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Transcendence results for periods of elliptic and abelian functions

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Introduction

The quest for transcendental numbers has always been a hard challenge: even nowadays the transcendence of several constants which are ubiquitous in mathematics is at a purely conjectural state and seems to be beyond of the reach of current methods. As a noticeable example, the numbers e and π have been proved transcendental only at the end of the nineteenth century, despite being widely studied and employed since much earlier. Our knowledge becomes even more blurred when one is interested in the algebraic independence of several numbers. At present, it is for instance an open problem to determine whether the aforementioned e and π are in fact algebraically independent or not.

The first methods to investigate the transcendence of some numbers involved a mixture of arithmetic and analytic techniques. This sophisticated machinery could be applied in practice only to special values of particularly well-behaved functions, such as the exponential one. A turning point in theory was reached through the adaptation of these methods to the study of numbers arising as periods of elliptic curves or, more generally, of Abelian varieties. In this thesis, we will go through some of the major results which have been obtained in this context, and we will eventually expose a contribution of ours which lies halfway between the framework of the exponential function and the one of the periods of Abelian varieties.

The first chapter will be devoted to a striking achievement by Chudnovsky dating back to 1976 concerning the existence of two algebraically independent numbers among the periods and quasi-periods of an elliptic curve over $\mathbb{C}$ with algebraic invariants. If we exclude the applications of Lindemann-Weierstraß Theorem to exponentials of algebraic numbers, this result enabled Chudnovsky to give the first examples of two explicit algebraically independent numbers of arithmetic interest, such as for instance π and $\Gamma(1/4)$. Although Chudnovsky's strategy has become a classic in transcendence proofs, the original paper [Chu76] in Russian has never been translated and only a rough sketch of the argument can be found in [Chu84]. We hope that our detailed exposition may serve as a possible reference on the subject.

In the second chapter we will turn to a generalization of these techniques to the case of periods of complex Abelian varieties. In particular, we will focus on a result obtained by Vasilev in 1996, which stands as a qualitative improvement of Chudnovsky's earlier work and is still the best achievement reached so far in this context. The main reference which we are going to follow is Vasilev's paper [Vas96].
Finally, the third and last chapter will revolve around an attempt of ours to apply these techniques to the case when some periods are replaced by their exponentials. Although we are not able to obtain a completely satisfactory result, we will nonetheless succeed in giving an algebraic independence criterion in most cases of interest. We will conclude our exposition by applying such criterion to values of the B-function at rational numbers.

Contents

Chapter 1

Periods of elliptic functions

1.1 Preliminaries

This section will be devoted to fix the common setting for the rest of our entire exposition. We will hence give some elementary definitions regarding finitely generated extensions of $\mathbb{Q}$, with a brief excursus about transcendence measures. We will then expose an elementary lemma in this context, upon which several constructions will be later based. We warn the reader that, whenever the terms *algebraic*, *transcendental*, *algebraically independent* and so forth are used with no reference to a base field, the latter is always intended to be $\mathbb{Q}$.

For a polynomial $P \in \mathbb{Z}[x]$, we define its *height*, denoted by $H(P)$, as the maximum modulus of its coefficients. We will write $L(P)$ for the *length* of P, that is, the sum of the absolute values of the coefficients of P. If $P \neq 0$, we also denote by $t(P)$ the *type* of P, which stands for the maximum between the degree and the logarithm of the height of P. We will now correctly generalize the definition of type of a polynomial to elements of finitely generated extensions of $\mathbb{Q}$.

Definition 1.1. A subfield K of $\mathbb{C}$ is *of finite type* over $\mathbb{Q}$ if it is a finite extension of $\mathbb{Q}(x_1, \dots, x_q)$ for some $x_1, \dots, x_q \in \mathbb{C}$ algebraically independent over $\mathbb{Q}$.

In this definition, we also admit the case $q = 0$, meaning that the set $\{x_1, \dots, x_q\}$ is empty, so K is a finite extension of $\mathbb{Q}$ embedded in $\mathbb{C}$. If K is an extension of $\mathbb{Q}$ of finite type, we denote by $\operatorname{trdeg}(K/\mathbb{Q})$ the transcendence degree of K over $\mathbb{Q}$, i.e. the highest cardinality of a set of elements of K which are algebraically independent over $\mathbb{Q}$. Let us set for short

$q = \text{trdeg}(K/\mathbb{Q})$ and let $x_1, \ldots, x_q \in K$ be algebraically independent over $\mathbb{Q}$. Since K is by definition a finite extension of $\mathbb{Q}(x_1, \ldots, x_q)$, by the primitive element Theorem there exists $y \in K$ algebraic over $\mathbb{Q}(x_1, \ldots, x_q)$ such that $K = \mathbb{Q}(x_1, \ldots, x_q, y)$. Up to multiplying y by the common denominator of the coefficients of its minimal polynomial over $\mathbb{Q}(x_1, \ldots, x_q)$, we may assume that y is integral over $\mathbb{Z}[x_1, \ldots, x_q]$.
A $(q+1)$-tuple $(x_1, \ldots, x_q, y)$ as the one just constructed will be called a *generating system* for K over $\mathbb{Q}$. Every extension K of $\mathbb{Q}$ of finite type admits a generating system, so that any $a \in K$ can be written uniquely in the form

$$a = \sum_{i=1}^{d} \frac{Q_i}{R_i} y^{i-1},$$

where $d = [K : \mathbb{Q}(x_1, \ldots, x_q)]$ is the degree of y over $\mathbb{Q}(x_1, \ldots, x_q)$ and, for any $i = 1, \ldots, d$, Q_i and R_i are relatively prime elements of $\mathbb{Z}[x_1, \ldots, x_q]$. We remark that, for $x_1, \ldots, x_q$ are algebraically independent over $\mathbb{Q}$, the ring $\mathbb{Z}[x_1, \ldots, x_q]$ is a unique factorization domain. We may thus consider the least common multiple P of the R_i's for $i = 1, \ldots, d$, so as to write $a \in K$ uniquely in the form

$$a = \frac{1}{P} \sum_{i=1}^{d} P_i y^{i-1},$$

where $P, P_1, \ldots, P_d$ lie in $\mathbb{Z}[x_1, \ldots, x_q]$.
At this point, for any $i = 1, \ldots, q$ we may define the degree of a in x_i as

$$\deg_{x_i} a := \max\{\deg_{x_i} P, \deg_{x_i} P_1, \ldots, \deg_{x_i} P_d\}.$$

Similarly, we set the type of $a \in K$ to be

$$t(a) := \max\{t(P), t(P_1), \ldots, t(P_d)\}.$$

We remark that this notion of *type* depends on the choice of a generating system for K over $\mathbb{Q}$. Furthermore, it enjoys the following properties:

Lemma 1.2. *Let K be an extension of $\mathbb{Q}$ of finite type and let $(x_1, \ldots, x_q, y)$ be a generating system for K over $\mathbb{Q}$.*

1. *For all $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}[x_1, \ldots, x_q, y]$ we have*

$$t(\alpha_1 + \cdots + \alpha_m) \leq \log m + \max_{i=1,\ldots,m} t(\alpha_i);$$

2. *There exists a constant $c > 0$, depending only on the chosen generating system, such that for all $a_1, \ldots, a_m \in K$ we have*

$$t(a_1 + \cdots + a_m) \leq c(t(a_1) + \cdots + t(a_m)),$$
$$t(a_1 \ldots a_m) \leq c(t(a_1) + \cdots + t(a_m));$$

3. For any $\alpha \in \mathbb{Z}[x_1, \dots, x_q, y]$ and $\beta \in \mathbb{Z}[x_1, \dots, x_q]$ with $\beta \neq 0$ we have

$$t\left(\frac{\alpha}{\beta}\right) \leq c \max\{t(\alpha), t(\beta)\}.$$

The proof of this Lemma, though elementary, requires some technical computations; a detailed exposition can be found in [Wal74, Lemme 4.2.5]. Since we will mainly deal with extensions of $\mathbb{Q}$ of transcendence degree $q = 1$, we will later go through the proof of this Lemma in this particular case.

We now briefly expose some quantitative results involving transcendence measures, which will be often recalled in the sequel. Let us consider a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \varphi(x, y)$, which for convenience we suppose to be defined only for $x \geq 1$ and $y \geq \log 16$. If $\omega \in \mathbb{C}$ is transcendental, we say that φ is a *transcendence measure* for ω if for all non-zero polynomials P with integer coefficients of degree at most n and height at most h we have $\log |P(\omega)| \geq -\varphi(n, \log h)$. Furthermore, a real number $\tau > 0$ is said to be a *transcendence type* for ω if there is a constant $c(\omega, \tau) > 0$ such that $c(\omega, \tau)(n + \log h)^\tau$ is a transcendence measure for ω.
With this definition, it is clear that if τ is a transcendence type for ω, then so is any $\tau' \geq \tau$. For this reason, in the literature the transcendence type of ω is often defined as the infimum of all these values τ. Nonetheless, we prefer to stick with our definition, for it is in general a hard question to determine whether this infimum is in fact a minimum or not.
The following two Propositions aim at proving that any transcendence type τ for a transcendental number ω must satisfy $\tau \geq 2$, and that this is essentially best possible in almost all the cases.

Proposition 1.3. *Let $\omega \in \mathbb{C}$ be transcendental. Then there exist two constants $c_1, c_2 > 0$ such that for any integer $T \geq c_1$ there is a non-zero polynomial P with integer coefficients which satisfies*

$$\deg P \leq 2T - 1, \qquad \log H(P) \leq T + \log 2, \qquad |P(\omega)| \leq e^{-c_2 T^2}.$$

Proof. For an arbitrary integer $T > 0$, set $n = 2T - 1$, $h = e^T$ and denote by Λ the set of all non-zero polynomials of degree at most n and height at most h. This set has cardinality $|\Lambda| = (2h + 1)^{n+1} - 1$. For any polynomial $P \in \Lambda$ we plainly have

$$|P(\omega)| \leq nh(\max\{1, |\omega|\})^n.$$

Let us consider the set $A = \{P(\omega) \mid P \in \Lambda\}$. Since ω is transcendental, the cardinality of A must coincide with the one of Λ. Moreover, by the above inequality the points of A lie in the square

$$\{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq nh(\max\{1,|\omega|\})^n,\ |\mathrm{Im}(z)| \leq nh(\max\{1,|\omega|\})^n\}.$$

We divide each side of this square into $\lfloor\sqrt{|\Lambda|-1}\rfloor$ congruent segments, which yield a grid made up of at most $|\Lambda|-1$ smaller squares. Thus, two distinct points of A, say $P_1(\omega)$ and $P_2(\omega)$, must lie in the same square. These then satisfy

$$|P_1(\omega)-P_2(\omega)| \leq \frac{2nh\max\{1,|\omega|\}^n\sqrt{2}}{\sqrt{(2h+1)^{n+1}-2}} \leq \frac{2nh\max\{1,|\omega|\}^n\sqrt{2}}{(2h)^{\frac{n+1}{2}}}$$
$$\leq e^{\log 2\sqrt{2}+\log n+\log h+n\log(1+|\omega|)-\frac{n+1}{2}\log(2h)}.$$

The leading term in this expression turns out to be $e^{-\frac{n+1}{2}\log(2h)}$, so by choosing T sufficiently large, say $T > c_1$, we may find a constant c_2 such that

$$|P_1(\omega)-P_2(\omega)| \leq e^{-c_2\frac{n+1}{2}\log h} \leq e^{-c_2T^2}.$$

The claim then follows by observing that $P_1 - P_2$ has degree at most n and height at most $2h$. □

Proposition 1.4. *Almost all complex numbers, with respect to the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$, have transcendence type $\leq 2+\varepsilon$ for any $\varepsilon > 0$.*

Proof. We adapt the proof given in [Amo90] to the one dimensional case. Let us fix $\omega \in \mathbb{C}$ and define $\tau(\omega)$ to be the infimum of the transcendence types of ω. One first observes that $\tau(\omega)$ coincides with the infimum of all the positive real numbers η for which there exists a constant $c(\omega,\eta) > 0$ such that for any algebraic number α we have

$$\log|\omega-\alpha| \geq -c(\omega,\eta)t(P_\alpha)^\eta,$$

where P_α is the minmal polynomial of α over $\mathbb{Q}$. The details of this step can be found for example in [Lan66, Chapter VI, Theorem 2].
We denote by λ the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. Let B signify the closed unit ball of $\mathbb{C}$ and set

$$\Lambda = \{\omega \in B \mid \tau(\omega) > 2\}.$$

It is enough to show that $\lambda(\Lambda) = 0$. Then

$$\Lambda \subseteq \bigcap_{s=2}^{\infty}\bigcup_{N=1}^{\infty}\bigcup_{\substack{f\in\mathbb{Z}[x]\\ \lfloor t(f)\rfloor=N}} A_f\left(e^{-sN^2}\right),$$

where $A_f(\varepsilon)$ consists of the set of $\omega \in B$ which have distance $\leq \varepsilon$ from one of the roots of f. Thus, $\lambda(A_f(\varepsilon)) \leq \pi\varepsilon^2 \deg f$, and so for any $s \geq 2$ and $N \geq 1$

$$\lambda\left(A_f\left(e^{-sN^2}\right)\right) \leq \pi e^{-2sN^2} \deg f.$$

The number of polynomials in $\mathbb{Z}[x]$ of type $\leq N$ is at most e^{2N^2}. This implies that for any $s \geq 2$ we have

$$\lambda(\Lambda) \leq \lambda\left(\bigcup_{N=1}^{\infty} \bigcup_{\substack{f \in \mathbb{Z}[x] \\ \lfloor t(f) \rfloor = N}} A_f\left(e^{-sN^2}\right)\right) \leq \sum_{N=1}^{\infty} \pi N e^{-2(s-1)N^2}.$$

The last sum is convergent and

$$\sum_{N \geq 1}^{\infty} N e^{-2(s-1)N^2} \leq \sum_{N=1}^{\infty} e^{-(2s-3)N^2} \leq \int_1^{\infty} e^{-(2s-3)x^2}\, dx \leq e^{\frac{1}{4s}} \sqrt{\frac{\pi}{2s-3}},$$

which tends to 0 as $s \to \infty$. □

As it is often the case, despite Proposition 1.4 it is in practice very complicated to estimate the transcendence type of a given number. To our knowledge, the only number that is known to have type $\leq 2+\varepsilon$ for any $\varepsilon > 0$ is π, for which Waldschmidt gave the following transcendence measure in [Wal78]:

$$\varphi(n, \log h) = 2^{40} n(\log h + n \log n)(1 + \log n).$$

In the same paper, other transcendence measures for classical numbers can be found, for instance connected with exponentials and logarithms. We content ourselves with exposing the one for π, for it will be of central importance later on.

Finally, we conclude this section with an evergreen tool in transcendence proofs, the so-called *Siegel's Lemma*, which, despite its elementary formulation, will be a crucial ingredient in all the constructions appearing throughout our whole discussion. This Lemma provides a non-explicit way to find non-trivial integer solutions to homogeneous linear systems with integer coefficients, and its proof is a mere application of the pigeonhole principle.

Lemma 1.5 (Siegel)**.** *Let m and n be positive integers with $m < n$. For all $i = 1, \dots, m$ and $j = 1, \dots, n$, let a_{ij} be integers with absolute value at most $A \geq 1$. Then there exist integers $x_1, \dots, x_n$, not all zero, with absolute value at most*

$$B = \left\lfloor (nA)^{\frac{m}{n-m}} \right\rfloor,$$

which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j = 0.$$

for any $i = 1, \dots, m$.

Proof. The number of n-tuples $(x_1, \dots, x_n)$ such that $0 \le x_j \le B$ for all $j = 1, \dots, n$ is $(B+1)^n$. For any such n-tuple and any $i = 1, \dots, m$, let us set $y_i = \sum_{j=1}^{n} a_{ij}x_j$. If we call $-V_i$ and W_i the sum of the negative and positive a_{ij}'s respectively, then $-V_iB \le y_i \le W_iB$. Each y_i runs therefore in a range of $(V_i + W_i)B + 1 \le nAB + 1$ possible values, so the possible m-tuples $(y_1, \dots, y_m)$ are $(nAB+1)^m$. We have $(B+1)^{n-m} > (nA)^m$, which yields $(B+1)^n > (nAB+1)^m$. It follows that the map which associates to each $(x_1, \dots, x_n)$ the corresponding $(y_1, \dots, y_m)$ cannot be injective. The difference of two n-tuples having the same image under such map is the sought solution to the homogeneous linear system of the Lemma. □

The version of Siegel's Lemma that we will now describe and that we will actually exploit is slightly more general, as it deals with extensions of $\mathbb{Q}$ of finite type. In spite of some technical complications, its proof remains essentially elementary, making its usefulness in our exposition even more remarkable and surprising.

Lemma 1.6. *Let K be an extension of $\mathbb{Q}$ of finite type and let $(x_1, \dots, x_q, y)$ be a generating system for K over $\mathbb{Q}$. Then there exists a constant $C > 0$ which enjoys the following property.*
Let n and r be positive integers with $n \ge 2r$ and consider $a_{ij} \in \mathbb{Z}[x_1, \dots, x_q, y]$, for $i = 1, \dots, n$, $j = 1, \dots, r$. Then there exist $\xi_1, \dots, \xi_n \in \mathbb{Z}[x_1, \dots, x_q, y]$, not all zero, such that for all $j = 1, \dots, r$

$$\sum_{i=1}^{n} \xi_i a_{ij} = 0 \quad \textit{and} \quad \max_{i=1,\dots,n} t(\xi_i) \le C\left(\max_{i,j} t(a_{ij}) + \log n\right).$$

Proof. Let us set $\delta = [K : \mathbb{Q}(x_1, \dots, x_q)]$. We wish to find a solution in $\mathbb{Z}[x_1, \dots, x_q, y]$ for the linear system

$$\sum_{i=1}^{n} \xi_i a_{ij} = 0 \quad \text{for } j = 1, \dots, r$$

in the unknowns $\xi_1, \dots, \xi_n$. Let us introduce for $i = 1, \dots, n$ and $l = 1, \dots, \delta$ the new unknowns $\eta_{ij} \in \mathbb{Z}[x_1, \dots, x_q]$ in such a way that

$$\xi_i = \sum_{l=1}^{\delta} \eta_{il} y^{l-1}.$$

Moreover, we may write $a_{ij} = \sum_{h=1}^{\delta} b_{ijh} y^{h-1}$ and $y^{\delta+u} = \sum_{k=1}^{\delta} \varepsilon_{uk} y^{k-1}$ for any non-negative integer $u \geq 0$, with b_{ijh} and ε_{kl} being suitable elements of $\mathbb{Z}[x_1, \dots, x_q]$. The initial linear system is then equivalent to solving

$$\sum_{i=1}^{n} \left(\sum_{h+l=k+1} b_{ijh} \eta_{il} + \sum_{u=0}^{\delta-2} \sum_{h+l=\delta+u+2} \varepsilon_{uk} b_{ijh} \eta_{il} \right) = 0$$

in the unknowns $\eta_{il} \in \mathbb{Z}[x_1, \dots, x_q]$. This argument enables us to reduce to a find a solution in $\mathbb{Z}[x_1, \dots, x_q]$ of a linear system with coefficients in $\mathbb{Z}[x_1, \dots, x_q]$ whose type is $\leq c_1 t(a_{ij})$ for some $c_1 > 0$ independent of the a_{ij}'s. In order to simplify notation, it then suffices to prove our statement for the case when $y \in \mathbb{Z}[x_1, \dots, x_q]$, that is, $K = \mathbb{Q}(x_1, \dots, x_q)$.
At this point, we may write

$$a_{ij} = \sum_{m_1=0}^{d_1-1} \cdots \sum_{m_q}^{d_q-1} a_{ijm} x_1^{m_1} \dots x_q^{m_q}$$

for suitable $a_{ijm} \in \mathbb{Z}$ with $m = (m_1, \dots, m_q)$ and $d_1, \dots, d_q \geq 0$. Consider an integer $c_2 > 0$ satisfying

$$c_2 \geq \left(\left(\frac{3}{2} \right)^{\frac{1}{q}} - 1 \right)^{-1}$$

and introduce the new unknowns $\xi_{i\mu} \in \mathbb{Z}$ for $i = 1, \dots, n$ and $\mu = (\mu_1, \dots, \mu_q)$ with $\mu_h = 1, \dots, c_2 d_h - 1$ for all $h = 1, \dots, q$, in such a way that

$$\xi_i = \sum_{\mu_1=0}^{c_2 d_1 - 1} \cdots \sum_{\mu_q=0}^{c_2 d_q - 1} \xi_{i\mu} x_1^{\mu_1} \dots x_q^{\mu_q}.$$

By the algebraic independence of $x_1, \dots, x_q$, the original system is then equivalent to the one given by

$$\sum_{i=1}^{n} \sum_{m+\mu=M} \xi_{i\mu} a_{ijm} = 0,$$

for all $M = (M_1, \dots M_q) \in \mathbb{Z}^q$ satisfying $0 \leq M_j \leq (c_2 + 1) d_j - 1$. This is a linear system with coefficients in $\mathbb{Z}$, consisting of $(c_2 + 1)^q d_1 \dots d_q r$ equations and $c_2^q d_1 \dots d_q n$ unknowns. Since

$$c_2^q d_1 \dots d_q n \geq \frac{4}{3} d_1 \dots d_q r,$$

Lemma 1.5 ensures the existence of a non-trivial solution $(\xi_{i\mu})_{i\mu}$ in $\mathbb{Z}$ for such linear system. Moreover, if A is the maximum modulus of the a_{ijm}'s, the modulus of each of the $\xi_{i\mu}$'s is bounded from above by

$$(c_2^q d_1 \dots d_q n A)^{\frac{(c_2+1)^q r}{c_2^q n-(c_2+1)^q r}} \leq \exp\left(c_3\left(\log A + \log d_1 \dots d_q + \log n\right)\right),$$

where c_3 is the absolute constant given by

$$c_3 = \frac{(c_2+1)^q}{2c_2 - (c_2+1)^q} + q \log c_2.$$

Since the maximum type of the a_{ij}'s is at most $\max\{d_1 + \dots + d_q, \log A\}$, the claim is established. □

1.2 Elliptic functions

Let us start by considering an elliptic curve E over $\mathbb{C}$ defined by the affine equation $y^2 = 4x^3 - g_2 x - g_3$ for suitable $g_2, g_3 \in \mathbb{C}$. It is well known that there exists a meromorphic function $\wp$ over $\mathbb{C}$, a so-called *Weierstraß $\wp$-function*, which yields a surjective homomorphism of complex Lie groups

$$\mathbb{C} \to E, \quad z \mapsto [\wp(z), \wp'(z), 1].$$

The kernel of this map is a lattice Λ in $\mathbb{C}$, hence E turns out to be isomorphic, as a complex Lie group, to a complex torus $\mathbb{C}/\Lambda$. Let us denote by ω_1 and ω_2 a pair of generators for the lattice Λ, chosen in such a way that $\mathrm{Im}\frac{\omega_1}{\omega_2} > 0$; these will be called *periods* of the elliptic curve E. We may then take $\wp$ to be defined by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

which converges absolutely and uniformly on $\mathbb{C} \setminus \Lambda$. Thus, $\wp$ has double poles at the lattice points and it is doubly periodic with respect to both ω_1 and ω_2. More generally, any meromorphic function which is doubly periodic in ω_1 and ω_2 is called an *elliptic function* for Λ. Of course, the derivative $\wp'$ of $\wp$ is elliptic for Λ, and it is a classical fact that the field of all elliptic functions for Λ precisely coincides with $\mathbb{C}(\wp, \wp')$.
A primitive of $-\wp$ is given by Weierstraß ζ-function, defined by the series

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

which converges absolutely and uniformly on $\mathbb{C} \smallsetminus \Lambda$. This function has simple poles at the lattice points and it is quasi-periodic with respect to Λ, in the sense that there is a $\mathbb{Z}$-linear map $\eta : \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \to \mathbb{C}$ such that for all $z \in \mathbb{C} \smallsetminus \Lambda$ and all $\omega \in \Lambda$

$$\zeta(z + \omega) = \zeta(z) + \eta(\omega).$$

We will usually denote by η also the $\mathbb{R}$-bilinear extension of η to $\Lambda \otimes \mathbb{R}$. The complex numbers $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$ are the *quasi-periods* of E. By integrating ζ along a fundamental parallelogram of Λ, Legendre's relation is easily deduced, that is,

$$\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i.$$

Finally, we recall the definition of Weierstraß σ-function, namely

$$\sigma(z) = z \prod_{\omega \in \Lambda \smallsetminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2},$$

which is an entire function having simple zeros at the points of Λ. Its logarithmic derivative coincides with ζ, hence

$$\sigma\zeta = \sigma', \quad \sigma^2\wp = (\sigma')^2 - \sigma''\sigma.$$

By the quasi-periodicity of ζ, we also deduce that for any $\omega \in \Lambda$

$$\sigma(z + \omega) = \pm\sigma(z)e^{\eta(\omega)\left(z + \frac{\omega}{2}\right)},$$

the sign being positive if and only if $\omega \in 2\Lambda$.

