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Primordial trispectra from gauge fields in Inflation

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Abstract

Modern cosmology is remarkably successful in describing the evolution of the Universe. Nevertheless, the standard Hot Big Bang model alone cannot account for several fundamental features of the observed Universe. The inflationary paradigm was introduced to address these shortcomings and has since become the leading framework for explaining the origin of primordial density perturbations and the formation of cosmic structures. Among the many inflationary scenarios proposed in the literature, models involving axion fields coupled to gauge fields have attracted considerable interest because of their rich phenomenology and their ability to generate distinctive observational signatures, including parity violation and primordial non-Gaussianity. A particularly notable example is Chromo Natural Inflation, in which an axion field acts as the inflaton and interacts with a non-Abelian SU(2) gauge field through a Chern–Simons coupling. This interaction gives rise to the production of chiral gravitational waves and other characteristic signatures. However, despite its theoretical appeal, the original Chromo Natural Inflation scenario is unable to simultaneously satisfy current observational constraints on the scalar (i.e. density) and tensor (i.e. gravitational waves) perturbation spectra. To overcome these difficulties, the model can be extended to Spectator Chromo Natural Inflation, where Inflation is driven by an independent inflaton field, while the axion–gauge sector acts as a spectator sector. While the tensor perturbations of this framework have been extensively investigated, the scalar sector remains relatively unexplored (in particular with regard to higher-order correlation functions), despite possessing a remarkably rich interaction structure.

The primary objective of this thesis is to investigate the scalar trispectrum (i.e. the Fourier counterpart of the 4-point correlation function) generated by axion fluctuations in Spectator Chromo Natural Inflation and to determine whether parity-violating signatures can arise purely from the scalar sector of the theory. To this end, the action governing scalar perturbations is expanded up to fourth order in fluctuations. After integrating out the non-dynamical degrees of freedom, the corresponding interaction Hamiltonian is derived and employed within the In-in formalism to compute the four-point correlation function. Particular attention is devoted to the contributions generated by scalar exchange and contact interaction diagrams, which constitute the sources of the scalar trispectrum considered in this work. The resulting trispectrum is then analyzed through its shape function, focusing on the equilateral momentum configuration. Numerical evaluations are performed to determine its kinematic dependence and to extract the associated non-linearity parameter t_{NL}, which characterizes the amplitude of the generated non-Gaussian signal in the regular tetrahedron limit.

The analysis leads to two main conclusions. First, no parity-violating signatures are found in the purely scalar trispectrum, because the correlation functions computed in this work do not contain the pseudoscalar momentum structures required to generate parity-odd contributions. Second, the scalar trispectrum generated by the considered interactions is found to be suppressed in the regular tetrahedron limit. The results presented in this thesis contribute to the phenomenological characterization of Spectator Chromo Natural Inflation beyond its tensor predictions and provide new theoretical insights into the structure of scalar primordial non-Gaussianities in axion–gauge inflationary models.

Contents

Introduction

During the twentieth century, the Hot Big Bang model emerged as the standard framework for describing the large-scale evolution of the Universe. Based on the principles of General Relativity and the Cosmological Principle, it provides a remarkably successful account of a wide range of cosmological observations, including the expansion of the Universe, the abundance of light elements produced during primordial nucleosynthesis, and the existence of the Cosmic Microwave Background. Despite its impressive observational achievements, the standard Hot Big Bang scenario leaves several fundamental questions unanswered. In particular, it does not provide a satisfactory explanation for a number of apparently fine-tuned features of the observed Universe. These shortcomings are grouped into three major problems: the horizon problem, the flatness problem, and the primordial relics problem. The solution to these problems came in the early 1980s with the introduction of the inflationary paradigm [1, 2, 3, 4]. Inflation postulates a period of accelerated expansion occurring in the very early Universe, prior to the conventional radiation-dominated era. Being originally proposed as a mechanism to resolve the shortcomings of the Hot Big Bang model, Inflation has gradually developed into the leading theoretical framework for describing the origin of the observable Universe: indeed, it provides a natural mechanism for generating the primordial perturbations that later evolved into the large-scale structure observed today. A particularly appealing feature of Inflation is its ability to connect quantum physics with cosmological observations. Quantum fluctuations generated during the inflationary epoch are stretched to astrophysical scales by the rapid expansion of spacetime, becoming the seeds of the density perturbations from which baryonic matter and dark matter subsequently cluster under the action of gravity. For this reason, Inflation currently represents the most successful and widely accepted framework for explaining both the early evolution of the Universe and the origin of cosmic structures.

Although accelerated expansion can, in principle, arise from a variety of theoretical mechanisms, Inflation is most commonly realized through the dynamics of scalar fields. The earliest models relied on the decay of a false vacuum state [1], while subsequent developments showed that a slowly rolling scalar field can provide a more realistic description of the inflationary phase [2]. In its simplest implementation, Inflation is driven by a canonical scalar field minimally coupled to gravity, although numerous extensions involving non-minimal couplings and modified gravitational sectors have also been proposed. Nevertheless, scalar-field models are not free from conceptual difficulties. One of the main challenges concerns the identity of the inflaton itself, which remains unknown from the perspective of particle physics. Moreover, successful reheating generally requires the inflaton to couple to Standard Model degrees of freedom, introducing a number of additional interactions whose origin is often unclear. These issues have motivated the exploration of alternative realizations of Inflation involving fields other than a single canonical scalar. Among these alternatives are models in which non-Abelian gauge fields actively participate in the inflationary dynamics. Such scenarios exploit the rich structure of gauge theories to generate novel phenomenological signatures. Another particularly attractive possibility is represented by models in which an axion field plays the role of the inflaton. Owing to the approximate shift symmetry that naturally protects the flatness of its potential, the axion provides a compelling candidate for driving a prolonged period of accelerated expansion. A notable example belonging to this class is Chromo Natural Inflation [5], where an axion field is coupled through a Chern-Simons interaction to an SU(2) gauge sector. The interaction between the axion and the gauge fields introduces an additional friction mechanism that allows slow-roll Inflation to be sustained even for sub-Planckian values of the axion decay constant. Furthermore, the gauge sector gives rise to a particularly rich phenomenology, including the production of chiral gravitational waves. The key ingredient is the isotropic background configuration of the SU(2) gauge field, which enables the gauge field to acquire effective tensor degrees of freedom. These tensor perturbations couple linearly to the tensor modes of the metric and can therefore act as an efficient source of primordial gravitational waves. A distinctive feature of this mechanism is that the two helicity states of the gauge field tensor perturbations evolve differently. Due to the parity-violating nature of the Chern-Simons interaction, one polarization undergoes a transient tachyonic instability around horizon crossing and is consequently amplified, while the opposite helicity remains essentially unaffected. Since the amplified gauge field tensor mode sources gravitational waves of the same helicity, the resulting gravitational waves are chiral [6]. This characteristic prediction has attracted considerable attention, as it provides a potentially observable signature capable of distinguishing these models from conventional inflationary scenarios. Unfortunately, studies have shown that the model is not observationally viable [6, 7]. A successful way to overcome this tension with observational bounds is to extend the theory by introducing an additional scalar field that is responsible for driving Inflation, while the axion-gauge sector plays only a spectator role. In this setup, the energy density of the Universe is dominated by the inflaton field, whereas the axion and the SU(2) gauge field remain dynamically active and continue to produce their distinctive phenomenology. This framework is known as Spectator Chromo Natural Inflation and will constitute the subject of investigation throughout this thesis.

The interest in this model is motivated by the presence of the Chern-Simons interaction, which couples the axion field χ to the non-Abelian gauge sector through the operator $\chi F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$, where $F^a_{\mu\nu}$ denotes the gauge field strength tensor and $\tilde{F}^{a\mu\nu} \propto \epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta}$ is its dual, with $\epsilon^{\mu\nu\alpha\beta}$ the totally antisymmetric Levi-Civita tensor. As discussed previously, this term provides the source of parity violation and is responsible for the chiral amplification of tensor fluctuations. Most of the existing literature has focused on the tensor sector; however, the gauge field perturbations also contain scalar degrees of freedom. This thesis focuses explicitly on characterizing the scalar fluctuations arising in the Spectator Chromo Natural Inflation scenario. Our investigation is further motivated by recent developments in the study of large-scale structure. In particular, two independent analyses [8, 9] claimed the detection of potential parity-violating signatures in the galaxy four-point correlation function extracted from the data release of the BOSS survey [10]. Although these claims generated considerable interest, subsequent investigations reached different conclusions. In particular, an independent analysis employing alternative mock catalogs found no statistically significant evidence for parity violation [11], a result that was later reinforced by additional studies such as [12]. Despite the absence of a definitive observational detection, the possibility of parity violation in large-scale structure remains a topic of significant theoretical interest. On the one hand, parity-violating observables are often associated with physics beyond the standard cosmological paradigm and can therefore provide a powerful probe of the early Universe. On the other hand, the next generation of galaxy surveys is expected to achieve substantially improved sensitivity, making it timely to investigate theoretical scenarios capable of generating such signatures. Moreover, parity-violating signals in scalar correlation functions are intrinsically linked to primordial non-Gaussianity [13], offering a complementary window into the dynamics of Inflation.

The primary goal of this thesis is therefore to compute the scalar trispectrum sourced

by axion fluctuations[1], and to determine whether the scalar sector of Spectator Chromo Natural Inflation can yield parity violation. Specifically, we evaluate the contributions associated with the interaction diagrams shown in Figure (3.1) and investigate if they produce parity-odd components in the resulting four-point function. Finally, following the derivation of the trispectrum, we will investigate its kinematic dependence by exploring a specific configuration in momentum space. An analysis of the shape topology, supported by numerical evaluations and graphical representations, will allow us to extract the amplitudes of the two contributions and, ultimately, to quantify the non-linearity parameter t_{NL}.
Natural units are adopted throughout this thesis, with $c = \hbar = 1$.

Outline

The thesis is structured as follows:

Chapter (1): We review the standard Hot Big Bang model and discuss the main conceptual issues that motivate the introduction of an inflationary phase. We then introduce the inflationary paradigm as a solution to these shortcomings and present a simple single-field slow-roll inflationary model; moreover we explore the generation and evolution of primordial quantum fluctuations in this context.

Chapter (2): We review the Chromo Natural and Spectator Chromo Natural Inflation. After introducing the theoretical framework of both models, we discuss their background dynamics and analyze the behavior of the corresponding cosmological perturbations.

Chapter (3): This chapter contains the first original contribution of this work. We begin by introducing the interaction diagrams relevant for the computation of the scalar trispectrum. We then derive the expansion of the Spectator Chromo Natural Inflation action up to fourth order in perturbations for the axion-gauge field sector and construct the corresponding Hamiltonian formulation. After reviewing the In-in formalism (which provides the theoretical framework for the computation of the correlation functions), we solve the equations of motion

[1]As will be shown throughout this work, the trispectrum, which is the Fourier transform of the connected four-point correlation function, constitutes the lowest-order purely scalar statistic capable of exhibiting parity-violating signatures.

for the relevant perturbations and derive the interaction Hamiltonians that enter the trispectrum calculation.

Chapter (4): We evaluate the two contributions to the scalar trispectrum considered in this work and investigate their properties in momentum space. Particular attention is devoted to the equilateral configuration, for which we derive an explicit expression for the parameter t_{NL} in the regular tetrahedron limit and discuss the corresponding shape function. Numerical results and graphical representations are presented in order to illustrate the amplitude and characteristic features of the generated non-Gaussian signal.

Chapter (5): Finally, we summarize the main results obtained throughout this work and discusses their implications for parity violation and primordial non-Gaussianity in Spectator Chromo Natural Inflation. We conclude by outlining possible directions for future investigations.

Chapter 1

Hot Big Bang Model

In this chapter, we will briefly review some basic notions about the standard Hot Big Bang (HBB) model. We begin by outlining its geometric framework and dynamical evolution, and then we will explore the shortcomings of the model, which ultimately led to the formulation of the inflationary paradigm.

1.1 The Friedmann-Lemaître-Robertson-Walker Metric

The starting point of Modern Cosmology is the assumption of the validity of the Cosmological Principle, which states that the Universe appears to be homogeneous and isotropic on large scales[1]. In particular, homogeneous means that the characteristics of the Universe are the same regardless of the point in which the observer is, while the isotropy condition is related to the fact that the properties of the Universe as observed in a certain direction are the same even if they were observed in another direction. But in order for the Cosmological Principle to be true, we have to consider a special type of observer: the comoving one. A comoving observer is the one who is at rest with respect to the cosmic fluid. This means that a comoving observer sees the Cosmic Microwave Background (CMB) radiation as isotropic, so he determines the same temperature in all directions (neglecting the intrinsic anisotropies of the CMB itself).
To describe a Universe with such properties, we can use the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, whose line element is given by

[1]Large scales stands for distances ≥ 100 Mpc.

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \right) \tag{1.1}$$

where t is the physical time, $a(t)$ is the scale factor, k is the spatial curvature and r, θ, ϕ are comoving spherical coordinates. These comoving terms do not depend on time, since they do not feel the expansion of the Universe; indeed, we can define the physical distance (the one subjected to the stretching of the Universe) as the comoving distance times the scale factor. From this we see the importance of the scale factor $a(t)$: varying it corresponds to the rescaling of all space. The parameter k is a constant, which can be always normalized to 0, + 1 or - 1. These values are related to a flat Universe, a closed or an open one, respectively.

1.2 The Friedmann Equations

Now we want to discuss the dynamics of the Universe, and a way to do it could be to take into account the variational principle: varying the action which describes the fields content of the Universe will give us the equation of motion. Unfortunately, we do not know the exact action of all the possible fields, so we should look for an alternative. Another approach consists in the consideration of Einstein's equations, which are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, G is the Newton constant and $T_{\mu\nu}$ is the stress-energy tensor. The Ricci tensor is the contraction of the first and third index of the Riemann tensor, which is

$$R^{\lambda}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\sigma\nu} - \partial_\nu \Gamma^{\lambda}_{\sigma\mu} + \Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\mu\sigma} \tag{1.3}$$

while the Ricci scalar is the contraction of the two remaining indices of the Ricci tensor. In order to determine $R_{\mu\nu}$ (and so R) we need the expression of the Christoffel symbols, which is

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\rho} \right) \tag{1.4}$$

Now, we can focus our attention on the stress-energy tensor $T_{\mu\nu}$: we do not know it, but we can recover its expression considering the symmetries of the FLRW metric (1.1); so, if we rely on the property of homogeneity and isotropy, we obtain a tensor that has the form of

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \tag{1.5}$$

This is the stress-energy tensor of a perfect fluid, where $g_{\mu\nu}$ is the metric, $P(t)$ and $\rho(t)$ are respectively the isotropic pressure and the energy density of the fluid and u_μ is the covariant expression for the four-velocity of the fluid $u^\nu = g^{\mu\nu}u_\mu$, such that $u_\mu u^\mu = -1$. We can rewrite the stress-energy tensor in the following form

$$T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Starting from equations (1.2), substituting in it the expressions (1.1) and (1.5), and considering the 00 and ij components, we obtain two equations. The first one is

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \tag{1.6}$$

where the dot is the derivative with respect to time and we have defined $H = \frac{\dot{a}}{a}$ as the Hubble rate. The second equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{1.7}$$

These are the so called Friedmann equations, which relate the energy density and the pressure of the fluid with the derivatives of the scale factor $a(t)$. We can also explore the conservation of energy-momentum through the Bianchi Identity $\nabla_\mu T^{\mu\nu} = 0$ in order to obtain the following continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{1.8}$$

We have three equations and three unknowns (ρ, P and a), but only equations (1.6) and (1.7) are independent. Thus, in order to close the system we need another equation, and what is usually done is to choose an equation of state. We know that this kind of equation can relate the isotropic pressure to different quantities, such as the temperature, the entropy or the energy density. We decide to use a barotropic equation, namely a relation between the pressure and the energy density of the type

$$P = \omega\rho, \tag{1.9}$$

with ω constant. Its value depends on the specific energy content we consider; taking into account ordinary elements for example, we have that $\omega = 0$ indicates non-relativistic pressureless matter (or dust), while a value of $\omega = \frac{1}{3}$ is related to the ultra-relativistic component (or radiation). All the other constituents which have $\omega < -\frac{1}{3}$ go under the name of dark energy.

1.2.1 Solution of Friedmann Equations

We can find a solution for the system made by the two independent Friedmann equations and the equation of state; the first step is to substitute (1.9) into (1.8)

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0 \tag{1.10}$$

Then we can solve (1.10) to find

$$\rho(t) = \rho_* \left(\frac{a_*}{a}\right)^{-3(1+w)} \tag{1.11}$$

where the subscripts $*$ indicate a certain reference time, such that $\rho_* = \rho(t_*)$ and $a_* = a(t_*)$. Next, we can consider $k = 0$ and, from equation (1.6), we obtain

$$a(t) = a_* \left(\frac{t}{t_*}\right)^{\frac{2}{3(1+w)}} \tag{1.12}$$

We have finally reached the expression for the scale factor, and we see that it depends on the energy content of the Universe through the presence of ω. From equation (1.11), we note that $\omega = -1$ is a particular value: indeed, if this is the case, we obtain $P = -\rho$, $\rho = constant$ and $a(t) = a_*^{ieH(t-t_*)}$. This is associated to a specific kind of dark energy, namely the cosmological constant Λ.

If we consider a Universe dominated by a single ordinary component, we obtain for the radiation ($\omega = \frac{1}{3}$)

$$a(t) = a_* \left(\frac{t}{t_*}\right)^{\frac{1}{2}} \tag{1.13}$$

while if we consider the dust ($\omega = 0$) we get

$$a(t) = a_* \left(\frac{t}{t_*}\right)^{\frac{2}{3}} \tag{1.14}$$

Moreover, regarding the energy density, for the radiation we obtain

$$\rho_r = \rho_{r*} \left(\frac{a_*}{a}\right)^4 \tag{1.15}$$

whereas for the matter

$$\rho_m = \rho_{m*} \left(\frac{a_*}{a}\right)^3 \tag{1.16}$$

We note that the two previous expressions are slightly different: both describe how energy gets diluted with the stretching of the Universe, but the behavior of ρ as a function of the scale factor is not the same. As a matter of fact, if we consider the expansion of the Universe in all the three spatial direction, we expect the energy density to be $\rho \propto a^{-3}$, and indeed we recover this in (1.16). We have the same effect for the relativistic particles as well, but there is also an additional term a^{-1} present due to the effect of the stretching of the wavelength of the radiation, i.e. we have the gravitational redshift phenomenon. For this reason we get an expression for the energy density $\rho \propto a^{-4}$ in (1.15).

We can now introduce a quantity related to the energy density. If we consider equation (1.6), we can define a particular value for the energy density by imposing $k = 0$. This value is named critical density ρ_c, that is the energy density which flattens the Universe. It is equal to

$$\rho_c \equiv \frac{3H^2}{8\pi G} \tag{1.17}$$

An important term related to ρ_c is the density parameter Ω, which is defined as

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{3H^2\rho}{8\pi G} \tag{1.18}$$

The density parameter is a dimensionless quantity that tells us how the actual energy density of the Universe ρ compares to the critical density ρ_c. Considering (1.18) and (1.17), we see that if $\Omega > 1$ we should get a closed Universe, while if $\Omega < 1$ the Universe is an open one. If instead $\Omega = 1$, we have $\rho = \rho_c$ and so a flat Universe[2].

[2]It is worth mentioning that it will never be possible to definitively confirm that the Universe is exactly flat: regardless of the precision achieved by future measurements, every observational result will be necessarily accompanied by some degree of uncertainty. Consequently, even if one

Before we start dealing with the shortcomings of the HBB model, we need to introduce some quantities, which are the Cosmological Horizon and the Hubble Radius.

1.2.2 Cosmological Horizon

The Cosmological Horizon is the physical distance that represents the radius of a region causally connected to a central observer; it is defined as

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} \tag{1.19}$$

We can determine the comoving equivalent of (1.19) as

$$l(t) = \int_0^t \frac{dt'}{a(t')} \tag{1.20}$$

If we insert (1.12) into (1.19), we find that

$$d_H(t) = \frac{3(1+\omega)}{1+3\omega} t \tag{1.21}$$

This allows us to say that the Cosmological Horizon is meaningful and finite only if $\omega > -\frac{1}{3}$ (and if d_H exists finite, it is called Particle Horizon). The latter requirement on ω means $P > -\frac{\varrho}{3}$ and thanks to (1.7), this leads to the following condition: $\ddot{a}(t) < 0$. So, we can conclude that the Cosmological Horizon exists finite if and only if the expansion of the Universe is decelerated. Moreover, we can also notice that regions separated by distance more than the Particle Horizon are "causally disconnected", meaning they can never communicate with each other.

1.2.3 Hubble Radius

The Hubble Radius is defined as

$$R_c(t) = \frac{1}{H(t)} = t_H \tag{1.22}$$

where t_H is the characteristic time of expansion of the Universe. We can again define the comoving counterpart of R_c as

$$r_c(t) = \frac{1}{aH} = \frac{1}{\dot{a}} \tag{1.23}$$

measures $\Omega = 1$ within experimental error, both open and closed geometries remain consistent with the data and cannot be ruled out.

Taking into account the expression for the Hubble rate, we can define the following relation between d_H and R_c

$$R_c(t) = \frac{3}{2}(1+\omega)t = \frac{1+3\omega}{2}d_H(t) \tag{1.24}$$

Moreover, considering a FLRW Universe, we can write $R_c \approx d_H$, and so the two quantities are interchangeable.

1.3 Shortcomings of the Hot Big Bang Model

The HBB model has yielded remarkably accurate predictions concerning the evolution of our Universe, successfully accounting for many of the key observational phenomena. Among its most notable successes there are the blackbody spectrum of the CMB, the primordial abundances of light elements, and the empirical validation of the Hubble–Lemaître law. However, despite its predictive power, there are some reasons to consider the model incomplete, as it suffers from several conceptual shortcomings. These limitations do not stem from internal inconsistencies of the theory itself, but rather from the necessity of imposing finely tuned initial conditions in order to account for certain observed features of the present-day Universe. These issues are the Horizon Problem, the Flatness Problem and the Primordial Relics Problem.

1.3.1 The Horizon Problem

The first thing to notice when we tackle this problem is that in the HBB model the comoving Hubble Radius is monotonically increasing. Indeed, we have

$$\dot{r}_c(t) = -\frac{\ddot{a}(t)}{\dot{a}(t)^2} \tag{1.25}$$

and we know that this time derivative is always positive, due to the fact that, considering ordinary equations of state, $\ddot{a}(t)$ is always negative. Therefore, if we define a general comoving scale λ, it follows that all such scales will eventually exceed the horizon r_c, and so they will cross and enter the horizon at some point in the evolution of the Universe. This means that the region of causal connection around the observer becomes bigger as time goes by.
The problem arises considering the CMB, so from an observational point of view. Indeed, we are able to observe regions on the last scattering surface that exhibit al-

most identical statistical properties [3] despite the fact that they were never in causal contact. These regions are separated by distances that exceed the maximum distance light could have traveled since the beginning of the Universe.
At this stage, two distinct approaches may be considered to address the Horizon Problem. The first is to postulate that the Universe was born with an extraordinarily high degree of homogeneity. However, such an assumption is generally regarded as unsatisfactory, as it just shifts the fine-tuning to the initial conditions without providing a physical explanation. A more compelling alternative is to identify a dynamical mechanism capable of driving the Universe toward homogeneity, starting from a wide range of initial configurations. This is precisely the role of Inflation, which introduces a phase of accelerated expansion in the early Universe. To understand how such expansion solves the Horizon Problem, it is sufficient to note that a positive acceleration of the scale factor, so $\ddot{a}(t) > 0$, leads to a shrinking comoving Hubble Radius, as shown in equation (1.25). If this inflationary regime persists for a sufficiently long duration, comoving scales that are entering the horizon today were once within causal contact, early in the accelerated phase. So, this mechanism naturally accounts for the high degree of homogeneity observed in the statistical properties of the CMB. The process is graphically represented in Figure (1.1).

Therefore, Inflation can solve the problem, but this solution is a viable option only under certain conditions. First of all, we know that the HBB model was very useful in explaining a large variety of phenomena, so we require Inflation to bring us back to the model after its ends. Then, it has to last enough time to take care of the Horizon Problem. Considering the latter requirement, it is helpful to introduce a new quantity, which quantifies the amount of expansion the Universe underwent during Inflation, measuring the logarithmic growth of the scale factor from an initial time t_i to a final time t_f.

$$N \equiv \int_{t_i}^{t_f} H(t)dt = \ln\left(\frac{a(t_f)}{a(t_i)}\right) = \ln\left(\frac{a_f}{a_i}\right) \tag{1.26}$$

This quantity is named number of e-folds N, because if we apply an exponential on both sides of equation (1.26), it reads also as

$$e^N = \frac{a_f}{a_i} \tag{1.27}$$

[3] For example, the same temperature up to extremely small fluctuations, of the order of 10^{-5}.

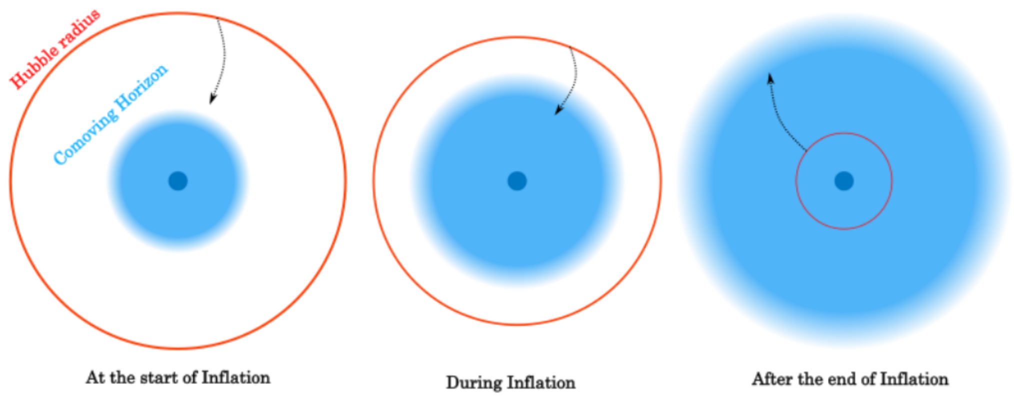


Figure 1.1: At the beginning, the entire region bounded by the Comoving Horizon lies within the Hubble Radius. During this phase, all regions of the area delimited by the horizon are causally connected, allowing sufficient time for them to reach thermal equilibrium and develop a high degree of homogeneity and isotropy. Once Inflation begins, the Hubble Radius (represented by the red curve) starts to decrease, continuing to shrink throughout the inflationary phase. Meanwhile, the observable region (illustrated as the blue area in the figure) grows as the Universe expands. Eventually, the Comoving Horizon exceeds the Hubble Radius, causing certain regions of the Universe to exit causal contact with one another, rendering communication between them impossible during that stage. After the inflationary epoch ends, the Hubble radius resumes its growth, and regions that had exited the horizon during Inflation eventually re-enter it, becoming observable once again. This is why regions that were causally connected now seem as they are causally disconnected [14].

that is the ratio between the two scale factors, which are related to the expansion of the Universe at t_i and t_f.
To address the Horizon Problem, it is sufficient to require that the comoving Hubble Radius at the beginning of Inflation was greater or equal to its today value. This condition ensures that the largest observable scale λ today was within causal connection prior to the beginning of Inflation. This requirement can be translated into the following condition:

$$r_c(t_0) = r_c(t_i) \tag{1.28}$$

where the subscript 0 means "today". If we multiply both sides for $a(t_0) = a_0$ and we use the expression (1.23), this relation translates into

$$\begin{aligned} H_0^{-1} &\leq \frac{a_0}{a_i} H_i^{-1} = \frac{a_0}{a_f}\frac{a_f}{a_i} H_i^{-1}, \\ \frac{a_f}{a_i} &\geq \frac{1}{a_0 H_0} H_i a_f \\ e^N &\geq \frac{T_0}{H_0}\frac{H_i}{T_f} \end{aligned} \tag{1.29}$$

where we used the approximation $T \propto a^{-1}$, with T being the temperature of the Universe. At the end we get

$$N \geq \ln\left(\frac{T_0}{H_0}\right) + \ln\left(\frac{H_i}{T_f}\right) \tag{1.30}$$

For the first term, we can consider $T_0 = 2.7K \simeq 10^{-13}$ GeV, $H_0 \simeq 10^{-42}$ GeV, so we obtain

$$\ln\left(\frac{T_0}{H_0}\right) \simeq 67 \tag{1.31}$$

Instead, the second term is model-dependent; however, a rough estimate can still be obtained under reasonable assumptions. If Inflation is approximated as a de Sitter phase [4] and if Inflation ends leaving the Universe in a radiation dominated state (providing the initial conditions for the HBB model to govern the subsequent evolution), then one can estimate $H_i \simeq H_f$ by applying the first Friedmann equation (1.6)

[4]This means that the Hubble parameter remains nearly constant throughout, i.e. $H_i \simeq H_f$.

$$\begin{aligned} H_f^2 &= \frac{8}{3}\pi G \rho_{r,f} \\ &= \frac{8}{3}\pi G \frac{\pi^2}{30} g_* T_f^4 \\ &\propto \frac{T_f^4}{M_{pl}^2} \end{aligned} \tag{1.32}$$

and substituting this relation into (1.30), the second term becomes $\ln(\frac{T_f}{M_{pl}})$ (where we have defined $M_{pl}^2 = \frac{1}{8\pi G}$ as the reduced Planck mass). In order to obtain a numerical estimate, one must assign a specific value to the temperature at the end of Inflation T_f (which is the reheating temperature); however, only upper and lower bounds on this quantity are currently known. Based on the absence of a detected primordial gravitational wave background, we can obtain the following condition

$$10^{-5} \leq \frac{T_f}{M_{pl}} \leq 1 \tag{1.33}$$

which means that

$$N \geq 67 + \ln\left(\frac{T_f}{M_P}\right) \approx 67 + [-11, 0] \Longrightarrow N \geq [56, 67] \tag{1.34}$$

At the end, we can say that Inflation has to last at least a number of e-folds $N \sim 60$ in order to solve the Horizon Problem.

1.3.2 The Flatness Problem

The second issue of the HBB model is the so called Flatness Problem, which is related to the curvature term. Indeed, considering the first Friedmann equation, we see that the first term in the right-hand side of (1.6) is proportional to a^{-3} or a^{-4} (depending on which ordinary cosmic fluid we are taking into account), while the second term, which is the curvature term, is proportional to a^{-2}. Therefore, as long as $k \neq 0$, there will be a moment such that the curvature term overcomes the one related to the energy density in (1.6), since the different behavior of the two terms as a function of the scale factor and, at the end, of the expansion of the Universe. To better understand the problem, we can use the density parameter (1.18), and plot its evolution as a function of time in Figure (1.2).

We can also notice, due to the same behavior mentioned above, that $\Omega = 1$ considering very early times, notwithstanding the value of the curvature k. In fact,

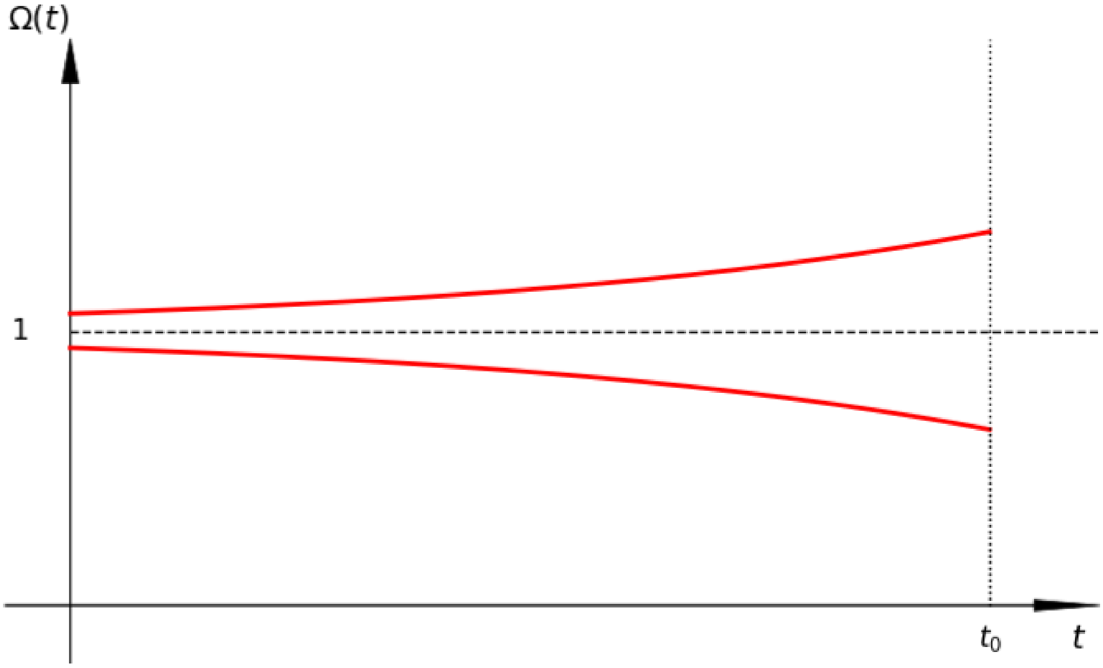


Figure 1.2: In this plot we see the time evolution of the density parameter Ω as a function of time. The upper part of the plot is related to a value of $k = +1$, while the lower part is related to $k = -1$, and the dashed line is for $k = 0$.

going backward in time corresponds to a decreasing scale factor, which in turn causes the energy density term in (1.6) to grow more rapidly than the curvature term. As a result, the contribution from curvature becomes negligible at early times, and Ω approaches unity, just as in the $k = 0$ case. However, as time passes, the scenario begins to depend sensitively on the value of k. Specifically, if $k = 0$, the energy density ρ exactly matches the critical density (1.17) at all times, forcing Ω to remain identically equal to 1. On the other hand, as previously noted, this no longer holds when $k \neq 0$. In this case we obtain

$$\Omega(t) - 1 = \frac{k}{a^2(t)H^2(t)} = k\ r_H^2(t) = \Omega_k(t) \tag{1.35}$$

Depending on whether k is positive or negative, the total density parameter eventually deviates from its initial value during cosmic evolution. However, current cosmological observations indicate that $|\Omega(t_0) - 1| < 10^{-3}$ at 95 % confidence level [15], implying that the Universe remains extremely close to spatial flatness today. So, this is the Flatness Problem: given that k should be different from zero[5], we ask ourselves how it is possible that the contribution of the curvature term to the total energy budget remains so small, even at present times.

[5]See footnote (2).

Again, we have two possible solutions. One possible explanation is that the initial value of Ω was extremely close to unity, meaning that the curvature contribution Ω_k was negligibly small, so that even today not enough time has passed for its effects to become significant. However, if we consider this to be true, we obtain a fine-tuning problem. Let us now examine this issue in more detail. The starting point is to note that the quantity $[\Omega^{-1} - 1]\rho a^2$ is constant. Indeed, starting from (1.35) and (1.6) we can write

$$\begin{aligned}
\Omega^{-1} &= \frac{a^2H^2}{k + a^2H^2} \Rightarrow \Omega^{-1} - 1 = \frac{-k}{k + a^2H^2} \\
\rho &= \frac{3M_{pl}^2 H^2 a^2 + 3kM_{pl}^2}{a^2} \Rightarrow \rho a^2 = 3M_{pl}^2\left(k + a^2H^2\right) \\
[\Omega^{-1} - 1]\rho a^2 &= \frac{-k[3M_{pl}^2\,(k + a^2H^2)]}{k + a^2H^2} \Rightarrow [\Omega^{-1} - 1]\rho a^2 = -3kM_{pl}^2
\end{aligned} \tag{1.36}$$

This allows us to write that

$$[\Omega(t)^{-1} - 1]\rho(t)a(t)^2 = [\Omega_0^{-1} - 1]\rho_0 a_0^2 \tag{1.37}$$

We are particularly interested in evaluating the quantity $\Omega(t)^{-1} - 1$ at early times, specifically during the radiation-dominated era. In this regime, the energy density can be expressed as $\rho_r = \rho_{eq}\left(\frac{a_{eq}}{a}\right)^4$, where as reference time we take t_{eq}, which is the time of equivalence between a matter and a radiation-dominated Universe. Then, equation (1.37) becomes

$$[\Omega(t)^{-1} - 1]\rho_{eq}\left(\frac{a_{eq}}{a(t)}\right)^4 \frac{a(t)^2}{\rho_0 a_0^2} = \Omega_0^{-1} - 1 \tag{1.38}$$

At this point, we introduce a significant simplification by assuming that the Universe is still matter-dominated today: we can then write $\rho_m(t) = \rho_0\left(\frac{a_0}{a(t)}\right)^3 \Rightarrow \rho_{eq} = \rho_0\left(\frac{a_0}{a_{eq}}\right)^3$. So we obtain

$$\begin{aligned}
[\Omega(t)^{-1} - 1]\left(\frac{a_0}{a_{eq}}\right)^3\left(\frac{a_{eq}}{a(t)}\right)^4\left(\frac{a(t)}{a_0}\right)^2 &= \Omega_0^{-1} - 1 \Rightarrow [\Omega(t)^{-1} - 1]\frac{a_0 a_{eq}}{a(t)^2} = \Omega_0^{-1} - 1 \\
[\Omega(t)^{-1} - 1]\left(\frac{a_{eq}}{a_0}\right)\left(\frac{a_0}{a(t)}\right)^2 &= \Omega_0^{-1} - 1 \Rightarrow [\Omega(t)^{-1} - 1] = [\Omega_0^{-1} - 1]\left(\frac{a(t)}{a_0}\right)^2(1 + z_{eq})
\end{aligned} \tag{1.39}$$

where we have introduced z_{eq}, which is the redshift associated to t_{eq}. Considering

$T \propto a^{-1}$ we get

$$[\Omega(t)^{-1} - 1] = [\Omega_0^{-1} - 1]\left(\frac{T_0}{T}\right)^2 (1 + z_{eq}) = [\Omega_0^{-1} - 1]\left(\frac{T_0}{T_{pl}}\right)^2 \left(\frac{T_{pl}}{T}\right)^2 (1 + z_{eq}) \tag{1.40}$$

and taking into account that $T_{pl} \simeq 10^{19}$ GeV and $(1+z_{eq}) \simeq 10^4$ we finally obtain

$$[\Omega(t_{pl})^{-1} - 1] = [\Omega_0^{-1} - 1]10^{-60} \Rightarrow \Omega(t_{pl})^{-1} = 1 + [\Omega_0^{-1} - 1]10^{-60} \tag{1.41}$$

The issue becomes evident: in order to account for the small value of Ω_k observed today, one would be required to fine-tune the initial value of Ω to unity with an extraordinary level of precision. Such a degree of fine-tuning appears highly unnatural, particularly when one considers that the range of admissible initial values constitutes a set of essentially null measure within the broader space of possibilities. So, we should look for another solution to the Flatness Problem. A possible answer could be to go back to (1.35) and take into account the effect of Inflation. If the Universe underwent a phase of exponential expansion, the contribution of the curvature term would be exponentially suppressed, rendering it negligible even up to the present time. This means that, regardless of the initial value of the total density parameter, the inflationary expansion would have dynamically driven it extremely close to unity. This mechanism naturally accounts for the near-flatness of the Universe observed today, even if the curvature is non-zero, and it does so without invoking any unnatural assumptions on the initial conditions. The time evolution of Ω during this phase is illustrated in Figure (1.3).

Again, we have to require Inflation to last enough to solve this problem. Therefore, we can impose the following relation

$$\frac{\Omega(t_i)^{-1} - 1}{\Omega(t_0)^{-1} - 1} \geq 1 \tag{1.42}$$

and then, from (1.37) we can write

$$\Omega(t_i)^{-1} - 1 = [\Omega(t_0)^{-1} - 1]\frac{\rho_0 a_0^2}{\rho_i a_i^2} = [\Omega(t_0)^{-1} - 1]\frac{\rho_0 a_0^2}{\rho_{eq} a_{eq}^2}\frac{\rho_{eq} a_{eq}^2}{\rho_f a_f^2}\frac{\rho_f a_f^2}{\rho_i a_i^2} \tag{1.43}$$

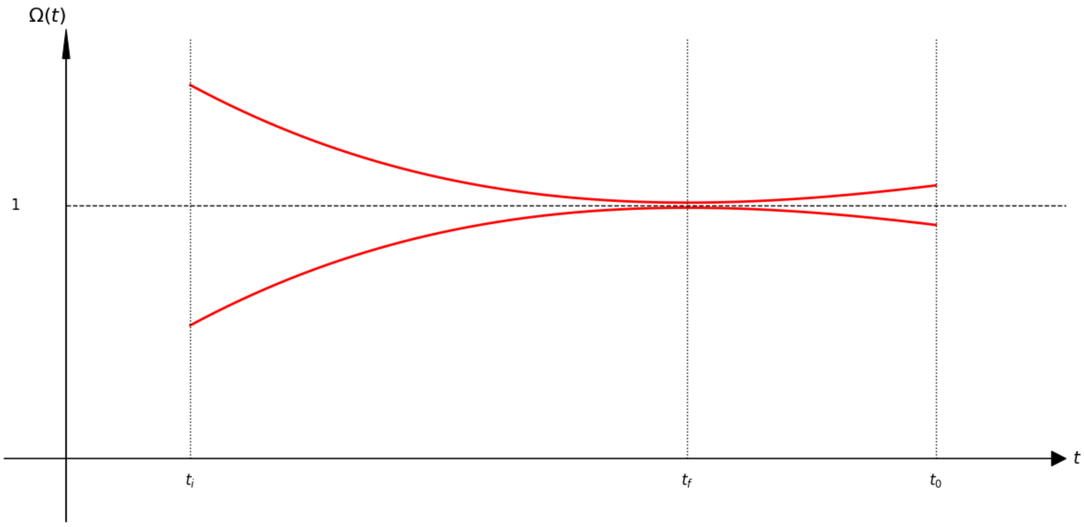


Figure 1.3: In this plot we see the time evolution of the density parameter Ω taking into account Inflation, which starts at t_i and ends at t_f. As we can notice by the plot, considering Inflation lead to a decrease of the value of Ω. The upper part of the plot is related to a value of $k = +1$, while the lower part is related to $k = -1$.

where we have introduced the factors $\rho_{eq} a_{eq}^2$ and $\rho_f a_f^2$, in order to make explicit the contributions from the three main cosmological epochs: the inflationary phase, the radiation-dominated era, and the matter-dominated era. Recalling (1.11), for a generic interval of time (t_i, t_j) we obtain

$$\frac{\rho_i a_i^2}{\rho_j a_j^2} = \left(\frac{a_i}{a_j}\right)^{-(1+3\omega)} \tag{1.44}$$

Substituting (1.44) in (1.43) we get

$$\Omega(t_i)^{-1} - 1 = [\Omega(t_0)^{-1} - 1] \left(\frac{a_0}{a_{eq}}\right)^{-1} \left(\frac{a_{eq}}{a_f}\right)^{-2} \left(\frac{a_f}{a_i}\right)^{-(1+3\omega_{inf})} = [\Omega(t_0)^{-1} - 1] \frac{a_f^2}{a_{eq} a_0} \left(\frac{a_f}{a_i}\right)^{-(1+3\omega_{inf})} \tag{1.45}$$

where we have introduced ω_{inf} (we do not know its exact value, but we know that it should be $\omega_{inf} < \frac{1}{3}$). We can now write

$$\left(\frac{a_f}{a_i}\right)^{-\left(1+3w_{inf}\right)} = e^{-N\left(1+3w_{inf}\right)} = \left(\frac{\Omega^{-1}\left(t_i\right)-1}{\Omega^{-1}\left(t_0\right)-1}\right) X \geq X,$$
$$X = \frac{a_{eq}a_0}{a_f^2} = \frac{a_{eq}}{a_0}\left(\frac{a_0}{a_f}\right)^2 = \left(1+z_{eq}\right)^{-1}\left(\frac{T_f}{T_{pl}}\frac{T_{pl}}{T_0}\right)^2 \approx 10^{64}\left(1+z_{eq}\right)^{-1}\left(\frac{T_f}{T_{pl}}\right)^2 \approx 10^{60}\left(\frac{T_f}{T_{pl}}\right)^2$$
$$N \geq N_{\min} = \frac{\ln X}{|1+3w_{inf}|} = \frac{60\left[2.3+\frac{1}{30}\ln\left(\frac{T_f}{T_{pl}}\right)\right]}{|1+3w_{inf}|} \tag{1.46}$$

Considering $\omega_{inf} \simeq 1$ and $10^{-5} \leq \frac{T_f}{M_{pl}} \leq 1$, we finally obtain a condition for the number of e-folds which reads $N \geq [57, 69]$. As the solution for the previous problem, we get a minimum number of e-folds of $N \sim 60$ to solve the Flatness Problem. Again, we stress that this result is model-dependent.

1.3.3 The Primordial Relics Problem

In the very early stages following the Big Bang, the Universe may have been described by a Grand Unified Theory (GUT), which is an extension of the Standard Model. Within a GUT framework, the Universe would have reached energies of the order $E_{GUT} \sim 10^{16}\ GeV \simeq 10^{30}K$, at which the electromagnetic, weak and strong nuclear interactions were unified into a single fundamental force. According to GUTs, as the Universe expanded and cooled below E_{GUT}, it underwent a symmetry-breaking phase transition. During this process, a variety of topological defects (such as domain walls, cosmic strings, and magnetic monopoles) could have been produced as relics of the broken symmetry. For example, magnetic monopoles are predicted to behave as point-like massive particles within the GUT framework. At the time of their formation, the abundance and energy density of these relics were expected to be significant, though still subdominant compared to the radiation energy density, and so the Universe in the early stage is still radiation-dominated. As the Universe evolved, these massive particles would have become non-relativistic, with their energy density scaling as (1.16), in contrast to the radiation component scaling as (1.15). As a result, relics like monopoles would eventually come to dominate the energy content of the Universe, leading to a scenario incompatible with current cosmological observations. In particular, the number density of magnetic monopoles today is constrained to be extremely small, with an upper limit of $n_M(t_0) \sim 10^{-19}cm^{-3}$ [14]. This contrast between theoretical expectations and observational constraints constitutes the so-called Primordial Relics Problem.

Inflation provides a natural solution to this issue. If magnetic monopoles (or other relics) were generated before or during the inflationary epoch, the subsequent rapid exponential expansion would have diluted their number density dramatically. As space expanded, monopoles were stretched across vast comoving volumes, reducing their present-day number density to such a low value that they would be virtually undetectable. This dilution mechanism elegantly explains the absence of observable primordial relics in the current Universe.

1.4 Inflation

In the previous section, we have seen that Inflation can naturally solve the shortcomings of the HBB model. In order to achieve this goal, Inflation must include a phase of accelerated expansion, where $\omega < -\frac{1}{3}$ and $\ddot{a}(t) > 0$. In particular, we have

$$\ddot{a} = (\dot{a})^{\cdot} = (\dot{Ha}) = \dot{a}H + a\dot{H} = a[H^2 + \dot{H}] \tag{1.47}$$

So, we can obtain $\ddot{a}(t) > 0$ in three different cases:

1. $\dot{H} < 0,\ |\dot{H}| < H^2$;
2. $\dot{H} = 0, \Rightarrow H = constant$;
3. $\dot{H} > 0$;

These three different configurations are related to three different values of ω:

1. $-1 < \omega < -\frac{1}{3} \Rightarrow a(t) \propto t^q$, where $q = \frac{2}{3(1+\omega)}$: this model is named Powerlaw Inflation;
2. $\omega = -1 \Rightarrow a(t) \propto e^{Ht}$, with H constant: this model is named de Sitter Inflation[6];
3. $\omega < -1 \Rightarrow a(t) \propto |t_{asy} - t|^q$, and $t_{asy} = t_* - \frac{2}{3(1+\omega)H(t_*)}, q < 0$: this model is named Pole Inflation;

As an example, we will study the simplest realization of Inflation, which is the single scalar field model.

