Università degli Studi di Padova

Dipartimento di Matematica *"Tullio Levi-Civita"*
Corso di Laurea Magistrale in Matematica

TESI DI LAUREA MAGISTRALE

Witten Deformations, Geometry and Dynamics

Relatori
Franco Cardin
Alberto Abbondandolo

Candidato
Leonardo Masci
Matricola 1155486

Anno Accademico 2017/2018 — Sessione di Laurea 14/12/2018

WITTEN DEFORMATIONS, GEOMETRY AND DYNAMICS

Leonardo Masci

Contents

Introduction

At the intersection of topology and dynamics, there is the classical question of how to tie the number of critical points of a function with the topology of the manifold on which it is defined. The first successful result was obtained by Marston Morse in the 30's, with his celebrated *Morse inequalities* [35]. These inequalities relate the number of critical points of a special class of functions, called *Morse functions*, with the Betti numbers of the manifold. Throughout the following decades, a great deal of work was put into generalizing the Morse inequalities, often by finding new techniques to derive them. This area of research led to some profound theorems in mathematics, like the Bott periodicity theorem [13] and Smale's proofs of the Poincaré conjecture in dimension ≥ 5 and of the h-cobordism theorem [42, 44].

In 1982, Edward Witten found two peculiar ways to prove the Morse inequalities [52]. Motivated by a simple model of supersymmetric quantum mechanics, he studied a one-parameter family of Laplace-like operators acting on the algebra of differential forms of a manifold. These operators are obtained by deforming the De Rham complex of differential forms by a Morse function and extracting a deformed Hodge-Laplace operator from it. He found that through the asymptotics of the spectrum of these operators, one could analytically recover the Morse inequalities for the deforming function. This analytical approach to a topological question was adapted to the realm of index theory for elliptic operators [8–12] and is a powerful asset for the derivation of explicit formulas in that theory. On the other hand, Witten introduced an algebraic technique, based on works of Thom and Smale [45, 46], that later proved to be of great versatility and power. He constructed a chain complex with the gradient flow lines of the Morse function, with which he found one could extract the Morse inequalities algebraically. The construction of this complex ultimately led Andreas Floer to formulate Floer homology [17, 18], one of the most successful tools in symplectic topology and geometry. Finally, Witten conjectured that his complex would arise naturally from the eigenforms of the family of deformed Laplacians. Such a conjecture was proved rigorously by Helffer and Sjöstrand using techniques from semi-classical analysis [26].

The goal of this thesis is to give an alternative, dynamical-flavored proof of Witten's conjectural correspondence between his analytical and algebraic techniques. To do this, we take a detour into Harvey and Lawson's Morse theory [21–23, 34]. The original motivation of Harvey-Lawson theory is to study the large-time limit of the pull-back of a flow on bundles over the manifold, which in dynamical systems theory, is called *transfer operator*. The transfer operator carries a lot of information on the dynamics of the flow which cannot be extracted efficiently from the flow acting on points [6]. Thus the study of the large-time limit of the transfer operator, when it exists, might shed light on the convergence of the system to an equilibrium state for the system. Harvey and Lawson study the limit for a class of hyperbolic flows, which contains the gradient flows of Morse functions. In the special case of the gradient flow of a Morse function, they also show how this large time limit may be expressed in terms of the stable and unstable manifolds of the gradient flow.

We will use this encoding of the large-time limit in the stable and unstable manifolds to define a map between the eigenforms of the deformed Laplacians and the complex constructed by Witten. By mimicking the semi-classical estimates of Helffer and Sjöstrand, we will then show that this map is a cochain isomorphism, obtaining a proof of the conjecture. The idea behind the use of Harvey-Lawson theory is that the deformed Laplacian can also be interpreted as a stochastic perturbation of the Lie derivative generating the flow, so that the asymptotics of its spectrum may also be understood in connection to the framework of *large deviations* from the classical paths of the system – the gradient flow lines.

Overview

1. In the first chapter we study the Witten Deformation of the De Rham complex of differential forms. We start from the general case, where the deformation is done with a vector field, and continue with the special case of the deformation with a gradient of a Morse function. In this special case we prove the first important theorem of the thesis, namely, we show that the spectrum of the Witten Laplacian acting on k-forms is in a one-to-one correspondence with the critical points of index k of the Morse function used in the deformation.
2. In the second chapter we define the "oriented Morse chain complex". This is a chain complex constructed with the orientations of the unstable manifolds of the critical points of a Morse function, where the differential is defined by appropriately counting gradient flow lines connecting two critical points whose index differs by one. We introduce the "Thom-Smale cellular filtration", which is a cellular filtration of the manifold based on the foliation by unstable submanifolds of the Morse function. Since this is a true cellular filtration, its cellular homology is computing the singular homology of the manifold. Then we show the second important theorem of the thesis: the oriented Morse complex is isomorphic to the cellular complex of this filtration, so it is computing the singular homology of the manifold. The Morse inequalities follow from this isomorphism.
3. In the third chapter we study Harvey-Lawson theory. We show that the large-time limit of the pull-back of a flow of a differential form exists in the complex of currents on the manifold – hence the limit exists in a distributional sense. This limit defines a map from the exterior bundle to the complex of currents, and it turns out to be a cochain map. Moreover, it can be expressed in terms of currents of integration over the stable and unstable manifolds. As a side note, this is also telling us that the equilibrium state of this very simple dynamical system will be concentrated around the stable and unstable manifolds of the flow. Finally, using the explicit form of this limit, we can construct a subcomplex of the complex of

currents which is isomorphic to the oriented Morse complex. This will be our "proxy complex" for the proof of Witten's conjecture.

4. In the last chapter we combine all of the above to show that the subcomplex of the deformed De Rham complex given by the eigenforms of the Witten Laplacian is isomorphic to the "proxy complex" of Harvey-Lawson theory, so in turn, to the oriented Morse complex. The idea of the proof is to adapt the heat kernel method of Bismut and approximate the eigenforms with "bump k-forms" whose traces on the stable spaces of critical points of index k are much easier to manage than the traces of the eigenforms.

Acknowledgements

I would like to thank my supervisors, Franco Cardin and Alberto Abbondandolo, for guiding me through the process of writing this thesis. Their interest and care towards my work have been precious. Their scientific and personal counsel has been invaluable. Thanks to you both, I have grown.

I would like to thank my extended family and friends for always being there for me, even in my periods of isolation. Their support has given me the strength to follow through and never falter. Words just don't cut it.

Lastly, but not least-ly, I want to thank all the people I've met from the *CRC/TRR 191* "Symplectic Structures in Geometry, Algebra and Dynamics". In my brief stays in Germany you've accepted me as if I were one of your own, and treated me as a friend. I hope one day I can return your acts of unconditional kindness.

1. Witten deformations of the De Rham complex

The Witten deformation of the De Rham complex, introduced by Witten in 1982 [52], is a family of cochain complexes dependent from a parameter, which are all quasi-isomorphic to the original, undeformed De Rham complex. The advantage introduced by this deformation is that for small values of the parameter, the Hodge theory of the deformed complex is simpler than the Hodge theory of the undeformed complex.

The contents of this chapter are loosely adapted from [53, Chapter 4 and 5]

1.1. *Witten deformations in general*

Let (M,g) be a Riemannian n-manifold. Consider the cochain complex $(\Omega^\bullet(M),d)$ of differential forms on M. This complex is quasi-isomorphic to the singular cochain complex with real coefficients, so that its cohomology is computing the real singular cohomology of the manifold M. We want to construct a family of complexes $(\Omega^\bullet(M),d_\nu)$ where $\nu\in\mathbb{R}$, which are quasi-isomorphic to the De Rham complex. See the Appendix, section A.1 for a review of Hodge theory and all the necessary notation we use in this chapter.

1.1.1. *Clifford actions*

Denote by $\Lambda^\bullet(V)$ the exterior algebra over a finite dimensional vector space V. For a vector $v\in V$ and a k-form $\theta\in\Lambda^k(V^*)$, denote by $i_v\theta\in\Lambda^{k-1}(V^*)$ the $(k-1)$-form obtained by evaluating θ on v. We call i_v the *interior product operator*. Now, if we consider the exterior bundle $\Lambda^\bullet(T^*M)$ and a $e\in TM$ such that $\pi_{TM}(e)=x$, we have a well defined interior product operator on $\Lambda^\bullet(T^*_xM)$ by setting

$$\Lambda^\bullet(T^*_xM)\ni\theta\mapsto i_e\theta=i_v\theta\in\Lambda^{\bullet-1}(T^*_xM) \tag{1.1.1}$$

where $v\in T_xM$ is such that in a trivializing chart $e=(x,v)$. Hence we may extend the definition of the interior product to an operation involving a vector field $X:M\to TM$ and a differential k-form $\theta\colon M\to\Lambda^k(T^*M)$ by setting $(i_X\theta)_x=i_{X(x)}\theta_x$.

Since we have fixed a Riemannian structure on M, for any $x\in M$, each $v\in T_xM$ has a corresponding $v^*\in T^*_xM$ such that for any $w\in T_xM$ it holds that $v^*(w)=g_x(v,w)$. Hence if we take a vector field $X:M\to TM$, we may define a dual 1-form $X^*\colon M\to T^*M$ by setting for every $x\in M$, $X^*_x(v)=g_x(X(x),v)$ for any $v\in T_xM$

Definition 1.1. Let $X : M \to TM$ be a vector field. Define the *Clifford operators* as follows:

$$\begin{aligned} c(X) : \Omega^\bullet(M) &\to \Omega^\bullet(M) \\ \theta &\mapsto c(X)(\theta) = X^* \wedge \theta - i_X\theta \\ \hat{c}(X) : \Omega^\bullet(M) &\to \Omega^\bullet(M) \\ \theta &\mapsto \hat{c}(X)(\theta) = X^* \wedge \theta + i_X\theta \end{aligned} \tag{1.1.2}$$

Remark 1.2. The Clifford operators are vector space endomorphisms of $\Omega^\bullet(M)$ of mixed degree, and not graded algebra morphisms. They interchange the even and odd spaces: if we define

$$\Omega^E(M) = \bigoplus_{j \text{ even}} \Omega^j(M), \quad \Omega^O(M) = \bigoplus_{j \text{ odd}} \Omega^j(M) \tag{1.1.3}$$

then we have

$$c(X)\big(\Omega^E(M)\big) \subseteq \Omega^O(M), \quad c(X)\big(\Omega^O(M)\big) \subseteq \Omega^E(M) \tag{1.1.4}$$

and the same for $\hat{c}$. In this sense we may say that the Clifford operators are Dirac-type operators.

PROPOSITION 1.3. *Let $X, Y : M \to TM$ be vector fields. Then the following identities hold:*

$$\begin{aligned} &[c(X)\circ c(Y) + c(Y)\circ c(X)]\theta = -2g(X,Y)\theta \\ &[\hat{c}(X)\circ \hat{c}(Y) + \hat{c}(Y)\circ \hat{c}(X)]\theta = 2g(X,Y)\theta \\ &c(X)\circ \hat{c}(Y) + \hat{c}(Y)\circ c(X) = 0 \end{aligned} \tag{1.1.5}$$

Proof. The proof is just a matter of computation.

$$\begin{aligned} c(X)\circ c(Y)\theta &= X^*\wedge Y^*\wedge\theta - X^*\wedge(i_Y\theta) - i_Y(X^*\wedge\theta) + i_Y i_X\theta \\ c(Y)\circ c(X)\theta &= Y^*\wedge X^*\wedge\theta - Y^*\wedge(i_X\theta) - i_X(Y^*\wedge\theta) + i_X i_Y\theta \end{aligned} \tag{1.1.6}$$

Hence in the sum the only non-canceling terms are

$$[c(X)\circ c(Y) + c(Y)\circ c(X)]\theta = -Y^*\wedge i_X\theta - i_Y(X^*\wedge\theta) - X^*\wedge i_Y\theta - i_X(Y^*\wedge\theta) \tag{1.1.7}$$

A quick computation in charts shows that $i_Y(X^*\wedge\theta) = i_X(Y^*\wedge\theta) = g(X,Y)\theta$ while the remaining terms cancel each other out. The rest of the proof is similar. □

1.1.2. Witten deformations

Recall that the presence of a metric on M induces an isomorphism $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ for all $0 \le k \le n$ which is an involution up to sign (see in Appendix, subsection A.1.1.2). Through this involution we may define an inner product on k-forms as

$$(\eta, \xi) = \int_M \eta \wedge *\xi \tag{1.1.8}$$

This induces a formal adjoint to the exterior differential, which we call *codifferential* $d^*\colon \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$, which is defined by imposing that $(d\eta,\xi) = (\eta, d^*\xi)$. The *Dirac operator* (Appendix, subsection A.1.1.5) is the partial differential operator $D = d + d^*$, which, like the Clifford operators, is a linear endomorphism which switches odd forms and even forms. For computations in charts, it is convenient to introduce the $C^\infty(M)$-valued inner product $\langle -,- \rangle\colon \Omega^k(M)\times\Omega^k(M) \to C^\infty(M)$, which is defined in the Appendix, equation (A.1.4). It characterizes the previous inner product by the fact that if μ_g is the Riemannian volume form of M,

$$\eta \wedge *\xi = \langle \eta, \xi \rangle \, \mu_g \tag{1.1.9}$$

Definition 1.4. Let $\nu > 0$ and $X\colon M \to TM$ be a vector field. To avoid degenerate situations, let's assume once and for all that X is a *transverse section*, that is, the image $X(M)$ in TM is transversal to the zero section.

1. The *Witten deformation* of the exterior differential operator is

$$d_\nu = d + \nu^{-1} X^* \wedge -\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M) \tag{1.1.10}$$

The adjoint operator with respect to the inner product $(-,-)$ is called the *Witten deformation* of the codifferential and has the expression

$$d_\nu^* = d^* + \nu^{-1} i_X\colon \Omega^\bullet(M) \to \Omega^{\bullet-1}(M) \tag{1.1.11}$$

2. The *Witten-Dirac* operator is

$$D_\nu = d_\nu + d_\nu^* = d + d^* + \nu^{-1}\hat{c}(X)\colon \Omega^\bullet(M) \to \Omega^\bullet(M) \tag{1.1.12}$$

3. The *Witten Laplacian* with respect to X is the operator

$$\Delta_\nu = D_\nu^2\colon \Omega^\bullet(M) \to \Omega^\bullet(M) \tag{1.1.13}$$

THEOREM 1.5. *Let $e_1,\dots,e_n$ be a local orthonormal frame for the tangent bundle. Let $\nabla\colon \Omega^0(M;TM) \to \Omega^1(M;TM)$ be the Levi-Civita connection and $\nabla_j = \nabla_{e_j}$ the covariant differentiation in direction e_j. We have the following* Bochner-type formula

$$\Delta_\nu = \Delta + \frac{1}{\nu}\sum_j c\left(e_j\right)\circ \hat{c}\left(\nabla_j X\right) + \frac{2g(X,X)}{\nu^2} \tag{1.1.14}$$

where $\Delta = D^2 = (d+d^*)^2$ is the usual Hodge-Laplacian.

Proof. We have Bochner-type expressions for the exterior differential and codifferential (Proposition A.7):

$$d = \sum_j \left(e_j \wedge -\right)\circ \nabla_j, \quad d^* = -\sum_j i_{e_j} \circ \nabla_j \tag{1.1.15}$$

which gives us the expression for the Dirac operator

$$D = \sum_j c\left(e_j\right) \circ \nabla_j \tag{1.1.16}$$

The claim now follows from an explicit calculation, using the properties of the Clifford operators. To shorten notation, denote by $[-,-]$ the commutator on linear operators and $\{-,-\}$ the anticommutator on linear operators.

$$\begin{aligned}
\Delta_\nu &= (D + \hat{c}(X))^2 = \Delta + \frac{1}{\nu}\{D, \hat{c}(X)\} + \frac{2g(X,X)}{\nu^2} = \\
&= \Delta + \frac{1}{\nu}\left[\sum_j \left\{c\left(e_j\right) \circ \nabla_j, \hat{c}(X)\right\}\right] + \frac{2g(X,X)}{\nu^2} = \\
&= \Delta + \frac{1}{\nu}\left[\sum_j c\left(e_j\right) \circ \nabla_j \circ \hat{c}(X) - \sum_j c\left(e_j\right) \circ \hat{c}(X) \circ \nabla_j\right] + \frac{2g(X,X)}{\nu^2} = \\
&= \Delta + \frac{1}{\nu}\sum_j c\left(e_j\right) \circ \left[\nabla_j, \hat{c}(X)\right] + \frac{2g(X,X)}{\nu^2}
\end{aligned} \tag{1.1.17}$$

To compute $\left[\nabla_j, \hat{c}(X)\right]$, we first prove the following

LEMMA 1.6 (Covariant derivative of dual 1-form). *Let X, Z be vector fields.*

$$\nabla_Z (X^*) = (\nabla_Z X)^* \tag{1.1.18}$$

Proof. The question is local, so we may pick an orthonormal frame $e_1, \ldots, e_n$ of TM and suppose $Z = e_j$. Now, $\nabla_Z (X^*)$ is a 1-form uniquely determined by the following equation: for any vector field Y

$$\nabla_Z (X^*) = Z(X^* Y) - X^*(\nabla_Z Y) = Z(X^* Y) - g(X, \nabla_Z Y) \tag{1.1.19}$$

But in charts

$$\begin{aligned}
Z(X^* Y) &= \frac{\partial}{\partial x^j}\left(g_{kl} X^k Y^l\right) = \frac{\partial g_{kl}}{\partial x^j} X^k Y^l + g_{kl}\frac{\partial X^k}{\partial x^j} Y^l + g_{kl} X^k \frac{\partial Y^l}{\partial x^j} = \\
&= \left(g_{il}\Gamma^i_{jk} + g_{ki}\Gamma^i_{jl}\right) X^k Y^l + g_{kl}\frac{\partial X^k}{\partial x^j} Y^l + g_{kl} X^k \frac{\partial Y^l}{\partial x^j} = \\
&= g_{kl}\left(\frac{\partial X^k}{\partial x^j} + \Gamma^k_{ji} X^i\right) Y^l + g_{kl}\left(\frac{\partial Y^k}{\partial x^j} + \Gamma^k_{ji} Y^i\right) X^l = \\
&= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\end{aligned} \tag{1.1.20}$$

Hence $\nabla_Z (X^*) Y = Z(X^* Y) - g(X, \nabla_Z Y) = g(\nabla_Z X, Y)$ for any vector field Y. By non-degeneracy of the Riemannian metric we conclude that $\nabla_Z (X^*) = (\nabla_Z X)^*$. □

So now we may compute the commutator. Notice that from the definition of connection on k-forms, it follows trivially that $\nabla_Z \circ i_X = i_X \circ \nabla_Z + i_{\nabla_Z X}$ on any k-form.

$$\begin{aligned}\left[\nabla_j, \hat{c}(X)\right]\theta &= \nabla_j(c(X)\theta) - c(X)\left(\nabla_j\theta\right) = \\ &= \nabla_j(X^* \wedge \theta) + \nabla_j(i_X\theta) - X^* \wedge \nabla_j\theta - i_X\left(\nabla_j\theta\right) = \\ &= \nabla_j(X^*) \wedge \theta + X^* \wedge \nabla_j\theta - X^* \wedge \nabla_j\theta + i_{\nabla_j X}\theta = \\ &= \left(\nabla_j X\right)^* \wedge \theta + i_{\nabla_j X}\theta = \hat{c}\left(\nabla_j X\right)\theta\end{aligned} \tag{1.1.21}$$

Finally

$$\Delta_\nu = \Delta + \frac{1}{\nu}\sum_j c\left(e_j\right)\circ\left[\nabla_j, \hat{c}(X)\right] + \frac{2g(X,X)}{\nu^2} = \Delta + \frac{1}{\nu}\sum_j c\left(e_j\right)\circ\hat{c}\left(\nabla_j X\right) + \frac{2g(X,X)}{\nu^2} \tag{1.1.22}$$

□

Remark 1.7. Notice that this explicit expression shows that Δ_ν does not respect the grading of $\Omega^\bullet(M)$ in general. It obviously does respect the parity of the k-forms. So a weaker preservation is in act: the $\mathbb{Z}/2$-grading of $\Omega^\bullet(M)$. In physicist terms, we say that Δ_ν is a *supersymmetry* operator.

1.1.3. Localizing close to invariant sets

The presence of a Riemannian metric on M induces a metric $\langle -, - \rangle$ on the whole tensor bundle $T^\bullet_\bullet(M)$ and thus also on the exterior bundles, both of k-forms and of k-vectors. In charts this metric has the very simple form

$$\langle T, S \rangle = g^{j_1 s_1} \dots g^{j_k s_k} g_{i_1 r_1} \dots g_{i_h r_h} T^{i_1 \dots i_h}_{j_1 \dots j_k} S^{s_1 \dots s_k}_{r_1 \dots r_h} \tag{1.1.23}$$

which is just the contraction of the indices of the tensors with each other. On the exterior bundle it coincides with the one defined in the Appendix. Denote by $||-||$ the L^2-type norm on the tensor bundle induced by integration of the metric.

Recall that we have assumed X to be a transverse section of the tangent bundle. This way, the zero set $Z(X)$ of X is a discrete set in M. The advantage of the Witten-Dirac operator over the classical Dirac operator itself is that for small ν, we have the following Poincaré inequality away from the zero set of X:

THEOREM 1.8. *Let $Z(X) \subset M$ be the zero locus of X on M. Let U be an open set containing $Z(M)$. There exist constants $C > 0$ and $\nu_0 > 0$ such that for any k-form θ with support disjoint from U and for any $\nu < \nu_0$ we have the following* Poincaré inequality*:*

$$||\theta|| \leq C\sqrt{\nu}\,||D_\nu\theta|| \tag{1.1.24}$$

Proof. Since we are interested in θ supported inside $M \setminus U$, there exists a small enough $C_1 > 0$ such that $||X||^2 \geq C_1/2$ for all $x \in M \setminus U$. Now, we can estimate the second term in formula (1.1.14) with a sup norm, so there exists a $C_2 > 0$ such that

$$||D_\nu\theta||^2 = (D_\nu\theta, D_\nu\theta) = (\Delta_\nu\theta, \theta) \geq \left(\frac{C_1}{\nu^2} - \frac{C_2}{\nu}\right)||\theta||^2 = \frac{C_1 - \nu C_2}{\nu^2}||\theta||^2 \tag{1.1.25}$$

so for small enough ν,

$$||\theta|| \leq \frac{\nu}{\sqrt{C_1 - \nu C_2}}||D_\nu\theta|| \leq C\sqrt{\nu}||D_\nu\theta|| \tag{1.1.26}$$

□

Remark 1.9. This theorem is telling us that away from the zero set of the vector field, the operator Δ_ν is injective. Hence the kernel of Δ_ν is comprised necessarily of forms with support concentrated around the zero set of the vector field.

Let $p \in Z(X)$ and let $U_p \subset M$ be a domain for a chart $(y_1, \ldots, y_n)$ centered in p for which the metric g is the flat metric

$$g|_{U_p} = \sum_{j=1}^{n} dy^j \otimes dy^j \tag{1.1.27}$$

For simplicity, assume that in this chart $X(y) = Ay$ for a linear operator $A \colon \mathbb{R}^n \to \mathbb{R}^n$ with nonzero determinant. Since we have chosen a flat chart, the Bochner-type formula (1.1.14) reads

$$\Delta_\nu = -\sum_{j=1}^{n}\left(\frac{\partial}{\partial y^j}\right)^2 + \frac{1}{\nu}\sum_{j=1}^{n} c\left(e_j\right) \circ \hat{c}\left(Ae_j\right) + \frac{1}{\nu^2}Ay \cdot Ay \tag{1.1.28}$$

Drawing from elementary quantum mechanics, this operator is close to the Hamiltonian of an n-dimensional quantum harmonic oscillator, where the action on a k-form θ is the action on a state comprising of k fermions. Namely,

$$\begin{aligned}
\Delta_\nu &= H_\nu + K_\nu \\
H_\nu &= -\sum_{j=1}^{n}\left(\frac{\partial}{\partial y^j}\right)^2 - \frac{1}{\nu}\operatorname{tr}\left[\sqrt{A^T A}\right] + \frac{1}{\nu^2}Ay \cdot Ay \\
K_\nu &= \frac{1}{\nu}\operatorname{tr}\left[\sqrt{A^T A}\right] + \frac{1}{\nu}\sum_{j=1}^{n} c\left(e_j\right) \circ \hat{c}\left(Ae_j\right)
\end{aligned} \tag{1.1.29}$$

Now, H_ν is the legitimate Hamiltonian of a quantum harmonic oscillator (sheared by A), whose spectral theory is very well known [20, Section 1.5]. The spectral theory for K_ν is studied in [41]. The Clifford operators in this interpretation act as creation and destruction operators. Particularly interesting for us is the following local expression of the kernel of Δ_ν:

PROPOSITION 1.10 ([53, Proposition 4.9]). *For any* $\nu > 0$, $\Delta_\nu|_{U_p}$ *is a positive operator with one dimensional kernel, of the following form*

$$\ker \Delta_\nu = \operatorname{span}\left\{ \exp\left(-\frac{|Ay|^2}{2\nu}\right)\rho \right\} \tag{1.1.30}$$

where ρ *is a norm-1 form in the kernel of* K_ν. *Moreover all nonzero eigenvalues of this operator are greater than* C/ν *for some positive constant* C.

1.1.4. *On the square of* d_ν

In this section we show under which conditions on the deformation vector field X the graded vector space $\Omega^\bullet(M)$ is a cochain complex when endowed with d_ν. First we must extend the operator $i_X : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ to a more general operator $i_{\mathbb{X}} : \Omega^\bullet(M) \to \Omega^{\bullet-h}(M)$ where $\mathbb{X}$ is an h-vector field, and then we borrow some terminology from fluid dynamics.

