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Signatures of graviton mass on the Stochastic Gravitational-Wave Background and implications for interferometers

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Abstract

Although astrophysical observations put strong constraints on the graviton mass in late Universe, there is still room for gravitons to acquire an heavy mass during its early stages. In this thesis we study the effect of a massive graviton on the Stochastic Gravitational Wave Background (SGWB) of Cosmological origin. At early time we consider a scenario where graviton mass during inflation originates, in an effective field theory approach, from a primordial mechanism of spontaneous symmetry breaking of space-diffeomorphisms; at late time instead it is considered a recent theory of massive gravity developed by De Felice and Mukohyama which minimally modifies the de Rham-Gabadadze-Tolley (dRGT) theory, and where the assumption of a Lorentz symmetry violation allows the propagation of only two tensor massive modes and ensures the stability of the solution on a Friedmann–Lemaître–Robertson–Walker (FLRW) background. Whereas the light graviton mass in late time modifies the graviton geodesics during its propagation, the heavy mass at early times strongly affects the primordial tensor power spectrum, pushing it toward a more blue tilt. Both these effects may leave distinct signatures on the angular correlators of the SGWB energy density. The analysis of the angular power spectrum is firstly performed analytically, and then numerically exploiting the publicly available code Cosmic Linear Anisotropy Solving System (CLASS), revealing different visible signatures arising from the graviton mass on early and late times on the large and medium scales between $\ell \sim 2$ and $\ell \sim 100$; this multipole domain overlaps with the range of scales where future interferometers as LISA and ET are expected to work, opening the doors for an exciting future. We have also explored the role of the graviton mass on the three-point function of the SGWB energy density, focusing both on primary and secondary non-Gaussianity effects.

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Introduction

Since the detection of the gravitational wave GW150914 from the merging of two black holes by the LIGO and VIRGO collaboration [1], the interest of the scientific community in gravitational waves has rapidly increased. The scenario is really promising for the years to come, when new interferometers, as LISA [11], Einstein Telescope [12], etc., are expected to reach the right sensitivity to detect gravitational waves of cosmological origin [14, 15, 16]. A detection of such gravitational waves would be a crucial test for any mechanism of generation of primordial perturbations, and especially for any inflationary model [16]. However it is likely that such detection of a cosmological SGWB will require the ability to distinguish it from the astrophysical background, which arises from the superposition of signals emitted by a number of unresolved sources. Among many techniques developed to distinguish between the various background, an important tool in this context is the study of its statistics, relying on the hope that future interferometers will allow for a sufficient angular resolution to detect anisotropies of the background. This discussion has been largely investigated yet in [59], but this is not the end of the story. Indeed, one of the most important implication of the GW150914 detection concerns the constraints on the speed of propagation of the gravitational waves. As they seems to propagate approximately at the speed of light, this turns out to be a very strict constraint on the graviton mass, which must be then approximately null. However a little not vanishing value is still not completely ruled out by this event, that set

$$m < 7.7 \times 10^{-23}\ \mathrm{eV}/c^2.$$

More recent and more stringent constraints were posed in later years from the study of planetary orbits (as INPOP17b [2]) and binary pulsars (as PSR J0737−3039A [3] and further developments in [6]), giving the current constraint

$$m < 2 \times 10^{-28}\ \mathrm{eV}/c^2.$$

In light of this constraint, there is still some room for massive gravitons. Actually there is much more. Astrophysical observations indeed can only investigate epochs very close to the present time with respect to the age of the Universe. Therefore, to be precise, it is only fair the say that the above bounds apply on gravitons generated during recent times. Next, one may be tempted to extend these bounds to all the history of the Universe, but this would be a strong assumption. Indeed, due to the expansion, the Universe went through a wide range of energies; while today we see the CMB temperature around the eV scale, it is commonly assumed that the Universe was around 10^{15} GeV during inflation. It's then very unlikely that the same theory of gravitation applies to the whole energy range, and strong deviations from GR are expected at primordial times, when the Universe was hotter. For this reason we cannot apply the above bound to gravitons generated during inflation, for which we really don't have any direct observation yet. Hence it is still open the possibility that gravitons acquire an heavy mass during inflation of the order of the energy driving the accelerated expansion, that is the Hubble rate. In order to avoid possible confusion between the two masses, we will refer with late (time) mass the tiny mass m characterizing graviton after the end of inflation, and early (time) mass the huge mass m_g they posses during this period. Whereas it was studied in literature how these masses modify the angular correlators for CMB anisotropies (see for example [60, 61, 62, 122]), much work has still to be done in the SGWB case, whew the path to study angular correlators has been well traced in [59]. Also on the gravitational wave background, the graviton masses may leave different and distinguishable signatures on the angular correlators which may become visible with future interferometers. The aim of this Thesis is to capture and characterize those signatures arising from the early and late time masses which are expected to fall in the range of sensitivity of interferometers like LISA and ET.

First of all one needs to contextualize the theoretical framework. As just said, we consider two different regimes of the Universe governed by two different and unconnected theories of gravity. The idea of massive gravity was firstly proposed by Fierz and Pauli [13] in 1939. Soon after its publication, it was shown that this theoy suffers of ghost instabilitiy [26], called Boulware-Deser ghost; moreover, two independent works of van Dam, Veltman [27] and Zakharov [28] showed that the massless limit of this theory is discontinuos: this is the so called vDVZ discontibuity. For this reason theories of massive gravity were put aside until 2010, when the illuminating works of de Rham, Gabadadze and Tolley (dRGT) [38, 29] shed to light the existence of ghost free theories of massive gravity. While the dRGT theory admits a Friedmann–Lemaître–Robertson–Walker (FLRW) solution [30], it was shown that all the homogeneous and isotropic background are unstable in the general dRGT because of the appearance of ghosts at non linear level [31]. There are several ways to cirvumvent this issue, as adding extra degrees of freedom [32], or considering its bi-gravity counterpart [33]. In this thesis we rather consider a theory of Lorentz-violating massive gravity recently proposed by De Felice and Mukohyama [56], where only two tensor degrees of freedom are propagated at any pertubative order, and where no ghosts or discontinuities appear. The theoretical framework of the Universe at early times is nicely described combining the tools of massive gravity theories with those of the Effective Field Theories of Inflation(EFTI). The Stückelberg formalism was already introduced in the EFTI where the inflaton dynamics is mimicked by a Goldstone boson which acts as a Stückelberg field for the broken time-reparametrization invariance (see [43, 45] for a detailed review). On top of that one may investigate the possibility to break space-reparametrizations as well. This scenario gives rise, in a bottom-up approach, to a new graviton mass term whose relative weight depends on the "amount" of violation of space-diffeomorphism invariance. The mechanism of spontaneous symmetry breaking introduces further three new Goldstone bosons [47, 46], corresponding to one scalar and two vector extra modes and recovering the total five degrees of freedom carried by a massive graviton. Within this wide scenario we consider two different branches. In one case the Goldstone bosons becomes so massive to be washed out during the expansion of the Universe; in the other all the extra modes are produced and propagate across the Universe until inflation ends. Both these branches provide distinctive features on the primordial power spectra which are the initial conditions for our problem.

We assume that the graviton population of the SGWB was formed contextually with the primordial metric perturbations. Then, once they were produced, they evolved propagating across the perturbed Universe. The evolution of the graviton distribution function is described by the Boltzmann equation on a linearly perturbed FLRW background (non linear effects will be only considered in the last chapter in the squeezed limit configuration for the three-point correlator), while the evolution of metric perturbations are controlled by the Einstein field equations. This approach is very similar to the one which is usually adopted in literature to analyze the CMB anisotropy [70, 63, 114]. The greatest difference between the two cases, besides the obvious observation that photons are massless, stands on the fact that the graviton population is expected to be collisionless, since the relevant collisions that gravitons can undergo are thought to be effective above the Planck energy scale. As a consequence, gravitons are not thermally distributed, and there are no clues about the functional structure of their distribution function. As will be seen later, this feature results in angular anisotropies that have an order one dependence on the GW frequency. This is in contrast with the CMB case, where the photon distribution density follows the well known Bose-Einstein distribution [66, 70, 63], for which this dependence only arises at second order in perturbation theory. The statistical analysis of the two and three-point correlation functions is performed at first perturbative order in harmonic space. Cosmological SGWB inherits its anisotropies both at its production and during its propagation across the perturbed Universe, providing respectively an initial condition contribution and scalar or tensor sourced terms. As a consequence of the non vanishing graviton mass at late times, both these effects lead to a graviton frequency dependence of the anisotropies, which instead is there only in the initial condition term in the massless case. In order to solve the system of differential equations, the explicit expressions for metric perturbations are needed. Since the model of massive gravity considered for late time Universe does not propagate extra scalar or vector modes, no deviation from GR are expected in these sectors, while the Einstein equations for tensor perturbations account for a new mass term.

The evaluation of the three-point function represents a crucial tool to test models, since non-Gaussianity is an unavoidable prediction which characterizes many mechanisms of generation of perturbations. Moreover, the three-point correlators are expected to receive no contributions from astrophysical sources; indeed, in light of the central limit theorem, it seems reasonable to expect the SGWB produced by incoherent astrophysical sources to be gaussian-distributed. Therefore, a measurement of non-Gaussianity would be a measurement of large scale coherency that would suggest a cosmological origin of the signal. These arguments support the three-point correlation functions as the preferred channel to select a cosmological signal of gravitational waves, and this is the reason why non-Gaussianity is so important. In theories of massive gravity there is the possibility to generate a non vanishing three-point correlation function between scalar and tensor modes [47]. This possibility however is not taken into account in this Thesis, but it could be an open opportunity for later works. Two cases in which non-Gaussianity can arise are analyzed. In the first case departure from Gaussianity is taken as an initial condition on the primordial perturbations, and in particular the attention is given to the so called local ansatz [48]. This is the simplest possible deviation from Gaussianity, where the primordial stochastic variable is expressed as a non-linear combination of an auxiliary gaussian variable. Then, a case of *secondary* non-Gaussianity is analyzed. This is the case when, even starting from a gaussian random field, non-Gaussian effects arise because the modes propagate through a perturbed Universe and combine together giving rise to a non-linear evolution described by General Relativity. The formulation pioneered by Weinberg [106] allows to evaluate the bispectra and the non-linear coefficient of the distribution in the simplest squeezed limit configuration, that is the situation where one of the three modes happens to have a much larger wavelength then the other two. Typically one assumes the two short wavelength modes to be inside the horizon, while the long wavelength mode remains outside, in such a way that they can be considered as independent modes, and the effects of the long-mode emerge, through an appropriate coordinate transformation, in the propagation of the short-modes only at second order. Of course, in all the mentioned analysis, particular attention is given to the effects and corrections arising from the non vanishing graviton mass. The work is organized as follows:

The first chapter is dedicated to a brief technical review on the motivations and the construction of theories o massive gravity. The aim of this introductory chapter is to pose a solid theoretical background to our arguments and to convince the reader of the consistency of these theories of modified gravity. Obviously this is far from being a complete discussion of the topic. This arguments open the room to the next Chapter (2), where the application of massive gravity to cosmological models is discussed. Both the mentioned cases of massive gravity during and after inflation are analyzed with the purpose to build the power spectra of the primordial fluctuations.

In Chapter (3), the collisionless Boltzmann equation for the gravition probability distribution density is built taking into account for a late time graviton mass contribution and solved at first order in perturbation theory. In order to connect with observable quantities, in Section (3.3) the graviton distribution is linked to the graviton energy density. Finally the solution is expanded in harmonic space and splitted in three different contributions sourced by scalar or tensor perturbations, or arising from an intrinsic perturbation in the initial configuration.

In Chapter (4) the angular power spectrum is computed exploiting the solution of the Einstein field equation for the scalar and tensor transfer functions.

Chapter (5) focuses on the analysis of non-Gaussianity, considering firstly a primordial source of non-Gaussianity in the local ansatz, and then a secondary source of non-Gaussianity arising at second order in perturbation theory from the propagation across the perturbed Universe.

Finally, in Chapter (6), we turn back to observations and show the numerical prediction for the angular power spectra computed in the previous chapters.

Units and Notation

It is useful at this point to clarify some of the notations that will be used later on. We adopt the convention for the metric signature $(-,+,+,+)$ and work in natural units such that $\hbar = c = k_b = 1$. It is useful to take in mind the following values[1] and conversion factors

$$\begin{aligned}
1\,\mathrm{Mpc} &= 1.56 \times 10^{38}\,\mathrm{GeV}^{-1}\;, \\
M_P &= 1.22 \times 10^{19}\,\mathrm{GeV}, \\
H_0 &= 2.1\,h \times 10^{-42}\,\mathrm{GeV} = 3.28\,h \times 10^{-4}\,\mathrm{Mpc}^{-1}, \\
\eta_0 &= 1.416 \times 10^{4}\,\mathrm{Mpc}, \\
\eta_{eq} &= 112.8\,\mathrm{Mpc},
\end{aligned}$$

where η_0 and η_{eq} denotes the conformal time today and at the epoch of matter-radiation equality respectively. The following rules for symmetrized and antisymmetrized indices on a generic tensor field $A_{\mu\nu}$ are used

$$\begin{aligned}
A_{(\mu\nu)} &= \frac{1}{2}\,(A_{\mu\nu} + A_{\nu\mu}), \\
A_{[\mu\nu]} &= A_{\mu\nu} - A_{\nu\mu}\;.
\end{aligned}$$

[1] These values are taken from the Plank collaboration [109] and the ones used in the publicly available code CLASS [111].

Moreover we will indifferently denote vectors both with a superscript arrow and the bold notation, that is $\vec{k} = \mathbf{k}$. We denote the contraction of two partial derivatives as $\Box = g_{\mu\nu}\partial^\mu\partial^\nu$, $\nabla^2 = g_{ij}\partial^i\partial^j$, while, for conciseness, we will sometimes denote the partial derivative with the usual comma notation, that is $\partial_i A = A_{,i}$. Finally, we will denote with a dot the partial derivative with respect to the coordinate time t, and with a prime the partial derivative with respect to the conformal time η. For a generic quantity A this means

$$\dot{A} \equiv \frac{\partial A}{\partial t}, \qquad\qquad A' \equiv \frac{\partial A}{\partial \eta}.$$

Chapter 1

Introduction to massive gravity

Within the framework of General Relativity (GR), the graviton is normally taken to be a massless spin-2 boson. Massive gravity extends this picture introducing a new dimensionful mass parameter m for gravitons. A leading principle in constructing massive gravity theories is that they should recover GR in the massless limit $m \to 0$. In the next chapters we will always verify that the massless limit recovers the expected results described by GR.

Motivations for massive gravity

Experimental bounds to the graviton mass derives from many observations, but they still leave room for a non-vanishing mass. Motivations for massive gravity can be found from:

- Gravitational waves: the graviton mass affects gravitational waves in two ways. The first intuitive effect is a modification of the speed of propagation of the wave. The graviton speed of propagation is obtained from the dispersion relation linking the frequency ω and the wavelength λ (or equivalently the wavenumber k) of the gravitational wave. The dispersion relation gets an additional contributions from the graviton mass according to

$$\omega^2 = c^2 k^2 + \frac{m^2 c^4}{\hbar^2}. \tag{1.1}$$

This relation is indeed consistent with the definition of the energy carried by the graviton $\hbar\omega$. The graviton group velocity v_g can be read from this expression by the definition

$$v_g \equiv \frac{d\omega}{dk} = c - \frac{m^2 c^5}{2\hbar^2 \omega^2} + \cdots < c. \tag{1.2}$$

As discussed in the introduction, this drop in the speed of propagation allows us to put contraints on the graviton mass. Moreover equation (1.2) tells us further that the graviton group velocity is frequency dependent, and that, when m is nonzero, low frequency gravitational waves travel more slowly than high frequency waves. This introduces a frequency dependent phase delay in the gravitational waveform, from which the LIGO and VIRGO collaboration placed the bound [5]

$$m < 7.7 \times 10^{-23} eV/c^2. \tag{1.3}$$

Second, a mass introduces new polarization states. Massless spin-2 particle have indeed two transverse polarization states, while massive spin-2 particles have five. This feature should be visible, for example, in the rate of spin-down of binary pulsars [4]. This rate is expected to be larger for massive gravity, because the system can radiate gravitational waves into the additional polarization states. As already mentioned, recent developments in experimental precision in measuring this effect have the bound [3, 6]

$$m < 2 \times 10^{-28} eV/c^2. \tag{1.4}$$

- Gravitational force: from the point of view of particle physics, a massive intermediate boson should also modify the laws of gravitation. Within the Newtonian approximation, we expect to recover the Newton's law at small distances, while the mass should induce a damping in the gravitational force for distances larger then the graviton Compton wavelength $\lambda_g = \hbar/mc$. The most reasonable potential which reproduces this trend is the Yukawa potential

$$V(r) = GM\frac{e^{-r/\lambda_g}}{r}, \tag{1.5}$$

where M denotes the mass of the source inducing the gravitational field, and G the Newton constant. Moreover the new polarization states bring new additional forces, often called *fifth forces*. All these contributions can be tested with solar system tests of planetary motion.

- Cosmic acceleration: in the last decades we have found a several number of proofs of the fact that the Universe is accelerating [9]. Within GR, this expansion can be explained assuming the existence of a cosmological constant Λ modifying the Einstein equations and entering as an additional contribution to the energy density ρ in the Friedman equations as (we consider for simplicity a negligible curvature)

$$H^2 = \frac{8\pi G}{3}\tilde{\rho} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}, \tag{1.6}$$

with H the Hubble parameter. The cosmological constant brings a constant contribution to the expansion rate, and then an exponential grow in the scale factor. Physically speaking, Λ can be interpreted as the vacuum energy density, which receives contributions from an intrinsic vacuum energy (usually called "bare") and from the vacuum energy of each particle fields:

$$\rho_{vac}^{obs} = \rho_{vac}^{bare} + \sum_i \alpha_i m_i^4, \tag{1.7}$$

where i runs over all the particle species in our model and α_i are numerical coefficients. Experimental data suggest it has the value $\rho_{vac}^{obs} \sim (10^{-12} GeV)^4$ [17]. This little value has to be compared with the huge amount of vacuum energy provided by the heaviest particle species, the top quark, with mass $m_t \sim 170 GeV/c^2$. This requires a fine cancellation between the bare vaccum contribution and the particle mass contributions to give the correct observed value ρ_{vac}^{obs} which goes under the name of "*cosmological constant problem*" (see [18] for further details). Massive gravity is able to provide alternative explanations for the Universe accelerated expansion. It is possible indeed for massive gravitons to *degravitate* a large cosmological constant [19], i.e. to decouple gravity from the vacuum energy on large scales as a consequence of the Yukawa suppression factor (1.5), such that we have an effect of screening of the cosmological constant; in massive gravity we might not see the full strength of the cosmological constant. Another possible scenario in massive gravity is called "*self-acceleration*", where gravitons, due to the self-interactions, form a condensate whose energy density drives the cosmic acceleration [20]

1.1 Fundamentals of massive gravity

For the sake of completeness we want to give some basic arguments which lead to the construction of theories of massive gravity. This introduction is not intended to be an exhaustive explanation of the subject, but rather a short discussion aiming to show how we can build a viable and consistent theory of massive gravity. If one is interested in further details, we recommend to look at [21] and [22], which are the main references for this section.

As said above, massive gravity is a modification of GR which consists in the introduction of a new graviton mass parameter. In order for the theory to be consistent we must always verify the massless limit $m \to 0$ to be a smooth limit, and that physical predictions converge to those of GR in this limit. In order to understand how to correctly perform this limit, it is instructive to start from the case of a spin-1 massive vector boson.

1.1.1 Consistent $m \to 0$ limit: the Stückelberg trick

The free theory of a massive spin-1 particle A_μ is described by the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{1.8}$$

with J^μ an external conserved current ($\partial_\mu J^\mu = 0$) and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field-strength tensor. The limit $m \to 0$ cannot be performed simply imposing $m = 0$ inside the Proca Lagrangian, because in this case the number of degrees of freedom would discontinuously change from 3, in the massive theory, to 2, in the massless one, and we would then effectively describe two different theories. This occurs because the massless theory enjoys an additional $U(1)$ gauge symmetry transforming the field as $A_\mu \to A_\mu + \partial_\mu\lambda$. In this transformation we have the freedom to choose the parameter λ to fix one of the three degrees of freedom.

The correct massless limit is instead performed by decoupling the third degrees of freedom from the system (for this reason it is usually called *decoupling limit*). To achieve this situation one follows the so-called *Stückelberg trick*: we introduce a new scalar field π in such a way that the new theory enjoys a new $U(1)$ symmetry but still being dynamically equivalent to the old theory. The Stückelberg field is then introduced via the replacement

$$A_\mu \underset{\text{replace}}{\longrightarrow} A_\mu + \frac{\partial_\mu \pi}{m}. \tag{1.9}$$

It worth stressing that this is not a gauge transformation, but a field redefinition leading to a new dynamically equivalent theory. After this substitution the Proca Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu\pi)^2 - m\pi\partial_\mu A^\mu + m^2 A^\mu A_\mu + A_\mu J^\mu - \frac{\pi\partial_\mu J^\mu}{m}, \tag{1.10}$$

and one can verify that this theory is symmetric under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\lambda, \qquad \pi \to \pi - m\lambda. \tag{1.11}$$

By fixing the gauge $\pi = 0$, we recover the original massive theory (1.8). This implies that the two theories are dynamically equivalent, and they both describe three degrees of freedom of a massive spin-1 particle in four dimensions. In the last expression one can appreciate the appearing of a new canonically normalized kinetic term for the scalar field π, which remains untouched in the limit $m \to 0$, where the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu\pi)^2 + A_\mu J^\mu. \tag{1.12}$$

This theory still describes three propagating degrees of freedom where π is completely decoupled from the system. The Stückelberg trick makes clear that gauge symmetries are nothing but redundancies of the theory. Any theory indeed can be made a gauge theory by introducing redundant variables, that are the Stückelberg fields. Having understood how to perform the massless limit, we can now focus on the spin-2 case.

1.1.2 Ghost modes

As we will see in the subsequent sections, a problem one usually has to deal with in building a massive theory of gravity is the appearance of ghost instabilities. In this section we want to explain rapidly when and why this issue arises.

In simple, a ghost is a field which enters in the Lagrangian with the wrong sign in the kinetic term. Let's for example consider the situation of two scalar fields ϕ and χ, with χ a ghost field:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \left(\frac{1}{2}(\partial\chi)^2 + \frac{1}{2}m^2_{\text{ghost}}\chi^2\right) - V(\phi,\chi). \tag{1.13}$$

It's corresponding Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\dot{\phi}^2 + (\partial_i\phi)^2\right) - \frac{1}{2}\left(\dot{\chi}^2 + (\partial_i\chi)^2 + m^2_{\text{ghost}}\chi^2\right) + V(\phi,\chi). \tag{1.14}$$

As one can immediately notice, the wrong sign in the kinetic of the ghost field implies that the energy is not bounded from below, which is a disaster for the construction of the ground state of the theory and the Fock space. From a quantum mechanics point of view, the potential $V(\phi,\chi)$ describes processes where the vacuum spontaneously generate ϕ and χ particles. Energy conservation imposes that the energy of particles must be grater the the ghost mass; therefore these processes of creation of particles have an infinite phase space allowed, and then also the decay rate is infinite. A possible loophole of these arguments is to say that our theory is trustable only below a certain cut-off scale $E_{c.o.}$. If $E_{c.o.} < m_{\text{ghost}}$, decay processes are just artifacts arising from using the Lagrangian (1.13) beyond its regime of validity, where some new physics is expected to become relevant.

Another example of ghost instability particularly relevant for our later discussion is the class of *Ostrogradsky ghosts*, which arise when higher derivative terms are present in the Lagrangian and in the equation of motion. For instance, let's consider the following simple example

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\phi\frac{\Box^2}{\Lambda^2}\phi - V(\phi). \tag{1.15}$$

A posteriori, one can verify that this Lagrangian is equivalent to

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \phi\Box\chi + \frac{1}{2}\Lambda^2\chi^2 - V(\phi), \tag{1.16}$$

where χ is another new scalar field. Indeed we can integrate out the field χ solving the equation of motion, which are simply

$$\chi = -\frac{1}{\Lambda^2}\Box^2\phi. \tag{1.17}$$

Inserting this solution inside (1.16) one immediately recovers (1.15). Now, performing the field redefinition $\phi = \tilde{\phi} - \chi$, the new Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}(\partial\tilde{\phi})^2 + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\Lambda^2\chi^2 - V(\tilde{\phi} - \chi). \tag{1.18}$$

In this form the ghost degree of freedom is manifest, since the field χ enters with the wrong sign in the kinetic term. Notice further that the field redefinition has produced a mass term for the ghost field

$$m_{\text{ghost}} = \Lambda. \tag{1.19}$$

1.1.3 Massless spin-2

The Lagrangian of a massless spin-2 particle can be constructed by hand in such a way to reproduce the linearized Einstein equations [24]

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\sigma h_{\nu\sigma} - \partial_\nu \partial^\sigma h_{\mu\sigma} = -\frac{1}{2M_P} T_{\mu\nu}, \tag{1.20}$$

with $h \equiv \eta^{\mu\nu} h_{\mu\nu}$, M_P the Planck mass and $T_{\mu\nu}$ the stress-energy tensor, which plays the role of an external source. The field $h_{\mu\nu}$ arises instead as a linear deviation of the physical metric $g_{\mu\nu}$ from the flat Minkowski metric $\eta_{\mu\nu}$, that is $g_{\mu\nu} = \eta_{\mu\nu} + M_P^{-1} h_{\mu\nu}$. A Lagrangian which brings (1.20) as equations of motion is

$$\mathcal{L} = -\frac{1}{4} h_{\mu\nu} \mathcal{E}^{\mu\nu}{}_{\alpha\beta} h^{\alpha\beta} + \frac{h_{\mu\nu} T^{\mu\nu}}{2M_P}, \tag{1.21}$$

where the Lichnerowicz operator $\mathcal{E}^{\mu\nu}{}_{\alpha\beta}$ is defined by its action on a symmetric tensor as

$$\begin{aligned} \mathcal{E}^{\alpha\beta}{}_{\mu\nu} h_{\alpha\beta} &= -\frac{1}{2}\Big(\Box h_{\mu\nu} - 2\partial_\lambda \partial_{(\mu} h^\lambda_{\nu)} + \partial_\mu \partial_\nu h + \eta_{\mu\nu}(\partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h)\Big) \\ &= -\frac{1}{2} \epsilon^{\alpha\gamma\sigma}_{\mu} \epsilon_{\nu\beta\lambda\sigma} \partial_\gamma \partial^\lambda h^\beta_\alpha. \end{aligned} \tag{1.22}$$

and enjoys the following properties (they are easy to verify using the expression of the Lichnerowicz operator in terms of the Levi-Civita tensors)

$$\begin{aligned} \mathcal{E}^{\mu\nu}{}_{\rho\sigma} = \mathcal{E}^{\nu\mu}{}_{\rho\sigma} = \mathcal{E}^{\mu\nu}{}_{\sigma\rho} = \mathcal{E}_{\rho\sigma}{}^{\mu\nu}, \\ \partial_\mu \mathcal{E}^{\mu\nu\rho\sigma} = \mathcal{E}^{\mu\nu\alpha\beta} \partial_\alpha \chi_\beta = 0, \end{aligned} \tag{1.23}$$

with χ_μ an arbitrary 1-form.

The action (1.21) is invariant under *linear diffeomorphisms*

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{1.24}$$

that is the generalization of a gauge transformation to the rank-2 tensor $h_{\mu\nu}$.

At the non-linear level instead, Einstein gravity is described by the Hilbert-Einstein action

$$S_{HE} = \frac{M_P^2}{2} \int d^4x \sqrt{-g} R, \tag{1.25}$$

with R the Ricci scalar. The invariance under (1.24) is replaced by invariance under general coordinate transformations, or *diffeomorphisms*, $x \to y = y(x)$, which transform the metric according to

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(y) = \frac{\partial x^\rho}{\partial y^\mu} \frac{\partial x^\sigma}{\partial y^\nu} g_{\rho\sigma}(x(y)). \tag{1.26}$$

Like in the spin-1 case, we expect that a mass term inside the action would spoil the invariance of the theory under diffeomorphisms.

1.1.4 Non-interacting massive spin-2 field

In order to insert a mass term for gravitons, we start from the linearized theory (1.21). In order to respect Lorentz invariance and to fulfill the correct mass dimension, we need an operator which is second order in the field and with all Lorentz indices contracted. The only possibility[1] is then the so called Fierz-Pauli action

$$\mathcal{L}_{FP} = -\frac{1}{4} h_{\mu\nu} \mathcal{E}^{\mu\nu}{}_{\alpha\beta} h^{\alpha\beta} - \frac{1}{8} m^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) + \frac{h_{\mu\nu} T^{\mu\nu}}{2M_P}. \tag{1.27}$$

The normalization of the mass term is chosen in such a way that the propagating degrees of freedom have mass m.

One can easily convince himself that the new mass term spoils the invariance under linear diffeomorphisms. Following the above discussion, gauge symmetry is erased by introducing a Stückelberg field. To respect the Lorentz structure, this must be a vector field with one index, and then

$$h_{\mu\nu} \underset{\text{replace}}{\longrightarrow} h_{\mu\nu} - \frac{1}{m} (\partial_\mu B_\nu + \partial_\nu B_\mu). \tag{1.28}$$

The multiplicative factor $1/m$ is chosen in such a way to produce a canonically normalized kinetic term in the Lagrangian. Defining for convenience the field-strength tensor $F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$, up to quadratic order, the action becomes

$$\begin{aligned} \mathcal{L}_{FP} = -\frac{1}{4} h_{\mu\nu} \mathcal{E}^{\mu\nu}{}_{\alpha\beta} h^{\alpha\beta} - \frac{1}{8} F_{\mu\nu} F^{\mu\nu} + \frac{m}{4} B_\nu (\partial_\mu h^{\mu\nu} - \partial^\nu h) - \frac{m^2}{8} (h_{\mu\nu} h^{\mu\nu} - h^2) \\ + \frac{h_{\mu\nu} T^{\mu\nu}}{2M_P} - \frac{B_\nu \partial_\mu T^{\mu\nu}}{m M_P}. \end{aligned} \tag{1.29}$$

In this case however, after the replacement (1.28), we are still not able to perform the decoupling limit, because the scalar degree of freedom with 0-helicity is still encoded inside the auxiliary massive vector field B_μ. This is exactly the same situation we described in Section (1.1.1). In order to isolate the scalar degree of freedom we introduce another Stückelberg field through

$$B^\mu \underset{\text{replace}}{\longrightarrow} B^\mu + \frac{\partial^\mu \pi}{m}, \tag{1.30}$$

where $\partial^\mu \pi = \eta^{\mu\nu} \partial_\nu \pi$. All in all, the linearized Fierz–Pauli action in terms of $h_{\mu\nu}$ and the Stückelberg fields A_μ and π is obtained from (1.27) performing the Stückelberg replacements

$$h_{\mu\nu} \underset{\text{replace}}{\longrightarrow} h_{\mu\nu} - \frac{2\partial_{(\mu} B_{\nu)}}{m} - \frac{2\partial_\mu \partial_\nu \pi}{m^2}, \tag{1.31}$$

and it becomes

$$\begin{aligned} \mathcal{L}_{FP} = -\frac{1}{4} h_{\mu\nu} \mathcal{E}^{\mu\nu}{}_{\alpha\beta} h^{\alpha\beta} - \frac{1}{8} F_{\mu\nu} F^{\mu\nu} + \frac{m}{4} B_\nu (\partial_\mu h^{\mu\nu} - \partial^\nu h) + \frac{1}{4} \partial_\nu \pi (\partial_\mu h^{\mu\nu} - \partial^\nu h) \\ - \frac{m^2}{8} (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{h_{\mu\nu} T^{\mu\nu}}{2M_P} - \frac{B_\nu \partial_\mu T^{\mu\nu}}{m M_P} - \frac{\partial_\nu \pi \partial_\mu T^{\mu\nu}}{m^2 M_P}. \end{aligned} \tag{1.32}$$

which, after integration by parts, reads

$$\begin{aligned} \mathcal{L}_{FP} = -\frac{1}{4} h_{\mu\nu} \mathcal{E}^{\mu\nu}{}_{\alpha\beta} h^{\alpha\beta} - \frac{m}{4} \partial_\mu B_\nu (h^{\mu\nu} - h\eta^{\mu\nu}) - \frac{1}{4} h_{\mu\nu} (\partial^\mu \partial^\nu \pi - \Box \pi \eta^{\mu\nu}) \\ - \frac{1}{8} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{8} (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{h_{\mu\nu} T^{\mu\nu}}{2M_P} - \frac{B_\nu \partial_\mu T^{\mu\nu}}{m M_P} - \frac{\partial_\nu \pi \partial_\mu T^{\mu\nu}}{m^2 M_P}. \end{aligned} \tag{1.33}$$

[1] Here we are arbitrary choosing the coefficients the terms h^2 and $h_{\mu\nu} h^{\mu\nu}$. This is indeed the only possible choice if we want to avoid ghost instabilities. Below we will give a brief motivation for this fact

In this form the kinetic term for the field π is hidden inside the mixing with $h_{\mu\nu}$. In order to extract the physical propagating degrees of freedom described by this theory we would like to diagonalize this mixing. This can be obtained by performing the shift

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \pi\eta_{\mu\nu}. \tag{1.34}$$

This way the linearized Fierz-Pauli action is

$$\begin{aligned}\mathcal{L}_{FP} = &-\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu}{}_{\alpha\beta}h^{\alpha\beta} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} - \frac{3}{4}(\partial_\mu\pi)^2 - \frac{1}{8}m^2(\tilde{h}^2_{\mu\nu} - \tilde{h}^2)\\ &+\frac{3}{2}m^2\pi^2 + \frac{3}{2}m^2\pi\tilde{h} - \frac{1}{2}m(\tilde{h}^{\mu\nu} - \tilde{h}\eta^{\mu\nu})\partial_{(\mu}B_{\nu)} + 3m\pi\partial_\alpha A^\alpha\\ &+\frac{h_{\mu\nu}T^{\mu\nu}}{2M_P} + \frac{\pi T}{2M_P} - \frac{B_\nu\partial_\mu T^{\mu\nu}}{mM_P} - \frac{\partial_\nu\pi\partial_\mu T^{\mu\nu}}{m^2M_P}.\end{aligned} \tag{1.35}$$

This decomposition allows us to identify the degrees of freedom (dofs) present in the massive gravity at linear level. The theory describes an helicity-2 state $\tilde{h}_{\mu\nu}$ with two dofs, an helicity-1 mode B_μ with again two dofs, and an helicity-0 mode π with one dof, leading to a total of five dofs, as expected for a massive spin-2 field in four dimensions.

At this point we are able to perform the massless limit $m \to 0$. Provided that the external mass source satisfies $m^{-1}\partial_\mu T^{\mu\nu} \to 0$ as $m \to 0$, the Fierz-Pauli Lagrangian in the decoupling limit becomes

$$\mathcal{L}_{FP} = -\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu}{}_{\alpha\beta}h^{\alpha\beta} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} - \frac{3}{4}(\partial\pi)^2 + \frac{\tilde{h}_{\mu\nu}T^{\mu\nu}}{2M_P} + \frac{\pi T}{2M_P}. \tag{1.36}$$

As expected, all the five dofs are still present in the massless limit and they happen to be all decoupled from each other.

Ghost modes: when we introduced the Fierz-Pauli mass term inside the lagrangian (1.27), we implicitly chose the relative coefficient between the terms h^2 and $h_{\mu\nu}h^{\mu\nu}$. This was done because one can show that any other choice would have brought to ghost instabilities, that is instabilities arising from the fact that the theory turns out to be not bounded from below. This situation can occur whenever our theory contains kinetic terms with the wrong sign or terms in the potential with higher order of derivatives (see [25] for a more detailed explanation). In order to understand this statement, let us consider the most generic case, introducing two arbitrary coefficients α and β such that the Fierz-Pauli Lagrangian is

$$\mathcal{L} = -\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu}{}_{\alpha\beta}h^{\alpha\beta} - \frac{1}{8}m^2\left(\alpha h_{\mu\nu}h^{\mu\nu} - \beta h^2\right) + \frac{h_{\mu\nu}T^{\mu\nu}}{2M_P}. \tag{1.37}$$

Applying as usual the Stückelberg procedure and introducing the new field by the replacement (1.31), the part of the Lagrangian which does not vanish in the massless limit is

$$\mathcal{L}^{m\to 0} = -\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu}{}_{\alpha\beta}h^{\alpha\beta} - h_{\mu\nu}\left(\partial^\mu\partial^\nu\pi + (\alpha-\beta)\eta^{\mu\nu}\Box\pi\right) - \frac{\alpha-\beta}{2m^2}(\Box\pi)^2. \tag{1.38}$$

The last term is dangerous for the stability of the theory because it involves an operator with higher order derivatives. This term brings the appearance of a new ghost field whose mass is found to be, retracing the steps outlined in (1.1.2),

$$m^2_{\text{ghost}} = \frac{3}{2}\frac{m^2}{\alpha-\beta}, \tag{1.39}$$

which diverges in the Fierz-Pauli potential, where $\alpha = \beta$. This is then the only viable potential which avoid ghost instabilities.

vDVZ Discontinuity: in Fierz-Pauli Lagrangian (1.36), the coupling

$$\frac{\pi T}{2M_P} \tag{1.40}$$

still survives in the decoupling limit. This leads to an apparent discrepancy with the GR prediction, which cannot be recovered in the limit $m \to 0$, since GR does not contain the π degree of freedom. This issue goes under the name "*van Dam-Veltman-Zakharov (vDVZ) discontinuity*". The solution to this problem was proposed by Vainshteinis, and it basically consists in the fact that the free theory is a bad approximation of GR in a regime that becomes larger and larger as we send $m \to 0$ (we will briefly return to this problem in Section (1.2.3). If the reader is interested in more details, we suggest to look at [21] and [25]).

1.2 Non-linear formulations of massive gravity

We now turn to the construction of a more realistic theory of interacting massive gravity, whose action assumes the most general form

$$S[g_{\mu\nu}] = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\left(R - \frac{m^2}{2}\mathcal{U}(g^{\mu\nu},\eta_{\mu\nu})\right) + S_{\text{matter}}[g_{\mu\nu},\psi], \tag{1.41}$$

where $\mathcal{U}(g^{\mu\nu},\eta_{\mu\nu})$ denotes a generic potential containing contractions of the dynamical metric $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$ which should reproduce the Fierz-Pauli potential at first order in the expansion of the physical metric. At higher order the guess of the potential is more subtle. Boulware and Deser [26] have shown that a simple covariant generalization of the Fierz-Pauli potential

$$\begin{aligned} \mathcal{U} &= \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}(H_{\mu\rho}H_{\nu\sigma} - H_{\mu\nu}H_{\rho\sigma}), \\ H_{\mu\nu} &\equiv g_{\mu\nu} - \eta_{\mu\nu} \end{aligned} \tag{1.42}$$

is not a viable solution, since it is plagued by ghost instabilities. The most general form of a BD-ghost free potential was given by de Rham, Gabadadze and Tolley in [29], and now goes under the name of dRGT theory:

$$\begin{aligned} \mathcal{U}_{\text{dRGT}} &= -\sum_{n=0}^{4}\alpha_n\mathcal{L}_n\left[\mathcal{K}(g^{\mu\nu},\eta_{\rho\sigma})\right], \\ \mathcal{K}(g,\eta)^\mu_\nu &\equiv \delta^\mu_\nu - \sqrt{g^{\mu\alpha}\eta_{\alpha\nu}}, \end{aligned} \tag{1.43}$$

with α_n free coefficients and the mass terms given by

$$\begin{aligned} \mathcal{L}_0[\mathcal{K}] &= 4! \\ \mathcal{L}_1[\mathcal{K}] &= 3!\,[\mathcal{K}], \\ \mathcal{L}_2[\mathcal{K}] &= 2!\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right), \\ \mathcal{L}_3[\mathcal{K}] &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{L}_4[\mathcal{K}] &= [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4], \end{aligned} \tag{1.44}$$

with $[\mathcal{K}] \equiv \mathcal{K}^\mu_\mu$.

For later discussions, it is crucial to mention that this theory admit an analogous formulation in the vielbein language. The idea in passing through the vielbein formalism consist in realizing that the square root structure of the dRGT potential can be linked to the definition of the vielbein variable

$$g_{\mu\nu} = \eta_{AB}\, e^A_\mu e^B_\nu\,, \tag{1.45}$$

with A, B flat indices. Then in the vielbein formalism, the dRGT action is proven to be [23]

$$\begin{aligned} S_{dRGT} = \frac{M_P^2}{2}\int d^4x(\det e)R[e] - \frac{M_P^2 m^2}{8}\sum_{n=0}^{4}\frac{\alpha_n}{n!\,(4-n)!} \\ \times \int \tilde{\epsilon}_{A_1A_2A_3A_4}\mathbf{1}^{A_1}\wedge\ldots\wedge\mathbf{1}^{A_n}\wedge\mathbf{e}^{A_{n+1}}\wedge\ldots\wedge\mathbf{e}^{A_4}, \end{aligned} \tag{1.46}$$

with the vielbein one-form $\mathbf{e}^A \equiv e^A_\mu dx^\mu$, and the identity vielbein $\mathbf{1}^A \equiv \delta^A_\mu dx^\mu$, which can be seen as the vielbein one-form for the flat background metric.

1.2.1 Counting degrees of freedom

In order to verify that this form of the potential provides a consistent ghost-free theory we will exploit the Arnowitt-Deser-Misne (ADM) split of the physical metric. In the ADM 3+1 approach to gravity [36], we foliate the spacetime with space-like hypersurfaces identifying a natural time-direction. The line element gets decomposed as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \end{aligned} \tag{1.47}$$

where N is called the *lapse* function and N^i the *shift vectors*. With this new decomposition the action is written as

$$S = \frac{M_P^2}{2}\int d^4x N\sqrt{\gamma}\left(R_3 + [K]^2 - [K^2] - \frac{m^2}{2}\mathcal{U}(N, N^i, \gamma_{ij})\right), \tag{1.48}$$

where R_3 denotes the Ricci scalar in three dimension for the spatial metric γ_{ij} and K_{ij} is the extrinsic curvature

$$K_{ij} \equiv \frac{1}{2N}\left(\dot{\gamma}_{ij} - 2D_{(i}N_{j)}\right), \tag{1.49}$$

where the dot denotes the usual time derivative $\dot{\gamma}_{ij} = \partial_t\gamma_{ij}$ and D_i stands for the covariant derivative with respect to the spatial metric γ_{ij}. Moreover in (1.48) it was used the usual notation for the contraction of the indices, but be aware that in this decomposition indices are contracted with the metric γ_{ij}, that is $[K] \equiv \gamma^{ij}K_{ij}$. Defining a three-dimensional momenta π_{ij} associated to the spatial metric field as

$$\pi^{ij} \equiv \frac{\delta\mathcal{L}_{HE}}{\delta\dot{\gamma}_{ij}} = \sqrt{\gamma}(K^{ij} - K\gamma^{ij}), \tag{1.50}$$

the action (1.48) can be recast in the so called *first order form* (see equation (3.13) of [36])

$$S[\gamma_{ij}, \pi^{ij}] = \int d^4x\left[\pi_{ij}\dot{\gamma}^{ij} - NC^0(\gamma,\pi) - N_iC^i(\gamma,\pi) - m^2\mathcal{H}_m(\gamma,\pi,N,N_i)\right], \tag{1.51}$$

with

$$\begin{aligned} C^0 &= -R_3 + \frac{1}{\gamma}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) \\ C^i &= 2D_j\left(\frac{\pi^{ij}}{\sqrt{\gamma}}\right). \end{aligned} \tag{1.52}$$

Varying the action with respect to γ_{ij} and π_{ij} we find twelve first order differential equations for the twelve variables, since both the metric tensor field and the associated momenta are symmetric tensors in three dimensions, and thus they both contain six variables. Therefore we need twelve initial condition to uniquely specify the dynamics of the system, corresponding to the six propagating degrees of freedom encoded in γ_{ij}. Actually this is not the real picture, because we might have external constraints which remove some dofs. This is indeed the case of GR with $m = 0$, where the coefficients C^0 and C^i enter as energy and momentum constraints respectively.

In the massive theory, we can minimize the variation of the action with respect to the lapse function (here for convenience we define $N^0 \equiv N$) and the shift vectors to obtain

$$\mathcal{E}^{(N)}_\mu(\gamma_{ij}, \pi^{ij}, N, N^i) \equiv \frac{\delta S}{\delta N^\mu} = C^\mu(\gamma_{ij}, \pi^{ij}) - m^2 \frac{\delta}{\delta N^\mu}\mathcal{H}_m(N, N^i, \gamma_{ij}, \pi^{ij}). \tag{1.53}$$

The solution of this equations will give the expressions for N^μ in terms of the potential H_m, but no constraints will be placed on the initial data. As a result, in this massive theory of gravity we effectively have 6 degrees of freedom, corresponding to five dofs of a massive graviton and a BD ghost dof.

However, for a particular choice of the potential $\mathcal{H}_m$, the equations (1.53) might not have solutions. To fix the idea, imagine to be able to solve only three of the four above equations

$$N^i = N^i(\gamma_{ij}, \pi^{ij}, N). \tag{1.54}$$

This allows us to eliminate the shifts from the action (1.51), which then assumes the general form

$$S = \int d^4x \left(\pi_{ij}\dot{\gamma}^{ij} - \tilde{\mathcal{H}}_m(\gamma_{ij}, \pi^{ij}, N)\right). \tag{1.55}$$

If $\tilde{\mathcal{H}}_m$ is a linear function of the lapse N, then it can be expanded in the form $\tilde{\mathcal{H}}_m = f(\gamma_{ij}, \pi^{ij}) + N\tilde{C}(\gamma_{ij}, \pi^{ij})$. Then the equation of motion for N brings an additional constraint which reduces the total number of dofs to five, corresponding to the massive graviton degrees of freedom[2].

Luckily, the solution (1.43) proposed by C. de Rham enjoys this feature. This can be seen working for example in *minisuperspace* [37], where the lapse N is taken to be homogeneous and the shifts N^i are null. Therefore the metric has the form

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{1.56}$$

and the action is automatically written in the form (1.55). If we want to reproduce the situation depicted above, in order to have the constraint, the action must be linear in the lapse N. If this is not the case, then BD-ghosts unavoidably arise. This test provides a quick and simple check to see whether our theory describes BD-ghost degrees of freedom. The existence of such constraint is a necessary but not sufficient condition for the safety of the theory, because the existence of the constraint in minisuperspace does not guarantee that the constraint exists beyond the minisuperspace. In minisuperspace the first order form of the action (1.51) becomes

$$S = \frac{M_P^2}{2}\int dt \left(\dot{a}\pi_a - N\frac{\pi_a^2}{12M_P^2 a} - m^2 M_P^2 \mathcal{U}_{m.s.}(N, a, \pi_a)\right), \tag{1.57}$$

[2] Actually one should also verify that this constraint is a *second order* constraint, that is its time derivative is null after applying the equations of motion and the first order constraints.

where $\pi_a \equiv 6M_P^2 N^{-1} a\dot{a}$. We just said that the naive covariantization (1.42) of the Fierz-Pauli potential is not a viable solution. Indeed, inserting the explicit expression of the metric in minisuperspace one obtains

$$\mathcal{U}_{m.s.} = N\left(\frac{3(a^2-1)(2a^2-1)}{4a}\right) + \frac{3a(a^2-1)}{N}, \tag{1.58}$$

which contains a non-linear term in N, pointing out the appearance of BD-ghost degrees of freedom.

Conversely, the ghost free potential (1.43) gives rise to linear potentials. For example, the mass term with $\alpha_2 = 1$, $\alpha_{3,4} = 0$ is

$$\mathcal{U}_{dRGT} = 3(a-1)a^2 + N(a-1)(2a-1)a, \tag{1.59}$$

which is clearly linear in N, and then it provides a consistent ghost free theory.

1.2.2 Decoupling limit

From now on we will focus on the theory (1.41) with potential (1.43). What we want to show is that the theory provides a smooth massless limit which recover GR predictions. Following the Stückelberg procedure for gravity, it is useful to recast the theory in a way where diffeomorphisms are restored. Diffeomorphism invariance is broken by the potential, and it can be restored introducing four new Stückelberg scalar fields ϕ^a. We then perform the replacement

$$\eta_{\mu\nu} \underset{\text{replace}}{\longrightarrow} \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \tag{1.60}$$

and the metric fluctuation becomes

$$H_{\mu\nu} = g_{\mu\nu} - \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \tag{1.61}$$

Then the action turns out to be invariant under a global symmetry in the field space

$$\phi^a \to \Lambda^a_b\phi^b, \tag{1.62}$$

and under coordinate transformations where $g_{\mu\nu}$ transforms as a tensor and ϕ^a as a scalar, that is

$$g_{\mu\nu} \to \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(x(y)), \qquad \phi^a \to \phi^a(x(y)). \tag{1.63}$$

This implies further that also the fluctuation $H_{\mu\nu}$ transforms as a rank-2 tensor.

Let us now turn to the discussion of our theory (1.41). We introduce first order perturbations around a Lorentz invariant background (which is the physically relevant choice if we want to recover the predictions of GR in solar system experiments)

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P}, \qquad \phi^a = x^a - \frac{B^a}{mM_P}. \tag{1.64}$$

With the choice of a Lorentz invariant background, the action is invariant under simultaneous Lorentz transformations of the Stückelberg fields and the coordinates

$$\phi^a \to \Lambda^a_b\phi^b, \qquad x^\mu \to \Lambda^\mu{}_\nu x^\nu. \tag{1.65}$$

This means that also B^a must transform as a four-vector under Lorentz transformations, and then we will usually write B^μ. The metric perturbation is

$$\begin{aligned} H_{\mu\nu} &= g_{\mu\nu} - \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \\ &= \frac{h_{\mu\nu}}{M_P} + \frac{2\partial_{(\mu}B_{\nu)}}{mM_P} + \frac{2\partial_\mu\partial_\nu\pi}{m^2M_P} \\ &\quad - \left(\frac{\partial_\mu B_\alpha}{mM_P} + \frac{\partial_\mu\partial_\alpha\pi}{m^2M_P}\right)\eta^{\alpha\beta}\left(\frac{\partial_\nu B_\beta}{mM_P} + \frac{\partial_\nu\partial_\beta\pi}{m^2M_P}\right), \end{aligned} \tag{1.66}$$

where the second line remembers the Stückelberg replacement in the Fierz-Pauli theory (1.31)

Interaction scales

Non-linearities in the potential will manifest themselves as interaction terms for the helicity -2, -1, -0 fields (respectively $h_{\mu\nu}$, B_μ and π). The generic form of the interaction terms can be guessed by dimensional analysis, having in mind that at low energies the dominant contributions come from the interactions at the lowest energy scale. Therefore dimensional analysis suggests that any interaction term should come in the form

$$\mathcal{L}^{(\mathrm{int})}_{n_h,n_B,n_\pi} \sim m^2M_P^2\left(\frac{h}{M_P}\right)^{n_h}\left(\frac{\partial B}{mM_P}\right)^{n_B}\left(\frac{\partial\partial\pi}{m^2M_P}\right)^{n_\pi} \tag{1.67}$$

at the scale

$$\Lambda_p = (m^{1-p}M_P)^{1/p}, \tag{1.68}$$

with

$$p = \frac{n_h + 2n_B + 3n_\pi - 4}{n_h + n_B + n_\pi - 2}. \tag{1.69}$$

Interactions of the form (1.67) have indeed mass dimension four and respect Lorentz invariance as long as all the indices are contracted. Since $m \ll M_P$, larger values of p correspond to interactions at lower scales.

Interactions at Λ_5: Λ_5 is the lowest possible energy scale at which interactions can occur, and it is realized when $n_h = n_B = 0$ and $n_\pi = 3$, that is by terms like

$$\mathcal{L}^{(\mathrm{int})}_{0,0,3} \sim \frac{1}{\Lambda_5^5}(\partial\partial\pi)^3. \tag{1.70}$$

The appearance of higher order derivatives is a signal of the presence of Ostrogradsky ghost instability. In order to avoid this dangerous situation, the ghost-free potentials (1.43) were constructed in such a way that the interactions at Λ_5 scale give rise to vanishing equations of motion, so that the equations of motion for π remain second order.

Interactions at Λ_3: In this case we have infinite ways to realize the interaction terms, but they are restricted to the forms

$$\mathcal{L}^{(\mathrm{int})}_{1,0,n_\pi} \sim \frac{1}{\Lambda_3^{3(n_\pi-1)}}h(\partial\partial\pi)^{n_\pi}, \qquad \frac{1}{\Lambda_3}(\partial B)^2(\partial\partial\pi)^{n_\pi}, \tag{1.71}$$

and always give rise to second order equations of motion. We have then infinite possible interactions at energies above Λ_3, but these are sub-dominant at low energies, and then in the following we will focus just on the Λ_3 interactions.

In this case the Λ_3 decoupling limit is

$$m \to 0, \qquad M_P \to \infty, \qquad \Lambda_3 = (m^2 M_P)^{1/3} \text{ fixed.} \tag{1.72}$$

The full computation of the Λ_3 decoupling limit massive lagrangian is quite long and it goes beyond the purposes of this introduction; if interested, one can find the full calculation explained in [21]. Basically one should apply the full Stukelberg replacement (1.66) inside the Fierz-Pauli action and expand the inverse metric and the determinant in powers of h. We just report the result

$$\mathcal{L}_{D.L.} = \mathcal{L}_{h\pi}[h_{\mu\nu}, \pi] + \mathcal{L}_{B\pi}[B_\mu, \pi] + \frac{h_{\mu\nu}T^{\mu\nu}}{2M_P}. \tag{1.73}$$

For our purposes, we are only interested in the scalar-tensor sector

$$\begin{aligned}
\mathcal{L}_{st} &= \mathcal{L}_{h\pi}[h_{\mu\nu}, \pi] + \frac{h_{\mu\nu}T^{\mu\nu}}{2M_P} \\
&= -\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu}{}_{\alpha\beta}h^{\alpha\beta} + \frac{1}{8}h^{\mu\nu}\left(2\alpha_2 X^{(1)}_{\mu\nu} + \frac{2\alpha_2 + 3\alpha_3}{\Lambda_3^3}X^{(2)}_{\mu\nu} + \frac{\alpha_3 + 4\alpha_4}{\Lambda_3^6}X^{(3)}_{\mu\nu}\right) \\
&\quad + \frac{h_{\mu\nu}T^{\mu\nu}}{2M_P},
\end{aligned} \tag{1.74}$$

where the X tensors are defined as

$$\begin{aligned}
X^{(1)\mu}_{\nu} &\equiv \delta^\mu_\nu[\Pi] - \Pi^\mu_\nu, \\
X^{(2)\mu}_{\nu} &\equiv ([\Pi]^2 - [\Pi^2])\delta^\mu_\nu - 2([\Pi]\Pi^\mu_\nu - (\Pi^2)^\mu_\nu), \\
X^{(3)\mu}_{\nu} &\equiv ([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3])\delta^\mu_\nu \\
&\quad -3([\Pi^2]\Pi^\mu_\nu - 2[\Pi](\Pi^2)^\mu_\nu - [\Pi^2]\Pi^\mu_\nu + 2(\Pi^3)^\mu_\nu),
\end{aligned} \tag{1.75}$$

with $\Pi^\mu_\nu \equiv \eta^{\mu\alpha}\partial_\alpha\partial_\nu\pi$. In this expression the helicity-2 and -0 modes are mixed. In order to diagonalize the kinetic terms of this sector of the theory we perform the field redefinition

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \pi\eta_{\mu\nu} - \frac{2\alpha_2 + 3\alpha_3}{2\Lambda_3^3}\partial_\mu\pi\partial_\nu\pi, \tag{1.76}$$

which extend the field shift (1.34) we introduced in the Fierz-Pauli theory adding additional non-linear interaction terms. As shown in [38], this shift lead to

$$\begin{aligned}
\mathcal{L}_{st} &= -\frac{1}{4}\tilde{h}\mathcal{E}\tilde{h} - \frac{3}{4}(\partial\pi)^2 - \frac{3}{4}\sum_{n=3}^{5}\frac{c_n}{\Lambda^{(n)}_{\text{Gal}}} + (\alpha_3 + 4\alpha_4)\tilde{h}_{\mu\nu}X^{(3),\mu\nu} \\
&\quad + \frac{\tilde{h}_{\mu\nu}T^{\mu\nu}}{2M_P} + \frac{\pi T}{2m^2M_P^2} - (2 + 3\alpha_3)\frac{\partial_\mu\pi\partial_\nu\pi T^{\mu\nu}}{2m^2M_P^2},
\end{aligned} \tag{1.77}$$

where the constants c_n are given by

$$\begin{aligned}
c_3 &= \frac{1}{2}(2 + 3\alpha_3) \\
c_4 &= \frac{3}{2}\left((2 + 3\alpha_3)^2 + 4(\alpha_3 + 4\alpha_4)^2\right) \\
c_5 &= \frac{5}{24}(2 + 3\alpha_2)(\alpha_3 + 4\alpha_4).
\end{aligned} \tag{1.78}$$

The Lagrangian terms $\mathcal{L}_{\rm Gal}$ are known as *Galileon interactions* [42] and thy are explicitly given by

$$\begin{aligned}
\mathcal{L}^{(3)}_{\rm Gal} &= (\partial\pi)^2[\Pi] \\
\mathcal{L}^{(4)}_{\rm Gal} &= (\partial\pi)^2([\Pi]^2-[\Pi^2]) \\
\mathcal{L}^{(5)}_{\rm Gal} &= (\partial\pi)^2([\Pi]^3-3[\Pi][\Pi^2]+2[\Pi^3]).
\end{aligned} \tag{1.79}$$

The upper index (n) refers to the order in π of each term. $\mathcal{L}^{(3)}$ is called the cubic Galileon, $\mathcal{L}^{(4)}$ the quartic Galileon, and $\mathcal{L}^{(5)}$ the quintic Galileon.

1.2.3 Vainshtein mechanism

Vainshtein developed a mechanism to solve the apparent vDVZ discontinuity. The coupling between the helicity-0 mode and matter in massive gravity produces a Vainshtein screening mechanism around a non trivial background. We want now to show explicitly how this mechanism works in the simplest case of a coupling between the cubic Galileon and matter:

$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 + \frac{1}{\Lambda_3^3}(\partial\pi)^2\Box\pi + \frac{\pi T}{2M_P}. \tag{1.80}$$

Let's then perturb this theory around a generic background $\bar{\pi}$:

$$\pi = \bar{\pi} + \varphi. \tag{1.81}$$

Up to quadratic order, the action for the perturbation has the form

$$\mathcal{L} \sim \left(1+\frac{\partial^2\bar{\pi}}{\Lambda_3^3}\right)(\partial\varphi)^2 + \frac{\varphi T}{M_P}. \tag{1.82}$$

Defining the factor

$$Z \sim 1 + \frac{\partial^2\bar{\pi}}{\Lambda_3^3}, \tag{1.83}$$

we can normalize the kinetic term to its canonical form performing the shift $\hat{\varphi} = \varphi/\sqrt{Z}$, obtaining

$$\mathcal{L} \sim (\partial\varphi)^2 + \frac{\hat{\varphi}T}{\sqrt{Z}M_P}. \tag{1.84}$$

This procedure has produced a new coupling between the helicity-0 mode and the matter component. When the background is large, that is $\partial^2\bar{\pi} \gg \Lambda_3^3$, then $Z \gg 1$, and the coupling to matter gets largely suppressed. Recalling the discussion at the end of Section (1.1.4), this was the dangerous term which did not vanish in the decoupling limit of the free theory, spoiling then the compatibility with GR. Non-linear interactions instead cause the coupling between helicity-0 mode and matter to become suppressed in the strong coupling regime. On the other hand, helicity-2 states behave as in GR, since their interactions come in at a much higher scales.

1.2.4 Boulware-Deser ghost

Our new understanding about the Vainshtein mechanism allows us to make a last comment about the issue concerning Boulware-Deser ghost degrees of freedom. We have just said in Section (1.2.2) that interactions at the scale Λ_5 come with a Lagrangian contribution

$$\sim \frac{1}{m^4 M_P}(\partial\partial\pi)^3. \tag{1.85}$$

and with the appearance of ghost degrees of freedom as a manifestation of the presence of higher order derivatives. Expanding π around a background

$$\pi = \bar{\pi} + \varphi, \tag{1.86}$$

the Lagrangian for φ has the form

$$\mathcal{L} \sim (\partial\varphi)^2 + \frac{(\partial\partial\bar{\pi})}{\Lambda_5^5}(\partial\partial\varphi)^2. \tag{1.87}$$

The fluctuation φ appears with higher derivatives in the Lagrangian, leading to a ghost mass

$$m^2_{\text{ghost}} \sim \frac{\Lambda_5^5}{(\partial\partial\bar{\pi})}. \tag{1.88}$$

This of course holds as long as $\partial\partial\bar{\pi} \leq \Lambda_5^3$ such that we are allowed to neglect interactions at larger scales. If the mass of the ghost were large, then we could neglect the ghost degree of freedom in a low energy effective theory, assuming that this theory is not trustalbe at energies comparable to m_{ghost} because new physics might arise in the full theory. However the mass of the ghost (1.87) is background dependent, and it decreases while the background becomes larger. Unluckily this is the necessary condition we found above to trigger the Vainshtein mechanism. These arguments show that the vDVZ discontinuity and the BD-ghost problems cannot be solved simultaneously. This is the reason why building a ghost-free theory of massive gravity (such as the theory (1.41) with the ghost-free potential (1.43)) is so important. The Fierz-Pauli theory is shown to be ghost free at quadratic order as long as the tuning condition $\alpha = \beta$ in (1.37) is satisfied. However, the situation becomes critical at non-linear level, where the massive spin-2 field is coupled to gravity, and the BD ghost cannot be avoided.

After this long digression, we will now move the discussion to define our theory of massive gravitons in a cosmological Universe and in particular to understand how cosmological perturbations propagates in this framework.

Chapter 2

Massive gravity in Cosmology

In our scenario we consider gravitational waves generated during the inflationary period, when the Universe was presumably about 10^{15} GeV hot. After their formation, being decoupled from any other particle species, they freely streamed across the Universe until today, at energies around the eV scale, when they eventually reach our interferometers. As just said, given the large range of energies involved in this scenario, it would be too hasty to formulate a unique theory of gravity describing the whole energy domain. For this reason two regimes are distinguished in this work: an high-energy regime during inflation, and a low-energy regime characterizing the entire period of propagation of graviton, from the end of inflation until today. In this Chapter we use the tools of massive gravity to study the evolution of cosmological perturbations at late time. After the pioneering work by de Rham, Gabadadze and Tolley (dRGT), it was clear that there exist viable theories of massive gravity which avoid the appearance of ghost modes. This was briefly summarised in the previous chapter. The next step is to test these theories on different backgrounds; in particular, for cosmological purposes, it is useful to apply them on a isotropic and homogeneous FLRW background metric. While the original theory defined on a FLRW physical background and a Minkowski fiducial metric does not allow for flat or closed Universes [40], these possibilities are erased if we consider a FLRW fiducial metric as well [39]. However, in the same work [39] it is shown that, at linear level, only the two tensor modes are propagated, while the helicity-0 and -1 modes remain non-dynamical. Soon after, De Felice, Gumrukcuoglu and Mukohyama argued, in an independent work [41], that all homogeneous and isotropic solutions of the dRGT theory in nonlinear massive gravity are unstable due to the appearance of ghost at non-linear level. Different viable solutions have been proposed so far either by abandoning the hypothesis of a homogeneous and isotropic background or adding new extra physical scalar or metric fields (the second case opens to the scenario of bigravity theories). But not everything is lost. In Section (2.3) of this chapter we review the new theory of Lorentz-violating massive gravity proposed by De Felice and Mukohyama [56], which is a viable solution of the problem constructed in such a way to propagate two only tensor degrees of freedom at fully non-linear level and that shares the same FLRW background equations of motion of the dRGT theory. This last feature allows to see this theory, called *Minimal theory of massive gravity*, as a stable non-linear completion of the dRGT solution.

Concerning the study of cosmological perturbation in the early Universe we followed a different approach. In this case there is the additional inflaton field to account for, and the Effective Field Theory approach seems more suitable to address the problem. In this context the graviton mass enters through a mechanism of spontaneous symmetry breaking of space-diffeomorphism invariance. The most simple way to realize this situation is to introduce three extra scalar field acquiring an explicit coordinate-dependent vacuum expectation values during inflation. Excitation around the vacuum give rise to massless Goldstone bosons, which are nothing but the Stückelberg fields restoring the space-time broken symmetries. We distinguish two cases; one where the new Goldstone bosons are so massive to be exponentially suppressed by the accelerated expansion of the Universe [46], and another where the new fields are dynamical and propagate the extra helicity-0 and helicity-1 modes of massive gravitons [47]. These two situations are described in Sections (2.1) and (2.2) respectively.

2.1 MG during inflation from Space-time Symmetry Breaking

In the past years the powerful tool of effective field theory has been successfully applied to the study of single field inflation (EFTI) [43, 44, 45]. General Relativity is a gauge theory build upon invariance under general diffeomorfisms

$$x^{\mu} \;\rightarrow\; x'^{\mu}(x^{\nu}). \tag{2.1}$$

During inflation time-reparametrization invariance

$$t \;\rightarrow\; t+\xi(x^{\mu}) \tag{2.2}$$

must be broken, because of the existence of an inflationary clock which controls how much time is left before inflation ends. Besides time-reparametrization symmetry breaking, since we don't have much information about the Universe at high energies, it is interesting to investigate the possibility that the Universe undergoes a spontaneous space-diffeomorphism symmetry breaking during inflation, that is

$$x^{i} \;\rightarrow\; x'^{i}(t,x^{j}). \tag{2.3}$$

Space diffeomorphisms are broken whenever there is a field that during inflation acquires a non vanishing *vev* depending on the spatial coordinates. We then present a model, studied in [46], where besides the time-reparametrization symmetry broken by the inflaton, all the space-rotation symmetries are spontaneously broken during inflation. Let's start considering a spatially flat background with the spatial metric

$$ds^2_{(3)} = a^2\delta_{ab}e^{(a)}_i e^{(b)}_j dx^i\, dx^j, \tag{2.4}$$

with a the scale factor and $e^{(a)}_i$ the vielbein representing the local SO(3) symmetry , where the indices i,j are space indices (curved indices), while a,b are the indices of the internal group (flat indices). In this framework, the local SO(3) invariance is broken by the choice of a preferred rigid spatial frame, as $e^{(a)}_i = \delta^a_i$. The minimal prescription to achieve this configuration is to introduce three new scalar fields responsible for the space-symmetry breaking. Analogously to the case of the inflaton, during the slow roll phase the theory spontaneously select a vacuum expectation value for the new three fields which breaks the local SO(3) symmetry, that is

$$\varphi^a = \delta^a_i x^i. \tag{2.5}$$

Excitation around the vacuum generate three massless Goldstone bosons π^a, which are nothing but the Stückelberg fields restoring, at non linear level, the symmetry under local space rotations. This is parametrized as

$$\varphi^a = \delta^a_i x^i + \pi^a. \tag{2.6}$$

Concerning the internal group, it is natural to assume some symmetries to restrict the structures of the possible operators entering in the action. We then impose the rotational SO(3) symmetry and the rescaling symmetry in the internal group

$$\varphi^a \to \Lambda^a_b \varphi^b, \qquad \varphi^a \to \lambda \varphi^a. \tag{2.7}$$

The usefulness of these symmetries will be clear in a while. As shown above, the breaking of spatial diffeomorphism manifestly appears with a new graviton mass term. Adapting the notation

$$Z^{ab} \equiv g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b, \qquad \bar{\delta}Z^{ab} \equiv \frac{Z^{ab}}{Z} - 3\frac{\delta_{cd}Z^{ac}Z^{bd}}{Z^2} \tag{2.8}$$

introduced in [53], the most general action for the inflaton ϕ and the metric $g^{\mu\nu}$ with broken spatial rotations can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_P^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{9}{8}M_P^2 m_g^2(\bar{\delta}Z^{ab})^2 + \dots\right], \tag{2.9}$$

with m_g a new parameter with mass dimension $[m_g] = [M]$ denoting the graviton mass during inflation, while the dots stand for higher order terms in the Stückelberg fields π^a and the metric $g^{\mu\nu}$. For the shortness of notation it was further defined $(\bar{\delta}Z^{ab})^2 = \delta_{ac}\delta_{bd}Z^{ab}Z^{cd}$. The numerical factor 9/8 in the last term is just a normalization factor introduced for later convenience. Speaking about the graviton mass, one can think that a certain relation between m_g and the inflaton ϕ exists, in such a way that the energy scale of the graviton mass is about the scale of inflation, that is approximately $10^{15} \div 10^{13}$GeV. At the exit of inflation, the inflaton field start decaying, and contextually the value of the graviton mass rapidly decreases going beneath the current astrophysical bounds. Even in absence of any functional dependence between the two, it is reasonable to maintain this mass scale, since it is the only energy scale characterizing the inflationary period. Notice further that by construction, the new term $\bar{\delta}Z^{ab}$ does not contribute to the background energy momentum tensor, such that the background inflationary dynamics is completely governed by the inflaton ϕ as in the standard single field model paradigm.

The inheritance of the mechanism of spontaneous symmetry breaking is visible in the appearance of three Nambu-Goldstone bosons π^a in the third term in the action. As one is used to do in the Higgs mechanism, it is possible to define a *unitary gauge* where these Goldstone boson are eaten by gravitons (the spatial part of the metric), setting then $\pi^a = 0$. This way gravitons develop one helicity-0 and two helicity-1 extra modes. However, thanks to Lorentz symmetry violation, the appearance of the extra modes can be avoided. Let's decompose the Stückelberg fields in the helicity basis

$$\pi^a = \rho(x)\delta^a_j x^j + \omega^{ab}(x)\delta_{bj}x^j, \tag{2.10}$$

where $\rho(x)$ is the helicity-0 mode and $\omega^{ab}(x)$, with a and b antisymmetric indices, the two helicity-1 modes. Considering long wavelength perturbations such that

$$\partial_i\rho \ll H\rho, \qquad \partial_i\omega^{ab} \ll H\omega^{ab}, \tag{2.11}$$

one can see that these perturbations can be reabsorbed through an infinitesimal symmetry transformation (2.7) at leading order in gradient expansion. Then, the three extra modes originating from symmetry breaking are not dynamical. Nonetheless, as will be clear in the following, they provide the source for a graviton mass term.

The same term contains interactions between gravitational perturbations and the Nambu-Goldstone bosons. However at high energies gravity decouple and these interactions can be neglected during inflation. In this regime the "*decoupling limit*" one can expand the mass term in the action at second order in π-fluctuations and obtain

$$S_\pi = \frac{9}{4}\Lambda^4 \int d^4x \left(\partial_i \pi^j \partial_i \pi^j + \frac{1}{3}(\partial_i \pi^a)^2\right) \sim \Lambda^4 k^2 \pi^i \pi^i, \tag{2.12}$$

with $\Lambda \equiv M_P m_g$. At lowest operator dimension, the symmetry pattern of our theory prevents the possibility for Goldstone bosons to have a kinetic term. This contribution must then be searched in higher order terms like $g^{\mu\nu}\partial_\mu \bar{\delta} Z^{ij} \partial_\nu \bar{\delta} Z^{ij}$, which provides

$$S_\pi \sim \Lambda^4 k^2 (\pi^i)^2 + \Lambda^2 k^2 (\dot{\pi}^i)^2. \tag{2.13}$$

In order to isolate a canonically normalized kinetic term one must redefine $\pi \to \Lambda k \pi^i$; this way the three Nambu Goldstone bosons enhance their mass to $m_\pi^2 \sim \Lambda^2$. Assuming that inflation and massive gravity are governed by the same underlying physics at high energies, then it is natural to identify the energy scale of the graviton mass with the energy scale of the mixing between gravity and inflation. This energy scale is encoded in the kinetic interaction term between the inflaton and the metric perturbation. In the same way we introduced the three fields φ^a for broken spatial rotations, one can regard the inflaton ϕ as the field which breaks time-reparametrization invariance selecting a preferred rest frame during inflation. The simplest choice is to consider that the inflaton acquires an explicit time-dependent background *vev*, as

$$\bar{\phi}(t) = t, \tag{2.14}$$

while excitation around the vev are parametrized by a new Stückelberg field π which non-linearly recovers time diffeomorphism invariance

$$\phi(t, x) = t + \pi(x). \tag{2.15}$$

Consider the (kinetic) inflaton sector of the action (2.9). At the background level it can be rewritten in the following useful way

$$S_\phi = \int d^4x \sqrt{-g}\, \frac{1}{2}\dot{\phi}^2 g^{00} = -\int d^4x \sqrt{-g} M_P^2 \dot{H} g^{00}, \tag{2.16}$$

where in the second step it was used the relation $\dot{\phi}^2 = -2M_P^2 \dot{H}$ of the standard inflationary paradigm. To reintroduce the Goldstone boson π, let's follow this procedure [44]. Under a broken time-diffeomorphism $t \to \tilde{t} = t + \xi^0(x)$, the time-time component of the metric transforms as

$$g^{00}(x) \to \tilde{g}^{00}(\tilde{x}(x)) = \frac{\partial \tilde{x}^0}{\partial x^\mu}\frac{\partial \tilde{x}^0}{\partial x^\nu} g^{\mu\nu}(x). \tag{2.17}$$

In terms of the transformed field, the action is

$$S_\phi = \int d^4\tilde{x} \sqrt{-\tilde{g}} M_P^2 \dot{H}(\tilde{t} - \xi^0(x(\tilde{x}))\, \frac{\partial x^0}{\partial \tilde{x}^\mu}\frac{\partial x^0}{\partial \tilde{x}^\nu} \tilde{g}^{\mu\nu}(\tilde{x}(x)). \tag{2.18}$$

The Goldstone boson enters when we promote the shift ξ^0 to a field through the substitution

$$\xi^0(x(\tilde{x})) \to -\tilde{\pi}(\tilde{x}), \tag{2.19}$$

thanks to which the action is

$$\begin{aligned} S_\phi &= \int d^4x\sqrt{-g}M_P^2\dot{H}(t+\pi(x))\,\frac{\partial(t+\pi(x))}{\partial x^\mu}\frac{\partial(t+\pi(x))}{\partial x^\nu}g^{\mu\nu}(x) \\ &= \int d^4x\sqrt{-g}M_P^2\dot{H}(t+\pi)\Big[(1+\dot{\pi})^2g^{00}+2(1+\dot{\pi})\partial_i\pi g^{0i}+\partial_i\pi\partial_j\pi g^{ij}\Big], \quad (2.20) \end{aligned}$$

where for the ease of notation the tilde has been dropped. The estimation of the mixing energy scale is very important in this context, since at energies above this threshold the dynamics of the Goldstone boson π and the metric fluctuations decouple. Indeed in equation (2.20) it is clear that the quadratic terms which mix π and $g_{\mu\nu}$ contain fewer derivatives than the kinetic term of π, so that they can be neglected above some high energy scale expressed by the coefficients of the mixing operators. Canonical normalization of the kinetic term requires the field redefinition $\pi_c = M_P\dot{H}^{1/2}\pi$, while in order to re-introduce the correct mass dimension of the metric fluctuation fields it is need to define $\delta g_c^{00} = M_P\delta g^{00}$. After these redefinitions the mixing term reads

$$M_P^2\dot{H}\dot{\pi}\delta g^{00} \sim \dot{H}^{1/2}\dot{\pi}_c\delta g_c^{00}. \quad (2.21)$$

And then finally the mixing energy is

$$E_{\text{mix}} \sim \dot{H}^{1/2} \sim \epsilon^{1/2}H, \quad (2.22)$$

where $\epsilon = -\dot{H}/H^2$ is the usual slow-roll parameter.