We will now go through some analytic properties of the functions just introduced, which will be used later on in this chapter.

Definition 1.7. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-zero entire function. The *order of growth* of f is defined as

$$\limsup_{R \to \infty} \frac{\log \log |f|_R}{\log R},$$

where $|f|_R$ denotes the maximum of f on the ball of radius R centred at the origin.

In other words, a non-zero entire function has order of growth $\leq \varrho$ for some positive real number ϱ if there exists a constant $c > 0$ such that for any sufficiently large $R > 0$ we have

$$|f|_R \leq e^{cR^\varrho}.$$

Lemma 1.8. *The function σ has order of growth ≤ 2.*

Proof. Let us fix $R > 0$ and pick any $w \in \mathbb{C}$ such that $|w| \leq R$. We may find unique $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1, t_2 < 1$ in such a way that $w = z + n_1\omega_1 + n_2\omega_2$, where $z = t_1\omega_1 + t_2\omega_2$ and $n_1, n_2 \in \mathbb{Z}$. We therefore have $|n_i| \leq \min\{|\omega_1|, |\omega_2|\}^{-1}R$. It follows that

$$\begin{aligned} |\sigma(w)| &= |\sigma(z)| e^{\mathrm{Re}\left((n_1\eta_1 + n_2\eta_2)\left(z + \frac{n_1\omega_1}{2} + \frac{n_2\omega_2}{2}\right)\right)} \\ &\leq |\sigma|_{|\omega_1|+|\omega_2|} e^{|n_1\eta_1 + n_2\eta_2|\left|z + \frac{n_1\omega_1}{2} + \frac{n_2\omega_2}{2}\right|} \\ &\leq |\sigma|_{|\omega_1|+|\omega_2|} e^{\frac{1}{\min\{|\omega_1|,|\omega_2|\}}(|\eta_1|+|\eta_2|)\left(\left|\frac{3\omega_1}{2}\right|+\left|\frac{3\omega_2}{2}\right|\right)R^2}, \end{aligned}$$

which proves the statement. □

Corollary 1.9. *All the derivatives of σ have order of growth ≤ 2.*

Proof. More generally, the order of growth of an entire function f is preserved under taking derivatives. Indeed, let us fix $R \geq 1$ and pick any $w \in \mathbb{C}$ with $|w| \leq R$. By Cauchy's integral formula, for any $m \geq 0$ we have

$$f^{(m)}(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{m+1}}\, dz,$$

where γ is the boundary of the ball of radius $2R$ centred at the origin. If f has order of growth ϱ, then there is an absolute constant $c > 0$ such that

$$|f^{(m)}|_R \leq \frac{1}{2\pi} \frac{2\pi R}{(2R-R)^{m+1}} |f|_{2R} \leq \frac{1}{2^{m+1}R^m} e^{c(2R)^\varrho} \leq e^{2^\varrho c R^\varrho},$$

hence the claim follows from the previous Lemma by specializing $f = \sigma$. □

Corollary 1.10. *Both $\sigma\zeta$ and $\sigma^2\wp$ are entire with order of growth ≤ 2.*

Proof. It follows immediately from the previous Corollary and the identities $\sigma\zeta = \sigma'$, $\sigma^2\wp = (\sigma')^2 - \sigma''\sigma$. □

We now turn to some technical results of key importance for the sequel.

Lemma 1.11. *The j-th derivative $\wp^{(j)}$ of $\wp$ can be expressed in the form*

$$\sum u(t, t', t'')\wp^t (\wp')^{t'} (\wp'')^{t''},$$

where the sum runs through all non-negative integers t, t', t'' such that

$$2t + 3t' + 4t'' = j + 2$$

and $u(t, t', t'')$ is a rational integer satisfying

$$|u(t, t', t'')| \leq 3^j (j+7)!.$$

Proof. By deriving the identity $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp - g_3$ three times, it is readily verified that $\wp'''(z) = 12\wp(z)\wp'(z)$. We now argue by induction on j, the cases $j = 0, 1, 2$ being trivially true. Suppose that $j \geq 3$ and that we may write

$$\wp^{(j-1)} = \sum v(s, s', s'')\wp^s(\wp')^{s'}(\wp'')^{s''}$$

with $2s + 3s'' + 4s'' = j + 1$ and $|v(s, s', s'')| \leq 3^{j-1}(j + 6)!$. By deriving this identity once, we get an expression for $\wp^{(j)}$ which is a sum, running over s, s', s'', in which each $v(s, s', s'')$ is multiplied by the term

$$s\wp^{s-1}(\wp')^{s'+1}(\wp'')^{s''} + s'\wp^s(\wp')^{s'-1}(\wp'')^{s''+1} + 12s''\wp^{s+1}(\wp')^{s'+1}(\wp'')^{s''-1}.$$

We therefore obtain an expression for $\wp^{(j)}$ as in the statement by setting

$$u(t, t', t'') = tv(t+1, t'-1, t'') + t'v(t, t'-1, t''+1) + 12t''v(t-1, t'-1, t''+1).$$

For the absolute value of $u(t, t', t'')$ we have indeed

$$\begin{aligned} |u(t, t', t'')| &\leq 3^{j-1}(j + 6)!(t + t' + 12t'') \leq 3^{j-1}(j + 6)!(6t + 9t' + 12t'') \\ &\leq 3^j(j + 6)!(j + 2) \leq 3^j(j + 7)!, \end{aligned}$$

which yields the claim. □

Proposition 1.12. *Let $k \geq 0$ be an integer. The j-th derivative of $\wp^k$ can be expressed in the form*

$$\sum v(t, t', t'')\wp^t(\wp')^{t'}(\wp'')^{t''},$$

where the sum runs through all non-negative integers t, t', t'' such that

$$2t + 3t' + 4t'' = j + 2k$$

and $v(t, t', t'')$ is a rational integer satisfying

$$|v(t, t', t'')| \leq j!c^{j+k}.$$

for some constant $c > 0$ independent of j and k.

Proof. The j-th derivative of $\wp^k$ can be expressed in the form

$$(\wp^k)^{(j)} = \sum_{i_1+\cdots+i_k=j} \binom{j}{i_1, \ldots, i_k} \wp^{(i_1)} \ldots \wp^{(i_k)}.$$

By substituting for all $h = 1, \ldots, k$ the expression for $\wp^{(i_h)}$ given by the previous Lemma, we obtain

$$(\wp^k)^{(j)} = \sum_{i_1+\cdots+i_k=j} \binom{j}{i_1, \ldots, i_k} \prod_{h=1}^{k} \sum u(t_{i_h}, t'_{i_h}, t''_{i_h}) \wp^{t_{i_h}} (\wp')^{t'_{i_h}} (\wp'')^{t''_{i_h}},$$

where the internal sum runs through all non-negative integers $t_{i_h}, t'_{i_h}, t''_{i_h}$ such that $2t_{i_h} + 3t'_{i_h} + 4t''_{i_h} = i_h + 2$. By summing these latter identities over $h = 1, \ldots, k$, we get

$$2\left(\sum_{h=1}^{k} t_{i_h}\right) + 3\left(\sum_{h=1}^{k} t'_{i_h}\right) + 4\left(\sum_{h=1}^{k} t''_{i_h}\right) = \left(\sum_{h=1}^{k} i_h\right) + 2k = j + 2k.$$

Let us now introduce indices t, t', t'', denoting non-negative integers such that $2t + 3t' + 4t'' = j + 2k$. We may then rewrite $(\wp^k)^{(j)}$ as

$$\sum_{t,t',t''} \sum_{i_1+\cdots+i_k=j} \binom{j}{i_1, \ldots, i_k} v_{i_1,\ldots,i_k}(t, t', t'') \wp^t (\wp')^{t'} (\wp'')^{t''},$$

where

$$v_{i_1,\ldots,i_k}(t, t', t'') = \sum \prod_{h=1}^{r} u(t_{i_h}, t'_{i_h}, t''_{i_h}),$$

the sum going through all non negative integers t_{i_h}, t'_{i_h} and t''_{i_h} for $h = 1, \ldots, k$ satisfying

$$\sum_{h=1}^{k} t_{i_h} = t, \quad \sum_{h=1}^{k} t'_{i_h} = t', \quad \sum_{h=1}^{k} t''_{i_h} = t'', \quad 2t_{i_h} + 3t'_{i_h} + 4t''_{i_h} = i_h + 2.$$

Now, given a triple (t, t', t'') as above, consider all possible sets made of k triples of non-negative integers, say $\underline{t}_1 = (t_1, t'_1, t''_1), \ldots, \underline{t}_k = (t_k, t'_k, t''_k)$, such that each triple is not $(0, 0, 0)$ and moreover

$$\sum_{h=1}^{k} t_h = t, \quad \sum_{h=1}^{k} t'_h = t', \quad \sum_{h=1}^{k} t''_h = t''.$$

For all $h = 1, \ldots, k$ we have $2t_h + 3t'_h + 4t''_h \geq 2$, so $i_{\underline{t}_h} = 2t_h + 3t'_h + 4t''_h - 2 \geq 0$, and furthermore

$$\sum_{h=1}^{k} i_{\underline{t}_h} = j.$$

With this notation, we gain the expression for $(\wp^k)^{(j)}$ given in the statement by setting

$$v(t,t',t'') = \sum_{\underline{t}_1,\dots,\underline{t}_k} \binom{j}{i_{\underline{t}_1},\dots,i_{\underline{t}_k}} \prod_{h=1}^{k} u(t_h,t'_h,t''_h).$$

It only remains to estimate $|v(t,t',t'')|$. First, by the previous Lemma

$$\begin{aligned}\left|\binom{j}{i_{\underline{t}_1},\dots,i_{\underline{t}_k}} \prod_{h=1}^{k} u(t_h,t'_h,t''_h)\right| &\le \frac{j!}{i_{\underline{t}_1}!\dots i_{\underline{t}_k}!}\prod_{h=1}^{k} 3^{i_{\underline{t}_h}}(i_{\underline{t}_h}+7)! \\ &= j!3^j \prod_{h=1}^{k}\frac{(i_{\underline{t}_h}+7)!}{i_{\underline{t}_h}!} = 3^j j! \prod_{h=1}^{k} 7!\binom{i_{\underline{t}_h}+7}{7} \\ &\le 3^j j!(7!)^k \prod_{h=1}^{k} 2^{i_{\underline{t}_h}+7} = 3^j j!(7!)^k 2^{j+7k}.\end{aligned}$$

On the other hand, the number of sets of k triples of the form $\underline{t}_1,\dots,\underline{t}_k$ as above is at most

$$\left|\binom{t+k-1}{t}\binom{t'+k-1}{t'}\binom{t''+k-1}{t''}\right| \le 2^{t+t'+t''+3k-3}.$$

Since $t \le (j+2k)/2$, $t' \le (j+2k)/3$ and $t'' \le (j+2k)/4$, we eventually deduce the estimate

$$|v(t,t',t'')| \le \frac{j!}{8}\left(3\cdot 2^{\frac{25}{12}}\right)^j \left(7!\cdot 2^{\frac{73}{6}}\right)^k,$$

which establishes the claim. □

Proposition 1.13. *Let $k \ge 0$ be an integer. The j-th derivative of ζ^k can be expressed in the form*

$$\sum u(\tau,t,t',t'')\zeta^\tau \wp^t (\wp')^{t'} (\wp'')^{t''},$$

where the sum runs through all non-negative integers τ,t,t',t'' such that

$$\tau + 2t + 3t' + 4t'' = j + k \quad \textit{and} \quad \tau \le k,$$

while $u(\tau,t,t',t'')$ denotes a rational integer with absolute value at most $j!c^{j+k}$ for some constant $c > 0$ independent of j and k.

Proof. We argue by induction on k. For $k = 1$, we have $\zeta^{(j)} = (-1)^j \wp^{(j-1)}$, so the induction basis is granted by Lemma 1.11. Assuming that the claim holds for $k - 1 \geq 1$, we have

$$\begin{aligned}(\zeta^k)^{(j)} = (\zeta\zeta^{k-1})^{(j)} &= \zeta(\zeta^{k-1})^{(j)} + \sum_{i=1}^{j} (-1)^i \binom{j}{i} \wp^{(i-1)} (\zeta^{k-1})^{(j-i)} \\ &= \zeta(\zeta^{k-1})^{(j)} + \sum_{i=1}^{j} (-1)^i \binom{j}{i} \sum vw\zeta^{\varrho_i} \wp^{s_i+r_i} (\wp')^{s'_i+r'_i} (\wp'')^{s''_i+r''_i},\end{aligned}$$

where $s_i, s'_i, s''_i, \varrho_i, r_i, r'_i, r''_i$ are non-negative integers with $2s_i+3s'_i+4s''_i = i+1$, $\varrho_i + 2r_i + 3r'_i + 4r''_i = j - i + k - 1$ and $r_i \leq k - 1$, while $v = v(s_i, s'_i, s''_i)$ is given by Lemma 1.11 and $w = w_i(\varrho_i, r_i, r'_i, r''_i)$ by induction hypothesis. Observe that for all $i = 1, \ldots, j$ we have

$$2\varrho_i + 2(s_i + r_i) + 3(s'_i + r'_i) + 4(s''_i + r''_i) = j - i + k - 1 + i - 1 = j + k.$$

Again by induction hypothesis, we may write

$$\zeta(\zeta^k)^{(j)} = \sum w_1(\tilde{\varrho}, \tilde{r}, \tilde{r}', \tilde{r}'')\zeta^{\tilde{\varrho}+1} \wp^{\tilde{r}} (\wp')^{\tilde{r}'} (\wp'')^{\tilde{r}''},$$

with $\tilde{\varrho}, \tilde{r}, \tilde{r}', \tilde{r}''$ non-negative integers such that $\tilde{\varrho}+2\tilde{r}+3\tilde{r}'+4\tilde{r}'' = j+k-1$ and $\tilde{\varrho} \leq k - 1$. Let us now consider τ, t, t', t'' as in the statement and introduce indices s, s', s'' denoting non-negative integers such that $s \leq t$, $s' \leq t'$ and $s'' \leq t''$, with $(s, s', s'') \neq 0$. We may then set $i_{s,s',s''} = 2s + 3s' + 4s'' - 1 \geq 0$. Thus, we obtain an expression for $(\zeta^k)^{(j)}$ as the one in the statement by defining $u(\tau, t, t', t'')$ to be

$$\tilde{w}(\tau - 1, t, t', t'') + \sum_{s,s',s''} (-1)^{i_{s,s',s''}} \binom{j}{i_{s,s',s''}} v(s, s', s'')w(\tau, t - s, t' - s', t'' - s''),$$

convening to set $w(\tau, t - s, t' - s', t'' - s'') = 0$ if $\tau = k$. The estimate for $|u|$ now follows from the one in Lemma 1.11 and from the fact that the above sum has at most $(t + 1)(t' + 1)(t'' + 1) \leq (j + k + 1)^3$ terms. □

We conclude this section with a remark of more arithmetic content regarding the values of $\wp$ at tosion points of the elliptic curve E. By a torsion point of E we mean a point of E of finite order, or, more loosely, any pre-image of such a point in $\mathbb{C}$ under the map $\mathbb{C} \to \mathbb{C}/\Lambda \cong E$. Thus, we shall speak of torsion points of E as any element of $\Lambda \otimes \mathbb{Q}$, to simplify notation.

Proposition 1.14. *Let $\omega \in \mathbb{C}$ be a non-trivial torsion point of E, so that $\omega \in \Lambda \otimes \mathbb{Q} \smallsetminus \Lambda$. Then $\wp(\omega)$ is algebraic over the field $\mathbb{Q}(g_2, g_3)$.*

Proof. Suppose that for a positive integer n we have $[n]([\wp(\omega), \wp'(\omega), 1]) = [0, 1, 0]$, where $[n]$ denotes the multiplication-by-n map on E. This plainly implies that the first component of $[n]([\wp(\omega), \wp'(\omega), 1])$ vanishes. Moreover, by the addition formulae for the coordinates of points of E, it is readily checked that this component is a non-zero rational expression in $\wp(\omega)$ with coefficients in $\mathbb{Q}(g_2, g_3)$. □

Corollary 1.15. *If ω is a non-trivial torsion point of E, then the set of all $\wp^{(j)}(\omega)$ for $j \geq 0$ lies in a finite extension of $\mathbb{Q}(g_2, g_3)$.*

Proof. By Proposition 1.14 and the equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, we deduce that $\wp'(\omega)$ and $\wp''(\omega)$ are algebraic over $\mathbb{Q}(g_2, g_3)$. The statement then follows by Proposition 1.12. □

By exploiting for example division polynomials, it would be possible to give more precise quantitative information about $\wp(\omega)$ for a non-trivial torsion point ω of E, for instance regarding its degree and height. For further details in this direction, we refer to [Bak68]. For the present exposition, the rather implicit description of the algebraicity of $\wp(\omega)$ given above will however suffice.

1.3 The auxiliary function

We shall now start addressing the core topic of this chapter, namely a striking result about algebraic independence of periods and quasi-periods of elliptic curves obtained by Chudnovsky in 1976. As in the previous section, we let E be an elliptic curve defined over $\mathbb{C}$ with invariants g_2, g_3, and ω_1, ω_2 be a pair of periods for E with corresponding quasi-periods η_1, η_2.
A first result concerning the transcendence properties of these invariants was published by Schneider in [Schn36] and reads as follows:

Theorem 1.16. *The numbers $\omega_1, \omega_2, \eta_1, \eta_2$ are transcendental over $\mathbb{Q}(g_2, g_3)$.*

Further generalizations were obtained by Baker in [Bak68] and [Bak69], who applied his method for the celebrated theorem on linear forms in logarithms of algebraic numbers to give effective quantitative contributions to the investigation of the transcendence of linear forms in $\omega_1, \omega_2, \eta_1, \eta_2$. Less than ten years later, a turning point along these lines was achieved by Chudnovsky, who managed to prove the following

Theorem 1.17. *At least two of the numbers*

$$g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2$$

are algebraically independent.

Historically, this result provided the first explicit examples of two algebraically independent numbers, left alone the ones arising in the context of the exponential function by means of Lindemann-Weierstraß Theorem, already known at the end of the nineteenth century. The rest of this chapter is devoted to the proof of Theorem 1.17 and to the exposition of some of its most noticeable applications. Theorem 1.17 first appeared in Russian in the paper [Chu76], which has never been translated. A rough sketch of the proof, which is going to be the main reference for our present exposition, was drafted by Chudnovsky himself in [Chu84].

We now start the proof of Theorem 1.17, which is structured as follows. The argument is by contradiction, thus supposing that any two numbers among $g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2$ are algebraically dependent. The present section will be then devoted to the exploitation of this assumption in order to construct a suitable auxiliary function Φ which vanishes with high multiplicity at all lattice points contained in some ball of sufficiently large radius. Afterwards, we will select a lattice point at which Φ vanishes with lower order, and we will be able to provide a description of the first non-vanishing derivative of Φ at this point as a polynomial $P(\pi)$ in π with integer coefficients. We will thus end this section by finding an explicit estimate for the degree and height of this polynomial.
In the next section, we will then derive a bound from above for $|P(\pi)|$ by analytic means, taking advantage of the large number of zeros of Φ. By taking into account the degree and height of P, this final estimate will eventually yield a high transcendence measure for π, which contradicts already known results regarding the type of transcendence of π.

Let us start by supposing that any two numbers among $g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2$ are algebraically dependent, that is, they generate an extension of $\mathbb{Q}$ of transcendence degree at most 1. We fix a torsion point $\omega \in \mathbb{C}$ of E of order at least 3, so that by Corollary 1.15 the numbers $\wp^{(j)}(\omega)$ lie in a finite extension of $\mathbb{Q}(g_2, g_3)$. Moreover, by Legendre's relation, π belongs to a finite extension K of $\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2)$, which we may suppose Galois over the latter field, up to passing to its normal closure. We also take K big enough so that it contains all the numbers $\wp^{(j)}(\omega)$ for $j \geq 0$. For π is transcendental, $\operatorname{trdeg}(K/\mathbb{Q}) = 1$. Let us therefore consider a number χ, integral over $\mathbb{Z}[\pi]$, such that $K = \mathbb{Q}(\pi, \chi)$. We call δ the degree of χ over $\mathbb{Q}(\pi)$.
In the sequel, we fix a positive real number $0 < \varepsilon < 1$ and we denote by $c_1, c_2, \dots$ positive constants that only depend on $g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, \chi,\ K$ and ε. Moreover, by N we signify a sufficiently large positive integer, and we

define the quantities

$$R = \left\lfloor N^{1-\frac{\varepsilon}{2}} \right\rfloor, \quad D = \left\lfloor N^{1-\frac{\varepsilon}{4}} \right\rfloor.$$

We first introduce our auxiliary function.

Proposition 1.18. *There exists an absolute constant $C \in \mathbb{Z}[\pi]$ and elements $E_h(\pi) = E_{h_1,h_2,h_3}(\pi) \in \mathbb{Z}[\pi]$ such that the function*

$$\Phi(z) = C^{6N} \sum_{h_1=0}^{\delta D} \sum_{h_2=0}^{\delta D} \sum_{h_3=0}^{\delta D} E_{h_1,h_2,h_3}(\pi) z^{h_1} \wp(z)^{h_2} \zeta(z)^{h_3}$$

satisfies

$$\Phi^{(j)}(\omega + n_1\omega_1 + n_2\omega_2) = 0$$

for all $n_1, n_2 = 0, \ldots, R-1$ and $j = 0, \ldots, N-1$. Moreover, $E_h(\pi)$, regarded as a polynomial in π, has type at most $c_1 N \log N$.