Definition 1.11. Let ξ be a k-form and $\mathbb{X}$ an h-vector field with $k \geq h$. The *inner product* of ξ with $\mathbb{X}$, $i_{\mathbb{X}}\xi$, is the $(k-h)$-form obtained by evaluating ξ on $\mathbb{X}$.

Definition 1.12. Consider a vector field X on a Riemannian manifold (M,g).

1. The *curl* of the vector field, $\operatorname{curl} X$, is the 2-form $\operatorname{curl} X = dX^*$.
2. The *vorticity* of the vector field, $\mathbb{V}X$, is the 2-vector field dual to its curl, that is, the unique 2-vector field such that $\operatorname{curl} X = \langle \mathbb{V}X, - \rangle$.

Borrowing terminology from classical 3D Riemannian geometry, we say that a vector field is *conservative* if its curl is zero.

Remark 1.13. By definition, $\langle \operatorname{curl} X \wedge -, - \rangle = \langle -, i_{\mathbb{V}X} - \rangle$. Moreover notice that when $\operatorname{curl} X = 0$, the dual 1-form to X defines a cohomology class $[X^*] \in H^1(M;\mathbb{R})$.

LEMMA 1.14. *Consider* $d_\nu = d + X^* \wedge -$ *and* $d_\nu^* = d^* + i_X$. *We have the following identities.*

$$\begin{aligned} d_\nu^2 &= \operatorname{curl} X \wedge - : \Omega^\bullet(M) \to \Omega^{\bullet+2}(M) \\ (d_\nu^*)^2 &= i_{\mathbb{V}X} : \Omega^\bullet(M) \to \Omega^{\bullet-2}(M) \end{aligned} \tag{1.1.31}$$

Proof. Indeed

$$\begin{aligned} d_\nu^2\theta &= d^2\theta + X^* \wedge d\theta + d(X^* \wedge \theta) + X^* \wedge X^* \wedge \theta = \\ &= X^* \wedge d\theta + dX^* \wedge \theta - X^* \wedge d\theta = \operatorname{curl} X \wedge \theta \end{aligned} \tag{1.1.32}$$

while by adjunction

$$\left\langle \theta, (d_\nu^*)^2 \xi \right\rangle = \left\langle d_\nu^2 \theta, \xi \right\rangle = \langle \operatorname{curl} X \wedge \theta, \xi \rangle = \langle \theta, i_{\mathbb{V}X}\xi \rangle \tag{1.1.33}$$

□

COROLLARY 1.15. *If X is conservative, then $(\Omega^\bullet(M), d_\nu)$ is a cochain complex for all $\nu > 0$.*

Remark 1.16. Notice that the above results show us that one may loosely think of d_ν as a connection, more than an exterior differentiation, where the curl form plays the rôle of the curvature 2-form. In the case of conservative forces, this "connection" is *flat*, and hence may be used to compute some sort of cohomology theory for M.

PROPOSITION 1.17. *Let X be conservative. Then we have an explicit formula for Δ_ν:*

$$\Delta_\nu = \Delta + \frac{1}{\nu}[L_X + L_X^*] + \frac{2g(X,X)}{\nu^2} \tag{1.1.34}$$

where L_X denotes the Lie derivative in direction X and L_X^* its formal adjoint with respect to the metric.

Proof. This is an explicit calculation involving all the adjunctions we have encountered so far. We shall use the Cartan formula $L_X = i_X \circ d + d \circ i_X$ and its adjoint version, $L_X^* = (X^* \wedge -) \circ d^* + d^* \circ (X^* \wedge -)$. Explicitly

$$\begin{aligned}
\Delta_\nu &= d_\nu^2 + (d_\nu^*)^2 + d_\nu d_\nu^* + d_\nu^* d_\nu = \\
&= dd^* + \nu^{-1} d i_X + \nu^{-1}(X^* \wedge -)d^* + \nu^{-2} X^* \wedge i_X + \\
&\qquad + d^* d + \nu^{-1} d^*(X^* \wedge -) + \nu^{-1} i_X d + \nu^{-2} i_X (X^* \wedge -) = \\
&= dd^* + d^* d + \nu^{-1}(d i_X + i_X d + d^*(X^* \wedge -) + (X^* \wedge -)d^*) + \\
&\qquad + \nu^{-2}(X^* \wedge i_X + i_X(X^* \wedge -)) = \\
&= \Delta + \nu^{-1}[L_X + L_X^*] + 2\nu^{-2} g(X,X)
\end{aligned} \tag{1.1.35}$$

□

1.2. *Deforming with a gradient field*

We have seen that if we deform the De Rham complex with a conservative vector field, the result is again a cochain complex. In this section we study deformations with a restricted subclass of conservative vector fields, the *gradient* vector fields, which admit a globally defined potential, hence whose dual 1-form is exact. In this case it is very easy to show that the deformed complexes are all isomorphic.

1.2.1. *Witten exterior differential revisited*

Recall that for a differentiable function $f : M \to \mathbb{R}$, the *gradient vector field* is the unique vector field $\nabla_g f$ such that $df = g(\nabla_g f, -)$. Denote again by d_ν, d_ν^*, D_ν and Δ_ν the deformed operators, where the deformation is by $X = \nabla_g f$. Notice that trivially X is a conservative field, since $dX^* = d(df) = 0$.

PROPOSITION 1.18. *$d_\nu = e^{-f/\nu} d e^{f/\nu}$ and $d_\nu^* = e^{f/\nu} d^* e^{-f/\nu}$.*

Proof.

$$\begin{aligned}\left(e^{-f/\nu} d e^{f/\nu}\right)\theta &= e^{-f/\nu} d\left(e^{f/\nu}\theta\right) = e^{-f/\nu}\left[e^{f/\nu}\nu^{-1} df \wedge \theta + e^{f/\nu} d\theta\right] = \\ &= \frac{1}{\nu} df \wedge \theta + d\theta = d_\nu \theta\end{aligned} \tag{1.2.1}$$

The other statement follows immediately by adjunction. □

Definition 1.19. The *Witten deformation of the De Rham complex* is the family of cochain complexes $\{(\Omega^\bullet(M), d_\nu)\}_{\nu>0}$.

PROPOSITION 1.20. *The complexes $(\Omega^\bullet(M), d)$ and $\{(\Omega^\bullet(M), d_\nu)\}_{\nu>0}$ are all isomorphic.*

Proof. Indeed the map is given by $\theta \mapsto e^{f/\nu}\theta$. Clearly this is an isomorphism for all $\nu > 0$ at each degree, since it's just the multiplication with an invertible element in the ring $C^\infty(M)$, and $\Omega^k(M)$ is a $C^\infty(M)$-module. Since $e^{f/\nu} d_\nu = e^{f/\nu} e^{-f/\nu} d e^{f/\nu} = d e^{f/\nu}$ it is also a cochain complex morphism. □

Remark 1.21. In the more general case of a conservative vector field, this proposition suggests how we might show that the deformed cochain complexes are *quasi*-isomorphic to the undeformed complex on M. Indeed, the dual 1-form θ to a conservative vector field X is closed, so we may pick the cover $\pi\colon E \to M$ of M for which its lift $\pi^*\theta$ is exact and has a potential f – e.g. the universal cover. Now, in E we may deform the De Rham complex $(\Omega^\bullet(E), \mathfrak{d})$ and obtain the isomorphic family of complexes $(\Omega^\bullet(E), \mathfrak{d}_\nu)$. To recover the cohomology of the space M we have that $H^k(M;R)$ is isomorphic to the equivariant cohomology $H^k(E;R)^G$ of E, where the group G is given by the deck transformations of the cover. So the problem of finding a quasi-isomorphism between $(\Omega^\bullet(M), d_\nu)$ and the undeformed complex is reduced to checking that the equivariant differential forms $(\Omega^\bullet(E)^G, \mathfrak{d})$ are preserved by the conjugation by $e^{f/\nu}$. This sort of reasoning fits in the scope of *Novikov theory* [37].

PROPOSITION 1.22. *We have the identity*

$$e^{f/\nu}\Delta_\nu e^{-f/\nu} = \Delta + \frac{2}{\nu} L_{\nabla_g f} \tag{1.2.2}$$

Proof. Indeed

$$\begin{aligned}e^{f/\nu}\Delta_\nu\left(e^{-f/\nu}\theta\right) &= d\left(d^*_{2f,\nu}\theta\right) + d^*\left(d_{2f,\nu}\theta\right) = \\ &= dd^*\theta + 2\nu^{-1} d\left(i_{\nabla_g f}\theta\right) + 2\nu^{-2} i_{\nabla_g f}(d\theta) + d^* d\theta = \\ &= \Delta\theta + 2\nu^{-1} L_{\nabla_g f}\theta\end{aligned} \tag{1.2.3}$$

where we have indicated by $d_{2f,\nu}$ the Witten deformation of d with respect to $2\nabla_g f$. □

Remark 1.23. Looking at $\hat{\Delta}_\nu = (\nu/2)\Delta_\nu$, we see that $e^{f/\nu}\hat{\Delta}_\nu e^{-f/\nu} = L_{\nabla_g f} + (\nu/2)\Delta$, which for ν small is a sort of *stochastic perturbation* of the Lie derivative, if thought of as the infinitesimal generator of a semigroup in $\Omega^\bullet(M)$. Moreover, the local expression for the kernel of Δ_ν found below shows that it is generated by "heat kernel forms", which we may think of as restrictions of the heat kernel to k-dimensional submanifolds in U_p.

1.2.2. *Localization close to critical points of a Morse function*

We want to study the kernel of the Witten laplacian in the gradient case.

Remark 1.24. What we are missing is a Hodge-type theorem that ties $\ker \Delta_\nu|_{\Omega^k}$ with $H^k(\Omega^\bullet(M), d_\nu)$. But we don't really need it, we already have plenty of information.

- Proposition 1.17 shows explicitly that Δ_ν respects the degree whenever X is conservative, so that in our case $\Delta_\nu \colon \Omega^k(M) \to \Omega^k(M)$.
- Proposition 1.20 shows that $\dim H^k(\Omega^\bullet(M), d_\nu) = \dim H^k(\Omega^\bullet(M), d)$, and that additionally they are isomorphic.
- Proposition 1.10 gives us an explicit expression for the kernel of the Witten Laplacian close to the critical points of f.

If we put stronger conditions on our gradient field, we can obtain considerable strength by combining the Propositions as we have done above.

From now on, assume that $f : M \to \mathbb{R}$ is a *Morse function*, that is, a function whose critical points are all non-degenerate. Elementary properties of Morse functions are spelled out in the Appendix, Section B.1. Namely, combining the Morse Lemma B.5 with the local expression of Proposition 1.10, we obtain the following

PROPOSITION 1.25 ([53, Proposition 5.4]). *Let $p \in \operatorname{Crit} f$. There exists a small enough chart $\left(U_p, \left(y^1, \dots, y^n\right)\right)$ centered at p such that*

$$\Delta_\nu = -\sum_{j=1}^{n} \left(\frac{\partial}{\partial y^j}\right)^2 + \frac{2}{\nu}\left(\sum_{j=1}^{\operatorname{ind} p} i_{\frac{\partial}{\partial y^j}}\left(dy^j \wedge -\right) + \sum_{j=\operatorname{ind} p}^{n} dy^j \wedge i_{\frac{\partial}{\partial y^j}} - \right) - \frac{n}{\nu} + \frac{2|y|^2}{\nu^2} \quad (1.2.4)$$

In this chart, Δ_ν acting on $\Gamma(\Lambda^\bullet(\mathbb{R}^n))$ is a non-negative operator with kernel

$$\ker \Delta_\nu = \operatorname{span}\left\{ \exp\left(-\frac{|y|^2}{2\nu}\right) dy^1 \wedge \cdots \wedge dy^{\operatorname{ind} p} \right\} \quad (1.2.5)$$

for all sufficiently small $\nu > 0$.

What we are saying is that the Witten Laplacian admits an approximation by *quantum harmonic oscillators* in a small enough chart. With this line of thought, we can show more – in charts, we can find the whole spectrum of the Witten Laplacian, following the original article [52]. First of all we may rewrite equation (1.1.28) in a Morse chart for f (see Theorem B.5 and use the identities in Proposition 1.3)

$$\sum_{j=1}^{n} \left\{ -\left(\frac{\partial}{\partial y^j}\right)^2 + \frac{2\left(y^j\right)^2}{\nu^2} \right\} + \frac{1}{\nu}\left\{ -\sum_{j=1}^{\operatorname{ind} p}\left[i_{\frac{\partial}{\partial y^j}}, dy^j \wedge - \right] + \sum_{j=\operatorname{ind} p+1}^{n} \left[i_{\frac{\partial}{\partial y^j}}, dy^j \wedge - \right] \right\} =$$
$$= \sum_{j=1}^{n} H_\nu^j - \sum_{j=1}^{\operatorname{ind} p} K_\nu^j + \sum_{j=\operatorname{ind} p+1}^{n} K_\nu^j \quad (1.2.6)$$

This time, the H_ν part is an isotropic quantum harmonic oscillator. Moreover, the harmonic oscillator part and the K_ν part commute: H_ν^j acts on the C^∞ coefficients of a form while K_ν^h on the basis chosen. So they may be simultaneously diagonalized and their eigenforms are thus the same. Precisely, the eigenfunctions of H_ν^j are proportional to $Y_{N_j}(y^j)\exp\left(-(y^j)^2/2\nu\right)$, where the Y_{N_j} are the Hermite polynomials, and $N_j \in \mathbb{N}$ is the first quantum number. The eigenvalues are $(1/\nu)(1+2N_j)$. For K_ν^j, it is easy to see that the eigenforms are the basis forms, while the eigenvalues are just ± 1. Indeed if I is a multi-index of length k,

$$\left[i_{\frac{\partial}{\partial y^j}}, dy^j \wedge -\right] dy^I = \begin{cases} dy^I, & j \in I \\ -dy^I, & j \notin I \end{cases} = \varepsilon_{jI} dy^I \tag{1.2.7}$$

where the symbol ε_{jI} is defined by that formula. This means that the eigenvalues of the Witten Laplacian acting on k-forms in this chart are

$$\alpha_{I,N} = \frac{1}{\nu}\sum_{j=1}^{n}\left(1+2N_j\right) - \sum_{j=1}^{\operatorname{ind} p}\varepsilon_{jI} + \sum_{j=\operatorname{ind} p+1}^{n}\varepsilon_{jI} \tag{1.2.8}$$

where I is a multi-index of length k. Hence for a form to be in the kernel, that is, have eigenvalue 0, it must be that $I = (1, \ldots, \operatorname{ind} p)$ and all the N_j must be zero. From this it follows that the kernel has the form of equation (1.2.5).

Remark 1.26. The expression (1.2.5) for the kernel shows that the harmonic forms for the Witten Laplacian are a sort of Thom forms [30, 31].

1.3. The spectrum of Δ_ν

The special local form of the Witten Laplacian and its kernel will enter in the computation of the spectrum of Δ_ν. In this section we seek the proof of the following

THEOREM 1.27. *For any $\varepsilon > 0$ there exists a $\nu_0 > 0$ such that whenever $\nu < \nu_0$, the number of eigenvalues of $\Delta_\nu|_{\Omega^k}$ in the interval $[0, \varepsilon]$ equals the number of critical points of f with index k.*

The proof will use the *heat kernel method* introduced by Bismut in [11].

1.3.1. Dimensional reduction

Definition 1.28. Let $p \in \operatorname{Crit}^k f$ and $(U_p, (y^1, \ldots, y^n))$ a chart centered in p. Denote by W the codomain of the chart. Consider an $a > 0$ such that we may fit a ball of radius

$4a$ in W. Let $\gamma\colon \mathbb{R}\to[0,1]$ be a bump function supported on $[-2a,2a]$, i.e. a smooth function such that $\gamma(z)=1$ if $|z|<a$ and 0 for $|z|>2a$. Define

$$\begin{aligned}\alpha_{p,\nu} &= \int_W \gamma(|y|)^2 \exp\left(-\frac{|y|^2}{\nu}\right) dy^1\wedge\cdots\wedge dy^n \\ \rho_{p,\nu} &= \frac{\gamma(|y|)}{\sqrt{\alpha_{p,\nu}}}\exp\left(-\frac{|y|^2}{2\nu}\right) dy^1\wedge\cdots\wedge dy^k\end{aligned} \tag{1.3.1}$$

We can see $\rho_{p,\nu}$ as a k-form on M with (compact) support contained in U_p.

Let $L^2\Omega^\bullet = L^2(M;\Lambda^\bullet(T^*M))$ be the completion of $\Omega^\bullet(M)$ under the Riemannian norm we've used up until now, and denote by $W^{1,2}\Omega^\bullet$ the subspace of forms of finite first Sobolev norm. Define

$$E_\nu = \bigoplus_{p\in\operatorname{Crit} f} \operatorname{span}\left(\rho_{p,\nu}\right) \tag{1.3.2}$$

Remark 1.29. By Proposition 1.25, $\rho_{p,\nu}\in\ker\Delta_\nu$ when restricted to the sections supported in the small ball $B_a(p)$. By construction, for any ν,

$$\dim E_\nu = \sum_{k\in\mathbb{N}} \sharp\left(\operatorname{Crit}^k f\right) \tag{1.3.3}$$

because for each critical point of index k there is a line in E_ν generated by the relative $\rho_{p,\nu}$. Hence E_ν is graded by the Morse index of the critical points.

Let $E_\nu^\perp$ be the orthogonal complement in $L^2\Omega^\bullet$ of E_ν, so that $L^2\Omega^\bullet = E_\nu\oplus E_\nu^\perp$. Relative to this splitting we can decompose the Witten-Dirac operator as follows

$$D_\nu = \begin{pmatrix} D_{\nu,11} & D_{\nu,12} \\ D_{\nu,21} & D_{\nu,22}\end{pmatrix} : E_\nu\oplus E_\nu^\perp \to E_\nu\oplus E_\nu^\perp \tag{1.3.4}$$

where, if $p_1\colon L^2\Omega^\bullet\to E_\nu$ and $p_2\colon L^2\Omega^\bullet\to E_\nu^\perp$ are the orthogonal projections, $D_{\nu,ij} = p_j\circ D_\nu\circ p_i$.

Remark 1.30. We have an explicit formula for p_1. Indeed if $\theta\in L^2\Omega^\bullet$, since we know explicitly the generators of E_ν, it is clear that

$$p_1\theta = \sum_{p\in\operatorname{Crit} f} \rho_{p,\nu}\left(\rho_{p,\nu},\theta\right) \tag{1.3.5}$$

The following norm estimates are the core of the proof. They show that some sort of dimensional reduction is going on, as $\nu\to 0$.

PROPOSITION 1.31. *We have the following estimates on $D_{\nu,ij}$:*

1. *For any $\nu>0$, $D_{\nu,11}=0$.*

2. *There exists a* $\nu_1 > 0$ *such that for any* $\theta \in E_\nu^\perp \cap W^{1,2}\Omega^\bullet$ *and any* $\theta' \in E_\nu$,

$$\begin{aligned} ||D_{\nu,12}\theta|| &\leq \nu||\theta|| \\ ||D_{\nu,21}\theta'|| &\leq \nu||\theta'|| \end{aligned} \tag{1.3.6}$$

for all $\nu < \nu_1$.

3. *There exists a* $\nu_2 > 0$ *and* $C > 0$ *such that for any* $\theta \in E_\nu^\perp \cap W^{1,2}\Omega^\bullet$

$$||D_\nu\theta|| \geq \frac{C}{\sqrt{\nu}}||\theta|| \tag{1.3.7}$$

for all $\nu < \nu_2$.

Proof. 1. We have $D_{\nu,11}\theta = p_1 D_\nu p_1\theta$. But $p_1\theta = \sum_{p\in\mathrm{Crit} f}\rho_{p,\nu}\left(\rho_{p,\nu},\theta\right)$, and since the inner product term is comprised of real numbers, all we have to consider is $D_\nu\rho_{p,\nu}$. Now, $D_\nu\rho_{\nu,p} \in \Omega^{\mathrm{ind}\,p-1}(M)\oplus\Omega^{\mathrm{ind}\,p+1}$ and it has compact support completely contained in U_p. But then $p_1 D_\nu p_1\theta = 0$.

2. We have that $D_{\nu,21}$ is the adjoint of $D_{\nu,12}$ so all we have to do is prove the first inequality. Now, since $\rho_{p,\nu}$ is compactly supported in U_p, by combining the explicit expression of $\rho_{p,\nu}$ and $D_\nu^2 = \Delta_\nu$ given by Proposition 1.25, for any $\theta \in E_\nu^\perp \cap W^{1,2}\Omega^\bullet$,

$$\begin{aligned} D_{\nu,12}\theta &= \sum_{p\in\mathrm{Crit} f}\rho_{p,\nu}\int_W \left\langle \rho_{p,\nu}, D_\nu\theta\right\rangle dy^1\wedge\cdots\wedge dy^n = \\ &= \sum_{p\in\mathrm{Crit} f}\rho_{p,\nu}\int_W \left\langle D_\nu\rho_{p,\nu},\theta\right\rangle dy^1\wedge\cdots\wedge dy^n = \\ &= \sum_{p\in\mathrm{Crit} f}\rho_{p,\nu}\int_W \left\langle D_\nu\left(\frac{\gamma(|y|)}{\sqrt{\alpha_{p,\nu}}}\exp\left(-\frac{|y|^2}{2\nu}\right)dy^1\wedge\cdots\wedge dy^{\mathrm{ind}\,p}\right),\theta\right\rangle d\mathrm{vol} \\ &= \sum_{p\in\mathrm{Crit} f}\rho_{p,\nu}\int_W \left\langle \frac{c(d\gamma(|y|))}{\sqrt{\alpha_{p,\nu}}}\exp\left(-\frac{|y|^2}{2\nu}\right)dy^1\wedge\cdots\wedge dy^{\mathrm{ind}\,p},\theta\right\rangle d\mathrm{vol} \end{aligned} \tag{1.3.8}$$

By construction, since $\gamma(|y|)$ is a bump function, $d\gamma$ vanishes in a small ball around p. Hence there exist $\nu_0 > 0, C_1 > 0$ and $C_2 > 0$ such that whenever $\nu < \nu_0$,

$$||D_{\nu,12}\theta|| \leq \frac{C_1}{\nu^{n/2}}e^{-\frac{C_2}{\nu}}||\theta|| \tag{1.3.9}$$

from which the inequality follows.

3. See [53, Proposition 4.12]. □

COROLLARY 1.32. *The operator* $D_{\nu,22}\colon E_\nu^\perp \cap W^{1,2}\Omega^\bullet \to E_\nu^\perp$ *is invertible.*

Proof. It suffices to show that there exists a $\nu_3 > 0$ and a $C > 0$ such that for every $\theta \in E_\nu^\perp \cap W^{1,2}\Omega^\bullet$ and any $\nu < \nu_3$,

$$||D_{\nu,22}\theta|| \geq C||\theta|| \tag{1.3.10}$$

But notice that on $E_\nu^\perp \cap W^{1,2}\Omega^\bullet$, we have $D_\nu = D_{\nu,12} + D_{\nu,22}$ and by inequalities (1.3.6, 1.3.7) this lets us conclude. □

COROLLARY 1.33. *The operator* $D_\nu(u) = D_{\nu,11} + D_{\nu,22} + u\left(D_{\nu,12} + D_{\nu,21}\right) : W^{1,2}\Omega^\bullet \to L^2\Omega^\bullet$ *is Fredholm for all* $0 \leq u \leq 1$.

Proof. The inequalities (1.3.6) show that $D_{\nu,12}$ and $D_{\nu,21}$ are compact operators. Hence combining this with inequality (1.3.7), we conclude. □

Remark 1.34. This last result is interesting in itself, because by the standard index theory of elliptic operators, the *index* $\operatorname{ind} D_\nu$ of D_ν is the Euler-Poincaré characteristic $\chi(M)$ of M – see Remark A.14 – so by homotopy invariance of the index of Fredholm operators, we see that it is also the index of $D_\nu(0) = D_{\nu,11}$. But $D_{\nu,11}$ is the zero operator which has full kernel E_ν, and we also know that $E_\nu \cap \Omega^k(M)$ has dimension the number of critical points of index k (Remark 1.29), so we obtain

$$\operatorname{ind} D_\nu = \operatorname{ind} D_{\nu,11} = \sum_{j \text{ even}} \dim\ker D_{\nu,11}|_{\Omega^j} - \sum_{j \text{ odd}} \dim\ker D_{\nu,11}|_{\Omega^j} = \sum_{k=1}^{n} (-1)^k \sharp\left(\operatorname{Crit}^k f\right) \tag{1.3.11}$$

We found out that the index of the Witten Dirac operator $\operatorname{ind} D_\nu$ ties a "length formula" for the total number of critical points of f to the "length formula" of the real cohomology of the manifold, which just gives $\chi(M)$. This is a result that hints towards the homological Morse theory of the next chapter.

1.3.2. Proof of the main theorem

Now, let $E_{\nu,\lambda}$ denote the eigenspace of D_ν in $L^2\Omega^\bullet$ with eigenvalue λ, and for $\varepsilon > 0$, define

$$E_\nu(\varepsilon) = \bigoplus_{\lambda \in [-\varepsilon,\varepsilon]} E_{\nu,\lambda} \tag{1.3.12}$$

with the relative orthogonal projector $p_\nu(\varepsilon) \colon L^2\Omega^\bullet \to E_\nu(\varepsilon)$. This is a finite dimensional subspace, since there can only be finitely many eigenvalues in $[-\varepsilon, \varepsilon]$.

The following result is telling us that the heat forms can be thought as *approximate eigenforms* for the Witten Dirac operator, and gives a crude estimate on how good such approximation is.

LEMMA 1.35. *There exists a* $\nu_3 > 0$ *and a* $C_1 > 0$ *such that for any* $\sigma \in E_\nu$

$$||p_\nu(\varepsilon)\sigma - \sigma|| \leq C_1 \nu ||\sigma|| \tag{1.3.13}$$

for all $\nu < \nu_3$.