[6]This model, which is often used to simplify computations, is not a viable model: in fact, due to the exponential expansion, Inflation do not get an end and it lasts forever.

1.4.1 Single Scalar Field

To introduce this framework, it is useful to begin with the idealized case of Inflation driven by a cosmological constant Λ, for which the equation of state parameter is $\omega = -1$. Λ can be associated with a stress-energy tensor of a perfect fluid

$$T^{\Lambda}_{\mu\nu} = (\rho_\lambda + P_\Lambda)u_\mu u_\nu + P_\Lambda g_{\mu\nu} = -\rho_\lambda g_{\mu\nu} \tag{1.48}$$

According to (1.48), one can reproduce the effects of a cosmological constant within the framework of particle physics if a given particle species possesses a stress-energy tensor $T_{\mu\nu}$ whose vacuum expectation value (VEV) satisfies the following condition

$$\langle T_{\mu\nu}\rangle = -\langle \rho_\lambda\rangle g_{\mu\nu} \tag{1.49}$$

This can be obtained if we take into account a real single scalar field $\varphi(\mathbf{x}, t)$. Indeed, we can consider for the field the following Lagrangian density

$$\mathcal{L}_\varphi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi) \tag{1.50}$$

which leads to the stress-energy tensor[7]

$$T^{\varphi}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\mathcal{L}_\varphi \tag{1.51}$$

If we suppose to have a ground state $\langle\varphi\rangle = constant$ in space and time, then the associated stress-energy tensor of the field becomes

$$\langle T^{\varphi}_{\mu\nu}\rangle = -V(\langle\varphi\rangle)g_{\mu\nu} \tag{1.52}$$

which in fact provides us with a relation similar to (1.49). However, until this moment we have not taken into account a crucial point of the theory, which is that Inflation has to come to an end. So, we need to add some dynamics, and the simplest way is to consider a background value for the scalar field that is time dependent, hence $\langle\varphi(t)\rangle \neq constant$.

[7] See Appendix (A.1) for the derivation.

1.4.2 The Dynamics of Inflation

In full generality, we can consider as a starting point for the theory the following action

$$S^{TOT} = S_{HE} + S_{\varphi} + S_m = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R + \mathcal{L}_{\varphi}\left[\varphi, \partial_{\mu}\varphi\right] + \mathcal{L}_{\text{fields}}\,) \tag{1.53}$$

where the first term in (1.53) is the Hilbert-Einstein action, the second one is the inflaton scalar field action, the last one is the action accounting for all the other fields [8] and $g = \det g_{\mu\nu}$. For the sake of simplicity, we will consider the last term subdominant, thus we can neglect it. In particular, we know that we can write $\mathcal{L}_{\varphi}$ as in (1.50) [9], where the form of the potential $V(\varphi)$ depends on the specific inflationary model under consideration. A commonly studied example is the quadratic potential, given by $V(\varphi) = \frac{1}{2}m_{\varphi}^2\varphi^2$, where $m_{\varphi} = \frac{\partial^2 V}{\partial\varphi^2}$ denotes the mass of the scalar field. Alternatively, the potential may include self-interaction terms, as in the case of a quartic potential of the form $V(\varphi) = \frac{\lambda}{4}\varphi^4$, where λ is a dimensionless self-coupling constant. Moreover, $V(\varphi)$ could also effectively account for other degrees of freedom present during Inflation. Anyway, we will use the first expression for the potential.

In order to study the dynamics for the scalar field φ, we can apply the variational principle

$$\frac{\delta S_{\varphi}}{\delta\varphi} = 0 \Rightarrow \Box\varphi = \frac{\partial V(\varphi)}{\partial\varphi} \tag{1.54}$$

where $\Box\varphi$ is the D'Alambertian operator applied to φ. From (1.54) we obtain the Klein-Gordon equation[10], namely the equation of motion for a scalar field in a curved spacetime

$$\ddot{\varphi}(\mathbf{x}, t) + 3H\dot{\varphi}(\mathbf{x}, t) - \frac{\nabla^2\varphi(\mathbf{x}, t)}{a(t)^2} = -\frac{\partial V(\varphi)}{\partial\varphi} \tag{1.55}$$

where $3H\dot{\varphi}(\mathbf{x}, t)$ is the friction term: due to this term, the field "feels" the expansion of the Universe.

[8] This action should take into account fermions, gauge fields, other scalars and all the fields which can be present other than the inflaton φ during Inflation.

[9] We are considering a minimally coupled field.

[10] See Appendix (A.2) for the derivation.

Now, we decompose the scalar field $\varphi(\mathbf{x}, t)$ into a classical background value and a quantum fluctuation around the classical trajectory

$$\varphi(\mathbf{x}, t) = \varphi(t) + \delta\varphi(\mathbf{x}, t) \tag{1.56}$$

This approach will be very useful considering perturbation theory, in fact if

$$\langle \delta\varphi^2 \rangle \ll \varphi^2(t) \tag{1.57}$$

we have that the fluctuations do not overwhelm the classical background, and so we can perform expansions in orders of the fluctuations[11].

From now on, we will focus our attention only on the background value for the field, therefore $\varphi = \varphi(t)$. This allows us to write (1.55) as

$$\ddot{\varphi}(t) + 3H\dot{\varphi}(t) = -\frac{\partial V(\varphi)}{\partial \varphi} \tag{1.58}$$

and the only non-vanishing component of the stress-energy tensor are

$$\begin{aligned} T^0_0 &= -\left(\frac{1}{2}\dot{\varphi}(t) + V(\varphi)\right) \equiv -\rho_\varphi \\ T^i_j &= \left(\frac{1}{2}\dot{\varphi}(t) - V(\varphi)\right)\delta^i_j \equiv P_\varphi \delta^i_j \end{aligned} \tag{1.59}$$

where ρ_φ is the energy density associated with the scalar field and P_φ is the isotropic pressure. We recall the very important relation between the energy density and the pressure, that is $P < -\frac{1}{3}\rho$, and we notice that, if we require the following condition

$$\frac{1}{2}\dot{\varphi}(t) \ll V(\varphi) \tag{1.60}$$

we can obtain $P_\varphi \simeq -V(\varphi) \simeq -\rho_\varphi$, and if we determine the equation of state parameter $\omega_\varphi = \frac{P_\varphi}{\rho_\varphi} \simeq -1$ we get that the scalar field $\varphi(t)$ behaves almost as an effective Λ, leading to a quasi-de Sitter phase, and thus to an inflating background. To fully describe the dynamics of $\varphi(t)$, in addition to (1.58) we can take into account also (1.6) and so the final system is

$$\begin{cases} H^2 = \frac{8\pi G}{3}\rho_\varphi = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi} + V(\varphi)\right) \\ \ddot{\varphi} + 3H\dot{\varphi} = -\frac{\partial V(\varphi)}{\partial \varphi} \end{cases} \tag{1.61}$$

[11] We have that, by definition, $\langle \delta\varphi \rangle = 0$

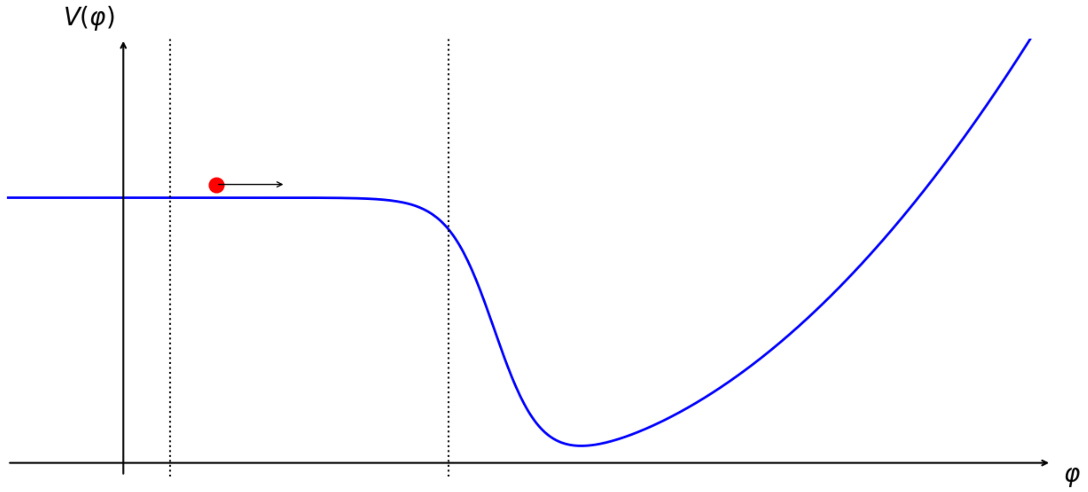


Figure 1.4: An example of $V(\varphi)$. Inflation takes place between the two dashed lines, where the potential is sufficiently flat.

1.4.3 Slow-Roll Regime

We have seen that, in order to have an inflationary period, the scalar field needs to fulfill the condition (1.60), and a simple way to achieve this is to consider a potential $V(\varphi)$ which is sufficiently flat. To demonstrate how this framework can give rise to Inflation (regardless of the initial conditions), we can study an initial configuration in which the conditions are not yet favorable for Inflation to occur. Specifically, we assume that the potential energy is negligible compared to the kinetic energy, i.e. $\frac{1}{2}\dot{\varphi}(t) \gg V(\varphi)$. In this regime, the dynamics are dominated by the kinetic term, and the Universe does not undergo accelerated expansion (indeed, we should get $P_\varphi \simeq \frac{1}{2}\dot{\varphi}(t) \simeq \rho_\varphi$, which leads to a value for the equation of state parameter $\omega_\varphi \simeq 1$). Moreover, recalling (1.11) we can write

$$\rho_\varphi(t) \propto a(t)^{-3(1+w_\varphi)} \simeq a(t)^6 \tag{1.62}$$

which, as we have mentioned above, is true in a kinetic dominated regime. But from (1.62) we easily see that the expansion of the Universe will damp the kinetic term for the energy density, leading to the potential term to dominate, and so to the condition (1.60) to be true, and Inflation can start. This example clearly demonstrates how this framework is an attractor solution, being that whatever are the initial conditions of kinetic and potential energy, the dynamics of the system eventually drive the evolution toward a regime in which the potential energy becomes dominant.

The relation (1.60) is named the first slow-roll condition, and it is related to the first slow-roll equation

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi}(t) + V(\varphi)\right) \simeq \frac{8\pi G}{3}V(\varphi) \tag{1.63}$$

We can also introduce a second slow-roll condition, which is

$$\ddot{\varphi}(t) \ll 3H\dot{\varphi}(t) \tag{1.64}$$

This condition controls the duration of Inflation, requiring also $\ddot{\varphi}(t)$ to be small enough; and it is related to the second slow-roll equation

$$3H\dot{\varphi}(t) \simeq -\frac{\partial V(\varphi)}{\partial \varphi} \tag{1.65}$$

This last equation can also be obtained if we notice that equation (1.58) recalls exactly the equation for a viscometer

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{\partial V(\varphi)}{\partial \varphi} \Longleftrightarrow \ddot{x} + 3\Gamma\dot{x} = F \tag{1.66}$$

with F constant. In this case we know that, after an initial transient of $\Delta t = \frac{1}{\Gamma}$, one gets to an asymptotic attractor solution

$$\dot{x} = \frac{F}{3\Gamma} \tag{1.67}$$

which is in fact the counterpart of equation (1.65).
To quantitatively characterize the slow-roll regime, it is customary to introduce the so-called slow-roll parameters. The first slow-roll parameter ϵ is defined as:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \tag{1.68}$$

which tells us how much H changes during Inflation. The relation between this parameter and the slow-roll conditions can be derived from the first slow-roll equation; if we derive with respect to time (1.63) (keeping the kinetic term) we obtain

$$2H\dot{H} = \frac{8\pi G}{3}\left[\dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi}\right] \tag{1.69}$$

where we have defined $V'(\varphi) = \frac{\partial V(\varphi)}{\partial \varphi}$. Then, we can substitute (1.58) and so we get

$$
\begin{aligned}
2H\dot{H} &= -8\pi G H \dot{\varphi}^2 \\
\dot{H} &= -4\pi G \dot{\varphi}^2
\end{aligned}
\tag{1.70}
$$

Using (1.68) we can write

$$
\epsilon = \frac{4\pi G \dot{\varphi}^2}{H^2} \simeq \frac{3}{2}\frac{\dot{\varphi^2}}{V} \tag{1.71}
$$

The right-hand side of the last equation can be interpreted as the ratio between the kinetic and potential energy and, taking into account (1.60), we can observe that slow-roll is possible only if $\epsilon \ll 1$. Instead, if we use the second equation (1.65) we get

$$
\epsilon \simeq \frac{3}{2}\frac{(V')^2}{9H^2 V} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{1.72}
$$

and so we get that if $\epsilon \ll 1$, the ratio $\frac{V'}{V}$ between the derivative of the potential and the potential itself is small, and consequently the potential is flat. We can obtain the same condition for ϵ also considering

$$
\ddot{a} = (\dot{a})^{\cdot} = (\dot{H a}) = \dot{a}H + a\dot{H} = aH^2 + a\dot{H} = aH^2(1 + \frac{\dot{H}}{H^2}) = aH^2(1-\epsilon) \tag{1.73}
$$

We know that in order to have Inflation we need $\ddot{a} > 0$, and this reflects into requiring $\epsilon \ll 1$. Moreover, from the last relation we can also determine that having $\epsilon = 1$ means that $\ddot{a} = 0$; so, whenever $\epsilon = 1$, Inflation ends.
The second slow-roll parameter is

$$
\eta \equiv -\frac{\ddot{\varphi}}{H\varphi} \tag{1.74}
$$

and from the second condition (1.64) we have $|\eta| \ll 1$. We can relate this parameter with the potential $V(\varphi)$ if we derive with respect to time the relation (1.65)

$$
\ddot{\varphi} = -\frac{V''\dot{\varphi}}{3H} + \frac{\dot{H}}{3H^2}V' \tag{1.75}
$$

We can plug this into (1.74) to obtain

$$
\eta \simeq \frac{V''}{3H^2} - \frac{\dot{H}}{H^2}\frac{V'}{3H\dot{\varphi}} \simeq \eta_V - \epsilon \tag{1.76}
$$

where we have used (1.65) in the last equation. In (1.76) we have defined

$$\eta_V \equiv \frac{V''}{3H^2} \simeq \frac{1}{8\pi G}\frac{V''}{V} \tag{1.77}$$

and having $\epsilon \ll 1$ and $|\eta| \ll 1$ means that also $|\eta_V| \ll 1$. We have found another relation between the potential and its derivative with the condition to be much smaller than one, but they are both necessary: the first one (1.72) is required for Inflation to start, the second relation (1.77) is needed to guarantee that Inflation lasts a sufficient amount of time.

1.4.4 Inflaton's Perturbations

In the last subsections, we have focused our attention towards the background of the scalar field, and how $\varphi(t)$ leads to a period of accelerated expansion, rendering the Universe nearly homogeneous and isotropic. However, the present-day Universe exhibits structure on a vast range of scales, from planetary systems to galaxy clusters. The seeds of these inhomogeneities can be traced back to quantum fluctuations of the inflaton field $\delta\varphi(\mathbf{x}, t)$. During Inflation, the physical wavelengths λ of these perturbations were stretched to super-horizon scales. Subsequently, during the matter-dominated era, these fluctuations began to grow under the influence of gravitational instability, and to form the structure we see today. Inflaton's perturbations are not the only ones which are present in the theory: $\delta\varphi(\mathbf{x}, t)$ will lead a perturbation in the stress-energy tensor $\delta T^{\varphi}_{\mu\nu}$ and, due to the Einstein equations, we will get a perturbation of the left-hand side of (1.2) and so of the metric $\delta g_{\mu\nu}$. Nevertheless, in the next computations we will deal only with the perturbations of the inflaton field $\delta\varphi(\mathbf{x}, t)$.
We can start from the equation of motion: we have seen that, in general, we can write the equation as (1.55), and if we substitute the expression of the field (1.56) we obtain:

$$\begin{cases} \delta\ddot{\varphi}(\mathbf{x}, t) + 3H\delta\dot{\varphi}(\mathbf{x}, t) - \frac{\nabla^2\delta\varphi(\mathbf{x},t)}{a(t)^2} = -V''\delta\varphi(\mathbf{x}, t) \\ \ddot{\varphi}(t) + 3H\dot{\varphi}(t) = -V' \end{cases} \tag{1.78}$$

The two equations are quite similar; so if we derive the second one with respect to time[12] we get

[12]We consider H as a constant.

$$\begin{cases} \delta\ddot{\varphi}(\mathbf{x},t) + 3H\delta\dot{\varphi}(\mathbf{x},t) - \frac{\nabla^2\delta\varphi(\mathbf{x},t)}{a(t)^2} = -V''\delta\varphi(\mathbf{x},t) \\ \dot{\ddot{\varphi}}(t) + 3H\ddot{\varphi}(t) = -V''\dot{\varphi}(t) \end{cases} \tag{1.79}$$

The two equations (the first for $\delta\varphi$ and the second for $\dot{\varphi}$) have the same structure if we neglect the laplacian term. If we go to Fourier space, we get

$$\frac{\nabla^2\delta\varphi}{a^2} \Rightarrow \frac{k^2}{a^2}\delta\varphi \tag{1.80}$$

where k is the Fourier variable. The term in (1.80) has to be compared to

$$\begin{aligned} H\delta\dot{\varphi} &\sim H^2\delta\varphi \\ \delta\ddot{\varphi} &\sim H^2\delta\varphi \end{aligned} \tag{1.81}$$

where we have used the following approximation $\frac{\partial}{\partial t} \simeq \frac{1}{\frac{1}{H}} = H$. We can neglect the laplacian term in (1.79) if

$$\frac{k^2}{a^2}\delta\varphi \ll H^2\delta\varphi \Rightarrow k^2 \ll a^2H^2 \tag{1.82}$$

This condition means, in comoving coordinates,

$$\lambda \gg \frac{1}{aH} = r_c \tag{1.83}$$

That is to say, we can neglect the laplacian term in (1.79) if we consider scales which are larger than the horizon, namely we should consider super-horizon scales. Returning to (1.79), given that the two equations coincide in the large-scale limit, the two solutions of the system may not be linearly independent. To verify whether $\delta\varphi \propto \dot{\varphi}$, one can compute the Wronskian

$$W(\delta\varphi,\dot{\varphi}) = \dot{\delta\varphi}\dot{\varphi} - \delta\varphi\dot{\dot{\varphi}} \tag{1.84}$$

If the result is zero, the two functions are linearly dependent. At linear order we obtain

$$\dot{W} \simeq -3HW \Rightarrow W \propto e^{-3Ht} \tag{1.85}$$

Since the Wronskian decreases exponentially with time, it is reasonable to assume that the two quantities are linearly dependent. Then, when W becomes negligible, the most general solution is the following [16]

$$\delta\varphi(\mathbf{x},t) = -\delta t(\mathbf{x})\dot{\varphi}(t) \tag{1.86}$$

therefore we can write (1.56) as

$$\varphi(\mathbf{x},t) = \varphi(t - \delta t(\mathbf{x})) \tag{1.87}$$

In fact, if we Taylor expand the previous relation to first order in δt, we get exactly (1.86) for $\delta\varphi(\mathbf{x},t)$. On sufficiently large scales, the inflaton field $\varphi(\mathbf{x},t)$ will assume the same background value $\varphi(t)$ across all spatial regions. As a consequence, each point in the Universe undergoes the same cosmological evolution, but at slightly different times.

1.4.5 Amplitude of the Perturbations

Now we want to determine a solution for the equation for the perturbations. In order to solve (1.79), we need to go to Fourier space; so, we can write

$$\delta\varphi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\varphi(k,t) \tag{1.88}$$

where, at the linear level, different modes as $\delta\varphi(k,t) = \delta\varphi_k(t)$ and $\delta\varphi(k',t) = \delta\varphi_{k'}(t)$ evolve independently, and $\delta\varphi_k^*(t) = \delta\varphi_{-k}(t)$, because $\delta\varphi_k(t)$ is real. The equation of motion for a given mode in Fourier space becomes

$$\delta\ddot{\varphi}_k(t) + 3H\delta\dot{\varphi}_k(t) - \frac{\nabla^2\delta\varphi_k(t)}{a^2} = -V''\delta\varphi_k(t) \tag{1.89}$$

Then, we can use conformal time τ, such that $d\tau = \frac{dt}{a(t)}$, and quantize the scalar field rescaled as

$$\delta\tilde{\varphi}(\mathbf{x},\tau) = a(t)\delta\varphi(\mathbf{x},\tau) \tag{1.90}$$

and so we get

$$\delta\tilde{\varphi}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3} \left[u_k(\tau) a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\tau) a_{\mathbf{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right] \tag{1.91}$$

where $a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger$ are the annihilation and the creation operators respectively, such that $a_{\mathbf{k}}|0\rangle = 0$ and $\langle 0|a_{\mathbf{k}}^\dagger = 0 \;\forall k$, being $|0\rangle$ the free vacuum state. We can define the commutation relations between the ladder operators as

$$[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = (2\pi)^3 \; \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \; [a^{\dagger}_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = 0 \tag{1.92}$$

and the normalization condition for the mode function $u_k(\tau)$ is

$$u_k^*(\tau) u_k'(\tau) - u_k'^*(\tau) u_k(\tau) = -i \tag{1.93}$$

We have that in flat spacetime

$$u_k(\tau) = \frac{e^{-i\omega_k \tau}}{\sqrt{2\omega_k}}, \quad \omega_k = \sqrt{k^2 + m^2} \tag{1.94}$$

so the solution is a plane wave. But in a curved spacetime this is not necessarily true; there is indeed an ambiguity in the choice of the vacuum state. A solution is that, at early times and at small scales $k \gg aH$, considering $\omega_k \simeq k$ we expect $u_k(\tau)$ to approach the flat spacetime form (1.94). These requirements lead to the Bunch-Davies vacuum choice [17], and so we can assume

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{1.95}$$

Now we can write equation (1.89) for u_k in Fourier space

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + a^2 \frac{\partial^2 V}{\partial \varphi^2} \right] u_k(\tau) = 0 \tag{1.96}$$

where now $'$ stands as $\frac{\partial}{\partial \tau}$. Equation (1.96) is the equation of motion for a harmonic oscillator with a frequency which is time dependent, due to the expansion of the Universe. Before deriving the most general solution, we begin by considering, for simplicity, the case of a massless scalar field evolving in de Sitter. This approximation is particularly insightful, as allows us to show the behavior of the fluctuations at super-horizon limit. Being the field massless, we have that $m_\varphi^2 = \frac{\partial^2 V}{\partial \varphi^2} = 0$, and considering de Sitter means

$$a(t) \propto e^{Ht} \Rightarrow \tau \propto -\frac{1}{H} e^{-Ht} = -\frac{1}{aH} \Rightarrow \frac{a''}{a} = \frac{2}{\tau^2} = 2a^2 H^2 \tag{1.97}$$

and so we can write (1.96) as

$$u_k''(\tau) + \left[k^2 - 2a^2 H^2 \right] u_k(\tau) = 0 \tag{1.98}$$

We can easily study this equation in the two limits: $k \gg aH$ and $k \ll aH$, which we show in Figure (1.5).

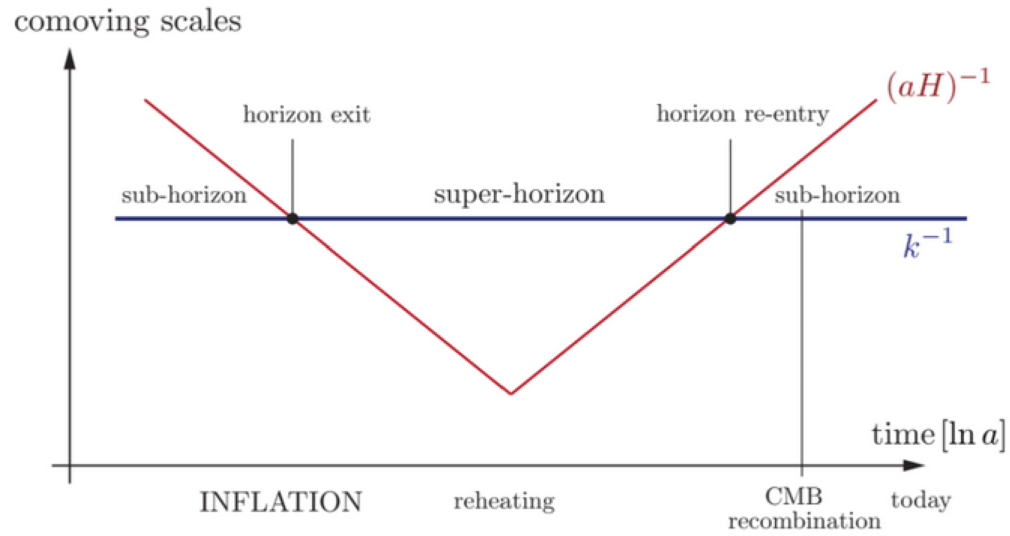


Figure 1.5: We see that, during Inflation, we can have two regimes: the sub-horizon or the super-horizon. The difference between the two is whether a generic constant comoving scale (in blue) is larger or smaller than Hubble Radius (in red) [18].

In the first case, which is named the sub-horizon regime [13], the second term on the left-hand side of (1.98) becomes equal to $k^2 u_k(\tau)$, and so we obtain $u_k''(\tau)+k^2u_k(\tau) = 0$. The solution for this equation is the Bunch-Davies vacuum (1.95). Then we can write

$$\delta\varphi_k^{sub} = \frac{u_k}{a} = \frac{e^{-ik\tau}}{a\sqrt{2k}} \tag{1.99}$$

From this expression, we can observe that the amplitude of the fluctuation exhibits an explicit dependence on the inverse of the scale factor. Consequently, during Inflation, the amplitude of the fluctuation decreases rapidly. In the second case, which is called the super-horizon regime, we can neglect k^2 in the second term of equation (1.98); this leads to $u_k''(\tau) - 2a^2H^2u_k(\tau) = 0$. This kind of equations are solved by

$$u_k(\tau) = B(k)a(\tau) + C(k)a^{-2}(\tau) \tag{1.100}$$

where $B(k)$, $C(k)$ are integration constants. The first term in (1.100) is related to a growing mode (due to the expansion of the Universe), while the second is a

[13] This is because $k \gg aH \iff \frac{1}{k} \ll \frac{1}{aH}$, and $\frac{1}{k} \propto \lambda$, which is a constant comoving scale, and $\frac{1}{aH} = r_c$, so we get $\lambda \ll r_c$

decaying mode, which we can neglect after a few e-folds. In the end, we get

$$\delta\varphi_k^{sup} = \frac{u_k}{a} = B(k) \tag{1.101}$$

We have found that, on super-horizon scales, the amplitude of the perturbation is constant. In order to determine it, we can perform a matching with the sub-horizon solution at the moment of horizon crossing: as we have said, from that point onward the amplitude remains effectively constant. At horizon crossing, the condition $k = aH$ holds, and thus:

$$|\delta\varphi_k| = |\delta\varphi_k^{sup}| = |B(k)| = |\delta\varphi_k^{sub}| = \left.\frac{1}{a\sqrt{2k}}\right|_{k=aH} = \frac{H}{\sqrt{2k^3}} \tag{1.102}$$

In general, when averaging over sub-horizon scales, one finds $\langle\delta\varphi\rangle_t$, where t denotes a macroscopic time interval. However, in an expanding background where the physical wavelength scales as $\lambda_{physical} \propto a \propto e^{Ht}$, the fluctuations no longer remain in their quantum vacuum state. Instead, they become frozen as they cross the horizon, leading to a non-zero expectation value $\langle\delta\varphi\rangle \neq 0$. In other words, quantum fluctuations of the inflaton field effectively transition into classical perturbations. This process can be interpreted as the generation of a net number of particles and is referred to as the gravitational amplification mechanism. It is conceptually analogous to the Schwinger effect [19], where electron–positron pairs are produced in strong electromagnetic fields. In the inflationary context, one may think of pairs of φ particles being created and subsequently stretched apart by the accelerated expansion, resulting in a net presence of such particles on super-horizon scales. The exact solution for a massless scalar field in a pure de Sitter background is given by [20]

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{1.103}$$

Now we can move to the more general scenario, namely we consider a massive scalar field in a quasi-de Sitter expansion. Indeed, we know that during Inflation the Hubble rate is not precisely constant, but it varies with time[14]. By making use of the definition of conformal time τ, one can demonstrate that, for small values of the slow-roll parameter ϵ, the scale factor takes the following form [20]

$$a(\tau) = -\frac{1}{H}\frac{1}{\tau^{1+\epsilon}} \tag{1.104}$$

[14]Recalling the definition of ϵ, we can write $\dot{H} = -\epsilon H^2$

This allows us to write the following

$$\frac{a''}{a} = \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right) \tag{1.105}$$

Then, thanks to the definition (1.77) we can also write

$$a^2 \frac{\partial^2 V}{\partial \varphi^2} = 3a^2 H^2 \eta_V \simeq \frac{3\eta_V}{\tau^2} \tag{1.106}$$

We are now able to rewrite equation (1.96) as[15]

$$u_k''(\tau) + \left[k^2 - \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon - \frac{3}{2}\eta_V\right)\right] u_k(\tau) = 0 \tag{1.107}$$

The last equation can be recast as

$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right] u_k(\tau) = 0 \tag{1.108}$$

where

$$\nu^2 = \frac{9}{4} + 3\epsilon - 3\eta_V \Rightarrow \nu \simeq \frac{3}{2} + \epsilon - \eta_V \tag{1.109}$$

Considering ν as a constant[16], (1.108) has the form of a Bessel equation; the solution for this type of equation is

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right] \tag{1.110}$$

where $c_1(k)$ and $c_2(k)$ are constants of integration, and $H_\nu^{(1)}(-k\tau)$ and $H_\nu^{(2)}(-k\tau)$ are the Hankel functions of the first and second kind. Again, we can impose, on sub-horizon scales ($k \gg aH$), the solution to be the Bunch-Davies vacuum (1.95) and, given that we can write

$$H_\nu^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)}, \; H_\nu^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{-i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \tag{1.111}$$

this fixes $c_1(k) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu + \frac{1}{2}\right)\frac{\pi}{2}}$ and $c_2(k) = 0$ (considering $x \equiv -\tau k$). We can then rewrite (1.110) as

[15] The massless result is obtained considering $\frac{\partial^2 V}{\partial \varphi^2} = 0$.

[16] We have that ν depends on ϵ and η_V, which in general can change in time; but at first order in slow-roll parameters we can consider them constant, and so ν.

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau) \tag{1.112}$$

Taking into account the super-horizon regime, we can use the following definition for the Hankel function

$$H_\nu^{(1)}(x \ll 1) \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right) x^{-\nu} \tag{1.113}$$

and so, at the end, we can write

$$u_k(\tau) = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu} \tag{1.114}$$

We are interested in the amplitude $|\delta\varphi_k|$; therefore, considering the relation $|\delta\varphi_k| = \frac{|u_k|}{a}$ and (1.114), we obtain (in the super-horizon limit $k \ll aH$)

$$|\delta\varphi_k| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \tag{1.115}$$

From (1.115) we immediately see that we recover the same result of a massless de Sitter expansion if we put $\epsilon = 0, \eta_V = 0 \Longleftrightarrow \nu = \frac{3}{2}$. Moreover, we notice that a quasi–de Sitter background introduces a slight scale dependence in the perturbation spectrum, of the form $k^{-\epsilon+\eta_V}$. This mild deviation from scale invariance is a distinctive prediction of inflationary models and serves as an important observational signature.
It is important to emphasize that the previous discussion applies to a massive scalar field evolving in a quasi–de Sitter background and, as such, does not fully capture the dynamics of inflaton perturbations. A complete treatment requires accounting for the coupled perturbations of both the inflaton field and the spacetime metric, with the latter that have been neglected here for the sake of simplicity. Nevertheless, it can be shown that including both scalar field and metric perturbations gives a result that is analogous to equation (1.115), but with a modified parameter $\nu \simeq \frac{3}{2} + 3\epsilon - \eta_V$.

1.4.6 Power Spectrum

We know that fluctuations of the inflaton field are inherently stochastic and can be described by a quantum field $\delta\varphi(\mathbf{x}, t)$, which encodes the amplitude of scalar perturbations of $\varphi(\mathbf{x}, t)$ at each point in spacetime. Although the VEV of $\delta\varphi$ vanishes when evaluated locally, non-trivial correlations can exist between field values at different spatial points. To quantify this statistical dependence, we introduce the two point correlation function, which characterizes how field fluctuations at two locations are statistically related. The general definition is

$$\xi(\mathbf{x},\mathbf{r}) = \langle \delta\varphi(\mathbf{x}+\mathbf{r},\tau)\varphi(\mathbf{x},\tau)\rangle \tag{1.116}$$

but, thanks to the homogeneity property, we have $\xi(\mathbf{x},\mathbf{r}) \Rightarrow \xi(\mathbf{r})$, and due to isotropy we can write $\xi(\mathbf{r}) \Rightarrow \xi(r)$, where $r = |\mathbf{r}|$. Then, we know that, using equations (1.91) and (1.92), we are able to write

$$\langle \delta\varphi_{k_1}\delta\varphi^*_{k_2}\rangle = (2\pi)^3\delta^{(3)}\left(\mathbf{k_1}-\mathbf{k_2}\right)|\delta\varphi_{k_1}|^2 \tag{1.117}$$

If we can take the Fourier transform of equation (1.116), we obtain

$$\begin{aligned}\xi(r) = \langle \delta\varphi(\mathbf{x}+\mathbf{r},\tau)\delta\varphi(\mathbf{x},\tau)\rangle &= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} e^{i\mathbf{k_1}(\mathbf{x}+\mathbf{r})} e^{\mathbf{k_2}\mathbf{x}} \langle \delta\varphi_{k_1}(\tau)\delta\varphi_{k_2}(\tau)\rangle \\ &= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} e^{i\mathbf{k_1}(\mathbf{x}+\mathbf{r})} e^{\mathbf{k_2}\mathbf{x}} (2\pi)^3\delta^{(3)}\left(\mathbf{k_1}+\mathbf{k_2}\right)|\delta\varphi_{k_1}|^2 \\ &= \int \frac{d^3k_1}{(2\pi)^3} e^{i\mathbf{k_1}\mathbf{r}} P\left(k_1\right)\end{aligned} \tag{1.118}$$

where we have defined the power spectrum $P(k) = |\delta\varphi_k|^2$. We can notice from equation (1.118) that the power spectrum is the Fourier transform of $\xi(r)$. In particular, on super-horizon scales we have

$$P(k) = |\delta\varphi_k|^2 = \left[\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}\right]^2 = \frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{3-2\nu} \tag{1.119}$$

The last equation tells us the power spectrum of quantum fluctuations $\delta\varphi$.

Chapter 2

Spectator Chromo Natural Inflation

2.1 Historical Development

In the previous chapter, we have seen that the inflationary paradigm is able to solve the shortcomings of the Hot Big Bang model, and that it is possible to achieve such a period of expansion of the Universe considering a single real scalar field. Nevertheless, generally theories which include only scalar fields lack protective symmetries capable of safeguarding their potentials against radiative corrections. Such corrections are usually large and tend to spoil the required flatness of the inflationary potential. Eliminating these corrections requires a significant degree of fine-tuning, which gives rise to a hierarchy problem. In the context of inflationary cosmology, this issue is commonly referred to as the η-problem, since the curvature of the potential is often parametrized by the slow-roll parameter η_V (1.77) in the literature. One particularly relevant framework proposed to overcome this problem is Natural Inflation [21], which addresses the issue by identifying the inflaton with an axion field χ[1], that has a typical axion potential $V(\chi) \sim \Big[1+\cos(\chi/f)\Big]$ (f being the decay constant of χ). In this scenario, the flatness and functional form of the potential are protected from dangerous quantum corrections by the axion's residual shift symmetry. However, a critical limitation of Natural Inflation arises when confronted with CMB observations: in fact, consistency with data requires f to be of the order of the Planck scale M_{pl} or larger[2] [23]. Unfortunately, it has proven unfeasible to realize a

[1] This was proposed also because axion-like particles are the simplest spin-zero degrees of freedom with a nontrivial radiatively stable potential.

[2] It is possible to write that [22]

$$f = \frac{2}{\sqrt{4(1-n_s)-r}} M_{pl} \tag{2.1}$$

setup with such constraint within an ultraviolet-complete model, like string theory for example [24]. The studies presented in [5, 25] explored the possibility that a sub-Planckian inflaton range could arise as a consequence of its interactions with a gauge field. In particular, [25] suggested a model where the pseudo-scalar inflaton is coupled to an Abelian U(1) gauge field A_μ with a non-vanishing vacuum expectation value. The interaction term between the inflaton and the gauge field $\chi F\tilde{F}$ (where F is the gauge field strength, and $\tilde{F}$ is its dual) is crucial to have Inflation, even in the presence of a steep potential. In fact, as χ rolls down its potential, it induces a time-dependent background for the gauge field, which in turn amplifies its vacuum fluctuations into physical excitations. The result of the production of gauge field quanta (which occurs at the expense of the inflaton's kinetic energy), is to effectively slow the inflaton motion. When the coupling between χ and A_μ is sufficiently strong, this dissipative effect can sustain a prolonged period of Inflation even in the case $f \ll M_{pl}$.
Moreover, this mechanism leads to a particularly interesting scenario for what concern perturbations: what happens is that, during the evolution of the inflaton, one polarization mode of the gauge field undergoes exponential amplification, while the opposite polarization remains suppressed. The enhanced polarization, before being diluted by the Universe's expansion, acts as a source for both scalar and tensor perturbations, through non-linear processes such as $\delta A+\delta A \Rightarrow \delta\chi$ with the inflaton field and $\delta A+\delta A \Rightarrow \delta g$, which sources metric perturbations, so gravitational waves. A related framework has been proposed in [5], where the Abelian U(1) gauge field is replaced by a non-Abelian SU(2) triplet of vector fields; this model goes under the name of Chromo Natural Inflation (CNI), which shows the same peculiarity.

2.2 Chromo Natural Inflation

The action of CNI is the following

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{pl}^2}{2}R - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}(\partial\chi)^2 - U(\chi) + \frac{\lambda}{4f}\chi F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right] \tag{2.2}$$

where χ is the pseudo-scalar axion, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc}A^b_\mu A^c_\nu$ is the field

where the inflationary observables n_s, r are the scalar spectral tilt and the tensor-to-scalar ratio, respectively; considering current experimental bounds on these observables, only values for $f \gtrsim M_pl$ are allowed.

strength of a SU(2)[3] gauge field A^a_μ and $\tilde{F}^{a\mu\nu} = \frac{1}{2\sqrt{-\tilde{g}}}\epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta}$ is its dual. The constant g is the gauge field coupling, not to be confused with the determinant of the spacetime metric $\tilde{g}$. The anti-symmetric tensors $\epsilon^{\mu\nu\alpha\beta}, \epsilon^{abc}$ are normalized such that $\epsilon^{0123} = +1$, $\epsilon^{123} = +1$, and we will use as line element $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 = a^2(\tau)[-d\tau + d\mathbf{x}^2]$[4]. We will assume a sinusoidal axion potential

$$U(\chi) = \mu^4\Big[1 + \cos(\chi/f)\Big] \tag{2.3}$$

which has an energy scale μ, and f is the decay constant of the axion, as before. It is worth noting that, although we have adopted the standard cosine potential for the axion, which is typically generated by non-perturbative effects arising from the interaction between the axion and an additional gauge sector, the mechanism under consideration does not depend on the potential having this specific functional form [5].

In general, for non-abelian gauge fields we can write

$$A_\mu = A^a_\mu T_a \tag{2.4}$$

where T_a is the group generator, and it satisfies the commutation relations

$$[J_a, J_b] = i f_{abc} J_c \tag{2.5}$$

where f_{abc} are the structure functions of the group. For the SU(2) group we have $T^a = \frac{\sigma^a}{2}$, where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

and $f_{abc} = \epsilon_{abc}$, the antisymmetric tensor.

The CNI action (2.2) is invariant under the local SU(2) transformation

$$\Lambda_\mu \to U\Lambda_\mu U^{-1} - \frac{i}{g_A}\left(\nabla_\mu U\right)U^{-1} \tag{2.7}$$

where

[3] We notice that, since every SU(N) group contains a SU(2) subgroup, it is always possible to assign a VEV to SU(N) gauge fields in a way that preserves rotational invariance, by identifying the global part of the SU(2) symmetry with the spatial rotational symmetry. So, in principle the proposal in [5] does not rely on any specific gauge group, but a SU(2) gauge field is chosen for concreteness.

[4] We will use greek letters $\mu, \nu, \rho, \ldots$ for space-time indices, $i, j, k, \ldots$ for space indices, and $a, b, c, \ldots$ for internal SU(2) indices.

$$U = e^{(-ig_A \beta x^\nu)}, \quad \beta = \beta^a \, \frac{\sigma^a}{2} \tag{2.8}$$

The last term in (2.2) is the Chern-Simons term, which represents the interaction between the axion and the gauge field. It is precisely this term that enables the background to sustain Inflation for a sufficiently large number of e-folds, despite the presence of a $f \ll M_{pl}$, which corresponds to a steeper potential. Similarly to the Abelian model presented in [25], the Chern-Simons term gives the additional friction in the axion dynamics needed to obtain an efficient slow-roll phase.

2.2.1 Background Dynamics

In this framework, in order to obtain an inflationary background driven by non-Abelian gauge fields and axion we need a homogeneous and rotationally invariant background configuration. This is due to the fact that the introduction of vector fields in the background would ordinarily break rotational invariance; however, this symmetry breaking can be compensated by a suitable global gauge transformation, thereby yielding a rotationally invariant solution[5]. The VEV of the gauge field is then specified by the classical configuration introduced in [26, 27] as

$$A_0^a = 0, \quad A_i^a = \delta_i^a a(t) Q(t) \tag{2.9}$$

where we have that the SU(2) index $a = \{1, 2, 3\}$, while the indices 0 and $i = \{1, 2, 3\}$ refer to the time and space components respectively. The scale factor $a(t)$ is included in the parametrization of the vector field VEV, as $Q(t)$ represents a slowly evolving quantity during the inflationary phase. The definition (2.9) implies that the SU(2) gauge field admits a configuration consistent with the homogeneity and isotropy of spacetime; this solution has been demonstrated to be dynamically stable and to exhibit an attractor behavior [28, 29]. Additionally, as shown in [30, 31], starting from an initial state effectively composed of multiple Abelian fields with vanishing VEVs, the chromo configuration with a non-zero gauge field VEV dynamically emerges during Inflation. Furthermore, we can notice that in the absence of the axion–gauge coupling, the gauge field vacuum expectation value decays within a Hubble time.

[5]Rotations in physical space can be compensated by corresponding rotations in gauge space, leaving the field configuration invariant. The SU(2) gauge symmetry plays a crucial role, as its global part can be identified with spatial rotations in three dimensions, ensuring the invariance of the configuration.

From the Einstein equations we get

$$\begin{aligned} 3M_{\rm pl}^2H^2 &= \frac{\dot{\chi}^2}{2} + U(\chi) + \frac{3}{2}\left(\dot{Q}+HQ\right)^2 + \frac{3}{2}g^2Q^4 \\ -2M_{\rm pl}^2\dot{H} &= \dot{\chi}^2 + \frac{2}{a^2}\left[\dot{(aQ)}\right]^2 + 2g^2Q^4 \end{aligned} \tag{2.10}$$

while the equations of motion for the gauge field and the axion are [5]

$$\begin{aligned} &\ddot{\chi} + 3H\dot{\chi} + U_\chi = -\frac{3g\lambda}{f}Q^2\left(\dot{Q}+HQ\right) \\ &\ddot{Q} + 3H\dot{Q} + \left(\dot{H}+2H^2\right)Q + 2g^2Q^3 = \frac{g\lambda}{f}\dot{\chi}Q^2 \end{aligned} \tag{2.11}$$

where $U_\chi = \partial_\chi U$. From the right-hand side of the first equation in (2.11) we can see the crucial element of this model. Indeed, this is the additional interaction term that enables slow-roll Inflation to occur even in the case of a sub-Planckian axion decay constant. Moreover, we can determine [32]

$$\rho_\chi = \frac{1}{2}\dot{\chi}^2 + U(\chi), \quad \rho_Q = \frac{3}{2}(\dot{Q}+HQ)^2 + \frac{3}{2}g^2Q^4 \tag{2.12}$$

$$p_\chi = \frac{1}{2}\dot{\chi}^2 - U(\chi), \quad p_Q = \frac{1}{2}(\dot{Q}+HQ)^2 + \frac{1}{2}g^2Q^4 \tag{2.13}$$

From equations (2.12) and (2.13), it is clear that the gauge sector exhibits a radiation-like equation of state $p_Q = \rho_Q/3$; as a consequence, the gauge field by itself cannot sustain Inflation.