Therefore, for what stated above, $m_g^2 \sim \Lambda_{\text{mix}}^2 \sim \dot{H}$. In the end, the masses of the Nambu-Goldstone bosons are estimated as $m_\pi^2 \sim M_P\dot{H}^{1/2} \gg H^2$. Therefore any excitation of this boson exponentially decays away during inflation because the Universe was not hot enough to source fluctuations of the π^i-fields. From the point of view of the EFTI one can think to integrate out the heavy fields and neglect higher order corrections in such a way that the three Nambu-Goldstone bosons can be completely neglected.

To be precise, the above discussion presented in [46] shows that extra modes are non-dynamical in the long wavelength regime. This situation is however more general. The fact that extra modes of the graviton are not produced in this model at any scale is more rigorously demonstrated in [62], where the full Hamiltonian analysis of the action (2.9) reveals the presence of two only degrees of freedom (the helicity-2 ones).

2.1.1 Cosmological perturbations

Let's consider a first order perturbed spacetime around a flat FLRW background which in full generality is described by the following metric elements[1]

$$\begin{aligned} g_{00} &= -a(\eta)^2\left[1-2\Phi(\eta,\vec{x})\right] \\ g_{0i} &= a(\eta)^2\omega_i(\eta,\vec{x}), \\ g_{ij} &= a(\eta)^2\left\{\left[1+2\Psi(\eta,\vec{x})\right]\delta_{ij}+\chi_{ij}(\eta,\vec{x})\right\}, \end{aligned} \quad (2.23)$$

where $a(\eta)$ is the scale factor in conformal time η, Φ and Ψ two scalar perturbations, while

$$\omega_i = \partial_i\omega'' + \omega_i^\perp,$$
$$\chi_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\chi'' + \partial_i\chi_j^\perp + \partial_j\chi_i^\perp + \chi_{ij}^T, \quad (2.24)$$

[1] This decomposition is allowed because the $SO(3)$ symmetry of the background spacetime is still unbroken.

with $\delta^{ij}\chi_{ij} = 0$. The superscript $''$ denotes the scalar components, while the superscript $\perp$ stands for the vector parts, and "T'' for the transverse and traceless tensor perturbation. Let us anticipate that in the subsequent chapters, in order to simplify the computations, it will be often used a particular gauge choice, the Poisson gauge, which simplifies the picture demanding one vector and two scalar metric perturbations to be vanishing. In particular it provides

$$\text{Poisson gauge:} \begin{cases} \omega'' = 0, \\ \chi'' = 0, \\ \chi_i^\perp = 0. \end{cases} \tag{2.25}$$

However, for the time being, the most useful gauge to adopt is the so called *unitary gauge*, where the perturbations of all Stückelberg fields are turned off.

Tensor-type perturbations

Let's consider a perturbed metric of the form

$$\begin{aligned} ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + \chi_{ij})\, dx^i dx^j\right], \\ \chi_i^i = \partial_i \chi_j^i = 0. \end{aligned} \tag{2.26}$$

The aim of this section is to investigate the contribution of the new term in (2.9) arising from the broken spatial SO(3) symmetry to the tensor metric perturbations. With this goal in mind it is useful to expand

$$\begin{aligned} (\bar{\delta} Z^{ab})^2 &= \frac{Z^{ab}Z^{cd}}{Z^2}\delta_{ac}\delta_{bd} - 6\frac{Z^{ab}Z^{cd}Z^{ef}}{Z^3}\delta_{ae}\delta_{cf}\delta_{bd} \\ &\quad +9\frac{Z^{ab}Z^{cd}Z^{ef}Z^{hl}}{Z^4}\delta_{bd}\delta_{fl}\delta_{ae}\delta_{ch}. \end{aligned} \tag{2.27}$$

Since we want to focus on the metric tensor fluctuations, in the decoupling limit we can forget about any contribution involving a Stückelberg field π^a. This way the expression for Z^{ab} greatly simplifies to[2]

$$Z^{ab} = g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b \approx g^{ab} = a^{-2}(\eta)\left(\delta^{ab} + \chi^{ab}\right), \tag{2.28}$$

while the trace must be taken contracting Z^{ab} with the Kronecker delta, because the indices a and b live in the flat internal group, and are then raised and lowered with the flat Minkowski metric; then

$$Z = Z^{ab}\delta_{ab} = g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi_a \approx g^a{}_a \approx 3. \tag{2.29}$$

This way one can easily evaluate separately each contribution to (2.27). Picking up just the terms which are quadratic in the fluctuation χ_{ij}[3], one finds

$$Z^{ab}Z^{cd}\delta_{ac}\delta_{bd} \approx g^{ab}g_{ab} \approx \chi_{ij}\chi^{ij}, \tag{2.30}$$
$$Z^{ab}Z^{cd}Z^{ef}\delta_{ae}\delta_{cf}\delta_{bd} \approx g^a{}_b\, g^b{}_c\, g^c{}_a \approx 3\chi_{ij}\chi^{ij}, \tag{2.31}$$
$$Z^{ab}Z^{cd}Z^{ef}Z^{hl}\delta_{ac}\delta_{bd}\delta_{ch}\delta_{fl} \approx g^{ab}g_{bc}g^{cd}g_{da} \approx 6\chi_{ij}\chi^{ij}. \tag{2.32}$$

[2]In this section we use the symbol $\approx$ to denote a relation which holds in the decoupling limit neglecting all the Nambu-Goldstone boson contributions

[3]Linear terms in the action do not bring any contribution to the equations of motion, while higher order terms are suppressed in amplitude as long as the perturbative regime holds

Summing up all these contributions equation (2.27) evaluates to

$$(\bar{\delta} Z^{ab})^2 \approx \frac{1}{9}\left[\chi_{ij}\chi^{ij} - \frac{18\chi_{ij}\chi^{ij}}{3} + \frac{54\chi_{ij}\chi^{ij}}{9}\right] = \frac{1}{9}\chi_{ij}\chi^{ij}. \tag{2.33}$$

As last step, one can expand the Ricci scalar as in [52] to obtain the final expression of the quadratic action for tensor modes in conformal time

$$S_T^{(2)} = \frac{M_P^2}{8}\int d^4x a^2(\eta)\left[\chi'_{ij}\chi^{ij\,\prime} - (k^2 + a^2(\eta)m_g^2)\chi_{ij}\chi^{ij}\right]. \tag{2.34}$$

The consequences of spatial SO(3) symmetry breaking are manifest in the fact that tensor modes acquire a new mass contribution. As done for the inflaton, one can solve the equations of motion for the tensor modes and compute the primordial power spectrum. The Euler-Lagrange equations for the action (2.34) are

$$\chi''_{ij}(\eta,\mathbf{k}) + 2\mathcal{H}\chi'_{ij}(\eta,\mathbf{k}) + (k^2 + m_g^2 a^2)\chi_{ij}(\eta,\mathbf{k}) = 0. \tag{2.35}$$

As before, it is convenient to decompose the field χ_{ij} into its Fourier modes projecting the solutions into the two helicity-2 states:

$$\chi_{ij}(x) = \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda=\pm 2}\epsilon^{\lambda}_{ij}(\mathbf{k})\chi_\lambda(\mathbf{k},\eta)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.36}$$

with λ the polarization index and $\epsilon^{\lambda}_{ij}(\mathbf{k})$ the polarization tensors satisfying

$$\epsilon^{\lambda}_{ii} = k^i\epsilon^{\lambda}_{ij} = 0, \qquad \epsilon^{\lambda *}_{ij}(\mathbf{k}) = \epsilon^{-\lambda}_{ij}(\mathbf{k}) = \epsilon^{\lambda}_{ij}(-\mathbf{k}), \qquad \epsilon^{\lambda}_{ij}\epsilon^{-\lambda'}_{ij} = 2\delta_{\lambda,\lambda'}. \tag{2.37}$$

Fourier modes can be expanded in the basis of positive and negative frequency solutions as

$$\chi_\lambda(\mathbf{k},\eta) = (2\pi)^{3/2}\left[b^{\lambda}_{k}\chi(k,\eta) + b^{\lambda\,\dagger}_{-k}\chi(k,\eta)^*\right], \tag{2.38}$$

where b^{λ}_{k} is the annihilation operator, and the positive frequency modes satisfy the Klein-Gordon normalization

$$\chi_k\chi_k^{*\prime} - \chi_k^*\chi_k' = \frac{i}{2a^2}. \tag{2.39}$$

Notice that in this decomposition the only quantity which contain a time-dependence is the mode $\chi(k,\eta)$. Therefore, defining the rescaled field $h(k,\eta) = a(\eta)\chi(k,\eta)$, equation (2.35) becomes

$$h''(k,\eta) + \left(k^2 - \frac{a''}{a} + a^2 m_g^2\right)h(k,\eta) = 0. \tag{2.40}$$

If we consider that inflation had been taking place with a quasi de Sitter expansion, the relation between conformal time and the scale factor is simply found as

$$\begin{aligned} d\eta &= e^{-H(1+\epsilon)t}dt \\ \eta &= -\frac{1}{aH(1+\epsilon)}, \end{aligned} \tag{2.41}$$

where H denotes the nearly constant Hubble parameter during the inflationary period; then the factor a''/a evaluates to

$$\frac{a''(\eta)}{a(\eta)} = \frac{2}{\eta^2}\left(1 + \frac{3}{2}\epsilon\right). \tag{2.42}$$

This way (2.40) becomes

$$h''(k,\eta) + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right] h(k,\eta) = 0, \tag{2.43}$$

where the new quantity $\nu^2 = \frac{9}{4} + 3\epsilon - \frac{m_g^2}{H^2}$ is introduced for later convenience in such a way to reproduce the Bessel equation whose solution is

$$h(k,\eta) = N\sqrt{-\eta}\left[c_1(k)H_\nu^{(1)}(-k\eta) + c_2(k)H_\nu^{(2)}(-k\eta)\right], \tag{2.44}$$

with $H_\nu^{(1)}(-k\eta)$ the Hankel function of the first species and order ν, and N a normalization factor which is set to $N = \sqrt{32\pi G} = 2/M_P$ by the normalization condition (2.39). Consistency with the Bunch-Davies vacuum choice in the regime $k\eta \gg 1$ demands $c_2 = 0$ and $c_1 = \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}$. At the end of the day, the solution reads

$$h(k,\eta) = \frac{\sqrt{\pi}}{M_P} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{-\eta} H_\nu^{(1)}(-k\eta), \tag{2.45}$$

which for the original field $\chi(k,\eta)$ is, at first order in slow-roll parameter expansion,

$$\chi(k,\eta) = -\frac{\sqrt{\pi}H\eta}{M_P} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{-\eta} H_\nu^{(1)}(-k\eta). \tag{2.46}$$

Hence the dimensionless power spectrum for tensor perturbations is

$$\begin{aligned} \mathcal{P}_\lambda(k) &= \frac{k^3}{2\pi^2}|\chi(k,\eta)|^2 \\ &= \frac{\bar{H}^2}{2\pi M_P^2}\left(\frac{k}{aH}\right)^3 \left|H_\nu^{(1)}\left(\frac{k}{aH}\right)\right|^2, \end{aligned} \tag{2.47}$$

with

$$\nu = \frac{3}{2}\sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{H^2} - 3\epsilon\right)}. \tag{2.48}$$

If we are interested in the primordial power spectrum generated during inflation, that is when the considered scale crossed the horizon for the first time, we must consider the above expression in the super-horizon regime, that is $k\eta \ll 1$. In this regime the Hankel function has the following limiting expression

$$H_\nu^{(1)}(z) \underset{z\ll 1}{\sim} \sqrt{\frac{2}{\pi}} e^{-i(\frac{\pi}{2})} 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right) z^{-\nu}. \tag{2.49}$$

Substituting this result in the above expression, the primordial tensor power spectrum becomes

$$\mathcal{P}_\lambda(k) = \frac{2^{2\nu-3}H^2}{\pi^2 M_P^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 \left(\frac{k}{aH}\right)^{n_T}, \tag{2.50}$$

with the tensor spectral index defined by

$$n_T \equiv 3 - 2\nu = 3\left[1 - \sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{H^2} - 3\epsilon\right)}\right]. \tag{2.51}$$

This expression generalizes the results of [34, 35] for a quasi-de Sitter expansion; one can verify indeed that the solutions match in the limit $\epsilon \to 0$. Notice that if

$$\frac{m_g^2}{H^2} > 3\epsilon, \tag{2.52}$$

the tensor power spectrum acquires a blue tilt which could be a smoking gun for both the inflationary period and massive gravity during the early universe. The blue tilt becomes unavoidable in the limiting case of a pure de Sitter stage, where the slow roll parameter vanishes. As a confirmation of this results, one can see that in the limit of small masses ($m_g^2/H^2 \ll 1$) ad pure de Sitter inflation, equation (2.50) recovers the ones shown in section (5.1) of ref. [16] in the limit for $c_T = 1$, that is

$$\mathcal{P}_\lambda(k) = \frac{\bar{H}^2}{2\pi^2 M_P^2}\left(\frac{k}{k_*}\right)^{n_T}, \tag{2.53}$$

with

$$n_T = \frac{2}{3}\frac{m_g^2}{\bar{H}^2}, \tag{2.54}$$

and k_* a reference scale defined by $k_* = (aH)_{h.c.}$ evaluated at the time of horizon crossing.

Scalar-type perturbations

Consistent with the Hamiltonian analysis of [62], no scalar degrees of freedom are expected to emerge in the gravity sector. Indeed the only gauge invariant scalar quantity described by the theory is found to be (see the Appendix of ref. [62] for a full calculation of the second order action)

$$\zeta \equiv \Phi + H\frac{\delta\phi}{\dot{\phi}}, \tag{2.55}$$

which is usually called *gauge invariant curvature perturbation*, and Ψ turns out to be a Lagrange multiplier for the theory. Scalar type perturbations are then directly linked to inflaton fluctuations $\delta\phi$, which are the only source to the energy momentum tensor during inflation. As already pointed out, the term proportional to $(\bar{\delta} Z^{ab})^2$ in the action (2.9) is decoupled from the inflaton sector. The inflaton dynamics is then completely untouched by symmetry breaking of the group SO(3). Therefore one can safely use the well known results for single field inflation widely studied in literature (see [75, 63, 54] for a review). For this reason we will not go through all the details, but just mention the most important results. In Section (4.3) we will learn how to relate this quantity to the scalar metric perturbations. For the time being we are just interested in evaluating the dimensionless power spectrum of primordial curvature perturbations. One can show further that the above relation on super horizon scales and in Poisson gauge [75] simplifies to

$$\zeta \simeq H\frac{\delta\phi}{\dot{\phi}}, \tag{2.56}$$

such that we are able to relate the power spectrum of the curvature perturbations to the power spectrum of the inflaton fluctuations in the the following way

$$\mathcal{P}_\zeta = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}|\delta\phi|^2 = \frac{k^3}{4\pi^2 M_P^2\epsilon}|\delta\phi|^2. \tag{2.57}$$

What is left to investigate then is the time evolution of the fluctuations of the inflaton field $\delta\phi$. This is encoded in the solution of the Euler-Lagrange equations for the action (2.9), which are [75]

$$\ddot{\delta\phi}(\mathbf{x},t) + 3H\dot{\delta\phi}(\mathbf{x},t) - \frac{\nabla}{a^2}\delta\phi(\mathbf{x},t) + \frac{\partial^2 V(\phi)}{\partial\phi^2}\delta\phi = 2\Psi\frac{\partial V(\phi)}{\partial\phi}. \tag{2.58}$$

As done for the tensor case, it is convenient to define a rescaled field $\hat{\delta\phi} = a(t)\delta\phi$, which in light of the prescription of second quantization can be written as

$$\hat{\delta\phi}(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[u_k(t)a_k e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(t)a_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{2.59}$$

Passing to conformal time the equations of motion in Fourier space are

$$u_k'' + \left[k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V(\phi)}{\partial\phi^2} - 6\epsilon\mathcal{H}^2\right]u_k = 0. \tag{2.60}$$

At this point one should recognize the equation (2.40) for tensor modes. Therefore one can use the same solution with a proper redefinition of the indices, that is

$$|\delta\phi_k| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}, \tag{2.61}$$

and

$$\frac{3}{2} - \nu \simeq \eta_v - 3\epsilon, \qquad \eta_v \equiv \frac{1}{3H^2}\frac{\partial^2 V(\phi)}{\partial\phi^2} \ll 1. \tag{2.62}$$

Finally the dimensionless power spectrum of the curvature perturbation is

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2 M_P^2\epsilon}\left(\frac{k}{aH}\right)^{n_s}, \tag{2.63}$$

with the scalar spectral index defined by

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}(k)}{d\ln k} = 3 - 2\nu = 2\eta_v - 6\epsilon. \tag{2.64}$$

Vector-type perturbations

Expanding the action (2.9) at second order in vector perturbations as in [62] one finds, in Fourier space,

$$S^{(2)}_{\text{vector}} = \frac{M_P^2}{4}\int d^3\mathbf{k}\,dt\left(\frac{1}{4}k^2a^3\dot{\omega}_i^\perp\dot{\omega}^{\perp\,i} - k^2a^2\chi_i^\perp\dot{\omega}^{\perp\,i} + k^2a\chi_i^\perp\chi^{\perp\,i}\right.$$
$$\left. - \frac{1}{4}m_g^2a^3\omega_i^\perp\omega^{\perp\,i}\right). \tag{2.65}$$

It is clear then that the modes $\chi^{\perp\,i}$ are not dynamical, since they don't have any kinetic term; then their equations of motion simply give the constraint

$$S_i = \frac{1}{2}a\dot{\omega}_i^\perp. \tag{2.66}$$

Once we insert this expression inside the vector action, it is straightforward to realize that the kinetic term for the modes $\chi_i^\perp$ cancels out as well. We indeed have

$$S^{(2)}_{\text{vector}} = -\frac{M_P^2}{16}\int d^3\mathbf{k}\,dt\,m_g^2a^3\omega_i^\perp\omega^{\perp\,i}. \tag{2.67}$$

Hence no propagating vector modes are present in the theory, as in the GR case.

2.2 MG during inflation with propagating extra modes

In the past years the powerful tool of effective field theory has been successfully applied to the study of single field inflation (EFTI). General Relativity is a gauge theory build upon invariance under general diffeomorfisms

$$x^\mu \;\to\; x'^\mu(x^\nu). \tag{2.68}$$

During inflation time-reparametrization invariance

$$t \;\to\; t + \xi(x^\mu) \tag{2.69}$$

must be broken, because of the existence of an inflationary clock which controls how much time is left before inflation ends. In single field models there is just one clock, which must be related to the dynamics of the inflaton field. From a larger perspective, at high energies gravity should decouple from the Universe, such that space-time diffeomorfisms reduces to an exact global symmetry. As soon as inflation starts, time reparametrization invariance breaks down giving rise to a massless Goldstone boson π. In the context of particle physics, the Goldstone boson equivalence theorem states that the amplitude for a process with the exchange of a longitudinally polarized massive boson is equivalent to an amplitude where the massive boson is replaced by a Goldstone boson. Consequence of this theorem, later applied to EFTI in ref. [44], is that the high energy dynamics of the inflaton field should be well described by the Goldstone π dynamics.

Besides time-reparametrization symmetry breaking, since we don't have much information about the Universe at high energies, it is interesting to investigate the possibility that the Universe undergoes a spontaneous space-diffeomorphism symmetry breaking during inflation, that is

$$x^i \;\to\; x'^i(t, x^j). \tag{2.70}$$

Space diffeomorphisms are broken whenever there is a field during inflation that acquire a non vanishing *vev* depending on the spatial coordinates. Therefore our model considers a situation where all the space-time diffeomorphisms are spontaneously broken during inflation. This configuration is achieved if the fields responsible for the breaking select a preferred direction or depend explicitly on space-time coordinates.

One can describe this situation following two different approaches. In the first approach one makes the hypothesis that there exist a gauge transformation where all the Goldstone boson emerging from the symmetry breaking can be set to zero. This is the so called "*unitary gauge*", and its viability demands that there are at most four Goldstone bosons which acquire a non vanishing vev. This is because a gauge transformation allows us to fix at most four functions of the coordinates; in doing so, no further gauge choice can be made on the perturbed metric[4]. Therefore, being a symmetric and transverse rank 2 tensor, the metric contains up to six degrees of freedom. This is indeed the expected number of degrees of freedom encoded in a propagating massive spin-2 particle. From the point of view of particle theory, a process similar to the Higgs mechanism is in place: the massless Goldstone bosons are eaten up by the graviton, which then becomes a massive gauge boson. At the end of the day, in this gauge all the propagating degrees of freedom are encoded inside the metric, which then should account for all the transverse (one helicity-2 state) and longitudinal polarizations (one helicity-1 and two helicity-0 states) of gravitons.

[4]More specifically, in the general treatment of GR, one usually introduce the de Donder gauge to remove four degrees of freedom from the metric (see ref. [68] for further details). In this case this condition cannot be applied since a gauge choice has just been made on the coordinates in such a way to eliminate the Goldstone bosons

While the unitary gauge provides a simple geometrical understanding of the additional propagating degrees of freedom, for the purposes of this thesis is convenient to consider the Goldstone bosons arising from space-time diffeomorphisms symmetry breaking as new propagating modes of our theory. The simplest choice one can make to introduce a preferred referential frame, is to align the coordinates with the background values of four new scalar fields

$$\bar{\phi}^0 = t, \qquad \bar{\phi}^i = \alpha x^i. \tag{2.71}$$

These are respectively the clock and the rules which break space-time reparametrization invariance during the inflationary period. The parameter α indicates the amount of space-reparametrization breaking. In order to restore the full diffeomorphism invariance of the theory, following the Stückelberg trick outlined in section (1.1), one introduces four Stückelberg fields π and σ^i

$$\phi^0 = t + \pi, \qquad \phi^i = \alpha(x^i + \sigma^i), \tag{2.72}$$

which must transform in such a way that the fields ϕ^0 and ϕ^i result invariant under space-time diffeomorphisms. In order to preserve homogeneity and isotropy of the Universe, we impose the additional symmetries of the fields under global rotations and translations

$$\phi^i \;\rightarrow\; O^i_j \phi^j \qquad \phi^i \;\rightarrow\; \phi^i + c^i, \tag{2.73}$$

with $O^i_j \in \mathrm{SO}(3)$ global. Moreover, in EFTI it is natural to assume the additional approximate time-shift symmetry

$$\phi^0 \;\rightarrow\; \phi^0 + c^0 \tag{2.74}$$

in order to prevent the coefficients of the action from being time-dependent. This way any new operator one can add to the Hilbert-Einstein action cannot involve the fields ϕ^μ ($\mu = 0, 1, 2, 3$) without their derivatives. The most general diffeomorphisms invariant action describing our system then is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 R + F\left(X, Y^i, Z^{ij}\right)\right], \tag{2.75}$$

with

$$\begin{aligned} X &= \partial_\mu \phi^0 \partial_\nu \phi^0 g^{\mu\nu}, \\ Y^i &= \partial_\mu \phi^0 \partial_\nu \phi^i g^{\mu\nu}, \\ Z^{ij} &= \partial_\mu \phi^i \partial_\nu \phi^j g^{\mu\nu}. \end{aligned} \tag{2.76}$$

and F an arbitrary function respecting the internal symmetry group of space rotations. Actually the arbitrariness on the choice of the function F is restricted by the fact the slow roll parameters must satisfy some conditions in order to be able to realize inflation. We consider a flat FLRW backgroung metric in conformal time

$$g_{\mu\nu} = a^2(\eta)\mathrm{diag}(-1, 1, 1, 1), \tag{2.77}$$

with $a(\eta)$ the scale factor. By definition, the energy momentum tensor is

$$\begin{aligned} T_{\mu\nu} &\equiv -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \\ &= g_{\mu\nu} F - 2(F_X \partial_\mu \phi^0 \partial_\nu \phi^0 + F_{Y^i} \partial_\mu \phi^0 \partial_\nu \phi^i + F_{Z^{ij}} \partial_\mu \phi^i \partial_\nu \phi^j). \end{aligned} \tag{2.78}$$

In this expression it was introduced the useful notation to indicate the partial derivative of the function F with a subscript, that is $F_X \equiv \partial F/\partial X$. Evaluated on the background

$$\bar{T}_{\mu\nu} = g_{\mu\nu}\bar{F} - 2(\bar{F}_X \delta^0_\mu \delta^0_\nu + \alpha \bar{F}_{Y^i} \delta^0_\mu \delta^i_\nu + \alpha^2 \bar{F}_{Z^{ij}} \delta^i_\mu \delta^j_\nu), \tag{2.79}$$

where the bar denotes the quantities evaluated on the background. Moreover, thanks to isotropy of the background

$$\bar{F}_{Y^i} = 0, \qquad \bar{F}_{Z^{ij}} = \bar{F}_Z \delta^{ij}. \tag{2.80}$$

Then one has

$$\begin{aligned} \bar{T}^0{}_0 &= -\rho = \bar{F} + 2\bar{F}_X \\ \bar{T}^i_j &= P\delta^i_j = (\bar{F} - 2\frac{\alpha^2}{a^2}\bar{F}_Z)\delta^i_j. \end{aligned} \tag{2.81}$$

The Friedmann background equations equations are

$$\begin{aligned} \mathcal{H}^2 &= \frac{a^2}{3M_P^2}\rho = -\frac{a^2}{3M_P^2}(\bar{F} + 2\bar{F}_X), \\ \mathcal{H}' &= -\frac{a^2}{6M_P^2}(\rho + 3P) = -\frac{a^2}{3M_P^2}(\bar{F} - \bar{F}_X - 3\frac{\alpha^2}{a^2}\bar{F}_Z). \end{aligned} \tag{2.82}$$

These expressions allow to evaluate the slow roll parameter ϵ as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{\mathcal{H}'}{\mathcal{H}^2} + 1 = 3\frac{\bar{F}_X + \frac{\alpha^2}{a^2}\bar{F}_Z}{\bar{F} + 2\bar{F}_X} = \frac{3\bar{X}\bar{F}_X - \bar{Z}\bar{F}_Z}{-\bar{F} + 2\bar{X}\bar{F}_X}, \tag{2.83}$$

where in the last step the background values

$$\bar{X} = -1, \qquad \bar{Y}^i = 0, \qquad \bar{Z}^{ij} = \frac{\alpha^2\delta^{ij}}{a^2(\eta)} \tag{2.84}$$

have been used. In order to realize a phase of flat slow roll one should verify $\epsilon \ll 1$, that in terms of the function F means

$$\left(\frac{\mathrm{d}\ln F}{\mathrm{d}\ln X}, \frac{\mathrm{d}\ln F}{\mathrm{d}\ln Z}\right) \ll 1. \tag{2.85}$$

It is interesting at this point to learn some physics about this model. The slow roll parameter ϵ is related to the speed of ticks of the inflationary clock. In order to reproduce a quasi de Sitter expansion, the ticks should run very slowly. The occurrence of this situation is controlled by the logarithmic derivative of the function F with respect to X and Z. This is because in our model we actually have two clocks, one encoded in X through the field $\bar{\phi}^0$ responsible of the time-reparametrization symmetry breaking, and one inside Z, since, even if it is related to space-diffeomorphisms breaking, $\bar{Z}^{ij}$ has an explicit time dependence through the scale factor $a(\eta)$. In order to ensure that the slow roll period lasts for enough amount of time, one should further require the parameter η to be very small, that is

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H} = 2\epsilon + \frac{6\bar{F}_{XZ} + 2\bar{Z}\bar{F}_Z + 2\bar{Z}^2\bar{F}_{Z^2}}{-3\bar{F}_X - \bar{Z}\bar{F}_Z} \ll 1. \tag{2.86}$$

The parameter η controls the time variation of the inflationary tick, and the condition $\eta \ll 1$ ensures that the rhythm of the clock does not change rapidly in time, in such a way to have a sufficiently long period of inflation.

2.2.1 Quadratic action for Stückelberg fields

In full generality, besides the expected interactions between the Stückelberg fields π and σ^i, the minimal coupling with the metric in the action (2.75) provides also interaction terms between these fields and the metric perturbations δg_{00}, δg_{0i} and δg_{ij}. However, on very high energies $E = k/a \gg H$, interaction terms turn out to be negligible. This fact is explicitly shown to hold for scalar modes in Section (2.2.2). In this limit the local space-time diffeomorphisms of GR reduce to the global symmetries of Lorentz boosts and translations of SR, while the Stückelberg fields become massless Goldstone boson with negligible interactions with the other fields. Therefore only self interactions betweeen π and σ^i should be accounted for in the action for the Stückelberg fields. On a linearly perturbed FLRW background, considering just the terms related to the graviton degrees of freedom

$$\begin{aligned}
X &= \partial_\mu\phi^0\partial_\nu\phi^0 g^{\mu\nu} = -(1+\dot{\pi})^2 + \frac{\partial_i\pi\partial^i\pi}{a^2}, \\
Y^i &= \partial_\mu\phi^0\partial_\nu\phi^i g^{\mu\nu} = -\alpha(1+\dot{\pi})\dot{\sigma}^i + \frac{\alpha}{a^2}\partial_j\pi(\delta^i_j + \partial_j\sigma^i), \\
Z^{ij} &= \partial_\mu\phi^i\partial_\nu\phi^j g^{\mu\nu} = -\alpha^2\dot{\sigma}^i + \frac{\alpha^2}{a^2}(\delta^{ij} + \partial^i\sigma^j + \partial^j\sigma^i + \partial_k\sigma^i\partial_k\sigma^j + \gamma^{ij}). \qquad (2.87)
\end{aligned}$$

Notice further that at first order

$$\bar{F}_{Z^{ij}} = \bar{F}_Z\,(\delta_{ij} + \gamma_{ij}). \qquad (2.88)$$

Then, the action for the graviton degrees of freedom up to second order is

$$\begin{aligned}
S &= \int d^4x\,a^4\left[\frac{1}{2}M_P^2 R + F_X\delta X + F_Z(\delta_{ij}+\gamma_{ij})\delta Z^{ij} + \frac{1}{2}\left(F_{X^2}\delta X\delta X + \right.\right. \\
&\quad \left.\left. +2F_{XZ}\delta X\delta Z + F_{Y^2}\delta Y\delta Y + (F_{ZZ}\delta_{ik}\delta_{jl} + F_{Z^2}\delta_{ij}\delta_{kl})\delta Z^{ij}\delta Z^{kl}\right)\right] \\
&= \int d^4x\,a^4\left[\frac{1}{2}M_P^2 R + \bar{F}_X\left(-\dot{\pi}^2 + \frac{\partial_i\pi\partial^i\pi}{a^2}\right) + \frac{1}{2}\bar{F}_{X^2}\left(2\dot{\pi} + \dot{\pi}^2)^2\right)\right. \\
&\quad + \bar{F}_Z\left(-\alpha^2\dot{\sigma}^2 + \frac{\alpha^2}{a^2}(\partial_k\sigma^i\partial_k\sigma^i + \gamma_{ij}\gamma^{ij})\right) + \bar{F}_{XZ}\left(-2\frac{\alpha^2}{a^2}(\partial_i\sigma^i)\dot{\pi}\right) \\
&\quad + \bar{F}_{Y^2}\frac{\alpha^2}{2}\left(-\dot{\sigma}^i - \dot{\pi}\dot{\sigma}^i + \frac{1}{a^2}(\partial^i\pi + \partial_k\partial^k\sigma^i)\right)^2 \\
&\quad + \frac{\alpha^4}{2a^4}(\partial^i\sigma^j + \partial^j\sigma^i + \partial_m\sigma^i\partial_m\sigma^j + \gamma^{ij})(\partial^k\sigma^l + \partial^l\sigma^k + \partial_n\sigma^k\partial_n\sigma^l + \gamma^{kl}) \\
&\quad \left. \times(\bar{F}_{ZZ}\delta_{ik}\delta_{jl} + \bar{F}_{Z^2}\delta_{ij}\delta_{kl})\right], \qquad (2.89)
\end{aligned}$$

where all the velocity terms were neglected, since first derivatives in the action do not contribute to the equations of motion. At this point it is useful to split the three Goldstone bosons σ^i into a vector and a scalar component through the decomposition

$$\sigma^i = \sigma^i_T + \frac{\partial^i\sigma_L}{\sqrt{-\nabla^2}}. \qquad (2.90)$$

Rearranging some terms in the above action, one can easily separate a scalar, a vector, and a tensor sector

$$\begin{aligned} S^{(S)} &= \int d^4x\, a^3 \Bigg[\left(-\bar{F}_X + 2\bar{F}_{X^2}\right) \dot{\pi}^2 + \left(\bar{F}_X + \frac{\alpha^2}{2a^2}\bar{F}_{Y^2}\right) \frac{\partial_i\pi\partial^i\pi}{a^2} \\ &\quad +\alpha^2 \left(-\bar{F}_Z + \frac{\bar{F}_{Y^2}}{2}\right)\dot{\sigma}_L^2 + \alpha^2\left(\bar{F}_Z + 2\frac{\alpha^2}{a^2}(\bar{F}_{ZZ} + \bar{F}_{Z^2})\right)\frac{\partial_i\sigma_L\partial^i\sigma_L}{a^2} \\ &\quad +4\frac{\alpha^2}{a^2}\bar{F}_{XZ}\sqrt{-\nabla^2}\dot{\pi}\sigma_L - \frac{\alpha^2}{a^2}\frac{\partial_i\dot{\sigma}_L\partial^i\pi}{\sqrt{-\nabla^2}}\Bigg], \end{aligned} \tag{2.91}$$

$$S^{(V)} = \int d^4x\, a^3\alpha^2 \left[\left(-\bar{F}_Z + \frac{\bar{F}_{Y^2}}{2}\right)\dot{\sigma}_T^i\dot{\sigma}_{i,T} + \left(\bar{F}_Z + 2\frac{\alpha^2}{a^2}\bar{F}_{ZZ}\right)\frac{\partial_j\sigma_T^i\partial_j\sigma_{i,T}}{a^2}\right], \tag{2.92}$$

$$S^{(T)} = \int d^4x\, \frac{a^3}{8}\left[M_P^2\left(\dot{\gamma}_{ij}\dot{\gamma}^{ij} - \frac{\partial_k\gamma_{ij}\partial^k\gamma^{ij}}{a^2}\right) + 8\alpha^2\left(\frac{\bar{F}_Z}{a^2} + \frac{\alpha^2\bar{F}_{ZZ}}{2a^4}\right)\gamma_{ij}\gamma^{ij}\right]. \tag{2.93}$$

The above expressions show clearly that this model propagates six dynamical degrees of freedom: two scalar modes, two helicities from a transverse vector, and two helicities from a traceless transfer tensor fluctuation. These are indeed the degrees of freedom carried by a massive graviton. Notice further that the two scalar modes interact in a non trivial way controlled by the amount of space-diffeomorphism breaking through the parameter α. This interaction introduce a second order correction to the curvature perturbation and, ultimately, to the primordial scalar power spectrum.

In single field models of inflation it is useful to introduce the gauge invariant curvature perturbation ζ. One would like then to find a relation between ζ and the scalar field π. The rigorous way to do that (see refs. [44, 43, 45]) is to perform a time diffeomorphism to go form the so called π-gauge, where the gauge is chosen in such a way that the spatial metric becomes $g_{ij} = a^2\delta_{ij}$, to the ζ-gauge, where the spatial metric is $g_{ij} = a^2e^{2\zeta}\delta_{ij}$ and $\pi = 0$. This condition is realized by performing the gauge transformation $x^\mu \to x^\mu - \xi^\mu$ with $\xi^0 = \pi$ and $\xi^i = 0$. At the same time the metric in the π-gauge transforms with the Lie derivative according to

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathscr{L}_\xi\, g^{(0)}_{\mu\nu} + o(\xi^2), \tag{2.94}$$

$$\mathscr{L}_\xi g_{\mu\nu} = g_{\mu\nu,\lambda}\xi^\lambda + g_{\lambda\nu}\xi_\mu{}^{,\lambda} + g_{\mu\lambda}\xi_\nu{}^{,\lambda}, \tag{2.95}$$

where $g^{(0)}_{\mu\nu}$ denotes the unperturbed FLRW metric. Hence

$$\tilde{g}_{ij} = a^2\delta_{ij} + 2a\dot{a}\pi\delta_{ij}, \tag{2.96}$$

which immediately identifies

$$\zeta(\vec{x}, t) = -H(t)\pi(\vec{x}, t). \tag{2.97}$$

However our model has two inflationary clocks, and non-adiabatic contributions can arise. Since the second clock originates from the space-diffeomorphism symmetry breaking, it is natural to expect that these terms are controlled by the parameter α. At lowest order, the full solution is (see ref. [47])

$$\zeta(\vec{x}, t) = \frac{H}{-M_P^2\dot{H}}\left[\left(-\bar{F}_X + \frac{\alpha^2\bar{F}_{Y^2}}{2a^2}\right)\pi + \alpha^2(2\bar{F}_Z - \bar{F}_{Y^2})\frac{\dot{\sigma}_L}{\sqrt{-\nabla^2}}\right]. \tag{2.98}$$

As a confirmation, one can verify, exploiting the Friedman equations, that the last expression reduces to (2.97) in the limit $\alpha \to 0$.

2.2.2 Primordial power spectra form extra modes

Scalar Power Spectrum

In (2.98) we have seen that the gauge invariant curvature perturbation $\zeta(\vec{x}, t)$ receive contributions from both the Goldstone bosons π and σ. In order to evaluate its power spectrum one needs to derive and solve the equations of motion for the Goldstone bosons. It is convenient then to introduce two new re-scaled field $\hat{\pi}$, $\hat{\sigma}$

$$\hat{\pi} = \sqrt{2(-\bar{F}_X + 2\bar{F}_{X^2})}\,\pi, \qquad \hat{\sigma} = \alpha\sqrt{2\left(\frac{\bar{F}_{Y^2}}{2a^2} - \frac{\bar{F}_Z}{a^2}\right)}\,\sigma_L, \tag{2.99}$$

such that the scalar sector of the action (2.91) is canonically normalized as

$$\begin{aligned} S^{(S)} = \int d^4x\, a^3 \Big[\frac{1}{2}\left(\dot{\hat{\pi}}^2 - c_\pi^2 \frac{\partial_i\hat{\pi}\partial^i\hat{\pi}}{a^2}\right) + \frac{1}{2}a^2\left(\dot{\hat{\sigma}}^2 - c_\sigma^2\frac{\partial_i\hat{\sigma}\partial^i\hat{\sigma}}{a^2}\right) + \\ + \alpha\lambda_1\sqrt{-\nabla^2}\,\dot{\hat{\pi}}\hat{\sigma} + \alpha\lambda_2\sqrt{-\nabla^2}\,\dot{\hat{\sigma}}\hat{\pi}\Big], \end{aligned} \tag{2.100}$$

with the sound speed $c_{\pi/\sigma}$ and the interaction coefficients $\lambda_{\pi/\sigma}$ defined by

$$c_\pi^2 = \frac{\bar{F}_X + \alpha^2\bar{F}_{Y^2}/2a^2}{\bar{F}_X - 2\bar{F}_{X^2}} \qquad \lambda_1 = \frac{2\bar{F}_{XZ}/a^2}{\sqrt{(-\bar{F}_X + 2\bar{F}_{X^2})(\bar{F}_{Y^2}/2a^2 - \bar{F}_Z/a^2)}}$$

$$c_\sigma^2 = \frac{\bar{F}_Z + 2\alpha^2\bar{F}_{ZZ}/a^2 + 2\alpha^2\bar{F}_{Z^2}/a^2}{F_Z - F_{Y^2}/2} \qquad \lambda_2 = \frac{-\bar{F}_{Y^2}/a^2}{2\sqrt{(-\bar{F}_X + 2\bar{F}_{X^2})(\bar{F}_{Y^2}/2a^2 - \bar{F}_Z/a^2)}}$$

Since we are assuming a tiny violation of space-diffeomorphism invariance, one can treat the parameter α as a small perturbative parameter and expand perturbatively the two point correlator making advantage of the *in-in formalism* to take into account of the interaction terms. At zero order then, only the field π sources the curvature perturbations. The equations of motion for the re-scaled field $\hat{\pi}$ are given by the Euler-Lagrange equation of the action (2.100), which are

$$\ddot{\hat{\pi}} + 3H\dot{\hat{\pi}} - c_\pi^2\frac{\nabla^2}{a^2}\pi = 0. \tag{2.101}$$

Passing to conformal time and defining the new variable $u_k(\eta) = a(\eta)\hat{\pi}(k,\eta)$ the equations of motion becomes

$$u_k'' + \left[c_\pi^2 k^2 - \frac{a''}{a}\right]u_k = 0. \tag{2.102}$$

At this point one can recognize again the form of the equation (2.40) for massless tensor modes and with the replacement $k \to c_\pi k$. Therefore we can straightforwardly consider the solution

$$|\hat{\pi}_k| \simeq \frac{H}{\sqrt{2c_\pi^3k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}, \tag{2.103}$$

with $3/2-\nu = -\epsilon$. Notice the difference with respect to (2.64). Here indeed, the symmetry pattern of the theory forbid the presence of any mass term and non-derivative interaction. Moreover in the present case we are working in the decoupling limit, where interactions between Goldstone bosons and metric fluctuations are neglected; this prevents from the presence of the additional term in (2.60). Therefore, in light of the considerations outlined in Section (2.1.1), on super-horizon scales these solution remain completely frozen at the value of the horizon crossing, that is when $k = aH$. Therefore the primordial power spectrum is

$$\langle \hat{\pi}_{\mathbf{k}_1} \hat{\pi}_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \hat{\mathcal{P}}_0, \tag{2.104}$$

$$\hat{\mathcal{P}}_0 = \frac{H_k^2}{4\pi^2 c_\pi^3}, \tag{2.105}$$

with $H_k = H(\eta_k)$ the Hubble parameter evaluated at the time when a certain mode with wavenumber k crossed the horizon. Now one can turn back to the power spectrum of the original field π by simply dividing for the multiplicative factor in (2.99)

$$\mathcal{P}_0 = \frac{H_k^2}{8\pi^2 c_\pi^3(-\bar{F}_X + 2\bar{F}_{X^2})} = \frac{H_k^2}{8\pi^2 c_\pi^3(-\bar{F}_X - \alpha^2 \bar{F}_{Y^2}/2a^2)}. \tag{2.106}$$

Notice that in the limit $\alpha \ll 1$, where we restore space-diffeomorphisms invariance, given the Einstein equation $\bar{F}_X = M_P^2 \dot{H}$, one recovers the prediction for the power spectrum of a massless single field inflation. Indeed

$$\mathcal{P}_0(\alpha \ll 1) \simeq \frac{H_k^2}{8\pi^2 c_\pi(-\dot{H}_k M_P^2)} = \frac{1}{8\pi^2 c_\pi M_P^2 \epsilon}. \tag{2.107}$$

Contributions to the power spectrum arising from the interaction terms can be computed with the *in-in formalism* [120]. At first order in the α parameter no contribution to the two point correlator is expected, since the action (2.100) does not contain any mass-like interaction term. The leading correction to the power spectrum is then given by

$$\delta\langle \hat{\pi}_{\mathbf{k}_1} \hat{\pi}_{\mathbf{k}_2(\eta)} \rangle = -\int_{\eta_{in}}^{\eta} d\eta_1 \int_{\eta_{in}}^{\eta_1} d\eta_2 \Big\langle \Big[\Big[\hat{\pi}^{(0)}_{\mathbf{k}_1} \hat{\pi}^{(0)}_{\mathbf{k}_2}(\eta), \mathcal{H}^{(2)}_{\text{int}}(\eta_1) \Big], \mathcal{H}^{(2)}_{\text{int}}(\eta_2) \Big] \Big\rangle, \tag{2.108}$$

with η_{in} the time when the interaction is turned on, and

$$\mathcal{H}^{(2)}_{\text{int}}(\eta) = \frac{\alpha}{(H\eta)^3} \left[\lambda_1 \int \frac{d^3k}{(2\pi)^3} \, k \, \hat{\sigma}^{(0)}_{\mathbf{k}} \hat{\pi}'^{(0)}_{-\mathbf{k}}(\eta) + \lambda_2 \int \frac{d^3k}{(2\pi)^3} \, k \, \hat{\pi}^{(0)}_{\mathbf{k}} \hat{\sigma}'^{(0)}_{-\mathbf{k}}(\eta) \right]. \tag{2.109}$$

Then one decomposes the Goldstone bosons π, σ into Fourier modes and inserts the solution of the equation of motion for the two fields. The computation has been already done in [47], with the result[5]

$$\frac{\delta\mathcal{P}}{\mathcal{P}_0} = \frac{3\alpha^2 \lambda_1 N_k \left[3\lambda_2 - \lambda_1 (3N_k + 6\gamma_E + 11 - \ln(64)) \right]}{c_\pi^2}, \tag{2.110}$$

with γ_E the Euler gamma function and N_k the number of e-folds the Universe expanded from the time of horizon crossing of mode with wavenumber k until the end of inflation. Notice that this new contribution does not depend on the scale k. In the limit for large N_k the most dominant contribution is

$$\frac{\delta\mathcal{P}}{\mathcal{P}_0} = -\frac{9\alpha^2 \lambda_1^2 N_k^2}{c_\pi^2}. \tag{2.111}$$

For small values of α, we have a simple relation between the gauge invariant curvature perturbation and the Goldstone boson π. This, in the end, leads to the power spectrum of curvature perturbation

$$\mathcal{P}_\zeta(k) \simeq \frac{H_k^2}{8\pi^2 M_P^2 \epsilon c_\pi} \left(1 - \frac{9\alpha^2 \lambda_1^2 N_k^2}{c_\pi^2} \right). \tag{2.112}$$

[5] For simplicity it is considered the case $c_\pi = c_\sigma$.

This is a quite interesting result, because it suggests the possibility of discriminating between the massive and massless case in physical observations. Indeed the factor N_k^2 introduces a new k-dependence in the primordial power spectrum which modifies the value of the primordial scalar spectral index. Indeed, making explicit the scale dependence,

$$\mathcal{P}_\zeta(k) \simeq \frac{H^2}{8\pi^2 M_P^2 \epsilon c_\pi} \left(\frac{k}{aH}\right)^{-2\epsilon} \left(1 - \frac{9\alpha^2\lambda_1^2 N_k^2}{c_\pi^2}\right). \tag{2.113}$$

By definition, the number of e-folds at the horizon-crossing time is

$$N_k = \int_{t_k}^{t_{end}} H(t)dt = \ln\left(\frac{a_{end}}{a_k}\right), \tag{2.114}$$

with t_k and t_{end} the coordinate time at the horizon crossing of the mode k and at the end of inflation respectively, while $a_{end} = a(t_{end})$, $a_k = a(t_k)$. Since the term with N_k^2 is already at second order in $alpha$-expansion, we can evaluate N_k at zero order in slow-roll parameter expansion. This means that $a_k = k/\bar{H}$ and

$$N_k^2 = -2\ln\left(\frac{k}{a_{end}\bar{H}}\right). \tag{2.115}$$

The last factor of equation (2.113) can be seen as the first order Taylor expansion of an exponential function. Inserting the above result one easily obtains

$$\mathcal{P}_\zeta(k) \simeq \frac{H^2}{8\pi^2 M_P^2 \epsilon c_\pi} \left(\frac{k}{aH}\right)^{-2\epsilon} \left(\frac{k}{a_{end}H}\right)^{\frac{18\alpha^2\lambda_1^2}{c_\pi^2}}, \tag{2.116}$$

and finally the scalar spectral index is

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}(k)}{d\ln k} = -2\epsilon + \frac{18\alpha^2\lambda_1^2}{c_\pi^2}. \tag{2.117}$$

Hence, the most important result is that the scalar power spectrum may acquire a blue tilt depending on the "amount" of violation of space-diffeomorphism invariance. This is similar to what it was shown in Section (2.1.1), where the primordial tensor power spectrum becomes blue for large values of the early time graviton mass. Moreover, both in the tensor and scalar cases, the primordial spectra becomes unavoidably blue in the limit of pure de Sitter stages during Inflation, where the slow-roll parameter ϵ vanishes.

Remember that in deriving the second order action (2.91), couplings between the Goldstone bosons and metric perturbations were neglected. In a sense, we are perturbing the fields over an unperturbed background. To eliminate this inconsistency one should take into account the perturbed metric around a FLRW background as in (2.23). As long as we are interested in the π-spectrum at the zero order expansion in α parameter, the only relevant difference arises from the term

$$\begin{aligned} X = \partial_\mu\phi^0\partial_\nu\phi^0 g^{\mu\nu} &= -(1+\dot{\pi})^2(1+2\Phi) + \frac{\partial_i\pi\partial^i\pi}{a^2}(1+2\Psi) \\ &= -1 - 2\dot{\pi} - 2\Phi - \dot{\pi}^2 - 4\dot{\pi}\Phi + \frac{\partial_i\pi\partial^i\pi}{a^2} + \dots \end{aligned} \tag{2.118}$$

where terms of third order in perturbations are understood inside the dots. This way the quadratic action for the rescaled Goldstone boson $\hat{\pi}$ at zero order in α becomes

$$S^{(\pi)} = \int d^4x\, a^3 \Big[\frac{1}{2}\left(\dot{\hat{\pi}}^2 - c_\pi^2\frac{\partial_i\hat{\pi}\partial^i\hat{\pi}}{a^2}\right) - c_\pi\dot{\hat{\pi}}\Phi\Big], \tag{2.119}$$

with

$$c_\pi = \frac{4\bar{F}_X}{\sqrt{2(-\bar{F}_X + 2\bar{F}_{X^2})}}. \tag{2.120}$$

The equations of motion for this action are

$$\ddot{\pi} + 3H\dot{\pi} - c_\pi^2 \frac{\nabla^2}{a^2}\pi = -c_\pi \dot{\Phi}. \tag{2.121}$$

As long as we are interested in the primordial power spectrum on super-horizon scales, the time derivative of the scalar metric fluctuation can be neglected. Hence we are back to the same equation (2.101), supporting the consistency of the previous analysis and our choice of neglecting the coupling between Goldstone bosons and metric perturbations.

Tensor Power Spectrum

In the tensor sector we have two propagating helicity-2 degrees of freedom with an additional mass term. This is the expected result of having introduced a non vanishing mass for gravitons. The equation of motions for this action were already studied in Section (2.1.1); in particular, the above action reproduces the one in equation (2.34) with a new redefinition of the mass parameter

$$m_g^2 = \frac{8}{M_P^2}\left(\frac{\alpha^2}{a^2}\bar{F}_Z + \frac{\alpha^4}{2a^4}\bar{F}_{ZZ}\right). \tag{2.122}$$

Hence the dimensionless power spectrum can be straightforwardly read from (2.50), that is

$$\mathcal{P}_\lambda(k) = \frac{2^{2\nu-3}H^2}{\pi^2 M_P^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 \left(\frac{k}{aH}\right)^{n_T}, \tag{2.123}$$

with a new definition of the parameters

$$\nu^2 = \frac{9}{4} + 3\epsilon - \frac{m_g^2}{H^2}, \tag{2.124}$$

$$n_T \equiv 3 - 2\nu = 3\left[1 - \sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{H^2} - 3\epsilon\right)}\right]. \tag{2.125}$$

As a last comment, we mention how these effects reflect on the tensor-to-scalar ratio. Because of the mass term, the scalar power spectrum decreases according to (2.112), while the tensor power spectrum acquires a non negligible time dependence encoded in the slow roll parameter η. Explicitly

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} = \frac{2^{2\nu+1}\epsilon c_\pi}{\left(1 - \frac{9\alpha^2\lambda_1^2 N_k^2}{c_\pi^2}\right)}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 \left(\frac{k}{aH}\right)^{n_T+2\epsilon}, \tag{2.126}$$

where $\mathcal{P}_T = \mathcal{P}_{+2} + \mathcal{P}_{-2} = 2\mathcal{P}_\lambda$ is the total tensor power spectrum, and the tensor spectral index n_T is defined in(2.51). This is a quite interesting result, since it shows our prediction for deviation from the usual single field inflation consistency relation. Notice that in the massless limit $n_T = -2\epsilon + o(\epsilon^2)$ and any time dependence of the ratio r goes away. Moreover, the usual single field inflation scenario is recovered taking the limit $\alpha \to 1$ and $c_\pi \to 1$. In this case, considering further $\nu \simeq 3/2$, we recover the expected consistency relation for slow roll inflation

$$r = 16\epsilon. \tag{2.127}$$

2.3 The minimal theory of massive gravity after inflation

In recent years, much effort in theoretical cosmology has been put in finding a viable theory of massive gravity in a Lorentz invariant way. These theories necessarily provide the graviton to have five physical degrees of freedom, but they are plagued by instabilities when applied on background space-times relevant for cosmology, as the FLRW one [55]. In this section we present a recent model developed by De Felice and Mukohyama [56, 57] which modifies GR in a minimal way without any extra scalar and vector propagating degree of freedom. The absence of the extra modes assures that the theory is stable in a FLRW background. It is worth to stress again that this theory is not in conflict with the one outlined in Section (2.1); We are looking at a very large range of energies, and it is really hard (if not impossible) to think that a unique theory would be able to describe gravitational interaction on such a wide domain. Therefore we are effectively describing this interaction in two limiting regimes through two different theories which are completely independent, and they are not required to match at certain energy scales as one usually does in building a low energy effective field theory.

In order to avoid the propagation of extra modes one has to give up Lorentz invariance. Lorentz violation is naturally confined in the gravity sector, and vanishes in the limit of zero graviton mass; on the matter sector, Lorentz violation induced by graviton loops should be suppressed by a factor m^2/M_P^2, where m is the graviton mass after inflation. Notice that this mass does not need to coincide with the one possessed by gravitons during inflation (m_g), since we are picturing the history of the Universe with two not communicating gravitational theories. In fact, for gravitons propagating across the Universe, we have recent astrophysical observations which tightly bound the value of the graviton mass to $m \leq 10^{-28}$ eV, which we should always have in mind. In the following, in order to highlight this important difference we will adopt two different notation to refer to the graviton mass: m_g for the graviton mass during inflation, and m for the graviton mass after inflation. In the following we review the main steps to build the minimal theory of massive gravity (MTMG thereafter); it is useful to proceed by steps, starting from a precursor theory and then applying some constraint to derive the final theory.

The precursor theory

Since the fundamental feature of this theory is to avoid propagation of extra modes, it is convenient to work in the ADM formalism [58] which allows a simple degrees of freedom counting. The basic variables of the theory are then the lapse N, the shift vector N^i, and the three-dimensional vielbeins e^I_j which allow to write the three-dimensional metric as

$$\gamma_{ij} = \delta_{IJ} e^I_i e^J_j, \tag{2.128}$$

where $I, J \in \{1, 2, 3\}$ are flat indices, while $i, j \in \{1, 2, 3\}$ are curved indices. Out of these variables, we can define a four-dimensional vielbein in such a way to put the four-dimensional metric in the usual form

$$g_{\mu\nu} = \eta_{\mathcal{AB}}\, e^{\mathcal{A}}_\mu e^{\mathcal{B}}_\nu, \tag{2.129}$$

with

$$\begin{aligned} g_{00} &= -N^2 + \gamma_{ij} N^i N^j, & g^{00} &= -N^{-2}, \\ g_{0i} &= g_{i0} = \gamma_{ij} N^j, & g^{0i} &= g^{i0} = N^i/N^2, \\ g_{ij} &= \gamma_{ij}, & g^{ij} &= \gamma^{ij} - (N^i N^j/N^2). \end{aligned}$$

This situation is realized choosing

$$\|e_\mu^{\mathcal{A}}\| = \begin{pmatrix} N & \vec{0}^T \\ e_i^I N^i & e_j^I \end{pmatrix}, \tag{2.130}$$

which is usually called ADM vielbein. In full generality, a four-dimensional vielbein has 16 independent components; our particular choice of the ADM vielbein contains just 13 of them, since three components, related to the boost parameters, are identically set to zero. Therefore the particular choice of (2.130) automatically select a preferred frame and thus explicitly breaks local Lorentz symmetry. Besides these dynamical quantities, the theory introduces a new four-dimensional non dynamical vielbein $E_\mu^{\mathcal{A}}$ out of non dynamical lapse M, shift M^i and three dimensional vielbein E_j^I. In analogy with the dynamical vielbein, the non dynamical counterpart is chosen in the form

$$\|E_\mu^{\mathcal{A}}\| = \begin{pmatrix} M & \vec{0}^T \\ E_i^I M^i & E_j^I \end{pmatrix}, \tag{2.131}$$

which defines a new non-dynamical metric as

$$\begin{aligned} f_{00} &= -N^2 + \tilde{\gamma}_{ij} M^i M^j, \\ f_{0i} &= f_{i0} = \tilde{\gamma}_{ij} M^j, \\ f_{ij} &= \tilde{\gamma}_{ij}. \end{aligned} \tag{2.132}$$

with

$$\tilde{\gamma}_{ij} = \delta_{IJ} E_i^I E_j^J. \tag{2.133}$$

The dual basis e_I^j and E_I^j is introduce in such a way that the three-dimensional vielbeins satisfy

$$e_k^I e_J^k = \delta_J^I, \qquad e_i^K e_K^j = \delta_j^i, \tag{2.134}$$

$$E_k^I E_J^k = \delta_J^I, \qquad E_i^K E_K^j = \delta_i^j, \tag{2.135}$$

while for the four-dimensional vielbeins

$$e_\mu^{\mathcal{A}} e_{\mathcal{B}}^\mu = \delta_{\mathcal{B}}^{\mathcal{A}}, \qquad e_\mu^{\mathcal{A}} e_{\mathcal{A}}^\nu = \delta_\mu^\nu, \tag{2.136}$$

$$E_\mu^{\mathcal{A}} E_{\mathcal{B}}^\mu = \delta_{\mathcal{B}}^{\mathcal{A}}, \qquad E_\mu^{\mathcal{A}} E_{\mathcal{A}}^\nu = \delta_\mu^\nu. \tag{2.137}$$

The precursor theory is then defined as a minimal modification of the dRGT action (1.46), by simply inserting the ADM vielbeins (2.130) as the physical vielbeins, and substituting the fiducial ones (2.131) in place of the identity vielbeins. This allows to define the theory of massive gravity on a generic background metric encoded in the choice of the fiducial variables, and eventually on a FLRW background. Therefore

$$S_{\text{pre}} = S_{dRGT}[e_i^I, E_j^J] = \frac{M_P^2}{2} \int d^4x \sqrt{-g} R[g] + \frac{M_P^2}{2} \mu^2 \sum_{n=0}^{4} \int d^4x c_n \mathcal{L}_n, \tag{2.138}$$

where μ denotes a graviton mass parameter, while $R[g]$ specifies the Ricci scalar for the physical metri $g_{\mu\nu}$, and

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{24}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mathcal{ABCD}}E^{\mathcal{A}}_{\mu}E^{\mathcal{B}}_{\nu}E^{\mathcal{C}}_{\rho}E^{\mathcal{D}}_{\sigma},\\
\mathcal{L}_1 &= \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mathcal{ABCD}}E^{\mathcal{A}}_{\mu}E^{\mathcal{B}}_{\nu}E^{\mathcal{C}}_{\rho}e^{\mathcal{D}}_{\sigma},\\
\mathcal{L}_2 &= \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mathcal{ABCD}}E^{\mathcal{A}}_{\mu}E^{\mathcal{B}}_{\nu}e^{\mathcal{C}}_{\rho}e^{\mathcal{D}}_{\sigma},\\
\mathcal{L}_3 &= \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mathcal{ABCD}}E^{\mathcal{A}}_{\mu}e^{\mathcal{B}}_{\nu}e^{\mathcal{C}}_{\rho}e^{\mathcal{D}}_{\sigma},\\
\mathcal{L}_4 &= \frac{1}{24}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mathcal{ABCD}}e^{\mathcal{A}}_{\mu}e^{\mathcal{B}}_{\nu}e^{\mathcal{C}}_{\rho}e^{\mathcal{D}}_{\sigma}.
\end{aligned} \tag{2.139}$$

One then can easily verify that our particular choice of the Lorentz symmetry breaking ADM vielbeins leads to

$$\begin{aligned}
S_{\rm pre} &= \frac{M_P^2}{2}\int d^4x\Big\{N\sqrt{\gamma}(R[\gamma]+K_{ij}K^{ij}-K^2)\\
&\quad -c_0\mu^2\sqrt{\tilde{\gamma}}M - c_1\mu^2\sqrt{\tilde{\gamma}}(N+MY_I{}^J)\\
&\quad -c_2\mu^2\sqrt{\tilde{\gamma}}\left[NY_I{}^I+\frac{M}{2}(Y_I{}^IY_I{}^J-Y_I{}^JY_J{}^I)\right]\\
&\quad -c_3\mu^2\sqrt{\gamma}(M+NX_I{}^I)-c_4\mu^2N\sqrt{\gamma}\Big\}.
\end{aligned} \tag{2.140}$$

with

$$X_I{}^J \equiv e_I{}^jE^J{}_j, \qquad Y_I{}^J \equiv E_I{}^je^J{}_j, \tag{2.141}$$

and the extrinsic curvature defined by

$$K_{ij} \equiv \frac{1}{2N}(\dot{\gamma}_{ij}-2D_{(i}N_{j)}) = \frac{1}{N}(\delta_{IJ}\dot{e}^I{}_ie^J{}_j - D_{(i}N_{j)}), \qquad K = K_{ij}\gamma^{ij}, \tag{2.142}$$

where D_I is the spatial covariant derivative compatible with γ_{ij}

Hamiltonian Analysis of the precursor theory Contrary to the dRGT case, the mass term of the precursor modified theory has a linear dependence on the lapses and no-dependence on the shift-vectors. This allows us to take N and N^i as Lagrange multipliers and fix four variables, leaving the theory with nine degrees of freedom $e^I{}_j$. In order to perform the physical degrees of freedom counting it is convenient to pass to the Hamiltonian formalism. The canonical momenta conjugate to the vielbeins variables are

$$\Pi_I{}^j \equiv \frac{\delta S_{\rm pre}}{\delta\dot{e}^I{}_j} = \frac{M_P^2N\sqrt{\gamma}}{2}\int d^4x\,[2K^{mn}-2K\gamma^{mn}]\frac{\delta K_{mn}}{\delta\dot{e}^I{}_j} = 2\pi^{jk}\delta_{IJ}e^J{}_k, \tag{2.143}$$

with

$$\pi^{ij} = \frac{M_P^2}{2}\sqrt{\gamma}(K^{ij}-K\gamma^{ij}). \tag{2.144}$$

Three primary constraints are directly obtained from the symmetric property of the extrinsic curvature

$$\mathcal{Q}_{[I,J]} \equiv \Pi^k_{[I}\delta_{J]K}e^K{}_k = \pi_{ij}e_I{}^{[i}e_J{}^{j]} = 0. \tag{2.145}$$

For the following discussions it is useful to write the extrinsic curvature in terms of the canonical variable only. Contacting the definition (2.144) with the spatial metric

$$\pi \equiv \pi^{ij}\gamma_{ij} = -M_P^2\sqrt{\gamma}K. \tag{2.146}$$

On the other hand, one can invert (2.143) as

$$\pi^{ij} = \frac{1}{2}\delta^{IJ}\Pi_I{}^i e_J{}^j. \tag{2.147}$$

Contacting again with the three-dimensional metric and comparing with (2.146), lead to

$$K = -\frac{1}{2}\Pi_I{}^i e^I{}_i. \tag{2.148}$$

Inserting this expression inside (2.144) together with (2.147), one eventually arrives to

$$\begin{aligned} K_{ij} &= \frac{2}{\sqrt{\gamma}M_P^2}\left[\gamma_{im}\gamma_{jn} - \frac{1}{2}\gamma_{ij}\gamma_{mn}\right]\pi^{mn} \\ &= \frac{1}{\sqrt{\gamma}M_P^2}\left[\gamma_{(k(i}\gamma_{j)l}\Pi_I{}^k\delta^{IJ}e_J{}^l - \frac{1}{2}\gamma_{kl}\Pi_K{}^k\delta^{KL}e_L{}^l\gamma_{ij}\right]. \end{aligned} \tag{2.149}$$

The Hamiltonian density of the precursor theory can be now introduced as the Legendre transform of the Lagrangian density (2.140), that is

$$\begin{aligned} \mathcal{H}_{\rm pre} &= \dot{e}_I{}^i\Pi^I{}_i - \frac{M_P^2}{2}\Big\{N\sqrt{\gamma}(R[\gamma] + K_{ij}K^{ij} - K^2) - c_0\mu^2\sqrt{\tilde{\gamma}}M \\ &\quad -c_1\mu^2\sqrt{\tilde{\gamma}}(N + MY_I{}^J) - c_2\mu^2\sqrt{\tilde{\gamma}}\left[NY_I{}^I + \frac{M}{2}(Y_I{}^IY_I{}^J - Y_I{}^JY_J{}^I)\right] \\ &\quad -c_3\mu^2\sqrt{\gamma}(M + NX_I{}^I) - c_4\mu^2N\sqrt{\gamma}\Big\}. \end{aligned} \tag{2.150}$$

Using the above relations, one can work out some term to find

$$K_{ij}K^{ij} - K^2 = \frac{4}{\gamma M_P^4}\left[\gamma_{im}\gamma_{jn} - \frac{1}{2}\gamma_{ij}\gamma_{mn}\right]\pi^{ij}\pi^{mn}, \tag{2.151}$$

$$\dot{e}^I{}_i\Pi_I{}^i = \frac{2N}{\sqrt{\gamma}M_P^2}\left[\gamma_{im}\gamma_{jn} - \frac{1}{2}\gamma_{ij}\gamma_{mn}\right]\pi^{ij}\pi^{mn} - 2\gamma_{jk}N^jD_i\pi^{ik}. \tag{2.152}$$

Summing up these results and rearranging all the terms in a more suitable way, the Hamiltonian of the precursor theory together with primary constraints is[6]

$$H^{(1)}_{\rm pre} = \int d^4x\left[-N\mathcal{R}_0 - N^i\mathcal{R}_i + \mu^2M\mathcal{H}_1 + \alpha_{MN}\mathcal{Q}^{[MM]}\right], \tag{2.153}$$

with

$$\begin{aligned} &\mathcal{R}_0 = \mathcal{R}_0^{GR} - \mu^2\mathcal{H}_0, \qquad \mathcal{R}_0^{GR} = \sqrt{\gamma}R[\gamma] - \frac{1}{\sqrt{\gamma}}\left(\gamma_{nl}\gamma_{mk} - \frac{1}{2}\gamma_{nm}\gamma_{kl}\right)\pi^{nm}\pi^{kl}, \\ &\mathcal{R}_i = \mathcal{R}_i^{GR} = 2\gamma_{ik}D_j\pi^{kj}, \\ &\mathcal{H}_0 = \sqrt{\tilde{\gamma}}\left(c_1 + c_2Y_I{}^I\right) + \sqrt{\gamma}\left(c_3X_I{}^I + c_4\right), \\ &\mathcal{H}_1 = \sqrt{\tilde{\gamma}}\left[c_1Y_I{}^I + \frac{c_2}{2}\left(Y_I{}^IY_J{}^J - Y_I{}^JY_J{}^I\right)\right] + c_3\sqrt{\gamma}, \\ &\mathcal{Q}^{[MN]} = e^M_j\Pi^j{}_I\delta^{IN} - e^N{}_j\Pi^j_I\delta^{IM}. \end{aligned} \tag{2.154}$$

[6]For the ease of notation, here and in the following the units are fixed such that $M_P^2 = 2$.

As already pointed out, since the the precursor Hamiltonian is linear in the lapse N and in the shift N^i, they can be treated as Lagrange multiplier, while[7]

$$\mathcal{R}_0 \approx 0, \qquad \mathcal{R}_i \approx 0 \tag{2.155}$$

are two additional primary constraints.

Secondary constraints

Secondary constraints may naturally arise by imposing the time conservation of primary constraints on the constrained surface:

$$\dot{\mathcal{Q}}^{[MN]} = \left\{\mathcal{Q}^{[MN]}, H^{(1)}_{\text{pre}}\right\}_{\mathcal{P}} \approx 0 \tag{2.156}$$

$$\dot{\mathcal{R}}_0 = \left\{\mathcal{R}_0, H^{(1)}_{\text{pre}}\right\}_{\mathcal{P}} + \frac{\partial \mathcal{R}_0}{\partial_t} \approx 0, \tag{2.157}$$

$$\dot{\mathcal{R}}_i = \left\{\mathcal{R}_i, H^{(1)}_{\text{pre}}\right\}_{\mathcal{P}} \approx 0, \tag{2.158}$$

where $\{\cdot,\cdot\}_{\mathcal{P}}$ denotes the Poisson bracket operator. Three secondary constraints arise from the first equation. Indeed

$$\dot{\mathcal{Q}}^{[MN]} = \int d^3x \left[\frac{\delta \mathcal{Q}^{[MN]}}{\delta e^I{}_i}\frac{\delta H^{(1)}_{\text{pre}}}{\delta \Pi_I{}^i} - \frac{\delta \mathcal{Q}^{[MN]}}{\delta \Pi_I{}^i}\frac{\delta H^{(1)}_{\text{pre}}}{\delta e^I{}_i}\right]. \tag{2.159}$$

Working out each operator one at a time, one finds

$$\begin{aligned}
\frac{\delta \mathcal{Q}^{[MN]}}{\delta e^I{}_i}\frac{\delta H^{(1)}_{\text{pre}}}{\delta \Pi_I{}^i} &= \frac{1}{2\sqrt{\gamma}}\left(\gamma_{ik}\gamma^{jl} - \frac{1}{2}\gamma^j_i\gamma^l_k\right)\Pi_J{}^i\Pi_L{}^k e^J{}_j e^I{}_l \delta_I^{[M}\delta^{N]L}. \\
\frac{\delta \mathcal{Q}^{[MN]}}{\delta \Pi_I{}^i} &= \frac{1}{2}\gamma^k_m\left[\delta^{MI}e^N{}_k - \delta^{NI}e^M{}_k\right].
\end{aligned} \tag{2.160}$$

Both these equations vanish whenever $\delta^{L[M}e^{N]}{}_i = 0$, which means when the condition

$$Y^{[MN]} \approx 0, \qquad Y^{MN} = \delta^{ML}Y_L{}^N \tag{2.161}$$

is satisfied. Now one may wonder how many constraints can be derived from (2.157) and (2.158). For this purpose one can compute the Poisson brackets among the primary constraints $\mathcal{R}_0$ and $\mathcal{R}_i$. Since

$$\begin{aligned}
&\{\mathcal{R}_0(x), \mathcal{R}_0(y)\} \approx 0, \\
&\{\mathcal{R}_j(x), \mathcal{R}_j(y)\} \approx 0, \\
&\{\mathcal{R}_0(x), \mathcal{R}_i(y)\} \not\approx 0,
\end{aligned}$$

then

$$\dot{\mathcal{R}}_0 \approx \int d^3x \Big[N^i\{\mathcal{R}_0, \mathcal{R}_i\} + \mu^2 M\{\mathcal{R}_0, \mathcal{H}_1\} + \alpha_{MN}\{\mathcal{R}_0, \mathcal{Q}^{[MN]}\}\Big] \approx -\partial_t\mathcal{R}_0, \tag{2.162}$$

$$\dot{\mathcal{R}}_i \approx \int d^3x \Big[-N\{\mathcal{R}_i, \mathcal{R}_0\} + \mu^2 M\{\mathcal{R}_i, \mathcal{H}_1\} + \alpha_{MN}\{\mathcal{R}_1, \mathcal{Q}^{[MN]}\}\Big] \approx 0. \tag{2.163}$$

[7] In this context the symbol $\approx$ denotes the weak equivalence sign, that is an equivalence which holds only on the constrained surface of the phase space.

Then equation (2.162) can be solved to fix one of the three components of the shift N^i in terms of the other variables. In the same way one of the equations (2.163) fixes the lapse N. This leaves with two equations which provide for two additional secondary constraints, which we symbolically denote with $\tilde{\mathcal{C}}_\tau$ ($\tau = 1, 2$). In the end the total precursor Hamiltonian is

$$H^{(2)}_{\text{pre}} = \int d^3x \left[-N\mathcal{R}_0 - N^i\mathcal{R}_i + \mu^2 M\mathcal{H}_1 + \alpha_{MN}\mathcal{Q}^{[MN]} + \beta_{MN}Y^{[MN]} + \tilde{\lambda}^\tau\tilde{\mathcal{C}}_\tau\right]. \quad (2.164)$$

All in all, we found 7 primary constraints ($\mathcal{R}_0, \mathcal{R}_i, \mathcal{Q}^{[MN]}$), and 5 secondary constraints ($Y^{[MN]}, \tilde{\mathcal{C}}_\tau$). By computing the Poisson brackets among all these quantities, one can straightforwardly realize that all these constraints are of second class. Therefore we end up with 12 second class constraints which provide the cancellation of twelve degrees of freedom from the Hamiltonian phase space. Since the lapse N and the shift N^i are interpreted as Lagrange multiplier, this is given by the variables ${e^I}_j$ and ${\Pi^I}_j$, which account for a total of 18 degrees of freedom. Finally, the number of physical degrees of freedom of the precursor theory is $\frac{1}{2}(18 - 12) = 3$.

