electric and magnetic couplings in 11 and 10 dimensional supergravity.

In 10 dimensional Type IIA supergravity $C^{(1)}$ couples (electrically) to a $D0$ brane and (magnetically) to a $D6$ brane; $C^{(3)}$ couples to a $D2$ and a $D4$ brane (D stays for Dirichlet). Also $B^{(2)}$ couples to a 1 dimensional and a 5 dimensional brane; the standard terminology is quite different in this case, as one calls them respectively $F1$ (also called fundamental string) and $NS5$. The reason for these different names will become clear dealing with string theory; the distinction between $F1$, $NS5$ and Dp branes will have a physical meaning.

In Type IIB supergravity, the new fields are $C^{(2)}$, which couples to a $D1$ and a $D5$ brane, and $C^{(4)}$, which couples both electrically and magnetically to a $D3$ brane. The field $C^{(0)}$ is a Lorentz scalar, and so it cannot produce charges on a brane. We note that a S-duality would change the role of $D1$ and $F1$ branes, as well as the role of $D5$ and $NS5$.

2.5 Solutions generation: some examples

In this section we will derive some non trivial solutions of the supergravity equations of motion, in 11 and 10 dimension. We focus here on a particular type of solutions, called *BPS* (Bogomolny-Prasad-Sommerfeld); they are purely bosonic solutions, where all the fermionic fields are not present, and they are invariant under some of the supersymmetries characterizing the theory. There are two methods that can be used in order to derive such solutions. The first and more direct one, although not solving directly the equations of motion, uses the symmetry that these solutions must have, in particular supersymmetry. The second method, which is more indirect, starting from a trivial solution derives other solutions by means of T and S dualities.

2.5.1 Direct method

The first solution we are looking for is that corresponding to a $M2$ brane in 11 dimensional supergravity. We know that such solution should exist, because the 3-form $A^{(3)}$ naturally couples to this type of brane. We want this brane to extend over the directions x^i $(i = 1, 2)$ and to be perpendicular to the directions x^a $(a = 3, \ldots, 10)$. Let us start from the following ansatz:

$$\begin{cases} ds^2 = Z(r)(-dt^2 + dx^i dx^i) + Y(r) dx^a dx^a \\ A^{(3)} = X(r) dt \wedge dx^1 \wedge dx^2 \\ \psi^\alpha_\mu = 0 \end{cases} \tag{2.37}$$

Here we have assumed that the solution would depend only on the 3 functions X, Y and Z, and that these function would depend only on the radial coordinate $r = \sqrt{(x^a)^2}$, where we have understood a sum over a from 3 to 10. This ansatz sounds reasonable; we will now see that a solution of this form is really allowed in 11 dimensional supergravity. We want this solution to be invariant under supersymmetry transformations; we will not derive their exact form, but only state that they are the following (using the vielbein formalism):

$$\begin{aligned} &\delta e_\mu{}^A = \bar{\epsilon}\gamma^A \psi_\mu \,, \\ &\delta A_{\mu\nu\rho} = -3\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} \,, \\ &\delta\psi_\mu = D_\mu \epsilon + \frac{1}{288}\left(\gamma_\mu{}^{\nu\rho\sigma\tau} F_{\nu\rho\sigma\tau} - 8\gamma^{\nu\rho\sigma} F_{\mu\nu\rho\sigma}\right)\epsilon \,, \end{aligned} \tag{2.38}$$

where, for simplicity, we have understood all spinor indices, and the flat index A runs from 0 to 10. A γ with more than one index must be intended as the antisymmetric product of γ matrices; for example $\gamma^{\nu\rho\sigma} = \gamma^{[\nu}\gamma^\rho\gamma^{\sigma]}$. The field strength $F = dA$ has only few non-trivial components, that are

$$F_{a12t} = \partial_a X(r) \,. \tag{2.39}$$

The fact that our solution has a vanishing gravitino, implies that $\delta e_\mu{}^A$ and $\delta A_{\mu\nu\rho}$ automatically vanish. Thus we must only check that also the variation of the gravitino is zero. We remember that the covariant derivative is defined in terms of the spin connection (see equation (2.10)):

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2}\omega_\mu{}^{AB} M_{AB} = \partial_\mu \epsilon + \frac{1}{4}\omega_\mu{}^{AB}\gamma_{AB}\,, \tag{2.40}$$

where we have identified the generators M_{AB} with $\frac{1}{2}\gamma_{AB}$. Our goal is to derive the spin-connection, and we do this using the vielbein postulate (2.13). We interpret the vielbeins as 1-forms e^A; their explicit form can be easily derived from the ansatz (2.37):

$$e^t = \sqrt{Z(r)}dt\,, \quad e^i = \sqrt{Z(r)}dx^i\,, \quad e^a = \sqrt{Y(r)}dx^a\,. \tag{2.41}$$

One can easily derive the expression of the spin connection components, using the vielbein postulate. The result is:

$$\omega_{ab} = \frac{1}{\sqrt{Y}}\partial_b\sqrt{Y}dx^a - \frac{1}{\sqrt{Y}}\partial_a\sqrt{Y}dx^b\,, \quad \omega_{ia} = \frac{1}{\sqrt{Y}}\partial_a\sqrt{Z}dx^i\,, \quad \omega_{ta} = -\frac{1}{\sqrt{Y}}\partial_a\sqrt{Z}dt\,. \tag{2.42}$$

Here we have written only the flat indices of the spin connection: this is because we see this components as 1-form. We now have to impose that the variation of the gravitino vanishes. Let us do it explicitly for the index $\mu = 1$:

$$0 = \partial_1\epsilon + \frac{1}{4}(\omega_1)_{AB}\gamma^{AB}\epsilon + \frac{1}{288}\left(4!\gamma_{\hat{1}}{}^{\hat{a}\hat{1}\hat{2}\hat{t}}F_{a12t} - \frac{8}{288}3!\gamma^{\hat{2}\hat{a}\hat{t}}F_{12at}\right)\epsilon\,, \tag{2.43}$$

where hatted indices of the gamma matrices should be intended as curved indices: one should express all in terms of gamma matrices with flat indices, by means of the appropriate vielbein. The first and the third term trivially vanish, and so we are left with

$$0 = \left(\frac{1}{2\sqrt{Y}}\partial_a\sqrt{Z}\gamma^{1a} - \frac{1}{6\sqrt{Y}}Z^{-1}\partial_a X\gamma^{2at}\right)\epsilon\,. \tag{2.44}$$

Multiplying this expression with γ^{1a} and using the Clifford algebra we arrive at

$$\frac{1}{3Z}\partial_a X\gamma^{012}\epsilon = \partial_a\sqrt{Z}\epsilon\,. \tag{2.45}$$

This is a sort of projection equation for the spinor ϵ: in fact we have that $(\gamma^{012})^2 = \mathbb{1}$. Thus it must be $\gamma^{012}\epsilon = \pm\epsilon$: these two possibilities are both possible, and correspond to a brane and its anti-brane. Here we choose the + sign, and the equation reduces to

$$X(r) = Z(r)^{\frac{3}{2}}\,. \tag{2.46}$$

Solving the same equation for $\mu = a$ one gets a link between the functions Y and X, in particular:

$$X(r) = Y(r)^{-3}\,. \tag{2.47}$$

This is all what we can say just using the supersymmetry. In order to go further, we cannot avoid solving an equation of motion, which we choose to be the equation of motion of the form A. It is simply the generalisation of Maxwell's equations in 4 dimensional electrodynamics, i.e.

$$d \star F = 0\,. \tag{2.48}$$

The correct definition for the Hodge dual of F is

$$(\star F)_{\mu_1\dots\mu_7} = \sqrt{-g}\epsilon_{\mu_1\dots\mu_7}{}^{\mu_8\dots\mu_{11}}F_{\mu_8\dots\mu_{11}}\,. \tag{2.49}$$

Thus we get $\star F = X^{-2}\partial_a X dx^{a_1}\wedge\dots\wedge dx^{a_7}$, with $a_1\dots a_7 \neq a$. The equation of motion is then equivalent to the Laplace equation for X^{-1}:

$$\partial^a\partial_a X^{-1} = 0,. \tag{2.50}$$

The solution of this equation is an harmonic function in 8 dimensions. We thus write $X^{-1} = 1+\frac{Q}{r^6}$, where the adding constant is fixed requiring that the metric is flat at infinity. The constant Q is precisely the electric charge corresponding to the $M2$ brane. We rewrite here the complete solution we have found:

$$\begin{cases} ds^2 = X(r)^{\frac{2}{3}}(-dt^2+dx^i dx^i) + X(r)^{-\frac{1}{3}}dx^a dx^a \\ A^{(3)} = X(r)dt\wedge dx^1\wedge dx^2 \\ \psi^\alpha_\mu = 0 \end{cases} \quad , \quad X(r) = \left(1+\frac{Q}{r^6}\right)^{-1}\,. \tag{2.51}$$

The fact that the metric and the A field depend on one single function $X(r)$ is due to the presence of supersymmetry. However, as we have seen for black holes in general relativity, the metric coefficients are related to the mass of the object generating that solution. Having a single function that determines the metric and the 3-form means that there is a precise relation between the mass and the charge of our solution. This properties, related to *BPS* solutions, is the analogous of the equality between mass and charge characterizing extremal black holes in general relativity: in fact these solutions are the correct generalisations of extremal solutions in general relativity. One can see it also from the form of the extremal Reissner–Nordstrøm black hole written in (1.20).

Once we have this $M2$ solution, it is quite simple to derive suitable solutions of Type IIA supergravity via dimensional reduction. There are two ways to do so, compactifying a coordinate x^i or a coordinate x^a. Let us first choose $y = x^1$ and apply the rules of equation (2.19); we find the following solution (where we have made a slight change of notation, and we have already turned to the string frame):

$$\begin{cases} ds^2 = Z(r)^{-1}(-dt^2+dx^2dx^2) + dx^a dx^a \\ e^\Phi = Z(r)^{-\frac{1}{2}} \\ B^{(2)} = -Z(r)^{-1}dt\wedge dx^2 \\ C^{(p)} = 0 \quad (p=1,3) \end{cases} \quad , \quad Z(r) = 1+\frac{Q}{r^6}\,. \tag{2.52}$$

This solution corresponds to a $F1$ fundamental string parallel to the x^2 direction, as can be seen from table 2.3. Obviously this is a Type IIA solution, because it was obtained via dimensional reduction. Obtaining this solution was straightforward, in that our $M2$ brane was parallel to the x^1 direction; therefore the brane was invariant under translations along x^1. If we now want to do the same for a direction perpendicular to the $M2$ brane, say x^3, we get some difficulties, because we have one single $M2$ brane located at $x^3 = 0$. The problem can be solved noting that the Laplace equation is linear, and so we can safely take a superposition of branes at different locations as a correct supersymmetric solution. Physically speaking, this is allowed because of the equality of mass and charge, that balances the attracting gravitational force and the repulsive gauge force between parallel branes. Suppose that we make a superposition of many branes, each one at position $x_i^3 = y_i$ and with charge Q; then the X function of equation (2.51) satisfies

$$X(r)^{-1} = 1 + Q \sum_i \frac{1}{|\vec{x} - \vec{x_i}|^6} \,. \tag{2.53}$$

Defining $r' = \sum_{a=4}^{10} (x^a)^2$, and letting the branes be continuously distributed along x^3, we have

$$X(r)^{-1} = 1 + Q \int_{-\infty}^{\infty} \frac{dy}{[r'^2 + (x^3 - y)^2]^3} = 1 + \frac{Q'}{r'^5} \,, \tag{2.54}$$

where Q' is proportional to Q (it is not important the right proportionality coefficient). Thus the solution corresponding to an infinite superposition of $M2$ branes is formally identical to (2.51), with the function $X(r)^{-1}$ replaced by (2.54).

We can now safely make a dimensional reduction along the x^3 direction. The result is a solution corresponding to a $D2$ brane parallel to the directions x^1 and x^2 which, when expressed in string frame, reads:

$$\begin{cases} ds^2 = Z(r)^{\frac{1}{2}}(-dt^2 + (dx^1)^2 + (dx^2)^2) + Z(r)^{\frac{1}{2}}(dx^a)^2 \\ e^{\Phi} = Z(r)^{\frac{1}{4}} \\ B^{(2)} = 0 = C^{(1)} \\ C^{(3)} = Z(r)^{-1} dt \wedge dx^1 \wedge dx^2 \end{cases} \,, \quad Z(r) = 1 + \frac{Q}{r^5} \,. \tag{2.55}$$

Here r is the radial direction in the 7 dimensional space orthogonal to the brane. Starting from these solutions for the $F1$ and the $D2$ brane, we can use T and S dualities in order to find other supergravity solutions in 10 dimensions. For example we can find the solution for a Type IIB fundamental string, making a T duality along, say, x^3. The result is formally identical to (2.52). Now one can make an S duality, thus obtaining a Type IIB $D1$ solution, and so on.

2.5.2 Indirect method

As anticipated, there is another method that can be used to derive the same supergravity solutions. One starts from a well known, almost trivial solution, and applies symmetries and dualities to construct the desired brane solutions. Let us now see an example. Our starting point is the 10 dimensional geometry

$$\begin{cases} ds^2 = -\left(1 - \frac{2M}{r^6}\right) dt^2 + \left(1 - \frac{2M}{r^6}\right)^{-1} (dx^a)^2 + dy^2 \qquad (a = 1, \ldots, 8) \\ \Phi = 0 \\ B^{(2)} = 0 = C^{(p)} \end{cases} \tag{2.56}$$

This solution is the generalisation of the Schwarzschild solution (where we have set $G = 1$) along the first 9 direction, and flat along the direction $x^9 = y$; thus it is for sure a solution of the Einstein equations in vacuum, and then a supergravity solution if all gauge fields vanish. This solution can be seen both in Einstein or in string frame, because the dilaton vanishes; it can also be seen both as a Type IIA or IIB solution, because all the gauge fields are trivial. The next step is to make a boost along the y direction:

$$y \to (\text{ch}\alpha) y + (\text{sh}\alpha) t \,, \quad t \to (\text{ch}\alpha) t + (\text{sh}\alpha) y \,. \tag{2.57}$$

This procedure gives another supergravity solution, because the supergravity action is Lorentz invariant. Notice however that when y is a compact direction this is a non-globally defined change of coordinates. This is the reason why we are actually constructing a different physical solution, and not rewriting the same solution in a different set of coordinates. The metric (2.56) becomes

$$ds^2 = dy^2\left(1+\frac{2M}{r^6}\mathrm{sh}^2\alpha\right) + dt^2\left(-1+\frac{2M}{r^6}\mathrm{ch}^2\alpha\right) + 2(\mathrm{ch}\alpha\mathrm{sh}\alpha)\frac{2M}{r^6}dydt + \left(1-\frac{2M}{r^6}\right)^{-1}(dx^a)^2\,. \tag{2.58}$$

This solution corresponds to a wave carrying momentum. The boost has the effect of adding a *charge* to the solution; in this case the charge is precisely the momentum of the wave P. If we now make a T duality along the y direction, we can get another solution with one charge; thus we rewrite the metric as

$$\begin{aligned} ds^2 = &\left(1+\frac{2M}{r^6}\mathrm{sh}^2\alpha\right)\left(dy + \frac{2M\mathrm{ch}\alpha\mathrm{sh}\alpha/r^6}{1+2M\mathrm{sh}^2\alpha/r^6}dt\right)^2 + \\ &+\left(1+\frac{2M}{r^6}\mathrm{sh}^2\alpha\right)^{-1}\left(-1+\frac{2M}{r^6}\right)dt^2 + \left(1-\frac{2M}{r^6}\right)^{-1}(dx^a)^2 \end{aligned}\,. \tag{2.59}$$

We are now ready to apply the formula (2.28), and we arrive at

$$\begin{cases} ds^2 = \left(1+\frac{2M}{r^6}\mathrm{sh}^2\alpha\right)^{-1}\left[dy^2 + \left(-1+\frac{2M}{r^6}\right)dt^2\right] + \left(1-\frac{2M}{r^6}\right)^{-1}(dx^a)^2 \\ e^{2\Phi} = \left(1+\frac{2M}{r^6}\mathrm{sh}^2\alpha\right)^{-1} \\ B^{(2)} = \frac{2M\mathrm{ch}\alpha\mathrm{sh}\alpha/r^6}{1+2M\mathrm{sh}^2\alpha/r^6}dt\wedge dy \\ C^{(p)} = 0 \end{cases} \tag{2.60}$$

This solution sound a bit strange, when compared with the ones we have obtained using the first method. This is because it is not a *BPS* solution. In order to get such a solution, one must take the so-called *BPS* limit, which consists on $M \to 0$ and $\alpha \to \infty$, while the combination $Me^{2\alpha}$ remains constant, precisely

$$Me^{2\alpha} = 2Q\,. \tag{2.61}$$

The *BPS* limit of the previous solution is then

$$\begin{cases} ds^2 = \left(1+\frac{Q}{r^6}\right)^{-1}(-dt^2+dy^2) + (dx^a)^2 \\ e^{2\Phi} = \left(1+\frac{Q}{r^6}\right)^{-\frac{1}{2}} \\ B^{(2)} = \frac{Q/r^6}{1+Q/r^6}dt\wedge dy = \left[1-\left(1+\frac{Q}{r^6}\right)^{-1}\right]dt\wedge dy \\ C^{(p)} = 0 \end{cases} \tag{2.62}$$

We notice that it coincide almost exactly with the solution of the fundamental string $F1$ along the y direction (cfr. equation (2.52)). The only difference is a constant shift in the expression of the B field. This should not worry us: in fact this constant difference is not physical, because the field strength is the same in both cases. The derivation of other solutions can be done using T and S dualities, as explained before.

2.6 The Strominger–Vafa black hole

All the geometries we have encountered up to now have a singularity at $r = 0$; notice that the singularity is not confined at one single point, but it is spread over the brane. One may think that these objects are black holes, but it turns out that they actually have a vanishing horizon area. Therefore it is impossible to interpret these solutions as black holes with thermodynamic properties, as we made in the previous chapter; in fact, assuming the validity of the relation between area and entropy, the entropy of such solutions would be zero. We can solve this problem considering solutions representing different types of branes in equilibrium with each other. In some cases it is possible to have a bound state of branes of different dimension, but not every possibility produces a supergravity solution.

We have seen that the solutions for $M2$, $F1$ and $D2$ depend on a single charge Q. Making bound states with more types of brane leads to solutions depending on more than one charge. Moreover each different type of brane halves the number of supersymmetries of the solution; while the solutions of the previous section were $\frac{1}{2}$-*BPS*, in this section we consider a three-charges, $\frac{1}{8}$-*BPS* solution. Doing so we will find that it has a non-vanishing horizon area, and can be safely considered as a black hole, the Strominger–Vafa black hole.

2.6.1 Derivation of the solution

During the derivation of this solution we will not solve the supersymmetry conditions, but we will follow the indirect method described in the previous section. The starting point is a 10 dimensional spacetime with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$. Let us call (t, x^i) the coordinates of the $\mathbb{R}^{1,4}$, y the coordinate of the circle S^1 and z^a the coordinates of the torus T^4. This particular choice of the spacetime allows us to easily make a dimensional reduction of the final solution; the black hole we will find can also be interpreted as a black hole in 5 dimensions. Let us now follow the derivation step by step.

- Let us start with a solution which is the analogous of the Schwarzschild one in the $\mathbb{R}^{1,4}$ directions, i.e.

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + dy^2 + \sum_a (dz^a)^2 \,. \tag{2.63}$$

  Here we have set $G = 1$ and $r^2 = (x^1)^2 + \ldots (x^4)^2$. Furthermore we have used polar coordinates instead of the cartesian coordinates x^i:

$$x^1 = r \sin\theta \cos\phi \,, \quad x^2 = r \sin\theta \sin\phi \,, \quad x^3 = r \cos\theta \cos\psi \,, \quad x^4 = r \cos\theta \sin\psi \,, \tag{2.64}$$

  where $\theta \in [0, \frac{\pi}{2}]$ and $\phi, \psi \in [0, 2\pi]$. The angular part of the metric is then

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \,. \tag{2.65}$$

  Here and in the following, fields like Φ, $B^{(2)}$ and $C^{(p)}$ should be considered as trivial, when not written explicitly.

- The second step is to perform a boost (with parameter α) along the y direction. The calculation is very similar to that one we have made in the previous section, and the result is

$$ds^2 = \left(1 + \frac{2M}{r^2}\text{sh}^2\alpha\right)\left(dy + \frac{2M\text{ch}\alpha\text{sh}\alpha/r^2}{1+2M\text{sh}^2\alpha/r^2}dt\right)^2 + \\ + \left(1 + \frac{2M}{r^2}\text{sh}^2\alpha\right)^{-1}\left(-1 + \frac{2M}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2 + \sum_a (dz^a)^2 \,. \tag{2.66}$$

With this step we have introduced the first of the 3 charges characterizing the final solution. We are left with a solution of Type IIA supergravity describing not a brane, but a wave with momentum along the y direction; we name it P_y.

- Let us now perform a T duality along the y direction. The result will be a solution of Type IIB supergravity describing a fundamental string; we denote it as $F1_y$. For simplicity let us use the definition

$$S_\alpha = \left(1 + \frac{2M}{r^2}\text{sh}^2\alpha\right) \,. \tag{2.67}$$

The solution for the fundamental string is given by

$$\begin{cases} ds^2 = S_\alpha^{-1}\left[dy^2 + \left(-1 + \frac{2M}{r^2}\right)dt^2\right] + \left(1 - \frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2 + \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ B^{(2)} = \frac{2M}{r^2}\text{ch}\alpha\text{sh}\alpha S_\alpha^{-1} dt \wedge dy \end{cases} \tag{2.68}$$

- The next step is to introduce the second charge, and we do so making another boost along y; let the parameter of this new boost be β. We are then left with a solution describing a string carrying momentum, denoted by the $F1_y$-P_y:

$$\begin{cases} ds^2 = S_\alpha^{-1}S_\beta\left(dy + \frac{2M\text{ch}\beta\text{sh}\beta/r^2}{1+2M\text{sh}^2\beta/r^2}dt\right)^2 + S_\alpha^{-1}S_\beta^{-1}\left(-1 + \frac{2M}{r^2}\right)dt^2 + \\ \qquad + \left(1 - \frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2 + \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ B^{(2)} = \frac{2M}{r^2}\text{ch}\alpha\text{sh}\alpha S_\alpha^{-1} dt \wedge dy \end{cases} \tag{2.69}$$

Here S_β is defined in the very same way as S_α was.