Proof. We want to use the spectral decomposition theorem for self-adjoint operators on Hilbert spaces on D_ν, but first we must show that its resolvent is well defined.

Let $S^1_\varepsilon = \{\lambda \in \mathbb{C} : |\lambda| = \varepsilon\}$ be the counter-clockwise oriented circle in $\mathbb{C}$ of radius ε. Let $\lambda \in S^1_\varepsilon$ and $0 < \nu < \nu_1 + \nu_2$, where ν_1 and ν_2 are the constants entering in Proposition 1.31. Then by the inequalities of Proposition 1.31, for any $\theta \in W^{1,2}\Omega^\bullet$,

$$\begin{aligned} ||(\lambda - D_\nu)\theta|| &\geq \frac{1}{2}\left\|\lambda p_1\theta - D_{\nu,12}\theta\right\| + \frac{1}{2}\left\|\lambda p_2\theta - D_{\nu,21}\theta - D_{\nu,22}\theta\right\| \geq \\ &\geq \frac{1}{2}\left[(\varepsilon-\nu)||p_1\theta|| + \left(\frac{C}{\sqrt{\nu}} - \varepsilon - \nu\right)||p_2\theta||\right] \end{aligned} \tag{1.3.14}$$

Hence there exists a $\nu_4 < \nu_1 + \nu_2$ and a $C_2 > 0$ such that for any $\theta \in W^{1,2}\Omega^\bullet$,

$$||(\lambda - D_\nu)\theta|| \geq C_2||\theta|| \tag{1.3.15}$$

for all $\nu < \nu_4$. Hence the operator $\lambda - D_\nu$ is invertible for small ν and for all $\lambda \in S^1_\varepsilon$, so that the resolvent $(\lambda - D_\nu)^{-1}$ is well defined. So by the standard spectral theorem for bounded self-adjoint operators on Hilbert spaces,

$$p_\nu(\varepsilon)\sigma - \sigma = \frac{1}{2\pi i}\int_{S^1_\varepsilon}\left[(\lambda - D_\nu)^{-1} - \lambda^{-1}\right]\sigma\, d\lambda \tag{1.3.16}$$

Using the splitting (1.3.4) and the fact that $D_{\nu,11} = 0$, we have

$$\left[(\lambda - D_\nu)^{-1} - \lambda^{-1}\right]\sigma = \lambda^{-1}[\lambda - D_\nu]^{-1}D_{\nu,21}\sigma \tag{1.3.17}$$

Finally using the estimate (1.3.6) for $D_{\nu,21}$ and the estimate (1.3.15), we have the estimate

$$\left\|\lambda^{-1}[\lambda - D_\nu]^{-1}D_{\nu,21}\sigma\right\| \leq C_2^{-1}\left\|D_{\nu,21}\sigma\right\| \leq C_2^{-1}\nu||\sigma|| \tag{1.3.18}$$

Combining (1.3.16) with (1.3.18) we conclude. □

We are ready for the proof of Theorem 1.27

Proof of Theorem 1.27. First of all, apply Lemma 1.35 to $\sigma = \rho_{p,\nu}$. When ν is small enough, it implies that the forms $p_\nu(\varepsilon)\rho_{p,\nu}$ for $p \in \operatorname{Crit} f$ are linearly independent. Hence there exists a $\nu_5 > 0$ such that for all $\nu < \nu_5$, $\dim E_\nu(\varepsilon) \geq \dim E_\nu$. Now, if the inequality were strict, there would exist at least a non-zero $\sigma \in E_\nu(\varepsilon)$ such that $(\sigma, \theta) = 0$ for all $\theta \in p_\nu(\varepsilon)(E_\nu)$. Equivalently there would be at least a non-zero $\sigma \in E_\nu(\varepsilon)$ such that

$$\left(\sigma, p_\nu(\varepsilon)\rho_{p,\nu}\right) = 0 \quad \forall p \in \operatorname{Crit} f \tag{1.3.19}$$

Now we argue by contradiction. Let σ be such a section. Then by combining equation

(1.3.5) and (1.3.19),

$$
\begin{aligned}
p_1\sigma &= \sum_{p\in\operatorname{Crit} f}\left(\sigma,\rho_{p,\nu}\right)\rho_{p,\nu} = \\
&= \sum_{p\in\operatorname{Crit} f}\left(\sigma,\rho_{p,\nu}\right)\rho_{p,\nu} - \sum_{p\in\operatorname{Crit} f}\left(\sigma, p_\nu(\varepsilon)\rho_{p,\nu}\right)p_\nu(\varepsilon)\rho_{p,\nu} = \\
&= \sum_{p\in\operatorname{Crit} f}\left(\sigma,\rho_{p,\nu}\right)\left[\rho_{p,\nu} - p_\nu(\varepsilon)\rho_{p,\nu}\right] + \sum_{p\in\operatorname{Crit} f}\left(\sigma,\rho_{p,\nu} - p_\nu(\varepsilon)\rho_{p,\nu}\right)p_\nu(\varepsilon)\rho_{p,\nu}
\end{aligned}
\tag{1.3.20}
$$

Applying Lemma 1.35 to the previous equation, we have that for all $\nu < \min\{\nu_3, \nu_5\}$

$$
||p_\nu(\varepsilon)\sigma|| \le C_1\nu||\sigma|| \implies ||p_2\sigma|| \ge ||\sigma|| - ||p_1\sigma|| \ge C_4||\sigma|| \tag{1.3.21}
$$

for some constant $C_4 > 0$. Now, using Proposition 1.31,

$$
\begin{aligned}
\frac{CC_4}{\sqrt{\nu}}||\sigma|| &\le ||D_\nu p_2\sigma|| = ||D_\nu\sigma - D_\nu p_1\sigma|| = ||D_\nu\sigma - D_{\nu,21}\sigma|| \le \\
&\le ||D_\nu\sigma|| + ||D_{\nu,21}\sigma|| \le ||D_\nu\sigma|| + \nu||\sigma||
\end{aligned}
\tag{1.3.22}
$$

Rearranging

$$
||D_\nu\sigma|| \ge \frac{CC_4 - \nu\sqrt{\nu}}{\sqrt{\nu}}||\sigma|| \implies ||\sigma|| \le \frac{\sqrt{\nu}}{CC_4 - \nu\sqrt{\nu}}||D_\nu\sigma|| \to 0 \text{ as } \nu \to 0 \tag{1.3.23}
$$

contradicting that σ is a nonzero element of $E_\nu(\varepsilon)$ for all ν.

We have shown that

$$
\dim E_\nu(\varepsilon) = \dim E_\nu = \sum_{k\in\mathbb{N}} \sharp\left(\operatorname{Crit}^k f\right) \tag{1.3.24}
$$

Denote by $q_k : L^2\Omega^\bullet \to L^2\Omega^k$ the projection from $L^2\Omega^\bullet$ to the Hilbert subspace $L^2\Omega^k$ of L^2 k-forms. Since Δ_ν respects the degree of the forms, if $\theta \in E_{\nu,\lambda}$, then

$$
\Delta_\nu q_k\theta = q_k D_\nu^2\theta = \lambda^2 q_k\theta \tag{1.3.25}
$$

so that if $\theta \in E_{\nu,\lambda}$, then $q_k\theta$ is a k-eigenform of Δ_ν of eigenvalue λ^2. In order to finish the proof, we must show that $\dim q_k E_\nu(\varepsilon) = \sharp\left(\operatorname{Crit}^k f\right)$.

By Lemma 1.35, if $p \in \operatorname{Crit}^k f$,

$$
\left\|q_k p_\nu(\varepsilon)\rho_{p,\nu} - \rho_{p,\nu}\right\| \le C_1\nu\left\|\rho_{p,\nu}\right\| \tag{1.3.26}
$$

Since $\rho_{p,\nu}$ and $\rho_{q,\nu}$ are linearly independent when $q \ne p$, by the previous inequality we conclude that also $q_k p_\nu(\varepsilon)\rho_{p,\nu}$ and $q_h p_\nu(\varepsilon)\rho_{q,\nu}$ are independent forms, for small enough ν. But this implies that $\dim q_k E_\nu(\varepsilon) \ge \sharp(\operatorname{Crit}^k f)$ for $0 \le k \le n$. Combined with the fact that $\dim E_\nu(\varepsilon)$ is the sum of such numbers (Remark 1.29), and $E_\nu(\varepsilon) = \bigoplus q_k E_\nu(\varepsilon)$, we conclude that $\dim E_\nu(\varepsilon) = \sharp(\operatorname{Crit}^k f)$. Finally since the eigenvalues of Δ_ν are the squares of the eigenvalues of D_ν with the same eigenforms, we conclude. $\square$

Remark 1.36. This theorem shows that the eigenvalues of Δ_ν are encoding dynamical information on f. In the next chapter, we will show how this dynamical information can be used to recover the topology of the underlying manifold. Hence the spectrum of the Witten Laplacian, just like any Laplacian, encodes topological information on the manifold. The difference with the usual Laplacian is that through the inequalities we showed in this section, we have gained control over the eigenvalues.

2. A homological approach to Morse theory

Morse theory is the study of the interplay between critical points of a real-valued function on a manifold and the topology of the manifold. It was fist formulated by Marston Morse in the early '30, but went under many stages of development, some of which are still being studied now. In this chapter we study a homological approach to Morse theory, the oriented Morse complex approach. The oriented Morse complex is a variant of the Morse complex which takes in account the orientations of the invariant submanifolds.

Introduction

The main goal of Morse theory is to find bounds on the number of critical points with fixed index of a Morse function on a compact manifold in terms of its Betti numbers. This bound is given in terms of the following

Morse inequalities. Let M be a closed Riemannian n-manifold and $f: M \to \mathbb{R}$ a Morse function. Denote by $\beta_k = \dim H^k(M;\mathbb{R})$ and $m_k = \sharp(\mathrm{Crit}^k f)$. There exists a polynomial $Q(t)$ with non-negative integer coefficients such that

$$\sum_{i=0}^{n} \beta_k t^k - \sum_{k=0}^{n} m_k t^k = (1+t)Q(t) \tag{2.0.1}$$

This equation is called the *strong Morse inequality*. For $t = -1$ the *weak Morse inequalities* follow:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k m_k \implies m_k \geq \beta_k \quad \forall k \tag{2.0.2}$$

The original topological approach to the Morse inequalities, devised by Morse himself [36] and called the *half-space* approach, uses handle-body decompositions of the manifold associated to the Morse function. Subsequent observations made by Thom [46], Milnor [33] and Smale [45] led to the acknowledgement that this kind of inequalities could be recovered by constructing a chain complex, quasi-isomorphic to the singular chain complex, whose building blocks are the critical points of fixed index. This observation gave rise to the many different homological approaches to Morse theory. In this section we develop one of these approaches.

2.1. Preliminaries on Morse-Smale functions

Definition 2.1. We say that a Morse function $f : M \to \mathbb{R}$ generates a *Morse-Smale negative gradient flow*, in short is a *Morse-Smale function*, if for every $p, q \in \operatorname{Crit} f$, the stable and unstable manifolds of its negative gradient flow, whenever they intersect, intersect transversally:

$$T_x M = T_x W^s(p) + T_x W^u(q) \quad \forall x \in W^s(p) \cap W^u(q) \tag{2.1.1}$$

LEMMA 2.2. *Let $p \in \operatorname{Crit}^k f$ and $q \in \operatorname{Crit}^h f$. Then $W^u(p) \cap W^s(q)$, if nonempty, is an invariant set which is the disjoint union of sumbanifolds of dimension $k - h$. Consequently if $h > k$, the intersection is empty.*

Proof. Both the stable and unstable manifolds are invariant, hence so is their intersection. Since the submanifolds are transverse, it is a standard application of the implicit function theorem that their intersection is a disjoint union of submanifolds of dimension $\dim W^u(p) + \dim W^s(q) - n = k + n - h - n = k - h$. □

COROLLARY 2.3. *If* $\operatorname{ind} p = \operatorname{ind} q + 1$ *then $W^u(p) \cap W^s(q)$ is comprised of a disjoint union of integral curves of $\nabla_g f$.*

Remark 2.4. By the stable manifold theorem [3, Theorem 7.2.9], since every critical point of f is a hyperbolic rest point for $-\nabla_g f$, the stable and unstable manifolds are images of an injective immersion of some $\mathbb{R}^k$. Hence they are orientable manifolds. Moreover, since their intersection is transverse, we have that $W^u(p)$ is the image of $\mathbb{R}^{\operatorname{ind} p}$ and $W^s(p)$ is the image of $\mathbb{R}^{n - \operatorname{ind} p}$.

Definition 2.5. Let $p \in \operatorname{Crit}^k f$ and $q \in \operatorname{Crit}^h f$. Define the *instanton set*

$$\hat{\mathfrak{I}}(p, q) = W^u(p) \cap W^s(q) \tag{2.1.2}$$

Being the transverse intersection of two orientable submanifolds, it is an orientable submanifold.

Remark 2.6. If $p \in \operatorname{Crit}^k f$ and $q \in \operatorname{Crit}^h f$ with $k < h$, then we have seen that $W^u(p) \cap W^s(q) = \varnothing$. If $k = h + 1$ then $\hat{\mathfrak{I}}(p, q) = \{\gamma \text{ integral curves} : \gamma_{-\infty} = p,\ \gamma_\infty = q\}$, a finite union of integral curves that mingle around p for infinite time, then whip towards q and mingle around it for another infinite amount of time. This is where the name "instanton" comes from.

The properties of Morse-Smale functions make them the main ingredient for our definition of the Morse complex. We already know that Morse functions are generic, so a natural question is whether we may generically choose a Riemannian metric which makes the gradient field of a differentiable function Morse-Smale. The answer has been settled in [43]

THEOREM 2.7 (Abundance of Morse-Smale functions). *Every real-valued differentiable function on M admits a sequence of Morse-Smale functions converging to it in the C^1 topology.*

So from now on, we fix a Morse-Smale function $f \in C^\infty(M)$.

2.1.1. *The compactified instanton moduli space*

In the construction of the Morse complex, the instanton set plays a key rôle in the definition of the boundary operator. To efficiently use it in the definition though, we need to "count" the gradient flow lines. Hence we can quotient out the action given by the gradient flow. This way the quotient set is comprised of unparametrized flow lines, which we can count.

Definition 2.8. Let $p \in \mathrm{Crit}^k f$ and $q \in \mathrm{Crit}^h f$ with $h < k$. Then $\hat{\mathfrak{I}}(p,q)$ carries a free action of $\mathbb{R}$ given by the reparametrization of the integral curves which foliate it. Define $\mathfrak{I}(p,q)$ as the quotient of $\hat{\mathfrak{I}}(p,q)$ by this action. We call $\mathfrak{I}(p,q)$ the *moduli space of instantons* between p and q.

PROPOSITION 2.9. *The instanton moduli space $\mathfrak{I}(p,q)$ is a smooth manifold with border, whose dimension is* $\operatorname{ind} p - \operatorname{ind} q - 1$.

Proof. This is obvious since the instanton set is a smooth manifold of dimension $\operatorname{ind} p - \operatorname{ind} q$ and the $\mathbb{R}$-action is free. □

Remark 2.10. In general, the instanton set $\hat{\mathfrak{I}}(p,q)$ might not be a closed set. Its closure might contain points which belong to an unstable manifold of a critical point of index intermediate between that of p and q, the so-called *broken flow lines* [4, Section 3.2]. Hence the moduli space $\mathfrak{I}(p,q)$ is not compact in general.

Definition 2.11. Let $p \in \mathrm{Crit}^k f$ and $q \in \mathrm{Crit}^h f$ with $h < k$. The *compact instanton moduli space* $\overline{\mathfrak{I}(p,q)}$ is the closure of $\mathfrak{I}(p,q)$:

$$\overline{\mathfrak{I}(p,q)} = \mathfrak{I}(p,q) \cup \bigcup_{\substack{p',q' \in \mathrm{Crit} f: \\ \operatorname{ind} q \leq \operatorname{ind} q' < \operatorname{ind} p' \leq \operatorname{ind} p}} \mathfrak{I}(p',q') \tag{2.1.3}$$

Only the critical points q', p' whose respective stable and unstable manifolds intersect contribute to the union. Moreover notice that this union is disjoint.

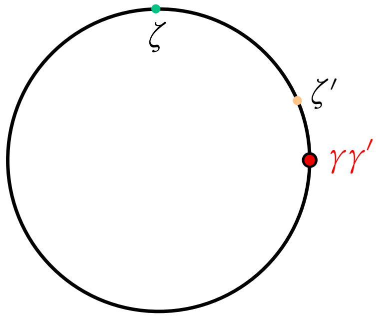
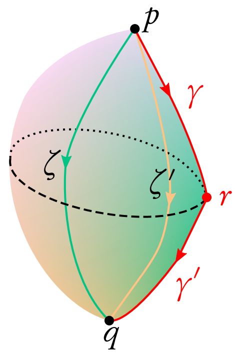
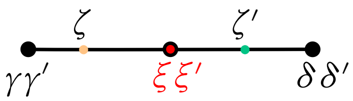
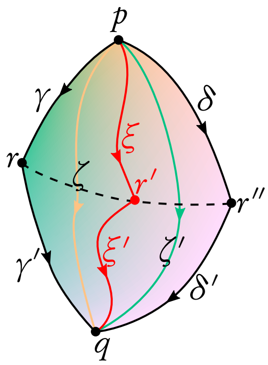
THEOREM 2.12 ([4, Theorem 3.2.2]). *The manifold $\overline{\mathfrak{I}(p,q)}$ is a compact manifold with corners*

Remark 2.13. A *manifold with corners* is a manifold whose boundary is itself a manifold with boundary of strictly lower dimension, whose boundary is a manifold with boundary of strictly lower dimension... until we exhaust the dimensions reaching 0.

COROLLARY 2.14. *The instantons connecting critical points of indices which differ by 1 come in a finite number.*

Proof. Indeed if $p, q \in \mathrm{Crit} f$ are such that $\operatorname{ind} p = \operatorname{ind} q + 1$, then the compact moduli space of instantons is a compact zero-dimensional space whose topology is induced by M. Such a space must be a finite set of points, otherwise the finite open subcover condition would not be satisfied. □

Table 2.1. – The topological structure of compactified instanton moduli spaces rules out the broken flow lines colored in red. The “large” points are points in the boundary, while the “smaller” points are examples of smooth instantons.

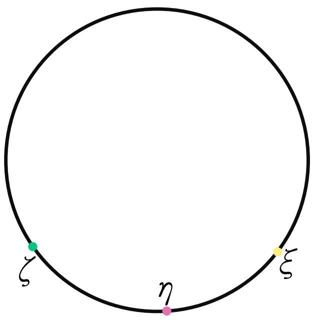
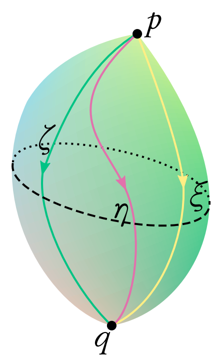
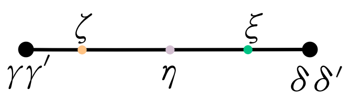
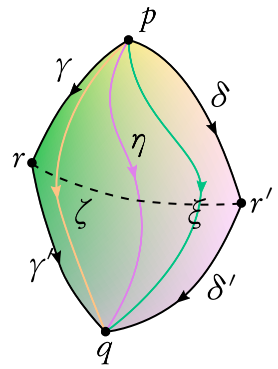
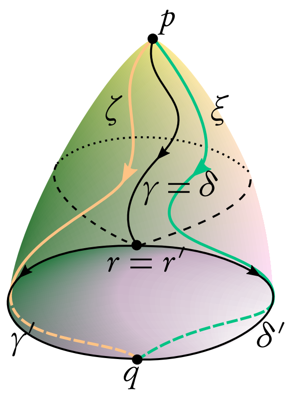
<table>
<tr><th>$\overline{\mathfrak{I}(p,q)}$</th><th>$\mathfrak{I}(p,q)$</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Corollary 2.15. *If* $\operatorname{ind} p = \operatorname{ind} q + 2$, *then* $\overline{\mathfrak{I}(p,q)}$ *is diffeomorphic to a* finite *disjoint union of circles and intervals.*

Proof. Indeed all one-dimensional manifolds with border are diffeomorphic to $[0,1]$. Moreover since the manifold must be compact, its connected components must be in a finite number. $\square$

Remark 2.16. The proof of the theorem actually shows something more strict [4, Proposition 3.2.8]. Precisely it says that every broken instanton admits an unique one-parameter family of legitimate flow lines converging to it. This rules out the *a priori* possible arrangements in Table 2.1, while it does not rule out the arrangements in Table 2.2.

Table 2.2. – Allowed arrangements of instantons. Even though in the compactified moduli space certain broken instantons are distinct, some of their branches might coincide, as for example in the last picture.

<table>
<tr><th>$\overline{\mathfrak{I}(p,q)}$</th><th>$\mathfrak{I}(p,q)$</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td rowspan="2">
</td><td>
</td></tr>
<tr><td>
</td></tr>
</table>

2.2. *Construction of the complex*

We are now ready to construct the oriented Morse complex. We will use the orientations of the unstable manifolds as a generating set for the degrees, following [2, Section A.3]. In this way, the degrees already carry the information on the orientations, which will be needed to define the boundary operator. Such choice will simplify the definition of the boundary operator. Compare with [4, Chapter 3]. As far as we know, this particular treatment is not present in the literature.

Definition 2.17. Let $p \in \operatorname{Crit}^k f$. Then $W^u(p)$ is an orientable submanifold of dimension k. Denote by o_p and $\hat{o}_p$ the two opposite orientations of $W^u(p)$. Define

$$C_k = \operatorname{span}_{\mathbb{Z}} \left\{ o_p : p \in \operatorname{Crit}^k f \right\} / \left\langle o_p + \hat{o}_p : p \in \operatorname{Crit}^k f \right\rangle \tag{2.2.1}$$

that is, the free abelian group generated on the orientations of the unstable submanifolds of dimension k, with the relation in which the two opposite orientations of the same unstable manifold are not seen as independent generators; one is the opposite of the other.

LEMMA 2.18. *C_k is a free abelian group of rank* $\operatorname{rk} C_k = \sharp(\operatorname{Crit}^k f)$.

Proof. The free abelian group generated by all orientations of critical points of index k has rank $2 \cdot \sharp(\operatorname{Crit}^k f)$, since it is generated by two orientations for each point. Then $\langle o_p + \hat{o}_p \rangle$ is free of rank $\sharp(\operatorname{Crit}^k f)$, meaning that we just kill half of the generators. The result is a free group of rank $\sharp(\operatorname{Crit}^k f)$. □

Construction. Consider a $p \in \operatorname{Crit}^k f$ and a $q \in \operatorname{Crit}^{k-1} f$. Then $\hat{\mathfrak{I}}(p,q)$ is a finite set of integral curves. If we fix an orientation o_p on $W^u(p)$ and we consider a $\gamma \in \hat{\mathfrak{I}}(p,q)$, then we may induce an orientation $\gamma_* o_p$ on $W^u(q)$ in the following way.

- First of all, notice that γ is itself an oriented submanifold, since we may orient it with the direction of the flow; namely, we say that the "positive" orientation of γ is the one in the direction $-\nabla_g f(\gamma_t)$. This works because the top exterior power of the tangent bundle to γ is the tangent bundle itself.
- Since $T_q W^u(q) \oplus T_q W^s(q) = T_q M$, a co-orientation of $W^s(q)$, i.e. an orientation of its normal bundle, is the same as an orientation of $W^u(q)$. Moreover, having supposed f to be Morse-Smale, $T_{\gamma_t} W^u(p)$ intersects transversly $T_{\gamma_t} W^s(q)$ for all t. It follows that we have the exact sequence

$$0 \to T_{\gamma_t} W^u(p) \cap T_{\gamma_t} W^s(q) \to T_{\gamma_t} W^u(p) \to T_{\gamma_t} M / T_{\gamma_t} W^s(q) \to 0 \tag{2.2.2}$$

  where the first (non-trivial) map is the inclusion and the following one the canonical projection.
- We have thus a "2 out of 3" principle: if two of the terms in the sequence above have a fixed orientation, then there is an unique orientation on the third which agrees with the exactness of the sequence.

- ⋄ Consequently we may define $\gamma_* o_p$ as the orientation on $W^u(q)$ which is precisely the co-orientation induced by o_p and the one on γ by this 2 out of 3 principle.

Definition 2.19. Define $\partial_k^C : C_k \to C_{k-1}$ on generators as

$$\partial_k o_p = \sum_{q \in \mathrm{Crit}^{k-1} f} \sum_{\gamma \in \mathfrak{I}(p,q)} \gamma_* o_p \tag{2.2.3}$$

THEOREM 2.20. *$\left(C_\bullet, \partial^C\right)$ is a chain complex.*

Ideas of the Proof. We have to show that $\partial_{k+1}^C \circ \partial_k^C = 0$. Consider $p \in \mathrm{Crit}^{k+1} f$. Being C_k free, we may write

$$\partial_{k+1}^C o_p = \sum_{q \in \mathrm{Crit}^{k} f} \sum_{\gamma \in \mathfrak{I}(p,q)} n_\gamma \left(o_p, o_q\right) o_q \tag{2.2.4}$$

for some $n_\gamma\left(o_p, o_q\right) \in \mathbb{Z}$. Now, since we imposed $\hat{o}_q = -o_q$,

$$n_\gamma\left(o_p, o_q\right) = \begin{cases} 1, & \gamma_* o_p = o_q \\ -1, & \gamma_* o_p = \hat{o}_q \end{cases} \tag{2.2.5}$$

This way

$$\partial_{k+1}^C \circ \partial_k^C o_p = \sum_{q \in \mathrm{Crit}^{k} f} \sum_{\gamma \in \mathfrak{I}(p,q)} \sum_{r \in \mathrm{Crit}^{k-1} f} \sum_{\zeta \in \mathfrak{I}(q,r)} n_\gamma\left(o_p, o_q\right) n_\zeta\left(o_q, o_r\right) o_r \tag{2.2.6}$$

The key of the proof is understanding what the number $n_\gamma\left(o_p, o_q\right) n_\zeta\left(o_q, o_r\right)$ represents as $r \in \mathrm{Crit}^{k-1} f$. So fix $r \in \mathrm{Crit}^{k-1} f$ and consider two instantons, $\gamma \in \mathfrak{I}(p,q)$ and $\zeta \in \mathfrak{I}(q,r)$. First of all, notice that γ followed by ζ is a broken flow line connecting p to r. Hence it is a point in the boundary of $\overline{\mathfrak{I}(p,r)}$. But since $\mathrm{ind}\, p - \mathrm{ind}\, r = 2$, $\overline{\mathfrak{I}(p,r)}$ is the union of intervals and circles. Ergo the only contributions to the boundary are the boundaries of the intervals, which present themselves in pairs. So to conclude it suffices to show that the two broken flow lines in one connected component of $\overline{\mathfrak{I}(p,r)}$ induce opposite orientations at their endpoint r – because this way the contributions as the sums range over the instantons all cancel out.