The minimal theory

So far, breaking Lorentz symmetry with the precursor Hamiltonian has removed two modes from the dRGT theory, leaving three only physical degrees of freedom. Our aim now is to remove an additional degree of freedom, while keeping the same background equation of motion of the dRGT theory. The MTMG is defined in the Hamiltonian language by imposing the four constraints

$$\mathcal{C}_0 \approx 0, \qquad \mathcal{C}_i \approx 0, \quad (2.165)$$

with

$$\mathcal{C}_i \approx \{\mathcal{R}^{GR}_i, H_1\}, \quad (2.166)$$

$$\mathcal{C}_0 \approx \{\mathcal{R}^{GR}_0, H_1\} - \mu^2\partial_t\mathcal{H}_0, \quad (2.167)$$

and

$$H_1 = \int d^3x\mu^2 M\mathcal{H}_1. \quad (2.168)$$

One can verify that the previous two constraints $\tilde{\mathcal{C}}_\tau$ are linear combinations of $\mathcal{C}_i$, such that only two new constraints are applied to the minimal theory with respect to the precursor one. In the end, this reduces the total number of physical degrees of freedom to two, which are the two tensor transverse and traceless modes[8]. The Hamiltonian of the minimal theory is then

$$\begin{aligned} H_{MTMG} = \int d^3x \big[& -N\mathcal{R}_0 - N^i\mathcal{R}_i + \mu^2 M\mathcal{H}_1 + \lambda\mathcal{C}_0 + \lambda^i\mathcal{C}_i + \\ & +\alpha_{MN}\mathcal{Q}^{[MN]} + \beta_{MN}Y^{[MN]}\Big], \end{aligned} \quad (2.169)$$

with $N, N^i, \lambda, \lambda^i, \alpha_{MN}, \beta_{MB}$ 14 Lagrange multipliers. Notice that on the constrained surface the Hamiltonian reduces to $H_{MTMG} \approx H_1$. The definitions (2.154) allow to compute

[8] Rigorously speaking, it was shown that the total number of physical degrees of freedom must not be greater the two. However, one can consider the Hamiltonian equations of motion for secondary constraints and realize that no tertiary constraint arise. This prove that the above ones are all the constraints provided by the minimal theory.

explicitly the constraints $\mathcal{C}_0$ and $\mathcal{C}_i$. In particular one can expand the Hamiltonian density

$$\mathcal{H}_1 = \sqrt{\tilde{\gamma}} E_I{}^i \left[c_1 e^I{}_i + \frac{c_2}{2} E_J{}^j (e^I{}_i e^J{}_j - e^I{}_j e^J{}_i) \right] + c_3 \sqrt{\gamma} \tag{2.170}$$

and compute

$$\begin{aligned}
\frac{\delta \mathcal{R}_0^{GR}(x)}{\delta \Pi^m_M(y)} &= \frac{1}{2\sqrt{\gamma}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) \delta^{IA}\delta^{LB} e_A{}^j e_B{}^l \Pi_I{}^i \delta_{LM} \gamma^{km} \delta^3(\mathbf{x}-\mathbf{y}), && (2.171)\\
\frac{\delta \mathcal{R}_i^{GR}(x)}{\delta \Pi^m_M(y)} &= \sqrt{\gamma}\gamma_{ik} D_j \left(\frac{\delta^{AM}\gamma^j_m e_A{}^k}{\sqrt{\gamma}} \right) \delta^3(\mathbf{x}-\mathbf{y}), && (2.172)\\
\frac{\delta \mathcal{H}_1(x)}{\delta e^M_m(y)} &= \left[\sqrt{\tilde{\gamma}} \left(c_1 \delta^J_I + c_2 (Y_L{}^L \delta^J_I - Y_I{}^J) \right) + c_3 \sqrt{\gamma} X_I{}^J \right] \delta^I_M E_J{}^m \delta^3(\mathbf{x}-\mathbf{y}), && (2.173)\\
\frac{\delta \mathcal{H}_1(x)}{\delta \Pi^m_M(y)} &= 0, && (2.174)\\
\partial_t \mathcal{H}_0 &= \left[\sqrt{\tilde{\gamma}} \left(c_1 \delta^J_I + c_2 (Y_L{}^L \delta^J_I - Y_I{}^J) \right) + c_3 \sqrt{\gamma} X_I{}^J \right] \gamma^{jl} E^J{}_j \partial_t E^L{}_l. && (2.175)
\end{aligned}$$

This way the constraints $\mathcal{C}_{0,i}$ can be written as

$$\begin{aligned}
\mathcal{C}_0 &= \mu^2 M W_I{}^J \left[\frac{1}{2} (\gamma_{ik} E_J{}^k e^I{}_j + \gamma_{jk} E_J{}^k e^I{}_i - \gamma_{ij} Y_J{}^I) \pi^{ij} - \sqrt{\gamma} H_J^{(f)\ I} \right], && (2.176)\\
\mathcal{C}_i &= -\mu^2 \sqrt{\gamma} D^j (M W_I^{\mathcal{I}J} Y_J{}^K \delta_{KL} e^I{}_i e^L{}_j, && (2.177)
\end{aligned}$$

where for convenience there were introduced the following definitions

$$\begin{aligned}
W_I{}^J &\equiv \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} \left[c_1 \delta^J_I + c_2 (Y_K{}^K \delta^J_I - Y_I{}^J) \right] + c_3 X_I{}^J, && (2.178)\\
H_J^{(f)\ I} &\equiv \frac{1}{M} E_J{}^l \partial_t E^I{}_l. && (2.179)
\end{aligned}$$

Metric formulation At this point one can turn back to the Lagrangian formalism of this new theory retracing backward the steps shown above to go to the Hamiltonian formalism. The calculation are a bit lengthy and cumbersome because of the additional constraints $\mathcal{C}_0$ and $\mathcal{C}_i$, but there are no new physically meaningful arguments to learn in this procedure. For these steps are omitted in this text, but the reader is encouraged to look at [57] if interested in more details. In few words, starting from (2.169), one should invert the Hamiltonian equations for $e^I{}_i$ to obtain an expression for $\Pi^I{}_i$. Then the Lagrangian density is found as the Legendre transform of the Hamiltonian density, as done in (2.150). For later discussions it is convenient to turn to the metric formulation of the Lagrangian. For this purpose let's introduce two time dependent external fields

$$\tilde{\gamma}_{ij} = \delta_{IJ} E^I{}_i E^J{}_j, \qquad \tilde{\zeta}^i{}_j = \frac{1}{M} E_I{}^i E^I{}_j. \tag{2.180}$$

Then we define the tensor $\mathcal{K}^m_n$ and its inverse $\mathcal{Y}^m_n$ as

$$\mathcal{K}^m_l \mathcal{K}^l_n = \tilde{\gamma}^{ms}\gamma_{sn}, \qquad \mathcal{Y}^m_l \mathcal{K}^l_n = \delta^m_n. \tag{2.181}$$

These condition are satisfied if the two tensor admit the vielbein representations

$$\mathcal{K}^m_n = E_M{}^m e^M{}_n, \qquad \mathcal{Y}^m_n = e_M{}^m E^M{}_n, \tag{2.182}$$

or, in metric formulation, if

$$\mathcal{K}^m_n = \left(\sqrt{\tilde{\gamma}^{-1}\gamma}\right)^m_n, \qquad \mathcal{Y}^m_n = \left(\sqrt{\tilde{\gamma}\gamma^{-1}}\right)^m_n. \tag{2.183}$$

Then the following tensor is introduced

$$\Theta^{ij} = \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}\Big\{c_1(\gamma^{il}\mathcal{K}^j_l + \gamma^{jl}\mathcal{K}^i_l) + c_2[\mathcal{K}(\gamma^{il}\mathcal{K}^j_l + \gamma^{jl}\mathcal{K}^i_l) - 2\tilde{\gamma}^{ij}\Big\} + 2c_3\gamma^{ij} \tag{2.184}$$

together with the four constraints

$$\begin{aligned} \bar{\mathcal{C}}_0 &= \mu^2 M\Big\{\frac{1}{2}K_{ij}\Theta^{ij} - \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}[c_i\tilde{\zeta} + c_2(\mathcal{K}\tilde{\zeta} - \mathcal{K}^m_n\tilde{\zeta}^n_m)] + c_3\mathcal{Y}^m_n\tilde{\zeta}^n_m\Big\}, &(2.185)\\ \mathcal{C}^n_i &= -\mu^2 M\Big\{[\frac{1}{2}(c_1 + c_2\mathcal{K})(\mathcal{K}^n_i + \gamma^{mn}\gamma_{li}\mathcal{K}^l_m) - c_2\tilde{\gamma}^{nl}\gamma_{li}] + c_3\delta^n_i\Big\}, &(2.186)\end{aligned}$$

and the full action of MTMG is, reinserting the Plank mass M_P,

$$\begin{aligned} S_{\rm MTMG} &= S_{\rm pre} + \frac{M_P^2}{2}\int d^4x\, N\sqrt{\gamma}\left(\frac{\mu^2}{4}\frac{M}{N}\lambda\right)^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\Theta^{ij}\Theta^{kl}\\ &\quad -\frac{M_P^2}{2}\int d^4x\,\sqrt{\gamma}\left[\lambda\bar{\mathcal{C}}_0 - (\mathcal{D}_n\lambda^i)\mathcal{C}^n_i\right] + S_{\rm mat}, \end{aligned} \tag{2.187}$$

with

$$\begin{aligned} S_{\rm pre} &= S_{GR} + \frac{M_P^2}{2}\sum_{i=1}^{4}\int d^4x\mathcal{S}_i,\\ S_{GR} &= \frac{M_P^2}{2}\int d^4xN\sqrt{\gamma}\left[R[\gamma] + K_{ij}K^{ij} - K^2\right],\\ \mathcal{S}_1 &= -\mu^2 c_1\sqrt{\tilde{\gamma}}(N + M\mathcal{K}),\\ \mathcal{S}_2 &= -\frac{1}{2}\mu^2 c_2\sqrt{\tilde{\gamma}}(2N\mathcal{K} + M\mathcal{K}^2 - M\tilde{\gamma}^{ij}\gamma_{ij}),\\ \mathcal{S}_3 &= -\mu^2 c_3\sqrt{\gamma}(M + N\mathcal{Y}),\\ \mathcal{S}_4 &= -\mu^2 c_4\sqrt{\gamma}N, \end{aligned} \tag{2.188}$$

and $S_{\rm mat}$ the action related to a matter component in the Universe. In the following we will abandon the full general treatment and specialize this theory to the case of a homogeneous and isotropic FLRW background.

2.3.1 MTMG on a FLRW background

The symmetric properties of homogeneity and isotropy of the background demands $g^{0i} = g^{i0} = N^i = 0$, while $\gamma_{ij} = a^2\delta_{ij}$ and $\tilde{\gamma}_{ij} = \tilde{a}^2\delta_{ij}$ with a and $\tilde{a}$ the scale factors for the dynamical and fiducial three-dimensional metrics respectively. These relations allow to explicitly express the tensor $\mathcal{K}^i_j$ and its inverse as

$$\mathcal{K}^i_j = \left(\frac{a}{\tilde{a}}\right)\delta^i_j, \qquad \mathcal{Y}^i_j = \left(\frac{\tilde{a}}{a}\right)\delta^i_j, \tag{2.189}$$

while the external field $\tilde{\zeta}^i_j$ and the extrinsic curvature become

$$\zeta^i_j = H_f\delta^i_j, \qquad K_{ij} = a^2H\delta_{ij}, \tag{2.190}$$

with

$$H \equiv \frac{\dot{a}}{aN}, \qquad H_f \equiv \frac{\dot{\tilde{a}}}{\tilde{a}M}. \tag{2.191}$$

Defining for convenience the quantity $X \equiv \tilde{a}/a$, the tensor Θ^{ij} admits a simple expression

$$\Theta^{ij} = 2\frac{\delta^{ij}}{a^2}\left(c_1X^2 + 2c_2X + c_3\right), \tag{2.192}$$

and then the constraint $\bar{\mathcal{C}}_0$ reduces to

$$\bar{\mathcal{C}}_0 = 3\mu^2 M\Big[H(c_1X^2 + 2c_2X + c_3) - H_f(c_1X^3 + 2c_2X^2 + c_3X)\Big]. \tag{2.193}$$

At this point the Friedmann background equations can be derived as the Einstein equations for the action (2.187). Since the GR sector is untouched by the theory, it is fair to think that no new contributions should appear in the Einstein tensor. On the contrary, the constraints imposed upon the theory enter in a non-trivial way into the energy-momentum tensor defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_m) = -\frac{2}{N\sqrt{\gamma}}\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_m), \tag{2.194}$$

where $\mathcal{L}_m$ contains all the terms inside (2.187) but the GR-contribution. Raising indices with the metric (2.129), one finds for the time component

$$\begin{aligned}
T^0{}_0 &= -\frac{1}{a^3}\frac{\delta}{\delta N}(\sqrt{-g}\mathcal{L}_m) = -\rho_g - \rho_\lambda - \rho_m, && (2.195)\\
\rho_g &= \frac{\mu^2 M}{2}(c_1X^3 + 3c_2X^2 + 3c_3X + c_4), && (2.196)\\
\rho_\lambda &= -\frac{3M_P^2\mu^2 M}{2N}\Big[\frac{\mu^2 M}{8N}(c_1X^2 + 2c_2X + c_3)^2\lambda^2 + \\
&\qquad\qquad + H(c_1X^2 + 2c_2X + c_3)\lambda\Big], && (2.197)\\
\rho_m &= \frac{1}{a^3}\frac{\delta}{\delta N}(\sqrt{-g}\mathcal{L}_{\rm mat}), && (2.198)
\end{aligned}$$

where ρ_g denotes the energy density contribution arising from the precursor action, while ρ_λ accounts for the contribution from the additional constraints. With this arguments the Friedmann equation is written as

$$E_0 \equiv 3M_P^2H^2 - \rho_m - \rho_\lambda - \rho_g = 0. \tag{2.199}$$

Moreover, varying the action with respect to λ, one obtains a further constraint on the equations of motions

$$\begin{aligned}
E_\lambda &\equiv \frac{\delta S_{\rm MTMG}}{\delta\lambda} = 0\\
&= \Big[\mu^2 M\lambda(c_1X^2 + 2c_2X + c_3) + 4N(H - H_fX)\Big](c_1X^2 + 2c_2X + c_3). \quad (2.200)
\end{aligned}$$

For the purposes of this review, it is not necessary to go through the whole dynamics of the action (2.187). What we really care the most is to understand how cosmological perturbations evolve and propagate across the Universe. For all the details the reader can look at [57]. Let's just mention that the Friedmann equation can be coupled with its derivative together with the Bianchi identity and (2.200) to give $\lambda = 0$ and $\rho_\lambda = P_\lambda = 0$. Then equation (2.200) separates between two branches of solutions.

- *Self-accelerating branch:* The variation of the action with respect to λ is solved by

$$c_1 X^2 + 2c_2 X + c_3 = 0, \tag{2.201}$$

which implies $X =$ constant and

$$\rho_g = \frac{\mu^2 M_P^2}{2}(c_4 - 3c_2 X^2 - 2c_1 X^3) = \text{constant}, \tag{2.202}$$

$$P_g \delta^i_j = -\frac{2}{Na^3}\gamma^{ik}\frac{\delta}{\delta\gamma_{kj}} = -\frac{[N(c_2X^2 + 2c_3X + c_4)]\mu^2 M_P^2}{2N}\delta^i_j, \tag{2.203}$$

$$w_g \equiv \frac{P_g}{\rho_g} = -1. \tag{2.204}$$

Then, at the background level, the new contributions arising from the additional constraints behave like a pure cosmological constant.

- *Normal branch:* after having set $\lambda = 0$, this branch corresponds to the solution of (2.200) for which

$$H = XH_f. \tag{2.205}$$

Since this time X is not required to be constant, even the energy density ρ_g can change in time, highlighting a different behavior with respect to the self-accelerating branch. From (2.196) it is possible to isolate a cosmological constant component ρ_Λ and a time dependent dark component ρ_X in the following way

$$\rho_g = \rho_\Lambda + \rho_X,$$

$$\rho_\Lambda = \frac{c_4\mu^2 M_P^2}{2}, \tag{2.206}$$

$$\rho_X = \frac{\mu^2 M_P^2}{2}(3c_3X + 3c_2X^2 + c_1X^3). \tag{2.207}$$

2.3.2 Cosmological perturbations in MTMG

The evolution equations for the perturbed fields are obtained by varying the action (2.187) expanded at second order in perturbation theory. In ref. [57] it is shown that the mentioned action does not bring significant differences with respect to the GR case in the scalar and vector sectors. This is quite expected, since the MTMG does not allow the propagation of extra graviton modes. For this reason we will skip the discussion about these modes, assuming that both scalars and vectors evolve exactly as in GR, and focusing on the tensor case. Notice firstly that no contributions are expected to arise from the Lagrange multipliers λ and λ_i. Indeed the only possibilities one has to build a scalar out of a tensor field, are to contract its free indices with the metric γ_{ij} or with its momentum k^i (i.e. with spatial derivatives in coordinate space). Since by definition tensor perturbations are transfer and traceless, both these contraction vanish. For the same reasons we cannot build a vector field out of a transverse and traceless tensor. Moreover one can prove that the constraints $\mathcal{C}_0$ and $\mathcal{C}_i^n$ do not receive any contribution from tensor perturbations, such that these terms can be safely forgotten when writing down the quadratic action for tensor modes. This is instead derived by expanding at second order each term of the precursor action. Therefore, without any computation, we already learn that this theory provides the same equation of motion for tensor modes of the dRGT theory, since only the precursor sector of the full theory is involved. Before expanding these terms it is necessary to write down the spatial vielbeins perturbed up to second order. Their expression can be derived from the definition (2.128), which, at second order in perturbation theory, demands

$$\gamma_{ij} = a^2(\delta_{ij} + \chi_{ij} + \frac{1}{2}\delta^{mn}\chi_{im}\chi_{jn}) = \delta_{IJ}e^I_i e^J_j. \tag{2.208}$$

In order to derive the expression of the vielbein, we start with the ansatz[9]

$$e^I{}_j = a\left(\delta^I_j + \alpha\delta^{Ik}\chi_{kj} - \beta\delta^{Ik}\delta^{mn}\chi_{km}\chi_{in}\right), \tag{2.209}$$

and solve (2.208) for the arbitrary coefficients α and β. One can verify that, up to second order, the above relation is satisfied by $\alpha = 1/2$ and $\beta = 1/8$, which leads to

$$e^I{}_j = a\left(\delta^I_j + \frac{1}{2}\delta^{Ik}\chi_{kj} + \frac{1}{8}\delta^{Ik}\delta^{mn}\chi_{km}\chi_{in}\right). \tag{2.210}$$

Being the fiducial metric unperturbed, the inverse fiducial vielbein is simply $E_I{}^j = \tilde{a}^{-1}\delta^j_I$, and then it is easy to compute the tensor

$$\begin{aligned}\mathcal{K}^m_n &= E_M{}^m e^M{}_n = X^{-1}\left(\delta^m_n + \frac{1}{2}\delta^{mk}\chi_{kn} + \frac{1}{8}\delta^{mk}\delta^{\alpha\beta}\chi_{\alpha k}\chi_{\beta n}\right)\\ \mathcal{K} &= \mathcal{K}^n_n = X^{-1}\left(3 + \frac{1}{8}\chi_{ij}\chi^{ij}\right).\end{aligned} \tag{2.211}$$

It is possible now to take its inverse by think of a simple Taylor expansion

$$\begin{aligned}\mathcal{Y}^m_n &= E_M{}^m e^M{}_n = X\left(\delta^m_n - \frac{1}{2}\delta^{mk}\chi_{kn} + \frac{1}{8}\delta^{mk}\delta^{\alpha\beta}\chi_{\alpha k}\chi_{\beta n}\right)\\ \mathcal{Y} &= \mathcal{Y}^n_n = X\left(3 + \frac{1}{8}\chi_{ij}\chi^{ij}\right).\end{aligned} \tag{2.212}$$

As consistency check one can verify that $\mathcal{K}^m_n\mathcal{Y}^n_r = \delta^m_r$ and $\mathcal{K}^m_l\mathcal{K}^l_n = X^{-2}(\delta^m_n + \delta^{mk}\chi_{nk} + \frac{1}{2}\delta^{mk}\delta^{ij}\chi_{ki}\chi_{nj}) = \tilde{\gamma}^{mk}\gamma_{nk}$ are satisfied. With these ingredients it is possible now to expand each term of the precursor action. Considering only quadratic terms in the tensor perturbations one obtains

$$\begin{aligned}\mathcal{S}_1 &= -\mu^2 c_1 a^3 X^3 M\left(\frac{X^{-1}}{8}\chi_{ij}\chi^{ij}\right) = -\frac{\mu^2 a^3 N}{8}c_1 r X^3\chi_{ij}\chi^{ij},\\ \mathcal{S}_2 &= -\frac{\mu^2 a^3}{2}X^2 c_2\left(\frac{N}{4} + \frac{MX^{-1}}{4}\right)\chi_{ij}\chi^{ij} = -\frac{\mu^2 a^3 N}{8}c_2 X^2(1+r)\chi_{ij}\chi^{ij},\\ \mathcal{S}_3 &= -\frac{\mu^2 a^3 N}{8}c_3 X\chi_{ij}\chi^{ij}\\ \mathcal{S}_4 &= 0.\end{aligned} \tag{2.213}$$

To be honest, we have been a bit fast in evaluating the terms $\mathcal{S}_3$ and $\mathcal{S}_4$, that need some more computations because of the determinant of the perturbed spatial metric, which actually may receive contributions from tensor perturbations. However, this is not the case, as we will prove now. Let's rewrite for convenience of notation

$$\gamma_{ij} \equiv a^2\bar{\gamma}_{ij} = a^2(\delta_{ij} + h_{ij}). \tag{2.214}$$

[9] Notice indeed that, since χ_{ij} is symmetric and traceless, the third term of equation (2.209) is the only independent combination of two filed fluctuations we can build, as one can realize by permuting the indices of the Kronecker deltas

Denoting $\bar{\gamma} = \det \bar{\gamma}$, the determinant of the perturbed metric can be expanded in Taylor series as

$$\bar{\gamma} = \det(\delta_{ij}) + h_{ij}\frac{\partial \bar{\gamma}}{\partial \bar{\gamma}_{ij}}|_{\bar{\gamma}_{ij}=\delta_{ij}} + \frac{1}{2!}h_{ij}h_{kl}\frac{\partial^2 \bar{\gamma}}{\partial \bar{\gamma}_{ij}\partial \bar{\gamma}_{kl}}|_{\bar{\gamma}_{ij}=\delta_{ij}} + o(h^3). \tag{2.215}$$

Using the well known property of the determinant $\ln(\det A) = \mathrm{tr}\,(\ln A)$, with A a generic squared matrix, one can verify that

$$\frac{\partial \bar{\gamma}}{\partial \bar{\gamma}_{ij}} = \bar{\gamma}\,\bar{\gamma}^{ij}, \tag{2.216}$$

while the trivial condition $A^{-1}A = 1$ implies

$$\frac{\partial \bar{\gamma}_{kl}}{\partial \bar{\gamma}_{ij}} = \delta_i^k \delta_j^l, \qquad \frac{\partial \bar{\gamma}^{kl}}{\partial \bar{\gamma}_{ij}} = -\bar{\gamma}^{ki}\bar{\gamma}^{lj}. \tag{2.217}$$

Finally (2.215) evaluates to

$$\bar{\gamma} = 1 + h_{ij}\delta^{ij} + \frac{1}{2}h_{ij}h_{kl}\left(\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl}\right). \tag{2.218}$$

Replacing $h_{ij} = \chi_{ij} + \frac{1}{2}\delta^{mn}\chi_{im}\chi_{jn})$ and exploiting the properties of tensor perturbations, this equation becomes

$$\bar{\gamma} = 1 + \frac{1}{2}\chi_{ij}\chi^{ij} + \frac{1}{2}(-\chi_{ij}\chi^{ij}) = 1, \tag{2.219}$$

and this proves the statement $\sqrt{\gamma} = a^3$.

Turning back to the quadratic action, nothing new happens in the GR-sector, which get expanded as usual as shown in [52]. In the end, summing up all the above terms, one can write the quadratic action for tensor perturbations as

$$S_T^{(2)} = \frac{M_P^2}{8}\int d^4x\, N a^3 \left[\frac{\dot{\chi}_{ij}\dot{\chi}^{ij}}{N^2} - \frac{\partial_k \chi_{ij}\partial^k \chi^{ij}}{a^2} - m^2 \chi_{ij}\chi^{ij}\right], \tag{2.220}$$

where the effective mass for tensor modes after inflation is defined as

$$m^2 = \frac{1}{2}\mu^2 X\left[c_2 X + c_3 + rX(c_1 X + c_2)\right], \tag{2.221}$$

and is valid in both the self accelerating and normal branches. Setting $N = 1$ in a FLRW Universe and and passing to conformal time, the action becomes

$$S_T^{(2)} = \frac{M_P^2}{8}\int d^4x\, a^2 \left[\chi'_{ij}\chi^{ij\,\prime} - (k^2 + a^2 m^2)\chi_{ij}\chi^{ij}\right], \tag{2.222}$$

which is exactly the same equation (2.34) with a different mass parameter. Then its solution can be taken on the same form of the one found in Section (2.1.1) by just replacing the correct definition of the mass. Notice that the mass parameter entering inside the equations of motion μ is different from the Lagrange parameter m. In the following, when we speak about the graviton mass after inflation, we refer to the parameter μ, which plays the role of an effective mass.

Chapter 3

Anisotropies in the SGWB with massive gravitons

The aim of this chapter is to study the origin of the anisotropies in the graviton population, and in particular how a possible not vanishing mass can affect the population. This work will focus only on the SGWB from cosmological origin, relying on the possibility, that we may reach with future interferometers, to distinguish them from the astrophysical GW background, which gets contributions from a huge population of unresolved sources. A way to discriminate between the two is shown in [65]; basically the distinction is based on a frequency separation, since the emitted frequency of any astrophysical source is constrained by its angular velocity. As one does for the CMB, the stochastic nature that characterizes the background of cosmological gravitational waves legitimizes the definition of a distribution function describing the graviton population, and to study the anisotropies of this distribution through an approach based on the Boltzmann equation. This equation indeed encodes the time evolution of the distribution function of a given particle species accounting for the Universe expansion and any possible interaction which can occur between the particle species and the thermal plasma. On the contrary, what distinguish gravitons from photons is the fact that they are collisionless and not thermally distributed. The latter feature results in a frequency dependence of the angular anisotropies which instead appears only at second order in the case of the CMB. Restricting the analysis to GWs at those scales that can be probed by our detectors, it is reasonable to focus only on large cosmological scales (usually one considers those scales which re-enter the horizon at late time, at least after the beginning of the matter domination in the Universe). This scales' selection provides a hierarchy $q \gg k$ between the GW comoving momentum q and the comoving momentum k of the large scales perturbations, since primordial GWs are supposed to carry an energy comparable with the energy scale of the Universe at the time they were formed. On the other hand, small scale perturbations are averaged out during the propagation of the GW, and thus they give no sensible contribution to the anisotropies. The path to follow is the same of the case of the CMB, as it is already outlined in many texts [70, 67]. Firstly, in section (3.1), we derive the collisionless Boltzmann equation for the graviton distribution function by considering just the contribution coming from the Liouville operator. Then, in section (3.2), we solve the Boltzmann equation through a perturbative approach, expanding the graviton distribution function around the backgroung FRW solution up to first order. As we will see, the first order solution is characterized by spatial fluctuations. In section (3.3) we show how to relate these fluctuations to physical observables, and in particular to the fluctuation in the energy density of the GWs.

Besides the energy density fluctuations, we are mostly interested in the produced anisotropy, since it is the real physical quantity we face with while dealing with interferometer observations. For this reason in the last section (3.4) of this chapter we prepare the field for the later studies of the n-point correlation functions. In particular we project the fluctuations on the orthonormal the basis of spherical harmonics, since, as we will see, they represent the preferential basis when working with angular correlators.

3.1 Boltzmann equation for massive gravitons

The statistical nature of the GWs generated by cosmological processes allows to define a graviton distribution function, from which one can eventually extrapolate any desired observable that characterizes the graviton population. Since the typical graviton production mechanisms we have in mind had been taking place below the Plank energy scale, and then during an epoch where the quantum gravitational effects are thought to be not very significant, the initial graviton population is not expected to be thermal, which on the contrary is the case of photons. This fact forbids us to give any guess about the shape of distribution function, which then will be treated in full generality as a function $f(x^\mu, p^\mu)$ with an implicit dependence on both the position x^μ and the momentum $p^\mu = dx^\mu/d\lambda$, with λ an affine parameter that parametrizes the GW geodesics. The evolution of the distribution function is governed by the Boltzmann equation:

$$\mathcal{L}[f] = \mathcal{C}[f(\lambda)] + \mathcal{I}[f(\lambda)]. \tag{3.1}$$

The collision term $\mathcal{C}[f(\lambda)]$ takes into account any interaction between gravitons and the thermal plasma. Since the thermal bath is mainly composed by photons, the graviton-photons interaction is thought to bring the dominant contribution to the collision term. However in [64] it has been proved that this interaction provides a very small contribution to the CMB anisotropy, and we can then neglect it for our purposes. Moreover, as explained in [68], a consistent definition of graviton beyond the framework of the quantum field theories is possible provided the possibility of gravitons' self-interactions. Nevertheless, these collisions are expected to be driven by a weak interaction whose coupling constant $\kappa = 2M_p^{-1}$ is many order of magnitude smaller then the typical coupling constants describing the other standard model interactions. Therefore any graviton self-interaction happens to be even more suppressed, and then can be neglected in our computations. The emissivity term $\mathcal{I}[f(\lambda)]$ instead considers both cosmological and astrophysical processes. However, as long as we focus on the stochastic GW background of cosmological origin, we can forget about the latter contribution, and treat the former as an initial condition on the distribution function. After these considerations we only remain with the Liouville operator $\mathcal{L} \equiv d/d\lambda$ describing the collisionless time evolution of the graviton distribution:

$$\frac{df}{d\lambda} = \frac{d\eta}{d\lambda}\frac{df}{d\eta} = 0. \tag{3.2}$$

Since the first factor on the right hand side is by definition the time component of the physical momentum p^μ, which in general is non vanishing, we can express the collisionless Boltzmann equation on a flat perturbed FLRW universe as

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial f}{\partial q}\frac{dq}{d\eta} + \frac{\partial f}{\partial n^i}\frac{dn^i}{d\eta} = 0, \tag{3.3}$$

where $\hat{n} \equiv \hat{p}$ identifies the direction of propagation of the GW, while $q \equiv |\vec{p}|a$ is the modulus of the comoving momentum $\vec{q}$, defined by

$$g_{ij}p^i p^j = \frac{q^2}{a^2}, \tag{3.4}$$

and the metric $g_{\mu\nu}$ is the usual perturbed metric on a FLRW background in the Poisson gauge

$$ds^2 = a^2(\eta)\left[-e^{2\Phi}d\eta^2 + \left(e^{-2\Psi}\delta_{ij} + \chi_{ij}\right)dx^i dx^j\right]. \tag{3.5}$$

Equation (3.3) contains many terms which arise from graviton relativistic dynamics, which then deserves a bit of attention. Denoting with m the mass charge of gravitons, we can exploit the mass-shell condition[1]

$$p^2 = g_{\mu\nu}p^\mu p^\nu = g_{00}(p^0)^2 + \frac{q^2}{a^2} = -m^2 \tag{3.6}$$

to obtain an expression for the time and spatial components of the physical momentum

$$p^0 = \frac{e^{-\Phi}}{a}\left(\frac{q^2}{a^2} + m^2\right)^{1/2} \equiv \frac{e^{-\Phi}}{a}E, \tag{3.7}$$

$$p^i = \frac{q}{a^2}n^i e^{\Psi}\left(1 - \frac{1}{2}\chi_{kl}n^k n^l\right), \tag{3.8}$$

having defined the graviton energy as

$$E = \sqrt{\frac{q^2}{a^2} + m^2}. \tag{3.9}$$

These results immediately allow us to express the factor $dx^i/d\eta$ appearing in (3.3):

$$\frac{dx^i}{d\eta} = \frac{p^i}{p^0} = \frac{q}{aE}n^i e^{\Psi+\Phi}\left(1 - \frac{1}{2}\chi_{kl}n^k n^l\right). \tag{3.10}$$

One can appreciate the physical meaning of this equation remembering that the scalar perturbations Φ and Ψ in the Poisson gauge acquire the meaning of gravitational potentials, and both have negative values on overdense regions as the effect of the gravitational redshift. Therefore (3.10) is telling us that gravitons are slowed down while traveling through overdense regions. However, since the zero-order distribution function of graviton is supposed to be isotropic and homogeneous, as it is defined on a FLRW background spacetime, the spatial derivative of the distribution function is at least a first-order term, and then we can safely retain only the zero-order contribution of (3.10) inside the Boltzmann equation.

Concerning the next term in (3.3), we can evaluate the factor $dq/d\eta$ exploiting the geodesic equation for gravitons that, by a simple reparametrization in terms of the conformal time, can be written as:

$$\frac{dp^0}{d\eta} + \Gamma^0_{\mu\nu}\frac{p^\mu p^\nu}{p^0} = 0. \tag{3.11}$$

Remembering (3.7), and defining $\mathcal{H} \equiv a'/a$, we can straightforwardly compute the first term

$$\frac{dp^0}{d\eta} = -\frac{d\Phi}{d\eta}\frac{E}{a}e^{-\Phi} - \frac{\mathcal{H}E}{a}e^{-\Phi} + \frac{e^{-\Phi}}{a}\frac{dE}{d\eta}, \tag{3.12}$$

such that the geodesic equation can now be rearranged in terms of the time derivative of the graviton energy:

$$\frac{dE}{d\eta} = \mathcal{H}E + E\frac{d\Phi}{d\eta} - \Gamma^0_{\mu\nu}\frac{p^\mu p^\nu}{p^0}ae^{\Phi}. \tag{3.13}$$

[1] The negative sign in front of the mass term comes from the conventional choice of using the mostly positive signature metric.

In Appendix (A) we have derived the linearly perturbed Christoffel symbols. We can now specialize the results (A.7) to our present case, where $h_{00} = -2\Phi$ and $h_{ij} = -2\Psi\delta_{ij} + \chi_{ij}$, while the mixed space-time components of the metric are all vanishing. Therefore the Christoffel symbols with fixed time upper index, up to first order, are:

$$\begin{aligned} \Gamma^0_{00} &= \mathcal{H} + \Phi', \qquad\qquad \Gamma^0_{0i} = \partial_i\Phi, \\ \Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + \frac{1}{2}\chi'_{ij} - \Psi'\delta_{ij} + \mathcal{H}\chi_{ij} - 2\mathcal{H}\delta_{ij}\Phi - 2\mathcal{H}\delta_{ij}\Psi, \end{aligned} \tag{3.14}$$

and the last terms of (3.13) read:

$$\begin{aligned} \Gamma^0_{00}\, p^0 a e^{\Phi} &= (\mathcal{H} + \Phi')E, \\ \Gamma^0_{0i}\, p^i a e^{\Phi} &= \frac{q}{a} n^i \partial_i \Phi, \\ \Gamma^0_{ij}\, \frac{p^i p^j}{p^0} a e^{\Phi} &= \frac{q^2}{a^2 E}\left(\mathcal{H} - \Psi' + \frac{1}{2}\chi'_{ij} n^i n^j\right), \end{aligned} \tag{3.15}$$

where all terms beyond the first order have been neglected. Summing up all the components we get[2]:

$$\begin{aligned} \frac{dE}{d\eta} &= E\left(\frac{d\Phi}{d\eta} - \Phi'\right) - 2\frac{q}{a} n^i \partial_i \Phi - \frac{q^2}{a^2 E}\left[\mathcal{H} - \Psi' + \frac{1}{2}\chi'_{ij} n^i n^j\right] \\ &= -\frac{q}{a} n^i \partial_i \Phi - \frac{q^2}{a^2 E}\left[\mathcal{H} - \Psi' + \frac{1}{2}\chi'_{ij} n^i n^j\right]. \end{aligned} \tag{3.17}$$

Lastly, differentiating the mass-shell condition (3.6) we obtain

$$E\,dE = p\,dp = \frac{q}{a^2} dq - \frac{q^2}{a^3} da, \tag{3.18}$$

in such a way that we are now able to transform the time derivative of the energy E in a time derivative of the (modulus of the) comoving momentum q. Indeed, inserting the last result inside (3.17), we easily arrive to

$$\frac{dq}{d\eta} = q\left[\Psi' - \frac{aE}{q} n^i \partial_i \Phi - \frac{1}{2}\chi'_{ij} n^i n^j\right]. \tag{3.19}$$

For what concerns the last term of the Boltzmann equation (3.3) instead, we can safely assume that it is an higher order contribution. Indeed we can consider that df/dn^i is a term of first order simply relying again on the fact that the distribution function f is defined on a flat FLRW background spacetime. On the other hand the factor $dn^i/d\eta$ is of first order as well, because in absence of perturbations any particle would pursue a straight path, then vanishing the derivative term. It's only because of the gravitational potentials Φ and Ψ and the tensor modes χ that a particle could deviate from its path; but this dependence on the perturbation modes immediately implies that this term could give non vanishing contributions at least at first order. All in all this whole last term turns out to be at least at second order.

[2] Notice that the $0i$-component must be counted twice, and that the total derivative of the scalar potential Φ appearing in (3.13) can be decomposed as

$$\frac{d\Phi}{d\eta} = \frac{\partial\Phi}{\partial\eta} + \frac{dx^i}{d\eta}\frac{\partial\Phi}{\partial x^i} = \Phi' + \frac{q}{aE} n^i \partial_i \Phi, \tag{3.16}$$

where the result (3.10) at zero order has been used in the second equality

At the end of the day, the results found so far, in particular (3.10) and (3.19), allows us to express the collisionless Boltzmann equation (3.3) in the more explicit form:

$$\frac{\partial f}{\partial \eta} + \frac{q}{aE} n^i \partial_i f + \left[\Psi' - \frac{aE}{q} n^i \partial_i \Phi - \frac{1}{2}\chi'_{ij} n^i n^j\right] q \frac{\partial f}{\partial q} = 0. \tag{3.20}$$

This result extends in a natural way the analysis of [63] performed for a scalar perturbed FLRW Universe. In order to appreciate the predictions of this result, we immediately notice that equation (3.20) recovers the collisionless Boltzmann equation for photons in the limit where $E = q/a$. It is easy to see that the ratio aE/q corresponds indeed to the graviton phase velocity (which is always grater the the light velocity), and that it approaches to the light velocity as the graviton mass m goes to zero. Denoting as v the graviton group velocity[3] defined by the relation $p = \gamma m v$ and recalling the definition of the relativistic factor $\gamma = (1 - v^2)^{-1/2}$, we can evaluate the ratio as

$$\frac{q}{aE} = \left(1 + \frac{a^2 m^2}{q^2}\right)^{-\frac{1}{2}} = \left(1 + \frac{1}{\gamma^2 v^2}\right)^{-\frac{1}{2}} = v, \tag{3.21}$$

where in the second step we used the definition of the physical momentum $p = q/a = \gamma m v$. As stated above, the first equality shows clearly that the massless limit sends the ratio q/aE, and thus the graviton velocity, to one, recovering then the collisionless Boltzmann equation for photons.

3.2 Formal solution

As we are perturbing the spacetime around the FLRW background, we can decompose the distribution function in an isotropic and homogeneous part and a perturbed one. At first order this decomposition reads:

$$f(\eta, x^i, q, n^i) = \bar{f}(q) + \delta f^{(1)}(\eta, x^i, q, n^i) = \bar{f}(q) - q\frac{\partial \bar{f}}{\partial q}\Gamma(\eta, x^i, q, n^i). \tag{3.22}$$

Inserting this decomposition inside (3.20), we can solve perturbatively the Boltzmann equation order by order. As expected, at zero order it gives

$$\frac{\partial \bar{f}}{\partial \eta} = 0. \tag{3.23}$$

From this equation we can already learn some interesting physics of gravitons. Indeed the equation is solved by any distribution function of the type $f = \bar{f}(q)$. Hence the the physical momentum of each graviton is expected to scale as a^{-1}, while the number density $n \propto \int d^3p \bar{f}(q)$ gets diluted as a^{-3} by the Universe expansion. This fact is directly supported by (3.19), which at zero order implies q being time independent, and then the mentioned scaling for the physical momentum p. These features are shared by photons as well, even though they are thermally distributed. Therefore we see that the scaling of the momentum with the scale factor a does not rely on the distribution being thermal, but it is instead a consequence of the free propagation through an expanding FLRW Universe. As one can see from equation (3.19), metric perturbations (as possible collision terms) break the free streaming of gravitons introducing a time dependence on its comoving momentum.

[3] In the following we will refer to the group velocity simply as the graviton velocity, while we specify the phase velocity when we will deal with it.

The first order Boltzmann equation instead becomes an equation for the anisotropy $\Gamma(\eta, x^i, q, n^i)$, that is:

$$\frac{\partial\Gamma}{\partial\eta} + \frac{q}{aE}n^i\frac{\partial\Gamma}{\partial x^i} = S(\eta, x^i, q, n^i), \tag{3.24}$$

where

$$S(\eta, x^i, q, n^i) = \Psi' - \frac{aE}{q}n^i\partial_i\Phi - \frac{1}{2}\chi'_{ij}n^in^j \tag{3.25}$$

denotes the source function that takes into account all the effects arising from the metric perturbations. Moreover notice that the the q-dependence of the source function is only contained inside the ratio aE/q. This dependence vanishes in the massless case, where the ratio turns out to be equal to 1. In order to solve this equation, it is convenient to pass to the momentum space operating a Fourier transform over the spatial coordinates on the anisotropy and source term:

$$\Gamma \equiv \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\Gamma(\eta, \vec{k}, q, \hat{n}), \tag{3.26}$$

$$S \equiv \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}S(\eta, \vec{k}, q, \hat{n}), \tag{3.27}$$

with

$$S(\eta, \vec{k}, q, \hat{n}) = \Psi' - ik\mu\frac{aE}{q}\Phi - \frac{1}{2}\chi'_{ij}n^in^j. \tag{3.28}$$

This way the first order Boltzmann equation in momentum space reads:

$$\Gamma' + ik\mu\frac{q}{aE}\Gamma = S, \tag{3.29}$$

having defined $\mu \equiv \hat{k}\cdot\hat{n}$ the cosine angle between the wave vector $\vec{k}$ of each Fourier mode and the direction of propagation $\hat{n}$ of the GW. Equation (3.29) is a first order differential equation, whose general solution can be written as

$$\Gamma(\eta, \vec{k}, q, \hat{n}) = \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')}\left[\Gamma(\eta', \vec{k}, q, \hat{n})\delta(\eta' - \eta_{in}) + S(\eta', \vec{k}, \hat{n})\right], \tag{3.30}$$

where, to simplify the notation, we have defined

$$l(\eta, \eta') \equiv \int_{\eta'}^{\eta} d\eta'' v(\eta'', q), \qquad v(\eta, q) \equiv \frac{q}{aE}, \tag{3.31}$$

accounting for a generic time dependence of the graviton velocity $v(\eta, q)$. Physically speaking, $l(\eta, \eta')$ is the distance traveled by gravitons from the time η' to η [4]. Again in the massless limit this distance turns out to be simply the difference between the two extreme values of the conformal time, as expected for photons. Notice the delta function multiplying the first term in (3.30) that ensures the right initial condition when $\eta = \eta_{in}$. One can

[4]One can further convince himself of this fact by considering the line element of a massive particle

$$ds^2 = -d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = -a^2d\eta^2 + a^2dl^2,$$

with τ the proper time of the particle. This expression can be rearranged in favor of the spatial distance

$$dl^2 = d\eta^2\left(1 - \frac{d\tau^2}{a^2d\eta^2}\right) = d\eta^2\left(1 - \gamma^{-2}\right) = v^2d\eta^2.$$

easily convince himself that the above solution is the correct one by simply performing the integration on the delta factor

$$\Gamma(\eta, \vec{k}, q, \hat{n}) = \Gamma(\eta_{in}, \vec{k}, q, \hat{n}) e^{-ik\mu l(\eta,\eta_{in})} + \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} S(\eta', \vec{k}, \hat{n}), \tag{3.32}$$

and evaluating its derivative with respect to the conformal time[5]

$$\begin{aligned}\Gamma' = &-ik\mu v \Gamma(\eta_{in}, \vec{k}, q, \hat{n}) e^{-ik\mu l(\eta,\eta_{in})} + S(\eta, \vec{k}, \hat{n}) e^{-ik\mu l(\eta,\eta)} \\ &-ik\mu \frac{q}{aE} \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu v(\eta,\eta')} S(\eta', \vec{k}, \hat{n}),\end{aligned} \tag{3.33}$$

which straightforwardly verifies (3.29) if we remember that $l(\eta, \eta) = 0$.

Now we would like to rewrite the formal solution in such a way to isolate the boundary conditions contribution from the ones describing the effects of the propagation of gravitons. For this purpose it is useful to work out the Φ-dependence appearing in (3.30) inside the source term as follows:

$$\begin{aligned}-\int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} ik\mu v^{-1} \Phi(\eta', \vec{k}) &= -\int_{\eta_{in}}^{\eta} d\eta' \frac{\partial e^{-ik\mu l(\eta,\eta')}}{\partial \eta'} v^{-2} \Phi(\eta', \vec{k}) \\ &= -e^{-ik\mu l(\eta,\eta')} v^{-2} \Phi(\eta', \vec{k})|_{\eta_{in}}^{\eta} + \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} \frac{\partial \left[v^{-2}\Phi(\eta', \vec{k})\right]}{\partial \eta'} \\ &= -v^{-2}\Phi(\eta, \vec{k}) + \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} \Bigg\{ v^{-2}\Phi(\eta', \vec{k}) \delta(\eta' - \eta_{in}) \\ &\qquad + \frac{\partial \left[v^{-2}\Phi(\eta', \vec{k})\right]}{\partial \eta'} \Bigg\}.\end{aligned} \tag{3.34}$$

The monopole term $v^{-2}\Phi(\eta, \vec{k})$ gives an isotropic contribution. Since our final purpose is to compute the angular power spectrum of the anisotropies, this term can be disregarded, and we can now express the formal solution in Fourier space in the form:

$$\begin{aligned}\Gamma(\eta, \vec{k}, q, \hat{n}) = \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} \Bigg\{ &\left[\Gamma(\eta', \vec{k}, q, \hat{n}) + v^{-2}\Phi(\eta', \vec{k})\right] \delta(\eta' - \eta_{in}) \\ &+ \frac{\partial \left[\Psi(\eta', \vec{k}) + v^{-2}\Phi(\eta', \vec{k})\right]}{\partial \eta'} - \frac{1}{2} n^i n^j \frac{\partial \chi_{ij}(\eta', \vec{k})}{\partial \eta'} \Bigg\}.\end{aligned} \tag{3.35}$$

The term multiplying the delta function is a boundary term that keeps memory of the initial conditions. In the massless case this is the only one that preserves the dependence on the GW's frequency q. Moreover, while the other term contain the dependence on the propagation direction only through the cosine angle $\mu \equiv \hat{k} \cdot \hat{n}$, in principle the same dependence in $\Gamma(\eta_{in}, \vec{k}, q, \hat{n})$ could be more general. However, being that term an initial condition term, the presence on such a dependence would imply the existence of an anisotropic mechanism of production of GW during primordial times across the whole Universe that would spoil the assumption of an exact FLRW background spacetime. For this reason in the following we will disregard the dependence on the director $\hat{n}$ in the initial condition term, that is we assume $\Gamma_{in} = \Gamma(\eta_{in}, \vec{k}, q)$. Notice further that a non vanishing graviton mass does not

[5] For ease of notation, the functional dependence of the graviton velocity $v(\eta, q)$ is here understood

introduce any additional directional dependence, as velocity correction terms only enter with the modulus of the comoving momentum q.

The form of equation (3.35) suggests to divide the solution as

$$\Gamma(\eta,\vec{k},q,\hat{n}) \equiv \Gamma_I(\eta,\vec{k},q,\hat{n}) + \Gamma_S(\eta,\vec{k},q,\hat{n}) + \Gamma_T(\eta,\vec{k},q,\hat{n}), \tag{3.36}$$

where the subscripts *I,S* and T denote respectively the *Initial, Scalar* and *Tensor* sourced terms, defined by

$$\Gamma_I(\eta,\vec{k},q,\hat{n}) = e^{-ik\mu l(\eta,\eta_{in})}\Gamma(\eta_{in},\vec{k},q), \tag{3.37}$$

$$\begin{aligned}\Gamma_S(\eta,\vec{k},q,\hat{n}) &= \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} \times \\ &\quad \times\left\{v^{-2}\Phi(\eta',\vec{k})\delta(\eta'-\eta_{in}) + \frac{\partial\left[\Psi(\eta',\vec{k})+v^{-2}\Phi(\eta',\vec{k})\right]}{\partial\eta'}\right\},\end{aligned} \tag{3.38}$$

$$\Gamma_T(\eta,\vec{k},q,\hat{n}) = -\frac{n^i n^j}{2}\int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')}\frac{\partial\chi_{ij}(\eta',\vec{k})}{\partial\eta'}. \tag{3.39}$$

As we will see, this decomposition will reveal itself very useful under the assumption that the three terms are not cross-correlated. In the end this assumption will result in three different sets of angular power spectra.

3.3 Energy density perturbations

The scalar and tensor metric perturbations give rise to fluctuations on the probability density distribution of the graviton population. These inhomogeneities, as we will see in this section, then reflect in perturbations of the SGWB energy density we may detect nowadays. This fact selects the energy density as the privileged channel to discover anisotropies and inhomogeneities in the SGWB, and lead us to the study of its perturbations. We define the SGWB energy density as

$$\begin{aligned}\rho_{GW}(\eta_0,\vec{x}) &\equiv \int d^3p\, E f(\eta_0,\vec{x},q,\hat{n}) \\ &= \int \frac{d^3q}{a^3}\sqrt{\frac{q^2}{a^2}+m^2} f(\eta_0,\vec{x},q,\hat{n}) \\ &= \int d\ln q\,\frac{q^4}{a^4}\sqrt{1+\frac{a^2m^2}{q^2}}\int d^2\hat{n} f(\eta_0,\vec{x},q,\hat{n}).\end{aligned} \tag{3.40}$$

Defining the spectral energy density

$$\Omega_{GW}(\eta_0,\vec{x},q) \equiv \frac{q^4}{a^4\rho_{crit,0}}\sqrt{1+\frac{a^2m^2}{q^2}}\int d^2\hat{n} f(\eta_0,\vec{x},q,\hat{n}), \tag{3.41}$$

where $\rho_{crit,0} = 3\mathcal{H}^2M_p^2/a^2$ denotes the present critical energy density, the SGWB energy density can be written in the more compact form

$$\rho_{GW}(\eta_0,\vec{x}) = \rho_{crit,0}\int d\ln q\,\Omega_{GW}(\eta_0,\vec{x},q). \tag{3.42}$$

For the sake of convenience it is common to introduce the quantity $\omega_{GW}(\eta_0,\vec{x},q,m,\hat{n})$ defined by

$$\Omega_{GW}(\eta_0,\vec{x},q) \equiv \int d^2\hat{n}\,\omega_{GW}(\vec{x},q,\hat{n}), \tag{3.43}$$

that is

$$\begin{aligned}
\omega_{GW}(\eta_0, \vec{x}, q, \hat{n}) &\equiv \frac{q^4}{a^4 \rho_{crit,0}} \sqrt{1 + \frac{a^2 m^2}{q^2}} f(\eta_0, \vec{x}, q, \hat{n}) \\
&= \frac{q^4}{a^4 \rho_{crit,0}} \sqrt{1 + \frac{a^2 m^2}{q^2}} \left[\bar{f}(\eta_0, q) - q \frac{\partial \bar{f}(\eta_0, q)}{\partial q} \Gamma(\eta_0, x^i, q, n^i) \right] \\
&= \frac{q^4}{a^4 \rho_{crit,0}} \sqrt{1 + \frac{a^2 m^2}{q^2}} \bar{f}(\eta_0, q) \left[1 - \frac{\partial \ln \bar{f}(\eta_0, q)}{\partial \ln q} \Gamma(\eta_0, x^i, q, n^i) \right] \\
&\equiv \bar{\omega}_{GW}(\eta_0, q) \left[1 + \frac{\delta\omega(\eta_0, \vec{x}, q, \hat{n})}{\bar{\omega}_{GW}(\eta_0, q)} \right], \qquad (3.44)
\end{aligned}$$

where in the last step we have separated the expression of $\omega_{GW}(\vec{x}, q, \hat{n})$ in an homogeneous component

$$\bar{\omega}_{GW}(\eta_0, q) \equiv \frac{q^4}{a^4 \rho_{crit,0}} \sqrt{1 + \frac{a^2 m^2}{q^2}} \bar{f}(q), \qquad (3.45)$$

and an inhomogeneous one

$$\delta_{GW}(\eta_0, \vec{x}, q, \hat{n}) \equiv \frac{\delta\omega(\eta_0, \vec{x}, q, \hat{n})}{\bar{\omega}_{GW}(\eta_0, q)} = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} \Gamma(\eta_0, x^i, q, n^i), \qquad (3.46)$$

which is usually called SGWB energy density contrast. With this decomposition we can compute the homogeneous contribution to the total spectral energy density as

$$\bar{\Omega}_{GW}(\eta_0, q) \equiv \int d^2\hat{n}\, \bar{\omega}_{GW}(\eta_0, q) = 4\pi \bar{\omega}_{GW}(\eta_0, q), \qquad (3.47)$$

while, inverting (3.45), the homogeneous part of the density distribution function can be written as

$$\bar{f}(q) = \frac{a^4 \rho_{crit,0}}{4\pi q^4} \left[1 + \frac{a^2 m^2}{q^2} \right]^{-1/2} \bar{\Omega}_{GW}(\eta_0, q). \qquad (3.48)$$

In order to write explicitly the SGWB density contrast, we consider its logarithmic derivative

$$\begin{aligned}
\frac{\partial \ln \bar{f}(q)}{\partial \ln q} &= -4 - \frac{1}{2} \frac{\partial \ln \left[1 + \frac{a^2 m^2}{q^2} \right]}{\partial \ln q} + \frac{\partial \ln \bar{\Omega}_{GW}(\eta_0, q)}{\partial \ln q} \\
&= -4 + \left(1 + \frac{q^2}{a^2 m^2} \right)^{-1} + \frac{\partial \ln \bar{\Omega}_{GW}(\eta_0, q)}{\partial \ln q}. \qquad (3.49)
\end{aligned}$$

Notice that the second term is a correction arising from a non vanishing graviton mass, and it is the only term carrying information about the mass. In the massless limit indeed this expression recover the result for the massless case shown in [59]. In the end the SGWB energy density contrast is

$$\delta_{GW}(\eta_0, \vec{x}, q, \hat{n}) = \left[4 - \frac{\partial \ln \bar{\Omega}_{GW}(\eta_0, q)}{\partial \ln q} - \left(1 + \frac{q^2}{a^2 m^2} \right)^{-1} \right] \Gamma(\eta_0, x^i, q, n^i). \qquad (3.50)$$

This expression clearly shows how the energy density contrast of the SGWB is related to the anisotropies generated by the metric perturbations. Any experimental measure of this contrast in the background of gravitational waves energy density would unequivocally confirm the presence of such a mechanism of production of perturbation.

Really we can do even more. The detection of the energy density contrast would give us information about the amplitude of primordial fluctuations, but how about their statistical properties? Indeed many models of production of anisotropies in the SGWB have been already proposed and studied, and we would like to have some arguments to establish their viability. A possible and powerful way to perform this selection is to study the statistical properties of the generated anisotropies. In particular we are interested in the correlation functions of the energy density contrast $\delta\omega_{GW}$, which, thanks to the relation (3.50), can be more easily studied in terms of correlators of the fluctuations Γ. With this purpose in mind, in the following of this chapter we will pose the basis for a statistical analysis of the SGWB; in particular we want to arrange the most convenient mathematical set up to apply to our analysis, which is to expand the fluctuations in multipoles.

3.4 Multipole expansion

As we will understand more deeply in chapter (4), the statistical features characterizing the pattern of fluctuations $\Gamma(\hat{n})$ on the sky are encoded inside the angular n-point correlation function

$$\langle \Gamma(\hat{n}_1)\Gamma(\hat{n}_2)\dots\Gamma(\hat{n}_n)\rangle, \tag{3.51}$$

where the brackets stand for the ensemble average. Actually it is worth mentioning that experimentally what we can observe is instead a spatial average over regions of the sky, since we have only one possible realization of our Universe. The best we can do is to is to study widely separated regions of the Universe that are causally disconnected, and consider them as different measurement from the same ensemble. By averaging over a sufficiently large volume, we expect the spatial average to approach the ensemble one. What we are doing then, in order to reconcile observations with theoretical predictions, is substituting a spatial average with an ensemble one. This assumption goes under the name of *Ergodic hypothesis* [67], and it is usually taken as an axiom in cosmology. Therefore in the following we will exploit this assumption and spatial and ensemble averages will be regarded as the same.

Unfortunately, in general angular correlation functions of the fluctuations at different angular scales seem not to be uncorrelated, and this would provide a huge complication in the statistical analysis of anisotropy. Hence it turns out to be useful to expand the fluctuation (3.35) in multipoles in the basis of spherical harmonics $Y_{\ell m}(\hat{n})$, which represents a complete set of eigenfunctions of the Laplace equation defined on the surface of a sphere. Since we are going to use intensively the properties of spherical harmonics, we found appropriate to give a brief review of their features in a dedicate Appendix (B). With the normalization condition $\int d^2\hat{n} Y_{\ell m} Y^*_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}$, the expansion reads

$$\Gamma(\hat{n}) = \sum_{\ell}\sum_{m=-\ell}^{\ell} \Gamma_{\ell m} Y_{\ell m}(\hat{n}), \qquad \Gamma_{\ell m} = \int d^2 n \Gamma(\hat{n}) Y^*_{\ell m}(\hat{n}). \tag{3.52}$$

As we will see, the angular spectra built from the coefficients $\Gamma_{\ell m}$ at different angular scales, that is at different ℓ, turns out to be uncorrelated if the perturbations are Gaussian fields (this comment will be more clear with the knowledge of chapter (4)). Even non-Gaussian fields are reasonably expected to be uncorrelated as long as the departure from Gaussianity is weak. The aim of this procedure is to relate the abstract anisotropy Γ to the observable coefficients $\Gamma_{\ell m}$. Expanding in momentum space, they are written

$$\Gamma_{\ell m} = \int d^2 n Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\Gamma_I(\eta,\vec{k},q,\hat{n}) + \Gamma_S(\eta,\vec{k},q,\hat{n}) + \Gamma_T(\eta,\vec{k},q,\hat{n})\right]$$
$$\equiv \Gamma_{\ell m,I} + \Gamma_{\ell m,S} + \Gamma_{\ell m,T}. \tag{3.53}$$

This way we have divided the problem in three different pieces, which we are now going to face separately one at a time.

3.4.1 Initial condition term

The above discussion about the $\hat{n}$ dependence of the initial condition term simplifies this contribution to

$$\Gamma_{\ell m,I} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x_0}} \Gamma(\eta_{in},\vec{k},q) \int d^2 n Y^*_{\ell m}(\hat{n}) e^{-ik\mu l(\eta_0,\eta_{in})}, \tag{3.54}$$

where η_0 denotes the present conformal time and x_0 our location within the chosen coordinate system, that may be possibly set as the origin of the coordinate system. We can rewrite the second integral remembering that $\mu = \hat{k}\cdot\hat{n}$ and by expanding the complex phase as

$$e^{-ikl(\eta_0,\eta_{in})\hat{k}\cdot\hat{n}} = \sum_l (-i)^\ell (2\ell+1) j_\ell \left[kl(\eta_0,\eta_{in})\right] P_\ell(\hat{k}\cdot\hat{n})$$
$$= 4\pi \sum_\ell \sum_{m=-\ell}^{\ell} (-i)^\ell j_\ell[kl(\eta_0,\eta_{in})] Y_{\ell m}(\hat{n}) Y^*_{\ell m}(\hat{k}), \tag{3.55}$$

with j_ℓ and P_ℓ respectively the spherical Bessel function and the Legendre polynomial of degree ℓ. This way, exploiting the normalization condition of the spherical harmonics, one readily arrives to

$$\Gamma_{\ell m,I} = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x_0}} \Gamma(\eta_{in},\vec{k},q) Y^*_{\ell m}(\hat{k}) j_\ell[kl(\eta_0,\eta_{in})]. \tag{3.56}$$

3.4.2 Scalar sourced term

Differently from the initial condition term, the scalar and tensor sourced terms are integrated contributions taking into account the whole period of propagation of the GW. As stated in the introduction of this chapter, small scale perturbations ($k \gg q$) get indeed diluted by the expansion of the Universe and averaged out during the graviton propagation. This allows to focus only on large scales perturbations, namely $k \ll q$, that re-enter the horizon at late time, at least after the epoch of matter-radiation equality. These large scale modes remain in linear regime even today, and this allows us to write them as the combination of a stochastic variable $\zeta(\vec{k})$, representing the primordial value of the scalar perturbation set by the primordial mechanisms (as inflation) that establish the initial conditions, times a transfer function $T(\eta,k)$[6] that describes the time evolution of the perturbation itself:

$$\Phi(\eta,\vec{k}) = T_\Phi(\eta,k)\zeta(\vec{k}), \qquad \Psi(\eta,\vec{k}) = T_\Psi(\eta,k)\zeta(\vec{k}). \tag{3.57}$$

[6] Notice that the transfer function depends just on the scale of the wavenumber, and not on its direction. This is indeed the case because an anisotropic evolution would spoil either the assumption of a FLRW background or the first order expansion.

As just said, the main mechanism responsible for the production of the SGWB is thought to be the inflation. In this mechanism there exists a preferential quantity in use to characterize the primordial scalar perturbation, that is the gauge invariant primordial curvature perturbation ζ defined by

$$\zeta \equiv \Phi + \mathcal{H}\frac{\delta\rho}{\rho'}, \tag{3.58}$$

with ρ the total energy density; this quantity indeed remains constant on super-horizon scales in absence of isocurvature perturbations, whose contribution is strongly constrained from CMB observations [98]. This feature turns out to be very useful in our particular case, and this is why it is common to take the primordial curvature perturbation as the stochastic variable ζ describing the primordial value of the scalar perturbations (see [67] for further details). Having introduced this new decomposition, the scalar sourced term becomes:

$$\Gamma_S(\eta, \vec{k}, q, \hat{n}) = \int_{\eta_{in}}^{\eta_0} d\eta' e^{-ik\mu l(\eta,\eta')}\mathcal{T}_S(\eta', k)\zeta(\vec{k}), \tag{3.59}$$

where we have defined

$$\mathcal{T}_S(\eta', k, q) = v^{-2}T_\Phi(\eta', k)\delta(\eta' - \eta_{in}) + \frac{\partial\left[T_\Psi(\eta', k) + v^{-2}T_\Phi(\eta', k)\right]}{\partial\eta'}. \tag{3.60}$$

Following the steps outlined in the previous section, we decompose the scalar sourced anisotropy in spherical harmonics and evaluate the coefficients as:

$$\begin{aligned}
\Gamma_{\ell m,S} &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int d^2n Y^*_{\ell m}(\hat{n}) \int_{\eta_{in}}^{\eta_0} d\eta' e^{-ik\mu l(\eta_0,\eta')}\mathcal{T}_S(\eta', k, q)\zeta(\vec{k}) \\
&= 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int_{\eta_{in}}^{\eta_0} d\eta' Y^*_{\ell m}(\hat{k}) j_\ell\left[kl(\eta_0, \eta')\right]\mathcal{T}_S(\eta', k, q)\zeta(\vec{k}) \\
&= 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\zeta(\vec{k})Y^*_{\ell m}(\hat{k}) \Bigg\{ v^{-2}T_\Phi(\eta_{in}, k) j_\ell\left[kl(\eta_0, \eta')\right] + \\
&\quad + \int_{\eta_{in}}^{\eta_0} d\eta' e^{-ik\mu l(\eta_0,\eta')} \frac{\partial\left[T_\Psi(\eta', k) + v^{-2}T_\Phi(\eta', k)\right]}{\partial\eta'} \Bigg\},
\end{aligned} \tag{3.61}$$

where in the second step we have used again (3.55) combined with the spherical harmonics normalization condition. Notice that the first end second term resemble respectively the Sachs-Wolfe and Integrated Sachs-Wolfe contribution of CMB with the additional velocity correction.

3.4.3 Tensor sourced term

The spherical harmonics formalism can now be applied to the last term concerning the contribution due to the propagation of GWs across large-scale tensor perturbations. Combining (3.53) and (3.39), the tensor spherical harmonics coefficients can be evaluated as:

$$\begin{aligned}
\Gamma_{\ell m,T} &= \int d^2n Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\Gamma_T(\eta_0, \vec{k}, q, \hat{n}) \\
&= -\int d^2n Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta_0,\eta')} \frac{n^i n^j}{2} \frac{\partial\chi_{ij}(\eta', \vec{k})}{\partial\eta'}.
\end{aligned} \tag{3.62}$$

Gravitational waves have in general two polarization states, that we represent in the circular right and left-handed basis:

$$\chi_{ij} = \chi_R e_{ij,R} + \chi_L e_{ij,L} = \chi_R \frac{e_{ij,+} + i e_{ij,\times}}{\sqrt{2}} + \chi_L \frac{e_{ij,+} - i e_{ij,\times}}{\sqrt{2}}, \tag{3.63}$$

where we have introduced the polarization tensors e_{ij}. Since tensor perturbations are defined to be symmetric, traceless and tranverse, these properties in momentum space are conserved inside the polarization tensor by

$$\begin{aligned} e_{ij} &= e_{ji}, \\ e^i_i &= 0, \\ k^j e_{ij} &= (0,0,0). \end{aligned} \tag{3.64}$$

As outlined in the scalar sourced case, we can encode the dynamical evolution of the tensor perturbations inside a transfer function $\chi(\eta, k)$ (equal for both the polarization states), which indeed evolves the primordial value of a stochastic variable $\xi(\vec{k})$ set by inflation (or any GW production mechanisms at work at early times):

$$\chi_\lambda = \chi(\eta, k)\xi_\lambda(\vec{k}) \qquad \lambda = R, L\,, \tag{3.65}$$

that allows to compactly write the tensor perturbation as

$$\chi_{ij} = \sum_{\lambda=R,L} e_{ij,\lambda}(\hat{k})\chi(\eta, k)\xi_\lambda(\vec{k}). \tag{3.66}$$

This time the integration over the director $\hat{n}$ seems more involved because of the presence of the factor $n^i n^j$ inside the integrand; thus we cannot proceed as before by exploiting the decomposition of the complex exponential. In order to understand the meaning of the subsequent computational steps, we briefly outline the path to follow: in the first place we will try compute the coefficient $\Gamma_T(\eta_0, \vec{k}, q, \hat{n})$ for a fixed direction $\hat{k}$ of the wave number of the scalar perturbations; as we will explicitly see, a suitable choice of the $\hat{k}$ direction allows a decomposition of the GW propagation director $\hat{n}$ which greatly simplify the problem. However the measure $d^2\hat{n}$ of the integral in (3.62) refers to any possible director $\hat{n}$ without any fixed orientation $\vec{k}$; in order to solve this mismatch we will apply a rotation (provided by a matrix $S(\Omega_k)$) on the integral over $d^2 n$ in such a way to orient the the director $\hat{n}$ to reproduce the situation with the fixed direction $\hat{k}$. In this new basis we will be then able to adopt the expression of $\Gamma_T(\eta_0, \vec{k}, q, \hat{n})$ found at the beginning with that fixed direction. Having posed the basis of the computations, let's start by fixing the direction of the wave vector. The easiest choice is to orient $\hat{k}$ along the z-axis, such that the symmetric, traceless and transverse conditions on the polarization tensors straightforwardly lead to the explicit expressions

$$e_{ij,+}(\hat{k}_z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{ij,\times}(\hat{k}_z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.67}$$

from which one can easily verify that the only non vanishing entries of the tensor modes are

$$\chi_{11} = -\chi_{22} = \chi(\eta, k)\frac{\xi_L(\vec{k}) + \xi_R(\vec{k})}{2} \tag{3.68}$$

$$\chi_{12} = \chi_{21} = \chi(\eta, k)\frac{\xi_L(\vec{k}) - \xi_R(\vec{k})}{2i}. \tag{3.69}$$

Now we introduce a polar coordinate system over which the director vector of GW propagation decomposes as

$$\hat{n} = \left(\sqrt{1-\mu_{k,n}^2}\cos\phi_{k,n}, \sqrt{1-\mu_{k,n}^2}\sin\phi_{k,n}, \mu_{k,n}\right), \tag{3.70}$$

where the subscript k, n reminds that we are working in a basis where $\vec{k}$ is oriented along the z-axis. Notice that the expression (3.70) is correctly normalized to 1 and that when $\hat{n}$ is parallel to $\hat{k}$ (i.e $\mu_{k,n} = 1$) it gives back $\hat{n} = (0, 0, 1)$, as expected. With these new ingredients we are now able to evaluate the integrand of (3.62):

$$\begin{aligned}
\frac{n^i n^j}{2}\chi'_{ij} &= \frac{1-\mu_{k,n}^2}{2}\left(\cos^2\phi_{k,n}\chi'_{11} + \sin^2\phi_{k,n}\chi'_{22} + 2\cos\phi_{k,n}\sin\phi_{k,n}\chi'_{12}\right) \\
&= \frac{1-\mu_{k,n}^2}{2}\chi'(\eta,k)\left[\frac{\xi_L(\vec{k})+\xi_R(\vec{k})}{4}\left(e^{2i\phi_{k,n}} + e^{-2i\phi_{k,n}}\right)\right. \\
&\qquad\qquad \left. -\frac{\xi_L(\vec{k})-\xi_R(\vec{k})}{4}\left(e^{2i\phi_{k,n}} - e^{-2i\phi_{k,n}}\right)\right] \\
&= \frac{1-\mu_{k,n}^2}{4}\chi'(\eta,k)\left[e^{2i\phi_{k,n}}\xi_R(\vec{k}) + e^{-2i\phi_{k,n}}\xi_L(\vec{k})\right].
\end{aligned} \tag{3.71}$$

This way the problem of finding the spherical harmonic coefficients $\Gamma_{\ell m}$ is solved for a fixed direction $\hat{k}$ by

$$\begin{aligned}
\Gamma_T(\eta_0, \vec{k}, q, \Omega_{k,n}) = &-\frac{1-\mu_{k,n}^2}{4}\left[e^{2i\phi_{k,n}}\xi_R(\vec{k}) + e^{-2i\phi_{k,n}}\xi_L(\vec{k})\right] \times \\
&\times \int_{\eta_{in}}^{\eta_0} d\eta\chi'(\eta,k)e^{-ik\mu l(\eta_0,\eta)},
\end{aligned} \tag{3.72}$$

$$\Gamma_{\ell m,T} = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\int d^2\Omega_n Y^*_{\ell m}(\Omega_n)\Gamma_T(\eta_0, \vec{k}, q, \Omega_n). \tag{3.73}$$

At this point, as anticipated, we have the mismatch between the innermost integral variable of (3.73) and the fixed orientation of $\hat{k}$ in (3.72). The latter is indeed valid only when the wave vector is oriented along the z-axis, while the former is an integration over the solid angle around a director $\hat{n}$ which is not fixed by any choice, and takes its general expression inside its integrand (that is it does not satisfy the decomposition (3.70)). This conflict can then be solved by rotating the integrand of the $\int d^2\Omega_n$ into a basis in which the direction $\hat{n}$ respect the mentioned decomposition (3.70). With this purpose in mind we introduce the rotation matrix

$$S(\Omega_k) \equiv \begin{pmatrix} \cos\theta_k\cos\phi_k & -\sin\phi_k & \sin\theta_k\cos\phi_k \\ \cos\theta_k\sin\phi_k & \cos\phi_k & \sin\theta_k\sin\phi_k \\ -\sin\theta_k & 0 & \cos\theta_k \end{pmatrix}, \tag{3.74}$$

such that:

$$\hat{k} = S(\Omega_k)\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \begin{pmatrix}\sin\theta_n\cos\phi_n\\ \sin\theta_n\sin\phi_n \\ \cos\phi_n\end{pmatrix} = S(\Omega_k)\begin{pmatrix}\sin\theta_{k,n}\cos\phi_{k,n}\\ \sin\theta_{k,n}\sin\phi_{k,n}\\ \cos\phi_{k,n}\end{pmatrix}. \tag{3.75}$$

Since the rotation matrix has unitary determinant (by definition a rotation matrix does not change the volume of the measure in the phase space), the measure $d^2\Omega_n = d^2\Omega_{k,n}$

remains unchanged under rotation. On the contrary the spherical harmonics transform according to:

$$Y^*_{\ell m}(\Omega_n) = \sum_{m'=-\ell}^{\ell} D^{(\ell)}_{mm'}(S(\Omega_k)) Y^*_{\ell m'}(\Omega_{k,n}), \tag{3.76}$$

where it has been introduced the Wigner rotation matrix as in Appendix (B)

$$D^{(\ell)}_{ms}(S(\Omega_k)) \equiv \sqrt{\frac{4\pi}{2\ell+1}}(-1)^s \ {}_{-s}Y^*_{\ell m}(\Omega_k) \tag{3.77}$$

in terms of the spin-weighted spherical harmonics

$$\begin{aligned} {}_{-s}Y^*_{\ell m}\ (\Omega_k) \equiv (-1)^m \sqrt{\frac{(\ell+m)!\,(\ell-m)!\,(2\ell+1)}{4\pi(\ell+s)!\,(\ell-s)!}} \sin^{2\ell}\left(\frac{\theta_k}{2}\right) \times \\ \times \sum_{r=0}^{\ell-s} \binom{\ell-s}{r} \binom{\ell+s}{r+s-m} (-1)^{\ell-r-s} e^{im\phi_k} \cot^{2r+s-m}\left(\frac{\theta_k}{2}\right). \end{aligned} \tag{3.78}$$

This way the solution (3.73) transforms into

$$\Gamma_{\ell m,T} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \sum_{m'=-\ell}^{\ell} D^{(\ell)}_{mm'}(S(\Omega_k)) \int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n}) \Gamma_T(\eta_0, \vec{k}, q, \Omega_{k,n}). \tag{3.79}$$

Now we see that the dangerous mismatch has been solved, and we are then able to compute the innermost integral by adopting the expression (3.72) for $\hat{k}$ fixed along the z-axis:

$$\begin{aligned} \mathcal{J} \equiv \int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n}) \Gamma_T(\eta_0, \vec{k}, q, \Omega_{k,n}) = -\int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n}) \left(\frac{1-\mu^2_{k,n}}{4}\right) \\ \times \left[e^{2i\phi_{k,n}} \xi_R(\vec{k}) + e^{-2i\phi_{k,n}} \xi_L(\vec{k})\right] \int_{\eta_{in}}^{\eta_0} d\eta \chi'(\eta, k) e^{-ik\mu_k l(\eta_0,\eta)}. \end{aligned} \tag{3.80}$$

Using the relation (B.4) between spherical harmonics and the Legendre polynomials, one can verify (see [46]) that

$$(1-\mu^2)e^{i\lambda\phi} = 4\sqrt{\frac{2\pi}{15}} Y_{2\lambda} \qquad (\lambda = \pm 2). \tag{3.81}$$

This way

$$\begin{aligned} \mathcal{J} = -\sqrt{\frac{2\pi}{15}} \int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n}) \int_{\eta_{in}}^{\eta_0} d\eta \chi'(\eta, k) e^{-ik\mu_k l(\eta_0,\eta)} \\ \times \left[Y_{2,2}\xi_R(\vec{k}) + Y_{2,-2}\xi_L(\vec{k})\right]. \end{aligned} \tag{3.82}$$

The mathematical results of integration of products of spherical harmonics [46] provide

$$\int d\Omega_{k,n} Y^*_{\ell,m'} Y_{2\pm2} e^{-i\mu_{k,n}x} = (-i)^{\ell-2} \delta_{m',\pm2} \sqrt{\frac{15}{8}(2\ell+1)\frac{(\ell+2)!}{(\ell-2)!}} \frac{j_\ell(x)}{x^2}, \tag{3.83}$$

such that the innermost integral becomes

$$\begin{aligned} \mathcal{J} = \int_{\eta_{in}}^{\eta_0} d\eta \chi'(\eta, k)(-i)^\ell \sqrt{4\pi(2\ell+1)\frac{(\ell+2)!}{(\ell-2)!}} \frac{1}{4} \frac{j_\ell(kl(\eta_0,\eta))}{k^2 l^2(\eta_0,\eta)} \\ \times \left\{\delta_{m',2}\, \xi_R(\vec{k}) + \delta_{m',-2}\, \xi_L(\vec{k})\right\}. \end{aligned} \tag{3.84}$$

Turning back to the angular coefficient, we can directly insert this result into (3.79) and obtain

$$\Gamma_{\ell m,T} = (-i)^{\ell}\sqrt{4\pi(2\ell+1)\frac{(\ell+2)!}{(\ell-2)!}}\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\int_{\eta_{in}}^{\eta_0} d\eta\chi'(\eta,k)\frac{j_\ell(kl(\eta_0,\eta))}{k^2l^2(\eta_0,\eta)} \\ \times\left\{D^{(\ell)}_{m2}(S(\Omega_k))\frac{\xi_R(\vec{k})}{4}+D^{(\ell)}_{m\bar{2}}(S(\Omega_k))\frac{\xi_L(\vec{k})}{4}\right\}. \tag{3.85}$$

In the end, inserting the relation (3.77) of the Wigner rotation matrix elements in terms of the spin-weighted spherical harmonics, we finally end up with:

$$\Gamma_{\ell m,T} = 4\pi(-i)^{\ell}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\int_{\eta_{in}}^{\eta_0} d\eta\chi'(\eta,k)\frac{j_\ell(kl(\eta_0,\eta))}{k^2l^2(\eta_0,\eta)} \\ \times\left\{{}_{-2}Y^*_{\ell m}(\Omega_k)\frac{\xi_R(\vec{k})}{4}+{}_{2}Y^*_{\ell m}(\Omega_k)\frac{\xi_L(\vec{k})}{4}\right\}, \tag{3.86}$$

which can be written more conveniently in terms of a transfer function as done for the scalar sourced term:

$$\Gamma_{\ell m,T} = 4\pi(-i)^{\ell}\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\mathcal{T}^T_\ell(k,\eta_0,\eta_{in},q) \\ \times\left\{{}_{-2}Y^*_{\ell m}(\Omega_k)\xi_R(\vec{k})+{}_{2}Y^*_{\ell m}(\Omega_k)\xi_L(\vec{k})\right\}, \tag{3.87}$$

with

$$\mathcal{T}^T_\ell(\eta_0,\eta_{in},k,q) \equiv \sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\frac{1}{4}\int_{\eta_{in}}^{\eta_0} d\eta\chi'(\eta,k)\frac{j_\ell(kl(\eta_0,\eta))}{k^2l^2(\eta_0,\eta)}. \tag{3.88}$$

3.4.4 Summary of the three contributions

In conclusion of this chapter we briefly summarize the results obtained in the previous sections. First of all we have decomposed the perturbation modes as

$$\Phi(\eta,\vec{k}) = T_\Phi(\eta,k)\zeta(\vec{k}), \qquad \Psi(\eta,\vec{k}) = T_\Psi(\eta,k)\zeta(\vec{k}) \\ \chi_{ij}(\eta,\vec{k}) = \chi(\eta,k)\left[e_{ij,R}(\hat{k})\xi_R(\vec{k})+e_{ij,L}(\hat{k})\xi_L(\vec{k})\right]. \tag{3.89}$$

Then we broke up the anisotropy $\Gamma(\eta,\vec{k},q,\hat{n})$ in three different contributions and expanded them in the spherical harmonics basis as

$$\Gamma(\hat{n}) = \sum_\ell\sum_{m=-\ell}^{\ell}\Gamma_{\ell m}Y_{\ell m}(\hat{n}), \qquad \Gamma_{\ell m} = \int d^2n\Gamma(\hat{n})Y^*_{\ell m}(\hat{n}), \tag{3.90}$$

$$\Gamma_{\ell m} \equiv \Gamma_{\ell m,I}+\Gamma_{\ell m,S}+\Gamma_{\ell m,T}. \tag{3.91}$$

For the three terms we found

$$\begin{aligned}\Gamma_{\ell m,I} &= 4\pi(-i)^{\ell}\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}Y^*_{\ell m}(\hat{k})j_\ell[kl(\eta_0,\eta_{in})]\Gamma(\eta_{in},\vec{k},q), \\ \Gamma_{\ell m,S} &= 4\pi(-i)^{\ell}\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}Y^*_{\ell m}(\hat{k})\mathcal{T}^S_\ell(\eta_0,\eta_{in},k,q)\zeta(\vec{k}), \\ \Gamma_{\ell m,T} &= 4\pi(-i)^{\ell}\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}_0}\sum_{\lambda=\pm2}{}_{-\lambda}Y^*_{\ell m}(\Omega_k)\xi_\lambda(\vec{k})\mathcal{T}^T_\ell(\eta_0,\eta_{in},k,q).\end{aligned} \tag{3.92}$$

where in the last equation (3.92) we have adopted the more compact convention such that $\lambda = 2$ for right polarization and $\lambda = -2$ for left polarization. The linear transfer functions appearing in the scalar and tensor sourced terms are respectively defined by

$$\begin{aligned}
\mathcal{T}_\ell^S(\eta_0, \eta_{in}, k, q) &\equiv v^{-2} T_\Phi(\eta_{in}, k) j_\ell \left[kl(\eta_0, \eta_{in}) \right] \\
&\quad + \int_{\eta_{in}}^{\eta_0} d\eta \frac{\partial \left[T_\Psi(\eta, k) + v^{-2} T_\Phi(\eta, k) \right]}{\partial \eta} j_\ell \left[kl(\eta_0, \eta) \right], \\
\mathcal{T}_\ell^T(\eta_0, \eta_{in}, k, q) &\equiv \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \frac{1}{4} \int_{\eta_{in}}^{\eta_0} d\eta\, \chi'(\eta, k) \frac{j_\ell(kl(\eta_0, \eta))}{k^2 l^2(\eta_0, \eta)},
\end{aligned} \tag{3.93}$$

with $v \equiv v(\eta, q) = q/aE$ the graviton group velocity.