- We now want to change perspective, applying an S duality. The resulting solution will be denoted by $D1_y$-P_y; it describes a $D1$ brane along the y direction carrying momentum. The solution is

$$\begin{cases} ds^2 = S_\alpha^{-\frac{1}{2}}S_\beta\left(dy + \frac{2M\text{ch}\beta\text{sh}\beta/r^2}{1+2M\text{sh}^2\beta/r^2}dt\right)^2 + S_\alpha^{-\frac{1}{2}}S_\beta^{-1}\left(-1 + \frac{2M}{r^2}\right)dt^2 + \\ \qquad + S_\alpha^{\frac{1}{2}}\left[\left(1 - \frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2 + \sum_a (dz^a)^2\right] \\ e^{2\Phi} = S_\alpha \\ C^{(2)} = -\frac{2M}{r^2}\text{ch}\alpha\text{sh}\alpha S_\alpha^{-1} dt \wedge dy \end{cases} \tag{2.70}$$

- The next step is to arrive at a solution describing a $D5$ brane instead of the $D1$. This can be done applying four T dualities, along the four directions of the torus T^4. The result is again a solution of Type IIB supergravity, which reads:

$$\begin{cases} ds^2 = S_\alpha^{-\frac{1}{2}} S_\beta \left(dy + \frac{2M\mathrm{ch}\beta\mathrm{sh}\beta/r^2}{1+2M\mathrm{sh}^2\beta/r^2} dt \right)^2 + S_\alpha^{-\frac{1}{2}} S_\beta^{-1} \left(-1 + \frac{2M}{r^2} \right) dt^2 + \\ \qquad + S_\alpha^{\frac{1}{2}} \left[\left(1 - \frac{2M}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] + S_\alpha^{-\frac{1}{2}} \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ C^{(6)} = -\frac{2M}{r^2} \mathrm{ch}\alpha\mathrm{sh}\alpha S_\alpha^{-1} dt \wedge dy \wedge dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \end{cases} \tag{2.71}$$

We notice that the solution is given in terms of the 6-form: the standard form describing a $D5$ brane is instead a 2-form, dual of the 6-form. The exact relation is given by

$$dC^{(6)} = \star dC^{(2)} \,. \tag{2.72}$$

The calculation is a bit complicated, and we will do it only when we will take the extremal limit. Let us now simply observe that the 2-form will have non zero component only along the angular coordinates ϕ and ψ. Thus we write:

$$C^{(2)} = f(r, \alpha, \beta) d\phi \wedge d\psi \,. \tag{2.73}$$

- Let us now turn to a description of a $NS5$ brane with momentum, performing an S duality. The result is a bit simpler, and reads:

$$\begin{cases} ds^2 = S_\beta \left(dy + \frac{2M\mathrm{ch}\beta\mathrm{sh}\beta/r^2}{1+2M\mathrm{sh}^2\beta/r^2} dt \right)^2 + S_\beta^{-1} \left(-1 + \frac{2M}{r^2} \right) dt^2 + \\ \qquad + S_\alpha \left[\left(1 - \frac{2M}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] + \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha \\ B^{(2)} = f(r, \alpha, \beta) d\phi \wedge d\psi \end{cases} \tag{2.74}$$

- A T duality along y allows us to have a solution describing two types of branes in equilibrium. We are left with a Type IIA solution, which we call $NS5_{y1234}$-$F1_y$:

$$\begin{cases} ds^2 = S_\beta^{-1} dy^2 + S_\beta^{-1} \left(-1 + \frac{2M}{r^2} \right) dt^2 + S_\alpha \left[\left(1 - \frac{2M}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] + \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha S_\beta^{-1} \\ B^{(2)} = f(r, \alpha, \beta) d\phi \wedge d\psi + \frac{2M\mathrm{ch}\beta\mathrm{sh}\beta/r^2}{1+2M\mathrm{sh}^2\beta/r^2} dt \wedge dy \end{cases} \tag{2.75}$$

- We now want to perform an S duality in order to have a solution describing the bound state of a $D1$ and a $D5$ brane. But the current one is a Type IIA solution, so we must first move to a Type IIB description. We do so making a T duality; we choose a particular direction along the torus, say z_1. Notice that this will not change the explicit form of the metric and the other fields, because $g_{z_1 z_1} = 1$. Anyway we have to remember that z_1 will be a privileged direction. After making this T duality we can perform the S duality and arrive to the solution $D5_{y1234}$-$D1_y$:

$$\begin{cases} ds^2 = S_\alpha^{-\frac{1}{2}} S_\beta^{-\frac{1}{2}} dy^2 + S_\alpha^{-\frac{1}{2}} S_\beta^{-\frac{1}{2}} \left(-1 + \frac{2M}{r^2}\right) dt^2 + \\ \qquad + S_\alpha^{\frac{1}{2}} S_\beta^{\frac{1}{2}} \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right] + S_\alpha^{-\frac{1}{2}} S_\beta^{\frac{1}{2}} \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} S_\beta \\ C^{(2)} = -f(r,\alpha,\beta) d\phi \wedge d\psi - \frac{2M \mathrm{ch}\beta \mathrm{sh}\beta / r^2}{1 + 2M \mathrm{sh}^2\beta / r^2} dt \wedge dy \end{cases} \tag{2.76}$$

- We are now ready for the last step, where we introduce the third charge. We do it again by performing a boost along the y direction, with parameter γ. Therefore we arrive to a solution describing the bound state of two type of branes with momentum, that we indicate as $D1$-$D5$-P. The final solution is

$$\begin{cases} ds^2 = S_\alpha^{-\frac{1}{2}} S_\beta^{-\frac{1}{2}} \left(dy + \frac{2M \mathrm{ch}\gamma \mathrm{sh}\gamma / r^2}{1 + 2M \mathrm{sh}^2\gamma / r^2} dt\right)^2 + \\ \qquad + S_\alpha^{\frac{1}{2}} S_\beta^{\frac{1}{2}} \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right] + S_\alpha^{-\frac{1}{2}} S_\beta^{\frac{1}{2}} \sum_a (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} S_\beta \\ C^{(2)} = -f(r,\alpha,\beta) d\phi \wedge d\psi - \frac{2M \mathrm{ch}\beta \mathrm{sh}\beta / r^2}{1 + 2M \mathrm{sh}^2\beta / r^2} dt \wedge dy \end{cases} \tag{2.77}$$

We are interested in the extremal limit of this solution; let us now make the limit $M \to 0$, $\alpha, \beta, \gamma \to \infty$ where

$$Me^{2\alpha} = 2Q_5 \,, \quad Me^{2\beta} = 2Q_1 \,, \quad Me^{2\gamma} = 2Q_P \,. \tag{2.78}$$

It is necessary to make this limit only at the end; in fact a boost acts trivially on a BPS solution, so we cannot make the limit before, separately for α, β and γ. The final result for our solution is the following:

$$\begin{cases} ds^2 = Z_1^{-\frac{1}{2}} Z_5^{-\frac{1}{2}} \left(-dt^2 + dy^2 + K(dt+dy)^2\right) + Z_1^{\frac{1}{2}} Z_5^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_3^2\right) + Z_1^{\frac{1}{2}} Z_5^{-\frac{1}{2}} \sum_a (dz^a)^2 \\ e^{2\Phi} = Z_1 Z_5^{-1} \\ C^{(2)} = -Q_5 \sin^2\theta d\phi \wedge d\psi + (1 - Z_2^{-1}) dt \wedge dy \end{cases} \tag{2.79}$$

where we have defined the function $Z_{1,5}$ and K as

$$Z_{1,5} = 1 + \frac{Q_{1,5}}{r^2} \,, \quad K = Z_P - 1 = \frac{Q_P}{r^2} \tag{2.80}$$

The suffix 1 or 5 stands for $D1$ or $D5$ brane; in fact the charge Q_5 is related to the boost with parameter α, that we have used to "construct" the $D5$ brane.

2.6.2 Properties of the black hole

We want to derive physical properties of this black hole, in particular its entropy. This is possible only in the Einstein frame of the metric, and not in the string frame (2.79). In Einstein frame the metric is given by

$$ds^2 = Z_1^{-\frac{3}{4}} Z_5^{-\frac{1}{4}} \left(-dt^2 + dy^2 + K(dt+dy)^2\right) + Z_1^{\frac{1}{4}} Z_5^{\frac{3}{4}} \left(dr^2 + r^2 d\Omega_3^2\right) + Z_1^{\frac{1}{4}} Z_5^{-\frac{1}{4}} \sum_a (dz^a)^2 \,. \quad (2.81)$$

The formula of the Bekenstein–Hawking entropy is valid not only in four but also in higher dimensions. One should consider that the Newton constant G (which appear in the Einstein–Hilbert Lagrangian) depends on the dimension of the spacetime. Let us now calculate the entropy of the Strominger–Vafa black hole both in 10 and in 5 dimensions, and compare the two results. In 10 dimensions the volume element depends on the determinant of the metric. We have

$$\sqrt{-g} = r^3 \sin\theta \cos\theta \sqrt{Z_1 Z_5 Z_P} \,. \quad (2.82)$$

The event horizon is located at $r = 0$ and its area is given by

$$A_{10} = V_5 \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta \sqrt{Q_1 Q_5 Q_P} = 2\pi^2 V_5 \sqrt{Q_1 Q_5 Q_P} \,, \quad (2.83)$$

where V_5 is the volume of the compactified $S^1 \times T^4$. Therefore the entropy of the black hole is symmetric in the three charges and, setting $\hbar = k_B = 1$, reads:

$$S = \frac{A_{10}}{4G_{10}} = \frac{V_5 \pi^2 \sqrt{Q_1 Q_5 Q_P}}{2G_{10}} \,. \quad (2.84)$$

The fact that the area (and also the entropy) is finite crucially depends on the number of charges of the solution; notice that with only two charges, the volume element would have vanished at the event horizon $r = 0$. The calculation of the entropy is possible also in 5 dimension; one must firstly calculate the form of the 5 dimensional metric derived from (2.81). It is not sufficient to cancel the terms related to the S^1 or the T^4; one should also multiply for suitable coefficients, so that the 5 dimensional action takes the right form of an Einstein–Hilbert action. In doing so one finds also the relation between the Newton constant in different dimension, i.e.

$$G_{10} = V_5 G_5 \,. \quad (2.85)$$

We do not follow each step explicitly, but we write only the final form of the metric in 5 dimensions, which is symmetric in the three charges:

$$ds^2 = -(Z_1 Z_5 Z_P)^{-\frac{2}{3}} dt^2 + (Z_1 Z_5 Z_P)^{\frac{1}{3}} (dr^2 + r^2 d\Omega_3^2) \,. \quad (2.86)$$

The event horizon of this geometry is located at $r = 0$, and its area is given by

$$A_5 = \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta \sqrt{Q_1 Q_5 Q_P} = 2\pi^2 \sqrt{Q_1 Q_5 Q_P} \,. \quad (2.87)$$

The only difference with the 10 dimensional area consists in the volume V_5; this is exactly balanced by the relation between G_{10} and G_5. Thus the entropy (which is a physical quantity) is the same when calculated in 10 or 5 dimension, as should be expected:

$$S = \frac{A_5}{4G_5} = \frac{\pi^2 \sqrt{Q_1 Q_5 Q_P}}{2G_5} = \frac{A_{10}}{4G_{10}} \,. \quad (2.88)$$

If one wants also to calculate the mass associated to this geometry, one should expand the g_{00} component of the metric in 5 dimension, i.e.

$$g_{00} = -(Z_1 Z_5 Z_P)^{-\frac{2}{3}} \simeq -1 + \frac{2}{3} \frac{Q_1 + Q_5 + Q_P}{r^2} \,. \quad (2.89)$$

A comparison with the formula given in the previous chapter shows that there is a precise relation between the mass and the charges of this solution:

$$M = \frac{Q_1 + Q_5 + Q_P}{4\pi G}\,. \tag{2.90}$$

It turns out that is mass is actually the smallest one compatible with these three charges: the Strominger–Vafa black hole is indeed extremal.

Let us now make a little observation on the construction of black holes in supergravity. We have seen that a solution with an event horizon with non-vanishing area exists if one considers three charges, i.e. a bound state of different types of branes. But we should remember that a solution with an event horizon represent a true black hole only when its mass is big enough to have the event horizon external to the object (notice that the horizon is at $r = 0$ only in this particular coordinate system). Equation (2.90) tells us that the mass is related to the charges: a big mass means a big value for the charges. Therefore wee see that a black hole is characterised by large values of its charges: this is possible if one considers the superposition of many branes, each one parallel to others of the same type. In conclusion, a black hole in supergravity can be constructed as a bound state of many branes, divided into a few sets of parallel branes of the same type.

In the following we will study in more details this black hole, which can naturally be seen as a five dimensional black hole. The construction of a black hole in four dimensions is also possible, but requires the presence of at least four charges. In the rest of the thesis we will try to derive some microstate geometries of the Strominger–Vafa black hole; the same goal would be more difficult in the case of a solution with four charges.

In order to understand how to deal with the microstate of a given supergravity solution, we need a theory of quantum gravity. String theory provides a consistent theory, and we are going to investigate how black holes arise in this theory. First of all, however, we need a brief introduction to the key concepts and results of string theory.

Chapter 3

A brief introduction to string theory

String theory is one of the most promising theory of quantum gravity. Therefore one should expect that a microscopic interpretation of the thermodynamic properties of black holes is possible thanks to this theory. Our final goal is to understand how a black hole arises in string theory, and to count (and identify) the microstates of a given black hole. In this chapter we briefly describe what string theory is, and the fundamental properties that we will need in the following. Some reviews on these arguments can be found, for example, in [17], [18], [19] and [20].

3.1 Bosonic string theory

The concept of string as a fundamental object is a natural generalisation of the concept of particle. While a particle is a pointlike object, the string (as suggested by its name) is a one dimensional object. The action of the free string theory is therefore just the obvious generalisation of that of a free particle. We remember that the latter (in D dimension) is given by

$$S = -m \int ds = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}} \qquad (\mu, \nu = 0, \ldots, D-1)\,, \tag{3.1}$$

where s is the proper time, while τ is a generic function parametrizing the worldline of the particle. This action has a problem: the presence of the square root makes the quantisation of this theory complicated. This can be solved by means of another action, which involves an auxiliary field e:

$$S' = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - em^2\right) \quad \text{where} \quad \dot{X}^2 = \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}\,. \tag{3.2}$$

This action gives the same equation of motion for the field X^μ as the action (3.1), when the auxiliary e is integrated away. Notice that this action does not have any problematic square root, and fits also for massless particles. We can also interpret the new field as some sort of (one dimensional) metric on the worldline; defining $e = \sqrt{-g}$, the action (3.2) becomes

$$S' = -\frac{1}{2}\int d\tau \sqrt{-g}\left(g^{-1}\dot{X}^2 - m^2\right)\,. \tag{3.3}$$

We can now turn to the generalisation of these actions in the case of string theory.

3.1.1 The Nambu–Goto action

We promote the field X^μ to depend not only on the time τ, but also on the coordinate σ parametrizing the string. As time passes the string describes a two dimensional surface, the so-called worldsheet; let us parametrise it via the coordinates $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. On the other hand, the space where the string lives is called the *target space*, and it is a D dimensional manifold with metric $\eta_{\mu\nu}$. The most natural generalisation of the action (3.1) is the so-called *Nambu–Goto action*, which reads (see [18])

$$S_{NG} = -\frac{T}{2}\int d^2\sigma \sqrt{-\det h_{ab}}\,, \tag{3.4}$$

where $a, b = 1, 2$ and h_{ab} is the induced metric on the worldsheet given by

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,. \tag{3.5}$$

The quantity T is the *tension* of the string, and it is the generalisation of the mass for a particle. It is conventionally related to another constant, the *universal Regge slope* α', through

$$T = \frac{1}{2\pi\alpha'}\,. \tag{3.6}$$

We see that while the action (3.1) for a point particle was proportional to the length $\int ds$, the Nambu–Goto action is proportional to the area of the worldsheet. This action has the same problem of the corresponding one for a particle: the presence of the square root makes the quantisation more difficult. Anyway we can derive the classical properties of string theory from this action; it is important to identify which are the symmetries of (3.4). First of all, we have Poincaré invariance of the spacetime: this symmetry does not depend on the worldsheet coordinates σ^a. Furthermore we have reparametrisation invariance of the coordinates σ^a; this reflects the fact that the worldsheet coordinates do not have physical meaning.

3.1.2 The Polyakov action

The problem of the square root in the Nambu–Goto action can be solved in a similar fashion as one does with the point particle. We introduce an additional field, the worldsheet metric g_{ab}, together with its inverse g^{ab}. We can then write the so-called *Polyakov action*, which is ([18])

$$S_P = -\frac{T}{2}\int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,, \tag{3.7}$$

where $g = \det g_{ab}$. This action is equivalent to the Nambu–Goto one, in the sense that the equations of motion for X^μ are the same in the two cases, when one uses the equation of motion for the metric g_{ab} deriving from the Polyakov action. We notice that (3.7) has the same symmetries of (3.4): the reparametrisation invariance can be seen as the diffeomorphisms invariance on the worldsheet. Furthermore, the Polyakov action has an additional symmetry, called *Weyl invariance*. A Weyl transformation corresponds to a “dilatation” of the metric coefficients dependent on the worldsheet point, and reads:

$$X^\mu(\sigma) \to X^\mu(\sigma)\,, \quad g_{ab}(\sigma) \to \Omega^2(\sigma) g_{ab}(\sigma)\,. \tag{3.8}$$

An important fact is that one can use these symmetries in order to fix three degrees of freedom; typically one goes in the so-called *conformal gauge*, which corresponds to the choice of a flat metric on the worldsheet: $g_{ab} = \eta_{ab} = \text{diag}(-1, 1)$.

Using this fact on can simply derive the equation of motion for the field X^μ, which reads:

$$\partial^a \partial_a X^\mu = 0 \tag{3.9}$$

where the worldsheet index a is raised and lowered by means of the metric g (in the same way we raise and lower spacetime indices $\mu, \nu, \ldots$ using the metric η). We have to remember that even in the harmonic gauge we have to impose the equations of motion for the metric g_{ab}, which should now be considered as constraints. Remembering that the energy-momentum tensor is given by the variation of the action with respect to the metric, we see that these constraints are nothing but

$$T_{ab} = \partial_a X^\mu \partial^a X_\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu = 0\,. \tag{3.10}$$

A closer analysis shows that these constraints have an important interpretation: the physical degrees of freedom describe only transverse oscillations of the string.

The Weyl invariance of the Polyakov action implies that string theory is a two dimensional conformal field theory on the worldsheet; this fact is important, in that the study of such a theory is really simplified by the presence of the conformal symmetry.

3.2 Open strings and D-branes

The actions given in the previous section describe the local properties of the fundamental string; it is also important to look at global features. We have essentially two types of strings, closed and open.

3.2.1 Closed and open strings

A closed string is parametrised by a coordinate σ in a compact domain, which is conventionally chosen to be $[0, 2\pi]$. An open string has instead two non-coinciding ends. One usually choose to parametrise the string using the coordinate $\sigma \in [0, \pi]$. In figure 3.1 we can schematically see the worldsheet of a closed and an open string, parametrised by σ and the time τ:

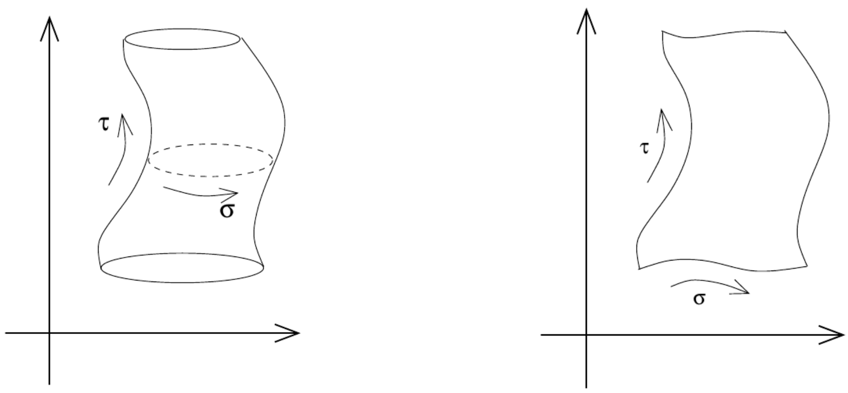

Figure 3.1: Closed (left) and open (right) string worldsheet.

In the open string case the derivation of the equation of motion requires appropriate boundary conditions; let us see it starting from the Polyakov action written in the harmonic gauge:

$$S = -\frac{T}{2} \int d^2\sigma \partial_a X^\mu \partial^a X_\mu\,. \tag{3.11}$$

We consider an open string evolving from some configuration at the initial time τ_i, to some other configuration at time τ_f: the variation of the action is then

$$\delta S = -T \int_{\tau_i}^{\tau_f} d\tau \int_o^{\pi} \partial_a X^\mu \partial^a \delta X_\mu = T \int_{\tau_i}^{\tau_f} d\tau \int_o^{\pi} \partial^a \partial_a X^\mu \delta X_\mu + \text{total derivative}\,. \tag{3.12}$$

We have two terms involving total derivatives with respect to the coordinates τ and σ respectively:

$$T \left[\int_0^{\pi} d\sigma \partial_\tau X^\mu \delta X_\mu \right]_{\tau=\tau_i}^{\tau=\tau_f} - T \left[\int_{\tau_i}^{\tau_f} d\tau \partial_\sigma X^\mu \delta X_\mu \right]_{\sigma=0}^{\sigma=\pi} . \tag{3.13}$$

The first term is of the kind one always gets when dealing with the least action principle; the second one is non trivial, and its vanishing gives rise to the requirement

$$\partial_\sigma X^\mu \delta X_\mu = 0 \qquad \text{at } \sigma = 0, \pi \tag{3.14}$$

There are essentially two ways in order to satisfy this relation:

- Neumann boundary conditions: $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$.
  We do not have any restriction on the position of the ends of the string; hence they can move freely. It turns out that both ends move with the speed of light.

- Dirichlet boundary conditions: $\delta X^\mu = 0$ at $\sigma = 0, \pi$.
  This means that the endpoints of the string lie at some fixed position in space ($X^\mu = c^\mu$).

3.2.2 Boundary conditions and D-branes

It is also possible to consider mixed boundary conditions, i.e. Neumann conditions for $\mu = 0, \dots p$ and Dirichlet conditions along the other directions ($\mu = p+1, \dots, D-1$). These conditions fix the end of the string to lie on a $p+1$ dimensional hypersurface in spacetime (notice that we have always Neumann condition on $\mu = 0$). Such an hypersurface is conventionally called *Dp-brane*, where D stands for Dirichlet and p indicates the spatial dimensions of the brane. The situation is represented in figure 3.2.

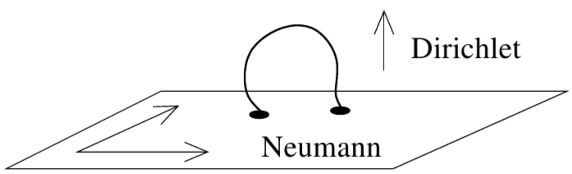

Figure 3.2: An open string with endpoints on a *Dp*-brane .

The two ends of the brane can lie on the same brane, or on two different branes; their dimension could also be different.

We then see that a theory of open strings naturally has to deal with D-branes. It turns out that branes should be considered as dynamical objects, in the same way as fundamental strings. The right action for a D-brane is the higher dimensional generalisation of the Nambu–Goto action, i.e.

$$D_{Dp} = -T_p \int d^{p+1}\xi \sqrt{-\det \gamma}\,. \tag{3.15}$$

where T_p is the tension of the brane, ξ^a ($a = 0, \dots, p$) are the coordinates of the worldvolume of the brane, and γ_{ab} is the pullback of the spacetime metric:

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} \,. \tag{3.16}$$

Anyway strings play a more fundamental role than branes; in fact only strings are characterized by the conformal invariance of the action. This symmetry plays a fundamental role, because it makes the theory much simpler. The consequence is that the spectrum of string theory is a discrete spectrum, and can be derived quite easily.