The proof of this fact is not trivial. The strategy is to dynamically conjugate the local picture to a generic "toy model" for which this property is easily verified. The analytical details of such a construction is carried out completely in [1, 51]. Here we just state the main results involved in the proof.

First of all, consider a neighborhood U_q of q. Around q we have the local stable and unstable manifolds respectively of r and p. Embed in $W^u(p)$ a small disc D^u of dimension $k+1 = \mathrm{ind}\, p$, and in $W^s(r)$ a small disc D^s of dimension $k-1 = \mathrm{ind}\, r$. We can evolve these two discs, D^u forwards and D^s backwards in time, until they intersect the chart U_q and each other. Denote by D_t^u and D_{-t}^s the connected components of such

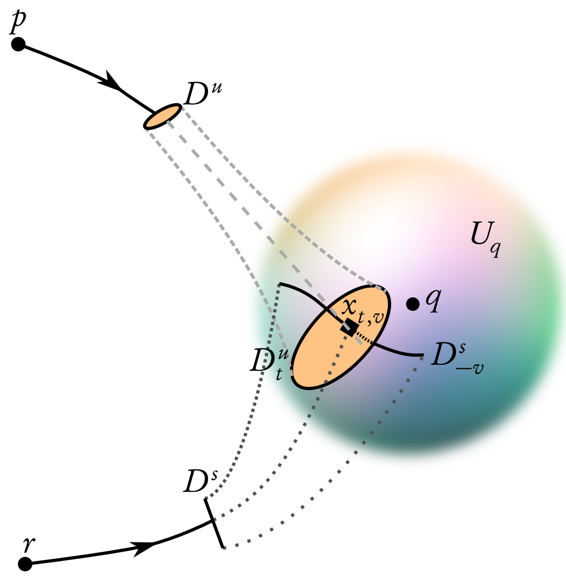


Figure 2.1. – Local picture around the mid-index critical point

evolved discs which intersect U_q and each other. It can be shown [51, Theorem 3.9] that for large enough $t, v \in (0, \infty)$, the intersection $D^u_t \cap D^s_{-v}$ consists of a single point $x_{t,v}$ (Figure 2.1). The assignment $(t, v) \mapsto x_{t,v} = h(t, v)$ is thus defined for (t, v) close to the "bottom-right corner" of the "square" $\mathbb{R} \cup \{\pm\infty\} \times \mathbb{R} \cup \{\pm\infty\}$. One may prolong this rule into an actual map $h : \mathbb{R}\cup\{\pm\infty\}\times\mathbb{R}\cup\{\pm\infty\} \to W$, which conjugates the negative gradient flow of f on the connected component W of $W^u(p) \cap W^s(r)$ in study to the *translation flow* $(\tau, (t, v)) \mapsto (t + \tau, v + \tau)$ on $\mathbb{R} \cup \{\pm\infty\} \times \mathbb{R} \cup \{\pm\infty\}$ [1, Theorem 9.1(*i*)]. Moreover, this map sends the broken instantons we are studying to the "corners" of the square [1, Theorem 9.1(*ii*)]. So the statement on orientations is reduced to the study of the degree of h, because the orientations induced by the translation flow on the "sides" are the obvious ones, and the sides are conjugated to the broken instantons in study. Our answer is in [1, Theorem 9.1(*iv*)]: the degree of h along the bottom-left corner is inverse to the degree along the upper-right corner. □

Remark 2.21. By using the *positive* gradient flow of f, we have that the instantons *raise* the index of the critical points – indeed, since for the positive gradient flow the stable/unstable spaces are the unstable/stable spaces of the negative gradient flow, we have that the dimension of the instanton moduli space is

$$\begin{aligned} \dim \mathfrak{I}(p, q) &= (\dim W^u(p) + \dim W^s(q) - n) - 1 = \\ &= n - \operatorname{ind} p + \operatorname{ind} q - n + 1 = \operatorname{ind} q - \operatorname{ind} p - 1 \end{aligned} \tag{2.2.7}$$

hence we have nonempty intersections only for critical points q such that $\operatorname{ind} q > \operatorname{ind} p$. We may repeat the same construction and obtain this time a cochain complex $(C^\bullet, d_C)$ where the groups are the same as above and the differential acts on generators as so

$$d^k_C o_p = \sum_{q \in \operatorname{Crit}^{k+1} f} \sum_{\gamma \in \mathfrak{I}(p,q)} \gamma_* o_p \tag{2.2.8}$$

2.3. Morse theory from the oriented Morse complex

In this subsection we define a cellular filtration of M, associated to our Morse-Smale function, and study its cellular complex. We then show that the oriented Morse complex is isomorphic to this cellular chain complex. This way, since cellular homology computes singular homology, we have gained a homological approach to proving the Morse inequalities.

2.3.1. Basic facts on cellular homology

A *cellular filtration* of a topological space X is a sequence $\{W_k\}_{k\in\mathbb{Z}}$ of subspaces of X such that

1. For every $k\in\mathbb{Z}$, $W_k\subset W_{k+1}$ and $\bigcup_{k\in\mathbb{Z}} W_k = X$. In short, $\{W_k\}_{k\in\mathbb{Z}}$ is a *filtration* of X.
2. Any singular simplex $\sigma:\Delta^n\to X$ is contained in some W_k.
3. If $p\neq k$ then $H_p\left(W_k, W_{k-1};\mathbb{Z}\right)=0$.

Let X be a topological space with a cellular filtration $\{W_k\}_{k\in\mathbb{Z}}$. By writing out the long relative exact sequences in homology associated to the pairs (W_k, W_{k-1}) and (W_{k-1}, W_k), we find

$$\begin{array}{ccccccc} \cdots \longrightarrow H_k\left(W_k, W_{k-1}\right) & \xrightarrow{\delta_k} & H_{k-1}\left(W_{k-1}\right) & \longrightarrow \cdots & & & \\ & & \| & & & & \\ & \cdots \longrightarrow & H_{k-1}\left(W_{k-1}\right) & \xrightarrow{i_*} & H_{k-1}\left(W_{k-1}, W_{k-2}\right) & \longrightarrow \cdots & \end{array} \tag{2.3.1}$$

where i_* is the morphism induced by the inclusion and δ is the connecting morphism making homology into a homological δ-functor. Composing these we have a morphism

$$\partial_k^E = i_*\circ\delta_k : H_k\left(W_k, W_{k-1}\right)\to H_{k-1}\left(W_{k-1}, W_{k-2}\right) \tag{2.3.2}$$

Moreover by exactness of the sequences above, $\partial_{k+1}^E\circ\partial_k^E = 0$. Hence setting $E_k = H_k\left(W_k, W_{k-1}\right)$, we have that $\left(E_\bullet,\partial^E\right)$ is a chain complex. The homology of this complex can be shown to be [25, Section 2.2]

$$H_l\left(E_\bullet,\partial^E\right)\cong H_l\left(X, W_{-1}\right) \tag{2.3.3}$$

In the case that interests us the cellular filtration is indexed by $\mathbb{N}$ and we set $W_{-1}=\varnothing$, so that the cellular homology computes precisely the singular homology of X with integer coefficients.

2.3.2. The Thom-Smale cellular filtration

Recall that $f: M\to\mathbb{R}$ is a Morse-Smale function and $\varphi:\mathbb{R}\times M\to M$ denotes its negative gradient flow.

Definition 2.22. Consider $p \in \operatorname{Crit} f$. Denote by $U_p(\varepsilon)$ an embedded open ball of dimension n and of radius ε, centered at p. For small enough ε we may always find such an embedded ball. Define

$$W_k = \bigcup_{\substack{p \in \operatorname{Crit} k \\ \operatorname{ind} p \leq k}} \varphi\Big([0,\infty) \times U_p(\varepsilon_p)\Big) \tag{2.3.4}$$

for some small positive numbers $\{\varepsilon_p : p \in \operatorname{Crit} f\}$ to be determined, and $W_{-1} = \varnothing$. We call $\{W_k\}$ the *Thom-Smale filtration* of M relative to f and $\{\varepsilon_p\}$.

Remark 2.23. Notice that since the rest points are all hyperbolic, W_k is an "infinitesimal thickening" of the unstable manifold. Indeed the ball will be squashed by φ^t in the stable direction and stretched in the unstable direction as $t \to \infty$. This way, by taking a small enough $\varepsilon > 0$ and $\varepsilon_p < \varepsilon$ for all p, the thickened unstable spaces relative to critical points of index k will not intersect any thickened unstable space relative to points of index greater than k:

LEMMA 2.24. *There exists an $\varepsilon > 0$ for which the following property is satisfied: If $\varepsilon_p < \varepsilon$ for all $p \in \operatorname{Crit} f$, then*

$$\varphi\Big([0,\infty) \times U_p\big(\varepsilon_p\big)\Big) \cap U_q\big(\varepsilon_q\big) = \varnothing \quad \forall q \in \operatorname{Crit} f : \operatorname{ind} q \geq \operatorname{ind} p \tag{2.3.5}$$

Proof. Assume by contradiction that for all $n \in \mathbb{N}$ there exist $p_n, q_n \in \operatorname{Crit} f$ with $\operatorname{ind} q_n \geq \operatorname{ind} p_n$ for which

$$\varphi\left([0,\infty) \times U_{p_n}\left(\frac{1}{n}\right)\right) \cap U_{q_n}\left(\frac{1}{n}\right) \neq \varnothing \tag{2.3.6}$$

Since M is compact, the critical points of f are finite, so we may assume $p_n = p$ and $q_n = q$ for all $n \in \mathbb{N}$, where $\operatorname{ind} q \geq \operatorname{ind} p$. But any point in $U_p(1/n)$, when evolved by φ^t, must approach some critical point of f. So what we are saying is that there is at least a flow line that is converging to some instanton, maybe broken, connecting p to q. But since $\operatorname{ind} p \leq \operatorname{ind} q$, by the Morse-Smale property there cannot be any such instanton. Hence the intersection must be empty. □

This property guarantees we may use the additivity axiom of singular homology to show that the Thom-Smale filtration is a cellular filtration.

THEOREM 2.25. *The Thom-Smale filtration is a cellular filtration of M.*

Proof. We must verify the three conditions in the definition of a cellular filtration.

1. Consider the cell W_{k+1}. Since the union in its definition is done over the critical points with indices *less or equal* to $k+1$, clearly it must contain W_k. Moreover, since any point in M is either a rest point of the gradient flow, or evolves towards a rest point by the gradient flow, the union of all cells is M. So $W_\bullet$ is a filtration.

2. Since M is foliated by the unstable manifolds, this filtration is actually an open cover. So any singular simplex must be contained in one of the cells.
3. We must compute $H_l\left(W_k, W_{k-1}\right)$. Now, W_k is the union of W_{k-1} and the thickening
$$V_k = \bigcup_{p\in\mathrm{Crit}^k f} \varphi\left([0,\infty)\times U_p\left(\varepsilon_p\right)\right) = \bigcup_{p\in\mathrm{Crit}^k f} V_k^p \tag{2.3.7}$$
so by excision we have
$$H_l\left(W_k, W_{k-1}\right) \cong H_l\left(V_k, W_{k-1}\cap V_k\right) \tag{2.3.8}$$
But by Lemma 2.24, if we pick all the radii ε_p of the balls smaller than a certain small $\varepsilon > 0$, we have that the union in equation (2.3.7) is actually a *disjoint* union, so
$$H_l\left(W_k, W_{k-1}\right) \cong H_l\left(V_k, W_{k-1}\cap V_k\right) \cong \bigoplus_{p\in\mathrm{Crit}^k f} H_l\left(V_k^p, W_{k-1}\cap V_k^p\right) \tag{2.3.9}$$
We are finished if we prove that the couple $(V_k^p, W_{k-1}\cap V_k^p)$ is homotopy equivalent to the couple $(B_0^k(1), S^{k-1})$, where $B_x^k(r)$ is the k-dimensional ball centered at x of radius r, because then
$$H_l\left(W_k, W_{k-1}\right) \cong \begin{cases} 0, & l\neq k \\ \mathbb{Z}^{\sharp\left(\mathrm{Crit}^k f\right)}, & l=k \end{cases} \tag{2.3.10}$$
This follows from the topological properties of the stable and unstable manifolds of a hyperbolic rest point. Namely, we have the chain of homotopy equivalences
$$\left(V_k^p, V_k^p\cap W_{k-1}\right) \sim \left(W^u(p)\times W^s(p), \partial W^u(p)\times W^s(p)\right) \sim \left(B_0^k(1), \partial B_0^k(1)\right) \tag{2.3.11}$$
where we have retracted the stable manifold – it's homeomorphic to a ball of dimension $n-k$, which is contractible. □

Remark 2.26. Notice that with equation (2.3.10) we have shown that the cellular homology in degree k, E_k, is isomorphic to the free abelian group over the critical points of index k. To regain a geometrical realization of this complex, we may think of it as equivalently generated by embedded discs D_p^k of dimension k, centered at $p\in\mathrm{Crit}^k f$ – these can be taken as the unstable manifolds $W^u(p)$. The problem with this geometrical realization is that it is not in general a CW complex.

COROLLARY 2.27. $H_l\left(E_\bullet, \partial^E\right) \cong H_l(M;\mathbb{Z})$

Remark 2.28. The Thom-Smale cellular filtration gives an explicit way to find handlebody decompositions of our manifolds, but where the attaching maps are in the smooth category, being defined dynamically. This allows an explicit inductive procedure to determine the diffeomorphism type of the manifold. For this reason, it has been of central importance in the works of Bott on the periodicity theorem [13] and Milnor on the h-cobordism theorem [33].

2.3.3. *An explicit isomorphism of homology theories*

Finally we are ready to prove the

THEOREM 2.29. $\left(C_\bullet, \partial^C\right) \cong \left(E_\bullet, \partial^E\right)$ *in the category of chain complexes.*

Proof. A bijection between generating sets is just sending each orientation o_p of the unstable manifold $W^u(p)$ to its critical point p, or equivalently to the disc D_p^k. Since we are working with free groups this suffices to define an isomorphism between all the degrees. So what we have to prove is that this isomorphism commutes with the boundary operators. We do this by showing that both boundary operators can be expressed in terms of *intersection numbers* [27, Section 5.2].

- In the case of the oriented Morse complex, the boundary operator can be expressed in terms of the numbers $n_\gamma\left(o_p, o_q\right)$, for $p \in \operatorname{Crit}^k f$ and $q \in \operatorname{Crit}^{k-1} f$ – defined in Theorem 2.20. These can not be directly interpreted as intersection numbers. But if we pick a number $\alpha \in (f(q), f(p))$ and we define $f^\alpha = f^{-1}(-\infty, \alpha]$, we may show that

$$\sum_{\gamma \in \mathfrak{I}(p,q)} n_\gamma\left(o_p, o_q\right) = \nu(W^u(p), W^s(p), f^\alpha) \tag{2.3.12}$$

  where ν is the intersection number of a family of transverse submanifolds (maybe with boundary) of M. Indeed, first of all notice that the intersection $W^u(p) \cap W^s(q) \cap f^\alpha$ is transverse – indeed the invariant submanifolds are parallel to $\nabla_g f$ while f^α is orthogonal to it – so that it is comprised of a finite number of points. Each of this points is the intersection of an instanton with f^α. We have found a bijection $\mathfrak{I}(p,q) \longleftrightarrow W^u(p) \cap W^s(q) \cap f^\alpha$. Now, the intersection number just counts the points of the intersection with a sign, where the sign is decided by assessing whether the orientation is respected or inverted at the intersection point. But the induced orientation by a $\gamma \in \mathfrak{I}(p,q)$ is precisely this number.
- This is the difficult part of the theorem, and we'll need a technical lemma proved in Appendix B.2.

  First of all, let ω_{n+1} be the standard orientation of $\mathbb{R}^{n+1}$ and σ_n the induced orientation on S^n. Having chosen these two orientations, we have that the boundary morphism $\delta'_{n+1} \colon H_{n+1}(D^{n+1}, \partial D^{n+1}) \to H_n(S^n)$ of the long relative exact sequence is an isomorphism mapping ω_{n+1} to σ_n. Let $p \in \operatorname{Crit}^k f$. By naturality we have the commuting square

$$\begin{array}{ccc} H_k\left(D^k, \partial D^k\right) & \xrightarrow{\theta_*^p} & H_k\left(W_k, W_{k-1}\right) \\ \Big\downarrow{\scriptstyle \delta'_k} & & \Big\downarrow{\scriptstyle \delta_k} \\ H_{k-1}\left(\partial D^k\right) & \xrightarrow{\alpha_*^p} & H_{k-1}\left(W_{k-1}\right) \end{array} \tag{2.3.13}$$

  where $\theta^p \colon \left(D^k, \partial D^k\right) \to \left(W_k, W_{k-1}\right)$ is an orientation preserving continuous map sending the disc into a small k-disc centered at p and embedded in $W^u(p)$, and α^p is its restriction to the boundary. Since δ'_k maps ω_k to σ_{k-1} and θ^p is orientation preserving, the parallel morphism δ_k must send the generator $\theta^p_* \omega_k$ to the generator $\alpha^p_* \sigma_k$ of $H_{k-1}(W_{k-1})$, meaning that $\partial^E_k = i_* \circ \delta_k$ must send $\theta^p_* \omega_k$ to $i_* \alpha^p_* \sigma_{k-1}$.

Now, since the flow is Morse-Smale, a k-disc embedded in $W^u(p)$ intersects finitely many stable manifolds of critical points of strictly lower index, in finitely many one-dimensional submanifolds, so the boundary of a k-disc intersects them in finitely many points, meaning that

$$(\alpha^p)^{-1}\left(\bigcup_{q \in \mathrm{Crit}^{k-1} f} W^s(q)\right) = \{\xi_1, \ldots, \xi_h\} \tag{2.3.14}$$

is a set of finite points in ∂D^k. This way we may embed small disjoint $(k-1)$-discs $A_1, \ldots, A_h$ in $\partial D^k = S^{k-1}$ with centers in the points $\xi_1, \ldots, \xi_h$.

Notice that $\alpha^p : \partial D^k \to W_{k-1}$ is attaching the boundary of the k-disc $W^u(p)$ to W_{k-1}. In particular we do not know how it is attaching it to the lower cells, but we may modify it so that we do know. To do this, consider the entrance time in the cell W_{k-2}, $t_{W_{k-2}} : M \to \mathbb{R}$, and the characteristic function of $\partial D_k \setminus \bigcup \mathring{A}_j$, $\chi : \partial D_k \to \{0, 1\}$. If $b^p : \partial D^k \to \mathbb{R}$ is a continuous function such that $b^p > \chi\left(t_{W_{k-2}} \circ \alpha^p\right)$, then α^p is homotopic to

$$\begin{aligned} \beta^p : \partial D^k &\to W_{k-1} \\ \xi &\mapsto \varphi^{b(\xi)}(\alpha(\xi)) \end{aligned} \tag{2.3.15}$$

Pictorially we're pushing inside W_{k-2} the part of the boundary of $W^u(p)$ which will eventually enter in W_{k-2}. This is legitimate because of the definition of the cells W_l in terms of the flow, and because the unstable manifolds are homotopic to discs. Moreover, since the two maps are homotopic, $\alpha^p_* = \beta^p_*$ on homology, so that $\theta^p_* \omega_k = i_* \beta^p_* \sigma_{k-1}$ as found above.

Using β^p we are in the hypotheses of Proposition B.11: defining

$$a^l : \left(D^{k-1}, \partial D^{k-1}\right) \to \left(\partial D^k, \partial D^k \setminus \bigcup_{j=1}^{h} \mathring{A}_j\right) \tag{2.3.16}$$

an orientation preserving homeomorphism sending the $(k-1)$-disc into A_l and $\hat{\imath} : \partial D^k \to \left(\partial D^k, \partial D^k \setminus \bigcup_{j=1}^{h} \mathring{A}_j\right)$ is the inclusion, it holds that

$$\hat{\imath}_* \sigma_{k-1} = \sum_{l=1}^{h} a^l_* \omega_{k-1} \tag{2.3.17}$$

from which it follows that

$$\partial^E_k \theta^p_* \omega_k = i_* \beta^p_* \sigma_{k-1} = \hat{\imath}_* \sigma_{k-1} = \sum_{l=1}^{h} a^l_* \omega_{k-1} = \sum_{l=1}^{h} a^l_* i_* \beta_* \sigma_{k-1} \tag{2.3.18}$$

Finally, recall how we defined the A_l: they are $(k-1)$-discs embedded in the sphere ∂D^k at the points which represent the intersection of $\alpha^p\left(\partial D^k\right)$ with the stable manifolds of critical points of index $(k-1)$. As such,

$$a_*^l \omega_{k-1} = \nu\left(W^u(p), W^s(q_l), \alpha^p\left(\partial D^k\right)\right) \theta_*^{q_l} \omega_{k-1} \tag{2.3.19}$$

where $q_l \in \operatorname{Crit}^{k-1} f$ is the point for which $(\alpha^p)^{-1}\left(W^u(p) \cap W^s(q) \cap \alpha^p\left(\partial D^k\right)\right) = \xi_l$. Combining equation (2.3.18) with equation (2.3.19) gives an expression of the boundary morphism in cellular homology in terms of explicit generators of $H_{k-1}\left(W_{k-1}, W_{k-2}\right)$ and intersection numbers.

We are ready to conclude, because the intersection numbers appearing in equation (2.3.19) are exactly the same as the intersection numbers appearing in equation (2.3.12). □

COROLLARY 2.30 (Morse inequalities). $H_l\left(C_\bullet, \partial^C\right) \cong H_l(M; \mathbb{Z})$

Remark 2.31. We could've skipped the proof of Theorem 2.20 since it is implied by Theorem 2.29.

3. Harvey & Lawson's Morse theory

In this chapter we study the existence of a certain type of limit associated with a suitable dynamical system. Apparently unrelated, in the case of a Morse-Smale gradient flow this limit is shown to encode the Morse theory of the manifold in the complex of deRham currents, and can be computed in terms of the integration-currents associated to stable and unstable manifolds. This theory is known as Harvey-Lawson theory. It shows in detail how the dynamics of Morse-Smale gradient flows are tied to the topology of the manifold.

Introduction and Motivation

When the orbits of a dynamical system have a complicated structure, and exponential separation of orbits occurs, it is hard to extract the large-time behaviour of the system simply by studying the flow acting on points. This problem is of high relevance in the mathematically sound approaches to thermodynamics and statistical mechanics, where one would like to show that the system is converging to some sort of thermal equilibrium. To extract information from the system, instead of working with points and their orbits, one may wish to study the pull-back – called *transfer operator* in this theory – of the dynamical system on geometric objects like functions or forms, the latter interpreted as correlation measures of multi-particle motions under the influence of the flow. In this case, the convergence of the limit is easier to show.

In the '80s and '90s, Bowen, Pollicott, Ruelle and others have devised an analytical technique to study such large-time limits [5, 14, 38, 40], based on the zeroes of a zeta function encoding the eigenvalues of the transfer operator associated to the dynamical system. Recently Baladi, Liverani, Tsujii and others have extended these techniques in the field of hyperbolic dynamics [7, 29, 47–50]. They refined the results of their predecessors – which were usually formulated, with great loss of information, using symbolic dynamics – by constructing appropriate anisotropic Sobolev spaces and consequently avoiding the symbolic dynamics.

In this chapter we study this sort of large-time limit for our Morse-Smale flows, from the geometric point of view of Harvey, Lawson and Minervini [21–23, 34]. It will turn out that the large-time limit exists in the space of currents, so in a distributional space, in accordance to the Sobolev spaces of the analytical counterpart. Then we will show how it relates to its Morse theory. In particular, it can be expressed as a chain in the oriented Morse complex.

3.1. Harvey-Lawson theory.

Let M be a manifold and $\varphi\colon \mathbb{R}\times M \to M$ a flow of diffeomorphisms.

Central question. For any k-form θ on M, consider the formal limit

$$\mathbf{P}(\theta) = \lim_{t\to\infty} (\varphi^t)^* \theta \tag{3.1.1}$$

When, and in which sense, does this limit exist?

We shall rely heavily on Appendix C.1. In particular, we will need the notion of current, its boundary and mass, and the homology theory associated with the complex of currents. For simplicity, we study the orientable case. The non-oriented case can be recovered by twisting the differential forms over the orientation bundle.