A key requirement for Inflation to take place in this framework is that the axion potential $U(\chi)$ constitutes the dominant contribution to the total energy density; so, from the first equation of (2.10) we have $3M_{pl}^2H^2 \simeq U(\chi)$. The last expression remains valid provided that all the other terms in (2.10) contribute only negligibly to the overall energy budget of the Universe. This allows us to introduce a set of small parameters

$$\epsilon_\chi \equiv \frac{\dot{\chi}^2}{2H^2M_{pl}^2}, \ \epsilon_Q \equiv \frac{\dot{Q}^2}{H^2M_{pl}^2}, \ \epsilon_1 \equiv \frac{Q^2}{M_{pl}^2}, \ \epsilon_B \equiv \frac{g^2Q^4}{H^2M_{pl}^2} \tag{2.14}$$

During the inflationary phase, all of these quantities are required to remain much smaller than unity. We can now seek slow-roll inflationary solutions to the system (2.11) by neglecting $\ddot{\chi}, \ddot{Q}$ and $\dot{H}$ [5]

$$\begin{aligned}\dot{\chi} &\simeq \frac{g\lambda f Q^2 H\left(\frac{2g^2Q^3}{H} - HQ - \frac{fU_\chi}{g\lambda Q^2}\right)}{3f^2H^2 + g^2\lambda^2 Q^4} \\ \dot{Q} &\simeq -\frac{HQ\left(2f^2H^2 + 2g^2f^2Q^2 + g^2\lambda^2Q^4\right) + \frac{g\lambda Q^2 fU_\chi}{3}}{3f^2H^2 + g^2\lambda Q^4}\end{aligned} \tag{2.15}$$

Then, if we consider as valid conditions[6] $3f^2H^2 \ll g^2\lambda^2Q^4$ and $\lambda^2Q^2 \gg 2f^2$, we can write

$$\begin{aligned}\dot{\chi} &\simeq \frac{fH}{g\lambda Q^2}\left(\frac{2g^2Q^3}{H} - HQ - \frac{fU_\chi}{g\lambda Q^2}\right) \\ \dot{Q} &\simeq -HQ - \frac{fU_\chi}{3g\lambda Q^2}.\end{aligned} \tag{2.16}$$

We notice that it is possible to write the last equation of (2.16) as an equation of motion for the gauge field if we define an effective potential U_{eff}

$$H\dot{Q} + \frac{\partial U_{\text{eff}}(Q)}{\partial Q} = 0, \quad U_{\text{eff}} \equiv \frac{H^2Q^2}{2} - \frac{fHU_\chi}{3g\lambda Q} \tag{2.17}$$

where U_{eff} is minimized by

$$Q_{\text{min}} = \left(\frac{\mu^4 \sin(\chi/f)}{3g\lambda H}\right)^{1/3} \tag{2.18}$$

When the gauge field attains this configuration (i.e., when $\dot{Q} \simeq 0$), the right-hand side of the first equation in (2.11) effectively balances the gradient of the axion potential on the left-hand side. As a result, the axion evolves very slowly, as its kinetic energy is predominantly transferred into the gauge sector. Rewriting the first equation in (2.16) by substituting Q_{min} produces an effective equation that describes the evolution of the axion in terms of the axion field alone [6]

$$\dot{\chi} \simeq \frac{2}{3^{2/3}}\left(\frac{f}{\lambda}\right)^{4/3} \frac{3(\frac{\lambda}{f})^{2/3}H^{8/3} + 3^{1/3}g^{4/3}(-U_\chi)^{2/3}}{g^{2/3}H^{1/3}(-U_\chi)^{1/3}} \tag{2.19}$$

This demonstrates that the dynamics of the axion is no longer governed by the gradient of its potential and the standard Hubble friction term, but instead proceed slowly on a flat effective potential arising from the interaction between the axion and the vacuum expectation value of the gauge field. From the last equation we are also able to determine the number of e-folds N [5]

[6]This parameter choice allow the value of $\frac{\lambda}{f}$ to be large.

$$N \simeq \int_{\frac{\chi_0}{f}}^{\pi} dx \, \frac{\frac{3^{1/3}}{2}\left(\frac{\mu}{M_pl}\right)^{4/3} g^{2/3}\lambda^{4/3}(\sin x)^{1/3}(1+\cos x)^{2/3}}{\left(\frac{\mu}{M_pl}\right)^{8/3}\lambda^{2/3}(1+\cos x)^{4/3} + 3^{2/3}g^{4/3}(\sin x)^{2/3}}, \tag{2.20}$$

where $x \equiv \dfrac{\chi}{f}$ and we assume that Inflation begins at χ_0. The parameter combination that maximizes the number of inflationary e-folds is given by $g^2/\lambda \simeq (\frac{\mu}{M_{pl}})^4/3$. Under this assumption, one can derive a simple expression for the maximum number of e-foldings $N_{max} \simeq 3\lambda/5$, where the numerical prefactor arises from integrating the combination of sine and cosine functions over the interval $x \in [0, \pi]$. It follows that values of $\lambda \sim 100$ are required in order to achieve $O(60)$ e-folds of Inflation.
Then, we can rewrite the second equation of (2.10) in order to obtain

$$\dot{H} = -\frac{1}{M_{pl}^2}\left\{\frac{\dot{\chi}^2}{2} + \frac{1}{a^2}\left[(a\dot{Q})\right]^2 + g^2Q^4\right\} \tag{2.21}$$

From the last expression we are able to observe that the slow-roll parameter $\epsilon_H = -\dot{H}/H^2$ emerges as a combination of all the small parameters defined in (2.14). However, when the gauge field relaxes to its minimum, given by equation (2.18), a hierarchy naturally develops between the fields and their time derivatives. For example, the second equation in (2.16) immediately shows that $\epsilon_Q \ll \epsilon_1$. Consequently, the relevant slow-roll parameters will be [6]

$$\epsilon_H \simeq \epsilon_1 + \epsilon_B = \frac{Q^2}{M_{pl}^2} + \frac{g^2Q^4}{H^2M_{pl}^2} \tag{2.22}$$

In addition, we have [6]

$$\eta \equiv \frac{\dot{\epsilon_H}}{H\epsilon_H} \simeq \frac{2g^2Q^4}{H^2M_{pl}^2} + \frac{\dot{Q}}{HM_{pl}^2\epsilon_H}\left(2Q + \frac{4g^2Q^3}{H^2}\right) \tag{2.23}$$

We can look to the way Inflation comes to an end. To do this, one can evaluate when $\epsilon_H = 1$: this condition admits a straightforward physical interpretation. Indeed, we have that the first term of (2.22) becomes negligible toward the conclusion of Inflation and can therefore be ignored. The numerator of the second term is instead associated with the energy density of the gauge field, while the denominator corresponds to the axion potential[7]. Inflation thus terminates when the gauge field energy density becomes comparable to the axion potential energy, namely when the Universe transitions from being vacuum energy dominated to radiation dominated.

[7] As we have previously stated, we can take $3M_{pl}^2H^2 \simeq U(\chi)$.

Since the gauge field energy remains small and approximately constant during Inflation, this occurs when the axion field reaches the minimum of its potential. Furthermore, we notice that in CNI, no assumptions are required regarding the specific region of the axion potential in which the field is initially located, as the system can be evolved over the entire interval $0 < \frac{\chi}{f} < \pi$. The degree to which the system approaches the point $\frac{\chi}{f} = \pi$ is determined by the first term in the equation (2.22). We can also define three parameters

$$\Lambda \equiv \frac{\lambda Q}{f}, \quad m_Q \equiv \frac{gQ}{H}, \quad \xi \equiv \frac{\lambda \dot{\chi}}{2fH} \tag{2.24}$$

and we see that the first one is related to the intensity of the interaction between the gauge field and the axion, while the second one is related to the VEV of the gauge field and the third one to the velocity of the axion. In particular, we have that the second parameter in (2.24) characterizes the mass of the gauge field fluctuations and controls their amplification [33]. Moreover, the third expression in (2.24) is a term which was first introduced in the models with the Abelian U(1) gauge field [34] and is often referred to as particle production parameter. We can understand why if we reconsider the process as follows: in the early stages of the evolution, when the axion rolls along the flatter region of its potential, the field χ remains nearly constant, resulting in a very small value of ξ. In this regime, the Chern–Simons term in (2.2) effectively reduces to a boundary term and decouples from the scalar field dynamics, so that the evolution coincides with that of Natural Inflation. As the axion velocity (and therefore ξ) increases, the interaction with the gauge sector becomes significant, leading to a transfer of energy from the axion to the gauge fields, as reflected by the source term in the first equation in (2.11). Taking into account the expression (2.18), we can write

$$\xi \simeq m_Q + \frac{1}{m_Q} \tag{2.25}$$

which is a good approximation during slow-roll. At the level of perturbation, the parameter ξ serves as an indicator of gauge quanta production. This mechanism proves to be highly efficient: as ξ grows with the axion velocity, gauge field fluctuations may become large enough for their backreaction on the background equations of motion to no longer be negligible. This regime, known as strong backreaction, has been the subject of analytical and numerical investigation [35].

2.2.2 CNI Perturbations

Now we can discuss the linear study for the perturbations of the model. The model involves 23 perturbations: one from the inflaton, 12 from the SU(2) vector field and 10 from the metric, which we decompose as follows [6]

$$
\begin{aligned}
\chi &= \bar{\chi} + \delta\chi, \\
A_0^a &= a\,(Y_a + \partial_a Y)\,, \quad A_i^a = a\left[(Q+\delta Q)\,\delta_{ai} + \partial_i(M_a + \partial_a M) + \epsilon_{iac}(U_c + \partial_c U) + t_{ia}\right], \\
g_{00} &= -a^2(1-2\phi), \quad g_{0i} = a^2\,(B_i + \partial_i B)\,, \quad g_{ij} = a^2\left[(1+2\psi)\,\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + h_{ij}\right]
\end{aligned}
\tag{2.26}
$$

The overall picture can therefore be summarized as follows

- Tensor modes: the perturbations t_{ia} and h_{ij}. We require these perturbations to be transverse ($\partial_i h_{ij} = \partial_i t_{ia} = \partial_a t_{ia} = 0$) and traceless ($t_{ii} = h_{ii} = 0$); because of these conditions, the tensor sector contains 4 perturbations.

- Vector modes: the perturbations Y_a, M_a, U_c, B_i, E_i. We impose them to be transverse ($\partial_i Y_i = \cdots = \partial_i E_i = 0$); this allows us to have 10 perturbations for the vector sector.

- Scalar modes: the remaining 9 perturbations, which are $\delta\chi, Y, \delta Q, M, U, \phi, B, \psi, E$.

We emphasize that the terminology "tensor/vector/scalar" is strictly appropriate for metric and inflaton perturbations, as it refers to the transformation properties of these modes under spatial rotations. We extend the same nomenclature to the perturbations of the gauge field, although, properly speaking, this classification is not entirely accurate, since for the decomposition the index employed is the SU(2) one. Nevertheless, this terminology remains convenient because the diagonal structure of the vector VEV (2.9), together with the imposed transversality conditions, ensures that the tensor, vector, and scalar sectors defined above remain decoupled at linear order.

We then perform a Fourier decomposition of the perturbation modes

$$\delta(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \delta(t, \mathbf{k}) \tag{2.27}$$

where δ generically denotes any perturbation variable, and analyze the theory in Fourier space. Given that the analysis is carried out at the linearized level, we consider perturbations with a fixed comoving momentum $\mathbf{k}$. Without loss of generality,

the momentum vector can be aligned along the z-axis. Starting from an arbitrary orientation of $\mathbf{k}$, one may always perform a spatial rotation such that $\mathbf{k} = k\ \hat{k}$, where $\hat{k}$ is the unit vector along the z-direction, and $k > 0$. Following this spatial rotation, the background configuration A_i^a is no longer proportional to δ_i^a; however, the original form for the VEV can be restored through a global SU(2) gauge transformation. As a consequence, one may consistently choose $k_x = k_y = 0$ without loss of generality. This choice considerably simplifies the algebraic analysis [6].
Moreover, it is necessary to eliminate the redundancies associated with general coordinate transformations and SU(2) gauge transformations. Under an infinitesimal coordinate transformation parameterized by $\xi^\mu = (\xi^0, \xi_i + \partial_i \xi)$, the metric perturbations transform as $\psi \to \psi - \mathcal{H}\xi^0$, $E \to E - \xi$ and $E_i \to E_i - \xi_i$. The residual freedom associated with the infinitesimal coordinate transformations can therefore be fixed by imposing $\psi = E = E_i = 0$. Similarly, under an infinitesimal SU(2) gauge transformation with parameter $\alpha^a = \epsilon_a + \partial_a \epsilon$, where ϵ_a is transverse, we get that U, U_i transform according to $U \to U - gQ\epsilon$, $U_i \to U_i + gQ$. Then, the SU(2) gauge freedom can be removed by fixing $U = U_i = 0$ [6].
These choices are motivated by two main reasons: first, it completely fixes the gauge freedom, and second, it preserves all the $\delta g_{0\mu}$ and δA_0^a components. These modes are non-dynamical, as they appear in the quadratic action without time derivatives, and can therefore be integrated out. With all these choices, the previous decomposition in (2.26) takes the explicit form [6]

$$
\begin{aligned}
\chi &= \bar{\chi} + \delta\chi \\
A_\mu^a &= a \begin{pmatrix} Y_1 & Q + \delta Q + t_+ & t_\times & \partial_z M_1 \\ Y_2 & t_\times & Q + \delta Q - t_+ & \partial_z M_2 \\ \partial_z Y & 0 & 0 & Q + \delta Q + \partial_z \partial_z M \end{pmatrix} \\
g_{\mu\nu} &= a^2 \begin{pmatrix} -1 + 2\phi & B_1 & B_2 & \partial_z B \\ B_1 & 1 + h_+ & h_\times & 0 \\ B_2 & h_\times & 1 - h_+ & 0 \\ \partial_z B & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{2.28}
$$

It can be shown that, at linear order, the scalar modes $(\delta\chi, Y, \delta Q, M, \phi, B)$, the vector modes $(Y_{1,2}, M_{1,2}, B_{1,2})$, and the tensor modes $(t_+, t_\times, h_+, h_\times)$ remain mutually decoupled. For this reason, the quadratic action for the perturbations decomposes into three independent sectors, which can be studied individually. If we consider one of these sectors, and we call $\mathcal{X}$ the vector of the perturbations of this section, we can carry out the transformation

$$\mathcal{X}_i = \mathcal{M}_{ij}\Delta_i \tag{2.29}$$

in such a way that the action for the sector can be written as

$$S = \frac{1}{2}\int d\tau\ d^3k\Big[\Delta'^{\dagger}\Delta' + \Delta'^{\dagger}K\Delta - \Delta^{\dagger}K\Delta' - \Delta^{\dagger}\Omega^2\Delta\Big] \tag{2.30}$$

where the matrices Ω and K are, respectively, a Hermitian and an anti-Hermitian ones, given that the action (2.30) is Hermitian.

Tensor Perturbations

In the first place, we can define the variables

$$\Delta_+ \equiv \begin{pmatrix} \frac{aM_{pl}}{\sqrt{2}}h_+ \\ \sqrt{2}at_+ \end{pmatrix}, \qquad \Delta_\times \equiv \begin{pmatrix} \frac{aM_{pl}}{\sqrt{2}}h_\times \\ \sqrt{2}at_\times \end{pmatrix} \tag{2.31}$$

which can be used to determine the left and right helicities

$$\Delta_{\rm L} \equiv \frac{\Delta_+ + i\Delta_\times}{\sqrt{2}}, \qquad \Delta_{\rm R} \equiv \frac{\Delta_+ - i\Delta_\times}{\sqrt{2}} \tag{2.32}$$

These definitions allow us to decompose the action governing the tensor perturbations into two independent sectors, corresponding to the left- and right-helicity doublets. The left-helicity sector is characterized by the following structure [6]

$$\begin{aligned}
K_{t,12} &= \frac{1}{M_{pl}}\left(Q' + \mathcal{H}Q\right),\\
\Omega^2_{t,11} &= k^2 - 2\mathcal{H}^2 - \frac{1}{M_{pl}^2}\left(Q' + \mathcal{H}Q\right)^2 + \frac{3g^2a^2Q^4}{M_{pl}^2} + \frac{\bar{\chi}'^2}{2M_{pl}^2}\\
\Omega^2_{t,12} &= ak\,\frac{2gQ^2}{M_{pl}} + \frac{\mathcal{H}}{M_{pl}}\left(Q' + \mathcal{H}Q\right) - g\lambda\,\frac{Q^2a\bar{\chi}'}{fM_{pl}}\\
\Omega^2_{t,22} &= k^2 - ak\left(2gQ + \frac{\lambda}{f}a\bar{\chi}'\right) + g\frac{\lambda}{f}Qa\bar{\chi}'
\end{aligned} \tag{2.33}$$

where $'$ denotes the derivative with respect to conformal time τ, and the index t stands for tensor. The corresponding action for the right-helicity doublet is obtained from (2.33) through the replacement $k \to -k$, reflecting the breaking of parity symmetry of the model. The interaction between gravitational waves and the tensor perturbations of the gauge field is suppressed by slow-roll parameters. Nevertheless, it is explicit that $\Omega^2_{t,22}$ (which is the effective squared frequency associated with the mode t_L) becomes negative over a finite interval around horizon crossing. This results in a tachyonic amplification of t_L during that interval and, as a consequence, in an enhancement of the left-handed gravitational-wave mode h_L. Such an instability does not arise in the right-helicity sector, because of the opposite sign of the linear term in k.

It is precisely this enhancement that generates a violation of parity invariance within the tensor sector of the model, thereby leading to the production of chiral gravitational waves.

Vector Perturbations

The study of the vector sector is not so enlightening because the perturbations quickly decay on super-horizon scales, so we will talk about it briefly.

Among the sixth vector modes, we have that only two are dynamical: B_1, B_2, Y_1, Y_2 are indeed non-dynamical, and they can be integrated out. After that, the remaining action is function only of the dynamical modes M_1, M_2. We can then define the canonical modes of the system V_1, V_2 thanks to the following relations

$$M_1 = F_1 V_1 + iF_2 V_2, \quad M_2 = iF_1 V_1 + F_2 V_2 \tag{2.34}$$

where we have that

$$F_{1,2} \equiv \frac{\sqrt{M_{pl}^2 \left(\pm k + gaQ\right)^2 + g^2 a^2 Q^2 \left(M_{pl}^2 + 2Q^2\right)}}{\sqrt{2} gk M_{pl} a^2 Q} \tag{2.35}$$

The upper $+$ sign refers to F_1, whereas the lower $-$ sign corresponds to F_2. We can then write

$$S_{2,\text{vector}} = \frac{1}{2} \int d\tau d^3k \left[|V_1'|^2 - \Omega^2_{v+} |V_1|^2 + |V_2'|^2 - \Omega^2_{v-} |V_2|^2 \right] \tag{2.36}$$

and it is possible to demonstrate that [6]

$$\frac{\Omega^2_{v\pm}}{a^2} = p^2 \pm \frac{\lambda}{f} \dot{\chi} p + \frac{M_{pl}^4 \left[c_4 p^4 \pm c_3 p^3 + c_2 p^2 \pm c_1 p + c_0\right]}{M_{pl}^2 p^2 \pm 2g M_{pl}^2 Q p + 2g^2 Q^2 \left(M_{pl}^2 + Q^2\right)} \tag{2.37}$$

where $p = k/a$ denotes the physical momentum, while c_i are some coefficients which depend on background quantities. From the expression (2.37) it is possible to see that the numerator is always positive, and we also have that

$$\Omega^2_{v+}(p) = \Omega^2_{v-}(-p) \tag{2.38}$$

Scalar Perturbations

From (2.28) we note that the scalar sector consists of six perturbation modes: the inflaton fluctuation $\delta\chi$, the three perturbations associated with the SU(2) gauge field, namely $Y, \delta Q$ and M, together with the two metric perturbations ϕ and B. As for the vector sector, not all of these perturbations are dynamical; in particular, we have that the modes Y, ϕ and B are non-dynamical. It is possible to demonstrate that the approximation $\phi = B = 0$ is well justified [6]; so, after integrating out the non-dynamical mode from the gauge sector Y, we have the action from the scalar sector as a function only of the dynamical modes $\delta\chi, \delta Q$ and M. Moreover, we can define these dynamical perturbations in terms of $\Delta = (\Delta_1, \Delta_2, \Delta_3)^T$

$$\begin{aligned} \delta\chi &= \frac{\Delta_1}{a} \\ \delta Q &= \frac{\Delta_2}{\sqrt{2}a} \\ M &= \frac{gaQ\Delta_2 + \sqrt{k^2 + 2g^2a^2Q^2}\Delta_3}{\sqrt{2}gk^2a^2Q} \end{aligned} \tag{2.39}$$

These definitions allow us to write the quadratic action for the scalar perturbations, up to boundary terms, as in the expression (2.30). The dynamics of the sector is encoded in the 3×3 matrices K_s and Ω^2_s, whose non-zero components are [6]

$$\begin{aligned} \frac{K_{s,12}}{a} &= \frac{g\lambda Q^2}{\sqrt{2}f} \\ \frac{K_{s,13}}{a} &= -\frac{g^2\lambda Q^3}{\sqrt{2}f\sqrt{p^2 + 2g^2Q^2}} \end{aligned} \tag{2.40}$$

and

$$
\begin{aligned}
\frac{\Omega^2_{s,11}}{a^2} &= p^2 + \frac{g^2\lambda^2 p^2 Q^4}{f^2\left(p^2+2g^2Q^2\right)} + \frac{g^2Q^4}{M_{pl}^2} + V_{,\chi\chi} - 2H^2 + \frac{\dot{\chi}^2}{2M_{pl}^2} + \frac{\left(\dot{Q}+HQ\right)^2}{M_{pl}^2} \\
\frac{\Omega^2_{s,12}}{a^2} &= \frac{3g\lambda HQ^2}{\sqrt{2}f} + \frac{\sqrt{2}g\lambda Q\dot{Q}}{f} \\
\frac{\Omega^2_{s,13}}{a^2} &= -\frac{\sqrt{2}\lambda}{f}\left[\frac{g^2HQ^3}{2\sqrt{p^2+2g^2Q^2}} + \frac{2p^4+3g^2p^2Q^2+4g^4Q^4}{2\left(p^2+2g^2Q^2\right)^{3/2}}\left(\dot{Q}+HQ\right)\right] \\
\frac{\Omega^2_{s,22}}{a^2} &= p^2 + 4g^2Q^2 - \frac{g\lambda Q\dot{\chi}}{f} \\
\frac{\Omega^2_{s,23}}{a^2} &= -\sqrt{p^2+2g^2Q^2}\left(2gQ - \frac{\lambda}{f}\dot{\chi}\right) \\
\frac{\Omega^2_{s,33}}{a^2} &= p^2 + \frac{4g^2Q^2(p^2+g^2Q^2)}{p^2+2g^2Q^2} - \frac{g\lambda p^2 Q\dot{\chi}}{f(p^2+2g^2Q^2)} + \frac{6g^2p^2(\dot{Q}+HQ)^2}{(p^2+2g^2Q^2)^2}
\end{aligned}
$$

A comprehensive analysis of the stability properties of the scalar sector has been presented in [6, 7]. Their results show that scalar perturbations develop an instability on sub-horizon scales whenever $m_Q < \sqrt{2}$; hence, the parameter m_Q has to be taken larger than $\sqrt{2}$ in order to have a stable inflationary background solution.
Further to this, authors of [6] found a fundamental tension within the model. In particular, no region of parameter space was found in which the predictions for the tensor-to-scalar ratio r, the scalar spectral index n_s, and the amplitude of the primordial power spectrum simultaneously satisfy current observational constraints. More specifically, obtaining a scalar spectral index compatible with observations requires relatively large values of the parameter m_Q. However, in this regime the coupling between the gauge-field fluctuations and tensor perturbations becomes sufficiently strong to generate an excessively large primordial gravitational-wave signal, leading to values of r that exceed observational bounds. In contrast, parameter choices that suppress the tensor sector fail to reproduce an acceptable scalar spectrum. As a result, the CNI scenario is unable to reconcile all cosmological observables simultaneously. This incompatibility with current CMB measurements leads to the conclusion that the model is observationally excluded. Different alternative formulations have been proposed to prevent the original scenario from being entirely ruled out by observational constraints. Various examples include the Higgsed CNI [36], Non-minimally Coupled CNI [37], and Spectator CNI [32] models; these extensions remain cosmologically viable while successfully preserving the distinctive signatures characteristic of the CNI framework. The specific model chosen for this thesis is the

Spectator CNI. This scenario is characterized by the introduction of a new scalar field that acts as the inflaton, whereas the original CNI sector is relegated to a purely spectator role.

2.2.3 Gauge-flation

An interesting feature regarding Chromo Natural Inflation is that it naturally reduces to a class of inflationary models which go under the name of Gauge-flation [26, 27]. This theory is closely related to the formulation presented in equation (2.2). However, in references [26, 27] the axion field is not introduced explicitly; instead, a higher-order interaction term for the gauge fields is considered. The Lagrangian of the model is

$$\mathcal{L}_{\mathrm{GF}} = \sqrt{-g}\left[-\frac{M_{pl}^2}{2}R - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{\kappa}{384}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a\right)^2\right] \tag{2.41}$$

A direct comparison between the two models immediately shows that the interaction term $(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a)^2$ is the one obtained by integrating out the axion field in CNI framework. Integrating out a field is justified only when that field can be assumed to reside at the minimum of its effective potential. Indeed, it is explicitly demonstrated in [38] that the Gauge-flation model corresponds to the CNI scenario in the limit where the axion lies very close to the minimum of its potential, so when $\chi \simeq \pi f$. In terms of the parameters of Chromo Natural Inflation, we can write $\kappa = \frac{3\lambda^2}{\mu^4}$ from which it follows that, in the large-axion regime $\chi \simeq \pi f$, Chromo Natural Inflation effectively reduces to Gauge-flation. In order for such a model to produce a sufficient amount of Inflation, the axion–gauge field coupling λ must be at least an order of magnitude larger than the minimum values required for CNI [39]. Moreover, it has also been demonstrated that Gauge-flation constitutes indeed a special case of Chromo Natural Inflation [39], as can be seen by comparing the trajectories of Gauge-flation with those of CNI. This follows from the fact that CNI encompasses a significantly larger parameter space, owing to the presence of two additional free parameters with respect to Gauge-flation, which provide greater model flexibility. As illustrated in Figure (2.1), the Gauge-flation trajectory coincides with a subset of the CNI trajectory, while CNI allows for a much broader range of evolutionary path.

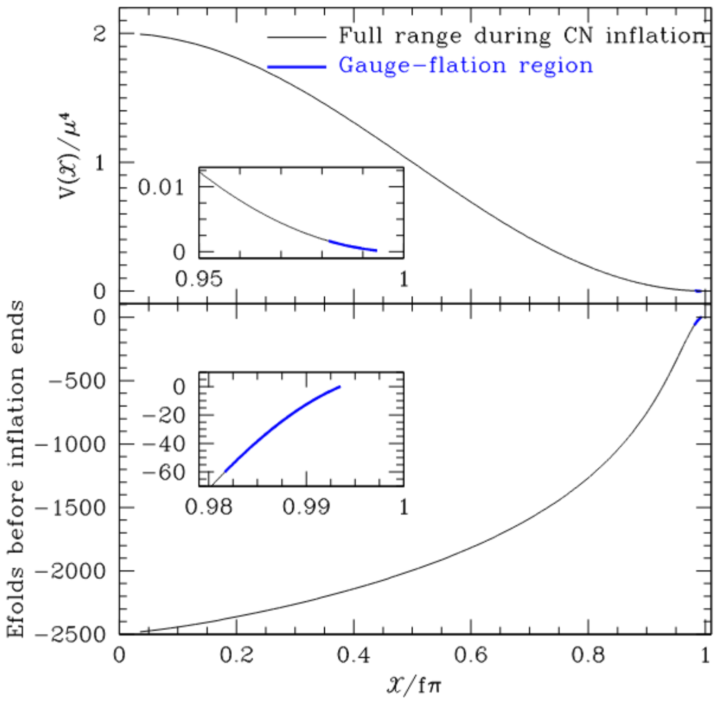


Figure 2.1: The full evolution of the axion field during Chromo Natural Inflation is displayed. The region corresponding to the Gauge-flation regime is highlighted in blue. The inset panels illustrate the portion of the axion's evolution over which the two models overlap [39].

2.3 The SCNI Model

As we have already mentioned, a solution to the shortcomings of the CNI is given by the extension of the model considering the introduction of an additional scalar field, ϕ, which plays the role of the inflaton. In this framework, the axion no longer drives Inflation directly, and so the axion-gauge field acts as a spectator sector. This modification preserves the phenomenology of CNI while avoiding tensions with observational constraints. The resulting scenario is called the Spectator Chromo Natural Inflation (SCNI) [32]. SCNI shares many similarities with the CNI scenario, so a part of the following analysis will refer directly to Section (2.2).

2.3.1 Background Dynamics

Due to the introduction of ϕ, the action of the model is [32]

$$\mathcal{S} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} - \frac{1}{2}(\partial\chi)^2 - U(\chi) + \frac{\lambda}{4f}\chi F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right] \tag{2.42}$$

As in the original CNI setup, the axion χ evolves along its potential $U(\chi)$, while the gauge field develops a homogeneous and isotropic background configuration, which is parametrized by a time-dependent function $Q(t)$(2.9). The inflaton field is assumed to dominate the energy density of the Universe through its potential $V(\phi)$, while the axion and the gauge field energy densities remain subdominant, and ϕ interacts only through gravitational interactions with the CNI sector. A new feature of the model is that now the axion is not required to roll throughout the entire inflationary epoch; despite this, the generation of observable signatures may occur during a limited interval in which both the axion and the inflaton are actively rolling. We denote by N_χ the phase in which χ rolls, and we have to take $N_\chi \gtrsim 10$, so that the produced signals extend over the range of cosmological scales accessible to observations. The rolling phase of the axion is then followed by approximately $60 - N_\chi$ additional e-folds of Inflation driven by the inflaton field.

Like we have done in (2.14), we can define a set of slow-roll parameters that characterize the evolution of the different sectors of the model

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}, \quad \epsilon_\phi \equiv \frac{\dot{\phi}^2}{2M_{pl}^2 H^2}, \quad \epsilon_\chi \equiv \frac{\dot{\chi}^2}{2M_{pl}^2 H^2}, \quad \epsilon_B \equiv \frac{g^2 Q^4}{M_{pl}^2 H^2}, \quad \epsilon_E \equiv \frac{(HQ+\dot{Q})^2}{H^2 M_{pl}^2} \tag{2.43}$$

where here we have introduced ϵ_ϕ and ϵ_E; moreover, we can set $\epsilon_A \equiv \epsilon_B + \epsilon_E$, which accounts for the total gauge field contribution. The dynamics of the background is governed by the classical equations of motion[8][40, 41]

$$\begin{aligned} &\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 \\ &\ddot{\chi} + 3H\dot{\chi} + U_\chi = -\frac{3g\lambda}{f} Q^2(\dot{Q} + HQ) \\ &\ddot{Q} + 3H(\dot{Q} + HQ) - H^2 Q(1 - \dot{H}/H^2) + 2g^2 Q^3 = \frac{g\lambda}{f}\dot{\chi}Q^2 \end{aligned} \tag{2.44}$$

These equations must be completed by a combination of the Einstein equations. A particularly convenient choice is one that can be expressed in terms of the slow-roll parameters (2.43), since it makes the relative contributions of the inflaton, axion, and gauge sectors to the inflationary dynamics transparent. So we have

$$\epsilon_H = \epsilon_\chi + \epsilon_\phi + \epsilon_A \tag{2.45}$$

Assuming a FLRW background together with the isotropic SU(2) gauge field configuration, the set of equations (2.44) and (2.45) provides a complete description of the background evolution. In addition, we can also define

$$\eta_H \equiv \epsilon_H - \frac{1}{2}\frac{\dot{\epsilon}_H}{H\epsilon_H}, \quad \eta_\phi \equiv M_{pl}^2 - \frac{1}{2}\frac{\dot{\epsilon}_H}{H\epsilon_H} \tag{2.46}$$

We can then write that the inflaton slow-roll parameter obeys [41]

$$\dot{\epsilon}_\phi \approx H\epsilon_\phi(6\epsilon_H - 4\eta_H + n_s - 1) \tag{2.47}$$

where we have that [41]

$$n_s - 1 \approx (4\eta_H - 2\eta_\phi - 2\epsilon_H)\cdot(1-\epsilon_H)^{-1} \approx 4\eta_H - 2\eta_\phi - 2\epsilon_H \tag{2.48}$$

where $n_s \simeq 0.965 \, \pm \, 0.004$ [15] is the scalar spectral index.
Once the slow-roll approximation is imposed, the second-derivative terms can be neglected, allowing the second and third equations of (2.44) to be recast into two

[8]We remind that X_y stands for the derivative of a function $X(y)$ with respect to its argument y.

independent equations of motion. The equation for the axion yields an important constraint on the magnitude of the coupling parameter λ. By employing the slow-roll expression for $\dot{\chi}$ and rewriting U_χ in terms of the remaining background quantities (under the assumption that Q has settled at a value for which the right-hand side of (2.15) vanishes) we obtain

$$\dot{\chi} \simeq \frac{2Hf}{\lambda}\Big(m_Q + m_Q^{-1}\Big) \tag{2.49}$$

The last expression shows that increasing λ suppresses the velocity of the axion field, effectively introducing an additional friction term in its dynamics. Conversely, in the regime of small λ, the axion evolves more rapidly. A rough estimate of the number of e-folds generated during the axion-driven stage can be obtained from (2.49) (considering $\dot{Q} = 0$): by evaluating the number of Hubble times required for χ to vary by an amount of order f, we find that [41]

$$N_\chi \sim \frac{\lambda}{2\Big(m_Q + m_Q^{-1}\Big)} \tag{2.50}$$

Demanding a value of $N_\chi \gtrsim 10$ implies the condition

$$\lambda \sim 2\Big(m_Q + m_Q^{-1}\Big)N_\chi \gtrsim 60 \tag{2.51}$$

A distinction within the SCNI framework can be made by considering the ratio $\frac{\epsilon_\phi}{\epsilon_A + \epsilon_\chi}$. Depending on the value of this quantity, two different realizations of the model emerge [41]. When the ratio is larger than one, one obtains the so-called "classic SCNI" branch, whereas values below unity correspond to the "mixed SCNI" branch. In the former case, the inflaton not only dominates the total energy density of the Universe but also provides the primary source of the inflationary expansion. In the latter, although the inflaton remains the dominant contribution to the energy density, the dynamics governing the inflationary evolution are controlled by the gauge sector. Despite these differences, both branches exhibit attractor behavior.
During the CNI phase, slow-roll solutions are characterized by a gauge field background Q that remains approximately constant and non-vanishing. Its value is determined by (2.15), which can be recast as the polynomial condition [41]

$$g^2\lambda^2Q^4 + 2g^2f^2Q^2 + 2f^2H^2 + \frac{g\lambda f}{3H}QU_\chi = 0 \tag{2.52}$$

Although (2.52) is quartic in Q, a more convenient formulation is obtained by considering it as a quadratic equation for Λ, while keeping all the other parameters fixed. In this form, the relation becomes

$$\Lambda^2 - \frac{\mu^4 f}{3H^3 m_Q^2/g}\,\Lambda + 2\left(1 + m_Q^2\right) = 0 \tag{2.53}$$

whose solutions are

$$\Lambda_{1,2} = \frac{\mu^4/f}{6m_Q^2 H^3/g}\left[1 \pm \sqrt{1 - 2\left(\mu^4/f\right)^{-2}\left(6m_Q^2 H^3/g\right)^2\left(1 + m_Q^{-2}\right)}\right] \tag{2.54}$$

From the last expression we can define a critical value $\Lambda_{crit} \equiv \sqrt{2}(1 + m_Q^{-2})^{1/2}$, obtained when the square root is zero; therefore, the larger root satisfies $\Lambda_1 \geq \Lambda_{crit} > \sqrt{2}$, while the smaller root obeys $\Lambda_2 \leq \Lambda_{crit}$. So, choosing a small Λ means taking Λ_2 as the solution that relates λ to μ and f. In addition, the solution Λ_2 may correspond to relatively small values of the coupling λ. In this regime, the additional friction generated by the gauge sector becomes less effective, causing the axion to roll more rapidly, and this can lead to a change of the predicted scalar power spectrum. Indeed, one obtains the constraint [41]

$$\Lambda \geq \Lambda_{\min} \equiv \frac{1 + m_Q^2}{g} 8\pi\sqrt{\mathcal{P}_\zeta}\left[1 - \frac{32\pi^2\mathcal{P}_\zeta}{g^2} m_Q^2(1 + m_Q^2)\right]^{-1/2} \tag{2.55}$$

where $\mathcal{P}_\zeta$ is the amplitude of the scalar spectrum. This adds a lower bound to Λ_2.

2.3.2 Constraints of the Model

As discussed in the literature [32, 40, 41], the parameter space of SCNI is subject to several theoretical and observational constraints, that restrict the range of viable configurations. We will focus our attention on the constraints relative to parameters that will largely occur during this work, which are m_Q and Λ (defined in (2.24)). Analyses of CNI have focused on the regime $\Lambda \gg \sqrt{2}$, where the parameter decouples from the problem. This limit is very attractive from a technical perspective, as it considerably simplifies the analytical treatment of the equations of motion for Q. Moreover, within the original CNI model, large values of Λ were required to sustain a sufficiently prolonged period of slow-roll Inflation [5, 6, 33]. The spectator realization of the SCNI model substantially modifies this picture: since the accelerated expansion is now driven by the additional inflaton field rather than by the axion–gauge sector itself, the requirement of large Λ is no longer tied to the duration of inflation. Therefore, the SCNI framework allows us to consider the range $\Lambda \ll \sqrt{2}$.

As in the CNI scenario, the absence of tachyonic instabilities in the scalar sector imposes a lower bound on the m_Q parameter, which is $m_Q > \sqrt{2}$. This condition is required at all values of Λ and guarantees the existence of a stable inflationary background characterized by a nearly scale-invariant spectrum of scalar perturbations. Also for the SCNI model violating this bound leads to the development of sub-horizon instabilities, signaling the breakdown of the classical solution.

Another crucial consistency condition concerns the production of inflaton perturbations sourced by the axion field. Specifically, as demonstrated in [40], integrating out the non-dynamical gauge perturbation δA_0^3 introduces new interaction terms beyond the standard, purely gravitational coupling expected between two scalar fields. Furthermore, in the relevant regime, the ratio of this induced coupling to the standard gravitational interaction scales as $\mathcal{O}(\Lambda^2)$. As a consequence, the authors analyze a one-loop diagram wherein the inflaton perturbation is sourced by the axion field, whose fluctuations are enhanced by the non-linear interactions with the unstable gauge tensor mode. They subsequently demonstrate the necessity of imposing an upper bound on the ratio between this non-linear and the linear contributions to the scalar power spectrum. This constraint is required for two reasons. First, it ensures that the total tensor-to-scalar ratio is not suppressed by an overproduction of sourced inflaton perturbations. Second, the sourced modes are expected to be highly non-Gaussian: if they dominate the power spectrum, they would violate the observational limits on primordial non-Gaussianity. However, when $\Lambda \gg 1$, this condition becomes so restrictive that it completely rules out the classic branch of the model [41]. Hence, to remain as model-agnostic as possible and avoid the issues caused by these non-linear quantum corrections, we choose to work in the $\Lambda \ll 1$ regime[9]. In this way, the bound becomes irrelevant, and no parameter space is excluded from the start.

Another important assumption we make is the neglect of the backreaction induced by the amplified tensor perturbations on the gauge field background. This approximation is crucial for maintaining analytical control over the system and significantly simplifies the study of the perturbation dynamics. Finally, considering all the vari-

[9] It is worth noting that choosing a small value for Λ places f within an order of magnitude of M_{pl}. Although this feature could potentially require a more careful assessment of the model's UV completion, it does not necessarily invalidate such a parameter choice.

ous constraints, it is possible to determine the range of values for m_Q and g, which is [40, 41]

$$\sqrt{2} < m_Q \lesssim 35\sqrt{g}, \quad 1.6 \cdot 10^{-3} \lesssim g \tag{2.56}$$

With this established, we can now summarize the parameter values that will be employed in the rest of the thesis[10]

$$\begin{aligned} &g = 0.014, \quad m_Q = 2.7, \quad \Lambda = 0.74, \\ &f = 0.13\ M_{pl}, \quad H = 8.2 \cdot 10^{-6} M_{pl}, \quad Q = 1.58 \cdot 10^{-3} M_{pl} \end{aligned} \tag{2.57}$$

2.3.3 SCNI Perturbations

Regarding the linear analysis of perturbations in the SCNI model, the study closely mirrors the one outlined for CNI in Section (2.2.2). Indeed, both the tensor and vector sectors remain completely identical. Therefore, we will briefly outline the differences arising in the scalar sector.
This sector is characterized by the inclusion of the inflaton field; thus, extending (2.26), the perturbation expansion must now include $\phi = \bar{\phi} + \delta\phi$. Again, it can be demonstrated that the non-dynamical metric modes provide additional contributions to the equations of motion that are suppressed by higher orders in the slow-roll parameters with respect to the leading terms, so these contributions can be safely neglected [32, 40]. For what concern the introduction of the inflaton into the system, it can be shown that the inflaton perturbation modifies the equations of motion of the other scalar modes solely through slow-roll suppressed terms. As a result, the inflaton perturbation affects the evolution of the CNI fields only negligibly [32, 40]. Then, it will not be considered in the analysis performed in the next chapter.

2.3.4 Curvature Perturbation

We briefly outline the relation between the spectator axion and the curvature perturbation ζ. In the spatially flat gauge, the latter is defined as

$$\zeta(\mathbf{x}, t) = -H(t)\frac{\delta\rho(\mathbf{x}, t)}{\dot{\rho}(t)} \tag{2.58}$$

where $\delta\rho$ is the energy density perturbation and $\rho(t)$ is the total background

[10] To determine Λ, we used the relation (2.55) along with the observed value $\mathcal{P}_\zeta^{obs} \simeq 2.1 \cdot 10^{-9}$ [15].

energy density. Under the assumptions of our model, the background energy density is dominated by the inflaton potential, allowing us to neglect the kinetic energies of both ϕ and χ. Considering the period during which the axion is in slow-roll, and taking into account only the inflaton and axion contributions, we obtain

$$\zeta = -H\frac{\delta\rho_\phi + \delta\rho_\chi}{\dot{V}(\phi)} \simeq -\frac{H}{\dot{\phi}}\delta\varphi - \frac{HU_\chi}{V_\phi\dot{\phi}}\delta\chi \tag{2.59}$$

Equation (2.59) indicates that the axion can source ζ through two distinct channels: indirectly, by sourcing inflaton perturbations, and directly, via the second term. The indirect contribution has been shown to be negligible in the literature [32, 40]. On the other hand, the direct contribution remains significant as long as the axion is slowly rolling.
However, this direct imprint is highly sensitive to the axion dynamics. If χ exits the slow-roll regime, approaches its potential minimum, and undergoes damped oscillations while Inflation is still ongoing, the second term in (2.59) rapidly decays [40, 42]. Despite this suppression during Inflation, the axion can still contribute to ζ through a curvaton-like mechanism [43, 44, 45, 46]. While conventionally ζ is conserved on super-horizon scales, the presence of the spectator field introduces isocurvature perturbations that can source curvature perturbations outside the horizon. If the axion survives to occupy a substantial fraction of the post-inflationary energy budget, its contribution is revitalized. For example, if χ oscillates around a quadratic minimum and decays after the inflaton has already decayed into radiation, assuming the sudden decay approximation, the resulting curvature perturbation is given by [42]

$$\zeta = \frac{2\rho_\chi}{4\rho_{\rm rad} + 3\rho_\chi}\frac{\delta\chi}{\chi}, \tag{2.60}$$

with $\rho_{rad} \propto a^{-4}$. This expression must be evaluated exactly at the time of the axion decay, after which its contribution to ζ becomes permanently frozen. Finally, an essential consistency check for this scenario would be needed to verify that this late-time sourcing mechanism does not spoil the agreement with the observed curvature power spectrum.

Chapter 3

Non-linear analysis

3.1 Parity Violation

It is well known that the SCNI model, similarly to the CNI framework, predicts parity violation within the tensor sector, as shown in Section (2.2.2). Specifically, gauge field tensor modes experience a transient growth in one of their polarizations; this affects the freeze-out values of the corresponding helicities of the metric tensor, thus leading to the production of chiral gravitational waves. Given these phenomenological implications, the tensor sector has been extensively investigated in the literature [32, 47, 48, 49]. For the purposes of this thesis, we have opted instead to investigate the purely scalar sector, which has received less attention. As a result, the following sections will exclusively consider the perturbations of the axion and the scalar modes of the gauge field.
As outlined in the Introduction, the aim of this thesis is to calculate the axion perturbation trispectrum and to determine whether parity violation manifests in this quantity. Evaluating a four-point correlation function is not merely a mathematical exercise, but rather a theoretical necessity, as we will now clarify [50, 51]. Starting from the Fourier expansion (2.27) of the axion perturbation $\delta\chi$, and making use of the reality condition for the field $\delta\chi(\mathbf{x}, t)$, we can state that $\delta\chi^*(\mathbf{k}, t) = \delta\chi(-\mathbf{k}, t)$. Therefore, an n-point correlation function, evaluated at a fixed time, under the parity operation $\mathbf{x} \to -\mathbf{x}$ and so $\mathbf{k} \to -\mathbf{k}$ transforms according to

$$\langle \Pi_i^n \, \delta\chi(\mathbf{k}_i, t)\rangle \to \langle \Pi_i^n \, \delta\chi(-\mathbf{k}_i, t)\rangle = \langle \Pi_i^n \, \delta\chi(\mathbf{k}_i, t)\rangle^* \tag{3.1}$$

It follows immediately that the real part of the correlator is parity-even, whereas the imaginary part changes sign under the transformation and therefore corresponds to a parity-odd contribution. Interestingly, parity-odd signatures can only arise in four-point and higher-order correlation functions. This can be understood from rotational invariance: in the case of the bispectrum, momentum conservation constrains the three wavevectors to lie in a common plane; after a parity transformation, the resulting configuration can always be mapped back onto the original one by an appropriate three-dimensional rotation. For this reason, no parity-odd structure can be constructed from three momenta alone in this case. The same conclusion can be reached from a more formal perspective. The three-point function may be written as

$$\langle \delta\chi(\mathbf{k}_1,t)\delta\chi(\mathbf{k}_2,t)\delta\chi(\mathbf{k}_3,t)\rangle = \frac{1}{(2\pi)^{3/2}}\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\ B(k_1,k_2,\mathbf{k}_1\cdot\mathbf{k}_2) \tag{3.2}$$

Since the variables k_1, k_2 and $\mathbf{k}_1 \cdot \mathbf{k}_2$ are all invariant under parity, the bispectrum B itself must be parity-even and therefore purely real. The situation changes at the level of the four-point function. With four independent momenta, one can construct additional scalar combinations involving triple products, such as $\mathbf{k_1}\cdot(\mathbf{k}_2\times\mathbf{k}_3)$, which reverse sign under parity transformations[1]. The presence of such pseudoscalar quantities allows the trispectrum (defined in (4.2)) to contain parity-odd components, making four-point statistics the lowest-order correlators capable of probing parity violation in primordial fluctuations. Consequently, a non-zero $\mathrm{Im}[T]$ is sensitive to parity violation.

In this thesis, we will focus on two specific diagrams to evaluate the trispectrum; namely, we will investigate the contributions arising from the scalar exchange diagram depicted in Figure (3.1a) and the contact interaction diagram illustrated in Figure (3.1b).

3.2 Lagrangian Computation

The starting point of the computation of this work is the expansion of the Lagrangian (2.42) up to the fourth order in perturbations for the axion-gauge field sector. For the terms which we are interested in we can write (see Appendix (B.1))

[1]Since the triple product can be written as $\mathbf{k_1}\cdot(\mathbf{k}_2\times\mathbf{k}_3) = \epsilon^{ijk}k_{1,i}k_{2,j}k_{3,k}$, parity-odd contributions are associated with the survival of an uncontracted Levi-Civita tensor in an interaction vertex or in the final correlation function.