Before proceeding with the proof of this Proposition, we catch up with the proof of Lemma 1.2 in the case of extensions of $\mathbb{Q}$ of finite type and transcendence degree 1. To our purposes, the following result will suffice:

Lemma 1.19. *Let us consider a polynomial $P(x, y) \in \mathbb{Z}[x, y]$ such that $\deg_y P \geq \delta$ and set $\beta = P(\pi, \chi)$. Set $d = \deg_\pi \chi^\delta$ and $H = H(\chi^\delta)$. Then*

$$\deg_\pi \beta \leq \deg_x P + (\deg_y P - \delta + 1)d,$$
$$H(\beta) \leq \left(\deg_x P + (\deg_y P - \delta + 1)d\right)(\deg_y P - \delta + 1)dH(P)H^{\deg_y P - \delta + 1}.$$

Proof. Let us first write

$$\chi^\delta = a_0 + a_1\chi + \cdots + a_{\delta-1}\chi^{\delta-1}$$

for some uniquely determined $a_0, \ldots, a_{\delta-1} \in \mathbb{Z}[\pi]$. Then we have

$$\chi^{\delta+1} = \chi^\delta \chi = a_0 a_{\delta-1} + (a_0 + a_1 a_{\delta-1})\chi + \cdots + (a_{\delta-1} + a_{\delta-2}a_{\delta-1})\chi^{\delta-2} + a_{\delta-1}^2 \chi^{\delta-1},$$

which shows that $\deg_\pi \chi^{\delta+1} \leq 2d$. By arguing inductively, it is then readily checked that for any $n \geq 0$ we have

$$\deg_\pi \chi^{\delta+n} \leq (n+1)d.$$

Let us now write

$$P(x, y) = b_0(x) + b_1(x)y + \cdots + b_{\deg_y P}(x)y^{\deg_y P}$$

for some $b_0, \dots, b_{\deg_y P} \in \mathbb{Z}[x]$. By our previous computations, we deduce that for any $h = \delta, \dots, \deg_y P$

$$\deg_\pi b_h(\pi)\chi^h = \deg_\pi b_h(\pi) + \deg_\pi \chi^h \leq \deg_x P + (h - \delta + 1)d,$$

where the first equality is justified by the fact that $b_h(\pi) \in \mathbb{Z}[\pi]$. Overall, we conclude that

$$\deg_\pi \beta \leq \deg_x P + (\deg_y P - \delta + 1)d.$$

A similar argument can be applied to the height. For $n \geq 1$ we easily compute by induction

$$H(\chi^{\delta+n}) \leq (n+1) \deg_\pi \chi^\delta H(\chi^\delta)^{n+1}.$$

Thus, for $h = \delta, \dots \deg_y P$

$$H(b_h(\pi)\chi^h) \leq (\deg_\pi b_h(\pi) + \deg_\pi \chi^h) H(b_h(\pi)) H(\chi^h),$$

whence the claim follows straightforwardly. □

Corollary 1.20. *There is a constant $c > 0$, only depending on χ, such that for any $P \in \mathbb{Z}[x, y]$ we have*

$$t(P(\pi, \chi)) \leq c\, t(P(x, y)).$$

We also recall some elementary estimates for the height of polynomials. If $P(x)$ and $Q(x)$ are polynomials with integer coefficients, then

$$H(PQ) \leq (\deg P + \deg Q) H(P) H(Q).$$

In particular, for any $k \geq 1$ we deduce the following upper bound for the height of the powers of $P(x)$:

$$H(P^k) \leq k! (\deg P)^k H(P)^k.$$

Analogous bounds apply to the case of polynomials in several indeterminates, provided we replace the $\deg P$ and $\deg Q$ by the sum of the degrees in each indeterminate.

We are now ready to have a closer look to Proposition 1.18. The construction of Φ as in the statement is based on the application of Siegel's Lemma 1.6. In order to achieve this, we first focus on those quantities that will arise as the coefficients of a linear system in $\mathbb{Z}[\pi, \chi]$, to which we will later apply the results of the previous sections.

Lemma 1.21. *There is an absolute constant $C \in \mathbb{Z}[\pi]$ enjoying the following property. For any $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ such that $0 \le k_1, k_2, k_3 \le D$, set*

$$F_k(z) := z^{k_1}\wp(z)^{k_2}\zeta(z)^{k_3}.$$

Then for each $n = (n_1, n_2)$, $n_1, n_2 = 0, \dots, R-1$ and $j = 0, \dots, N-1$

$$P_{njk}(\pi, \chi) := C^{6N} F_k^{(j)}(\omega + n_1\omega_1 + n_2\omega_2) \in \mathbb{Z}[\pi, \chi],$$
$$t(P_{njk}) \le c_2 N \log N.$$

Proof. We start by observing that for $j = 0, \dots, N-1$

$$F_k^{(j)}(z) = \sum_{j_1+j_2+j_3=j} \binom{j}{j_1, j_2, j_3} (z^{k_1})^{(j_1)} (\wp(z)^{k_2})^{(j_2)} (\zeta(z)^{k_3})^{(j_3)}.$$

Observe that $\zeta(\omega) \in \mathbb{Q}\eta_1 + \mathbb{Q}\eta_2$, by the quasi-periodicity of ζ. Thus, by Lemma 1.15 we may choose C to be a denominator in $\mathbb{Q}(\pi, \chi)$ for each of the numbers

$$\omega, \omega_1, \omega_2, \eta_1, \eta_2, \wp(\omega), \wp'(\omega), \wp''(\omega), \zeta(\omega).$$

To simplify notation, we introduce the quantities

$$a_1 = (z^{k_1})^{(j_1)}|_{z=\omega+n_1\omega_1+n_2\omega_2}, \quad a_2 = (\wp(z)^{k_2})^{(j_2)}|_{z=\omega+n_1\omega_1+n_2\omega_2},$$
$$a_3 = (\zeta(z)^{k_3})^{(j_3)}|_{z=\omega+n_1\omega_1+n_2\omega_2}.$$

If $k_1 \ge j_1$, we have

$$C^{k_1-j_1} a_1 = \frac{k_1!}{j_1!} (C\omega + n_1 C\omega_1 + n_2 C\omega_2)^{k_1-j_1}.$$

This yields a polynomial expression in $\mathbb{Z}[\pi, \chi]$ for $C^{k_1-j_1}a_1$ which has degree in π and χ at most $c_3(k_1 - j_1) \le c_3 D$ and height at most $c_4 k_1! R^{k_1-j_1} \le \exp(c_4 D \log D + c_4 D \log R)$. In the latter inequality we are taking advantage of the fact that the term related to the degree in the classical estimate for the height of a power of a polynomial is essentially irrelevant in this computation. Furthermore, by the periodicity of $\wp$ and Proposition 1.12, we infer that

$$C^{j_2+2k_2} a_2 = C^{j_2+2k_2} \sum u(t, t', t')\wp(\omega)^t \wp'(\omega)^{t'} \wp''(\omega)^{t''},$$

where the sum runs through all non-negative integers t, t', t'' which satisfy $2t + 3t' + 4t'' = j_2 + 2k_2$, while $u(t, t', t'')$ is an integer of absolute value no larger than $j_2! c_5^{j_2+k_2}$. This therefore yields a polynomial expression in $\mathbb{Z}[\pi, \chi]$ for $C^{j_2+2k_2}a_2$ with degree in π and χ at most $c_6(t + t' + t'') \le c_6(N + 2D)$

and height at most $j_2!c_5^{j_2+k_2}c_7^{t+t'+t''} \leq \exp(c_8 N \log N)$.
Finally, by the quasi-periodicity of ζ together with Proposition 1.13, we get

$$a_3 = \sum u(\tau, t, t', t')(\zeta(\omega) + n_1\eta_1 + n_2\eta_2)^\tau \wp(\omega)^t \wp'(\omega)^{t'} \wp''(\omega)^{t''}$$

with the sum running through all non-negative integers τ, t, t', t'' which satisfy $\tau + 2t + 3t' + 4t'' = j_3 + k_3$ with $\tau \leq k_3$, while $u(\tau, t, t', t'')$ is an integer with absolute value at most $j_3!c_9^{j_3+k_3}$. Thus, we have a polynomial expression for $C^{j_3+k_3}a_3$ with degree in π and χ at most $c_{10}(j_3 + k_3) \leq c_{10}N$ and height at most $j_3!c_9^{j_3+k_3}(c_{10}R)^\tau c_{11}^{\tau+t+t'+t''} \leq \exp(c_{12}N \log N)$.
Overall, the term C^{6N} provides a suitable power of C so as to be a denominator for $a_1a_2a_3$ in $\mathbb{Q}(\pi, \chi)$. Thus, $C^{6N}a_1a_2a_3$ has degree in π and χ at most $c_{13}N$ and logarithm of the height at most $c_{14}N \log N$. These estimates carry through to sums of $C^{6N}a_1(j_1)a_2(j_2)a_3(j_3)$ for any j_1, j_2, j_3 with $j_1+j_2+j_3 = j$, up to modifying the absolute constants. Indeed, the overall number of terms in such sum is

$$\binom{j+2}{3} = \frac{(j+2)(j+1)j}{6} \leq c_{15}j^3 \leq c_{15}N^3,$$

which does not affect the upper bound for the height, up to absolute constants. Overall, we obtain a polynomial expression for $P(x, y)$ in $\mathbb{Z}[x, y]$ of type $\leq c_{16}N \log N$ such that $P(\pi, \chi) = P_{njk}(\pi, \chi)$. By Corollary 1.20, we conclude that $t(P_{njk}) \leq c_2 N \log N$. □

We are now ready to complete the proof of Proposition 1.18. Let us consider the function

$$\widetilde{\Phi}(z) = C^{6N} \sum_{k_1=0}^{D} \sum_{k_2=0}^{D} \sum_{k_3=0}^{D} \widetilde{E}_{k_1,k_2,k_3}(\pi, \chi) z^{k_1} \wp(z)^{k_2} \zeta(z)^{k_3},$$

for suitable $\widetilde{E}_k(\pi, \chi)$ to be defined. We remark that the $P_{njk}(\pi, \chi)$'s as in the previous Lemma make up the coefficients of the linear system

$$\widetilde{\Phi}^{(j)}(\omega + n_1\omega_1 + n_2\omega_2) = 0, \quad j = 0, \ldots, N-1,\ n_1, n_2 = 0, \ldots, R-1,$$

the unknowns being the $\widetilde{E}_k$'s. Notice that this system consists of NR^2 equations and $(D + 1)^3$ unknowns. Since

$$NR^2 \leq N^{3-\varepsilon} < N^{3-\frac{3\varepsilon}{4}} \leq (D + 1)^3,$$

Siegel's Lemma 1.6 ensures the existence of a non-trivial solution for this linear system with components in $\mathbb{Z}[\pi, \chi]$. These components, namely the $\widetilde{E}_k$'s, have a type which satisfies

$$t(\widetilde{E}_k(\pi, \chi)) \leq c_{17}N \log N.$$

We now aim at removing χ from these coefficients; this will eventually lead to the function $\Phi(z)$ as in Proposition 1.18. Thus, consider the polynomial

$$A(x_1, x_2, x_3) = \sum_{k_1=0}^{D} \sum_{k_2=0}^{D} \sum_{k_3=0}^{D} \widetilde{E}_k(\pi, \chi) x_1^{k_1} x_2^{k_2} x_3^{k_3}.$$

Let us call $\delta = [\mathbb{Q}(\pi, \chi) : \mathbb{Q}(\pi)]$ and let $\sigma_1, \ldots, \sigma_\delta$ be the Galois automorphisms of $\mathbb{Q}(\pi, \chi)$ over $\mathbb{Q}(\pi)$. The norm of $A(x_1, x_2, x_3)$ over $\mathbb{Q}(\pi)$, denoted by $N_{\mathbb{Q}(\pi,\chi)/\mathbb{Q}(\pi)} A$, precisely coincides with

$$\prod_{i=1}^{\delta} \left(\sum_k \sigma_i(\widetilde{E}_k) x_1^{k_1} x_2^{k_2} x_3^{k_3} \right) = \sum_{h_1=0}^{\delta D} \sum_{h_2=0}^{\delta D} \sum_{h_3=0}^{\delta D} E_h(\pi) x_1^{h_1} x_2^{h_2} x_3^{h_3}, \quad \text{with}$$

$$E_h(\pi) = \sum_{\kappa_1, \ldots, \kappa_\delta} \prod_{i=1}^{\delta} \sigma_i(\widetilde{E}_{\kappa_i}),$$

where the κ_i's go through all the triples $(\kappa_{i1}, \kappa_{i2}, \kappa_{i3})$ with $0 \leq \kappa_{ij} \leq D$ such that $h_j = \kappa_{1j} + \cdots + \kappa_{\delta j}$ for all $j = 1, 2, 3$. It is apparent that $E_h(\pi) \in \mathbb{Z}[\pi]$, for it is fixed by all the Galois automorphisms of $\mathbb{Q}(\pi, \chi)$ over $\mathbb{Q}(\pi)$. Moreover, by exploiting the same arguments as in the previous Lemma, it is readily checked that the type of the $E_h(\pi)$'s is at most $c_1 N \log N$.
The function

$$\Phi(z) = N_{\mathbb{Q}(\pi,\chi)/\mathbb{Q}(\pi)} A(z, \wp(z), \zeta(z))$$

now satisfies all conditions in Proposition 1.18. Indeed, the type of its coefficients has just been verified, and the identities

$$\Phi^{(j)}(\omega + n_1\omega_1 + n_2\omega_2) = 0, \quad j = 0, \ldots, N-1, \; n_1, n_2 = 0, \ldots, R-1,$$

hold true, since they are already satisfied by $\widetilde{\Phi}$ and the ratio $\Phi/\widetilde{\Phi}$ is holomorphic outside Λ.

Now that we have constructed our auxiliary function, we first check that it does not vanish identically. This is an immediate consequence of the following

Lemma 1.22. *The three functions z, $\wp(z)$ and $\zeta(z)$ are algebraically independent over $\mathbb{C}$.*

Proof. We start by observing that z and $\wp$ are algebraically independent over $\mathbb{C}$. For let us consider a polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(z, \wp(z)) = 0$ for any $z \in \mathbb{C} \setminus \Lambda$. Then the polynomial $P(x, \wp(z))$ has infinitely many roots, namely $z + \omega$ for all $\omega \in \Lambda$, so it vanishes identically. If we write

$P(x,y) = \sum_{m\geq 0} p_m(y)x^m$, this implies that $p_m(\wp(z)) = 0$ for all m. Since $\wp$ is a non-constant holomorphic function on $\mathbb{C} \smallsetminus \Lambda$, its image is open in $\mathbb{C}$, so by the identity principle $p_m = 0$ for all m, whence $P = 0$.
Let us now assume that z, $\wp$ and ζ are algebraically dependent over $\mathbb{C}$. Since z and $\wp$ are algebraically independent, ζ must be algebraic over $\mathbb{C}(z,\wp)$. In particular, it also lies in a finite extension of $F = \mathbb{C}(z,\wp,\wp')$. Let Q be its minimal polynomial over F, and write $Q(x) = q_0 + q_1 x + \cdots + x^d$ for suitable $q_0, \ldots, q_{d-1} \in F$, setting $q_d = 1$. By taking the derivative of the identity $Q(\zeta(z)) = 0$, we obtain

$$\sum_{m=0}^{d-1} \left(q'_m - (m+1)q_{m+1}\wp\right)\zeta^m = 0.$$

Since the field F is closed under taking derivatives, this is a polynomial relation for ζ over F of degree $d-1$, so by minimality of Q we must have $q'_m = (m+1)q_{m+1}\wp$ for all $m = 0, \ldots d-1$. Since $q_d = 1$, in particular $q'_{d-1} = d\wp$, which shows that $q_{d-1} = -d\zeta + k$ for some $k \in \mathbb{C}$. From this identity we deduce that $\zeta \in \mathbb{C}(z,\wp,\wp')$. We may therefore write

$$\zeta(z) = \frac{a(z)}{b(z)}$$

for some $a(z), b(z) \in \mathbb{C}(\wp,\wp')[z]$ with $b(z) \neq 0$. Let c and d be the degrees in z of $a(z)$ and $b(z)$ respectively and call a_c and b_d their correspondent leading coefficients.
By the quasi-periodicity of ζ, for any $\omega \in \Lambda$ we have

$$\frac{a(z)}{b(z)} + \eta(\omega) = \zeta(z) + \eta(\omega) = \zeta(z+\omega) = \frac{a(z+\omega)}{b(z+\omega)}.$$

By specializing $\omega = n\omega_i$ for $i = 1, 2$ and $n \in \mathbb{Z}$, $n > 0$, we obtain

$$f_n(z) = \frac{a(z+n\omega_i)}{nb(z+n\omega_i)} - \frac{a(z)}{nb(z)} = \eta_i.$$

We restrict f_n to the points of $\mathbb{C}$ where the coefficients of a and b have no poles and b is non-zero. Notice that these points exist for we are excluding from $\mathbb{C}$ only countably many elements. Thus, $f_n(z)$ is a rational function in n with complex coefficients for any fixed $z \in \mathbb{C}$. We may then compute

$$0 \neq \eta_i = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{a_c n^c \omega_i^c}{b_d n^{d+1} \omega_i^d}.$$

In order for this limit to be finite and non-zero, we must have $c = d+1$, in which case we obtain

$$\frac{\eta_1}{\omega_1} = \frac{a_c(z)}{b_d(z)} = \frac{\eta_2}{\omega_2}.$$

This is however in contrast with Legendre's relation. □

As a result, we may find a positive integer N_0 satisfying the following property. Set for short $R_0 = \lfloor (N_0+1)^{1-\varepsilon/2} \rfloor$. For all $n_1, n_2 = 0, \ldots, \lfloor N_0^{1-\varepsilon/2} \rfloor - 1$ and any $j = 0, \ldots, N_0 - 1$ we have

$$\Phi^{(j)}(\omega + n_1\omega_1 + n_2\omega_2) = 0,$$

but there are $j_0 \in \{0, \ldots, N_0 - 1\}$ and $m_1, m_2 \in \{0, \ldots, R_0 - 1\}$ such that the point $z_0 = \omega + m_1\omega_1 + m_2\omega_2$ is a zero of multiplicity j_0 for Φ. This reads that $\Phi^{(j)}(z_0) = 0$ for all $j = 0, \ldots, j_0 - 1$, but $\Phi^{(j_0)}(0) \neq 0$. Notice that $N_0 \geq N$ by Proposition 1.18.
By combining Lemma 1.21 and the estimates on the type of the coefficients of the $E_h(\pi)$'s, it is then readily checked that

$$P(\pi, \chi) := \Phi^{(j_0)}(z_0)$$

is a non-zero element in $\mathbb{Z}[\pi, \chi]$, whose type satisfies $t(P) \leq c_{18} N_0 \log N_0$.
Our next purpose is to take advantage of the large number of zeros of Φ in the ball of radius R_0 centred at the origin in order to provide a sharp upper bound for $|P(\pi, \chi)|$. This will be achieved in the next section by analytic means and will eventually allow us to derive a high measure of transcendence for π, which is known to have type of transcendence $\leq 2 + \varepsilon$ for any $\varepsilon > 0$.

1.4 Analytic part of the proof

In order to derive an upper bound for $|P(\pi, \chi)|$, we begin with clearing out the denominator of the auxiliary function Φ in the field of meromorphic functions over $\mathbb{C}$ and giving an estimate for its order of growth.

Lemma 1.23. *The function $\Psi(z) = \sigma(z)^{3\delta D} \Phi(z)$ is entire and satisfies for any $\varrho \geq N_0^{\frac{3\varepsilon}{8}} (\log N_0)^2$*

$$|\Psi|_\varrho \leq e^{c_{19} N_0^{1-\frac{\varepsilon}{4}} \varrho^2}.$$

Proof. It is clear that Ψ is holomorphic on $\mathbb{C} \setminus \Lambda$. Moreover, ζ and $\wp$ have respectively simple and double poles at each lattice point, so $\Phi(z)$ has poles of order at most $3\delta D$ at all elements of Λ. Since σ has simple zeros at these

same points, the claim about Ψ being entire is established.
Let us fix $h_1, h_2, h_3 \in \{0, \ldots, \delta D\}$. By Corollary 1.10, we have

$$|\sigma^{2\delta D}\wp^{h_2}|_\varrho \leq e^{c_{20}D\varrho^2}, \qquad |\sigma^{\delta D}\zeta^{h_3}|_\varrho \leq e^{c_{21}D\varrho^2}.$$

We already computed that $t(E_h(\pi)) \leq c_{18}N_0 \log N_0$, which implies that

$$|E_h(\pi)| \leq \deg E_h |H(E_h)||\pi|^{\deg E_h} \leq e^{c_{22}N_0 \log N_0}.$$

Since the sum defining Φ consists of at most $(D+1)^3$ terms, the estimate of the claim is readily established. □

Since z_0 is a zero of order j_0 for Φ, we have

$$\lim_{z \to z_0} \frac{\Phi(z)}{(z - z_0)^{j_0}} = \frac{\Phi^{(j_0)}(z_0)}{j_0!} = \frac{P(\pi, \chi)}{j_0!}.$$

As a result, it follows that

$$P(\pi, \chi) = \frac{j_0!}{\sigma^{3\delta D}(z_0)} \lim_{z \to z_0} \frac{\Psi(z)}{(z - z_0)^{j_0}}.$$

We now introduce the main analytic tool that will enable us to give an upper bound for $|P(\pi, \chi)|$.

Lemma 1.24 (Schwarz). *Let f be a holomorphic function on an open subset of $\mathbb{C}$ containing the ball B_1 of radius $\varrho_1 > 0$ centred at the origin. Suppose that f has n zeros, counted with multiplicity, in the ball B_2 of radius $\varrho_2 < \frac{1}{3}\varrho_1$ centred at the origin. Then*

$$|f|_{\varrho_2} \leq \left(\frac{3\varrho_2}{\varrho_1}\right)^n |f|_{\varrho_1}.$$

Schwarz's Lemma therefore results in a sharpening of the estimates given by the classical maximum modulus principle, yielding a more accurate upper bound for $|f|_{\varrho_1}$ according to the number of zeros of f in a relatively small ball around the origin.