3.1.1. Finite volume flows and the current equations.

Let M be a closed, oriented Riemannian manifold of dimension n, and $\varphi\colon \mathbb{R}\times M \to M$ a flow generated by a vector field X. Let $\Delta \subset M\times M$ be the diagonal and $P_t = \{(\varphi^t(x), x) : x\in M\}$ the "reversed graph" of φ^t. Consider the "graph cylinder"

$$\mathscr{G}_t = \{(s, \varphi^s(x), x) : 0 \le s \le t,\ x\in M\} \subset \mathbb{R}\times M\times M \tag{3.1.2}$$

$\mathscr{G}_t$ is a smooth compact submanifold with boundary of $\mathbb{R}\times M\times M$, where the boundary is $\{0\}\times\Delta \cup \{t\}\times P_t$. This way $\mathscr{G}_t$ defines a current in $\mathbb{R}\times M\times M$ by integration, with boundary $\partial[\![\mathscr{G}_t]\!] = [\![\{0\}\times\Delta]\!] - [\![\{t\}\times P_t]\!]$. Let $\mathrm{pr}\colon \mathbb{R}\times M\times M \to M\times M$ be the projection and define $G_t = \mathrm{pr}_*[\![\mathscr{G}_t]\!]$ to be the push-forward current. Then since push-forward and boundary commute, we have the equation of currents

$$\partial G_t = [\![\Delta]\!] - [\![P_t]\!] \tag{3.1.3}$$

LEMMA 3.1. *Let $\Phi\colon \mathbb{R}\times M \to M\times M$ be the smooth map $(s,x)\mapsto(\varphi^s(x),x)$. Then $G_t = \Phi_*[\![[0,t]\times M]\!]$.*

Proof. Indeed $\mathscr{G}_t = [0,t]\times\Phi([0,t]\times M)$. Then $\mathrm{pr}(\mathscr{G}_t) = \Phi([0,t]\times M)$. So as currents $G_t = \mathrm{pr}_*[\![\mathscr{G}_t]\!] = \Phi_*[\![[0,t]\times M]\!]$. □

Remark 3.2. The map Φ is an immersion away from the rest points of φ^t, which are the zeroes $Z(X)\subset M$ of X. So Φ is an immersion on $\mathbb{R}\times(M\setminus Z(X))$.

Definition 3.3. The flow φ is said to be a *finite volume flow* if $[0,\infty)\times(M\setminus Z(X))$ has finite volume with respect to the Riemannian metric $\Phi^*(g\times g)$.

LEMMA 3.4. *If φ is the negative gradient flow of a Morse function, then it is of finite volume.*

Proof. $Z(-\nabla_g f) = \operatorname{Crit} f$ is a finite set of points so since M is compact and φ is complete, $\Phi([0,\infty)\times(M\setminus Z(X)))$ is a compact subset of $M\times M$, hence of finite volume. □

THEOREM 3.5. *Let φ be a finite volume flow and G_t be as above. Then the limits $P=\lim_{t\to\infty}[\![P_t]\!]$ and $G=\lim_{t\to\infty}G_t$ exist in mass and satisfy the equation*

$$\partial G = [\![\Delta]\!] - P \tag{3.1.4}$$

Proof. Define $G=\Phi_*[\![[0,\infty)\times M]\!]$. The mass of G_t and G are respectively the volume of $\Phi([0,t]\times M)$ and the volume of $\Phi([0,\infty)\times M])$ which are both finite by hypothesis. Hence $||G_t||_M$ is bounded as $t\to\infty$, which means that there is an increasing sequence of times t_k such that $G_{t_k}\xrightarrow{*}G$. Moreover clearly $||G_t||_M\to||G||_M$. So we have that $G_{t_k}\xrightarrow{\mathrm{M}}G$ as $k\to\infty$. Now, if we take any increasing sequence of times, we can repeat the argument and find a subsequence converging in mass to G, so every sequence admits a subsequence with a further subsequence converging in mass to the same limit. This means G_t converges in mass to G as $t\to\infty$. Combining this with equation (3.1.3) and the continuity of ∂, we conclude that $P_t\xrightarrow{\mathrm{M}}P$ where $P=[\![\{(\varphi^s(x),x): s\in[0,\infty),\ x\in M\}]\!]$. The equation (3.1.4) is now clear. □

3.1.2. Operator equations.

In this section we transform currents on products to operators from forms to currents. This is because our limit (3.1.1) is given in form of an operator $\mathbf{P}$ on the space of forms, and we will show that it lands generally in the space of currents.

Let M,N be closed manifolds of dimension n and n' respectively. Denote by π_M and π_N the projections from the product $N\times M$ to each of the factors.

Definition 3.6. Consider an l-current $K\in\mathcal{D}_l(N\times M)$. Then for any $k\le l$, K defines a linear operator $\mathbf{K}\colon \Omega^{n'-k}(N)\to\mathcal{D}_{l-k}(M)$ by the formula

$$\mathbf{K}(\theta)=(\pi_M)_*(K\llcorner\pi_N^*\theta)\quad\forall\theta\in\Omega^{n'-k}(N) \tag{3.1.5}$$

Explicitly, on an $(n-(l-k))$-form $\xi\in\Omega_c^k(M)$, we have

$$\mathbf{K}(\theta)\xi = K(\pi_N^*\theta\wedge\pi_M^*\xi) \tag{3.1.6}$$

When we have a linear continuous operator $\mathbf{G}\colon\Omega^{n'-k}(N)\to\mathcal{D}_{l-k}(M)$ which comes from a current $G\in\mathcal{D}_l(N\times M)$, we call G the *kernel* (in the sense of integrals) of $\mathbf{G}$.

LEMMA 3.7. *For any l-current K on the product, the operator $\mathbf{K}$ is linear and continuous when we endow the forms with the C^∞ topology and currents with the weak topology.*

Proof. Indeed $K,(\pi_M)_*,\pi_N^*,\llcorner$ are all linear. Weak continuity follows immediately from continuity of K as a linear operator. □

Example. Let $\varphi\colon M \to N$ be a differentiable map. Consider the reversed graph $P = \{(\varphi(x), x) : x \in M\} \subset N \times M$, as we did above. Then $[\![P]\!]$ is a current on $N \times M$, in general of mixed degrees, and defines an operator $\mathbf{P}\colon \Omega^\bullet(N) \to \mathscr{D}_\bullet(M)$, again in general of mixed degrees. Computing:

$$\mathbf{P}(\theta)\xi = \int_P \pi_N^*\theta \wedge \pi_M^*\xi = \int_M \varphi^*\theta \wedge \xi = \mathbf{I}^k(\varphi^*\theta)\xi \tag{3.1.7}$$

Hence $\mathbf{P} = \mathbf{I}^k \circ \varphi^*$. In particular if $\varphi = \mathrm{id}_M$, then $P = \Delta$ is the diagonal and $\mathbf{P} = \mathbf{I}^k$ is the injection of forms into currents (see Proposition C.9).

PROPOSITION 3.8. *Let K be a current on $N \times M$ of degree l. Consider, for every $k \le l$, the operator $\mathbf{K}\colon \Omega^{n'-k}(N) \to \mathscr{D}_{l-k}(M)$. Then we have the identity*

$$d_{\mathscr{D}} \circ \mathbf{K} + \mathbf{K} \circ d = \partial \mathbf{K} \tag{3.1.8}$$

that is, ∂K is its kernel in the sense of integrals.

Proof. It's a matter of computation. Let $\theta \in \Omega^{n'-k}(N)$ and $\xi \in \Omega^{n-(l-k)}(M)$.

$$\begin{aligned}\big(\partial \circ \mathbf{K} + (-1)^{k+1}\mathbf{K} \circ d\big)(\theta)\xi &= (\partial \circ \mathbf{K})(\theta)\xi + (-1)^{k+1}(\mathbf{K} \circ d)(\theta)\xi = \\ &= \partial\,(K(\pi_N^*\theta \wedge \pi_M^*\xi)) + (-1)^{k+1}K(\pi_N^* d\theta \wedge \pi_M^*\xi) = \\ &= K\big(d\,(\pi_N^*\theta \wedge \pi_M^*\xi) + (-1)^{k+1} d\,\pi_N^*\theta \wedge \pi_M^*\xi\big) = \\ &= K(\pi_N^*\theta \wedge \pi_M^* d\xi) = \mathbf{K}(\theta)(d\xi) = (\partial \mathbf{K})(\theta)\xi\end{aligned} \tag{3.1.9}$$

□

THEOREM 3.9. *Let $\varphi\colon \mathbb{R} \times M \to M$ be a finite volume flow. As in Theorem 3.5, consider the currents G_t, $[\![P_t]\!]$ on $M \times M$, and the respective operators $\mathbf{G}_t, \mathbf{P}_t\colon \Omega^\bullet(M) \to \mathscr{D}_\bullet(M)$. For all t we have*

$$d_{\mathscr{D}} \circ \mathbf{G}_t + \mathbf{G}_t \circ d = \mathbf{I} - \mathbf{P}_t \tag{3.1.10}$$

Moreover the limits $\lim_{t\to\infty} \mathbf{G}_t = \mathbf{G}$ *and* $\lim_{t\to\infty} \mathbf{P}_t = \mathbf{P}$ *exist and satisfy the equation*

$$d_{\mathscr{D}} \circ \mathbf{G} + \mathbf{G} \circ d = \mathbf{I} - \mathbf{P} \tag{3.1.11}$$

Proof. First of all notice that $\mathbf{P}_t(\theta) = \mathbf{I} \circ (\varphi^t)^*\,\theta$. So the existence of the limit for $t \to \infty$ is the answer to our original question. The existence of these limits is just a restatement of Theorem 3.5, while the operator equations are just the combination of Proposition 3.8 and equations (3.1.3, 3.1.4). □

Remark 3.10. By Theorem C.19, **I** is a quasi-isomorphism. Since **P** is chain-homotopic to it, also **P** is a quasi-isomorphism. Hence we found out that our correlation operator is cohomological in nature, and by it we may actually compute the cohomology of the manifold.

3.2. Currents in Morse theory.

Now let M be compact and φ be the *positive* Morse-Smale gradient flow of a Morse function f. We choose the positive gradient flow because later on we'll want to relate the cohomology theories. The flow is finite-volume, and Theorem 3.9 holds. This means that the correlation integral

$$\Omega^k(M)\times\Omega^{n-k}(M)\ni(\theta,\xi)\mapsto\int_M(\varphi^t)^*\theta\wedge\psi=(\varphi^t)^*\theta\smile\xi=\theta\smile_t\xi \tag{3.2.1}$$

for $t\to\infty$ induces a perfect pairing between the cohomologies. Notice that $\smile_t$ can also be thought as a deformation of the cup product structure on the cohomology ring of the manifold. We found out that the large-time limit actually is described completely geometrically. We can actually find a more explicit form for $\mathbf{P}$ in the case of a Morse-Smale gradient flow, which will show us how the long-time behaviour of the flow is encoded in the oriented Morse cochain complex.

3.2.1. Structure of the correlation current

Recall the basic properties of dynamics of Morse functions, this time applied to the *positive* gradient flow, or equivalently to the negative gradient flow of $-f$. Namely, if $p\in\operatorname{Crit}f$ then $\dim W^u(p)=n-\operatorname{ind}p$ and $\dim W^s(p)=\operatorname{ind}p$. Hence every point on M belongs to some instanton connecting a critical point of index k to a critical point of index $l>k$; finally in a Morse chart centered at a critical point p of index k, the expression for f is

$$\mathbb{R}^{n-k}\times\mathbb{R}^k\ni(u,v)\mapsto f(u,v)=f(0,0)+|u|^2-|v|^2 \tag{3.2.2}$$

because for the positive gradient flow, it is the positive definite part which is unstable.

Definition 3.11. Define the relation on M

$$x\prec y \iff \text{there is a (not necessarily) broken instanton connecting } x \text{ to } y \tag{3.2.3}$$

Then, by the dynamical properties of Morse-Smale flows and the topology of the instanton moduli spaces, $\prec$ is a partial order on M.

By the properties of Morse-Smale flows, if $x=p\in\operatorname{Crit}^k f$ and $y=q\in\operatorname{Crit}^l f$, then $p\prec q$ only if $k<l$. Hence the points $\{q\in\operatorname{Crit}^l f: p\prec q\}$ are precisely those critical points whose stable manifold intersects $W^u(p)$.

In this subsection we seek the proof of the following

THEOREM 3.12. *Consider the current P of the preceding theorems, satisfying the equations $\partial G=[\![\Delta]\!]-P$. Then*

$$P=\sum_{p\in\operatorname{Crit}f}[\![W^u(p)]\!]\otimes[\![W^s(p)]\!] \tag{3.2.4}$$

Proof. This is [22, Theorem 3.3]. We shall prove it in Lemmata. But first recall the definitions of the currents P, G:

$$G = \lim_{t\to\infty} G_t, \quad G_t = \mathrm{pr}_* [\![\mathcal{G}_t]\!], \quad \mathcal{G}_t = \{(s, \varphi^s(x), x) : 0 \le s \le t,\ x \in M\} \subset \mathbb{R} \times M \times M$$
$$P = \lim_{t\to\infty} [\![P_t]\!], \quad P_t = \{(\varphi^t(x), x) : x \in M\} \subset M \times M \tag{3.2.5}$$

and the fundamental equation

$$\partial G = [\![\Delta]\!] - P \tag{3.2.6}$$

where $\Delta \subset M \times M$ is the diagonal.

LEMMA 3.13. *Let $p \in \mathrm{Crit}^k f$. Define*

$$\widetilde{W^s(p)} = \bigcup_{\substack{q\in \mathrm{Crit}^l f,\\ p\prec q}} W^s(q) \tag{3.2.7}$$

which is just the union of the stable manifolds of the critical points q with the property that $W^s(q) \cap W^u(p) \neq \varnothing$. Then

$$\mathrm{supp}\, P \subset \bigcup_{p\in \mathrm{Crit} f} W^u(p) \times \widetilde{W^s(p)} \tag{3.2.8}$$

Proof. Since $P = \lim_{t\to\infty} [\![P_t]\!]$, we have that $(y, x) \in \mathrm{supp}\, P$ if and only if there exist sequences $x_j \to x$ in M and $s_j \to \infty$ in $\mathbb{R}$ such that, setting $y_j = \varphi^{s_j}(x_j)$, it holds that $(y_j, x_j) \to (y, x)$, because $(y_j, x_j) \in P_{s_j}$ for all $j \in \mathbb{N}$. So assume $(y, x) \in \mathrm{supp} P$, and consider the sequence $\{(y_j, x_j)\} \subset M \times M$. Since M is compact, every point evolves towards some critical point under the flow φ. Moreover, if γ_j denotes the trajectory connecting x_j to y_j, we know that γ_j must necessarily converge towards a (possibly) broken instanton $\gamma \subset W^u(p) \cap W^s(q)$ for some critical points $p, q \in \mathrm{Crit} f$ satisfying $\mathrm{ind}\, p < \mathrm{ind}\, q$. Finally by construction, both of the limits x and y must be in γ. Hence we have the claim if we set $p = \lim_{j\to\infty} \varphi^{s_j}(x)$, because this way $W^s(q) \subset \widetilde{W^s(p)}$. □

LEMMA 3.14. *Let*

$$\Sigma = \{(p, p) : p \in \mathrm{Crit} f\} \cup \bigcup_{\substack{(p,q)\in(\mathrm{Crit} f)^2\\ p\prec q}} W^s(p) \times W^u(q) \tag{3.2.9}$$

Then Σ is a finite disjoint union of submanifolds of dimension at most $n-1$.

Proof. We know that $\dim W^u(p) \times W^s(q) = \mathrm{ind}\, p + n - \mathrm{ind}\, q$. Since we're only considering the couples such that $p \prec q$, it must be that $\mathrm{ind}\, q - \mathrm{ind}\, p \le -1$. That the union is disjoint is clear, and finiteness is simply because the critical points are in a finite number. □

LEMMA 3.15. *Consider the set*

$$T=\{(y,x):x\in M\setminus \operatorname{Crit} f,\ \exists t\in(0,\infty)\ s.t.\ y=\varphi^t(x)\}\subset M^2\setminus\Sigma \tag{3.2.10}$$

Then T is an embedded submanifold, its closure $\overline{T}$ in $M^2\setminus\Sigma$ is a smooth embedded submanifold and as a current

$$\partial\left[\!\left[\overline{T}\right]\!\right]=[\![\Delta]\!]-\sum_{p\in\operatorname{Crit} f}[\![W^u(p)]\!]\otimes[\![W^s(q)]\!] \tag{3.2.11}$$

Proof. First of all notice that

$$T=\bigcup_{t\in(0,\infty)}G_t\setminus\Sigma\implies[\![T]\!]=G-[\![\Sigma]\!] \tag{3.2.12}$$

Suppose $(\bar{y},\bar{x})\in\overline{T}\setminus T$. Then either $\bar{y}=\bar{x}\implies(\bar{y},\bar{x})\in\Delta$ or $(\bar{y},\bar{x})\in W^u(p)\times W^s(p)$ for a $p\in\operatorname{Crit} f$, because the $(\bar{y},\bar{x})\notin\Delta$ are the points for which $\bar{y}=\lim_{t\to\infty}\varphi^t(\bar{x})$, so the reasoning of Lemma 3.13 applies without much change.

- If $(\bar{y},\bar{x})\in\Delta$ then clearly in a small open set around $(\bar{y},\bar{x})$ the set $\bar{T}$ is a smooth manifold with boundary Δ. This is because we're away from critical points.
- If $(\bar{y},\bar{x})\in W^u(p)\times W^s(p)$ for a $p\in\operatorname{Crit} f$, then defining $\psi_s(y,x)=(\varphi^{-s}(y),\varphi^s(x))$ it is clear that for any neighborhood U of (p,p), there exists an $\bar{s}\in(0,\infty)$ such that $\psi_s(\bar{y},\bar{x})\in U$ for all $s\geq\bar{s}$. Moreover ψ_s is a diffeomorphisms which leaves $W^u(p)\times W^s(p)$ invariant, and ψ_s^{-1} maps T into T.

So if we assume that the claim holds in a neighborhood of $(p,p)\in(\operatorname{Crit} f)^2$, by means of ψ_s we can show that in a neighborhood of $(\bar{y},\bar{x})\in W^u(p)\times W^s(p)$ the boundary

$$\partial\left[\!\left[\overline{T}\right]\!\right]=-[\![W^u(p)]\!]\otimes[\![W^s(p)]\!] \tag{3.2.13}$$

meaning it suffices to prove the claim in a neighborhood of a $(p,p)\in(\operatorname{Crit} f)^2$. But in a neighborhood of p we can always find a chart for which f is a quadratic form and its flow is simply exponential dilation/contraction – a Morse chart in which the metric is flat. We call this sort of chart a *flat Morse chart*. Explicitly this is a chart U centered at p with coordinates $(u,v)\in\mathbb{R}^{n-\operatorname{ind} p}\times\mathbb{R}^{\operatorname{ind} p}$ in which

$$f(u,v)=f(0,0)+|u|^2-|v|^2,\quad \varphi^s(u,v)=\left(e^s u,e^{-s}v\right) \tag{3.2.14}$$

The fact that the "unstable coordinates" are the first $n-\operatorname{ind} p$ is because, again, we are looking at the *positive* gradient flow of f. Around $(p,p)\in(\operatorname{Crit} f)^2$, there is a chart $\mathscr{U}$ such that

$$T\cap\mathscr{U}=\{(\bar{u},\bar{v},u,v):\bar{u}=e^t u,\ \bar{v}=e^{-t}v\ \exists t\in(0,\infty)\} \tag{3.2.15}$$

so that the closure is comprised of the points such that $\bar{u}=su$ and $\bar{v}=sv$ for some $0\leq s\leq 1$. This is clearly a smooth submanifold away from $(p,p)=(0,0)$, and its boundary is comprised of the points $\{\bar{u}=0,v=0\}$ which is the set $\mathscr{U}\cap(W^u(p)\times W^s(p)\cup\Delta)$.

□

We need one final technical lemma.

LEMMA 3.16 ([16, p. 4.1.15]). *Let N be an oriented k-submanifold of $\mathbb{R}^n$ of locally finite Hausdorff measure. Suppose that* $\operatorname{supp} \partial [\![N]\!] \subset \mathbb{R}^l$ *for some $l < k-1$. Then $\partial [\![N]\!] = 0$.*

We are now ready for the proof. By combining Lemmata 3.13 and 3.15, we have

$$\operatorname{supp}\left(P - \sum_{p \in \operatorname{Crit} f} [\![W^u(p)]\!] \otimes [\![W^s(p)]\!]\right) \subset \Sigma \tag{3.2.16}$$

But then by Lemma 3.16 applied to Lemma 3.14, $\partial [\![\Sigma]\!] = 0$. By the fundamental current equation $\partial G = [\![\Delta]\!] - P$, on the one hand

$$\partial \left[\!\!\left[\overline{T}\right]\!\!\right] = \partial G - \partial [\![\Sigma]\!] = [\![\Delta]\!] - P \tag{3.2.17}$$

while on the other, by Lemma 3.15

$$\partial \left[\!\!\left[\overline{T}\right]\!\!\right] = [\![\Delta]\!] - \sum_{p \in \operatorname{Crit} f} [\![W^u(p)]\!] \otimes [\![W^s(p)]\!] \tag{3.2.18}$$

from which we conclude

$$P = \sum_{p \in \operatorname{Crit} f} [\![W^u(p)]\!] \otimes [\![W^s(p)]\!] \tag{3.2.19}$$

□

COROLLARY 3.17. *The correlation operator* $\mathbf{P}$ *is of the form*

$$\mathbf{P}(\theta) = \sum_{p \in \operatorname{Crit}^{\deg \theta} f} \left(\int_{W^s(p)} \theta\right) [\![W^u(p)]\!] \tag{3.2.20}$$

Proof. Indeed by Theorem 3.12 for any $\xi \in \Omega^{n-\deg\theta}(M)$

$$\mathbf{P}(\theta)\xi = P(\pi_M^* \theta \wedge \pi_M^* \xi) = \sum_{p \in \operatorname{Crit}^{\deg \theta} f} ([\![W^s(p)]\!] \theta)([\![W^u(p)]\!] \xi) \tag{3.2.21}$$

□

Remark 3.18. Dynamically, this equation is telling us something very transparent. Thinking of θ as a continuous distribution function of initial states which are concentrated around some $\deg\theta$-dimensional submanifold, we're saying that the long-time evolution of this distribution of states will have support concentrated around the unstable manifolds, with "residues" the total quantity of states on the stable manifolds of the appropriate index. We can think of this new distribution of states as the *equilibrium state* which is reached by the system. The possibility to render explicit the correlation functional is thus a consequence of the simplicity of the dynamics of Morse-Smale gradient flows. Similar statements can be made for different types of flows, as done in the articles cited in the introduction.

3.2.2. *Recovering Morse theory*

Theorem 3.12 shows that the correlation operator $\mathbf{P}$ has image contained in $\hat{\mathscr{S}} = \operatorname{span}_{\mathbb{R}}(\llbracket W^u(p) \rrbracket : p \in \operatorname{Crit} f)$. Moreover, $\hat{\mathscr{S}} = \hat{\mathscr{S}}^\bullet$ has an obvious grading given by the index of the critical points: $\hat{\mathscr{S}}^k = \operatorname{span}_{\mathbb{R}}\left(\llbracket W^u(p) \rrbracket : p \in \operatorname{Crit}^k f\right)$. So the natural question is whether we can use this complex to compute the cohomology of the manifold.

PROPOSITION 3.19. $\left(\hat{\mathscr{S}}^\bullet, d_{\mathscr{D}}\right)$ *is a subcomplex of* $(\mathscr{D}^\bullet(M), d_{\mathscr{D}})$.

Proof. Since $\dim W^u(p) = n - \operatorname{ind} p$, $\llbracket W^u(p) \rrbracket$ defines a $(\operatorname{ind} p)$-current, so clearly $\hat{\mathscr{S}}^k \subset \mathscr{D}^k(M)$ is a subspace. So it suffices to show that $\partial\left(\hat{\mathscr{S}}^k\right) \subset \hat{\mathscr{S}}^{k+1}$. But the boundary of $W^u(p)$ is the (stratified) union of unstable manifolds of points of strictly larger index ([28, Proposition 2] or for a more refined result [34]), so as a current $\partial \llbracket W^u(p) \rrbracket$ is a chain in $\mathscr{S}^{k+1}$. □

Now, recall the definition of the oriented Morse cochain complex $(C^\bullet, d_C)$ given in Remark 2.21, and notice that there is an obvious morphism of vector spaces, defined on generators as

$$\begin{aligned} \hat{\iota}^k : \hat{\mathscr{S}}^k &\to C^k \otimes \mathbb{R} \\ \llbracket W^u(p) \rrbracket &\mapsto o_p \end{aligned} \tag{3.2.22}$$

This is an isomorphism. We would like for it to extend into a full cochain isomorphism – more so, we would like for it to extend to a morphism of cochain complexes of *abelian groups*, so that we may recover the full integral cohomology of M.

Definition 3.20. Define the abelian groups

$$\mathscr{S}^k = \operatorname{span}_{\mathbb{Z}}\left\{\llbracket W^u(p) \rrbracket : p \in \operatorname{Crit}^k f\right\} \tag{3.2.23}$$

Then $\mathscr{S}^\bullet$ is a sublattice of $\hat{\mathscr{S}}$.

THEOREM 3.21. $\left(\mathscr{S}^k, d_{\mathscr{D}}\right)$ *is a cochain complex of abelian groups, isomorphic to* $(C^\bullet, d_C)$.