3.5 Isocurvature perturbations

In single slow-roll inflationary models, the inflaton fluctuations seed perturbation in the total energy density after the end of inflation. However, this mechanism involve the total energy density, and is not capable of distinguish between different components of the Universe. Hence only adiabatic perturbations are generated by a single slow-rolling inflaton field. These are characterized by the condition that the relative ratios in the number densities n between different species remain unperturbed (see [99, 100]). This is the so called *adiabatic condition*

$$\delta \left(\frac{n_X}{n_Y} \right), \tag{3.94}$$

where X and Y denote two arbitrary particle species. Since any perturbation in the number density correspond to a perturbation in the energy density, isocurvature perturbations are related to spatial curvature perturbations through the Einstein's equations. Defining

$$\delta_X \equiv \frac{\delta \rho_X}{\rho_X}, \tag{3.95}$$

the adiabatic condition reads (see [63] for a derivation)

$$\frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu = \frac{1}{3}\delta_b = \frac{1}{3}\delta_{cdm}. \tag{3.96}$$

On the contrary, *isocurvature perturbations* change the relative number densities among the particle species without bringing any curvature perturbation[7]. Then they can be parametrized by

$$S_{X,Y} = \frac{\delta n_X}{n_X} - \frac{\delta n_Y}{n_Y} = \frac{\delta_X}{1+w_X} - \frac{\delta_Y}{1+w_Y}. \tag{3.97}$$

It is convenient at this point to choose a species of reference; usually one considers the photons as the reference species, such that

$$S_b \equiv \delta_b - \frac{3}{4}\delta_\gamma, \qquad S_c \equiv \delta_c - \frac{3}{4}\delta_\gamma, \qquad S_\nu \equiv \frac{3}{4}\delta_\nu - \frac{3}{4}\delta_\gamma. \tag{3.98}$$

Adiabatic perturbation are then defined by $S_b = S_{cdm} = S_\nu = 0$. In the most general picture, one can decompose a general perturbation in its adiabatic and the three isocurvature modes. It is worth mentioning that this decomposition is not time-invariant. Indeed

[7] For this reason they are also called *entropy perturbations*.

primordial isocurvature modes can give rise, during the evolution, to an adiabatic contribution if the energy densities of the various species evolve differently in time, because in such case the balance that ensured an unpertubed total energy density is lost.

The golden channel to study isocurvature perturbations is through the CMB anisotropies, since they leave a distinctive signature on the angular spectra which can be easily discriminate from the adiabatic contribution. Indeed, as shown in ref. [99, 100], whereas adiabatic modes introduce a cosine oscillation in the baryon-photon fluid, the isocurvature ones enters with a sine oscillatory phase. In the former case the oscillation gives rise to a first peak in the angular power spectrum centered at $\ell \simeq 220$, while, if one considers only isocurvature modes, the same peak would have been expected at $\ell \simeq 330$. Observations tell us that the position of the first peak is quite well in agreement with the prediction in absence of isocurvature modes. This argument suggests that, if they exist, isocurvature perturbations must be subdominant with respect to the adiabatic ones. This motivates why we neglected these kind of perturbations so far, and in the following we will pursue our discussions assuming implicitly that isocurvature modes are not relevant for our observables.

Chapter 4

Statistical analysis of the SGWB from Gaussian perturbations

In the previous chapter we have derived the Boltzmann equation describing the evolution of the graviton population and computed the anisotropy originating from three different contributions, an initial term, a scalar sourced term and a tensor sourced one. With these results we are now in a position to attempt a statistical analysis of the SGWB. On a fundamental ground, the statistical features of the graviton population background relies on the statistical properties of the stochastic variables ζ and ξ_λ. The easiest assumption is to take the perturbations as Gaussian random fields, that is fields whose probability density function is a normal distribution

$$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-x_0)^2}{2\sigma^2}}, \tag{4.1}$$

with σ^2 the variance of the distribution. This assumption is motivated, in the behalf of the central limit theorem [73, 77], by assuming each perturbation mode being independent from the others. The fact that many physical processes in nature, since most of them emerge as an average effect of independent small scale processes, manifest this behavior can be seen as a phenomenological confirmation of the reasonability of this assumption. This is indeed the case of inflation [75], which succeeds in generating primordial perturbations on small scales beneath the horizon, where the microphysics if effective. Moreover the Gaussian distribution describes the ground state of quantum harmonic oscillators [83], that is the fundamental basis with which we quantize the inflaton field [82]. Therefore a Gaussian distribution reasonably explains also the quantum origins of perturbations.

A crucial feature of the Gaussian distribution is that its shape remains invariant under any general linear transformation [77]. Linear transformations affect the distribution by changing its mean value and variance, but do not spoil the shape of the distribution. This is the reason why linear perturbations preserve the Gaussianity of the distribution through the Universe evolution; matter of facts, in cosmology linear perturbations are always identified with Gaussian random fields, while higher order terms are regarded as non-Gaussian corrections. These arguments encourage us to pursue, in a first study, a statistical analysis of the SGWB anisotropies considering the metric perturbation to be Gaussian distributed. The chapter is then organized as follows.

The first section (4.1) is dedicated to a brief review of the definition and the properties of a general Gaussian random field, so that a more deep understanding about the hypothesis underlying the Gaussian assumption can be gained.

As we will show, in this case the only relevant statistical quantity is the 2-point correlation function, or, in momentum space, the power spectrum. This is the main topic we discuss in section (4.2), starting from its definition and features deriving from the assumption of statistical isotropy. With this knowledge in mind we proceed in evaluating the 2-point function starting from the results (3.92). This analysis largely follows the path of [59], and naturally extends the results accounting for a non vanishing graviton mass. Assuming that the three terms corresponding to the initial condition and the sourced terms are not cross correlated, the final correlator will result in three different contributions whose explicit calculation is performed by steps.

Firstly, in sections (4.3) and (4.4), we evaluate the transfer functions $T_\Phi(\eta, k)$, $T_\Psi(\eta, k)$ and $\chi(\eta, k)$ defined in (3.89), which then will allow us to obtain an explicit expression for the linear functions $\mathcal{T}_\ell^{S,T}(\eta_0, \eta_{in}, k, q)$ in (3.93).

Once these functions are known we will have all the ingredients to evaluate the three contribution to the 2-point correlator, which we will perform one at a time in the last section (4.5).

4.1 Gaussian random fields

Before exploring the angular power spectrum, we find useful to give a brief review of the concept of gaussian random variable, so that the reader could possibly understand better how sources of non gaussianity, which we will study later, could arise within a cosmological framework. In this section we will mainly refer to [63] and [67]; for some further details about cosmological applications of these topics we encourage to give a look at [74] as well.

A function $G(\vec{x})$[1] is said to be a random field if for any x it assumes random values g, and if it exists a distribution function

$$F_{1,\dots,n}(g_1, \dots, g_n) = F\left[G(\vec{x}_1) < g_1, \dots, G(\vec{x}_n) < g_n\right] \tag{4.2}$$

well defined for any n. Each value g_i that the random field may assumes at the point $\vec{x}_i$ is said to be a representation of the field $G(\vec{x})$, while the the set of all the possible representation forms the ensemble. The mathematical properties of the distribution function and of the statistical quantities we will define later on are largely described in [76]. At any point $\vec{x}_i$ there must be defined a probability density function $p_i(g_i)$ encoding the probability of the random field G to acquire the value g_i in $\vec{x}_i$. By definition the probability density function is related to the distribution function by

$$p_i(g_i) = \frac{dF_i(g_i)}{dg_i}. \tag{4.3}$$

As in [76], we then define the expectation value of the random field as the ensemble average

$$\langle G(\vec{x}_i) \rangle \equiv \int_\Omega g_i p_i(g_i)\, dg_i, \tag{4.4}$$

where Ω indicates the ensemble. Since this quantity happens to vanish in most cases when dealing with cosmological perturbations, a more statistically relevant quantity in cosmology is the joint probability

$$p_{ij}(g_i, g_j)\, dg_i dg_j \tag{4.5}$$

[1] Notice that we are restricting the coordinate space to the only spatial coordinates, since the stochastic properties of any cosmological quantity are defined at fixed time.

of finding the random field $G(\vec{x}_i)$ with a value g_i given that in the point x_j the random field assumes the value g_j. Then we define the two point correlation function as the expectation value of the combination $G(\vec{x}_i)G(\vec{x}_j)$, that is:

$$\xi(\vec{x}_i, \vec{x}_j) \equiv \langle G(\vec{x}_i)G(\vec{x}_j)\rangle \equiv \int_\Omega g_i g_j p_{ij}(g_i, g_j)\, dg_i dg_j. \tag{4.6}$$

In the same way we can generalize this definition for the N point correlation function as

$$\begin{aligned}\xi^{(N)}(\vec{x}_1, \dots, \vec{x}_N) &\equiv \langle G(\vec{x}_1), \dots, G(\vec{x}_N)\rangle \\ &\equiv \int_\Omega g_1 \dots g_N p_{1,\dots,N}(g_1, \dots, g_N) dg_1 \dots dg_N.\end{aligned} \tag{4.7}$$

Obviously, when the different realizations are independent, the joint probability reduces simply to the product of each individual probability density functional (in this case the distribution is said to be Poissonian); as a consequence if the realizations are independent the N point correlation function reduces to the product of the expectation value of the N random fields. Another very useful statistical quantity is the ensemble variance σ, which is essentially a measure of the deviation from a Poissonian distribution, defined by

$$\sigma^2(\vec{x}_1, \vec{x}_2) \equiv \langle G(\vec{x}_1)G(\vec{x}_2)\rangle - \langle G(\vec{x}_1)\rangle\langle G(\vec{x}_2)\rangle. \tag{4.8}$$

In many practical purposes, statistical analysis is often more manageable in Fourier space. Therefore we introduce the Fourier transform of the random field as:

$$G(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{G}(\vec{k})e^{i\vec{k}\cdot\vec{x}}, \qquad \tilde{G}(\vec{k}) = \int d^3x\, G(\vec{x})e^{-i\vec{k}\cdot\vec{x}}, \tag{4.9}$$

and the reality condition

$$\tilde{G}^*(k) = \tilde{G}(-k) \tag{4.10}$$

holds.

Gaussian distribution

What has been shown so far is totally general and relevant for any distribution function F. From now on instead give up the general discussion in order to focus our attention on the particular case of the Normal Gaussian distribution. The physical assumption that characterizes a Gaussian random field is that the phases of different Fourier modes are random and uncorrelated. Therefore, on behalf of the central limit theorem, the infinite sum of the various Fourier modes will tend to be normally distributed, and ultimately, the integral expansion in Fourier modes will results in a Gaussian random field.

An alternative more formal, but nevertheless quite elegant, way to define a Gaussian random field is to proceed by an axiomatic approach:

i. A Gaussian random field is a random field whose Fourier modes have no correlation except for the reality condition (4.10). The fact that all the modes are uncorrelated forces the two point correlation function to be of the form

$$\langle\tilde{G}(\vec{k})\tilde{G}^*(\vec{k}')\rangle = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')P_G(\vec{k}), \tag{4.11}$$

with $P_G(\vec{k})$ a function to be determined called power spectrum.

ii. The *odd-number* correlators of any Gaussian random field vanish[2]

$$\langle \tilde{G}(\vec{k})\rangle = \dots = \langle \tilde{G}(\vec{k}_1)\dots\tilde{G}(\vec{k}_{2n+1})\rangle = 0. \tag{4.12}$$

iii. The 4-point correlators can be written in terms of the 2-point correlators by

$$\begin{aligned}\langle \tilde{G}(\vec{k}_1)\tilde{G}(\vec{k}_2)\tilde{G}(\vec{k}_3)\tilde{G}(\vec{k}_4)\rangle &= \langle \tilde{G}(\vec{k}_1)\tilde{G}(\vec{k}_2)\rangle\langle \tilde{G}(\vec{k}_3)\tilde{G}(\vec{k}_4)\rangle \\ &\quad +\langle \tilde{G}(\vec{k}_1)\tilde{G}(\vec{k}_3)\rangle\langle \tilde{G}(\vec{k}_2)\tilde{G}(\vec{k}_4)\rangle \\ &\quad +\langle \tilde{G}(\vec{k}_1)\tilde{G}(\vec{k}_4)\rangle\langle \tilde{G}(\vec{k}_2)\tilde{G}(\vec{k}_3)\rangle, \end{aligned} \tag{4.13}$$

and in the same way we can decompose all the *even-number* correlators.

In other words, the Fourier coefficients of a Gaussian random field are minimally correlated provided the requirement of the reality condition. As a consequence all the stochastic properties of a Gaussian field are completely determined by its power spectrum, which then represent the fundamental statistical quantity to study. Before investigating the structure of the power spectrum, notice that these axioms reconstruct the expected situation for a Gaussian random variable. Indeed the n-point correlator reproduce exactly the expression for the n-moments of a gaussian distribution, as reported in [78]. This uniquely determines the density probability function $p_i(g_i)$ that in the end will assume the form of a normal distribution

$$p(g_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{g_i^2}{2\sigma^2}}, \tag{4.14}$$

where, in virtue of the first two axioms, the ensemble variance simplifies to

$$\sigma^2 \equiv \sigma^2(\vec{x},\vec{x}) = \langle |G(\vec{x})|^2\rangle = \int \frac{d^3k}{(2\pi)^3} P_G(\vec{k}), \tag{4.15}$$

and turns out to be position independent thanks to the delta function in (4.11).

Let us now turn back to the problem of the evaluation of the function $P_G(\vec{k})$. In a cosmological framework the random field is usually taken to be statistically homogeneous and isotropic, that is the probability densities related to each realization are invariant under translation and rotations. Under the assumption of statistical homogeneity, the two point correlation function in coordinate space turns out to have a simpler dependence on coordinates, that is

$$\xi(\vec{x}_1,\vec{x}_2) = \xi(\vec{x}_1 - \vec{x}_2). \tag{4.16}$$

In momentum space the 2-point correlator is

$$\begin{aligned}\langle \tilde{G}(\vec{k})\tilde{G}(\vec{k}')\rangle &= \int d^3x \int d^3x' \langle G(\vec{x})G(\vec{x}')\rangle e^{-i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \\ &= \int d^3x \int d^3x' \xi(\vec{x}' - \vec{x}) e^{-i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \\ &= \int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} \int d^3z\, \xi(\vec{z}) e^{i\vec{k}'\cdot\vec{z}}. \end{aligned} \tag{4.17}$$

[2] Actually $\langle \tilde{G}(\vec{k})\rangle$ is not a correlator, and in principle it could be different from zero. However, given a random field with a non vanishing mean value, it is always possible to define a new field with zero mean simply by shifting the old one as $\tilde{H}(\vec{k}) \equiv \tilde{G}(\vec{k}) - \langle \tilde{G}(\vec{k})\rangle$. This is what is usually done in cosmology, where we prefer to work with perturbation with zero mean. Any non zero-centered perturbation can be reabsorbed in the background Universe. For this reason in this work we will always assume $\langle \tilde{G}(\vec{k})\rangle = 0$.

Then, exploiting the integral representation of the delta function, we finally arrive to (4.11) with the definition of the power spectrum

$$P_G(\vec{k}) \equiv \int d^3x\, \xi(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}, \tag{4.18}$$

which is nothing but the Fourier transform of the 2-point correlation function. However in many cases it is more convenient to work with the so called dimensionless power spectrum defined as

$$\mathcal{P}_G(\vec{k}) \equiv \frac{k^3}{2\pi^2} P_G(\vec{k}). \tag{4.19}$$

As a last comment we want to interpret these results within the framework of quantum field theories [84]. What we are going to argue is that a Gaussian random field can only originate from a free theory. A powerful tool to compute the connected[3] correlation functions of any order n is the so called *in-in formalism* as shown in [80, 81, 88]. As usual we divide the Hamiltonian of the system in a free term and an interacting one as

$$H = H_0 + H_{int}. \tag{4.20}$$

In the interaction picture [82] we can evolve the vacuum state with a Schrodinger like equation such that the n-point correlation function of a generic quantum field $\varphi(\eta, \vec{k})$ becomes[4]

$$\langle \varphi^n(\eta) \rangle = \langle 0(\eta_{in}) | U_{int}^{-1}(\eta, \eta_{in}) \varphi^n(\eta) U_{int}(\eta, \eta_{in}) | 0(\eta_{in}) \rangle, \tag{4.21}$$

where η_{in} denotes the initial time when the interaction is turned on; notice that the vacuum state $|0(\eta)\rangle$ is the one of the interacting theory, but, for continuity, it must coincide with the free vacuum at initial time, when the interaction was turned off[5]. The evolution operator is defined by

$$U_{int}(\eta, \eta_{in}) = T\, e^{-i\int_{\eta_{in}}^{\eta} H_{int}(\eta')d\eta'}, \tag{4.22}$$

with T the time ordering operator. To first order (4.21) reads

$$\langle \varphi^n(\eta) \rangle = -i \int_{\eta_{in}}^{\eta} d\eta' \, \langle 0 | T \left[\varphi^n(\eta), H_{int}(\eta') \right] | 0 \rangle. \tag{4.23}$$

In a free theory the only term we can treat as an interaction is the mass term $\mathcal{H}_{int} \propto m^2\varphi^2/2$, and

$$\begin{aligned} H_{int}(\eta) &= \frac{m^2}{2} \int d^3x \varphi^2 = \frac{m^2}{2} \int d^3x \int d^3p\, d^3p' e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} \varphi(\vec{p}, \eta) \varphi(\vec{p}', \eta) \\ &= \frac{m^2}{2} \int d^3p\, d^3p' (2\pi)^3 \delta^3(\vec{p} + \vec{p}') \varphi(\vec{p}, \eta) \varphi(\vec{p}', \eta). \end{aligned} \tag{4.24}$$

The rigorous computation of the connected correlation functions descends from (4.21), but, since we are not interested in the exact result, we will settle for just an estimate of the first term of the commutator, that is

$$\langle \varphi^n(\eta) \rangle \sim -im^2 \int_{\eta_{in}}^{\eta} d\eta' \int d^3p\, d^3p' \delta^3(\vec{p} + \vec{p}') \langle 0 | T \varphi^n(\eta) \varphi(\vec{p}, \eta') \varphi(\vec{p}', \eta') | 0 \rangle. \tag{4.25}$$

[3] In accordance with the common QFT costum, we say that a correlator is "connected" if it cannot be factorized as products of correlation function of lower order.

[4] For the ease of notation we are omit the dependence on the momenta, while we are making explicit the dependence on the time fixed by our observation.

[5] For simplicity, if there is no chance of confusion, we will denote the free vacuum just as $|0\rangle$

The time ordering product can now be substituted by a sum of normal ordering products, denoted by N, thanks to the Wick theorem [82]

$$\begin{aligned} T[\varphi(\eta_1)\dots\varphi(\eta_n)] =\ & N[\varphi(\eta_1)\dots\varphi(\eta_n)] \\ & + \sum_{\text{1 contr.}} N[\underline{\varphi(\eta_1)\varphi(\eta_2)}\dots\varphi(\eta_n)] \\ & + \dots \\ & + \sum_{\text{all contr.}} N[\underline{\varphi(\eta_1)\varphi(\eta_2)}\dots\underline{\varphi(\eta_i)\dots\varphi(\eta_n)}], \end{aligned} \tag{4.26}$$

where the contraction

$$\underline{\varphi(\vec{k},\eta)\varphi(\vec{k}',\eta')} = \langle 0|\varphi(\vec{k},\eta)\varphi(\vec{k}',\eta')|0\rangle \tag{4.27}$$

must involve different times. However, any normal ordering product where we have at least one uncontracted field gives zero contribution once we apply it to the vacuum $|0\rangle$. This is indeed the case because, in the context of second quantization, the scalar field φ is written as a superposition of ladder operators. The normal ordering acts by moving the annihilation operators on the right and the creation operators on the left, making then those contributions to vanish. Therefore the only non trivial term of (4.26) is the last one, where all the field involved in the time ordering operator are contracted. Turning back to (4.25), now we see that, for a free theory, the Wick theorem selects the 2-point function as the only non vanishing one. In other term (4.25) is non zero only for $n = 2$, because in the other case there will always be at least one uncontracted field left. This is indeed in accordance with the fact that the statistical properties of a Gaussian random field are completely characterized by the two point function. Moreover one can show [87] that any non connected correlation function can be decomposed as a sum of products of connected functions. Since the only non vanishing connected function is the 2-point function, then we immediately have the result that any higher-order correlator can be written in terms of the 2-point one. Nevertheless, in the case n-*odd*, there is no way to decompose the correlator in terms of 2-point functions only, and this implies than any *odd*-point correlator automatically vanishes. All these results perfectly recover the axioms (4.11), (4.12), (4.13) that we posed to define a Gaussian random field. In the end we have proved that Gaussianity originates from the random field being a free theory.

4.2 GW Angular Power spectrum

After this long parenthesis, let's now turn back to the problem of evaluation of statistical measurable quantities. In order to take contact with observations, our final purpose is to obtain an expression for the angular correlators. Indeed, as seen in the previous section, the intrinsic nature of the perturbations implies $\langle \Gamma_{\ell m}\rangle = 0$, and therefore the only meaningful quantity to study is the two point function, which, on the behalf of the rotational invariance, is restricted to the form

$$\langle \Gamma_{\ell m}\Gamma_{\ell' m'}\rangle \equiv \delta_{\ell\ell'}\delta_{mm'}\tilde{C}_\ell, \tag{4.28}$$

where the quantity $\tilde{C}_\ell = \langle |a_{\ell m}|^2\rangle$ is called spectrum of the SGWB anisotropy, and it is the crucial quantity that encodes the information about how much fluctuations is there on a given angular scale ℓ. Actually we should be aware that this expression holds as long as we are neglecting explicit $\hat{n}$-dependence inside the initial condition contribution (3.37). This is indeed the only term in which such an explicit dependence could possibly arise, while in the other contributions the dependence in contained just inside the combination

$\mu = \hat{k} \cdot \hat{n}$ confined inside the exponential factors, as we have seen in (3.38) and (3.39). Such an angular dependence would reflect in an overall statistical anisotropy, and ultimately on a more general dependence of the angular correlators on the multipole indices ℓ and m. However, for the reasons mentioned above, this work will focus only on the statistically isotropic case. As briefly said before, this assumption drops any dependence of the spectrum $\tilde{C}_\ell$ on the index m. One can easily convince himself that this is indeed the case by thinking at the intuitive meaning of the indices of the monopole expansion: ℓ tells us the angular scales we are considering, while m contains information about the orientation of the spherical harmonics, that is about the direction of our observation. The assumption of statistical isotropy implies that our observations of angular correlators should be orientation-independent, that means, at the end of the day, m-independent. A more rigorous proof was given by Komatsu in [79]. Statistical isotropy demands the correlators to be invariant under rotations. Denoting with $D = D(\alpha, \beta, \gamma)$ a rotation matrix for the Euler angles α, β and γ, this symmetry reads

$$\langle D\Gamma(\hat{n}_1) D\Gamma(\hat{n}_2) \dots D\Gamma(\hat{n}_n) \rangle = \langle \Gamma(\hat{n}_1)\Gamma(\hat{n}_2) \dots \Gamma(\hat{n}_n) \rangle. \tag{4.29}$$

Remembering the multiple expansion

$$\Gamma(\hat{n}) = \sum_\ell \sum_{m=-\ell}^{\ell} \Gamma_{\ell m} Y_{\ell m}(\hat{n}), \tag{4.30}$$

we need to understand how the rotation matrix D applies to the spherical harmonics. As shown in [63, 79], this transformation is given by

$$DY_{\ell m}(\hat{n}) = \sum_{m'=-\ell}^{\ell} D^{(\ell)}_{m'm} Y_{\ell m'}(\hat{n}), \tag{4.31}$$

where we have introduced the Wigner matrix elements $D^{(\ell)}_{m'm} = \langle \ell, m' | D | \ell, m \rangle$. In the context of quantum mechanics these matrix elements describe the rotation of an initial state whose angular momentum is represented by the quantum number ℓ and m into a final state where these quantum number are transformed into ℓ and m'. Inserting the multiple expansion inside (4.29), the isotropy condition becomes

$$\begin{aligned}\sum_{\ell_1 m_1 \dots \ell_n m_n} \langle \Gamma_{\ell_1 m_1} Y_{\ell_1 m_1}(\hat{n}_1) \dots \Gamma_{\ell_n m_n} Y_{\ell_n m_n}(\hat{n}_n) \rangle &= \sum_{\ell_1 m_1 \dots \ell_n m_n} \langle \Gamma_{\ell_1 m_1} \dots \Gamma_{\ell_n m_n} \rangle \\ &\times \sum_{m'_1 \dots m'_n} D^{(\ell)}_{m'_1 m_1} Y_{\ell_1 m'_1}(\hat{n}_1) \dots D^{(\ell)}_{m'_n m_n} Y_{\ell_n m'_n}(\hat{n}_n).\end{aligned} \tag{4.32}$$

In order to extrapolate the n-point correlators the trick is to multiply both the right and left side of the equation for the spherical harmonic $Y^*_{L_i M_i}$ and to integrate over $d^2\hat{n}_i$ to reproduce the condition for the normalization of the spherical harmonics (B.7). Iterating this procedure for $i = 1, \dots, n$, we end up with

$$\langle \Gamma_{\ell_1 m_1} \Gamma_{\ell_2 m_2} \dots \Gamma_{\ell_n m_n} \rangle = \sum_{m'_1 \dots m'_n} \langle \Gamma_{\ell_1 m'_1} \dots \Gamma_{\ell_n m'_n} \rangle D^{(\ell)}_{m'_1 m_1} D^{(\ell)}_{m'_2 m_2} \dots D^{(\ell)}_{m'_n m_n}. \tag{4.33}$$

Focusing on the 2-point function

$$\langle \Gamma_{\ell_1 m_1} \Gamma_{\ell_2 m_2} \rangle = \sum_{m'_1 m'_2} \langle \Gamma_{\ell_1 m'_1} \Gamma_{\ell_2 m'_2} \rangle D^{(\ell)}_{m'_1 m_1} D^{(\ell)}_{m'_2 m_2}. \tag{4.34}$$

This relation is the manifestation of rotational invariance. From this equation, we now look for a rotationally invariant representation of the angular spectrum. Let's then take (4.28) as an ansatz, i.e. $\langle \Gamma_{\ell_1 m_1'} \Gamma_{\ell_2 m_2'} \rangle = \tilde{C}_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$. Then the relation (4.34) becomes

$$\langle \Gamma_{\ell_1 m_1} \Gamma_{\ell_2 m_2} \rangle = \tilde{C}_{\ell_1} \delta_{\ell_1 \ell_2} \sum_{m_1'} D^{(\ell)}_{m_1' m_1} D^{(\ell)}_{m_2' m_2} = \tilde{C}_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}. \tag{4.35}$$

In the end we have shown that $\tilde{C}_\ell$ is rotationally invariant and that rotational symmetry demands the 2-point function to be of the form of (4.28).

Looking at the solution for the $\Gamma_{\ell m}$ coefficients in (3.92), we see that there are just four statistical operators which are sensitive to the ensemble average operator defined in (4.28), that are $\Gamma(\eta_{in}, \vec{k}, q)$, $\zeta(\vec{k})$, $\xi_L(\vec{k})$, and $\xi_R(\vec{k})$. All the other operators are deterministic quantities either encoding the time evolution of large scale modes (transfer functions) or projecting the GW anisotropies in the harmonic space (spherical harmonic functions). Assuming that the four stochastic variables are statistically uncorrelated, the only non vanishing 2-point functions, defined in relation to their dimensionless power spectra, are:

$$\begin{aligned} \langle \Gamma(\eta_{in}, \vec{k}, q) \Gamma^*(\eta_{in}, \vec{k}', q) \rangle &= \frac{2\pi^2}{k^3} \mathcal{P}_I(q,k) (2\pi)^3 \delta\left(\vec{k} - \vec{k}'\right) \\ \langle \zeta(\vec{k}) \zeta^*(\vec{k}') \rangle &= \frac{2\pi^2}{k^3} \mathcal{P}_\zeta(k) (2\pi)^3 \delta\left(\vec{k} - \vec{k}'\right) \\ \langle \xi_\lambda(\vec{k}) \xi^*_{\lambda'}(\vec{k}') \rangle &= \frac{2\pi^2}{k^3} \mathcal{P}_\lambda(k) (2\pi)^3 \delta_{\lambda\lambda'} \delta\left(\vec{k} - \vec{k}'\right). \end{aligned} \tag{4.36}$$

These expressions can now be used combined with the results (3.92) to evaluate the angular correlator (4.28), which, thanks to the above assumption, can be split into three different contributions:

$$\tilde{C}_\ell = \tilde{C}_{\ell,I} + \tilde{C}_{\ell,S} + \tilde{C}_{\ell,T}. \tag{4.37}$$

At this point one can appreciate the great benefit of having decomposed the anisotropies in the above three terms. Moreover, since the three contributions to the coefficients $\Gamma_{\ell m}$ share the same functional form, we can perform the computation for just one term, and then easily extend the result for the other two. Let's then consider the contribution to the angular correlator arising from the initial condition term:

$$\begin{aligned} \langle \Gamma_{\ell m, I} \Gamma^*_{\ell' m', I} \rangle &= (4\pi)^2 (-i)^{\ell - \ell'} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int \frac{d^3k'}{(2\pi)^3} e^{-i\vec{k}'\cdot\vec{x}_0} Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}') \\ &\quad \times j_\ell[kl(\eta_0, \eta_{in})] j_{\ell'}[kl(\eta_0, \eta_{in})] \langle \Gamma(\eta_{in}, \vec{k}, q) \Gamma(\eta_{in}, \vec{k}', q) \rangle \\ &= (4\pi)^2 (-i)^{\ell - \ell'} \int \frac{d^3k}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_I(q,k) Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}) \\ &\quad \times j_\ell[kl(\eta_0, \eta_{in})] j_{\ell'}[kl(\eta_0, \eta_{in})]. \end{aligned} \tag{4.38}$$

Decomposing the integral measure as $d^3k = k^2 dk d^2\hat{k}$ and exploiting the orthonormality of the spherical harmonics, the integral reduces to

$$\begin{aligned} \langle \Gamma_{\ell m, I} \Gamma^*_{\ell' m', I} \rangle &= \delta_{\ell\ell'} \delta_{mm'} (4\pi)^2 \int \frac{k^2 dk}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_I(q,k) \left[j_\ell[kl(\eta_0, \eta_{in})] \right]^2 \\ &= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int \frac{dk}{k} P_I(q,k) \left[j_\ell[kl(\eta_0, \eta_{in})] \right]^2. \end{aligned} \tag{4.39}$$

Notice that, as expected, any dependence on the index m has dropped out after the implementation of the orthonormality relation, since this dependence is contained only inside the spherical harmonics. A more general dependence of the initial fluctuation Γ_I on the director $\hat{n}$ would have introduced a more complex dependence of the integrand of (4.38) on the versor $\hat{k}$, preventing this great simplification. The initial condition spectrum is increased by the presence of a non vanishing mass, since this information only enters inside the argument of the spherical Bessel function. In particular the argument is maximized (and then the spherical Bessel function minimized) in the massless case, where the graviton speed equals the light velocity and the distance $l(\eta_0, \eta_{in})$ reduces to the time difference $\eta_0 - \eta_{in}$.

As stressed above, thanks to the fact that all the three $\Gamma_{\ell m}$ coefficients listed in (3.92) share identical structures, we can immediately extend the last computation to the scalar and tensor sourced contributions. All we have to take care about is to replace the spherical Bessel function and the dimensionless power spectrum $\mathcal{P}_I(q,k)$ appearing in (4.39) with the appropriate linear transfer function and power spectra for the scalar and tensor sourced terms. This procedure straightforwardly leads to:

$$\langle \Gamma_{\ell m,S} \Gamma^*_{\ell' m',S} \rangle = \delta_{\ell\ell'} \delta_{mm'} 4\pi \int \frac{dk}{k} \mathcal{P}_\zeta(k) |\mathcal{T}^S_\ell(\eta_0, \eta_{in}, k, q)|^2 \tag{4.40}$$

for the 2-point correlator scalar sourced term, and

$$\langle \Gamma_{\ell m,T} \Gamma^*_{\ell' m',T} \rangle = \delta_{\ell\ell'} \delta_{mm'} 4\pi \int \frac{dk}{k} \left[\mathcal{P}_2(k) + \mathcal{P}_{-2}(k)\right] |\mathcal{T}^T_\ell(\eta_0, \eta_{in}, k, q)|^2 \tag{4.41}$$

for the tensor sourced one. Notice that both these expressions do not show a graviton mass dependence, as it is confined inside the linear transfer functions (3.93). This concludes the problem of the evaluation of the 2-point correlators, whose angular spectra defined in (4.28) are listed below to summarize the previous computations.

$$\begin{aligned}
\tilde{C}_{\ell,I} &= 4\pi \int \frac{dk}{k} |j_\ell[kl(\eta_0, \eta_{in})]|^2 \mathcal{P}_I(q,k), \\
\tilde{C}_{\ell,S} &= 4\pi \int \frac{dk}{k} |\mathcal{T}^S_\ell(\eta_0, \eta_{in}, k, q)|^2 \mathcal{P}_\zeta(k), \\
\tilde{C}_{\ell,T} &= 4\pi \int \frac{dk}{k} |\mathcal{T}^T_\ell(\eta_0, \eta_{in}, k, q)|^2 \sum_{\lambda=\pm 2} \mathcal{P}_\lambda(k).
\end{aligned} \tag{4.42}$$

It is worth to mention one more time that all the three contribution in (4.42) contain a q-dependence. This is due to the graviton mass, which affects the linear transfer functions by adding a graviton velocity correction. In the massless case this velocity equals the light velocity, and then the correction factor goes to one. As a result, in the massless case, one would find that only the initial condition term maintains the dependence on the GW frequency inside the dimensionless power spectrum $\mathcal{P}_I(q,k)$.

4.3 Scalar transfer functions

In order to evaluate the scalar transfer functions, let's consider a FLRW background spacetime perturbed by only scalar perturbations:

$$ds^2 = a^2(\eta) \left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij} dx^i dx^j\right]. \tag{4.43}$$

Let's suppose further that the energy density budget of the Universe is dominated by relativistic and non-relativistic particles and by a cosmological constant. If we additionally assume that the cosmic fluid of the Universe is perfect, then we can immediately write down the energy-momentum tensor elements remembering that that the matter content is pressureless, while for the cosmological constant $\rho_\Lambda = -p_\Lambda$:

$$\begin{aligned} T^0_0 &= -\rho_m(1+\delta_m) - \rho_r(1+\delta_r) - \rho_\Lambda, \\ T^i_j &= \left(\frac{1}{3}\rho_r(1+\delta_r) - \rho_\Lambda\right)\delta^i_j, \end{aligned} \tag{4.44}$$

where $\delta_{r,m} \equiv \delta\rho_{r,m}/\rho_{r,m}$ represent the first order radiation or matter perturbation accounting for both baryonic and non baryonic particles, while v is the fluid particle velocity. The time evolution of cosmological perturbations is governed by Einstein equations, which at zero order lead to the background Friedman equations:

$$\begin{gathered} \frac{a'^2}{a^4} = \frac{1}{3M_p^2}\left[\rho_\Lambda + \rho_m + \rho_r\right], \\ \rho_m' + 3\mathcal{H}\rho_m = 0, \qquad \rho_r' + 4\mathcal{H}\rho_r = 0, \qquad \rho_\Lambda' = 0, \\ \frac{a''}{a} = \frac{a^2}{6M_p^2}(\rho - 3p) = \frac{a^2}{6M_p^2}\left(4\rho_\Lambda + \rho_m\right), \end{gathered} \tag{4.45}$$

with M_p the Plank mass and $\mathcal{H} \equiv a'/a$. In order to see the effects of cosmological perturbation we need to go at least at first order. The computation of the perturbed Einstein tensor elements has been performed in Appendix (A). For our purposes we just need to consider the spatial terms; therefore we can specialize (A.21) to our present case, where the only non vanishing perturbed metric elements are $h_{00} = -2\Phi$ and $h_{ij} = -2\Psi\delta_{ij}$. This way the linearized spatial Einstein equations are :

$$\begin{aligned} a^2\delta G^i_j &= \left[2\Psi'' + 4\mathcal{H}\Psi' + 2\mathcal{H}\Phi' + 4\frac{a''}{a}\Phi - 2\mathcal{H}^2\Phi + \nabla^2(\Phi - \Psi)\right]\delta^i_j - \partial^i\partial_j(\Phi - \Psi) \\ &= \frac{a^2}{M_P^2}\delta T^i{}_j = \frac{a^2}{3M_P^2}\rho_r\delta_r. \end{aligned} \tag{4.46}$$

This equation can be split in a diagonal and in an off-diagonal contribution. The latter is immediately solved if

$$\Phi = \Psi. \tag{4.47}$$

An analytic solution of the diagonal part is not achievable in full generality. For this reason we simplify the problem considering firstly the situation where the contribution to the total energy density from radiation is dominating over the other components.

Radiation domination The solution of the Einstein equation (4.61) during radiation domination is well known. The reader can find this solution in many text; for example, in ref. [70] it is shown that the solution is

$$\Phi(k,\eta) = 3\Phi_{in}(k)\frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}. \tag{4.48}$$

On super-horizon ($k\eta \ll 1$) scales the scalar perturbation is well described by its primordial value, while on sub-horizon scales ($k\eta \gg 1$) the perturbations undergo a damped oscillation with increasing frequency.

It is useful now to link the primordial value of the scalar perturbation $\Phi_{in}(k)$ to the gauge invariant quantity $\zeta_{in}(k)$. For this purpose one can take advantage one more time of the linearized Einstein equations to rewrite the gauge invariant curvature perturbation in a more convenient way. In particular, from the Einstein tensor (A.19), the time-time component of the Einstein equation at first order reads:

$$6\mathcal{H}^2\left(\Phi+\frac{1}{\mathcal{H}}\Phi'\right)+2k^2\Phi=-\frac{a^2}{M_p^2}\delta T^0{}_0=-\frac{a^2}{M_p^2}(\rho_m\delta_m+\rho_r\delta_r). \tag{4.49}$$

At early times the Universe was dominated by the ultra-relativistic component, and then, inserting the first Friedman equation (4.45) inside (4.49), one gets:

$$\delta_r\simeq\delta_{tot}\simeq-2\left(\Phi+\frac{1}{\mathcal{H}}\Phi'\right)-\frac{2M_P^2}{a^2\rho_r}k^2\Phi. \tag{4.50}$$

Finally this result can be used with the continuity equation of (4.45) to rewrite (3.58) in the desired form as follows

$$\begin{aligned}\zeta&=\Phi-\frac{\delta}{3(1+w)}\\&=\Phi+\frac{2}{3(1+w)}\left[\frac{a}{a'}\Phi'+\Phi\left(1+\frac{M_P^2}{a^2\rho_r}k^2\right)\right].\end{aligned} \tag{4.51}$$

When the radiation component dominates in the Universe, the equation of state imposes $w=1/3$. Moreover, since we are interested in physical scales that happen to be above the horizon at the end of inflation, we can neglect their first derivatives at initial time and contextually neglect all the terms proportional to k^2 as well. Applying all these considerations to (4.51), the curvature perturbation at initial time is given by

$$\zeta_{in}\simeq\frac{3}{2}\Phi_{in}, \tag{4.52}$$

and the scalar metric perturbation is

$$\Phi(k,\eta)=2\frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right)-\frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}\zeta_{in}(k). \tag{4.53}$$

Then, remembering the definitions (3.89):

$$T_\Phi^{\rm rad}=T_\Psi^{\rm rad}=\frac{2}{3}g_{\rm rad}(k,\eta), \tag{4.54}$$

with

$$g_{\rm rad}(k,\eta)=3\,\frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right)-\frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}. \tag{4.55}$$

The scalar transfer function can be rewritten in terms of observable quantities exploiting $\rho_r(\eta_i)a^4(\eta_i)=\rho_0$ and the definition of critical energy density today $\rho_{cr,0}=3M_P^2H_0^2$:

$$T_\Phi^{\rm rad}=T_\Psi^{\rm rad}=\frac{2}{3}g_{\rm rad}(k,\eta). \tag{4.56}$$

In order to easily understand the behavior of the growing rate with time, Figure (4.1) shows the trend of the function $g_{rad}(k,\eta)$ for a fixed scale $k\simeq10^4H_0$.

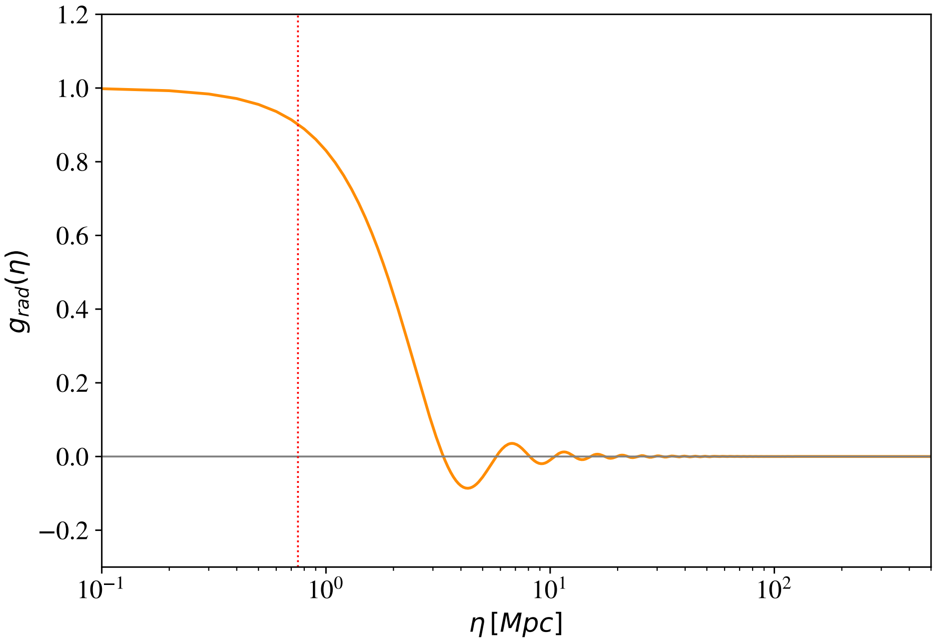

Figure 4.1: *Scalar growing rate during radiation domination. This function remain approximately constant until the mode under consideration re-enter the horizon. This event is higlighted by a vertical red dotted line corresponding to* $k\eta \simeq \sqrt{3}$*. Once the mode has re-entered the horizon, it soon starts decaying with an oscillating fashion.*

Matter domination Scalar modes which are outside the horizion at the end of the radiation domination epoch are frozen to their primordial value at the entering of the matter domination era. This paragraph considers only such scales, since those that crossed the horizon during the radiation era are washed out by quantum effects during the expansion of the Universe. After the epoch of equality, the radiation component provides a negligible contribution to the total energy density. In this case the condition (4.47) can be inserted inside (4.46) to give the desired time evolution equation for the scalar perturbations:

$$\Phi'' + 3\mathcal{H}\Phi' + \left(2\frac{a''}{a} - \mathcal{H}^2\right)\Phi = 0, \tag{4.57}$$

whose formal solution can be found in the form

$$\Phi = C_1 D_1(a) + C_2 D_2(a), \tag{4.58}$$

with C_1 and C_2 integration constants. The path to follow in order to specify the form of each solution, $D_1(a)$ and $D_2(a)$, is outlined in Appendix (D). The first solution has to be guessed, and in particular it seems reasonable to look for a solution built as a combination of the relevant quantities entering in (4.57), that are a and $\mathcal{H}$. The easiest function we can think to build with those quantities is $\Phi \propto a^n\mathcal{H}$, with n an integer number. Inserting this ansatz inside (4.57), we can solve the differential equation to obtain the viable values for n. By a direct computation:

$$\begin{aligned} 0 &= \mathcal{H}'' + 3(n+1)\mathcal{H}\mathcal{H}' + 2\frac{a''}{a}\mathcal{H} + (n^2+3n-1)\mathcal{H}^3 \\ &= \frac{a'''}{a} + (3n+1)\mathcal{H}\left(\frac{a''}{a} - \mathcal{H}^2\right) + \frac{a''}{a}\mathcal{H} + (n^2+3n-1)\mathcal{H}^3, \end{aligned} \tag{4.59}$$

where in the second step we directly inserted the expression for the first and second derivatives of $\mathcal{H}$. The third derivative of the scale factor can be obtained by deriving the last equation of (4.45). Since the energy density of the cosmological constant does not depend on time, this reads:

$$\frac{a'''}{a} = 3\mathcal{H}\frac{a''}{a} - \frac{a^2}{2M_p^2}\mathcal{H}\rho_m. \tag{4.60}$$

Using further the Friedman equation for the second derivative of the scale factor, (4.59) simplifies to

$$\begin{aligned} 0 &= (3n+5)\frac{a''}{a} - \frac{a^2}{M_p^2}\frac{\rho_m}{2} + (n^2-2)\mathcal{H}^2 \\ &= (3n+5)\frac{1}{6}(4\rho_\Lambda + \rho_m) - \frac{\rho_m}{2} + (n^2-2)\frac{1}{3}(\rho_\Lambda + \rho_m). \end{aligned} \tag{4.61}$$

This equation can now be split in a system of two equations, one for the matter energy density and the other for the cosmological constant energy density, that is:

$$\begin{cases} (3n+5) - 3 + 2(n^2-2) = 0, & \text{for } \rho_m \\ 4(3n+5) + 2(n^2-2) = 0, & \text{for } \rho_\Lambda \end{cases}. \tag{4.62}$$

The system is immediately solved by the value $n = -2$. Therefore, in the end, the desired solution we were looking for is

$$D_1(a) \propto \frac{\mathcal{H}}{a^2}. \tag{4.63}$$

Following the arguments of [71], in Appendix (D) we have shown how to find the second solution of an homogeneous second order differential equation once the first solution is already known. Specializing (D.7) to our case, the second solution is obtained in the form

$$D_2(a) \propto D_1(a) \int d\eta \, \frac{e^{-\int d\eta \, 3\mathcal{H}}}{D_1{}^2(a(\eta))}, \tag{4.64}$$

where in the denominator of the integrand we have made explicit the dependence of the scale factor on the conformal time η just to make clear that even the denominator is involved by the integral operator, and cannot be simplified with the factor outside the integral. Now we can recast the solution in a more simple form. The integral entering in the exponential factor can be directly computed through a change of integration variable:

$$-\int d\eta \, 3\mathcal{H} = -3\int \frac{d\tilde{a}}{\tilde{a}} = -3\ln a, \tag{4.65}$$

having set the integration constant to zero, since it would give rise to a solution proportional to $D_1(a)$. This way, applying the same change of variables to the integral in (4.64) and substituting the solution (4.63), we arrive to:

$$D_2(a) \propto \frac{\mathcal{H}}{a^2} \int d\eta \, \frac{a^{-3}}{a^{-4}\mathcal{H}^2} = \frac{\mathcal{H}}{a^2} \int d\tilde{a} \, \frac{1}{\tilde{\mathcal{H}}^3}, \tag{4.66}$$

where in the last step we defined $\tilde{\mathcal{H}} \equiv \tilde{a}'/\tilde{a}$. Summing up, this procedure provides two linearly independent solutions to (4.57) of the form:

$$D_1(a) \propto \frac{\mathcal{H}}{a^2}, \qquad D_2(a) \propto \frac{\mathcal{H}}{a^2} \int d\tilde{a} \frac{1}{\tilde{\mathcal{H}}^3}. \tag{4.67}$$

The solution $D_1(a)$ represents a decay mode, and it happens to decay very rapidly in time, hence it can be safely neglected. Then the scalar metric perturbation is

$$\Phi = C(k)\frac{\mathcal{H}}{a^2}\int d\tilde{a}\frac{1}{\tilde{\mathcal{H}}^3}. \tag{4.68}$$

Remember that our aim is to obtain an expression for the transfer scalar functions as defined in (3.57). In order to reach our purpose, we thus need to separate the primordial value of the scalar perturbation from its temporal evolution. In order to evaluate the primordial value of Φ it is useful to firstly make explicit the a-dependence contained inside the Hubble parameter $\mathcal{H}$. This dependence can be directly read from the first Friedman equation in (4.45) rewritten as

$$\frac{\mathcal{H}}{a} = \frac{1}{\sqrt{3}M_p}\sqrt{\rho_\Lambda + \frac{\rho_{m,0}}{a^3}} = \frac{\sqrt{\rho_\Lambda}}{\sqrt{3}M_p}\sqrt{1 + \frac{\Omega_{m,0}}{(1-\Omega_{m,0})a^3}} = \frac{\sqrt{\rho_\Lambda}}{\sqrt{3}M_p}\sqrt{\frac{a^3+r}{a^3}}, \tag{4.69}$$

with the definitions

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{cr,0}}, \qquad r \equiv \frac{\Omega_{m,0}}{1-\Omega_{m,0}}, \tag{4.70}$$

where as usual $\rho_{cr,0} = 3M_p^2\mathcal{H}^2/a^2$ denotes the critical energy density today. Therefore, enclosing any constant term inside the proportionality factor $C(k)$, the scalar perturbation becomes:

$$\begin{aligned}
\Phi &= C(k)\sqrt{\frac{a^3+r}{a^5}}\int_0^a dx\left(\frac{x^3+r}{x}\right)^{-3/2} \\
&= C(k)\sqrt{\frac{a^3+r}{a^5}}\,\frac{2a^2\sqrt{\frac{a}{a^3+r}}\,{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a^3}{r}\right)}{5r} \\
&= C(k)\frac{2}{5r}\,{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a^3}{r}\right).
\end{aligned} \tag{4.71}$$

As mentioned at the beginning of this computation, this solution is really reliable only when matter and the cosmological constant dominate in the Universe. This means that this solution is valid only for $\eta > \eta_{eq}$, while at earlier epochs one should consider the solution (4.48). At the epoch of matter-radiation equality the two solution must match in order to preserve the continuity of the solution. Modes that happen to be above the sound horizon at the epoch of equality are frozen to their primordial value, and then the matching condition easily imposes

$$\begin{aligned}
C(k) &= \frac{5r}{2}\left[{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a_{eq}^3}{r}\right)\right]^{-1}\Phi_{in}(k) \\
&= \frac{5r}{2}\left[{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a_{eq}^3}{r}\right)\right]^{-1}\frac{2}{3}\zeta_{in}(k).
\end{aligned} \tag{4.72}$$

This result completely defines the expression for the gravitational potential Φ, which can be written as:

$$\Phi(k,\eta) = \frac{{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a^3}{r}\right)}{{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a_{eq}^3}{r}\right)}\Phi_{in}(k) = \frac{2}{3}g_{\Lambda,\mathrm{m}}(\eta)\,\zeta_{in}(k), \tag{4.73}$$

where the normalized growing rate $g_{\Lambda,\text{matt}}(\eta)$ is defined as

$$g_{\Lambda,\text{m}}(\eta) \equiv \frac{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a^3}{r}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}. \tag{4.74}$$

In the end, the expression (4.73) automatically defines the transfer functions T_Φ and T_Ψ to be

$$T_\Phi^{\Lambda,\text{m}}(\eta, k) = T_\Psi^{\Lambda,\text{m}}(\eta, k) = \frac{2}{3} g_{\Lambda,m}(\eta). \tag{4.75}$$

The time evolution of the growing rate $g(\eta)$ is completely understood only once the explicit expression for the scale factor $a(\eta)$ is given. This relation is better derived solving the Friedman equation with respect to the coordinate time t, and then establishing the link between t and the conformal time η. Considering a ΛCDM model with negligible curvature and radiation energy contribution, the Friedman equation in conformal time is

$$a^2\mathcal{H}^2 = \frac{a^4}{3M_P^2}\left(\rho_\Lambda + \frac{\rho_{m,0}}{a^3}\right) = \frac{\rho_{m,0}}{3M_P^2}\, a\left(1 + a^3 r\right). \tag{4.76}$$

Performing an integral on both sides

$$\int_{a_{eq}}^{a} \frac{da}{\sqrt{a(1+a^3 r)}} = \sqrt{\frac{\rho_{m,0}}{3M_P^2}} \int_{\eta_{eq}}^{\eta} d\eta' \tag{4.77}$$

leads to

$$\eta(a) = \eta_{eq} + \frac{2\sqrt{a}}{H_0\sqrt{\Omega_{m,0}}}\, {}_2F_1\left[\frac{1}{6}, \frac{1}{2}; \frac{7}{6}; -ra^3\right] - \frac{2\sqrt{a_{eq}}}{H_0\sqrt{\Omega_{m,0}}}\, {}_2F_1\left[\frac{1}{6}, \frac{1}{2}; \frac{7}{6}; -ra_{eq}^3\right]. \tag{4.78}$$

This relation cannot be inverted analytically. For this reason we attempted a numerical approach, tabulating the values of $\eta(a)$ as a function of the scale factor a starting from the epoch of equality; then this relation is plotted and inverted with Python to obtain the desired expression of $a(\eta)$. The results of this procedure are shown in Figure (4.2). The form of the fitting function of the second plot was guessed by considering that during matter domination the scale factor scales as $a(\eta) \propto \eta^2$, while when Λ dominates $a(\eta) \propto -\eta^{-1}$; then it is reasonable to use the fit function

$$a(\eta) = c_1 + c_2\left(\frac{\eta}{\eta_0}\right)^{-1} + c_3\left(\frac{\eta}{\eta_0}\right)^2. \tag{4.79}$$

Using the Plank data (ref. [109] for the values of the Hubble constant today ($H_0 = 67.36\ \text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$) and for the matter density parameter ($\Omega_{m,0} = 0.315$), the optimal parameters given by the fit function are

$$\begin{aligned} c_1 &= -7.524 \times 10^{-3} \\ c_2 &= 2.546 \times 10^{-4} \\ c_3 &= 9.871 \times 10^{-1} \end{aligned}$$

At this point then, the growing rate $g(\eta)$ is perfectly understood and it is given, as a function of the conformal time, by

$$g_{\Lambda,\text{m}}(\eta) \equiv \frac{{}_2F_1\left[\frac{1}{3}, 1; \frac{11}{6}; -r^{-1}\left(c_1 + c_2\left(\frac{\eta}{\eta_0}\right)^{-1} + c_3\left(\frac{\eta}{\eta_0}\right)^2\right)^3\right]}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}. \tag{4.80}$$

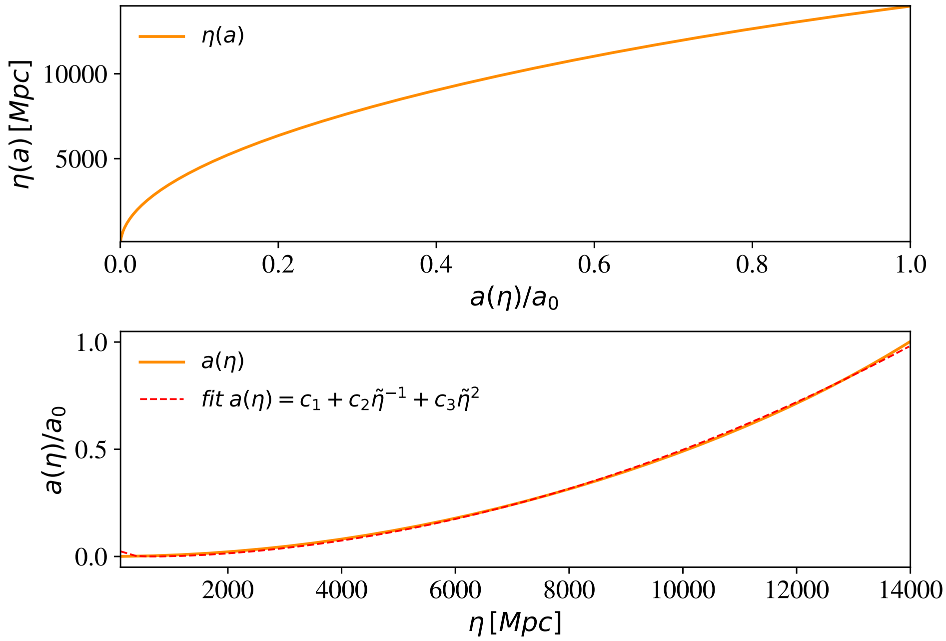

Figure 4.2: *Scale factor evolution in conformal time in matter and Λ dominated Universe. In the upper panel it is shown the evolution of the conformal time $\eta(a)$ (in Mpc) as a function of the scale factor a, (equation (4.78)), considering as reference values $r = 0.46$, $\Omega_{m,0} = 0.315$, $\eta_{eq} = 112.8\,Mpc$, and $a_{eq} = (1 + z_{eq})^{-1} = 2.94 \times 10^{-4}$. In the lower panel this relation is inverted graphically simply exchanging the two axis. The fit function reproduces very well the exact function along the whole time domain.*

The result is plotted in Figure (4.3).

To summarize the above results, the scalar metric perturbations can be written in the following way

$$\Phi(k, \eta) = \Psi(k, \eta) = \frac{2}{3}\left[1 + \frac{a_i^2}{9H_0^2}k^2\right]^{-1} g_{\Lambda,m}(\eta)\zeta_{in}(k), \tag{4.81}$$

with

$$g(\eta) = \begin{cases} 3\,\dfrac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3} & \eta < \eta_{eq} \\ \dfrac{{}_2F_1\left[\frac{1}{3}, 1; \frac{11}{6}; -r^{-1}\left(c_1 + c_2\left(\frac{\eta}{\eta_0}\right)^{-1} + c_3\left(\frac{\eta}{\eta_0}\right)^2\right)^3\right]}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)} & \eta > \eta_{eq} \end{cases}. \tag{4.82}$$

4.4 Tensor transfer function

The evolution equation for the tensor transfer function are obtained minimizing the action

$$S_T^{(2)} = \frac{M_P^2}{8}\int d^4x\, a^2\left[\chi_{ij}'\chi^{ij\,\prime} - (k^2 + a^2m^2)\chi_{ij}\chi^{ij}\right] \tag{4.83}$$

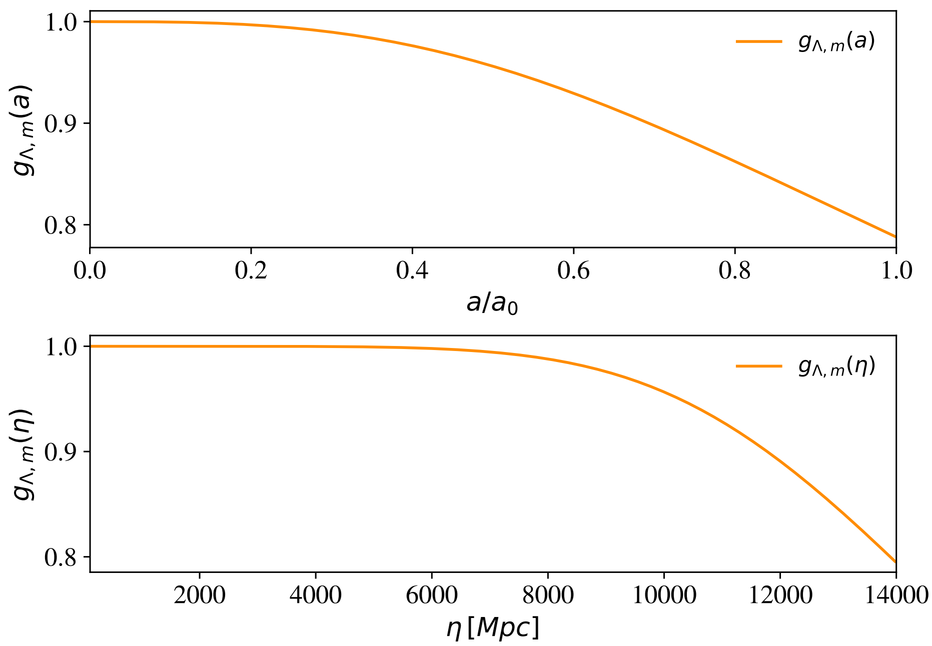

Figure 4.3: *Upper panel: solution for the growing rate $g(a)$ as a function of the scale factor from the exact formula (4.74). Lower panel: approximate solution of $g(\eta)$ using the previous fit function to replace the scale factor dependence with the conformal time one.*

found in Section (2.3.2). The Euler Lagrange equations for this action are

$$\chi''(\eta, k) + 2\mathcal{H}\chi'(\eta, k) + (k^2 + m^2 a^2)\chi(\eta, k) = 0. \tag{4.84}$$

In order to solve this equation in a semi-analytical way, we firstly have to make explicit the dependence of the scale factor a on the conformal time η. This relation can be directly obtained working out the Friedman equations in the two regimes

Radiation domination The last Friedman equation (4.45) in this regime reads

$$a'' = 0 \qquad \rightarrow \qquad a(\eta) = c_1\eta + c_2. \tag{4.85}$$

The integration constant c_2 vanishes under the requirement $a(0) = 0$, while c_1 is fixed by the first Friedman equation

$$a'^2(0) = \frac{1}{3M_P^2}\rho a^4 = H_0^2. \tag{4.86}$$

Therefore

$$a(\eta) = H_0\eta. \tag{4.87}$$

The Einstein equation (4.84) then becomes

$$\chi'' + \frac{2}{\eta}\chi' + (k^2 + m^2 H_0^2\eta^2)\chi = 0. \tag{4.88}$$

In the regime of small masses, that is $m^2a^2 \ll k^2$, equation (4.88) immediately reproduces the spherical Bessel equation (C.1) with $\ell = 0$. Therefore the solution in this regime is

$$\chi(k, \eta) = j_0(k\eta) \qquad\qquad m^2a^2 \ll k^2. \tag{4.89}$$

In the opposite regime, i.e. $m^2a^2 \gg k^2$, it is convenient to introduce a new variable $x \equiv mH_0\eta^2/2$. This definition sets the following rules for differentiation

$$\begin{aligned} \frac{d}{d\eta} &= \sqrt{2mH_0x}\frac{d}{dx} \\ \frac{d^2}{d\eta^2} &= mH_0\frac{d}{dx} + 2mH_0x\frac{d^2}{dx^2}, \end{aligned} \tag{4.90}$$

and then the Einstein equation becomes

$$\ddot{\chi} + \frac{3}{2x}\dot{\chi} + \left(\frac{k^2}{2mH_0x} + 1\right)\chi = 0. \tag{4.91}$$

In the large masses approximation the first term inside the parenthesis can be neglected, and the equation is solved by

$$\chi(k,x) = c_1x^{-1/4}J_{\frac{1}{4}}(x) + c_2x^{-1/4}Y_{\frac{1}{4}}(x). \tag{4.92}$$

Requiring $\chi(k,0) = 1$, since at initial time all the modes under consideration are outside the sound horizon, imposes $c_2 = 0$. The same condition fixes the value of the constant c_1. One can see this expanding the Bessel function with the Frobenius method (see ref. [105] for further details) in the limit for $x \to 0$

$$J_\alpha(x) \underset{x\to 0}{=} \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n+\alpha}\right]. \tag{4.93}$$

Then

$$\begin{aligned} \chi(k,x) &\underset{x\to 0}{=} x^{-1/4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(\Gamma\left(n+\frac{1}{4}+1\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{4}}\right] \\ &= x^{-1/4}\left[\frac{1}{\Gamma\left(\frac{5}{4}\right)}\left(\frac{x}{2}\right)^{\frac{1}{4}} + o(x^{9/4})\right] \\ &= \frac{1}{2^{1/4}\Gamma\left(\frac{5}{4}\right)} + o(x^2), \end{aligned} \tag{4.94}$$

which implies

$$c_1 = 2^{1/4}\Gamma\left(\frac{5}{4}\right). \tag{4.95}$$

Turning back to the original time η, and exploiting the relation between the Bessel function $J_n(x)$ and the spherical Bessel ones

$$J_n(x) = \sqrt{\frac{2x}{\pi}}j_{n-\frac{1}{2}}(x), \tag{4.96}$$

the solution is

$$\chi(k,\eta) = \frac{2^{3/4}}{\sqrt{\pi}}\Gamma\left(\frac{5}{4}\right)\left(\frac{mH_0\eta^2}{2}\right)^{1/4} j_{-\frac{1}{4}}\left(\frac{mH_0\eta^2}{2}\right) \qquad m^2a^2 \gg k^2. \tag{4.97}$$

A simple way to interpolate the solutions in the two regimes is to take their products, that is

$$\chi(k,\eta) = \frac{2^{3/4}}{\sqrt{\pi}}\Gamma\left(\frac{5}{4}\right)\left(\frac{mH_0\eta^2}{2}\right)^{1/4} j_{-\frac{1}{4}}\left(\frac{mH_0\eta^2}{2}\right) j_0(k\eta). \tag{4.98}$$

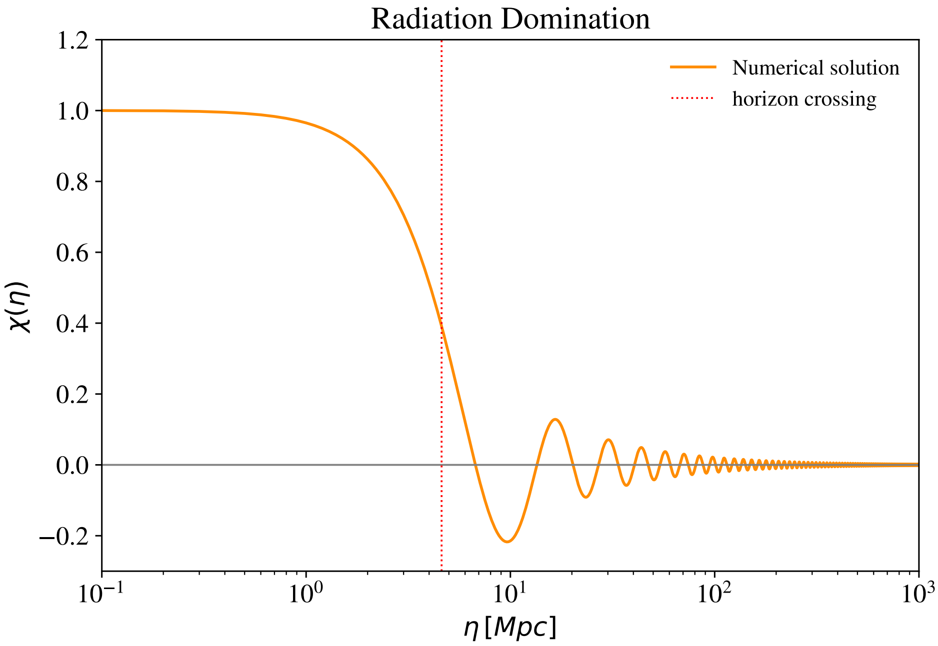

Figure 4.4: *Numerical solution of the Einstein equation* (4.88) *for tensor modes in radiation domination with* $m = H_0 = k/2000$. *The tensor transfer function remains nearly frozen to one until the mode re-enters the horizon. This event is roughly captured by equation* (4.100) *and illustrated in this picture with a vertical dashed line. After the mode has re-entered the horizon it soon starts decaying with a damped oscillatory trend.*

This solution shows an oscillating behavior with a time dependent frequency very similar to the solution found for the scalar growing rate during radiation domination. The numerical exact solution is shown in Figure (4.4) The horizon re-entering condition can be obtained by directly comparing the scale of interest with the size of the comoving Hubble radius

$$r_H(\eta) \equiv \frac{1}{\mathcal{H}} = \frac{a}{a'}. \tag{4.99}$$

Moreover, one should further take into account the mass term contribution, which, at the end of the day, brings to the condition

$$k^2 + m^2 H_0^2 \eta_*^2 = \frac{1}{\eta_*}. \tag{4.100}$$

Matter domination For large enough scales and sufficiently tiny graviton masses, tensor perturbation modes may survive the damping oscillation until the epoch of matter-radiation equivalence. In matter domination, since $\rho_m = \rho_{m,0}/a^3$, one can again integrate twice the last equation of (4.45) to get:

$$a'(\eta) = c_1 + \frac{\rho_{m,0}}{6M_p^2}\eta, \tag{4.101}$$

$$a(\eta) = c_2 + c_1\eta + \frac{\rho_{m,0}}{12M_p^2}\eta^2, \tag{4.102}$$

with c_1 and c_2 integration constants. Imposing the constraint $a(0) = 0$, we can immediately set $c_2 = 0$. Furthermore, inserting (4.102) inside the first Friedman equation (4.45), one can readily verify that this equation is compatible with (4.101) only if $c_1 = 0$ as well. Therefore, at the end of the day the relation between the scale factor and the conformal time during matter domination is:

$$a(\eta) = \frac{\rho_{m,0}}{12M_p^2}\eta^2 \simeq \frac{H_0^2}{4}\eta^2, \tag{4.103}$$

with $H_0 = \dot{a}(t_0)/a(t_0)$ the present Hubble parameter computed with respect to the coordinate time under the usual assumption of matter domination. Omitting the functional dependence of the tensor mode for the ease of notation, equation (4.84) specializes to

$$\chi'' + \frac{4}{\eta}\chi' + \left(k^2 + \frac{m^2H_0^4}{16}\eta^4\right)\chi = 0. \tag{4.104}$$

This equation can be solved in two extreme regimes according to the value of the graviton mass with respect to its wavenumber. In the limit for small masses, that is $m^2a^2 \ll k^2$, we can neglect the second term in the bracket. The field χ behaves then as massless spin-2 field, and, according to [69], (4.104) in the massless limit has two exact solutions:

$$\begin{aligned}
\chi(\eta, k) &= b_1\chi_1(\eta, k) + b_2\chi_2(\eta, k), \\
\\
\chi_1(\eta, k) &= \frac{1}{(k\eta)^2}\left[\frac{\sin(k\eta)}{k\eta} - \cos(k\eta)\right], \\
\chi_2(\eta, k) &= \frac{1}{(k\eta)^2}\left[\frac{\cos(k\eta)}{k\eta} + \sin(k\eta)\right].
\end{aligned} \tag{4.105}$$

Continuity with the solution in radiation domination demands $\chi(\eta_{eq}, k) = 1$; this situation can be always realized setting $b_2 = 0$, and b_1 an appropriate normalization factor. Moreover we recognize in the first solution the typical form of the spherical Bessel function of first degree (C.2). All in all, the total solution can be written as:

$$\chi(k, \eta) \propto \frac{j_1(k\eta)}{k\eta}. \tag{4.106}$$

One can verify that is indeed the correct solution by a direct substitution inside (4.104); denoting with x the combination $x = k\eta$, the wave equation reduces to

$$\begin{aligned}
0 &= \frac{k}{\eta}\frac{d^2j_1(x)}{dx^2} - \frac{2}{\eta^2}\frac{dj_1(x)}{dx} + \frac{2}{\eta^3}j_1(x) + \frac{4}{\eta^2}\frac{dj_1(x)}{dx} - \frac{4}{k\eta^3}j_1(x) + \frac{k^2}{k\eta}j_1(x) \\
&= \frac{k}{\eta}\left[\frac{d^2j_1(x)}{dx^2} + \frac{2}{x}\frac{dj_1(x)}{dx} + \left(1 - \frac{2}{x^2}\right)j_1(x)\right],
\end{aligned} \tag{4.107}$$

which is identically zero since the term inside the square brackets is exactly the spherical Bessel equation (C.1) with $\ell = 1$.