Proof. Let $a_1, \ldots, a_n$ be zeros of f in B_2. Then the function

$$g(z) = f(z) \prod_{i=1}^{n} \frac{1}{z - a_i}$$

is holomorphic on B_1, so we may apply the maximum modulus principle to deduce that

$$|g|_{\varrho_2} \leq |g|_{\varrho_1} = |f|_{\varrho_1} \max_{|z|=\varrho_1} \prod_{i=1}^{n} \left| \frac{1}{z-a_i} \right| \leq |f|_{\varrho_1} \left(\frac{1}{\varrho_1 - \varrho_2} \right)^n.$$

On the other hand, we have

$$|g|_{\varrho_2} = |f|_{\varrho_2} \max_{|z|=\varrho_2} \prod_{i=1}^{n} \left| \frac{1}{z-a_i} \right| \geq \frac{|f|_{\varrho_2}}{(2\varrho_2)^n}.$$

Since by assumption $\varrho_1 - \varrho_2 > \varrho_1 - \frac{1}{3}\varrho_1 = \frac{2}{3}\varrho_1$, this chain of inequalities finally yields

$$|f|_{\varrho_2} \leq \left(\frac{2\varrho_2}{\varrho_1 - \varrho_2} \right)^n |f|_{\varrho_1} \leq \left(\frac{3\varrho_2}{\varrho_1} \right)^n |f|_{\varrho_1}.$$

□

Turning back to our situation, let us set $c_{23} = |\omega| + |\omega_1| + |\omega_2|$, so that z_0 lies in the ball centred at the origin of radius

$$\varrho_2 := c_{23} R_0 = c_{23} \left\lfloor (N_0+1)^{1-\frac{\varepsilon}{2}} \right\rfloor.$$

Up to taking N sufficiently large, in this ball Φ has by construction at least

$$N_0 \left\lfloor N_0^{1-\frac{\varepsilon}{2}} \right\rfloor^2 \geq N_0 (N_0-1)^{2-\varepsilon} \geq \frac{1}{2} N_0^{3-\varepsilon}$$

zeros, counted with multiplicities. Let us now set

$$\varrho_1 := N_0^{1-\frac{3\varepsilon}{8}}.$$

By Schwarz's Lemma 1.24 we infer that

$$|\Psi|_{\varrho_2} \leq |\Psi|_{\varrho_1} \left(\frac{3c_{23}R_0}{N_0^{1-\frac{3\varepsilon}{8}}} \right)^{\frac{1}{2}N_0^{3-\varepsilon}} \leq |\Psi|_{\varrho_1} (c_{24}N_0)^{-\frac{\varepsilon}{16}N_0^{3-\varepsilon}} \leq |\Psi|_{\varrho_1} e^{-c_{25}N_0^{3-\varepsilon}\log N_0}.$$

Observe now that by Lemma 1.23 we have

$$|\Psi|_{\varrho_1} \leq e^{c_{19}N_0^{3-\varepsilon}}.$$

The dominating term in the upper bound given by Schwarz's Lemma turns therefore out to be the one related to the zeros of Ψ, so that

$$|\Psi|_{\varrho_2} \leq e^{c_{26}N_0^{3-\varepsilon}\log N_0}.$$

At this point, we may go back to estimating $|P(\pi,\chi)|$. Since z_0 lies in the ball of radius ϱ_2 centred at the origin, we get

$$|P(\pi,\chi)| \le \frac{j_0!}{|\sigma^{3\delta D}(z_0)|}\left|\frac{\Psi(z)}{(z-z_0)^{j_0}}\right|_{\varrho_2} \le \frac{1}{|\sigma^{3\delta D}(z_0)|}e^{-c_{27}N_0^{3-\varepsilon}\log N_0}.$$

Thus, it remains only to bound $|\sigma^{3\delta D}(z_0)|$ from below. Let us recall that $z_0 = \omega + m_1\omega_1 + m_2\omega_2$ for some $m_1, m_2 \in \{0,\dots,R_0-1\}$, hence

$$|\sigma(z_0)| = |\sigma(\omega)|e^{\mathrm{Re}\left((m_1\eta_1+m_2\eta_2)\left(\omega+\frac{m_1}{2}\omega_1+\frac{m_2}{2}\omega_2\right)\right)}.$$

Observe that $\mathrm{Re}\left((m_1\eta_1+m_2\eta_2)\left(\omega+\frac{m_1}{2}\omega_1+\frac{m_2}{2}\omega_2\right)\right) = \alpha-\beta$, where

$$\begin{aligned}
\alpha &:= (m_1\mathrm{Re}\,\eta_1 + m_2\mathrm{Re}\,\eta_2)\left(\mathrm{Re}\,\omega + \frac{m_1}{2}\mathrm{Re}\,\omega_1 + \frac{m_2}{2}\mathrm{Re}\,\omega_2\right),\\
\beta &:= (m_1\mathrm{Im}\,\eta_1 + m_2\mathrm{Im}\,\eta_2)\left(\mathrm{Im}\,\omega + \frac{m_1}{2}\mathrm{Im}\,\omega_1 + \frac{m_2}{2}\mathrm{Im}\,\omega_2\right).
\end{aligned}$$

It follows that $\alpha \ge -c_{28}R_0^2$ and $\beta \le c_{29}R_0^2$, which yields $\alpha-\beta \ge -c_{30}R_0^2$. As a result, we infer that

$$|\sigma^{3\delta D}(z_0)| \ge e^{-c_{31}DR_0^2} \ge e^{-c_{31}N_0^{3-\frac{5\varepsilon}{4}}}.$$

Since $\frac{5\varepsilon}{4} > \varepsilon$, we eventually reach the upper bound

$$|P(\pi,\chi)| \le e^{-c_{32}N_0^{3-\varepsilon}\log N_0}.$$

At this point, we would like to combine this with the fact that $P(\pi,\chi)$ has type at most $c_{19}N_0\log N_0$ to derive a transcendence measure for π. In order to achieve this, there is only one step missing, namely to obtain a polynomial in $\mathbb{Z}[\pi]$, therefore removing χ. We may accomplish this by taking the norm of $P(\pi,\chi)$ over $\mathbb{Q}(\pi)$, exactly as we did before for the $E_h(\pi)$'s.
Let us remind that we called $\sigma_1,\dots,\sigma_\delta$ the Galois automorphism of $\mathbb{Q}(\pi,\chi)$ over $\mathbb{Q}(\pi)$. We first wish to estimate the type of $Q(\pi) = N_{\mathbb{Q}(\pi,\chi)/\mathbb{Q}(\pi)}(P(\pi,\chi))$. Let us write $P(\pi,\chi) = a_0 + a_1\chi + \cdots + a_{\delta-1}\chi^{\delta-1}$ for some uniquely determined $a_0,\dots,a_{\delta-1} \in \mathbb{Z}[\pi]$. We set $I := \{0,\dots,\delta-1\}$ and denote by $\mathfrak{S}_\delta$ the symmetric group over δ elements. Thus, $\mathfrak{S}_\delta$ acts over I^δ by permuting components and we may form the factor set $I^\delta/\mathfrak{S}_\delta$. Then we have

$$Q(\pi) = \sum_{(i_1,\dots,i_\delta)\in I^\delta} a_{i_1}\dots a_{i_\delta}\prod_{j=1}^{\delta}\sigma_j(\chi)^{i_j} = \sum_{\alpha\in I^d/\mathfrak{S}_\delta} a_\alpha \sum_{(i_1,\dots,i_\delta)\in\alpha}\prod_{j=1}^{\delta}\sigma_j(\chi)^{i_j},$$

where we put for the sake of brevity $a_\alpha := a_{i_1} \dots a_{i_\delta}$ for an arbitrarily chosen representative $(i_1, \dots, i_\delta)$ of α. Notice that for any $\alpha \in I^\delta / \mathfrak{S}_\delta$

$$\sum_{(i_1,\dots,i_\delta)\in\alpha} \prod_{j=1}^{\delta} \sigma_j(\chi)^{i_j} \in \mathbb{Z}[\pi],$$

and its degree and height do not depend on N. It is then readily seen that the type of $Q(\pi) \in \mathbb{Z}[\pi]$ satisfies $\leq c_{33} N_0 \log N_0$.
On the other hand, we can easily derive an effective upper bound for $|Q(\pi)|$. For any Galois automorphism σ of $\mathbb{Q}(\pi, \chi)$ over $\mathbb{Q}(\pi)$ we have

$$|\sigma(P(\pi, \chi))| \leq \sum_{i=0}^{\delta-1} |a_i||\sigma(\chi)^i| \leq e^{c_{34} N_0 \log N_0},$$

by taking advantage of the estimate for the type of $P(\pi, \chi)$. Since we have already seen that $|P(\pi, \chi)| \leq e^{-c_{32} N_0^{3-\varepsilon} \log N_0}$, we obtain

$$0 < \left|N_{\mathbb{Q}(\pi,\chi)/\mathbb{Q}(\pi)}(P(\pi, \chi))\right| = \prod_{i=1}^{\delta} |\sigma_i(P(\pi, \chi))| \leq e^{-c_{35} N_0^{3-\varepsilon} \log N_0}.$$

Summing up, we have constructed a polynomial Q with integer coefficients which enjoys the following properties:

$$0 < |Q(\pi)| \leq e^{-c_{35} N_0^{3-\varepsilon} \log N_0}, \qquad t(Q) \leq c_{33} N_0 \log N_0.$$

This would imply that π has transcendence type $\geq 3 - 2\varepsilon$, which contradicts the transcendence measure for π given in [Wal78] and exposed at the beginning of this chapter.

1.5 Corollaries

We shall now go through some of the most noticeable consequences of Theorem 1.17, starting with an immediate Corollary covering a fairly broad amount of cases of interest.

Corollary 1.25. *If E has algebraic invariants g_2, g_3, then there are two algebraically independent numbers among*

$$\omega_1, \omega_2, \eta_1, \eta_2.$$

Before applying this result to some concrete instances of elliptic curves, we expose a striking sharpening which can be obtained in the case of complex multiplication

Corollary 1.26. *If E has algebraic invariants g_2, g_3 and complex multiplication, then for any period ω of E the numbers ω and π are algebraically independent.*

Proof. Since E complex multiplication, the ratio $\tau = \omega_1/\omega_2$ is an algebraic number. As exposed in [Mas75a, Lemma 3.1], there exist non-zero coprime integers A and B together with an algebraic number α such that

$$A\eta_1 - B\tau\eta_2 = \alpha\omega_2$$

Moreover, by Legendre's relation

$$\eta_2 = \frac{1}{\tau}\eta_1 + \frac{2\pi i}{\tau\omega_2}.$$

By combining these two identities, we get

$$\eta_1 = \frac{1}{A-B}\left(\frac{2\pi i B}{\omega_2} + \alpha\omega_2\right),$$

which shows that η_1 is algebraic over $\mathbb{Q}(\omega_2, \pi)$. One similarly proves that η_2 is algebraic over the latter field, so that the transcendence degree of $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ coincides with the one of $\mathbb{Q}(\omega_2, \pi)$. The claim then follows by means of Corollary 1.25. □

We may now list some interesting applications connected with values of the Γ-function. Let E be an elliptic curve admitting complex multiplication in the ring of integers of an imaginary quadratic field $\mathbb{Q}(\tau)$. It is well known that there are exactly as many isomorphism classes of such elliptic curves as the class number of $\mathbb{Q}(\tau)$. Moreover, E has automorphisms different from ± 1 if and only if the ring of integers of $\mathbb{Q}(\tau)$ is either $\mathbb{Z}[i]$ or $\mathbb{Z}[\varrho]$, where ϱ is a third root of unity. In both cases, $\mathbb{Q}(\tau)$ has class number 1, so we may find exactly two elliptic curves, up to isomorhpism, which have automorhpism group larger then $\{\pm 1\}$ and complex multiplication in the ring of integers of an imaginary quadratic field.
The one of these with endomorphism ring isomorphic to $\mathbb{Z}[i]$ is given by the affine equation

$$y^2 = 4x^3 - 4x.$$

As in [Wal99, Appendix 1], it is possible to compute that a pair of fundamental periods is given by

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} \quad \text{and} \quad \omega_2 = i\omega_1.$$

The other elliptic curve in which we are interested, having endomorphism ring isomorphic to $\mathbb{Z}[\varrho]$, is given by the affine equation

$$y^2 = 4x^3 - 4,$$

while a pair of fundamental periods can be chosen to be

$$\omega_1 = \frac{\Gamma(1/3)^3}{\sqrt[3]{16}\pi} \quad \text{and} \quad \omega_2 = \varrho\omega_1,$$

provided that we embed $\mathbb{Q}(\varrho)$ in $\mathbb{C}$ so as to identify ϱ with $e^{\frac{2\pi i}{3}}$. Corollary 1.26 then yields the following

Corollary 1.27. *π is algebraically independent from both $\Gamma(1/4)$ and $\Gamma(1/3)$.*

A more general treatment of the periods of elliptic curves with complex multiplication in the ring of integers of an imaginary quadratic field can be achieved by means of Chowla-Selberg formula. In this context, we briefly summarize a result obtained in [ChoSel67].

Proposition 1.28. *Let E be an elliptic curve having complex multiplication in the ring of integers of an imaginary quadratic field K. Then a fundamental period of E is given by*

$$\omega = \alpha\sqrt{\pi}\prod_{a=0}^{d}\Gamma\left(\frac{a}{d}\right)^{\frac{w\chi(a)}{4h}},$$

where α is a non-zero algebraic number, d is the discriminant of K, w is the number of roots of unity of K, h is the class number of K and $\chi = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol.

A straightforward application of Corollary 1.26 then yields the following

Corollary 1.29. *With notation as in the previous Proposition, the numbers*

$$\pi \quad \text{and} \quad \prod_{a=0}^{d}\Gamma\left(\frac{a}{d}\right)^{\chi(a)}$$

are algebraically independent.

By exploiting the identity

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right) = \frac{\pi}{\sin\left(\frac{\pi}{6}\right)} = 2\pi,$$

one also deduces the algebraic independence of π and $\Gamma(1/6)$. Further results concerning transcendence properties of Γ-values have been established by generalizing Theorem 1.17 to Abelian varieties, which will be the main focus of the next chapter.

Chudnovsky's result also leads to some remarkable applications connected to the j-function. We recall that the latter is a weight 0 modular function of level $\mathrm{SL}_2(\mathbb{Z})$ and it is defined by

$$j = 1728 \frac{E_4^3}{E_4^3 - E_6^3},$$

where E_4 and E_6 are the Eisenstein series of weight 4 and 6 respectively. It is a classical fact that for any quadratic irrational $\tau \in \mathbb{C}$ in the upper half plane, $j(\tau)$ is algebraic and, in fact, an algebraic integer. A discussion about these results can be found for instance in [Sil94, Chapter 2]. For the remaining values of j at algebraic points, we have the following

Proposition 1.30. *For any algebraic number α with positive imaginary part, other than a quadratic irrational, $j(\alpha)$ is transcendental.*

This result follows directly from Schneider-Lang's Theorem; a complete exposition can be found for instance in [Bak75, Chapter 6]. Other questions, on which Chudnovsky's result sheds some light, are related to the transcendence of the values of the derivative of the j-function. As explained in [Lan71], for any τ in the upper half plane we have the relation

$$j'(\tau) = 18 \frac{\omega_1^2}{2\pi i} \frac{g_3}{g_2} j(\tau),$$

where ω_1, g_2 and g_3 are the usual quantities referred to an elliptic curve whose j-invariant coincides with $j(\tau)$. As a by-product of Corollary1.26, if such curve has complex multiplication, so if τ is quadratic irrational, then ω_1 and π are algebraically independent, hence ω_1^2/π is transcendental. As a result, we obtain the following

Corollary 1.31. *For any τ quadratic irrational with positive imaginary part such that $j'(\tau) \neq 0$, π and $j'(\tau)$ are algebraically independent.*

Apart from the case of complex multiplication, it is conjectured that $j'(\tau)$ is transcendental whenever $j(\tau)$ is algebraic and $j'(\tau) \neq 0$.

Chapter 2

Periods of abelian functions

2.1 Complex Abelian varieties

In this chapter, we wish to expose a generalization of the techniques exploited in proving Theorem 1.17 to the case of complex Abelian varieties. Since the theory of Abelian varieties is extremely broad, we will spend this initial section for introducing the main context of the rest of chapter, focusing only on the theory of theta functions and eventually exposing some technical results needed for later reference. We will follow closely [Lan82]. The main subject of this chapter, which consists in a result by Vasilev in 1996 concerning the algebraic independence of periods of complex Abelian varieties, will be described in the next section, after the necessary prerequisites have been properly settled.

Let us start with an integer $g \geq 1$ and let us consider a lattice $\Lambda \subseteq \mathbb{C}^g$. The main question that we will now address is to understand when it is possible to embed the torus $\mathbb{C}^g/\Lambda$ in some complex projective space. If this is the case, by Chow's Theorem the image of $\mathbb{C}^g/\Lambda$ coincides with the $\mathbb{C}$-points of a projective variety, which we call a complex Abelian variety. In order to achieve this, we will start by describing some functions which are going to play a key role in the construction of this embedding.

Definition 2.1. A *theta function* on $\mathbb{C}^g$ with respect to Λ is a meromorphic function ϑ, not identically zero, which satisfies for all $z \in \mathbb{C}^g$ and $\omega \in \Lambda$

$$\vartheta(z+\omega) = \vartheta(z)e^{2\pi i(L(z,\omega)+J(\omega))}$$

for some $L : \mathbb{C}^g \times \Lambda \to \mathbb{C}$, $\mathbb{C}$-linear in the first argument, and some function $J : \Lambda \times \Lambda \to \mathbb{C}$. The pair (L, J) is called the *type* of ϑ.

It is apparent that J is defined only modulo a $\mathbb{Z}$-valued function on Λ. Moreover, this definition actually imposes some restrictions on L and J, which we now briefly investigate. By computing $\vartheta(z + \omega_1 + \omega_2)$ for any $\omega_1, \omega_2 \in \Lambda$ via the above formula, one easily checks that

$$L(\omega_1, \omega_2) \equiv L(\omega_2, \omega_1) \pmod{\mathbb{Z}},$$
$$L(z, \omega_1 + \omega_2) \equiv L(z, \omega_1) + L(z, \omega_2) \pmod{\mathbb{Z}}.$$

For L is $\mathbb{C}$-linear in the first argument, the second congruence is in fact an equality, showing that L is $\mathbb{R}$-linear in the second argument. We may therefore extend L by continuity to a map $\mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first argument and $\mathbb{R}$-linear in the second one.

Before proceeding in our discussion, let us give a first elementary example. Let q be a quadratic form on $\mathbb{C}^g$, λ a $\mathbb{C}$-linear map on $\mathbb{C}^g$ and $c \in \mathbb{C}$ any complex number. The function

$$\vartheta(z) = e^{2\pi i(q(z) + \lambda(z) + c)}$$

is a theta function, which is called *trivial.* If b is the bilinear form associated with q, then the type (L, J) of ϑ is given by

$$L(z, \omega) = 2b(z, \omega), \quad J(\omega) = b(\omega, \omega) + \lambda(\omega) + c.$$

These theta functions make up a multiplicative group, and they will serve for normalization purposes. We will say that two theta functions are equivalent if their quotient is a trivial theta function.
Notice that if ϑ is an entire theta function having no zeros, then it must be trivial. Indeed, in this case ϑ admits a logarithm, so we may write

$$\vartheta(z) = e^{2\pi i f(z)}$$

for some entire function f. It then follows that $f(z + \omega) - f(z) = L(z, \omega) + J(\omega)$, so all second order partial derivatives of f are constant. Thus, f is a quadratic polynomial and ϑ must be trivial.

After these considerations, we introduce two $\mathbb{R}$-bilinear forms which are of central importance in this discussion. First, for any $z, w \in \mathbb{C}^g$ we define

$$E(z, w) = L(z, w) - L(w, z).$$

The condition $L(\omega_1, \omega_2) \equiv L(\omega_2, \omega_1) \pmod{\mathbb{Z}}$ implies that E takes on integral values on $\Lambda \times \Lambda$. Since the latter is a basis over $\mathbb{R}$ of $\mathbb{C}^g \times \mathbb{C}^g$ and L is $\mathbb{R}$-bilinear, E is in fact an alternating $\mathbb{R}$-bilinear map which is also real valued. Finally, let us consider for any $z, w \in \mathbb{C}^g$

$$H(x, y) = E(ix, y) + iE(x, y).$$

A standard computation shows that H is a hermitian form.

Definition 2.2. A theta function ϑ is called *normalized* if the following conditions hold:

1. $L(z, w) = -\frac{i}{2} H(z, w)$;
2. The function $K(\omega) = J(\omega) - \frac{1}{2} L(\omega, \omega)$ is real valued.

The reason for this technical definition will appear shortly. In the meanwhile, we observe that for a normalized theta function the fundamental equation reads

$$\vartheta(z+\omega) = \vartheta(z) e^{2\pi i(-\frac{i}{2} H(z,\omega) - \frac{i}{4} H(\omega,\omega) + K(\omega))}.$$

Given any theta function, it is always possible to normalize it by multiplying by trivial theta functions, as shown by the following

Proposition 2.3. *In any equivalence class of theta functions modulo trivial ones, there exists a normalized theta function, unique up to a non-zero constant factor.*

Proof. This is a straightforward reduction via multiplication by suitable trivial theta functions. For a complete argument, we refer to [Lan82, Chapter VI, Theorem 1.3]. □

The main reason which justifies the interest in normalized entire theta functions results turns out to be the fact that their associated hermitian form is non-negative, though not necessarily positive definite, as we now show.

Proposition 2.4. *If ϑ is a normalized entire theta function on $\mathbb{C}^g$ with respect to Λ, then its associated hermitian form H is non-negative.*

Proof. We first prove that there is an absolute constant $C > 0$ such that for all $z \in \mathbb{C}^g$

$$|\vartheta(z)| \leq C e^{\frac{\pi}{2} H(z,z)}.$$

Indeed, the function $g(z) = \vartheta(z) \exp(-\frac{\pi}{2} H(z, z))$ satisfies for any $\omega \in \Lambda$

$$g(z+\omega) = g(z) e^{i\pi(E(z,\omega)+K(\omega))}.$$

Since the term $E(z, \omega) + K(\omega)$ in the exponent in the right hand side is real, we infer that $|g(z + \omega)| = |g(z)|$. Thus, the function $|g|$ is periodic, and therefore bounded for it is also continuous. This proves the claimed inequality.

Let us now suppose that there is $w_0 \in \mathbb{C}^g$ such that $H(w_0, w_0) < 0$. By continuity, $H(w,w) < 0$ for all w in a suitable neighbourhood of w_0. The function $z \mapsto \vartheta(zw)$ is entire and by the above inequality it tends to 0 as z tends to infinity. It follows therefore that such function is identically zero, so in particular it vanishes when computed at $z = 1$. As a result, ϑ is zero in a neighbourhood of w_0, whence also $\vartheta = 0$, a contradiction. □

The next point in our discussion is to determine when the hermitian form associated with a normalized entire theta function ϑ is in fact positive definite. If this is the case, we say that ϑ is non-degenerate. This special kind of functions will play a crucial role in the attempt of embedding the torus $\mathbb{C}^g/\Lambda$ in a projective space.

Proposition 2.5. *Let ϑ be a normalized entire theta function on $\mathbb{C}^g$ with respect to Λ, with associated Hermitian form H. Let N_H be the null space of H, that is, the subspace of $\mathbb{C}^g$ consisting of the $z \in \mathbb{C}^g$ such that $H(z,z) = 0$. Then*

1. *The values of ϑ only depend on the cosets of N_H in $\mathbb{C}^g$;*
2. *The image of the lattice Λ in $\mathbb{C}^g/N_H$ is discrete.*

Proof. Let us fix $w \in \mathbb{C}^g$ and $w_0 \in N_H$. For any $z \in \mathbb{C}^g$ we have

$$H(w + zw_0, w + zw_0) = H(w,w).$$

The same estimate as in the previous Proposition then yields

$$|\vartheta(w + zw_0)| \leq Ce^{\frac{\pi}{2}H(w,w)}.$$

Thus, the function $z \mapsto \vartheta(w + zw_0)$ is entire and bounded, hence constant. By evaluating it at $z = 0$, we see that it coincides with $\vartheta(w)$ and the first point is established.
Let us now consider $\omega_1, \ldots, \omega_r \in \Lambda$ whose residue classes $\overline{\omega}_1, \ldots, \overline{\omega}_r$ generate $\mathbb{C}^g/N_H$ as a real vector space. We may find $\varepsilon > 0$ such that for all $z \in \mathbb{C}^g$ in the ball of radius ε centred at the origin we have $|E(z, \omega_i)| < \frac{1}{2}$ for each $i = 1, \ldots, r$. Let $\overline{\omega}$ be an element of $\mathbb{C}^g/N_H$ which has a lift $\omega \in \Lambda$. If $\overline{\omega}$ lies in the ball of radius ε centred at the origin of $\mathbb{C}^g/N_H$, we may choose ω to lie in the corresponding ball in $\mathbb{C}^g$. Then $E(\omega, \omega_i) = 0$, for it is an integer. As a result, $\overline{\omega}$ is orthogonal to $\overline{\omega}_1, \ldots, \overline{\omega}_r$, whence $\overline{\omega} = 0$. This proves that the image of Λ in $\mathbb{C}^g/N_H$ is discrete. □

As a consequence of this Proposition, a normalized entire theta function ϑ naturally induces a theta function $\overline{\vartheta}$ on the quotient space $\mathbb{C}^g/N_H$ with respect to the lattice given by the projection of Λ modulo N_H. The hermitian form $\overline{H}$ associated with $\overline{\vartheta}$ is then the one induced by H. It is readily checked that $\overline{H}$ is always positive definite, so that $\overline{\vartheta}$ is non-degenerate. Thus, asking whether a normalized entire theta function is degenerate or not is tantamount to wondering whether it can be viewed as a theta function on a proper quotient of $\mathbb{C}^g$.