Proof. We must show that $d_{\mathscr{D}}\mathscr{S}^\bullet \subset \mathscr{S}^\bullet$. Let $p \in \operatorname{Crit}^k f$. Clearly, since $\hat{\mathscr{S}}$ is a cochain complex of vector spaces,

$$d_{\mathscr{D}} \llbracket W^u(p) \rrbracket = \sum_{q \in \operatorname{Crit}^{k+1} f} n_{p,q} \llbracket W^u(q) \rrbracket \tag{3.2.24}$$

where $n_{p,q} \in \mathbb{R}$. We are done if we show that $n_{p,q}$ is the intersection number of $W^u(p)$ with $W^s(q)$ as defined in Theorem 2.29, because then it is automatically an integer, and moreover $\hat{\iota}$ restricts to a cochain isomorphism on $\mathscr{S}$. To do this, consider a flat Morse chart around p and define the *local (un)stable manifold*

$$W^u_\varepsilon(p) = \{(u, 0) : |u| < \varepsilon\}, \quad W^s_\varepsilon(p) = \{(0, v) : |v| < \varepsilon\} \tag{3.2.25}$$

It is a basic result in the theory of hyperbolic dynamical systems that for any $\varepsilon > 0$

$$W^s(p) = \bigcup_{0<t<\infty} \varphi^{-t}(W^s_\varepsilon(p)), \quad W^u(p) = \bigcup_{0<t<\infty} \varphi^{t}(W^u_\varepsilon(p)) \tag{3.2.26}$$

Now, from this fact it follows that for any form of the appropriate degree,

$$(-1)^{\operatorname{ind} p} d\,[\![W^u(p)]\!]\,\theta = \int_{W^u(p)} d\theta = \lim_{t\to\infty} \int_{\partial\varphi^t(W^u(\varepsilon))} \theta \tag{3.2.27}$$

To prove our claim it suffices to take θ with support concentrated around a fixed $q \in \operatorname{Crit}^{k+1} f$. Close to q and for large $t > 0$, $\varphi^t(W^u(\varepsilon))$ is a disjoint finite union of submanifolds with boundary, all transversal to $W^s(q)$, one for each instanton $\gamma \in \hat{\mathfrak{I}}(p,q)$. Pictorially we're flowing an $(n - \operatorname{ind} p)$-disc until its "tendrils" intersect $W^s(q)$. By the properties of Morse-Smale flows we know that there is a time for which they do, and when they do they do it transversally. Finally the intersection is one-dimensional. So the boundary-component containing some $\gamma \in \hat{\mathfrak{I}}(p,q)$ of $\varphi^{-t}(W^u(\varepsilon))$ must converge to $\pm W^u(q)$ where the sign is determined by whether γ flips or respects orientations. This is what we wanted to prove. □

To understand the rôle of the correlation operator in all this, equation (3.1.11) combined with Proposition C.18 tells us that

$$\begin{aligned}\partial \circ \mathbf{P} - (-1)^{k+1}\mathbf{P}\circ d &= (-1)^{k+1}\partial \circ \mathbf{G}\circ d - \partial\circ\mathbf{I} - (-1)^{k+1}\partial\circ\mathbf{G}\circ d + (-1)^{k+1}\mathbf{I}\circ d = \\ &= \partial\circ\mathbf{I} - (-1)^{k+1}\mathbf{I}\circ d = 0\end{aligned} \tag{3.2.28}$$

which shows that $d_{\mathscr{D}} \circ \mathbf{P} = \mathbf{P}\circ d$. This means that there is a well-defined induced morphism at the level of cohomology

$$\mathbf{P}^l_*\colon H^l(\Omega^\bullet(M), d) \to H^l(\mathscr{D}^\bullet, d_{\mathscr{D}}) \tag{3.2.29}$$

which is also an isomorphism, since $\mathbf{P}$ is chain-homotopic to $\mathbf{I}$. This morphism factors through the inclusion $i_*\colon H^l(\mathscr{S}_\bullet, d_{\mathscr{D}}) \to H^l(\mathscr{D}_\bullet, d_{\mathscr{D}})$, that is, there is a commuting triangle

$$\begin{tikzcd} H^l(\Omega^\bullet(M),d) \arrow[rr, "\mathbf{P}^l_*"] \arrow[dr, "\mathbf{Q}^l_*"'] & & H^l(\mathscr{D}^\bullet, d_{\mathscr{D}}) \\ & H^l\left(\hat{\mathscr{S}}^\bullet, d_{\mathscr{D}}\right) \arrow[ur, "i^l_*"'] & \end{tikzcd} \tag{3.2.30}$$

The morphisms $\mathbf{P}^l_*$ and i^l_* are both monomorphisms so $\mathbf{Q}^l_*$ must be too.

PROPOSITION 3.22 ([22, Theorem 4.2]). *For any l, $\mathbf{Q}^l_*$ is also an epimorphism.*

Finally, to recover the integral cohomology, it suffices to consider the subspace $\Omega^\bullet_{\mathbb{Z}}(M) = \mathbf{P}^{-1}(\mathscr{S}^\bullet)$ of smooth forms with integer residues over the stable spaces. Since $\mathbf{P}$ is a cochain morphism, this is a cochain subcomplex of the full De Rham complex. The previous proposition implies immediately that we have an isomorphism $H^l(\mathscr{S}^\bullet, d_{\mathscr{D}}) \cong H^l(\Omega^\bullet_{\mathbb{Z}}(M), d)$.

4. From Witten deformations to the Oriented Morse Complex

In the first chapter, we tied the spectrum of the Witten Laplacian with the critical points of the Morse function used to deform the De Rham complex. In the second chapter, we used the critical points of a Morse function to construct a complex which we have shown to be quasi-isomorphic to the singular chain complex of the manifold. In this chapter we will use the Harvey-Lawson theory of the previous chapter to tie the Witten deformation with the Morse theory of the deforming function, on the level of cochain complexes. Specifically, we'll construct a finite-dimensional subcomplex of the deformed L^2–De Rham complex which we'll show to be isomorphic, and not just quasi-isomorphic, to the oriented Morse cochain complex.

4.1. The Witten Instanton complex

In this section we show how we can construct a finite-dimensional subcomplex of the deformed De Rham complex which carries all the cohomological information contained in the full deformed complex. To do this, we'll use the Morse theory of the deforming function.

Recall that for a Morse function f, the Witten deformation of the De Rham complex is the cochain complex $(\Omega^\bullet(M), d_\nu)$ where $d_\nu = e^{-f/\nu} d e^{f/\nu}$ (Definition 1.19). This deformed complex is isomorphic to the undeformed De Rham complex (Proposition 1.20) so its cohomology recovers the cohomology of M with real coefficients.

Recall that the Witten Laplacian on the L^2-forms is the elliptic self-adjoint bounded operator $\Delta_\nu = D_\nu^2 = (d_\nu + d_\nu^*)^2$. The spectrum of Δ_ν on $L^2\Omega^\bullet(M)$ has a special form (Theorem 1.27): for any $\varepsilon > 0$ there exists a $\nu_0 > 0$ such that whenever $\nu < \nu_0$, the number of eigenvalues between 0 and ε of Δ_ν acting on k-forms is equal to the number of critical points of f of index k.

4.1.1. Construction of the complex

Definition 4.1. Let $0<\varepsilon<1$. Let $F^k_{\nu,\lambda}\leq L^2\Omega^k(M)$ be the eigenspace of $\Delta_\nu|_{\Omega^k}$ relative to an eigenvalue $\lambda\geq 0$. Define the vector space

$$F^k_\nu(\varepsilon)=\bigoplus_{\lambda\in[0,\varepsilon]}F^k_{\nu,\lambda} \tag{4.1.1}$$

Remark 4.2. Recall from Subsection 1.3.2 the space $E_\nu(\varepsilon)$ made from the direct sum of eigenspaces of the deformed Dirac operator D_ν with eigenvalues in $[-\varepsilon,\varepsilon]$. Then, since $\Delta_\nu=D^2_\nu$ has the same eigenvectors as D_ν, with squared eigenvalues, we have $E_\nu(\varepsilon)\cap L^2\Omega^k(M)\cong F^k_\nu(\varepsilon^2)$.

LEMMA 4.3. *For any $\varepsilon>0$ there exists a $\nu_0>0$ such that whenever $\nu<\nu_0$,*

$$\dim F^k_\nu(\varepsilon)=\sharp\left(\mathrm{Crit}^k f\right) \tag{4.1.2}$$

Proof. Follows immediately from Theorem 1.27. □

LEMMA 4.4. *For small enough $\nu>0$, the graded vector space $(F^\bullet_\nu(\varepsilon),d_\nu)$ is a cochain complex.*

Proof. Let $\theta\in F^k_\nu(\varepsilon)$. Then we may write

$$\theta=\sum_{p\in\mathrm{Crit}^k f}\theta_p\omega^\nu_p \tag{4.1.3}$$

for $\theta_p\in\mathbb{R}$ and ω^ν_p such that $\Delta_\nu\omega^\nu_p=\lambda_p\omega^\nu_p$, because for small enough ν the eigenforms of Δ_ν with eigenvalue smaller than ε are in one-to-one correspondence with $\mathrm{Crit}^k f$. So we have

$$d_\nu\theta=\sum_p\theta_p d_\nu\omega^\nu_p \tag{4.1.4}$$

But clearly, $\Delta_\nu\circ d_\nu=d_\nu\circ\Delta_\nu$. Hence $d_\nu\omega^\nu_p$ is a $(k+1)$-eigenform of Δ_ν with the same eigenvalue $\lambda_p<\varepsilon$. This means that $d_\nu\theta\in F^{k+1}_\nu(\varepsilon)$. □

Definition 4.5. Let $\nu>0$ be such that Lemma 4.3 holds. We call $(F^\bullet_\nu(\varepsilon),d_\nu)$ the *Witten instanton complex*.

Remark 4.6. The Witten instanton complex is a subcomplex of the full deformed De Rham cochain complex on M. Moreover it is a finite-dimensional subcomplex. It carries a natural projector

$$P^\bullet_\nu(\varepsilon)\colon\Omega^\bullet(M)\to F^\bullet_\nu(\varepsilon) \tag{4.1.5}$$

By Remark 4.2, $P_\nu(\varepsilon^2)=p_\nu(\varepsilon)$ where $p_\nu(\varepsilon)$ is the spectral projector of Lemma 1.35. As such, it satisfies the same inequality (1.3.13). The different notation is for psychological reasons – the projector $p_\nu(\varepsilon)$ was projecting onto the eigenspaces of D_ν.

4.2. From Witten to Morse via Harvey & Lawson

From now on, for fixed $\varepsilon > 0$, we always assume that the $\nu > 0$ in the Witten instanton complex is small enough for Lemma 4.3 to hold. Recall the subcomplex $\hat{\mathscr{S}}^\bullet$ of the cochain complex of currents $\mathscr{D}^\bullet(M)$ defined in Subsection 3.2.2. This is the vector space generated by the currents of integration over the unstable manifolds, graded by the Morse index.

Definition 4.7. Define the map

$$\begin{aligned} \mathscr{P}_\nu^k : \Omega^k(M) &\to \hat{\mathscr{S}}^k \\ \theta &\mapsto \mathbf{P}\left(e^{\frac{f}{\nu}}\theta\right) \end{aligned} \tag{4.2.1}$$

In this section, we want to show the following

THEOREM 4.8. *$\mathscr{P}_\nu^\bullet : F_\nu^\bullet(\varepsilon) \to \hat{\mathscr{S}}^\bullet$ is a cochain isomorphism.*

4.2.1. Preliminary results

PROPOSITION 4.9. *$\mathscr{P}_\nu^\bullet$ is a cochain morphism between $(\Omega^\bullet(M), d_\nu)$ and $\left(\hat{\mathscr{S}}^\bullet, d_{\mathscr{D}}\right)$.*

Proof. First of all, the codomain is indeed $\hat{\mathscr{S}}^k$ because by Corollary 3.17

$$\mathbf{P}\left(e^{\frac{f}{\nu}}\theta\right) = \sum_{p\in \mathrm{Crit}^k f} \left(\int_{W^s(p)} e^{\frac{f}{\nu}}\theta\right) [\![W^u(p)]\!] \tag{4.2.2}$$

Now we must show that $\mathscr{P}_\nu \circ d_\nu = d_{\mathscr{D}} \circ \mathscr{P}_\nu$. Let φ^t denote the *positive* gradient flow of the Morse function f. By definition of $\mathbf{P}$

$$\mathscr{P}_\nu(d_\nu\theta) = \lim_{t\to\infty} (\varphi^t)^*\left[e^{\frac{f}{\nu}} d_\nu\theta\right] = \lim_{t\to\infty} (\varphi^t)^*\left[e^{\frac{f}{\nu}}\cdot e^{-\frac{f}{\nu}} d\left(e^{\frac{f}{\nu}}\theta\right)\right] = \lim_{t\to\infty} d\left[(\varphi^t)^*\left(e^{\frac{f}{\nu}}\theta\right)\right] \tag{4.2.3}$$

So if β is any $(n-k)$-form,

$$\begin{aligned} \mathscr{P}_\nu(d_\nu\theta)(\beta) &= \lim_{t\to\infty}\int_M d\left[(\varphi^t)^*\left(e^{\frac{f}{\nu}}\theta\right)\right]\wedge\beta = \lim_{t\to\infty}\int_M (-1)^{n-k+1}\left[(\varphi^t)^*\left(e^{\frac{f}{\nu}}\theta\right)\right]\wedge d\beta = \\ &= (-1)^{n-k+1}\partial\mathscr{P}_\nu(\theta)(\beta) = d_{\mathscr{D}}\mathscr{P}_\nu(\theta)(\beta) \end{aligned} \tag{4.2.4}$$

□

PROPOSITION 4.10. *There is a well-defined and unique extension of $\mathscr{P}_\nu^\bullet$ to $L^2\Omega^\bullet(M)$.*

Proof. By Proposition D.1 we have to show that $\mathscr{P}_\nu^\bullet$ is continuous when we endow $\Omega^\bullet(M)$ with the C^∞ topology and $\hat{\mathscr{S}}^\bullet$ with the mass norm topology. By Proposition D.2, the kernel calculus operators are indeed continuous on $\mathscr{D}^\bullet(M)$, moreso on a finite dimensional subspace. Thus all we have to show is the continuity in the C^∞ topology of the map

$$\begin{gathered} \Omega^\bullet(M) \to \Omega^\bullet(M) \\ \theta \mapsto e^{\frac{f}{\nu}}\theta \end{gathered} \tag{4.2.5}$$

but $e^{f/\nu} \in C^\infty(M)$ so this is just multiplication with an invertible element of $\Omega^0(M)$, which is clearly a continuous operation – it's a linear automorphism. Hence $\mathscr{P}_\nu^\bullet = \mathbf{P} \circ e^{f/\nu}\cdot$ is continuous. □

The previous two propositions show that $\mathscr{P}_\nu^\bullet$ is a well defined cochain complex morphism between $(L^2\Omega^\bullet(M), d_\nu)$ and $(\hat{\mathscr{S}}^\bullet, d_{\mathscr{D}})$.

Now we concentrate on the restriction on $F_\nu^\bullet(\varepsilon)$, which we denote in the same way. Recall that we are supposing $\nu > 0$ small enough so that the dimension of $F_\nu^k(\varepsilon)$ is the number of critical points of index k. The last preliminary result we need is a refinement of the estimate in Lemma 1.35. Recall the *heat forms* of Subsection 1.3.1.

PROPOSITION 4.11. *For $1 \leq \alpha \leq \infty$, let $\|\cdot\|_{W^{1,\alpha}}$ and $\|\cdot\|_\alpha$ be respectively the α-th Sobolev norm and the L^α-norm induced on differential forms by the Riemannian metric. There exists constants $C, c > 0$ such that for small enough $\nu > 0$*

$$\left\| P_\nu^k(\varepsilon)\rho_{p,\nu} - \rho_{p,\nu} \right\|_\infty \leq C e^{-\frac{c}{\nu}} \tag{4.2.6}$$

Proof. Since the eigenforms of Δ_ν are precisely the eigenforms of D_ν, we may proceed by spectral decomposition of D_ν as in Lemma 1.35 – we can do this for small $\nu > 0$:

$$\begin{aligned} P_\nu^k(\varepsilon)\rho_{p,\nu} - \rho_{p,\nu} &= \frac{1}{2\pi i}\int_{S^1_\varepsilon} \left[(\lambda - D_\nu)^{-1} - \lambda^{-1}\right]\rho_{p,\nu} d\lambda = \\ &= \frac{1}{2\pi i}\int_{S^1_\varepsilon} (\lambda - D_\nu)^{-1}\frac{D_\nu \rho_{p,\nu}}{\lambda} d\lambda \end{aligned} \tag{4.2.7}$$

We'll use this expression to show the claim. Fix $\alpha > 1$. For small $\nu > 0$, $D_\nu \rho_{p,\nu} = 0$ because of the explicit expressions in a small chart around p of D_ν and $\rho_{p,\nu}$. This means that, since the coefficients of $\rho_{p,\nu}$ are proportional to $\exp(-|y|^2/2\nu)$, the L^α-norm:

$$\|D_\nu \rho_{p,\nu}\|_\alpha \leq C_{\alpha,a} e^{-c_a/\nu} \tag{4.2.8}$$

for some positive constants $c_a > 0$ and $C_{a,\alpha} > 0$ depending on the radius $2a$ of the ball supporting $\rho_{p,\nu}$ and the L^α-norm of $\rho_{p,\nu}$ – we're just estimating the "L^α-residue" of

$D_\nu \rho_{p,\nu}$ on the spherical shell $B_{2a}(p) \setminus B_a(p)$ along these lines:

$$\begin{aligned}\left\| D_\nu \rho_{p,\nu} \right\|_\alpha &= \left(\int_M \left\langle D_\nu \rho_{p,\nu}, D_\nu \rho_{p,\nu} \right\rangle^{\frac{\alpha}{2}} \mu_g \right)^{\frac{1}{\alpha}} = \\ &= \left(\int_{B_{2a}(p) \setminus B_a(p)} \left\langle \Delta_\nu \rho_{p,\nu}, \rho_{p,\nu} \right\rangle^{\frac{\alpha}{2}} d\mathrm{vol} \right)^{\frac{1}{\alpha}} \leq C_{a,\alpha} e^{-\frac{c_a}{\nu}}\end{aligned} \tag{4.2.9}$$

Since D_ν is an elliptic first order operator, we can apply the following *bootstrapping argument*. By elliptic regularity [15], there exists a constant $C_1 > 0$ such that for any $\theta \in \Omega^\bullet(M)$

$$||\theta||_{W^{1,\alpha}} \leq C_1 \left(||D_\nu \theta||_{\alpha-1} + ||\theta||_2 \right) \tag{4.2.10}$$

But $||D_\nu \theta||_{\alpha-1} \leq ||(\lambda - D_\nu)\theta||_{\alpha-1} + (C_2/\nu)||\theta||_{\alpha-1}$ for some $C_2 > 0$, hence by using the elliptic regularity (4.2.10) inductively on the term $(C_2/\nu)||\theta||_{\alpha-1} \leq (C_2/\nu)||\theta||_{W^{1,\alpha-1}}$ we find there is a positive constant $C > 0$ such that

$$||\theta||_{W^{1,\alpha}} \leq \frac{C}{\nu^\alpha} \left(||(\lambda - D_\nu)\theta||_{\alpha-1} + ||\theta||_2 \right) \tag{4.2.11}$$

On the other hand, since the resolvent of D_ν is well defined (recall the proof of Lemma 1.35), we have the inequality

$$\left\| (\lambda - D_\nu)^{-1} \theta \right\|_2 \leq C' ||\theta||_2 \tag{4.2.12}$$

Now apply inequality (4.2.11) to $(\lambda - D_\nu)^{-1}\theta$: with inequality (4.2.12), there is a $C'' > 0$ such that for small enough $\nu > 0$

$$\begin{aligned}\left\| (\lambda - D_\nu)^{-1} \theta \right\|_{W^{1,\alpha}} &\leq \frac{C}{\nu^\alpha} \left(||\theta||_{\alpha-1} + \left\| (\lambda - D_\nu)^{-1} \theta \right\|_2 \right) \leq \\ &\leq \frac{C}{\nu^\alpha} \left(||\theta||_{\alpha-1} + C' ||\theta||_2 \right) \leq \frac{C''}{\nu^\alpha} ||\theta||_{\alpha-1}\end{aligned} \tag{4.2.13}$$

Finally, we may combine inequalities (4.2.9) and (4.2.13) to conclude that for small enough $\nu > 0$, there exists positive constants $C, c > 0$ such that for any $\alpha > 1$

$$\left\| (\lambda - D_\nu)^{-1} D_\nu \rho_{p,\nu} \right\|_\alpha \leq C \exp\left(-\frac{c}{\nu} \right) \tag{4.2.14}$$

By Sobolev inequality we obtain the L^∞-estimate we were looking for:

$$\left\| (\lambda - D_\nu)^{-1} D_\nu \rho_{p,\nu} \right\|_\infty \leq C \exp\left(-\frac{c}{\nu} \right) \tag{4.2.15}$$

The claim follows by combining this inequality with the spectral representation (4.2.7).
$\square$

4.2.2. Proof of Theorem 4.8

Proof. Since $\mathscr{P}_\nu^\bullet$ is a cochain complex morphism, it suffices to show is that $\mathscr{P}_\nu^k$ is an isomorphism. All we have to show is that $\mathscr{P}_\nu^k$ is injective, because $\dim F_\nu^k(\varepsilon) = \dim \hat{\mathscr{S}}^k = \sharp(\mathrm{Crit}^k f)$. To do this, we pick bases of the spaces and work with the relative matrix representation of $\mathscr{P}_\nu^k$. So let ω_p^ν be the eigenform of Δ_ν, which in a small Morse chart $\left(U_p, (u,v) \in \mathbb{R}^{n-k} \times \mathbb{R}^k\right)$ around $p \in \mathrm{Crit}^k f$ can be written as (recall Subsection 1.2.2)

$$\omega_p^\nu = \left[\prod_{j=1}^k Y_{N_j}\left(v^j\right)\right] \cdot \exp\left(-\frac{|v|^2}{2\nu}\right) dv^1 \wedge \cdots \wedge dv^k \tag{4.2.16}$$

where for the Hermite polynomials Y_{N_j} we must constrain the quantum numbers N_j to be such that the eigenvalue $\lambda_p^\nu > 0$ relative to ω_p^ν satisfies $\lambda_p^\nu < \varepsilon$. We know that for ν small, the set $\{\omega_p^\nu : p \in \mathrm{Crit}^k f\}$ is a basis for $F_\nu^k(\varepsilon)$. Hence in this basis, we may represent $\mathscr{P}_\nu^k$ as a matrix (recall that we're looking at the *positive* gradient flow, so $\dim W^s(p) = \mathrm{ind}\, p$)

$$S(\nu) = \left(s_{pq}(\nu)\right)_{(p,q)\in\left(\mathrm{Crit}^k f\right)^2}, \quad s_{pq}(\nu) = \int_{W^s(q)} e^{\frac{f}{\nu}} \omega_p^\nu \tag{4.2.17}$$

Set $s(\nu) = \min\left\{s_{pp}^\nu : p \in \mathrm{Crit}^k f\right\}$. If we show that for any $q \neq p$, $s_{pq}(\nu) = o\left(s\left(\nu\right)\right)$ as $\nu \to 0$, we are done, because then $s_{pp}(\nu)/s(\nu) > 1$ as $\nu \to 0$ while $s_{pq}(\nu)/s(\nu) = o(1)$ as $\nu \to 0$ so that

$$\frac{S(\nu)}{s(\nu)} = \mathrm{Diag}\left(s_{pp}(\nu)/s(\nu) : p \in \mathrm{Crit}^k f\right) + R(\nu), \quad ||R(\nu)|| \to 0 \text{ as } \nu \to 0 \tag{4.2.18}$$

which implies that $S(\nu)$ is invertible as $\nu \to 0$. To do this, we'll have to use the heat forms $\rho_{p,\nu}$. Notice that for small enough ν, in $F_\nu^k(\varepsilon)$ the forms $P_\nu^k(\varepsilon)\rho_{p,\nu}$ and ω_p^ν are parallel. By Proposition 4.11, if in the definition of $s_{pq}(\nu)$ we substitute $\rho_{p,\nu}$ to ω_p^ν, we commit an error of order $o(\exp(-c/\nu))$:

$$s_{pq}(\nu) = \int_{W^s(q)} e^{\frac{f}{\nu}} \omega_p^\nu = \int_{W^s(q)} e^{\frac{f}{\nu}} \rho_{p,\nu} + o\left(\exp\left(-\frac{c}{\nu}\right)\right) \tag{4.2.19}$$

But since $\rho_{p,\nu}$ has support contained in a small ball around p, its integral over $W^s(q)$ is zero whenever $p \neq q$! We found our first estimate

$$s_{pq}(\nu) = o\left(\exp\left(-\frac{c}{\nu}\right)\right) \tag{4.2.20}$$

and automatically, since $\rho_{p,\nu}$ has nonzero integral over $W^s(p)$, we find the second estimate

$$s_{pp}(\nu) = o\left(\exp\left(\frac{c}{\nu}\right)\right) \tag{4.2.21}$$

which is telling us that the diagonal greatly dominates the nondiagonal terms. We have finished the proof. $\square$

Remark 4.12. From a quantum mechanical point of view, the situation is as follows. The vanishing parameter ν represents the Planck constant, so the limit $\nu \to 0$ is just the semi-classical limit. The elliptic operator Δ_ν is a Schödinger operator with potential $||\nabla f||^2$. Its extension on the whole exterior algebra of the manifold is interpreted as a "supersymmetric extension" of the standard Witten Laplacian, that is, acting on functions. Finally, the matrix $S(\nu)$ is a sort of "interaction matrix" between the wells given by the critical points of f of the same index: its coefficients give us the probability of tunneling from a well to the other. So in the semi-classical limit we have found the usual exponential decay of the tunneling rates (4.2.20). Moreover, the Morse-Smale request on the flow, which doesn't appear explicitly in the proof but is needed, is equivalent to imposing that potential wells of higher index don't resonate with the wells of lower index.

A. Appendix to Chapter 1

A.1. Review of Hodge theory

Let (M, g) be an oriented Riemannian n-manifold. Let $\Omega^\bullet(M)$ be the dg-algebra of differential forms on M. We will denote multi-indexes by capital letters $(I, J, K\ldots)$. Given a local chart, we will denote the induced basis on k-forms by $\left\{dx^I : |I| = k\right\}$, where whenever $I = (i_1, \ldots, i_k)$, we mean $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

Notice that the non-orientable case may be recovered without any extra work by twisting $\Omega^\bullet$ over the orientation bundle. The choice of orienting M is thus just to simplify notation.