On the other hand, in the opposite regime for large masses ($m^2a^2 \gg k^2$), the mass term in the brackets of equation (4.104) represents the dominant contribution. In order to solve the Einstein equation in this limit, let's consider the following trick. We apply the change of variable $x = \frac{mH_0^2\eta^3}{12}$ such that

$$\begin{aligned}
\frac{d}{d\eta} &= \left(\frac{9mH_0^2}{4}\right)^{1/3} x^{2/3}\frac{d}{dx}, \\
\frac{d^2}{d\eta^2} &= \left(\frac{9mH_0^2}{4}\right)^{2/3} x^{2/3}\left(\frac{2}{3}x^{-1/3}\frac{d}{dx} + x^{2/3}\frac{d^2}{dx^2}\right).
\end{aligned} \tag{4.108}$$

This way equation (4.104) becomes:

$$\ddot{\chi}\left(\frac{9mH_0^2x^2}{4}\right)^{2/3}+\dot{\chi}\left[\frac{2}{3}+\frac{4}{3}\right]x^{1/3}\left(\frac{9mH_0^2}{4}\right)^{2/3}+\left[k^2+\left(\frac{9mH_0^2x^2}{4}\right)^{2/3}\right]\chi=0$$
$$\ddot{\chi}+\frac{2}{x}\dot{\chi}+\left[k^2\left(\frac{9mH_0^2}{4}\right)^{-2/3}x^{-4/3}+1\right]\chi=0$$
$$\ddot{\chi}+\frac{2}{x}\dot{\chi}+\left[1+\frac{16k^2}{m^2H_0^4\eta^4}\right]\chi=0 \tag{4.109}$$

where the dots in $\dot{\chi}$ and $\ddot{\chi}$ are used to denote the first and second derivative with respect to x respectively. Now we take the limit for large masses such that the second term in the bracket can be neglected and we recognize again the form of a Bessel function of order zero. Therefore the solution in this regime can be approximated as

$$\chi\propto j_0\left(\frac{mH_0^2\eta^3}{12}\right). \tag{4.110}$$

The two solution show a similar behavior. They both approach to a constant value at early times, while they provide a suppressed and damped oscillating behavior at late time, as a symptom of the horizon re-entering process. A possible way to interpolate the solutions in the two regimes is to treat the mass contribution as a modulation of the massless one, that is we consider their product:

$$\chi(\eta,k)\simeq A\,\frac{j_1(k\eta)}{k\eta}j_0\left(\frac{mH_0^2\eta^3}{12}\right). \tag{4.111}$$

with A a real coefficient set by requiring the matching with the solution in the radiation era at the time of matter-radiation equivalence (the reader can find the explicit expression in ref. [60]. The trend of this solution is shown in Figure (4.5). Taking the limit for $\eta\to 0$ on the spherical Bessel functions of first and zero order (C.2), one can easily realize that this is indeed the correct normalization. Thanks to the properties of the solutions in the two regimes, their product shows a constant trend at early times as well, as it should be. This behavior gets broken once the cosmological scale k re-enters inside the horizon. In matter domination the comoving Hubble radius increase as $r_H(\eta)=\eta^2/2$, and the condition for the horizon crossing becomes

$$k^2+\frac{1}{16}m^2H_0^4\eta_*^6=\frac{2}{\eta_*^2}. \tag{4.112}$$

This is an interesting result; the main effect of a non vanishing graviton mass is visible in the horizon re-entering process. In particular the mass moves up this event, massive modes happen to cross the horizon earlier. As a consequence, small scale massive tensor modes happen to be even more suppressed, since they live within the horizon longer. Equation (4.112) has six roots, but the only positive and real one is

$$\eta_*=\frac{1}{3^{1/3}}\left[\frac{\left[\sqrt{3\mu^6\left(k^6+27\mu^2\right)}+9\mu^4\right]^{1/3}}{\mu^2}-\frac{3^{1/3}k^2}{\left[\sqrt{3\mu^6\left(k^6+27\mu^2\right)}+9\mu^4\right]^{1/3}}\right]^{1/2}, \tag{4.113}$$

where, for the ease of notation, it was defined an effective mass $\mu\equiv mH_0/4$.

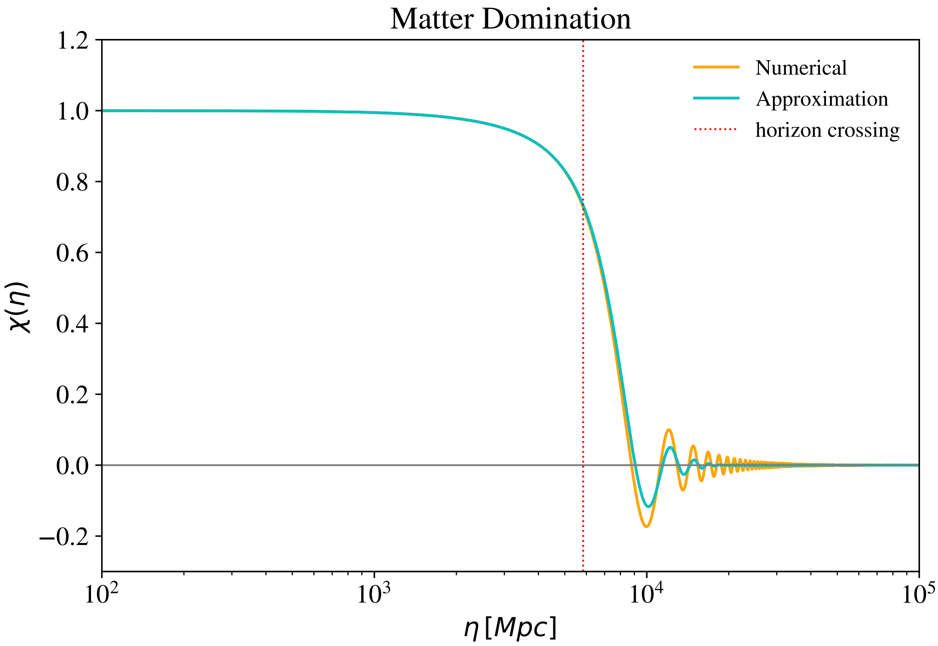

Figure 4.5: *Tensor transfer function in matter domination as function of time with* $m = k = H_0$*. The plot shows that tensor modes remain nearly fronzen to their primordial value until the occurrence of the horizon crossing process (higlighted by a dotted red vertical line taken from the solution of the condition* (4.112)*). After this event, the perturbation modes soon decay and undergo a rapid oscillation process averaged to zero.*

In order to see more clearly the behavior of the interpolate solution (4.84), let's consider the superhorizon regime, namely $k\eta \ll 1$, and expand the massless contribution up to the second order. This way, making explicit the zero order Bessel function,

$$\chi(\eta, k) \sim \frac{12 \sin\left(\frac{mH_0^2\eta^3}{12}\right)}{mH_0^2\eta^3}\left(1 - \frac{(k\eta)^2}{10}\right) + O(k^4). \tag{4.114}$$

Few comments are now in order. First of all this expression makes clear that in the massless case the transfer function outside the horizon behaves nearly like a constant. When we turn on the graviton mass, the sinusoidal factor introduces a time dependent modulation on the transfer function. The frequency of these modulations increases with the mass. This directly reflects the effect of the graviton mass on the horizon re-entering process stated above. Indeed, the more is the the mass, the earlier a given cosmological scale will re-enter the horizon, or, in other words, the more rapidly the modulation factor in (4.114) will force the transfer function to oscillate and damp. The limiting mass which distinguishes between the two regimes is conventionally defined by $m\eta^3/12 = k\eta$, that is when the argument of the massive and massless solutions equate. Significant mass contributions becomes evident over this threshold mass. All these features are captured in Figure (4.6)

Λ domination The last case to analyze is the one where the contribution of the cosmological constant dominates. This epoch refers to very to very late times, since Λ started dominating at redshift $z_{\Lambda_{eq}} \simeq 0.4$, which means $a(\eta_{\Lambda_{eq}})/a_0 \simeq 0.71$. As in the previous cases, one can find the scale factor evolution integrating the Friedman equation (4.45),

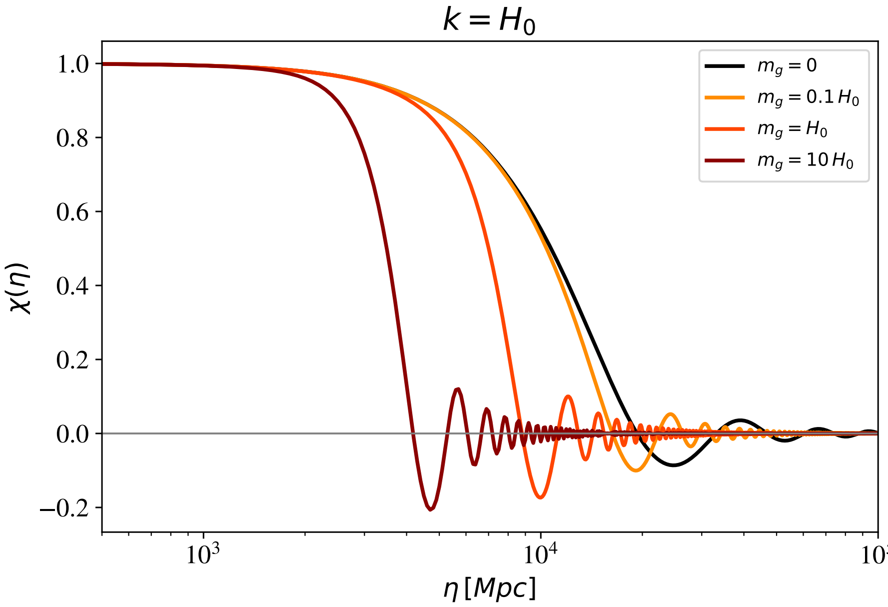

Figure 4.6: *Tensor transfer function in matter domination for different values of the late graviton mass m and with fixed wavenumber $k = H_0$ as function of conformal time. The plot shows clearly that the heavier the mass the sooner a certein perturbation mode cross the horizon. Moreover, massive modes undergo a more rapid oscillation when they leave beneath the horizon.*

which in this case leads to

$$\int \frac{da}{a^2} = \sqrt{\frac{\rho_\Lambda}{3M_P^2}} \int d\eta, \tag{4.115}$$

with solution

$$a(\eta) = -\frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\eta^{-1}. \tag{4.116}$$

Then the Einstein equation (4.84) becomes

$$\chi'' + \frac{2}{\eta}\chi' + \left(k^2 + \frac{m^2}{H_0^2(1-\Omega_{m,0})}\eta^{-2}\right)\chi = 0. \tag{4.117}$$

Again the easiest path is to consider separately the regimes for large or small graviton masses and then interpolate the two solutions. In the latter case the equation simplifies to

$$\chi'' + \frac{2}{\eta}\chi' + k^2\chi = 0, \tag{4.118}$$

which is identical to the radiation domination case (4.88) with solution

$$\chi(k,\eta) \propto j_0(k\eta) \qquad\qquad m^2a^2 \ll k^2. \tag{4.119}$$

In the opposite regime the mass term survives, and the solution is readily found to be

$$\chi(\eta) = c_1\eta^{\frac{1}{2}\left(-\sqrt{1-4\tilde{m}_0^2}-1\right)} + c_2\eta^{\frac{1}{2}\left(\sqrt{1-4\tilde{m}_0^2}-1\right)}. \tag{4.120}$$

Since the first term happens to decay more rapidly the the other one, one can safely decide to neglect this contribution at late times, and then to set $c_1 = 0$. The other constant is fixed by the matching condition with the solution in matter domination. This way the complete interpolating solution is found as

$$\chi(\eta) = B\, j_0(k\eta)\,(\eta)^{\frac{1}{2}\left(\sqrt{1-4\tilde{m}_0^2}-1\right)}\,. \tag{4.121}$$

To sum up the results of this section, the tensor transfer function in the three different regimes is found as

$$\chi(k,\eta) = \begin{cases} 2^{1/4}\Gamma\left(\frac{5}{4}\right)\left(\frac{mH_0\eta^2}{2}\right)^{1/4} j_{-\frac{1}{4}}\left(\frac{mH_0\eta^2}{2}\right) j_0(k\eta), & \text{if} \quad \eta < \eta_{eq}, \\ A\,\frac{j_1(k\eta)}{k\eta} j_0\left(\frac{mH_0^2\eta^3}{12}\right), & \text{if} \quad \eta_{eq} < \eta < \eta_{\Lambda_{eq}}, \\ B\, j_0(k\eta)\,(\eta)^{\frac{1}{2}\left(\sqrt{1-4\tilde{m}_0^2}-1\right)}, & \text{if} \quad \eta > \eta_{\Lambda_{eq}}. \end{cases} \tag{4.122}$$

4.5 Explicit computation of the angular power spectra

The discussion of the previous sections provide us all the tools to attempt an explicit evaluation of the linear transfer functions (3.93), and, ultimately, of the angular spectra (4.42). As usual we divide this procedure in three step, one for each contribution.

4.5.1 Initial condition angular spectrum $\tilde{C}_{\ell,I}$

For a start, let's consider the initial condition contribution

$$\tilde{C}_{\ell,I} = 4\pi \int \frac{dk}{k} |j_\ell[kl(\eta_0,\eta_{in})]|^2 \mathcal{P}_I(q,k). \tag{4.123}$$

In order to evaluate this expression we firstly need a guess about the dimensionless power spectrum $\mathcal{P}_I(q,k)$. The easiest case is to assume an Harrison-Zel'dovich spectrum, that is we assume that the dimensionless power spectrum does not depend on the scale k at all. This way the computation simplifies to

$$\begin{aligned} \tilde{C}_{\ell,I} &= 4\pi\mathcal{P}_I(q) \int \frac{dk}{k} |j_\ell[kl(\eta_0,\eta_{in})]|^2 \\ &= 4\pi\mathcal{P}_I(q) \int \frac{dx}{x} j_\ell(x)^2. \end{aligned} \tag{4.124}$$

Having in mind the known solution for the CMB anisotropy, we immediately recognize in this expression a contribution coming from the Sachs Wolfe effect, that is the contribution of the gravitational redshift occurring at the time of the GW formation (which is manifest in the dependence from the initial time η_{in}). At this point the integral can be evaluated by exploiting the property (C.8) of the spherical Bessel functions:

$$\int \frac{dx}{x} j_\ell(x)^2 = 2^{-3}\pi \frac{\Gamma(\ell)\Gamma(2)}{\Gamma(\ell+2)\Gamma^2\left(\frac{3}{2}\right)} = \frac{2^{-1}}{\ell(\ell+1)}, \tag{4.125}$$

where in the last step we used the properties of the Euler gamma functions:

$$\Gamma(n) = (n-1)!\,, \qquad \Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}, \tag{4.126}$$

for n any non-negative integer. Therefore, the final result for the spectra of the initial condition with the assumption of a Harrison Zel'dovich spectrum is

$$\tilde{C}_{\ell,I} = \frac{2\pi \mathcal{P}_I(q)}{\ell(\ell+1)}. \tag{4.127}$$

Notice that the initial condition contribution gets no corrections from the graviton mass. This seems a reasonable result, since we expect the mass to affect the propagation of gravitons, and not the way they were produced at the initial time. Anyway this term still preserves the dependence on the graviton frequency q, regardless the presence of a non vanishing mass.

We can now try to extend these result to allow a more general class of dimensionless power spectra. We consider the case of a constant spectral index n_I such that the dimensionless power spectrum obeys the power law

$$\mathcal{P}_I(k) = \mathcal{P}(k_0)\left(\frac{k}{k_0}\right)^{n_I-1}, \tag{4.128}$$

where k_0 denotes the pivot scale of the spectrum. In this case the angular spectra evaluate to

$$\begin{aligned}\tilde{C}_{\ell,I} &= 4\pi \mathcal{P}_I(q,k_0)\int \frac{dk}{k}\left(\frac{k}{k_0}\right)^{n_I-1} |j_\ell[kl(\eta_0,\eta_{in})]|^2 \\ &= 4\pi \mathcal{P}_I(q,k_0)\left(\frac{1}{k_0 l(\eta_0,\eta_{in})}\right)^{n_I-1}\int dx\, x^{n_I-2} j_\ell(x)^2.\end{aligned} \tag{4.129}$$

Using again (C.8) to evaluate the integral, we obtain the full general expression

$$\tilde{C}_{\ell,I} = \frac{\pi^2}{2}\frac{\Gamma\left(\ell+\frac{n_I}{2}-\frac{1}{2}\right)}{\Gamma\left(\ell+\frac{5}{2}-\frac{n_I}{2}\right)}\frac{\Gamma(3-n_I)}{\Gamma^2\left(2-\frac{n_I}{2}\right)}\mathcal{P}_I(q,k_0)\left(\frac{2}{k_0 l(\eta_0,\eta_{in})}\right)^{n_I-1}. \tag{4.130}$$

4.5.2 Scalar sourced angular spectrum $\tilde{C}_{\ell,S}$

The final result of section (4.3) allows to write an explicit expression for the linear scalar transfer function (3.93), while the angular transfer function reads:

$$\begin{aligned}\mathcal{T}_\ell^S(\eta_0,\eta_{in},k,q) = \frac{2}{3}\Bigg\{ & v^{-2} g(k,\eta_{in}) j_\ell\left[kl(\eta_0,\eta_{in})\right] + \\ & + \int_{\eta_{in}}^{\eta_0} d\eta \frac{\partial\left[(1+v^{-2})g(k,\eta)\right]}{\partial\eta} j_\ell\left[kl(\eta_0,\eta)\right]\Bigg\},\end{aligned} \tag{4.131}$$

with

$$g(k,\eta) = \begin{cases} g_{\rm rad}(k,\eta) = 3\,\dfrac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}, & \text{if} \quad \eta < \eta_{eq}, \\ g_{\Lambda,\rm m}(\eta) = \dfrac{{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a^3}{r}\right)}{{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-\frac{a_{eq}^3}{r}\right)}, & \text{if} \quad \eta > \eta_{eq}. \end{cases} \tag{4.132}$$

and $g(k,\eta_{in}) = g_{\rm rad}(k,\eta_{in})$. The first term of equation (4.131) represents an intrinsic fluctuation in the energy density at primordial times, when the gravitational waves were generated. In the CMB literature this effect is commonly called *Sachs Wolfe* contribution. The second term instead can be compared with the *Integrated Sachs Wolfe* effect, that is an integrated gravitational-redshift effect accounting for all the history of the Universe from the GW formation until today, and it is sensitive to any temporal variation of the scalar potentials or the graviton phase velocity. The above expression can now be inserted inside the spectra for the scalar sourced contribution (4.42).

$$\begin{aligned}\tilde{C}_{\ell,S} &= 4\pi \int \frac{dk}{k} |\mathcal{T}_\ell^S(\eta_0, \eta_{in}, k, q)|^2 \mathcal{P}_\zeta(k) \\ &= \frac{16\pi}{9} \int \frac{dk}{k} \Bigg| v^{-2} g_{\rm rad}(k, \eta_{in}) j_\ell \left[k l(\eta_0, \eta_{in}) \right] + \\ &\qquad + \int_{\eta_{in}}^{\eta_0} d\eta \frac{\partial \left[(1+v^{-2}) g(\eta) \right]}{\partial \eta} j_\ell \left[k l(\eta_0, \eta) \right] \Bigg|^2 \mathcal{P}_\zeta(k). \end{aligned} \tag{4.133}$$

As before, we now perform a change of variable inside the integral and pass to the dimensionless variable $x = kl(\eta_0, \eta_{in})$. Notice that the definition (3.31) of the quantity $l(\eta, \eta')$ allows the decomposition $l(\eta_0, \eta) = l(\eta_0, \eta_{in}) - l(\eta, \eta_{in})$. This way (4.133) becomes

$$\begin{aligned}\tilde{C}_{\ell,S} = \frac{16\pi}{9} \int \frac{dx}{x} \mathcal{P}_\zeta \left(\frac{x}{l(\eta_0, \eta_{in})} \right) \Bigg\{ & v^{-2} g_{\rm rad} \left(\frac{x}{l(\eta_0, \eta_{in})}, \eta_{in} \right) j_\ell(x) \\ & + \int_{\eta_{in}}^{\eta_0} d\eta \frac{\partial \left[(1+v^{-2}) g(\eta) \right]}{\partial \eta} j_\ell \left[x \left(1 - \frac{l(\eta, \eta_{in})}{l(\eta_0, \eta_{in})} \right) \right] \Bigg\}^2. \end{aligned} \tag{4.134}$$

This expression can be compared again with the known results of the CMB anisotropy. It is instructive to look at the two contribution separately to understand how each term contributes to the total angular power spectrum.

Sachs-Wolfe contribution

The term of the angular transfer function corresponding to the the Sachs-Wolfe contribution is

$$\mathcal{T}_{\ell,S}^{SW}(\eta_0, \eta_{in}, k, q) = \frac{2}{3} v^{-2} g_{\rm rad}(k, \eta_{in}) j_\ell \left[k l(\eta_0, \eta_{in}) \right]. \tag{4.135}$$

Hence, considering a power law scale-dependence of the primordial power spectrum as

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k_0) \left(\frac{k}{k_0} \right)^{n_s - 1}, \tag{4.136}$$

with n_s the scalar spectral index and k_0 a reference scale, the angular power spectrum is

$$\tilde{C}_{\ell,S}^{SW} = \frac{16\pi}{9} v^{-4} \frac{\mathcal{P}_\zeta(k_0)}{x_0^{n_s-1}} \int dx\, x^{n_s-2} g_{\rm rad} \left(\frac{x}{l(\eta_0, \eta_{in})}, \eta_{in} \right)^2 j_\ell(x)^2, \tag{4.137}$$

where $x_0 \equiv k_0 l(\eta_0, \eta_{in})$. As long as we are interested in large angular scales, that is low multipoles ℓ, since η_{in} is fixed, the most dominant contribution to the angular transfer function comes from the large linear scales k. This is because small scales are suppressed by the spherical Bessel function, which selects wavenumber of the order $\ell \sim kl(\eta_0, \eta_{in})$. Hence, one can assume that, for low multipoles, the scales that contribute the most to the angular transfer function happened to be out of the horizon at initial time. In this regime the growing rate $g_{\rm rad}(k, \eta_{in}) \simeq 1$. Then, for large scales,

$$\begin{aligned}\tilde{C}^{SW}_{\ell,S} &= \frac{16\pi}{9} v^{-4} \mathcal{P}_\zeta(k_0) \left(\frac{1}{k_0 l(\eta_0, \eta_{in})}\right)^{n_s-1} \int dx\, x^{n_s-2} j_\ell(x)^2 \\ &= \frac{2\pi^2}{9} v^{-4} \mathcal{P}_\zeta(k_0) \left(\frac{2}{k_0 l(\eta_0, \eta_{in})}\right)^{n_s-1} \frac{\Gamma\left(\ell + \frac{n_s}{2} - \frac{1}{2}\right)}{\Gamma\left(\ell + \frac{5}{2} - \frac{n_s}{2}\right)} \frac{\Gamma(3-n_s)}{\Gamma^2\left(2 - \frac{n_s}{2}\right)}. \end{aligned} \tag{4.138}$$

In the easiest Harrison-Zel'dovich case the result simplifies to

$$\frac{\ell(\ell+1)}{2\pi} \tilde{C}^{SW}_{\ell,S} = \frac{4}{9} v^{-4} \mathcal{P}_\zeta, \tag{4.139}$$

that describes the usual Sachs-Wolfe Plateau which the reader may be familiar with from the results of CMB anisotropy.

As we will see, the tensor sourced spectrum is subdominant with respect to the initial and scalar sourced one, which therefore provide the most contribution to the total spectrum. Hence we can already read the leading contributions to the angular spectrum, in the simplest case of an Harrison-Zel'dovich spectrum, from (4.127) and (4.139), which together constitute the overall Sachs Wolfe effect of SGWB, that is:

$$\frac{\ell(\ell+1)}{2\pi} \tilde{C}_\ell \simeq \frac{\ell(\ell+1)}{2\pi} \left[\tilde{C}_{\ell,I} + \tilde{C}^{SW}_{\ell,S}\right] \simeq \mathcal{P}_I(q) + \left(\frac{2}{3}\right)^2 v^{-4} \mathcal{P}_\zeta. \tag{4.140}$$

This result looks very similar to the one obtained in [59]. The only difference relies on the graviton group velocity v, and the massless result is straightforwardly recovered in the limit $v \to 1$.

Integrated Sachs-Wolfe contribution

The ISW effect arise from the time variation of the gravitational potentials encoded in the second term of (4.134). One should expect that the cross product of the two terms gives a negligible contribution, since it involves an integral of two spherical Bessel functions with different arguments, such that it is probable that they interfere in a distructive way. At the level of the angular transfer function, the ISW contribution comes from

$$\mathcal{T}^{ISW}_{\ell,S}(\eta_0, \eta_{in}, k, q) = \frac{2}{3} \int_{\eta_{in}}^{\eta_0} d\eta \frac{\partial\left[(1+v^{-2}) g(k,\eta)\right]}{\partial\eta} j_\ell\left[k l(\eta_0, \eta)\right]. \tag{4.141}$$

Assuming for simplicity that gravitons propagate with constant velocity and considering the limit $\eta_{in} \ll \eta_0$, the ISW contribution to the angular power spectrum is

$$\tilde{C}^{ISW}_{\ell,S} = \frac{16\pi(1+v^{-2})}{9} \int \frac{dx}{x} \mathcal{P}_\zeta\left(\frac{x}{v\eta_0}\right) \left\{\int_{\eta_{eq}}^{\eta_0} d\eta \frac{dg(k,\eta)}{d\eta} j_\ell\left[x\left(1 - \frac{\eta}{\eta_0}\right)\right]\right\}^2, \tag{4.142}$$

with $x = kl(\eta_0, \eta_{in}) \approx kv\eta_0$. In order to estimate this effect it is necessary to understand the derivative of the growing rate $g(\eta)$ appearing inside the second term of the angular transfer function (4.131) (for simplicity in the following the graviton group velocity v will be retained as constant in time). As seen above, the behavior of scalar modes is very different during radiation or matter domination. In order to reach a semi-analytical expression it is fair to discriminate between modes on large scales that re-enter the horizon during matter domination and the small scales that re-enter when ultra-relativistic species were dominating the energy budget of the Universe.

Large scales: since these scales have remained frozen to their primordial value throughout the radiation domination epoch, the derivative of the growing rate vanishes for $\eta < \eta_{eq}$. On the contrary, once the Universe enters in the matter domination epoch, all the scales happen to decrease according to (4.73) with the growing rate (4.74) defined as

$$g(\eta) = g_{\Lambda,\mathrm{m}}(\eta) = \frac{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a^3}{r}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}. \tag{4.143}$$

Its conformal time derivative can be decomposed in the following way

$$\frac{\partial g(\eta)}{\partial \eta} = \frac{dg(a)}{da}\frac{da(\eta)}{d\eta}. \tag{4.144}$$

The first factor can be treated analytically and evaluates to

$$\frac{dg(a)}{da} = -\frac{6a^2}{11r}\frac{{}_2F_1\left(\frac{4}{3}, 2; \frac{17}{6}; -\frac{a^3}{r}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}. \tag{4.145}$$

For what concerns the time derivative of the scale factor instead, we do not have an exact analytic expression of $a(\eta)$. The best approach is to use the numerical fit function solution (4.79) to obtain

$$\frac{da(\eta)}{d\eta} = -\frac{c_2\eta_0}{\eta^2} + 2c_3\frac{\eta}{\eta_0^2}. \tag{4.146}$$

Then finally

$$\begin{aligned}\frac{dg(\eta)}{d\eta} = -\frac{6}{11r}\left(-\frac{c_2\eta_0}{\eta^2} + 2c_3\frac{\eta}{\eta_0^2}\right)\left[c_1 + c_2\left(\frac{\eta_0}{\eta}\right) + c_3\left(\frac{\eta}{\eta_0}\right)^2\right]^2 \\ \times\frac{{}_2F_1\left(\frac{4}{3}, 2; \frac{17}{6}; -\frac{\left(c_1+\frac{c_2}{\eta}+c_3\eta^2\right)^3}{r}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}.\end{aligned} \tag{4.147}$$

This function is really complicated and hard to integrate. For this reason some approximations are necessary to proceed with an analytic evaluation. In the first place, looking at the magnitude of the parameters involved in the fitting function for the scale factor (4.79), one can convince himself that the by far dominant contribution arise from the matter component. This situation is achievable by simply neglecting the terms proportional to c_1 and c_2, which leads to

$$\frac{dg(\eta)}{d\eta} = -\frac{12c_3^3}{11r\eta_0^6}\eta^5\frac{{}_2F_1\left(\frac{4}{3}, 2; \frac{17}{6}; -\frac{c_3^3\eta^6}{r\eta_0^6}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}. \tag{4.148}$$

Figure (4.7) shows the derivative of the growing rate considering both the two situation, the one with dust and Λ components, and the one with matter only. Moreover, for a rough estimation of the order of magnitude of this effect, one can try to replace the hypergeometrical functions with their averaged value. Indeed, as shown in Figure (4.8), the ratio of these functions provides a weak damping at late times (with values included between [0.3, 1]) which does not spoil the most important polynomial behavior. The average of this

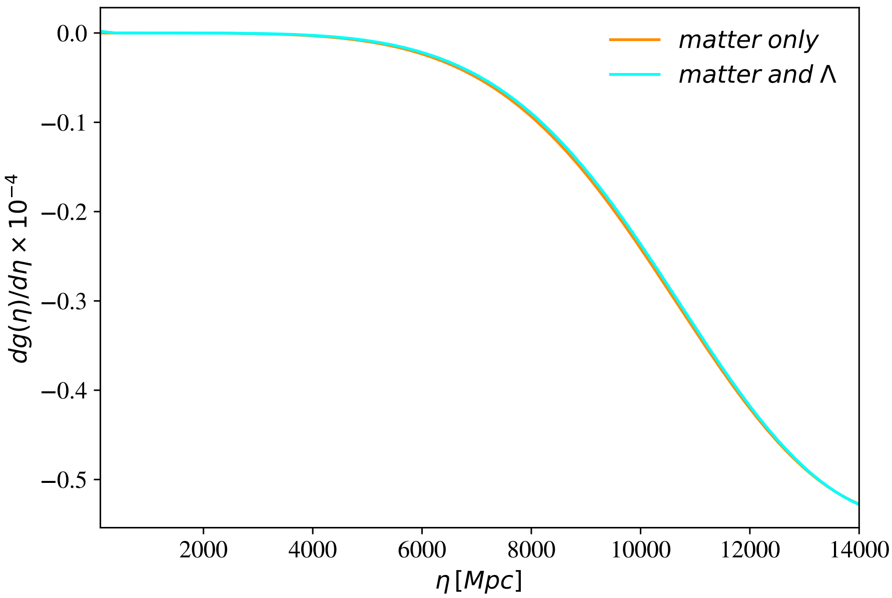

Figure 4.7: *Comparison between the full solution* (4.147) *considering both the matter and the* Λ *contribution, and the simplified solution* (4.148) *taking into account for only the matter component. The two solution are almost identical.*

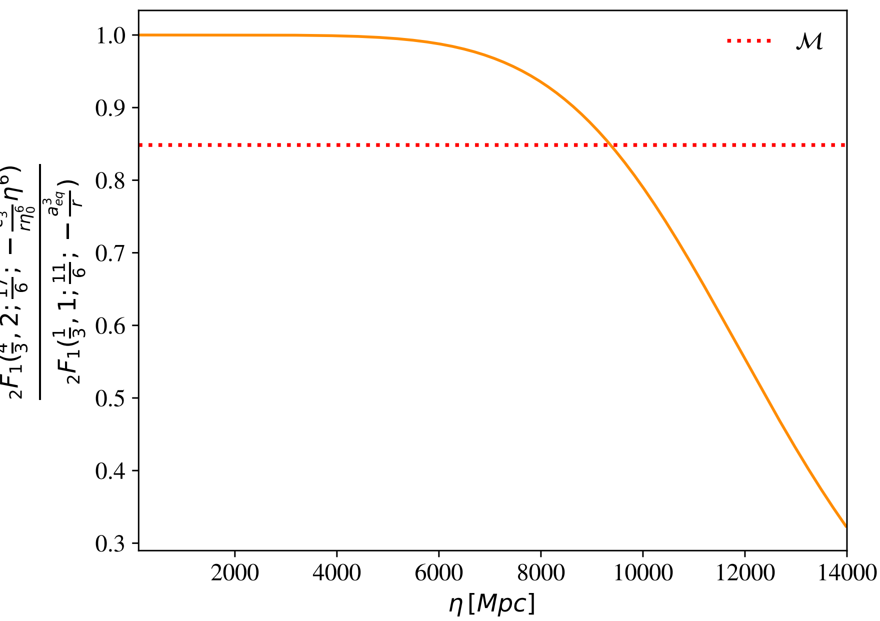

Figure 4.8: *Averaging process of the hypergeometric function* ${}_2F_1\left(\frac{4}{3}, 2; \frac{17}{6}; -\frac{c_3^3\eta^6}{r\eta_0^6}\right)$

function is given, in virtue of the integral mean value theorem, by

$$\mathcal{M} = \frac{1}{\eta_0 - \eta_{eq}} \int_{\eta_{eq}}^{\eta_0} d\eta \frac{{}_2F_1\left(\frac{4}{3}, 2; \frac{17}{6}; -\frac{c_3^3\eta^6}{r\eta_0^6}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)} = 0.8484. \tag{4.149}$$

Here is was chosen to take the estimate value for the conformal time at matter radiation equivalence η_{eq} from the input parameters of the public code CLASS [111] for computation of the CMB anisotropy ($\eta_{eq} = 112.800\,\mathrm{Mpc}$), while the present conformal time is computed from (4.78) setting $a(\eta_0) = 1$, with the result: $\eta_0 = 13982.6\,\mathrm{Mpc}$. At the end of the day, the final approximated estimate for the derivative of the growing rate is

$$\frac{dg(\eta)}{d\eta} = -\frac{12c_3^3}{11r\eta_0^6}\mathcal{M}\eta^5. \tag{4.150}$$

At this point one can attempt the evaluation of the time integral in (4.142) by hand:

$$\begin{aligned}\int_{\eta_{eq}}^{\eta_0} d\eta \frac{dg(\eta)}{d\eta} j_\ell\left[x\left(1-\frac{\eta}{\eta_0}\right)\right] &= -\frac{12c_3^3}{11r\eta_0^6}\mathcal{M}\int_{\eta_{eq}}^{\eta_0} d\eta\, \eta^5 j_\ell\left[x\left(1-\frac{\eta}{\eta_0}\right)\right] \\ &= -\frac{12c_3^3}{11r}\frac{\mathcal{M}}{x}\int_0^x dy\, \left(1-\frac{y}{x}\right)^5 j_\ell(y),\end{aligned} \tag{4.151}$$

where in the last step the following changes of variables were applied

$$\tilde{\eta} \equiv \frac{\eta}{\eta_0} \qquad y \equiv x(1-\tilde{\eta}). \tag{4.152}$$

Looking at the parenthesis in the second line of (4.151), one immediately realizes that the term y/x is always subdominant with respect to the term 1, since the integration variable y runs from 0 to x. Therefore, considering only the leading term, the integral evaluates to

$$\begin{aligned}\int_0^x dy\, j_\ell(y) &= \sqrt{\pi}2^{-l-2}x^{\ell+1}\Gamma\left(\frac{\ell+1}{2}\right)\,{}_1\tilde{F}_2\left(\frac{\ell+1}{2}; \ell+\frac{3}{2}, \frac{\ell+3}{2}; -\frac{x^2}{4}\right) \\ &= \sqrt{\pi}2^{-\ell-2}x^{\ell+1}\Gamma\left(\frac{\ell+1}{2}\right)\frac{{}_1F_2\left(\frac{\ell+1}{2}; \ell+\frac{3}{2}, \frac{\ell+3}{2}; -\frac{x^2}{4}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(\frac{\ell+3}{2}\right)}.\end{aligned} \tag{4.153}$$

Therefore

$$\begin{aligned}\tilde{C}_{\ell,S}^{ISW} \simeq\ & \frac{16\pi^2(1+v^{-2})c_3^6\mathcal{M}^2}{121\,r^2 2^{2\ell}}\frac{\Gamma^2\left(\frac{\ell+1}{2}\right)}{\Gamma^2\left(\ell+\frac{3}{2}\right)\Gamma^2\left(\frac{\ell+3}{2}\right)} \\ & \int dx\, x^{2\ell-1}\mathcal{P}_\zeta\left(\frac{x}{v\eta_0}\right)\,{}_1F_2^2\left(\frac{\ell+1}{2}; \ell+\frac{3}{2}, \frac{\ell+3}{2}; -\frac{x^2}{4}\right).\end{aligned} \tag{4.154}$$

At this point one cannot go further by analytical approaches. The integral of the hypergeometric function can be evaluated with numerical methods for any given value of the multipole ℓ and the scalar spectral index n_s.

Small scales: small scale perturbation re-enter the horizon before the matter-radiation equality, and hence they follow a time growth given by (4.55). As one can see in Figure (4.1), the solution remains constant as long as the perturbation remains out of the Hubble horizon, while it undergoes a rapid decay during the re-entering process. Once the mode has crossed the horizon, it soon start oscillating with zero average. Therefore one can approximate the derivative of the growing rate $g_{\rm rad}(k,\eta)$ as a Dirac delta function centered at the time of horizon crossing, that is when $k\eta_* \simeq \sqrt{3}$. Then the ISW contribution simplifies to

$$\tilde{C}^{ISW}_{\ell,S} = \frac{16\pi(1+v^{-2})}{9}\int\frac{dx}{x}\mathcal{P}_\zeta\left(\frac{x}{v\eta_0}\right)j_\ell\left[x\left(1-\frac{\eta_*}{\eta_0}\right)\right]^2. \tag{4.155}$$

This expression resembles the one of the SW term, but here there is a more tricky x-dependence inside the spherical Bessel function. Indeed the horizon crossing time η_* depends on the variable x via

$$\frac{\eta_0}{\eta_*} = \frac{x}{v\sqrt{3}}+1, \tag{4.156}$$

and then

$$\tilde{C}^{ISW}_{\ell,S} = \frac{16\pi(1+v^{-2})}{9}\int\frac{dx}{x}\mathcal{P}_\zeta\left(\frac{x}{v\eta_0}\right)j_\ell\left[\frac{x^2}{x+v\sqrt{3}}\right]^2. \tag{4.157}$$

The numerical solutions of the integrated Sachs-Wolfe effect, which we will discuss in Section (6.2), are in agreement with the last result (4.157) only for very large multipoles, namely $\ell > 1000$, that are inaccessible to our future GW interferometers. For lower multipoles instead, the numerical solution in not in agreement with (4.154); more efforts are then needed to develop an efficient analytic computation of the ISW contribution.

4.5.3 Tensor sourced angular spectrum $\tilde{C}_{\ell,T}$

In light of the result of section (4.4), and remembering the definitions (4.42) and (3.93), the tensor sourced contribution to the angular spectrum is:

$$\begin{aligned}\tilde{C}_{\ell,T} &= 4\pi\int\frac{dk}{k}|\mathcal{T}^T_\ell(\eta_0,\eta_{in},k,q)|^2\sum_{\lambda=\pm2}\mathcal{P}_\lambda(k)\\ &= \frac{\pi}{4}\frac{(\ell+2)!}{(\ell-2)!}\int\frac{dk}{k}\left|\int_{\eta_{in}}^{\eta_0}d\eta\,\chi'(\eta,k)\frac{j_\ell[kl(\eta_0,\eta)]}{k^2l^2(\eta_0,\eta)}\right|^2\sum_{\lambda=\pm2}\mathcal{P}_\lambda(k).\end{aligned} \tag{4.158}$$

As done for the scalar case, one could separate between two distinct regimes, considering large scales that re-enter the horizon during matter domination, and small scales which re-enter before the time of equivalence, that is during the radiation era. However, looking at the solution for the tensor transfer function in these regimes (4.122) and remembering the discussion of Section (4.4)[6], one can treat both the two cases at the same time with a simple argument. Since for large arguments the growing rate oscillates very rapidly, it can be averaged out to zero, while there is a steep variation when the modes pass from super- to sub-horizon regime. Then it is reasonable to take the time derivative of the transfer $\chi(k,\eta)$ as a Dirac delta function centered at the time of horizon crossing. What distinguish between the two regimes is the condition which defines this event. For large scales which re-enter during matter domination it is given by (4.112):

$$k^2\eta_*^2 = 2 - \frac{1}{16}m^2H_0^4\eta_*^6. \tag{4.159}$$

while for scales that re-enter during the radiation epoch the condition is (4.100)

$$k^2 + m^2H_0^2\eta_*^2 = \frac{1}{\eta_*}. \tag{4.160}$$

[6]Indeed the form of the solutions for the tensor transfer function in both the two regimes is really similar to the behavior of the scalar transfer function during radiation domination. Then, all the three situation can be treated is an analogous way.

Moreover, as done several times so far, one can consider $\eta_{in} \approx 0$ and $\eta_0 \approx \infty$, such that the angular spectrum becomes

$$\tilde{C}_{\ell,T} \simeq \frac{\pi}{4}\frac{(\ell+2)!}{(\ell-2)!}\int \frac{dk}{k}\frac{j_\ell^2[kl(\eta_0,\eta_*)]}{k^4 l^4(\eta_0,\eta_*)}\sum_{\lambda=\pm 2}\mathcal{P}_\lambda(k). \tag{4.161}$$

In this situation one should remember that the horizon crossing time η_* depends on the scale k under consideration. However this dependence is suppressed inside the spherical Bessel function, such that it is a good approximation to retain η_* as k-independent. Considering a power law dependence for the primordial spectrum $\mathcal{P}_\lambda(k)$ as

$$\mathcal{P}_\lambda(k) = \mathcal{P}_\lambda(k_0)\left(\frac{k}{k_0}\right)^{n_T}, \tag{4.162}$$

and introducing the new integral variable $x \equiv kl(\eta_0,\eta_*)$, one gets

$$\begin{aligned}\tilde{C}_{\ell,T} &\simeq \frac{\pi}{4}\frac{(\ell+2)!}{(\ell-2)!}\frac{\sum_{\lambda=\pm 2}\mathcal{P}_\lambda(k_0)}{k_0^{n_T} l(\eta_0,\eta_*)^{n_T}}\int dx\, x^{n_T-5} j_\ell^2(x)\\ &\simeq \frac{2^{n_T}\pi^2}{512}\frac{\sum_{\lambda=\pm 2}\mathcal{P}_\lambda(k_0)}{k_0^{n_T} l(\eta_0,\eta_*)^{n_T}}\frac{(\ell+2)!}{(\ell-2)!}\frac{\Gamma\left(\ell+\frac{n_T}{2}-2\right)}{\Gamma\left(\ell-\frac{n_T}{2}+4\right)}\frac{\Gamma(6-n_T)}{\Gamma^2\left(\frac{7}{2}-\frac{n_T}{2}\right)}.\end{aligned} \tag{4.163}$$

In section (2.1.1) it was shown the most general expression for the tensor primordial power spectrum. In particular we obtained

$$n_T \equiv 3 - 2\nu = 3\left[1 - \sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{\bar{H}^2} - 3\epsilon\right)}\right]. \tag{4.164}$$

The effects of graviton mass are visible only inside the tensor spectral index and in the quantity $l(\eta_0,\eta_*)$ that, besides for the graviton group velocity, contains a mass dependence inside the time of horizon crossing η_*. As one can see in (4.164), the presence of a graviton mass induces a blue tilt in the primordial tensor power spectrum. For a scale invariant primordial power spectrum ($n_T = 0$), the final result becomes

$$\begin{aligned}\tilde{C}_{\ell,T} &\simeq \frac{\pi^2}{512}\sum_{\lambda=\pm 2}\mathcal{P}_\lambda(k_0)\frac{(\ell+2)!}{(\ell-2)!}\frac{\Gamma(\ell-2)}{\Gamma(\ell+4)}\frac{\Gamma(6)}{\Gamma^2\left(\frac{7}{2}\right)}\\ &\simeq \frac{\pi}{15}\sum_{\lambda=\pm 2}\mathcal{P}_\lambda(k_0)\frac{1}{(\ell+3)(\ell-2)}.\end{aligned} \tag{4.165}$$

4.5.4 Summary

The most important results of this chapter are now summerized. The scalar source contribution to the anisotropy are divided in a SW (Sachs-Wolfe) and an ISW (Integrated Sachs-Wolfe) effects:

$$\tilde{C}_{\ell,S}^{SW} = \frac{16\pi}{9}v^{-4}\frac{\mathcal{P}_\zeta(k_0)}{x_0^{n_s-1}}\int dx\, x^{n_s-2} g_{\rm rad}\left(\frac{x}{l(\eta_0,\eta_{in})},\eta_{in}\right)^2 j_\ell(x)^2, \tag{4.166}$$

$$\tilde{C}_{\ell,S}^{ISW} = \frac{16\pi(1+v^{-2})}{9}\int\frac{dx}{x}\mathcal{P}_\zeta\left(\frac{x}{v\eta_0}\right)\left\{\int_{\eta_{eq}}^{\eta_0} d\eta\frac{dg(\eta)}{d\eta}j_\ell\left[x\left(1-\frac{\eta}{\eta_0}\right)\right]\right\}^2, \tag{4.167}$$

with the growing rate evaluated in two different regimes as

$$g(\eta) = \begin{cases} g_{\rm rad}(\eta) = 3\,\dfrac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{\sqrt{k\eta}}{3}\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\frac{k\eta}{\sqrt{3}}}, & \text{if} \quad \eta < \eta_{eq}, \\ g_{\Lambda,\rm m}(\eta) = \dfrac{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a^3}{r}\right)}{{}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -\frac{a_{eq}^3}{r}\right)}, & \text{if} \quad \eta > \eta_{eq}. \end{cases} \tag{4.168}$$

It is particularly interesting to evaluate the Sachs-Wolfe contribution in the limit of large angular scales, where the angular power spectrum reduces to

$$\tilde{C}_{\ell,S}^{SW} = \frac{2\pi^2}{9} v^{-4} \mathcal{P}_\zeta(k_0) \left(\frac{2}{k_0 l(\eta_0, \eta_{in})}\right)^{n_s-1} \frac{\Gamma\left(\ell + \frac{n_s}{2} - \frac{1}{2}\right)}{\Gamma\left(\ell + \frac{5}{2} - \frac{n_s}{2}\right)} \frac{\Gamma(3-n_s)}{\Gamma^2\left(2 - \frac{n_s}{2}\right)}. \tag{4.169}$$

which in the Harrison-Zel'dovich case $n_s = 1$ recovers the familiar Sachs-Wolfe Plateau

$$\frac{\ell(\ell+1)}{2\pi} \tilde{C}_{\ell,S}^{SW} = \frac{4}{9} v^{-4} \mathcal{P}_\zeta. \tag{4.170}$$

The contribution arising from tensor perturbations instead amounts to a unique approximated term

$$\tilde{C}_{\ell,T} = \frac{2^{n_T}\pi^2}{512} \frac{\sum_{\lambda=\pm 2} \mathcal{P}_\lambda(k_0)}{k_0^{n_T} l(\eta_0, \eta_*)^{n_T}} \frac{(\ell+2)!}{(\ell-2)!} \frac{\Gamma\left(\ell + \frac{n_T}{2} - 2\right)}{\Gamma\left(\ell - \frac{n_T}{2} + 4\right)} \frac{\Gamma(6-n_T)}{\Gamma^2\left(\frac{7}{2} - \frac{n_T}{2}\right)}, \tag{4.171}$$

with η_* the time of horizon crossing defined by (4.159) or (4.160) depending on the multipole ℓ. In the simplest Harrison-Zel'dovich case with $n_T = 0$ one finds

$$\frac{\ell(\ell+1)}{2\pi} \tilde{C}_{\ell,T} = \frac{1}{30} \sum_{\lambda=\pm 2} \mathcal{P}_\lambda(k_0) \frac{\ell(\ell+1)}{(\ell+3)(\ell-2)}. \tag{4.172}$$

All in all, the late time graviton mass m enters in the velocity factor v, for the scalar sourced angular spectrum, and in the time of tensor modes horizon crossing η_* for the tensor case. The information about the heavy graviton mass instead is contained inside the scalar and spectral indices, that in full generality were found to be

$$n_s - 1 = \frac{d \ln \mathcal{P}(k)}{d \ln k} = -2\epsilon + \frac{18\alpha^2\lambda_1^2}{c_\pi^2}, \tag{4.173}$$

$$n_T = 3\left[1 - \sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{H^2} - 3\epsilon\right)}\right]. \tag{4.174}$$

4.6 Vector contributions

In Section (3.4.3), it was shown how the tensor sourced contribution to the density fluctuations can be expanded in multipole and split in a product between a transfer function and a primordial stochastic field. It is straightforward then to apply the vary same results to the case of vector perturbations if we turn for a moment into synchronous gauge, that is

$$\begin{aligned} \delta g_{0i}^{(V)} &= 0 \\ \delta g_{ij}^{(V)} &= a^2(\eta)\left[\partial_i \chi_j^\perp + \partial_j \chi_i^\perp\right]. \end{aligned} \tag{4.175}$$

The perturbed Boltzmann equation for vector fluctuations then is

$$\frac{\partial f}{\partial \eta} + \frac{q}{aE} n^i \partial_i f - \frac{1}{2} C'_{ij} n^i n^j q \frac{\partial f}{\partial q} = 0, \tag{4.176}$$

with $C_{ij} \equiv \partial_i \chi_j^\perp + \partial_j \chi_i^\perp$ a symmetric traceless but not divergence-less tensor. This expression is exactly the same Boltzmann equation for tensor perturbation upon the replacement $\chi'_{ij} \to C'_{ij}$, such that we can safely follow the line of all the computations done above for tensors. However, first of all, it is convenient to introduce a gauge invariant quantity to work with. The most general perturbed metric around a FLRW background contains two vector quantities. The gauge freedom allows to set one vector degree of freedom, such that there is one only independent gauge invariant vector quantity we can build out of metric perturbations; this is the vector contribution to the *extrinsic curvature*:

$$V_i = \omega_i^\perp - \chi_i^{\perp\prime}. \tag{4.177}$$

Hence, working in the synchronous gauge and defining

$$V_{ij} \equiv -\partial_i \chi_j^{\perp\prime} - \partial_j \chi_i^{\perp\prime}, \tag{4.178}$$

the Boltzmann equation is written in term of the extrinsic curvature as

$$\frac{\partial f}{\partial \eta} + \frac{q}{aE} n^i \partial_i f + \frac{1}{2} n^i n^j V_{ij}\, q \frac{\partial f}{\partial q} = 0. \tag{4.179}$$

The transition to Fourier space is performed through

$$\begin{aligned} V_{ij}(\vec{x}, \eta) &= -i \int \frac{d^3k}{(2\pi)^3} \left[k_i \chi_j^{\perp\prime}(\eta, k) + k_j \chi^{\perp\prime}(\eta, k) \right] e^{\mathbf{k}\cdot\mathbf{x}} \\ &= i \int \frac{d^3k}{(2\pi)^3} \tilde{V}_{ij}(k, \eta) e^{\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \tag{4.180}$$

The source function seeded by vector fluctuations then is

$$S^{(V)}(\eta, \vec{k}, q, n^i) = i \frac{1}{2} \tilde{V}_{ij} n^i n^j, \tag{4.181}$$

and the fluctuations in the energy density distribution correspondingly are

$$\Gamma_V(\eta, \vec{k}, q, \hat{n}) = i \frac{n^i n^j}{2} \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} \tilde{V}_{ij}(\eta', \vec{k}). \tag{4.182}$$

The spherical harmonics formalism can now be applied to expand the fluctuations in multipoles as:

$$\begin{aligned} \Gamma_{\ell m, V} &= \int d^2n Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \Gamma_V(\eta_0, \vec{k}, q, \hat{n}) \\ &= i \int d^2n Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta_0,\eta')} \frac{n^i n^j}{2} \tilde{V}_{ij}(\eta', \vec{k}). \end{aligned} \tag{4.183}$$

In order to make the formalism of vector compatible with the one used with tensor perturbations, let's follow these arguments. First of all it is useful to project the vector metric perturbations $\chi_i^\perp$ into helicity modes

$$\chi_i^\perp(\mathbf{k}, \eta) = \sum_{\lambda=R,L} \epsilon_{i,\lambda}(\hat{\mathbf{k}}) \chi^\perp(\mathbf{k}, \eta)_\lambda. \tag{4.184}$$

Assuming that a decomposition between the time evolution and the stochastic primordial value of the vector modes is allowed,

$$\chi_i^{\perp}(\mathbf{k},\eta) = \chi^{\perp}(k,\eta) \sum_{\lambda=R,L} \epsilon_{i,\lambda}(\hat{\mathbf{k}})\,\mathcal{V}_\lambda(\mathbf{k}), \tag{4.185}$$

$$V_i(\mathbf{k},\eta) = -\chi^{\perp\prime}(k,\eta) \sum_{\lambda=R,L} \epsilon_{i,\lambda}(\hat{\mathbf{k}})\,\mathcal{V}_\lambda(\mathbf{k}) = V(k,\eta) \sum_{\lambda=R,L} \epsilon_{i,\lambda}(\hat{\mathbf{k}})\,\mathcal{V}_\lambda(\mathbf{k}). \tag{4.186}$$

with $\mathcal{V}_\lambda$ a stochastic random field encoding the statistical primordial features of the extrinsic curvature. Considering modes propagating along the $\hat{z}$ direction, a possible basis for the polarization vectors is

$$\epsilon_{i,R}(\hat{\mathbf{k}}_z) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}, \qquad \epsilon_{i,L}(\hat{\mathbf{k}}_z) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}. \tag{4.187}$$

satisfying $k^i\epsilon_{i,R/L} = 0$. Therefore one can define a polarization tensor with two indices as

$$\tilde{V}_{ij} = k\,V(k,\eta)\left[\mathcal{O}_{ij}^R\mathcal{V}_R + \mathcal{O}_{ij}^L\mathcal{V}_L\right] = C(k,\eta)\left[\mathcal{O}_{ij}^R\mathcal{V}_R + \mathcal{O}_{ij}^L\mathcal{V}_L\right] \tag{4.188}$$

$$\mathcal{O}_{ij}^{(R/L)}(\hat{\mathbf{k}}_z) \equiv \hat{\mathbf{k}}_i\epsilon_j^{(R/L)}(\hat{\mathbf{k}}_z) + \hat{\mathbf{k}}_j\epsilon_i^{(R/L)}(\hat{\mathbf{k}}_z). \tag{4.189}$$

Substituting the explicit expression of the polarization vectors

$$\mathcal{O}_{ij}^{(R)}(\hat{\mathbf{k}}_z) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&i\\1&i&0\end{pmatrix}, \qquad \mathcal{O}_{ij}^{(L)}(\hat{\mathbf{k}}_z) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&-i\\1&-i&0\end{pmatrix}. \tag{4.190}$$

As done for tensors, it is convenient to introduce the $+/\times$ polarization basis as

$$\tilde{V}_{ij} = C(k,\eta)\left[\mathcal{O}_{ij}^R\mathcal{V}_R + \mathcal{O}_{ij}^L\mathcal{V}_L\right] = C(k,\eta)\left[\mathcal{O}_{ij}^{+}\frac{\mathcal{V}_R+\mathcal{V}_L}{\sqrt{2}} + \mathcal{O}_{ij}^{\times}\frac{\mathcal{V}_R-\mathcal{V}_L}{\sqrt{2}i}\right], \tag{4.191}$$

with

$$\begin{aligned}\mathcal{O}_{ij}^{+}(\hat{\mathbf{k}}_z) &= \frac{1}{2}\left(\mathcal{O}_{ij}^R + \mathcal{O}_{ij}^L\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\\ \mathcal{O}_{ij}^{\times}(\hat{\mathbf{k}}_z) &= \frac{1}{2i}\left(\mathcal{O}_{ij}^R - \mathcal{O}_{ij}^L\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}.\end{aligned} \tag{4.192}$$

Adopting the usual decomposition of the unit director vector (3.70)

$$\hat{n} = \left(\sqrt{1-\mu_{k,n}^2}\cos\phi_{k,n}, \sqrt{1-\mu_{k,n}^2}\sin\phi_{k,n}, \mu_{k,n}\right), \tag{4.193}$$

one can compute

$$\frac{n^in^j}{2}\tilde{V}_{ij} = \frac{\mu_{k,n}\sqrt{1-\mu_{k,n}^2}}{2}\left[\mathcal{V}_R e^{-i\phi_{k,n}} + \mathcal{V}_L e^{i\phi_{k,n}}\right]C(k,\eta). \tag{4.194}$$

At this point one can exploit one more time the relation between spherical harmonics and the Legendre polynomials (B.4) to compute

$$\begin{aligned}Y_{2,\lambda} &= -\sqrt{\frac{15}{8\pi}}\sin\phi_{n,k}\cos\phi_{n,k}e^{\lambda i\phi_{n,k}}\\ &= -\sqrt{\frac{15}{8\pi}}\mu_{k,n}\sqrt{1-\mu_{k,n}^2}e^{\lambda i\phi_{n,k}} \qquad (\lambda = \pm 1).\end{aligned} \tag{4.195}$$

Hence,

$$\frac{n^i n^j}{2}\tilde{C}_{ij} = -\sqrt{\frac{2\pi}{15}}\left[\mathcal{V}_R Y_{2,-1} + \mathcal{V}_L Y_{2,1}\right] C(k,\eta), \tag{4.196}$$

and the vector fluctuations (4.182) for a fixed momentum $\mathbf{k} \parallel \hat{\mathbf{z}}$, becomes

$$\Gamma_V(\eta,\vec{k},q,\hat{n}) = -i\sqrt{\frac{2\pi}{15}}\left[\mathcal{V}_R Y_{2,-1} + \mathcal{V}_L Y_{2,1}\right]\int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} C(k,\eta) \tag{4.197}$$

$$\Gamma_{\ell m,V} = \int d^2n Y^*_{\ell m}(\hat{n})\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\Gamma_V(\eta_0,\vec{k},q,\hat{n}). \tag{4.198}$$

In order to make the two expressions compatible, the trick to use is to rotate the integration variable $d^2\Omega_n$ in such a way to pass into the coordinate system where $\mathbf{k} \parallel \hat{\mathbf{z}}$. As shown in Section (3.4.3), this procedure transforms the spherical harmonics according to (3.76),

$$\Gamma_{\ell m,V} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\sum_{m'=-\ell}^{\ell} D^{(\ell)}_{mm'}(S(\Omega_k))\int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n})\Gamma_V(\eta_0,\vec{k},q,\Omega_{k,n}). \tag{4.199}$$

with the Wigner rotation matrix defined by

$$D^{(\ell)}_{ms}(S(\Omega_k)) \equiv \sqrt{\frac{4\pi}{2\ell+1}}(-1)^s\ {}_{-s}Y^*_{\ell m}(\Omega_k). \tag{4.200}$$

Now we are allowed to use the expression (4.197) inside the above integral, which leads to

$$\begin{aligned}\mathcal{J}_V &\equiv \int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n})\Gamma_V(\eta_0,\vec{k},q,\Omega_{k,n}) \\ &= -i\int d^2\Omega_{k,n} Y^*_{\ell m'}(\Omega_{k,n})\sqrt{\frac{2\pi}{15}}\left[\mathcal{V}_R Y_{2,-1} + \mathcal{V}_L Y_{2,1}\right]\int_{\eta_{in}}^{\eta} d\eta' e^{-ik\mu l(\eta,\eta')} C(k,\eta). \end{aligned}\tag{4.201}$$

Again the integration is performed exploiting the properties of the spherical harmonics [46]:

$$\int d\Omega_{k,n} Y^*_{\ell,m'} Y_{2\pm1} e^{-i\mu_{k,n}x} = (-i)^{\ell-1}\delta_{m',\pm1}\sqrt{\frac{3}{2}(2\ell+1)\frac{(\ell+1)!}{(\ell-1)!}}\,\frac{j_\ell(x)}{x}. \tag{4.202}$$

This way the innermost integral becomes:

$$\begin{aligned}\mathcal{J}_V = -(-i)^\ell \int_{\eta_{in}}^{\eta} d\eta' C(k,\eta')\sqrt{\frac{\pi}{5}(2\ell+1)\frac{(\ell+1)!}{(\ell-1)!}}\,\frac{j_\ell(kl(\eta_0,\eta'))}{kl(\eta_0,\eta')} \\ \times\left[\mathcal{V}_R(\mathbf{k})\delta_{m',-1} + \mathcal{V}_L(\mathbf{k})\delta_{m',1}\right]. \end{aligned}\tag{4.203}$$

Inserting this result in the above equation (4.199)

$$\begin{aligned}\Gamma_{\ell m,V} &= -(-i)^\ell\sqrt{\frac{\pi}{5}(2\ell+1)\frac{(\ell+1)!}{(\ell-1)!}}\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\int_{\eta_{in}}^{\eta} d\eta' C(k,\eta') \\ &\quad\times\frac{j_\ell(kl(\eta_0,\eta))}{kl(\eta_0,\eta)}\left[\mathcal{V}_R(\mathbf{k})D^{(\ell)}_{m,-1}(S(\Omega_k)) + \mathcal{V}_L(\mathbf{k})D^{(\ell)}_{m1}(S(\Omega_k))\right] \\ &= (-i)^\ell\sqrt{\frac{4\pi^2}{5}\frac{(\ell+1)!}{(\ell-1)!}}\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0}\int_{\eta_{in}}^{\eta} d\eta' C(k,\eta') \\ &\quad\times\frac{j_\ell(kl(\eta_0,\eta'))}{kl(\eta_0,\eta')}\left[{}_1Y^*_{\ell m}\mathcal{V}_R(\mathbf{k}) + {}_{-1}Y^*_{\ell m}\mathcal{V}_L(\mathbf{k})\right]. \end{aligned}\tag{4.204}$$

From this expression we can isolate an angular transfer function as

$$\Gamma_{\ell m,V} = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \mathcal{T}_\ell^V(k,\eta_0,\eta_{in},q) \times \left\{ {}_1Y^*_{\ell m}(\Omega_k)\mathcal{V}_R(\vec{k}) + {}_{-1}Y^*_{\ell m}(\Omega_k)\mathcal{V}_L(\vec{k}) \right\}, \tag{4.205}$$

with

$$\mathcal{T}_\ell^V(k,\eta,\eta_{in},q) = \sqrt{\frac{1}{20}}\sqrt{\frac{(\ell+1)!}{(\ell-1)!}} \int_{\eta_{in}}^{\eta} d\eta' C(k,\eta') \frac{j_\ell(kl(\eta_0,\eta'))}{kl(\eta_0,\eta)}. \tag{4.206}$$

Angular power spectrum In order to compute the two-point function of the vector sourced fluctuations we consider as usual

$$\langle \Gamma_{\ell m,V}\Gamma_{\ell' m',V}\rangle = \delta_{\ell,\ell'}\delta_{m,m'}\tilde{C}_{\ell,V} \tag{4.207}$$

$$\langle \mathcal{V}_\tau(\mathbf{k})\mathcal{V}^*_{\tau'}(\mathbf{k}')\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_\tau(k)(2\pi)^3\delta_{\tau\tau'}\delta\left(\mathbf{k}-\mathbf{k}'\right), \tag{4.208}$$

where for convenience we denoted the two polarization states as $R = -1$ and $L = 1$, and the primordial power spectrum $\mathcal{P}_\tau(k)$ was computed in (F.11). Analogously the two point correlator is

$$\langle \Gamma_{\ell m,V}\Gamma^*_{\ell' m',V}\rangle = \delta_{\ell,\ell'}\delta_{m,m'}\, 4\pi \int \frac{dk}{k}\left[\mathcal{P}_{-1}(k)+\mathcal{P}_1(k)\right]\left|\mathcal{T}_\ell^V(k,\eta_0,\eta_{in},q)\right|^2, \tag{4.209}$$

which means

$$\tilde{C}_{\ell,V} = 4\pi \int \frac{dk}{k}\left|\mathcal{T}_\ell^V(k,\eta_0,\eta_{in},q)\right|^2 \sum_{\tau=\pm1}\mathcal{P}_\tau(k), \tag{4.210}$$

with

$$\mathcal{T}_\ell^V(k,\eta,\eta_{in},q) = \sqrt{\frac{1}{20}}\sqrt{\frac{(\ell+1)!}{(\ell-1)!}} \int_{\eta_{in}}^{\eta} d\eta'\, k\, V(k,\eta') \frac{j_\ell(kl(\eta_0,\eta'))}{kl(\eta_0,\eta)}. \tag{4.211}$$

What we still don't know is the expression for the primordial power spectrum $\mathcal{P}_\tau(k)$ and the transfer function $C(k,\eta)$. The first one is usually guessed as a power law, as we have seen in the case of scalar and tensor perturbations. The transfer function instead must be studied starting from the Einstein equation for vector modes.

4.6.1 Einstein equations for vector modes

As commented in Section (2.3.2), none modification of General Relativity is expected in the vector sector from our minimal theory of massive gravity, since no extra vector modes are propagated. Interestingly this is also the case for other massive gravity theories, as bimetric theories [115]. Hence one can work with the usual Einstein equations of GR, which in terms of gauge invariant quantities, and in absence of an anisotropic stress, are [119]:

$$\partial_{(i}V'_{j)} + 2\mathcal{H}\partial_{(i}V_{j)} = 0, \tag{4.212}$$

$$\nabla^2 V_i - \frac{2a^2}{M_P^2}(p+\rho)\Omega_i = 0, \tag{4.213}$$

where Ω_i is a matter-related gauge invariant vector variable called "*vorticity*" and defined by

$$\Omega_i \equiv v_i^{(V)} - \omega_i^{\perp}, \tag{4.214}$$

and $v_i^{(V)}$ the vector perturbation to the fluid velocity. In momentum space the above equations are

$$V_i' + 2\mathcal{H}V_i = 0, \tag{4.215}$$

$$k^2 V_i = -\frac{2a^2}{M_P^2}(p+\rho)\Omega_i. \tag{4.216}$$

The first equation is straightforwardly solved by $V(k,\eta) \propto a^{-2}(\eta)$. Concerning the vorticity, due to the strong bounds on the graviton mass in late Universe posed by astrophysical GW detection, one can fairly think that through the whole history of propagation, gravitons after inflation have always been highly ultra-relativistic. For that species then, both pressure and radiation dilute due to the expansion as $\rho \propto p \propto a^{-4}(\eta)$, providing the vorticity to be a constant quantity $\Omega_i \propto -M_P^2 k^2$:

$$V(k,\eta) \propto \frac{1}{k^2 a^2(\eta)}, \qquad \Omega_i(k,\eta) \propto \text{constant}. \tag{4.217}$$

A case to study: Primordial Magnetic Field

Recent observations of galaxies and clusters at redshift $z \sim 0.7 - 2.0$ have shown the existence of magnetic fields with magnitude $\mathcal{O}(10^{-6})\,\mathrm{G}$. A possible scenario to realize this situation is an amplification mechanism of the magnetic field taking place through the evolution of the Universe prior to galaxy formation. The seeds for the magnetic field can be produced both during the inflationary epoch [116], and after cosmic phase transitions [117], and during recombination [118]. Before neutrino decoupling, the Universe is dominated by a fluid of ultrarelativistic particles. Along this epoch baryons are tightly coupled to the fluid and there is no possibility for this fluid to create any anisotropic stress. Hence, in this period, total anisotropic stress comes from only a primordial magnetic field (PMF thereafter). This field survives until neutrino decoupling, and, for this arch of time, it is a source of metric perturbations via the Einstein equation. Its effects can be inscribed in an additional contribution to the stress-energy tensor, and in particular an anisotropic contribution. Scalar and tensor modes on super-horizons scale acquire an additional logarithmic contribution depending on the time τ_B when the PMF turned on and the time τ_ν of neutrino decoupling [46]. In this section however we are mostly interested in the vector contributions.

Let's consider a stochastic PMF $B^i(\mathbf{k},\eta)$. As the Universe expands, magnetic field lines are simply conformally diluted due to flux conservation $B^i(\mathbf{k},\eta) = B^i(\mathbf{k})/a^2(\eta)$, and the energy momentum tensor is

$$\begin{aligned} T^0_{\ 0} &= -\frac{1}{8\pi a^4}B^2(\mathbf{x}) = -\rho_\gamma(\eta)\Delta_B^2(x^\mu), \\ T^0_{\ i} &= 0, \\ T^i_{\ j} &= \frac{1}{4\pi a^4}\left[\frac{B^2(\mathbf{x})}{2}\delta^i_j - B^i(\mathbf{x})B_j(\mathbf{x})\right] = \rho_\gamma(\eta)\left[\Delta_B(x^\mu)\delta^i_j + \Pi^i_{B\,j}(x^\mu)\right]. \end{aligned} \tag{4.218}$$

Now, following [119], we can take its vector contribution by applying the projector operators

$$P_{ij} \equiv \delta_{ij} - \hat{k}_i\hat{k}_j \tag{4.219}$$

on the magnetic stress-energy tensor

$$\Pi_{ij}^{(V)} = \left(P_i^n \hat{k}_j + P_j^n \hat{k}_i\right) \hat{k}^m T_{B,mn}. \tag{4.220}$$

Out of this expression we can build a rank-1 quantity via contracting with the unit vector $\hat{k}_i$

$$\Pi_i^{(V)} = \Pi_{ij}^{(V)} \hat{k}^j = P_i^n \hat{k}^m T_{B,mn}. \tag{4.221}$$

This way the Einstein equation (4.215) becomes

$$V_i' + 2\mathcal{H}V_i = -\frac{2}{M_P^2}\frac{a^2\Pi_i^{(V)}(\mathbf{k},\eta)}{k} = -\frac{2}{M_P^2}\frac{\Pi_i^{(V)}(\mathbf{k})}{a^2 k}, \tag{4.222}$$

where in the second step we used the fact that the stress energy tensor depends quadratically on the PMF, so that it has an explicit time dependence through the scale factor as $\Pi_i(\mathbf{k},\eta) = \Pi_i(\mathbf{k})/a^4(\eta)$. The complete solution then is

$$V_i(\mathbf{k},\eta) = -\frac{2}{M_P^2}\frac{\Pi_i^{(V)}(\mathbf{k})\eta}{a^2 k}, \tag{4.223}$$

while vorticity derives from the Poisson-like equation (4.216)

$$\Omega_i(\mathbf{k},\eta) \propto -\frac{M_P^2}{2\rho_{\gamma,0}} a^2 k^2\, V_i = \frac{1}{\rho_{\gamma,0}} k \Pi_i^{(V)}(\mathbf{k})\eta. \tag{4.224}$$

In radiation domination $a \propto \eta$, and then vector perturbations decay as $\sim a^{-1}$. Therefore the interesting effect provided by a magnetic field is to slow down the decay of vector modes (remember that without any anisotropic source $V_i \propto a^{-2}$). In the end, a MPF produce an enhancement on the angular power spectra which may be visible or not depending on the size of the PMF itself.

Chapter 5

Statistical analysis of non-Gaussianity in the SGWB

As we have shown in the previous chapter, the easiest and quite reasonable assumption for the primordial perturbations is to consider them as Gaussian distributed. As long as we consider linear perturbations, the Gaussian distribution will not change its shape. However deviations from Gaussianity can appear whenever we turn on higher order terms in the definition of the stochastic random fields, since non linear transformations spoil the trend of the normal distribution. The situation where non linearity is set by the primordial perturbation at initial time is usually referred as *primordial* non-Gaussianity [49, 50, 51]. Actually, since the Einstein field equations are non linear, the system can evolve toward a non Gaussian distribution even starting from a perfectly Gaussian initial condition. Usually one refers to this situation as *secondary* non-Gaussianity [15]. In the latest years many models have been proposed to give predictions about possible sources of non-Gaussianity [49, 89, 90]. Gaussianity is indeed a very restrictive prescription, as one parameter (the variance, since the mean value can be always set to zero as we discussed in (4.1)) encodes all the statistical features of the field, and all the three axioms of chapter (4) must be satisfied. On the other hand non-Gaussianity can be anything else, and this fact opens to a very wide landscape for many new theories. Moreover in many inflationary models, non-Gaussianity is not only possible, but it is also an unavoidable prescription. This is for example the case for the theory of space-diffeomorphism symmetry breaking during inflation [47] that we analyzed in chapter (2). Non-Gaussianity is then becoming more and more interesting during the last years, since it a crucial test to establish the validity of any models of generation of primordial perturbations. These arguments give us motivations to study the statistical features of the SGWB that would point out a deviation from Gaussianity. This whole analysis follows the line of [59], and extends those results accounting for a non vanishing late time graviton mass.

In section (5.1) we introduce the definition of bispectrum and decompose the 3-point function in the basis of spherical harmonics. Then we will review the properties of the bispectrum under rotations which will be eventually useful to compute the explicit expression of the correlators for the three contributions, the initial condition and the scalar and tensor sourced terms.

In order to simplify the picture, we will focus just on the case where no extra graviton degrees of freedom are produced during inflation. In (5.2) we introduce non-Gaussianity in the form of the local ansatz. Moreover we will consider only the scalar contribution, since the tensor one is expected to be subdominant by far. With the local ansatz we will then proceed on the explicit evaluation of the bispectra for the scalar sourced contribution and the initial condition one.

The last section (5.3) is instead dedicated to an interesting example of secondary non-Gaussianity. We will indeed show how the coupling between long and short wavelength modes can give rise to a non linear evolution which ultimately will produce a non vanishing 3-point function in the squeezed limit.

5.1 GW bispectra and transformation properties

As we will show soon, proceeding in the same line of section (4.2), the assumption of statistical isotropy of the correlation functions demands the 3-point correlator to assume the form

$$\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle = \langle B_{\ell_1\ell_2\ell_3}\rangle \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{5.1}$$

where the spatially averaged quantity $\langle B_{\ell_1\ell_2\ell_3}\rangle$ is called angular avaraged bispectrum (or more briefly just bisprectrum) in analogy with the angular power spectrum $\tilde{C}_\ell$ we studied in the previous chapter. The angular average is here inserted to reconcile with an experimental setting: observationally we have just one realization of our Universe contributing to the ensemle avarage, and then our measurements of the bispectrum are so noisy that we would like to average them somehow. Relying on the ergodic principle and on statistical isotropy, what we usually do in experiments is to avarage the spectrum over different orientations on the sky, that is over the index m_i. The matrix appearing in (5.1) instead denotes the Wigner 3-j symbol, whose properties and geometric interpretation are shown in Appendix (E); the angular momenta ℓ_1, ℓ_2, ℓ_3 represent the three sides of a triangle, and must therefore satisfy the triangle constraints (E.4). Moreover if we further require the correlation functions to be invariant under parity, from the transformation laws (E.5) and (E.6), it immediately derives that it must hold $\ell_1+\ell_2+\ell_3 = \text{even}$. An intuitive and very enlightening representation of the situation is again given in [79]. Imagine to consider two states with angular momenta (ℓ_1, m_1) and (ℓ_2, m_2), and to combine them to form a coupled state with angular momentum (ℓ_3, m_3). All together they form a triangle in the angular momentum space whose orientation is represented by m_1, m_2, m_3. When we apply a rotation to the system, the Wigner 3-j symbol transforms the m's directions in such a way to preserve the triangle conditions (E.4), that is preserving the triangle configuration. At the same time, the assumption of statistical isotropy demands the angular averaged bispectrum to provide the same amplitude regardless of its orientation. Therefore, in a sense, we can think at the averaged bispectrum as the area of the triangle, which does not change if we rotate the triangle, while the Wigner 3-j symbol describes its orientation in terms of the azimuthal angle dependence.