We are now ready to tackle the question of whether the torus $\mathbb{C}^g/\Lambda$ can be analytically embedded in a projective space. First, let us consider an entire theta function ϑ_0 on $\mathbb{C}^g$ with respect to Λ. We denote by $\mathcal{L}(\vartheta_0)$ the set of all entire theta functions of the same type as ϑ_0. Since these functions share the same type, $\mathcal{L}(\vartheta_0) \cup \{0\}$ is plainly a complex vector space. As shown in [Lan82, Chapter VI, Theorem 3.1], if ϑ_0 is non-degenerate the dimension of $\mathcal{L}(\vartheta_0) \cup \{0\}$ over $\mathbb{C}$ is finite and coincides with the square root of the determinant of the associated $\mathbb{R}$-bilinear form E with respect to the $\mathbb{R}$-basis of $\mathbb{C}^g$ given by Λ.
Let us now consider a basis $\vartheta_0, \vartheta_1, \ldots, \vartheta_m$ for $\mathcal{L}(\vartheta_0) \cup \{0\}$ over $\mathbb{C}$. We denote by $V(\vartheta_0, \ldots, \vartheta_m)$ the set of common zeros of $\vartheta_0, \ldots, \vartheta_m$ in $\mathbb{C}^g$. Notice that the fundamental formula for theta functions ensures that $V(\vartheta_0, \ldots, \vartheta_m)$ is invariant modulo Λ. Since $\vartheta_0, \ldots, \vartheta_m$ have the same type, we obtain a well-defined map

$$\varphi_{\vartheta_0} : {(\mathbb{C}^g \setminus V(\vartheta_0, \ldots, \vartheta_m))}\big/_{\Lambda} \longrightarrow \mathbb{P}^m, \quad z \longmapsto [\vartheta_0(z), \ldots, \vartheta_m(z)].$$

The main result in this context is given by the following

Theorem 2.6 (Lefschetz)**.** *Suppose that ϑ_0 is an entire non-degenerate theta function. Then the map $\varphi_{\vartheta_0^3}$ induced by $\mathcal{L}(\vartheta_0^3)$ is everywhere defined, and it is an analytic embedding of $\mathbb{C}^g/\Lambda$ into a projective space.*

Proof. We refer to [Lan82, Chapter VI, Theorem 6.1] or to [Mum74]. □

Let us observe that $\mathbb{C}^g/\Lambda$ is compact, while the standard metric topology on $\mathbb{P}^m$ is Hausdorff. Thus, if there exists an entire non-degenerate theta function, then the image of $\mathbb{C}^g/\Lambda$ in $\mathbb{P}^m$ via the above embedding is closed in $\mathbb{P}^m$. By Chow's Theorem, this implies that $\mathbb{C}^g/\Lambda$ may be identified with an algebraic subvariety of $\mathbb{P}^m$.

Definition 2.7. Let Λ be a lattice in $\mathbb{C}^g$. The complex torus $\mathbb{C}^g/\Lambda$ is called a *complex Abelian variety* if it can be analytically embedded in a projective space.

In view of this definition, Lefschetz's Theorem yields a criterion to determine when a torus $\mathbb{C}^g/\Lambda$ can be given the structure of a complex Abelian variety via an embedding into a projective space, the key point being the existence of an entire non-degenerate theta function. However, so far we have not given any example of explicit constructions of these functions for suitable lattices $\Lambda \subseteq \mathbb{C}^g$. Since a complete exposition would lead us too far afield from our purposes, we content ourselves with giving some references on the subject. Historically, the whole theory of theta functions arose naturally from the study of Jacobian varieties. Given a compact Riemann surface R of genus $g \geq 1$, it is possible to construct a complex torus $\mathbb{C}^g/\Lambda$ for a suitable lattice Λ by exploiting de Rham duality. The torus $\mathbb{C}^g/\Lambda$ obtained via this procedure is usually called the *Jacobian variety* associated with R and it is possible to explicitly construct an entire non-degenerate theta function on $\mathbb{C}^g$ with respect to Λ, thus turning $\mathbb{C}^g/\Lambda$ into a complex Abelian variety. A detailed description can be found for instance in [Mum82, Chapter 2, §2].

We now turn to the study of the function theory on a complex Abelian variety V of dimension g. Let Λ be a lattice associated with V, in the sense that the complex torus $\mathbb{C}^g/\Lambda$ can be identify with V via an embedding into a projective space. From now forth, we will always restrict ourselves to the case in which this embedding is realized by means of Lefschetz's Theorem, that is, we will always assume that there is an entire non-degenerate theta function ϑ_0 on $\mathbb{C}^g$ with respect to Λ.

Definition 2.8. A function $A : \mathbb{C}^g \to \mathbb{C}$ is called *abelian* with respect to Λ if it is either 0 or the quotient of two entire theta functions of the same type.

If A is an abelian function, then it is meromorphic and periodic with respect to Λ. Abelian functions for Λ make up a field, which is referred to as the *function field* of V. We now give a description of the latter.

Proposition 2.9. *The function field of V is a finite extension of a purely transcendental extension of $\mathbb{C}$ of transcendence degree g.*

Proof. We first construct g algebraically independent abelian functions. Up to replacing ϑ_0 by its cube, by Lefschetz's Theorem we have an embedding

$$\varphi_{\vartheta_0} : \mathbb{C}^g/\Lambda \longrightarrow \mathbb{P}^m, \quad z \longmapsto [\vartheta_0(z), \dots, \vartheta_m(z)],$$

where $\vartheta_0, \dots, \vartheta_m$ is a basis of $\mathcal{L}(\vartheta_0) \cup \{0\}$ over $\mathbb{C}$. After a change of projective coordinates, we may assume that the point $[1, 0, \dots, 0] \in \mathbb{P}^m$ lies in the image of φ_{ϑ_0}. Consider now the affine open subset of $\mathbb{P}^m$ given by

$$U_0 = \{[x_0, \dots, x_m] \in \mathbb{P}^m \mid x_0 \neq 0\},$$

which is isomorphic as a variety to the complex affine space $\mathbb{A}^m$, and therefore homeomorphic to $\mathbb{C}^m$. If $z_0 \in \mathbb{C}^g/\Lambda$ is such that $\varphi_{\vartheta_0}(z_0) = [1, 0, \ldots, 0]$, then the map

$$\mathbb{C}^g\!/\!\Lambda \cap \varphi_{\vartheta_0}^{-1}(U_0) \longrightarrow \mathbb{C}^g, \quad z \longmapsto \left(\frac{\vartheta_1(z)}{\vartheta_0(z)}, \ldots, \frac{\vartheta_m(z)}{\vartheta_0(z)}\right)$$

is a local chart around z_0, when restricting its codomain to its image. Since the torus $\mathbb{C}^g/\Lambda$ has dimension g as a complex manifold, we may find g abelian functions $A_1, \ldots, A_g$ which are local analytic coordinates at z_0, and these are algebraically independent.
It is now possible to prove that the function field of V is a finite extension of $\mathbb{C}(A_1, \ldots, A_g)$ by some combinatorial arguments together with elementary results on the dimension of $\mathcal{L}(\vartheta_0) \cup \{0\}$. A complete exposition is given in [Lan82, Chapter VI, Corollary 2]. □

We remark that, up to applying a suitable translation to the torus $\mathbb{C}^g/\Lambda$, we may suppose that $z_0 = 0$. Under this assumption, we have $\varphi_{\vartheta_0}(0) = [1, 0, \ldots, 0]$, so the proof of the previous Proposition implies in particular that $\vartheta_0(0) \neq 0$, while $A_1(0) = \cdots = A_g(0) = 0$. Moreover, from the fact that φ_{ϑ_0} is an embedding, one also deduces that the Jacobian of the map $z \mapsto (A_1(z), \ldots, A_g(z))$ is non-zero at $z = 0$.
We now wish to define quasi-periodic functions and the period matrix for a complex Abelian variety V. As a piece of notation, for any $i = 1, \ldots, g$ we write ∂_i for the standard derivation $\partial/\partial z_i$ on $\mathbb{C}^g$, endowed with coordinate functions $z_1, \ldots, z_g$. Furthermore, for a derivation ∂ of any order and a holomorphic function f we write $\partial \log f = (\partial f)/f$ for the corresponding logarithmic derivative of f. The next result provides a basis for the complex vector space $H^1_{dR}(V, \mathbb{C})$ of the equivalence classes of first order meromorphic differentials of the second kind, *i.e.* with vanishing residue at any point, modulo exact differentials.

Proposition 2.10. *The meromorphic differentials of the second kind*

$$dz_1,\ \ldots,\ dz_g,\ d\partial_1 \log \vartheta_0,\ \ldots,\ d\partial_g \log \vartheta_0$$

make up a basis for $H^1_{dR}(V, \mathbb{C})$ *over* $\mathbb{C}$.

Proof. We follow the argument of [Gri02]. Let us set for short $\varphi_i = dz_i$ and $\varphi_{g+i} = d\partial_i \log \vartheta_0$ for $i = 1, \ldots, g$. Fix a basis $\lambda_1, \ldots, \lambda_{2g}$ of Λ as a $\mathbb{Z}$-module. These differentials are Λ-invariant and may therefore be viewed as meromorphic differentials on V. It then suffices to prove the non-degeneracy of the $2g \times 2g$ matrix Ω whose entries are given by the so-called periods

$$\omega_{ij} = \int_0^{\lambda_j} \varphi_i, \quad i, j = 1, \dots, 2g.$$

As in [Lan66, Chapter VI, Theorem 3.2], it is possible to replace $\lambda_1, \dots, \lambda_{2g}$ by a Frobenius basis $e_1, \dots, e_g, v_1, \dots, v_g$, that is, $e_1, \dots, e_g$ make up a basis for $\mathbb{C}^g$ over $\mathbb{C}$ and for all $j = 1, \dots, g$ we have

$$\vartheta_0(z + e_j) = \vartheta_0(z), \qquad \vartheta_0(z + v_j) = \vartheta_0(z)e^{c_j z_j + d_j}$$

for some $c_j \neq 0$. With respect to this basis, for $i, j = 1, \dots, g$ we have $\omega_{ij} = e_{ij}$, so the $g \times g$ minor of Ω given by the first g rows and the first g columns is non-zero. Let us now fix $i \in \{1, \dots, g\}$. For any $j = 1, \dots, g$, ϑ_0 is periodic with respect to e_j, so $\omega_{ij} = 0$. If $j = 1, \dots, 2g$, then

$$\begin{aligned} \omega_{ij} &= \frac{\partial_i \vartheta_0(v_j)}{\vartheta_0(v_j)} - \frac{\partial_i \vartheta_0(0)}{\vartheta_0(0)} = \frac{1}{\vartheta_0(0)} \left(\frac{\partial_i \vartheta_0(0)}{e^{c_j v_{ij} + d_j}} - \partial_i \vartheta_0(0) \right) \\ &= \frac{1}{\vartheta_0(0)} \left(\frac{\partial_i \vartheta_0(0) e^{c_j v_{ij} + d_j}}{e^{c_j v_{ij} + d_j}} + \frac{\vartheta_0(0) e^{c_j z_j + d_j} \delta_{ij} c_j}{e^{c_j v_{ij} + d_j}} - \partial_i \vartheta_0(0) \right) = \delta_{ij} c_j, \end{aligned}$$

where δ_{ij} is Kronecker's delta. Since the c_j's are non-zero, we conclude that Ω is non-singular, as desired. □

The non-degenerate matrix Ω appearing in this proof is called the *period matrix* of V. Notice that the columns of the submatrix consisting of the first g rows of Ω are a basis for the lattice Λ as a $\mathbb{Z}$-module; its entries are usually referred to as the *periods* of V, while the remaining entries of Ω are the *quasi-periods* of V.
From the point of view of the function theory of V, we have constructed $2g$ meromorphic functions

$$H_1 = z_1, \ \dots, \ H_g = z_g, \ H_{g+1} = \partial_1 \log \vartheta_0, \ \dots, \ H_{2g} = \partial_g \log \vartheta_0$$

which are *quasi-periodic* with respect to Λ, in the sense that for the basis $\lambda_1, \dots, \lambda_{2g}$ of Λ given by the upper half of the matrix Ω the identity

$$H_i(z + \lambda_j) = H(z) + \omega_{ij}$$

holds true for all $i, j = 1, \dots, 2g$. Moreover, we remark that by construction ϑ_0 is a denominator for all the abelian and quasi-periodic functions so far constructed, in the sense that $\vartheta_0 A_1, \dots, \vartheta_0 A_g, \vartheta_0 H_1, \dots, \vartheta_0 H_{2g}$ are entire.
By exploiting their periodicity and quasi-periodicity, it is also possible to

check that the functions $A_1, \dots, A_g, H_1, \dots, H_{2g}$ are algebraically independent over $\mathbb{C}$. In general, proofs for algebraic independence of this kind of functions, though more manageable than the quest for algebraic independent numbers, become particularly complicated for the case of several variables. A satisfactory result for our purposes is [BroKub77, Corollary 7], which also deals with exponential functions, appearing later on in our discussion.

We finally conclude the exposition of this setting with some remarks of arithmetic nature. The further assumption that we will make on V is that it be defined over a number field K, in the sense that $V = X(\mathbb{C})$ for some projective variety X over K. Since X has dimension g, the function field $K(X)$ of X coincides with $K(R_1, \dots, R_g, S)$ for some rational functions $R_1, \dots, R_g, S$ on X with S algebraic over $K(R_1, \dots, R_g)$ and $R_1, \dots, R_g$ algebraically independent over K.
Given any basis $\varphi_1, \dots, \varphi_g$ of $\Omega^1_{X/K}(X)$, there is a unique map $\pi : \mathbb{C}^g \to V$ such that $\pi^*\varphi_i = dz_i$ for all $i = 1, \dots, g$. It is also possible to choose π in such a way that it yields an embedding of $V \hookrightarrow \mathbb{P}^m$ defined over K for some $m \geq 1$. Moreover, we may always recover an entire non-degenerate theta function ϑ_0 with respect to Λ in such a way that this embedding is induced by ϑ_0 as in Lefschetz's Theorem. For further details, we refer to [BirLan92].
By composing $R_1, \dots, R_g, S$ with this embedding, we obtain abelian functions $A_1, \dots, A_g, B$ on V such that B is algebraic over $K(A_1, \dots, A_g)$ and the latter field has transcendence degree g. The differential $d : \mathcal{O}_X \to \Omega^1_{X/K}$ induces a map $K(X) \to \Omega^1_{X/K}(X) \otimes_{\mathcal{O}_X(X)} K(X)$, which in turn implies that the field $K(A_1, \dots, A_g, B)$ is closed under $\frac{\partial}{\partial z_i}$.
We remark that the denominators of $A_1, \dots, A_g$ are entire theta functions on $\mathbb{C}^g$ with respect to Λ, since they are homogeneous polynomial expressions in theta functions of the same type. The product of these denominators is therefore a theta function with respect to Λ, which we will denote by ϑ.
By similar considerations it is possible to construct quasi-periodic functions $H_1, \dots, H_{2g}$ starting with a basis for $H^1_{dR}(V, K)$ and following the reasoning proposed before. One eventually deduces that the field generated over K by $A_1, \dots, A_g, B$ and $H_1, \dots, H_{2g}$ is closed under taking derivatives. Up to multiplying ϑ by some theta function, we may assume that ϑ is a denominator for $H_1, \dots, H_{2g}$ as well.
The last result that we will assume is concerned with more quantitative information. As a piece of notation, for a vector $\mu \in \mathbb{Z}^g$, $\mu = (\mu_1, \dots, \mu_g)$, we set $|\mu| = \sum_{i=1}^{g} |\mu_i|$.

Lemma 2.11. *The functions $A_1, \dots, A_g$ and $H_1, \dots, H_{2g}$ can be expanded at the origin in power series of the form*

$$\sum_{|\mu|\geq 1} b_\mu z_1^{\mu_1} \dots z_g^{\mu_g},$$

where $z = (z_1, \dots, z_g) \in \mathbb{C}^g$, $\mu = (\mu_1, \dots, \mu_g) \in \mathbb{Z}^g$ and $\mu_i \geq 0$ for all $i = 1, \dots, g$. Furthermore, the coefficients b_μ lie in K and enjoy the following properties:

1. *There exists a positive integer c_1 such that the maximum modulus of the conjugates of b_μ satisfies $\leq e^{c_1|\mu|}$;*

2. *there exists a positive integer c_2 such that*

$$(3|\mu|)!c_2^{|\mu|}b_\mu$$

is an algebraic integer.

Proof. We refer to [Schn41, §2, Hilf 5, 6]. □

2.2 The auxiliary function

For the rest of this chapter, we let $g \geq 1$ be an integer and V be a complex Abelian variety of dimension g defined over a number field K satisfying the assumptions of the previous section. Our aim is to describe how to obtain transcendence results concerning the periods and the quasi-periods of V, that is, of the entries of the matrix Ω described in the preceding section. It is apparent that the case $g = 1$ boils down to the one of elliptic curves, which has already been treated extensively in the first chapter.
A starting point in the quest for transcendence properties of the period matrix of V is to be found in [Schn41], where Schneider managed to show that no row of Ω has only algebraic components. After proving Theorem 1.17, Chudnovsky was able to generalize such result to the case of complex Abelian varieties, thereby proving that among the entries of V there are at least two algebraically independent numbers.
Further progress in the subject was made by Vasilev, who succeeded in showing in 1996 the following

Theorem 2.12. *Any $g + 1$ distinct rows of the matrix Ω taken together contain at least two algebraically independent numbers.*

At present, Vasilev's contribution is still the sharpest result obtained so far concerning the algebraic independence of the entries of the period matrix Ω. We will now expose the proof of Theorem 2.12 proposed by Vasilev in [Vas96], which will be the main reference for the rest of the chapter.
The core strategy of the proof will follow closely the main techniques that we exposed in the first chapter, with some major technical complications due to the presence of several complex variables. We will therefore construct an auxiliary function vanishing with high multiplicity at several points of the lattice Λ defining V by means of Siegel's Lemma. The present section revolves around the construction of such function and it will finish with a couple of Lemmas which allowed Vasilev to render Chudnovsky's argument independent of known transcendence measures, as it was instead the case for Theorem 1.17.
In the next section, the final part of the proof, essentially analytic in nature, will be carried out. The main tools to deal with the case of several complex variables are not as elementary as the ones appearing in the previous chapter and they rely on the Bombieri-Lang version of Schwarz's Lemma. In the last section we will eventually deal with some applications of Theorem 2.12 for Γ-values at rational numbers along the lines of the ones proposed in the first chapter.

We first introduce some notation. The standard coordinates in $\mathbb{C}^g$ will be given by $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$. For an integral vector $k = (k_1, \ldots, k_g) \in \mathbb{Z}^g$, we set

$$|k| := \sum_{j=1}^{g} |k_j|, \quad \|k\| := \max_{j=1,\ldots,g} |k_j|, \quad k \cdot \lambda := \sum_{j=1}^{g} k_j \lambda_j, \quad z^k = z_1^{k_1} \ldots z_g^{k_g}.$$

Let $P \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a polynomial and set for short $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$. We will write $\deg P$ for the degree of P, while the notation $\deg_x P$ will refer to the degree of P in x, and similarly for $\deg_y P$. The maximum modulus of the coefficients of P, the *height* of P, will be denoted as usual by $H(P)$. The sum of the moduli of the coefficients of P, the *length* of P, will be denoted by $L(P)$.
Given a differentiation operator

$$\partial = \frac{\partial^m}{\partial^{m_1} \ldots \partial^{m_g}},$$

its order will be denoted by $|\partial| = m = m_1 + \cdots + m_g$. The notation $|\partial| = 0$ signifies that ∂ is the identity operator. For an entire function $f : \mathbb{C}^g \to \mathbb{C}$, we write $|f|_R$ for the maximum modulus of f in the ball of radius $R \geq 0$

centered at the origin.
By $c_1, c_2, \dots$ we mean positive constants that only depend on the functions $A_1, \dots, A_g, H_1, \dots, H_{2g}$ and ϑ.

We are now ready to begin with the proof of Theorem 2.12. Without loss of generality, it is not restrictive to consider only the case of the first $g+1$ rows of the matrix Ω. By the aforementioned result in [Schn41], at least one of the numbers ω_{ij} for $i = 1, \dots, g+1$, $j = 1, \dots, 2g$ is transcendental, say ω. Arguing by contradiction, let us suppose that the entries of the first $g+1$ rows of Ω have transcendence degree strictly less than 2, so they generate a finite extension of $\mathbb{Q}(\omega)$. By the primitive element theorem, there is a complex number χ, integral over the ring $\mathbb{Z}[\omega]$, such that the field $\mathbb{Q}(\omega, \chi)$ contains both K and ω_{ij} for the above ranges of i and j. Set for short $\delta := [K : \mathbb{Q}]$ and fix an integral basis $\alpha_1, \dots, \alpha_\delta$ of the ring of integers of K over $\mathbb{Q}$.
Let us now consider a sufficiently large integer N and define the quantities

$$D = N^{6g}, \qquad R = N^{3g+1}, \qquad T = N^{6g+1}.$$

We may now introduce our auxiliary function.

Proposition 2.13. *There exists a constant $C(\omega)$ only depending on the ω_{ij}'s, ω, $\alpha_1, \dots, \alpha_\delta$ and χ and there exist numbers $E_{l\nu}(\omega) \in \mathbb{Z}[\omega]$ not all zero such that the function*

$$\Phi(z) := C(\omega)^D (3T)! c_2^T \sum_{\|l\| \le D} \sum_{\|\nu\| \le D} E_{l\nu}(\omega) H_1^{l_1}(z) \dots H_{g+1}^{l_{g+1}}(z) A_1^{\nu_1}(z) \dots A_g^{\nu_g}(z),$$

where $l = (l_1, \dots, l_{g+1}) \in \mathbb{Z}_{\ge 0}^{g+1}$ and $\nu = (\nu_1, \dots, \nu_g) \in \mathbb{Z}_{\ge 0}^g$, satisfies the conditions

$$\partial \Phi(k \cdot \lambda) = 0 \quad \text{for } |\partial| \le T,\ \|k\| \le R.$$

Furthermore, the $E_{l\nu}(\omega)$'s, as polynomials in ω, may be chosen to have degree $\le c_3 D$ and length $\le e^{c_4(T \log T + D \log R)}$.