(a) Scalar Exchange (b) Contact Interaction

Figure 3.1: Diagrams of the four-point function that we compute. The external solid line represents the axion perturbation $\delta\chi$. The internal dashed line shows the scalar perturbation of the gauge field δQ.

$$\mathcal{S}^{CS} = \int d^4x - \frac{\lambda}{2f}\partial_\mu\chi\ \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma - \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\right) \tag{3.3}$$

for the Chern-Simons term and

$$\begin{aligned}\mathcal{S}^{K} = \int d^4x\ \sqrt{\tilde{g}}\Big\{&\frac{1}{2a^2}(\partial_0A^a_i\partial_0A^a_i - 2\partial_0A^a_i\partial_iA^a_0 + \partial_iA^a_0\partial_iA^a_0) - \frac{1}{2a^4}(\partial_iA^a_k\partial_iA^a_k - \partial_iA^a_k\partial_kA^a_i)\\ &- \frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_i - \partial_iA^a_0A^d_0A^e_i)\Big] + \frac{1}{a^4}\Big(g\epsilon^{ade}\partial_iA^a_kA^d_iA^e_k\Big) + \frac{g^2}{2a^2}(A^b_0A^c_iA^b_0A^c_i - A^b_0A^c_iA^c_0A^b_i)\\ &- \frac{g^2}{4a^4}(A^b_iA^c_kA^b_iA^c_k - A^b_iA^c_kA^c_iA^b_k)\Big\}\end{aligned} \tag{3.4}$$

for the gauge kinetic term. Considering the perturbation expansion for the axion and gauge fields given in (2.26), and recalling that the background quantities $\bar{\chi}$ and $\bar{A}^a_\mu$ (defined in (2.9)) are purely time-dependent, our goal is to determine the Lagrangian contributions $\mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$, with the subscripts indicating the perturbative order. For convenience, we report the scalar gauge field perturbations of interest here

$$\begin{aligned}\delta A^a_0(x,t) &= a\partial_aY(x,t)\\ \delta A^a_i(x,t) &= a\left[\delta Q(x,t)\delta^a_i + \partial_i\partial_aM(x,t)\right]\end{aligned} \tag{3.5}$$

and we know that the perturbations $\delta\chi, \delta Q$ and M are dynamical modes, while the mode Y is non-dynamical. The presence of this non-dynamical mode complicates our calculations. Indeed, since we aim to reach the fourth order in perturbations, it is not sufficient to determine the expression for Y in terms of the dynamical modes only at the linear order. Specifically, considering the structure of the model's

Lagrangian, we must expand Y up to the second order, given that the Lagrangian (3.47) contains at most quadratic terms in this non-dynamical field. For instance, the second order Lagrangian $\mathcal{L}_2$ contains terms such as $\frac{1}{2}(\partial_i\partial_a Y)^2$; when expanding $Y = Y^{(1)} + Y^{(2)}$, the quadratic operator yields third order contributions of the form $(\partial_i\partial_a Y^{(1)})(\partial_i\partial_a Y^{(2)})$ and fourth order contributions of the form $(\partial_i\partial_a Y^{(2)})(\partial_i\partial_a Y^{(2)})$. It is worth noting that it is not necessary to push the expansion of Y up to the third order. The only possible contributions involving $Y^{(3)}$ would arise from the quadratic part of the Hamiltonian (producing terms like $\sim Y^{(3)}Y^{(1)}$ and $\sim Y^{(3)}\delta A$). However, these contributions are proportional to the linear equations of motion for Y evaluated at $Y = Y^{(1)}$, and therefore vanish identically.
Hence, our procedure will be as follows: we will first derive the general expression for the Hamiltonian $\mathcal{H}(\delta\chi, \delta Q, M, Y)$, then determine the constraint $Y = Y^{(1)} + Y^{(2)}$ in terms of the dynamical modes, and finally substitute this expansion back into $\mathcal{H}(\delta\chi, \delta Q, M)$ to extract the interaction terms of interest.

3.2.1 Second Order

We now want to determine the expression for the perturbed Lagrangian $\mathcal{L}_2$; we start from the Chern-Simons term, with $\mu = i$. We can write

$$-\frac{\lambda}{2f}\partial_i\chi\,\epsilon^{i\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma - \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\right) = -\frac{\lambda}{2f}\partial_i\chi\,\epsilon^{i\nu\rho\sigma}A^a_\nu\partial_\rho A^a_\sigma + \frac{\lambda}{2f}\frac{g}{3}\partial_i\chi\,\epsilon^{i\nu\rho\sigma}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma \tag{3.6}$$

If we substitute the expression for the perturbed fields, from the first term we get

$$\begin{aligned}
&-\frac{\lambda}{2f}\partial_i\delta\chi\,\epsilon^{i\nu\rho\sigma}\left(\bar{A}^a_\nu\partial_\rho\delta A^a_\sigma + \delta A^a_\nu\partial_\rho\bar{A}^a_\sigma\right) = -\frac{\lambda}{2f}\partial_i\delta\chi\left[\epsilon^{ij\rho\sigma}(aQ)\delta^a_j\partial_\rho\delta A^a_\sigma + \epsilon^{ik0j}\delta A^a_k\partial_0(aQ)\delta^a_j\right]\\
&= -\frac{\lambda}{2f}\partial_i\delta\chi\left[\epsilon^{iak0}a(aQ)\partial_k\partial_a Y + \epsilon^{ia0j}(aQ)\partial_0\delta A^a_j + \epsilon^{ik0a}\delta A^a_k\partial_0(aQ)\right]\\
&= -\frac{\lambda}{2f}\partial_i\delta\chi\left\{(aQ)\epsilon^{ia0j}\partial_0[a(\delta Q\delta^a_j + \partial_a\partial_j M)] + \partial_0(aQ)\epsilon^{ik0a}[a(\delta Q\delta^a_k + \partial_a\partial_k M)]\right\} = 0
\end{aligned} \tag{3.7}$$

where we have used the anti-symmetric property of the Levi-Civita tensor and the fact that partial derivatives commute to set the result to 0.[2] While for the second term we obtain

[2]We will use this often in the next computations.

$$
\begin{aligned}
&\frac{\lambda}{2f}\frac{g}{3}\partial_i\delta\chi\ \epsilon^{i\nu\rho\sigma}\epsilon^{abc}\left(\delta A^a_\nu\bar{A}^b_\rho\bar{A}^c_\sigma+\bar{A}^a_\nu\delta A^b_\rho\bar{A}^c_\sigma+\bar{A}^a_\nu\bar{A}^b_\rho\delta A^c_\sigma\right)=\frac{\lambda}{2f}\frac{g}{3}\partial_i\delta\chi\ 3\epsilon^{i\nu\rho\sigma}\epsilon^{abc}\delta A^a_\nu\bar{A}^b_\rho\bar{A}^c_\sigma\\
&=\frac{\lambda}{2f}g\partial_i\delta\chi\ \epsilon^{i0jk}\epsilon^{abc}\delta A^a_0(aQ)\delta^b_j(aQ)\delta^c_k=\frac{\lambda}{2f}ag(aQ)^2\partial_i\delta\chi\ \epsilon^{i0jk}\epsilon^{ajk}\partial_aY=-\frac{\lambda}{f}ag(aQ)^2\partial_i\delta\chi\partial_iY
\end{aligned}
\tag{3.8}
$$

where, due to the Levi-Civita tensor property, we find that the index μ must be equal to 0. Instead, if we consider $\mu=0$ in (3.3), we get

$$
-\frac{\lambda}{2f}\partial_0\chi\,\epsilon^{0\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma-\frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\right)=-\frac{\lambda}{2f}\partial_0\chi\,\epsilon^{0\nu\rho\sigma}A^a_\nu\partial_\rho A^a_\sigma+\frac{\lambda}{6f}g\partial_0\chi\,\epsilon^{0\nu\rho\sigma}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma
\tag{3.9}
$$

At first, we take into account the axion perturbation and a gauge field perturbation; so, from the first term we obtain

$$
\begin{aligned}
&-\frac{\lambda}{2f}\partial_0\delta\chi\ \epsilon^{0\nu\rho\sigma}\left(\bar{A}^a_\nu\partial_\rho\delta A^a_\sigma+\delta A^a_\nu\partial_\rho\bar{A}^a_\sigma\right)=-\frac{\lambda}{2f}\partial_0\delta\chi\left[\epsilon^{0jki}(aQ)\delta^a_j\partial_k\delta A^a_i+\epsilon^{0i0j}\delta A^a_i\partial_0(aQ)\delta^a_j\right]\\
&=-\frac{\lambda}{2f}\partial_0\delta\chi\epsilon^{0aki}a(aQ)\partial_k(\delta Q\delta^a_i+\partial_a\partial_iM)=0
\end{aligned}
\tag{3.10}
$$

while from the second we have

$$
\begin{aligned}
&\frac{\lambda}{2f}\frac{g}{3}\partial_0\delta\chi\ \epsilon^{0\nu\rho\sigma}\epsilon^{abc}\left(\delta A^a_\nu\bar{A}^b_\rho\bar{A}^c_\sigma+\bar{A}^a_\nu\delta A^b_\rho\bar{A}^c_\sigma+\bar{A}^a_\nu\bar{A}^b_\rho\delta A^c_\sigma\right)=\frac{\lambda}{2f}\frac{g}{3}\partial_0\delta\chi\ 3\epsilon^{0\nu\rho\sigma}\epsilon^{abc}\delta A^a_\nu\bar{A}^b_\rho\bar{A}^c_\sigma\\
&=\frac{\lambda}{2f}g\partial_0\delta\chi\ \epsilon^{0ijk}\epsilon^{abc}\delta A^a_i(aQ)\delta^b_j(aQ)\delta^c_k=\frac{\lambda}{2f}ag(aQ)^2\partial_0\delta\chi\ \epsilon^{0ijk}\epsilon^{ajk}(\delta Q\delta^a_i+\partial_a\partial_iM)\\
&=\frac{\lambda}{2f}ag(aQ)^2\partial_0\delta\chi\ [6\delta Q+2\delta^i_a(\partial_a\partial_iM)]=\frac{\lambda}{f}ag(aQ)^2\partial_0\delta\chi\ (3\delta Q+\Delta M)
\end{aligned}
\tag{3.11}
$$

Now instead we consider two gauge field perturbations, and we obtain from the first term in (3.9)

$$
\begin{aligned}
&-\frac{\lambda}{2f}\partial_0\bar{\chi}\ \epsilon^{0\nu\rho\sigma}\delta A^a_\nu\partial_\rho\delta A^a_\sigma=-\frac{\lambda}{2f}a^2\partial_0\bar{\chi}\epsilon^{0ijk}(\delta Q\delta^a_i+\partial_a\partial_iM)\partial_k(\delta Q\delta^a_j+\partial_a\partial_jM)\\
&=-\frac{\lambda}{2f}a^2\partial_0\bar{\chi}\epsilon^{0ijk}(\delta Q\partial_k\partial_i\partial_jM+\partial_i\partial_jM\partial_k\delta Q+\partial_a\partial_iM\partial_k\partial_a\partial_jM)=0
\end{aligned}
\tag{3.12}
$$

and from the second we get

$$\begin{aligned}
&\frac{\lambda}{2f}\frac{g}{3}\partial_0\bar{\chi}\ \epsilon^{0\nu\rho\sigma}\epsilon^{abc}\left(\delta A^a_\nu\delta A^b_\rho\bar{A}^c_\sigma+\bar{A}^a_\nu\delta A^b_\rho\delta A^c_\sigma+\delta A^a_\nu\bar{A}^b_\rho\delta A^c_\sigma\right)=\frac{\lambda}{2f}\frac{g}{3}\partial_0\bar{\chi}\ 3\epsilon^{0ij\sigma}\epsilon^{abc}\delta A^a_i\delta A^b_j\bar{A}^c_\sigma\\
&=\frac{\lambda}{2f}g\partial_0\bar{\chi}\ \epsilon^{0ijk}\epsilon^{abc}\delta A^a_i\delta A^b_j(aQ)\delta^c_k=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}\ \epsilon^{0ijk}\epsilon^{abk}(\delta Q\delta^a_i+\partial_a\partial_iM)(\delta Q\delta^b_j+\partial_b\partial_jM)\\
&=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}\ \epsilon^{0ijk}\epsilon^{abk}(\delta Q^2\delta^a_i\delta^b_j+\delta Q\delta^a_i\partial_b\partial_jM+\partial_a\partial_iM\delta Q\delta^b_j+\partial_a\partial_iM\partial_b\partial_jM)\\
&=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}[6\delta Q^2+(2\delta^j_b)\delta Q\partial_b\partial_jM+(2\delta^i_a)\partial_a\partial_iM\delta Q+(\delta^{jb}\delta^{ia}-\delta^{ja}\delta^{ib})\partial_a\partial_iM\partial_b\partial_jM]\\
&=\frac{\lambda}{f}a^2g(aQ)\partial_0\bar{\chi}(3\delta Q^2+2\delta Q\Delta M)
\end{aligned}\tag{3.13}$$

where we have used

$$\begin{aligned}
&\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}(\delta^{jb}\delta^{ia}-\delta^{ja}\delta^{ib})\partial_a\partial_iM\partial_b\partial_jM=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}(\partial_i\partial_iM\partial_j\partial_jM-\partial_j\partial_iM\partial_i\partial_jM)\\
&=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}[\partial_j(\partial_i\partial_iM\partial_jM)-\partial_j\partial_i\partial_iM\partial_jM-\partial_j(\partial_j\partial_iM\partial_iM)+\partial_j\partial_j\partial_iM\partial_iM]\\
&=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}[-\partial_j(\partial_j\partial_i\partial_iMM)+\partial_j\partial_j\partial_i\partial_iMM+\partial_i(\partial_j\partial_j\partial_iMM)-\partial_i\partial_j\partial_j\partial_iMM]\\
&=\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}[\Delta^2MM-\Delta^2MM]
\end{aligned}\tag{3.14}$$

but this is equal to 0, due to the fact that terms like $\frac{\lambda}{2f}a^2g(aQ)\partial_0\bar{\chi}\partial_j(\partial_i\partial_iM\partial_jM)$ are a total derivative, and therefore a boundary term when integrated[3]. So the final contribution from Chern-Simons is

$$\mathcal{L}_2^{CS}=-\frac{\lambda}{f}ag(aQ)^2\partial_i\delta\chi\partial_iY+\frac{\lambda}{f}ag(aQ)^2\partial_0\delta\chi\ (3\delta Q+\Delta M)+\frac{\lambda}{f}a^2g(aQ)\partial_0\bar{\chi}(3\delta Q^2+2\delta Q\Delta M)\tag{3.15}$$

The next step is to consider the kinetic part of the gauge field; the first term in (3.4) gives us

[3] We will use this again in the following computations.

$$\frac{1}{2a^2}(\partial_0\delta A_i^a\partial_0\delta A_i^a - 2\partial_0\delta A_i^a\partial_i\delta A_0^a + \partial_i\delta A_0^a\partial_i\delta A_0^a) = \frac{1}{2a^2}\{\partial_0[a(\delta Q\delta_i^a + \partial_i\partial_a M)]\partial_0[a(\delta Q\delta_i^a + \partial_i\partial_a M)] - 2\partial_0\left(a\delta Q\delta_i^a + a\partial_i\partial_a M\right)\partial_i(a\partial_a Y) + \partial_i(a\partial_a Y)\partial_i(a\partial_a Y)\} = \frac{1}{2a^2}\{[\dot a(\delta Q\delta_i^a + \partial_i\partial_a M) + a(\delta\dot Q\delta_i^a + \partial_i\partial_a\dot M)]^2 - 2\left(\dot a\delta Q\delta_i^a + a\delta\dot Q\delta_i^a + \dot a\partial_i\partial_a M + a\partial_i\partial_a\dot M\right)a\partial_i\partial_a Y + a\partial_i\partial_a Y a\partial_i\partial_a Y\} = \frac{1}{2a^2}[\dot a^2(3\delta Q^2 + 2\delta Q\Delta M + \partial_i\partial_a M\partial_i\partial_a M) + a^2(3\delta\dot Q^2 + 2\delta\dot Q\Delta\dot M + \partial_i\partial_a\dot M\partial_i\partial_a\dot M) + 2a\dot a(3\delta Q\delta\dot Q + \delta Q\Delta\dot M + \delta\dot Q\Delta M + \partial_i\partial_a M\partial_i\partial_a\dot M) - 2a(\dot a\delta Q\partial_a\partial_a Y + a\delta\dot Q\partial_a\partial_a Y + \dot a\partial_i\partial_a M\partial_i\partial_a Y + a\partial_i\partial_a\dot M\partial_i\partial_a Y) + a^2(\partial_i\partial_a Y)^2]$$
$$= \frac{1}{2a^2}[\dot a^2(3\delta Q^2 + 2\delta Q\Delta M + \partial_i\partial_a M\partial_i\partial_a M) + a^2(3\delta\dot Q^2 + 2\delta\dot Q\Delta\dot M + \partial_i\partial_a\dot M\partial_i\partial_a\dot M) + 2a\dot a(3\delta Q\delta\dot Q + \delta Q\Delta\dot M + \delta\dot Q\Delta M + \partial_i\partial_a M\partial_i\partial_a\dot M)] + (-H\delta Q - \delta\dot Q)\Delta Y + (-H\partial_i\partial_a M - \partial_i\partial_a\dot M)\partial_i\partial_a Y + \frac{1}{2}(\partial_i\partial_a Y)^2 = \frac{1}{2}H^2(3\delta Q^2 + 2\delta Q\Delta M + \partial_i\partial_a M\partial_i\partial_a M) + \frac{1}{2}(3\delta\dot Q^2 + 2\delta\dot Q\Delta\dot M + \partial_i\partial_a\dot M\partial_i\partial_a\dot M) + H(3\delta Q\delta\dot Q + \delta Q\Delta\dot M + \delta\dot Q\Delta M + \partial_i\partial_a M\partial_i\partial_a\dot M) - (H\delta Q + \delta\dot Q)\Delta Y - (H\partial_i\partial_a M + \partial_i\partial_a\dot M)\partial_i\partial_a Y + \frac{1}{2}(\partial_i\partial_a Y)^2 \tag{3.16}$$

We note that the third term of the last expression can be written as (multiplied by $\sqrt{\bar g} = a^3$)

$$a^3H\left(3\delta Q\delta\dot Q + \delta\dot Q\Delta M + \delta Q\Delta\dot M + \partial_i\partial_a M\partial_i\partial_a\dot M\right) = a^3H\partial_0\left(\frac{3}{2}\delta Q^2 + \delta Q\Delta M + \frac{1}{2}\partial_i\partial_a M\partial_i\partial_a M\right) = \partial_0\left[(a^3H)\left(\frac{3}{2}\delta Q^2 + \delta Q\Delta M + \frac{1}{2}\partial_i\partial_a M\partial_i\partial_a M\right)\right] - \partial_0(a^3H)\left(\frac{3}{2}\delta Q^2 + \delta Q\Delta M + \frac{1}{2}\partial_i\partial_a M\partial_i\partial_a M\right) \tag{3.17}$$

We can neglect the first term, being a total derivative. Then we can write $\partial_0(a^3H) = 3a^2\dot aH + a^3\dot H = a^3(3H^2 + \dot H) = a^3H^2(3-\epsilon) \simeq 3a^3H^2$ and thus obtain $-\frac{3}{2}a^3H^2\left(3\delta Q^2 + 2\delta Q\Delta M + \partial_i\partial_a M\partial_i\partial_a M\right)$. If we divide this term by a^3 and substitute it back in (3.16) we obtain the contribution

$$- H^2(3\delta Q^2 + 2\delta Q\Delta M + \partial_i\partial_a M\partial_i\partial_a M) + \frac{1}{2}(3\delta\dot Q^2 + 2\delta\dot Q\Delta\dot M + \partial_i\partial_a\dot M\partial_i\partial_a\dot M) - (H\delta Q + \delta\dot Q)\Delta Y - (H\partial_i\partial_a M + \partial_i\partial_a\dot M)\partial_i\partial_a Y + \frac{1}{2}(\partial_i\partial_a Y)^2 \tag{3.18}$$

From the second term in (3.4) we get

$$
\begin{aligned}
&-\frac{1}{2a^4}(\partial_i\delta A^a_k\partial_i\delta A^a_k-\partial_i\delta A^a_k\partial_k\delta A^a_i)=-\frac{1}{2a^4}\{[a\partial_i(\delta Q\delta^a_k+\partial_k\partial_a M)]^2-[a\partial_i(\delta Q\delta^a_k\\
&+\partial_k\partial_a M)][a\partial_k(\delta Q\delta^a_i+\partial_i\partial_a M)]\}=-\frac{1}{2a^4}\{a^2[3(\partial_i\delta Q)^2+(\partial_i\partial_a\partial_k M)^2+2\partial_i\delta Q\partial_i\partial_a\partial_a M]\\
&-a^2(\partial_a\delta Q\partial_a\delta Q+\partial_i\delta Q\partial_i\partial_a\partial_a M+\partial_k\delta Q\partial_k\partial_a\partial_a M+\partial_i\partial_a\partial_k M\partial_i\partial_a\partial_k M)\}=-\frac{1}{a^2}(\partial_i\delta Q)^2
\end{aligned}
\tag{3.19}
$$

and from the third one we have

$$
\begin{aligned}
&-\frac{g}{a^2}\epsilon^{ade}(\partial_0\bar{A}^a_i\delta A^d_0\delta A^e_i+\partial_0\delta A^a_i\delta A^d_0\bar{A}^e_i-\partial_i\delta A^a_0\delta A^d_0\bar{A}^e_i)=-\frac{g}{a^2}\epsilon^{ade}\Big\{a^2\partial_0(aQ)\delta^a_i\partial_d Y(\delta Q\delta^e_i\\
&+\partial_i\partial_e M)+a\partial_0[a(\delta Q\delta^a_i+\partial_i\partial_a M)]\partial_d Y(aQ)\delta^e_i-a^2\partial_i\partial_a Y\partial_d Y(aQ)\delta^e_i\Big\}\\
&=-\frac{g}{a^2}\epsilon^{ade}\Big\{a^2\partial_0(aQ)\partial_d Y(\delta Q\delta^e_a+\partial_a\partial_e M)+a\partial_0[a(\delta Q\delta^a_e+\partial_e\partial_a M)]\partial_d Y(aQ)\\
&-a^2\partial_e\partial_a Y\partial_d Y(aQ)\Big\}=0
\end{aligned}
\tag{3.20}
$$

while from the fourth term we obtain

$$
\begin{aligned}
&\frac{g}{a^4}\epsilon^{ade}(\partial_i\bar{A}^a_k\delta A^d_i\delta A^e_k+\partial_i\delta A^a_k\bar{A}^d_i\delta A^e_k+\partial_i\delta A^a_k\delta A^d_i\bar{A}^e_k)\\
&=\frac{g}{a^4}\epsilon^{ade}[a^2\partial_i(\delta Q\delta^a_k+\partial_k\partial_a M)(aQ)\delta^d_i(\delta Q\delta^e_k+\partial_e\partial_k M)+a^2\partial_i(\delta Q\delta^a_k+\partial_k\partial_a M)(\delta Q\delta^d_i\\
&+\partial_i\partial_d M)(aQ)\delta^e_k]=\frac{g}{a^2}(aQ)\epsilon^{ade}[\partial_d\delta Q\partial_e\partial_a M+\partial_d\partial_e\partial_a M\delta Q+\partial_d\partial_k\partial_a M\partial_e\partial_k M\\
&+\partial_d\partial_e\partial_a M\delta Q+\partial_i\partial_e\partial_a M\partial_d\partial_i M)]=\frac{g}{a^2}(aQ)\epsilon^{ade}[\partial_d\delta Q\partial_e\partial_a M+2\partial_d\partial_e\partial_a M\delta Q\\
&+\partial_d\partial_k\partial_a M\partial_e\partial_k M+\partial_i\partial_e\partial_a M\partial_d\partial_i M)]=0
\end{aligned}
\tag{3.21}
$$

From the fifth term in (3.4) we have

$$
\begin{aligned}
&\frac{g^2}{2a^2}\Big(\delta A^b_0\bar{A}^c_i\delta A^b_0\bar{A}^c_i-\delta A^b_0\bar{A}^c_i\delta A^c_0\bar{A}^b_i\Big)=\frac{g^2}{2a^2}\Big[\delta A^b_0(aQ)\delta^c_i\delta A^b_0(aQ)\delta^c_i-\delta A^b_0(aQ)\delta^c_i\delta A^c_0(aQ)\delta^b_i\Big]\\
&=\frac{g^2}{2a^2}(aQ)^2\Big[3\delta A^b_0\delta A^b_0-\delta A^i_0\delta A^i_0\Big]=\frac{g^2}{a^2}(aQ)^2\delta A^b_0\delta A^b_0=g^2(aQ)^2\partial_b Y\partial_b Y
\end{aligned}
\tag{3.22}
$$

while from the last one we get

$$
\begin{aligned}
&-\frac{g^2}{4a^4}(\delta A_i^b\delta A_k^c\bar{A}_i^b\bar{A}_k^c+\delta A_i^b\bar{A}_k^c\delta A_i^b\bar{A}_k^c+\delta A_i^b\bar{A}_k^c\bar{A}_i^b\delta A_k^c+\bar{A}_i^b\delta A_k^c\delta A_i^b\bar{A}_k^c+\bar{A}_i^b\bar{A}_k^c\delta A_i^b\delta A_k^c\\
&+\bar{A}_i^b\delta A_k^c\bar{A}_i^b\delta A_k^c-\delta A_i^b\delta A_k^c\bar{A}_i^c\bar{A}_k^b-\delta A_i^b\bar{A}_k^c\delta A_i^c\bar{A}_k^b-\delta A_i^b\bar{A}_k^c\bar{A}_i^c\delta A_k^b-\bar{A}_i^b\delta A_k^c\delta A_i^c\bar{A}_k^b\\
&-\bar{A}_i^b\bar{A}_k^c\delta A_i^c\delta A_k^b-\bar{A}_i^b\delta A_k^c\bar{A}_i^c\delta A_k^b)=-\frac{g^2}{2a^4}[2\delta A_i^b\delta A_k^c(aQ)\delta_i^b(aQ)\delta_k^c+\delta A_i^b\delta A_i^b(aQ)\delta_k^c(aQ)\delta_k^c\\
&-\delta A_i^b\delta A_k^c(aQ)\delta_i^c(aQ)\delta_k^b-\delta A_i^b\delta A_i^c(aQ)\delta_k^b(aQ)\delta_k^c-\delta A_i^b\delta A_k^b(aQ)\delta_i^c(aQ)\delta_k^c]\\
&=-\frac{g^2(aQ)^2}{2a^4}[2\delta A_b^b\delta A_c^c+3\delta A_i^b\delta A_i^b-\delta A_c^b\delta A_b^c-\delta A_i^k\delta A_i^k-\delta A_c^b\delta A_c^b]\\
&=-\frac{g^2(aQ)^2}{a^2}(\delta Q\delta_b^b+\partial_b\partial_bM)(\delta Q\delta_c^c+\partial_c\partial_cM)=-\frac{g^2(aQ)^2}{a^2}[9\delta Q^2+6\delta Q\Delta M+(\Delta M)^2]
\end{aligned}
\tag{3.23}
$$

In the end, for the kinetic term we can write

$$
\begin{aligned}
&\frac{\mathcal{L}_2^K}{\sqrt{\tilde{g}}}=-H^2(3\delta Q^2+2\delta Q\Delta M+\partial_i\partial_aM\partial_i\partial_aM)+\frac{1}{2}(3\delta\dot{Q}^2+2\delta\dot{Q}\Delta\dot{M}+\partial_i\partial_a\dot{M}\partial_i\partial_a\dot{M})\\
&-(H\delta Q+\delta\dot{Q})\Delta Y-(H\partial_i\partial_aM+\partial_i\partial_a\dot{M})\partial_i\partial_aY+\frac{1}{2}(\partial_i\partial_aY)^2-\frac{1}{a^2}(\partial_i\delta Q)^2\\
&+g^2(aQ)^2\partial_bY\partial_bY-g^2Q^2[9\delta Q^2+6\delta Q\Delta M+(\Delta M)^2]
\end{aligned}
\tag{3.24}
$$

Moreover, from the axion kinetic term present in (2.42)

$$
\begin{aligned}
&\frac{\mathcal{L}_2^\chi}{\sqrt{-\tilde{g}}}=-\frac{1}{2}(\partial\delta\chi)^2=-\frac{1}{2}g^{\mu\nu}\partial_\mu\delta\chi\partial_\nu\delta\chi=-\frac{1}{2}\Big[g^{00}(\partial_0\delta\chi)^2+g^{ij}(\partial_i\delta\chi\partial_j\delta\chi)\Big]\\
&=-\frac{1}{2}\Big[-(\partial_0\delta\chi)^2+\frac{1}{a^2}\delta^{ij}(\partial_i\delta\chi\partial_j\delta\chi)\Big]=\frac{1}{2}(\partial_0\delta\chi)^2-\frac{1}{2a^2}(\partial_i\delta\chi)^2
\end{aligned}
\tag{3.25}
$$

We can also expand the potential of the axion around the background $\bar{\chi}$ as

$$
U(\chi)=U(\bar{\chi})+U'(\bar{\chi})\delta\chi+\frac{1}{2}U''(\bar{\chi})\delta\chi^2+\frac{1}{3!}U'''(\bar{\chi})\delta\chi^3+\frac{1}{4!}U''''(\bar{\chi})\delta\chi^4+\dots \tag{3.26}
$$

As a result, we can write the final expression for the perturbed Lagrangian at the second order

$$\mathcal{L}_2 = \mathcal{L}_2^{CS} + \mathcal{L}_2^K + \mathcal{L}_2^\chi - \sqrt{-\tilde{g}}\; U(\chi) = -\frac{\lambda}{f} ag(aQ)^2 \partial_i \delta\chi \partial_i Y + \frac{\lambda}{f} ag(aQ)^2 \partial_0 \delta\chi\; (3\delta Q + \Delta M)$$
$$+ \frac{\lambda}{f} a^2 g(aQ) \partial_0 \bar{\chi} (3\delta Q^2 + 2\delta Q \Delta M) - a^3 H^2 (3\delta Q^2 + 2\delta Q \Delta M + \partial_i \partial_a M \partial_i \partial_a M) + \frac{a^3}{2} (3\delta\dot{Q}^2$$
$$+ 2\delta\dot{Q}\Delta\dot{M} + \partial_i \partial_a \dot{M} \partial_i \partial_a \dot{M}) - a^3 (H\delta Q + \delta\dot{Q}) \Delta Y - a^3 (H \partial_i \partial_a M + \partial_i \partial_a \dot{M}) \partial_i \partial_a Y$$
$$+ \frac{a^3}{2} (\partial_i \partial_a Y)^2 - a(\partial_i \delta Q)^2 + a^3 g^2 (aQ)^2 \partial_b Y \partial_b Y - g^2 a^3 Q^2 [9\delta Q^2 + 6\delta Q \Delta M + (\Delta M)^2]$$
$$+ \frac{1}{2} a^3 (\partial_0 \delta\chi)^2 - \frac{1}{2} a (\partial_i \delta\chi)^2 - \frac{a^3}{2} U''(\bar{\chi}) \delta\chi^2 \tag{3.27}$$

3.2.2 Third Order

We now turn our attention to the third order Lagrangian $\mathcal{L}_3$. Starting from the Chern-Simons term with $\mu = i$, the first expression in (3.6) yields

$$-\frac{\lambda}{2f} \partial_i \delta\chi\; \epsilon^{i\nu\rho\sigma} (\delta A^a_\nu \partial_\rho \delta A^a_\sigma) = -\frac{\lambda}{2f} \partial_i \delta\chi a^2 [\epsilon^{ij0k} (\delta Q \delta^a_j + \partial_j \partial_a M) \partial_0 (\delta Q \delta^a_k + \partial_k \partial_a M)$$
$$+ \epsilon^{i0kj} \partial_a Y \partial_k (\delta Q \delta^a_j + \partial_j \partial_a M) + \epsilon^{ijk0} (\delta Q \delta^a_j + \partial_j \partial_a M) \partial_k \partial_a Y]$$
$$= -\frac{\lambda}{2f} \partial_i \delta\chi a^2 [\epsilon^{ij0k} (\delta Q \partial_0 \partial_j \partial_k M + \partial_j \partial_k M \partial_0 \delta Q + \partial_j \partial_a M \partial_0 \partial_a \partial_k M) + \epsilon^{i0kj} (\partial_j Y \partial_k \delta Q$$
$$+ \partial_a Y \partial_k \partial_j \partial_a M) + \epsilon^{ijk0} (\delta Q \partial_k \partial_j Y + \partial_j \partial_a M \partial_k \partial_a Y)] = 0 \tag{3.28}$$

while from the second term we get

$$\frac{\lambda}{2f} \frac{g}{3} \partial_i \delta\chi\; \epsilon^{i\nu\rho\sigma} \epsilon^{abc} \left(\delta A^a_\nu \delta A^b_\rho \bar{A}^c_\sigma + \bar{A}^a_\nu \delta A^b_\rho \delta A^c_\sigma + \delta A^a_\nu \bar{A}^b_\rho \delta A^c_\sigma \right) = \frac{\lambda}{2f} \frac{g}{3} \partial_i \delta\chi\; 3\epsilon^{i\nu\rho\sigma} \epsilon^{abc} \delta A^a_\nu \delta A^b_\rho \bar{A}^c_\sigma$$
$$= \frac{\lambda}{2f} g \partial_i \delta\chi\; \epsilon^{i\nu\rho k} \epsilon^{abc} \delta A^a_\nu \delta A^b_\rho (aQ) \delta^c_k = \frac{\lambda}{2f} a^2 g(aQ) \partial_i \delta\chi\; \epsilon^{abk} [\epsilon^{i0jk} \partial_a Y (\delta Q \delta^b_j + \partial_j \partial_b M)$$
$$+ \epsilon^{ij0k} (\delta Q \delta^a_j + \partial_j \partial_a M) \partial_b Y] = \frac{\lambda}{f} a^2 g(aQ) \partial_i \delta\chi\; \epsilon^{abk} \epsilon^{i0jk} \partial_a Y (\delta Q \delta^b_j + \partial_j \partial_b M)$$
$$= \frac{\lambda}{f} a^2 g(aQ) \partial_i \delta\chi\; \epsilon^{ajk} \epsilon^{i0jk} \partial_a Y \delta Q + \frac{\lambda}{f} a^2 g(aQ) \partial_i \delta\chi\; \epsilon^{abk} \epsilon^{i0jk} \partial_a Y \partial_j \partial_b M$$
$$= \frac{\lambda}{f} a^2 g(aQ) [-2\delta^i_a \partial_i \delta\chi \partial_a Y \delta Q - (\delta^{bj} \delta^{ai} - \delta^{aj} \delta^{bi}) \partial_i \delta\chi \partial_a Y \partial_j \partial_b M]$$
$$= -2\frac{\lambda}{f} a^2 g(aQ) \partial_a \delta\chi \partial_a Y \delta Q - \frac{\lambda}{f} a^2 g(aQ) (\partial_i \delta\chi \partial_i Y \Delta M - \partial_b \delta\chi \partial_j Y \partial_b \partial_j M) \tag{3.29}$$

If instead we consider $\mu = 0$ we obtain, from the first term in (3.9)

$$-\frac{\lambda}{2f}\partial_0\delta\chi\ \epsilon^{0ikj}\left(\delta A_i^a\partial_k\delta A_j^a\right) = -\frac{\lambda}{2f}a^2\partial_0\delta\chi\epsilon^{0jki}(\delta Q\delta_i^a + \partial_i\partial_a M)\partial_k(\delta Q\delta_j^a + \partial_j\partial_a M)$$
$$= -\frac{\lambda}{2f}a^2\partial_0\delta\chi\epsilon^{0jki}(\delta Q\partial_k\partial_i\partial_j M + \partial_i\partial_j M\partial_k\delta Q + \partial_i\partial_a M\partial_k\partial_j\partial_a M) = 0 \tag{3.30}$$

and from the second term

$$\frac{\lambda}{2f}\frac{g}{3}\partial_0\delta\chi\ \epsilon^{0\nu\rho\sigma}\epsilon^{abc}\left(\delta A_\nu^a\delta A_\rho^b\bar{A}_\sigma^c + \bar{A}_\nu^a\delta A_\rho^b\delta A_\sigma^c + \delta A_\nu^a\bar{A}_\rho^b\delta A_\sigma^c\right) = \frac{\lambda}{2f}\frac{g}{3}\partial_0\delta\chi\ 3\epsilon^{0ij\sigma}\epsilon^{abc}\delta A_i^a\delta A_j^b\bar{A}_\sigma^c$$
$$= \frac{\lambda}{2f}g\partial_0\delta\chi\ \epsilon^{0ijk}\epsilon^{abc}\delta A_i^a\delta A_j^b(aQ)\delta_k^c = \frac{\lambda}{2f}a^2g(aQ)\partial_0\delta\chi\ \epsilon^{0ijk}\epsilon^{abk}(\delta Q\delta_i^a + \partial_a\partial_i M)(\delta Q\delta_j^b + \partial_b\partial_j M)$$
$$= \frac{\lambda}{2f}a^2g(aQ)\partial_0\delta\chi\ \epsilon^{0ijk}\epsilon^{abk}(\delta Q^2\delta_i^a\delta_j^b + \delta Q\delta_i^a\partial_b\partial_j M + \partial_a\partial_i M\delta Q\delta_j^b + \partial_a\partial_i M\partial_b\partial_j M)$$
$$= \frac{\lambda}{2f}a^2g(aQ)\partial_0\delta\chi[6\delta Q^2 + (2\delta_b^j)\delta Q\partial_b\partial_j M + (2\delta_a^i)\partial_a\partial_i M\delta Q + (\delta^{jb}\delta^{ia} - \delta^{ja}\delta^{ib})\partial_a\partial_i M\partial_b\partial_j M]$$
$$= \frac{\lambda}{f}a^2g(aQ)\partial_0\delta\chi(3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}\partial_i\partial_i M\partial_j\partial_j M - \frac{1}{2}\partial_j\partial_i M\partial_i\partial_j M)$$
$$= \frac{\lambda}{f}a^2g(aQ)\partial_0\delta\chi[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2] \tag{3.31}$$

We also have to consider the configuration in which we have three gauge field perturbations from the Chern-Simons term

$$\frac{\lambda}{2f}\frac{g}{3}\partial_0\bar{\chi}\ \epsilon^{0\nu\rho\sigma}\epsilon^{abc}\left(\delta A_\nu^a\delta A_\rho^b\delta A_\sigma^c\right) = \frac{\lambda}{2f}\frac{g}{3}\partial_0\bar{\chi}\ a^3\epsilon^{0ijk}\epsilon^{abc}[(\delta Q\delta_i^a + \partial_a\partial_i M)(\delta Q\delta_j^b$$
$$+ \partial_b\partial_j M)(\delta Q\delta_k^c + \partial_c\partial_k M)] = \frac{\lambda}{2f}\frac{g}{3}\partial_0\bar{\chi}\ a^3(6\delta Q^3 + 2\delta Q^2\Delta M + 2\delta Q^2\Delta M$$
$$+ \epsilon^{0ajk}\epsilon^{abc}\delta Q\partial_b\partial_j M\partial_c\partial_k M + 2\delta Q^2\Delta M + \epsilon^{0ibk}\epsilon^{abc}\delta Q\partial_a\partial_i M\partial_c\partial_k M$$
$$+ \epsilon^{0ijc}\epsilon^{abc}\delta Q\partial_a\partial_i M\partial_b\partial_j M + \epsilon^{0ijk}\epsilon^{abc}\partial_a\partial_i M\partial_b\partial_j M\partial_c\partial_k M)$$
$$= \frac{\lambda}{6f}\partial_0\bar{\chi}\ a^3g(6\delta Q^3 + 6\delta Q^2\Delta M + 3\epsilon^{0ibk}\epsilon^{abc}\delta Q\partial_a\partial_i M\partial_c\partial_k M)$$
$$= \frac{\lambda}{2f}a^3g\ \partial_0\bar{\chi}\{2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q[(\Delta M)^2 - (\partial_a\partial_c M)^2]\} \tag{3.32}$$

where again we have integrated by part and made explicit the product between the Levi-Civita tensors in order to simplify the expression.
Hence, the total contribution from the Chern-Simons term reads

$$\mathcal{L}_3^{CS} = -2\frac{\lambda}{f}a^2 g(aQ)\partial_a\delta\chi\partial_a Y\delta Q - \frac{\lambda}{f}a^2 g(aQ)(\partial_i\delta\chi\partial_i Y\Delta M - \partial_b\delta\chi\partial_j Y\partial_b\partial_j M)$$
$$+ \frac{\lambda}{f}a^2 g(aQ)\partial_0\delta\chi[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2] + \frac{\lambda}{2f}a^3 g\ \partial_0\bar{\chi}\{2\delta Q^3$$
$$+ 2\delta Q^2\Delta M + \delta Q[(\Delta M)^2 - (\partial_a\partial_c M)^2]\} \tag{3.33}$$

Regarding the expansion of the kinetic term up to the third order, the third term in (3.4) provides

$$-\frac{g}{a^2}\epsilon^{ade}(\partial_0\delta A_i^a\delta A_0^d\delta A_i^e - \partial_i\delta A_0^a\delta A_0^d\delta A_i^e) = -\frac{g}{a^2}\epsilon^{ade}\{\partial_0[a(\delta Q\delta_i^a + \partial_i\partial_a M)]a\partial_d Y[a(\delta Q\delta_i^e$$
$$+ \partial_i\partial_e M)] - a\partial_i\partial_a Y a\partial_d Y[a(\delta Q\delta_i^e + \partial_i\partial_e M)]\} = -\frac{g}{a^2}\epsilon^{ade}\{[\dot{a}(\delta Q\delta_i^a + \partial_i\partial_a M)$$
$$+ a(\delta\dot{Q}\delta_i^a + \partial_i\partial_a\dot{M})]a\partial_d Y[a(\delta Q\delta_i^e + \partial_i\partial_e M)] - a^3(\partial_e\partial_a Y\partial_d Y\delta Q + \partial_i\partial_a Y\partial_d Y\partial_i\partial_e M)\}$$
$$= -ga\ \epsilon^{ade}\{[H(\delta Q\delta_i^a + \partial_i\partial_a M) + (\delta\dot{Q}\delta_i^a + \partial_i\partial_a\dot{M})](\partial_d Y\delta Q\delta_i^e + \partial_d Y\partial_i\partial_e M)$$
$$- \partial_i\partial_a Y\partial_d Y\partial_i\partial_e M\} = -ga\ \epsilon^{ade}[H(\delta Q\partial_d Y\partial_a\partial_e M + \partial_e\partial_a M\partial_d Y\delta Q + \partial_i\partial_a M\partial_d Y\partial_i\partial_e M)$$
$$+ \delta\dot{Q}\partial_d Y\partial_a\partial_e M + \partial_e\partial_a\dot{M}\partial_d Y\delta Q + \partial_i\partial_a\dot{M}\partial_d Y\partial_i\partial_e M] = 0 \tag{3.34}$$

and from the fourth one we get

$$\frac{g}{a^4}\epsilon^{ade}\partial_i\delta A_k^a\delta A_i^d\delta A_k^e = \frac{g}{a}\epsilon^{ade}\partial_i(\delta Q\delta_k^a + \partial_k\partial_a M)(\delta Q\delta_i^d + \partial_d\partial_i M)(\delta Q\delta_k^e + \partial_e\partial_k M)$$
$$= \frac{g}{a}\epsilon^{ade}[\partial_d\delta Q^2\partial_e\partial_a M + \partial_i\delta Q\partial_d\partial_i M\partial_e\partial_a M + \partial_d\partial_e\partial_a M\delta Q^2 + \partial_d\partial_k\partial_a M\delta Q\partial_e\partial_k M$$
$$+ \partial_i\partial_e\partial_a M\partial_d\partial_i M\delta Q + \partial_i\partial_k\partial_a M\partial_d\partial_i M\partial_e\partial_k M] = 0 \tag{3.35}$$

From the fifth term we obtain

$$\frac{g^2}{a^2}\left(\delta A_0^b\delta A_i^c\delta A_0^b\bar{A}_i^c - \delta A_0^b\delta A_i^c\delta A_0^c\bar{A}_i^b\right) = \frac{g^2}{a^2}\Big[a^3\partial_b Y\partial_b Y(\delta Q\delta_i^c + \partial_c\partial_i M)(aQ)\delta_i^c$$
$$- a^3\partial_b Y\partial_c Y(\delta Q\delta_i^c + \partial_c\partial_i M)(aQ)\delta_i^b\Big] = ag^2(aQ)(3\partial_b Y\partial_b Y\delta Q + \partial_b Y\partial_b Y\Delta M - \partial_b Y\partial_b Y\delta Q$$
$$- \partial_b Y\partial_c Y\partial_b\partial_c M) = ag^2(aQ)(2\partial_b Y\partial_b Y\delta Q + \partial_b Y\partial_b Y\Delta M - \partial_b Y\partial_c Y\partial_b\partial_c M) \tag{3.36}$$

and the last term produces the following

$$
\begin{aligned}
&- \frac{g^2}{4a^4}(\delta A_i^b \bar{A}_k^c \delta A_i^b \delta A_k^c + \delta A_i^b \delta A_k^c \bar{A}_i^b \delta A_k^c + \bar{A}_i^b \delta A_k^c \delta A_i^b \delta A_k^c + \delta A_i^b \delta A_k^c \delta A_i^b \bar{A}_k^c \\
&- \delta A_i^b \bar{A}_k^c \delta A_i^c \delta A_k^b + \delta A_i^b \delta A_k^c \bar{A}_i^c \delta A_k^b + \bar{A}_i^b \delta A_k^c \delta A_i^c \delta A_k^b + \delta A_i^b \delta A_k^c \delta A_i^c \bar{A}_k^b) \\
&= -\frac{g^2}{a}(aQ)\delta_i^b[(\delta Q \delta_k^c + \partial_c \partial_k M)(\delta Q \delta_i^b + \partial_b \partial_i M)(\delta Q \delta_k^c + \partial_c \partial_k M) \\
&- (\delta Q \delta_k^c + \partial_c \partial_k M)(\delta Q \delta_i^c + \partial_c \partial_i M)(\delta Q \delta_k^b + \partial_b \partial_k M)] \\
&= -\frac{g^2}{a}(aQ)\{9\delta Q^3 + 3\delta Q^2 \Delta M + 3\delta Q^2 \Delta M + \delta Q(\Delta M)^2 + 3\delta Q^2 \Delta M + 3\delta Q(\partial_c \partial_k M)^2 \\
&+ \delta Q(\Delta M)^2 + \Delta M(\partial_c \partial_k M)^2 - [3\delta Q^3 + \delta Q^2 \Delta M + \delta Q^2 \Delta M + \delta Q(\partial_k \partial_i M)^2 + \delta Q^2 \Delta M \\
&+ \delta Q(\partial_i \partial_k M)^2 + \delta Q(\partial_i \partial_c M)^2 + \partial_i \partial_k M \partial_c \partial_k M \partial_c \partial_i M]\} = -\frac{g^2}{a}(aQ)[6\delta Q^3 + 6\delta Q^2 \Delta M \\
&+ 2\delta Q(\Delta M)^2 + \Delta M(\partial_c \partial_k M)^2 - \partial_i \partial_k M \partial_c \partial_k M \partial_c \partial_i M]
\end{aligned} \tag{3.37}
$$

This allows us to write define

$$
\begin{aligned}
\frac{\mathcal{L}_3^K}{\sqrt{\bar{g}}} &= a g^2 (aQ)(2\partial_b Y \partial_b Y \delta Q + \partial_b Y \partial_b Y \Delta M - \partial_b Y \partial_c Y \partial_b \partial_c M) - \frac{g^2}{a}(aQ)[6\delta Q^3 + 6\delta Q^2 \Delta M \\
&+ 2\delta Q(\Delta M)^2 + \Delta M(\partial_c \partial_k M)^2 - \partial_i \partial_k M \partial_c \partial_k M \partial_c \partial_i M]
\end{aligned} \tag{3.38}
$$

Thus, the final expression for the perturbed Lagrangian at the third order is

$$
\begin{aligned}
\mathcal{L}_3 &= \mathcal{L}_3^{CS} + \mathcal{L}_3^K - \sqrt{-\bar{g}}\ U(\chi) = -2\frac{\lambda}{f}a^2 g(aQ)\partial_a \delta\chi \partial_a Y \delta Q - \frac{\lambda}{f}a^2 g(aQ)(\partial_i \delta\chi \partial_i Y \Delta M \\
&- \partial_b \delta\chi \partial_j Y \partial_b \partial_j M) + \frac{\lambda}{f}a^2 g(aQ)\partial_0 \delta\chi[3\delta Q^2 + 2\delta Q \Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i \partial_j M)^2] \\
&+ \frac{\lambda}{2f}a^3 g\ \partial_0 \bar{\chi}\{2\delta Q^3 + 2\delta Q^2 \Delta M + \delta Q[(\Delta M)^2 - (\partial_a \partial_c M)^2]\} + a^4 g^2 (aQ)(2\partial_b Y \partial_b Y \delta Q \\
&+ \partial_b Y \partial_b Y \Delta M - \partial_b Y \partial_c Y \partial_b \partial_c M) - a^2 g^2 (aQ)[6\delta Q^3 + 6\delta Q^2 \Delta M + 2\delta Q(\Delta M)^2 \\
&+ \Delta M(\partial_c \partial_k M)^2 - \partial_i \partial_k M \partial_c \partial_k M \partial_c \partial_i M] - \frac{a^3}{6}U'''(\bar{\chi})\delta\chi^3
\end{aligned} \tag{3.39}
$$