On the same line of reasoning of the 2-point case, the expression for the anisotropies (3.92) make clear that we can relate the 3-point correlators of the fluctuations $\Gamma_{\ell m}$ to the correlators of the four stochastic variables $\Gamma(\eta_{in}, \vec{k}, q)$, $\zeta(\vec{k})$ and $\xi_\lambda(\vec{k})$ which are the quantities encoding all the statistical properties of the primordial perturbations. Following the *in-in formalism* presented in section (4.1) one can easily verify that a non vanishing 3 point correlator can arise from a *three*-self interaction term in the interacting Hamiltonian describing the stochastic field dynamics. In analogy to the power spectrum, it is common to write the 3-point correlators as

$$\begin{aligned} \langle \Gamma(\eta_{in}, \vec{k}, q)\Gamma^*(\eta_{in}, \vec{k}', q)\Gamma^*(\eta_{in}, \vec{k}'', q)\rangle &= B_I(q, k, k', k'')(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}''), \\ \langle \zeta(\vec{k})\zeta(\vec{k}')\zeta(\vec{k}'')\rangle &= B_\zeta(k, k', k'')(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}''), \\ \langle \xi_\lambda(\vec{k})\xi_{\lambda'}(\vec{k}')\xi_{\lambda''}(\vec{k}'')\rangle &= B_\lambda(\vec{k}, \vec{k}', \vec{k}'')\delta_{\lambda\lambda'}\delta_{\lambda\lambda''}(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}''), \end{aligned} \tag{5.2}$$

where the delta functions enforce the invariance under translation. The quantities $B_I(q,k,k',k'')$, $B_\zeta(k,k',k'')$ and $B_\lambda(\vec{k},\vec{k}',\vec{k}'')$ are called bispectrum of the initial condition, scalar and tensor modes respectively. Notice that the tensor bispectrum maintains the dependence on the wavevectors' directions, since, as we will see later, it transforms in a non trivial way under rotation of the vectors $\vec{k}$.

As usual we have assumed for simplicity that the scalar and tensor modes are not cross correlated. Before entering into the details of the computation of the bispectra, we want to linger for a while on their properties under rotations which motivate the expressions (5.1) and (5.2).

5.1.1 Rotation of the GW direction of propagation

First of all we want to prove that (5.1) provides a rotational invariant expression for the 3-point function, as stated at the beginning of this section. More precisely in this context the invariance refers to rotations of the momentum director $\hat{n}$ which describes the direction of propagation of the GW; in other words the correlators are expected to be independent on the orientation of our observation. As usual we impose this invariance by the constraining the correlator with (4.33). Specializing to the 3-point case it reads

$$\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle = \sum_{m_1' m_2' m_3'} \langle \Gamma_{\ell_1 m_1'}\Gamma_{\ell_2 m_2'}\Gamma_{\ell_3 m_3'}\rangle D^{(\ell_1)}_{m_1' m_1} D^{(\ell_2)}_{m_2' m_2} D^{(\ell_3)}_{m_3' m_3}. \tag{5.3}$$

Using the relation (E.13) for the composition of two rotation operators, it becomes

$$\begin{aligned}\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle = &\sum_{m_1' m_2' m_3'} \langle \Gamma_{\ell_1 m_1'}\Gamma_{\ell_2 m_2'}\Gamma_{\ell_3 m_3'}\rangle \sum_{LMM'}(2L+1)\\ &\times \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1' & m_2' & M' \end{pmatrix}\begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} D^{(L)*}_{M'M} D^{(\ell_3)}_{m_3' m_3}.\end{aligned} \tag{5.4}$$

At this point we introduce (5.1) as an ansatz for the 3-point correlator

$$\begin{aligned}\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle = \tilde{b}_{\ell_1\ell_2\ell_3} &\sum_{m_1' m_2' m_3'} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1' & m_2' & m_3' \end{pmatrix} \sum_{LMM'}(2L+1)\\ &\times \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1' & m_2' & M' \end{pmatrix}\begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} D^{(L)*}_{M'M} D^{(\ell_3)}_{m_3' m_3}.\end{aligned} \tag{5.5}$$

and we want to verify that also the rotated correlator recovers the same expression. Exploiting the orthogonality condition (E.10)

$$\sum_{m_1 m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{\ell_3\ell_3'}\delta_{m_3 m_3'}}{2\ell_3+1}, \tag{5.6}$$

the equation simplifies to

$$\begin{aligned}\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle &= \tilde{b}_{\ell_1\ell_2\ell_3} \sum_{m_3' LMM'} \delta_{\ell_3 L}\delta_{m_3' M'} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} D^{(L)*}_{M'M} D^{(\ell_3)}_{m_3' m_3}\\ &= \tilde{b}_{\ell_1\ell_2\ell_3} \sum_{m_3' M} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & M \end{pmatrix} D^{(\ell_3)*}_{m_3' M} D^{(\ell_3)}_{m_3' m_3}.\end{aligned} \tag{5.7}$$

Finally, using the orthonormality condition of the rotation matrix

$$\sum_m D^{(\ell)*}_{m'm} D^{(\ell)}_{m''m} = \delta_{m'm''}, \tag{5.8}$$

we arrive to

$$\langle \Gamma_{\ell_1 m_1} \Gamma_{\ell_2 m_2} \Gamma_{\ell_3 m_3} \rangle = \langle B_{\ell_1 \ell_2 \ell_3} \rangle \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{5.9}$$

which is exactly (5.1). All in all we have demonstrated that the angular averaged bispectrum $\langle B_{\ell_1\ell_2\ell_3}\rangle$ is symmetric under rotation of the director $\hat{n}$, and that the expression (5.1) provides an orientational invariant definition for the 3-point correlation function. As we are interested in the bispecrum, it could be useful to invert the relation (5.1). Exploiting again the orthogonality condition (E.11), we can isolate the averaged bispectrum as

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \langle \Gamma_{\ell_1 m_1} \Gamma_{\ell_2 m_2} \Gamma_{\ell_3 m_3} \rangle = \langle B_{\ell_1 \ell_2 \ell_3} \rangle. \tag{5.10}$$

In practical computation anyway it could be useful to rewrite the Wigner symbols in a more suitable form. One possible simplification comes from (E.14), thanks to which

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = {} & \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}} \\ & \times \int d^2\hat{n}\, Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n}) Y_{\ell_3 m_3}(\hat{n}). \end{aligned} \tag{5.11}$$

Indeed, defining further the azimuthally avaraged harmonic transform

$$e_\ell(\hat{n}) \equiv \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m=-\ell}^{\ell} \Gamma_{\ell m} Y_{\ell m}(\hat{n}), \tag{5.12}$$

we can express the averaged bispectrum as

$$\langle B_{\ell_1 \ell_2 \ell_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \int \frac{d^2\hat{n}}{4\pi} \langle e_{\ell_1}(\hat{n}) e_{\ell_2}(\hat{n}) e_{\ell_3}(\hat{n}) \rangle. \tag{5.13}$$

This expression is computationally very efficient, since the quantities $e_\ell(\hat{n})$ can be easily evaluated with the spherical harmonic transform, while the integral simply represents an average over the full sky. Moreover, from the Clebsch-Gordan tables one can verify that the Wigner 3-j symbols appearing in (5.13) admits an analytic expression in the case $\ell_1 + \ell_2 + \ell_3 =$ even and $m_i = 0$ for $i = 1, 2, 3$:

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\sum_{i=1}^{3} - \frac{\ell_i}{2}} & \frac{\sqrt{(-\ell_1+\ell_2+\ell_3)!\,(\ell_1-\ell_2+\ell_3)!\,(\ell_1+\ell_2-\ell_3)!}}{\left(\frac{-\ell_1+\ell_2+\ell_3}{2}\right)!\left(\frac{\ell_1-\ell_2+\ell_3}{2}\right)!\left(\frac{\ell_1+\ell_2-\ell_3}{2}\right)!} \times \\ & \times \frac{\left(\sum_{i=1}^{3} \frac{\ell_i}{2}\right)!}{\sqrt{\left(\sum_{i=1}^{3} \ell_i + 1\right)!}}. \end{aligned} \tag{5.14}$$

For convenience it is common to define the Gaunt integral as

$$\begin{aligned} \mathcal{G}^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} & \equiv \int d^2\hat{n}\, Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n}) Y_{\ell_3 m_3}(\hat{n}) \\ & = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{5.15}$$

in terms of which the 3-point correlation function is written

$$\begin{aligned}\langle \Gamma_{\ell_1 m_1}\Gamma_{\ell_2 m_2}\Gamma_{\ell_3 m_3}\rangle &= \langle B_{\ell_1\ell_2\ell_3}\rangle \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ &= \langle B_{\ell_1\ell_2\ell_3}\rangle \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}}\mathcal{G}^{m_1 m_2 m_3}_{\ell_1\ell_2\ell_3}. \end{aligned} \tag{5.16}$$

The Gaunt integral is by construction invariant under both even and odd permutation and different from zero when ℓ_{tot} = even, as one can easily verify from the properties of the Wigner 3-j symbols in (E). Moreover it encodes the fundamental geometric properties of the averaged bispectrum, as it is non vanishing only when the triangle conditions (E.4) is satisfied.

5.1.2 Rotation of the GW wave vector

The isotropy requirement of the correlation functions implies the angular averaged bispectrum to be invariant under rotations of the versor $\hat{n}$, but it does not tell anything about the transformation properties of the bispectrum under rotation of the wavevectors $\vec{k}$. Notice indeed that the tensor sourced 3-point correlator (5.2) preserves a vectorial dependence on the vector $\vec{k}$, since, as we will prove in this section, it does not transform as a scalar under rotations [94].

The case where the bispectrum is built on scalar stochastic variables is trivial. Indeed these quantities, by definition, do not change under rotation of the wavevector, and this property is of course preserved by the two and three point correlation function. Therefore we can focus just on the tensor mode correlator. Let us recall the tensor decomposition (3.89):

$$\chi_{ij}(\eta,\vec{k}) = \chi(\eta,k)\sum_{\lambda} e_{ij,\lambda}(\hat{k})\xi_\lambda(\vec{k}), \tag{5.17}$$

with $e_{ij,\lambda}$ the polarization tensors in a generic basis. Usually one defines the polarization basis by introducing an auxiliary unit vector $\hat{e}_z$ oriented along the GW direction of propagation, which is often taken to coincide with the z-axis of a cartesian system, and then completing the 3-dimensional basis by adding two orthogonal vectors. Our goal is to find the transformation properties of the stochastic variable $\xi_\lambda(\vec{k})$, knowing that $\chi_{ij,\lambda}(\eta,\vec{x})$ is a rank 2 tensor quantity with definite transformation properties. For this purpose it is convenient to focus on the polarization tensors' features firstly. In [94] it was shown that, under a rotation R of the wave vector $\vec{k}$, the polarization tensors do not transform simply as a rank 2 tensor, but they further acquire an additional phase due to the fact that the fixed vector $\hat{e}_z$ does not change under the rotation of the wave vector. More precisely they found

$$e_{ij,\lambda}(R\hat{k}) = e^{-2i\lambda\gamma[\hat{k},R]}R_{ik}R_{jl}e_{kl,\lambda}(\hat{k}), \tag{5.18}$$

where $\gamma[\hat{k},R]$ is the rotation angle transforming the orthogonal vector basis. Defining for convenience the metric

$$\Pi_{ij}(\hat{k}) \equiv \delta_{ij} - \hat{k}_i\hat{k}_j, \tag{5.19}$$

and denoting the by $(\vec{a}\cdot\vec{b})_\Pi \equiv a_i b_j \Pi_{ij}$ the product of two tensorial quantities with the metric Π_{ij}, the angle $\gamma[\hat{k},R]$ is defined by the relations

$$\cos\gamma[\hat{k},R] \equiv \frac{(\hat{e}_z\cdot\hat{e}'_z)_\Pi}{\sqrt{(\hat{e}_z\cdot\hat{e}_z)_\Pi(\hat{e}'_z\cdot\hat{e}'_z)_\Pi}} \qquad \sin\gamma[\hat{k},R] \equiv \frac{\hat{k}\cdot(\hat{e}_z\times\hat{e}'_z)}{\sqrt{(\hat{e}_z\cdot\hat{e}_z)_\Pi(\hat{e}'_z\cdot\hat{e}'_z)_\Pi}}. \tag{5.20}$$

At the end of the day, the important lesson is that, in order to make χ_{ij} transform as a rank 2 tensor, in virtue of (5.17) and (5.18), one has to impose that the stochastic modes transform as

$$\xi_\lambda(R\vec{k}) = e^{2i\lambda\gamma[\hat{k},R]}\xi_\lambda(\vec{k}) \tag{5.21}$$

to compensate the additional phase introduced by the polarization tensors. From (5.20) one can see that $\gamma[-\hat{k}, R] = -\gamma[\hat{k}, R]$. As a consequence the power spectrum for the tensor mode is rotationally invariant. Indeed, denoting the rotated momenta as $\vec{k}'_i = R\vec{k}_i$, the rotation acts on the 2-point correlator as

$$\begin{aligned}
\langle\xi_{\lambda_1}(\vec{k}'_1)\xi_{\lambda_2}(\vec{k}'_2)\rangle &= e^{2i\sum_{i=1}^{2}\lambda_i\gamma[\hat{k}_i,R]}\langle\xi_{\lambda_1}(\vec{k}_1)\xi_{\lambda_2}(\vec{k}_2)\rangle \\
&= (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)\delta_{\lambda_1\lambda_2}P_{\lambda_1}(k_1)e^{2i\lambda_1(\gamma[\hat{k}_1,R]+\gamma[\hat{k}_2,R]})\\
&= (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)\delta_{\lambda_1\lambda_2}P_{\lambda_1}(k_1).
\end{aligned} \tag{5.22}$$

In the last step the exponential phase vanishes thanks to the delta function, which demands $\vec{k}_1 = -\vec{k}_2$. In order recover the expected expression we must rotate the vectors in the Dirac delta. For any matrix $A \in \mathbb{R}^{n\times n}$, it holds

$$\delta(A\vec{x}) = \frac{1}{|\det(A)|}\delta(\vec{x}), \qquad \vec{x}\in\mathbb{R}^n. \tag{5.23}$$

Since rotation matrices are orthogonal matrices with unitary determinant, it follows immediately that $\delta(R\vec{x}) = \delta(\vec{x})$. This way the last line of (5.22) becomes

$$\langle\xi_{\lambda_1}(\vec{k}'_1)\xi_{\lambda_2}(\vec{k}'_2)\rangle = (2\pi)^3\delta(\vec{k}'_1+\vec{k}'_2)\delta_{\lambda_1\lambda_2}P_{\lambda_1}(k_1). \tag{5.24}$$

Hence this expression explicitly demonstrates that the power spectrum for the tensor stochastic modes is rotationally invariant, and this justifies the fact that it depends only on the modulus of the momenta, as we assumed in (4.36). On the other hand, this trick is not working for the bispectrum case. The fact that we have three momenta forbids the huge simplification carried by the delta function. Indeed it holds

$$\begin{aligned}
\langle\xi_{\lambda_1}(\vec{k}'_1)\xi_{\lambda_2}(\vec{k}'_2)\xi_{\lambda_3}(\vec{k}'_3)\rangle &= e^{2i\sum_{i=1}^{3}\lambda_i\gamma[\hat{k}_i,R]}\langle\xi_{\lambda_1}(\vec{k}_1)\xi_{\lambda_2}(\vec{k}_2)\xi_{\lambda_3}(\vec{k}_3)\rangle \\
&= e^{2i\sum_{i=1}^{3}\lambda_i\gamma[\hat{k}_i,R]}(2\pi)^3\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\delta_{\lambda_1\lambda_2}\delta_{\lambda_1\lambda_3}B_{\lambda_1}(\vec{k}_1,\vec{k}_2,\vec{k}_3)\\
&= e^{2i\sum_{i=1}^{3}\lambda_i\gamma[\hat{k}_i,R]}(2\pi)^3\delta(\vec{k}'_1+\vec{k}'_2+\vec{k}'_3)\delta_{\lambda_1\lambda_2}\delta_{\lambda_1\lambda_3}B_{\lambda_1}(\vec{k}_1,\vec{k}_2,\vec{k}_3),
\end{aligned} \tag{5.25}$$

where in the last step we exploit again the invariance of the delta function under rotation. Now, as before, the above equation can be rewritten as a relation between the bispectra. However this time we have no arguments to make the exponential phase vanish, hence the relation reads

$$B_{\lambda_1}(\vec{k}'_1,\vec{k}'_2,\vec{k}'_3) = e^{2i\sum_{i=1}^{3}\lambda_i\gamma[\hat{k}_i,R]}B_{\lambda_1}(\vec{k}_1,\vec{k}_2,\vec{k}_3). \tag{5.26}$$

Therefore we have obtained the important result that the tensor modes' bispectrum transforms in a non trivial way under rotation. As a manifestation of this features, the tensor bispectrum conserves the vectorial dependence on the momenta.

After this technical digression, we have now all the necessary tools to attempt the computation of the 3-point correlators, considering each contribution one at a time.

5.1.3 Initial condition term

The method we pursue to compute the angular bispectrum is the same we used for the angular power spectrum. Combining the solution of the multipole expansion (3.92) with the definition of the bispectrum (5.2), the 3-point correlator of the anisotropy is written as

$$\begin{aligned}\langle \Gamma_{\ell m,I}\Gamma_{\ell' m',I}\Gamma_{\ell'' m'',I}\rangle = (4\pi)^3(-i)^{\ell+\ell'+\ell''}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3k''}{(2\pi)^3}\\ B_I(q,k,k',k'')(2\pi)^3\delta(\vec{k}+\vec{k}'+\vec{k}'')Y^*_{\ell m}(\hat{k})Y^*_{\ell' m'}(\hat{k}')Y^*_{\ell'' m''}(\hat{k}'')\\ j_\ell\left[kl(\eta_0,\eta_{in})\right]j_{\ell'}\left[k'l(\eta_0,\eta_{in})\right]j_{\ell''}\left[k''l(\eta_0,\eta_{in})\right].\end{aligned} \tag{5.27}$$

In the 2-point case we exploited the properties of the delta function to reduce the number of integration variables such that only one was remaining. In the present case this procedure fails, because the delta function involves three momenta. It turns out to be rather more convenient the following trick. We use the integral representation for the delta function and expand the complex phase with (3.55):

$$\begin{aligned}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3) &= \int\frac{d^3y}{(2\pi)^3}e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot\vec{y}}\\ &= \int_0^\infty dy\,y^2\int d\Omega_y\prod_{i=1}^{3}\left[2\sum_{L_iM_i}i^{L_i}j_{L_i}(k_iy)Y^*_{L_iM_i}(\Omega_y)Y_{L_iM_i}(\hat{k}_i)\right].\end{aligned} \tag{5.28}$$

Then (5.27) becomes

$$\begin{aligned}\langle \Gamma_{\ell m,I}\Gamma_{\ell' m',I}\Gamma_{\ell'' m'',I}\rangle = (4\pi)^3(-i)^{\ell+\ell'+\ell''}\int\frac{k^2dk}{(2\pi)^3}\int\frac{k'^2dk'}{(2\pi)^3}\int\frac{k''^2dk''}{(2\pi)^3}\\ B_I(q,k,k',k'')(2\pi)^3 j_\ell\left[kl(\eta_0,\eta_{in})\right]j_{\ell'}\left[k'l(\eta_0,\eta_{in})\right]j_{\ell''}\left[k''l(\eta_0,\eta_{in})\right]\\ \int d\Omega_k\int d\Omega_{k'}\int d\Omega_{k''}Y^*_{\ell m}(\hat{k})Y^*_{\ell' m'}(\hat{k}')Y^*_{\ell'' m''}(\hat{k}'')\\ \int_0^\infty dy\,y^2 d\Omega_y\,8\sum_{LML'M'L''M''}i^{L+L'+L''}j_L(ky)Y^*_{LM}(\Omega_y)Y_{LM}(\hat{k})\\ j_{L'}(k'y)Y^*_{L'M'}(\Omega_y)Y_{L'M'}(\hat{k}')j_{L''}(k''y)Y^*_{L''M''}(\Omega_y)Y_{L''M''}(\hat{k}'').\end{aligned} \tag{5.29}$$

Since the spherical harmonics $Y_{L_iM_i}(\hat{k}_i)$ in the last line does not depend on the vector $\vec{y}$, we can move them inside the integrals in $d\Omega_{k_i}$ and exploit the normalization condition(B.7) to get

$$\begin{aligned}\int d\Omega_k d\Omega_{k'}d\Omega_{k''}Y^*_{\ell m}(\hat{k})Y^*_{\ell' m'}(\hat{k}')Y^*_{\ell'' m''}(\hat{k}'')Y_{LM}(\hat{k})Y_{L'M'}(\hat{k}')Y_{L''M''}(\hat{k}'')\\ = \delta_{\ell L}\delta_{mM}\delta_{\ell' L'}\delta_{m'M'}\delta_{\ell'' L''}\delta_{m''M''}.\end{aligned} \tag{5.30}$$

This result brings a huge simplification inside the integral (5.29), which indeed becomes

$$\begin{aligned}\langle \Gamma_{\ell m,I}\Gamma_{\ell' m',I}\Gamma_{\ell'' m'',I}\rangle = (8\pi)^3(-i)^{\ell+\ell'+\ell''}\int\frac{k^2dk}{(2\pi)^3}\int\frac{k'^2dk'}{(2\pi)^3}\int\frac{k''^2dk''}{(2\pi)^3}\\ B_I(q,k,k',k'')(2\pi)^3 j_\ell\left[kl(\eta_0,\eta_{in})\right]j_{\ell'}\left[k'l(\eta_0,\eta_{in})\right]j_{\ell''}\left[k''l(\eta_0,\eta_{in})\right]\\ \int_0^\infty dy\,y^2 j_\ell(ky)j_{\ell'}(k'y)j_{\ell''}(k''y)\int d\Omega_y\,Y^*_{\ell m}(\Omega_y)Y^*_{\ell' m'}(\Omega_y)Y^*_{\ell'' m''}(\Omega_y).\end{aligned} \tag{5.31}$$

In the last line we recognize the definition (5.15) of the Gaunt integral. Finally, simplifying the multiplicative factors, we arrive to

$$\begin{aligned}\langle \Gamma_{\ell m,I}\Gamma_{\ell' m',I}\Gamma_{\ell'' m'',I}\rangle = \mathcal{G}^{mm'm''}_{\ell\ell'\ell''}&\int_0^\infty dr\, r^2 B_I(q,k,k',k'')\\ &\frac{2}{\pi}\int dk\, k^2 j_\ell\left[kl(\eta_0,\eta_{in})\right] j_\ell(kr)\frac{2}{\pi}\int dk'\, k'^2 j_\ell\left[k'l(\eta_0,\eta_{in})\right] j_\ell(k'r)\\ &\frac{2}{\pi}\int dk''\, k''^2 j_\ell\left[k''l(\eta_0,\eta_{in})\right] j_\ell(k''r),\end{aligned} \tag{5.32}$$

which can be recast more compactly as

$$\begin{aligned}\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,I}\rangle = \mathcal{G}^{mm'm''}_{\ell\ell'\ell''}\int_0^\infty dr\, r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_i k_i^2 j_{\ell_i}\left[k_i l(\eta_0,\eta_{in})\right] j_{\ell_i}(k_i r)\right]&\\ B_I(q,k,k',k'').&\end{aligned} \tag{5.33}$$

5.1.4 Scalar sourced term

Since the fluctuation related to the scalar sourced term share the same functional structure of the initial condition one in the multipole basis, (3.92), we can readily extend the results of the previous section in the same way as we have done in the 2-point case (4.2). Therefore, by substituting the bispectra for the scalar modes and inserting the scalar linear transfer function (3.93), the 3-point correlator is evaluated as

$$\begin{aligned}\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,S}\rangle = \mathcal{G}^{mm'm''}_{\ell\ell'\ell''}\int_0^\infty dr\, r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_i k_i^2 \mathcal{T}^S_{\ell_i}(\eta_0,\eta_{in},k_i,q) j_{\ell_i}(k_i r)\right]&\\ B_\zeta(k,k',k'').&\end{aligned} \tag{5.34}$$

5.1.5 Tensor sourced term

The last case to study is the tensor sourced one. This time the computation is a bit more involved, since, as already seen, the tensorial bispectrum is not invariant under rotations, that is it depends on the directions of the wavevectors $\vec{k}$. This additional dependence spoils the huge simplification coming from the solid angle integration of the spherical harmonics. Therefore we cannot consider the above solution for the initial condition case to be valid also for the tensor sourced one. Let's then start as usual by the definition of the tensor sourced fluctuation (3.92) and evaluate the 3-point correlator as

$$\begin{aligned}\left\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,T}\right\rangle = \sum_{\lambda=\pm 2}(4\pi)^3(-i)^{\ell_1+\ell_2+\ell_3}\prod_{i=1}^{3}\left[\int\frac{k_i^2 dk_i}{(2\pi)^3}\mathcal{T}^T_{\ell_i}(\eta_0,\eta_{in},k,q)\right]&\\ \left\langle\prod_{i=1}^{3}\int d\Omega_{k_i}\xi_\lambda(\vec{k}_i)\,{}_{-\lambda}Y^*_{\ell_i m_i}(\Omega_{k_i})\right\rangle.&\end{aligned} \tag{5.35}$$

At this point we see that we cannot proceed by performing the integrals on the solid angle. However [96] shows a clever trick to evaluate the ensemble average in the second line. Since the stochastic tensor modes ξ_λ describe a quantum field (the graviton field) with spin $\lambda = 2$, we can project them on the basis of spin-weighted spherical harmonics (B.25) of spin $-\lambda$ and write

$$\xi_\lambda(\vec{k}) = \sum_{\ell m}\xi^{(\lambda)}_{\ell m}(k)\,{}_{-\lambda}Y_{\ell m}(\hat{k}). \tag{5.36}$$

This way, thanks to the normalization condition of the spin-weighted spherical harmonics, the integral over the solid angle evaluates to

$$\int d\Omega_{k_i}\xi_\lambda(\vec{k}_i)\,{}_{-\lambda}Y^*_{\ell_i m_i}(\Omega_{k_i}) = \sum_{\ell m}\xi^{(\lambda)}_{\ell m}\int d\Omega_{k_i}\,{}_{-\lambda}Y_{\ell m}(\hat{k})_{-\lambda}Y^*_{\ell_i m_i}(\Omega_{k_i}) = \xi^{(\lambda)}_{\ell_i m_i}. \quad (5.37)$$

Therefore the second line of (5.35) is nothing but the 3-point correlator of the coefficients $\xi^{(\lambda)}_{\ell_i m_i}$, that is

$$\Big\langle\prod_{i=1}^{3}\int d\Omega_{k_i}\xi_\lambda(\vec{k}_i)\,{}_{-\lambda}Y^*_{\ell_i m_i}(\Omega_{k_i})\Big\rangle = \Big\langle\prod_{i=1}^{3}\xi^{(\lambda)}_{\ell_i m_i}(k_i)\Big\rangle. \quad (5.38)$$

Under the assumption of statistical isotropy, as we have seen in section (5.1), the 3-point correlators are orientation independent and can be always written in the form (5.1). Adopting the notation introduced by [97] we define

$$\Big\langle\prod_{i=1}^{3}\xi^{(\lambda)}_{\ell_i m_i}(k_i)\Big\rangle \equiv (2\pi)^3\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}. \quad (5.39)$$

Exploiting the orthogonality condition (E.11) of the Wigner 3-j symbols we can invert this relation to

$$\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) = \sum_{m_1m_2m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\frac{1}{(2\pi)^3}\Big\langle\prod_{i=1}^{3}\xi^{(\lambda)}_{\ell_i m_i}(k_i)\Big\rangle \quad (5.40)$$

$$= \sum_{m_1m_2m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\frac{1}{(2\pi)^3}\prod_{i=1}^{3}\int d\Omega_{k_i}\,{}_{-\lambda}Y^*_{\ell_i m_i}\Big\langle\prod_{i=1}^{3}\xi_\lambda(\vec{k}_i)\Big\rangle. \quad (5.41)$$

Remembering the definition (5.1) for the bispectrum, the expression (5.35) for the 3-point correlator immediately allows an expression for the tensor sourced averaged bispectrum as

$$\Big\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,T}\Big\rangle = \langle B_{\ell_1\ell_2\ell_3,T}\rangle\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}, \quad (5.42)$$

with

$$\langle B_{\ell_1\ell_2\ell_3}\rangle = \left[\prod_{i=1}^{3}(-i)^{\ell_i}\int\frac{k_i^2dk_i}{(2\pi)^2}\mathcal{T}^T_{\ell_i}(\eta_0,\eta_{in},k,q)\right](4\pi)^3\sum_{\lambda=\pm2}\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3). \quad (5.43)$$

In order to obtain an expression with the same functional structure of (5.33) and (5.34), let us work out a bit the Wigner 3-j symbols. From the definition (5.15) of the Gaunt integral it derives

$$\Big\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,T}\Big\rangle = \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-1}\sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}}\langle B_{\ell_1\ell_2\ell_3}\rangle$$

$$= \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-1}\sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}}(2\pi)^3$$

$$\left[\prod_{i=1}^{3}4\pi(-i)^{\ell_i}\int\frac{k_i^2dk_i}{(2\pi)^2}\mathcal{T}^T_{\ell_i}(\eta_0,\eta_{in},k,q)\right]\sum_{\lambda=\pm2}\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3). \quad (5.44)$$

Finally, defining

$$\tilde{\mathcal{F}}^{\lambda}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) \equiv \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}} \\ (2\pi)^3 \mathcal{F}^{\lambda}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3), \tag{5.45}$$

we recover the desired form for the 3-point correlation function

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i,T} \right\rangle = \mathcal{G}^{m_1m_2m_3} \left[\prod_{i=1}^{3} 4\pi(-i)^{\ell_i} \int \frac{k_i^2 dk_i}{(2\pi)^3} \mathcal{T}^{T}_{\ell_i}(\eta_0,\eta_{in},k,q) \right] \\ \sum_{\lambda=\pm 2} \tilde{\mathcal{F}}^{\lambda}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3). \tag{5.46}$$

Let us conclude this section by showing another additional ways to express the quantity $\tilde{\mathcal{F}}^{\lambda}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)$ which may be useful in the subsequent computations. By inserting the definition (5.40), it reads

$$\tilde{\mathcal{F}}^{\lambda}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sqrt{\frac{4\pi}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}} \\ \sum_{m_1m_2m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left[\prod_{i=1}^{3} \int d\Omega_{k_i}\, {}_{-\lambda}Y^{*}_{\ell_i m_i} \right] \left\langle \prod_{i=1}^{3} \xi_\lambda(\vec{k_i}) \right\rangle. \tag{5.47}$$

Rearranging some multiplicative factor

$$\tilde{\mathcal{F}}^{\lambda}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) = \sqrt{4\pi} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sum_{m_1m_2m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ \left[\prod_{i=1}^{3} \int d\Omega_{k_i} \frac{{}_{-\lambda}Y^{*}_{\ell_i m_i}}{\sqrt{2\ell_i+1}} \right] \left\langle \prod_{i=1}^{3} \xi_\lambda(\vec{k_i}) \right\rangle. \tag{5.48}$$

5.1.6 Summary of the three contribution

At this point we have all the tools to attempt an explicit evaluation of the three point functions and, more generally, of any non-Gaussian contribution. Before proceeding in the computation it may be useful to tidy up the important results we derived in this section and that we will use in the following sections. Expanding the anisotropies in multipoles, with the assumption of statistical isotropy, and assuming further that there is no statistical correlation between scalar and tensor modes, the only non vanishing 3-point function in the spherical harmonics basis are

$$\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,I}\rangle = \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\int_0^\infty dr\, r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_ik_i^2 j_{\ell_i}\left[k''l(\eta_0,\eta_{in})\right]j_{\ell_i}(k_ir)\right] B_I(q,k,k',k''),$$

$$\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,S}\rangle = \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\int_0^\infty dr\, r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_ik_i^2 \mathcal{T}^S_{\ell_i}(\eta_0,\eta_{in},k_i,q)j_{\ell_i}(k_ir)\right] B_\zeta(k,k',k''),$$

$$\Big\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,T}\Big\rangle = \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\left[\prod_{i=1}^{3}4\pi(-i)^{\ell_i}\int\frac{k_i^2dk_i}{(2\pi)^3}\mathcal{T}^T_{\ell_i}(\eta_0,\eta_{in},k,q)\right]\sum_{\lambda=\pm2}\tilde{\mathcal{F}}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3), \tag{5.49}$$

with the definition

$$\tilde{\mathcal{F}}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) = \sqrt{4\pi}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-1}\sum_{m_1m_2m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\left[\prod_{i=1}^{3}\int d\Omega_{k_i}\frac{{}_{-\lambda}Y^*_{\ell_im_i}}{\sqrt{2\ell_i+1}}\right]\Big\langle\prod_{i=1}^{3}\xi_\lambda(\vec{k}_i)\Big\rangle. \tag{5.50}$$

Again we note that all the three contributions depend on the graviton momentum q. However only the initial condition contribution maintains this dependence in the massless limit, since this dependence is intrinsic in the definition of the stochastic variable $\Gamma(\eta_{in},\vec{k},q)$; hence this dependence is reflected in the angular bispectrum. On the contrary in the other two cases the q dependence arises inside the linear transfer functions from the velocity correction terms, which tend to one in the limit $m\to 0$. For the ease of notation, as presented by [101], sometimes it could be convenient to define the reduced bispectrum $\tilde{b}_{\ell_1\ell_2\ell_3}$ as

$$\Big\langle\prod_{i=1}^{3}\Gamma_{\ell_im_i}\Big\rangle \equiv \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\tilde{b}_{\ell_1\ell_2\ell_3}. \tag{5.51}$$

By comparison with the expressions (5.49), we can identify three different contributions

$$\begin{aligned}
\tilde{b}_{\ell_1\ell_2\ell_3,I} &= \int_0^\infty dr\, r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_ik_i^2 j_{\ell_i}\left[k_il(\eta_0,\eta_{in})\right]j_{\ell_i}(k_ir)\right]B_I(q,k_1,k_2,k_3),\\
\tilde{b}_{\ell_1\ell_2\ell_3,S} &= \int_0^\infty dr\, r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_ik_i^2 \mathcal{T}^S_{\ell_i}(\eta_0,\eta_{in},k_i,q)j_{\ell_i}(k_ir)\right]B_\zeta(k_1,k_2,k_3),\\
\tilde{b}_{\ell_1\ell_2\ell_3,T} &= \left[\prod_{i=1}^{3}4\pi(-i)^{\ell_i}\int\frac{k_i^2dk_i}{(2\pi)^2}\mathcal{T}^T_{\ell_i}(\eta_0,\eta_{in},k,q)\right]\sum_{\lambda=\pm2}\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3),\\
&= \frac{4}{\pi^2}\sum_{\lambda=\pm2}\sum_{m_i}\left[\prod_{i=1}^{3}\frac{(-i)^{\ell_i}}{2\ell_i+1}\int dk_ik_i^2\mathcal{T}^T_{\ell_i}(k_i)\,{}_{-\lambda}Y^*_{\ell_im_i}(\Omega_{k_i})\right]\\
&\quad\mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}^{-2}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_\lambda(\vec{k}_1,\vec{k}_2,\vec{k}_3),
\end{aligned} \tag{5.52}$$

where in the last expression we have made explicit the factor $\mathcal{F}^\lambda_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)$ through (5.50) combined with the definition (5.15) of the Gaunt integral.

5.2 Explicit evaluation of the scalar sourced bispectrum for local non-Gaussianity

In order to give an estimate of the SGWB angular bisprectrum we focus just on the initial condition and the scalar sourced contributions, since we expect to find a hierarchy between the three contributions similar to the one found for the angular spectrum. In particular in this section we will start from the latter, since it is the more complete and instructive case, and then we will trivially extend the results to the former. If we restrict the analysis to the case where no graviton extra modes are produced, we do not expect to see any deviation arising from the early graviton mass term. The theory (2.9) is indeed untouched by the additional mass term in the scalar sector. Hence we are allowed to proceed in the stardand wat, forgetting about the heavy graviton mass during inflation. Among the many models of non-Gaussian perturbations, the most simple one is the so called local model. Motivations for this model rely on the dynamics of the inflationary mechanism [102]. The production mechanism of perturbations provided by the inflation naturally selects the large scale modes as the most important contributions. On such scales spatial gradients are expected to become more and more negligible with respect to the Hubble rate expansion. Neglecting spatial gradients means to apply a coarse graining procedure on sufficiently large patches of the Universe (large at least as the size of the Hubble parameter), and to treat the evolution "locally" on the different patches, which then evolve as effectively separated Universes. This picture lead to the study of local models where the primordial metric perturbations are described by local functions, that is functions defined on a single point of the spacetime.

5.2.1 Parametrization of local non-Gaussianity

As stated at the beginning of this chapter, a linear function of Gaussian fields is Gaussian itself; hence source of non Gaussianity would arises from a non linear perturbation. The first and dominant correction is then expected to arise from a quadratic term, which is conventionally introduced as

$$\Phi(\vec{x}) = \Phi_g(\vec{x}) + f_{NL}\left[\Phi_g^2(\vec{x}) - \langle\Phi_g^2\rangle\right]. \tag{5.53}$$

In this expression $\Phi_g(\vec{x})$ denotes the gaussian contribution to the scalar metric perturbation, while the factor f_{NL} is a constant space-independent parameter which describes the amount of deviation from a Gaussian distribution. The ensemble averaged quantity $\langle\Phi_g^2\rangle$ is required to ensure the scalar perturbations to be averaged to zero, as expected.

During the radiation dominated era, on large scale it holds $\Phi \simeq (2/3)\zeta$; one can then parametrize the non-Gaussianity in terms of the gauge invariant curvature perturbation as

$$\zeta(\vec{x}) = \zeta_g(\vec{x}) + \frac{2}{3}f_{NL}\left[\zeta_g^2(\vec{x}) - \langle\zeta_g^2\rangle\right]. \tag{5.54}$$

By definition the ensemble averaged term is the Fourier antitransform of the power spectrum. Indeed

$$\langle\zeta_g^2(\vec{x})\rangle = \int \frac{d^3p\,d^3q}{(2\pi)^6} e^{i(\vec{p}+\vec{q})\cdot\vec{x}}\langle\zeta_g(\vec{p})\zeta_g(\vec{q})\rangle = \int \frac{d^3p}{(2\pi)^3}P_\zeta(p). \tag{5.55}$$

Therefore, in momentum space

$$\begin{aligned}\zeta(\vec{k}) &= \int d^3x e^{-i\vec{k}\cdot\vec{x}} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \zeta_g(\vec{p}) \\ &+ \frac{2}{3} f_{NL} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left[\int \frac{d^3p d^3q}{(2\pi)^6} e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \zeta_g(\vec{p})\zeta_g(\vec{q}) - \int \frac{d^3p}{(2\pi)^3} P_\zeta(p) \right] \\ &= \zeta_g(\vec{k}) + \frac{2}{3} f_{NL} \int \frac{d^3p}{(2\pi)^3} \left[\zeta_g(\vec{p})\zeta_g(\vec{k}-\vec{p}) - (2\pi)^3 \delta(\vec{k}) P_\zeta(p) \right], \end{aligned} \tag{5.56}$$

such that $\langle \zeta(\vec{k}) \rangle = 0$ is obviously preserved. The first statistical quantity which may receives contributions from the non linearity of the scalar perturbation is the 2-point function, which becomes

$$\begin{aligned}\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2) \rangle &= \langle \zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2) \rangle \\ &+ \frac{2}{3} f_{NL} \int \frac{d^3p}{(2\pi)^3} \langle \zeta_g(\vec{k}_1)\zeta_g(\vec{p})\zeta_g(\vec{k}_2-\vec{p}) \rangle + \vec{k}_1 \leftrightarrow \vec{k}_2 \\ &- \frac{2}{3} f_{NL} \int d^3p \langle \zeta(\vec{k}_1) \rangle \delta(\vec{k}_2) P_\zeta(p) + \vec{k}_1 \leftrightarrow \vec{k}_2. \end{aligned} \tag{5.57}$$

At this point one can appreciate the simplification coming from the local Gaussianity assumption. Indeed the 2-point correlator is completely determined by the 2 and 3-point correlator of a gaussian random variable. Therefore we can use the result of section (4.1) to perform the present computation, and in particular the axioms (4.11), (4.12) and (4.13). Since all the *odd*-point correlator functions of gaussian random fields are null, the only non trivial term is the first one, which is nothing but the 2-point correlator of a gaussian random variable:

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2) \rangle = \langle \zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2) \rangle = (2\pi)^3 \delta(\vec{k}_1+\vec{k}_2) P_\zeta(\vec{k}_1). \tag{5.58}$$

This is a quite remarkable result, since it shows that the 2-point function is not affected, in the context of local models, by the primordial non-Gaussianity of the stochastic perturbation fields. Therefore the first channel capable of detecting a deviation from Gaussianity is the 3-point correlator. Let's then compute

$$\begin{aligned}\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle &= \langle \zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2)\zeta_g(\vec{k}_3) \rangle \\ &+ \frac{2}{3} f_{NL} \int \frac{d^3p}{(2\pi)^3} \langle \zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2)\zeta_g(\vec{p})\zeta_g(\vec{k}_3-\vec{p}) \rangle + 2\,\text{perms.} \\ &- \frac{2}{3} f_{NL} \int d^3p\, \langle \zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2) \rangle \delta(\vec{k}_3) P_\zeta(p) - 2\,\text{perms.}, \end{aligned} \tag{5.59}$$

where the abbreviation "perms" stands for permutations of the three momenta $\vec{k}_i$. Again we are in the nice position where all the quantities contributing to the 3-point correlators are correlation functions of gaussian fields, and then they can be evaluated exploiting the axioms (4.11), (4.12) and (4.13). The first line trivially vanishes since it involves an odd number of random fields. The 4-point correlator inside the integral instead can be written in term of the 2-point correlator functions by (4.13), which we report here for the sake of clarity

$$\begin{aligned}\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\zeta(\vec{k}_4) \rangle &= \langle \zeta(\vec{k}_1)\zeta(\vec{k}_2) \rangle \langle \zeta(\vec{k}_3)\zeta(\vec{k}_4) \rangle + \langle \zeta(\vec{k}_1)\zeta(\vec{k}_3) \rangle \langle \zeta(\vec{k}_2)\zeta(\vec{k}_4) \rangle \\ &+ \langle \zeta(\vec{k}_1)\zeta(\vec{k}_4) \rangle \langle \zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle. \end{aligned} \tag{5.60}$$

This way (5.59) becomes

$$\begin{aligned}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle &= \frac{2}{3}f_{NL}\int\frac{d^3p}{(2\pi)^3}\Big[\langle\zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2)\rangle\langle\zeta_g(\vec{p})\zeta_g(\vec{k}_3-\vec{p})\rangle\\&+\langle\zeta_g(\vec{k}_1)\zeta_g(\vec{p})\rangle\langle\zeta_g(\vec{k}_2)\zeta_g(\vec{k}_3-\vec{p})\rangle\\&+\langle\zeta_g(\vec{k}_1)\zeta_g(\vec{k}_3-\vec{p})\rangle\langle\zeta_g(\vec{k}_2)\zeta_g(\vec{p})\rangle\Big]+2\,\text{perms.}\\&-\frac{2}{3}f_{NL}\int d^3p\,\langle\zeta_g(\vec{k}_1)\zeta_g(\vec{k}_2)\rangle\delta(\vec{k}_3)P_\zeta(p)-2\,\text{perms.}\,,\end{aligned}\tag{5.61}$$

From the definition of power spectrum

$$\langle\zeta_g(\vec{p})\zeta_g(\vec{k}_3-\vec{p})\rangle=(2\pi)^3\delta(\vec{k}_3)P_\zeta(p).\tag{5.62}$$

the first and the last lines simplify, leaving

$$\begin{aligned}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle &= \frac{2}{3}f_{NL}(2\pi)^3P(k_1)P(k_2)\int d^3p\Big[\delta(\vec{k}_1+\vec{p})\delta(\vec{k}_2+\vec{k}_3-\vec{p})\\&+\delta(\vec{k}_1+\vec{k}_3-\vec{p})\delta(\vec{k}_2+\vec{p})\Big]+2\,\text{perms.}\\&=\frac{4}{3}f_{NL}(2\pi)^3P(k_1)P(k_2)\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)+2\,\text{perms.}\,,\end{aligned}\tag{5.63}$$

In terms of the dimensionless power spectrum (4.19), making explicit each term, we arrive to

$$\begin{aligned}\langle\prod_{i=1}^{3}\zeta(\vec{k}_i)\rangle &= \frac{4}{3}f_{NL}4\pi^4(2\pi)^3\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\\&\left[\frac{\mathcal{P}_\zeta(k_1)\mathcal{P}_\zeta(k_2)}{k_1^3k_2^3}+\frac{\mathcal{P}_\zeta(k_1)\mathcal{P}_\zeta(k_3)}{k_1^3k_3^3}+\frac{\mathcal{P}_\zeta(k_2)\mathcal{P}_\zeta(k_3)}{k_2^3k_3^3}\right].\end{aligned}\tag{5.64}$$

Finally, by comparing this expression with the definition (5.2), we obtain the expression for the scalar sourced bisprectrum we were looking for:

$$B_\zeta(k_1,k_2,k_3)=\frac{4}{3}f_{NL}\left[\frac{2\pi^2}{k_1^3}\mathcal{P}_\zeta(k_1)\frac{2\pi^2}{k_2^3}\mathcal{P}_\zeta(k_2)+2\,\text{perms.}\,,\right]\tag{5.65}$$

5.2.2 Scalar sourced reduced bispectrum and 3-point correlator

This last result (5.65) can now be used to evaluate the reduced bispectrum (5.52) as

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} &= \int_0^\infty dr\,r^2\prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_ik_i^2\mathcal{T}^S_{\ell_i}(\eta_0,\eta_{in},k_i,q)j_{\ell_i}(k_ir)\right]\times\\&\times\frac{4}{3}f_{NL}\left[\frac{2\pi^2}{k_1^3}\mathcal{P}_\zeta(k_1)\frac{2\pi^2}{k_2^3}\mathcal{P}_\zeta(k_2)+2\,\text{perms.}\right],\end{aligned}\tag{5.66}$$

where scalar sourced angular transfer function (4.131) is

$$\begin{aligned}\mathcal{T}^S_\ell(\eta_0,\eta_{in},k,q) &= v^{-2}\frac{2}{3}g(k,\eta_{in})j_\ell\left[kl(\eta_0,\eta_{in})\right]\\&+\frac{2}{3}\int_{\eta_{in}}^{\eta_0}d\eta\frac{\partial\left[(1+v^{-2})g(\eta)\right]}{\partial\eta}j_\ell\left[kl(\eta_0,\eta)\right],\end{aligned}\tag{5.67}$$

with $g(k,\eta_{in}) = g_{\rm rad}(k,\eta_{in})$ as initial condition. In analogy with the results for the CMB case, one is lead to expect a really tiny signal from a three-point angular correlator. For this reason one can just focus on the dominant Sachs-Wolfe contribution, leading to

$$\tilde{b}_{\ell_1\ell_2\ell_3,S} = \int_0^\infty dr\, r^2 \prod_{i=1}^{3} \left[\frac{2}{\pi}\int dk_i k_i^2 v^{-2} \frac{2}{3} g_{\rm rad}(k_i,\eta_{in}) j_{\ell_i}\left[k_i l(\eta_0,\eta_{in})\right] j_{\ell_i}(k_i r)\right] \times$$
$$\times \frac{4}{3} f_{NL}\left[\frac{2\pi^2}{k_1^3}\mathcal{P}_\zeta(k_1)\frac{2\pi^2}{k_2^3}\mathcal{P}_\zeta(k_2) + 2\,\text{perms.}\right]. \quad (5.68)$$

For the sake of simplicity, on the following the case of an Harrison-Zel'dovich spectrum will be considered. The final result will eventually be generalized to a general power law scale dependence.

Harrison-Zel'dovich spectrum In this case any dependence of the dimensionless power sprectrum from the momenta is neglected. Given our low resolution in experiments with interferometers, one can fairly restrict the study to the case of low multipoles, that is large scales, where the growing rate can be fairly approximated to $g_{\rm rad}(k,\eta_{in}) \simeq 1$. Then

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{4}{3} f_{NL}\left(\frac{4}{3\pi}\right)^3 (2\pi^2)^2 \mathcal{P}_\zeta^2 v^{-6} \int_0^\infty dr\, r^2 \int \frac{dk_1}{k_1} j_{\ell_1}\left[k_1 l(\eta_0,\eta_{in})\right] j_{\ell_1}(k_1 r) \\ &\int \frac{dk_2}{k_2} j_{\ell_2}\left[k_2 l(\eta_0,\eta_{in})\right] j_{\ell_2}(k_2 r) \int_{|k_1-k_2|}^{k_1+k_2} dk_3\, k_3^2 j_{\ell_3}\left[k_3 l(\eta_0,\eta_{in})\right] j_{\ell_3}(k_3 r) \\ &+2\,\text{permutations}. \end{aligned} \quad (5.69)$$

The assumption of a k-independent power spectrum implicitly selects those cosmological scales which where already outside the horizon at initial time (otherwise, beneath the horizon, the power spectrum could not pursue a constant trend). This forces $k_i \ll \eta_{in}^{-1}$ as upper bound on the wavenumber, and then the first two integral should run up to $k_i^{\max} \simeq 1/\eta_{in}$. However, for sufficiently initial times, it is fair to set $\eta_{in} \simeq 0$, sending then the upper bound to infinity. This approximation is quite reasonable since the first two integrals are dominated in the regime of small momenta thanks to both the power law factor k_i^{-1} and the spherical Bessel functions. The extremes of the last integral instead are set to ensure that the triangular inequality $|k_i - k_j| \leq k_k \leq k_i + k_j$ for $i,j,k = 1,2,3$ is always verified. Let's now rewrite these integrals by performing two consecutive changes of variables, firstly $x_i \equiv k_i l(\eta_0,\eta_{in})$

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{1024\pi}{81} f_{NL}\mathcal{P}_\zeta^2 v^{-6}\int_0^\infty dr\, r^2 \int \frac{dx_1}{x_1} j_{\ell_1}(x_1)\, j_{\ell_1}\left(x_1 \frac{r}{l(\eta_0,\eta_{in})}\right) \\ &\int \frac{dx_2}{x_2} j_{\ell_2}(x_2)\, j_{\ell_2}\left(x_2\frac{r}{l(\eta_0,\eta_{in})}\right) \\ &\frac{1}{l^3(\eta_0,\eta_{in})}\int_{|x_1-x_2|}^{x_1+x_2} dx_3\, x_3^2 j_{\ell_3}(x_3)\, j_{\ell_3}\left(x_3\frac{r}{l(\eta_0,\eta_{in})}\right) + 2\,\text{perms.}, \end{aligned} \quad (5.70)$$

and then $y \equiv r/l(\eta_0,\eta_{in})$:

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{1024\pi}{81} f_{NL}\mathcal{P}_\zeta^2 v^{-6}\int_0^\infty dy\, y^2 \int \frac{dx_1}{x_1} j_{\ell_1}(x_1)\, j_{\ell_1}(x_1 y) \\ &\int \frac{dx_2}{x_2} j_{\ell_2}(x_2)\, j_{\ell_2}(x_2 y)\int_{|x_1-x_2|}^{x_1+x_2} dx_3\, x_3^2 j_{\ell_3}(x_3)\, j_{\ell_3}(x_3 y) + 2\,\text{perms.}. \end{aligned} \quad (5.71)$$

It is useful to take advantage of the relation between the spherical Bessel functions $j_\ell(x)$ and the ordinary Bessel functions $J_\ell(x)$

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x) \tag{5.72}$$

to write

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{128\pi^4}{81} f_{NL}\mathcal{P}_\zeta^2 v^{-6} \int_0^\infty dy\,\sqrt{y} \int \frac{dx_1}{x_1^2} J_{\ell_1+\frac{1}{2}}(x_1)\, J_{\ell_1+\frac{1}{2}}(x_1 y)\\ &\int \frac{dx_2}{x_2^2} J_{\ell_2+\frac{1}{2}}(x_2)\, J_{\ell_2+\frac{1}{2}}(x_2 y)\\ &\int_{|x_1-x_2|}^{x_1+x_2} dx_3\, x_3 J_{\ell_3+\frac{1}{2}}(x_3)\, J_{\ell_3+\frac{1}{2}}(x_3 y) + 2\,\text{perms.}.\end{aligned} \tag{5.73}$$

These integrals cannot be solved analytically yet; some more reasonable approximations are needed. Thinking about the geometrical interpretation of the three point function, the bispectrum can be linked to the surface of the triangle built from the momenta $\vec{k}_i$. Therefore it is reasonable to assume that the most contribution to the bisprectrum should come from the geometrical configuration that maximizes the area of the triangle. Fixed the basis k_3, the geometrical configuration that maximizes the area is realized by the isosceles triangle[1], that is when $k_1 \approx k_2$, or equally $x_1 \approx x_2$. This fact suggest to set to zero the lower integration extreme of the innermost integral. Moreover, as the gravitons stream freely without efficient collisional processes, the integrals should be dominated by $k \approx (\theta l(\eta_0, \eta_{in}))^{-1} \approx \ell / l(\eta_0, \eta_{in})$, where θ denotes the angular separation between the events under consideration. In terms of the dimensionless variable x, this implies that the dominant contribution should come from $x_i \approx \ell_i$. All in all the most dominant contribution to the reduced bespectrum is approximated to

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} \simeq &\frac{128\pi}{81} f_{NL}\mathcal{P}_\zeta^2 v^{-6} \int_0^\infty dy\,\sqrt{y} \int \frac{dx_1}{x_1^2} J_{\ell_1+\frac{1}{2}}(x_1)\, J_{\ell_1+\frac{1}{2}}(x_1 y)\\ &\int \frac{dx_2}{x_2^2} J_{\ell_2+\frac{1}{2}}(x_2)\, J_{\ell_2+\frac{1}{2}}(x_2 y)\\ &\int_0^{\ell_1+\ell_2} dx_3\, x_3 J_{\ell_3+\frac{1}{2}}(x_3)\, J_{\ell_3+\frac{1}{2}}(x_3 y) + 2\,\text{perms.}.\end{aligned} \tag{5.74}$$

In this way the three integrals have becomed independent, and then they can be computed separately one at a time. In [103] it is shown how to perform the integration in dx_1 and dx_2. In general they proved that

$$\begin{aligned}\int_0^\infty dx\, x^{-s} J_\mu(ax) J_\nu(bx) = 2^{-s} b^\nu a^{s-\nu-1} &\frac{\Gamma\left(\frac{\mu+\nu-s+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+s+1}{2}\right)}\\ &{}_2F_1\left[\frac{\nu-\mu-s+1}{2}, \frac{\nu+\mu-s+1}{2}; \nu+1; \frac{b^2}{a^2}\right],\end{aligned} \tag{5.75}$$

[1]The condition comes from the particular setting we can choose to adopt for our integration. Consider the innermost integral in dx_3. Imagine to take a circle whose diameter coincides with the fixed basis x_3. Out of this basis imagine to build a triangle whose sides are represented by x_1 and x_2. This way we are constructing a triangle inscribed in a circle. From a basic geometrical discussion, one can prove that the isosceles triangle is really the configuration that maximizes the triangle area. Then eventually the integration is performed by running the basis x_3 to infinity, and then, at the same time, expanding the circle containing the triangle.

when the conditions

$$\mathrm{Re}(\mu+\nu-s) > -1, \qquad 0 < b < a. \tag{5.76}$$

are satisfied. The term ${}_2F_1[a,b;c;d]$ indicates the Gauss's hypergeometric function (see [104] for further details about generalized hypergeometric functions). In our case, remembering that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$$\tilde{\mathcal{I}}_\ell(y<1) \equiv \int_0^\infty \frac{dx}{x^2} J_{\ell+\frac{1}{2}}(x) J_{\ell+\frac{1}{2}}(yx) \;=\; \frac{y^{\ell+\frac{1}{2}}}{2\sqrt{\pi}} \frac{\Gamma(\ell)}{\Gamma(\ell+\frac{3}{2})}\, {}_2F_1\left[-\frac{1}{2}, \ell; \ell+\frac{3}{2}; y^2\right] \tag{5.77}$$

for

$$\ell > 0, \qquad y < 1. \tag{5.78}$$

In the opposite case when $y > 1$ the result is easily obtained by exploiting the symmetry of the integrand under the exchanging of the order of the two Bessel functions, since they are the same. Performing the change of variable $z = yx$, the integral $\mathcal{I}(y)$ becomes

$$\tilde{\mathcal{I}}(y>1) = y \int_0^\infty \frac{dz}{z^2} J_{\ell+\frac{1}{2}}\left(\frac{1}{y} z\right) J_{\ell+\frac{1}{2}}(z). \tag{5.79}$$

This way one immediately sees that the integral in dz exactly reproduce the one of the previous case; then the same solution (5.77) applies under the substitution $y \to 1/y$, that is

$$\tilde{\mathcal{I}}_\ell(y>1) = y\, \frac{y^{-\ell-\frac{1}{2}}}{2\sqrt{\pi}} \frac{\Gamma(\ell)}{\Gamma(\ell+\frac{3}{2})}\, {}_2F_1\left[-\frac{1}{2}, \ell; \ell+\frac{3}{2}; \frac{1}{y^2}\right]. \tag{5.80}$$

The last case, when $y = 1$, can be better solved by transforming back the Bessel functions into the spherical ones with (5.72):

$$\mathcal{I}_\ell(y=1) = \frac{2}{\pi} \int \frac{dx}{x} j_\ell^2(x) = 2^{-2} \frac{\Gamma(\ell)}{\Gamma(\ell+2)} \frac{\Gamma(2)}{\Gamma^2\left(\frac{3}{2}\right)} = \frac{1}{\pi\ell(\ell+1)}, \tag{5.81}$$

where in the second equality we have used (C.8). Summarizing the results

$$\tilde{\mathcal{I}}_\ell(y) = \begin{cases} \dfrac{y^{\ell+\frac{1}{2}}}{2\sqrt{\pi}} \dfrac{\Gamma(\ell)}{\Gamma(\ell+\frac{3}{2})}\, {}_2F_1\left[-\dfrac{1}{2}, \ell; \ell+\dfrac{3}{2}; y^2\right], & y < 1, \\ \dfrac{1}{\pi\ell(\ell+1)}, & y = 1, \\ \dfrac{y^{-\ell+\frac{1}{2}}}{2\sqrt{\pi}} \dfrac{\Gamma(\ell)}{\Gamma(\ell+\frac{3}{2})}\, {}_2F_1\left[-\dfrac{1}{2}, \ell; \ell+\dfrac{3}{2}; \dfrac{1}{y^2}\right], & y > 1. \end{cases} \tag{5.82}$$

Figure (5.1) shows the behavior of these solutions, which will be useful later on. Concerning the dx_3 integral in (5.74), it can be solved analytically to

$$\begin{aligned} \tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3}(y) &\equiv \int_0^{\ell_1+\ell_2} dx_3 x_3 J_{\ell_3+\frac{1}{2}}(x_3) J_{\ell_3+\frac{1}{2}}(yx_3) \\ &= \frac{\ell_1+\ell_2}{1-y^2} \Big\{ y J_{\ell_3+\frac{1}{2}}(\ell_1+\ell_2) J_{\ell_3-\frac{1}{2}}[(\ell_1+\ell_2)y] \\ &\quad - J_{\ell_3-\frac{1}{2}}(\ell_1+\ell_2) J_{\ell_3+\frac{1}{2}}[(\ell_1+\ell_2)y] \Big\}. \end{aligned} \tag{5.83}$$

At the end of the day, summing up the last results, the reduced bispectrum becomes

$$\tilde{b}_{\ell_1\ell_2\ell_3} = \frac{128\pi}{81} f_{NL} \mathcal{P}_\zeta^2 v^{-6} \int_0^\infty dy \sqrt{y}\, \tilde{\mathcal{I}}_{\ell_1}(y) \tilde{\mathcal{I}}_{\ell_2}(y) \tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3}(y) + 2\,\text{perms.}. \tag{5.84}$$

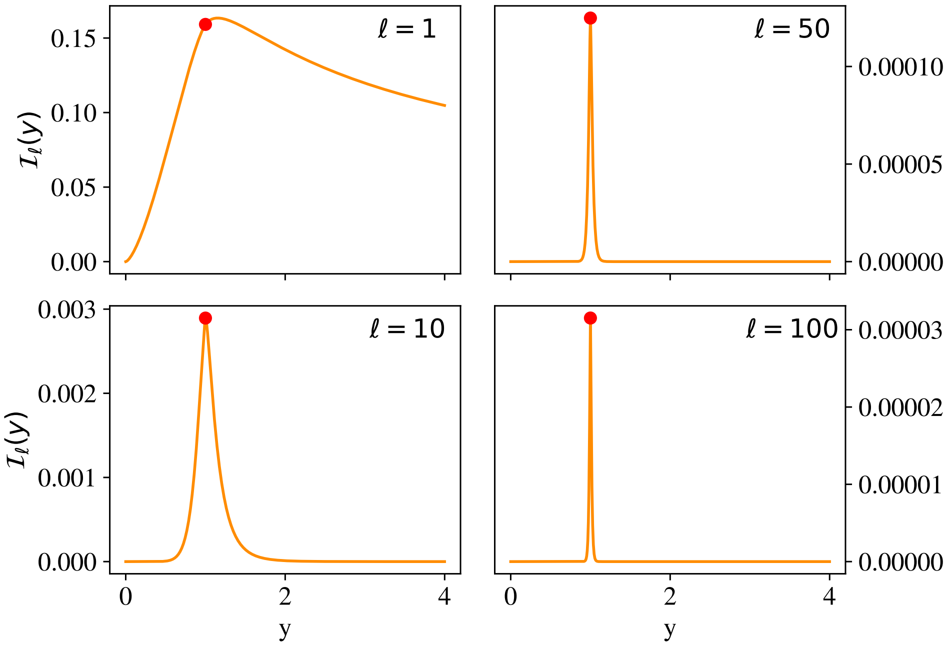

Figure 5.1: *Plot of the integral $\tilde{\mathcal{I}}(y)$ for different values of the multipole ℓ. The red dot represents the solution* (5.82) *for $y = 1$, while the orange line on the left and right show respectively the same solution for $y < 1$ and $y > 1$. The three solution perfectly match at the discontinuity point $y = 1$, as it should be in order to preserve the continuity of the function $\tilde{\mathcal{I}}(y)$. Notice that this function has a maximum around $y = 1$, which becomes more and more peaked as the multipole ℓ increases*

For the same geometrical reasons outlined above, the most dominant configuration for the bispectrum is expected to be the equilateral one, i.e. when $\ell_1 = \ell_2 = \ell_3 = \ell$. In this case the three terms with permuted momenta give the same contribution, and they can then be summed to

$$\tilde{b}_{\ell\ell\ell} = \frac{128\pi^4}{27} f_{NL} \mathcal{P}_\zeta^2 v^{-6} \int_0^\infty dy \sqrt{y}\, \tilde{\mathcal{I}}_\ell(y) \tilde{\mathcal{I}}_\ell(y) \tilde{\mathcal{I}}_{\ell\ell\ell}(y). \tag{5.85}$$

This integral can now be computed numerically to give an estimation of the equilateral reduced bispectrum, and, ultimately, of the equilateral bispectrum as well. Anyway we can roughly understand the trend of the integral in a simple way by looking for its dominant contribution. Remembering the definitions (5.77) and (5.83), one can convince himself that the integrand of (5.85) is mostly dominated by $y = 1$ since, when this condition is not verified, the Bessel functions appearing inside $\mathcal{I}_\ell$ and $\mathcal{I}_{\ell\ell\ell}$ are out of phase and they interfere in a destructive way. On the contrary, when $y = 1$ the peaks and the troughs of both the Bessel functions involved in each term occur at the same position giving rise to a constructive interference. A more convincing proof in support of these arguments can be found in Figure (5.2). For $y = 1$ a simple expression for $\mathcal{I}_\ell$ holds, given by the second line of (5.82), and then one expects the integral to be proportional to $\propto \ell^{-2}(\ell+1)^{-2}$; hence

$$\tilde{b}_{\ell\ell\ell} \approx \frac{128\pi^2}{27} f_{NL} \mathcal{P}_\zeta^2 v^{-6} \frac{\tilde{\mathcal{I}}_{\ell\ell\ell}(1)}{\ell^2(\ell+1)^2}. \tag{5.86}$$

Recalling the the result (4.139) for the Sachs Wolfe contribution to the scalar source spectrum

$$\frac{\ell(\ell+1)}{2\pi} \tilde{C}_{\ell,S}^{SW} = \frac{4}{9} v^{-4} P_\zeta, \tag{5.87}$$

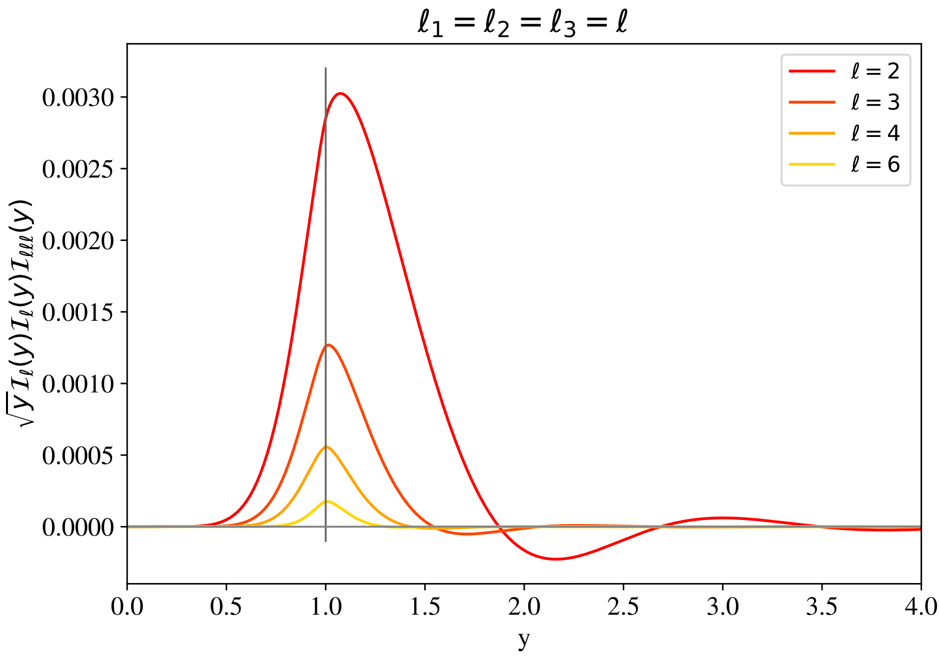

Figure 5.2: *Integrand function of equation* (5.85). *The purpose of this plot is to show the validity of the approximation in considering the function as strongly peaked around $y = 1$. This approximation increases its precision with larger values of the multipole ℓ*

the 3-point correlation function finally is

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell m_i,S} \right\rangle^{(SW)} \approx \mathcal{G}^{m_1 m_2 m_3}_{\ell\ell\ell} 6 f_{NL} v^2 \tilde{\mathcal{I}}_{\ell\ell\ell}(1) \tilde{C}^{(SW)\,2}_{\ell,S}, \tag{5.88}$$

where $\tilde{\mathcal{I}}_{\ell\ell\ell}(1)$ can be evaluated numerically from (5.83).

One can then quite easily extend this result to a general configuration simply considering that in general $\mathcal{I}_{\ell_i}(1) = \frac{1}{\pi \ell_i(\ell_i+1)}$ and retracing the arguments outlined above. This way, from (5.84),

$$\begin{aligned} \tilde{b}_{\ell_1\ell_2\ell_3} &\approx \frac{128\pi^2}{81} f_{NL} \mathcal{P}_\zeta^2 v^{-6} \frac{1}{\ell_1(\ell_1+1)} \frac{1}{\ell_2(\ell_2+1)} \tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3} + 2\,\text{perms} \\ &\approx 2 f_{NL} v^2 \tilde{C}^{SW}_{\ell_1} \tilde{C}^{SW}_{\ell_2} \tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3}(1) + 2\,\text{perms}.\,. \end{aligned} \tag{5.89}$$

Hence

$$\begin{aligned} \left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i,S} \right\rangle \approx \mathcal{G}^{m_1 m_2 m_3}_{\ell_1\ell_2\ell_3} 2 f_{NL} v^2 \Big[& \tilde{C}^{(SW)}_{\ell_1,S} \tilde{C}^{(SW)}_{\ell_2,S} \tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3}(1) \\ & + \tilde{C}^{(SW)}_{\ell_2,S} \tilde{C}^{(SW)}_{\ell_3,S} \tilde{\mathcal{I}}_{\ell_2\ell_3\ell_1}(1) + \tilde{C}^{(SW)}_{\ell_3,S} \tilde{C}^{(SW)}_{\ell_1,S} \tilde{\mathcal{I}}_{\ell_3\ell_1\ell_2}(1) \Big]. \end{aligned} \tag{5.90}$$

The terms $\tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3}(1)$ are oscillating factors which suppress some geometrical configurations of the triangle $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$, and they come from the imposition of the triangular constraint on the innermost integral of (5.69). In the next paragraph it will be shown how it is possible, in a rough approximation, to get rid of these terms.