3.3 String theory spectrum

We are studying string theory because it is a consistent theory of quantum gravity; therefore we are interested not only in the classical theory of strings, as described by the Nambu–Goto (or Polyakov) action. We must turn to the quantisation of the theory. The starting point is to express the classical fields in terms of their Fourier modes. It is useful to introduce the coordinates $\sigma^\pm$:

$$\sigma^\pm = \tau \pm \sigma = \sigma^0 \pm \sigma^1 \,. \tag{3.17}$$

The equations of motion (3.9) read simply $\partial_+\partial_- X^\mu = 0$. In the case of closed strings the most general solution of these equation is

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \,, \tag{3.18}$$

where X^μ_L and X^μ_R are arbitrary functions, subjected to the constraint (3.10) and to the periodicity condition

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma + 2\pi) \,. \tag{3.19}$$

The most general periodic solution is conventionally expanded in Fourier modes in the following way:

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}^\mu_n e^{-in\sigma^+} \,, \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-} \,. \end{aligned} \tag{3.20}$$

It is also useful to define the zero modes as

$$\alpha^\mu_0 = \widetilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu \,. \tag{3.21}$$

One can then see that the constraints (3.10) can be written as

$$L_n = \widetilde{L}_n = 0 \quad n \in \mathbb{Z} \,. \tag{3.22}$$

The quantities L_n and $\widetilde{L}_n$ are the oscillator modes of the constraints. They are nothing but the oscillator modes of the energy-momentum tensor, the explicit expression of which is

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha^\mu_{n-m}\alpha^\nu_m \eta_{\mu\nu} \,. \tag{3.23}$$

The expression for $\widetilde{L}_n$ is the same, with the modes $\widetilde{\alpha}^\mu_n$ instead of α^μ_n. The constraints arising from L_0 and $\widetilde{L}_0$ are simply connected to the square of the spacetime momentum p^μ, which is nothing else than the rest mass of a particle. These two constraints are named *level matching conditions*, and read:

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} = \frac{4}{\alpha'} \sum_{n>0} \widetilde{\alpha}^\mu_{-n} \widetilde{\alpha}^\nu_n \eta_{\mu\nu} \,. \tag{3.24}$$

The quantisation of the theory starts from the observation that the modes α_n and $\widetilde{\alpha}_n$ (for every μ) can be promoted, up to some normalisation, to be annihilation (for $n > 0$) and creation (for $n < 0$) operators. One effect is that (3.24) receive an additional contribution coming from the normal ordering of these operators; the true level matching conditions are

$$M^2 = \frac{4}{\alpha'} \left(-1 + \sum_{n>0} \alpha^\mu_{-n} \alpha^\nu_n \eta_{\mu\nu} \right) = \frac{4}{\alpha'} \left(-1 + \sum_{n>0} \widetilde{\alpha}^\mu_{-n} \widetilde{\alpha}^\nu_n \eta_{\mu\nu} \right) \,. \tag{3.25}$$

The space of quantum states can be constructed as a Fock space, starting from the family of vacuum states $|0;p\rangle$, defined by

$$\begin{cases} \alpha^\mu_n |0;p\rangle = \widetilde{\alpha}^\mu_n |0;p\rangle = 0 \quad (n > 0) \\ \sqrt{\dfrac{2}{\alpha'}} \alpha^\mu_0 |0;p\rangle = \sqrt{\dfrac{2}{\alpha'}} \widetilde{\alpha}^\mu_0 |0;p\rangle = p^\mu |0;p\rangle \end{cases} \tag{3.26}$$

We notice that we have a vacuum state for each possible momentum vector p^μ. The construction of the Fock space continues applying the creation operators to these vacuum states. Anyway this is not the end of the story, because one must also impose the validity of the constraints. There are two ways in order to do so: one is to solve the constraints in the classical theory, and then quantise; this is called lightcone quantisation. The other is to first quantise the theory, and then solve the constraint at the quantum level; it is called covariant quantisation.

We do not follow explicitly any of these two approaches, but only notice that the constraint has the effect of eliminating two possible directions of oscillations (only transverse fluctuation are possible). A remarkable result of the quantisation is that a consistent quantum theory of strings is possible only if the dimension D of the spacetime is $D = 26$. Furthermore, the vacuum state turns out to be a tachyon, i.e. it is characterised by $M^2 < 0$. The presence of this tachyon is indeed a problem of this (bosonic) string theory; we will see later that it can be solved turning to superstring theory, i.e. introducing fermions on the worldsheet.

The first exited states are obtained applying one right-moving and one left-moving oscillator:

$$\widetilde{\alpha}^i_{-1} \alpha^j_{-1} |0;p\rangle \quad \text{with } i,j = 1, \ldots, 24 \,. \tag{3.27}$$

Here we have renamed the oscillators in such a way that the transverse oscillations are associated to the indices i and j. These states are characterised by $M^2 = 0$ and can be associated to massless fields in spacetime; in fact they can be decomposed in representations of the little group $SO(24)$ in the following way:

- a symmetric traceless field $g_{ij}(X)$, which we interpret as the metric;
- an antisymmetric fields $B_{ij}(X)$, called Kalb–Ramond field;
- a scalar field $\Phi(X)$, the dilaton.

We see that the massless field content of the theory reproduces some of the fields we encountered in the study of supergravity in 10 dimensions. Anyway we do not have either any field analogous to the gauge forms $C^{(p)}$, or any fermionic field.

One can also consider more than one excitation, thus getting massive fields; we do not consider them here. We notice however that their mass must be comparable to the Planck mass, and thus it is natural to think that they would not contribute to any process characterised by "normal" energy, i.e. very smaller than the Planck scale.

In the case of open strings there are some differences both in the classical theory and in the quantisation. The resulting spectrum, anyway, can be derived in a similar way. Open strings are characterized by only one set of oscillators: this means that the massless spectrum of the theory will be constituted by a particle with the properties of a photon. However, it should be noticed that even a theory of open strings requires the presence of gravity, because it naturally describes also closed string, coming from the interaction of open ones.

3.4 String theory as conformal field theory

In this section we look in more detail at the conformal symmetry characterizing the Polyakov action. We will introduce some notations and state some results that will be useful in the following.

3.4.1 Conformal group and algebra

A conformal (or Weyl) transformation produces a scaling of the metric, dependent on the worldsheet point:

$$g_{ab}(\sigma) \to \Omega^2(\sigma) g_{ab}(\sigma) \,. \tag{3.28}$$

Roughly speaking, such a transformation has the effect of deforming in an arbitrary way (preserving the angles) the worldsheet; an example is shown in figure 3.3.

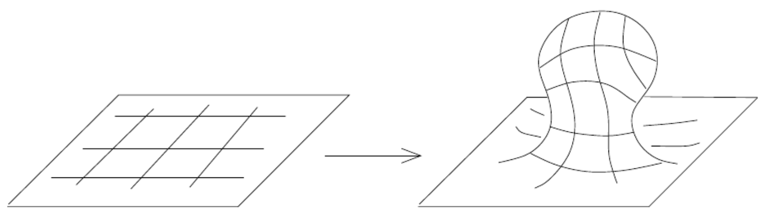

Figure 3.3: An example of a Weyl transformation.

In order to study a bit closer the properties of a 2 dimensional conformal field theory, it is useful to deal with an Euclidean worldsheet, the coordinate of which are $(\sigma^1, \sigma^2) = (\sigma, i\tau)$. One usually defines also the complex coordinates

$$z = \sigma^1 + i\sigma^2 \,, \quad \bar{z} = \sigma^1 - i\sigma^2 \,. \tag{3.29}$$

Conformal invariance in 2 dimension allows us to work always with flat metric; this means that we can fix a gauge in such a way that the worldsheet metric is

$$ds^2 = -d\tau^2 + d\sigma^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz d\bar{z} \,. \tag{3.30}$$

In complex coordinates conformal transformations of flat space are very simple, and correspond to any holomorphic change of coordinates, i.e.

$$z \to z' = f(z) \,, \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}) \,, \tag{3.31}$$

where the functions $f(z)$ and $\bar{f}(\bar{z})$ are arbitrary. We see that the set of these transformations has the structure of a group, the so-called *conformal group*. The corresponding algebra is characterised by an infinite set of generators, which are conventionally defined as

$$l_n = -z^{n+1}\partial_z \,, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \,. \tag{3.32}$$

One can easily see that these generators satisfy the following commutation relations:

$$[l_m, l_n] = (m-n)l_{m+n} \,, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \,, \quad [\bar{l}_m, l_n] = 0 \,. \tag{3.33}$$

The generators l_n and $\bar{l}_n$ define two identical independent subalgebras, each of one is called *Witt algebra*. When these generators are promoted to quantum operators, it turns out that the commutation relations must be modified in the following way:

$$[l_m, l_n] = (m-n)l_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \,, \tag{3.34}$$

and similarly for the generators $\bar{l}_m$. This algebra is called *Virasoro algebra*; the new term is called central extension, and c is the *central charge* of the theory. This name reflects the fact that c commutes with all the generators l_m and $\bar{l}_m$. Another important subalgebra is the one generated by l_0 and $l_{\pm 1}$; the corresponding group is called *restricted conformal group*, which is isomorphic to $SL(2, \mathbb{C})$. The transformations of this group can be parametrised by

$$z \to \frac{az+b}{cz+d} \,, \quad \text{where } a, b, c, d \in \mathbb{C} \,, \; ad - bc = 1 \,. \tag{3.35}$$

This group contains translations and scaling transformations as special cases. An analogous subgroup can be defined for transformations of the antiholomorphic variable $\bar{z}$.

3.4.2 Primary fields

An important object in conformal field theory is the energy-momentum tensor T_{ab}, which is defined as

$$T_{ab} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{ab}} \,, \tag{3.36}$$

where the coefficient 4π is a matter of convention. A conformal transformation is characterised by a variation of the metric of the form $\delta g_{ab} = \epsilon g_{ab}$. Imposing that the action is invariant under such a transformation implies that

$$0 = \delta S = \int d^2\sigma \frac{\partial S}{\partial g^{ab}} \delta g_{ab} = -\frac{1}{4\pi} \int d^2\sigma \sqrt{g} \epsilon T^a{}_a \,, \tag{3.37}$$

which means that the energy-momentum tensor is traceless. When expressed in complex coordinates, this properties is equivalent to $T_{z\bar{z}} = T_{\bar{z}z} = 0$. On the other hand, the conservation equation $\partial^a T_{ab} = 0$ implies that

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0 \,. \tag{3.38}$$

This means that the energy-momentum tensor is completely determined only by an holomorphic and an antiholomorphic function:

$$T_{zz}(z,\bar{z}) \equiv T(z)\,, \quad T_{\bar{z}\bar{z}}(z,\bar{z}) \equiv \bar{T}(\bar{z})\,. \tag{3.39}$$

The energy-momentum tensor is an example of field; a conformal field theory is characterised by an infinite set of fields. This set contains also derivatives of other fields. Among this set we can define the class of *quasi-primary fields*, that transform under restricted conformal transformation as

$$\Phi(z,\bar{z}) \to \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \Phi(z',\bar{z}')\,. \tag{3.40}$$

h and $\bar{h}$ are not the conjugate of each other; they are real valued numbers, called *conformal weights*. One can see that the energy-momentum tensor is a quasi primary field: in particular $T(z)$ has conformal weights $(h,\bar{h}) = (2,0)$, while $\bar{T}(\bar{z})$ has conformal weights $(0,2)$.

There are also the so-called *primary fields*, which transform as in (3.40) for all conformal transformations, not only the restricted ones. It turns out that $T(z)$ and $\bar{T}(\bar{z})$ are not primary fields. Quasi-primary and in particular primary fields are the building blocks of any conformal field theory; the concept and transformation rules of primary fields can be seen as analogous to those of tensors.

Looking at the Polyakov action (3.7), one can show that X^μ is not primary. Anyway, if we express the theory in complex coordinates, we can see that $\partial_z X^\mu$ and $\partial_{\bar{z}} X^\mu$ are primary fields with conformal weights $(1,0)$ and $(0,1)$ respectively. Other primary fields deriving from the bosonic string theory are the *vertex operators*

$$V_a(z,\bar{z}) =: e^{ia_\mu X^\mu(z,\bar{z})} :\,, \tag{3.41}$$

where : : indicates the normal ordered product. Such operators are primary with conformal weights $(\frac{a^2}{2}, \frac{a^2}{2})$, and are useful in the calculation of Green functions and amplitudes. In many applications one often has to calculate correlators of fields like

$$\langle \mathcal{O}_1(z)\mathcal{O}_2(w)\ldots\rangle\,. \tag{3.42}$$

Fortunately products (and hence correlators) of primary and quasi-primary fields are greatly simplified by the presence of the conformal symmetry. For example, the two point function is completely determined only by the conformal weights of the two quasi-primary fields h_1 and h_2:

$$\langle \mathcal{O}_1(z)\mathcal{O}_2(w)\rangle = \frac{d_{1,2}\ \delta_{h_1,h_2}}{(z-w)^{h_1+h_2}}\,, \tag{3.43}$$

where $d_{1,2}$ is a constant factor, depending on the normalization of the fields. Notice that the correlator is non-vanishing only if the conformal weights of the two fields are equal. Even the form of other correlators is greatly simplified by the conformal symmetry; we remand to [18] and [19] for details.

3.4.3 The ghost system

The conformal symmetry implies also that a completely consistent theory at quantum level requires the presence of ghosts. This is completely analogous to the ghosts appearing when dealing with theories characterized by gauge symmetries. In the case of bosonic string theory one has to add a term to the action of the form:

$$S_{gh} = \frac{1}{2\pi}\int d^2\sigma \sqrt{g} b_{ab} D^a c^b\,, \tag{3.44}$$

where D^a is the covariant derivative coming from the metric g. When expressed in conformal gauge and in complex coordinates, this action reads

$$S_{gh} = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c})\,. \tag{3.45}$$

The equations of motion deriving from S_{gh} imply that b and c are holomorphic fields, while $\bar{b}$ and $\bar{c}$ are antiholomorphic. The ghosts satisfy anticommuting relations and are characterized by conformal weights $h_b = (2, 0)$ and $h_c = (-1, 0)$.

The presence of ghosts is not only a matter of formality, but it has important physical consequences. In particular they are important for the correct calculation of string amplitudes. We will see it in the following.

3.5 Interactions and amplitudes

Once we have identified how particles arise from strings, we would also like to understand how the theory describes interactions. The simplest case is the following: we consider two closed strings which enter in contact, interact, and then exit again the interaction region as isolated closed strings. A diagram corresponding to this process is shown in figure 3.4.

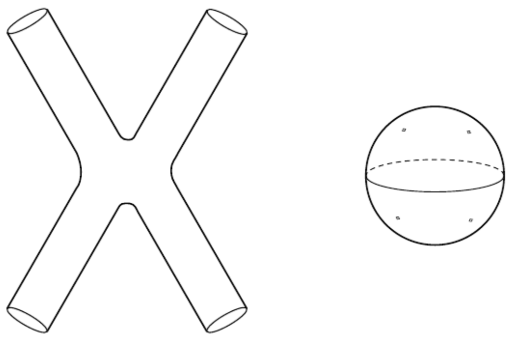

Figure 3.4: The tree-level diagram describing the four-point scattering of closed strings (left) and a conformally equivalent diagram (right).

We can use the conformal invariance of the theory in order to deform our diagram, obtaining a simpler equivalent description. The final result in our case is a sphere with four *punctures*, corresponding to the four initial or final states. We thus see that the contribution of an external state reduces to an insertion of a local operator on the sphere. This properties is deeply related to the conformal invariance of the theory, and gives rise to the so-called *state-operator correspondence*. In the case of bosonic string theory, it turns out that these local operators are related to the vertex operators defined in (3.41), with the quantity a_μ being the momentum of the outgoing string. Schematically the m-point amplitude is given by a path integral of the form

$$\mathcal{A}^{(m)}(p_1, \ldots, p_m) \propto \int \mathcal{D}X\; e^{-S} \prod_{i=1}^{m} W(p_i)\,, \tag{3.46}$$

where the operators $W(p_i)$ are given by

$$W(p_i) \propto \int d^2z V_{p_i}(z, \bar{z})\,. \tag{3.47}$$

We see that we integrate over all possible position of the punctures on the sphere. However, the invariance under the restricted conformal group $SL(2,\mathbb{C})$ implies that we can fix the position of three of the punctures on the sphere, while we have to integrate over all the possible position of the other punctures.

Figure 3.4 shows the tree level contribution to the four points amplitude: the same physical process has contributions coming from a worldsheet with more complicated topologies. It turns out that one has to sum over all possible different topologies, which are typically determined by the genus of the surface. The first contributions to the four points amplitudes are given in figure 3.5.

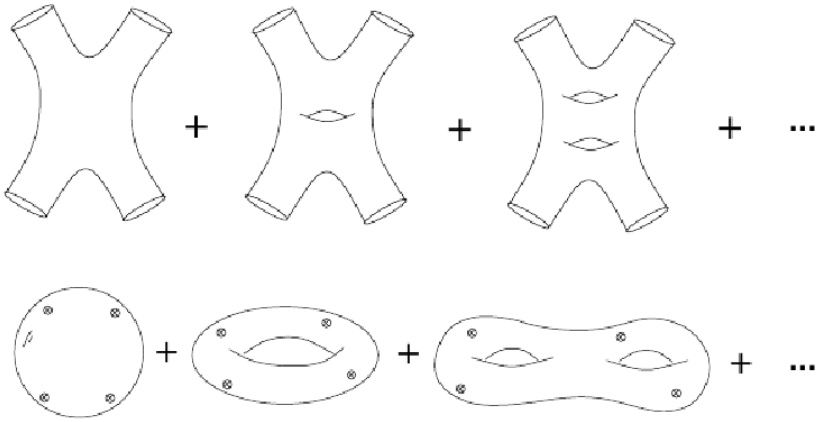

Figure 3.5: Diagrams describing the four-point scattering of closed strings (up) and their conformally equivalent diagrams (down).

The analogous of the loop expansion in quantum field theory is this expansion over all types of topology, which is weighted by powers of the coupling constant g_s.

If one wants to calculate an amplitude corresponding to a process involving open strings, the situation is slightly different. An open string stretches between two branes: the worldsheet corresponding to a tree-level scattering is conformally equivalent to a disk, where suitable boundary conditions must be imposed at the contour. The interaction with another open strings can be encoded in the insertion of a suitable vertex operator at the boundary. It is also possible to consider the interaction of an open and a closed string: in this case the situation is described by a disk with an insertion of a vertex operator in the interior of the disk. This is the situation in which we are going to interest in the following.

3.6 Superstring theories

The theory we have analysed up to now involves only the bosonic fields X^μ on the worldsheet; for this reason it is called *bosonic* string theory. We have seen that it gives rise to a quantum massless spectrum which consists only on bosonic fields on the spacetime: the metric, the Kalb–Ramond field and the dilaton. We have also seen that this theory is problematic, due to the presence of the tachyon. Furthermore a connection with supergravity is not possible: we have no fermions in the theory, and also the dimension of spacetime (26) is inconsistent with a supergravity theory. On the other hand, the massless spectrum of the bosonic string theory does not have either any fermionic field, or any gauge form field. For these reasons we turn to the analysis of the superstring theory (in particular Type II).

The starting point is to introduce fermionic fields at the level of the worldsheet: we will see that this fact will solve the problems explained above. We would like to have supersymmetry even at the level of the worldsheet, thus we introduce D spinors ψ^μ (along with their conjugate $\bar{\psi}^\mu$). An appropriate action for the fermions is the following one ([22]):

$$S_\psi = \frac{i}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\bar{\psi}^\mu \gamma^a \partial_a \psi_\mu \, . \tag{3.48}$$

where γ^a are the gamma matrices in two dimensions. A closer analysis of the action would show that one must impose periodic or antiperiodic boundary conditions for the combinations $\psi^\mu_\pm = \psi^0 \pm \psi^1$. Periodic conditions give rise to the so called *R sector*, while antiperiodic conditions correspond to the *NS sector*:

$$\psi^\mu_\pm(\sigma + 2\pi) = \begin{cases} \psi^\mu_\pm(\sigma) & \text{(R sector)} \\ -\psi^\mu_\pm(\sigma) & \text{(NS sector)} \end{cases} \tag{3.49}$$

We have in total four combinations, because one can choose different types of boundary conditions on ψ^μ_+ and ψ^μ_-. Using the point of view of conformal field theory, one can see that ψ^μ and $\bar{\psi}^\mu$ are primary operators with conformal dimension $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively.

A further step is to combine bosons (X^μ) and fermions (ψ^μ) in the same theory, with an action that should be supersymmetric: the resulting theory is Type II superstring, the action of which is

$$S_{II} = \frac{1}{2\pi\alpha'} \int \sqrt{-g} \left[g^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{i}{2} \bar{\psi}^\mu \gamma^a \partial_a \psi_\mu + \frac{i}{2} \left(\chi_a \gamma^b \gamma^a \psi^\mu \right) \left(\partial_b X_\mu - \frac{i}{4} \chi_b \psi_\mu \right) \right] . \tag{3.50}$$

Here χ_a is the supersymmetric partner of the worldsheet metric g_{ab}, the *gravitino*. This action is invariant under a supersymmetry transformation that involves X^μ and ψ^μ, which we indicate as (1,0). There is also invariance under a supersymmetry involving X^μ and $\bar{\psi}^\mu$, which we call (0,1). In summary, we say that our theory has (1,1) supersymmetry.

This theory, as the bosonic field theory, possesses reparametrisation and conformal invariance. One can use them in order to fix some degrees of freedom, in the very same way we have done introducing the conformal gauge for the Polyakov action. An useful choice is to impose a gauge in which

$$g_{ab} = \eta_{ab} \quad \text{and} \quad \chi_a = 0 \, . \tag{3.51}$$

This choice is called *superconformal gauge*. One should also remember that the equations of motion of these degrees of freedom must be imposed; we are then left with the constraints:

$$T_{ab} = 0 \quad \text{and} \quad G_a = 0 \tag{3.52}$$

where T_{ab} is the energy-momentum tensor, while G_a is its supersymmetric partner, called supercurrent. The expressions for these fields in superconformal gauge, using complex coordinates, are the following ones:

$$T(z) = -\frac{1}{4\alpha'} \partial X^\mu \partial X_\mu - \frac{1}{2} \psi^\mu \partial \psi_\mu \, , \quad G(z) = i\psi^\mu \partial X_\mu \, , \tag{3.53}$$

and similarly for the antiholomorphic components $\bar{T}(\bar{z})$ and $\bar{G}(\bar{z})$.

3.6.1 The superstring spectrum

The quantisation of the theory is conceptually identical to the bosonic case; we have to deal both with bosonic and fermionc oscillators, and impose the constraints corresponding to the vanishing of T_{ab} and G_a. An important feature is that the consistency of the theory requires the dimension of the spacetime to be 10. The derivation of the spectrum of the theory is similar to the case of the bosonic string, but consistency reasons drive to take a further restriction of the Fock space, using the so-called *GSO projection*. Roughly speaking, physical states are only those constructed applying an odd number of fermionic creation operators to a vacuum state. A first consequence is that we are left with no tachyon in the spectrum; the first physical excited states correspond to massless particles on the spacetime. Let us see which is the field content of the massless spectrum, dividing it in the four sectors corresponding to the different ways one can choose the boundary conditions.

- *NS-NS* sector: the massles spectrum is identical to that of the bosonic field theory (now in 10 dimensions), i.e. it consists on the scalar dilaton Φ, the Kalb–Ramond antysymmetric field $B_{\mu\nu}$, and the symmetric traceless graviton $g_{\mu\nu}$.
- *NS-R* sector: the spectrum consists on two fermionic fields, the dilatino and the gravitino (they are the supersymmetric partner of the dilaton and the graviton respectively).
- *R-NS* sector: the spectrum is identical to the *NS-R* sector
- *R-R* sector: in this sector we have again bosonic fields but it is important the way one makes the *GSO* projection. There are two ways to do so, and then we are left with two different possible theories, called Type IIA and IIB superstring theories. In Type IIA we have a 1-form and a 3-form, while in Type IIB we have a 0-form, a 2-form and a self-dual 4-form.

We see that the field content of Type II theories exactly reproduces the Type II supergravity theories we have encountered in the previous chapter; this is an important result of superstring theory. This correspondence between superstring and supergravity is not limited to the massless spectrum of the theory; one can show that the supergravity actions can be derived from the superstring action, taking an appropriate low energy limit (see [17] for details).