3.2.3 Fourth order

The last contribution we need is the fourth order perturbation Lagrangian $\mathcal{L}_4$; we start from the Chern-Simons term, as usual considering initially $\mu = i$; the second term of (3.6) gives us

$$\frac{\lambda}{2f}\frac{g}{3}\partial_i\delta\chi\ \epsilon^{i\nu\rho\sigma}\epsilon^{abc}\delta A^a_\nu\delta A^b_\rho\delta A^c_\sigma = \frac{\lambda}{2f}\frac{g}{3}a^3\partial_i\delta\chi[\epsilon^{i0jk}\epsilon^{abc}\partial_a Y(\delta Q\delta^b_j + \partial_j\partial_b M)(\delta Q\delta^c_k + \partial_c\partial_k M)$$
$$+ \epsilon^{ij0k}\epsilon^{abc}(\delta Q\delta^a_j + \partial_j\partial_a M)\partial_b Y(\delta Q\delta^c_k + \partial_c\partial_k M) + \epsilon^{ijk0}\epsilon^{abc}(\delta Q\delta^a_j + \partial_j\partial_a M)(\delta Q\delta^b_k + \partial_c\partial_b M)\partial_c Y]$$
$$= \frac{\lambda}{2f}ga^3\partial_i\delta\chi[\epsilon^{i0jk}\epsilon^{abc}\partial_a Y(\delta Q\delta^b_j + \partial_j\partial_b M)(\delta Q\delta^c_k + \partial_c\partial_k M)] = \frac{\lambda}{2f}ga^3\partial_i\delta\chi(\epsilon^{i0bc}\epsilon^{abc}\partial_a Y\delta Q^2$$
$$+ \epsilon^{i0bk}\epsilon^{abc}\partial_a Y\delta Q\partial_c\partial_k M + \epsilon^{i0jc}\epsilon^{abc}\partial_a Y\delta Q\partial_b\partial_j M + \epsilon^{i0jk}\epsilon^{abc}\partial_a Y\partial_j\partial_b M\partial_c\partial_k M)$$
$$= \frac{\lambda}{2f}ga^3\partial_i\delta\chi[-2\partial_i Y\delta Q^2 - (\delta^{ia}\delta^{kc} - \delta^{ic}\delta^{ka})\partial_a Y\delta Q\partial_c\partial_k M - (\delta^{jb}\delta^{ia} - \delta^{ja}\delta^{ib})\partial_a Y\delta Q\partial_b\partial_j M$$
$$+ \epsilon^{i0jk}\epsilon^{abc}\partial_a Y\partial_j\partial_b M\partial_c\partial_k M] = \frac{\lambda}{2f}ga^3\partial_i\delta\chi[-2\partial_i Y\delta Q^2 - 2(\partial_i Y\delta Q\Delta M - \partial_a Y\delta Q\partial_i\partial_a M)$$
$$+ \epsilon^{i0jk}\epsilon^{abc}\partial_a Y\partial_j\partial_b M\partial_c\partial_k M] \tag{3.40}$$

while if we consider $\mu = 0$ we have

$$\frac{\lambda}{2f}\frac{g}{3}\partial_0\delta\chi\ \epsilon^{0\nu\rho\sigma}\epsilon^{abc}\left(\delta A^a_\nu\delta A^b_\rho\delta A^c_\sigma\right) = \frac{\lambda}{2f}\frac{g}{3}a^3\partial_0\delta\chi\ \epsilon^{0ijk}\epsilon^{abc}[(\delta Q\delta^a_i + \partial_a\partial_i M)(\delta Q\delta^b_j$$
$$+ \partial_b\partial_j M)(\delta Q\delta^c_k + \partial_c\partial_k M)] = \frac{\lambda}{6f}ga^3\partial_0\delta\chi[6\delta Q^3 + 2\delta Q^2\Delta M + 2\delta Q^2\Delta M$$
$$+ \epsilon^{0ajk}\epsilon^{abc}\delta Q\partial_b\partial_j M\partial_c\partial_k M + 2\delta Q^2\Delta M + \epsilon^{0ibk}\epsilon^{abc}\delta Q\partial_a\partial_i M\partial_c\partial_k M$$
$$+ \epsilon^{0ijc}\epsilon^{abc}\delta Q\partial_a\partial_i M\partial_b\partial_j M + \epsilon^{0ijk}\epsilon^{abc}\partial_b\partial_j M\partial_a\partial_i M\partial_c\partial_k M]$$
$$= \frac{\lambda}{2f}ga^3\partial_0\delta\chi[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2$$
$$+ \frac{1}{3}\epsilon^{0ijk}\epsilon^{abc}\partial_b\partial_j M\partial_a\partial_i M\partial_c\partial_k M] \tag{3.41}$$

Hence, we can write the total part from the Chern-Simons term as

$$\mathcal{L}^{CS}_4 = \frac{\lambda}{2f}ga^3\partial_i\delta\chi[-2\partial_i Y\delta Q^2 - 2(\partial_i Y\delta Q\Delta M - \partial_a Y\delta Q\partial_i\partial_a M) + \epsilon^{i0jk}\epsilon^{abc}\partial_a Y\partial_j\partial_b M\partial_c\partial_k M]$$
$$+ \frac{\lambda}{2f}ga^3\partial_0\delta\chi[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2 + \frac{1}{3}\epsilon^{0ijk}\epsilon^{abc}\partial_b\partial_j M\partial_a\partial_i M\partial_c\partial_k M] \tag{3.42}$$

Considering the kinetic term of the gauge field instead, we obtain the following from the fifth term

$$
\frac{g^2}{2a^2}\left(\delta A_0^b\delta A_i^c\delta A_0^b\delta A_i^c - \delta A_0^b\delta A_i^c\delta A_0^c\delta A_i^b\right) = \frac{1}{2}g^2a^2\Big[\partial_bY\partial_bY(\delta Q\delta_i^c + \partial_c\partial_iM)(\delta Q\delta_i^c + \partial_c\partial_iM)
- \partial_bY\partial_cY(\delta Q\delta_i^b + \partial_b\partial_iM)(\delta Q\delta_i^c + \partial_c\partial_iM)\Big] = \frac{1}{2}g^2a^2(3\partial_bY\partial_bY\delta Q^2 + 2\partial_bY\partial_bY\delta Q\Delta M
+ \partial_bY\partial_bY\partial_i\partial_cM\partial_i\partial_cM - \partial_iY\partial_iY\delta Q^2 - \partial_bY\partial_cY\delta Q\partial_b\partial_cM - \partial_bY\partial_cY\delta Q\partial_b\partial_cM
- \partial_bY\partial_cY\partial_b\partial_iM\partial_i\partial_cM) = \frac{1}{2}g^2a^2(2\partial_bY\partial_bY\delta Q^2 + 2\partial_bY\partial_bY\delta Q\Delta M + (\partial_bY\partial_i\partial_cM)^2
- 2\partial_bY\partial_cY\delta Q\partial_b\partial_cM - \partial_bY\partial_cY\partial_b\partial_iM\partial_i\partial_cM) \tag{3.43}
$$

and from the last one in (3.4)

$$
- \frac{g^2}{4a^4}(\delta A_i^b\delta A_k^c\delta A_i^b\delta A_k^c - \delta A_i^b\delta A_k^c\delta A_i^c\delta A_k^b) = -\frac{g^2}{4}[(\delta Q\delta_i^b + \partial_b\partial_iM)(\delta Q\delta_k^c + \partial_c\partial_kM)(\delta Q\delta_i^b
+ \partial_b\partial_iM)(\delta Q\delta_k^c + \partial_c\partial_kM) - (\delta Q\delta_i^b + \partial_b\partial_iM)(\delta Q\delta_k^c + \partial_c\partial_kM)(\delta Q\delta_i^c + \partial_c\partial_iM)(\delta Q\delta_k^b
+ \partial_b\partial_kM)] = -\frac{g^2}{4}\{9\delta Q^4 + 3\delta Q^3\Delta M + 3\delta Q^3\Delta M + \delta Q^2(\Delta M)^2 + 3\delta Q^3\Delta M
+ 3\delta Q^2(\partial_c\partial_kM)^2 + \delta Q^2(\Delta M)^2 + \delta Q\Delta M(\partial_c\partial_kM)^2 + 3\delta Q^3\Delta M + \delta Q^2(\Delta M)^2
+ 3\delta Q^2(\partial_b\partial_iM)^2 + \delta Q\Delta M(\partial_b\partial_iM)^2 + \delta Q^2(\Delta M)^2 + \delta Q\Delta M(\partial_c\partial_kM)^2 + \delta Q\Delta M(\partial_b\partial_iM)^2
+ (\partial_b\partial_iM)^2(\partial_c\partial_kM)^2 - [3\delta Q^4 + \delta Q^3\Delta M + \delta Q^3\Delta M + \delta Q^2(\partial_b\partial_cM)^2 + \delta Q^3\Delta M
+ \delta Q^2(\partial_c\partial_kM)^2 + \delta Q^2(\partial_k\partial_cM)^2 + \delta Q\partial_c\partial_kM\partial_c\partial_bM\partial_b\partial_kM + \delta Q^3\Delta M + \delta Q^2(\partial_b\partial_iM)^2
+ \delta Q^2(\partial_b\partial_iM)^2 + \delta Q\partial_b\partial_iM\partial_k\partial_iM\partial_b\partial_kM + \delta Q^2(\partial_k\partial_iM)^2 + \delta Q\partial_b\partial_cM\partial_k\partial_cM\partial_b\partial_kM
+ \delta Q\partial_b\partial_iM\partial_b\partial_cM\partial_c\partial_iM + \partial_b\partial_iM\partial_k\partial_cM\partial_c\partial_iM\partial_b\partial_kM]\} = -\frac{g^2}{4}[6\delta Q^4 + 8\delta Q^3\Delta M
+ 4\delta Q^2(\Delta M)^2 + 4\delta Q\Delta M(\partial_c\partial_kM)^2 + (\partial_b\partial_iM)^2(\partial_c\partial_kM)^2 - 4\delta Q\partial_c\partial_kM\partial_c\partial_bM\partial_b\partial_kM
- \partial_b\partial_iM\partial_k\partial_cM\partial_c\partial_iM\partial_b\partial_kM] \tag{3.44}
$$

Therefore, we can write the contribution from the kinetic term as

$$
\frac{\mathcal{L}_4^K}{\sqrt{\tilde{g}}} = \frac{1}{2}g^2a^2(2\partial_bY\partial_bY\delta Q^2 + 2\partial_bY\partial_bY\delta Q\Delta M + (\partial_bY\partial_i\partial_cM)^2 - 2\partial_bY\partial_cY\delta Q\partial_b\partial_cM
- \partial_bY\partial_cY\partial_b\partial_iM\partial_i\partial_cM) - \frac{g^2}{4}[6\delta Q^4 + 8\delta Q^3\Delta M + 4\delta Q^2(\Delta M)^2 + 4\delta Q\Delta M(\partial_c\partial_kM)^2
+ (\partial_b\partial_iM)^2(\partial_c\partial_kM)^2 - 4\delta Q\partial_c\partial_kM\partial_c\partial_bM\partial_b\partial_kM - \partial_b\partial_iM\partial_k\partial_cM\partial_c\partial_iM\partial_b\partial_kM] \tag{3.45}
$$

This allows us to formulate the final expression for the Lagrangian perturbed at the fourth order

$$\mathcal{L}_4 = \mathcal{L}_4^{CS} + \mathcal{L}_4^{K} - \sqrt{-\tilde{g}}\ U(\chi) = \frac{\lambda}{2f} g a^3 \partial_i \delta\chi [-2\partial_i Y \delta Q^2 - 2(\partial_i Y \delta Q \Delta M - \partial_a Y \delta Q \partial_i \partial_a M)$$
$$+ \epsilon^{i0jk}\epsilon^{abc}\partial_a Y \partial_j \partial_b M \partial_c \partial_k M] + \frac{\lambda}{2f} g a^3 \partial_0 \delta\chi [2\delta Q^3 + 2\delta Q^2 \Delta M + \delta Q (\Delta M)^2 - \delta Q(\partial_c \partial_k M)^2$$
$$+ \frac{1}{3}\epsilon^{0ijk}\epsilon^{abc}\partial_b \partial_j M \partial_a \partial_i M \partial_c \partial_k M] + \frac{1}{2} g^2 a^5 (2\partial_b Y \partial_b Y \delta Q^2 + 2\partial_b Y \partial_b Y \delta Q \Delta M + (\partial_b Y \partial_i \partial_c M)^2$$
$$- 2\partial_b Y \partial_c Y \delta Q \partial_b \partial_c M - \partial_b Y \partial_c Y \partial_b \partial_i M \partial_i \partial_c M) - \frac{g^2 a^3}{4}[6\delta Q^4 + 8\delta Q^3 \Delta M + 4\delta Q^2 (\Delta M)^2$$
$$+ 4\delta Q \Delta M (\partial_c \partial_k M)^2 + (\partial_b \partial_i M)^2 (\partial_c \partial_k M)^2 - 4\delta Q \partial_c \partial_k M \partial_c \partial_b M \partial_b \partial_k M$$
$$- \partial_b \partial_i M \partial_k \partial_c M \partial_c \partial_i M \partial_b \partial_k M] - \frac{a^3}{24} U''''(\bar{\chi})\delta\chi^4 \tag{3.46}$$

The sum of all the terms we have calculated provides us with the final expression of the total perturbed Lagrangian $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$

$$\mathcal{L} = -\frac{\lambda}{f} a g (aQ)^2 \partial_i \delta\chi \partial_i Y + \frac{\lambda}{f} a g (aQ)^2 \partial_0 \delta\chi\ (3\delta Q + \Delta M) + \frac{1}{2} a^3 (\partial_0 \delta\chi)^2 - \frac{1}{2} a (\partial_i \delta\chi)^2$$
$$- \frac{a^3}{2} U''(\bar{\chi})\delta\chi^2 + \frac{\lambda}{f} a^2 g (aQ) \partial_0 \bar{\chi} (3\delta Q^2 + 2\delta Q \Delta M) - a^3 H^2 (3\delta Q^2 + 2\delta Q \Delta M$$
$$+ \partial_i \partial_a M \partial_i \partial_a M) + \frac{a^3}{2}(3\delta\dot{Q}^2 + 2\delta\dot{Q}\Delta\dot{M} + \partial_i \partial_a \dot{M} \partial_i \partial_a \dot{M}) - a^3 (H\delta Q + \delta\dot{Q})\Delta Y$$
$$- a^3 (H \partial_i \partial_a M + \partial_i \partial_a \dot{M}) \partial_i \partial_a Y + \frac{a^3}{2}(\partial_i \partial_a Y)^2 - a(\partial_i \delta Q)^2 + a^3 g^2 (aQ)^2 \partial_b Y \partial_b Y$$
$$- g^2 a^3 Q^2 [9\delta Q^2 + 6\delta Q \Delta M + (\Delta M)^2] - 2\frac{\lambda}{f} a^2 g (aQ) \partial_a \delta\chi \partial_a Y \delta Q$$
$$- \frac{\lambda}{f} a^2 g (aQ)(\partial_i \delta\chi \partial_i Y \Delta M - \partial_b \delta\chi \partial_j Y \partial_b \partial_j M) + \frac{\lambda}{f} a^2 g (aQ) \partial_0 \delta\chi [3\delta Q^2 + 2\delta Q \Delta M$$
$$+ \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i \partial_j M)^2] + \frac{\lambda}{2f} a^3 g\ \partial_0 \bar{\chi} \{2\delta Q^3 + 2\delta Q^2 \Delta M + \delta Q [(\Delta M)^2 - (\partial_a \partial_c M)^2]\}$$

$$
\begin{aligned}
&+ a^4 g^2 (aQ)(2\partial_b Y \partial_b Y \delta Q + \partial_b Y \partial_b Y \Delta M - \partial_b Y \partial_c Y \partial_b \partial_c M) - a^2 g^2 (aQ)[6\delta Q^3 + 6\delta Q^2 \Delta M \\
&+ 2\delta Q(\Delta M)^2 + \Delta M(\partial_c \partial_k M)^2 - \partial_i \partial_k M \partial_c \partial_k M \partial_c \partial_i M] + \frac{\lambda}{2f} g a^3 \partial_i \delta\chi[-2\partial_i Y \delta Q^2 \\
&- 2(\partial_i Y \delta Q \Delta M - \partial_a Y \delta Q \partial_i \partial_a M) + \epsilon^{i0jk}\epsilon^{abc}\partial_a Y \partial_j \partial_b M \partial_c \partial_k M] + \frac{\lambda}{2f} g a^3 \partial_0 \delta\chi[2\delta Q^3 \\
&+ 2\delta Q^2 \Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c \partial_k M)^2 + \frac{1}{3}\epsilon^{0ijk}\epsilon^{abc}\partial_b \partial_j M \partial_a \partial_i M \partial_c \partial_k M] \\
&+ \frac{1}{2} g^2 a^5 (2\partial_b Y \partial_b Y \delta Q^2 + 2\partial_b Y \partial_b Y \delta Q \Delta M + (\partial_b Y \partial_i \partial_c M)^2 \\
&- 2\partial_b Y \partial_c Y \delta Q \partial_b \partial_c M - \partial_b Y \partial_c Y \partial_b \partial_i M \partial_i \partial_c M) - \frac{g^2 a^3}{4}[6\delta Q^4 + 8\delta Q^3 \Delta M + 4\delta Q^2 (\Delta M)^2 \\
&+ 4\delta Q \Delta M(\partial_c \partial_k M)^2 + (\partial_b \partial_i M)^2 (\partial_c \partial_k M)^2 - 4\delta Q \partial_c \partial_k M \partial_c \partial_b M \partial_b \partial_k M \\
&- \partial_b \partial_i M \partial_k \partial_c M \partial_c \partial_i M \partial_b \partial_k M] - \frac{a^3}{6} U'''(\bar{\chi})\delta\chi^3 - \frac{a^3}{24} U''''(\bar{\chi})\delta\chi^4
\end{aligned}
\tag{3.47}
$$

From the final expression for the Lagrangian in (3.47), it is evident that no parity-violating structures are present: since the Levi-Civita tensors appear only in pairs, they can be reduced to products of Kronecker deltas; for this reason, no explicit triple product of the form $\mathbf{k_1} \cdot (\mathbf{k}_2 \times \mathbf{k}_3)$ arises from these interactions. Therefore, we can deduce that parity violation cannot occur in the scalar sector alone. This expectation will be definitively confirmed by the explicit computation of the trispectra.

3.3 In-in Formalism

In the present work, to study the correlation functions, we employ the In-in formalism, originally developed by Schwinger and Keldysh [52, 53]. Although conceptually analogous to the standard framework of quantum field theory, this formalism presents several important differences. For example, unlike the usual QFT approach, where boundary conditions are imposed on asymptotic in- and out-states at early and late times, here one specifies the initial conditions only in the asymptotic past, namely in the regime $k\tau \ll 1$.

In particular, we are not interested in the computation of S-matrix elements, but rather in the evaluation of expectation values of field operators at the end of Inflation. If we define the operator Q, in the Heisenberg picture we have [54, 55]

$$
\langle Q \rangle \equiv \langle \Omega | Q(t_f) | \Omega \rangle \tag{3.48}
$$

where t_f denotes the time at which Inflation ends, while $|\Omega\rangle$ is the vacuum state of the interacting theory in the far past t_0. The operator is a product of fields $\delta\pi_a$ and conjugate momenta $\delta\phi_a$ (where the subscript a identifies different fields), which are the perturbations associated to the canonical variables $\pi_a(\mathbf{x},t)$ and $\delta\phi(\mathbf{x},t)$ (3.52). These variables satisfy the canonical commutation relations

$$[\phi_a(\mathbf{x},t),\, \pi_b(\mathbf{y},t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad [\phi_a(\mathbf{x},t),\, \phi_b(\mathbf{y},t)] = 0 = [\pi_a(\mathbf{x},t),\, \pi_b(\mathbf{x},t)] \tag{3.49}$$

and the equations of motion

$$\dot{\phi}_a(\mathbf{x},t) = i[H[\phi(t),\pi(t)],\, \phi_a(\mathbf{x},t)], \qquad \dot{\pi}_a(\mathbf{x},t) = i[H[\phi(t),\pi(t)],\, \pi_a(\mathbf{x},t)] \tag{3.50}$$

where H is the Hamiltonian of the system, defined by

$$H[\phi(t),\pi(t)] = \int d^3x \; \mathcal{H}[\phi_a(\mathbf{x},t),\pi_a(\mathbf{x},t)] \tag{3.51}$$

where we have suppressed the index a which is summed over. We can decompose the fields in their time-dependent background and the perturbations

$$\phi_a(\mathbf{x},t) = \bar{\phi}_a(\mathbf{x},t) + \delta\phi_a(\mathbf{x},t), \quad \pi_a(\mathbf{x},t) = \bar{\pi}_a(\mathbf{x},t) + \delta\pi_a(\mathbf{x},t) \tag{3.52}$$

This allows us to write conditions (3.49) as

$$[\delta\phi_a(\mathbf{x},t),\, \delta\pi_b(\mathbf{y},t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad [\delta\phi_a(\mathbf{x},t),\, \delta\phi_b(\mathbf{y},t)] = 0 = [\delta\pi_a(\mathbf{x},t),\, \delta\pi_b(\mathbf{x},t)] \tag{3.53}$$

We can now expand the Hamiltonian (3.51) as

$$\begin{aligned} H[\phi(t),\pi(t)] &= \bar{H} + \delta H + \delta^2 H + \dots \\ &= H[\bar{\phi}(t),\bar{\pi}(t)] + \sum_a \int d^3x \frac{\partial\mathcal{H}}{\partial\bar{\phi}_a(\mathbf{x},t)}\delta\phi_a(\mathbf{x},t) + \sum_a \int d^3x \frac{\partial\mathcal{H}}{\partial\bar{\pi}_a(\mathbf{x},t)}\delta\pi_a(\mathbf{x},t) \\ &+ \tilde{H}[\delta\phi(t),\delta\pi(t);t] \end{aligned} \tag{3.54}$$

where $\tilde{H}$ indicates terms second and higher orders in perturbations. By means of the last equation, (3.53) and the equations of motion for the background

$$\dot{\bar{\phi}}_a(\mathbf{x},t) = \frac{\partial \mathcal{H}}{\partial \bar{\pi}_a}, \quad \dot{\bar{\pi}}_a(\mathbf{x},t) = -\frac{\partial \mathcal{H}}{\partial \bar{\phi}_a} \tag{3.55}$$

the equations (3.50) become

$$\delta\dot{\phi}_a(\mathbf{x},t) = i[\tilde{H}[\delta\phi,\delta\pi;t],\delta\phi_a(\mathbf{x},t)] \qquad \delta\dot{\pi}_a(\mathbf{x},t) = i[\tilde{H}[\delta\phi,\delta\pi;t],\delta\pi_a(\mathbf{x},t)] \tag{3.56}$$

This clearly shows that the time evolution for the perturbations $\delta\phi_a$ and $\delta\pi_a$ is caused by $\tilde{H}$.

The next step is to define an operator formalism: in order to do so, we star from a time-evolution operator $U(t,t_0)$, which allows us to write the solutions for (3.56) as

$$\delta\phi_a(\mathbf{x},t) = U^{-1}(t,t_0)\delta\phi_a(\mathbf{x},t_0)U(t,t_0), \quad \delta\pi_a(\mathbf{x},t) = U^{-1}(t,t_0)\delta\pi_a(\mathbf{x},t_0)U(t,t_0) \tag{3.57}$$

where U satisfies

$$\frac{d}{dt}U(t,t_0) = -i\tilde{H}[\delta\phi(t_0),\delta\pi(t_0);t]U(t,t_0) \tag{3.58}$$

with the initial condition $U(t_0,t_0) = 1$. In order to develop a systematic perturbative expansion, we decompose the Hamiltonian $\tilde{H}$ into two separate contributions

$$\tilde{H}[\delta\phi(t),\delta\pi(t);t] = H_0[\delta\phi(t),\delta\pi(t);t] + H_I[\delta\phi(t),\delta\pi(t);t] \tag{3.59}$$

The first term corresponds to the quadratic kinematic Hamiltonian and governs the leading-order evolution of the perturbed fields. The fields evolving under H_0 are referred to as interaction picture fields, and we will denoted them by a superscript I. Indeed, we can write

$$\delta\dot{\phi}^I_a(\mathbf{x},t) = i[H_0[\delta\phi^I(t),\delta\pi^I(t);t],\delta\phi^I_a(\mathbf{x},t)], \quad \delta\dot{\pi}^I_a(\mathbf{x},t) = i[H_0[\delta\phi^I(t),\delta\pi^I(t);t],\delta\pi^I_a(\mathbf{x},t)] \tag{3.60}$$

It is straightforward to determine the solutions of these equations as

$$\delta\phi_a^I(\mathbf{x},t) = U_0^{-1}(t,t_0)\delta\phi_a(\mathbf{x},t_0)U_0(t,t_0), \quad \delta\pi_a^I(\mathbf{x},t) = U_0^{-1}(t,t_0)\delta\pi_a(\mathbf{x},t_0)U_0(t,t_0) \tag{3.61}$$

where, again, U_0 satisfies

$$\frac{d}{dt}U_0(t,t_0) = -iH_0[\delta\phi(t_0),\delta\pi(t_0);t]U_0(t,t_0) \tag{3.62}$$

with $U_0(t_0,t_0) = 1$.
The strategy is to encode the leading kinematic dynamics into the interaction-picture operators, while treating the effects of the interactions perturbatively through an expansion in powers of the interaction Hamiltonian H_I. With this in mind, from (3.48) we can write

$$\begin{aligned}
&\langle\Omega|Q[\delta\phi_a(\mathbf{x},t),\delta\pi_a(\mathbf{x},t)]|\Omega\rangle = \langle\Omega|U^{-1}(t,t_0)Q[\delta\phi_a(\mathbf{x},t_0),\delta\pi_a(\mathbf{x},t_0)]U(t,t_0)|\Omega\rangle \\
&= \langle\Omega|F^{-1}(t,t_0)U_0^{-1}(t,t_0)Q[\delta\phi_a(\mathbf{x},t_0),\delta\pi_a(\mathbf{x},t_0)]U_0(t,t_0)F(t,t_0)|\Omega\rangle \\
&= \langle\Omega|F^{-1}(t,t_0)Q[\delta\phi_a^I(\mathbf{x},t),\delta\pi_a^I(\mathbf{x},t)]F(t,t_0)|\Omega\rangle
\end{aligned} \tag{3.63}$$

where we have introduced

$$F(t,t_0) \equiv U_0^{-1}(t,t_0)U(t,t_0) \tag{3.64}$$

Considering (3.58), (3.62) and (3.59) leads us to

$$\begin{aligned}
\frac{d}{dt}F(t,t_0) &= -iU_0^{-1}(t,t_0)H_I[\delta\phi(t_0),\delta\pi(t_0);t]U_0(t,t_0)F(t,t_0) \\
&= -iH_I[\delta\phi^I(t),\delta\pi^I(t);t]F(t,t_0) \\
&\equiv -iH_I(t)F(t,t_0)
\end{aligned} \tag{3.65}$$

with $F(t_0,t_0) = 1$. The solution to (3.65) results in

$$F(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t)dt\right) \tag{3.66}$$

where we have that the operator T signifies that all the time variables must be time-ordered. Finally, being $\bar{T}$ the anti time-ordering operator, the correlation function (3.48) can be written as [54, 55]

$$
\begin{aligned}
\langle Q\rangle &= \langle\Omega|F^{-1}(t,t_0)Q^I(t)F(t,t_0)|\Omega\rangle \\
&= \langle\Omega|\left[\overline{T}\exp\left(i\int_{t_0}^{t}H_I(t)dt\right)\right]Q^I(t)\left[T\exp\left(-i\int_{t_0}^{t}H_I(t)dt\right)\right]|\Omega\rangle
\end{aligned}
\tag{3.67}
$$

We note that in the last expression all the field perturbations in Q^I and in H_I are in the interaction picture.

Now we can deal with the vacuum choice of the interacting theory: the goal is to express it in terms of the vacuum of the free theory. Contrary to the standard situation encountered in scattering theory, where the interacting and free vacua are generally different, the dynamics considered here does not generate nontrivial vacuum fluctuations through the interaction terms. This property follows directly from the identity $F^{-1}F = \mathbb{I}$. As a consequence, the interacting vacuum state $|\Omega\rangle$ entering in (3.67) can be consistently identified with the Bunch–Davies vacuum $|0\rangle$ (1.95). Another delicate point concerns the behavior in the asymptotic past of $H_I(t)$ appearing in (3.66). In fact, we have that the corresponding integrand becomes highly oscillatory due to presence of the mode functions in it, which behave as plane wave $\propto e^{-ik\tau}$. These rapid oscillations effectively average out their contribution to the integral. In the case of the Bunch–Davies vacuum, this prescription can be implemented rigorously by slightly deforming the integration contour according to $\tau_0 \to -\infty(1+i\epsilon)$, or equivalently by performing a Wick rotation $\tau \to i\tau$[4]. The introduction of an imaginary component transforms the oscillatory behavior into an exponentially damped one, thereby rendering the integral well defined.

Lastly, we can turn to the treatment of the time-ordered integrals appearing in the perturbative expansion. We have that (3.67) can be expanded following two distinct but equivalent approaches [55, 56]. The first possibility consists in the expansion of the exponential operator in (3.66). We get that the nth order contribution takes the form

[4] We will make use of the Wick rotation later in our computations

$$
\begin{aligned}
& i^n(-1)^{n/2}\int_{t_0}^{t} d\bar{t}_1\int_{t_0}^{\bar{t}_1} d\bar{t}_2\cdots\int_{t_0}^{\bar{t}_{n/2-1}} d\bar{t}_{n/2}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n/2-1}} dt_{n/2} \\
&\quad\times\langle 0|H_I(\bar{t}_{n/2})\cdots H_I(\bar{t}_1)Q_I(t)H_I(t_1)\cdots H_I(t_{n/2})|0\rangle \\
&+2\,\mathrm{Re}\sum_{m=1}^{n/2} i^n(-1)^{m+n/2}\int_{t_0}^{t} d\bar{t}_1\int_{t_0}^{\bar{t}_1} d\bar{t}_2\cdots\int_{t_0}^{\bar{t}_{n/2-1-m}} d\bar{t}_{n/2-m}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n/2-1+m}} dt_{n/2+m} \\
&\quad\times\langle 0|H_I(\bar{t}_{n/2-m})\cdots H_I(\bar{t}_1)Q_I(t)H_I(t_1)\cdots H_I(t_{n/2+m})|0\rangle
\end{aligned}
\tag{3.68}
$$

for even n, while we have

$$
\begin{aligned}
& 2\,\mathrm{Re}\sum_{m=1}^{(n+1)/2} i^n(-1)^{m+(n-1)/2}\int_{t_0}^{t} d\bar{t}_1\cdots\int_{t_0}^{\bar{t}_{(n-1)/2-m}} d\bar{t}_{(n+1)/2-m}\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t_{(n-3)/2+m}} dt_{(n-1)/2+m} \\
&\quad\times\langle 0|H_I(\bar{t}_{(n+1)/2-m})\cdots H_I(\bar{t}_1)Q_I(t)H_I(t_1)\cdots H_I(t_{(n-1)/2+m})|0\rangle
\end{aligned}
\tag{3.69}
$$

for an odd value of n. In this representation, each term contains two distinct sets of multiple integrals: one originating from F^{-1} and the other from F. We note that, while the integrations within each set are individually time-ordered (or anti-time-ordered), no ordering relation exists between the two sets. This formulation is commonly referred to as the factorized form. The second representation can be obtained by reorganizing the factorized expression in such a way that all time variables become time-ordered and all contributions are combined into a single multiple integral. The resulting nth order term is given by

$$
i^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\langle 0|[H_I(t_n),[H_I(t_{n-1}),\cdots,[H_I(t_1),Q_I(t)]\cdots]]|0\rangle
\tag{3.70}
$$

This second formulation is known as the commutator form. Although both representations are entirely equivalent, in this thesis we will focus exclusively on the factorized form (3.69).

In general, in order to solve these integral we have to deal with time-ordered and anti-time-ordered products of field operators. A considerable simplification to these computations is provided by introducing a different ordering prescription, namely the normal ordering N. By definition, the normal ordering operator rearranges the field operators such that all annihilation operators are placed to the right and all creation operators to the left. Since operators of the same type mutually commute, their relative ordering within each group is irrelevant. The usefulness of this prescription becomes immediately apparent when computing vacuum expectation values. Indeed, because $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$, any normal-ordered operator containing uncontracted fields has a vanishing expectation value in the vacuum state. To properly apply this result to our computations, we introduce Wick's theorem [57], which provides a connection between time-ordered products and normal-ordered products. Schematically, the theorem allows a time-ordered product to be expressed as a sum of normal-ordered products supplemented by all possible contractions. For example, if we consider four fields F_1, F_2, F_3 and F_4 we get

$$T[F_1F_2F_3F_4] = N[F_1F_2F_3F_4] + \sum_{i,j,k,l} \wick{\c F_i \c F_j} N[F_kF_l] + \sum_{i,j,k,l} \wick{\c1 F_i \c1 F_j \c2 F_k \c2 F_l} \tag{3.71}$$

where in the second term on the right-hand side of the last expression we have that the fields $F_k, F_l \neq F_i, F_j$. The term $\wick{\c F_1 \c F_2} = \langle 0|T[F_1F_2]|0\rangle$ is the contraction between the fields F_1 and F_2. Since a contraction is a c-number quantity, it can be extracted from the normal-ordering symbol. An important consequence of Wick's theorem is that, when evaluating vacuum expectation values, only the terms in which all fields are fully contracted contribute. Therefore, in practical computations of correlation functions, one only needs to retain the fully contracted contributions generated by Wick's expansion, which reduces the number of nonvanishing terms.

3.4 Hamiltonian Computation

As discussed in Section (3.3), the application of the In-in formalism requires the explicit determination of the Hamiltonian contributions. For this reason, we now transition to the Hamiltonian formulation of the system. The complete derivation of the total Hamiltonian density is presented in the Appendix (B.2).

3.4.1 Mode Functions

Within the framework of the In-in formalism, we have introduced the quadratic free Hamiltonian H_0 (B.24), which will be employed to derive the equations of motion for the perturbed fields.

We begin by considering the contribution associated with the perturbation $\delta\chi$ [5], for which we obtain the following expression in Fourier space

$$H_{0,\delta\chi} = \int d^3k \; \frac{1}{2a^3}|\delta\pi_{1,k}|^2 + \left[\frac{a}{2}k^2 + \frac{a^3}{2}U'' + \frac{a}{4}\left(\frac{\lambda}{f}Q\right)^2 k^2\right]|\delta\chi_k|^2 \tag{3.72}$$

By means of Hamilton's equations

$$\delta\dot{\chi}_k = \frac{\partial H_{0,\delta\chi}}{\partial \delta\pi^*_{1,k}} = \frac{\delta\pi_{1,k}}{a^3}, \quad \delta\dot{\pi}_{1,k} = -\frac{\partial H_0}{\partial \delta\chi^*_k} = -2\left[\frac{a}{2}k^2 + \frac{a^3}{2}U''\delta\chi_k + \frac{a}{4}\left(\frac{\lambda}{f}Q\right)^2 k^2\right]\delta\chi_k \tag{3.73}$$

we can derive the equation of motion

$$\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + m_\chi\chi_k + c_{s1}^2\frac{k^2}{a^2}\delta\chi_k = 0 \tag{3.74}$$

where we have defined $m_\chi = U''$ and $c_{s1}^2 = \left[1 + \frac{1}{2}\left(\frac{\lambda}{f}Q\right)^2\right] \simeq 1.3$ as the modified speed of sound squared for the axion. Then, we can move to the conformal time τ, change variable $\delta\chi_k(\tau) = \frac{X_k(\tau)}{a}$, and decompose the Fourier component

$$X_k(\tau) = x_k(\tau)\;\hat{a}_{\mathbf{k}} + x^*_{-k}(\tau)\;\hat{a}^\dagger_{-\mathbf{k}} \tag{3.75}$$

where the annihilation and creation operators satisfy the relations[6]

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \quad [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0 \tag{3.76}$$

Using the expression $a(\tau) \simeq -\frac{1}{H\tau(1-\epsilon_H)}$ for the scale factor, we obtain the following equation of motion

$$x''_k + \left(c_{s1}^2 k^2 - \frac{\nu_1^2 - \frac{1}{4}}{\tau^2}\right)x_k = 0 \tag{3.77}$$

where we have defined $\nu_1 = \sqrt{\frac{9}{4} + 3\epsilon_H - \frac{m_\chi^2}{H^2}} \simeq \frac{3}{2} + \epsilon_H - \frac{m_\chi^2}{3H^2}$.

As discussed in Section (1.4.5), the solution to equation (3.77) is known to take the form given in (1.112); it follows that the solution is

[5]All fields and their corresponding conjugate momenta are defined in the interaction picture. For notational simplicity, we will henceforth omit the superscript I.

[6]We note that, due to the convention adopted for the Fourier transform in (2.27), the $(2\pi)^3$ factor is not present.

$$u_k(\tau) = \frac{x_k(\tau)}{a} = \frac{\sqrt{\pi}}{2} H(-\tau)^{3/2} e^{i\frac{\pi}{2}(\nu_1+\frac{1}{2})} H^{(1)}_{\nu_1}(-c_{s1}k\tau) \tag{3.78}$$

Thereafter, for δQ we can write

$$H_{0,\delta Q} = \int d^3k \; \frac{1}{4a^3}|\delta\pi_{2,k}|^2 + \Big(a^3 B + ak^2 + \frac{aH^2k^2}{4g^2Q^2}\Big)|\delta Q_k|^2 \tag{3.79}$$

where we have defined

$$B = \frac{9}{2}\Big(\frac{\lambda}{f}gQ^2\Big)^2 - 3\frac{\lambda}{f}gQ\partial_0\bar{\chi} + 9g^2Q^2 + 3H^2 = \frac{9}{2}\Lambda^2 g^2Q^2 - 6H\Big(m_Q + \frac{1}{m_Q}\Big)gQ + 9g^2Q^2$$
$$+ 3H^2 = \Big(\frac{9}{2}\Lambda^2 m_Q^2 - 6m_Q^2 - 6 + 9m_Q^2 + 3\Big)H^2 \tag{3.80}$$

where we used the third term in (2.24) and (2.25) to rewrite the second term. Throughout this analysis, we treat this parameter as constant. We can write Hamilton's equation

$$\delta\dot{Q}_k = \frac{\partial H_{0,\delta Q}}{\partial \delta\pi^*_{2,k}} = \frac{\delta\pi_{2,k}}{2a^3}, \quad \delta\dot{\pi}_{2,k} = -\frac{\partial H_{0,\delta Q_k}}{\partial \delta Q^*_k} = -2\Big(a^3B + ak^2 + \frac{aH^2k^2}{4g^2Q^2}\Big)\delta Q_k \tag{3.81}$$

to obtain the equation of motion

$$\delta\ddot{Q}_k + 3H\delta\dot{Q}_k + B\;\delta Q_k + \frac{k^2}{a^2}\delta Q_k + \frac{H^2k^2}{4a^2g^2Q^2}\delta Q_k = 0 \tag{3.82}$$

As in the previous analysis, we perform the transformation to conformal time and redefine the perturbation variable according to $\delta Q_k = \frac{Z_k}{\sqrt{2}a}$, with

$$Z_k(\tau) = z_k(\tau)\;\hat{a}_{\mathbf{k}} + z^*_k(\tau)\;\hat{a}^\dagger_{\mathbf{k}} \tag{3.83}$$

We can rewrite the equation of motion in terms of the new canonical variable as

$$z''_k + \Big(c^2_{s2}k^2 - \frac{\nu_2^2 - \frac{1}{4}}{\tau^2}\Big)z_k = 0 \tag{3.84}$$

where we have defined $c^2_{s2} = \Big(1 + \frac{1}{4m_Q^2}\Big) \simeq 1.03$. As in the previous case, we introduce the auxiliary parameter ν; however, in the present situation it becomes purely imaginary due to the fact that $B = 36,8\; H^2$ [7]. We have

[7] The value of B is obtained by using the values specified in (2.57).

$$\begin{aligned}\nu_2 &= \sqrt{\frac{9}{4} + 3\epsilon_H - \frac{B}{H^2} - \frac{2B\epsilon_H}{H^2}} = \sqrt{\frac{9}{4} - \frac{B}{H^2} + \epsilon_H\Big(3 - \frac{2B}{H^2}\Big)} \\ &\simeq \sqrt{\frac{9}{4} - \frac{B}{H^2}} + \epsilon_H \frac{\Big(3 - \frac{2B}{H^2}\Big)}{2\sqrt{\frac{9}{4} - \frac{B}{H^2}}}\end{aligned} \tag{3.85}$$

Thus, by defining $\nu_2 = i\mu_2$, we can write $\mu_2 = \sqrt{\frac{B}{H^2} - \frac{9}{4}} - \epsilon_H \frac{\Big(3 - \frac{2B}{H^2}\Big)}{2\sqrt{\frac{B}{H^2} - \frac{9}{4}}}$.

The solution to equation (3.84) is

$$v_k(\tau) = \frac{z_k(\tau)}{\sqrt{2}a} = \sqrt{\frac{\pi}{2}}\frac{H}{2} e^{-(\mu_2 - \frac{i}{2})\frac{\pi}{2}} (-\tau)^{3/2} H^{(1)}_{i\mu_2}(-c_{s2}k\tau) \tag{3.86}$$

Lastly, for the perturbation M we have the following

$$H_{0,M} = \int d^3k \; \frac{3g^2(aQ)^2 + k^2}{4a^3g^2(aQ)^2k^4} |\delta\pi_{3,k}|^2 + \Big(a^3k^4D + \frac{aH^2k^6}{4g^2Q^2}\Big)|M_k|^2 \tag{3.87}$$

where we have defined the constant

$$\begin{aligned}D &= \frac{1}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 + H^2 + g^2Q^2 = \frac{1}{2}\Lambda^2 g^2Q^2 + H^2 + g^2Q^2 \\ &= \Big(\frac{1}{2}\Lambda^2 m_Q^2 + 1 + m_Q^2\Big)H^2\end{aligned} \tag{3.88}$$

The relation in (3.87) is non-standard and differs from the previous ones. To simplify it, we can examine the coefficient of $|\delta\pi_{3,k}|^2$. Looking at the denominator, a comparison can be made between k^2 and $3g^2(aQ)^2$. We can have two limits: $k^2 > 3g^2(aQ)^2$ and $k^2 < 3g^2(aQ)^2$, which correspond to $k^2/(aH)^2 > 3m_Q^2$ and $k^2/(aH)^2 < 3m_Q^2$ respectively. Given that $\sqrt{3}m_Q \simeq 4.7$, these two conditions roughly identify two kinematic regimes: the first is associated with the sub-horizon limit, while the second includes approximately the horizon-crossing and super-horizon phases. As a first approximation, we decide to focus on the second regime, so $k^2 < 3g^2(aQ)^2$. This choice allows us to drop the k^2 term in the denominator, meaning that (3.87) can be rewritten as

$$H_{0,M} = \int d^3k \; \frac{3}{4a^3k^4} |\delta\pi_{3,k}|^2 + \Big(a^3k^4D + \frac{aH^2k^6}{4g^2Q^2}\Big)|M_k|^2 \tag{3.89}$$

We may then apply Hamilton's equations to derive

$$\dot{M}_k = \frac{\partial H_{0,M}}{\partial \delta\pi^*_{3,k}} = \frac{3\delta\pi_{3,k}}{2a^3k^4}, \quad \delta\dot{\pi}_{3,k} = -\frac{\partial H_{0,M}}{\partial M^*_k} = -2\Big(a^3D + \frac{aH^2k^2}{4g^2Q^2}\Big)k^4M_k \tag{3.90}$$

which together determine the corresponding equation of motion

$$\ddot{M}_k + 3H\dot{M}_k + 3\Big(D + \frac{H^2k^2}{4a^2g^2Q^2}\Big)M_k = 0 \tag{3.91}$$

The parentheses in the above equation contain the ratio between k^2 and $g^2(aQ)^2$. Consistent with our adopted approximation, and taking into account that $D = 10.3\ H^2$ [8], this bracket is primarily dictated by D, which represents the effective mass. Because the mass term outweighs the kinetic one, the resulting mode function for M would be characterized by severe oscillations. We therefore choose to neglect this mode moving forward; physically, this approximation is justified by the fact that the contribution of such rapidly oscillating function to the in-in time integrals is expected to average out and be suppressed.

3.4.2 Interaction Terms

Our goal is to determine the trispectrum represented by Figure (3.1). Given that the perturbations under consideration are $\delta\chi$ e δQ, we must isolate the bilinear term proportional to $\delta\chi\delta Q$ from H^I_2 and the quartic term proportional to $\delta\chi\delta Q^3$ from H^I_4 for the contact interaction diagram[9]; instead, for the scalar exchange one we need the cubic term proportional to $\delta\chi^2\delta Q$ from H^I_3.
As anticipated in Section (3.2), we have also extra terms to take into account: for the diagram (3.1b), the cubic interaction Hamiltonian H^I_3 can also generate an effective quartic contribution due to the presence of the non-dynamical mode Y. Specifically, in (B.26) we have a term $\propto \delta\chi Y\delta Q$ and a term $\propto YY\delta Q$. For the former, it is required the part of $Y^{(1)} \propto \delta Q$, while for the latter we need a combination of $Y^{(2)}$ and $Y^{(1)}$. In particular, since $Y^{(2)}$ is proportional only to $(\delta Q)^2$, its insertion alongside the $\delta\chi$-dependent part of $Y^{(1)}$ induces a quartic interaction. Additionally, further contributions involving $Y^{(1)} \propto \delta Q$ arise directly from H^I_4.
Regarding the second diagram (3.1a), the only way to generate the $\delta\chi^2\delta Q$ term is by evaluating the $\delta\chi Y\delta Q$ and $YY\delta Q$ interactions within H^I_3. This requires taking

[8]The value of D is obtained by using the values specified in (2.57).

[9]We point out that this does not represent the only possible contribution to the contact interaction diagram. However, for the purposes of this study, we will restrict our attention solely to the diagram arising from these vertices.

$Y^{(1)} \propto \delta\chi$.