Power-law spectrum In this paragraph the scale-dependence of the power spectrum is taken as a power law in reference to a pivot scale k_0

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k_0)\left(\frac{k}{k_0}\right)^{n_s-1} . \tag{5.91}$$

As long as the scalar spectral index does not deviate much from the the value $n_s = 1$, we can trace back the previous computations applying the same approximations. Therefore (5.69) immediately generalizes to

$$\begin{aligned}
\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{4}{3}f_{NL}\left(\frac{4}{3\pi}\right)^3 (2\pi^2)^2\mathcal{P}(k_0)^2_\zeta v^{-6}\int_0^\infty dr\, r^2 \\
&\int \frac{dk_1}{k_1}\left(\frac{k_1}{k_0}\right)^{n_s-1} j_{\ell_1}\left[k_1 l(\eta_0,\eta_{in})\right] j_{\ell_1}(k_1 r) \\
&\int \frac{dk_2}{k_2}\left(\frac{k_2}{k_0}\right)^{n_s-1} j_{\ell_2}\left[k_2 l(\eta_0,\eta_{in})\right] j_{\ell_2}(k_2 r) \\
&\int_{|k_1-k_2|}^{k_1+k_2} dk_3\, k_3^2 j_{\ell_3}\left[k_3 l(\eta_0,\eta_{in})\right] j_{\ell_3}(k_3 r) \quad + 2\,\text{permutations.}
\end{aligned} \tag{5.92}$$

As before, it is convenient to perform two changes of coordinate, $x_i = k_i l(\eta_0, \eta_{in})$ and $y = r/l(\eta_0, \eta_{in})$. This way

$$\begin{aligned}
\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{1024\pi}{81}f_{NL}\mathcal{P}^2_\zeta(k_0)\frac{v^{-6}}{\left[k_0 l(\eta_0,\eta_{in})\right]^{2n_s-2}}\int_0^\infty dy\, y^2 \\
&\int dx_1\, {x_1}^{n_s-2} j_{\ell_1}(x_1)\, j_{\ell_1}(x_1 y) \\
&\int dx_2\, {x_2}^{n_s-2} j_{\ell_2}(x_2)\, j_{\ell_2}(x_2 y) \\
&\int_{|x_1-x_2|}^{x_1+x_2} dx_3\, x_3^2 j_{\ell_3}(x_3)\, j_{\ell_3}(x_3 y) + 2\,\text{perms.}.
\end{aligned} \tag{5.93}$$

Transforming the spherical Bessel function $j_\ell(x)$ into the ordinary ones $J_\ell(x)$ through (5.72)

$$\begin{aligned}
\tilde{b}_{\ell_1\ell_2\ell_3,S} = &\frac{128\pi^4}{81}f_{NL}\mathcal{P}^2_\zeta(k_0)\frac{v^{-6}}{\left[k_0 l(\eta_0,\eta_{in})\right]^{2n_s-2}}\int_0^\infty dy\, \sqrt{y} \\
&\int dx_1\, {x_1}^{n_s-3} J_{\ell_1+\frac{1}{2}}(x_1)\, J_{\ell_1+\frac{1}{2}}(x_1 y) \\
&\int dx_2\, {x_2}^{n_s-3} J_{\ell_2+\frac{1}{2}}(x_2)\, J_{\ell_2+\frac{1}{2}}(x_2 y) \\
&\int_{|x_1-x_2|}^{x_1+x_2} dx_3\, x_3 J_{\ell_3+\frac{1}{2}}(x_3)\, J_{\ell_3+\frac{1}{2}}(x_3 y) + 2\,\text{perms.}.
\end{aligned} \tag{5.94}$$

As commented above, the dominant contribution to the innermost integral comes from the isosceles triangle configuration, that is when $x_1 \approx x_2$, while, as long as the modes propagate freely in the Universe, it can be assumed that $x_i \approx \ell_i$. This way the three integrals in (5.94) are independent and can be computed separately one at a time. The innermost integral is the same appearing in the Harrison-Zel'dovich case, that is

$$\begin{aligned}\mathcal{J}_{\ell_1\ell_2\ell_3}(y) &\equiv \int_0^{\ell_1+\ell_2} dx_3 x_3 J_{\ell_3+\frac{1}{2}}(x_3) J_{\ell_3+\frac{1}{2}}(yx_3) \\ &= \frac{\ell_1+\ell_2}{1-y^2}\Big\{ y J_{\ell_3+\frac{1}{2}}(\ell_1+\ell_2) J_{\ell_3-\frac{1}{2}}[(\ell_1+\ell_2)y] \\ &\quad - J_{\ell_3-\frac{1}{2}}(\ell_1+\ell_2) J_{\ell_3+\frac{1}{2}}[(\ell_1+\ell_2)y]\Big\},\end{aligned} \tag{5.95}$$

while the other integrals share the same structure of the previous case but with a different power of the integral variable x_i; therefore they can be solved using the same general relation (5.75) with $\mu = \nu = \ell_i + 1/2$ and $s = 3 - n_s$:

$$\begin{aligned}\mathcal{J}_\ell(y<1) &\equiv \int_0^\infty dx\, x^{n_s-3} J_{\ell+\frac{1}{2}}(x) J_{\ell+\frac{1}{2}}(yx) \\ &= \frac{2^{n_s-3} y^{\ell+\frac{1}{2}}\, \Gamma\left(\ell-\frac{1}{2}+\frac{n_s}{2}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(2-\frac{n_s}{2}\right)}\, {}_2F_1\left[\frac{n_s-2}{2}, \frac{2\ell+n_s-1}{2}; \ell+\frac{3}{2}; y^2\right]\end{aligned} \tag{5.96}$$

for

$$\ell > \frac{1-n_s}{2}, \qquad y < 1. \tag{5.97}$$

The case $y > 1$ can be reduced to the above one by performing the change of variable $z = yx$, which easily brings, as we have seen before, to

$$\mathcal{J}(y,\, y>1) = y\, \mathcal{J}(\frac{1}{y},\, y>1), \tag{5.98}$$

and then

$$\mathcal{J}_\ell(y>1) = \frac{2^{n_s-3} y^{\frac{1}{2}-\ell}\, \Gamma\left(\ell-\frac{1}{2}+\frac{n_s}{2}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(2-\frac{n_s}{2}\right)}\, {}_2F_1\left[\frac{n_s-2}{2}, \frac{2\ell+n_s-1}{2}; \ell+\frac{3}{2}; \frac{1}{y^2}\right]. \tag{5.99}$$

The case $y = 1$ is better solved by transforming back the Bessel functions into the spherical ones with (5.72):

$$\mathcal{J}_\ell(y=1) = \int dx\, x^{n_s-3} J^2_{\ell+\frac{1}{2}}(x) = \frac{2}{\pi}\int dx\, x^{n_s-2} j^2_\ell(x). \tag{5.100}$$

This expression simplifies using (C.8) to

$$\mathcal{J}_\ell(y=1) = 2^{n_s-3}\frac{\Gamma\left(\ell+\frac{n_s}{2}-\frac{1}{2}\right)}{\Gamma\left(\ell+\frac{5}{2}-\frac{n_s}{2}\right)}\frac{\Gamma(3-n_s)}{\Gamma^2\left(2-\frac{n_s}{2}\right)}. \tag{5.101}$$

Summarizing the results:

$$\mathcal{J}_\ell(y) = \begin{cases} \dfrac{2^{n_s-3} y^{\ell+\frac{1}{2}}\, \Gamma\left(\ell-\frac{1}{2}+\frac{n_s}{2}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(2-\frac{n_s}{2}\right)}\, {}_2F_1\left[\dfrac{n_s-2}{2}, \dfrac{2\ell+n_s-1}{2}; \ell+\dfrac{3}{2}; y^2\right], & y<1, \\[2ex] 2^{n_s-3}\dfrac{\Gamma\left(\ell+\frac{n_s}{2}-\frac{1}{2}\right)}{\Gamma\left(\ell+\frac{5}{2}-\frac{n_s}{2}\right)}\dfrac{\Gamma(3-n_s)}{\Gamma^2\left(2-\frac{n_s}{2}\right)}, & y=1, \\[2ex] \dfrac{2^{n_s-3} y^{\frac{1}{2}-\ell}\, \Gamma\left(\ell-\frac{1}{2}+\frac{n_s}{2}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(2-\frac{n_s}{2}\right)}\, {}_2F_1\left[\dfrac{n_s-2}{2}, \dfrac{2\ell+n_s-1}{2}; \ell+\dfrac{3}{2}; \dfrac{1}{y^2}\right], & y>1. \end{cases}$$

Collecting these results the reduced bispectrum finally reads

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3} &= \frac{128\pi^4}{81} f_{NL}\mathcal{P}_\zeta^2(k_0)\frac{v^{-6}}{[k_0 l(\eta_0,\eta_{in})]^{2n_s-2}}\int_0^\infty dy\sqrt{y}\,\mathcal{J}_{\ell_1}(y)\mathcal{J}_{\ell_2}(y)\mathcal{J}_{\ell_1\ell_2\ell_3}(y)\\ &\quad +2\,\text{perms.}\,,\end{aligned} \tag{5.102}$$

and is ready to be evaluated with numerical approaches. Alternatively they can be introduced some further assumptions to simplify the analytical solution. As sketched above, it is reasonable to think that the integrand of (5.102) is highly peaked at $y = 1$, since it is the case where the Bessel functions happen to interfere in a constructive way, and then

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3} &\simeq \frac{128\pi^4}{81} f_{NL}\mathcal{P}_\zeta^2(k_0)\frac{v^{-6}}{[k_0 l(\eta_0,\eta_{in})]^{2n_s-2}}\mathcal{J}_{\ell_1}(1)\mathcal{J}_{\ell_2}(1)\mathcal{J}_{\ell_1\ell_2\ell_3}(1)\\ &\quad +2\,\text{perms.}\,.\end{aligned} \tag{5.103}$$

Again we note that, in the case $y = 1$, the expression (5.101) for $\mathcal{J}_\ell(y)$ can be linked to the Sachs-Wolfe contribution to the scalar source angular power spectrum (4.138)

$$\tilde{C}_{\ell,S}^{SW} = \frac{2\pi^2}{9}v^{-4}\mathcal{P}_\zeta(k_0)\left(\frac{2}{k_0 l(\eta_0,\eta_{in})}\right)^{n_s-1}\frac{\Gamma\left(\ell+\frac{n_s}{2}-\frac{1}{2}\right)}{\Gamma\left(\ell+\frac{5}{2}-\frac{n_s}{2}\right)}\frac{\Gamma(3-n_s)}{\Gamma^2\left(2-\frac{n_s}{2}\right)} \tag{5.104}$$

through

$$\mathcal{J}_\ell(1) = \frac{9}{8\pi^2}v^4\mathcal{P}_\zeta^{-1}(k_0)\left(\frac{1}{k_0 l(\eta_0,\eta_{in})}\right)^{1-n_s}\tilde{C}_{\ell,S}^{SW}\,. \tag{5.105}$$

This way the reduced bispectrum becomes

$$\tilde{b}_{\ell_1\ell_2\ell_3} \simeq 2v^2 f_{NL}\tilde{C}_{\ell_1}^{SW}\tilde{C}_{\ell_2}^{SW}\tilde{\mathcal{J}}_{\ell_1\ell_2\ell_3}(1) + 2\,\text{perms.}\,. \tag{5.106}$$

Hence

$$\begin{aligned}\left\langle\prod_{i=1}^{3}\Gamma_{\ell_i m_i,S}\right\rangle &\simeq \mathcal{G}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3}2v^2 f_{NL}\left[\tilde{C}_{\ell_1,S}^{(SW)}\tilde{C}_{\ell_2,S}^{(SW)}\tilde{\mathcal{J}}_{\ell_1\ell_2\ell_3}(1)\right.\\ &\quad\left.+\tilde{C}_{\ell_2,S}^{(SW)}\tilde{C}_{\ell_3,S}^{(SW)}\tilde{\mathcal{J}}_{\ell_2\ell_3\ell_1}(1)+\tilde{C}_{\ell_3,S}^{(SW)}\tilde{C}_{\ell_1,S}^{(SW)}\tilde{\mathcal{J}}_{\ell_3\ell_1\ell_2}(1)\right].\end{aligned} \tag{5.107}$$

This expression is completely analogous to the one obtained with the assumption of an Harrison-Zel'dovich spectrum; the only difference stands in the definition of the angular spectra which are taken here in the more general form (4.138) The oscillating terms $\tilde{\mathcal{J}}_{\ell_3\ell_1\ell_2}(1)$ (and equivalently the oscillating terms $\tilde{\mathcal{I}}_{\ell_1\ell_2\ell_3}(1)$ in the Harrison-Zel'dovich case as well) can be eliminated if we consider the usual isosceles triangle configuration and if we make the upper extreme of the integral run to infinity. This procedure is justified by the fact that the integrand is dominated by the smallest values of the wave number k, and then the upper region of the integration domain is not really relevant. This approximation becomes more and more valid as the angular scales we are considering decrease, that is as ℓ increases. Then the innermost integral can be evaluated exploiting the property (C.9) of the spherical Bessel functions as

$$\int dk_3 k_3^2 j_{\ell_3}[k_3 l(\eta_0,\eta_{in})]j_{\ell_3}(k_3 r)\Big|_{\ell_3\gg 1} = \frac{\pi}{2}\frac{\delta(l(\eta_0,\eta_{in})-r)}{r^2}. \tag{5.108}$$

This result is quite useful, since now we are able to perform directly the integral over dr in (5.92):

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} \simeq \frac{512\pi^2}{81} f_{NL} v^{-6} & \int_0^\infty dr\, \delta(l(\eta_0,\eta_{in}) - r) \\ & \int \frac{dk_1}{k_1} j_{\ell_1}\left[k_1 l(\eta_0,\eta_{in})\right] j_{\ell_1}(k_1 r)\mathcal{P}_\zeta(k_1) \\ & \int \frac{dk_2}{k_2} j_{\ell_2}\left[k_1 l(\eta_0,\eta_{in})\right] j_{\ell_2}(k_2 r)\mathcal{P}_\zeta(k_2) + 2\,\text{perms.}. \end{aligned} \tag{5.109}$$

Notice that in this expression we have left implicit the form of the primordial power spectrum $P_\zeta(k)$ in such a way to highlight that the following results are completely general. Performing the integration in dr and rearranging some multiplicative factor

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} \approx \frac{32}{81} v^2 f_{NL} & \left(4\pi v^{-4} \int \frac{dk_1}{k_1} j^2_{\ell_1}\left[k_1 l(\eta_0,\eta_{in})\right]\mathcal{P}_\zeta(k_1)\right) \\ & \left(4\pi v^{-4} \int \frac{dk_2}{k_2} j^2_{\ell_2}\left[k_2 l(\eta_0,\eta_{in})\right]\mathcal{P}_\zeta(k_2)\right) + 2\,\text{perms.}. \end{aligned} \tag{5.110}$$

Remembering the expression (4.134) for the scalar sourced spectrum, the bispectrum can be written

$$\tilde{b}_{\ell_1\ell_2\ell_3,S} \approx 2v^2 f_{NL}\left[\tilde{C}^{SW}_{\ell_1,S}\tilde{C}^{SW}_{\ell_2,S} + \tilde{C}^{SW}_{\ell_1,S}\tilde{C}^{SW}_{\ell_3,S} + \tilde{C}^{SW}_{\ell_2,S}\tilde{C}^{SW}_{\ell_3,S}\right], \tag{5.111}$$

while the 3-point correlator is

$$\left\langle \prod_{i=1}^{3} \Gamma_{\ell_i m_i,S}\right\rangle \approx \mathcal{G}^{m_1 m_2 m_3}_{\ell_1\ell_2\ell_3} 2v^2 f_{NL}\left[\tilde{C}^{SW}_{\ell_1,S}\tilde{C}^{SW}_{\ell_2,S} + \tilde{C}^{SW}_{\ell_1,S}\tilde{C}^{SW}_{\ell_3,S} + \tilde{C}^{SW}_{\ell_2,S}\tilde{C}^{SW}_{\ell_3,S}\right]. \tag{5.112}$$

This is a remarkable result. We have found that the 3-point correlator characterizing the non-Gaussianity of the SGWB is completely determined by its power spectrum, which arises at the gaussian level, and a constant factor f_{NL}. This is indeed quite expected, since the local ansatz which parametrizes the departure from Gaussianity implies 3-point correlation function to be decomposed in terms of correlator of gaussian random variables (5.61), while the 2-point functions turns out to be untouched by non linear effects (5.58). It is worth to stress one more time that the expressions (5.111) and (5.112) are fully general, that is they are valid for any choice of the primordial curvature power spectrum. Differences between models are eventually encoded inside the expression of the angular power spectrum, which should recover (4.139) in the Harrison-Zel'dovich case, and (4.138) for a more general power-law spectrum.

5.2.3 Initial condition term

In complete analogy one decomposes the initial condition stochastic variable $\Gamma(\eta_{in},\vec{k},q)$ as

$$\begin{aligned}\Gamma(\eta_{in},\vec{k},q) = \Gamma_g(\eta_{in},\vec{k},q) + \frac{2}{3}\tilde{f}_{NL}\int \frac{d^3p}{(2\pi)^3}\Big[& \Gamma_g(\eta_{in},\vec{p},q)\Gamma_g(\eta_{in},\vec{k}-\vec{p},q) \\ & -(2\pi)^3\delta(\vec{k})P_I(p,q)\Big], \end{aligned} \tag{5.113}$$

where the factor 2/3 is just conventional so as to reproduce the same form of the curvature perturbation decomposition (5.56). Following the same steps outlined above for the scalar sourced case, one eventually ends up with

$$B_I(q,k_1,k_2,k_3) = \frac{4}{3} f_{NL}\left[\frac{2\pi^2}{k_1^3}\mathcal{P}_I(q,k_1)\frac{2\pi^2}{k_2^3}\mathcal{P}_I(q,k_2) + 2\,\text{perms.}\right], \tag{5.114}$$

while, from (5.52), the initial condition reduced bisprectrum is

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,I} &= \int_0^\infty dr\, r^2 \prod_{i=1}^{3}\left[\frac{2}{\pi}\int dk_i k_i^2 j_{\ell_i}[k_i l(\eta_0,\eta_{in})] j_{\ell_i}(k_i r)\right] \\ &\frac{4}{3}\tilde{f}_{NL}\left[\frac{2\pi^2}{k_1^3}\mathcal{P}_I(q,k_1)\frac{2\pi^2}{k_2^3}\mathcal{P}_I(q,k_2) + 2\,\text{perms.}\right]. \end{aligned} \tag{5.115}$$

Now one immediately recognizes the same expression found in (5.68); the only difference between the two is a constant factor $(2/3)v^{-2}$ which can be factorized out of the integral. This is a nice situation, since one can take advantage of all the result of the previous section. In particular, (5.110) becomes to

$$\begin{aligned}\tilde{b}_{\ell_1\ell_2\ell_3,S} &\approx \frac{4}{3}\tilde{f}_{NL}\left(4\pi\int\frac{dk_1}{k_1}j_{\ell_1}^2\left[k_1 l(\eta_0,\eta_{in})\right]\mathcal{P}_I(q,k_1)\right) \\ &\left(4\pi\int\frac{dk_2}{k_2}j_{\ell_2}^2\left[k_2 l(\eta_0,\eta_{in})\right]\mathcal{P}_I(q,k_2)\right) + 2\,\text{perms.}. \end{aligned} \tag{5.116}$$

Recognizing the expression (4.123) of the initial condition spectrum, the bispectrum evaluates to

$$\tilde{b}_{\ell_1\ell_2\ell_3,S} \approx \frac{4}{3}\tilde{f}_{NL}\left[\tilde{C}_{\ell_1,I}\tilde{C}_{\ell_2,I} + \tilde{C}_{\ell_1,I}\tilde{C}_{\ell_3,I} + \tilde{C}_{\ell_2,I}\tilde{C}_{\ell_3,I}\right]. \tag{5.117}$$

and the 3-point function

$$\left\langle \prod_{i=1}^{3}\Gamma_{\ell_i m_i,I}\right\rangle \approx \mathcal{G}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3}\frac{4}{3}\tilde{f}_{NL}\left[\tilde{C}_{\ell_1,I}\tilde{C}_{\ell_2,I} + \tilde{C}_{\ell_1,I}\tilde{C}_{\ell_3,I} + \tilde{C}_{\ell_2,I}\tilde{C}_{\ell_3,I}\right]. \tag{5.118}$$

5.3 Secondary Non-Gaussianity in the SGWB: squeezed limit configuration

Another possibility to generate a non vanishing n-point correlator even in absence of intrinsic primordial non-Gaussianity relies on the non-linear gravitational effects on the propagation of interacting GWs. In particular this section shows that the bispectra of the GW energy density contrast (3.46) acquires a non vanishing value in the squeezed limit, that is the limit where one of the three modes involved in the 3-point function is outside the horizon, while the other two modes are taken to be at small scales, i.e. $k_3 \ll k_{1,2}$. Among the many possible configurations of the 3-point functions which are very difficult to study in full generality at small scales, this situation represents a rather simple case, since long wavelength modes are not expected to affect any physical process; then the effect of the long wavelength mode will be only detectable in the way the short scale 2-point function is observed. Following the discussion firstly proposed by Weinberg in [106], and then resumed in [107], we will indeed show that the long wavelength perturbation mode can be reabsorbed with a coordinate transformation which ultimately will give rise to a second-order modulation of the short wavelength modes. All this discussion follows quite faithfully the arguments presented in [59].

Before entering in the details of this discussion let us resume some definitions of the section (3.3) and set some new useful conventions. The energy density of the SGWB is computed in terms of integral of the fundamental quantity $\omega_{GW}(\eta_0,\vec{x},q,m,\hat{n})$, which in momentum space is

$$\omega_{GW}(\eta,\vec{k},q,\hat{n}) = \bar{\omega}_{GW}(\eta,q)\left[1+\delta_{GW}(\eta,\vec{k},q,\hat{n})\right], \tag{5.119}$$

where the energy density contrast δ_{GW} is the quantity governing the small anisotropies in the SGWB defined as

$$\delta_{GW}(\eta, \vec{k}, q, \hat{n}) = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} \Gamma(\eta, \vec{k}, q, \hat{n}). \tag{5.120}$$

In the following the focus will be posed only on the scalar adiabatic contribution to metric perturbations in the usual Newtonian gauge, with the further assumption that there is no anisotropic stress. Basing on the discussion of (4.5.2), the scalar perturbations are parametrized as

$$\Phi(\eta, \vec{k}) = \Psi(\eta, \vec{k}) = \frac{2}{3} g(\eta) \zeta(\vec{k}). \tag{5.121}$$

With this decomposition, the scalar sourced fluctuation Γ_S, which is the only contribution we are considering in this discussion, reads

$$\begin{aligned}
\Gamma_S(\eta, \vec{k}, q, \hat{n}) = \frac{2}{3} \int_{\eta_{in}}^{\eta_0} d\eta' e^{-ik\mu l(\eta, \eta')} \Bigg[& v^{-2} g(\eta') \delta(\eta' - \eta_{in}) \\
& + \frac{\partial \left[(1 + v^{-2}) g(\eta') \right]}{\partial \eta'} \Bigg] \zeta(\vec{k}) \\
& \equiv T_S(\eta, k, q, \mu) \zeta(\vec{k}),
\end{aligned} \tag{5.122}$$

so that the density contrast becomes

$$\delta_{GW}(\eta, \vec{k}, q, \hat{n}) = -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k, q, \mu) \zeta(\vec{k}). \tag{5.123}$$

For what concerns the statistical correlators, in the following, for later convenience, it will be adopted the prime $'$ sign convention to understand the factor $(2\pi)^3 \delta(\sum \vec{k}_i)$. Denoting with $\mathcal{P}_\Gamma$ the dimensionless power spectrum arising from the 2-point correlators of the scalar sourced fluctuations Γ_S:

$$\begin{aligned}
\left\langle \Gamma_S(\eta, \vec{k}_1, q, \hat{n}) \Gamma_S(\eta, \vec{k}_2, q, \hat{n}) \right\rangle' &= \frac{2\pi^2}{k_1^3} \mathcal{P}_\Gamma(\eta, k_1, q, \hat{n}) \\
&= \frac{2\pi^2}{k_1^3} |T_S(\eta, k_1, q, \mu_1)|^2 \mathcal{P}_\zeta(k_1),
\end{aligned} \tag{5.124}$$

and

$$\begin{aligned}
\mathcal{P}_\Gamma(\eta, k, q, \mu) &= |T_S(\eta, k, q, \mu)|^2 \mathcal{P}_\zeta(k) \\
\mathcal{P}_{\delta_{GW}}(\eta, k, q, \mu) &= \left| \frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k, q, \mu) \right|^2 \mathcal{P}_\zeta(k).
\end{aligned} \tag{5.125}$$

In matter domination all these expression greatly simplify, since $g(\eta) \approx 1$ and the transfer function T_S, assuming for simplicity the graviton velocity to be constant, reduces to

$$T_S = \frac{2}{3} v^{-2} e^{-ik\mu l(\eta, \eta')}, \tag{5.126}$$

so that $|T_S|^2 = 4/(9v^2)$ without any dependence on η, k, μ. All these new convention prepare the field to study the 2-point correlation function of the SGWB anisotropies at small scales k, and in particular how the presence of a long scale mode $\zeta_L \equiv \zeta(\vec{k}_l)$, with $k_L \ll k$, modulates it. At a later stage, this modulation will affect higher order correlators as well, giving rise, as an example, to a non vanishing 3-point function. As anticipated, the path to follow basically consists in reabsorbing the long mode in a coordinate transformation, and then in studying the evolution of the small wavelength modes in this new coordinate system.

5.3.1 Long wavelength mode as coordinate transformation

The aim of this approach, introduced in [106], is to generate second order perturbations in the small wavelength modes by hiding the large ones inside a coordinate transformation on the first order perturbed Universe. At first order, the metric for a long wavelength scalar perturbation $\Phi_L \equiv \Phi(\vec{k}_L)$ and $\Psi_L \equiv \Psi(\vec{k}_L)$[2] is

$$ds^2 = a^2(\eta)\left[-(1+2\Phi_L)d\eta^2 + (1-2\Psi_L)\delta_{ij}dx^i dx^j\right], \tag{5.127}$$

Let's then consider the following coordinate transformation

$$\begin{aligned} \tilde{\eta} &= \eta + \epsilon(\eta)\zeta_L & \eta &= \tilde{\eta} - \epsilon(\tilde{\eta})\zeta_L, \\ \tilde{x}^i &= x^i(1-\lambda\zeta_L) & x^i &= \tilde{x}^i(1+\lambda\zeta_L), \end{aligned} \tag{5.128}$$

with λ an arbitrary constant and $\epsilon(\eta)$ an arbitrary time dependent function. This change of coordinates acts on the scale factor as

$$a(\tilde{\eta}) = a(\eta)\left(1 + \mathcal{H}\epsilon(\eta)\zeta_L\right), \tag{5.129}$$

so that in the new coordinate system the line element reads

$$\begin{aligned} ds^2 &= a^2(\tilde{\eta})(1-2\epsilon(\eta)\mathcal{H}\zeta_L) \\ &\quad \left[-(1+2\Phi_L)(1-2\epsilon'\zeta_L)d\tilde{\eta}^2 + (1-2\Psi_L)(1+2\lambda\zeta_L)\delta_{ij}d\tilde{x}^i d\tilde{x}^j\right] \\ &= a^2(\tilde{\eta})\left[-(1+2\tilde{\Phi}_L)d\tilde{\eta}^2 + (1-2\tilde{\Psi}_L)\delta_{ij}d\tilde{x}^i d\tilde{x}^j\right]. \end{aligned} \tag{5.130}$$

Therefore the coordinate transformation (5.128) preserve the Poisson gauge structure of the metric with the redefinition of the long perturbation modes

$$\begin{aligned} \tilde{\Phi}_L &= \Phi_L - \epsilon'\zeta_L - \mathcal{H}\epsilon\zeta_L \\ \tilde{\Psi}_L &= \Psi_L - \lambda\zeta_L + \mathcal{H}\epsilon\zeta_L. \end{aligned} \tag{5.131}$$

Now it is possible to 'gauge away' the long wavelength modes with a proper gauge choice of the parameter ϵ and λ, such that $\tilde{\Phi}_L = \tilde{\Psi}_L = 0$. We then set the two parameter so that to satisfy the relations

$$\begin{aligned} \Phi_L &= (\epsilon' + \mathcal{H}\epsilon)\zeta_L, \\ \Psi_L &= (\lambda - \mathcal{H}\epsilon)\zeta_L. \end{aligned} \tag{5.132}$$

As a result, in the new coordinates, we have a pure FLRW Universe, with no long wavelength perturbations. As shown in [106], the Einstein field equation with $k_L = 0$ are always invariant under this gauge transformation. However, for it to acquire a physical meaning, we should verify that thy are consistent with those piece of the Einstein equation that vanish in the limit $k_L \to 0$ and that could then spoil the invariance. These requirements will then reveal themselves as constraints on the two parameter. The first condition can be easily read from the off diagonal space-space components of the Einstein equation (4.46), which in momentum space read

$$k^i k_j(\Phi - \Psi) = 0. \tag{5.133}$$

[2]Since we are assuming k_L to be very small, we can consider the long wavelength perturbations to be space independent

This equation obviously vanishes in the limit $k \to 0$, while it can be extended for any value of k provided $\Phi = \Psi$, and then, from (5.132), it yields

$$\epsilon' + 2\mathcal{H}\epsilon = \lambda. \tag{5.134}$$

This is a first order differential equation with non homogeneous coefficients. In [71] it is proved that such equation admits a solution in the form

$$\epsilon(\eta) = e^{-f(\eta)} \int_{\eta_*}^{\eta} d\eta' e^{f(\eta)} \lambda + c e^{-f(\eta)}, \tag{5.135}$$

with η_* a reference initial time and c an integration constant, while

$$f(\eta) = 2 \int_{\eta_*}^{\eta} d\eta' \mathcal{H}(\eta'). \tag{5.136}$$

Opening the Hubble parameter in terms of the scale factors, this expression can be directly integrated to $f(\eta) = 2\ln(a(\eta)) - 2\ln(a(\eta_*))$. Therefore for sufficiently initial times, since $\ln(a(\eta_*)) \ll 0$, the solution reads

$$\epsilon = \frac{1}{a^2(\eta)} \int_{\eta_*}^{\eta} d\eta \, a^2(\eta')\lambda. \tag{5.137}$$

A second condition instead derives from the space-time components of the Einstein equations. Equating the Einstein tensor components (A.20) to the stress energy tensor (4.44) leads to

$$k_i(\Psi' + \mathcal{H}\Phi) = -\frac{a^2}{2M_p^2}(\rho + p)k_i v = (\mathcal{H}' - \mathcal{H}^2)k_i v, \tag{5.138}$$

where in the second step the first and third Friedman equations (4.45) were used. Inserting the conditions (5.132), this equation is satisfied by

$$v = -\epsilon\zeta_L. \tag{5.139}$$

Moreover, the gauge transformation applies on the energy density through the Lie derivative [75] as

$$\delta\tilde{\rho} = \delta\rho + \rho_0'\epsilon\zeta_L, \tag{5.140}$$

where the subscript denotes the background quantities. Since the transformed spacetime is defined in such a way to be an unperturbed pure FLRW Universe, one can take $\delta\tilde{\rho} = 0$. Then

$$\delta\rho = -\rho_0'\epsilon\zeta_L. \tag{5.141}$$

Finally, from the definition of the gauge invariant curvature perturbation (4.51)

$$\zeta_L = \Phi_L + \mathcal{H}\frac{\delta\rho}{\rho_0'} = \Psi_L - \mathcal{H}\epsilon\zeta_L, \tag{5.142}$$

which is consistent with (5.132) provided $\lambda = 1$. An alternative way to obtain the same result is to consider the limit $k \to 0$, where [106] proves that $v = \delta\rho/\rho_0'$; then, thanks to (5.139), one can turn back to (5.142). To summarize, the conditions on the parameter ϵ and λ to be consistent with the Einstein equation are

$$\begin{aligned} \lambda &= 1 \\ \epsilon(\eta) &= \frac{1}{a^2(\eta)} \int_{\eta^*}^{\eta} d\eta' a^2(\eta'). \end{aligned} \tag{5.143}$$

Notice further that all these results, if inserted inside (5.132), are consistent with the fact that in the Poisson gauge $\Phi = \Psi$. If the equation of state $p = w\rho$ provides a constant factor w, in which case $a \propto \eta^{2/(1+3w)}$ and $\mathcal{H} = 2/[\eta(1+3w)]$, then

$$\begin{aligned}\epsilon(\eta) &= \frac{1+3w}{5+3w}\eta, \\ \mathcal{H}\epsilon &= \frac{2}{5+3w}.\end{aligned} \tag{5.144}$$

After having understood how to hide the long-wavelength modes inside the coordinate transformation, we want to use it to study the second-order evolution of the short-wavelength modes. Let's then consider a perturbed metric in the $(\tilde{\eta}, \tilde{x}^i)$ coordinate system where only short-wavelength perturbations appear explicitly, the long-wavelength ones being encoded inside the coordinates. In the usual Poisson gauge

$$ds^2 = a^2(\tilde{\eta})\left[-(1+2\tilde{\Phi}_S)d\tilde{\eta}^2 + (1-2\tilde{\Psi}_S)\delta_{ij}d\tilde{x}^i d\tilde{x}^j\right]. \tag{5.145}$$

Using again the coordinate transformation (5.128), we can express the short-wavelength potentials $\tilde{\Phi}_S$ and $\tilde{\Psi}_S$ in the original system (η, x^i):

$$\begin{aligned}\tilde{\Phi}_S(\tilde{\eta}, \tilde{x}^i) &= \Phi_S + \epsilon\zeta_L\frac{\partial\Phi_S}{\partial\eta} - \lambda\zeta_L x^i\frac{\partial\Phi_S}{\partial x^i}, \\ \tilde{\Psi}_S(\tilde{\eta}, \tilde{x}^i) &= \Psi_S + \epsilon\zeta_L\frac{\partial\Psi_S}{\partial\eta} - \lambda\zeta_L x^i\frac{\partial\Psi_S}{\partial x^i}.\end{aligned} \tag{5.146}$$

For the ease of notation, here the dependence on the original coordinate (η, x^i) is understood, while the dependence on the transformed ones is left explicit. Applying the transformation to the entire metric (5.145), one ends up with a second order perturbed metric which can be written in the usual Poisson gauge form performing the following steps[3]

$$\begin{aligned}ds^2 &= a^2(\eta)(1+2\mathcal{H}\epsilon\zeta_L) \\ &\quad \left[-(1+2\tilde{\Phi}_S)(1+2\epsilon'\zeta_L)d\eta^2 + (1-2\tilde{\Psi}_S)(1-2\lambda\zeta_L)\delta_{ij}dx^i dx^j\right] \\ &= a^2(\eta)\left[-(1+2\tilde{\Phi}_S)(1+2\Phi_L)d\eta^2 + (1-2\tilde{\Psi}_S)(1-2\Psi_L)\delta_{ij}dx^i dx^j\right] \\ &= a^2(\eta)\left[-(1+2\hat{\Phi}_S)d\eta^2 + (1-2\hat{\Psi}_S)\delta_{ij}dx^i dx^j\right],\end{aligned} \tag{5.147}$$

with the definitions

$$\begin{aligned}\hat{\Phi}_S &= \Phi_S + \Phi_L + 2\Phi_S\Phi_L + \epsilon\zeta_L\frac{\partial\Phi_S}{\partial\eta} - \lambda\zeta_L x^i\frac{\partial\Phi_S}{\partial x^i}, \\ \hat{\Psi}_S &= \Psi_S + \Psi_L - 2\Psi_S\Psi_L + \epsilon\zeta_L\frac{\partial\Psi_S}{\partial\eta} - \lambda\zeta_L x^i\frac{\partial\Psi_S}{\partial x^i}.\end{aligned} \tag{5.148}$$

This is the remarkable result which was already anticipated. These expression indeed show that in the squeezed limit the physical effects of the long wavelength mode Φ_L simply amount to a modulation of of the short wavelength ones, introducing a non linear evolution

[3] In this expression we are going to neglect terms like ζ_L^2 since at second order the give rise to constant terms only, which are meaningless since they can always be reabsorbed with a field shifting. Indeed the long wavelength modes can be assumed to be spatial independent while we focus on measurements on small scales.

through the coupling between the two different modes. In the end, this coupling will be responsible for a modification of the 2-point function of the small-wavelength perturbations and for a non vanishing squeezed limit of the 3-point function.

For the subsequent discussion it will be useful to understand how the spatial coordinate shifting (5.128) applies when working in the Fourier space. Let's then consider the transformation acting on a generic function $f(x^i)$ of the spatial coordinate

$$f(x^i) \to f(x^i(1-\lambda\zeta_L)). \tag{5.149}$$

In the Fourier space the function $f(x^i)$ is expanded as

$$f(x^i) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{f}(k^i), \tag{5.150}$$

while the space shifted function reads

$$\begin{aligned} f(x^i(1-\lambda\zeta_L)) &= \int \frac{d^3k}{(2\pi)} e^{i\vec{k}\cdot\vec{x}(1-\lambda\zeta_L)} f(k^i) \\ &= \int \frac{d^3k}{(2\pi)} e^{i\vec{k}\cdot\vec{x}} \left[(1+3\lambda\zeta_L)\tilde{f}(k^i(1+\lambda\zeta_L))\right], \end{aligned} \tag{5.151}$$

where in the second step a simple shift in the integration variable $k^i \to k^i(1+\lambda\zeta_L)$ was applied and expanded at first order in ζ_L. Therefore one learns that in momentum space the spatial coordinate transformation acts in the following way

$$\tilde{f}(k^i) \to (1+3\lambda\zeta_L)\tilde{f}(k^i(1+\lambda\zeta_L)), \tag{5.152}$$

or, expanding at first order,

$$\tilde{f}(k^i) \to \tilde{f}(k^i) + 3\lambda\zeta_L\tilde{f}(k^i) + \lambda\zeta_L k^m \frac{\partial\tilde{f}(k^i)}{\partial k^m}. \tag{5.153}$$

What is still left to understand is how the coordinate transformation (5.128) acts on the momentum q and on its director n^i. The GW wave momentum components have already been written in (3.7), (3.8). Neglecting tensor perturbations, they are

$$p^0 = \frac{e^{-\hat{\Phi}_S}}{a}\left(\frac{q^2}{a^2}+m^2\right)^{1/2} = \frac{e^{-\hat{\Phi}_S}}{a}E, \tag{5.154}$$

$$p^i = \frac{q}{a^2} n^i e^{\hat{\Psi}_S}. \tag{5.155}$$

Under the coordinate transformation, the four-momentum transform as a vector

$$\tilde{p}^\mu = \frac{\partial\tilde{x}^\mu}{\partial x^\nu} p^\nu, \tag{5.156}$$

with

$$\frac{\partial\tilde{x}^0}{\partial x^\nu} = (1+\epsilon'\zeta_L)\delta^0_\nu, \qquad \frac{\partial\tilde{x}^i}{\partial x^\nu} = (1-\lambda\zeta_L)\delta^i_\nu. \tag{5.157}$$

The transformation on the time component of the four momentum then implies

$$\frac{e^{-\tilde{\Phi}_S}}{\tilde{a}}\tilde{E} = (1+\epsilon'\zeta_L)\frac{e^{-\hat{\Phi}_S}}{a}E. \tag{5.158}$$

Remembering the definitions (5.148), and expanding this condition at first order in the long wavelength perturbations,

$$\tilde{E} = \frac{\tilde{a}}{a}(1+\epsilon'\zeta_L)e^{-\Phi_L}E = \frac{\tilde{a}}{a}(1+\epsilon'\zeta_L)(1-\Phi_L)E, \tag{5.159}$$

that, in terms of the comoving momentum q, reads

$$\begin{aligned}
\tilde{q}^2 &= \left(\frac{\tilde{a}}{a}\right)^4 (1+2\epsilon'\zeta_L - 2\Phi_L)q^2 + \left(\frac{\tilde{a}}{a}\right)^4 a^2m^2(1+2\epsilon'\zeta_L - 2\Phi_L) - a^2m^2 \\
&= (1+4\mathcal{H}\epsilon\zeta_L + 2\epsilon'\zeta_L - 2\Phi_L)q^2 + 2a^2m^2(2\mathcal{H}\epsilon\zeta_L + \epsilon'\zeta_L - \Phi_L) \\
&= \left[1+2\left(1-\frac{3}{5}g(\eta)\right)\zeta_L\right]q^2 - 2a^2m^2\left(1+2\epsilon\mathcal{H} - \frac{3}{5}g(\eta)\right)\zeta_L,
\end{aligned} \tag{5.160}$$

and then

$$\tilde{q} = \left[1+\left(1-\frac{3}{5}g(\eta)\right)\zeta_L\right]q - \frac{a^2m^2}{q}\left(1+2\epsilon\mathcal{H} - \frac{3}{5}g(\eta)\right)\zeta_L. \tag{5.161}$$

Notice that the graviton mass induces an additional term to the zero component momentum transformation, while in the massless case this expression recovers the result of [59]. Concerning the spatial components of the four momentum, with the same arguments the change of coordinate applies in this way

$$\frac{\tilde{q}}{\tilde{a}^2}\tilde{n}^i = (1-\lambda\zeta_L)\frac{q}{a^2}n^i e^{\Psi_L}, \tag{5.162}$$

which in terms of the director $\tilde{n}^i$ becomes

$$\begin{aligned}
\tilde{n}^i &= \frac{\tilde{a}^2}{a^2}\frac{q}{\tilde{q}}(1-\lambda\zeta_L)(1+\Psi_L)n^i \\
&= (1+2\mathcal{H}\epsilon\zeta_L)\left[1-\left(1-\frac{3}{5}g(\eta)\right)\zeta_L + \frac{a^2m^2}{q}\left(1+2\epsilon\mathcal{H}-\frac{3}{5}g(\eta)\right)\zeta_L\right] \\
&\quad (1-\lambda\zeta_L)(1+\Psi_L)n^i \\
&= \left[1-2\left(1-\frac{3}{5}g(\eta)-\mathcal{H}\epsilon\right)\zeta_L + \frac{a^2m^2}{q}\left(1+2\epsilon\mathcal{H}-\frac{3}{5}g(\eta)\right)\zeta_L\right]n^i.
\end{aligned} \tag{5.163}$$

To summarize the last results in the more compact form

$$\begin{aligned}
\tilde{q} &= \left[1+\left(\beta_q(\eta) - \frac{a^2m^2}{q^2}\beta_m(\eta)\right)\zeta_L\right]q, \\
\tilde{n}^i &= \left[1+\left(\beta_n(\eta) + \frac{a^2m^2}{q}\beta_m(\eta)\right)\zeta_L\right]n^i,
\end{aligned} \tag{5.164}$$

with the definitions

$$\begin{aligned}
\beta_q(\eta) &= 1-\frac{3}{5}g(\eta), \\
\beta_n(\eta) &= -2\left(1-\frac{3}{5}g(\eta)-\mathcal{H}\epsilon\right), \\
\beta_m(\eta) &= 1-\frac{3}{5}g(\eta)+2\epsilon\mathcal{H}.
\end{aligned} \tag{5.165}$$

In matter domination they evaluate to $\beta_q = 2/5$, $\beta_n = 0$, and $\beta_m = 6/5$.

5.3.2 Coordinate transformation of the GW energy distribution

At this point we are ready to apply the previous results to our case of interest and to study the transformation properties of the energy density of the GW, which is completely characterized by the quantity ω_{GW}. As seen in section (3.3), this function can be split in an homogeneous and a non-homogeneous contributions

$$\omega_{GW}(\eta, k^i, q, n^i) = \bar{\omega}_{GW}(\eta, q)[1 + \delta_{GW}(\eta, k^i, q, n^i)], \tag{5.166}$$

with

$$\begin{aligned} \bar{\omega}_{GW}(\eta, q) &= \frac{q^4}{a^4(\eta)\rho_{crit}}\sqrt{1 + \frac{a^2m^2}{q^2}}\bar{f}(q), \\ \delta_{GW}(\eta, \vec{k}, q, n^i) &= -\frac{\partial \ln \bar{f}(\eta, q)}{\partial \ln q}\Gamma_S(\eta, \vec{k}, q, n^i). \end{aligned} \tag{5.167}$$

It is necessary now to understand the behavior of each contribution under the transformation (5.128), starting from the background quantities $\bar{\omega}_{GW}$ and $\bar{f}(q)$. For the homogeneous energy density one can write

$$\bar{\omega}_{GW}(\tilde{\eta}, \tilde{q}) = \bar{\omega}(\eta, q) + \frac{\partial \bar{\omega}_{GW}}{\partial \eta}\delta\eta + \frac{\partial \bar{\omega}_{GW}}{\partial q}\delta q, \tag{5.168}$$

with

$$\delta\eta = \epsilon\zeta_L, \qquad \delta q = q\left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\zeta_L. \tag{5.169}$$

Computing explicitly the derivative terms

$$\begin{aligned} \bar{\omega}_{GW}(\tilde{\eta}, \tilde{q}) &= \bar{\omega}_{GW}(\eta, q) - 4\mathcal{H}\bar{\omega}_{GW}\delta\eta + \frac{\bar{\omega}_{GW}}{q}\left[4 + \frac{\partial \ln \bar{f}(q)}{\partial \ln q} - \frac{1}{1 + \frac{q^2}{a^2m^2}}\right]\delta q \\ &= \bar{\omega}_{GW}(\eta, q)\left\{1 - 4\mathcal{H}\epsilon\zeta_L + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\right. \\ &\qquad \left. \times \left[4 + \frac{\partial \ln \bar{f}(q)}{\partial \ln q} - \frac{1}{1 + \frac{q^2}{a^2m^2}}\right]\zeta_L\right\}. \end{aligned} \tag{5.170}$$

To understand the transformation properties of the logarithmic derivative of the background distribution function it is convenient to proceed by steps. As a preliminary study, let's focus on the differential operator. From (5.164), it derives, at first order in ζ_L,

$$\ln\tilde{q} = \ln q + \ln\left[1 + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\zeta_L\right] = \ln q + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\zeta_L. \tag{5.171}$$

Differentiating with fixed time

$$\mathrm{d}\ln\tilde{q} = \left(1 + 2\frac{a^2m^2}{q^2}\beta_m\zeta_L\right)\mathrm{d}\ln q. \tag{5.172}$$

Let's now turn to the transformation of the zero order distribution function:

$$\begin{aligned} \bar{f}(\tilde{q}) &= \bar{f}(q) + \frac{\partial \bar{f}(q)}{\partial q}q\zeta_L\left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) \\ &= \bar{f}(q)\left[1 + \frac{\partial \ln \bar{f}(q)}{\partial \ln q}\left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\zeta_L\right]. \end{aligned} \tag{5.173}$$

Considering now the expansion at first order of the logarithm of this expression and taking its logarithmic derivative, one finds

$$
\begin{aligned}
\frac{\partial \ln \bar{f}(\tilde{q})}{\partial \ln \tilde{q}} &= \left(1-2\frac{a^2m^2}{q^2}\beta_m\zeta_L\right)\left[\frac{\partial \ln \bar{f}(q)}{\partial \ln q}+\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\zeta_L\right. \\
&\qquad\left.+\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\left(2\frac{a^2m^2}{q^2}\beta_m\right)\zeta_L\right] \\
&= \left(1-2\frac{a^2m^2}{q^2}\beta_m\zeta_L\right)\left[\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\left(1+2\frac{a^2m^2}{q^2}\beta_m\zeta_L\right)\right. \\
&\qquad\left.+\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\zeta_L\right] \\
&= \frac{\partial \ln \bar{f}(q)}{\partial \ln q}+\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\zeta_L. \qquad (5.174)
\end{aligned}
$$

Lastly, let's consider the scalar sourced fluctuation Γ_S. Using the previous result (5.152), the change of coordinate applies as:

$$
\begin{aligned}
\Gamma_S(\tilde{\eta},\tilde{k}^i,\tilde{q},\tilde{n}^i) = \Gamma_S\Bigg(&\eta+\epsilon(\eta)\zeta_L,\vec{k}(1+\zeta_L),\left[1+\left(\beta_q(\eta)-\frac{a^2m^2}{q^2}\beta_m(\eta)\right)\zeta_L\right]q, \\
&\left[1+\left(\beta_n(\eta)+\frac{a^2m^2}{q}\beta_m(\eta)\right)\zeta_L\right]n^i\Bigg)(1+3\zeta_L). \qquad (5.175)
\end{aligned}
$$

Expanding at first order in the long wavelength modes, and omitting the Γ-dependence on the original quantities (η, k^i, q, n^i) and the conformal time dependence of the β and ϵ factors, we have

$$
\begin{aligned}
\Gamma_S(\tilde{\eta},\tilde{k}^i,\tilde{q},\tilde{n}^i) &= (1+3\zeta_L)\Gamma_S+\frac{\partial \Gamma_S}{\partial \eta}\epsilon\zeta_L+k^i\frac{\partial \Gamma_S}{\partial k^i}\zeta_L \\
&+\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \Gamma_S}{\partial \ln q}\zeta_L+\left(\beta_n+\frac{a^2m^2}{q}\beta_m\right)n^j\frac{\partial \Gamma_S}{\partial n^j}\zeta_L. \qquad (5.176)
\end{aligned}
$$

The energy density contrast $\tilde{\delta}_{GW} \equiv \delta_{GW}(\tilde{\eta},\tilde{k}^i,\tilde{q},\tilde{n}^i)$ becomes

$$
\begin{aligned}
\tilde{\delta}_{GW} &= -\frac{\partial \ln \bar{f}(\tilde{\eta},\tilde{q})}{\partial \ln \tilde{q}}\Gamma_S(\tilde{\eta},\tilde{k}^i,\tilde{q},\tilde{n}^i) \\
&= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\left[1+\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\zeta_L\right] \\
&\quad \Gamma_S\left[1+3\zeta_L+\frac{\partial \ln \Gamma_S}{\partial \eta}\epsilon\zeta_L+k^i\frac{\partial \ln \Gamma_S}{\partial k^i}\zeta_L+\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \ln \Gamma_S}{\partial \ln q}\zeta_L\right. \\
&\quad\left.+\left(\beta_n+\frac{a^2m^2}{q^2}\beta_m\right)n^i\frac{\partial \ln \Gamma_S}{\partial n^i}\zeta_L\right] \\
&= \delta_{GW}+\delta_{GW}\zeta_L\left[3+\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)+\frac{\partial \ln \Gamma_S}{\partial \eta}\epsilon\right. \\
&\quad+k^i\frac{\partial \ln \Gamma_S}{\partial k^i}+\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \ln \Gamma_S}{\partial \ln q} \\
&\quad\left.+\left(\beta_n+\frac{a^2m^2}{q^2}\beta_m\right)n^i\frac{\partial \ln \Gamma_S}{\partial n^i}\right]. \qquad (5.177)
\end{aligned}
$$

With this last ingredient the energy density $\tilde{\omega}_{GW} \equiv \omega_{GW}(\tilde{\eta}, \tilde{k}^i, \tilde{q}, \tilde{n}^i)$ finally evaluates to:

$$\begin{aligned}\tilde{\omega}_{GW} &= \bar{\omega}_{GW}(\tilde{\eta}, \tilde{q})[1 + \tilde{\delta}_{GW}(\tilde{\eta}, \tilde{k}^i, \tilde{q}, \tilde{n}^i)] \\ &= \bar{\omega}_{GW} \left\{ 1 - 4\mathcal{H}\epsilon\zeta_L + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) \left[4 + \frac{\partial \ln \bar{f}(q)}{\partial \ln q} - \frac{1}{1 + \frac{q^2}{a^2m^2}}\right] \zeta_L \right\} \\ &\quad \left\{ 1 + \delta_{GW} + \delta_{GW}\zeta_L \left[3 + \frac{\partial \ln q}{\partial \ln \bar{f}(q)} \frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) \right.\right. \\ &\quad + \frac{\partial \ln \Gamma_S}{\partial \eta}\epsilon + k^i \frac{\partial \ln \Gamma_S}{\partial k^i} + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) \frac{\partial \ln \Gamma_S}{\partial \ln q} \\ &\quad \left.\left. + \left(\beta_n + \frac{a^2m^2}{q^2}\beta_m\right) n^i \frac{\partial \ln \Gamma_S}{\partial n^i} \right] \right\}. \end{aligned} \tag{5.178}$$

Rearranging some terms, the final result is

$$\begin{aligned}\tilde{\omega}_{GW} &= \bar{\omega}_{GW}(\eta, q) \left\{ 1 + \delta_{GW} + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) \left[4 + \frac{\partial \ln \bar{f}(q)}{\partial \ln q} - \frac{1}{1 + \frac{q^2}{a^2m^2}}\right] \zeta_L \right. \\ &\quad - 4\mathcal{H}\epsilon(\eta)\zeta_L + \delta_{GW}\zeta_L \left[3 + \frac{\partial \ln q}{\partial \ln \bar{f}(q)} \frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) + \right. \\ &\quad + \frac{\partial \ln \Gamma_S}{\partial \eta}\epsilon(\eta) + k^i \frac{\partial \ln \Gamma_S}{\partial k^i} + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right) \frac{\partial \ln \Gamma_S}{\partial \ln q} \\ &\quad \left.\left. + \left(\beta_n + \frac{a^2m^2}{q^2}\beta_m\right) n^i \frac{\partial \ln \Gamma_S}{\partial n^i} \right] \right\}. \end{aligned} \tag{5.179}$$

This is a first remarkable result. Indeed this expression clearly shows that, even in absence of intrinsic small wavelength anisotropies (i.e. $\delta_{GW} = 0$), there is a modulation of the background energy density provided by the coupling with the long wavelength mode ζ_L. As an example, in matter domination, the energy density in absence of intrinsic anisotropies gets modulated by the long mode as

$$\begin{aligned}\tilde{\omega}_{GW} = \bar{\omega}_{GW}(\eta, q) \left[1 + \frac{2}{5}\left(1 - \frac{3a^2m^2}{q^2}\right) \left[\frac{\partial \ln \bar{f}(q)}{\partial \ln q} - \frac{1}{1 + \frac{q^2}{a^2m^2}}\right] \zeta_L \right. \\ \left. - \frac{24a^2m^2}{5q^2}\zeta_L \right]. \end{aligned} \tag{5.180}$$

The modulation in the energy density is controlled by the q-dependence of the graviton distribution function, as found in [59], and further by the dimensionless ratio a^2m^2/q^2 encoding the information about the non vanishing graviton mass.

5.3.3 Squeezed limit of the two and three point correlation functions

As anticipated, the final purpose is to evaluate the correlation functions of the energy density contrast δ_{GW} at small scales, and in particular to understand how a long wavelength mode influences them. The simplest case is the 2-point correlators of the small-wavelength modes modulated by the long wavelength one, which can be directly read from (5.179). In a compact way, at first order in ζ_L, one can write it as

$$\left\langle \tilde{\delta}_{GW}(\vec{k}_1)\tilde{\delta}_{GW}(\vec{k}_2) \right\rangle' = (1 + \mathcal{M}\zeta_L) \left\langle \delta_{GW}(\vec{k}_1)\delta_{GW}(\vec{k}_2) \right\rangle'. \tag{5.181}$$

Be careful that the quantity $\tilde{\delta}_{GW}(\vec{k})$ is the energy density contrast modulated by the long wavelength mode but evaluated in the original coordinate system (η, x^i, q, n^i), which is the first order contribution of the last result in (5.179). However those terms which are not proportional to δ_{GW} do not contribute at first order to the 2-point functions, since they give rise to terms proportional to the mean value of the energy contrast $\langle \delta_{GW} \rangle$ which is null by definition. Therefore the only effective contributions to the 2-point correlators come from the last line of (5.177). In the previous expression it was defined the modulating factor $\mathcal{M}$ as

$$
\begin{aligned}
\mathcal{M} =\, & 6 + 2\left(\beta_q - \frac{a^2 m^2}{q^2}\beta_m\right)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} + \epsilon(\eta)\frac{\partial \ln \left\langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right\rangle'}{\partial \eta} \\
& + k_1^i \frac{\partial \ln \left\langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right\rangle'}{\partial k_1^i} + k_2^i \frac{\partial \ln \left\langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right\rangle'}{\partial k_2^i} \\
& + \left(\beta_q - \frac{a^2 m^2}{q^2}\beta_m\right)\frac{\partial \ln \left\langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right\rangle'}{\partial \ln q} \\
& + \left(\beta_n + \frac{a^2 m^2}{q^2}\beta_m\right) n^i \frac{\partial \ln \left\langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right\rangle'}{\partial n^i}.
\end{aligned} \tag{5.182}
$$

The second line is justified by the obvious observation that

$$
k_1^i \frac{\partial \ln \Gamma_S(\vec{k}_1)}{\partial k_1^i} = k_1^i \frac{\partial \left(\ln \Gamma_S(\vec{k}_1) + \ln \Gamma_S(\vec{k}_2)\right)}{\partial k_1^i} = k_1^i \frac{\partial \ln \left(\Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right)}{\partial k_1^i}. \tag{5.183}
$$

More shortly, one can rewrite the modulation factor is in the the following way

$$
\begin{aligned}
\mathcal{M} =\, & 6 + 2\left(\beta_q - \frac{a^2 m^2}{q^2}\beta_m\right)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} \\
& + \left[\epsilon(\eta)\frac{\partial}{\partial \eta} + k_1^i \frac{\partial}{\partial k_1^i} + k_2^i \frac{\partial}{\partial k_2^i} + \left(\beta_q - \frac{a^2 m^2}{q^2}\beta_m\right)\frac{\partial}{\partial \ln q}\right. \\
& \left. + \left(\beta_n + \frac{a^2 m^2}{q^2}\beta_m\right) n^i \frac{\partial}{\partial n^i}\right] \ln \left\langle \Gamma_S(\vec{k}_1)\Gamma_S(\vec{k}_2)\right\rangle'.
\end{aligned} \tag{5.184}
$$

Let's now take in consideration the power spectrum relative to the energy density contrast defined in (5.124) and (5.125). The intention is to study how the long wavelength mode modulates the results; therefore as usual one starts from the modulated spectrum in the transformed coordinates

$$
\mathcal{P}_{\tilde{\delta}_{GW}}(\tilde{\eta}, \tilde{k}, \tilde{q}, \tilde{n}^i) = \left| \frac{\partial \ln \bar{f}(\tilde{q})}{\partial \ln \tilde{q}} T_S(\tilde{\eta}, \tilde{k}, \tilde{q}, \tilde{n}^i)\right|^2 \mathcal{P}_\zeta(\tilde{k}), \tag{5.185}
$$

and then performs the inverse coordinate transformation (5.128) to come back to the original coordinate system. The transformation of the distribution function was already written in (5.174), while for the dimensionless scalar soured power spectrum and the transfer function T_S one finds respectively

$$\begin{aligned}
\mathcal{P}_\zeta(\tilde{k}) &= \mathcal{P}_\zeta(k)\left(1+\frac{\partial\ln\mathcal{P}_\zeta(k)}{\partial\ln k}\zeta_L\right), \\
T_S(\tilde{\eta},\tilde{k},\tilde{q},\tilde{n}^i) &= T_S\left[\frac{\partial\ln T_S}{\partial\eta}\epsilon\zeta_L + k\frac{\partial\ln T_S}{\partial k}\zeta_L + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial\ln T_S}{\partial\ln q}\zeta_L\right. \\
&\quad \left. + \left(\beta_n + \frac{a^2m^2}{q}\beta_m\right)n^i\frac{\partial\ln T_S}{\partial n^i}\zeta_L\right]. \qquad (5.186)
\end{aligned}$$

Inserting these results inside (5.185), and defining $\mathcal{P}_{\delta_{GW}} \equiv \mathcal{P}_{\delta_{GW}}(\eta,k,q,n^i)$ for convenience of notation, we arrive to

$$\begin{aligned}
\mathcal{P}_{\tilde{\delta}_{GW}}(\eta,k,q,n^i) = \mathcal{P}_{\delta_{GW}}\Bigg\{1+\zeta_L\Bigg[&\frac{\partial\ln\mathcal{P}_\zeta(k)}{\partial\ln k} \\
&+2\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial\ln q}{\partial\ln\bar{f}(q)}\frac{\partial^2\ln\bar{f}(q)}{\partial(\ln q)^2} \\
&+\epsilon\frac{\partial\ln|T_S|^2}{\partial\eta}+\frac{\partial\ln|T_S|^2}{\partial\ln k}+\left(\beta_q-\frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial\ln|T_S|^2}{\partial\ln q} \\
&+\left(\beta_n+\frac{a^2m^2}{q}\beta_m\right)n^i\frac{\partial\ln|T_S|^2}{\partial n^i}\Bigg]\Bigg\}, \qquad (5.187)
\end{aligned}$$

where as usual the dependence on the original coordinates was omitted. This important result shows that the long wavelength modes modulate the dimensionless power spectrum of the energy density contrast δ_{GW} through three different effects:

i. The first contribution comes from the scale dependence of the dimensionless power spectrum relative to the gauge invariant curvature perturbation.

ii. A second effect comes from the q-dependence of the background graviton distribution function through the combination of its first and second logarithmic derivatives.

iii. The third effect instead derives from the time, scale, momentum and direction dependence of the transfer function T_S.

As a last comment, it is worth understanding the role played by a non vanishing graviton mass. The first correction brought by the mass consist in an additional contribution to the β factor, as one can understand remembering (5.164). Moreover the graviton mass adds a non trivial q-dependence of the transfer function T_S confined inside the factor $l(\eta,\eta')$; this ultimately gives rise to a richer structure of the density contrast power spectrum, adding a derivative term $\partial\ln|T_S|^2/\partial\ln q$ which is not there in the massless case, as one can verify looking at [59].

Finally the result for the power spectrum can be used to study the modulation of the long mode on the 3-point correlation function. Let's start noticing that we can always write the large scale limit of the energy contrast δ_{GW} in the form

$$\tilde{\delta}_{GW}(\eta,k_3^i,q,n^i) = -\frac{\partial\ln\bar{f}(q)}{\partial\ln q}T_S(\eta,k_3^i,q,n_3^i)\zeta(\vec{k}_3). \qquad (5.188)$$

The long scale mode (in the following we are going to assume $k_3 \ll k_{1,2}$) indeed does not receive an extra modulation, being itself responsible for the modulation of the small scale modes, and then it assumes the same form in both the 'tilde' and 'untilde' basis. Moreover the huge scale separation between the large and small wavelength modes, suggests to think that they are non causally connected; while the small scales are beneath the horizon, as long as the long mode remain outside the horizon it cannot interfere significantly with our observable Universe. Hence we do not expect the statistics of the two scale modes to be relevantly coupled. However we don't want to treat them as completely independent perturbations, but rather we consider a weak dependence. These arguments allow the decomposition of the 3-point correlator in the following way:

$$\begin{aligned}
\lim_{k_3\to 0}\ &\langle \tilde{\delta}_{GW}(\vec{k}_1)\tilde{\delta}_{GW}(\vec{k}_2)\tilde{\delta}_{GW}(\vec{k}_3)\rangle = \\
&= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\Big\langle \langle \tilde{\delta}_{GW}(\vec{k}_1)\tilde{\delta}_{GW}(\vec{k}_2)\rangle \zeta(\vec{k}_3)\Big\rangle \\
&= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\Big\langle \langle \delta_{GW}(\vec{k}_1)\delta_{GW}(\vec{k}_2)\rangle (1+\mathcal{M}\zeta_L)\zeta(\vec{k}_3)\Big\rangle \\
&= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\mathcal{M}\Big\langle \langle \delta_{GW}(\vec{k}_1)\delta_{GW}(\vec{k}_2)\rangle \zeta_L\zeta(\vec{k}_3)\Big\rangle . \qquad (5.189)
\end{aligned}$$

where in the second line the relation (5.181) was used. This is the final result for the 3-point correlation function of the energy density contrast in the squeezed limit. It is common to parametrize the amount of the correlation by defining a non-linear parameter $f_{NL}^{\delta_{GW}}$ such that

$$\lim_{k_3\to 0}\langle \delta_{GW}(\vec{k}_1)\delta_{GW}(\vec{k}_2)\delta_{GW}(\vec{k}_3)\rangle = f_{NL}^{\delta_{GW}}\left(\frac{4\pi^4}{k_1^3k_3^3}\right)\mathcal{P}_{\delta_{GW}}(k_1)\mathcal{P}_\zeta(k_3). \qquad (5.190)$$

Using again the transformation properties of the distribution function (5.174) and the transfer function (5.186), it is also possible to give a prediction about the value of the non-linear parameter $f_{NL}^{\delta_{GW}}$. Indeed from the two transfromation properties

$$\langle \tilde{\delta}_{GW}(\vec{k}_1)\tilde{\delta}_{GW}(\vec{k}_2)\tilde{\delta}_{GW}(\vec{k}_3)\rangle = (1+o(\zeta_L))\langle \delta_{GW}(\vec{k}_1)\delta_{GW}(\vec{k}_2)\delta_{GW}(\vec{k}_3)\rangle. \qquad (5.191)$$

On the other hand, from (5.187),

$$\begin{aligned}
\lim_{k_3\to 0}&\langle \tilde{\delta}_{GW}(\vec{k}_1)\tilde{\delta}_{GW}(\vec{k}_2)\tilde{\delta}_{GW}(\vec{k}_3)\rangle = \\
&= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\Big\langle \langle \tilde{\delta}_{GW}(\vec{k}_1)\tilde{\delta}_{GW}(\vec{k}_2)\rangle \zeta(\vec{k}_3)\Big\rangle \\
&= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\frac{4\pi^4}{k_1^3k_3^3}\mathcal{P}_{\delta_{GW}}(k_1)\mathcal{P}_\zeta(k_3) \\
&\left[\frac{\partial \ln \mathcal{P}_\zeta(k)}{\partial \ln k} + 2\left(\beta_q - \frac{m^2}{q^2}\beta_m\right)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} + \epsilon\frac{\partial \ln |T_S|^2}{\partial \eta}\right. \\
&\left. + \frac{\partial \ln |T_S|^2}{\partial \ln k} + \left(\beta_q - \frac{m^2}{q^2}\beta_m\right)\frac{\partial \ln |T_S|^2}{\partial \ln q} + \left(\beta_n + \frac{m^2}{q}\beta_m\right)n^i\frac{\partial \ln |T_S|^2}{\partial n^i}\right]. \qquad (5.192)
\end{aligned}$$

Therefore, at first order the non-linear parameter reads

$$\begin{aligned}
f_{NL}^{\delta_{GW}} &= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\left[\frac{\partial \ln \mathcal{P}_\zeta(k)}{\partial \ln k}\right. \\
&+2\left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2} + \epsilon\frac{\partial \ln |T_S|^2}{\partial \eta} + \frac{\partial \ln |T_S|^2}{\partial \ln k} \\
&\left. + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \ln |T_S|^2}{\partial \ln q} + \left(\beta_n + \frac{a^2m^2}{q}\beta_m\right)n^i\frac{\partial \ln |T_S|^2}{\partial n^i}\right]. \qquad (5.193)
\end{aligned}$$

By construction, this quantity receives the same contributions studied for the density contrast power spectrum. In pure matter domination this expression greatly simplifies since $\beta_q = \mathcal{H}\epsilon = 2/5$, $\beta_n = 0$, and $\beta_m = 6/5$, while the transfer function is estimated to be

$$T_S(\eta, k_3^i, q, n_3^i) = \frac{3}{5} v^{-2} e^{-ik_3\mu l(\eta,\eta_{in})} \underset{k_3\to 0}{\simeq} \frac{3}{5} v^{-2} = \frac{3}{5}\left(1 + \frac{m^2 a^2}{q^2}\right). \tag{5.194}$$

Therefore, neglecting any time dependence of the group velocity v,

$$\begin{aligned} f_{NL}^{\delta_{GW}} &= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q}\frac{3}{5} v^{-2}\left[\frac{\partial \ln \mathcal{P}_\zeta(k)}{\partial \ln k} - \frac{8}{5}\left(1 - \frac{3a^2m^2}{q^2}\right)\frac{1}{1 + \frac{q^2}{a^2m^2}}\right] \\ &\quad -\frac{12}{25} v^{-2}\left(1 - \frac{3a^2m^2}{q^2}\right)\frac{\partial^2 \ln \bar{f}(q)}{\partial (\ln q)^2}. \end{aligned} \tag{5.195}$$

5.4 The need to go further

Only the situation where no extra graviton's degrees of freedom propagates was taken into account so far. One can then start to think how to address the same problem considering these extra modes as well. Unfortunately the computations are rather complicated and go beyond the purposes of this text. Nonetheless it is useful and instructive to set the starting framework and comment some results for possible future developments. What changes in this new situation is that we are not allowed anymore to consider the local ansatz for the scalar perturbations. The bispectrum must be instead evaluated rigorously with the *in-in formalism.* Remembering the discussion of Section (2.1.1), in order to compute the primordial power spectra we expanded the action up to second order in Goldstone bosons. Analogously, the bispectrum is sourced by three-point interactions, such that the expansion up to the third order is required. In doing this many interactions terms between the Goldstone bosons appear. In ref. [47] it is found that, up to second order in the α parameter, the new interaction terms are

$$\begin{gathered} \frac{8\alpha^2 \bar{F}_{X^2 Z}}{3a^2}\dot{\pi}^2 \partial_i \sigma^i, \\ \alpha^2 \bar{F}_{Y^2}\left(\dot{\pi}\dot{\sigma}_i\dot{\sigma}^i - \frac{\dot{\pi}\dot{\sigma}^i\partial_i\pi}{a^2} + \frac{\gamma_{ij}\dot{\sigma}^i\partial^j\pi}{a^2} - \frac{\gamma_{ij}\partial^i\pi\partial^j\pi}{a^4} - \frac{\dot{\sigma}^i\partial_j\sigma_i\partial^j\pi}{a^2} + \frac{\partial_j\sigma_i\partial^i\pi\partial^j\pi}{a^4}\right), \\ \frac{2}{3}\alpha^2 \bar{F}_{Y^2 X}\left(-\dot{\pi}\dot{\sigma}^i\dot{\sigma}_i + \frac{2\dot{\pi}\dot{\sigma}^i\partial_i\pi}{a^2}\right). \end{gathered} \tag{5.196}$$

All these terms provide different sources for the bispectra and open to the possibility of cross colletation between different modes. These are particularly relevant for unveiling the nature of the graviton mass term, since these contributions are expected to vanish in the limit $\alpha \to 0$. For example in [47] they found that in the squeezed limit configuration with $k_1 = k_L \to 0$, the scalar bispectrum is

$$\begin{aligned} \langle\prod_{i=1}^{3}\zeta_{\mathbf{k}_i}\rangle = -H^3\langle\prod_{i=1}^{3}\pi_{\mathbf{k}_i}\rangle &= -\frac{\alpha^2 H^8}{16 c_\pi^{10} k_L^3 k_S^3}\sqrt{2(-\bar{F}_X + 2\bar{F}_{X^2})}\ln\left(\frac{k_L}{k_S}\right) \times \\ &\quad \times \left[9c_\pi^2\lambda_1\lambda_4 + (3\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)c_\pi^2\hat{S}_2 - 27\lambda_1\lambda_2\hat{S}_1\right], \end{aligned} \tag{5.197}$$

with

$$\hat{S}_1 = -\cos^2\theta, \qquad \hat{S}_2 = 1 - 2\cos^2\theta,$$

θ being the angle between the long and the short wavelength modes, and λ_i, $i = 1, 2, 3, 4$ four coefficients built upon derivatives of the the term $F(X, Y^i, Z_{ij})$. Interesting features of this solution that worth to be mentioned are the log-enhancement $\ln(k_L/k_S)$ which we encountered even in the angular power spectrum, and the non trivial angular dependence inside the terms $\hat{S}_{1,2}$. Besides this three-point correlator, one can investigate many other terms. In [47] for example the cross correlator between a long wavelength tensor mode and two short wavelength scalar modes is computed. It is clear that this scenario leaves a wide open panorama where many models can be tested and hopefully compared with future observations; this may be an interesting topics for future developments.

Chapter 6

SGWB Angular Power Spectrum

All the previous analysis was dedicated to obtain theoretical predictions for angular correlators of the SGWB. In particular we found deviations from GR predictions both at the time of generation of the perturbations and through their propagation until today. In the present Chapter we want to turn back to physical observations and quantify graviton mass effects. First of all, in Section (6.1), we link the graviton density fluctuations to the true observable, that is the energy density contrast. This relation was left implicit in Section (3.3); now we want to evaluate explicitly the factor relating the two quantities in order to estimate the amplitude of the correlators we expect in our interferometers. After this short digression, the angular power spectra are analyzed. Since a fully analytic solution describing the angular spectra is not accessible, we take advantage of the publicly available codes CLASS [111] and hi_class [112, 113] to attempt a numerical solution. Actually these codes were meant to analyze CMB anisotropy, so that few modifications had to be applied to the source code in order to adapt it to consider SGWB. Both the scalar and tensor sourced cases are then plotted and discussed.

6.1 GW energy density

Although it is easier to work with the distribution density fluctuation Γ, one should always remember that our physical observable when dealing with anisotropies in the SGWB is the energy contrast δ_{GW} defined in section (3.3). It was shown that the two quantities are related by

$$\delta_{GW}(\eta_0, \vec{x}, q, \hat{n}) = \left[4 - \frac{\partial \ln \bar{\Omega}_{GW}(\eta_0, q)}{\partial \ln q} - \left(1 + \frac{q^2}{a^2 m^2}\right)^{-1}\right] \Gamma(\eta_0, x^i, q, n^i). \tag{6.1}$$

The first step is to obtain an explicit expression for the gravitational wave energy density per logaritmic frequency interval Ω_{GW} defined in (3.42). Adopting the convention of ref. [108], the energy density carried by gravitational waves perturbations today is defined as

$$\rho_{GW}(\eta_0) = T^{00}_{GW} = \frac{M_P^2}{4} \frac{\langle \chi'_{ij}(\vec{x}, \eta_0) \chi'_{ij}(\vec{x}, \eta_0) \rangle}{a^2(\eta_0)}. \tag{6.2}$$

On the other and, working out the tensor equations of motion (4.84), one can easily verify that the re-scaled tensor field $h_{ij}(\mathbf{x}, \eta) = a(\eta)\chi_{ij}(\mathbf{x}, \eta)$ satisfies, in momentum space, the Klein-Gordon equation

$$h''_{ij} + \left(q^2 - \frac{a''}{a} + a^2 m^2\right) h_{ij} = 0. \tag{6.3}$$

Therefore its solution can be written as a superposition of positive and negative energy plane waves

$$h_{ij}(\mathbf{x},\eta) = \sum_{\lambda=+,\times} \int \frac{d^3q}{(2\pi)^{3/2}} \left[h_{q,\lambda}(\eta)e^{i\mathbf{qx}}a_{q,\lambda} + h^*_{q,\lambda}(\eta)e^{-i\mathbf{qx}}a^\dagger_{q,\lambda}\right] e_{ij,\lambda}(\hat{q}). \tag{6.4}$$

and the same form of the solution must be shared by the original perturbation χ_{ij}:

$$\chi_{ij}(\mathbf{x},\eta) = \sum_{\lambda=+,\times} \int \frac{d^3q}{(2\pi)^{3/2}} \left[\chi_{q,\lambda}(\eta)e^{i\mathbf{qx}}a_{q,\lambda} + \chi^*_{q,\lambda}(\eta)e^{-i\mathbf{qx}}a^\dagger_{q,\lambda}\right] e_{ij,\lambda}(\hat{q}). \tag{6.5}$$

with the appropriate consistency condition relating the two Fourier modes

$$h_\lambda(q,\eta) = \frac{\chi_\lambda(q,\eta)}{a(\eta)}. \tag{6.6}$$

In the framework of second quantization, the functions $a_{q,\lambda}$ and $a^+_{q,\lambda}$ becomes the ladder operators, and they satisfy the canonical commutation relations

$$\begin{aligned} \left[\hat{a}_{q,\lambda}, \hat{a}^+_{q',\lambda'}\right] &= \delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{q}-\mathbf{q'}), \\ \left[\hat{a}_{q,\lambda}, \hat{a}_{q',\lambda'}\right] &= \left[\hat{a}^+_{q,\lambda}, \hat{a}^+_{q',\lambda'}\right] = 0. \end{aligned} \tag{6.7}$$

This description in terms of harmonic oscillator correspond to a particular choice of the vacuum state, called Bunch-Davies vacuum [108]

$$\chi_\lambda(q,\eta) = \frac{e^{-iq\eta}}{a\sqrt{2q}}. \tag{6.8}$$

Inserting the decomposition (3.89) inside (6.5) and exploiting the commutation relations (6.7) one eventually obtains

$$\begin{aligned} \rho_{GW}(\eta_0) &= \frac{M_P^2}{4a_0^2(2\pi)^3} \int d^3\mathbf{q} \left[\chi'(q,\eta_0)\right]^2 |\xi_\lambda(q)|^2 \\ &= \frac{M_P^2}{8\pi^2 a_0^2} \int_0^\infty dq q^2 \left[\chi'(q,\eta_0)\right]^2 |\xi_\lambda(q)|^2 \\ &= \frac{\rho_{c,0}}{12H_0^2 a_0^2} \frac{1}{2\pi^2} \int_0^\infty \frac{dq}{q} q^3 \left[\chi'(q,\eta_0)\right]^2 |\xi_\lambda(q)|^2. \end{aligned} \tag{6.9}$$

Comparing the last expression with (3.42), the energy density reads

$$\Omega_{GW}(q) = \frac{1}{12H_0^2 a_0^2} \left[\chi'(q,\eta_0)\right]^2 \mathcal{P}_\lambda(q). \tag{6.10}$$

At the present time, we can consider the solution for the tensor transfer function $\chi(q,\eta)$ during matter domination, which was evaluated in (4.111) as

$$\chi(\eta,q) \simeq 3\,\frac{j_1(q\eta)}{q\eta} j_0\left(\frac{mH_0^2\eta^3}{12}\right). \tag{6.11}$$

However, in our experiments, the scale we can probe are those which are beneath the horizon at the present time. For this reason the above solution must be considered in the sub-horizon regime, where the transfer function shows a rapid oscillating behavior. It is common (see ref. [108]) to approximate this oscillation considering $\langle\chi'(q,\eta)\rangle \simeq q\chi(q,\eta)$. Moreover, from an observational point of view, the exact solution of the transfer function $\chi(\eta_0)$ is not really meaningful at late time, because of the oscillations that rapidly change its value. What we really observe is an average of the value of the transfer function; then an averaging procedure on the spherical Bessel function is needed. For this purpose it is useful to expand them in trigonometric functions as

$$\langle\left[\chi'(q,\eta_0)\right]^2\rangle \simeq q^2 \frac{9}{(q\eta_0)^4}\left(\frac{\sin(q\eta_0)}{q\eta_0} - \cos(q\eta_0)\right)^2 j_0^2\left(\frac{mH_0^2\eta^3}{12}\right). \tag{6.12}$$

On sub-horizon one can take the limit $q\eta \gg 1$, and then, averaging the cosine function, it brings

$$\langle\left[\chi'(q,\eta_0)\right]^2\rangle \simeq \frac{9}{2q^2\eta_0^4} j_0^2\left(\frac{mH_0^2\eta^3}{12}\right). \tag{6.13}$$

The same arguments applies for those modes which re-enter the horizon before the time of matter-radiation equivalence. In this case however the solution for the transfer function requires a bit of care in order to match the transition from radiation domination and matter domination. The full solution for the massless case is shown in ref. [108]. In the present case it generalizes to

$$\chi(q,\eta) = A(q)\frac{j_1(q\eta)}{q\eta} j_0\left(\frac{mH_0^2\eta^3}{12}\right) + B(q)\frac{y_1(q\eta)}{q\eta} y_0\left(\frac{mH_0^2\eta^3}{12}\right), \tag{6.14}$$

with

$$\begin{aligned} A(q) &= \frac{3}{2} - \frac{\cos(2q\eta_{eq})}{2} + \frac{\sin(2q\eta_{eq})}{q\eta_{eq}}, \\ B(q) &= \frac{1}{q\eta_{eq}} - q\eta_{eq} - \frac{\sin(2q\eta_{eq})}{2} - \frac{\cos(q\eta_{eq})}{q\eta_{eq}}. \end{aligned} \tag{6.15}$$

Since in this regime $q\eta_{eq} < 1$, the most dominant term in the transfer function is

$$\begin{aligned} \chi(q,\eta) &= -q\eta_{eq}\frac{y_1(q\eta)}{q\eta} y_0\left(\frac{mH_0^2\eta^3}{12}\right) \\ &= \frac{\eta_{eq}}{q^2\eta^2}\left[\sin(q\eta) + \frac{\cos(q\eta)}{q\eta}\right] y_0\left(\frac{mH_0^2\eta^3}{12}\right). \end{aligned} \tag{6.16}$$

Considering now the derivative of this expression at late time with the same approximation $\langle\chi'(q,\eta)\rangle \simeq q\chi(q,\eta)$ and performing the average procedure over oscillations, one eventually ends up with

$$\langle\left[\chi'(q,\eta)\right]^2\rangle \simeq \frac{\eta_{eq}^2}{2\eta_0^2} y_0^2\left(\frac{mH_0^2\eta^3}{12}\right). \tag{6.17}$$

Summing up the results in the two limiting case, the tensor transfer function can be written in a compact way as

$$\langle\left[\chi'(q,\eta)\right]^2\rangle \underset{q\eta_0\gg 1}{\simeq} \begin{cases} \dfrac{9}{2q^2\eta_0^4} j_0^2\left(\dfrac{mH_0^2\eta^3}{12}\right), & \text{if} \quad q < \eta_{eq}^{-1}, \\ \dfrac{\eta_{eq}^2}{2\eta_0^2} y_0^2\left(\dfrac{mH_0^2\eta^3}{12}\right), & \text{if} \quad q > \eta_{eq}^{-1}, \end{cases} \tag{6.18}$$

where $j_0(x)$ and $y_0(x)$ denote respectively the spherical Bessel and Neumann functions.