In order to prove this Proposition, we intend to interpret the vanishing conditions on Φ as a homogeneous linear system in the $E_{l\nu}$'s so as to apply Siegel's Lemma. It is therefore convenient to separately study first the quantities that shall arise as coefficients of this linear system, which is the aim of the following

Lemma 2.14. *For $k = (k_1, \dots, k_{2g}) \in \mathbb{Z}^g$ and l, ν as above, define*

$$F_{kl\nu}(z) := (3T)! c_2^T \prod_{i=1}^{g+1} H_i^{l_i}(z + k \cdot \lambda) \prod_{s=1}^{g} A_s^{\nu_s}(z).$$

If ∂ is a differential operator of order $|\partial| \leq T$, then

$$\partial F_{kl\nu}(0) = \sum_{\kappa} q_\kappa \prod_{i=1}^{g+1} \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}}$$

where $\kappa = (\kappa_{ij})_{ij}$, $\kappa_{ij} \in \{0, \dots, D\}$ for all $i = 1, \dots, g+1$, $j = 1, \dots, 2g$, while q_κ is an algebraic integer satisfying

$$q_\kappa = \sum_{m=1}^{\delta} q_{m\kappa} \alpha_m \quad \textit{for some } q_{m\kappa} \in \mathbb{Z} \textit{ with } |q_{m\kappa}| \leq e^{c_5(T \log T + D \log R)}.$$

Proof. By the quasi-periodicity of H_i, we have

$$\begin{aligned} H_i^{l_i}(z + k \cdot \lambda) &= \left(H_i(z) + \sum_{j=1}^{2g} k_j \omega_{ij} \right)^{l_i} \\ &= \sum_{m_0 + \dots + m_{2g} = l_i} \binom{l_i}{m_0; \dots; m_{2g}} H_i^{m_0}(z) \prod_{j=1}^{2g} (k_j \omega_{ij})^{m_j} \\ &= \sum_{m_1 + \dots + m_{2g} \leq l_i} \varphi_{i,m}(z) \prod_{j=1}^{2g} \omega_{ij}^{m_j}. \end{aligned}$$

Let now κ range through the $(g+1) \times 2g$ matrices with non-negative integer coefficients κ_{ij} such that for $i = 1, \dots, g+1$ we have $\kappa_{i1} + \dots + \kappa_{i,2g} \leq l_i$. When taking the product over $i = 1, \dots, g+1$ in the above equalities, we obtain

$$\prod_{i=1}^{g+1} H_i^{l_i}(z + k \cdot \lambda) = \sum_{\kappa} \psi_\kappa(z) \prod_{i=1}^{g+1} \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}},$$

the ψ_κ's being suitable sums of products of the $H_1, \dots, H_{g+1}$ in such a way that the exponent of H_i does not exceed l_i. Notice that $\kappa_{ij} \leq D$, since $l_i \leq D$ for all $i = 1, \dots, g+1$. Thus, Lemma 2.11 shows that the coefficients of the Taylor expansion of $F_{kl\nu}(z)$ around 0 all lie in the field generated by K and the ω_{ij}'s. More precisely, if we write

$$\partial = \frac{\partial^t}{\partial^{t_1} \dots \partial^{t_g}},$$

the coefficient in this series corresponding to

$$\frac{z_1^{t_1} \dots z_g^{t_g}}{t_1! \dots t_g!}$$

coincides with $\partial F_{kl\nu}(0)$ and by the previous computations has the form

$$\partial F_{kl\nu}(0) = \sum_{\kappa} q_\kappa \prod_{i=1}^{g+1} \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}}$$

for some integers $q_\kappa \in K$. Furthermore, it turns out that q_κ is a sum of suitable products of the coefficients of the Taylor expansion of $H_1, \ldots, H_{2g}$ and $A_1, \ldots, A_g$ whose total order is $\leq |\partial|$, by Cauchy's formula for the product of series. As a result, by the second part of Lemma 2.11 the term $(3T)!c_2^T$ is a denominator for all such coefficients, so q_κ is an algebraic integer. Hence, we have

$$q_\kappa = \sum_{m=1}^{\delta} q_{m\kappa} \alpha_m$$

for some $q_{1\kappa}, \ldots, q_{\delta\kappa} \in \mathbb{Z}$.
Let us now write

$$F_{kl\nu}(z) = \sum_{\mu \in \mathbb{Z}_{\geq 0}^g} \sum_{\kappa} \xi_{\kappa\mu} \prod_{i=1}^{g+1} \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}} z^\mu.$$

Let $\widetilde{F}_{kl\nu}(z)$ be the formal power series obtained from $F_{kl\nu}$ by replacing each $\xi_{\kappa\mu}$ with the maximum modulus of its Galois conjugates over $\mathbb{Q}$. The following inequalities between formal power series in z^μ are intended to summarize all corresponding inequalities between th moduli of coefficients of the same order:

$$\begin{aligned}
\widetilde{F}_{kl\nu}(z) &\leq (3T)!c_2^T \prod_{i=1}^{g+1} \left(R \sum_{j=1}^{2g} \omega_{ij} + \sum_{|\mu| \geq 1} e^{c_5|\mu|} z^\mu \right)^D \left(\sum_{|\mu| \geq 1} e^{c_5|\mu| z^\mu} \right)^{|\nu|} \\
&\leq (3T)!c_2^T R^{(g+1)D} \prod_{i=1}^{g+1} (\omega_{i1} + \cdots + \omega_{i,2g})^D \left(1 + \sum_{|\mu| \geq 1} e^{c_6|\mu|} z^\mu \right).
\end{aligned}$$

For any κ, let $\widetilde{q_\kappa}$ denote one of the Galois conjugates of q_κ with maximal modulus. The previous inequalities then show that

$$|q_{m\kappa}| \leq c_7 |\widetilde{q_\kappa}| \leq t_1! \ldots t_g!(3T)!c_2^T R^{nD} e^{c_8 T} \leq e^{c_5(T \log T + D \log R)},$$

which yields the claim. The fact that $|q_{m\kappa}| \leq c_7|\widetilde{q_\kappa}|$ is a general fact, which we prove in the next Lemma for the sake of clarity. □

Lemma 2.15. *Let K be a finite Galois extension of $\mathbb{Q}$ of degree δ and let $\alpha_1, \ldots, \alpha_\delta$ be an integral basis of K over $\mathbb{Q}$. Let $\beta \in K$ be an algebraic integer and write*

$$\beta = \sum_{i=1}^{\delta} b_i \alpha_i$$

for some $b_i \in \mathbb{Z}$. If $\widetilde{\beta}$ is one of the Galois conjugates of β over $\mathbb{Q}$ with maximal modulus, then for all $i = 1, \ldots, \delta$ we have

$$|b_i| \le c|\widetilde{\beta}|,$$

where the constant $c > 0$ only depends on the choice of an integral basis of K over $\mathbb{Q}$.

Proof. We follow the proof proposed in [Shid87, Chapter 3, §8, Lemma 12]. Let $\sigma_1, \ldots, \sigma_n$ be the Galois automorphisms of K over $\mathbb{Q}$. We regard the equations

$$\sigma_j(\beta) = b_1\sigma_j(\alpha_1) + \cdots + b_\delta \sigma_j(\alpha_\delta) \quad \text{for } j = 1, \ldots, \delta$$

as a linear system in the unknowns $b_1, \ldots, b_\delta$. Let A be the matrix associated with this linear system, whose determinant therefore coincides with the square root of the discriminant of K over $\mathbb{Q}$, up to the sign. Let us set for short $\underline{\beta} := (\sigma_1(\beta), \ldots, \sigma_\delta(\beta))$ and $\underline{b} := (b_1, \ldots, b_\delta)$. We plainly have

$$\underline{b} = A^{-1}\underline{\beta} = \frac{1}{\det A} A^* \underline{\beta},$$

where A^* is the transposed cofactor matrix of A. Let us denote by $\|A^*\|$ the maximum modulus of the entries of A^*, id est, the maximum modulus of all the $(\delta - 1) \times (\delta - 1)$ minors of A. Then we have

$$|b_i| \le \frac{\delta \|A^*\|}{\det A} |\widetilde{\beta}|,$$

which yields the statement. □

Lemma 2.16. *Under the assumptions of Lemma 2.14, there exists a constant $C(\omega)$ depending only on $\alpha_1, \ldots, \alpha_\delta$, ω, χ and on the ω_{ij}'s such that*

$$P_{\partial kl\nu}(\omega, \chi) := C(\omega)^D \partial F_{kl\nu}(0) \in \mathbb{Z}[\omega, \chi],$$

where $P_{\partial kl\nu}(\omega, \chi)$, as a polynomial in ω and χ, satisfies the conditions

$$\deg_\chi P_{\partial kl\nu} \le c_9, \quad \deg_\omega P_{\partial kl\nu} \le c_{10} D, \quad L(P_{\partial kl\nu}) \le e^{c_{11}(T \log T + D \log R)}.$$

In this Lemma, we are assuming the usual conventions for the degree in χ and ω and for the length of elements in $\mathbb{Z}[\omega, \chi]$, which we briefly recall. Since χ is integral over $\mathbb{Z}[\omega]$, of course the degree of $P_{\partial kl\nu}$ in χ is not well defined. However, $P_{\partial kl\nu}$ has a unique representation as a polynomial in χ with coefficients in $\mathbb{Z}[\omega]$ if we admit only the first $[\mathbb{Q}(\omega, \chi) : \mathbb{Q}(\omega)]$ non-negative powers of χ, and it is with respect to this representation that we define $\deg_\chi P_{\partial kl\nu}$, $\deg_\omega P_{\partial kl\nu}$ and $L(P_{\partial kl\nu})$. The inequality $\deg_\chi P_{\partial kl\nu} \leq c_9$ is then straightforward, since $[\mathbb{Q}(\omega, \chi) : \mathbb{Q}(\omega)]$ does not depend on N.

Proof. Let us choose $C_0(\omega) \in \mathbb{Z}[\omega]$ in such a way that

$$C_0(\omega)\alpha_1, \ldots, C_0(\omega)\alpha_\delta, C_0(\omega)\omega_{ij} \quad \text{for } i = 1, \ldots, g+1,\ j = 1, \ldots, 2g$$

are polynomials in $\mathbb{Z}[\omega, \chi]$, ad let e' denote the maximum of their degrees in ω. Setting $C(\omega) \coloneqq C_0(\omega)^{(2g(g+1)+1)}$, we have

$$P_{\partial kl\nu}(\omega, \chi) \coloneqq C(\omega)^D \partial F_{kl\nu}(0) \in \mathbb{Z}[\omega, \chi].$$

Since by Lemma 2.14

$$\partial F_{kl\nu}(0) = \sum_\kappa q_\kappa \prod_{i=1}^{g+1} \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}}$$

with $\kappa_{ij} \leq D$, we deduce that

$$C(\omega)^{1+\sum_{i,j}\kappa_{ij}} \partial F_{kl\nu}(0)$$

admits a representation as a polynomial in ω and χ of degree in ω at most $(2g(g+1)+1)e'D$ and of degree in χ at most $(2g(g+1)+1)\delta D$. Set for short

$$a \coloneqq 1 + \sum_{i,j} \kappa_{ij}.$$

By Lemma 1.19, it follows that

$$\deg_\omega C(\omega)^a \partial F_{kl\nu}(0) \leq (2g^2+2g+1)e'D + ((2g^2+2g+1)\delta D - \delta + 1)d \leq cD$$

for some absolute constant $c > 0$. Finally, $\deg_\omega P_{\partial kl\nu}$ is obtained from $\deg_\omega C(\omega)^a \partial F_{kl\nu}(0)$ by adding a term which plainly grows linearly in D, so we may conclude that $\deg_\omega P_{\partial kl\nu} \leq c_{10} D$, as desired.
The claim on the length of $P_{\partial kl\nu}$ can be established in a completely analogous way by exploiting the estimates on the $q_{m\kappa}$'s provided by Lemma 2.14. $\square$

Proposition 2.13 now follows easily. We consider the auxiliary function

$$\Phi(z) := C(\omega)^D (3T)! c_2^T \sum_{\|l\| \le D} \sum_{\|\nu\| \le D} E_{l\nu}(\omega) H_1^{l_1}(z) \dots H_{g+1}^{l_{g+1}}(z) A_1^{\nu_1}(z) \dots A_g^{\nu_g}(z),$$

with $C(\omega)$ given by the previous Lemma. The conditions

$$\partial\Phi(k \cdot \lambda) = 0 \quad \text{for } |\partial| \le T,\ \|k\| \le R.$$

as in the statement are equivalent to a homogeneous linear system in the $E_{l\nu}$, whose coefficients coincide with the $C(\omega)^D P_{\partial k l \nu}$'s introduced in the previous Lemma. The number of unknowns of this system is $D^{2g+1} = N^{12g^2+6g}$, while the one of equations is $\le c_{12} T^g R^{2g} = c_{12} N^{12g^2+3g}$. By the version of Siegel's Lemma exposed in [Bro75, Lemma 5.2], it is possible to find a non-zero solution $E_{l\nu}(\omega) \in \mathbb{Z}[\omega]$ satisfying

$$\deg E_{l\nu} \le c_3 D, \qquad L(E_{l\nu}) \le e^{c_4(T \log T + D \log R)}.$$

Let us remark that $\Phi(z)$ does not vanish identically due to the algebraic independence of $H_1, \dots, H_{g+1}, A_1, \dots, A_g$.
We will now pursue the idea of bounding from above the number of zeros of Φ on the period lattice.

Lemma 2.17. *If $P(x_1, \dots, x_g)$ is a non-zero polynomial, then the multiplicity of the zero of the function $P(A_1(z), \dots, A_g(z))$ at the origin does not exceed the degree of P.*

Proof. By the inverse function Theorem, the function $z \mapsto (A_1(z), \dots, A_g(z))$ is invertible in a suitable neighbourhood of the origin, since its Jacobian is non-singular at the origin. This implies that the order of vanishing of $P(A_1(z), \dots, A_g(z))$ at the origin does not exceed the one of the function $\mathbb{C}^g \to \mathbb{C}$, $w \mapsto P(w)$, which is at most the degree of P. □

Proposition 2.18. *Let us consider a polynomial*

$$P(x, y) \in \mathbb{C}[x_1, \dots, x_{2g}, y_1, \dots, y_g]$$

and set $d_1 := \deg_x P$, $d_2 := \deg_y P$. Suppose that the function

$$F(z) := P(H_1(z), \dots, H_{2g}(z), A_1(z), \dots, A_g(z))$$

satisfies the conditions $\partial F(k \cdot \lambda) = 0$ for all k, ∂ such that $\|k\| \le \frac{d_1+1}{2}$ and $|\partial| \le d_2$. Then the polynomial P is identically zero.

Proof. Let us write $P(x,y) = \sum_l P_l(y)x^l$ for $P_l(y) \in \mathbb{C}[y]$, $l \in \mathbb{Z}^{2g}_{\geq 0}$, $\|l\| \leq d_1$ and set for short $F_l(z) \coloneqq P_l(A_1(z), \dots, A_g(z))$. By Lemma 2.17, it suffices to show that for each l we have $\partial F_l(0) = 0$ whenever $|\partial| \leq d_2$.
We argue by induction on r ranging from 0 to d_2. Let us therefore suppose that $\partial F_l(0) = 0$ for all l and for $|\partial| < r$. If now $|\partial| = r$, by induction hypothesis for any function $f : \mathbb{C}^g \to \mathbb{C}$ holomorphic at the origin

$$\partial \left(F_l(z) f(z) \right)|_{z=0} = (\partial F_l(0)) f(0).$$

Since F_l is abelian, if f is more generally holomorphic at $k \cdot \lambda$, we have

$$\partial \left(F_l(z) f(z) \right)|_{z=k\cdot\lambda} = (\partial F_l(0)) f(k \cdot \lambda).$$

Hence, a straightforward computation leads to

$$0 = \partial F(k \cdot \lambda) = \partial \left(\sum_l F_l(z) \prod_{i=1}^{g+1} H_i^{l_i}(z) \right) \Bigg|_{z=k\cdot\lambda} = \sum_l \partial F_l(0) \prod_{i=1}^{g+1} H_i^{l_i}(k \cdot \lambda)$$
$$= \sum_l \partial F_l(0) \prod_{i=1}^{g+1} (k_1 \omega_{i1} + \cdots + k_{2g}\omega_{i,2g})^{l_i}.$$

This implies that the polynomial

$$R(s_1, \dots, s_{g+1}) \coloneqq \sum_l (\partial F_l(0)) s^l$$

vanishes at each $k \cdot \tau$ for $\|k\| \leq \frac{d_1+1}{2}$ and $\tau_i \coloneqq (\omega_{i1}, \dots, \omega_{i,2g}) \in \mathbb{C}^{g+1}$ for $i = 1, \dots, g+1$. Since the period matrix is non-degenerate, the t_i's are a basis for the whole $\mathbb{C}^{g+1}$. We conclude that $R(s)$ is identically zero by [Schm80, Chapter 6, §8, Lemma 8A], so $\partial_l F(0) = 0$ for all l, as desired. □

Corollary 2.19. *The function $\Phi(z)$ defined in Proposition 2.13 cannot vanish with multiplicity $T+1$ at all points $k \cdot \lambda$ with $\|k\| \leq R^2$.*

Proof. If it were so, the previous Proposition would imply that the $E_{l\nu}(\omega)$'s all vanish, since $D < T$ and $D < R^2$, in contrast with Proposition 2.13. □

As a consequence of this Corollary, we infer the existence of $R_0 > 0$, k_0 and ∂_0 satisfying the relations

$$R < \|k_0\| = R_0 + 1 \leq R^2, \qquad |\partial_0| \leq T$$

and moreover such that $\partial \Phi(k \cdot \lambda) = 0$ for $|\partial| \leq T$ and $\|k\| \leq R_0$, while $k_0 \cdot \lambda$ is a zero of multiplicity exactly $|\partial_0| + 1$ for $\Phi(z)$. In particular, $\partial_0 \Phi(k_0 \cdot \lambda) \neq 0$, and we set for short

$$P(\omega, \chi) \coloneqq \partial_0 \Phi(k_0 \cdot \lambda) \in \mathbb{Z}[\omega, \chi].$$

With the notation of Lemma 2.14, we have

$$P(\omega,\chi) = C(\omega)^D (3T)! \sum_{l,\nu} E_{l\nu}(\omega)\partial_0 F_{k_0 l\nu}(0)$$

By exploiting the estimates for $\deg E_{l\nu}$ and $L(E_{l\nu})$ provided by Proposition 2.13 and the ones for $\partial_0 F_{k_0 l\nu}(0)$ given in Lemma 2.16, it readily follows that

$$\deg_\chi P \leq c_9, \quad \deg_\omega P \leq c_{13} D, \quad L(P) \leq e^{c_{14}(T\log T + D\log R)}.$$

Notice that these estimates can be checked without reducing the degree in χ, for $E_{l\nu} \in \mathbb{Z}[\omega]$.
The next step is to derive an estimate from above for $|P(\omega,\chi)|$ by taking advantage of the high number of zeros of $\Phi(z)$. This will be achieved via analytic means, and it will be the object of the next section.

2.3 Analytic part of the proof

We begin our discussion with an elementary Lemma on the order of growth of the main functions involved in the proof.

Lemma 2.20. *Let $G(z)$ be one of the functions $1, A_1, \ldots, A_g, H_1, \ldots, H_{g+1}$. Then ϑG is entire and for any $\varrho > 0$*

$$|\vartheta(z) G(z)|_\varrho \leq e^{c_{15}\varrho^2}$$

Proof. By definition of theta function, for all $j = 1, \ldots, 2g$ we may find $u_{j1}, \ldots, u_{jg}, v_j \in \mathbb{C}$ satisfying

$$\vartheta(z + \lambda_j) = \vartheta(z) \exp(2\pi i (u_{j1} z_1 + \cdots + u_{jg} z_g + v_j)).$$

Let us denote by U the $2g \times g$ complex matrix with entries the u_{ij}'s and by V the vector $(v_1, \ldots, v_{2g}) \in \mathbb{C}^{2g}$. With this notation, for any $k \in \mathbb{Z}^{2g}$ we have

$$\vartheta(z + k\cdot\lambda) = \vartheta(z) \exp\left(2\pi i ({}^t k U z + V \cdot k)\right).$$

Let us now pick $z \in \mathbb{C}^g$ with $|z| \leq \varrho$. We may write $z = w + k\cdot\lambda$ for some $k \in \mathbb{Z}^g$ and w in the fundamental parallelogram generated by $\lambda_1, \ldots, \lambda_{2g}$. If M denotes the maximum of $|\vartheta|$ on such parallelogram, we have

$$|\vartheta(z)| = |\vartheta(w)| \exp\left(\mathrm{Re}(2\pi i\, {}^t k U z + 2\pi i V \cdot k)\right) \leq M \exp\left(|2\pi\, {}^t k U z + 2\pi V \cdot k|\right).$$

Let L be the minimum modulus of the λ_j's, so that $\|k\| \leq \frac{1}{L}\varrho$. If $\|U\|$ denotes the maximum modulus of the u_{ij}'s, then

$$\left|{}^t kUz\right| \leq \frac{2g^2\|U\|}{L}\varrho^2, \quad |V \cdot k| \leq 2g\|V\|\varrho,$$

and the claim for $G = 1$ follows. The remaining cases can be treated analogously, by exploiting the fact that G is either abelian or quasi-periodic. □

We now turn to the main analytic result that will enable us to derive a useful upper bound for the quantity $P(\omega, \chi)$ introduced in the previous section, taking advantage of the high number of zeros of the auxiliary function $\Phi(z)$.

Proposition 2.21. *There exist positive constants c_{16}, c_{17} and c_{18}, only depending on $\lambda_1 \dots, \lambda_{2g}$, such that for any $T \geq 1$, $\varrho \geq 1$ and $\varrho_1 \geq c_{16}\varrho$ and any entire function $f : \mathbb{C}^g \to \mathbb{C}$ satisfying the equation $\partial f(k \cdot \lambda) = 0$ for $|\partial| < T$ and $\|k\| \leq \varrho$ the following inequality holds:*

$$|f|_{\varrho_1} \leq |f|_{c_{17}\varrho_1} \exp(-c_{18}TR^2).$$

This Proposition stands as a generalization of Schwarz's Lemma to several complex variables. While the case of a single variable is essentially based on the maximum modulus principle, some further refinements, first introduced by Bombieri and Lang, are needed for this more general setting. We now expose the proof of Proposition 2.21, following [Mas75b, Lemma 7].
Let $f : \mathbb{C}^g \to \mathbb{C}$ be a holomorphic function and let W be its set of zeros. For any $z \in \mathbb{C}^g$ and $\varrho \in \mathbb{R}$, $\varrho > 0$, denote by $B(z, \varrho)$ the closed ball of $\mathbb{C}^g$ centred at z of radius ϱ. Moreover, we write $\mathcal{H}$ for the $2g-2$ real dimensional Hausdorff measure over $\mathbb{C}^g$. We then define the total multiplicity function of f as

$$\mathcal{M}(z, \varrho) := \frac{(g-1)!}{\pi^{g-1}\varrho^{2g-2}}\mu\left(B(z, \varrho)\right),$$

where μ is the measure induced by $\mathcal{H}$ on W, in such a way that $\mu(B(z, \varrho)) = \mathcal{H}(B(z, \varrho) \cap W)$. This function measures the set of zeros inside $B(z, \varrho)$, normalized with respect to the volume of the $(g-1)$-st dimensional sphere. We advise the reader that this notation for the total multiplicity function is not standard; we prefer to stick with the symbols just introduced because the traditional notation risks creating confusion with the rest of the proof of our main result.
The function $\varrho \mapsto \mathcal{M}(z, \varrho)$ is non-decreasing for any fixed $z \in \mathbb{C}^g$, so we may define

$$\mathcal{M}(z) = \lim_{\varrho \to 0} \mathcal{M}(z, \varrho).$$

As shown in [Bom70, Proposition 3], $\mathcal{M}(z)$ is in fact a non-negative integer equal to the order of vanishing of f at z. $\mathcal{M}(z,\varrho)$ is the main tool to develop Schwarz's Lemma in several variables, as it is apparent from the following:

Lemma 2.22. *For any $\varrho > 0$ we have*

$$|f|_\varrho \leq e^{-\frac{1}{4}\mathcal{M}(0,\varrho)}|f|_{24g\varrho}.$$

Proof. See [Bom70, Proposition 4] and [Mas76, Lemma 18]. □

To establish Proposition 2.21, it therefore remains to bound $\mathcal{M}(0,\varrho)$ from below. First, we may find some $\sigma \leq 1$ independent of ϱ such that the balls $B(k\cdot\lambda, s)$ for $\|k\| \leq \varrho$ are pairwise disjoint. Notice that the independence of σ from ϱ is granted by the fact that the $k\cdot\lambda$'s are points of a lattice in $\mathbb{C}^g$. Moreover, the hypothesis $\varrho \geq 1$ also ensures that $B(k\cdot\lambda,\sigma) \subseteq B(0,2\varrho)$. We may then compute

$$\begin{aligned}\mathcal{M}(0,2\varrho) &= \frac{(g-1)!}{\pi^{g-1}(2\varrho)^{2g-2}}\mu\left(B(0,2\varrho)\right) \geq \frac{(g-1)!}{\pi^{g-1}(2\varrho)^{2g-2}}\sum_{\|k\|\leq\varrho}\mu\left(B(k\cdot\lambda,\sigma)\right)\\ &\geq \left(\frac{\sigma}{2\varrho}\right)^{2g-2}\sum_{\|k\|\leq\varrho}\mathcal{M}(k\cdot\lambda) = \left(\frac{\sigma}{2}\right)^{2g-2}T\varrho^2.\end{aligned}$$

This finally establishes Proposition 2.21 by taking $c_{16} = 2$, $c_{17} = 48g$ and $c_{18} = (\sigma/2)^{2g-2}$.