3.6.2 The superghost system

The conformal symmetry of the theory implies also the presence of a second ghost system, related to the fermions ψ^μ and $\bar{\psi}^\mu$. While the ghosts of the bosonic string theory b and c were anticommuting, these new ghosts, which we call β and γ, commute. The action of this ghost system is completely analogous to (3.54):

$$S'_{gh} = \frac{1}{2\pi}\int d^2z(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma})\,. \tag{3.54}$$

The conformal weights, however, are different:

$$h_\beta = \frac{3}{2} \quad \text{and} \quad h_\gamma = -\frac{1}{2}\,. \tag{3.55}$$

For practical applications, such as the calculation of string amplitudes, it is useful to parametrize these ghosts in terms of other fields, using the so-called *bosonisation*. We thus introduce new fields, ϕ and χ, such that

$$\beta = e^{-\phi}e^{\xi}\partial\xi \quad \text{and} \quad \gamma = e^{-\xi}e^{\phi}\,. \tag{3.56}$$

With the use of these new fields, the expressions for the vertex operators become simpler; we will use them explicitly in the following. For further details on the ghost systems and their bosonisation, we remand to [21].

What we have described in this chapter is, with no doubt, a brief and incomplete introduction to string theory; we have just introduced the concepts and results which will be needed in the following. In the next chapter we will use string theory in order to understand the microscopic nature of black holes: how does a black hole arise in string theory and what is its relation with the corresponding “classical” supergravity solution.

Chapter 4

Making black holes in string theory

In the previous chapter we have introduced some notions of string theory; we are now ready to understand what is a black hole from the point of view of this theory. We are going to see whether this theory can find an answer to the puzzles we have described in chapter 1, in particular the information paradox.

In chapter 2 we have seen how a black hole can be constructed in supergravity, taking suitable bound states of branes. The terminology for branes we have introduced can now be interpreted in the language of string theory. The $F1$-brane is the fundamental string; it should be considered as the fundamental object of the theory. Dp-branes (with p generic) are D-branes; this means that they can be considered as dynamical objects, and also be interpreted as surfaces where the endpoints of fundamental open strings lie. The $NS5$-brane, on the other hand, is not a D-brane; hence it should be considered as a fundamentally different object. It is actually the magnetic dual of the fundamental string: it is a solitonic object of the theory. One can think at the $NS5$-brane as the analogous of a magnetic monopole.

We have derived many supergravity solutions corresponding to bound states of branes, even if not all of them were black holes. It is natural to think that these solutions are also superstring solutions (at least in the low energy limit), but there could be a problem. The fact that we can find such a solution does not mean that the corresponding singularity is allowed in string theory. We have to be sure of the existence of a suitable microscopic source generating the solution. This fact becomes clear when we look in more detail at some particular examples. In the following we will consider only solutions with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$, because this is the topology of the Strominger–Vafa black hole solution, in which we are ultimately interested.

4.1 The single-charge solution

The first kind of solution involves only one charge; let us start from the solution corresponding to a fundamental string $F1$. Let us take a Type IIA solution, where we compactify the y direction along the circle S^1. The explicit form of the solution was derived in (2.52). We remember here only the expression of the metric (in string frame):

$$ds^2 = Z(r)^{-1}(-dt^2 + dy^2) + dx^i dx^i + dz^a dz^a \,, \quad Z(r) = 1 + \frac{Q}{r^2} \,, \tag{4.1}$$

where $i, a = 1, \ldots, 4$. Notice that we have a different power of r in the function Z, due to the compactification of the torus T^4. The singularity of this solution is located at $r = 0$, but the area of the horizon vanishes. From the point of view of string theory, this sounds reasonable: we can think that the source is a string wrapped (possibly more than once) around the circle S^1. The tension of the string causes it to shrink; thus the string collapse at the position $r = 0$, which is consistent with the metric (4.1). The vanishing of the area, and consequently of the entropy, of this solution is consistent with the microscopic count of state we can made. In fact a single string, when thought in superstring theory, is in an oscillator ground state: its degeneracy coincide with the number of the zero modes of this theory. If we sum all the degrees of freedom, we obtain 128 bosonic and 128 fermionic states; thus, from a microscopic point of view, we would have an entropy $S_{micro} = \ln(256)$. However, this entropy does not increase with the winding number n_1: in the macroscopic limit ($n_1 \to \infty$) we must associate a null entropy to this system. This result agrees with the vanishing of the area of this solution.

One can also look at this solution from another perspective, writing it in another duality frame. An S-duality allows us to derive a solution corresponding to a $D1$-brane. The corresponding metric in string frame is

$$ds^2 = Z(r)^{-\frac{1}{2}}(-dt^2 + dy^2) + Z(r)^{\frac{1}{2}}(dx^i dx^i + dz^a dz^a)\,, \quad Z(r) = 1 + \frac{Q}{r^2}\,. \tag{4.2}$$

The singularity is again located at $r = 0$; this is consistent with the fact that the brane wraps around the circle S^1, and its tension causes it to shrink. From a fundamental point of view we can think at this system as an open string stretched between the $D1$-brane. Hence the geometry comes from a disk amplitude with boundary conditions relative to a $D1$-brane; we will calculate explicitly this amplitude in the next chapter.

Another equivalent picture for this system is obtained applying a T duality for each compact direction of the torus T^4. In such a way we derive a solution corresponding to a $D5$-brane. The metric of such solution is, in string frame,

$$ds^2 = Z(r)^{-\frac{1}{2}}(-dt^2 + dy^2) + Z(r)^{\frac{1}{2}}(dx^i dx^i) + Z(r)^{-\frac{1}{2}}(dz^a dz^a)\,, \quad Z(r) = 1 + \frac{Q}{r^2}\,. \tag{4.3}$$

We will calculate explicitly the corresponding string amplitude in the following.

4.2 The two-charge solution

The second type of solution involves a bound states of two charges. A solution of this kind corresponds for example to a fundamental string $F1$ carrying momentum. The metric can be read from (2.69), when the extremal limit is taken; the result is the following:

$$\begin{aligned} &ds^2 = Z(r)^{-1}(-dt^2 + dy^2 + K(r)(dt + dy)^2) + dx^i dx^i + dz^a dz^a\,, \\ &Z(r) = 1 + \frac{Q_1}{r^2}\,, \quad K(r) = \frac{Q_P}{r^2}\,. \end{aligned} \tag{4.4}$$

Again the singularity is located at $r = 0$, and the area of the horizon vanishes. If we think at the source of this solution as a string carrying momentum, we immediately see that it can not be confined at the origin. In fact we have seen that only transverse vibrations of a string are allowed; therefore a string carrying momentum must bend away from its central axis. We must conclude that the singularity of this solution is not allowed in string theory, because there is not any source that can produce it. We may ask what kind of solution produces a fundamental string with momentum, i.e. with a non trivial profile; the resulting metric is known, and reads in string frame([22]):

$$ds^2 = Z^{-1}(-dt^2 + dy^2 + K(dt + dy)^2 + 2A_i dx^i(dt + dy)) + dx^i dx^i + dz^a dz^a\,, \tag{4.5}$$

where the functions Z, K, and A_i have the following expressions:

$$
\begin{aligned}
Z(\vec{x},y,t) = 1+\frac{Q_1}{|\vec{x}-\vec{f}(t+y)|^2}\,, \quad K(\vec{x},y,t) = \frac{Q_1|\dot{\vec{f}}(t+y)|^2}{|\vec{x}-\vec{f}(t+y)|^2}\,,\\
A_i(\vec{x},y,t) = -\frac{Q_1\dot{f}_i(t+y)}{|\vec{x}-\vec{f}(t+y)|^2}\,.
\end{aligned}
\tag{4.6}
$$

The functions f_i represent the components of the profile of the string, which is not confined at $r=0$. The dependence of the profile on the combination $t+y$ is a consequence of supersymmetry: the solution could depend on $t+y$ or $t-y$, but not on both. The choice of using $t+y$ is just a matter of convention. The metric is singular along the curve $x_i = f_i(t+y)$: this curve represents the position of the string.

The macroscopic solution is obtained adding more than one of such solutions, corresponding to more strands of the same string winding along the circle. The metric looks the same, while the functions Z, K, and A_i involves sum of different terms like the ones written in equation (4.5). If we sum a great number of such strands, we can approximate the sum with an integral; the resulting functions are then

$$
\begin{aligned}
Z(\vec{x},t) = 1+\frac{Q_1}{L}\int_0^L\frac{dv}{|\vec{x}-\vec{f}(t+y)|^2}\,, \quad K(\vec{x},t) = \frac{Q_1}{L}\int_0^L\frac{dv\;|\dot{\vec{f}}(t+y)|^2}{|\vec{x}-\vec{f}(t+y)|^2}\,,\\
A_i(\vec{x},t) = -\frac{Q_1}{L}\int_0^L\frac{dv\;\dot{f}_i(t+y)}{|\vec{x}-\vec{f}(t+y)|^2}\,.
\end{aligned}
\tag{4.7}
$$

We can use dualities in order to obtain analogous actual solution in other frames. Applying an S duality, for example, one obtains a $D1$-P solution. Performing also T dualities we can derive also a $D5$-P solution. In these cases the functions f_i represent the profile of the D-branes: we can think on an open string with endpoints on these branes. Now the branes are not located at $r=0$, but extend in the non-compact direction x^i.

Using also T dualities, we can derive the solution corresponding to a $D1$-$D5$ bound state, where the $D1$ wraps around the S^1, while the $D5$ wraps around the S^1 and the torus T^4. In this case the two branes do not carry momentum; hence their tension causes them to collapse to the origin $r=0$. Therefore in this case the functions f_i do not have the interpretation of profile of the branes. The explicit derivation of the geometry can be made following the same passages we used when we derived the supergravity solution for the Strominger–Vafa black hole; schematically they are

$$
(D5-P)\;\xrightarrow{S}\;(NS5-P)\;\xrightarrow{T_y}\;(NS5-NS1)\;\xrightarrow{T_{z_1}}\;(NS5-NS1)\;\xrightarrow{S}\;(D5-D1)\,. \tag{4.8}
$$

Notice that the presence of a T duality along y requires a smearing along this direction. The starting point must be a solution independent on y; we must use the function Z, K and A_i as in (4.7), where the dependence on y is integrated away. The calculation is pretty long; we only state that the resulting metric is the following:

$$
ds^2 = (Z_1Z_5)^{-\frac{1}{2}}[-(dt-A_idx^i)^2+(dy+B_idx^i)^2]+(Z_1Z_5)^{\frac{1}{2}}(dx^idx^i)+\left(\frac{Z_1}{Z_5}\right)^{\frac{1}{2}}(dz^adz^a)\,. \tag{4.9}
$$

The functions Z_1, Z_5, A_i and B_i are given by

$$Z_1(\vec{x},v) = 1 + \frac{Q}{L}\int_0^L \frac{dv|\dot{\vec{f}}(v)|^2}{|\vec{x}-\vec{f}(v)|^2}\,, \quad Z_5(\vec{x},v) = 1 + \frac{Q}{L}\int_0^L \frac{dv}{|\vec{x}-\vec{f}(v)|^2}\,,$$
$$A_i(\vec{x},v) = -\frac{Q}{L}\int_0^L \frac{dv\ \dot{f}_i(v)}{|\vec{x}-\vec{f}(v)|^2}\,, \quad dB = -\star_4\, dA\,, \tag{4.10}$$

where $\star_4$ is the Hodge dual operator in the non-compact directions $x^1,\ldots,x^4$.

An important particularity of this duality frame is the fact that the geometry is completely smooth. The singularities at $x_i = f_i(t+y)$ can be cancelled by a change of coordinates.

4.2.1 Counting the microstates

The next question is how to count the number of microstates of this system, and find whether it gives rise to a vanishing entropy, as expected on account of the vanishing area of the horizon. We make this count thinking at a free string, i.e. in the limit where the coupling constant g_s is very small. However, this count makes sense, because for BPS bound states it is independent on the value of g_s. Therefore the result we are going to find is valid also in the "black hole limit", where g_s is big. We are going to calculate the number of microstates in the $F1$-P duality frame; the result is the same in all the other equivalent duality frames.

Let us take a long string winding n_1 times around S^1, and with n_P units of momentum. Let us assume that the circle has radius R: hence the total length of the string is $L_T = 2\pi R n_1$. The total momentum of the string is

$$P = \frac{n_P}{R} = \frac{2\pi n_1 n_P}{L_T}\,. \tag{4.11}$$

We know that to each excitation of the Fourier mode k is associated a momentum

$$p_k = \frac{2\pi k}{L_T}\,. \tag{4.12}$$

We can think at the total momentum being distributed among the transverse directions of vibration. Let us suppose that we have m_i units of the Fourier mode k_i; comparing (4.11) and (4.12) we can write

$$\sum_i m_i k_i = n_1 n_P\,. \tag{4.13}$$

This means that the degeneracy of the system is given by the number of partitions of $n_1 n_P$. One can show that this partition is roughly given by $e^{\sqrt{n_1 n_P}}$. One should also consider that this momentum is partitioned among the 8 bosonic and 8 fermionic vibration modes. Remembering that the partition functions of a boson and a fermion with energy e_k are given by

$$Z_k^B = \sum_{m_k=0}^{\infty} e^{-\beta m_k e_k} = \frac{1}{1-e^{-\beta e_k}}\,, \quad Z_k^B = \sum_{m_k=0}^{1} e^{-\beta m_k e_k} = 1 + e^{-\beta e_k}\,, \tag{4.14}$$

one can see that a fermionic degrees of freedom count as half a bosonic one. In fact, summing over k and approximating the sum with an integral we get

$$\log Z^B = \log\left(\sum_k Z_k^B\right) \simeq 2\log Z^F = 2\log\left(\sum_k Z_k^F\right)\,. \tag{4.15}$$

Taking into account all the proportionality factors, the final result (see [22]) gives a non-vanishing entropy:

$$S_{micro} = 2\pi\sqrt{2}\sqrt{n_1 n_P}\,. \tag{4.16}$$

We notice here an interesting puzzle: the microscopic count of states does not reproduce the Bekestein–Hawking entropy derived from the classical solution (4.4). We have to remember that the latter has not an allowed singularity in string theory, even if it is a solution of the low energy supergravity equation of motion away from $r = 0$. We should think at this solution as a statistical ensemble describing only the common behaviour, far from the source, of the actual microstates of the system. The fact that such solution has vanishing horizon area is due to the higher order (in the curvature) terms in the supergravity action; the Bekenstein–Hawking entropy, in fact, was proven in the approximation where the action is the Einstein–Hilbert one. When also higher powers of R are taken into account, one gets the right matching with the microscopic count of states.

4.3 The three-charge black hole solution

We have already seen that the construction of a real black hole requires the presence of a bound state with three charges. We have derived, in the previous chapter, the solution corresponding to the $D1$-$D5$-P bound state (equation (2.79)). We remember here only the metric (in string frame), which reads:

$$ds^2 = Z_1^{-\frac{1}{2}} Z_5^{-\frac{1}{2}} \left(-dt^2 + dy^2 + K(dt+dy)^2\right) + Z_1^{\frac{1}{2}} Z_5^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_3^2\right) + Z_1^{\frac{1}{2}} Z_5^{-\frac{1}{2}} \sum_a (dz^a)^2\,, \tag{4.17}$$

where the functions Z_1, Z_5 and K are related to the three charges by

$$Z_{1,5} = 1 + \frac{Q_{1,5}}{r^2}\,, \quad K = Z_P - 1 = \frac{Q_P}{r^2}\,. \tag{4.18}$$

This solution represents a physical black hole, because the area of the horizon does not vanish; we have already calculated its value, and derived the Bekenstein–Hawking entropy

$$S_{Bek} = \frac{V_5 \pi^2 \sqrt{Q_1 Q_5 Q_P}}{2G_{10}} = \frac{\pi^2 \sqrt{Q_1 Q_5 Q_P}}{2G_5}\,. \tag{4.19}$$

A calculation of the total number of microstates is possible in an analogous way as we did in the two-charge case. Adding a charge means that one should now find the number of partitions of $n_1 n_5 n_P$; the resulting entropy is then proportional to $\sqrt{n_1 n_5 n_P}$. The three numbers are related to the three macroscopic charges $Q_{1,5}$ and Q_P; when all the proportionality factors are taken into account, one finds a perfect matching between the entropy calculated in this way and the Bekenstein–Hawking one (see for example [22]). This is a remarkable and fundamental result, which supports the validity of string theory as a quantum gravity theory.

Differently from the two-charge solution, in this case we expect such an agreement. In fact, the three-charge solution is indeed a black hole; the problem of the vanishing Bekenstein–Hawking entropy does not affect the three-charge solution. While the two-charge classical solution (4.4) fails to reproduce the number of microstates, the black hole classical solution (4.17) succeed in doing this. The non-vanishing area of the horizon, corresponding to a non-zero entropy, encodes the information about the degeneracy of the solution. In this case the higher order corrections to the Einstein–Hilbert action are not necessary for a correct matching with the microscopic count of states.

Finding correct expressions for actual string theory solutions, corresponding to microstates of this system, can be made. The situation is obviously more complicated than the two-charge system; we postpone the analysis of possible microstates of the three-charge system to the next chapters. We can however make some important considerations. We expect the microstate geometry to have no horizon; otherwise, to this horizon it would be associated an entropy, in contrast to the fact that we are considering a microstate. A geometry with horizon is necessarily a classical solution, a "statistical ensemble" of all the actual microstate solution of the system.

4.4 The *fuzzball* proposal

We have seen that string theory allows us to find solutions corresponding to microstates of a system, in particular microstates of a black hole. The natural question is whether these microstates can solve the problems explained in the end of chapter 1. In particular we may ask if we can find a satisfying answer to the black hole information paradox. The problem is a non-trivial one, and its solution is controversial. There is a conjecture, the so-called *fuzzball proposal*, that tries to explain where the derivation of the spectrum of the Hawking radiation fails.

We describe here briefly the key features of this conjecture. Reviews of these concepts can be found in [4] or [22]. The basic idea is the following: not only the number of microstates matches exactly with the Bekenstein–Hawking entropy, but the explicit form of these microstates is responsible for the solution of the information paradox. The reason is that the differences between two different microstates, and between a microstate and the classical solution, are spread over an extended region, with dimension comparable with the size of the horizon. This means that the creation of particle pairs near the horizon is influenced by the precise form of the microstate we are considering; therefore the emitted radiation must retain in some way the information relative to each microstate. Hawking's calculation was based on the assumption that the microstate were not distinguishable near the horizon: if this is not true, the information paradox could vanish.

The idea underlying this concept of fuzzball can be understood looking at the two-charge solution $D1$-$D5$. We have seen that the microstates are horizonless solutions characterised by the functions f_i; each solution has the same behaviour at infinity, but they are different in the region near the center of the hole at $r = 0$. While the latter had a singularity at the origin, the actual microstate solutions do not have any singularity, but they end with some sort of smooth "cap". The situation is explained in figure 4.1.

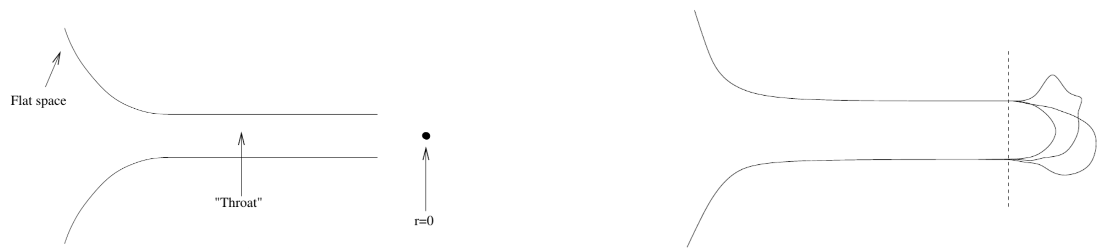

Figure 4.1: Schematic representation of the classical solution $F1$-P (left) and some of its actual microstates (right).

It can be shown that the typical dimension of these caps grows with the charges Q_1 and Q_P.

The size of the horizon of the black hole represents the size under which the differences between microstates become sensible; this is the reason why the horizon would encode in some way informations about the microstates.

We stress here that even if the fuzzball proposal is a tantalizing and promising solution to the black hole information paradox, it is not entirely accepted. One problem is the extension of the concept from the two-charge system (which is relatively simple) to the three-charge black hole. Another problem is that the nature of the Strominger–Vafa black hole microstates is not completely understood. In particular it is has not been proved yet that all the microstates of this black hole can indeed be represented by horizonless smooth solutions. It is also not clear if all these microstates will differ from the classical solution at horizon scales. Work has yet to be made in order to understand whether this conjecture works or fails.

In the following we will try to understand how to derive the geometry of microstates from string theory amplitudes. Firstly we will start from the simple single-charge system, but the final goal is to arrive to the derivation of microstates of the three-charge black hole.

In two-charge system (and also in the single-charge one) the derivation of the geometries from string amplitude will not reveal anything new; the classification of the microstates has already been made by means of solutions such as (4.5). Anyway it is a non-trivial check to verify that the microscopical interpretation given by strings gives the right supergravity solution we expect. The situation become more complicated in the three-charge case. Here we do not have a systematic way to classify all the microstates of a given classical solution; the calculation of the string amplitudes is one of the few methods that can be used to investigate the nature of the microstates of this system.

CHAPTER 5

Geometry of D-brane bound states from string amplitudes

We have seen that it is possible to identify the geometries corresponding to the microstates of a given classical solution, at least for the two-charge system. In this chapter we are particularly interested in solutions expressed in duality frames involving bound states of D-brane. This is because D-branes can be considered as surfaces where the endpoints of an open string lie. We can use this picture to derive the metric from amplitudes that describe emission of closed strings from D-branes. We think at the D-branes as the surfaces where the endpoints of an open string lie. The interaction of a closed string with this open one is the process we are interested in. The situation is schematically represented in figure 5.1.

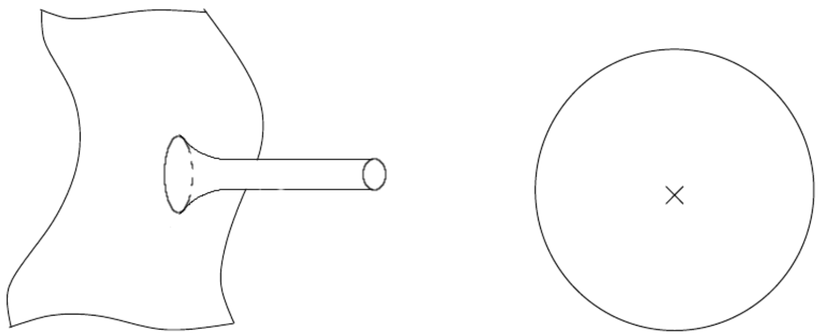

Figure 5.1: Diagram representing an open-closed string interaction (left) and its conformal equivalent (right).

It is clear that such a diagram can be conformally deformed into a disk with a puncture in its interior; to this puncture is associated a closed string vertex operator, of the same kind one gets when dealing with closed-closed string interactions. The amplitude will also depend on the particular branes on which the open string ends; their contribution will be encoded in suitable boundary condition at the border of the disk.

In this chapter we will explain how to calculate such amplitudes, and compare the results with known supergravity solutions. We start with the single-charge solution, but the final goal is to arrive to the three-charge black hole. In this chapter we review known results, while in the next we will turn to a calculation of another type of amplitude, which will give a new microstate of the Strominger–Vafa black hole. All the example we treat are solutions with topology $\mathbb{R}^{1,4} \times S^1 \times T^4$. We will restrict for simplicity only on the bosonic fields relative to the

NS-*NS* sector: the graviton $g_{\mu\nu}$, the dilaton Φ and the Kalb–Ramond field $B_{\mu\nu}$.

An important observation is that our procedure for calculating the disk amplitude will automatically give the metric in Einstein frame, where the dilaton and the metric are correctly separated; in order to make a comparison with the corresponding supergravity solution, we will have to express the latter in Einstein frame. The disk amplitude will not give the complete metric $g_{\mu\nu}$, but its deviation from the flat metric, i.e. $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Furthermore, if one wants to recover the complete metric, one should sum over an infinite number of configuration, with arbitrary number of borders: figure 5.1 represents only the tree-level contribution to the process.