Contributions for the Contact Interaction Diagram

Starting with the quadratic interaction, and recalling that we want to isolate the $\delta\chi\delta Q$ mixing term, (B.25) yields

$$\int d^3k - 3\frac{\lambda}{f} g Q^2 a^3 \delta\dot{\chi}_k \delta Q_{-k} - \frac{\lambda H a k^2}{2fg} \delta\chi_k \delta Q_{-k} = H_a + H_b \tag{3.92}$$

In order to reduce the combinatorial complexity of the In-in integrals, we first compare the strength of the two interactions. Evaluating the couplings at horizon crossing[10], we find that the first coupling is approximately a factor of 40 larger than the second. For this reason, as a first-order approximation, we choose to neglect the subdominant contribution and restrict our analysis to the H_a term only. Therefore we have

$$H_2^{Int}(\tau) = H_a = -3\frac{\lambda}{f} g Q^2 a^2 \int d^3k \delta\chi'_k \delta Q_{-k} \tag{3.93}$$

Concerning the quartic interaction, the pure contribution arising from the dynamical modes from H_4^I (B.27) reads

$$\int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3} - \frac{\lambda}{f} g a^3 \delta\dot{\chi}_k \delta Q_q \delta Q_p \delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.94}$$

When accounting for the non-dynamical field, several distinct contributions emerge. From H_4^I, we obtain, considering the term $\propto \delta\chi Y^{(1)} \delta Q^2$

$$\int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3} \frac{\lambda a H}{2fgQ^2} (\mathbf{k}\cdot\mathbf{q})\ \delta\chi_k \delta Q_q \delta Q_p \delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.95}$$

where we used $Y^{(1),\delta Q} = -\frac{H}{2g^2(aQ)^2}\delta Q$ and from the term $\propto Y^{(1)}Y^{(1)}\delta Q^2$ (taking also into account $Y^{(1),\delta\chi} = \frac{\lambda}{2fa^2g}\delta\chi$) we get

[10] We expect the massive mode δQ to survive for a few e-folds after horizon crossing before decaying. As a consequence, evaluating these couplings at horizon crossing can provide a reliable estimate of their physical impact.

$$\int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3}\ \frac{g^2a^5}{2}\Big[-4(\mathbf{k}\cdot\mathbf{q})\frac{\lambda H}{4fa^2g^3(aQ)^2}\delta\chi_k\delta Q_q\delta Q_p\delta Q_l\Big]\delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) =$$
$$= \int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3} - \frac{\lambda a H}{2fgQ^2}(\mathbf{k}\cdot\mathbf{q})\ \delta\chi_k\delta Q_q\delta Q_p\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.96}$$

and they cancel each other. Moreover, additional contributions arise from H_3^I. Within these terms, we must evaluate $Y^{(2)} \propto \delta Q^2$, which, after straightforward algebra, yields the following expression

$$Y_r^{(2)} = \int \frac{d^3p\ d^3q}{(2\pi)^{3/2}}\frac{H}{2g^2a^2Q^3}\delta Q_p\delta Q_q\ \delta^{(3)}(\mathbf{q}+\mathbf{p}-\mathbf{r}) \tag{3.97}$$

This allows us to determine the first term proportional to $\delta\chi Y^{(2)}\delta Q$

$$\int \frac{d^3k\ d^3r\ d^3l}{(2\pi)^{3/2}} - \frac{2\lambda a^3 gQ}{f}(\mathbf{k}\cdot\mathbf{q})\delta\chi_k Y_r^{(2)}\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{r}+\mathbf{l})$$
$$= \int \frac{d^3k\ d^3r\ d^3p\ d^3q\ d^3l}{(2\pi)^3} - \frac{\lambda H a}{fgQ^2}(\mathbf{k}\cdot\mathbf{q})\delta\chi_k\delta Q_p\delta Q_{q-p}\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{r}+\mathbf{l})\delta^{(3)}(\mathbf{q}+\mathbf{p}-\mathbf{r})$$
$$= \int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3} - \frac{\lambda H a}{fgQ^2}(\mathbf{k}\cdot\mathbf{q})\delta\chi_k\delta Q_p\delta Q_{q-p}\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.98}$$

By analogous reasoning, the term proportional to $2Y^{(2)}Y^{(1)}\delta Q$ takes the form

$$\int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3}\ 4a^5g^2Q\frac{\lambda H}{4fa^4g^3Q^3}(\mathbf{k}\cdot\mathbf{q})\delta\chi_k\delta Q_p\delta Q_{q-p}\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l})$$
$$= \int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3}\ \frac{\lambda H a}{fgQ^2}(\mathbf{k}\cdot\mathbf{q})\delta\chi_k\delta Q_p\delta Q_{q-p}\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.99}$$

As a result of this, the total contribution for the quartic term $\delta\chi\delta Q^3$ in conformal time τ is

$$H_4^{Int}(\tau) = -\frac{\lambda}{f}ga^2\int \frac{d^3k\ d^3q\ d^3p\ d^3l}{(2\pi)^3}\delta\chi_k'\delta Q_q\delta Q_p\delta Q_l\ \delta^{(3)}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.100}$$

Hence, the essential components required for the contact interaction diagram are H_2^{Int} and H_4^{Int}.

Contributions for the Scalar Exchange Diagram

To extract the contributions proportional to $\delta\chi^2\delta Q$, we start from H_3^I and we substitute $Y^{(1),\delta\chi} = \frac{\lambda}{2fa^2g}\delta\chi$ in it. We obtain from $\delta\chi Y^{(1)}\delta Q$

$$\int \frac{d^3q\ d^3p\ d^3l}{(2\pi)^{3/2}} - \left(\frac{\lambda}{f}\right)^2 aQ(\mathbf{p}\cdot\mathbf{q})\delta\chi_p\delta\chi_q\delta Q_l\ \delta^{(3)}(\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.101}$$

while from $Y^{(1)}Y^{(1)}\delta Q$ we find

$$\begin{aligned} &\int \frac{d^3q\ d^3p\ d^3l}{(2\pi)^{3/2}}\ 2a^4g^2aQ\left(\frac{\lambda}{2fa^2g}\right)^2(\mathbf{p}\cdot\mathbf{q})\delta\chi_p\delta\chi_q\delta Q_l\ \delta^{(3)}(\mathbf{q}+\mathbf{p}+\mathbf{l}) \\ &= \int \frac{d^3q\ d^3p\ d^3l}{(2\pi)^{3/2}}\ \frac{1}{2}\left(\frac{\lambda}{f}\right)^2 aQ(\mathbf{p}\cdot\mathbf{q})\delta\chi_p\delta\chi_q\delta Q_l\ \delta^{(3)}(\mathbf{q}+\mathbf{p}+\mathbf{l}) \end{aligned} \tag{3.102}$$

Summing these contributions yields

$$H_3^{Int}(\tau) = -\frac{1}{2}\left(\frac{\lambda}{f}\right)^2 aQ\int \frac{d^3q\ d^3p\ d^3l}{(2\pi)^{3/2}}\ (\mathbf{p}\cdot\mathbf{q})\delta\chi_p\delta\chi_q\delta Q_l\ \delta^{(3)}(\mathbf{q}+\mathbf{p}+\mathbf{l}) \tag{3.103}$$

This represents the final expression required to evaluate the trispectrum described by diagram (3.1a).

Chapter 4

Trispectra

Before going into details about the computations, it is important to highlight that the full trispectrum can be decomposed in two distinct contributions: the connected and disconnected components. The first contribution vanishes for purely Gaussian field fluctuations, and therefore constitutes a measure of primordial non-Gaussianity. The second contribution is the disconnected part, and is obtained by summing over all possible ways of combining the four fields into pairs, with each pair generating a copy of the power spectrum. From a physical perspective, the disconnected contribution carries no information beyond that already encoded in the power spectrum. So, for the purposes of our analysis, we will strictly focus on the connected part.
To evaluate the correlator, we employ the factorized expression introduced in (3.69), which involves multiple terms depending on the number of vertices present in the diagram. For the following calculations we will work in the exact de Sitter limit, setting $\epsilon_H = 0$, and we will treat the perturbation $\delta\chi$ as massless. Thus, considering (3.78), the index of the mode function reduces to $\nu = 3/2$. This simplification allows us to write

$$u_k(\tau) = \frac{iH}{\sqrt{2c_{s1}^3 k^3}}(1 + ic_{s1}k\tau)e^{-ic_{s1}k\tau} \tag{4.1}$$

Regarding the perturbation δQ, we will use the mode function defined in (3.86), with $\mu_2 = \sqrt{\frac{B}{H^2} - \frac{9}{4}} = 5.88$. To simplify the notation, and since there is no risk of confusion, we will refer to this parameter simply as μ. All the subsequent numerical evaluations have been carried out using the symbolic computation software Mathematica.
The trispectrum for the axion is defined as

$$\langle \delta\chi_{k_1}\delta\chi_{k_2}\delta\chi_{k_3}\delta\chi_{k_4}\rangle = (2\pi)^{-3}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\; T(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3},\mathbf{k_4}) \tag{4.2}$$

We can introduce the dimensionless form factor $\mathcal{T}(k_1,k_2,k_3,k_4,k_{12},k_{14})$ [58]

$$T(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3},\mathbf{k_4}) = \mathcal{P}^3_{\delta\chi}\frac{1}{\prod_{i=1}^4 k_i^3}\mathcal{T}, \quad \mathcal{T} = \mathcal{T}_{SE}+\mathcal{T}_{CI} \tag{4.3}$$

where $\mathcal{T}_{SE}$ and $\mathcal{T}_{CI}$ are the scalar exchange and contact interaction contribution, respectively, while $\mathcal{P}_{\delta\chi}$ is the dimensionless power spectrum of the axion. It is related to the power spectrum $P_{\delta\chi}$ by the following relation

$$\mathcal{P}_{\delta\chi} = \frac{k^3}{2\pi^2}P_{\delta\chi}(k) = \frac{H^2}{4\pi^2 c_{s1}^3}\Xi \tag{4.4}$$

The power spectrum of the axion is defined in the Appendix (C.1).

4.1 Scalar Exchange

As previously discussed, our starting point is (3.69); applied to the diagram (3.1a), this yields

$$\begin{aligned}
&i^2(-1)\int_{t_0}^{t} d\tilde{t}_1\int_{t_0}^{t} dt_1\langle H_I(\tilde{t}_1)Q_I(t)H_I(t_1)\rangle + 2\,\mathrm{Re}\left[i^2(-1)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\langle Q_I(t)H_I(t_1)H_I(t_2)\rangle\right]\\
&=\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\langle H_I(\tilde{\tau}_1)Q_I(\tau)H_I(\tau_1)\rangle\\
&-2\,\mathrm{Re}\left[\int_{\tau_0}^{\tau}d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1}d\tau_2 a(\tau_2)\langle Q_I(\tau)H_I(\tau_1)H_I(\tau_2)\rangle\right]
\end{aligned} \tag{4.5}$$

In particular, substituting the relevant expressions we can write

$$\begin{aligned}
&\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\langle H_3^{Int}(\tilde{\tau}_1)\delta\chi_{k_1}(\tau)\delta\chi_{k_2}(\tau)\delta\chi_{k_3}(\tau)\delta\chi_{k_4}(\tau)H_3^{Int}(\tau_1)\rangle\\
&-2\,\mathrm{Re}\left[\int_{\tau_0}^{\tau}d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1}d\tau_2 a(\tau_2)\langle\delta\chi_{k_1}(\tau)\delta\chi_{k_2}(\tau)\delta\chi_{k_3}(\tau)\delta\chi_{k_4}(\tau)H_3^{Int}(\tau_1)H_3^{Int}(\tau_2)\rangle\right]
\end{aligned} \tag{4.6}$$

Thus, two separate contributions emerge: one from the first line and one from the second one. We will explicitly derive the result for the first line only, relegating

the evaluation of the second line to Appendix (C.2).
To evaluate the correlator, we must compute the field contractions. As explained in Section (3.3), the use of normal ordering allows us to compute exclusively products of contractions. In particular, we can employ the following decomposition

$$\delta\chi(\mathbf{k},\tau) \equiv \delta\chi^{+} + \delta\chi^{-} = u_k(\tau)\ \hat{a}_{\mathbf{k}} + u^{*}_{-k}(\tau)\ \hat{a}^{\dagger}_{-\mathbf{k}} \tag{4.7}$$

A contraction between two fields ($\delta\chi(\mathbf{k_1},\tau_1)$ on the left and $\delta\chi(\mathbf{k_2},\tau_2)$ on the right) gives

$$\Big[\delta\chi^{+}(\mathbf{k_1},\tau_1),\delta\chi^{-}(\mathbf{k_2},\tau_2)\Big] = u_{k_1}(\tau_1)u^{*}_{-k_2}(\tau_2)\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}) \tag{4.8}$$

and the same applies to the perturbation δQ. Therefore, considering a specific configuration for k_i, the correlator in the first line of (4.6) can be rewritten as

$$\begin{aligned}
&\langle\delta\chi(\mathbf{q_1},\tilde{\tau}_1)\delta\chi(\mathbf{q_2},\tilde{\tau}_1)\delta Q(\mathbf{q_3},\tilde{\tau}_1)\delta\chi(\mathbf{k_1},\tau)\delta\chi(\mathbf{k_2},\tau)\delta\chi(\mathbf{k_3},\tau)\delta\chi(\mathbf{k_4},\tau)\delta\chi(\mathbf{p_1},\tau_1)\delta\chi(\mathbf{p_2},\tau_1)\delta Q(\mathbf{p_3},\tau_1)\rangle\\
&=\Big[\delta\chi^{+}(\mathbf{q_1},\tilde{\tau}_1),\delta\chi^{-}(\mathbf{k_1},\tau)\Big]\Big[\delta\chi^{+}(\mathbf{q_2},\tilde{\tau}_1),\delta\chi^{-}(\mathbf{k_2},\tau)\Big]\Big[\delta Q^{+}(\mathbf{q_3},\tilde{\tau}_1),\delta Q^{-}(\mathbf{p_3},\tau_1)\Big]\\
&\quad\times\Big[\delta\chi^{+}(\mathbf{k_3},\tau),\delta\chi^{-}(\mathbf{p_1},\tau_1)\Big]\Big[\delta\chi^{+}(\mathbf{k_4},\tau),\delta\chi^{-}(\mathbf{p_2},\tau_1)\Big]
\end{aligned} \tag{4.9}$$

in addition to this specific term, one must account for the 23 other contributions arising from the permutations of the momenta k_i. Thereby, the overall contribution to the first line of (4.6) reads

$$\begin{aligned}
&\frac{1}{4}\left(\frac{\lambda}{f}\right)^4 Q^2\int\frac{d^3q_1\ d^3q_2\ d^3q_3}{(2\pi)^{3/2}}\int\frac{d^3p_1\ d^3p_2\ d^3p_3}{(2\pi)^{3/2}}(\mathbf{q_1}\cdot\mathbf{q_2})(\mathbf{p_1}\cdot\mathbf{p_2})\\
&\times\delta^{(3)}(\mathbf{q_1}+\mathbf{q_2}+\mathbf{q_3})\ \delta^{(3)}(\mathbf{p_1}+\mathbf{p_2}+\mathbf{p_3})\int_{\tau_0}^{\tau}d\tilde{\tau}_1 a^2(\tilde{\tau}_1)\int_{\tau_0}^{\tau}d\tau_1 a^2(\tau_1)\\
&\times\langle\delta\chi_{q_1}(\tilde{\tau}_1)\delta\chi_{q_2}(\tilde{\tau}_1)\delta Q_{q_3}(\tilde{\tau}_1)\delta\chi_{k_1}(\tau)\delta\chi_{k_2}(\tau)\delta\chi_{k_3}(\tau)\delta\chi_{k_4}(\tau)\delta\chi_{p_1}(\tau_1)\delta\chi_{p_2}(\tau_1)\delta Q_{p_3}(\tau_1)\rangle\\
&+23\text{ permutations of }\mathbf{k_i}\\
&=\frac{1}{4(2\pi)^3}\left(\frac{\lambda}{f}\right)^4 Q^2 u^{*}_{k_1}(\tau)u^{*}_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)(\mathbf{k_1}\cdot\mathbf{k_2})(\mathbf{k_3}\cdot\mathbf{k_4})\\
&\times\int_{\tau_0}^{\tau}d\tilde{\tau}_1 a^2(\tilde{\tau}_1)u_{k_1}(\tilde{\tau}_1)u_{k_2}(\tilde{\tau}_1)v_{k_1+k_2}(\tilde{\tau}_1)\int_{\tau_0}^{\tau}d\tau_1 a^2(\tau_1)u^{*}_{k_3}(\tau_1)u^{*}_{k_4}(\tau_1)v^{*}_{k_3+k_4}(\tau_1)\\
&\times\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}
\end{aligned} \tag{4.10}$$

Dealing with the time integrals, we will take $\tau = 0$ and $\tau_0 = -\infty$; moreover, we introduce the variables $x \equiv -c_{s2}k_1\tau,\ c_s \equiv \frac{c_{s1}}{c_{s2}} = 1.11$, dimensionless momenta $k_i^* = \frac{k_i}{k_1}$ normalized by k_1 and $k_{34} \equiv k_3 + k_4,\ k_{12} \equiv k_1 + k_2$[1]. An additional technical remark concerns the implementation of the contour deformation used to regulate the ultraviolet behavior of the time integrals. While a small tilt of the integration contour provides a straightforward analytical prescription, its numerical implementation is often computationally inefficient. For this reason, in this thesis we will adopt a more effective approach based on a Wick rotation [56]. As discussed in Section (3.3), the integration variables can be analytically continued into the complex plane according to the transformation $x \to \pm iy$[2], thus rotating the integration contour away from the real axis. Under this change of variables, the oscillatory factors that characterize the mode functions in the asymptotic past are converted into exponentially suppressed terms. As a consequence, the highly oscillatory behavior of the integrand, which would otherwise prevent numerical convergence, is effectively removed. So, the total expression we obtain from (4.6) is

$$
\begin{aligned}
\langle \delta\chi_{k_1}\delta\chi_{k_2}\delta\chi_{k_3}\delta\chi_{k_4}\rangle_{SE} = {}& \frac{1}{(2\pi)^3}\left(\frac{\lambda}{f}\right)^4 Q^2(\mathbf{k_1}\cdot\mathbf{k_2})(\mathbf{k_3}\cdot\mathbf{k_4})\frac{H^6}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)^3}\frac{1}{16c_{s1}^3c_{s2}k_1} \\
&\left\{\frac{1}{\pi}\int_0^{+\infty} d\tilde{y}_1\int_0^{+\infty} dy_1 (\tilde{y}_1)^{-\frac{1}{2}}(1+c_sk_1^*\tilde{y}_1)(1+c_sk_2^*\tilde{y}_1)e^{-c_s\tilde{y}_1(k_1^*+k_2^*)}\right. \\
&\times K_{i\mu}(k_{34}^*\tilde{y}_1)(y_1)^{-\frac{1}{2}}(1+c_sk_3^*y_1)(1+c_sk_4^*y_1)e^{-c_sy_1(k_3^*+k_4^*)}K_{i\mu}(k_{34}^*y_1) \\
&-\mathrm{Re}\left[\int_0^{+\infty} dy_1\int_{y_1}^{+\infty} dy_2 (y_1)^{-\frac{1}{2}}(1+c_sk_1^*y_1)\right. \\
&\times(1+c_sk_2^*y_1)e^{-c_sy_1(k_1^*+k_2^*)}e^{-\frac{\pi\mu}{2}}H_{i\mu}^{(1)}(-ik_{34}^*y_1)(y_2)^{-\frac{1}{2}}(1+c_sk_3^*y_2) \\
&\left.\left.\times(1+c_sk_4^*y_2)e^{-c_sy_2(k_3^*+k_4^*)}K_{i\mu}(k_{34}^*y_2)\right]\right\}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) \\
&+ 23 \text{ permutations of } \mathbf{k_i}
\end{aligned}
\tag{4.11}
$$

where we used the relations $H_{i\mu}^{(1)}(ix) = \frac{2}{\pi i}e^{\frac{\pi\mu}{2}}K_{i\mu}(x)$ and $H_{-i\mu}^{(2)}(-ix) = -\frac{2}{\pi i}e^{\frac{\pi\mu}{2}}K_{i\mu}(x)$, and the subscript SE stands for Scalar Exchange. By defining

[1] It is possible to write $(\mathbf{k_i}\cdot\mathbf{k_j}) = \frac{1}{2}(k_{ij}^2 - k_i^2 - k_j^2)$

[2] We note that the sign of y is chosen to be either plus or minus to ensure that the corresponding integrand is decaying as $y \to -\infty$.

$$\begin{aligned}
\mathcal{I}_1 =& \Bigg[\int_0^{+\infty} d\tilde{y}_1 \int_0^{+\infty} dy_1 (\tilde{y}_1)^{-\frac{1}{2}} (1 + c_s k_1^* \tilde{y}_1)(1 + c_s k_2^* \tilde{y}_1) e^{-c_s \tilde{y}_1 (k_1^* + k_2^*)} \\
&\times K_{i\mu}(k_{34}^* \tilde{y}_1)(y_1)^{-\frac{1}{2}} (1 + c_s k_3^* y_1)(1 + c_s k_4^* y_1) e^{-c_s y_1 (k_3^* + k_4^*)} K_{i\mu}(k_{34}^* y_1) \Bigg] \\
\mathcal{I}_2 =& - \mathrm{Re} \Bigg[\int_0^{+\infty} dy_1 \int_{y_1}^{+\infty} dy_2 (y_1)^{-\frac{1}{2}} (1 + c_s k_1^* y_1) \\
&\times (1 + c_s k_2^* y_1) e^{-c_s y_1 (k_1^* + k_2^*)} e^{-\frac{\pi\mu}{2}} H_{i\mu}^{(1)}(-i k_{34}^* y_1)(y_2)^{-\frac{1}{2}} (1 + c_s k_3^* y_2) \\
&\times (1 + c_s k_4^* y_2) e^{-c_s y_2 (k_3^* + k_4^*)} K_{i\mu}(k_{34}^* y_2) \Bigg]
\end{aligned} \tag{4.12}$$

the contribution of the scalar exchange diagram to the trispectrum can be expressed as

$$\begin{aligned}
\mathcal{T}_{SE} &= \frac{\pi^6}{\Xi^3} \frac{1}{c_{s2} k_1} \left(\frac{\lambda}{f}\right)^2 \Lambda^2 \left[(\mathbf{k_1} \cdot \mathbf{k_2})(\mathbf{k_3} \cdot \mathbf{k_4}) \left(\frac{1}{\pi} \mathcal{I}_1 + \mathcal{I}_2 \right) + 23 \text{ permutations of } \mathbf{k_i} \right] \\
&= C_{SE} \frac{1}{k_1} \left[(\mathbf{k_1} \cdot \mathbf{k_2})(\mathbf{k_3} \cdot \mathbf{k_4}) \left(\frac{1}{\pi} \mathcal{I}_1 + \mathcal{I}_2 \right) + 23 \text{ permutations of } \mathbf{k_i} \right]
\end{aligned} \tag{4.13}$$

4.2 Contact Interaction

Regarding the diagram represented in Figure (3.1b), the general formula for $n = 4$ is given by

$$\begin{aligned}
&\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1) \int_{\tau_0}^{\tilde{\tau}_1} d\tilde{\tau}_2 a(\tilde{\tau}_2) \int_{\tau_0}^{\tau} d\tau_1 a(\tau_1) \int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2) \langle H_I(\tilde{\tau}_2) H_I(\tilde{\tau}_1) Q_I(\tau) H_I(\tau_1) H_I(\tau_2) \rangle \\
&- 2\mathrm{Re} \Big[\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1) \int_{\tau_0}^{\tau} d\tau_1 a(\tau_1) \int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2) \int_{\tau_0}^{\tau_2} d\tau_3 a(\tau_3) \langle H_I(\tilde{\tau}_1) Q_I(\tau) H_I(\tau_1) H_I(\tau_2) H_I(\tau_3) \rangle \Big] \\
&+ 2\mathrm{Re} \Big[\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1) \int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2) \int_{\tau_0}^{\tau_2} d\tau_3 a(\tau_3) \int_{\tau_0}^{\tau_3} d\tau_4 a(\tau_4) \langle Q_I(\tau) H_I(\tau_1) H_I(\tau_2) H_I(\tau_3) H_I(\tau_4) \rangle \Big]
\end{aligned} \tag{4.14}$$

where, in this case, the interaction vertex H_4^{Int} can appear in one of the four H_I, while the remaining three H_I should be replaced by the H_2^{Int}. As a consequence, for each of the three lines in (4.14), we obtain four distinct contributions, arising from the four possible time-orderings of H_4^{Int}. Focusing on the first line, we can apply a useful simplification: the contribution featuring H_4^{Int} in the first position is exactly the complex conjugate of the one with H_4^{Int} in the fourth position (and a similar relation holds between the second and third positions). As a result, the total contribution from the first line can be rewritten as:

$$
\begin{aligned}
&2\mathrm{Re}\Big[\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tilde{\tau}_1} d\tilde{\tau}_2 a(\tilde{\tau}_2)\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2)\langle H_4^{Int}(\tilde{\tau}_2)H_2^{Int}(\tilde{\tau}_1)\delta\chi^4 H_2^{Int}(\tau_1)H_2^{Int}(\tau_2)\rangle \\
&+\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tilde{\tau}_1} d\tilde{\tau}_2 a(\tilde{\tau}_2)\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2)\langle H_2^{Int}(\tilde{\tau}_2)H_4^{Int}(\tilde{\tau}_1)\delta\chi^4 H_2^{Int}(\tau_1)H_2^{Int}(\tau_2)\rangle\Big]
\end{aligned}
\tag{4.15}
$$

For the sake of clarity, we report only the derivation for the first line of (4.15), relegating all the contributions from the remaining terms to Appendix (C.2). Substituting the expressions for H_2^{Int} and H_4^{Int} into the first line yields

$$
\begin{aligned}
&3!\cdot 2\mathrm{Re}\Big[27\Big(\frac{\lambda}{f}g\Big)^4 Q^6\int\frac{d^3p_1d^3p_2d^3p_3d^3p_4}{(2\pi)^3}\int d^3q\int d^3l\int d^3r\ \delta^3(\mathbf{p_1}+\mathbf{p_2}+\mathbf{p_3}+\mathbf{p_4}) \\
&\times\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a^3(\tilde{\tau}_1)\int_{\tau_0}^{\tilde{\tau}_1} d\tilde{\tau}_2 a^3(\tilde{\tau}_2)\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2) \\
&\times\langle\delta\chi'(p_1,\tilde{\tau}_2)\delta Q(p_2,\tilde{\tau}_2)\delta Q(p_3,\tilde{\tau}_2)\delta Q(p_4,\tilde{\tau}_2)\delta\chi'(q,\tilde{\tau}_1)\delta Q(-q,\tilde{\tau}_1)\delta\chi(k_1,\tau)\delta\chi(k_2,\tau) \\
&\times\delta\chi(k_3,\tau)\delta\chi(k_4,\tau)\delta\chi'(l,\tau_1)\delta Q(-l,\tau_1)\delta\chi'(r,\tau_2)\delta Q(-r,\tau_2)\rangle\Big]+23\text{ permutations of }\mathbf{k_i} \\
&=\frac{324}{(2\pi)^3}\Big(\frac{\lambda}{f}g\Big)^4 Q^6\ \mathrm{Re}\Big[u_{k_1}^*(\tau)u_{k_2}^*(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a^3(\tilde{\tau}_1)v_{k_2}^*(\tilde{\tau}_1)u'_{k_2}(\tilde{\tau}_1) \\
&\times\int_{\tau_0}^{\tilde{\tau}_1} d\tilde{\tau}_2 a^3(\tilde{\tau}_2)u'_{k_1}(\tilde{\tau}_2)v_{k_2}(\tilde{\tau}_2)v_{k_3}(\tilde{\tau}_2)v_{k_4}(\tilde{\tau}_2)\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)v_{k_3}^*(\tau_1)u_{k_3}^{*\prime}(\tau_1) \\
&\times\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)v_{k_4}^*(\tau_2)u_{k_4}^{*\prime}(\tau_2)\ \delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\Big]+23\text{ permutations of }\mathbf{k_i}
\end{aligned}
\tag{4.16}
$$

Here, the overall 3! factor acts as a symmetry factor, accounting for the 3! possible ways to contract the internal momenta of the δQ perturbations. By inserting the mode functions, the final contribution from the first line of (4.15) becomes

$$
\begin{aligned}
&-\frac{1}{(2\pi)^3}\frac{81}{64}\Big(\frac{\lambda}{f}g\Big)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\Big(\frac{2}{\pi}\Big)^2\frac{c_{s1}^2}{c_{s2}^7k_1^5} \\
&\times\mathrm{Re}\Big[\int_0^{+\infty} dy_1\int_{y_1}^{+\infty} dy_2(y_1)^{-\frac{1}{2}}K_{i\mu}(k_3^*y_1)e^{-c_sk_3^*y_1}(y_2)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_2)e^{-c_sk_4^*y_2} \\
&\times\int_0^{+\infty} d\tilde{y}_1\int_{\tilde{y}_1}^{+\infty} d\tilde{y}_2(\tilde{y}_1)^{-\frac{1}{2}}e^{-\frac{\pi\mu}{2}}H_{-i\mu}^{(2)}(ik_2^*\tilde{y}_1)e^{-c_sk_2^*\tilde{y}_1}(\tilde{y}_2)^{\frac{5}{2}}K_{i\mu}(k_2^*\tilde{y}_2)K_{i\mu}(k_3^*\tilde{y}_2)K_{i\mu}(k_4^*\tilde{y}_2)e^{-c_sk_1^*\tilde{y}_2}\Big] \\
&+23\text{ permutations of }\mathbf{k_i}
\end{aligned}
\tag{4.17}
$$

where we have applied the same change of variables introduced in the previous section.
Numerical analysis conducted via Mathematica indicates that the dominant contribution actually arises from the third line of (4.14); therefore, we will focus our attention on this specific line, which as said provides four terms. Due to the structure of the In-in formalism, each of these terms involves a nested integral over four time domains. Because the integrands are highly oscillatory, we found that numerical stability is only achieved for the contribution where H_4^{Int} appears in the first position. Consequently, to provide an estimate that is partial but still highly indicative of the underlying physics, we have decided to retain exclusively this specific term to represent the contact interaction diagram in the remainder of this work. Thus, the final contribution takes the form

$$\begin{aligned}\langle\delta\chi_{k_1}\delta\chi_{k_2}\delta\chi_{k_3}\delta\chi_{k_4}\rangle_{CI} = &-\frac{1}{(2\pi)^3}\frac{81}{64}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\ \mathrm{Re}&\left[\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2\int_{y_2}^{+\infty}dy_3\int_{y_3}^{+\infty}dy_4\right.\\ &\times(y_1)^{\frac{5}{2}}e^{-\frac{3\pi\mu}{2}}H_{i\mu}^{(1)}(-ik_2^*y_1)H_{i\mu}^{(1)}(-ik_3^*y_1)H_{i\mu}^{(1)}(-ik_4^*y_1)e^{-c_sk_1^*y_1}\\ &\times(y_2)^{-\frac{1}{2}}K_{i\mu}(k_2^*y_2)e^{-c_sk_2^*y_2}(y_3)^{-\frac{1}{2}}K_{i\mu}(k_3^*y_3)e^{-c_sk_3^*y_3}\\ &\left.\times(y_4)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_4)e^{-c_sk_4^*y_4}\right]\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\\ &+23\text{ permutations of }\mathbf{k_i}\end{aligned} \tag{4.18}$$

where the subscript CI stands for Contact Interaction. If we define

$$\begin{aligned}\mathcal{I}_3 = -\,\mathrm{Re}&\left[\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2\int_{y_2}^{+\infty}dy_3\int_{y_3}^{+\infty}dy_4\;(y_1)^{\frac{5}{2}}e^{-\frac{3\pi\mu}{2}}H_{i\mu}^{(1)}(-ik_2^*y_1)\right.\\ &\times H_{i\mu}^{(1)}(-ik_3^*y_1)H_{i\mu}^{(1)}(-ik_4^*y_1)e^{-c_sk_1^*y_1}(y_2)^{-\frac{1}{2}}K_{i\mu}(k_2^*y_2)e^{-c_sk_2^*y_2}(y_3)^{-\frac{1}{2}}K_{i\mu}(k_3^*y_3)e^{-c_sk_3^*y_3}\\ &\left.\times(y_4)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_4)e^{-c_sk_4^*y_4}\right]\end{aligned} \tag{4.19}$$

the corresponding contribution from the contact interaction diagram reads

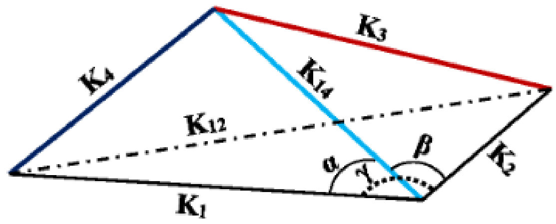


Figure 4.1: Schematic representation of the closed tetrahedron formed by the momentum vectors $\mathbf{k}_i$ [59].

$$\begin{aligned}\mathcal{T}_{CI} &= \frac{\pi^6}{\Xi^3}\frac{81c_{s1}^5}{c_{s2}^7 k_1^5}\left(\frac{\lambda}{f}\right)^2\Lambda^2 m_Q^4\prod_{i=1}^{4}k_i^2\Big(\mathcal{I}_3 + 23\text{ permutations of }\mathbf{k_i}\Big)\\ &= C_{CI}\frac{1}{k_1^5}\prod_{i=1}^{4}k_i^2\Big(\mathcal{I}_3 + 23\text{ permutations of }\mathbf{k_i}\Big)\end{aligned} \tag{4.20}$$

4.3 Plots

In this section, we analyze the momentum dependence of the trispectrum by studying the shape functions of $\mathcal{T}_{SE}$ and $\mathcal{T}_{CI}$. Before discussing specific momentum configurations, it is useful to recall that these shape functions depend on six independent momentum variables, namely $k_1, k_2, k_3, k_4, k_4, k_{12}$ and k_{14}. These quantities must satisfy a set of geometrical constraints in order to define a physically admissible tetrahedral configuration in momentum space, as illustrated in Figure (4.1). The first requirement is related to the geometry at a given vertex of the tetrahedron. Following the convention of [58], we can introduce the three angles

$$\begin{aligned}\cos(\alpha) &= \frac{k_1^2 + k_{14}^2 - k_4^2}{2k_1 k_{14}}\\ \cos(\beta) &= \frac{k_2^2 + k_{14}^2 - k_3^2}{2k_2 k_{14}}\\ \cos(\gamma) &= \frac{k_1^2 + k_2^2 - k_{12}^2}{2k_1 k_2}\end{aligned} \tag{4.21}$$

One finds that they must satisfy the condition $\cos(\alpha-\beta) \geq \cos(\gamma) \geq \cos(\alpha+\beta)$, which is equivalent to

$$1 - \cos^2(\alpha) - \cos^2(\beta) - \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) \geq 0 \tag{4.22}$$

A second set of constraints arises from the requirement that all momentum vectors form valid triangles. As a consequence, the momenta must satisfy the corresponding triangle inequalities

$$\begin{aligned} &k_1 + k_4 > k_{14}, \qquad k_1 + k_2 > k_{12}, \qquad k_2 + k_3 > k_{14} \\ &k_1 + k_{14} > k_4, \qquad k_1 + k_{12} > k_2, \qquad k_2 + k_{14} > k_3 \\ &k_4 + k_{14} > k_1, \qquad k_2 + k_{12} > k_1, \qquad k_3 + k_{14} > k_2 \end{aligned} \tag{4.23}$$

It is worth noting that the final inequality involving k_3, k_4 and k_{12} is automatically fulfilled once (4.22) and (4.23) are imposed. One may further introduce the relation

$$k_{13} \equiv |\mathbf{k}_1 + \mathbf{k}_3| = \sqrt{k_1^2 + k_2^2 + k_3^2 + k_4^2 - k_{12}^2 - k_{14}^2}. \tag{4.24}$$

which provide a convenient parametrization for k_{13}.
Having established the allowed geometric domain, now we can proceed to the analysis of the shape functions.
Unlike the bispectrum (where the involvement of only three momenta simplifies the analysis, as its momentum dependence is fully captured by the shape of a closed triangle) the trispectrum relies on the geometry of a quadrilateral in momentum space. Thus, characterizing the four-point function is intrinsically more complex: while a triangle is uniquely determined by three parameters, defining a closed quadrilateral requires six independent variables, as previously said. This higher dimensionality leads to a richer variety of possible momentum shapes. As a result, unlike the bispectrum which enjoys standardized parametrizations, the study of the trispectrum in the literature is characterized by several different conventions and shape classifications. In this work, we choose to focus our analysis specifically on the equilateral configuration of the trispectrum. This decision is motivated by the fact that in similar inflationary models (specifically those featuring an Abelian gauge field U(1)) primordial non-Gaussianities are often maximized in equilateral configurations (as stated in [60, 40] for the bispectrum and in [51] for the trispectrum).
Indeed, to characterize the amplitude of the non-Gaussian signal associated with each equilateral trispectrum component, following the approach in [58], we introduce the estimator $t_{NL}^{\delta\chi}$[3]

[3]While [58] defines the parameter t_{NL} in association with the curvature perturbation ζ, in our case it is formulated for the scalar field χ. This distinction introduces some differences; most notably, our parameter is not dimensionless, but rather carries dimensions of M_{pl}^{-2}.

$$\langle \delta\chi_{k_1}\delta\chi_{k_2}\delta\chi_{k_3}\delta\chi_{k_4}\rangle_{\text{component}} \xrightarrow[\text{limit}]{\text{RT}} (2\pi)^{-3}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\mathcal{P}^3_{\delta\chi}\frac{1}{k^9}\, t^{\delta\chi}_{NL} \quad (4.25)$$

where the subscript RT denotes the evaluation in the regular tetrahedron configuration $k_1 = k_2 = k_3 = k_4 = k_{12} = k_{14} \equiv k$. In general, the parameter t_{NL} plays a role analogous to that of the non-linearity parameter f_{NL} commonly used in bispectrum analyses. Unlike the bispectrum case, however, where the normalization of f_{NL} is conventionally fixed with reference to the local template, here we adopt a normalization consistent with (4.25).
Applying (4.25) to (4.2) allows us to determine the contribution for $t^{\delta\chi}_{NL}$, which are

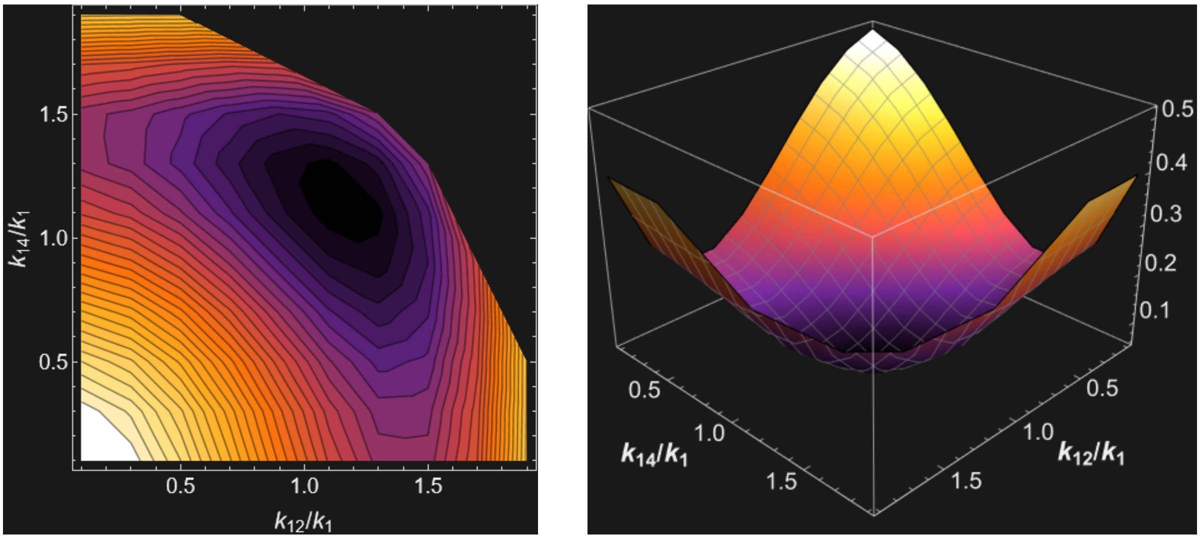
$$t^{\delta\chi}_{NL,SE} = 0.138\ C_{SE}, \quad t^{\delta\chi}_{NL,CI} = 1.47 \cdot 10^{-6}\ C_{CI} \quad (4.26)$$

To compare the two parameters, we can evaluate their ratio

$$\frac{t^{\delta\chi}_{NL,SE}}{t^{\delta\chi}_{NL,CI}} \simeq 13 \quad (4.27)$$

Thereby, it follows that, in the regular tetrahedron configuration, the dominant contribution to the non-Gaussianity of the axion trispectrum arises from the scalar exchange component, which exceeds the contact interaction contribution by approximately one order of magnitude.
We now proceed to visualize the shape functions. Because plotting the full six-dimensional parameter space is impossible, we restrict our analysis to representative limits that reduce the dimensionality of the problem. In all plots, the shape function is set to zero whenever the chosen momenta fail to form a closed tetrahedron. In particular, we examine two limits. The first limit is called equilateral configuration [61, 62, 58], where all external momenta have the same magnitude, so we have $k_1 = k_2 = k_3 = k_4 = k$. In Figure (4.2) we have plotted $\mathcal{T}_{SE}/C_{SE}$ as functions of k_{12}/k_1 and k_{14}/k_1. We did not generate an analogous plot for $\mathcal{T}_{CI}$ since it is independent of both k_{12} and k_{14}. Hence, plotting this quantity would have yielded a constant value across all configurations of k_{12}/k_1 and k_{14}/k_1 that form a closed tetrahedron. From the analysis of (4.2b), it is evident that the interaction amplitude exhibits a minimum, precisely for $k_{12}/k_1 \simeq 1.16$ and $k_{14}/k_1 \simeq 1.16$. These values are not casual; rather, they correspond to the geometric configuration of maximal symmetry. Indeed, by setting the internal diagonals k_{12}, k_{13}, and k_{14} equal to each other, and recalling that $k_1 = k_2 = k_3 = k_4 = k$, we obtain $k_{12} = k_{13} = k_{14} = \sqrt{4/3}\ k \simeq 1.16\ k$.

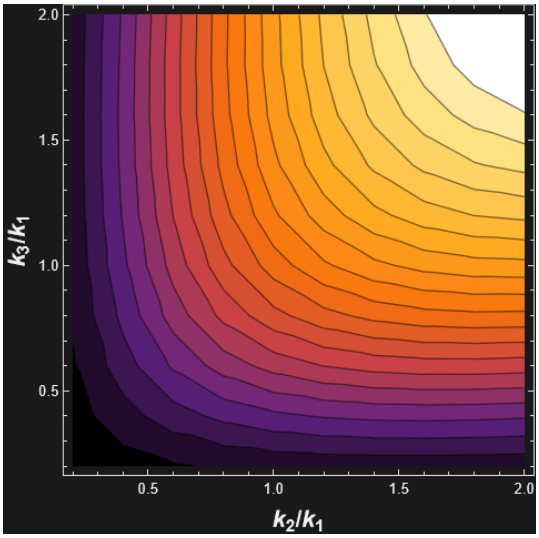
This result demonstrates that the scalar exchange contribution to the trispectrum does not generate an equilateral-type non-Gaussianity. Furthermore, having established in (4.27) that the scalar exchange provides the dominant contribution in the regular tetrahedron limit, we can conclude that the trispectrum under consideration is suppressed in this limit. In contrast, the signal grows significantly as one moves away from the center, forming peaks near the boundaries of the kinematic space. Specifically, a sharp enhancement emerges in the limit where $k_{12} \to 0$ and $k_{14} \to 0$. We can therefore conclude that the dominant non-Gaussian signal is not of the regular tetrahedron type, and that a natural next step would be to investigate the local configuration, which is best suited for capturing the physics of these limits.



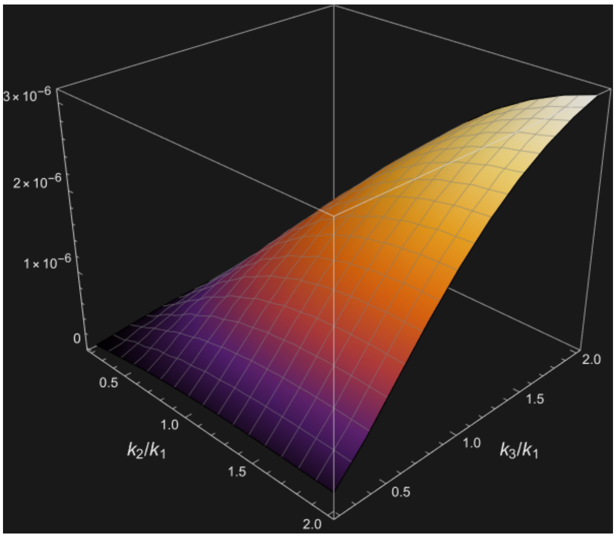
(a) Density plot (b) Shape

Figure 4.2: Plots for the scalar exchange contribution in the equilateral limit. Left: Density plot of the shape of the scalar exchange trispectrum as a function of k_{12}/k_1 and k_{14}/k_1. Right: The shape $\mathcal{T}_{SE}/C_{SE}$ of the scalar exchange trispectrum as a function of k_{12}/k_1 and k_{14}/k_1.

In addition, for $\mathcal{T}_{CI}$, we have chosen a second configuration, which is obtained by setting $k_1 = k_4$. This limit is similar to what the authors of [61, 62, 58] refer to as the specialized planar limit, with one key distinction: since the contact interaction contribution does not depend on the internal diagonals k_{12} and k_{14}, but only on the magnitude of k_i, the plotted figure remains valid for any 3D spatial configuration. The shape function $\mathcal{T}_{CI}/C_{CI}$ is shown in Figure (4.3) as a function of k_2/k_1 and k_3/k_1. Regarding this specific shape, one can observe that the amplitude of the interaction vanishes in the limits where $k_2 \to 0$ or $k_3 \to 0$, as expected. Moreover, unlike the previous case, this shape does not exhibit any valleys; instead, it displays a smooth and monotonic growth towards configurations where $k_2/k_1 \to 2$ and $k_3/k_1 \to 2$.



(a) Density plot



(b) Shape

Figure 4.3: Plots for the contact interaction contribution in the limit $k_1 = k_4$. Left: Density plot of the shape of the contact interaction trispectrum as a function of k_2/k_1 and k_3/k_1. Right: The shape $\mathcal{T}_{CI}/C_{CI}$ of the contact interaction trispectrum as a function of k_2/k_1 and k_3/k_1.

Chapter 5

Conclusions

In this thesis, to provide the necessary theoretical background, we first presented the standard Hot Big Bang cosmology and discussed the inflationary paradigm as a solution to its main shortcomings. We then introduced the framework of single-field slow-roll Inflation and the generation of primordial quantum fluctuations. Next, we reviewed and analyzed the Chromo Natural Inflation model along with its perturbations, subsequently turning our attention to the study of the central focus of this work: Spectator Chromo Natural Inflation (SCNI). The analysis provided the basis for the original contribution of this thesis, namely the investigation of whether parity-violating signatures can arise in SCNI in purely scalar correlation functions. Since parity-odd features cannot appear in scalar two- or three-point statistics, our attention focused on the scalar trispectrum, namely the Fourier transform of the connected four-point correlation function. To this end, we identified the relevant interaction diagrams and derived the perturbative expansion of the axion-gauge sector of the SCNI action up to fourth order in the fluctuations. After introducing the In-in formalism, we derived the free and interaction Hamiltonians, H_0 and H_I, respectively. The former was used to obtain the equations of motion and mode functions for the perturbations, while the latter provided the interaction vertices entering the computation of the axion trispectrum. Particular care was devoted to the treatment of the non-dynamical mode in the gauge sector, whose elimination allowed us to derive the effective interaction terms required for a consistent treatment of the diagrams shown in Figure (3.1). We then evaluated the contributions associated with the scalar exchange and contact interaction diagrams and analyzed their properties in the equilateral momentum configuration. Numerical computations were used to generate plots of the corresponding shape functions across two distinct kinematic limits and estimate the non-linearity parameter $t_{NL}^{\delta\chi}$, which corresponds to the overall amplitude of the generated non-Gaussian signal in the regular tetrahedron limit.