A.1.1. Hodge duality

Hodge duality is an encoding of Poincaré duality at the level of differential forms. As such, it relates k-forms to $(n-k)$-forms with a process of integration. What we need of Hodge duality is the Hodge star, which we'll use extensively in computations, and the adjoints of operators under the Hodge pairing.

A.1.1.1. Constructing inner products

We have a natural volume form defined by $\mu = \sqrt{|\det g|}\, dx^1 \wedge \cdots \wedge dx^n$. We may define an interior product on k-forms in steps: define the inner product of 1-forms $[\alpha, \beta]_x = g_x(\alpha^*, \beta^*)$, where α^* is the vector field such that $g(\alpha^*, -) = \alpha$, which gives a $C^\infty(M)$ function. Let ζ, η be decomposable k-forms

$$\begin{aligned} \zeta &= \alpha_1 \wedge \cdots \wedge \alpha_k \\ \eta &= \beta_1 \wedge \cdots \wedge \beta_k \end{aligned} \tag{A.1.1}$$

The interior product of ζ and η is

$$\langle \zeta, \eta \rangle = \det\left([\alpha_i, \beta_j]\right) \tag{A.1.2}$$

This too is a $C^\infty(M)$ function. The full-fledged interior product $\langle -, - \rangle : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \Omega^0(M)$ is defined by extension through linearity starting from a basis: if

$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $dx^J = dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ then we have

$$\begin{aligned} \left\langle dx^I, dx^J \right\rangle &= \det \begin{pmatrix} g_x\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{j_1}}\right) & \cdots & g_x\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{j_k}}\right) \\ \vdots & & \vdots \\ g_x\left(\frac{\partial}{\partial x^{i_k}}, \frac{\partial}{\partial x^{j_1}}\right) & \cdots & g_x\left(\frac{\partial}{\partial x^{i_k}}, \frac{\partial}{\partial x^{j_k}}\right) \end{pmatrix} = \\ &= \det \begin{pmatrix} g_{i_1 j_1}(x) & \cdots & g_{i_1 j_k}(x) \\ \vdots & & \vdots \\ g_{i_k j_1}(x) & \cdots & g_{i_k j_k}(x) \end{pmatrix} = \det g_{IJ}(x) \end{aligned} \tag{A.1.3}$$

where g_{IJ} is the "sub-matrix" of entries $(i \in I, j \in J)$ of g. Finally if in some chart $\zeta = \zeta_I dx^I$ $\eta = \eta_J dx^J$ (Einstein summation),

$$\langle \zeta, \eta \rangle = \zeta_I \eta_J \left\langle dx^I, dx^J \right\rangle = \sum_{IJ} \zeta_I \eta_J \det g_{IJ} = \zeta_I \eta^I \tag{A.1.4}$$

A.1.1.2. Hodge Star, Hodge Duality, L^2 Norm

Definition A.1. The Hodge dual of a k-form ζ is the unique $(n-k)$-form $*\zeta$ such that for any k-form η one has

$$\eta \wedge *\zeta = \langle \eta, \zeta \rangle \, \mu \tag{A.1.5}$$

This defines a map which turns out to be an isomorphism, $*\colon \Omega^\bullet(M) \to \Omega^{n-\bullet}(M)$. We can find an explicit formula for the Hodge star of a k-form, once local coordinates are introduced.

PROPOSITION A.2. *Fix a local coordinate system. For any k-form $\zeta = \zeta_I dx^I$, we have*

$$*\zeta = \sqrt{|\det g|} \sum_{|I|=k} \sum_{|J|=n-k} \zeta_I \varepsilon(IJ) dx^J \tag{A.1.6}$$

where $\varepsilon(IJ)$ is the sign of the permutation $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) \mapsto (1 \ldots n)$ and we put $\varepsilon(IJ) = 0$ whenever $I \cap J \neq \varnothing$.

Proof. First of all, from the definition of interior product we have that $*$ acts on the basis dx^I as follows:

$$*dx^I = \sqrt{|\det g|} \sum_{|J|=n-k} \varepsilon(IJ) dx^J \tag{A.1.7}$$

Now by linearity we obtain immediately for any k-form

$$\zeta = \sum_{|I|=k} \zeta_I dx^I \implies *\zeta = \sum_I \zeta_I * dx^I = \sqrt{|\det g|} \sum_{|I|=k} \sum_{|J|=n-k} \zeta_I \varepsilon(IJ) dx^J \tag{A.1.8}$$

□

Remark A.3. With some effort one may use this local expression to check that $*$ is an involution up to sign, in the sense that on k-forms $** = (-1)^{nk+1}\mathrm{id}$.

Definition A.4. We may define an L^2-type norm on k-forms through Hodge duality, coming from the following pairing:

$$(\zeta,\eta)=\int_M \zeta\wedge *\eta=\int_M \zeta_x\cdot_{g_x}\eta_x d\mu(x)\Longrightarrow ||\zeta||^2=\int_M \zeta\wedge *\zeta \qquad \text{(A.1.9)}$$

This norm extends the L^2 norm defined on the $C^\infty(M)$ functions with respect to the measure associated to the volume form μ.

A.1.1.3. The codifferential and a Weitzenböck formula

Definition A.5. The adjoint of the exterior derivative under the L^2-pairing $(-,-)$ is called the *codifferential*:

$$d^*:\Omega^\bullet(M)\to\Omega^{\bullet-1}(M),\ \text{such that } (d\eta,\zeta)=(\eta,d^*\zeta) \qquad \text{(A.1.10)}$$

PROPOSITION A.6. *It holds that*

$$d^*=(-1)^{n(k+1)+1}*d* \qquad \text{(A.1.11)}$$

Proof. Indeed if $\eta,\zeta\in\Omega^k(M)$, and we take the above formula as a *redefinition* of d^*, using the fact that $**=\pm\mathrm{id}$

$$\begin{aligned} d(\eta\wedge *\zeta) &= d\eta\wedge *\zeta+(-1)^k\eta\wedge d*\zeta=\\ &= d\eta\wedge *\zeta+(-1)^{k-nk-k)}\eta\wedge **d*\zeta=d\eta\wedge *\zeta-\eta\wedge *d^*\zeta \end{aligned} \qquad \text{(A.1.12)}$$

so since M has no border, integrating over M the left and right hand side we obtain

$$0=\int_M d(\eta\wedge *\zeta)=\int_M d\eta\wedge *\zeta-\int_M \eta\wedge *d^*\zeta=(d\eta,\zeta)-(\eta,d^*\zeta) \qquad \text{(A.1.13)}$$

so by uniqueness of the adjoint we conclude. □

There is an additional formula for the codifferential, which involves the Levi-Civita connection on M, and makes explicit the dependence of d^* from the metric. Consider the Levi-Civita connection ∇ on TM. We know we may extend it uniquely to a connection over the exterior bundle, which we still denote by ∇.

PROPOSITION A.7. *Let $e_1,\dots,e_n$ be a local orthonormal frame for the tangent bundle. Set $\nabla_j=\nabla_{e_i}$. Then if $e_1^*,\dots,e_n^*$ is the dual frame for T^*M, the exterior derivative can be expressed as*

$$d=\sum_{j=1}^n\left(e_j^*\wedge -\right)\circ\nabla_j:\Omega^\bullet(M)\to\Omega^{\bullet+1}(M) \qquad \text{(A.1.14)}$$

Proof. The question is manifestly local, so we can and will choose any suitable chart. The most suitable is a *normal* coordinate system, where the Christoffel symbols vanish and ∇_i are derivations in the canonical directions. Now the claim is trivial. □

COROLLARY A.8 (Weitzenböck formula). *The codifferential, in the notations above, satisfies*

$$d^* = -\sum_l i_{e_l} \circ \nabla_l : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M) \tag{A.1.15}$$

Proof. Follows immediately from the fact that $X^* \wedge -$ is formally adjoint to i_X. □

A.1.1.4. *Further adjoint operators*

Thanks to the Hodge star we may dualize two very important operators.

LEMMA A.9. *For a vector field X, i_X is adjoint to $X^* \wedge -$. In particular $*^{-1} i_X * = X^* \wedge -$*

Proof. In charts: let J be a multi-index of order $k-1$, $\xi = \xi_{jJ} dx^j \wedge dx^J$, and $\eta = \eta_J dx^J$.

$$\langle (X^* \wedge \eta), \xi \rangle = X_j \eta_J \xi^{jJ} = \eta_J X_j \xi^{jJ} = \langle \eta, (i_X \xi) \rangle \tag{A.1.16}$$

□

LEMMA A.10. *The adjoint operator L_X^* to L_X with respect to $(-,-)$ has the explicit expression*

$$L_X^* = d^* \circ (X^* \wedge -) + (X^* \wedge -) \circ d^* \tag{A.1.17}$$

Proof. Immediate from the preceding lemma. □

A.1.1.5. *The Hodge Laplacian*

Through the codifferential one may define the main ingredient of our treatment, the Hodge-Laplace operator, also called form Laplacian:

$$\Delta = dd^* + d^*d : \Omega^\bullet(M) \to \Omega^\bullet(M) \tag{A.1.18}$$

Remark A.11. Notice that also $\Delta = (d + d^*)^2$ since $d^2 = d^{*2} = 0$. The operator

$$D = d + d^* : \Omega^\bullet(M) \to \Omega^\bullet(M) \tag{A.1.19}$$

is called the *Dirac operator*. It is a vector space endomorphism but not a graded algebra morphism. Indeed it exchanges the odd degree with the even degree, in the following sense: if we define

$$\Omega^E(M) = \bigoplus_{j \text{ even}} \Omega^j(M), \quad \Omega^O(M) = \bigoplus_{j \text{ odd}} \Omega^j(M) \tag{A.1.20}$$

then we have

$$(d + d^*)\big(\Omega^E(M)\big) \subseteq \Omega^O(M), \quad (d + d^*)\big(\Omega^O(M)\big) \subseteq \Omega^E(M) \tag{A.1.21}$$

Denote by D^E and D^O the restriction to the even and odd spaces of the Dirac operator D.

A.1.1.6. Laplace-Beltrami versus Hodge-Laplace

The Hodge-Laplace operator is the most general realization of the Laplacian on Riemannian manifolds. An intermediate step could be the Laplace-Beltrami operator: if ∇ is the Levi-Civita connection associated to the Riemannian metric g, then the Laplace-Beltrami operator $\Delta_g : C^\infty(M) \to C^\infty(M)$ is given by $\Delta_g = \operatorname{tr}(\nabla^2)$.

Fortunately, Δ_g differs from $\Delta\colon \Omega^0(M) \to \Omega^0(M)$ only by sign. Indeed, notice that $\Delta f = dd^* f + d^* df = d^* df = - * d * df$, so:

$$\begin{aligned}
(*df)_J &= \sqrt{|\det g|} \sum_i \frac{\partial f}{\partial i} \varepsilon(iJ) dx^J, \\
(d * df)_{iJ} &= \sum_j \varepsilon(iJ) \frac{\partial}{\partial x^j} \left(\sqrt{|\det g|} \frac{\partial f}{\partial x^i} dx^j \wedge dx^J \right) = \\
&= \frac{1}{\sqrt{|\det g|}} \sum_j \frac{\partial}{\partial x^j} \left(\sqrt{|\det g|} \frac{\partial f}{\partial x^j} \right) \mu_{iJ}, \\
*\mu = 1 \implies \Delta f &= - \frac{1}{\sqrt{|\det g|}} \sum_j \frac{\partial}{\partial x^j} \left(\sqrt{|\det g|} \frac{\partial f}{\partial x^j} \right) = - \Delta_g f
\end{aligned} \tag{A.1.22}$$

Notice that the positive operator is $-\Delta_g$, and so Δ is elliptic as an operator on Ω^0. It turns out that it is still elliptic on all its higher-dimensional domains.

A.1.2. Hodge theorem

The topological importance in the Hodge Laplacian on k-forms is the fact that its kernel is isomorphic to the k-th De Rham cohomology group of M, giving an analytical tool to compute the cohomology of the manifold.

THEOREM A.12 (Hodge decomposition). *Consider the Hodge Laplacian $\Delta = dd^* + d^* d$ acting on k-forms.*

$$\Omega^k(M) = \ker \Delta \oplus \operatorname{im} \Delta \tag{A.1.23}$$

Furthermore, we have $\operatorname{im} \Delta = \operatorname{im} d \oplus \operatorname{im} d^*$ *for the adequate degrees.*

Idea of Proof. The idea is to pass to the Hilbert space of L^2 forms $L^2\Omega^k(M)$, which is just the completion of $\Omega^k(M)$ with the L^2-norm we have used. On this space, Δ becomes an honest self-adjoint operator, so we may use the standard decomposition theorems for self-adjoint operators. □

COROLLARY A.13 (Hodge theorem). *Consider the Hodge Laplacian $\Delta = dd^* + d^* d$ acting on k-forms.*

$$\ker \Delta \cong H^k(M; \mathbb{R}) \tag{A.1.24}$$

Proof. Let $\theta \in \Omega^k(M)$. If $\theta \in \ker \Delta$. Then

$$(d\theta, d\theta) + (d^*\theta, d^*\theta) = ((dd^* + d^*d)\theta, \theta) = 0 \tag{A.1.25}$$

hence $d\theta = 0$. Moreover, if $\theta, \theta' \in \ker \Delta$ are such that $\theta - \theta' = d\omega$, the Hodge decomposition immediately implies that $\theta = \theta'$. Conversely if $d\theta = 0$, the Hodge decomposition implies that $\theta = \omega + d\omega'$ with $\omega \in \ker \Delta$, so that passing to the cohomology θ and ω represent the same class. □

Remark A.14. This theorem has a deeper interpretation in term of *indices* of partial differential operators over manifolds. Define the *analytical index* of the Dirac operator D as $\operatorname{ind} D = \dim \ker D^E - \dim \ker D^O$. Thanks to the treatment above, one may check that $\ker D = \ker(d + d^*) = \ker \Delta$, so that the Hodge theorem implies

$$\sum_{j=0}^{n} \dim H^j(M; \mathbb{R}) = \sum_{j=0}^{n} \dim \ker D = \sum_{j \text{ even}} \dim \ker D^E|_{\Omega^j} - \sum_{j \text{ odd}} \dim \ker D^O|_{\Omega^j} = \operatorname{ind} D \tag{A.1.26}$$

but the left-most term is just the Euler characteristic $\chi(M)$ of M. This sort of statement can be greatly generalized to elliptic complexes over manifolds, resulting in the celebrated *Atyah-Singer index theorem*.

B. Appendix to Chapter 2

B.1. *Dynamical properties of Morse-Smale functions*

Let (M,g) be a Riemannian n-manifold, not necessarily closed or orientable, and $f : M \to \mathbb{R}$ a differentiable function. Denote by $df : M \to T^*M$ the differential of f. We may define the *gradient* $\nabla_g f : M \to TM$ as the vector field satisfying

$$df_x = g_x\left(\nabla_g f(x), -\right)\ \forall x \in M \tag{B.1.1}$$

The central object of study in this section is the ODE in M

$$\frac{dx_t}{dt} = -\nabla_g f(x_t) \tag{B.1.2}$$

and the flow of diffeomorphisms $\varphi : \mathbb{R} \times M \to M$ it generates, called *gradient flow of* f.

LEMMA B.1. *A gradient flow never has periodic orbits.*

Proof. Indeed along integral curves, f always decreases. □

Let $\operatorname{Crit} f$ be the set of zeroes of $\nabla_g f$ on M and $p \in \operatorname{Crit} f$. The *stable* and *unstable* sets at p are the subsets of M defined as

$$\begin{aligned} W^u(p) &= \left\{x \in M : \lim_{t\to\infty} \varphi^{-t}(x) = p\right\} \\ W^s(p) &= \left\{x \in M : \lim_{t\to\infty} \varphi^{t}(x) = p\right\} \end{aligned} \tag{B.1.3}$$

In general the stable and unstable sets are not submanifolds.

B.1.1. Morse functions

Definition B.2. A function $f : M \to \mathbb{R}$ is a Morse function if every critical point $p \in \operatorname{Crit} f$ is hyperbolic, that is, the Hessian $\operatorname{Hess} f(p)$ is a non-degenerate bilinear form at p for every $p \in \operatorname{Crit} f$.

Remark B.3. Since every rest point $p \in \operatorname{Crit} f$ is hyperbolic, the Stable Manifold Theorem [3, Theorem 7.2.9] tells us that both of the stable and unstable sets for all critical points $p \in \operatorname{Crit} f$, are actually immersed submanifolds, homeomorphic to discs: they are images of injective immersions of the stable/unstable *spaces* of the linearization of the system at p.

Definition B.4. Define $\operatorname{ind} p\colon \operatorname{Crit} f \to \mathbb{N}$ as the number of negative eigenvalues of the Hessian of f at p. Denote by $\operatorname{Crit}^k f$ the set of critical points p of f of index $\operatorname{ind} p = k$.

By what we have just said, this is equal to the dimension of the unstable manifold for the negative gradient flow at a critical point. Morse functions have very rigid local properties, which mostly derive from the following

THEOREM B.5 (Morse lemma [32, Lemma 2.2]). *For any critical point $p \in \operatorname{Crit} f$ there exists a local chart $\left(U_p, (y^1, \dots, y^n)\right)$ centered at p for which*

$$f(y) = f(0) - \sum_{j=1}^{\operatorname{ind} p} \left(y^j\right)^2 + \sum_{j=\operatorname{ind} p+1}^{n} \left(y^j\right)^2 \tag{B.1.4}$$

LEMMA B.6. *Let $K \Subset M$ be compact. The cardinality of* $\operatorname{Crit} f \cap K$ *for a Morse function f is finite.*

Proof. By the Morse lemma non-degenerate critical points are isolated. It follows that f may only have a discrete set of critical points. Since K is compact, one may use a cover condition to prove that every discrete set made of isolated points is finite, so that $\operatorname{Crit} f \cap K$ is finite. □

Definition B.7. Let $f : M \to \mathbb{R}$ be a smooth function. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$ is a *Palais-Smale* sequence for f if for some $c \in \mathbb{R}$ we have that

$$f(x_n) \to c \text{ and } df_{x_n}\left(\nabla_g f(x_n)\right) = \left\|\nabla_g f(x_n)\right\|^2 \to 0 \tag{B.1.5}$$

The function f satisfies the *Palais-Smale condition* (PS) if every Palais-Smale sequence for f is compact, hence admits a converging subsequence.

Remark B.8. If M is a compact manifold then the Palais-Smale condition is always satisfied.

LEMMA B.9. *Let $f : M \to \mathbb{R}$ be a Morse function with complete gradient flow satisfying the (PS) condition. Then for every $x \in M$, either $\varphi^t(x) \to p \in \operatorname{Crit} f$ as $t \to \infty$ or such limit does not converge.*

The following is proved in [32, Corollary 6.8], and shows that we are in a generic situation.

THEOREM B.10 (Abundance of Morse functions). *The set of Morse functions on M is open and dense in $C^\infty(M)$ with the uniform topology.*

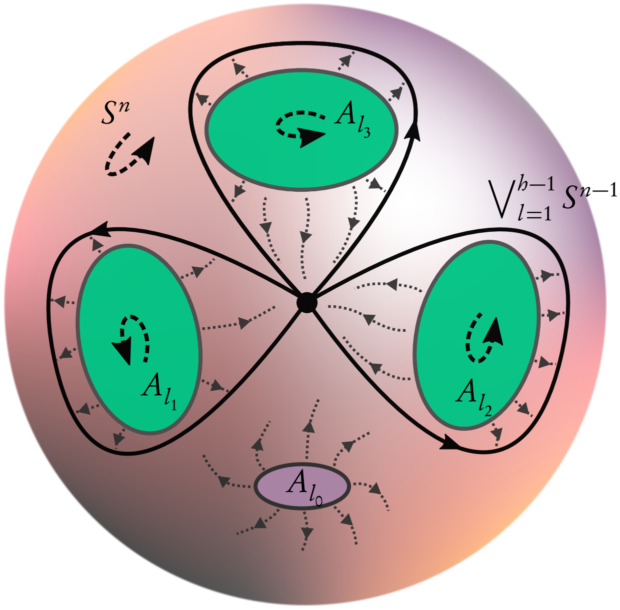


Figure B.1. – Retracting the punctured sphere on a wedge of lower-dimensional spheres. The disc A_{l_0} punctures the sphere making it an n-disc, which then retracts on the boundary of the remaining discs. The orientation on the discs A_l, induced by the ambient orientation of S^n, in turn induces an orientation on the bouquet of spheres.

B.2. *Technical lemma for Theorem 2.29*

PROPOSITION B.11. *Let $A_1, \dots, A_h$ be n-discs in S^n, embedded disjointly. Consider some orientation-preserving homeomorphisms*

$$a^l : (D^n, \partial D^n) \to \left(S^n, S^n \setminus \bigcup_{j=1}^{h} \mathring{A}_j \right) \quad \forall l \in \{1, \dots, h\} \tag{B.2.1}$$

which send D^n to the disc A_l. Let σ_n denote the standard orientation on S^n induced by the standard orientation ω_{n+1} of $\mathbb{R}^{n+1}$. Finally let $\hat{\imath} : (S^n, \varnothing) \to \left(S^n, S^n \setminus \bigcup_{j=1}^{h} \mathring{A}_j\right)$ be the inclusion. Then we have the following expression for $\hat{\imath}_* : H_n(S^n) \to H_n\left(S^n, S^n \setminus \bigcup_{j=1}^{h} \mathring{A}_j\right)$:

$$\hat{\imath}_* \sigma_n = \sum_{l=1}^{h} a_*^l \omega_n \tag{B.2.2}$$

Proof. We can excise $S^n \setminus \bigcup_j A_j$ from $\left(S^n, S^n \setminus \bigcup_j \mathring{A}_j\right)$ and so we obtain that

$$H_l\left(S^n, S^n \setminus \bigcup_j \mathring{A}_j\right) \cong H_l\left(\bigsqcup_j A_j, \bigsqcup_j \partial A_j\right) \cong \bigoplus_j H_l\left(A_j, \partial A_j\right) \quad \forall l \in \{0, \dots, n\} \tag{B.2.3}$$

Now, since a^l is orientation preserving, clearly it is an isomorphism onto $H_n(A_l, \partial A_l)$. So we have that $\{a_*^1\omega_n, \ldots, a_*^h\omega_n\}$ is a free generating set for the nth homology group of the punctured sphere. Hence we have an expression of the type

$$\hat{\imath}_*\sigma_n = \sum_{l=1}^{h} r^l a_*^l \omega_n \tag{B.2.4}$$

for some integers r^l. The claim is thus that these integers are all 1. Denote by $\hat{\imath}^l : (S^n, \varnothing) \to \left(S^n, S^n \setminus \mathring{A}_l\right)$ the inclusion. Notice that then

$$\hat{\imath}_* = \bigoplus_{l=1}^{h} \hat{\imath}_*^l : H_n(S^n) \to H_n\left(S^n, S^n \setminus \bigcup \mathring{A}_l\right) \cong \bigoplus_l H_n(A_l, \partial A_l) \tag{B.2.5}$$

so it suffices to show that $\hat{\imath}_*^1\sigma_n = a_*^1\omega_n$. But if we look at the standard orientation of $\mathbb{R}^{n+1}$, this is the generator $\omega_{n+1} \in H_{n+1}\left(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\}\right)$ which by definition is precisely the image of $\sigma_n \in H_n(S^n)$ under the natural isomorphism $H_{n+1}\left(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\}\right) \cong H_n(S^n)$ given by the long exact relative homology sequence of the couple $\left(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\}\right)$. Moreover, the standard orientation ω_{n+1} can be obtained from the standard orientation ω_n of $\mathbb{R}^n$, seen as a subspace of $\mathbb{R}^{n+1}$, by choosing the positive direction of the normal line in the decomposition $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$. This also orients $(D^n, \partial D^n)$ if we embed it in the $\mathbb{R}^n$ component. So since the homeomorphisms a^l we chose were orientation preserving, the previous reasoning forces the equality we want to prove. $\square$

C. Appendix to Chapter 3

C.1. On currents

Here I revise the basic elements we need on the theory of currents to understand the statements of Harvey & Lawson's theory. I follow [16, 19, 39] and my notes from Roberto Monti's course in Calculus of Variations. There can be two conventions when defining currents: the one of the geometric measure theorists is to call a k-current a bounded linear functional on compactly-supported k-forms, and this gives a homological theory; for Harvey and Lawson theory it is more convenient to define a k-current as a bounded linear functional on compactly-supported $(n-k)$-forms, giving a cohomological theory. We follow the latter route.

C.1.1. Basic definitions

Let M be an n-dimensional smooth manifold. Denote by $\Omega^k(M)$ the space of smooth k-forms on M, and $\Omega^k_c(M)$ the subspace of compactly supported k -forms. For $\theta \in \Omega^k_c(M)$ denote by $\operatorname{supp}\theta$ the support of the k-form θ. A current relates to k-forms in the same way a distribution relates to a function. Hence we must endow $\Omega^k_c(M)$ with a suitable family of seminorms. These seminorms are the direct generalization to sections of a vector bundle of the seminorm on the "test functions" used in distribution theory. For brevity, I use multi-index notation and the Einstein convention on summations.

Definition C.1. Let $\theta \in \Omega^k_c(M)$. Consider an atlas $\mathfrak{U}$ of M. Let $\theta = \theta_I(x)dx^I$ be the coordinate expression of θ in a chart $U \in \mathfrak{U}$ containing its support.

$$||\theta||_{C^h} = \sum_I \sup\left\{\left|\frac{\partial^h \theta_j(x)}{\partial x_j}\right| : j_l \in I,\ x \in \operatorname{supp}\theta\right\} \tag{C.1.1}$$

or in words, the sum of the sup norms of the h-th derivatives of the coordinate functions on the support of the form. Equivalently we say that a sequence $\{\theta_l\}_{l\in\mathbb{N}} \subset \Omega^k(M)$ *converges in C^h seminorm* to $\theta \in \Omega^k_c(M)$ if there is a compact set $K \Subset M$ containing all the supports of the θ_l and moreover, in one (and then any) coordinate chart on K, the h-th derivatives of the coordinates of the θ_l converge in sup norm to the h-th derivatives of θ.