5.1 The single-charge solution

5.1.1 The $D1$ solution

Let us start with the simplest case: the single-charge solution. We are interested in the solution expressed in the $D1$ duality frame, in such a way that we can compute the outgoing fields via a disk amplitude. We have seen that the classical solution was an allowed one in string theory, so we expect the disk amplitude to reproduce it. From a supergravity point of view, the $D1$ solution can be obtained from the $F1$ one using an S duality. The metric has already been calculated in (4.2); we write here also the other *NS*-*NS* fields:

$$\begin{cases} ds^2 = Z(r)^{-\frac{1}{2}}(-dt^2 + dy^2) + Z(r)^{\frac{1}{2}}(dx^i dx^i + dz^a dz^a) \\ e^{2\Phi} = Z(r) \\ B^{(2)} = 0 \end{cases} , \quad Z(r) = 1 + \frac{Q}{r^2} . \tag{5.1}$$

In order to express the metric in Einstein frame we have to remember that $(g_E)_{\mu\nu} = e^{-\frac{\Phi}{2}}(g_s)_{\mu\nu}$. The $D1$ metric is then

$$ds_E^2 = Z(r)^{-\frac{3}{4}}(-dt^2 + dy^2) + Z(r)^{\frac{1}{4}}(dx^i dx^i + dz^a dz^a) . \tag{5.2}$$

We are going to calculate the tree-level amplitude corresponding to this system; this means that we will not derive the complete solution, but only the leading terms. The topology expansions is equivalent to an expansion of the solution at small charges and large distance r; in fact the charge is proportional to the number of D-branes, which is in turn proportional to the number of boundaries of the diagram. For dimensional reasons, the topology expansion is then an expansion with parameter $\frac{Q}{r^2}$. Therefore we have to compare the tree-level string amplitude result with the first term of the expansion of the solution (with the metric written in Einstein frame). Such an expansion can be derived easily, and leads to

$$h_{tt} = -h_{yy} \simeq \frac{3}{4}\frac{Q}{r^2} , \quad h_{ii} = h_{aa} \simeq \frac{1}{4}\frac{Q}{r^2} , \quad \Phi \simeq \frac{1}{2}\frac{Q}{r^2} , \quad B_{\mu\nu} = 0 . \tag{5.3}$$

In order to calculate the disk amplitude, we have to understand how the presence of the D-brane reflects into suitable boundary condition on the disk. In the previous chapter we have defined a D-brane as a surface along which one has Neumann boundary conditions for an open string, while in the other directions one has Dirichlet boundary conditions. Expressed in imaginary coordinates, these conditions are the following:

$$\begin{array}{ll} \partial X^\mu = \bar{\partial} X^\mu & \text{Neumann boundary conditions, along the brane,} \\ \partial X^\mu = -\bar{\partial} X^\mu & \text{Dirichelet boundary conditions, perpendicular to the brane,} \end{array} \tag{5.4}$$

where we have defined $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. When dealing with superstring theory, it is also important to understand the boundary conditions relative to the spinor fields. In order to derive it, it is sufficient to notice that the action is supersymmetric; a supersymmetry transformation relates ψ^μ with ∂X^μ and $\bar{\psi}^\mu$ with $\bar{\partial} X^\mu$. This means that the boundary conditions can be expressed as follows:

$$\begin{aligned} &\psi^\mu = \bar{\psi}^\mu && \text{Neumann boundary conditions, along the brane,} \\ &\psi^\mu = -\bar{\psi}^\mu && \text{Dirichelet boundary conditions, perpendicular to the brane.} \end{aligned} \tag{5.5}$$

To be precise, this is true for the *NS* sector, while in the *R* sector we would have an extra minus sing. It is convenient to write equation (5.5) in matrix notation as $\bar{\psi}^\mu = \mathcal{R}^\mu{}_\nu \psi^\nu$; the matrix $\mathcal{R}$ encodes all the useful informations about the boundary conditions, and hence all the informations about the *D*-brane characterizing the system. In the case of the *D*1-brane, the matrix $\mathcal{R}$ is simply given by

$$(\mathcal{R}_{D1})^\bullet{}_\bullet = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix} . \tag{5.6}$$

Here we intend the coordinate of the spacetime to be in the following order: (t, y, x^i, z^a).

The second ingredient needed for the calculation of the string amplitude is the vertex operator, that we call W; we deal here with the vertex operator in the *NS-NS* sector. There are some technicalities concerning the right choice of this vertex operator. First of all one should look for a superstring vertex operator; furthermore there are several equivalent pictures in which a given superstring vertex operator can be written. Anyway the insertion of a vertex operator into a disk amplitude gives vanishing result, unless it saturates the superghost charge of the system. One can then show that the vertex operator must be taken in the (-2) picture. We do not derive the explicit expression, and remand to a detailed reference ([21]). We can take our vertex operator as

$$W_{NSNS} = c(z)\bar{c}(\bar{z})\mathcal{G}_{\mu\nu}\psi^\mu(z)e^{-\phi(z)}\bar{\psi}^\nu(\bar{z})e^{-\bar{\phi}(\bar{z})}e^{ik_\mu X^\mu} , \tag{5.7}$$

where c and $\bar{c}$ are the ghosts related to the bosons X^μ, while ϕ and $\bar{\phi}$ are those appearing in the bosonised form of the superghosts β and γ. The vector k_μ is the momentum of the outgoing field; we always take k_a to be zero, because the z^a directions are compactified on the torus T^4. The fact that the supergravity solution depends only on the coordinates x^i implies that we have to integrate our string amplitude over the coordinates t and y or, alternatively, u and v. The term $e^{ik\cdot X}$ will contribute only through the zero modes u_0 and v_0; thus we will have a term like

$$\mathcal{A} \propto \int du_0 e^{ik_u u_0} \propto \delta(k_u) . \tag{5.8}$$

The same is valid for v. We see that the independence of the solution on u and v reflects on the vanishing of the corresponding components of the momentum k_u and k_v.

$\mathcal{G}_{\mu\nu}$ is a sort of polarisation tensor, and can be decomposed into three parts, corresponding to the dilaton, the metric and the Kalb–Ramond field. The decomposition is determined by the following relations (see [23]):

$$\begin{aligned} &\mathcal{G}^{(\Phi)}_{\mu\nu} = \frac{1}{2\sqrt{2}}(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu) , \quad k_\mu l^\mu = 1 , \quad l_\mu l^\mu = 0 , \\ &\mathcal{G}^{(h)}_{\mu\nu} = \mathcal{G}^{(h)}_{\nu\mu} , \quad \mathcal{G}^{(h)}_{\mu\nu}\eta^{\mu\nu} = 0 = \mathcal{G}^{(h)}_{\mu\nu}k^\mu \\ &\mathcal{G}^{(B)}_{\mu\nu} = -\mathcal{G}^{(B)}_{\nu\mu} . \end{aligned} \tag{5.9}$$

In the following we will always assume that even the vector l^μ has non-vanishing components only along the x^i directions.

We are now ready for the explicit calculation of the amplitude, which reads

$$\mathcal{A}_{NSNS} = \int \frac{dzd\bar{z}}{V_{gauge}} \langle W_{NSNS}(z,\bar{z})\rangle \,, \tag{5.10}$$

where z and $\bar{z}$ should be intended as independent variable, and V_{gauge} takes into account the invariance under reparametrisation and Weyl transformations. For our purpose it is not important to keep all the multiplicative factors. That is the reason why we can avoid the calculation of all the correlators of the fields: they will give terms depending on the integration variables, which give constant factors when the integral is taken. All these factors can be calculated with an explicit study of the conformal field theories of the bosonic, fermionic and ghost systems; we will not do it for simplicity, and restrict to the analysis of the spacetime structure of the amplitude. First of all, the contribution of the ghosts will give just constant factors. Regarding the contribution of $\langle e^{ik_\mu X^\mu}\rangle = \langle e^{ik_i X^i}\rangle$, only the zero modes of X can contribute. Assuming that the brane is located at $r = 0$, we then see that this term is simply 1. For the fermionic fields, we can write

$$\langle \psi^\mu \bar{\psi}^\nu \rangle = \langle \psi^\mu \mathcal{R}^\nu{}_\lambda \psi^\lambda \rangle \propto \mathcal{R}^\nu{}_\lambda \eta^{\mu\lambda} \,. \tag{5.11}$$

Summing up, and avoiding the constant multiplicative factors, we get the following expression for the amplitude:

$$\mathcal{A}_{NSNS} \propto \mathcal{G}_{\mu\nu} \mathcal{R}^\nu{}_\lambda \eta^{\mu\lambda} = \mathcal{G}_{\mu\nu}\mathcal{R}^{\mu\nu} \,. \tag{5.12}$$

We see from the last expression that it is useful to calculate the matrix of boundary conditions $\mathcal{R}$ with both indices up, where the second index is raised using the flat metric. This matrix is simply given by

$$(\mathcal{R}_{D1})^{\bullet\bullet} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix} . \tag{5.13}$$

The next step is to distinguish from this amplitude the contribution relative to the various fields. We immediately notice that the Kalb–Ramond field will not receive any contribution from this amplitude, because the matrix $\mathcal{R}$ is symmetric. The amplitude given by the dilaton is

$$\mathcal{A}^{(\Phi)} = \mathcal{G}^{(\Phi)}_{\mu\nu}\mathcal{R}^{\mu\nu} \propto \frac{1}{2\sqrt{2}}(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu)\mathcal{R}^{\mu\nu} = \frac{1}{2\sqrt{2}}(-6 + 2k_i l^i) = -\sqrt{2} \,. \tag{5.14}$$

From this amplitude one gets a prediction for the Fourier transform of the dilaton, using the formula

$$\Phi(k) = \frac{1}{\sqrt{2}}\mathcal{A}^{(\Phi)} \propto -1 \,. \tag{5.15}$$

The factor $\sqrt{2}$ can be derived if one studies in detail the relative normalisation between the dilaton and the graviton. In order to derive the contributions to the metric one has to subtract properly the contribution coming from the dilaton. The expression for $h_{\mu\nu}(k)$ should satisfy an orthogonality condition like $h^{\mu\nu}\mathcal{G}^{(\Phi)}_{\mu\nu} = 0$. The contribution of the dilaton must be proportional to the flat metric $\eta_{\mu\nu}$; therefore the right expression must the following (see [23]):

$$h_{\mu\nu}(k) = \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}\cdot\mathcal{G}^{(\Phi)}}{\eta\cdot\mathcal{G}^{(\Phi)}}\eta_{\mu\nu} \,. \tag{5.16}$$

We can easily calculate

$$\eta \cdot \mathcal{G}^{(\Phi)} = \frac{1}{2\sqrt{2}}(10 - 2k_i l^i) = 2\sqrt{2}\,. \tag{5.17}$$

We then arrive to the final expression for the deviation from the metric

$$h_{\mu\nu} \propto \mathcal{R}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\,, \tag{5.18}$$

which leads to the following expressions:

$$h_{tt}(k) = -h_{yy}(k) \propto -\frac{3}{2}\,, \quad h_{ii}(k) = h_{aa}(k) \propto -\frac{1}{2}\,. \tag{5.19}$$

We are left with the metric coefficient in the momentum space; we have to take the Fourier transform, in order to get the corresponding expression in the coordinate space. This operation depends on the number of non-compact directions of the solution: in our case the correct expression for a generic tensor field $a_{\mu_1\ldots\mu_n}$ is given by

$$a_{\mu_1\ldots\mu_n}(x) = \int \frac{d^4k}{(2\pi)^4}\left(-\frac{i}{k^2}\right) a_{\mu_1\ldots\mu_n}(k) e^{-ik\cdot x} \tag{5.20}$$

Notice that there is the insertion of a free propagator which, in 4 dimensions and in the so-called *de Donder gauge*, is proportional to k^{-2}. Remembering that

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{k^2} = -\frac{1}{4\pi^2}\frac{1}{r^2}\,, \tag{5.21}$$

we can write down the final expressions for the fields derived from the disk amplitude:

$$h_{tt} = -h_{yy} \simeq -\frac{3}{2}\frac{Q'}{r^2}\,, \quad h_{ii} = h_{aa} \simeq -\frac{1}{2}\frac{Q'}{r^2}\,, \quad \Phi \simeq -\frac{Q'}{r^2}\,, \quad B_{\mu\nu} = 0\,, \tag{5.22}$$

where we have indicated with Q' the overall multiplicative constant. We use the symbol $\simeq$ because we should remember that our amplitude is a tree-level process, and gives only the first term of the expansion of the fields at long distances (or at small charge). If one wants a better approximation, one has to add the loop contributions. A comparison with equation (5.3) shows that our calculation correctly reproduces the leading term of the expansion of the supergravity solution, provided that we identify $Q = -2Q'$. Notice that it is very important that our calculation succeeds in reproducing the relative proportionality factors between all the metric coefficients and the dilaton.

5.1.2 The $D5$ solution

Another test is to verify if this method correctly reproduces the supergravity solution of a $D5$-brane, wrapped around the S^1 and the T^4. From a supergravity point of view it can be derived from the $D1$ solution applying a T duality along each of the z^a directions. The result (in string frame) is the following:

$$\begin{cases} ds^2 = Z(r)^{-\frac{1}{2}}(-dt^2 + dy^2) + Z(r)^{\frac{1}{2}}(dx^i dx^i) + Z(r)^{-\frac{1}{2}}(dz^a dz^a) \\ e^{2\Phi} = Z(r)^{-1} \\ B^{(2)} = 0 \end{cases} \quad , \quad Z(r) = 1 + \frac{Q}{r^2}\,. \tag{5.23}$$

The metric in Einstein frame reads

$$ds^2 = Z(r)^{-\frac{1}{4}}(-dt^2 + dy^2) + Z(r)^{\frac{3}{4}}(dx^i dx^i) + Z(r)^{-\frac{1}{4}}(dz^a dz^a)\,. \tag{5.24}$$

The long distance (or small charge) behaviour can be easily calculated, and gives

$$h_{tt} = -h_{yy} \simeq \frac{1}{4}\frac{Q}{r^2}\,, \quad h_{ii} \simeq \frac{3}{4}\frac{Q}{r^2}\,, \quad h_{aa} \simeq -\frac{1}{4}\frac{Q}{r^2}\,, \quad \Phi \simeq -\frac{1}{2}\frac{Q}{r^2}\,, \quad B_{\mu\nu} = 0\,. \tag{5.25}$$

Looking at the string amplitude, the only difference is the matrix of Boundary conditions. Along the T^4 we have to change from Dirichlet to Neumann boundary conditions: the matrix $\mathcal{R}$ takes the form

$$(\mathcal{R}_{D5})^{\bullet}{}_{\bullet} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}\,, \quad (\mathcal{R}_{D5})^{\bullet\bullet} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}\,. \tag{5.26}$$

The difference reflects on the fact that now $\mathcal{A}^{(\Phi)}$ changes sign; this means that our prediction for the dilaton is

$$\Phi(k) = \frac{1}{\sqrt{2}}\mathcal{A}^{(\Phi)} \propto 1\,, \tag{5.27}$$

and that the metric coefficients are given by

$$h_{\mu\nu}(k) = \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}\cdot\mathcal{G}^{(\Phi)}}{\eta\cdot\mathcal{G}^{(\Phi)}}\eta_{\mu\nu} \propto \mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,. \tag{5.28}$$

The final expressions for the metric coefficients and the dilaton are

$$h_{tt} = -h_{yy} \simeq -\frac{1}{2}\frac{Q'}{r^2}\,, \quad h_{ii} \simeq -\frac{3}{2}\frac{Q'}{r^2}\,, \quad h_{aa} \simeq \frac{1}{2}\frac{Q'}{r^2}\,, \quad \Phi \simeq \frac{Q'}{r^2}\,. \tag{5.29}$$

We can see that these coefficients coincide exactly with the ones in (5.25); the only requirement is again the identification $Q = -2Q'$.

5.2 The two-charge solution

Let us now consider some more complicated supergravity solutions, involving two charges. This step is very important in order to arrive to the analysis of the three-charge black hole.

5.2.1 The $D1$-P solution

The first duality frame we consider is $D1$-P. This differs from the $D1$ solution because the brane is not fixed in space, but has a non trivial profile, due to the presence of momentum. The profile of the brane is transverse to it; let us also assume that it is non-trivial only along the non-compact directions x_i. The profile can be described by a set of functions $f_i(t+y)$; the dependence on $t+y$ is due to the fact that the momentum produces waves on the brane travelling with the speed of light. An analogous treatment is possible considering functions of $t-y$. We remember that a dependence on $t+y$ and $t-y$ separately is not possible, because we are considering *BPS* states. The $D1$-P solution is known, and can be derived from the $F1$-P

using an S duality. The result, expressed in terms of the lightcone coordinates $v = t + y$ and $u = t - y$ is the following:

$$\begin{cases} ds^2 = Z^{-\frac{1}{2}} dv(-du + Kdv + 2A_i dx^i) + Z^{\frac{1}{2}}(dx^i dx^i + dz^a dz^a) \\ e^{2\Phi} = Z \\ B^{(2)} = 0 \end{cases} , \tag{5.30}$$

where

$$Z(\vec{x}, v) = 1 + \frac{Q}{|\vec{x} - \vec{f}(v)|^2}, \quad K(\vec{x}, v) = \frac{Q|\dot{f}(v)|^2}{|\vec{x} - \vec{f}(v)|^2}, \quad A_i(\vec{x}, v) = -\frac{Q\dot{f}_i(v)}{|\vec{x} - \vec{f}(v)|^2}. \tag{5.31}$$

We consider here a single strand of the brane; if one wants the solution for more strands, it is sufficient to superpose many single-strand solutions. Written in Einstein frame, the metric is the following:

$$ds^2 = Z^{-\frac{3}{4}} dv(-du + Kdv + 2A_i dx^i) + Z^{\frac{1}{4}}(dx^i dx^i + dz^a dz^a). \tag{5.32}$$

In this case an expansion at small Q is different from an expansion at large r; In fact we have another dimensional quantity, which is the profile f_i. The correct expansion is then an expansion of parameter $\frac{Q}{|x_i - f_i|^2}$ One finds the following coefficients for the deviation from the metric and the other *NS*-*NS* fields:

$$\begin{gathered} h_{vv} \simeq \frac{Q|\dot{f}|^2}{|\vec{x} - \vec{f}|^2}, \quad h_{vu} \simeq \frac{3}{8}\frac{Q}{|\vec{x} - \vec{f}|^2}, \quad h_{vi} \simeq -\frac{Q\dot{f}_i}{|\vec{x} - \vec{f}|^2}, \quad h_{ii} = h_{aa} \simeq \frac{1}{4}\frac{Q}{|\vec{x} - \vec{f}|^2} \\ \Phi \simeq \frac{1}{2}\frac{Q}{|\vec{x} - \vec{f}|^2}, \quad B_{\mu\nu} = 0. \end{gathered} \tag{5.33}$$

If we want to calculate the string amplitude, we have to consider that the matrix of boundary conditions changes; this is due to the fact that the boundary (i.e. the surface of the brane) is not fixed in space. A derivation of the correct expression for the matrix $\mathcal{R}$ is possible, but we write here only the result (see [24]):

$$\begin{aligned} (\mathcal{R}_{D1_f})^{\bullet}{}_{\bullet} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\dot{f}(v)|^2 & 1 & -4\dot{f}_i(v) & 0 \\ 2\dot{f}_i(v) & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}, \\ (\mathcal{R}_{D1_f})^{\bullet\bullet} &= \begin{pmatrix} -2|\dot{f}(v)|^2 & -\frac{1}{2} & 2\dot{f}_i(v) & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 2\dot{f}_i(v) & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}. \end{aligned} \tag{5.34}$$

The order of the coordinates in which the matrices are written is (v, u, x^i, z^a). We can notice that this matrix satisfy an important consistency condition: the border of the disk is the locus of points where $z = \bar{z}$. We also expect the energy momentum tensor to satisfy $\bar{T} = T$ at the boundary. We then have to check whether the boundary conditions on ψ^μ and X^μ are such that $T = \bar{T}$, where we take the expression of (3.53) for the energy momentum tensor. When a non trivial profile of the brane is present, the boundary conditions are

$$\bar{\psi}^\mu = \mathcal{R}^\mu{}_\nu \psi^\nu, \quad \bar{\partial} X^\mu = \mathcal{R}^\mu{}_\nu \partial X^\nu - 8\alpha' \delta^\mu_u \ddot{f}_j \psi^j \psi^v. \tag{5.35}$$

Using these expressions one can verify that the matrix (5.34) satisfies the condition $\bar{T} = T$ at the boundary.

The calculation of the amplitude has another new ingredient; the term $\langle e^{ik\cdot X}\rangle$ gives a contribution $e^{ik\cdot f}$, coming from the zero mode of X. Hence the amplitude reads

$$\mathcal{A}_{NSNS} \propto \mathcal{G}_{\mu\nu}\mathcal{R}^{\mu\nu}e^{ik\cdot f}\,. \tag{5.36}$$

The part of the amplitude corresponding to the dilaton is $\mathcal{A}^{(\Phi)} \propto -\sqrt{2}e^{ik\cdot f}$ (and hence $\Phi(k) \propto -e^{ik\cdot f}$), while the deviation from the metric is derived from

$$h_{\mu\nu}(k) = e^{ik\cdot f}\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}\cdot\mathcal{G}^{(\Phi)}}{\eta\cdot\mathcal{G}^{(\Phi)}}\eta_{\mu\nu}\right) \propto e^{ik\cdot f}\left(\mathcal{R}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\right)\,. \tag{5.37}$$

Finally one has to take a Fourier transform, in order to get the result in coordinate space. It is useful to notice that we always have an integral of the form

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-f)}}{k^2} = -\frac{1}{4\pi^2}\frac{1}{|\vec{x}-\vec{f}|^2}\,. \tag{5.38}$$

The final result of the calculation is given by the following expressions, where we introduce an overall multiplicative constant Q':

$$\begin{gathered} h_{vv} \simeq -2\frac{Q'|\dot{f}|^2}{|\vec{x}-\vec{f}|^2}\,, \quad h_{vu} \simeq -\frac{3}{4}\frac{Q'}{|\vec{x}-\vec{f}|^2}\,, \quad h_{vi} \simeq 2\frac{Q'\dot{f}_i}{|\vec{x}-\vec{f}|^2}\,, \quad h_{ii} = h_{aa} \simeq -\frac{1}{2}\frac{Q'}{|\vec{x}-\vec{f}|^2} \\ \Phi \simeq -\frac{Q'}{|\vec{x}-\vec{f}|^2}\,, \quad B_{\mu\nu} = 0\,. \end{gathered} \tag{5.39}$$

We notice that there is a perfect matching with (5.33), provided that $Q = -2Q'$.

If one wants to derive the macroscopic solution from string amplitude, one should add by hand the solution corresponding to different strands of the brane. Notice that the solution for a single strand has explicit dependence on the coordinate v; it should be argued that for this reason we should have considered a momentum with $k_v \neq 0$. This is indeed true, but we can notice that the momentum appears in the amplitude only through the term $e^{ik\cdot X}$; therefore, allowing for a $k_v \neq 0$ would not have changed our result.