Throughout the analysis, several approximations were adopted. In particular, the contribution of the gauge field scalar perturbation M to the diagrams was neglected, as its dynamics is dominated by the effective mass term in the regime of interest, leading to a oscillatory suppressed contribution to the relevant time integrals. Furthermore, for the bilinear interaction, only the term associated with the largest coupling at horizon crossing was retained (since we expect the massive mode to survive, before decaying, for a few e-folds after horizon crossing), and, in the numerical study of the contact interaction contribution, a single term out of the complete set was considered because of numerical instabilities; nevertheless, we expect this specific component to be qualitatively representative of the full physical result. While these approximations may affect the quantitative value of the resulting trispectrum, they are not expected to alter the qualitative conclusions reached in this work, including those regarding parity violation.

The main results of this thesis are two. First, we demonstrated that the purely scalar sector of Spectator Chromo Natural Inflation does not generate parity-violating signatures at the level of the axion trispectrum. More precisely, the correlation functions computed in this work do not contain the pseudoscalar momentum structures required to produce parity-odd contributions. This result is particularly interesting because it demonstrates that the parity violation introduced by the Chern-Simons interaction does not automatically propagate to all sectors of the theory. Rather, its observational consequences depend crucially on the nature of the degrees of freedom involved in the correlation function under consideration. The second result concerns the amplitude of the scalar trispectrum itself. In the regular tetrahedron limit studied in this work, the corresponding trispectrum under consideration is found to be suppressed with respect to other configurations.

These findings naturally suggest several directions for future investigation. The first one would be to extend the analysis beyond the configurations considered so far, e.g. investigating the trispectrum across alternative kinematic configurations such as the local shape, and analyzing its behavior in the collapsed limit $k_{12} \to 0$ (especially in light of the features suggested by the numerical results presented in the plot (4.2b)). A second natural extension would be to perform a fully systematic calculation by relaxing the approximations adopted here. This would include incorporating the scalar mode M, accounting for all relevant interaction vertices, evaluating the complete set of contact interaction contributions and and considering additional diagrams predicted by the model. Such an analysis would provide a more accurate determination of the amplitude and shape of the scalar trispectrum.

Finally, although our results indicate that parity violation is absent in the purely scalar sector of the theory, they do not exclude the possibility that parity-odd signatures may arise in scalar observables through interactions with tensor degrees of freedom. Given that only the left-handed gauge field tensor mode t_L experiences a substantial tachyonic enhancement, it is plausible that parity-violating contributions could be transferred to the axion trispectrum through mixed scalar-tensor interactions. A comprehensive study of all interaction channels involving $\delta\chi$ and the enhanced tensor mode t_L therefore represents a particularly promising direction for future work. Such an investigation could clarify whether the parity violation characteristic of the tensor sector can ultimately leave observable imprints in scalar higher-order correlation functions.

Appendices

Appendix A

A.1 Computation of the Stress-energy Tensor

In order to obtain the stress-energy tensor $T^{\varphi}_{\mu\nu}$ for the scalar field $\varphi(\mathbf{x}, t)$, we can use its general definition $T^{\varphi}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\varphi}}{\delta g^{\mu\nu}}$. Being $S_{\varphi} = \int d^4x\sqrt{-g}\mathcal{L}_{\varphi}$ and $\mathcal{L}_{\varphi} = -\frac{1}{2}g^{\mu\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi - V(\varphi)$, we obtain

$$T^{\varphi}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}\mathcal{L}_{\varphi})}{\partial g^{\mu\nu}}\right] = -2\frac{\partial\mathcal{L}_{\varphi}}{\partial g^{\mu\nu}} - \frac{2}{\sqrt{-g}}\mathcal{L}_{\varphi}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -2\frac{\partial\mathcal{L}_{\varphi}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\varphi} = \partial_{\mu}\varphi\partial_{\nu}\varphi + g_{\mu\nu}\mathcal{L}_{\varphi} \tag{A.1}$$

where we have used

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}\ g_{\mu\nu} \tag{A.2}$$

The last relation can be derived from the general matrix relation $Tr(\log M) = \log(\det M)$, where we can take $M = g^{\mu\nu}$; in particular, we get

$$Tr(\log\ g^{\mu\nu}) = \log(\det g^{\mu\nu}) \tag{A.3}$$

Then, using the logarithmic derivative expression $\delta\log M = M^{-1}\delta M$, the fact that the trace is linear and being $g = \det g_{\mu\nu} \Rightarrow \frac{1}{g} = \det g^{\mu\nu}$, we can write

$$\begin{gathered} Tr\ (\delta\log\ g^{\mu\nu}) = Tr\ (g_{\mu\nu}\delta g^{\nu\sigma}) = \delta\ \log\ (\det g^{\mu\nu}) = \frac{\delta\det g^{\mu\nu}}{\det g_{\mu\nu}} = \frac{\delta(1/g)}{1/g} = -\frac{\delta g}{g}, \\ \Rightarrow Tr\ (g_{\mu\nu}\delta g^{\nu\sigma}) = g_{\mu\nu}\delta g^{\mu\nu} = -\frac{\delta g}{g} \Rightarrow \delta g = -g\ g_{\mu\nu}\delta g^{\mu\nu} \end{gathered} \tag{A.4}$$

Let us now consider a variation of $\sqrt{-g}$:

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = +\frac{1}{2}\frac{1}{\sqrt{-g}}g\ g_{\mu\nu}\delta g^{\mu\nu} \tag{A.5}$$

This yields

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = +\frac{1}{2}\frac{1}{\sqrt{-g}}g\ g_{\mu\nu} = -\frac{\sqrt{-g}}{2}g_{\mu\nu} \tag{A.6}$$

where we used the fact that $\frac{g}{\sqrt{-g}} = -\sqrt{-g}$, which is the correct solution of $(g/\sqrt{-g})^2 = -g$ because we know g has to be negative.

A.2 Klein-Gordon Equation for a Scalar Field in a Curved Spacetime

From equation (1.54) we can obtain the the equation of motion for φ. Indeed, we can write

$$\begin{aligned}
\frac{\delta S_\varphi}{\delta\varphi} &= -\int d^4x\ \delta\left[\sqrt{-g}\ V(\varphi)\right] + \frac{1}{2}\delta\left[\sqrt{-g}\ g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right] \\
&= -\int d^4x\sqrt{-g}\ \frac{\partial V(\varphi)}{\partial\varphi}\delta\varphi + \sqrt{-g}\ g^{\mu\nu}\delta\left(\partial_\mu\varphi\right)\partial_\nu\varphi \\
&= -\int d^4x\sqrt{-g}\ \frac{\partial V(\varphi)}{\partial\varphi}\delta\varphi + \partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\delta\varphi\partial_\nu\varphi\right) - \partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\varphi\right)\delta\varphi \\
&= -\int d^4x\left[\sqrt{-g}\ \frac{\partial V(\varphi)}{\partial\varphi} - \partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\varphi\right)\right]\delta\varphi = 0
\end{aligned} \tag{A.7}$$

where we put $\partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\delta\varphi\partial_\nu\varphi\right) = 0$ because it is a boundary term. So, from (A.7) we obtain (considering $\delta\varphi \neq 0$)

$$\sqrt{-g}\ \frac{\partial V(\varphi)}{\partial\varphi} - \partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\varphi\right) = 0 \Rightarrow \frac{\partial V(\varphi)}{\partial\varphi} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\varphi\right) = \Box\varphi \tag{A.8}$$

where $\Box\varphi$ is the D'Alembertian operator applied to φ, and in a curved spacetime reads

$$
\begin{aligned}
\Box\varphi =& \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\varphi\right) = \frac{1}{a^3}\partial_0\left(g^{00}a^3\partial_0\varphi\right) + \frac{1}{a^3}\partial_i\left(g^{ii}a^3\partial_i\varphi\right) \\
=& \frac{1}{a^3}\partial_0\left(-a^3\partial_0\varphi\right) + \frac{1}{a^3}\partial_i\left(a\ \partial_i\varphi\right) = -\ddot{\varphi} - \frac{3a^2\dot{a}\dot{\varphi}}{a^3} + \frac{\nabla^2\varphi}{a^2} = -\ddot{\varphi} - 3H\dot{\varphi} + \frac{\nabla^2\varphi}{a^2}
\end{aligned}
\tag{A.9}
$$

At the end, we can write $\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2\varphi}{a^2} = -\frac{\partial V(\varphi)}{\partial\varphi}$, which is exactly equal to (1.55).

Appendix B

B.1 Expansion of the Action Terms

We want to write explicitly the Chern-Simons term and the kinetic term for the gauge field from the action (2.42). Starting from the Chern-Simons term we have

$$\mathcal{S}^{CS} = \int d^4x \, \frac{\lambda}{8f}\chi \, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \int d^4x \, \frac{\lambda}{8f}\chi \, \epsilon^{\mu\nu\rho\sigma}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$
$$- g\epsilon^{abc} A^b_\mu A^c_\nu)(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho - g\epsilon^{abc} A^b_\rho A^c_\sigma) = \int d^4x \, \frac{\lambda}{8f}\chi \, \epsilon^{\mu\nu\rho\sigma}\Big[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho)$$
$$- g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\epsilon^{ade} A^d_\rho A^e_\sigma) - g(\epsilon^{abc} A^b_\mu A^c_\nu)(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho) + g^2(\epsilon^{abc} A^b_\mu A^c_\nu)(\epsilon^{ade} A^d_\rho A^e_\sigma)\Big]$$
$$= \int d^4x \, \frac{\lambda}{8f}\chi \, \epsilon^{\mu\nu\rho\sigma}\Big[(\partial_\mu A^a_\nu \partial_\rho A^a_\sigma - \partial_\mu A^a_\nu \partial_\sigma A^a_\rho - \partial_\nu A^a_\mu \partial_\rho A^a_\sigma + \partial_\nu A^a_\mu \partial_\sigma A^a_\rho)$$
$$- g\epsilon^{ade}(\partial_\mu A^a_\nu A^d_\rho A^e_\sigma - \partial_\nu A^a_\mu A^d_\rho A^e_\sigma) - g\epsilon^{abc}(A^b_\mu A^c_\nu \partial_\rho A^a_\sigma - A^b_\mu A^c_\nu \partial_\sigma A^a_\rho) + g^2 \epsilon^{abc}\epsilon^{ade} A^b_\mu A^c_\nu A^d_\rho A^e_\sigma\Big]$$
$$= \int d^4x \, \frac{\lambda}{8f}\chi \, \epsilon^{\mu\nu\rho\sigma}\Big[4(\partial_\mu A^a_\nu \partial_\rho A^a_\sigma) - 2g\epsilon^{ade}(\partial_\mu A^a_\nu A^d_\rho A^e_\sigma) - 2g\epsilon^{abc}(A^b_\mu A^c_\nu \partial_\rho A^a_\sigma)$$
$$+ g^2 \epsilon^{abc}\epsilon^{ade} A^b_\mu A^c_\nu A^d_\rho A^e_\sigma\Big] \tag{B.1}$$

Then, for the first term of the last expression we can write

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^a_\nu \partial_\rho A^a_\sigma = \epsilon^{\mu\nu\rho\sigma}\partial_\mu(A^a_\nu \partial_\rho A^a_\sigma) - \epsilon^{\mu\nu\rho\sigma} A^a_\nu \partial_\mu \partial_\rho A^a_\sigma = \epsilon^{\mu\nu\rho\sigma}\partial_\mu(A^a_\nu \partial_\rho A^a_\sigma) \tag{B.2}$$

thanks to the property of the anti-symmetric tensor. Instead, for the second term we have

$$
\begin{aligned}
&- 2g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{ade}\partial_\mu(A^a_\nu A^d_\rho A^e_\sigma) = -2g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{ade}[\partial_\mu A^a_\nu A^d_\rho A^e_\sigma + A^a_\nu \partial_\mu A^d_\rho A^e_\sigma + A^a_\nu A^d_\rho \partial_\mu A^e_\sigma] \\
&= -6g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{ade}\partial_\mu A^a_\nu A^d_\rho A^e_\sigma
\end{aligned}
\tag{B.3}
$$

so we can write $-2g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{ade}\partial_\mu A^a_\nu A^d_\rho A^e_\sigma = -\frac{2}{3}g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{ade}\partial_\mu(A^a_\nu A^d_\rho A^e_\sigma)$. Moreover, if we relabel the dummy indices of the third term and apply the symmetries of the Levi-Civita tensor, we can write $-2g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{abc}(A^b_\mu A^c_\nu \partial_\rho A^a_\sigma) = -2g\ \epsilon^{\rho\sigma\mu\nu}\epsilon^{ade}(A^d_\rho A^e_\sigma \partial_\mu A^a_\nu) = -2g\ \epsilon^{\mu\nu\rho\sigma}\epsilon^{ade}(A^d_\rho A^e_\sigma \partial_\mu A^a_\nu)$; this explicitly shows that the second and third terms are equivalent. Proceeding to the fourth term in (B.1), we find

$$
\begin{aligned}
g^2\epsilon^{\mu\nu\rho\sigma}\epsilon^{abc}\epsilon^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma &= g^2\epsilon^{\mu\nu\rho\sigma}(\delta^{bd}\delta^{ce} - \delta^{cd}\delta^{be})A^b_\mu A^c_\nu A^d_\rho A^e_\sigma \\
= g^2\epsilon^{\mu\nu\rho\sigma}(A^b_\mu A^c_\nu A^b_\rho A^c_\sigma - A^b_\mu A^c_\nu A^c_\rho A^b_\sigma) &= 2g^2\epsilon^{\mu\nu\rho\sigma}A^b_\mu A^c_\nu A^b_\rho A^c_\sigma = 0
\end{aligned}
\tag{B.4}
$$

owing to the property of the anti-symmetric tensor. So finally we can write

$$
\begin{aligned}
&\mathcal{S}^{CS} = \int d^4x\ \frac{\lambda}{8f}\chi\epsilon^{\mu\nu\rho\sigma}\left[4\partial_\mu(A^a_\nu\partial_\rho A^a_\sigma) - \frac{4}{3}g\epsilon^{abc}\partial_\mu(A^a_\nu A^b_\rho A^c_\sigma)\right] = \int d^4x\ \frac{\lambda}{2f}\chi\epsilon^{\mu\nu\rho\sigma}\partial_\mu\Big(A^a_\nu\partial_\rho A^a_\sigma \\
&- \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\Big) = \int d^4x\ \frac{\lambda}{2f}\partial_\mu\left[\chi\epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma - \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\right)\right] \\
&- \int d^4x\ \frac{\lambda}{2f}\partial_\mu\chi\ \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma - \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\right) \\
&= \int d^4x - \frac{\lambda}{2f}\partial_\mu\chi\ \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma - \frac{g}{3}\epsilon^{abc}A^a_\nu A^b_\rho A^c_\sigma\right)
\end{aligned}
\tag{B.5}
$$

where in the last step we have eliminated the total derivative.
Next, we expand the kinetic term of the gauge field from (2.42) to obtain

$$
\begin{aligned}
&\mathcal{S}^K = \int d^4x - \sqrt{-\tilde{g}}\ \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} = \int d^4x - \sqrt{-\tilde{g}}\ \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \\
&= \int d^4x - \sqrt{\tilde{g}}\ \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}\Big[(\partial_\mu A^a_\nu\partial_\rho A^a_\sigma - \partial_\mu A^a_\nu\partial_\sigma A^a_\rho - \partial_\nu A^a_\mu\partial_\rho A^a_\sigma + \partial_\nu A^a_\mu\partial_\sigma A^a_\rho) \\
&- 2g\epsilon^{ade}(\partial_\mu A^a_\nu A^d_\rho A^e_\sigma - \partial_\nu A^a_\mu A^d_\rho A^e_\sigma) + g^2\epsilon^{abc}\epsilon^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma\Big]
\end{aligned}
\tag{B.6}
$$

Looking at the first term in the last expression, we can write

$$
\begin{aligned}
&-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}\Big[(\partial_\mu A^a_\nu\partial_\rho A^a_\sigma-\partial_\mu A^a_\nu\partial_\sigma A^a_\rho-\partial_\nu A^a_\mu\partial_\rho A^a_\sigma+\partial_\nu A^a_\mu\partial_\sigma A^a_\rho)\Big]=-\frac{1}{4}\Big\{g^{00}g^{00}\Big[(\partial_0A^a_0\partial_0A^a_0\\
&-\partial_0A^a_0\partial_0A^a_0-\partial_0A^a_0\partial_0A^a_0+\partial_0A^a_0\partial_0A^a_0)\Big]+2g^{00}g^{ij}\Big[(\partial_0A^a_i\partial_0A^a_j-\partial_0A^a_i\partial_jA^a_0-\partial_iA^a_0\partial_0A^a_j\\
&+\partial_iA^a_0\partial_jA^a_0)\Big]+g^{ij}g^{kl}\Big[(\partial_iA^a_k\partial_jA^a_l-\partial_iA^a_k\partial_lA^a_j-\partial_kA^a_i\partial_jA^a_l+\partial_kA^a_i\partial_lA^a_j)\Big]\Big\}\\
&=-\frac{1}{4}\Big[0-2\frac{1}{a^2}\delta^{ij}(\partial_0A^a_i\partial_0A^a_j-\partial_0A^a_i\partial_jA^a_0-\partial_iA^a_0\partial_0A^a_j+\partial_iA^a_0\partial_jA^a_0)\\
&+\frac{1}{a^2}\delta^{ij}\frac{1}{a^2}\delta^{kl}(\partial_iA^a_k\partial_jA^a_l-\partial_iA^a_k\partial_lA^a_j-\partial_kA^a_i\partial_jA^a_l+\partial_kA^a_i\partial_lA^a_j)\Big]\\
&=-\frac{1}{4}\Big[-2\frac{1}{a^2}(\partial_0A^a_i\partial_0A^a_i-\partial_0A^a_i\partial_iA^a_0-\partial_iA^a_0\partial_0A^a_i+\partial_iA^a_0\partial_iA^a_0)+\frac{1}{a^4}(\partial_iA^a_k\partial_iA^a_k-\partial_iA^a_k\partial_kA^a_i\\
&-\partial_kA^a_i\partial_iA^a_k+\partial_kA^a_i\partial_kA^a_i)\Big]=\frac{1}{2a^2}(\partial_0A^a_i\partial_0A^a_i-2\partial_0A^a_i\partial_iA^a_0+\partial_iA^a_0\partial_iA^a_0)-\frac{1}{4a^4}(\partial_iA^a_k\partial_iA^a_k\\
&-\partial_iA^a_k\partial_kA^a_i-\partial_kA^a_i\partial_iA^a_k+\partial_kA^a_i\partial_kA^a_i)=\frac{1}{2a^2}(\partial_0A^a_i\partial_0A^a_i-2\partial_0A^a_i\partial_iA^a_0+\partial_iA^a_0\partial_iA^a_0)\\
&-\frac{1}{2a^4}(\partial_iA^a_k\partial_iA^a_k-\partial_iA^a_k\partial_kA^a_i)
\end{aligned}
\tag{B.7}
$$

For the second term we get

$$
\begin{aligned}
&\frac{1}{2}g^{\mu\rho}g^{\nu\sigma}\Big[g\epsilon^{ade}(\partial_\mu A^a_\nu A^d_\rho A^e_\sigma-\partial_\nu A^a_\mu A^d_\rho A^e_\sigma)\Big]=\frac{1}{2}\Big\{g^{00}g^{00}\Big[g\epsilon^{ade}(\partial_0A^a_0A^d_0A^e_0-\partial_0A^a_0A^d_0A^e_0)\Big]\\
&+g^{00}g^{ij}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_j-\partial_iA^a_0A^d_0A^e_j)\Big]+g^{ij}g^{00}\Big[g\epsilon^{ade}(\partial_iA^a_0A^d_jA^e_0-\partial_0A^a_iA^d_jA^e_0)\Big]\\
&+g^{ij}g^{kl}\Big[g\epsilon^{ade}(\partial_iA^a_kA^d_jA^e_l-\partial_kA^a_iA^d_jA^e_l)\Big]\Big\}=\frac{1}{2}\Big\{0-\frac{1}{a^2}\delta^{ij}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_j-\partial_iA^a_0A^d_0A^e_j)\Big]\\
&-\frac{1}{a^2}\delta^{ij}\Big[g\epsilon^{ade}(\partial_iA^a_0A^d_jA^e_0-\partial_0A^a_iA^d_jA^e_0)\Big]+\frac{1}{a^2}\delta^{ij}\frac{1}{a^2}\delta^{kl}\Big[g\epsilon^{ade}(\partial_iA^a_kA^d_jA^e_l-\partial_kA^a_iA^d_jA^e_l)\Big]\Big\}\\
&=\frac{1}{2}\Big\{-\frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_i-\partial_iA^a_0A^d_0A^e_i)\Big]-\frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_iA^a_0A^d_iA^e_0-\partial_0A^a_iA^d_iA^e_0)\Big]\\
&+\frac{1}{a^4}\Big[g\epsilon^{ade}(\partial_iA^a_kA^d_iA^e_k-\partial_kA^a_iA^d_iA^e_k)\Big]\Big\}=\frac{1}{2}\Big\{-\frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_i-\partial_iA^a_0A^d_0A^e_i)\Big]\\
&+\frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_iA^a_0A^e_iA^d_0-\partial_0A^a_iA^e_iA^d_0)\Big]+\frac{1}{a^4}\Big[g\epsilon^{ade}(\partial_iA^a_kA^d_iA^e_k+\partial_iA^a_kA^e_kA^d_i)\Big]\Big\}\\
&=-\frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_i-\partial_iA^a_0A^d_0A^e_i)\Big]+\frac{1}{a^4}\Big(g\epsilon^{ade}\partial_iA^a_kA^d_iA^e_k\Big)
\end{aligned}
\tag{B.8}
$$

Instead, for the last term we have

$$
\begin{aligned}
&-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}g^2\epsilon^{abc}\epsilon^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma = -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}g^2(\delta^{bd}\delta^{ce}-\delta^{be}\delta^{cd})A^b_\mu A^c_\nu A^d_\rho A^e_\sigma \\
&= -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}g^2(A^b_\mu A^c_\nu A^b_\rho A^c_\sigma - A^b_\mu A^c_\nu A^c_\rho A^b_\sigma) = -\frac{1}{4}g^2\Big[g^{00}g^{00}(A^b_0A^c_0A^b_0A^c_0 - A^b_0A^c_0A^c_0A^b_0) \\
&+ g^{00}g^{ij}(A^b_0A^c_iA^b_0A^c_j - A^b_0A^c_iA^c_0A^b_j) + g^{ij}g^{00}(A^b_iA^c_0A^b_jA^c_0 - A^b_iA^c_0A^c_jA^b_0) \\
&+ g^{ij}g^{kl}(A^b_iA^c_kA^b_jA^c_l - A^b_iA^c_kA^c_jA^b_l)\Big] = -\frac{1}{4}g^2\Big[-\frac{1}{a^2}\delta^{ij}(A^b_0A^c_iA^b_0A^c_j - A^b_0A^c_iA^c_0A^b_j) \\
&- \frac{1}{a^2}\delta^{ij}(A^b_iA^c_0A^b_jA^c_0 - A^b_iA^c_0A^c_jA^b_0) + \frac{1}{a^2}\delta^{ij}\frac{1}{a^2}\delta^{kl}(A^b_iA^c_kA^b_jA^c_l - A^b_iA^c_kA^c_jA^b_l)\Big] \\
&= -\frac{1}{4}g^2\Big[-\frac{1}{a^2}(A^b_0A^c_iA^b_0A^c_i - A^b_0A^c_iA^c_0A^b_i) - \frac{1}{a^2}(A^b_iA^c_0A^b_iA^c_0 - A^b_iA^c_0A^c_iA^b_0) \\
&+ \frac{1}{a^4}(A^b_iA^c_kA^b_iA^c_k - A^b_iA^c_kA^c_iA^b_k)\Big] = -\frac{1}{4}g^2\Big[-\frac{1}{a^2}(A^b_0A^c_iA^b_0A^c_i + A^b_iA^c_0A^b_iA^c_0 \\
&- 2A^b_0A^c_iA^c_0A^b_i) + \frac{1}{a^4}(A^b_iA^c_kA^b_iA^c_k - A^b_iA^c_kA^c_iA^b_k)\Big] \\
&= \frac{g^2}{2a^2}(A^b_0A^c_iA^b_0A^c_i - A^b_0A^c_iA^c_0A^b_i) - \frac{g^2}{4a^4}(A^b_iA^c_kA^b_iA^c_k - A^b_iA^c_kA^c_iA^b_k)
\end{aligned}
\tag{B.9}
$$

Therefore, the final expression for the gauge field kinetic term is given by

$$
\begin{aligned}
\mathcal{S}^K = &\int d^4x\,\sqrt{\tilde{g}}\Big\{\frac{1}{2a^2}(\partial_0A^a_i\partial_0A^a_i - 2\partial_0A^a_i\partial_iA^a_0 + \partial_iA^a_0\partial_iA^a_0) - \frac{1}{2a^4}(\partial_iA^a_k\partial_iA^a_k - \partial_iA^a_k\partial_kA^a_i) \\
&- \frac{1}{a^2}\Big[g\epsilon^{ade}(\partial_0A^a_iA^d_0A^e_i - \partial_iA^a_0A^d_0A^e_i)\Big] + \frac{1}{a^4}\Big(g\epsilon^{ade}\partial_iA^a_kA^d_iA^e_k\Big) + \frac{g^2}{2a^2}(A^b_0A^c_iA^b_0A^c_i - A^b_0A^c_iA^c_0A^b_i) \\
&- \frac{g^2}{4a^4}(A^b_iA^c_kA^b_iA^c_k - A^b_iA^c_kA^c_iA^b_k)\Big\}
\end{aligned}
\tag{B.10}
$$

B.2 Perturbed Hamiltonian and Non-dynamical Mode

In this section, we construct the Hamiltonian density $\mathcal{H}$; we start by computing the conjugate momentum densities $\delta\pi_i$, where i identifies one field among $\delta\chi, \delta Q, M$

$$\delta\pi_1 = \frac{\partial\mathcal{L}}{\partial\delta\dot{\chi}} = a^3\delta\dot{\chi} + \frac{\lambda}{f}ag(aQ)^2(3\delta Q + \Delta M) + \frac{\lambda}{f}a^2g(aQ)\Big[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2$$
$$-\frac{1}{2}(\partial_i\partial_j M)^2\Big] + \frac{\lambda}{2f}ga^3\Big[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2$$
$$+\frac{1}{3}(\Delta M)^3 + \frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2\Big] \tag{B.11}$$

$$\delta\pi_2 = \frac{\partial\mathcal{L}}{\partial\delta\dot{Q}} = 3a^3\delta\dot{Q} + a^3\Delta\dot{M} - a^3\Delta Y; \tag{B.12}$$

$$\delta\pi_3 = \frac{\partial\mathcal{L}}{\partial\dot{M}} - \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\dot{M})}\right) + \partial_i\partial_j\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j\dot{M})}\right)$$
$$= \partial_i\partial_j(\delta_i^a\delta_j^b\, a^3\delta\dot{Q} + a^3\delta_j^a\partial_i\partial_a\dot{M} - a^3\partial_i\partial_a Y\delta_j^a) = a^3\Delta\delta\dot{Q} + a^3\Delta^2\dot{M} - a^3\Delta^2 Y \tag{B.13}$$

Then, from these definitions, we obtain

$$\delta\dot{\chi} = \frac{1}{a^3}\delta\pi_1 - \frac{\lambda}{f}gQ^2(3\delta Q + \Delta M) - \frac{\lambda}{f}gQ\Big[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2\Big]$$
$$-\frac{\lambda}{2f}g\Big[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2 + \frac{1}{3}(\Delta M)^3$$
$$+\frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2\Big] \tag{B.14}$$

while for the fields $\delta\dot{Q}, \dot{M}$, after straightforward algebraic manipulations, we can write the following expression

$$\delta\dot{Q} = \frac{\delta\pi_2}{2a^3} - \frac{\delta\pi_3}{2a^3\Delta} \tag{B.15}$$

$$\dot{M} = \frac{3}{2a^3}\frac{\delta\pi_3}{\Delta^2} - \frac{1}{2a^3}\frac{\delta\pi_2}{\Delta} + Y; \tag{B.16}$$

Now we can work out the Hamiltonian density in terms of $\delta\pi_1, \delta\pi_2, \delta\pi_3$ and $\delta\chi, \delta Q, M$ and Y

$$
\begin{aligned}
\mathcal{H} = {} & \delta\pi_1\delta\dot{\chi} + \delta\pi_2\delta\dot{Q} + \delta\pi_3\dot{M} - \mathcal{L} = \frac{\delta\pi_1^2}{a^3} - \frac{\lambda}{f}gQ^2\delta\pi_1(3\delta Q + \Delta M) - \frac{\lambda}{f}gQ\delta\pi_1\Big[3\delta Q^2 \\
& +2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2\Big] - \frac{\lambda}{2f}g\delta\pi_1\Big[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 \\
& -\delta Q(\partial_c\partial_k M)^2 + \frac{1}{3}(\Delta M)^3 + \frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2\Big] \\
& + \frac{\delta\pi_2^2}{2a^3} - \frac{\delta\pi_2}{2a^3}\frac{\delta\pi_3}{\Delta} + \frac{3\delta\pi_3}{2a^3}\frac{\delta\pi_3}{\Delta^2} - \frac{\delta\pi_3}{2a^3}\frac{\delta\pi_2}{\Delta} + Y\delta\pi_3 + \frac{\lambda}{f}ag(aQ)^2\partial_i\delta\chi\partial_i Y \\
& - \frac{\lambda}{f}gQ^2\delta\pi_1(3\delta Q + \Delta M) + \left(\frac{\lambda}{f}gQ^2\right)^2 a^3(3\delta Q + \Delta M)^2 + \left(\frac{\lambda}{f}g\right)^2(aQ)^3[3\delta Q^2 + 2\delta Q\Delta M \\
& + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2](3\delta Q + \Delta M) + \frac{1}{2}\left(\frac{\lambda}{f}gQ\right)^2 a^3[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 \\
& - \delta Q(\partial_c\partial_k M)^2 + \frac{1}{3}(\Delta M)^3 + \frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2](3\delta Q + \Delta M) \\
& - \frac{1}{2a^3}\delta\pi_1^2 - \frac{a^3}{2}\Big\{\left(\frac{\lambda}{f}gQ^2\right)^2(3\delta Q + \Delta M)^2 + \left(\frac{\lambda}{f}gQ\right)^2[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 \\
& - \frac{1}{2}(\partial_i\partial_j M)^2]^2 - \frac{2}{a^3}\frac{\lambda}{f}gQ^3\delta\pi_1(3\delta Q + \Delta M) - \frac{2}{a^3}\frac{\lambda}{f}gQ\delta\pi_1[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 \\
& - \frac{1}{2}(\partial_i\partial_j M)^2] - \frac{1}{a^3}\frac{\lambda}{f}g\delta\pi_1[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2 + \frac{1}{3}(\Delta M)^3 \\
& + \frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2] + 2\left(\frac{\lambda}{f}g\right)^2 Q^3(3\delta Q + \Delta M)[3\delta Q^2 \\
& + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2] + Q^2\left(\frac{\lambda}{f}g\right)^2(3\delta Q + \Delta M)[2\delta Q^3 + 2\delta Q^2\Delta M \\
& + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2 + \frac{1}{3}(\Delta M)^3 + \frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2]\Big\} \\
& + \frac{1}{2}a(\partial_i\delta\chi)^2 + \frac{a^3}{2}U''(\bar{\chi})\delta\chi^2 - \frac{\lambda}{f}a^2g(aQ)\partial_0\bar{\chi}(3\delta Q^2 + 2\delta Q\Delta M) + a^3H^2(3\delta Q^2 \\
& + 2\delta Q\Delta M + \partial_i\partial_a M\partial_i\partial_a M) - \frac{3a^3}{2}\left(\frac{\delta\pi_2^2}{4a^6} + \frac{1}{4a^6}\frac{\delta\pi_3}{\Delta}\frac{\delta\pi_3}{\Delta} - \frac{\delta\pi_2}{2a^6}\frac{\delta\pi_3}{\Delta}\right) - a^3\Big(\frac{3\delta\pi_2}{4a^6}\frac{\delta\pi_3}{\Delta} - \frac{\delta\pi_2^2}{4a^6} \\
& + \frac{\delta\pi_2}{2a^3}\Delta Y - \frac{3}{4a^6}\frac{\delta\pi_3}{\Delta}\frac{\delta\pi_3}{\Delta} + \frac{\delta\pi_2}{4a^6}\frac{\delta\pi_3}{\Delta} - \frac{\Delta Y}{2a^3}\frac{\delta\pi_3}{\Delta}\Big) - \frac{a^3}{2}\Big(\frac{9\delta\pi_3}{4a^6}\frac{\delta\pi_3}{\Delta^2} - \frac{3\delta\pi_3}{4a^6}\frac{\delta\pi_2}{\Delta} + \frac{3}{2a^3}Y\delta\pi_3 \\
& -\frac{3\Delta\delta\pi_2}{4a^6}\frac{\delta\pi_3}{\Delta^2} + \frac{1}{4a^6}\Delta\delta\pi_2\frac{\delta\pi_2}{\Delta} - \frac{Y}{2a^3}\Delta\delta\pi_2 + \frac{3}{2a^3}\frac{\delta\pi_3}{\Delta^2}\Delta^2 Y - \frac{1}{2a^3}\frac{\delta\pi_2}{\Delta}\Delta^2 Y + \Delta^2 Y\ Y\Big) \\
& + a^3H\delta Q\Delta Y + \frac{\delta\pi_2}{2}\Delta Y - \frac{\delta\pi_3}{2\Delta}\Delta Y + a^3H\partial_i\partial_a M\partial_i\partial_a Y + \frac{3}{2}\frac{\delta\pi_3}{\Delta^2}\Delta^2 Y - \frac{1}{2}\frac{\delta\pi_2}{\Delta}\Delta^2 Y + a^3Y\Delta^2 Y \\
& - \frac{a^3}{2}(\partial_i\partial_a Y)^2 + a(\partial_i\delta Q)^2 - a^3g^2(aQ)^2\partial_b Y\partial_b Y + g^2a^3Q^2[9\delta Q^2 + 6\delta Q\Delta M + (\Delta M)^2] \\
& + 2\frac{\lambda}{f}a^2g(aQ)\partial_a\delta\chi\partial_a Y\delta Q + \frac{\lambda}{f}a^2g(aQ)(\partial_i\delta\chi\partial_i Y\Delta M - \partial_b\delta\chi\partial_j Y\partial_b\partial_j M)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda}{f}gQ\delta\pi_1\left[3\delta Q^2+2\delta Q\Delta M+\frac{1}{2}(\Delta M)^2-\frac{1}{2}(\partial_i\partial_j M)^2\right]\\
&+\left(\frac{\lambda}{f}gQ\right)^2 a^3Q(3\delta Q+\Delta M)\,[3\delta Q^2+2\delta Q\Delta M+\frac{1}{2}(\Delta M)^2-\frac{1}{2}(\partial_i\partial_j M)^2]\\
&+\left(\frac{\lambda}{f}gQ\right)^2 a^3\,[3\delta Q^2+2\delta Q\Delta M+\frac{1}{2}(\Delta M)^2-\frac{1}{2}(\partial_i\partial_j M)^2]^2\\
&-\frac{\lambda}{2f}a^3g\ \partial_0\bar{\chi}\{2\delta Q^3+2\delta Q^2\Delta M+\delta Q[(\Delta M)^2-(\partial_a\partial_c M)^2]\}\\
&-a^4g^2(aQ)(2\partial_bY\partial_bY\delta Q+\partial_bY\partial_bY\Delta M-\partial_bY\partial_cY\partial_b\partial_c M)\\
&+a^2g^2(aQ)[6\delta Q^3+6\delta Q^2\Delta M+2\delta Q(\Delta M)^2+\Delta M(\partial_c\partial_k M)^2-\partial_i\partial_k M\partial_c\partial_k M\partial_c\partial_i M]\\
&-\frac{\lambda}{2f}ga^3\partial_i\delta\chi[-2\partial_iY\delta Q^2-2(\partial_iY\delta Q\Delta M-\partial_aY\delta Q\partial_i\partial_a M)+\epsilon^{i0jk}\epsilon^{abc}\partial_aY\partial_j\partial_b M\partial_c\partial_k M]\\
&-\frac{\lambda}{2f}g\delta\pi_1\,[2\delta Q^3+2\delta Q^2\Delta M+\delta Q(\Delta M)^2-\delta Q(\partial_c\partial_k M)^2+\frac{1}{3}(\Delta M)^3\\
&+\frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M)-(\Delta M)(\partial_j\partial_k M)^2]\\
&+\left(\frac{\lambda}{f}gQ\right)^2\frac{a^3}{2}(3\delta Q+\Delta M)\,[2\delta Q^3+2\delta Q^2\Delta M+\delta Q(\Delta M)^2-\delta Q(\partial_c\partial_k M)^2+\frac{1}{3}(\Delta M)^3\\
&+\frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M)-(\Delta M)(\partial_j\partial_k M)^2]-\frac{1}{2}g^2a^5(2\partial_bY\partial_bY\delta Q^2+2\partial_bY\partial_bY\delta Q\Delta M\\
&+(\partial_bY\partial_i\partial_c M)^2-2\partial_bY\partial_cY\delta Q\partial_b\partial_c M-\partial_bY\partial_cY\partial_b\partial_i M\partial_i\partial_c M)\\
&+\frac{g^2a^3}{4}[6\delta Q^4+8\delta Q^3\Delta M+4\delta Q^2(\Delta M)^2+4\delta Q\Delta M(\partial_c\partial_k M)^2+(\partial_b\partial_i M)^2(\partial_c\partial_k M)^2\\
&-4\delta Q\partial_c\partial_k M\partial_c\partial_b M\partial_b\partial_k M-\partial_b\partial_i M\partial_k\partial_c M\partial_c\partial_i M\partial_b\partial_k M]+\frac{a^3}{6}U'''(\bar{\chi})\delta\chi^3+\frac{a^3}{24}U''''(\bar{\chi})\delta\chi^4
\end{aligned}
\tag{B.17}
$$

At this stage, as discussed in Section (3.2), our goal is to determine the explicit expression for the non-dynamical variable Y, so as to express the Hamiltonian solely in terms of the dynamical degrees of freedom. To this end, we begin by isolating the terms in the quadratic perturbed Hamiltonian density that are proportional to Y

$$
\begin{aligned}
\mathcal{H}_{2,Y}=&+\frac{1}{4}Y\delta\pi_3+\frac{3}{4}\Delta^2Y\frac{\delta\pi_3}{\Delta^2}+\frac{1}{4}Y\Delta\delta\pi_2-\frac{1}{4}\Delta^2Y\frac{\delta\pi_2}{\Delta}+\frac{a^3}{2}Y\Delta^2Y+\frac{\lambda}{f}gQ^2a^3\partial_i\delta\chi\partial_iY\\
&+a^3H\delta Q\Delta Y+a^3HM\Delta^2Y-\frac{a^3}{2}Y\Delta^2Y-a^3g^2(aQ)^2(\partial_iY)^2
\end{aligned}
\tag{B.18}
$$

The corresponding constraint equation is then obtained by imposing $\frac{\delta H_{2,Y}}{\delta Y}=0$. We have, for the terms in (B.18)

$$\int d^3x\,\frac{1}{4}\delta\pi_3\delta Y + \frac{3}{4}\delta(\Delta^2 Y)\frac{\delta\pi_3}{\Delta^2} = \int d^3x\,\frac{1}{4}\delta\pi_3\delta Y + \frac{3}{4}\Delta^2\frac{\delta\pi_3}{\Delta^2}\delta Y = \int d^3x\,\delta\pi_3\delta Y;$$
$$\int d^3x\,\frac{1}{4}\Delta\delta\pi_2\delta Y - \frac{1}{4}\delta(\Delta^2 Y)\frac{\delta\pi_2}{\Delta} = \int d^3x\,\frac{1}{4}\Delta\delta\pi_2\delta Y - \frac{1}{4}\Delta^2\frac{\delta\pi_2}{\Delta}\delta Y = 0;$$
$$\int d^3x\,\frac{\lambda}{f}gQ^2a^3\partial_i\delta\chi\delta(\partial_i Y) + a^3H\delta Q\delta(\Delta Y) + a^3H\partial_i\partial_a M\delta(\partial_i\partial_a Y) - a^3g^2(aQ)^2\delta(\partial_b Y)^2$$
$$= \int d^3x - \frac{\lambda}{f}gQ^2a^3\Delta\delta\chi\partial_i Y + a^3H\Delta\delta Q\delta Y + a^3H\Delta^2 M\delta Y + 2a^3g^2(aQ)^2\Delta Y\delta Y; \tag{B.19}$$

In deriving the previous expressions, we have made use of integration by parts. Imposing the condition $\frac{\delta H_{2,Y}}{\delta Y} = 0$ leads to

$$\int d^3x\left[\delta\pi_3 - \frac{\lambda}{f}ag(aQ)^2\Delta\delta\chi + a^3H\Delta\delta Q + a^3H\Delta^2 M + 2a^3g^2(aQ)^2\Delta Y\right]\delta Y = 0 \tag{B.20}$$

The last equation is satisfied by setting the terms in squared parentheses equal to zero

$$\begin{aligned}&\delta\pi_3 - \frac{\lambda}{f}ag(aQ)^2\Delta\delta\chi + a^3H\Delta\delta Q + a^3H\Delta^2 M + 2a^3g^2(aQ)^2\Delta Y = 0\\ &2a^3g^2(aQ)^2\Delta Y = -\delta\pi_3 + \frac{\lambda}{f}ag(aQ)^2\Delta\delta\chi - a^3H\Delta\delta Q - a^3H\Delta^2 M\end{aligned} \tag{B.21}$$

We can therefore define the following relation

$$Y^{(1)} = \frac{-\delta\pi_3 + \frac{\lambda}{f}ag(aQ)^2\Delta\delta\chi - a^3H\Delta\delta Q - a^3H\Delta^2 M}{2a^3g^2(aQ)^2\Delta} \tag{B.22}$$

where the superscript (1) denotes that the quantity is the first order perturbation expression. We can now substitute this expression back into the Hamiltonian in order to obtain a formulation of H_2 that depends only on the dynamical modes. In particular, by considering the quadratic Hamiltonian, we obtain

$$
\begin{aligned}
H_2 &= \int d^3x\, \mathcal{H}_2 = \int d^3k\; \frac{1}{2a^3}|\delta\pi_{1,k}|^2 + \frac{3}{8a^3}|\delta\pi_{2,k}|^2 + \frac{3}{8a^3k^4}|\delta\pi_{3,k}|^2 + \frac{3}{8a^3k^4}|\delta\pi_{3,k}|^2 \\
&- \frac{1}{8a^3}|\delta\pi_{2,k}|^2 + \frac{9a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2|\delta Q_k|^2 + \frac{a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 k^4|M_k|^2 - \frac{\lambda}{f}gQ^2\delta\pi_{1,k}(3\delta Q_{-k} - k^2M_{-k}) \\
&- 3a^3\left(\frac{\lambda}{f}gQ^2\right)^2 k^2\delta Q_k M_{-k} + \frac{3}{4a^3k^2}\delta\pi_{2,k}\delta\pi_{3,-k} + \frac{1}{8a^3k^2}\delta\pi_{3,k}\delta\pi_{2,-k} - \frac{3}{8a^3k^2}\delta\pi_{2,k}\delta\pi_{3,-k} \\
&+ \frac{1}{4}\delta\pi_{3,k}Y_{-k} + \frac{3}{4}\delta\pi_{3,k}Y_{-k} - \frac{k^2}{4}Y_{-k}\delta\pi_{2,k} + \frac{k^2}{4}Y_{-k}\delta\pi_{2,k} + \frac{\lambda}{f}gQ^2a^3k^2\delta\chi_kY_{-k} \\
&+ \frac{a}{2}k^2|\delta\chi_k|^2 + \frac{a^3}{2}U''|\delta\chi_k|^2 - 3\frac{\lambda}{f}gQa^3\partial_0\bar{\chi}|\delta Q_k|^2 + 2\frac{\lambda}{f}gQa^3\partial_0\bar{\chi}k^2\delta Q_kM_{-k} \\
&- 2a^3H^2k^2\delta Q_kM_{-k} + a^3H^2k^4|M_k|^2 - a^3Hk^2\delta Q_kY_{-k} + a^3Hk^4M_kY_{-k} + ak^2|\delta Q_k|^2 \\
&- a^3g^2(aQ)^2k^2|Y_k|^2 + 9a^3g^2Q^2|\delta Q_k|^2 + a^3g^2Q^2k^4|M_k|^2 - 6a^3g^2Q^2k^2\delta Q_kM_{-k} \\
&+ 3a^3H^2|\delta Q_k|^2 = \int d^3k\; \frac{1}{2a^3}|\delta\pi_{1,k}|^2 + \frac{1}{4a^3}|\delta\pi_{2,k}|^2 + \frac{3}{4a^3k^4}|\delta\pi_{3,k}|^2 \\
&+ \left[\frac{9a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 - 3\frac{\lambda}{f}gQa^3\partial_0\bar{\chi} + ak^2 + 9a^3g^2Q^2 + 3a^3H^2\right]|\delta Q_k|^2 \\
&+ \left[\frac{a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 k^4 + a^3H^2k^4 + a^3g^2Q^2k^4\right]|M_k|^2 + \left(\frac{a}{2}k^2 + \frac{a^3}{2}U''\right)|\delta\chi_k|^2 \\
&+ \frac{1}{2a^3k^2}\delta\pi_{2,k}\delta\pi_{3,-k} + \delta\pi_{3,k}Y_{-k} - \frac{\lambda}{f}gQ^2\delta\pi_{1,k}(3\delta Q_{-k} - k^2M_{-k}) + \frac{\lambda}{f}gQ^2a^3k^2\delta\chi_kY_{-k} \\
&+ \left[2\frac{\lambda}{f}gQa^3\partial_0\bar{\chi}k^2 - 3a^3\left(\frac{\lambda}{f}gQ^2\right)^2 k^2 - 2a^3H^2k^2 - 6a^3g^2Q^2k^2\right]\delta Q_kM_{-k} \\
&- a^3Hk^2\delta Q_kY_{-k} + a^3Hk^4M_kY_{-k} - a^3g^2(aQ)^2k^2|Y_k|^2;
\end{aligned}
\tag{B.23}
$$

from which the quadratic kinematic Hamiltonian H_0 can be identified as

$$
\begin{aligned}
H_0 &= \int d^3x\ \mathcal{H}_0 = \int d^3k\ \frac{1}{2a^3}|\delta\pi_{1,k}|^2 + \frac{1}{4a^3}|\delta\pi_{2,k}|^2 + \frac{3}{4a^3k^4}|\delta\pi_{3,k}|^2 \\
&+ \left[\frac{9a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 - 3\frac{\lambda}{f}gQa^3\partial_0\bar{\chi} + ak^2 + 9a^3g^2Q^2 + 3a^3H^2\right]|\delta Q_k|^2 \\
&+ \left[\frac{a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 k^4 + a^3H^2k^4 + a^3g^2Q^2k^4\right]|M_k|^2 + \left(\frac{a}{2}k^2 + \frac{a^3}{2}U''\right)|\delta\chi_k|^2 \\
&+ \frac{|\delta\pi_{3,k}|^2}{2a^3g^2(aQ)^2k^2} + \frac{a}{2}\left(\frac{\lambda}{f}Q\right)^2 k^2|\delta\chi_k|^2 + \frac{aH^2k^2}{2g^2Q^2}|\delta Q_k|^2 \\
&+ \frac{aH^2k^6}{2g^2Q^2}|M_k|^2 + \left[\frac{|\delta\pi_{3,k}|^2}{4a^6g^4(aQ)^4k^4} + \left(\frac{\lambda}{2fa^2g}\right)^2|\delta\chi_k|^2 + \left(\frac{H}{2g^2(aQ)^2}\right)^2|\delta Q_k|^2\right. \\
&\left.+ \left(\frac{Hk^2}{2g^2(aQ)^2}\right)^2|M_k|^2\right]\left[-a^3g^2(aQ)^2k^2\right] = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{2a^3}|\delta\pi_{1,k}|^2 + \frac{1}{4a^3}|\delta\pi_{2,k}|^2 \\
&+ \frac{3}{4a^3k^4}|\delta\pi_{3,k}|^2 + \left[\frac{9a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 - 3\frac{\lambda}{f}gQa^3\partial_0\bar{\chi} + ak^2 + 9a^3g^2Q^2 + 3a^3H^2\right]|\delta Q_k|^2 \\
&+ \left[\frac{a^3}{2}\left(\frac{\lambda}{f}gQ^2\right)^2 k^4 + a^3H^2k^4 + a^3g^2Q^2k^4\right]|M_k|^2 + \left(\frac{a}{2}k^2 + \frac{a^3}{2}U''\right)|\delta\chi_k|^2 \\
&+ \frac{|\delta\pi_{3,k}|^2}{4a^3g^2(aQ)^2k^2} + \frac{a}{4}\left(\frac{\lambda}{f}Q\right)^2 k^2|\delta\chi_k|^2 + \frac{aH^2k^2}{4g^2Q^2}|\delta Q_k|^2 + \frac{aH^2k^6}{4g^2Q^2}|M_k|^2
\end{aligned}
\tag{B.24}
$$