We are now ready to head towards the conclusion of the proof. We first set $\Theta(z) := \vartheta(z)^{(2g+1)D}$, so that the function

$$\Psi(z) := \Theta(z)\Phi(z)$$

is entire. Since $\|k_0\| = R_0 + 1$, the point $k_0\cdot\lambda$ lies in the ball centred at the origin of radius $\sqrt{g}c(R_0+1) \leq c_{19}R_0$, where c is the maximum modulo of λ_j for $j = 1,\ldots,2g$. It is not restrictive to assume $c_{19} \geq c_{18}$, in such a way that by Proposition 2.21

$$|\Psi|_{c_{19}R_0} \leq |\Psi|_{c_{20}R_0}\exp\left(-c_{18}TR_0^2\right).$$

Let us now observe that by Lemma 2.20 $\Psi(z)$ is a sum of D^{2g+1} functions of order of growth 2, each one raised at most to its $(2g+1)D$-th power. Indeed, the coefficients $E_{l\nu}(\omega)$ satisfy

$$|E_{l\nu}(\omega)| \leq |L(E_{l\nu})|\max\{|\omega|,1\}^{\deg E_{l\nu}} \leq \exp\left(c_{21}(T\log T + D\log R + D)\right),$$

so the inequality $T < DR^2$ yields that the leading term in $|\Psi(z)|$ is given by the powers of ϑA_i and ϑH_j for $i = 1, \ldots, g$, $j = 1, \ldots, g+1$. Thus,

$$|\Psi|_{c_{20}R_0} \leq \exp\left(c_{22}DR_0^2\right).$$

By combining these two results, we therefore get

$$|\Psi|_{c_{19}R_0} \leq \exp\left(c_{22}DR_0^2 - c_{18}TR_0^2\right) \leq \exp\left(-c_{23}TR_0^2\right).$$

A similar inequality carries through to $\partial_0\Psi(k_0 \cdot \lambda)$. Indeed, by Cauchy's formula

$$\frac{\partial^t \Psi(z)}{\partial z_1^{t_1} \ldots \partial z_g^{t_g}} = \frac{t_1! \ldots t_g!}{(2\pi i)^g} \int_{\gamma_1} \cdots \int_{\gamma_g} \frac{\Psi(\zeta)}{(\zeta_1 - z_1)^{t_1+1} \ldots (\zeta_g - z_g)^{t_g+1}} \, d\zeta,$$

where γ_i denotes a circle centred at z_i of radius ≤ 1 for all $i = 1, \ldots, g$. Hence, we derive the inequality

$$|\partial_0\Psi(k_0 \cdot \lambda)| \leq T! \exp(-c_{23}TR_0^2) \leq \exp(-c_{24}TR_0^2).$$

On the other hand, let us recall that $\partial\Phi(k_0 \cdot \lambda) = 0$ whenever $|\partial| < |\partial_0|$, which implies that $\partial_0\Psi(k_0 \cdot \lambda) = \Theta(k_0 \cdot \lambda)\partial_0\Phi(k_0 \cdot \lambda)$. Since $\vartheta(0) \neq 0$, we have

$$|\Theta(k_0 \cdot \lambda)| = \left|\vartheta(0)^{(2g+1)D} \exp(2\pi i(2g+1)D({}^t kU\lambda + V \cdot k))\right| \geq \exp(c_{25}DR_0^2).$$

All in all, we derive the following upper bound for $|P(\omega, \chi)|$:

$$\begin{aligned} |P(\omega, \chi)| = \frac{|\partial_0\Psi(k_0 \cdot \lambda)|}{|\Theta(k_0 \cdot \lambda)|} &\leq \exp\left(c_{25}DR_0^2 - c_{24}TR_0^2\right) \\ &\leq \exp\left(-c_{26}TR_0^2\right) \leq \exp\left(-c_{26}TR^2\right). \end{aligned}$$

By making explicit the dependence on N, we eventually constructed a polynomial expression $P(\omega, \chi) \in \mathbb{Z}[\omega, \chi]$ satisfying

$$\begin{gathered} \deg_\omega P \leq c_{13}N^{6g}, \quad L(P) \leq \exp\left(c_{14}N^{6g+1}\log N\right), \\ 0 < |P(\omega, \chi)| \leq \exp\left(c_{26}N^{12g+3}\right). \end{gathered}$$

By taking the norm of $P(\omega, \chi)$ over $\mathbb{Q}(\omega)$, we get a polynomial $Q(\omega) \in \mathbb{Z}[\omega]$. In order to gather information about the degree and length of Q, as well as about $|Q(\omega)|$, we may argue exactly as in the very end of the proof of Theorem 1.17. It is then readily seen that $Q(\omega) \in \mathbb{Z}[\omega]$ satisfies

$$\begin{gathered} \deg Q \leq c_{27}N^{6g}, \quad L(Q) \leq \exp\left(c_{28}N^{6g+1}\log N\right), \\ 0 < |Q(\omega, \chi)| \leq \exp\left(-c_{29}N^{12g+3}\right). \end{gathered}$$

Summing up, for every N sufficiently large we have constructed a polynomial $Q(\omega) \in \mathbb{Z}[\omega]$ satisfying the above conditions.
Contrary to the previous chapter, it is no longer possible to derive a contradiction by means of transcendence measures. Indeed, although it would be possible to prove that π is generated by the entries of the period matrix Ω, the restriction to the first $g+1$ rows of Ω prevents us from choosing $\omega = \pi$. We therefore need to rely on a classical transcendence criterion due to Gelfond.

Proposition 2.23. *Let us consider two increasing unbounded sequences $\{a_n\}_n$ and $\{b_n\}_n$ of real numbers for which there exist two constants c, d such that*

$$a_{n+1} \leq ca_n, \quad b_{n+1} \leq db_n$$

for every $n \geq 0$. Let $\alpha \in \mathbb{C}$. Suppose that there is a sequence $\{P_n\}_n$ of non-zero polynomials with integer coefficients such that P_n has degree at most a_n and logarithm of the height at most b_n, and moreover

$$|P(\xi)| \leq e^{-a_n((2c+1)a_n+(c+d+1)b_n)}.$$

Then $P_n(\xi) = 0$ for all sufficiently large n. In particular, ξ is algebraic.

Proof. We refer to [Wal74, Chapitre 5]. □

Since ω is transcendental, this Proposition yields at once the desired contradiction. We remark that the growth conditions on the sequences $\{a_n\}_n$ and $\{b_n\}_n$ are satisfied, in our setting, thanks to Proposition 2.18. Indeed, the latter allows us to circumvent the inconvenience appeared in the first chapter of replacing N by some implicit $N_0 \geq N$ whose modulus could not be controlled.

2.4 Corollaries

We now derive some consequences of Theorem 2.12, mainly connected with values of B- and Γ-functions at rational points.
Let us start by considering the curve $y^2 = 1 - x^n$ for some $n \geq 3$. As shown in [Wal79, §5.2g], this curve has genus $g = \lfloor (n-1)/2 \rfloor$ and the period matrix Ω can be taken as

$$\omega_{ij} = \alpha_j B\left(\frac{k(i)}{n}, \frac{1}{2}\right) \quad \text{for } i, h = 1, \ldots, 2g,$$

where $\alpha_1, \ldots, \alpha_{2g}$ are some algebraic numbers lying in the n-th cyclotomic field, while $k(i) = i$ if n is odd or if n is even and $i \leq g$, while $k(i) = i+1$ if

n is even and $i \geq g+1$.
It is well known since [Schn41] that every row of the matrix Ω contains a transcendental number, so that all numbers in the set

$$S = \left\{ B\left(\frac{k}{n}, \frac{1}{2}\right) \middle| k = 1, \ldots, n-1, \, k \neq \frac{n}{2} \right\}$$

are transcendental. By considering other curves it is actually possible to see that $B(a,b)$ is transcendental for any choice of $a, b \in \mathbb{Q} \smallsetminus \mathbb{Z}$. By selecting now $g+1$ rows from Ω, Vasilev's result immediately yields the following:

Corollary 2.24. *Any subset of S of $\lfloor (n+1)/2 \rfloor$ elements contains two algebraically independent numbers.*

Let us now recall the classical identities

$$B\left(a, \frac{1}{2}\right) = 2^{2a-1} \frac{\Gamma^2(a)}{\Gamma(2a)}, \qquad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)},$$

holding for any $a \in \mathbb{Q} \smallsetminus \mathbb{Z}$. For $\sin(\pi a)$ is algebraic whenever a is rational, it turns out that the numbers S lie in a finite extension of the field generated over $\mathbb{Q}$ by

$$\sqrt{\pi}, \, \Gamma\left(\frac{1}{n}\right), \ldots, \Gamma\left(\frac{g}{n}\right).$$

This observation readily yields an algebraic independence result for values

Corollary 2.25. *At least two of the numbers*

$$\pi, \, \Gamma\left(\frac{1}{n}\right), \ldots, \Gamma\left(\frac{g}{n}\right)$$

are algebraically independent.

For instance, applying this Corollary for $n = 3$ and $n = 4$ and recalling the identities

$$\Gamma\left(\frac{1}{3}\right) = \frac{1}{2^{2/3}\pi^{1/2}} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{2}{3}\right) = 2^{1/3} \left(\frac{\pi}{3}\right)^{1/2} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right)^{-1},$$

one easily deduces the algebraic independence of π from each of the numbers

$$\Gamma\left(\frac{1}{6}\right), \Gamma\left(\frac{1}{4}\right), \Gamma\left(\frac{1}{3}\right), \Gamma\left(\frac{2}{3}\right), \Gamma\left(\frac{3}{4}\right), \Gamma\left(\frac{5}{6}\right).$$

We now expose a slightly more sophisticated refinement of this arguments.

Corollary 2.26. *Let $n \geq 3$ be an integer and set $m = \lfloor (n+3)/4 \rfloor$. Suppose that $l_1, \dots, l_m$ are distinct numbers of the set $\{1, 2, \dots, n-1\}$ such that*

1. *$l_j \neq n - l_k$ for all $j, k \in \{1, \dots, m\}$;*

2. *for any $s \in \{1, \dots, m-1\}$ there exists a k such that*

$$\{2l_s\}_n \in \left\{ \frac{n}{2}, l_k, n - l_k \right\},$$

where $\{2l_s\}_n$ is the smallest non-negative residue of $2l_s$ modulo n.

Then there are at least two algebraically independent numbers among

$$\pi, \Gamma\left(\frac{2l_m}{n}\right), \Gamma\left(\frac{l_1}{n}\right), \dots, \Gamma\left(\frac{l_m}{n}\right).$$

Proof. We first prove the existence of $l_1, \dots, l_m$. For l_1 it is possible to choose any number between 1 and $n-1$ different from $\frac{n}{2}$. Arguing inductively, suppose that for some $r \in \{1, \dots, m-1\}$ there are distinct numbers $l_1, \dots, l_r \in \{1, \dots, n-1\}$ satisfying the conditions in the statement for $j, k \leq r$ and $s \leq r-1$. It follows that the cardinality of the set

$$M_r = \left\{ \frac{n}{2}, l_1, \dots, l_r, n - l_1, \dots, n - l_r \right\}$$

is at most $2r + 1 \leq 2m - 1 < n - 1$. As a result, its complement M_r^c in $\{1, \dots, n-1\}$ is non-empty. If $\{2l_r\}_n \in M_r$, we may define l_{r+1} to be any number of M_r. Otherwise, if $\{2l_r\}_n \in M_r^c$, then we choose $l_{r+1} = \{2l_r\}_n$ or $l_{r+1} = n - \{2l_r\}_n$. In this way the numbers $l_1, \dots, l_r$ are pairwise distinct and satisfy the two conditions of the statement, thus yielding the existence of the desired $l_1, \dots, l_m$ by induction.
Now, the first condition on the l_i's implies that $n/2, l_1, \dots, l_m, n-l_1, \dots, n-l_m$ are pairwise distinct. Setting $g = \lfloor (n-1)/2 \rfloor$, we have $2m > g \geq$, hence by Corollary 2.24 we deduce that at least two numbers in the set

$$S = \left\{ B\left(\frac{l_1}{n}, \frac{1}{2}\right), \dots, B\left(\frac{l_m}{n}, \frac{1}{2}\right), B\left(\frac{n-l_1}{n}, \frac{1}{2}\right), \dots, B\left(\frac{n-l_m}{n}, \frac{1}{2}\right) \right\}$$

are algebraically independent. Observe that for any $i = 1, \dots, m$ we have

$$\begin{aligned} B\left(\frac{l_i}{n}, \frac{1}{2}\right) &= 2^{\frac{2l_i}{n}-1} \Gamma^2\left(\frac{l_i}{n}\right) \Gamma\left(\frac{2l_i}{n}\right)^{-1} \\ &= 2^{\frac{2l_i}{n}-1} \left(\frac{2l_i}{n} - 1\right) \dots \left(\frac{\{2l_i\}_n}{n} + 1\right) \frac{\{2l_i\}_n}{n} \Gamma^2\left(\frac{l_i}{n}\right) \Gamma\left(\frac{\{2l_i\}_n}{n}\right)^{-1} \end{aligned}$$

by exploiting the functional equation $z\Gamma(z) = \Gamma(z+1)$. Furthermore, the identity $\Gamma(z)\Gamma(z+1) = \pi/\sin(\pi z)$ allows us to see that

$$\frac{n}{n-2l_i} B\left(\frac{n-l_i}{n}, \frac{1}{2}\right) = \frac{\sin(2\pi l_i/n)}{\sin^2(\pi l_i/n)} \pi \Gamma^2\left(\frac{l_i}{n}\right) \Gamma\left(\frac{2l_i}{n}\right)^{-1}.$$

By using once again the second property of $l_1, \ldots, l_m$, we see that the set S is contained in a finite algebraic extension of the field generated over $\mathbb{Q}$ by

$$\pi, \Gamma\left(\frac{2l_m}{n}\right), \Gamma\left(\frac{l_1}{n}\right), \ldots, \Gamma\left(\frac{l_m}{n}\right),$$

which establishes the claim. □

As an application, by choosing $n = 15, m = 4, l_1 = 1, l_2 = 2, l_3 = 4, l_4 = 8$, we see for example that among the numbers

$$\pi, \Gamma\left(\frac{1}{15}\right), \Gamma\left(\frac{2}{15}\right), \Gamma\left(\frac{4}{15}\right), \Gamma\left(\frac{8}{15}\right)$$

there are two algebraically independent ones.

Remark 2.27. Theorem 2.12 naturally leads to some questions about possible improvements in this direction. First, we notice that, without imposing restrictions to the genus g of the curve, this result is best possible in terms of the number of rows considered. Indeed, choosing only g rows would not work for the case $g = 1$, since for instance the periods of an elliptic curve with complex multiplication have algebraic ratio.
Another question is related to the possibility of increasing the transcendence degree of the field generated by periods and quasi-periods. Again, this is not possible in general, for it would not apply to the one-dimensional case.
It remains an open problem to determine whether these two problems can be overcome by imposing some conditions on g. It has been shown that the number of algebraically independent coordinates of periods and quasi-periods of a simple complex abelian variety of dimension g can be comparatively small with respect to its dimension even in the case of complex multiplication. For instance, Shimura in [Shim79] provides an example with $g = 2^{n-1}$ in which there are at most $n+1$ algebraically independent numbers among the entries of the period matrix. In the same context, see also [Rib80].

Chapter 3

A result involving exponential and abelian functions

3.1 Main result

We will now turn to an attempt of exploiting the techniques studied so far in order to obtain a mixed transcendence result for both the periods of a complex Abelian variety and their exponentials. Historically, the first results about transcendental numbers revolved around the exponential function, which appeared particularly suited for the application of the methods that we have been investigating until now. We will therefore try to address the question of whether it is possible to combine the framework of periods of complex Abelian varieties with the more classical one of the exponential function in the context of transcendence proofs. In this section, we will obtain a partial answer to this, so as to discuss some applications in the next section.

Let us consider a complex Abelian variety V satisfying the same assumptions of the previous chapter. Let Ω be its period matrix, with components ω_{ij} for $i, j = 1, \ldots, 2g$. Let us then fix g non-zero complex numbers $\xi_1, \ldots, \xi_g$ and define the $g \times 2g$ matrix E whose i-th row, for $i = 1, \ldots, g$, coincides with

$$\left(e^{\xi_i \omega_{i1}}, \ldots, e^{\xi_i \omega_{i,2g}}\right).$$

Pick two integers m, n such that $1 \leq m \leq 2g$ and $1 \leq n \leq g$. Let us choose m rows of the matrix Ω and n rows of the matrix E, say the rows $i_1, \ldots, i_n$ for $1 \leq i_1, \ldots, i_n \leq g$. Let S be the set made up of the entries of the chosen rows, together with $\xi_{i_1}, \ldots, \xi_{i_n}$. According to [Schn41], each row of Ω cannot have only algebraic entries, so the field $\mathbb{Q}(S)$ has transcendence degree at least 1 over $\mathbb{Q}$.
The aim of the present chapter is to prove the following

Theorem 3.1. *Suppose that $\mathbb{Q}(S)$ has transcendence degree 1 over $\mathbb{Q}$. If $2m + n > 2g$, then any transcendence type τ of a transcendental number ω in $\mathbb{Q}(S)$ satisfies*

$$\tau \geq 2 + \frac{2m + n - 2g}{2g + n}.$$

For the proof, we will apply the same transcendence techniques as the ones introduced in the previous chapters. In the shape of the auxiliary function that we will introduce, some exponential functions related to the rows of the matrix E will appear. The analytic part of the proof remains essentially untouched, since these exponential terms have lower order of growth than the abelian and quasi-periodic functions. However, their presence badly affects the arithmetic arguments in the proofs of the preceding chapters, since it tends to increase the type of the final polynomial, thereby weakening the final measure of transcendence. We will therefore suitably adapt our auxiliary function so as to restore a transcendence type bounded away from 2. After the proof, in the next section we will discuss some applications of Theorem 3.1.

Let us now start the proof of Theorem 3.1. Without loss of generality, we may suppose that the chosen rows are the first m of Ω and the first n of E. Let K be a number field, embedded in $\mathbb{C}$, containing all the coefficients of the Taylor expansion of $H_1, \ldots, H_m$ and $A_1, \ldots, A_g$ around 0, as in Lemma 2.11. We may assume that K is a Galois extension of $\mathbb{Q}$, up to passing to its normal closure. Set moreover $\delta := [K : \mathbb{Q}]$ and let $\alpha_1, \ldots, \alpha_\delta$ be an integral basis for the ring of integers of K. We fix a transcendental number $\omega \in \mathbb{Q}(S)$ and we suppose that all numbers in S are algebraic over $\mathbb{Q}(\omega)$. By the primitive element Theorem, we may find a complex number χ, integral over $\mathbb{Z}[\omega]$, such that $\mathbb{Q}(\omega, \chi)$ contains both $\mathbb{Q}(S)$ and K. We let N signify a sufficiently large positive integer and we write $c_1, c_2, \ldots$ for positive constants depending only on S.
We set for short $b = 2m + n - 2g$, which is positive by hypothesis, and we let a be a real number such that

$$a > \frac{2g + n}{2m + n - 2g}$$

We then choose a real number ε satisfying

$$0 < \varepsilon < \min\left\{\frac{b}{2m + n}, \frac{ab - 2g - n}{a(m + 1) + g(a + 1)}\right\}.$$

For ε in this range of values, it is always possible to find $\delta > 0$ such that

$$\varepsilon < \delta < \min\left\{\frac{b - \varepsilon g}{m}, \frac{ab - 2g - n + \varepsilon(a(m + n - g) - g)}{a(2m + n) + m}\right\}.$$

We define the quantities

$$\begin{aligned} r &= m + n, \\ t &= 2g + \varepsilon(g + n + m) - \delta n + n, \\ d &= t - r\varepsilon, \\ d_0 &= t - r(\varepsilon + 1 - \delta), \end{aligned}$$

which have been chosen in such a manner that they satisfy

$$\begin{cases} gt + 2gr = nd_0 + (g + m)d, \\ 0 < d_0 < d < t < d_0 + r, \\ \left(2 + \frac{1}{a}\right) d_0 + \frac{1}{a} r < t. \end{cases}$$

We finally set

$$R = N^r, \quad T = N^t, \quad D = N^d, \quad D_0 = \lfloor N^{d_0} \log N \rfloor.$$

We are ready to introduce our auxiliary function.

Proposition 3.2. *There exists a constant $C(\omega)$ only depending on S, ω, $\alpha_1, \ldots, \alpha_\delta$ and χ and there exist numbers $E_{hl\nu}(\omega) \in \mathbb{Z}[\omega]$ not all zero satisfying the following property. Consider the function*

$$\Phi(z) := C(N) \sum_{\|h\| \le \delta D_0} \sum_{\|l\| \le \delta D} \sum_{\|\nu\| \le \delta D} E_{hl\nu}(\omega) \prod_{r=1}^{n} e^{h_r \xi_r z_r} \prod_{i=1}^{m} H_i^{l_i}(z) \prod_{s=1}^{g} A_s^{\nu_s}(z),$$

with

$$C(N) = C(\omega)^{nT + 2gnD_0R + 2gmD + 1} (3T)! c_2^T,$$

c_2 as in Lemma 2.11, $h = (h_1, \ldots, h_n) \in \mathbb{Z}_{\ge 0}^n$, $l = (l_1, \ldots, l_m) \in \mathbb{Z}_{\ge 0}^m$ and $\nu = (\nu_1, \ldots, \nu_g) \in \mathbb{Z}_{\ge 0}^g$. Then Φ satisfies the conditions

$$\partial \Phi(k \cdot \lambda) = 0 \quad \text{for } |\partial| \le T, \ \|k\| \le R.$$

Moreover, the $E_{hl\nu}(\omega)$'s, as polynomials in ω, satisfy

$$t(E_{hl\nu}) \le c_3 D_0 R \log(D_0 R).$$

For the proof of this Proposition, we rely once again on Siegel's Lemma. We first deal with the quantities that will serve as coefficients for a linear system to which we will apply Siegel's Lemma.