5.2.2 The *D*5-*P* solution

A completely analogous treatment can be made for the $D5$-P case. The supergravity solution can be obtained applying four T dualities along the z^a directions. The result is the following:

$$\begin{cases} ds^2 = Z^{-\frac{1}{2}}dv(-du + Kdv + 2A_i dx^i) + Z^{\frac{1}{2}}(dx^i dx^i) + Z^{-\frac{1}{2}}(dz^a dz^a) \\ e^{2\Phi} = Z^{-1} \\ B^{(2)} = 0 \end{cases}\,, \tag{5.40}$$

where the functions Z, K and A_i are identical to those appearing in the $D1$-P solution. Regarding the matrix of boundary conditions $\mathcal{R}$, it is simply given by

$$(\mathcal{R}_{D5_f})^{\bullet}{}_{\bullet} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\dot{f}(v)|^2 & 1 & -4\dot{f}_i(v) & 0 \\ 2\dot{f}_i(v) & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}, \qquad (\mathcal{R}_{D5_f})^{\bullet\bullet} = \begin{pmatrix} -2|\dot{f}(v)|^2 & -\frac{1}{2} & 2\dot{f}_i(v) & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 2\dot{f}_i(v) & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}. \tag{5.41}$$

The only difference with respect to $\mathcal{R}_{D1_f}$ is a sign along the directions of the torus. It can be checked that also this matrix satisfy the consistency condition $\bar{T} = T$ at the boundary of the disk. The calculation proceeds as before; we write directly the result for the coefficients $h_{\mu\nu}$, Φ and $B_{\mu\nu}$, which is

$$\begin{gathered} h_{vv} \simeq -2\frac{Q'|\dot{f}|^2}{|\vec{x}-\vec{f}|^2}, \quad h_{vu} \simeq -\frac{1}{4}\frac{Q'}{|\vec{x}-\vec{f}|^2}, \quad h_{vi} \simeq 2\frac{Q'\dot{f}_i}{|\vec{x}-\vec{f}|^2}, \\ h_{ii} \simeq -\frac{3}{2}\frac{Q'}{|\vec{x}-\vec{f}|^2}, \quad h_{aa} \simeq \frac{1}{2}\frac{Q'}{|\vec{x}-\vec{f}|^2}, \quad \Phi \simeq -\frac{1}{2}\frac{Q}{|\vec{x}-\vec{f}|^2}, \quad B_{\mu\nu} = 0\,. \end{gathered} \tag{5.42}$$

Again we have a perfect matching with the small charge behaviour of the supergravity fields (with the metric expressed in Einstein frame), provided that $Q = -2Q'$.

5.2.3 The $D1$-$D5$ solution

The two-charge system has another interesting duality frame: we can consider the bound state of two different types of brane, in particular $D1$ (wrapped along S^1) and $D5$ (wrapped along S^1 and T^4). The microstate metric was derived in (4.9). Her we need also the other *NS*-*NS* fields, which are the following ones (see [22] and [25]):

$$\begin{cases} ds^2 = (Z_1 Z_5)^{-\frac{1}{2}}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + (Z_1 Z_5)^{\frac{1}{2}}(dx^i dx^i) + \left(\dfrac{Z_1}{Z_5}\right)^{\frac{1}{2}}(dz^a dz^a) \\ e^{2\Phi} = \dfrac{Z_1}{Z_5} \\ B^{(2)} = 0 \end{cases} \tag{5.43}$$

The functions Z_1, Z_5, A_i and B_i are given by

$$\begin{gathered} Z_1(\vec{x}, v) = 1 + \frac{Q}{L}\int_0^L \frac{dv|\dot{\vec{f}}(v)|^2}{|\vec{x}-\vec{f}(v)|^2}, \quad Z_5(\vec{x}, v) = 1 + \frac{Q}{L}\int_0^L \frac{dv}{|\vec{x}-\vec{f}(v)|^2}, \\ A_i(\vec{x}, v) = -\frac{Q}{L}\int_0^L \frac{dv\ \dot{f}_i(v)}{|\vec{x}-\vec{f}(v)|^2}, \quad dB = -\star_4 dA\,, \end{gathered} \tag{5.44}$$

where $\star_4$ is the Hodge dual operator in the non-compact directions $x^1, \ldots, x^4$. Notice that the dependence on v has been integrated away: this is because we had to perform a smearing along the y direction in order to apply the T duality needed to arrive to this solution.

The functions $f_i(v)$ do not have the interpretation of profile of the brane; in fact the latter do not carry momentum, and hence they lie at a fixed position (in particular at $r = 0$). When expressed in Einstein frame the metric reads

$$ds^2 = (Z_1)^{-\frac{3}{4}}(Z_5)^{-\frac{1}{4}}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + (Z_1)^{\frac{1}{2}}(Z_5)^{\frac{3}{2}}(dx^i dx^i) + \left(\frac{Z_1}{Z_5}\right)^{\frac{1}{4}}(dz^a dz^a) \tag{5.45}$$

It turns out that this time the suitable expansion of the metric is that with parameter $\frac{Q}{r^2}$; in fact, the quantities $f^i(v)$ do not have here a true physical meaning. It is useful to notice that at long distances we can approximate the expressions:

$$\begin{gathered} \frac{Q}{L}\int_0^L \frac{dv}{|\vec{x}-\vec{f}(v)|^2} \simeq \frac{Q}{r^2}\,, \\ \frac{Q}{L}\int_0^L \frac{dv\dot{f}_i(v)}{|\vec{x}-\vec{f}(v)|^2} \simeq \frac{Q}{L}\int_0^L dv\dot{f}_i(v)\left(\frac{1}{r^2}+2\frac{x\cdot f}{r^4}\right) = \frac{2Q}{L}\int_0^L dv\dot{f}_i(v)\left(\frac{x\cdot f}{r^4}\right). \end{gathered} \tag{5.46}$$

In order to simplify the calculations, we define the quantities

$$\hat{f}_{ij} = \frac{1}{L}\int_0^L dv\ \dot{f}_i f_j = -\hat{f}_{ji}\,, \quad \hat{F} = \frac{1}{L}\int_0^L dv\ |\dot{f}(v)|^2\,; \tag{5.47}$$

notice that they depend only on the functions $f_i(v)$, but are independent on v. We then perform an expansion at long distances; we obtain that:

$$Z_1 \simeq 1 + \frac{Q\hat{F}}{r^2}\,, \quad Z_5 \simeq 1 + \frac{Q}{r^2}\,, \quad A_i \simeq -2Q\hat{f}_{ij}\frac{x^j}{r^4}\,, \quad \mathcal{A} \simeq -2Q\hat{f}_i\frac{x^i}{r^4}\,. \tag{5.48}$$

The expression for the dual B_i is

$$B_i \simeq -Q\epsilon_{ij}{}^{kl}\hat{f}_{kl}\frac{x^j}{r^4}\,. \tag{5.49}$$

Using these expressions we can finally derive the expansion of the metric and the dilaton at long distances. The result is:

$$\begin{gathered} h_{tt} = -\,h_{yy} \simeq \frac{1}{4}\frac{Q}{r^2}(3\hat{F}+1)\,, \quad h_{ii} \simeq \frac{1}{4}\frac{Q}{r^2}(\hat{F}+3)\,, \quad h_{aa} \simeq \frac{1}{4}\frac{Q}{r^2}(\hat{F}-1)\,, \\ h_{ti} \simeq -2\frac{Q}{r^4}\hat{f}_{ij}x^j\,, \quad h_{yi} \simeq -\frac{Q}{r^4}\epsilon_{ij}{}^{kl}\hat{f}_{kl}x^j\,, \quad \Phi \simeq \frac{1}{2}\frac{Q}{r^2}(\hat{F}-1)\,. \end{gathered} \tag{5.50}$$

The $D1$-$D5$ solution seems to be complicated; notice however that the metric coefficients we have found depend only on the quantities Q, $\hat{F}$ and $\hat{f}_{ij}$. Our goal is to reproduce the behaviour of the metric using the method of calculating a disk amplitude.

Having a bound state of two different types of brane means that an open string has three possibilities: the endpoints can lie both on the $D1$, both on the $D5$, or in different branes. We have to calculate all these three different contributions, and to sum them at the end. The first two possibilities correspond simply to a single charge system; we can draw the two diagrams we have to calculate, denoting differently the two types of brane (figure 5.2).

These diagrams have already been calculated when dealing with single-charge solutions. We rewrite here the result they give, indicating with two different symbols Q_1 and Q_5 the charges of the two branes. Equations (5.3) and (5.25) give

$$h_{tt} \simeq -h_{yy} = \frac{1}{4}\frac{3Q_1+Q_5}{r^2}\,, \quad h_{ii} \simeq \frac{1}{4}\frac{Q_1+3Q_5}{r^2}\,, \quad h_{aa} \simeq \frac{1}{4}\frac{Q_1-Q_5}{r^2}\,, \quad \Phi \simeq \frac{1}{2}\frac{Q_1-Q_5}{r^2} \tag{5.51}$$

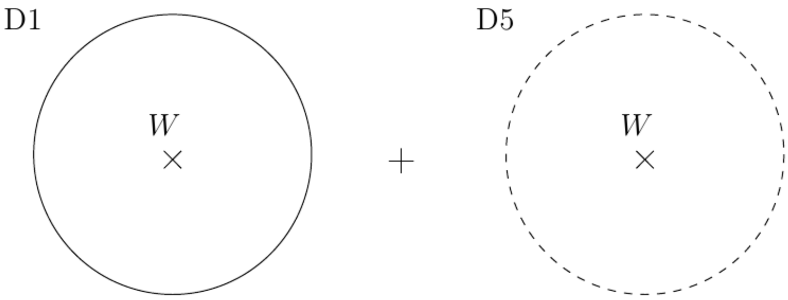

Figure 5.2: The first two contributions to the amplitude of a $D1$-$D5$ system.

We immediately see that these terms correspond exactly to the ones we derived in (5.50), when we make the identifications

$$Q\hat{F} = Q_1 \qquad Q = Q_5\,; \tag{5.52}$$

we are allowed to do this, because Q and $\hat{F}$ are independent of each other, as should be expected for the charges of two independent branes Q_1 and Q_5.

We now have to add the contribution of a disk amplitude with mixed boundary conditions, i.e. corresponding to an open string with one end on the $D1$-brane and the other on the $D5$-brane. This contribution is what makes this system different from a trivial superposition of the single charge solutions, but a real bound state. From an operative point of view, we have to insert two new *twisted vertex operators* V_μ and $V_{\bar{\mu}}$ on the boundary, which allow for a change of boundary condition. Schematically the situation is represented in figure 5.3.

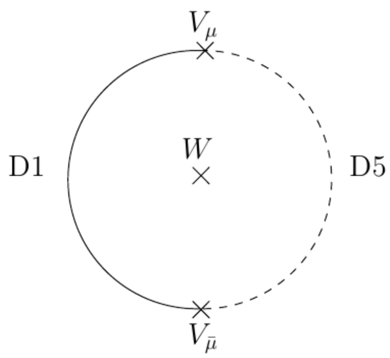

Figure 5.3: The simplest contribution to the amplitude of a $D1$-$D5$ system with mixed boundary conditions.

The expressions for these two additional vertex operator is (we use the same conventions of [25] and [26])

$$V_\mu = \mu^A e^{-\frac{\phi}{2}} S_A \Delta\,, \quad V_{\bar{\mu}} = \bar{\mu}^A e^{-\frac{\phi}{2}} S_A \Delta\,. \tag{5.53}$$

Here μ^A and $\bar{\mu}^A$ are spinors with definite chirality, say positive. They are also matrices with respectively $n_1 \times n_5$ and $n_5 \times n_1$ components: they take into account all the possible branes on which the endpoints of the string can lie. These matrices are usually indicated with the name *Chan–Paton factors*. The field ϕ is related to the superghost system, while Δ is the bosonic twist operator, which changes the boundary conditions for the bosonic fields. S_A are the $SO(1,5)$

spin fields: they are fields related to the fermions ψ^a and $\bar{\psi}^a$ (where a takes the values t, y and $i = 1, \ldots, 4$). Their precise definition is related to the bosonisation of the spinors; one usually defines the complex spinors

$$\begin{aligned} \Psi^1 &= \psi^t + i\psi^y\,, \quad \Psi^2 = \psi^1 + i\psi^2\,, \quad \Psi^3 = \psi^3 + i\psi^4 \\ \bar{\Psi}^1 &= \psi^t - i\psi^y\,, \quad \bar{\Psi}^2 = \psi^1 - i\psi^2\,, \quad \bar{\Psi}^3 = \psi^3 - i\psi^4\,. \end{aligned} \tag{5.54}$$

The bosonisation is made introducing three bosons h_I such that $\Psi^I = e^{ih_I}$ and $\bar{\Psi}^I = e^{-ih_I}$. The spin fields are then defined in the following way:

$$S_A = S^{\vec{\epsilon}_A} = e^{\frac{i}{2}\epsilon^I_A h_I}\,, \tag{5.55}$$

where the vector $\vec{\epsilon}_A$ is given by:

$$\vec{\epsilon}_A = \{(- - -), (- + +), (+ - +), (+ + -)\}\,. \tag{5.56}$$

The other possible combinations of $+$ and $-$ are not considered; in fact S_A is contracted with μ^A in the twisted vertex operator. Thus the total number of possibilities (8) must be reduced by a factor 2.

We do not enter in further details explaining why the twisted vertex operators take this form; a complete analysis is beyond the purpose of this thesis. We only give the necessary rules needed to compute the amplitude. The latter will be given by

$$\mathcal{A}_{NSNS} = \int \frac{dzd\bar{z}dz_1dz_2}{V_{gauge}} \langle V_\mu(z_1) W_{NSNS}(z,\bar{z}) V_{\bar{\mu}}(z_2)\rangle\,, \tag{5.57}$$

where z_1 and z_2 have to be chosen on the boundary, and so they are real coordinates ($\bar{z}_{1,2} = z_{1,2}$). The insertion of the twisted operators has a side effect: both V_μ and $V_{\bar{\mu}}$ carry $-\frac{1}{2}$ superghost charge. This means that the vertex operator W has to be taken in a picture such that it has -1 superghost charge. There is more than one way to write down a suitable operator of this type; we choose an expression that will simplify the calculation:

$$W_{NSNS} = \mathcal{G}_{\mu\nu} \left(\partial X^\mu - \frac{i}{2} k_\rho \psi^\rho \psi^\mu\right) e^{i\frac{k}{2}\cdot X}\bigg|_z \bar{\psi}^\nu e^{-\bar{\phi}} e^{i\frac{k}{2}\cdot \bar{X}}\bigg|_{\bar{z}}\,. \tag{5.58}$$

Ignoring the normalisation factors, the interesting part of the amplitude is

$$\mathcal{A}_{NSNS} \propto \mathcal{G}_{\mu\nu} \bar{\mu}^A \mu^B k_\sigma \int dzd\bar{z}dz_1dz_2 \langle S_A(z_1) \psi^\sigma(z) \psi^\mu(z) \bar{\psi}^\nu(\bar{z}) S_B(z_2)\rangle\,. \tag{5.59}$$

Notice that the possible contraction $\langle \partial X^\mu e^{i\frac{k}{2}\cdot X}\rangle \propto k^\mu$ does not contribute, because of the transversality condition $\mathcal{G}_{\mu\nu} k^\mu = 0$. The product of the two Chan–Paton factors can be decomposed in a basis constituted by products of gamma matrices (those corresponding to the $SO(1,5)$) as follows

$$\bar{\mu}^A \mu^B = v_\mu (C\Gamma^\mu)^{[AB]} + \frac{1}{3!} v_{\alpha\beta\gamma} (C\Gamma^{\alpha\beta\gamma})^{(AB)}\,, \tag{5.60}$$

where C is the charge conjugation matrix, and the Greek indices take the values t, y and $i = 1, \ldots, 4$. The fact that $\bar{\mu}^A$ and μ^B have definite chirality implies that terms with an even number of gamma matrices are absent, and that $v_{\mu\nu\rho}$ is a self-dual 3-form. In the following we will ignore the contribution of the 1-form v_μ, and we will consider only the component of $v_{\mu\nu\rho}$ with one index along the t, y directions, and the other two along the $\mathbb{R}^4$. With these assumptions we will be able to reproduce the coefficients (5.50); other possibilities give rise to different classes of microstates of the system $D1$-$D5$ (see [25] for an example). The self-duality of $v_{\mu\nu\rho}$ gives the following expression:

$$v_{yij} = \frac{1}{2}\epsilon_{ij}{}^{kl} v_{tkl} \,. \tag{5.61}$$

We have also to impose boundary condition for the spinor field $\bar{\psi}$. We thus obtain

$$\langle S_A(z_1)\psi^\sigma\psi^\mu|_z\bar{\psi}^\nu(\bar{z})S_B(z_2)\rangle = \mathcal{R}^\nu{}_\lambda\langle S_A(z_1)\psi^\sigma\psi^\mu|_z\psi^\nu(\bar{z})S_B(z_2)\rangle \propto \mathcal{R}^\nu{}_\lambda(\Gamma^{\sigma\mu\lambda}C^{-1})_{AB} \,. \tag{5.62}$$

An awkward problem for the calculation is the choice of the matrix of boundary conditions. Should we use the one corresponding to a $D1$-brane, or to a $D5$? It is an important consistency check to verify that both matrices give the same amplitude. Let us use, say, $\mathcal{R}_{D1}$ as defined in (5.6). We are now ready for the explicit calculation of the disk amplitude:

$$\begin{aligned} \mathcal{A} &\propto \mathcal{G}_{\mu\nu}\mathcal{R}^\nu{}_\lambda v_{\alpha\beta\gamma}k_\sigma(\Gamma^{\sigma\mu\lambda}C^{-1})_{AB}(C\Gamma^{\alpha\beta\gamma})^{(AB)} = \\ &= \mathcal{G}_{\mu\nu}\mathcal{R}^\nu{}_\lambda v_{\alpha\beta\gamma}k_\sigma Tr[\Gamma^{\sigma\mu\lambda}\Gamma^{\alpha\beta\gamma}] \propto \mathcal{G}^{\mu\nu}\mathcal{R}_\nu{}^\rho k^\sigma v_{\mu\rho\sigma} \,. \end{aligned} \tag{5.63}$$

Such an amplitude does not give any further contribution to the dilaton. In fact:

$$\mathcal{A}^{(\Phi)} \propto (\eta^{\mu\nu} - k^\mu k^\nu - k^\nu k^\mu)\mathcal{R}_\nu{}^\rho k^\sigma v_{\mu\rho\sigma} = k^\sigma v_{\mu\rho\sigma}(\mathcal{R}^{\mu\rho} - k^\mu l^\nu \mathcal{R}_\nu{}^\rho - l^\mu k^\nu \mathcal{R}_\nu{}^\rho) \,. \tag{5.64}$$

The first two terms vanish automatically, due to the total antisymmetry of $v_{\mu\rho\sigma}$ and the symmetry of $\mathcal{R}^{\mu\rho}$. The third term is again zero because:

$$k^\nu\mathcal{R}_\nu{}^\rho = k^i\mathcal{R}_i{}^\rho \propto k^\rho \,. \tag{5.65}$$

This amplitude will only give contributions to the metric and, possibly, the Kalb–Ramond field. Remembering that the non-vanishing component of $\mathcal{R}_\nu{}^\rho$ are

$$\mathcal{R}_t{}^t = \mathcal{R}_y{}^y = 1 \,, \quad \mathcal{R}_i{}^j = -\delta_i^j \,, \quad \mathcal{R}_a{}^b = -\delta_a^b \,, \tag{5.66}$$

we obtain the following expression:

$$\begin{aligned} \mathcal{A} &\propto k^j[\mathcal{G}^{ti}\mathcal{R}_i{}^k v_{tkj} + \mathcal{G}^{yi}\mathcal{R}_i{}^k v_{ykj} + \mathcal{G}^{it}\mathcal{R}_t{}^t v_{itj} + \mathcal{G}^{iy}\mathcal{R}_y{}^y v_{iyj}] = \\ &= k^j[v_{tij}(-\mathcal{G}^{ti} - \mathcal{G}^{it}) + v_{yij}(-\mathcal{G}^{yi} - \mathcal{G}^{iy})] \,. \end{aligned} \tag{5.67}$$

Notice that the matrix $\mathcal{R}_{D5}$ would have given the same amplitude, as should be expected for consistency. This amplitude involves only the symmetric part of $\mathcal{G}^{\mu\nu}$; this means that it gives no contribution to the B field. It is useful to define the two tensor $\hat{h}$ and $\hat{b}$ int the following way:

$$\mathcal{G}^{\mu\nu} = \hat{h}^{\mu\nu} + \frac{1}{\sqrt{2}}\hat{b}^{\mu\nu} \,. \tag{5.68}$$

Therefore the amplitude becomes simply

$$\mathcal{A} \propto -2k^j[\hat{h}^{ti}v_{tij} + \hat{h}^{yi}v_{yij}] \,. \tag{5.69}$$

It turns out that the correct Fourier modes of the metric and Kalb–Ramond field are given by

$$h_{\mu\nu}(k) = \frac{1}{2}\frac{\delta\mathcal{A}}{\delta\hat{h}^{\mu\nu}}\ (\mu < \nu) \,, \quad h_{\mu\mu}(k) = \frac{1}{2}\frac{\delta\mathcal{A}}{\delta\hat{h}^{\mu\mu}} \,, \quad B_{\mu\nu}(k) = \frac{\delta\mathcal{A}}{\delta\hat{b}^{\mu\nu}}\ (\mu < \nu) \,, \tag{5.70}$$

where the index μ in $h_{\mu\mu}$ is not summed. In our case we get the following two contributions:

$$h_{ti}(k) \propto -k^j v_{tij}\,, \quad h_{yi}(k) \propto -k^j v_{yij} = -\frac{1}{2} k^j \epsilon_{ij}{}^{kl} v_{tkl}\,. \tag{5.71}$$

For the first time we encounter a dependence on the momentum k. In order to make the Fourier transform and get the expressions in coordinate space, we remember the integral

$$\int \frac{d^4k}{(2\pi)^4} \left(-\frac{i}{k^2}\right) k^j e^{-ik\cdot x} = -\frac{\partial}{\partial x_j} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{-ik\cdot x} = -\frac{1}{2\pi^2} \frac{x^j}{r^4}\,. \tag{5.72}$$

Calling Q' an overall proportionality constant we get the following predictions:

$$h_{ti} = -\frac{Q'}{r^4} v_{tij} x^j\,, \quad h_{yi} = -\frac{Q'}{2r^4} \epsilon_{ij}{}^{kl} v_{tkl} x^j\,. \tag{5.73}$$

We can see that an exact matching with the expansion of the supergravity geometry (5.50) is possible, if we make the identification

$$Q \hat{f}_{ij} = \frac{1}{2} Q' v_{tij}\,. \tag{5.74}$$

This identification is consistent, because both $\hat{f}_{ij}$ and v_{tij} are arbitrary, subjected only to the antisymmetry condition in the indices i and j.

5.3 The three-charge black hole solution

We are now ready for the analysis of some microstates of the three-charge Strominger–Vafa black hole. The classical supergravity solution in the $D1$-$D5$-P duality frame has been derived in equation (2.79); we rewrite it here for convenience (only the NS-NS fields):

$$\begin{cases} ds^2 = Z_1^{-\frac{1}{2}} Z_5^{-\frac{1}{2}} \left(-dt^2 + dy^2 + K(dt+dy)^2\right) + Z_1^{\frac{1}{2}} Z_5^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_3^2\right) + Z_1^{\frac{1}{2}} Z_5^{-\frac{1}{2}} \sum_a (dz^a)^2 \\ e^{2\Phi} = Z_1 Z_5^{-1} \\ B^{(2)} = 0 \end{cases} \tag{5.75}$$

The functions Z_1, Z_5 and K are related to the three charges by the relations

$$Z_{1,5}(r) = 1 + \frac{Q_{1,5}}{r^2}\,, \quad K(r) = \frac{Q_P}{r^2}\,. \tag{5.76}$$

We expect that this time we have more possibilities to construct a microstate, due to the presence of one more charge. From a microscopic point of view, we have to imagine a bound state of $D1$ and $D5$ branes carrying momentum, i.e. not localised in space, but having with a non trivial profile. We have to use the ingredients we have learned in the study of the two charges system, both in the $D1$-P (or $D5$-P) and $D1$-$D5$ frame.