This family of seminorms induces the C^∞-topology on $\Omega^k_c(M)$, which is characterized sequentially as follows: a sequence of smooth compactly supported k-forms $\{\theta_l\}_{l\in\mathbb{N}}$ converges to $\theta \in \Omega^k(M)$ if and only if there exists a compact $K \Subset M$ containing all the supports of θ, θ_l and the derivatives of any order of the coordinates of the θ_l in any chart converge uniformly to the derivatives of any order of the coordinates of θ.

Remark C.2. One has to show that this topology does not depend on the particular atlas chosen, and that the space $\Omega^k_c(M)$ is a Fréchet space under the topology generated by the family of seminorms $\{||-||_{C^h}\}_{h\in\mathbb{N}}$. These facts are true and descend a little more easily from the next

Remark C.3. Take an atlas $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of M, without loss of generality such that it trivializes the k-th exterior bundle, and consider the isomorphism, given by the coordinate charts:

$$\Omega^k_c(U_\alpha) \xrightarrow{\sim} C^\infty_c(\varphi_\alpha(U_\alpha))^{\binom{n}{k}} \tag{C.1.2}$$

For any open subset $V \subset \mathbb{R}^n$ we have the "test function topology" on $C^\infty_c(V)$, so that $C^\infty_c(\varphi_\alpha(U_\alpha))^{\binom{n}{k}}$ has the product topology. Now, we may include

$$\Omega^k_c(M) \hookrightarrow \prod_{\alpha\in A} C^\infty_c(\varphi_\alpha(U_\alpha))^{\binom{n}{k}} \tag{C.1.3}$$

and endow $\Omega^k_c(M)$ with the induced topology from the inclusion, where the r.h.s. has again the product topology. One may show that the choice of the atlas is inessential, and that this topology is the same as the one above.

Definition C.4. A *k-current* on M is a continuous linear functional $T\colon \Omega^{n-k}_c(M) \to \mathbb{R}$. Explicitly, it is a linear functional on the space of $(n-k)$-forms with compact supports, such that for all compact $K \Subset M$, there exists a $C > 0$ such that

$$|T\theta| \leq C||\theta||_{C^h} \ \forall h \in \mathbb{N}\ \forall \theta \in \Omega^{n-k}_c(M) : \operatorname{supp}\theta \subset K \tag{C.1.4}$$

The vector space of k-currents on M is denoted by $\mathscr{D}_k(M)$.

Example. Let $\Sigma \subset M$ be an oriented k-submanifold. Then there is naturally an $(n-k)$-current $[\![\Sigma]\!] : \Omega^k_c(M) \to \mathbb{R}$ defined by integration:

$$[\![\Sigma]\!]\theta = \int_\Sigma \theta \tag{C.1.5}$$

Linearity is obvious, while continuity is guaranteed by the finiteness of the Hausdorff measure of $\Sigma \cap \operatorname{supp}\theta$ in one (then any) chart – we can estimate the integral with the Hausdorff measure of $\Sigma \cap \operatorname{supp}\theta$ times the sup norm of the coefficients of θ, which is finite being the form smooth.

LEMMA C.5. *The space $\mathscr{D}_0$ is the space of Schwartz distributions on M.*

Remark C.6. Some authors say that currents are "k-forms with distributional coefficients". Currents and forms behave quite differently under morphisms though, since forms naturally pull back while currents naturally push forward.

Definition C.7. Let $U \subset M$ be a chart with coordinates $(x^1, \dots, x^n)$. For a multi-index I of length k, define the current $e_{(k),I} \in \mathscr{D}_k(U)$ as

$$e_{(k),I} dx^J = \delta^J_I \quad \forall J : |J| = n-k \tag{C.1.6}$$

Then every current $T \in \mathscr{D}_k(M)$ can be written locally as $T = T^I e_{(k),I}$ for Schwartz distributions $\{T^I : I \text{ multi-index of length } k\}$. We call these distributions the *coefficients* of T in the chart U.

Definition C.8. Let T be a k-current. The *support* of T is

$$\operatorname{supp} T = M \setminus \bigcup \{U \subset M \text{ open} : T\theta = 0\ \forall \theta \in \Omega^k_c(M) \text{ s.t. } \operatorname{supp}\theta \subset U\} \tag{C.1.7}$$

We say that T is a current of compact support, $T \in \mathscr{D}^c_k(M)$, if its support is compact.

PROPOSITION C.9 (Poincaré duality). *Let M be oriented, or $\Omega^\bullet(M)$ twisted over the orientation bundle. There is a linear continuous injective morphism $\mathbf{I}^k : \Omega^k(M) \to \mathscr{D}_k(M)$.*

Proof. In the hypotheses of the proposition we can integrate n-forms over M. Define

$$\begin{aligned} \mathbf{I}^k : \Omega^k_c(M) &\to \mathscr{D}_k(M) \\ \theta &\mapsto \mathbf{I}^k\theta = \theta \smile - = [\![M]\!](\theta \wedge -) : \Omega^{n-k}_c \to \mathbb{R} \end{aligned} \tag{C.1.8}$$

We could call this the geometrical realization of the Poincaré duality, and it is just the linear morphism given by cupping with a general k-form. It is well defined because we integrate against an $(n-k)$-form of compact support, so the domain is allowed to be without compact supports. Linearity is clear. Injectivity goes as follows: if $\mathbf{I}^k\theta = 0 \in \mathscr{D}_k(M)$ this means that $\theta \smile \xi = 0$ for all $\xi \in \Omega^{n-k}_c(M)$, but this is possible only for $\theta = 0$, being the cup product a non-degenerate pairing. For continuity, since $[\![M]\!]$ is a current, it suffices to show that if $\xi_l \to \xi$ in Ω^{n-k}_c then also $\theta \wedge \xi_l \to \theta \wedge \xi$ in Ω^n_c. But $\theta \wedge \xi_l - \theta \wedge \xi = \theta \wedge (\xi_l - \xi)$, so all we have to show is that if $\xi_l \to 0$ in $\Omega^{n-k}_c(M)$ then also $\theta \wedge \xi_l \to 0$ in $\Omega^n_c(M)$ – simply stated, that the wedge product is continuous. But this is trivial because if in charts we have $\theta = \theta_I dx^I$ and $\xi_l = \xi_{l,J} dx^J$, then $\theta \wedge \xi_l = \theta_I \xi_{l,J} dx^I \wedge dx^J$, so if $\xi_{l,J}$ and all its derivatives go to zero, by Leibniz also $\theta_I \xi_{l,J}$ does. □

C.1.2. Modes of convergence, Operations

Since $\mathscr{D}_k(M)$ is the topological dual of $\Omega^{n-k}_c(M)$, we immediately have the notion of weak convergence of k-currents.

Definition C.10. A sequence $\{T_j\}_{j\in\mathbb{N}} \subset \mathscr{D}_k(M)$ is said to *converge weakly* to a current $T \in \mathscr{D}_k(M)$, denoted $T_j \overset{*}{\rightharpoonup} T$ if for all $\theta \in \Omega^{n-k}_c(M)$ we have $T_j\theta \to T\theta$.

Moreover, we can endow $\Omega^{n-k}_c(M)$ with a norm, so that we also have the operator norm on $\mathscr{D}_k(M)$, and the respective notion of convergence.

Definition C.11. 1. Let $\theta \in \Omega_c^l(M)$. The *comass* of θ is

$$||\theta||^M = \sup_{x \in \operatorname{supp}\theta} \sup_{\tau \in \Lambda_l(T_xM)} \frac{|\theta(\tau)|}{||\tau||} \tag{C.1.9}$$

where the norm of an l-vector $\tau \in \Lambda_l(T_xM)$ is just its norm as a vector in $\mathbb{R}^{\binom{n}{l}}$. This is a sort of operator norm on l-forms, if we think of them as operators on l-vector fields.

2. Let $T \in \mathscr{D}_k(M)$ be a current. The *mass* of T is

$$||T||_M = \sup\left\{|T\theta| : \theta \in \Omega_c^{n-k}(M),\ ||\theta||^M \leq 1\right\} \tag{C.1.10}$$

This is not always finite. We denote by $\mathscr{M}_k(M)$ the subspace of currents of finite mass.

3. We say that a sequence of currents $\{T_j\}_{j\in\mathbb{N}} \subset \mathscr{D}_k$ *converges in mass*, denoted $T_j \xrightarrow{\mathrm{M}} T$, if the sequence $||T_j - T||_M \to 0$ in $\mathbb{R}$. Clearly if $T_j \xrightarrow{\mathrm{M}} T$ then $T_j \xrightarrow{*} T$.

Example. The mass of the current of integration $[\![\Sigma]\!]$ of a submanifold is its Hausdorff measure. If it is a compact submanifold, then it is surely finite. The converse is not always true…

Remark C.12. The advantage of mass lies in its compactness and lower semicontinuity properties, essential for the calculus of variations. Namely,

1. If a sequence of currents is bounded in mass, then it is weakly relatively compact. In particular if a sequence is bounded in mass, it admits a subsequence which converges weakly.
2. The functional $||-||_M : \mathscr{D}_k(M) \to [0,\infty]$ is lower semicontinuous, because the mass is a dual norm. Hence it satisfies the direct method of the calculus of variations; if we are looking for the minima of mass over a compact subset of currents, then it always is reached. This is the content of the weak form of the Plateau problem.

We define operations on currents which are dual to the operations on differential forms. To fix notations, if $\mathbb{X} \in \Gamma(\Lambda_h(TM))$ is an h-vector field on M and θ is a k-form with $k \geq h$, then define the inner product $i_{\mathbb{X}}\theta$ as the $(k-h)$-form obtained by applying θ to $\mathbb{X}$.

Definition C.13. Let $T \in \mathscr{D}_k$.

1. Let $\xi \in \Omega^h(M)$, where $h \leq k$. Define the *inner product current* $T \llcorner \xi \in \mathscr{D}_{k-h}(M)$ as $(T \llcorner \xi)\theta = T(\xi \wedge \theta)$. Notice that since the test form has compact support, the form with which we contract T doesn't have to have compact support.
2. Let $\mathbb{X} \in \Gamma(\Lambda_h(TM))$ be an h-vector field on M. Define the *exterior product current* $T \wedge \mathbb{X} \in \mathscr{D}_{k+h}(M)$ as $(T \wedge \mathbb{X})\theta = T(i_{\mathbb{X}}\theta)$.
3. Let N be another smooth manifold and $f : M \to N$ a differentiable function. The *push-forward* current $f_*T \in \mathscr{D}_k(N)$ is the current on N defined by $f_*T(\theta) = T(f^*\theta)$.

Remark C.14. To be sure that the results of these operations are currents, one must check that they are continuous. This follows from the continuity of the dual operations on forms.

LEMMA C.15. *Let $f : M \to N$. Then if $\Sigma \hookrightarrow M$ is an oriented submanifold of M, $f_* [\![\Sigma]\!] = [\![f(\Sigma)]\!]$ (we're identifying Σ with the image of its immersion).*

Proof. Integrating a form on Σ means integrating the pull-back of the immersion $\iota \colon \Sigma \hookrightarrow M$. But then

$$f_* [\![\Sigma]\!](\theta) = [\![\Sigma]\!](f^*\theta) = \int_\Sigma \iota^* f^* \theta = \int_\Sigma (f \circ \iota)^* \theta = [\![f(\Sigma)]\!](\theta) \tag{C.1.11}$$

□

C.1.3. Cohomology of currents

Definition C.16. Let $T \in \mathscr{D}_k(M)$. The *boundary* of T is the $(k+1)$-current $\partial T \in \mathscr{D}_{k+1}(M)$ defined as $\partial T(\theta) = T(d\theta)$. Clearly $\partial^2 T = 0$. The *coboundary* of T is the $(k+1)$-current $d_{\mathscr{D}} T = (-1)^{k+1} \partial T$.

We have that $(\mathscr{D}_\bullet(M), d_{\mathscr{D}})$ is a cochain complex, so we may compute its cohomology.

Definition C.17. Define the cohomology groups

$$H^k_{\mathscr{D}}(M; \mathbb{R}) = \ker d^k_{\mathscr{D}} / d^{k-1}_{\mathscr{D}} \left(\mathscr{D}_{k-1}(M) \right) \tag{C.1.12}$$

Example. If Σ is an orientable submanifold with boundary of M then by the Stokes theorem it is clear that $\partial [\![\Sigma]\!] = [\![\partial \Sigma]\!]$. As a corollary,

PROPOSITION C.18. *Let M be oriented, or the differential forms twisted over the orientation bundle. $\mathbf{I}^\bullet$ is a cochain morphism: $d_{\mathscr{D}} \circ \mathbf{I}^k = \mathbf{I}^k \circ d$.*

Proof. Indeed if $\theta \in \Omega^k(M)$ and $\xi \in \Omega_c^{n-(k+1)}(M)$, since M does not have a boundary, we have by Stokes formula that

$$\int_M d(\theta \wedge \xi) = \int_{\partial M = \varnothing} \theta \wedge \xi = 0 \tag{C.1.13}$$

But then $\int_M d\theta \wedge \xi = (-1)^{k+1} \int_M \theta \wedge d\xi$. Hence

$$\partial \mathbf{I}^k \theta(\xi) = \mathbf{I}^k \theta(d\xi) = (-1)^{k+1} [\mathbf{I}^k(d\theta)](\xi) \tag{C.1.14}$$

□

The Poincaré duality morphism is an injective cochain morphism. It is natural to ask ourselves if it is an isomorphism at the level of cohomology. The main result is the following

THEOREM C.19 ([19, Section 5.3.2, Proposition 1]). *Every class in* $H^k_{\mathscr{D}}(M;\mathbb{R})$ *admits a representative of the form* $\mathbf{I}^k\theta$ *where* $\theta \in \Omega^k(M)$ *is a* closed *form.*

The proof of the theorem goes through a regularization argument, where one shows that for each current $T \in \mathscr{D}_k(M)$ we can pick a sequence of smooth k-forms θ_l such that $\mathbf{I}\theta_l \to T$ in mass. Moreover if T is closed, the forms θ_l can be chosen closed. With these facts, combined with the key Proposition C.18, one shows that each closed current is cohomologous to a current which is the Poincaré image of a closed form.

COROLLARY C.20. *The cohomology of currents in* M *is isomorphic to the deRham cohomology of* M.

Remark C.21. There is a more elegant but less direct proof of this fact, which I found in the appendix of [24]. They show that the constant sheaf $\underline{\mathbb{R}}$ has an acyclic resolution in terms of the sheaf of germs of currents. This implies that the cohomology of that complex computes the real singular cohomology of M. This point of view simplifies the original treatment of [16] in the recovering of the integer singular cohomology of M by means of integral currents.

D. Appendix to Chapter 4

D.1. More on operators defined by currents

Let M be an n-manifold. Recall the C^∞ topology on the compactly supported forms of Definition C.1 and the mass norm on currents of Definition C.11.

PROPOSITION D.1. *Consider an operator* $\mathbf{P}\colon \Omega_c^k(M) \to \mathscr{D}^k(M)$ *which is continuous if we endow the spaces respectively with the* C^∞ *topology and the mass topology. Then for each* $1 \leq p < \infty$ *there exists an unique linear extension* $\widehat{\mathbf{P}}\colon L^p\Omega^k(M) \to \mathscr{D}^k(M)$ *which is continuous if we endow the spaces respectively with the* L^p *norm topology and the mass norm topology.*

Proof. By the Hahn-Banach theorem, if we show that $\mathbf{P}$ is continuous also in the L^2-topology on $\Omega_c^k(M)$ – which is thus a *non-closed* linear subspace of $L^p\Omega^k(M)$ – then there exists an extension to the whole L^p space. Moreover, since $\Omega_c^k(M)$ is dense, this extension $\widehat{\mathbf{P}}$ is necessarily unique. But clearly, since $\mathbf{P}$ is C^∞-continuous, it is L^∞-continuous, because the C^∞ topology is the initial topology with respect to the topology generated by the sup norms of the coefficients and all their derivatives. It follows that $\mathbf{P}$ is also L^p-continuous for all $1 \leq p \leq \infty$. □

The following proposition refines the tautological weak continuity of the operators of the kernel calculus shown in Lemma 3.7. By the previous proposition, it implies we may extend them to the L^2-forms.

PROPOSITION D.2. *Let* N *be an* n'*-manifold and* $K \in \mathscr{D}^l(N \times M)$*. For* $k \leq l$*, consider the linear operator* $\mathbf{K}\colon \Omega_c^{n'-k}(N) \to \mathscr{D}^{l-k}$ *given by* $\theta \mapsto \mathbf{K}(\theta) = (\pi_M)_* K \llcorner \pi_N^*\theta$*. Then* $\mathbf{K}$ *is continuous when we endow* $\Omega_c^{n'-k}(N)$ *with the* C^∞ *topology and* $\mathscr{D}^{l-k}$ *with the mass norm topology.*

Proof. We have to show that if $\theta_l \to \theta$ in the C^∞-topology, then $||\mathbf{K}(\theta_l) - \mathbf{K}(\theta)||_M \to 0$ in $\mathbb{R}$. Being $\mathbf{K}$ linear, it suffices to show that if $\theta_l \to 0$ in C^∞ then $||\mathbf{K}(\theta_l)||_M \to 0$. Now,

$$||\mathbf{K}(\theta_l)||_M = \sup\left\{|K(\pi_N^*\theta \wedge \pi_M^*\xi)| : ||\xi||^M \leq 1\right\} \tag{D.1.1}$$

and by writing the definition of the comass norm, it is clear that the ξ which have comass less than 1 must have coefficients bounded by 1. Hence the forms $\pi_N^*\theta \wedge \pi_M^*\xi \to 0$ in C^∞ and by continuity of K, this means that $|\mathbf{K}(\theta_l)\xi| \to 0$ in $\mathbb{R}$. So also the sup must go to zero, from which we conclude. □

Bibliography

[1] A. Abbondandolo and P. Majer. *A Morse complex for infinite dimensional manifolds. I. Adv. Math.* 197.(**2**) (2005), pp. 321–410.

[2] A. Abbondandolo and M. Schwarz. *The role of the Legendre transform in the study of the Floer complex of cotangent bundles*. *Comm. Pure Appl. Math.* 68.(**11**) (2015), pp. 1885–1945.

[3] R. Abraham and J. E. Marsden. *Foundations of mechanics.* Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978, pp. xxii+m–xvi+806.

[4] M. Audin and M. Damian. *Morse theory and Floer homology.* Universitext. Springer, London; EDP Sciences, Les Ulis, 2014, pp. xiv+596.

[5] V. Baladi, J.-P. Eckmann, and D. Ruelle. *Resonances for intermittent systems*. *Nonlinearity* 2.(**1**) (1989), pp. 119–135.

[6] V. Baladi. *Dynamical zeta functions and dynamical determinants for hyperbolic maps.* Vol. 68. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, 2018, pp. xv+291.

[7] V. Baladi and M. Tsujii. *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*. *Ann. Inst. Fourier (Grenoble)* 57.(**1**) (2007), pp. 127–154.

[8] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators.* Vol. 298. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992, pp. viii+369.

[9] J.-M. Bismut. *The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem*. *J. Funct. Anal.* 57.(**1**) (1984), pp. 56–99.

[10] J.-M. Bismut. *The Atiyah-Singer theorems: a probabilistic approach. II. The Lefschetz fixed point formulas*. *J. Funct. Anal.* 57.(**3**) (1984), pp. 329–348.

[11] J.-M. Bismut. *The Witten complex and the degenerate Morse inequalities*. *J. Differential Geom.* 23.(**3**) (1986), pp. 207–240.

[12] J.-M. Bismut. *Localization formulas, superconnections, and the index theorem for families*. *Comm. Math. Phys.* 103.(**1**) (1986), pp. 127–166.

[13] R. Bott. *An application of the Morse theory to the topology of Lie-groups*. *Bull. Soc. Math. France* 84 (1956), pp. 251–281.

[14] R. Bowen and B. Marcus. *Unique ergodicity for horocycle foliations*. *Israel J. Math.* 26.(**1**) (1977), pp. 43–67.

[15] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599.

[16] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969, pp. xiv+676.

[17] A. Floer. *Morse theory for Lagrangian intersections*. *J. Differential Geom.* 28.(**3**) (1988), pp. 513–547.

[18] A. Floer. *Witten's complex and infinite-dimensional Morse theory*. *J. Differential Geom.* 30.(**1**) (1989), pp. 207–221.

[19] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations. I.* Vol. 37. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xxiv+711.

[20] J. Glimm and A. Jaffe. *Quantum physics.* Second. Springer-Verlag, New York, 1987, pp. xxii+535.

[21] F. R. Harvey and H. B. Lawson Jr. *Morse theory and Stokes' theorem*. Surv. Differ. Geom. Int. Press, Somerville, MA, 2000, pp. 259–311.

[22] F. R. Harvey and H. B. Lawson Jr. *Finite volume flows and Morse theory*. *Ann. of Math. (2)* 153.(**1**) (2001), pp. 1–25.

[23] F. R. Harvey and G. Minervini. *Morse Novikov theory and cohomology with forward supports*. *Math. Ann.* 335.(**4**) (2006), pp. 787–818.

[24] R. Harvey and J. Zweck. *Stiefel-Whitney currents*. *J. Geom. Anal.* 8.(**5**) (1998), pp. 809–844.

[25] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002, pp. xii+544.

[26] B. Helffer and J. Sjöstrand. *Puits Multiples en Mechanique Semi-Classique IV – Etude du Complexe de Witten*. *Communications in Partial Differential Equations* 10.(**3**) (1985), pp. 245–340. eprint: `https://doi.org/10.1080/03605308508820379`.

[27] M. W. Hirsch. *Differential topology.* Vol. 33. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994, pp. x+222.

[28] F. Laudenbach. *On the Thom-Smale complex. Astérisque* (**205**) (1992), p. 235.

[29] C. Liverani. *Fredholm determinants, Anosov maps and Ruelle resonances*. *Discrete Contin. Dyn. Syst.* 13.(**5**) (2005), pp. 1203–1215.

[30] V. Mathai. *Heat kernels and Thom forms*. *J. Funct. Anal.* 104.(**1**) (1992), pp. 34–46.

[31] V. Mathai and D. Quillen. *Superconnections, Thom classes, and equivariant differential forms*. *Topology* 25.(**1**) (1986), pp. 85–110.

[32] J. Milnor. *Morse theory*. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963, pp. vi+153.

[33] J. Milnor. *Lectures on the h-cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965, pp. v+116.

[34] G. Minervini. *A current approach to Morse and Novikov theories*. *Rend. Mat. Appl. (7)* 36.(**3-4**) (2015), pp. 95–195.

[35] M. Morse. *The critical points of functions and the calculus of variations in the large*. *Bull. Amer. Math. Soc.* 35.(**1**) (1929), pp. 38–54.

[36] M. Morse. *The calculus of variations in the large*. Vol. 18. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1996, pp. xii+368.

[37] S. P. Novikov. *Multivalued functions and functionals. An analogue of the Morse theory*. *Dokl. Akad. Nauk SSSR* 260.(**1**) (1981), pp. 31–35.

[38] M. Pollicott. *Meromorphic extensions of generalised zeta functions*. *Invent. Math.* 85.(**1**) (1986), pp. 147–164.

[39] G. de Rham. *Differentiable manifolds*. Vol. 266. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1984, pp. x+167.

[40] D. Ruelle. *Resonances for Axiom A flows*. *J. Differential Geom.* 25.(**1**) (1987), pp. 99–116.

[41] M. Shubin. *Novikov inequalities for vector fields*. *The Gelfand Mathematical Seminars, 1993–1995*. Gelfand Math. Sem. Birkhäuser Boston, Boston, MA, 1996, pp. 243–274.

[42] S. Smale. *Generalized Poincaré's conjecture in dimensions greater than four*. *Ann. of Math. (2)* 74 (1961), pp. 391–406.

[43] S. Smale. *On gradient dynamical systems*. *Ann. of Math. (2)* 74 (1961), pp. 199–206.

[44] S. Smale. *On the structure of manifolds*. *Amer. J. Math.* 84 (1962), pp. 387–399.

[45] S. Smale. *Differentiable dynamical systems*. *Bull. Amer. Math. Soc.* 73 (1967), pp. 747–817.

[46] R. Thom. *Sur une partition en cellules associée à une fonction sur une variété*. *C. R. Acad. Sci. Paris* 228 (1949), pp. 973–975.

[47] N. Viet Dang and G. Riviere. *Spectral analysis of Morse-Smale gradient flows*. *ArXiv e-prints* (May 2016). arXiv: `1605.05516 [math.DS]`.

[48] N. Viet Dang and G. Riviere. *Pollicott-Ruelle spectrum and Witten Laplacians*. *ArXiv e-prints* (Sept. 2017). arXiv: `1709.04265 [math.DS]`.

[49] N. Viet Dang and G. Riviere. *Spectral analysis of Morse-Smale flows I: construction of the anisotropic spaces*. *ArXiv e-prints* (Mar. 2017). arXiv: `1703.08040 [math-ph]`.

[50] N. Viet Dang and G. Riviere. *Spectral analysis of Morse-Smale flows II: resonances and resonant states*. *ArXiv e-prints* (Mar. 2017). arXiv: `1703.08038 [math-ph]`.

[51] J. Weber. *The Morse-Witten complex via dynamical systems*. *Expo. Math.* 24.(**2**) (2006), pp. 127–159.

[52] E. Witten. *Supersymmetry and Morse theory*. *J. Differential Geom.* 17.(**4**) (1982), 661–692 (1983).

[53] W. Zhang. *Lectures on Chern-Weil theory and Witten deformations*. Vol. 4. Nankai Tracts in Mathematics. World Scientific Publishing Co., Inc., River Edge, NJ, 2001, pp. xii+117.