Lemma 3.3. *For $k = (k_1, \ldots, k_{2g}) \in \mathbb{Z}^g$ and h, l, ν as above with $\|h\| \leq D_0$ and $\|l\|, \|\nu\| \leq D$, define*

$$F_{khl\nu}(z) := \prod_{r=1}^{n} e^{h_r \xi_r (z_r + (k\cdot\lambda)_r)} \prod_{i=1}^{m} H_i^{l_i}(z + k\cdot\lambda) \prod_{s=1}^{g} A_s^{\nu_s}(z),$$

and let ∂ be a differential operator of order $|\partial| \leq T$. Then there is a constant $C(\omega) \in \mathbb{Z}[\omega]$ depending only on S, ω, $\alpha_1, \ldots, \alpha_\delta$ and χ such that

$$P_{\partial khl\nu}(\omega, \chi) := C(N)\partial F_{khl\nu}(0) \in \mathbb{Z}[\omega, \chi],$$

with $C(N)$ defined as in Proposition 3.2, satisfies

$$t(P_{\partial khl\nu}(\omega, \chi)) \leq c_4 D_0 R \log(D_0 R).$$

Proof. As we have seen in Lemma 2.14, we have

$$\prod_{i=1}^{m} H_i^{l_i}(z + k\cdot\lambda) = \sum_{\kappa} \psi_\kappa(z) \prod_{i=1}^{m}\prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}},$$

the ψ_κ's being suitable sums of products of the $H_1, \ldots, H_m$ in such a way that the exponent of H_i does not exceed l_i. Let us also write

$$\psi'_\kappa(z) = \psi_\kappa(z) \prod_{s=1}^{g} A_s^{\nu_s}(z),$$

with Taylor expansion around 0 given by

$$\psi'_\kappa(z) = \sum_q \beta_{\kappa,q} \frac{z_1^{q_1} \ldots z_g^{q_g}}{q_1! \ldots q_g!}.$$

for some algebraic integers $\beta_{\kappa,q} \in K$. By Lemma 2.14, the components of $\beta_{\kappa,q}$ with respect to the integral basis $\alpha_1, \ldots, \alpha_\delta$ have modulus $\leq c_5 e^{T\log T}$. Furthermore, we have

$$\begin{aligned}\prod_{r=1}^{n} e^{h_r \xi_r (z_r + (k\cdot\lambda)_r)} &= \prod_{r=1}^{n} e^{h_r \xi_r z_r} \prod_{r=1}^{n}\prod_{j=1}^{2g} \left(e^{\xi_r \omega_{rj}}\right)^{h_r k_j} \\ &= \prod_{r=1}^{n}\prod_{j=1}^{2g} \left(e^{\xi_r \omega_{rj}}\right)^{h_r k_j} \sum_p \gamma_p \frac{z_1^{p_1} \ldots z_g^{p_g}}{p_1! \ldots p_g!},\end{aligned}$$

where $p = (p_1, \dots, p_g) \in \mathbb{Z}^g_{\geq 0}$ and $\gamma_p = 0$ if $p_j \neq 0$ for some $j = n+1, \dots, g$, while $\gamma_p = \prod_{r=1}^n h_r^{p_r} \xi_r^{p_r}$ otherwise. It follows that

$$\prod_{r=1}^n e^{h_r \xi_r z_r} \prod_{i=1}^m H_i^{l_i}(z + k \cdot \lambda) \prod_{s=1}^g A_s^{\nu_s}(z) =$$

$$= \left(\sum_p \gamma_p \frac{z_1^{p_1} \dots z_g^{p_g}}{p_1! \dots p_g!} \right) \left(\sum_\kappa \sum_q \beta_{\kappa,q} \prod_{i=1}^m \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}} \frac{z_1^{q_1} \dots z_g^{q_g}}{q_1! \dots q_g!} \right) =$$

$$= \sum_{p,q} \sum_\kappa \gamma_p \beta_{\kappa,q} \prod_{i=1}^m \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}} \frac{z_1^{p_1+q_1} \dots z_g^{p_g+q_g}}{p_1! q_1! \dots p_g! q_g!} =$$

$$= \sum_{p,q} \sum_\kappa \gamma_p \beta_{\kappa,q} \binom{p_1+q_1}{p_1} \dots \binom{p_g+q_g}{p_g} \prod_{i=1}^m \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}} \frac{z_1^{p_1+q_1} \dots z_g^{p_g+q_g}}{(p_1+q_1)! \dots (p_g+q_g)!}.$$

The t-th term in the Taylor expansion of $F_{khl\nu}$ at 0 therefore coincides with

$$\sum_{p+q=t} \sum_\kappa \beta_{\kappa,q} \prod_{a=1}^g \binom{p_a+q_a}{p_a} \prod_{r=1}^n h_r^{p_r} \xi_r^{p_r} \prod_{i=1}^m \prod_{j=1}^{2g} \omega_{ij}^{\kappa_{ij}} \prod_{r=1}^n \prod_{j=1}^{2g} \left(e^{\xi_r \omega_{rj}}\right)^{h_r k_j} \frac{z_1^t \dots z_g^t}{t_1! \dots t_g!}$$

This t-th coefficient is nothing but $\partial F_{khl\nu}(0)$ for $\partial = \partial^t / \partial^{t_1} \dots \partial^{t_g}$ divided by $t_1! \dots t_g!$. First, we need to find a suitable quantity that clears out denominators. The $\beta_{\kappa,q}$ have already been treated in Lemma 2.14, and to force them into $\mathbb{Z}[\omega, \chi]$ we need to multiply by a factor $C(\omega)(3T)! c_2^T$, where $C(\omega)$ is a denominator for $\alpha_1, \dots, \alpha_\delta$. We may then choose $C(\omega)$ in such a way that it be a denominator also for all the numbers in S. Thus

$$P_{\partial khl\nu}(\omega, \chi) = C(\omega)^{nT + 2gmD + 2gnD_0R + 1} (3T)! c_2^T \partial F_{khl\nu}(0) \in \mathbb{Z}[\omega, \chi].$$

In view of Corollary 1.20, we wish to compute the degree in ω and χ and the height of the above polynomial expression for $P_{\partial khl\nu}(\omega, \chi)$. First, we remark that $C(\omega)(3T)! c_2^T \beta_{\kappa,q}$ has bounded degree in ω and by the computations in Lemma 2.16 logarithm of the height $\leq c_6 T \log T$. Moreover,

$$\log \left(\prod_{a=1}^g \binom{p_a+q_a}{p_a} \right) \leq \log \left(2^{gT}\right) \leq c_7 T, \quad \log \left(\prod_{r=1}^n h_r^{p_r} \right) \leq c_8 T \log D_0,$$

and these terms only contribute with their height, being constant in ω. Furthermore, $C(\omega)^{nT} \prod_{r=1}^n \xi_r^{p_r}$ yields a polynomial expression in $\mathbb{Z}[\omega, \chi]$ with both degree in ω and χ satisfying $\leq c_9 T$, while logarithm of the height $\leq c_{10} T \log T$. For the product of the ω_{ij}'s we have degree in ω and $\chi \leq c_{11} D$

and logarithm of the height $\leq c_{12} D \log D$, while for the product of the $e^{\xi_r \omega_{rj}}$ we get degrees $\leq c_{13} D_0 R$ and logarithm of the height $\leq c_{14} D_0 R \log(D_0 R)$. These computations are easily checked by recalling that for a polynomial $Q \in \mathbb{Z}[x]$ and any integer $s \geq 0$

$$H(Q^s) \leq s!(\deg Q)^s H(Q)^s$$

and noticing that in our situation the leading term is always the factorial one. Overall, we conclude that for the above polynomial expression of $P_{\partial khl\nu}(\omega, \chi)$ the following inequality holds:

$$t(P_{\partial khl\nu}(\omega, \chi)) \leq c_{15} D_0 R \log(D_0 R).$$

The claim is then established by means of Corollary 1.20. □

After these technical computations, we are now in the position of proving Proposition 3.2. We first consider the function

$$\widetilde{\Phi}(z) := C(N) \sum_{\|h\| \leq D_0} \sum_{\|l\| \leq D} \sum_{\|\nu\| \leq D} \widetilde{E}_{hl\nu}(\omega, \chi) \prod_{r=1}^{n} e^{h_r \xi_r z_r} \prod_{i=1}^{m} H_i^{l_i}(z) \prod_{s=1}^{g} A_s^{\nu_s}(z),$$

for some $\widetilde{E}_{hl\nu}(\omega, \chi) \in \mathbb{Z}[\omega, \chi]$ not all zero to be chosen in such a way that

$$\partial \widetilde{\Phi}(k \cdot \lambda) = 0 \quad \text{for } |\partial| \leq T \text{ and } \|k\| \leq R.$$

We may regard these conditions as a linear system in the $\widetilde{E}_{hl\nu}$'s whose coefficients coincide with the $P_{\partial khl\nu}(\omega, \chi)$ of Lemma 3.3. Since $T^g R^{2g} < D_0^n D^{g+m}$, Siegel's Lemma 1.6 ensures the existence of the desired $\widetilde{E}_{hl\nu}$, and in combination with Lemma 3.3 it also yields

$$t(\widetilde{E}_{hl\nu}(\omega, \chi)) \leq c_{16} D_0 R \log(D_0 R).$$

At this point, we may remove χ from the coefficients $\widetilde{E}_{hl\nu}(\omega, \chi)$ by applying the same procedure as in the first chapter, that is, by multiplying $\widetilde{\Phi}$ by all those functions obtained from $\widetilde{\Phi}$ by replacing the $\widetilde{E}_{hl\nu}(\omega, \chi)$ with their Galois conjugates over $\mathbb{Q}(\omega)$. This eventually leads to the auxiliary function Φ described in Proposition 3.2.

As shown in [BroKub77, Corollary 7], the functions $A_1, \dots, A_g$, $H_1, \dots, H_m$ and $e^{\xi_1 z_1}, \dots, e^{\xi_n z_n}$ are algebraically independent, whence $\Phi(z)$ does not vanish identically. As a result, we can find an integer $N_0 \geq N$ such that

$$\partial \Phi(k \cdot \lambda) = 0 \quad \text{for } \|k\| \leq N_0^r,\ |\partial| \leq N_0^t,$$

but there are k_0, ∂_0 with $\|k_0\| \leq (N_0+1)^r$, $|\partial_0| < (N_0+1)^t$ such that

$$\partial\Phi(k_0 \cdot \lambda) = 0 \quad \text{for } |\partial| < |\partial_0|,$$
$$\partial_0\Phi(k_0 \cdot \lambda) \neq 0.$$

We set for short $R_0 = (N_0+1)^r$ and $T_0 = (N_0+1)^t$. The argument proposed in Lemma 3.3 shows that

$$P(\omega, \chi) = \partial_0\Phi(k_0 \cdot \lambda) \in \mathbb{Z}[\omega, \chi]$$

is a non-zero element of $\mathbb{Z}[\omega, \chi]$ of type

$$t(P(\omega, \chi)) \leq c_{17} D_0 R_0 \log(D_0 R_0).$$

The final step consists in estimating $|P(\omega, \chi)|$ from above. As it has been customary so far, we are going to achieve this goal by analytic means. The same tools exploited in the previous chapter can be applied essentially unchanged. Indeed, the functions $e^{\xi_1 z_1}, \ldots, e^{\xi_n z_n}$ have a lower order of growth than the quasi-periodic functions, so they shall not harm the estimates that were established in Vasilev's proof of Theorem 2.12.
Let us consider the function

$$\Psi(z) = \vartheta_0(z)^{(g+m)\delta D}\Phi(z),$$

which is entire, since ϑ_0 is a denominator for the abelian and quasi-periodic functions appearing in the expression of Φ. Since $k_0 \cdot \lambda$ is contained in the ball centred at the origin of radius $\leq c_{18}R_0$, by Proposition 2.21 we infer that

$$|\Psi|_{c_{19}R_0} \leq |\Psi|_{c_{20}R_0} e^{-c_{21}T_0R_0^2}.$$

The coefficients $E_{hl\nu}$ have modulus at most $e^{c_{22}D_0R\log(D_0R)}$ by Lemma 3.3. The exponential terms $e^{\xi_1 z_1}, \ldots, e^{\xi_n z_n}$ plainly have order of growth 1, while Lemma 2.20 implies that $\vartheta_0, \vartheta_0 A_1, \ldots, \vartheta_0 A_g, \vartheta_0 H_1, \ldots, \vartheta_0 H_m$ all have order of growth ≤ 2. As a result, it turns out that

$$|\Psi|_{c_{20}R_0} \leq e^{c_{22}(D_0R_0\log(D_0R) + DR_0^2)},$$

and therefore

$$|\Psi|_{c_{19}R_0} \leq e^{-c_{23}T_0R_0^2}.$$

By Cauchy's estimate, this upper bound remains essentially unaltered for the derivatives of Ψ, as we have already seen in the previous chapter. Hence

$$|\partial_0\Psi|_{c_{19}R_0} \leq e^{-c_{24}T_0R_0^2}.$$

Since $\vartheta_0(0) \neq 0$, it follows that $|\vartheta_0(k_0 \cdot \lambda)^{(g+m)\delta D}| \geq e^{c_{25} D R_0^2}$. This finally yields

$$|P(\omega, \chi)| = \frac{|\partial_0 \Psi(k_0 \cdot \lambda)|}{|\vartheta_0(k_0 \cdot \lambda)^{(g+m)\delta D}|} \leq e^{-c_{26} T_0 R_0^2}.$$

By taking the norm of $P(\omega, \chi)$ over $\mathbb{Q}(\omega)$ as we did in the preceding chapters, we eventually obtain a polynomial Q with integer coefficients satisfying

$$0 < |Q(\omega)| \leq e^{-c_{27} T_0 R_0^2}, \qquad t(Q) \leq c_{28} D_0 R_0 \log(D_0 R_0).$$

Since $(D_0 R_0 \log(D_0 R_0))^{2+\frac{1}{a}} < T_0 R_0^2$, we conclude that ω has the desired transcendence type.

3.2 Some applications

Let us now comment on some features of Theorem 3.1. By Proposition 1.4, almost all transcendental numbers have transcendence type $\leq 2 + \varepsilon$ for any $\varepsilon > 0$. It should therefore be expected that in almost all cases Theorem 3.1 yields in fact the existence of two algebraically independent numbers in S, provided $2m + n > 2g$. Another reason why it seems likely to turn this Theorem into the form $\mathrm{trdeg}(\mathbb{Q}(S)/\mathbb{Q}) \geq 2$ is that it is possible to see that π belongs to the field generated by the entries of Ω, by exploiting a version of Legendre's relation for complex Abelian varieties. Thus, in case π can actually be generated by fewer than $g + 1$ rows of Ω, we deduce the existence of two algebraically independent numbers among the ones in S when selecting precisely those rows.
The main problem in deducing that $\mathrm{trdeg}(\mathbb{Q}(S)/\mathbb{Q}) \geq 2$ is the fact that we have not been able to apply Gelfond's criterion at the end of the proof. In order to do so, we should find an upper bound for the number of zeros of the auxiliary function Φ in a ball of radius a power of N, as it had been done in Vasilev's proof via Proposition 2.18. Unfortunately, the exponential terms appearing in the expression for Φ do not allow a clear generalization of Vasilev's arguments.
We mention that Theorem 3.1 is already present in [Chu84] in the form $\mathrm{trdeg}(\mathbb{Q}(S)/\mathbb{Q}) \geq 2$ for the case of elliptic curves, thus with $g = 1$. However, the proof is only roughly sketched, and there is no reference on how to make it independent of the final transcendence type. Moreover, other results in [Chu84] are quoted without any proof, and they appeared in the literature only much later with complete proofs by other authors. The result by Vasilev that we exposed in the second chapter is an example of these, as pointed out in [Gri02]. In view of all this, we prefer to discuss some corollaries and applications of Theorem 3.1 without assuming Chudnovsky's stronger result

claimed in [Chu84]. Anyway, we found no reference in the literature concerning results akin to Theorem 3.1 for the general case of complex Abelian varieties.

Let us consider the case of elliptic curves for $g = 1$.

Corollary 3.4. *Let E be an elliptic curve over $\mathbb{C}$ with algebraic invariants. Let ω_1, ω_2 be a pair of fundamental periods for E with η_1, η_2 their associated quasi-periods. Choose any non-zero complex number $\xi \in \mathbb{C}$. If the numbers*

$$\omega_1,\ \omega_2,\ \xi,\ e^{\xi},\ e^{\xi\frac{\omega_1}{\omega_2}}$$

are algebraically independent, then any transcendence type τ of one of these numbers satisfies

$$\tau \geq 2 + \frac{1}{3}.$$

The same statement holds for the numbers

$$\eta_1,\ \eta_2,\ \xi,\ e^{\xi},\ e^{\xi\frac{\omega_1}{\omega_2}}.$$

Proof. It is enough to apply Theorem 3.1 to the first or the second row of the period matrix of E while choosing $\xi_1 = \frac{\xi}{\omega_1}$. □

Corollary 3.5. *In the notation of Corollary 3.4, there are two algebraically independent numbers in each of the sets*

$$\left\{\omega_1,\ \omega_2,\ \log\pi,\ \pi,\ \pi^{\frac{\omega_1}{\omega_2}}\right\}, \quad \left\{\eta_1,\ \eta_2,\ \log\pi,\ \pi,\ \pi^{\frac{\omega_1}{\omega_2}}\right\}.$$

Proof. We apply Corollary 3.4 with $\xi = \log\pi$ and take advantage of the fact that π is known to have transcendence type $\leq 2+\varepsilon$ for any $\varepsilon > 0$. For a list of known transcendence types, we refer to [Wal78]. □

Corollary 3.6. *The statement of Corollary 3.4 applies to each of the set of numbers*

$$\left\{\omega_1,\ \omega_2,\ e,\ e^{\frac{\omega_1}{\omega_2}}\right\}, \quad \left\{\eta_1,\ \eta_2,\ e,\ e^{\frac{\omega_1}{\omega_2}}\right\}.$$

The case of complex multiplication is not particularly interesting in this last Corollary. Indeed, under this assumption ω_1/ω_2 would be an imaginary quadratic algebraic number, so e and e^{ω_1/ω_2} would be algebraically independent by Lindemann-Weierstraß Theorem.

Corollary 3.7. *Let $\alpha \neq 1$ be any non-zero algebraic number. Then the statement of Corollary 3.4 applies to each set of numbers*

$$\left\{\omega_1,\ \omega_2,\ \log\alpha,\ \alpha^{\frac{\omega_1}{\omega_2}}\right\},\quad \left\{\eta_1,\ \eta_2,\ \log\alpha,\ \alpha^{\frac{\omega_1}{\omega_2}}\right\}.$$

To give some concrete examples, we may consider the elliptic curves that we have already described in the first chapter. Let $\varrho = e^{2\pi i/3}$ be a primitive third root of unity. Then the statement of Corollary 3.4 holds for each set of numbers

$$\left\{\Gamma\left(\frac{1}{4}\right),\ e^{\Gamma\left(\frac{1}{4}\right)},\ e^{i\Gamma\left(\frac{1}{4}\right)}\right\},\quad \left\{\Gamma\left(\frac{1}{3}\right),\ e^{\Gamma\left(\frac{1}{3}\right)},\ e^{\varrho\Gamma\left(\frac{1}{3}\right)}\right\}.$$

Moreover, there are two algebraically independent numbers in both the sets

$$\left\{\Gamma\left(\frac{1}{4}\right),\ \pi,\ e^{\pi}\right\},\quad \left\{\Gamma\left(\frac{1}{3}\right),\ \pi,\ e^{\pi},\ e^{\varrho\pi}\right\}.$$

Notice that the case of $\Gamma(\frac{1}{4})$, π and e^{π} is particularly fortunate due to Euler's identity $e^{\pi i} = -1$. We remark that for the last two sets of numbers there are way sharper results thanks to a celebrated Theorem by Nesterenko, who managed to show that the sets of numbers

$$\left\{\Gamma\left(\frac{1}{4}\right),\ \pi,\ e^{\pi}\right\},\quad \left\{\Gamma\left(\frac{1}{3}\right),\ e^{\pi},\ e^{\pi\sqrt{3}}\right\}$$

have transcendence degree 3. Results connected with Nesterenko's techniques are at present the only ones concerning the algebraic independence of three numbers, left alone Lindemann-Weierstraß Theorem. Nesterenko's arguments make use of Eisenstein series and special values of modular functions; we refer to [Nes96] for more details.

We now pass to study some examples of applications to complex Abelian varieties. For an integer $N \geq 3$, let us consider the curve $y^2 = 1 - 4x^N$, which has genus $\lfloor\frac{N-1}{2}\rfloor$. Let us also write $\zeta = e^{\frac{2\pi i}{N}}$. As shown in [Lan66, Chapter V], the period lattice of the Jacobian variety associated with this curve has generators given by the vectors

$$\lambda_j = \left(\ldots, \zeta^{kj}\left(1-\zeta^k\right)^2 \frac{1}{N} B\left(\frac{k}{N}, \frac{k}{N}\right), \ldots\right)$$

for $j = 0, \ldots, N-1$, the components running over $k = 1, \ldots, \lfloor\frac{N-1}{2}\rfloor$, together with the vector

$$\left(\ldots, \left(1-\zeta^k\right) \frac{1}{N} B\left(\frac{k}{N}, \frac{k}{N}\right), \ldots\right).$$

An analogous expression applies to the quasi-periods, provided we let k run from $\lfloor \frac{N+1}{2} \rfloor$ to $N-1$.
One may now apply Theorem 3.1 in order to derive results of algebraic independence for B-values, for instance by exploiting the fact that for any $a \in \mathbb{Q} \setminus \mathbb{Z}$ the number $B(a,a)B(1-a,1-a)$ is a non-zero algebraic multiple of π, using computations as the ones in the previous chapter. Let us go through some examples of these arguments.

Corollary 3.8. *For any non-zero complex number ξ, there are two algebraically independent numbers among*

$$B\left(\frac{1}{12},\frac{1}{12}\right),\ B\left(\frac{5}{12},\frac{5}{12}\right),\ \pi,\ \xi,\ e^{\xi},\ e^{i\sqrt{3}\xi}.$$

Proof. We apply Theorem 3.1 to the complex Abelian variety described above for $N = 12$ with the following choices. We choose the rows of the period matrix whose components are algebraic multiples of

$$B\left(\frac{1}{12},\frac{1}{12}\right),\ B\left(\frac{5}{12},\frac{5}{12}\right),\ B\left(\frac{7}{12},\frac{7}{12}\right),\ B\left(\frac{11}{12},\frac{11}{12}\right).$$

As for the matrix E defined before Theorem 3.1, we choose its fourth row, and we also pick

$$\xi_4 = \xi\left(\left(1-\zeta^4\right)^2 \frac{1}{12} B\left(\frac{4}{12},\frac{4}{12}\right)\right)^{-1},$$

where $\zeta = e^{\frac{2\pi i}{12}}$. It is readily checked that the products

$$B\left(\frac{1}{12},\frac{1}{12}\right)B\left(\frac{11}{12},\frac{11}{12}\right),\quad B\left(\frac{5}{12},\frac{5}{12}\right)B\left(\frac{7}{12},\frac{7}{12}\right)$$

are non-zero algebraic multiples of π. Thus, π belongs to the field generated by S, with S defined as in Theorem 3.1. As a result, $\operatorname{trdeg}(\mathbb{Q}(S)/\mathbb{Q}) \geq 2$. The numbers of S appearing in the matrix E are of the form $e^{\xi\zeta^{4j}}$ for some integers j, together with $e^{\xi(1-\zeta^4)^{-1}}$. Moreover, $\zeta^4 = \varrho$, where $\varrho = e^{\frac{2\pi i}{3}}$, while $(1-\zeta^4)$ is a primitive sixth root of unity. Hence, these numbers turn out to be

$$e^{\xi},\ e^{\xi\varrho},\ e^{\xi\varrho^2},\ e^{\xi\varrho^{-1/2}}.$$

Since $\varrho = (-1+i\sqrt{3})/2$, it follows that the transcendence degree of $\mathbb{Q}(S)$ coincides with the one of the field generated over $\mathbb{Q}$ by the numbers in the statement, which is therefore proved. □

A similar strategy, choosing the third row of the matrix E, allows for example to prove that there are at least two algebraically independent numbers among

$$B\left(\frac{1}{12},\frac{1}{12}\right),\ B\left(\frac{5}{12},\frac{5}{12}\right),\ \pi,\ \xi,\ e^{\xi},\ e^{i\xi}.$$

By choosing $\xi = \pi^2$, one deduces for instance the existence of two algebraically independent numbers among

$$B\left(\frac{1}{12},\frac{1}{12}\right),\ B\left(\frac{5}{12},\frac{5}{12}\right),\ \pi,\ e^{\pi^2},\ (-1)^{\pi}.$$

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