In this section we analyse the direct generalisation of the microstate we have studied when dealing with the $D1$-$D5$ solution. We consider the branes having a profile along $\mathbb{R}^4$, i.e. along the non-compact directions. In chapter 6 we will turn to a different kind of microstate of the Strominger–Vafa black hole, involving a profile along the compact directions of the torus T^4.

5.3.1 The $D1$-$D5$-P solution with profile along $\mathbb{R}^4$

As we did with the $D1$-$D5$ bound state, we have to sum the three contributions coming from the diagrams represented in figure 5.2 and 5.3; the only difference is the non-trivial profile of the brane, which reflects in a different matrix $\mathcal{R}$ for the boundary conditions. The contributions coming from the first two diagrams are simply the sum of the geometries corresponding to a $D1$-P and a $D5$-P microstate. Let us concentrate on the new contributions, coming from the third diagram. We can safely say that the amplitude looks like the case without momentum, with an extra $e^{ik\cdot f}$ factor coming from the term $\langle e^{ik\cdot X}\rangle$:

$$\mathcal{A} \propto \mathcal{G}^{\mu\nu}\mathcal{R}_\nu{}^\rho k^\sigma v_{\mu\rho\sigma} e^{ik\cdot f} \,. \tag{5.77}$$

It is useful to work with the lightcone coordinates $v = t + y$ and $u = t - y$; the matrices of boundary conditions are the following (they can be derived from (5.34) and (5.41))

$$(\mathcal{R}_{D1_f})_\bullet{}^\bullet = \begin{pmatrix} 1 & 4|\dot f(v)|^2 & 2\dot f_i(v) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4\dot f_i(v) & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix} , \tag{5.78}$$

$$(\mathcal{R}_{D5_f})_\bullet{}^\bullet = \begin{pmatrix} 1 & 4|\dot f(v)|^2 & 2\dot f_i(v) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4\dot f_i(v) & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} . \tag{5.79}$$

Again, the order of the coordinate is (v, u, x^i, z^a). It turns out that both matrices give the same result when inserted in the amplitude, provided that the branes have the same profile $f_i(v)$. The situation of branes with different profiles seems to be more complicated: we neglect this case here. Furthermore we can see that the amplitude gives no further contribution to the dilaton.

Thanks to the fact that the matrix $\mathcal{R}$ has explicit dependence on v, we expect to derive a solution which is dependent on v, as we found in the $D1$-P or $D5$-P case. For consistency we must allow the momentum k to have a non vanishing component k_v, or equivalently k^u. However, when an integration over v is taken, the contribution coming from the presence of k_v will vanish. Let us do the explicit calculation:

$$\begin{aligned} \mathcal{A} \propto & e^{ik\cdot f} k^l [\mathcal{G}^{v\mu}\mathcal{R}_\mu{}^j v_{vjl} + \mathcal{G}^{u\mu}\mathcal{R}_\mu{}^j v_{ujl} + \mathcal{G}^{j\mu}\mathcal{R}_\mu{}^v v_{jvl} + \mathcal{G}^{j\mu}\mathcal{R}_\mu{}^u v_{jul}] + e^{ik\cdot f} k^u \mathcal{G}^{i\mu}\mathcal{R}_\mu{}^j v_{iju} = \\ =& e^{ik\cdot f} k^l [(\mathcal{G}^{v\mu}\mathcal{R}_\mu{}^j - \mathcal{G}^{j\mu}\mathcal{R}_\mu{}^v) v_{vjl} + (\mathcal{G}^{u\mu}\mathcal{R}_\mu{}^j - \mathcal{G}^{j\mu}\mathcal{R}_\mu{}^u) v_{ujl}] + e^{ik\cdot f} k^u \mathcal{G}^{i\mu}\mathcal{R}_\mu{}^j v_{iju} = \\ =& e^{ik\cdot f} k^l [-(\mathcal{G}^{vj} + \mathcal{G}^{jv}) v_{vjl} - (\mathcal{G}^{uj} + \mathcal{G}^{ju}) v_{ujl} - 4|\dot f|^2 \mathcal{G}^{jv} v_{ujl} + 4\dot f_i \mathcal{G}^{ji} v_{ujl} + \\ & + 2\dot f^j \mathcal{G}^{uv} v_{ujl} + 2\dot f^j \mathcal{G}^{vv} v_{vjl}] + e^{ik\cdot f} k^u [-\mathcal{G}^{ij} v_{iju} + 2\dot f^j \mathcal{G}^{iv} v_{iju}] \,. \end{aligned} \tag{5.80}$$

Notice that this amplitude will produce non-trivial contributions to the Kalb–Ramond field. In fact, using the definitions of $\hat h^{\mu\nu}$ and $\hat b^{\mu\nu}$, we can write

$$\begin{aligned} \mathcal{A}^{(h)} \propto e^{ik\cdot f} k^l \Big[& -2\hat h^{vj} v_{vjl} - 2\hat h^{uj} v_{ujl} - 4|\dot f|^2 \hat h^{jv} v_{ujl} + 4\sum_{i<j} \hat h^{ij} (v_{uil}\dot f_j + v_{ujl}\dot f_i) + \\ & + 2\hat h^{uv} v_{uil} \dot f^i + 2\hat h^{vv} v_{vjl} \dot f^j \Big] + e^{ik\cdot f} k^u [2\dot f^j \hat h^{vi}] \,, \end{aligned} \tag{5.81}$$

$$\begin{aligned} \mathcal{A}^{(B)} \propto & e^{ik\cdot f} \frac{k^l}{\sqrt{2}} \Big[-4|\dot f|^2 \hat b^{jv} v_{ujl} + 4\sum_{i<j} \hat b^{ij} (v_{uil}\dot f_j - v_{ujl}\dot f_i) + 2\hat b^{uv} v_{uil}\dot f^i \Big] + \\ & + e^{ik\cdot f} \Big[-\sqrt{2}\sum_{i<j} \hat b^{ij} v_{iju} - \sqrt{2}\dot f^j \hat b^{vi} v_{iju} \Big] \,. \end{aligned} \tag{5.82}$$

Notice that we have written in the amplitude $\hat{h}_{ij}$ and $\hat{b}_{ij}$ only with $i < j$; in fact these are the independent coefficients, the other being determined by the symmetry (or antisymmetry) conditions. From this amplitude we can derive the expressions for the Fourier modes of the metric and the Kalb–Ramond field, using

$$h_{\mu\nu}(k) = \frac{1}{2}\frac{\delta\mathcal{A}}{\delta\hat{h}^{\mu\nu}}\ (\mu < \nu)\,, \quad h_{\mu\mu}(k) = \frac{1}{2}\frac{\delta\mathcal{A}}{\delta\hat{h}^{\mu\mu}}\,, \quad B_{\mu\nu}(k) = \frac{\delta\mathcal{A}}{\delta\hat{b}^{\mu\nu}}\ (\mu < \nu)\,. \tag{5.83}$$

The resulting coefficients $h_{\mu\nu}(k)$ and $B_{\mu\nu}(k)$ are

$$\begin{aligned}
&h_{vv}(k) \propto 2k^l v_{vil}\dot{f}^i e^{ik\cdot f}\,, \quad h_{vu} \propto k^l v_{uil}\dot{f}^i e^{ik\cdot f}\,,\\
&h_{vi}(k) \propto (-k^l v_{vil} - 2|\dot{f}|^2 k^l v_{uil})e^{ik\cdot f} + \dot{f}^j k^u v_{iju} e^{ik\cdot f}\,,\\
&h_{ui}(k) \propto -k^l v_{uil} e^{ik\cdot f}\,, \quad h_{ij}(k) \propto 2k^l(v_{uil}\dot{f}_j + v_{ujl}\dot{f}_i)e^{ik\cdot f}\,,\\
&B_{vu}(k) \propto -\sqrt{2}k^l v_{uil}\dot{f}^i e^{ik\cdot f}\,, \quad B_{vi}(k) \propto 2\sqrt{2}k^l v_{uil}|\dot{f}|^2 e^{ik\cdot f} - \sqrt{2}\dot{f}^j k^u v_{iju} e^{ik\cdot f}\,,\\
&B_{ij}(k) \propto 2\sqrt{2}k^l(v_{uil}\dot{f}_j - v_{ujl}\dot{f}_i)e^{ik\cdot f} - \sqrt{2}k^u v_{iju} e^{ik\cdot f}\,.
\end{aligned} \tag{5.84}$$

We should also remember that the duality condition (5.61) when written in lightcone coordinates reads

$$v_{vij} = \frac{1}{2}\epsilon_{ij}{}^{kl} v_{vkl}\,, \quad v_{uij} = -\frac{1}{2}\epsilon_{ij}{}^{kl} v_{ukl}\,. \tag{5.85}$$

When we take the Fourier transform, the dependence $k^l e^{ik\cdot f}$ becomes a factor proportional to $(x^l - f^l(v))|\vec{x} - \vec{f}(v)|^{-4}$. But we have also the presence of terms with k^u, appearing in h_{vi}, B_{vi} and B_{ij}. Remembering that $\eta^{uv} = \eta^{vu} = -2$ we have

$$k^u = \eta^{uv} k_v = -2k_v\,. \tag{5.86}$$

It is also possible to relate k_v to the components of the momentum k_i. This is because all the coefficients of the metric and the B field will depend on v only through the combination $x^i - f^i(v)$. When a Fourier transform is taken, this fact reflects in the following relation (see [24]):

$$k_v = -\dot{f}^i(v) k_i\,. \tag{5.87}$$

When we sum this contributions with those coming from the diagrams with only one type of boundary, we expect to obtain the long distance behaviour of a supergravity solution.

Contrary to the two-charge system, there is not a systematic way to construct all the supergravity solutions representing the microstates. This is because using dualities we can not reach a simpler configuration, as the $F1$-P was for the two-charges case. This means that our predictions coming from the string amplitude cannot be directly compared with a known supergravity solution. Here we see the importance of the calculation of the string amplitude corresponding to this microstate.

We can anyway check the consistency of our prediction, i.e. verify that is satisfies the supergravity equations of motion, which are Einstein's equations of motion and the supersymmetry relations. Obviously we can do this only in a first order approximation, because we have a prediction only for the large distance behaviour of our fields. This procedure has also the problem that all bosonic fields are needed, not only those of the *NS-NS* sector, but also the *R-R* ones. In [26] this has been done; here we do not go further in the investigation of this microstate and refer to [26] for a complete analysis.

Chapter 6

A new microstate of the Strominger–Vafa black hole

In this chapter we consider another type of microstate of the three-charge Strominger–Vafa black hole; the calculation of the corresponding amplitude has not been made yet. We work again in the duality frame $D1$-$D5$-P, but this time with different assumptions on the setup of the source.

The idea is to see whether we can reproduce a known supergravity solution, derived in [27], which break the symmetry of the torus T^4. This solution is a good candidate to be a microstate of the Strominger–Vafa black hole. We are interested to understand whether this is indeed the case, trying to understand its microscopical interpretation. A natural guess is that this source comes from a bound state of branes with profile along a direction of the torus (let it be z_1). To be precise, it is not possible for the $D5$-brane to have a non trivial profile along z_1; in fact the brane is wrapped around the torus, and its physical oscillations can only manifest in the orthogonal space (the non compact $\mathbb{R}^4$). It turns out that such a "profile" along the torus has to be interpreted as a gauge field propagating on the brane ([24]).

In this chapter we start from the analysis of the supergravity solution, and then we make the calculation of the string disk amplitude. At the end we compare the two results and check whether there is a correspondence or not.

6.1 Analysis and expansion of the supergravity solution

There are several methods that allow us to derive new supersymmetric solution starting from a known one; this is what was made in [27]. The deformation of a $D1$-$D5$ background provides a method to introduce one more charges in the solution. We thus expect the result to be a possible microstate of the Strominger–Vafa black hole. In the following we assume that $Q_1 = Q_5 = Q$: we thus expect the dilaton to vanish. In fact the supersymmetric solution we consider has vanishing dilaton and Kalb–Ramond field. Adapting the solution to our conventions, the microstate of [27] reads

$$ds^2 = -\frac{1}{H}\left[du + A\right]\left[dv + B\right] + Hf\left[\frac{dr^2}{r^2+a^2} + d\theta^2\right] + H\left[r^2 c_\theta^2 d\psi^2 + (r^2+a^2) s_\theta^2 d\phi^2\right] + dz^a dz^a\,, \tag{6.1}$$

where

$$A = \frac{aQ}{f}\{s_\theta^2 d\phi + c_\theta^2 d\psi\} + \Phi(\vec{x}, v) dz_1\,, \quad B = \frac{aQ}{f}\{s_\theta^2 d\phi - c_\theta^2 d\psi\}\,, \quad f = r^2 + a^2 c_\theta^2\,, \quad H = 1 + \frac{Q}{f}\,. \tag{6.2}$$

The solution is actually supersymmetric if $\Phi(\vec{x}, v)$ takes the form

$$\Phi(\vec{x}, v) = \sum_{n=-\infty}^{\infty} c_n e^{-in\frac{v}{R_y}} \left(\frac{r^2}{r^2 + a^2}\right)^{\frac{|n|}{2}} , \tag{6.3}$$

where R_y is the radius of the circle S^1 and the coefficients c_n satisfy $(c_n)* = c_{-n}$. This solution is written in a set of coordinates that resembles the polar coordinates of $\mathbb{R}^4$. The latter are given by:

$$x^1 = \tilde{r}\cos\tilde{\theta}\cos\psi\,, \quad x^2 = \tilde{r}\cos\tilde{\theta}\sin\psi\,, \quad x^3 = \tilde{r}\sin\tilde{\theta}\cos\phi\,, \quad x^4 = \tilde{r}\sin\tilde{\theta}\sin\phi\,. \tag{6.4}$$

The coordinates r and θ are related to $\tilde{r}$ and $\tilde{\theta}$ by the relations

$$r^2\cos^2\theta = \tilde{r}^2\cos^2\tilde{\theta}\,, \quad (r^2 + a^2)\sin^2\theta = \tilde{r}^2\sin^2\tilde{\theta}\,. \tag{6.5}$$

Notice however that the two set of coordinates coincide at infinity, where $r^2 \gg a^2$. Furthermore we have introduced the notation $c_\theta^2 = \cos^2\theta$, $s_\theta^2 = \sin^2\theta$.

Our goal is to expand this geometry at large distances ($r \to 0$); in fact this will be the appropriate expansion in order to compare the result with the string amplitude. As one goes to infinity Φ approaches a regular function, that we call $g(v)$. In the large distance limit we have $r^2 + a^2 \simeq r^2$, and the metric becomes:

$$ds^2 \simeq -(du + g(v)dz_1)dv + \left[dr^2 + r^2 d\theta^2 + r^2 c_\theta^2 d\psi^2 + r^2 s_\theta^2 d\phi^2\right] + dz^a dz^a\,. \tag{6.6}$$

From this expression we see again that, in the large distance limit, we can safely treat r, θ, ϕ and ψ as polar coordinates. This metric is flat, although not written in the "standard" coordinates. In order to remove the extra term $-g(v)dz_1 du$ we perform the following diffeomorphism:

$$z_1' = z_1 - \frac{1}{2}\int g(v)dv, \quad u' = \lambda\left[u + \frac{1}{4}\int g(v)^2 dv\right], \quad v' = \frac{v}{\lambda}\,. \tag{6.7}$$

There is a subtlety here: y is a compact coordinate, i.e. we have the identification $y \approx y + 2\pi R_y$. We must require that the coordinate t' satisfies $t'(y = 0) = t'(y = 2\pi R_y)$. This fixes the value of λ to:

$$\lambda^{-2} = 1 - \frac{1}{8\pi R_y}\int_0^{2\pi R_y} g^2(v)dv\,. \tag{6.8}$$

The resulting metric at infinity is now given by:

$$ds^2 \simeq -du'dv' + dx^i dx^i + dz_1' dz_1' + dz_2 dz_2 + dz_3 dz_3 + dz_4 dz_4\,, \tag{6.9}$$

which is the standard form for a flat metric expressed in lightcone coordinates. In order to find a correct expansion comparable with the result of the string amplitude, we must first apply the same diffeomorphism to the complete metric (6.1). Let us define $\bar{A}$ as

$$\bar{A} = A - \Phi(\vec{x}, v)dz_1 = \frac{aQ}{f}\{s_\theta^2 d\phi + c_\theta^2 d\psi\}\,. \tag{6.10}$$

The metric after the diffeomorphism reads

$$\begin{aligned} ds^2 = &-\frac{1}{H}\left[\frac{du}{\lambda} - \frac{\lambda}{4}g^2 dv + \bar{A} + \Phi dz^1 + \frac{\lambda}{2}\Phi g dv\right][\lambda dv + B] + Hf\left[\frac{dr^2}{r^2 + a^2} + d\theta^2\right] \\ &+ H\left[r^2 c_\theta^2 d\psi^2 + (r^2 + a^2)s_\theta^2 d\phi^2\right] + dz^a dz^a + \lambda g dz^1 dv + \frac{\lambda^2}{4}g^2 dv^2\,. \end{aligned} \tag{6.11}$$

This is the complete solution expressed in standard coordinates, i.e. such that the metric at infinity approaches the flat metric written in standard form. We then perform an expansion of this metric at large r: remember that the function $\Phi(\vec{x}, v)$ approaches the function $g(v)$. The resulting coefficients are:

$$\begin{aligned}
g_{vv} &= -\frac{1}{H}\left(-\frac{\lambda^2}{4}g^2 + \frac{\lambda^2}{2}\Phi g\right) + \frac{\lambda^2 g^2}{4} \simeq (1 - H^{-1})\frac{\lambda^2 g^2}{4} \simeq \lambda^2 g^2 \frac{Q}{4r^2}\,, \\
g_{vu} &= \frac{1}{2}\left(-\frac{1}{H}\right) \simeq -\frac{1}{2} + \frac{Q}{2r^2}\,, \\
g_{vz_1} &= \frac{1}{2}\left(-\frac{1}{H}\Phi\lambda + g\lambda\right) \simeq \frac{1}{2}\lambda g(1 - H^{-1}) \simeq \lambda g \frac{Q}{2r^2}\,, \\
g_{ij} &= H\delta_{ij} \simeq \delta_{ij} + \frac{Q}{r^2}\delta_{ij}\,, \\
g_{ab} &= \delta_{ab}\,,
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
g_{v\phi} &= \frac{1}{2}\left(-\frac{1}{H}\right)\left(-\frac{\lambda}{4}g^2 + \frac{\lambda}{2}\Phi g + \lambda\right)\frac{aQ}{f}s_\theta^2 \simeq -\lambda\frac{aQ}{2r^2}s_\theta^2\left(\frac{g^2}{4} + 1\right)\,, \\
g_{v\psi} &= \frac{1}{2}\left(-\frac{1}{H}\right)\left(-\frac{\lambda}{4}g^2 - \frac{\lambda}{2}\Phi g + \lambda\right)\frac{aQ}{f}c_\theta^2 \simeq -\lambda\frac{aQ}{2r^2}c_\theta^2\left(-\frac{g^2}{4} + 1\right)\,, \\
g_{u\phi} &= \frac{1}{2}\left(-\frac{1}{H}\right)\frac{1}{\lambda}\frac{aQ}{f}s_\theta^2 \simeq -\frac{1}{\lambda}\frac{aQ}{2r^2}s_\theta^2\,, \\
g_{u\psi} &= \frac{1}{2}\left(\frac{1}{H}\right)\frac{1}{\lambda}\frac{aQ}{f}c_\theta^2 \simeq \frac{1}{\lambda}\frac{aQ}{2r^2}c_\theta^2\,, \\
g_{\phi z_1} &= \frac{1}{2}\left(-\frac{1}{H}\right)\Phi\frac{aQ}{f}s_\theta^2 \simeq -g\frac{aQ}{2r^2}s_\theta^2\,, \\
g_{\psi z_1} &= \frac{1}{2}\left(\frac{1}{H}\right)\Phi\frac{aQ}{f}c_\theta^2 \simeq g\frac{aQ}{2r^2}c_\theta^2\,.
\end{aligned} \tag{6.13}$$

Here we have divided the coefficients in two groups: those with no index along the $\mathbb{R}^4$ and those with one index equal to one of the two angles ϕ and ψ. All the coefficients that are not explicitly written, are intended to be zero. We remember again that the other *NS-NS* fields (Φ and $B_{\mu\nu}$) of this solution vanish.

We are now ready for the calculation of the string disk amplitude, and then for the comparison of the two results. Notice that the coefficient in (6.12) and (6.13) have explicit dependence on v (through the function $g = g(v)$); we expect to derive the same dependence from the calculation of the string amplitude.

6.2 Calculation of the string disk amplitude

The situation is completely analogous to the case we have studied in last chapter; the only difference is the profile function $f(v)$. We guess that we can derive our solution using a function that is non-vanishing only along the direction z^1. We have to sum three contribution, because the open string can have:

1. both ends on a D1-brane
2. both ends on a D5-brane
3. one end on a D1-brane and the other on a D5-brane

Let us analyse all these three contributions separately. We work with lightcone coordinates, and use a generic profile function $f^a(v)$ with components along the torus. At the end we will restrict to a profile only along z^1. We remember that the flat metric written in lightcone coordinates reads

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}. \tag{6.14}$$

The coordinates are always intended in the order (v, u, x^i, z^a).