This term will be used to derive the equations of motion for the perturbations. In addition, we also require the explicit expressions for H_2^I, H_3^I and H_4^I. To determine them, one should replace $\delta\pi_i$ with $\delta\varphi_i$ using the relation $\delta\dot{\varphi}_{i,k} = \frac{\partial H_0}{\partial\delta\pi_{i,k}^*}$, which was derived in Section (3.4). For the sake of brevity, we illustrate this substitution for H_2^I, whereas for the remaining terms we only report the resulting Hamiltonian densities. Thus, the quadratic interaction Hamiltonian is given by

$$
\begin{aligned}
H_2^I =& \int d^3k \ \frac{1}{2a^3k^2}\delta\pi_{2,k}\delta\pi_{3,-k} + \delta\pi_{3,k}Y_{-k} - \frac{\lambda}{f}gQ^2\delta\pi_{1,k}(3\delta Q_{-k} - k^2 M_{-k}) \\
&+ \frac{\lambda}{f}gQ^2a^3k^2\delta\chi_k Y_{-k} + \left[2\frac{\lambda}{f}gQa^3\partial_0\bar{\chi}k^2 - 3a^3\left(\frac{\lambda}{f}gQ^2\right)^2k^2 - 2a^3H^2k^2 - 6a^3g^2Q^2k^2\right]\delta Q_k M_{-k} \\
&- a^3Hk^2\delta Q_k Y_{-k} + a^3Hk^4M_kY_{-k} - a^3g^2(aQ)^2k^2|Y_k|^2 = \\
=& \int d^3k \ \frac{2}{3k^2}a^3\delta\dot{Q}_k\dot{M}_{-k} + \frac{2}{3}a^3\dot{M}_k\frac{\left[\frac{\lambda}{f}ag(aQ)^2k^2\delta\chi_{-k} - a^3Hk^2\delta Q_{-k} + a^3Hk^4M_{-k}\right]}{2a^3g^2(aQ)^2k^2} \\
&- \frac{\lambda}{f}gQ^2a^3\delta\dot{\chi}_k(3\delta Q_{-k} - k^2M_{-k}) + \frac{\lambda}{f}gQ^2a^3k^2\delta\chi_k\frac{\left[\frac{2}{3}a^3\dot{M}_{-k} - a^3Hk^2\delta Q_{-k} + a^3Hk^4M_{-k}\right]}{2a^3g^2(aQ)^2k^2} \\
&+ \left[2\frac{\lambda}{f}gQa^3\partial_0\bar{\chi}k^2 - 3a^3\left(\frac{\lambda}{f}gQ^2\right)^2k^2 - 2a^3H^2k^2 - 6a^3g^2Q^2k^2\right]\delta Q_k M_{-k} \\
&- a^3Hk^2\delta Q_k\frac{\left[\frac{2}{3}a^3\dot{M}_{-k} + \frac{\lambda}{f}ag(aQ)^2k^2\delta\chi_{-k} + a^3Hk^4M_{-k}\right]}{2a^3g^2(aQ)^2k^2} \\
&+ a^3Hk^4M_k\frac{\left[\frac{2}{3}a^3\dot{M}_{-k} + \frac{\lambda}{f}ag(aQ)^2k^2\delta\chi_{-k} - a^3Hk^2\delta Q_{-k}\right]}{2a^3g^2(aQ)^2k^2} \\
&- a^3g^2(aQ)^2k^2\Bigg\{\frac{2}{3}a^3\dot{M}_k\frac{\left[\frac{\lambda}{f}ag(aQ)^2k^2\delta\chi_{-k} - a^3Hk^2\delta Q_{-k} + a^3Hk^4M_{-k}\right]}{(2a^3g^2(aQ)^2k^2)^2} \\
&+ \frac{\lambda}{2a^2fg}k^2\delta\chi_k\frac{\left[\frac{2}{3}a^3\dot{M}_{-k} - a^3Hk^2\delta Q_{-k} + a^3Hk^4M_{-k}\right]}{2a^3g^2(aQ)^2k^2} \\
&- a^3Hk^2\delta Q_k\frac{\left[\frac{2}{3}a^3\dot{M}_{-k} + \frac{\lambda}{f}ag(aQ)^2k^2\delta\chi_{-k} + a^3Hk^4M_{-k}\right]}{(2a^3g^2(aQ)^2k^2)^2} \\
&+ a^3Hk^4M_k\frac{\left[\frac{2}{3}a^3\dot{M}_{-k} + \frac{\lambda}{f}ag(aQ)^2k^2\delta\chi_{-k} - a^3Hk^2\delta Q_{-k}\right]}{(2a^3g^2(aQ)^2k^2)^2}\Bigg\} \\
=& \int d^3k \ \frac{2}{3k^2}a^3\delta\dot{Q}_k\dot{M}_{-k} - \frac{\lambda}{f}gQ^2a^3\delta\dot{\chi}_k(3\delta Q_{-k} - k^2M_{-k}) - \frac{\lambda Hak^2}{2fg}\delta\chi_k\delta Q_{-k} \\
&+ \frac{\lambda Hak^4}{2fg}\delta\chi_kM_{-k} + \left[2\frac{\lambda}{f}gQa^3\partial_0\bar{\chi}k^2 - 3a^3\left(\frac{\lambda}{f}gQ^2\right)^2k^2 - 2a^3H^2k^2 - 6a^3g^2Q^2k^2\right. \\
&\left. - \frac{H^2k^4}{2g^2(aQ)^2}\right]\delta Q_kM_{-k} + \frac{\lambda}{3fg}a\dot{M}_k\delta\chi_{-k} - \frac{H}{g^2Q^2}a\dot{M}_k\delta Q_{-k} + \frac{Hk^2}{3g^2Q^2}a\dot{M}_kM_{-k}
\end{aligned}
\tag{B.25}
$$

while the cubic and quartic interaction Hamiltonian density, in real space, are

$$\mathcal{H}^I_3 = -\frac{\lambda}{f}gQ\delta\pi_1\Big[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2\Big]$$
$$+ \left(\frac{\lambda}{f}gQ\right)^2 a^3 Q(3\delta Q + \Delta M)\Big[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2\Big]$$
$$+ 2\frac{\lambda}{f}a^2 g(aQ)\partial_a\delta\chi\partial_a Y\delta Q + \frac{\lambda}{f}a^2 g(aQ)(\partial_i\delta\chi\partial_i Y\Delta M - \partial_b\delta\chi\partial_j Y\partial_b\partial_j M)$$
$$- \frac{\lambda}{f}a^2 g(aQ)\partial_0\delta\chi[3\delta Q^2 + 2\delta Q\Delta M + \frac{1}{2}(\Delta M)^2 - \frac{1}{2}(\partial_i\partial_j M)^2] - \frac{\lambda}{2f}a^3 g\ \partial_0\bar{\chi}\{2\delta Q^3 + 2\delta Q^2\Delta M$$
$$+ \delta Q[(\Delta M)^2 - (\partial_a\partial_c M)^2]\} - a^4 g^2(aQ)(2\partial_b Y\partial_b Y\delta Q + \partial_b Y\partial_b Y\Delta M - \partial_b Y\partial_c Y\partial_b\partial_c M)$$
$$+ a^2 g^2(aQ)[6\delta Q^3 + 6\delta Q^2\Delta M + 2\delta Q(\Delta M)^2 + \Delta M(\partial_c\partial_k M)^2 - \partial_i\partial_k M\partial_c\partial_k M\partial_c\partial_i M] \tag{B.26}$$

$$\mathcal{H}^I_4 = -\frac{\lambda}{2f}g\delta\pi_1\Big[2\delta Q^3 + 2\delta Q^2\Delta M + \delta Q(\Delta M)^2 - \delta Q(\partial_c\partial_k M)^2$$
$$+ \frac{1}{3}(\Delta M)^3 + \frac{2}{3}(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) - (\Delta M)(\partial_j\partial_k M)^2\Big]$$
$$+ \left(\frac{\lambda}{f}gQ\right)^2\frac{a^3}{2}\Big[15\delta Q^4 + 20\delta Q^3\Delta M + 12\delta Q^2(\Delta M)^2 - 6\delta Q^2(\partial_c\partial_k M)^2 + 4\delta Q(\Delta M)^3$$
$$- 6\delta Q\Delta M(\partial_i\partial_j M)^2 + 2\delta Q(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M) + \frac{7}{12}(\Delta M)^4 + \frac{1}{4}(\partial_i\partial_j M)^4$$
$$- \frac{3}{2}(\Delta M)^2(\partial_i\partial_j M)^2 + \frac{2}{3}\Delta M(\partial_k\partial_i M)(\partial_i\partial_j M)(\partial_j\partial_k M)\Big] - \frac{\lambda}{2f}ga^3\partial_i\delta\chi[-2\partial_i Y^{(1)}\delta Q^2$$
$$- 2(\partial_i Y^{(1)}\delta Q\Delta M - \partial_a Y^{(1)}\delta Q\partial_i\partial_a M) + \epsilon^{i0jk}\epsilon^{abc}\partial_a Y^{(1)}\partial_j\partial_b M\partial_c\partial_k M]$$
$$- \frac{1}{2}g^2 a^5[2\partial_b Y^{(1)}\partial_b Y^{(1)}\delta Q^2 + 2\partial_b Y^{(1)}\partial_b Y^{(1)}\delta Q\Delta M + (\partial_b Y^{(1)}\partial_i\partial_c M)^2$$
$$- 2\partial_b Y^{(1)}\partial_c Y^{(1)}\delta Q\partial_b\partial_c M - \partial_b Y^{(1)}\partial_c Y^{(1)}\partial_b\partial_i M\partial_i\partial_c M] + \frac{g^2 a^3}{4}[6\delta Q^4 + 8\delta Q^3\Delta M$$
$$+ 4\delta Q^2(\Delta M)^2 + 4\delta Q\Delta M(\partial_c\partial_k M)^2 + (\partial_b\partial_i M)^2(\partial_c\partial_k M)^2 - 4\delta Q\partial_c\partial_k M\partial_c\partial_b M\partial_b\partial_k M$$
$$- \partial_b\partial_i M\partial_k\partial_c M\partial_c\partial_i M\partial_b\partial_k M] + \frac{a^3}{24}U''''(\bar{\chi})\delta\chi^4 \tag{B.27}$$

As explained in Section (3.2), additional contributions required to determine the full cubic and quartic interaction Hamiltonian arises from the second order perturbation expression $Y^{(2)}$. In order to compute this term, it is necessary to include the contribution of H_3. To do this, we first isolate the terms in the cubic perturbed Hamiltonian density that are proportional to Y

$$\mathcal{H}_{3,Y} = 2g\frac{\lambda}{f}a^2(aQ)\partial_a\delta\chi\partial_a Y\delta Q + g\frac{\lambda}{f}a^2(aQ)(\partial_i\delta\chi\partial_i Y\Delta M - \partial_b\delta\chi\partial_j Y\partial_b\partial_j M)$$
$$- a^4g^2(aQ)(2\partial_b Y\partial_b Y\delta Q + \partial_b Y\partial_b Y\Delta M - \partial_b Y\partial_c Y\partial_c\partial_b M) \tag{B.28}$$

Similarly to the previous case, the expression for Y can be obtained by imposing $\frac{\delta(H_{2,Y}+H_{3,Y})}{\delta Y} = 0$; considering (B.28), we get

$$\begin{aligned}
&\int d^3x\, 2g\frac{\lambda}{f}a^2(aQ)\partial_a\delta\chi\delta Q\delta(\partial_a Y) = \int d^3x - 2g\frac{\lambda}{f}a^2(aQ)\partial_a(\partial_a\delta\chi\delta Q)\delta Y\\
&\int d^3x\, \frac{\lambda}{f}ga^2(aQ)\partial_i\delta\chi\Delta M\delta(\partial_i Y) = \int d^3x - \frac{\lambda}{f}ga^2(aQ)\partial_i(\partial_i\delta\chi\Delta M)\delta Y\\
&\int d^3x - \frac{\lambda}{f}a^2g(aQ)\partial_b\delta\chi\partial_b\partial_j M\delta(\partial_j Y) = \int d^3x\, \frac{\lambda}{f}ga^2(aQ)\partial_j(\partial_b\delta\chi\partial_b\partial_j M)\delta Y\\
&\int d^3x - 2a^4g^2(aQ)\delta(\partial_b Y\partial_b Y)\delta Q = \int d^3x - 4a^4g^2(aQ)\partial_b Y\delta Q\delta(\partial_b Y)\\
&= \int d^3x\, 4a^4g^2(aQ)\partial_b(\partial_b Y\delta Q)\delta Y\\
&\int d^3x - a^4g^2(aQ)\delta(\partial_b Y\partial_b Y)\Delta M = \int d^3x - 2a^4g^2(aQ)\partial_b Y\Delta M\delta(\partial_b Y)\\
&= \int d^3x\, 2a^4g^2(aQ)\partial_b(\partial_b Y\Delta M)\delta Y\\
&\int d^3x\, a^4g^2(aQ)\delta(\partial_b Y\partial_c Y\partial_c\partial_b M) = \int d^4x\, 2a^4g^2(aQ)\partial_c Y\partial_c\partial_b M\delta(\partial_b Y)\\
&= \int d^3x - 2a^4g^2(aQ)\partial_b(\partial_c Y\partial_b\partial_c M)\delta Y
\end{aligned} \tag{B.29}$$

Summing all the relevant contributions and setting the variation to zero, we obtain

$$\int d^3x\Big[- 2g\frac{\lambda}{f}a^2(aQ)(\Delta\delta\chi\delta Q + \partial_a\delta\chi\partial_a\delta Q) - \frac{\lambda}{f}ga^2(aQ)(\Delta\delta\chi\Delta M + \partial_i\delta\chi\partial_i\Delta M)$$
$$+ \frac{\lambda}{f}ga^2(aQ)(\partial_j\partial_b\delta\chi\partial_b\partial_j M + \partial_b\delta\chi\partial_b\Delta M) + 4a^4g^2(aQ)(\Delta Y\delta Q + \partial_b Y\partial_b\delta Q)$$
$$+ 2a^4g^2(aQ)(\Delta Y\Delta M + \partial_b Y\partial_b\Delta M) - 2a^4g^2(aQ)(\partial_b\partial_c Y\partial_c\partial_b M + \partial_c Y\partial_c\Delta M)\Big]\delta Y = 0 \tag{B.30}$$

Taking into account this result with (B.20), we can write

$$\int d^3x \Big[\delta\pi_3 - \frac{\lambda}{f} ag(aQ)^2 \Delta\delta\chi + a^3 H\Delta\delta Q + a^3 H\Delta^2 M + 2a^3 g^2 (aQ)^2 \Delta Y$$
$$- 2g\frac{\lambda}{f} a^2(aQ)\left(\Delta\delta\chi\delta Q + \partial_a\delta\chi\partial_a\delta Q\right) - \frac{\lambda}{f} ga^2(aQ)\left(\Delta\delta\chi\Delta M + \partial_i\delta\chi\partial_i\Delta M\right)$$
$$+ \frac{\lambda}{f} ga^2(aQ)\left(\partial_j\partial_b\delta\chi\partial_b\partial_j M + \partial_b\delta\chi\partial_b\Delta M\right) + 4a^4 g^2(aQ)\Big(\Delta Y^{(1)}\delta Q \tag{B.31}$$
$$+\partial_b Y^{(1)}\partial_b\delta Q\Big) + 2a^4 g^2(aQ)\Big(\Delta Y^{(1)}\Delta M + \partial_b Y^{(1)}\partial_b\Delta M\Big)$$
$$- 2a^4 g^2(aQ)\Big(\partial_b\partial_c Y^{(1)}\partial_c\partial_b M + \partial_c Y^{(1)}\partial_c\Delta M\Big)\Big]\delta Y = 0$$

Solving for $\Delta Y = \Delta(Y^{(1)} + Y^{(2)})$ we get

$$\Delta Y = \frac{1}{2a^3 g^2(aQ)^2}\Big[- \delta\pi_3 + \frac{\lambda}{f} ag(aQ)^2\Delta\delta\chi - a^3 H\Delta\delta Q - a^3 H\Delta^2 M$$
$$+ 2g\frac{\lambda}{f} a^2(aQ)\left(\Delta\delta\chi\delta Q + \partial_a\delta\chi\partial_a\delta Q\right) + \frac{\lambda}{f} ga^2(aQ)\left(\Delta\delta\chi\Delta M + \partial_i\delta\chi\partial_i\Delta M\right)$$
$$- \frac{\lambda}{f} ga^2(aQ)\left(\partial_j\partial_b\delta\chi\partial_b\partial_j M + \partial_b\delta\chi\partial_b\Delta M\right)$$
$$- 4a^4 g^2(aQ)\Big(\Delta Y^{(1)}\delta Q + \partial_b Y^{(1)}\partial_b\delta Q\Big) - 2a^4 g^2(aQ)\Big(\Delta Y^{(1)}\Delta M + \partial_b Y^{(1)}\partial_b\Delta M\Big)$$
$$+ 2a^4 g^2(aQ)\Big(\partial_b\partial_c Y^{(1)}\partial_c\partial_b M + \partial_c Y^{(1)}\partial_c\Delta M\Big)\Big] \tag{B.32}$$

Substituting the expression for $Y^{(1)}$ (B.22), one can observe that the second and third lines of (B.32) cancel against the contribution proportional to $\delta\chi$ in (B.22). As a consequence, it follows that $Y^{(2)} \propto (\delta A)^2$, where δA denotes a perturbation of the gauge sector.

We do not report the full expression for $Y^{(2)}$, as it is not particularly illuminating and, for the purposes of the present computation, only a subset of its terms will be required.

Appendix C

C.1 Power Spectrum of the Axion

The two point function for $\delta\chi$ is defined by the expectation value of $\delta\chi^2$, therefore it can be expressed as

$$\begin{aligned}
\langle\delta\chi^2\rangle &= \langle 0|\delta\chi^2|0\rangle \\
&+\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\langle H_2^{Int}(\tilde{\tau}_1)\delta\chi^2 H_2^{Int}(\tau_1)\rangle \\
&-2\,\mathrm{Re}\left[\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2)\langle\delta\chi^2 H_2^{Int}(\tau_1)H_2^{Int}(\tau_2)\rangle\right]
\end{aligned} \tag{C.1}$$

where the second and the third lines represent the leading order corrections. The first line yields the standard result

$$\langle\delta\chi_{k_1}\delta\chi_{k_2}\rangle = u_{k_1}(0)u^*_{-k_2}(0)\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}) = \frac{H^2}{2c_{s1}^3k^3}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}) \tag{C.2}$$

To evaluate the second and third lines, we apply the same methodology introduced in Section (4.1). Specifically, the second line reads

$$\begin{aligned}
&\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\langle H_2^{Int}(\tilde{\tau}_1)\delta\chi^2 H_2^{Int}(\tau_1)\rangle \\
&= 18\left(\frac{\lambda}{f}\right)^2 g^2Q^4u^*_{k_1}(0)u_{k_1}(0)\int_{-\infty}^{0} d\tilde{\tau}_1 a^3(\tilde{\tau}_1)u'_{k_1}(\tilde{\tau}_1)v_{k_1}(\tilde{\tau}_1) \\
&\quad\times\int_{-\infty}^{0} d\tau_1 a^3(\tau_1)v^*_{k_1}(\tau_1)u^{*\prime}_{k_1}(\tau_1)\ \delta^3(k_1+k_2) \\
&= \frac{H^2}{2c_{s1}^3k_1^3}\frac{9c_{s1}}{2\pi c_{s2}^2}\Lambda^2m_Q^2\int_0^{+\infty} d\tilde{y}_1\int_0^{+\infty} dy_1(\tilde{y}_1)^{-\frac{1}{2}} \\
&\quad\times K_{i\mu}(k_1^*\tilde{y}_1)e^{-c_sk_1^*\tilde{y}_1}(y_1)^{-\frac{1}{2}}K_{i\mu}(k_1^*y_1)e^{-c_sk_1^*y_1}\ \delta^3(k_1+k_2)
\end{aligned} \tag{C.3}$$

while the third line takes the form

$$
\begin{aligned}
&-2\,\mathrm{Re}\left[\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2)\langle\delta\chi^2 H_2^{Int}(\tau_1)H_2^{Int}(\tau_2)\rangle\right]\\
&=-4\,\mathrm{Re}\left[9\Big(\frac{\lambda}{f}\Big)^2 g^2Q^4 u_{k_1}(0)u_{k_1}(0)\int_{-\infty}^{0} d\tau_1 a^3(\tau_1)u^{*\prime}_{k_1}(\tau_1)v_{k_1}(\tau_1)\right.\\
&\left.\times\int_{-\infty}^{\tau_1} d\tau_2 a^3(\tau_2)u^{*\prime}_{k_1}(\tau_2)v^*_{k_1}(\tau_2)\;\delta^3(k_1+k_2)\right]\\
&=-\frac{H^2}{(2c_{s1}^3)k_1^3}\frac{9c_{s1}}{2c_{s2}^2}\Lambda^2 m_Q^2\,\mathrm{Re}\left[\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2(y_1)^{-\frac{1}{2}}H^{(1)}_{i\mu}(-ik_1^*y_1)\right.\\
&\left.\times\, e^{-c_sk_1^*y_1}(y_2)^{-\frac{1}{2}}e^{-\pi\mu/2}K_{i\mu}(k_1^*y_2)e^{-c_sk_1^*y_2}\;\delta^3(k_1+k_2)\right]
\end{aligned}
$$

Finally, after computing the integrals with the aid of Mathematica, the complete result is given by

$$
\begin{aligned}
&\langle\delta\chi^2\rangle=\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})P_{\delta\chi}(k)\\
&=\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})\frac{H^2}{2c_{s1}^3k^3}\left(1+0.06\frac{c_{s1}}{c_{s2}^2}\Lambda^2m_Q^2\right)=\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})\frac{H^2}{2c_{s1}^3k^3}\Xi
\end{aligned}
\tag{C.4}
$$

where we have that $k_1=k_2=k$ and $\Xi\equiv\left(1+0.06\frac{c_{s1}}{c_{s2}^2}\Lambda^2m_Q^2\right)=1.26$.

C.2 Details on the Contributions to the Trispectrum

In this appendix, we report the results for the trispectrum contributions arising from both diagrams shown in Figure (3.1).

C.2.1 Scalar Exchange Contributions

Regarding the scalar exchange, we present the contribution associated with

$$-2\,\mathrm{Re}\left[\int_{\tau_0}^{\tau} d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2)\langle\delta\chi_{k_1}(\tau)\delta\chi_{k_2}(\tau)\delta\chi_{k_3}(\tau)\delta\chi_{k_4}(\tau)H_3^{Int}(\tau_1)H_3^{Int}(\tau_2)\rangle\right] \tag{C.5}$$

After substituting the mode functions, we obtain

$$\begin{aligned}
&-\frac{1}{2(2\pi)^3}\left(\frac{\lambda}{f}\right)^4 Q^2\,\mathrm{Re}\Bigg[u_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)(\mathbf{k_1}\cdot\mathbf{k_2})(\mathbf{k_3}\cdot\mathbf{k_4})\\
&\quad\times\int_{\tau_0}^{\tau} d\tau_1 a^2(\tau_1)u^*_{k_1}(\tau_1)u^*_{k_2}(\tau_1)v_{k_1+k_2}(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a^2(\tau_2)u^*_{k_3}(\tau_2)u^*_{k_4}(\tau_2)v^*_{k_3+k_4}(\tau_2)\Bigg]\\
&\quad\times\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}\\
&= -\frac{1}{(2\pi)^3}\left(\frac{\lambda}{f}\right)^4 Q^2(\mathbf{k_1}\cdot\mathbf{k_2})(\mathbf{k_3}\cdot\mathbf{k_4})\frac{H^6}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)^3}\frac{1}{16c_{s1}^3c_{s2}k_1}\\
&\mathrm{Re}\Bigg[\int_0^{+\infty} dy_1\int_{y_1}^{+\infty} dy_2 (y_1)^{-\frac{1}{2}}(1+c_sk_1^*y_1)\\
&\quad\times(1+c_sk_2^*y_1)e^{-c_sy_1(k_1^*+k_2^*)}e^{-\frac{\pi\mu}{2}}H^{(1)}_{i\mu}(-ik^*_{34}y_1)(y_2)^{-\frac{1}{2}}(1+c_sk_3^*y_2)\\
&\quad\times(1+c_sk_4^*y_2)e^{-c_sy_2(k_3^*+k_4^*)}K_{i\mu}(k^*_{34}y_2)\Bigg]\;\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})\\
&\quad+ 23 \text{ permutations of } \mathbf{k_i}
\end{aligned} \tag{C.6}$$

C.2.2 Contact Interaction Contributions

For the contact interaction term, there are ten distinct contributions in total: two from the first line, four from the second, and four from the third line in (4.14). Since the first result for the first line was already provided in Section (4.2), we explicitly outline the remaining nine contributions below.
The second term for the first line reads

$$\frac{324}{(2\pi)^3}\left(\frac{\lambda}{f}g\right)^4 Q^6\,\mathrm{Re}\left[u^*_{k_1}(\tau)u^*_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\right.$$
$$\times\int_{\tau_0}^{\tau}d\tilde{\tau}_1 a^3(\tilde{\tau}_1)v^*_{k_1}(\tilde{\tau}_1)v_{k_3}(\tilde{\tau}_1)v_{k_4}(\tilde{\tau}_1)u'_{k_2}(\tilde{\tau}_1)\int_{\tau_0}^{\tilde{\tau}_1}d\tilde{\tau}_2 a^3(\tilde{\tau}_2)u'_{k_1}(\tilde{\tau}_2)v_{k_1}(\tilde{\tau}_2)$$
$$\left.\times\int_{\tau_0}^{\tau}d\tau_1 a^3(\tau_1)v^*_{k_3}(\tau_1)u^{*\prime}_{k_3}(\tau_1)\int_{\tau_0}^{\tau_1}d\tau_2 a^3(\tau_2)v^*_{k_4}(\tau_2)u^{*\prime}_{k_4}(\tau_2)\right]$$
$$\times\,\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}$$
$$=-\frac{1}{(2\pi)^3}\frac{81}{64}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\left(\frac{2}{\pi}\right)^2\frac{c_{s1}^2}{c_{s2}^7k_1^5}$$
$$\times\,\mathrm{Re}\left[\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2(y_1)^{-\frac{1}{2}}K_{i\mu}(k_3^*y_1)e^{-c_sk_3^*y_1}(y_2)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_2)e^{-c_sk_4^*y_2}\right.$$
$$\times\int_0^{+\infty}d\tilde{y}_1\int_{\tilde{y}_1}^{+\infty}d\tilde{y}_2(\tilde{y}_1)^{\frac{5}{2}}e^{-\frac{\pi\mu}{2}}H^{(2)}_{-i\mu}(ik_1^*\tilde{y}_1)K_{i\mu}(k_3^*\tilde{y}_1)K_{i\mu}(k_4^*\tilde{y}_1)e^{-c_sk_2^*\tilde{y}_1}$$
$$\left.\times\,(\tilde{y}_2)^{-\frac{1}{2}}K_{i\mu}(k_1^*\tilde{y}_2)e^{-c_sk_1^*\tilde{y}_2}\right]\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}\tag{C.7}$$

On the other hand, for the second line we have

$$-2\mathrm{Re}\left[\int_{t_0}^{t}d\tilde{t}_1\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3\langle H_I(\tilde{t}_1)O_I(t)H_I(t_1)H_I(t_2)H_I(t_3)\rangle\right]=$$
$$=\int_{\tau_0}^{\tau}d\tilde{\tau}_1 a(\tilde{\tau}_1)\int_{\tau_0}^{\tau}d\tau_1 a(\tau_1)\int_{\tau_0}^{\tau_1}d\tau_2 a(\tau_2)\int_{\tau_0}^{\tau_2}d\tau_3 a(\tau_3)\langle H_I(\tilde{\tau}_1)O_I(\tau)H_I(\tau_1)H_I(\tau_2)H_I(\tau_3)\rangle\tag{C.8}$$

where, as explained in the main text, one of the four H_I operators is H_4^{Int}, while the remaining three are H_2^{Int}. This yields four contributions; the first one (featuring H_4^{Int} in place of the first H_I) is

$$
\begin{aligned}
&-\frac{324}{(2\pi)^3}\left(\frac{\lambda}{f}g\right)^4 Q^6\,\mathrm{Re}\Bigg[u^*_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\\
&\times\int_{\tau_0}^{\tau}d\tilde{\tau}_1 a^3(\tilde{\tau}_1)u'_{k_1}(\tilde{\tau}_1)v_{k_2}(\tilde{\tau}_1)v_{k_3}(\tilde{\tau}_1)v_{k_4}(\tilde{\tau}_1)\int_{\tau_0}^{\tau}d\tau_1 a^3(\tau_1)v^*_{k_2}(\tau_1)u^{*\prime}_{k_2}(\tau_1)\\
&\times\int_{\tau_0}^{\tau_1}d\tau_2 a^3(\tau_2)v^*_{k_3}(\tau_2)u^{*\prime}_{k_3}(\tau_2)\int_{\tau_0}^{\tau_2}d\tau_3 a^3(\tau_3)v^*_{k_4}(\tau_3)u^{*\prime}_{k_4}(\tau_3)\Bigg]\\
&\quad\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}\\
&=\frac{1}{(2\pi)^3}\frac{81}{63}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\left(\frac{2}{\pi}\right)^3\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\
&\times\mathrm{Re}\Bigg[i\int_0^{+\infty}d\tilde{y}_1(\tilde{y}_1)^{\frac{5}{2}}K_{i\mu}(k_2^*\tilde{y}_1)K_{i\mu}(k_3^*\tilde{y}_1)K_{i\mu}(k_4^*\tilde{y}_1)e^{-c_sk_1^*\tilde{y}_1}\\
&\times\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2\int_{y_2}^{+\infty}dy_3(y_1)^{-\frac{1}{2}}K_{i\mu}(k_2^*y_1)e^{-c_sk_2^*y_1}\\
&\times(y_2)^{-\frac{1}{2}}K_{i\mu}(k_3^*y_2)e^{-c_sk_3^*y_2}(y_3)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_3)e^{-c_sk_4^*y_3}\Bigg]\\
&\times\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}
\end{aligned}
\tag{C.9}
$$

The second contribution is

$$
\begin{aligned}
&-\frac{324}{(2\pi)^3}\left(\frac{\lambda}{f}g\right)^4 Q^6\,\mathrm{Re}\Bigg[u^*_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\\
&\times\int_{\tau_0}^{\tau}d\tilde{\tau}_1 a^3(\tilde{\tau}_1)u'_{k_1}(\tilde{\tau}_1)v_{k_1}(\tilde{\tau}_1)\int_{\tau_0}^{\tau}d\tau_1 a^3(\tau_1)v_{k_3}(\tau_1)u^{*\prime}_{k_2}(\tau_1)v^*_{k_1}(\tau_1)v_{k_4}(\tau_1)\\
&\times\int_{\tau_0}^{\tau_1}d\tau_2 a^3(\tau_2)v^*_{k_3}(\tau_2)u^{*\prime}_{k_3}(\tau_2)\int_{\tau_0}^{\tau_2}d\tau_3 a^3(\tau_3)v^*_{k_4}(\tau_3)u^{*\prime}_{k_4}(\tau_3)\Bigg]\\
&\quad\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}\\
&=\frac{1}{(2\pi)^3}\frac{81}{63}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\left(\frac{2}{\pi}\right)\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\
&\times\mathrm{Re}\Bigg[\int_0^{+\infty}d\tilde{y}_1(\tilde{y}_1)^{-\frac{1}{2}}K_{i\mu}(k_1^*\tilde{y}_1)e^{-c_sk_1^*\tilde{y}_1}\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2\int_{y_2}^{+\infty}dy_3\\
&\times(y_1)^{\frac{5}{2}}e^{-\pi\mu}K_{i\mu}(ik_1^*y_1)H^{(1)}_{i\mu}(-ik_3^*y_1)H^{(1)}_{i\mu}(-ik_4^*y_1)e^{-c_sk_2^*y_1}\\
&\times(y_2)^{-\frac{1}{2}}K_{i\mu}(ik_3^*y_2)e^{-c_sk_3^*y_2}(y_3)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_3)e^{-c_sk_4^*y_3}\Bigg]\\
&\times\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4})+23\text{ permutations of }\mathbf{k_i}
\end{aligned}
\tag{C.10}
$$

whereas the third one is

$$
\begin{aligned}
&-\frac{324}{(2\pi)^3}\left(\frac{\lambda}{f}g\right)^4 Q^6\,\mathrm{Re}\Bigg[u^*_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\\
&\times\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a^3(\tilde{\tau}_1)u'_{k_1}(\tilde{\tau}_1)v_{k_1}(\tilde{\tau}_1)\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)v_{k_2}(\tau_1)u^{*\prime}_{k_2}(\tau_1)\\
&\times\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)v^*_{k_1}(\tau_2)v^*_{k_2}(\tau_2)u^{*\prime}_{k_3}(\tau_2)v_{k_4}(\tau_2)\int_{\tau_0}^{\tau_2} d\tau_3 a^3(\tau_3)v^*_{k_4}(\tau_3)u^{*\prime}_{k_4}(\tau_3)\Bigg]\\
&\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}\\
&=\frac{1}{(2\pi)^3}\frac{81}{63}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\left(\frac{2}{\pi}\right)\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\
&\times\mathrm{Re}\Bigg[\int_0^{+\infty} d\tilde{y}_1(\tilde{y}_1)^{-\frac{1}{2}}K_{i\mu}(k_1^*\tilde{y}_1)e^{-c_sk_1^*\tilde{y}_1}\int_0^{+\infty} dy_1\int_{y_1}^{+\infty} dy_2\int_{y_2}^{+\infty} dy_3\\
&\times(y_1)^{-\frac{1}{2}}e^{-\pi\mu}H^{(1)}_{i\mu}(-ik_2^*y_1)e^{-c_sk_2^*y_1}(y_2)^{\frac{5}{2}}K_{i\mu}(ik_1^*y_2)K_{i\mu}(ik_2^*y_2)\\
&\times H^{(1)}_{i\mu}(-ik_4^*y_2)e^{-c_sk_3^*y_2}(y_3)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_3)e^{-c_sk_4^*y_3}\Bigg]\\
&\times\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}
\end{aligned}
\tag{C.11}
$$

The fourth and final contribution reads

$$
\begin{aligned}
&-\frac{324}{(2\pi)^3}\left(\frac{\lambda}{f}g\right)^4 Q^6\,\mathrm{Re}\Bigg[u^*_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\\
&\times\int_{\tau_0}^{\tau} d\tilde{\tau}_1 a^3(\tilde{\tau}_1)u'_{k_1}(\tilde{\tau}_1)v_{k_1}(\tilde{\tau}_1)\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)v_{k_2}(\tau_1)u^{*\prime}_{k_2}(\tau_1)\\
&\times\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)v_{k_3}(\tau_2)u^{*\prime}_{k_3}(\tau_2)\int_{\tau_0}^{\tau_2} d\tau_3 a^3(\tau_3)v^*_{k_1}(\tau_3)v^*_{k_2}(\tau_3)v^*_{k_3}(\tau_3)u^{*\prime}_{k_4}(\tau_3)\Bigg]\\
&\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}\\
&=\frac{1}{(2\pi)^3}\frac{81}{63}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\left(\frac{2}{\pi}\right)\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\
&\times\mathrm{Re}\Bigg[\int_0^{+\infty} d\tilde{y}_1(\tilde{y}_1)^{-\frac{1}{2}}K_{i\mu}(k_1^*\tilde{y}_1)e^{-c_sk_1^*\tilde{y}_1}\int_0^{+\infty} dy_1\int_{y_1}^{+\infty} dy_2\int_{y_2}^{+\infty} dy_3\\
&\times(y_1)^{-\frac{1}{2}}e^{-\pi\mu}H^{(1)}_{i\mu}(-ik_2^*y_1)e^{-c_sk_2^*y_1}(y_2)^{-\frac{1}{2}}H^{(1)}_{i\mu}(-ik_3^*y_2)e^{-c_sk_3^*y_2}\\
&\times(y_3)^{\frac{5}{2}}K_{i\mu}(k_1^*y_3)K_{i\mu}(k_2^*y_3)K_{i\mu}(k_3^*y_3)e^{-c_sk_4^*y_3}\Bigg]\\
&\times\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}
\end{aligned}
\tag{C.12}
$$

The third line is given by

$$2\mathrm{Re}\left[\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \int_{t_0}^{t_3} dt_4 \langle O_I(t) H_I(t_1) H_I(t_2) H_I(t_3) H_I(t_4)\rangle\right] =$$
$$= \int_{\tau_0}^{\tau} d\tau_1 a(\tau_1) \int_{\tau_0}^{\tau_1} d\tau_2 a(\tau_2) \int_{\tau_0}^{\tau_2} d\tau_3 a(\tau_3) \int_{\tau_0}^{\tau_3} d\tau_4 a(\tau_4) \langle O_I(\tau) H_I(\tau_1) H_I(\tau_2) H_I(\tau_3) H_I(\tau_4)\rangle \tag{C.13}$$

Following the same principle applied to the second line, we identify four additional contributions. The first is

$$\begin{aligned}
&2\,\mathrm{Re}\Bigg[162\left(\frac{\lambda}{f}g\right)^4 Q^6 u_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\\
&\quad\times\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)u^{*\prime}_{k_1}(\tau_1)v_{k_2}(\tau_1)v_{k_3}(\tau_1)v_{k_4}(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)v^*_{k_2}(\tau_2)u^{*\prime}_{k_2}(\tau_2)\\
&\quad\times\int_{\tau_0}^{\tau_2} d\tau_3 a^3(\tau_3)v^*_{k_3}(\tau_3)u^{*\prime}_{k_3}(\tau_3)\int_{\tau_0}^{\tau_3} d\tau_4 a^3(\tau_4)v^*_{k_4}(\tau_4)u^{*\prime}_{k_4}(\tau_4)\Bigg]\\
&\quad\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}\\
&= -\frac{1}{(2\pi)^3}\frac{81}{64}\left(\frac{\lambda}{f}g\right)^4 Q^6 \frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\frac{c_{s1}^2}{c_{s2}^7 k_1^5}\\
&\mathrm{Re}\Bigg[\int_0^{+\infty} dy_1 \int_{y_1}^{+\infty} dy_2 \int_{y_2}^{+\infty} dy_3 \int_{y_3}^{+\infty} dy_4\\
&\quad\times (y_1)^{\frac{5}{2}} e^{-\frac{3\pi\mu}{2}} H^{(1)}_{i\mu}(-ik_2^* y_1) H^{(1)}_{i\mu}(-ik_3^* y_1) H^{(1)}_{i\mu}(-ik_4^* y_1) e^{-c_s k_1^* y_1}\\
&\quad\times (y_2)^{-\frac{1}{2}} K_{i\mu}(k_2^* y_2) e^{-c_s k_2^* y_2} (y_3)^{-\frac{1}{2}} K_{i\mu}(k_3^* y_3) e^{-c_s k_3^* y_3}\\
&\quad\times (y_4)^{-\frac{1}{2}} K_{i\mu}(k_4^* y_4) e^{-c_s k_4^* y_4}\Bigg]\ \delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}
\end{aligned} \tag{C.14}$$

while the second is

$$
\begin{aligned}
&2\,\mathrm{Re}\left[162\left(\frac{\lambda}{f}g\right)^4 Q^6 u_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\right.\\
&\quad\times\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)u_{k_1}^{*\prime}(\tau_1)v_{k_1}(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)u_{k_2}^{*\prime}(\tau_2)v_{k_1}^{*}(\tau_2)v_{k_3}(\tau_2)v_{k_4}(\tau_2)\\
&\quad\left.\times\int_{\tau_0}^{\tau_2} d\tau_3 a^3(\tau_3)v_{k_3}^{*}(\tau_3)u_{k_3}^{*\prime}(\tau_3)\int_{\tau_0}^{\tau_3} d\tau_4 a^3(\tau_4)v_{k_4}^{*}(\tau_4)u_{k_4}^{*\prime}(\tau_4)\right]\\
&\quad\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}\\
&= -\frac{1}{(2\pi)^3}\frac{81}{64}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\
&\mathrm{Re}\left[\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2\int_{y_2}^{+\infty}dy_3\int_{y_3}^{+\infty}dy_4\right.\\
&\quad\times(y_1)^{-\frac{1}{2}}e^{-\frac{3\pi\mu}{2}}H_{i\mu}^{(1)}(-ik_1^*y_1)e^{-c_sk_1^*y_1}(y_2)^{\frac{5}{2}}K_{i\mu}(k_1^*y_2)\\
&\quad\times H_{i\mu}^{(1)}(-ik_3^*y_2)H_{i\mu}^{(1)}(-ik_4^*y_2)e^{-c_sk_2^*y_2}(y_3)^{-\frac{1}{2}}K_{i\mu}(k_3^*y_3)e^{-c_sk_3^*y_3}\\
&\quad\left.\times(y_4)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_4)e^{-c_sk_4^*y_4}\right]\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}
\end{aligned}
\tag{C.15}
$$

The third turns out to be

$$
\begin{aligned}
&2\,\mathrm{Re}\left[162\left(\frac{\lambda}{f}g\right)^4 Q^6 u_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau)\right.\\
&\quad\times\int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)u_{k_1}^{*\prime}(\tau_1)v_{k_1}(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)u_{k_2}^{*\prime}(\tau_2)v_{k_2}(\tau_2)\\
&\quad\left.\times\int_{\tau_0}^{\tau_2} d\tau_3 a^3(\tau_3)u_{k_3}^{*\prime}(\tau_3)v_{k_1}^{*}(\tau_3)v_{k_2}^{*}(\tau_3)v_{k_4}(\tau_3)\int_{\tau_0}^{\tau_3} d\tau_4 a^3(\tau_4)v_{k_4}^{*}(\tau_4)u_{k_4}^{*\prime}(\tau_4)\right]\\
&\quad\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}\\
&= -\frac{1}{(2\pi)^3}\frac{81}{64}\left(\frac{\lambda}{f}g\right)^4 Q^6\frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\frac{c_{s1}^2}{c_{s2}^7k_1^5}\\
&\mathrm{Re}\left[\int_0^{+\infty}dy_1\int_{y_1}^{+\infty}dy_2\int_{y_2}^{+\infty}dy_3\int_{y_3}^{+\infty}dy_4\right.\\
&\quad\times(y_1)^{-\frac{1}{2}}e^{-\frac{3\pi\mu}{2}}H_{i\mu}^{(1)}(-ik_1^*y_1)e^{-c_sk_1^*y_1}(y_2)^{-\frac{1}{2}}H_{i\mu}^{(1)}(-k_2^*y_2)\\
&\quad\times e^{-c_sk_2^*y_2}(y_3)^{\frac{5}{2}}K_{i\mu}(k_1^*y_3)K_{i\mu}(k_2^*y_3)H_{i\mu}^{(1)}(-ik_4^*y_3)e^{-c_sk_3^*y_3}\\
&\quad\left.\times(y_4)^{-\frac{1}{2}}K_{i\mu}(k_4^*y_4)e^{-c_sk_4^*y_4}\right]\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}
\end{aligned}
\tag{C.16}
$$

Finally, the fourth one is

$$
\begin{aligned}
&2\,\mathrm{Re}\Bigg[162\Big(\frac{\lambda}{f}g\Big)^4 Q^6 u_{k_1}(\tau)u_{k_2}(\tau)u_{k_3}(\tau)u_{k_4}(\tau) \\
&\quad\times \int_{\tau_0}^{\tau} d\tau_1 a^3(\tau_1)u^{*\prime}_{k_1}(\tau_1)v_{k_1}(\tau_1)\int_{\tau_0}^{\tau_1} d\tau_2 a^3(\tau_2)u^{*\prime}_{k_2}(\tau_2)v_{k_2}(\tau_2) \\
&\quad\times \int_{\tau_0}^{\tau_2} d\tau_3 a^3(\tau_3)u^{*\prime}_{k_3}(\tau_3)v_{k_3}(\tau_3)\int_{\tau_0}^{\tau_3} d\tau_4 a^3(\tau_4)v^*_{k_1}(\tau_4)v^*_{k_2}(\tau_4)v^*_{k_3}(\tau_4)u^{*\prime}_{k_4}(\tau_4)\Bigg] \\
&\quad \delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i} \\
&= -\frac{1}{(2\pi)^3}\frac{81}{64}\Big(\frac{\lambda}{f}g\Big)^4 Q^6 \frac{H^2}{(2c_{s1}^3)^2}\frac{1}{(k_1k_2k_3k_4)}\frac{c_{s1}^2}{c_{s2}^7k_1^5} \\
&\mathrm{Re}\Bigg[\int_0^{+\infty} dy_1\int_{y_1}^{+\infty} dy_2\int_{y_2}^{+\infty} dy_3\int_{y_3}^{+\infty} dy_4 (y_1)^{-\frac{1}{2}}e^{-\frac{3\pi\mu}{2}}H^{(1)}_{i\mu}(-ik_1^*y_1)e^{-c_sk_1^*y_1} \\
&\quad\times (y_2)^{-\frac{1}{2}}H^{(1)}_{i\mu}(-k_2^*y_2)e^{-c_sk_2^*y_2}(y_3)^{-\frac{1}{2}}H^{(1)}_{i\mu}(-ik_3^*y_3)e^{-c_sk_3^*y_3}(y_4)^{\frac{5}{2}}K_{i\mu}(k_1^*y_4) \\
&\quad\times K_{i\mu}(k_2^*y_4)K_{i\mu}(k_3^*y_4)e^{-c_sk_4^*y_4}\Bigg]\,\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}) + 23 \text{ permutations of } \mathbf{k_i}
\end{aligned}
\tag{C.17}
$$

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