6.2.1 Open string with both ends on a $D1$-brane

What we have to do is to calculate the amplitude corresponding to a closed string emitted from an open one stretching between D-1 branes carrying momentum. The key ingredient, as always, is the matrix of boundary conditions $\mathcal{R}$. When dealing with a $D1$-brane, the direction of the torus are perpendicular to the brane itself, as the non-compact directions x^i are. We thus expect the matrix $\mathcal{R}$ to be completely analogous to (5.34), i.e.

$$(\mathcal{R}_{D1})_{\bullet\bullet} = \begin{pmatrix} -2|\dot{f}|^2 & -\frac{1}{2} & 0 & 2\dot{f}^a \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 2\dot{f}^a & 0 & 0 & -\mathbb{1} \end{pmatrix}, \quad (\mathcal{R}_{D1})_{\bullet}{}^{\bullet} = \begin{pmatrix} 1 & 4|\dot{f}|^2 & 0 & 2\dot{f}^a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & -4\dot{f}^a & 0 & -\mathbb{1} \end{pmatrix}. \tag{6.15}$$

The supergravity solution depends on x^i and v; thus we take the momentum k having components $k_\mu = (k_v, 0, k_i, 0)$, or equivalently $k^\mu = (0, k^u, k^i, 0)$. Anyway, looking at the fact that we assume $f^i(v) = 0$ together with (5.87) implies that we have

$$k_v = 0 \Longleftrightarrow k^u = 0\,. \tag{6.16}$$

The calculation goes in the very same way as in the case of a profile along $\mathbb{R}^4$; the only difference is the matrix of boundary conditions, and the term $\langle e^{ik\cdot X}\rangle$ which gives a trivial contribution, due to the fact that the brane is located at $r = 0$ along the non-compact directions. Therefore we have the following amplitude (in the $NSNS$ sector):

$$\mathcal{A} \propto \mathcal{G}^{\mu\nu}\mathcal{R}_{\mu\nu}\,. \tag{6.17}$$

Notice that the symmetry of $\mathcal{R}_{\mu\nu}$ implies that we do not have any contribution to the Kalb–Ramond field. On the other hand we have a contribution to the dilaton, which is

$$\mathcal{A}^{(\Phi)} \propto \mathcal{G}^{\mu\nu}_{(\Phi)}\mathcal{R}_{\mu\nu} = \frac{1}{2\sqrt{2}}(\eta^{\mu\nu} - k^\mu l^\nu - k^\nu l^\mu)\mathcal{R}_{\mu\nu} = -\sqrt{2}\,. \tag{6.18}$$

From this amplitude we can derive a prediction for the dilaton. In particular

$$\Phi(k) = \frac{1}{\sqrt{2}}\mathcal{A}^{(\Phi)} \propto -1\,. \tag{6.19}$$

We then have to subtract properly the contribution of the dilaton, in order to obtain the coefficients of the metric. The result is

$$h_{\mu\nu}(k) \propto \mathcal{R}_{\mu\nu} - \frac{\mathcal{G}_\phi \cdot \mathcal{R}}{\mathcal{G}_\phi \cdot \eta}\eta_{\mu\nu} = \mathcal{R}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\,. \tag{6.20}$$

Therefore we obtain the following coefficients for $h_{\mu\nu}(k)$:

$$h_{vv}(k) \propto -2|\dot{f}(v)|^2\,, \quad h_{vu}(k) \propto -\frac{3}{4}\,, \quad h_{va}(k) \propto 2\dot{f}^a(v)\,, \quad h_{ii}(k) = h_{aa}(k) \propto -\frac{1}{2}\,. \tag{6.21}$$

The coefficients are not dependent on k; performing the Fourier transform, and calling Q_1' an overall multiplicative factor, we get the following (first order) predictions:

$$h_{vv} \simeq -2|\dot{f}(v)|^2 \frac{Q_1'}{r^2}\,, \quad h_{vu} \simeq -\frac{3}{4}\frac{Q_1'}{r^2}\,, \quad h_{va} \simeq 2\dot{f}^a(v)\frac{Q_1'}{r^2}\,, \quad h_{ii} = h_{aa} \simeq -\frac{1}{2}\frac{Q_1'}{r^2}\,. \tag{6.22}$$

We make the same Fourier transform for the dilaton, obtaining

$$\Phi \simeq -\frac{Q_1'}{r^2}\,. \tag{6.23}$$

6.2.2 Open string with both ends on a $D5$-brane

Let us now consider what happens when the open string has both the endpoints on a $D5$-brane. In this case the function $f^a(v)$ does not have the interpretation of the profile of the brane, because the latter wraps around the torus. Anyway it make sense to consider the analogous of the matrix (6.15), which can be obtained using T-dualities. The result is ([24]):

$$(\mathcal{R}_{D5})_{\bullet\bullet} = \begin{pmatrix} -2|\dot{f}|^2 & -\frac{1}{2} & 0 & -2\dot{f}^a \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 2\dot{f}^a & 0 & 0 & \mathbb{1} \end{pmatrix}\,, \quad (\mathcal{R}_{D5})_{\bullet}{}^{\bullet} = \begin{pmatrix} 1 & 4|\dot{f}|^2 & 0 & -2\dot{f}^a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & -4\dot{f}^a & 0 & \mathbb{1} \end{pmatrix}\,. \tag{6.24}$$

It can be checked that also this matrix satisfies $\bar{T} = T$ at the boundary. If one compares this expression with the one given in [24], one notice that there is the difference of the sign of the profile $f^a(v)$. Obviously there is nothing physical in this change of sign. Anyway (6.24) is the correct sign if we have to compare a $D1$-brane and a $D5$-brane. This fact will be very important when dealing with the amplitude with mixed boundary condition. The fact that (6.15) and (6.24) are consistent with each other can be seen from the calculation of the monodromy matrix (see [26]), defined as $M^\mu{}_\nu = (\mathcal{R}_{D5}^{-1}\mathcal{R}_{D1})^\mu{}_\nu$. Making the calculation one see that the result is $M^\mu{}_\nu = \mathrm{diag}(1, 1, \mathbb{1}, -\mathbb{1})$, which is the came result one can obtain from the other system studied in the previous chapter. This monodromy matrix make the worldsheet very simpler; one can say that in this case the $D1$ and $D5$ branes has the same "profile". A change of sign in (6.24) would have produced a non-diagonal monodromy matrix, and a much more complicated situation.

The form of the amplitude is the same of the case with the $D1$-brane. We can see that the contribution of the dilaton is

$$\mathcal{A}^{(\Phi)} \propto \mathcal{G}^{\mu\nu}_{(\Phi)}\mathcal{R}_{\mu\nu} = \frac{1}{2\sqrt{2}}(\eta^{\mu\nu} - k^\mu l^\nu - k^\nu l^\mu)\mathcal{R}_{\mu\nu} = \sqrt{2}\,. \tag{6.25}$$

Notice that the matrix $\mathcal{R}_{\mu\nu}$ is not symmetric: this means the amplitude gives rise to a non-trivial prediction for the Kalb–Ramond field. The correct normalisation is given by

$$h_{\mu\nu}(k) \propto \mathcal{R}_{\mu\nu} - \frac{\mathcal{G}_\phi \cdot \mathcal{R}}{\mathcal{G}_\phi \cdot \eta}\eta_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,, \quad B_{\mu\nu}(k) \propto \frac{1}{2}(\mathcal{R}_{\mu\nu} - \mathcal{R}_{\nu\mu})\,. \tag{6.26}$$

Performing the Fourier expansion, and calling Q_5' the overall constant, we get the following (first order) predictions for $h_{\mu\nu}$ and $B_{\mu\nu}$:

$$h_{vv} \simeq -2|\dot{f}(v)|^2 \frac{Q_5'}{r^2}\,, \quad h_{vu} \simeq -\frac{1}{4}\frac{Q_5'}{r^2}\,, \quad h_{ii} \simeq -\frac{3}{2}\frac{Q_5'}{r^2}\,, \quad h_{aa} \simeq -\frac{1}{2}\frac{Q_5'}{r^2}\,, \tag{6.27}$$

$$B_{va} \simeq -2\dot{f}^a(v)\frac{Q_5'}{r^2}\,. \tag{6.28}$$

We can do the same for the dilaton, and we obtain

$$\Phi \simeq \frac{Q_5'}{r^2}\,. \tag{6.29}$$

6.2.3 Open string with ends on different types of brane

The third process we have to consider is that involving mixed boundary conditions. In this case we have to insert two twisted vertex operators at the boundary of the disk, and to choose the open string vertex operator with a -1 superghost charge. The situation is very similar to the other $D1$-$D5$-P microstate we have studied in the previous chapter. The calculation of the amplitude goes in the same way as in the case of profile along $\mathbb{R}^4$, and one arrives to the following $NSNS$ amplitude:

$$\mathcal{A} \propto k^\sigma \mathcal{G}^{\mu\nu}\mathcal{R}_\nu{}^\rho v_{\mu\rho\sigma}\,, \tag{6.30}$$

where we have that $\langle e^{ik\cdot X}\rangle$ gives here a contribution equal to 1. The product of the two Chan–Paton factors $\bar{\mu}$ and μ involves the self-dual form $v_{\mu\nu\rho}$; from now on we assume that only the following components are non-vanishing:

$$v_{v12} = v_{v34}\,, \quad v_{u12} = -v_{u34}\,, \tag{6.31}$$

where the equalities follow from the self-duality conditions (5.85). Thus the amplitude reads

$$\mathcal{A} \propto k^l(\mathcal{G}^{v\mu}\mathcal{R}_\mu{}^i - \mathcal{G}^{i\mu}\mathcal{R}_\mu{}^v)v_{vil} + k^l(\mathcal{G}^{u\mu}\mathcal{R}_\mu{}^i - \mathcal{G}^{i\mu}\mathcal{R}_\mu{}^u)v_{uil}\,. \tag{6.32}$$

Using the explicit expression for the matrix $\mathcal{R}$ we have that

$$\begin{aligned} &\mathcal{G}^{v\mu}\mathcal{R}_\mu{}^i = -\mathcal{G}^{vi}\,, \quad \mathcal{G}^{i\mu}\mathcal{R}_\mu{}^v = \mathcal{G}^{iv}\,, \quad \mathcal{G}^{u\mu}\mathcal{R}_\mu{}^i = -\mathcal{G}^{ui}\,, \\ &\mathcal{G}^{i\mu}\mathcal{R}_\mu{}^u = 4|\dot{f}|^2\mathcal{G}^{iv} + \mathcal{G}^{iu} - 4\dot{f}^a\mathcal{G}^{ia}\,. \end{aligned} \tag{6.33}$$

Notice that if we use $\mathcal{R}_{D5}$ instead of $\mathcal{R}_{D1}$ we get exactly the same result. This is an important consistency check, and supports our choice of the sign of $f^a(v)$ in equation (6.24). Substituting into (6.32) we get:

$$\mathcal{A} \propto k^l(-\mathcal{G}^{vi} - \mathcal{G}^{iv})v_{vil} + k^l(-\mathcal{G}^{ui} - \mathcal{G}^{iu} - 4|\dot{f}|^2\mathcal{G}^{iv} + 4\dot{f}^a\mathcal{G}^{ia})v_{uil}\,. \tag{6.34}$$

We use the fact that $\mathcal{G}^{\mu\nu} = \hat{h}^{\mu\nu} + \frac{1}{\sqrt{2}}\hat{b}^{\mu\nu}$ and express the amplitude in terms only of the independent components ($\mu < \nu$):

$$\begin{aligned} \mathcal{A} \propto\, &\hat{h}^{vi}(-2k^l v_{vil} - 4k^l|\dot{f}|^2 v_{uil}) + \hat{h}^{ui}(-2k^l v_{uil}) + \hat{h}^{ia}(4k^l\dot{f}^a v_{uil}) + \\ &+ \hat{b}^{vi}\left(2\sqrt{2}k^l|\dot{f}|^2 v_{uil}\right) + \hat{b}^{ia}\left(2\sqrt{2}k^l\dot{f}^a v_{uil}\right)\,. \end{aligned} \tag{6.35}$$

From this last equation we can derive first order predictions for $h_{\mu\nu}(k)$ and $B_{\mu\nu}(k)$. We also reduce to the case $f^a = (f, 0, 0, 0)$; the result is

$$\begin{aligned} &h_{vi}(k) \propto -k^l v_{vil} - 2k^l\dot{f}^2 v_{uil}\,, \quad h_{ui}(k) \propto -k^l v_{uil}\,, \quad h_{iz_1}(k) \propto 2k^l\dot{f}v_{uil}\,, \\ &B_{vi}(k) \propto 2\sqrt{2}k^l\dot{f}^2 v_{uil}\,, \quad B_{iz_1}(k) \propto 2\sqrt{2}k^l\dot{f}v_{uil}\,. \end{aligned} \tag{6.36}$$

Making the Fourier transform, the dependence on k^l become a dependence on $\frac{x^l}{r^4}$. calling K an overall constant we arrive at the following coefficients:

$$h_{vi} \simeq -Kv_{vil}\frac{x^l}{r^4} - 2K\dot{f}^2 v_{uil}\frac{x^l}{r^4}\,, \quad h_{ui} \simeq -Kv_{uil}\frac{x^l}{r^4}\,, \quad h_{iz_1} \simeq 2K\dot{f}v_{uil}\frac{x^l}{r^4}\,, \tag{6.37}$$

$$B_{vi} \simeq 2\sqrt{2}K\dot{f}^2 v_{uil}\frac{x^l}{r^4}\,, \quad B_{iz_1} \simeq 2\sqrt{2}K\dot{f}v_{uil}\frac{x^l}{r^4}\,. \tag{6.38}$$

6.3 Comparison between the supergravity geometry and the string amplitude

At the beginning of the chapter, we said that our goal was to compare the string amplitude with the supergravity solution of [27]. We immediately see that this will not be possible: the supergravity solution has vanishing Kalb–Ramond field, while our amplitude predicts $B_{\mu\nu} \neq 0$. Anyway, this fact simply means that the supergravity solution does not have the simple microscopic interpretation we have thought.

Nevertheless, n this section we inspect how far we can go in the comparison of the expansion of the supergravity solution with the prediction deriving from the string disk amplitude. First of all we have to sum the terms in (6.22) with the similar ones calculated in (6.27). We have to impose that the charges of the two types of brane are equal; let us then assume that $Q'_1 = Q'_5 = Q'$. Specialising to the case where $f^a = (f, 0, 0, 0)$ we obtain

$$h_{vv} \simeq -4\dot{f}^2(v)\frac{Q'}{r^2}\,, \quad h_{vu} \simeq -\frac{Q'}{r^2}\,, \quad h_{vz_1} \simeq 2\dot{f}(v)\frac{Q'}{r^2}\,, \quad h_{ij} \simeq -2\frac{Q'}{r^2}\delta_{ij}\,, \quad h_{aa} \simeq 0\,. \tag{6.39}$$

To get the prediction for the dilaton we have to sum (6.23) and (6.29):

$$\Phi \simeq \frac{-Q'_1 + Q'_5}{r^2} = 0\,. \tag{6.40}$$

The Kalb–Ramond field receives contributions only from (6.28); we rewrite it here for convenience:

$$B_{vz_1} \simeq -2\dot{f}(v)\frac{Q'_5}{r^2}\,. \tag{6.41}$$

We immediately see that we have a match for the prediction $\Phi \simeq 0$. Let us now turn to the comparison of the metric coefficients (6.39) and (6.12); let us do it step by step: We immediately see that we have a perfect agreement regarding the coefficient $h_{ab} \simeq 0$.
Let us turn to the coefficient h_{uv}: we have to compare $h_{vu} = -\frac{Q'}{r^2}$ with $h_{vu} = \frac{Q}{2r^2}$. The two agree if we impose

$$Q = -2Q'\,. \tag{6.42}$$

With this identification the agreement between $h_{ij} = -2\frac{Q'}{r^2}\delta_{ij}$ and $h_{ij} = \frac{Q}{r^2}\delta_{ij}$ is guaranteed.
We now turn to the comparison of the coefficient h_{vz_1}. From the string amplitude (6.39) we get $h_{vz_1} = 2\dot{f}\frac{Q'}{r^2} = -\dot{f}\frac{Q}{r^2}$. Comparing to (6.12) we have to impose the relation:

$$\dot{f}(v) = -\frac{1}{2}\lambda g(v)\,. \tag{6.43}$$

The last term h_{vv} should be completely determined now. In fact from the string amplitude we got $h_{vv} = -4\dot{f}^2\frac{Q'}{r^2}$. Using the identifications (6.42) and (6.43) this should be equal to $h_{vv} = 2\dot{f}^2\frac{Q}{r^2} = \lambda^2 g^2 \frac{Q}{2r^2}$. Now we see a problem: we have a factor 2 of difference with the supersymmetric solution (6.12). This problem is probably related to the fact that our string amplitude predicts a non-vanishing Kalb–Ramond field ((6.41)), while the supersymmetric geometry has $B_{\mu\nu} = 0$.

The comparison of the other metric coefficient (6.37) and (6.13) requires a change of coordinates in the $\mathbb{R}^4$ directions. We use the convention of (6.4). We notice however that we have other non-vanishing predictions for the Kalb–Ramond field, given in equation (6.38). In order to make the comparison of the metric, let us start from the simpler coefficient, i.e. h_{ui}. Our goal is to derive h_{ur}, $h_{u\theta}$, $h_{u\phi}$ and $h_{u\psi}$. The first one is (thanks to the fact that $h_{ui} = g_{ui}$)

$$h_{ur} = h_{ui}\frac{dx^i}{dr} = h_{ui}\frac{x^i}{r} \simeq -Kx^i x^l \frac{v_{uil}}{r^5} = 0\,, \tag{6.44}$$

where we use the fact that $v_{uil} = -v_{uli}$. For the second one we notice that

$$\frac{dx^{1,2}}{d\theta} = -x^{1,2}\tan\theta \qquad \frac{dx^{3,4}}{d\theta} = x^{3,4}\cot\theta\,. \tag{6.45}$$

Using also the assumption (6.31), we get

$$\begin{aligned} h_{u\theta} = h_{ui}\frac{dx^i}{d\theta} &= \tan\theta(-x^1 h_{u1} - x^2 h_{u2}) + \cot\theta(x^3 h_{u3} + x^4 h_{u4}) \simeq \\ &\simeq K\tan\theta\left(x^1 x^2\frac{v_{u12}}{r^4} + x^2 x^1\frac{v_{u21}}{r^4}\right) - K\cot\theta\left(x^3 x^4\frac{v_{u34}}{r^4} + x^4 x^3\frac{v_{u43}}{r^4}\right) = 0\,. \end{aligned} \tag{6.46}$$

We see that the string amplitude correctly predicts $h_{ur} = 0 = h_{u\theta}$, in agreement with the supersymmetric solution. Let us now turn to the calculation of $h_{u\phi}$, using the fact that

$$\frac{dx^{1,2}}{d\phi} = 0 \qquad \frac{dx^3}{d\phi} = -x^4 \qquad \frac{dx^4}{d\phi} = x^3\,, \tag{6.47}$$

we arrive at the following result:

$$h_{u\phi} = h_{ui}\frac{dx^i}{d\phi} = -x^4 h_{u3} + x^3 h_{u4} = K(x^4)^2\frac{v_{u34}}{r^4} - K(x^3)^2\frac{v_{u43}}{r^4} = K\frac{v_{u34}}{r^2}s_\theta^2\,. \tag{6.48}$$

In an analogous way we find also that

$$h_{u\psi} = h_{ui}\frac{dx^i}{d\phi} = -x^2 h_{u1} + x^1 h_{u2} = K(x^2)^2\frac{v_{u12}}{r^4} - K(x^1)^2\frac{v_{u21}}{r^4} = K\frac{v_{u12}}{r^2}c_\theta^2\,. \tag{6.49}$$

Using also $v_{u12} = -v_{u34}$ we see that the result actually agrees with (6.13), provided that:

$$K v_{u34} = -\frac{aQ}{2\lambda}\,. \tag{6.50}$$

We now turn to the analysis of the h_{iz_1} terms. From (6.37) we see that $h_{iz_1} = -2\dot{f}h_{ui}$. Using the identification (6.43) we should have $h_{iz_1} = \lambda g h_{ui}$. This is indeed the case, as we can see looking at (6.13). The last terms are h_{vi}. From (6.37) we see that h_{vi} has two contribution, one proportional to $x^l v_{uil}$ and one to $x^l v_{vil}$. We can use the same analysis we made for h_{ui}, and conclude that $h_{vr} \simeq 0 \simeq h_{v\theta}$. Using the identifications (6.43) and (6.50) we get

$$h_{v\phi} = K\frac{v_{v34}}{r^2}s_\theta^2 - \lambda\frac{aQ}{4r^2}g^2 s_\theta^2\,, \quad h_{v\psi} = K\frac{v_{v12}}{r^2}c_\theta^2 + \lambda\frac{aQ}{4r^2}g^2 c_\theta^2\,. \tag{6.51}$$

We now remember that $v_{v12} = v_{v34}$; looking at the first term (the one not proportional to g^2), we can make it agree with (6.13) if we impose

$$K v_{v34} = -\frac{\lambda aQ}{2}\,. \tag{6.52}$$

We see that we have again a factor 2 of difference in the terms proportional to g^2.

Let us now draw the conclusions from what we have found in this chapter. We tried to find the microscopical origin of the supersymmetric solution (6.1); anyway, we found that the string amplitude we have calculated fails to reproduce completely the solution. We can find a match between many metric coefficients, but not all; we have found some factors 2 of difference. This problem is related to the incongruence we have found for the Kalb–Ramond field; it is possible that the two solution are somehow related to each other, even if they are not exactly the same.

In fact, changing the gauge fields of a supersymmetric solution induces some changes in the metric, if one wants to get another supersymmetric solution.

It is natural to think that in order to obtain (6.1) one has to take a more complicated worldsheet setup: probably a "fine-tuning" of the microscopic situation is needed to make the Kalb–Ramond field vanish.

One could continue the analysis checking whether the predictions coming from the string amplitude satisfy the supersymmetry equation of motion at first order (as should be expected for consistency): in order to do so, however, one needs also the R-R bosonic fields. One could also try to deform in some way the solution (6.1), in such a way to derive a supersymmetric solution with a non trivial Kalb–Ramond field. We think that such a deformation could be responsible for all the mismatches we have described.

Conclusions

In this work we presented the derivation of black hole microstates through the calculation of string amplitudes.

We have described the properties of black holes in general relativity, the uniqueness theorems and the thermodynamic properties. These two facts give rise to non-trivial puzzles, such as the information paradox and the nature of the microstates responsible for the macroscopical entropy of a black hole.

We have then turned to the study of black holes in supergravity, in particular in 11 and 10 dimensions; one has to consider suitable bound states of different types of branes, possibly carrying momentum. Furthermore we have explained why a real black hole with non vanishing horizon area needs the presence of at least three charges. In particular we have investigated the properties of the Strominger–Vafa black hole, which is a three-charge system in a 5-dimensional spacetime, and can be considered (in 10 dimensions) as the bound state of $D1$ and $D5$ branes carrying momentum.

In order to give a microscopical interpretation to the properties of black holes, and to solve the related problems, one has to turn to a quantum gravity theory. String theory provides such a theory, and we turned to a brief study of its fundamental properties. It is then possible to understand which sources can produce a solution with singularity, and to count all the possible microstates related to the same classical solution. We have seen how this count correctly reproduces the entropy of the Strominger–Vafa black hole ([3]); furthermore we have described a proposed solution to the information problem, the so-called fuzzball conjecture ([22]).

We have analysed the derivation of the microstate geometries from string amplitudes. The idea comes from the interpretation of D-branes as surfaces where the endpoints of open strings lie. These open strings can couple to closed ones, whose massless excitations describe the metric and other supergravity fields. By computing the amplitude corresponding to this process one can derive (at least in the large distance limit) the supergravity solution corresponding to the bound states of D-branes. We expect this solution to describe a microstate of the corresponding black hole.

We have reviewed the single-charge and the two-charge systems; we found that there is a perfect matching between the amplitude predictions and some particular known supergravity solutions. We have done the calculation only for a restricted set of bosonic fields, those appearing in the *NS*-*NS* superstring sector. The analysis of the other fields has been made in the literature ([23], [24], [25]), resulting again in a perfect matching with the corresponding supergravity solution. In the case of the three-charge black hole the situation is more complicated, and a complete characterization of the microstates is still lacking. For this reason, the calculation of string amplitudes provides one of the few methods to derive the supergravity solution describing the microstates (at least in the large distance limit). We have done this for a particular microstate; the complete analysis of this solution, and the proof that it is indeed supersymmetric can be found in [26].

The new contribution of this thesis was the calculation of the amplitude of a string configuration which should reproduce a different type of microstate of the Strominger–Vafa black hole. The goal was to compare the amplitude prediction with a known supergravity solution, obtained in [27], which is a candidate to be a microstates of the black hole. This calculation, however, shows a non perfect matching between the amplitude predictions and the supergravity solution; the reason is probably that we get a different microstate of the same black hole. What should be done now, is to extend the calculation of the amplitude for all the other bosonic fields, belonging to the R-R superstring sector; one can also compare these fields with the corresponding ones in [27]. A non-trivial check is the verification that our solution coming from the string amplitude is supersymmetric; in order to do so, one should verify that our first order predictions satisfy the supersymmetry equations at first order. For this purpose all bosonic fields are needed.

This result shows that it is not clear the connection between string configurations and supersymmetric solutions. Given the supergravity solution, we have done the simplest possible ansatz for the corresponding worldsheet setup; however, we have seen that simple configurations on the worldsheet do not correspond to simple supergravity solution. We think that a suitable "fine-tuning" of the microscopic configuration is needed in order to get the desired supergravity solution, and make some supergravity fields (like the B field) vanish. More investigation is necessary to arrive to a complete understanding of the microscopical intepretation of black holes.

There are several ways one could complete and generalize the work described in this thesis. First of all it would be interesting to analyse more string configurations and derive the corresponding supersymmetric solutions, in order to have a complete description of the microstates of a given classical solution, in particular black holes. Progresses have been done in the case of the Strominger–Vafa black hole. It would also be interesting to deal with the four-charge black holes in four dimensions.

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