The dimensionless power spectrum at the epoch of inflation for massive tensor modes, $\mathcal{P}_\lambda^{\rm inf}(q)$, was obtained in (2.50), which we report here for convenience

$$\mathcal{P}_\lambda(q) = \frac{2^{2\nu-3}H^2}{\pi^2 M_P^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2\left(\frac{q}{aH}\right)^{n_T}, \tag{6.19}$$

with

$$n_T \equiv 3-2\nu = 3\left[1-\sqrt{1-\frac{4}{9}\left(\frac{m_g^2}{H^2}-3\epsilon\right)}\right]. \tag{6.20}$$

Summing up all the contributions, the energy density parameter is

$$\Omega_{GW}(q) \simeq \frac{2^{2\nu-3}H^2}{24H_0^2\eta_0^2\pi^2M_P^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2\left(\frac{q}{aH}\right)^{n_T}\begin{cases}\dfrac{9}{q^2\eta_0^2}\,j_0^2\left(\dfrac{mH_0^2\eta^3}{12}\right), & q<\eta_{eq}^{-1},\\ \eta_{eq}^2\,y_0^2\left(\dfrac{mH_0^2\eta^3}{12}\right), & q>\eta_{eq}^{-1},\end{cases} \tag{6.21}$$

and its logaritmic derivative is

$$\frac{\partial\ln\Omega_{GW}(q)}{\partial\ln q} = \begin{cases} n_T-2 & q<\eta_{eq}^{-1}\\ n_T & q>\eta_{eq}^{-1}\end{cases}. \tag{6.22}$$

At the end of the day the proportionality factor between the density contrast and the fluctuation Γ is

$$\begin{aligned}\alpha &\equiv 4-\frac{\partial\ln\bar\Omega_{GW}(\eta_0,q)}{\partial\ln q}-\left(1+\frac{q^2}{a^2m^2}\right)^{-1}\\ &= \begin{cases}4-n_T+2-\left(1+\dfrac{q^2}{a^2m^2}\right)^{-1}, & \text{if}\quad q<\eta_{eq}^{-1},\\ 4-n_T-\left(1+\dfrac{q^2}{a^2m^2}\right)^{-1}, & \text{if}\quad q>\eta_{eq}^{-1}.\end{cases}\end{aligned} \tag{6.23}$$

For any practical purposes, the last term of this equation can be safely neglected. Indeed it is reasonable to pose an upper bound on the physical momentum $p_{\max}\sim 10^{15}$GeV, which correspond to the estimated temperature of the Universe during inflation (all estimated values refer to ref. [66]). Gravitons are produced with wide range of physical momenta up to $p_{\max}$ which correspond to a comoving momentum $q_{\max}\sim 10^{-11}$GeV. In light of the astrophysical bounds on the graviton mass at late times, one can consider $m\lesssim 10^{-38}$GeV. Therefore, evaluated at the present time, this factor approximately gives $q^2/a_0^2m^2\sim 10^{54}$, providing then an highly suppressed term. Given these considerations, one can take

$$\alpha = \begin{cases}6-3\left(1-\sqrt{1-\dfrac{4m^2}{9\bar H^2}}\right), & \text{if}\quad q<\eta_{eq}^{-1},\\ 4-3\left(1-\sqrt{1-\dfrac{4m^2}{9\bar H^2}}\right), & \text{if}\quad q>\eta_{eq}^{-1}.\end{cases} \tag{6.24}$$

As an example, one can see what happens considering $m^2/H^2\approx 1$. In this case the tensor spectral index evaluates to $n_T\simeq 0.764$ and the proportionality factor to

$$\alpha = \begin{cases}5.236, & \text{if}\quad q<\eta_{eq}^{-1},\\ 3.236, & \text{if}\quad q>\eta_{eq}^{-1}.\end{cases} \tag{6.25}$$

6.2 Numerical results

In this section, numerical results for the angular correlators are shown. For this purpose it was used the publicly available code CLASS [111] and its extension hi_class [112, 113]. Actually the code is meant to study the CMB, then some modifications are needed in order to apply it to the case of the SGWB. These modifications are shown in Appendix (G). The quantity we are interested in are the angular power spectra in the scalar and tensor sourced case, that is

$$\begin{aligned}
\tilde{C}_{\ell,S} &= 4\pi \int \frac{dk}{k} |\mathcal{T}_\ell^S(\eta_0, \eta_{in}, k, q)|^2 \mathcal{P}_\zeta(k), \\
\tilde{C}_{\ell,T} &= 4\pi \int \frac{dk}{k} |\mathcal{T}_\ell^T(\eta_0, \eta_{in}, k, q)|^2 \sum_{\lambda=\pm 2} \mathcal{P}_\lambda(k),
\end{aligned} \tag{6.26}$$

with the angular transfer functions

$$\begin{aligned}
\mathcal{T}_\ell^S(\eta_0, \eta_{in}, k, q) &\equiv v^{-2} T_\Phi(\eta_{in}, k) j_\ell \left[k l(\eta_0, \eta_{in}) \right] \\
&\quad + \int_{\eta_{in}}^{\eta_0} d\eta \frac{\partial \left[T_\Psi(\eta, k) + v^{-2} T_\Phi(\eta, k) \right]}{\partial \eta} j_\ell \left[k l(\eta_0, \eta) \right], \\
\mathcal{T}_\ell^T(\eta_0, \eta_{in}, k, q) &\equiv \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \frac{1}{4} \int_{\eta_{in}}^{\eta_0} d\eta \, \chi'(\eta, k) \frac{j_\ell(k l(\eta_0, \eta))}{k^2 l^2(\eta_0, \eta)}.
\end{aligned} \tag{6.27}$$

6.2.1 Graviton mass effects on the scalar sourced angular spectrum

In the scalar case one can see that the graviton mass enters in the angular transfer function just with a multiplicative factor related to the graviton velocity. This factor is present, in different ways, both in the Sachs-Wolfe and integrated Sachs-Wolfe terms. In principle, what one should expect is to see a gain in the angular spectra for lower velocities, which corresponds to higher values of the graviton mass. This behavior is indeed captured in both the two contributions shown in Figures (6.1) and (6.2) In Figures (6.1),(6.2) the difference between the graviton velocity and the speed of light was highly overestimated for the pedagogical purpose to show how the late graviton mass m would modify the shape of the SGWB angular scalar sourced spectra. Indeed, astrophysical observations set strong bounds on the graviton mass at late times (see ref. [10]); in light of these bounds one can reasonably take $m \lesssim 10^{-28}\,\text{eV}$ [3, 6]. On the other hand the averaged present grativon energy can be estimated by red-shifting its value from the time of production until today. Assuming that at initial time gravitons were produced with $E_{\text{inf}} \sim 10^{13}\,\text{GeV}$ when the Universe was $a_0/a_{\text{inf}} \sim 10^{26}$ times smaller, this gives

$$E_0 = E_{inf} \left(\frac{a_{\text{inf}}}{a_0} \right) \sim 10^{-11}\,\text{GeV},$$

and the displacement between the graviton and light velocities in this estimation is bounded by

$$c^2 - v^2 = \frac{m^2}{E^2} \lesssim 10^{-54}, \tag{6.28}$$

which brings a completely inappreciable correction to the scalar angular spectra. Therefore we do not expect to see any difference arising in this sector from the massive and massless case. The interesting feature to notice in the SW contribution is the decreasing trend of

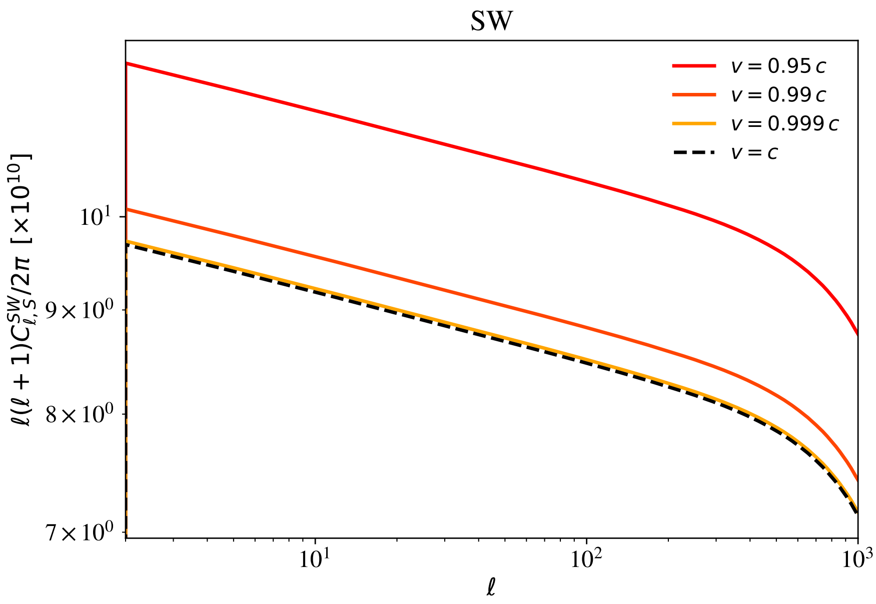

Figure 6.1: *Sachs-Wolfe contribution to the SGWB scalar sourced angular power spectrum with different values of the late time graviton velocity. The velocities are taken arbitrarily in order to highlight how the graviton mass modifies the angular specra. In a real situation, due to the strong astrophysical bounds, we don't expect to be sensitive to this effect. The plots show a slowly decreasing behavior caused by the growing rate $g_{rad}(k,\eta_{in})$ that suppresses the small scales.*

the curve, which is different from the classical behavior characterizing CMB anisotropies. This peculiar trend is well motivated by the analytic expression (4.133)

$$\tilde{C}_{\ell,S} = \frac{16\pi}{9}\int \frac{dk}{k}\left| v^{-2} g_{\rm rad}(k,\eta_{in}) j_\ell\left[kl(\eta_0,\eta_{in})\right]\right|^2 \mathcal{P}_\zeta(k). \tag{6.29}$$

In this equation the growing rate $g_{rad}(k,\eta_{in})$ acts as a low-pass filter, selecting those scales which at initial time happened to be outside the horizon. On the other hand, the spherical Bessel function , for fixed multipole ℓ, gets more contribution from those scales such that $\ell \simeq kl(\eta_0,\eta_{in})$. Therefore, increasing the multipole ℓ, means to consider larger wavenumbers k which are closer to the horizon at initial epoch; this means that larger multipoles ℓ are slowly suppressed by the growing rate $g_{\rm rad}(k,\eta_{in})$.

An analytic solution for the ISW term is much harder to find. Anyway we can still comment in a qualitative way the behavior shown in Figure (6.2) to learn some physics. This picture clearly shows two different regimes for low and high multipoles. In order to understand the origin of this splitting, let's first remember the analytic expression of the ISW contribution

$$\tilde{C}_{\ell,S}^{ISW} = \frac{16\pi(1+v^{-2})}{9}\int \frac{dx}{x}\mathcal{P}_\zeta\left(\frac{x}{v\eta_0}\right)\left\{\int_{\eta_{eq}}^{\eta_0} d\eta \frac{dg(k,\eta)}{d\eta} j_\ell\left[x\left(1-\frac{\eta}{\eta_0}\right)\right]\right\}^2. \tag{6.30}$$

The two different regimes arise from the definition of the growing rate $g(k,\eta)$, for which in Section (4.3) we found two different expressions in radiation domination or in a Universe dominated by pressureless dust and a cosmological constant. Large scales, i.e. low ℓ, are frozen during radiation domination, and re-enter the horizon after matter-radiation equality. The growing rate for these modes is defined in (4.74). Figure (4.3) shows that

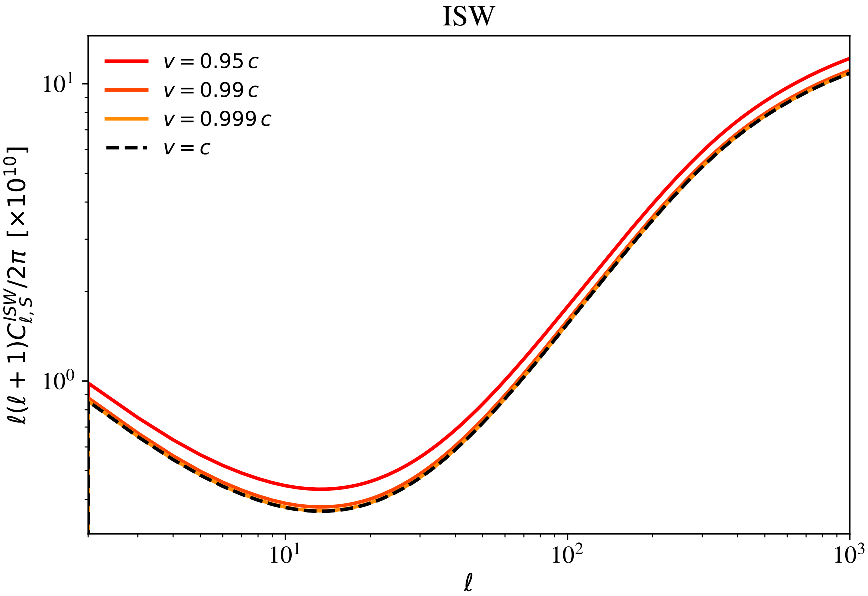

Figure 6.2: *Integrated Sachs-Wolfe contribution to the scalar sourced angular power spectrum with different values of arbitrarily chosen late time graviton velocity. Two regimes can be distinguished: low multipoles ($\ell < 20$) that receive contributions from those modes which re-enter during matter domination, and large mutipoles ($\ell > 50$) which are sourced by those modes which re-enter when the Universe was still dominated by radiation.*

the growing rate after the epoch of equality is k-independent and very slowly changing in time. Hence large wavelength modes brings a really tiny contribution to the angular power spectra. On the opposite, short wavelength modes re-enter during radiation domination, where the growing rate has a steep decay in correspondence to the horizon crossing. This modes then can brings a much larger contribution. The separation between the two regimes can be approximately evaluated considering modes that re-enter at the time of matter-radiation equality. Remembering (4.55), this is realized for modes such that $k\eta_{eq} \approx 1$. On the other hand, the spherical Bessel function selects $k\eta_0 \approx \ell$. Combining these two conditions one has[1]

$$\ell \approx \frac{\eta_0}{\eta_{eq}} \approx 100, \tag{6.31}$$

which identifies quite well the multipole scale at which the two regimes intersect. Moreover, very small scales cross the horizon very early, such that $\eta/\eta_0 \ll 1$. In this regime the above expression (6.30) simplifies to

$$\tilde{C}^{ISW}_{\ell,S} \simeq \frac{16\pi(1+v^{-2})}{9} \int \frac{dx}{x} \mathcal{P}_\zeta \left(\frac{x}{v\eta_0}\right) j_\ell(x)^2, \tag{6.32}$$

which has exactly the same structure of the SW term on large scales. Indeed, if we compare the two Figures (6.1) and (6.2), we can notice that the SW term on large scales and the ISW term on small scales bring contributions of the same order.

On the contrary, much more visible effects can arise taking into account the graviton mass during inflation. In Section (2.2.2) it was derived the power spectrum for the primordial curvature perturbation. The intriguing result is that this quantity gets contributions

[1] The value of the conformal time at the epoch of equality and today are reported in the Introduction. They are taken from [111].

from both the inflaton field and the scalar extra mode of the massive graviton, giving rise to a scalar spectral index

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}(k)}{d \ln k} = -2\epsilon + \frac{18\alpha^2 \lambda_1^2}{c_\pi^2}. \tag{6.33}$$

This shows that Lorentz-symmetry violation during inflation drives the primordial scalar spectrum toward a blue tilt (Planck measurements however ensure that a blue spectrum is not achievable). Although knowing that a relation exists between the early graviton mass parameter m_g and the symmetry violation parameter α, since these quantities are linked via model-dependent relation (2.122), we are not able to connect them in full generality without fixing a model. Hence we chose to treat them as independent parameters. In Figure (6.3) the total scalar sourced angular power spectrum for different values of n_s is shown. Plack CMB measurements bound the scalar spectral index to be $n_s = 0.965 \pm 0.04$

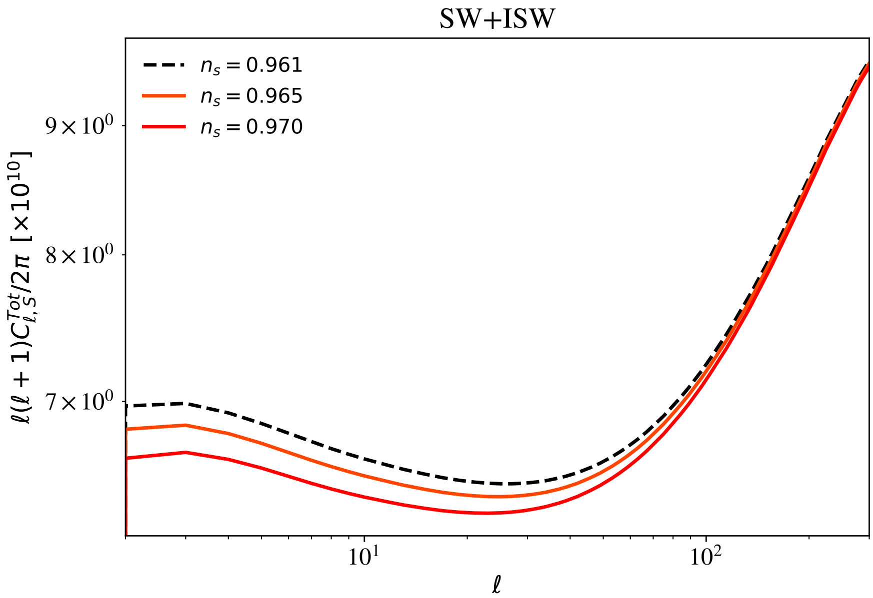

Figure 6.3: *Scalar sourced angular power spectrum for different values of the scalar spectral index. The scalar spectral index is directly related, through other model-dependent coefficients, to the parameter α controlling the amount of space-diffeomorphism violation during inflation in the gravity sector. The more amount of violation, the more the priordial spectrum is pushed toward a blue tilt and angular power spectrum is suppressed at large scales. Here the values of n_s are arbitrarily taken within the CMB data range [109] $n_s = 0.965 \pm 0.004$ just to show how space-diffeomorphism symmetry violation modifies the angular spectra. If, through an independent experiment, we will succeed in measuring ϵ, we will be able to fix the GR prediction and parametrize any possible deviation with equation* (6.33).

[109]. However, in equation (6.33) the two parameters ϵ and α are degenerate, and we are not able to discriminate between the two contributions at this level. If we were able to measure, with enough precision, the slow-roll parameter ϵ in an independent experiment, this analysis could unveil the amount of contribution attributable to a mechanism of space-diffeomorphism violation during inflation. Here we arbitrarily assumed that the value predicted by the model with $\alpha = 0$ for the scalar spectral index is $n_s(\alpha = 0) = 0.961$ (the dashed black line in Figure (6.3)), and we make vary the symmetry-violation contribution within the range $n_s = 0.965 \pm 0.004$ measured by Planck experiment [109], relying on the reasonable assumption that the α correction should be of the order of the slow-roll

parameter. It worth to notice that more evident signatures are visible on the lowest angular multipoles, until $\ell \sim 100$. This corresponds to the range where future interferometers as LISA and ET are expected to work [16, 15].

6.2.2 Graviton mass effects on the tensor sourced angular spectrum

The effects of the graviton mass are more evident in the tensor sector. As seen in Chapter (2), the mass term explicitly modifies the equations of motion for tensor modes providing deviation from the GR predictions arising both from the generation mechanism during inflation and from their propagation across the Universe. We studied these effects separately in order to understand better how they enter in the angular correlators. In Figure (6.4) it is shown the tensor sourced angular power spectrum taking into account only the late time graviton mass effect, which modifies the Einstein equations of motion as

$$\chi''(\eta,k) + 2\mathcal{H}\chi'(\eta,k) + (k^2 + m^2a^2)\chi(\eta,k) = 0. \tag{6.34}$$

All the current observations are in agreement with a single field slow-roll inflation. Assum-

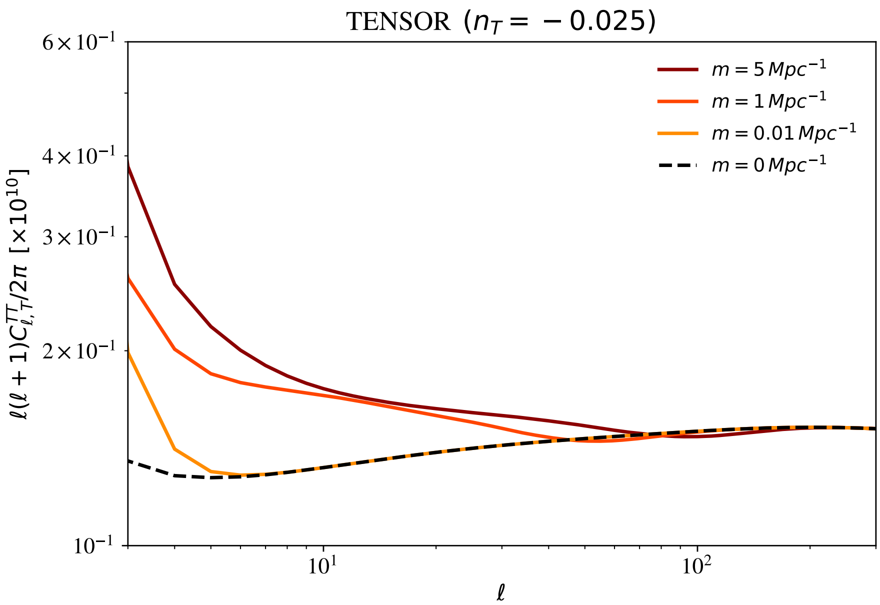

Figure 6.4: *Tensor sourced angualr power spectrum for different values of the late time graviton mass m and with fixed tensor spectral index $n_T = -0.025$ ($r = 0.2$). We chose the values of the graviton mass within the permitted range allowed by astrophysical bounds. Notice that, according to* (4.172)*, only the solutions with $\ell \geq 3$ must be considered, and that the value of the late graviton mass cosidered in this plot are well below our current bounds.*

ing the single slow-roll scenario, the Planck observations [109] and the BICEP2 and Keck Array Collaborations [110] put strong upper bound on the tensor-to-scalar ratio $r < 0.06$. Nonetheless, for pedagogical reasons, we take $r \simeq 0.2$ in order to emphasize the interesting signatures that are expected to arise in the tensor sourced angular power spectrum[2] and considered then the value of the tensor spectral index predicted by the slow-roll inflation scenario, that is $n_T = -r/8 \simeq -0.025$ [75]. Finally we modified the Einstein equations

[2]Anyway, it is still possible that inflation didn't happened with a single slow rolling field, and the possibility to have a larger value for r then the one predicted by the standard single slow-roll field paradigm is not completely ruled out today.

inside the CLASS code to reproduce (6.34) (see Appendix (G) for a detailed explanation of the modification applied to the code). The plots (6.4) must be compared with the semi-analytic solution (4.171); Given the small value of the tensor spectral index, we can fairly look at the particular solution in the scale invariant case ($n_T = 0$):

$$\frac{\ell(\ell+1)}{2\pi}\tilde{C}_{\ell,T} = \frac{1}{30}\sum_{\lambda=\pm 2}\mathcal{P}_\lambda(k_0)\,\frac{\ell(\ell+1)}{(\ell+3)(\ell-2)}. \tag{6.35}$$

The trends of the curves are perfectly described by the analytic solution. This is a strong proof in support of our analytic analysis. The piece of information not captured by equation (6.35) is that regarding the effects of the late time graviton mass. This is because of our rough approximation in replacing the time derivative of the tensor transfer function with a Dirac delta peaked at the horizon crossing time. This approximation looses tracks of the graviton mass, which only remains encoded in the primordial power spectrum and goes completely away in the limit $n_T = 0$. Nonetheless we can give a reasonable qualitative explanation of the fact that large scales, i.e. low multipoles, enhance their amplitude for larger values of the late graviton mass at fixed n_T. As already stressed, the more the mass, the earlier tensor modes re-enter the horizon. This means that massive modes remain frozen outside the horizon for shorter times, turning on the tensor angular transfer function (3.88), which depends on the derivative of the linear tensor transfer function, for longer times. On smaller scales this effect becomes weaker because small wavelength gravitons are so energetic that the mass term becomes more and more negligible. The net effect is then a peculiar signature on low multipoles; the heavier the mass the stronger the signature. More specifically, signatures of late masses of order $m \simeq 1\,\mathrm{Mpc}^{-1}$ are expected to be visible until $\ell \simeq 40 \div 50$, while masses of the order $m \simeq 0.01\,\mathrm{Mpc}^{-1}$ leave a distinct signature only below $\ell \leq 5$. This argument promotes the tensor sector as the golden channel to unveil a possible graviton mass in late times. Recent observations of binary pulsars have put the current contraint [3, 6, 10]

$$m \leq 2\times 10^{-28}\,\mathrm{eV}/c^2 \sim 3\times 10^{1}\,\mathrm{Mpc}^{-1} \qquad (95\,\%\,\mathrm{C.L.}). \tag{6.36}$$

In Figure (6.4) one can see that even much smaller values, around the inverse megaparsec scale, would bring appreciable effects.

The most evident signature comes from the graviton mass in the primordial stages. As it was shown in Section (2.1.1), the early graviton mass enters in the primordial power spectra modifying the tensor spectra index according to

$$n_T \equiv 3 - 2\nu = 3\left[1 - \sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{\bar{H}^2} - 3\epsilon\right)}\right]. \tag{6.37}$$

This is a modified expression of the consistency relation telling us that for large enough values of the early graviton mass, the tensor spectral index acquires a positive value, and the primordial power spectrum becomes blue tilted. In the following table we list some values that will be used later:

ϵ	$m_g^2/\bar{H}^2$	n_T
0.0125	1	0.730
0.0125	0.1	0.042
0.0125	3ϵ	0.0
0.0125	0	−0.025

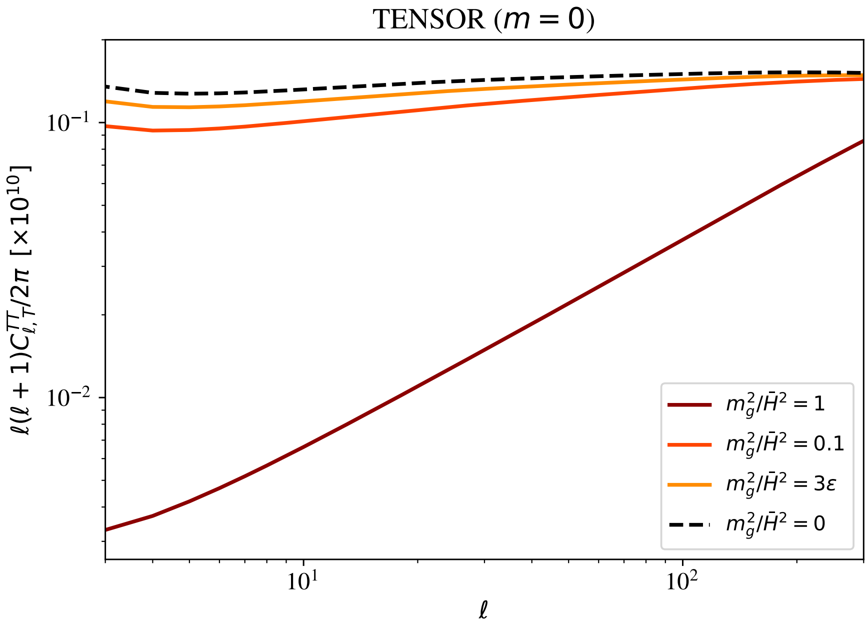

Figure 6.5: *Tensor sourced angualr power spectrum for different values of the early time graviton mass m_g ($r = 0.2$). For small values of m_g the curves nearly follows the predicted Harrison-Zel'dovich trend, while deviations become significant for $m_g^2 \sim \bar{H}^2$, where the mentioned trend is completely spoiled.*

The first evident effect shown in Figure (6.5) is the suppression in amplitude of the more massive modes on large scales. Moreover, while lighter modes nearly maintain the expected scale invariant trend discussed above, modes with large enough masses ($m_g^2 \sim \bar{H}^2$) show a more and more increasing deviation with a rapid and appreciable growth at all the scales from $\ell = 3$ up to $\ell \simeq 500$, falling in the visible range of LISA and ET.

6.2.3 Graviton mass effects on the total angular spectrum

Dealing with real observations, it is not possible to separate between the scalar sourced and the tensor sourced contributions to angular power spectrum. Hence it is more useful to look for the whole angular spectrum taking into account all the sources. This is shown in Figures (6.6) and (6.7), where the effects due to the late and early graviton masses are considered respectively. In the first case only the tensor sector provides some appreciable deviation from GR. As a result, possible distinctive signature may be visible on very low multipoles ($\ell \leq 5 \div 20$ depending on the value of m). In the second case instead, deviations arise both from the tensor and scalar sectors. Comparing with the previous figure, in this case signatures are visible also for larger multipoles (until $\ell \sim 300$ in the more optimistic case $m_g^2 \simeq \bar{H}^2$). Last thing to notice, comparing the above plots, is that late and early graviton masses have very different implications on low multipoles angular spectrum. Indeed, while it was already noticed that larger late graviton masses increase the angular spectrum, early graviton masses tend to damp it.

At this point one may wonder if there is a way to disentangle between the effects arising from the early and late graviton masses in the tensor sector. The answer is already written in the previous figures, and consists in a multipole separation between the two effects. This separation is better shown in Figure (6.8). While the effects of the graviton late mass

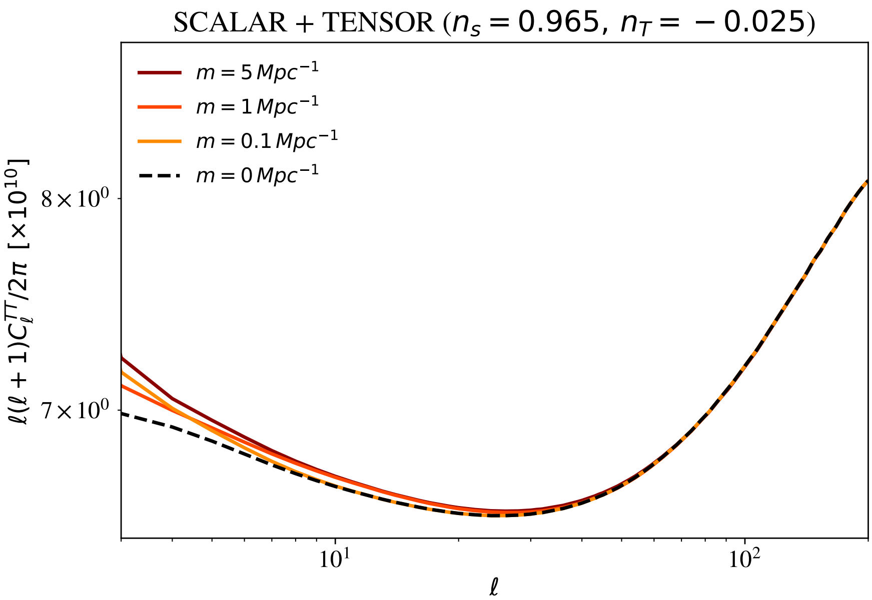

Figure 6.6: *Scalar and Tensor sourced angular power spectrum for different values of the late time graviton masses and fixed $m_g^2 = 0$ ($r = 0.2$). Since it was already pointed out that deviation from the light speed are completely negligible at this level, in these plot it was fairly taken $v = c$. The signatures present in the tensor case survive in the total angular spectrum on very low multipoles.*

are confined on the very largest scales, the early graviton mass survives even for larger multipoles. For practical purposes one can then treat the effects of a late graviton mass as an higher order correction which superimposes on the largest scales with the more evident modification brought by the early graviton mass. Notice further that the signatures of the late mass becomes more and more weak for larger values of the tensor spectral index. This is because the enhancement brought by the late mass is balanced by the factor

$$\frac{\sum_{\lambda=\pm2}\mathcal{P}_\lambda(k_0)}{k_0^{n_T}\,l(\eta_0,\eta_*)^{n_T}} \tag{6.38}$$

entering in the tensor sourced angular spectrum (4.171). For $n_T = 0.730$ the late time graviton mass effects become completely irrelevant.

In the case modifications from GR come with the appearance of the extra modes, on top of the just studied effect, one has to add the signatures coming from the change in the scalar spectral index. In Figure (6.9) we address this analysis considering the effect of a modified scalar spectral index having fixed the early graviton mass to $m_g^2 = \bar{H}^2$, and hence $n_T = 0.730$. It may be an interesting development for future works to analyze some particular models of inflation with space-diffeomorphism violation and production of extra degrees of freedom in such a way to fix the relation between m_g and α and pursue the same analysis in term of one parameter only.

To conclude, we come back to the more strict prediction of single field slow roll inflation model. In this case one has to consider the bound $r < 0.06$ [110]. As one can see in Figure (6.10), because of this condition, signatures arising from the tensor sector become weaker in the total SGWB angular power spectrum. The same figure shows clearly one more time the different contributions coming from the late and early graviton masses. While the former increases the the angular spectrum in the rage $\ell \leq 10$, the latter damps it for far larger multipoles, providing possibly visible effects until $\ell \sim 100$. Evident signatures could

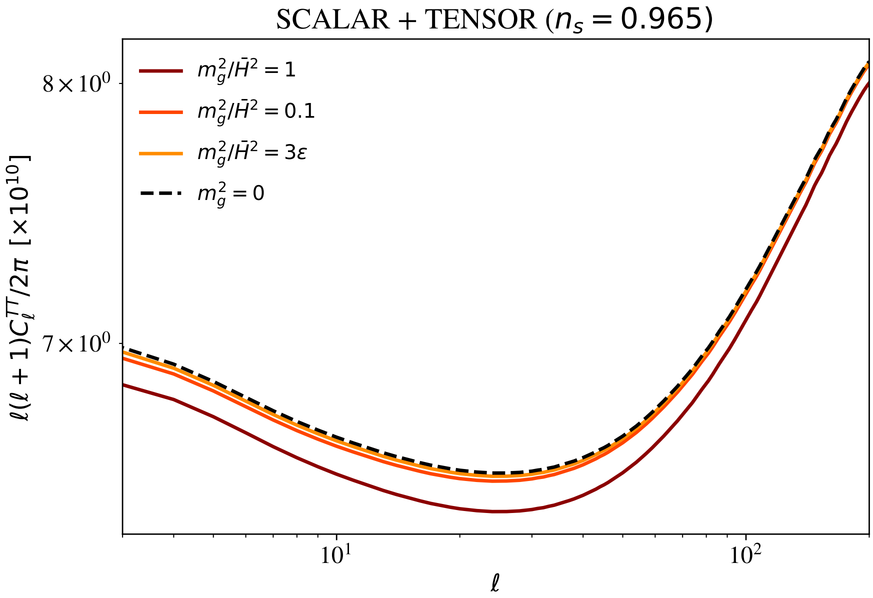

Figure 6.7: *Scalar and Tensor sourced angular power spectrum for different values of the early graviton masses with fixed late graviton mass $m = 0$. Only the effect on the tensor spectral indices are considered. The signatures in this case are much more manifest and survive for larger multipoles.*

also arise in the case where graviton extra modes are produced; as highlighted by Figure (6.10), these signatures are indeed enhanced for smaller values of r.

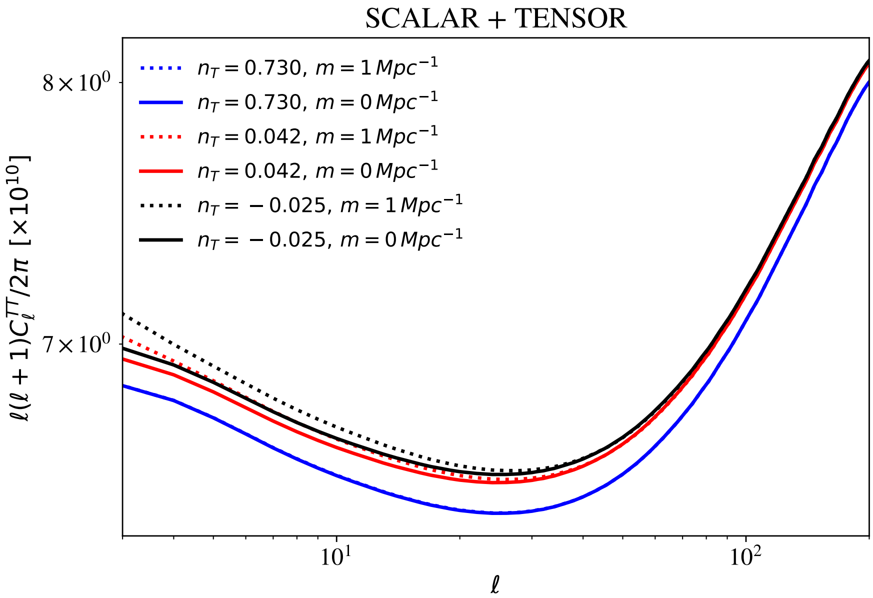

Figure 6.8: *Comparison between the early and late graviton mass effects in the total angular power spectrum ($r = 0.2$). The late graviton mass effects are weaker then the early mass effects, and are visible up to $\ell \sim 20$). Early mass effects instead can survive until $\ell \sim 100$ for large enough mass.*

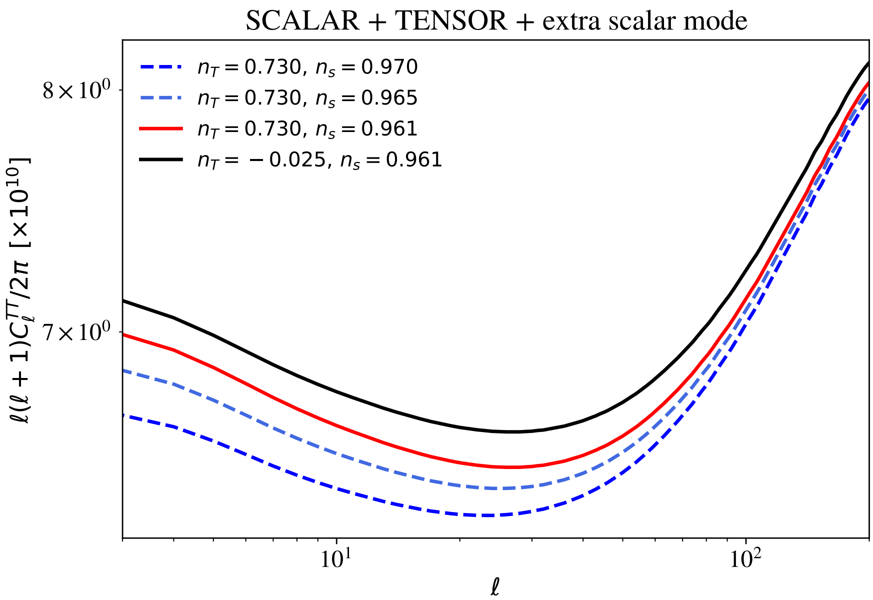

Figure 6.9: *Scalar and Tensor sourced angular power spectrum for different values of the scalar spectral index n_s with fixed value of the late graviton mass $m = 0$ ($r = 0.2$). The black line shows the GR prediction, while the solid red line the modification due to the mass term in the tensor modes (it is considered the case $m_g^2 = \bar{H}^2$). Dotted lines show deviations arising from the extra scalar sector.*

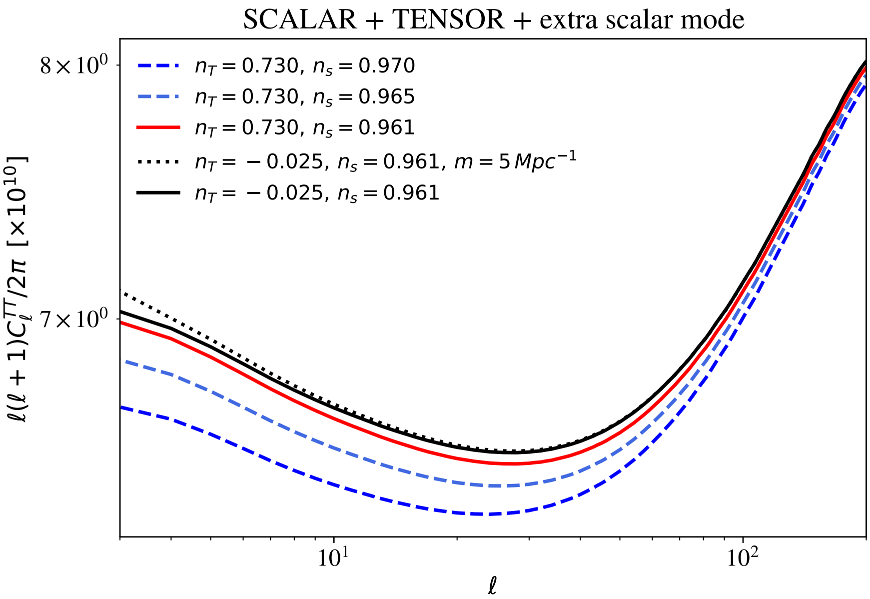

Figure 6.10: *Comparison between the effects arising from the late and early graviton masses and from violation of space-diffeomorphism invariance during inflation in the single slow-roll paradigm ($r = 0.06$). Where it is not specified, the late graviton mass m is taken to be vanishing. The signatures on the tensor sector are damped in the total SGWB angular power spectrum because of the little value of the tensor-to-scalar-ratio r, but they still may bring visible deviation from GR prredictions; variation in n_s due to symmetry violation, if this phenomenum really happend, is likely to be the most evident effect in the total angular spectrum.*

Conclusions

The high sensitivity we expect to reach with future Gravitational Waves interferometers as LISA and ET promotes the Stochastic Gravitational Wave Background to be one of the most interesting and promising laboratories to test many theories of gravity. In this thesis we focused in particular on the signatures of a graviton mass term arising from different models of massive gravity. We depicted a scenario considering two distinct moments: one during inflation, where the graviton is allowed to acquire an heavy mass of the order $m_g^2 \sim \bar{H}^2$, and one after inflation, where astrophysical bounds constraint the late time graviton mass to be $m \leq 10^{-28}\,\mathrm{eV}$ [3]. Whereas for the latter case we adopted a viable model of massive gravity developed by De Felice and Mukhoyama where only two massive tensor degrees of freedom are propagated, we considered two different models for the Universe during inflation. Both these models describe the physics of inflation through an effective field theory approach where spatial-diffeomorphism invariance is broken in addition to the time-reparametrization invariance broken by the inflaton field. The two theories differ for the fact that in one case the graviton develops all its five degrees of freedom, while in the other the scalar and vector extra modes are too massive and undergo an exponential dilution due to the expansion of the Universe, hence becoming completely negligible. In both the two cases, tensor modes acquire a mass term which increase the tensor spectral index of the primordial tensor power spectrum

$$n_T = 3\left[1 - \sqrt{1 - \frac{4}{9}\left(\frac{m_g^2}{H^2} - 3\epsilon\right)}\right], \tag{6.39}$$

pushing it toward a blue tilt.

Having depicted the full theoretical setting, in Chapter (3) we solved the Boltzmann equations for the fluctuations of the graviton distribution function, and expanded them in the multipole basis. One can notice, already at this level, the effect of the late time graviton mass; the angular transfer functions (3.93) show indeed a peculiar dependence on the graviton momentum which goes away in the massless limit; in general such a dependence is instead present in the initial condition term regardless to the graviton mass. The solutions of the Boltzmann equations are then coupled with the Einstein equations for scalar and tensor modes to compute the two-points angular correlators. Besides the initial condition term, an analytic approximate solution is only available for the tensor sourced SGWB angular power spectrum and for the Sachs-Wolfe contribution to the scalar sourced SGWB angular power spectrum in the low multipole regime. Our approximate solution for the tensor sourced angular power spectrum has an explicit dependence on the horizon re-entering time; this is where the late time graviton mass dependence arises in this case, because the mass term shift the horizon re-entering event to earlier times. In the scalar sourced case instead, the late time graviton mass only enters modifying the graviton speed volocity. This correction is however shown to be completely negligible for reasonable values of the late time graviton mass m. We have found that the more important signatures arise from the early time graviton mass, which at the level of the angular correlators is hidden inside the tensor spectral indices of the primordial power spectra. To confirm the validity of our analytic results we explicitly verified that the well known expression of the Sachs-Wolfe contribution in the Harrison-Zel'dovich limit is recovered in the massless limit [59]. For the tensor sourced angular spectrum instead we are not aware of any paper showing a similar result. We have found a completely new result, never encountered also in CMB literature

$$\tilde{C}_{\ell,T} \propto \frac{1}{(\ell+3)(\ell-2)}. \tag{6.40}$$

The same Boltzman formalism used for scalar and tensor perturbations is then introduced also for vectors obtaining a reasonable result similar to its CMB analogous [114]. No additional new effects are expected from the late graviton mass. In the end of this topic, also the possibility to have a primordial magnetic field is investigated. Even in the presence of a primordial vector field setting a non vanishing value for the primordial power spectrum of vector perturbation, we have very little hope to detect signatures arising from vector perturbations, because vector modes decay with the inverse squared power of the scale factor through the whole expansion of the Universe as long as vorticity is conserved. The presence of a magnetic field prior to structure formation would give us better chances, because it has the effect to slow down the damping of vector perturbation during their evolution, increasing the expected angular power spectrum today. The possibility to be sensitive to such signatures strongly depends on the amplitude of the primordial power spectrum, for which we attempted a calculation in the Appendix (F), and on the presence of a strong magnetic field, but the discussion must still be investigated in deep, and it may require further efforts in future works.

In Chapter (5) the analysis of non-Gaussianity is performed. A bit of care is needed in the three-points correlators of the tensor sourced fluctuations, because the tensor bispectrum transforms non trivially under rotation of the wavevectors. The explicit evaluation of the bispectrum in full generality is a really subtle topic; we considered the simplest case focusing on scalar perturbations and working on very large scales, where we are allowed to consider the Sachs-Wolfe contribution only. In the scenario without any extra graviton degree of freedom, primary non-Gaussianity arises for example taking the curvature perturbation in the form of the local ansatz. This particular choice greatly simplifies the computation, and brings to the three-points correlator (5.112), which has the same form of the one found in [59]. None deviation from the standard GR case is then expected in this term. The analysis is instead more interesting in the case of secondary non-Gaussianity. In this case deviation from Gaussianity arises from a non linear evolution of long and short wavelength modes. We used the Weinberg coordinate transformation to explicit the interaction between these modes, finding

$$\lim_{k_3\to 0}\langle \delta_{GW}(\vec{k}_1)\delta_{GW}(\vec{k}_2)\delta_{GW}(\vec{k}_3)\rangle = f_{NL}^{\delta_{GW}}\left(\frac{4\pi^4}{k_1^3 k_3^3}\right)\mathcal{P}_{\delta_{GW}}(k_1)\mathcal{P}_\zeta(k_3), \tag{6.41}$$

with

$$\begin{aligned} f_{NL}^{\delta_{GW}} &= -\frac{\partial \ln \bar{f}(q)}{\partial \ln q} T_S(\eta, k_3^i, q, n_3^i)\left[\frac{\partial \ln \mathcal{P}_\zeta(k)}{\partial \ln k}\right. \\ &\quad +2\left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \ln q}{\partial \ln \bar{f}(q)}\frac{\partial^2 \ln \bar{f}(q)}{\partial(\ln q)^2} + \epsilon\frac{\partial \ln|T_S|^2}{\partial \eta} + \frac{\partial \ln|T_S|^2}{\partial \ln k} \\ &\quad \left. + \left(\beta_q - \frac{a^2m^2}{q^2}\beta_m\right)\frac{\partial \ln|T_S|^2}{\partial \ln q} + \left(\beta_n + \frac{a^2m^2}{q}\beta_m\right) n^i \frac{\partial \ln|T_S|^2}{\partial n^i}\right]. \end{aligned} \tag{6.42}$$

this solution highlights the appearance of completely new terms which vanish in the massless limit, and that may bring significant contribution to the non linear parameter $f_{NL}^{\delta_{GW}}$. Particularly interesting is the term $\partial \ln|T_S|^2/\partial \ln q$, which comes out from the new dependence of the linear transfer function on the graviton momentum q. The case where the graviton mass brings the graviton extra degrees of freedom with itsef is much more complicated. In that case the bispectrum is source by three-points interactions at the Lagrangian level, and have to be computed with the *in-in formalism.* These are found after a proper expansion of the action at third order in Goldstone bosons. Non-Gaussianity in this case could be a very useful tool to constraint some parameters which define the model.

Finally, the numerical analysis of the SGWB two-points angular correlators performed with the program *hi_class* is developed in the last Chapter (6). As expected, no interesting signatures of late time graviton mass are produced on the scalar sourced angular power spectrum. Nonetheless the plots are instructive to identify how the SW and ISW contributions enter differently inside the angular power spectrum. In particular one can see that on very large scales the SW contribution is dominant, while the ISW contribution increase on smaller scales, becoming relevant for $\ell \sim 50$. The transition between the two regimes coincides with the fact that large scale re-enter the horizon during matter domination, where the slow damping provides for just a tiny contribution to the ISW, while small scales cross the horizon in the radiation epoch, with a rapidly changing growing rate. Visible signature of massive modes arise instead from the extra scalar graviton degree of freedom, which modifies the definition of the scalar spectral index as in (6.33). The amount of violation of space-diffeomorphism is controlled by a parameter α; the more α, the more the sourced angular power spectrum gets suppressed on low multipoles. Without specifying a model, we are not able to give predictions on such effect, but visible modifications are shown even for deviation of the spectral index of the order of the slow-roll parameter ϵ. A similar behavior is shown by the tensor sourced angular power spectrum, where the value of the early time graviton mass m_g was compared with the energy scale of the Universe during inflation $\bar{H}$. The numerical analysis supports our analytical solution, showing a trend in agreement with (6.40). Moreover, besides the effect of the early time graviton mass, the tensor sector reveals itself to be sensitive also to the effect of the late time mass. These are weaker but still maybe visible effects which increase the tensor sourced angular power spectrum on the largest scales. As a consequence, for large enough values of the early graviton mass around $m_g^2 \sim \bar{H}^2$, the effects of the late time graviton mass become completely irrelevant. The possibility that these signatures survive in the total angular power spectrum is strongly related to the value of the tensor-to-scalar ration r. Although BICEP2 and Keck Array Collaborations [110], in accordance with Planck measurement [109], have set $r < 0.06$ for single field slow roll inflation, we firstly considered a more optimistic situation with $r = 0.2$; this is done for pedagogical reason, because a larger value for r enhances the signatures arising from the tensor sector. We can then summarize our findings in the following way: an heavy graviton mass during inflation provides a visible damping of the angular power spectrum on low multipoles; the damping is enhanced in the case where the gravity sector propagates an additional scalar degree of freedom; these two effects are a little counterbalanced by a possible late time graviton mass. In this regard it has to be said that late time graviton mass effects becomes less and less important as the early time graviton mass increase. This was shown in Figure (6.8) and motivated by the factor

$$\tilde{C}_{\ell,T} \propto \frac{\sum_{\lambda=\pm 2} \mathcal{P}_\lambda(k_0)}{k_0^{n_T} l(\eta_0, \eta_*)^{n_T}} \tag{6.43}$$

emerging from the primordial tensor power spectrum. Interestingly, the curves one can obtain considering all these effects are not uniquely determined. The three contribution may indeed combine in many different ways, such that different curves may overlaps in some multipole ranges. This calls for the need to be sensitive to the largest possible multipole range, in order to be able to disentangle different combinations of the three mentioned contributions. Finally we compared the contributions arising from both the space-diffeomorphism breaking and the late/early graviton masses within the bounds of the single field slow roll inflation, that is setting $r = 0.06$.

Future developments

In the main text we sometimes mentioned the possibility to proceed deeper in some arguments in future works. We want to resume these possibilities of challenging and intriguing future developments. The first step to connect our analysis with observations is to quantify the sensitivity of future GW interferometers to the effects studied in this thesis. The idea is to build an estimator in order to determine the Signal to Noise ratio (SNR) of such effects considering the sensitivity level of LISA and ET.

Concerning more theoretical aspects, in Chapter (6) it was mentioned that the modification in the scalar spectral index n_s and the early time graviton mass m_g are degenerate in the angular power spectra, in the sense that they affect the power spectra simultaneously providing similar effects; hence we are not able to discriminate between the two. The path to follow to eliminate this degeneracy is to focus on particular models, where the relation between m_g and the parameter α

$$m_g^2 = \frac{8}{M_P^2}\left(\frac{\alpha^2}{a^2}\bar{F}_Z + \frac{\alpha^4}{2a^4}\bar{F}_{ZZ}\right) \tag{6.44}$$

is fixed by the choice of the operator $F(X, Y^i, Z^{ij})$. Having fixed a model we should be able to parametrize both the variation in the scalar spectral index and the graviton mass with one parameter only, namely m_g^2 or α[3]. The analysis of non-Gaussianity plays a crucial role in this sense, because predictions about three-points interaction terms could allow to constraint some coefficients of the theory in an effective field theory approach. This analysis is in part performed in [47], but it is far from being completed; hence much more efforts are needed to completely describe the theory. Another possibility to disentangle between n_T and n_s could be to study polarization of the angular power spectrum. It is known indeed that E-modes and B-modes select respectively scalar and tensor perturbation contributions, so that the two effects may be split. The study of B-modes could be interesting also for a deeper analysis of the late graviton mass effects. Since these effects arise only in the tensor sector, they are suppressed in the total (temperature) angular power spectrum by the fact that the amplitude of primordial tensor power spectrum is more then one order of magnitude lower then the corresponding amplitude for the primordial scalar power spectrum ($r < 0.06$ in the standard single field slow-roll inflation). If we have a way to separate between scalar and tensor perturbations, we get rid of this suppression, reaching better chances to discover such effects.

Another open question concerns vector modes. This argument was very little studied so far because the standard slow-roll inflationary paradigm does not provide any velocity field, and because vector modes decay during the expansion of the Universe as $V \propto a^{-2}$, so that they are expected to be negligible today. However now possibilities arise in massive gravity, because the extra vector degree of freedom is able, in principle, to set the

[3] It is important to stress that the modifications in the scalar sector arise from the symmetry breaking pattern, and not from the graviton mass.

primordial vector perturbations forming the primordial vector power spectrum. There are no constraints in the theory about the amplitude of such primordial power spectrum, so that the possibility to have non vanishing effects today is still open and has to face with observations. As already commented, the chances to have a significant contribution from vector perturbations increase in the presence of a strong magnetic field. This is indeed a source for vorticity, which increases in time, and slows down the vector's decay to $V \propto a^{-1}$.

Appendices

Appendix A

Linearised Einstein equations

In this chapter of the Appendix we want to derive the linearised Einstein tensor in the framework of standard cosmology, since it is widely used inside the text. Let $\bar{g}_{\mu\nu}$ be the FLRW metric describing the background spacetime. Let's then introduce a first order perturbation on the background, defining then the physical spacetime as the difference

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{A.1}$$

with $|\bar{g}_{\mu\nu}| \gg |\delta g_{\mu\nu}|$. In full generality the physical metric has 10 degrees of freedom, and, in a generic gauge, can be written as

$$g_{\mu\nu} = a^2(\eta) \begin{Bmatrix} -[1-2\Phi(\eta,\vec{x})] & \omega_i(\eta,\vec{x}) \\ \omega_i(\eta,\vec{x}) & \delta_{ij}[1+2\Psi(\eta,\vec{x})] + \chi_{ij}(\eta,\vec{x}) \end{Bmatrix}, \tag{A.2}$$

The perturbations of the metric directly reflect on the perturbed Einstein tensor, and then the field equations for gravity. In order to derive the linearised Einstein tensor we proceed by steps starting from the perturbed Christoffel symbols.

Perturbed Christoffel symbols: Inserting the decomposition (A.1) inside the definition of the Christoffel symbols, one can verify that

$$\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) = \bar{\Gamma}^\mu_{\nu\rho} + \delta\Gamma^\mu_{\nu\rho}, \tag{A.3}$$

$$\begin{aligned}\delta\Gamma^\mu_{\nu\rho} &= \frac{1}{2}\bar{g}^{\mu\sigma}(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma}) + \frac{1}{2}\delta g^{\mu\sigma}(\bar{g}_{\sigma\nu,\rho} + \bar{g}_{\sigma\rho,\nu} - \bar{g}_{\nu\rho,\sigma}) \\ &= \frac{1}{2}\bar{g}^{\mu\sigma}(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} - 2\delta g_{\sigma\alpha}\bar{\Gamma}^\alpha_{\nu\rho}).\end{aligned} \tag{A.4}$$

Let's define the perturbed contribution $h_{\mu\nu}$ to the physical metric by

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}). \tag{A.5}$$

with $\eta_{\mu\nu}$ the flat Minkowski metric. From the definition (A.3), one can easily verify that, at zero order, the only non vanishing Christoffel symbols are

$$\bar{\Gamma}^0_{00} = \frac{a'}{a}, \qquad \bar{\Gamma}^0_{ij} = \frac{a'}{a}\delta_{ij}, \qquad \bar{\Gamma}^i_{0j} = \frac{a'}{a}\delta^i_j, \tag{A.6}$$

while the perturbed contributions are:

$$
\begin{aligned}
\delta\Gamma^0_{00} &= -\frac{1}{2}h'_{00}, \qquad \delta\Gamma^0_{i0} = -\frac{1}{2}(h_{00,i} - 2\mathcal{H}h_{0i}),\\
\delta\Gamma^i_{00} &= h'_{i0} + \mathcal{H}h_{i0} - \frac{1}{2}h_{00,i} \qquad \delta\Gamma^i_{j0} = \frac{1}{2}(h'_{ij} + h_{i0,j} - h_{0j,i}),\\
\delta\Gamma^0_{ij} &= -\frac{1}{2}(h_{0i,j} + h_{0j,i} - h'_{ij} - 2\mathcal{H}h_{ij} - 2\mathcal{H}\delta_{ij}h_{00}),\\
\delta\Gamma^i_{jk} &= \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i} - 2\mathcal{H}\delta_{jk}h_{i0}),
\end{aligned} \tag{A.7}
$$

where it has been defined the conformal time Hubble factor $\mathcal{H} \equiv a'/a$.

Perturbed Ricci tensor At linear order also the Ricci tensor can be decomposed in an unperturbed component and a perturbed one. Starting from its definition

$$
\begin{aligned}
R_{\mu\nu} &\equiv \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho} \equiv \bar{R}_{\mu\nu} + \delta R_{\mu\nu},\\
\bar{R}_{\mu\nu} &= \bar{\Gamma}^\rho_{\mu\nu,\rho} - \bar{\Gamma}^\rho_{\mu\rho,\nu} + \bar{\Gamma}^\rho_{\mu\nu}\bar{\Gamma}^\sigma_{\rho\sigma} - \bar{\Gamma}^\rho_{\mu\sigma}\bar{\Gamma}^\sigma_{\nu\rho},\\
\delta R_{\mu\nu} &= \delta\Gamma^\rho_{\mu\nu,\rho} - \delta\Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\mu\nu}\delta\Gamma^\sigma_{\rho\sigma} + \delta\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\delta\Gamma^\sigma_{\nu\rho} - \delta\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}.
\end{aligned} \tag{A.8}
$$

The unperturbed Ricci tensor elements can be easily computed from the explicit expression of the zero order Christoffel symbols. The only non vanishing elements are

$$
\bar{R}_{00} = 3\left(\mathcal{H}^2 - \frac{a''}{a}\right), \qquad \bar{R}_{ij} = \delta_{ij}\left(\mathcal{H}^2 + \frac{a''}{a}\right). \tag{A.9}
$$

Concerning the perturbed part, we can compute individually each element of the Ricci tensor by opening the contracted indeces. In particular for the time-time component we have:

$$
\begin{aligned}
\delta R_{00} &= \delta\Gamma^k_{00,k} - \delta\Gamma^k_{0k,0} + \bar{\Gamma}^0_{00}\delta\Gamma^i_{0i} + \bar{\Gamma}^i_{00}(\delta\Gamma^k_{ik} - \delta\Gamma^0_{0i}) + \bar{\Gamma}^i_{0i}\delta\Gamma^0_{00} + \bar{\Gamma}^k_{ik}\delta\Gamma^i_{00}\\
&\quad -\bar{\Gamma}^i_{0k}\delta\Gamma^k_{0i} - \bar{\Gamma}^0_{0i}\delta\Gamma^i_{00} - \bar{\Gamma}^k_{0i}\delta\Gamma^i_{0k}\\
&= \delta\Gamma^k_{00,k} - \delta\Gamma^k_{0k,0} - \mathcal{H}\delta\Gamma^i_{0i} + 3\mathcal{H}\delta\Gamma^0_{00}.
\end{aligned} \tag{A.10}
$$

where in the second step we used the explicit values for the unperturbed Christoffel symbols. Inserting then the expression of the perturbed ones, we arrive to

$$
\delta R_{00} = -\frac{1}{2}\nabla^2 h_{00} - \frac{3}{2}\mathcal{H}h'_{00} + h'_{k0,k} + \mathcal{H}h_{k0,k} - \frac{1}{2}(h''_{kk} + \mathcal{H}h'_{kk}). \tag{A.11}
$$

In the same spirit one can compute the other matrix elements as well. The computation is a bit lengthy and tedious, therefore we will only show the result. At the end of the day one can verify that

$$
\delta R_{0i} = -\mathcal{H}h_{00,i} - \frac{1}{2}(\nabla^2 h_{0i} - h_{k0,ik}) + \left(\frac{a''}{a} + \mathcal{H}^2\right)h_{0i} - \frac{1}{2}(h'_{kk,i} - h'_{ki,k}), \tag{A.12}
$$

$$
\begin{aligned}
\delta R_{ij} &= \frac{1}{2}h_{00,ij} + \frac{\mathcal{H}}{2}h'_{00}\delta_{ij} + \left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{00}\delta_{ij} + \frac{1}{2}h''_{ij} + \mathcal{H}h'_{ij}\\
&\quad -\mathcal{H}h_{k0,k}\delta_{ij} - \frac{1}{2}(h'_{0i,j} + h'_{0j,i}) - \mathcal{H}(h_{0i,j} + h_{0j,i}) + \frac{\mathcal{H}}{2}h'_{kk}\delta_{ij}\\
&\quad -\frac{1}{2}(\nabla^2 h_{ij} - h_{ki,kj} - h_{kj,ki} + h_{kk,ij}) + \left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}.
\end{aligned} \tag{A.13}
$$

Perturbed Ricci scalar: having computed the perturbed Ricci tensor elements, we are now able to compute the Ricci scalar at first order in metric perturbations. By definition the Ricci scalar is given by

$$\begin{aligned} R &\equiv g^{\mu\nu}R_{\mu\nu} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}\bar{R}_{\mu\nu} \equiv \bar{R} + \delta R, \\ \delta R &= \bar{g}^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}\bar{R}_{\mu\nu}. \end{aligned} \tag{A.14}$$

In order to go on we need to understand the behavior of $\delta g^{\mu\nu}$. The decomposition (A.1) implies that, since both the background and the physical metrics by definition satisfy $\bar{g}_{\mu\rho}\bar{g}^{\rho\nu} = \delta^\mu_\nu$ and $g_{\mu\rho}g^{\rho\nu} = \delta^\mu_\nu$:

$$(\bar{g}^{\mu\rho} + \delta g^{\mu\rho})(\bar{g}^{\rho\nu} + \delta g^{\rho\nu}) = \bar{g}^{\mu\rho}\bar{g}_{\rho\nu} \quad \rightarrow \quad \delta g^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\delta g_{\rho\sigma}. \tag{A.15}$$

This property allows to rewrite the perturbed Ricci scalar in terms of all known quantities:

$$\delta R = -\frac{1}{a^2}\delta R_{00} + \frac{1}{a^2}\delta^{ij}\delta R_{ij} - a^2 h_{\rho\sigma}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\bar{R}_{\mu\nu}. \tag{A.16}$$

Substituting the expression for the perturbed and unperturbed Ricci tensor elements, one eventually arrives to

$$\begin{aligned} a^2\delta R = \nabla^2 h_{00} + 3\mathcal{H}h'_{00} + 6\frac{a''}{a}h_{00} - 2h'_{k0,k} - 6\mathcal{H}h_{k0,k} \\ +h''_{kk} + 3\mathcal{H}h'_{kk} - \nabla^2 h_{kk} + h_{kl,kl}. \end{aligned} \tag{A.17}$$

Perturbed Einstein tensor: at this point we can collect all the above results to compute the Einstein tensor elements. It is common to work with mixed indices, for which we raise one index with

$$\begin{aligned} G^\mu_\nu &= g^{\mu\rho}R_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu R = \bar{g}^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\bar{R} + \bar{g}^{\mu\rho}\delta R_{\rho\nu} + \delta g^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\delta R, \\ \bar{G}^\mu_\nu &= \bar{R}^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\bar{R}, \\ \delta G^\mu_\nu &= \bar{g}^{\mu\rho}\delta R_{\rho\nu} - \bar{g}^{\mu\tau}\bar{g}^{\rho\sigma}\delta g_{\tau\sigma}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\delta R. \end{aligned} \tag{A.18}$$

Inserting the results for the Ricci tensor and scalar we obtain[1]

$$\begin{aligned} a^2\delta G^0_0 &= -\delta R_{00} - h_{00}\bar{R}_{00} - \frac{a^2}{2}\delta R \\ &= -3\mathcal{H}^2 h_{00} + 2\mathcal{H}h_{k0,k} - \mathcal{H}h'_{kk} + \frac{1}{2}\nabla^2 h_{kk} - \frac{1}{2}h_{kl,kl} \end{aligned} \tag{A.19}$$

for the time-time component, and

$$a^2\delta G^0_i = -\delta R_{0i} + \bar{R}_{ji}h_{0j} = -\mathcal{H}h_{00,i} - \frac{1}{2}(\nabla^2 h_{0i} - h_{k0,ik}) - \frac{1}{2}(h'_{kk,i} - h'_{ki,k}) \tag{A.20}$$

for the time-space components, while the space-space components are:

$$\begin{aligned} 2a^2\delta G^i_j &= \delta R_{ij} - g^{kl}a^2 h_{il}\bar{R}_{kj} - \frac{1}{2}\delta^i_j a^2\delta R \\ &= \Big[-4\frac{a''}{a}h_{00} - 2\mathcal{H}h'_{00} - \nabla^2 h_{00} + 2\mathcal{H}^2 h_{00} - 2\mathcal{H}h'_{kk} + \nabla^2 h_{kk} - h_{kl,kl} \\ &\quad +2h'_{k0,k} + 4\mathcal{H}h_{k0,k} - h''_{kk}\Big]\delta^i_j + h_{00,ij} - \nabla^2 h_{ij} + h_{ki,kj} + h_{kj,ki} \\ &\quad -h_{kk,ij} + h''_{ij} + 2\mathcal{H}h'_{ij} - (h'_{0i,j} + h'_{0j,i}) - 2\mathcal{H}(h_{0i,j} + h_{0j,i}). \end{aligned} \tag{A.21}$$

[1] Be careful that in the following repeated indeces are always meant to be contracted, even if they are both covariant or controvariant

Appendix B

Spherical harmonics

In this appendix chapter we want to review briefly the main features of the spherical harmonics. These are eigenfunctions of the Laplacian operator on the sphere. In quantum mechanics this operator is associated to the square of the orbital angular momentum. Spherical harmonics are proved to be an orthonormal set, and therefore any function on the sphere can be expressed as a linear combination of them. In order to write down the eigenvalue equation defining the spherical harmonics, let's start by the line element of the unit sphere in spherical coordinate:

$$ds_S^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{B.1}$$

Through the definition of the Laplacian operator in a general relativistic framework

$$\nabla^2 = \frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu\nu}\sqrt{-g}\,\partial_\mu\right), \tag{B.2}$$

the spherical harmonics are defined as the functions $Y(\theta,\phi)$ solving the eigenvalue equation

$$\nabla^2 Y(\theta,\phi) = \left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\phi^2\right] Y(\theta,\phi) = \lambda Y(\theta,\phi). \tag{B.3}$$

This equation can be solved with the assumption that the θ and ϕ dependences can be factorized in two distinct functions. With this assumption, the solution for the spherical harmonics is given by

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}, \tag{B.4}$$

in terms of the Legendre polynomials

$$P_\ell^m(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}[(x^2-1)^\ell], \tag{B.5}$$

with the convention

$$P_\ell^{-m} = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}P_\ell^m. \tag{B.6}$$

The fact that the spherical harmonics form an orthonormal system derives immediately from the normalization condition

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}. \tag{B.7}$$

Moreover the completeness relation

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)Y_{\ell m}(\theta',\phi') = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') \tag{B.8}$$

allows to expand any square-integrable function $f(\theta,\phi)$ as:

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_{\ell m}(\theta,\phi) \tag{B.9}$$

in a unique way.

B.0.1 Spin-weighted spherical harmonics

In the previous section we have seen how spherical harmonics allow to expand scalar quantities. These are quantities $f(\hat{n})$ that remain invariant under rotation, that is

$$f(\hat{n}) \to f'(\hat{n}') = f(\hat{n}), \qquad \hat{n}' = R\hat{n}, \tag{B.10}$$

with R a rotation matrix. decomposing the scalar function as (B.9), this relation reads:

$$\sum_{\ell m} a'_{\ell m}Y_{\ell m}(\hat{n}') = \sum_{\ell m} a_{\ell m}Y_{\ell m}(\hat{n}). \tag{B.11}$$

Using the properties of the spherical harmonics under rotation:

$$Y_{\ell m}(R\hat{n}) = \sum_{m'=-\ell}^{\ell} D^{(\ell)}_{m'm}(R^{-1})Y_{\ell m'}(\hat{n}), \tag{B.12}$$

where $D^{(\ell)}_{m'm}$ are the elements of the Wigner D-matrix which we will define later, one can readily verify that the coefficients $a_{\ell m}$ transform as:

$$a'_{\ell m} = \sum_{m'} D^{(\ell)}_{m'm}(R)a_{\ell m'}. \tag{B.13}$$

As a consequence, it is easy to infer that the angular power spectra are direction independent, that is rotationally invariant:

$$C'_\ell = \langle a'_{\ell m}a^{*'}_{\ell m}\rangle = \sum_{MM'} D^{(\ell)}_{mM}D^{(\ell)*}_{mM'}\langle a_{\ell M}a^{*'}_{\ell M}\rangle = C_\ell \sum_{MM'} D^{(\ell)}_{mM}D^{(\ell)*}_{mM'} = C_\ell, \tag{B.14}$$

since the Wigner D-matrices are unitary matrices. Therefore we proved that, as long as $f(\hat{n})$ is a scalar quantity, its angular power spectrum is invariant under rotation. Actually, if one remembers the discussion of section (4.2), it is exactly the assumption of statistical isotropy that allows to express the 2-point correlator function as

$$\langle a_{\ell m}a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell. \tag{B.15}$$

What about non scalar quantities? In general one can see that if we expand non-zero spin quantities with the basis of spherical harmonics, they would provide power spectra which depend on the orientation of the coordinate system, breaking the rotational invariance. In order to avoid this problem, it is common to introduce the spin-weighted spherical harmonics. In general, if η is a spin-s quantity, then it transforms as

$$\eta(\hat{n}) \to \eta'(\hat{n}') = e^{si\psi}\eta(\hat{n}), \tag{B.16}$$

under rotation of an angle ψ around the propagation direction $\hat{n}$. In order to get rotational invariant quantities, one defines the spin rising and lowering differential operator respectively as:

$$\eth\eta = -(\sin\theta)^s \left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right][(\sin\theta)^{-s}\eta], \tag{B.17}$$

$$\bar{\eth}\eta = -(\sin\theta)^{-s} \left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right][(\sin\theta)^{s}\eta]. \tag{B.18}$$

These operators owe their name to the fact that $\eth\eta$ turns out to be a spin-$(s-1)$ quantity, while $\bar{\eth}\eta$ a spin-$(s-1)$ one, and they are nothing but covariant derivative on the sphere. With these new ingredients one defines the spin-s spherical harmonics simply by applying s-times the rising or lowering spin operator on the usual spherical harmonics, that is

$${}_sY_{\ell m} = \sqrt{\frac{(\ell-s)!}{(\ell+s)!}}\,\eth^s Y_{\ell m} \qquad (0 \le s \le \ell), \tag{B.19}$$

$${}_sY_{\ell m} = (-1)^s\sqrt{\frac{(\ell+s)!}{(\ell-s)!}}\,\bar{\eth}^s Y_{\ell m} \qquad (-\ell \le s \le 0), \tag{B.20}$$

where the root factors are there just to provide the correct normalization. By expanding explicitly the operators one can possibly recover the expression (3.78). Notice that, as expected, the spin-s spherical harmonics reduce to the usual ones when $s = 0$. For this reason on can regards the spherical harmonics just as a particular case of the more general spin-weighted ones. Indeed even the latter share the properties of orthonormality

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi\, {}_sY_{\ell m}(\theta,\phi)\, {}_sY^*_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'} \tag{B.21}$$

and completeness

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} {}_sY_{\ell m}(\theta,\phi)\, {}_sY_{\ell m}(\theta',\phi') = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi'). \tag{B.22}$$

As a consequence any spin-s quantinty can be expressed as a combination of spin-weighted spherical harmonics as

$$\eta(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \eta_{\ell m}\; {}_sY_{\ell m}(\theta,\phi). \tag{B.23}$$

Moreover one can show [86] that the complex conjugation acts on the spin-weighted functions as

$${}_sY^*_{\ell m}(\theta,\phi) = (-1)^{s+m}_{-s}Y_{\ell -m}(\theta,\phi). \tag{B.24}$$

As a consequence any spin-s quantity can be equally expanded on a basis of spin weighted spherical harmonics of spin s, as (B.23), or of spin $-s$, i.e.

$$\eta(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \tilde{\eta}_{\ell m}\; {}_{-s}Y_{\ell m}(\theta,\phi). \tag{B.25}$$

In conclusion of this appendix we mention that the spin-weighted spherical harmonics enter in the definition of the Wigner D-matrix used in (B.12) as follows:

$$D^{(\ell)}_{ms}(\phi,\theta,\psi) = \sqrt{\frac{4\pi}{2\ell+1}}\; {}_sY_{\ell -m}(\theta,\phi)e^{is\psi}, \tag{B.26}$$

with (ϕ,θ,ψ) the Euler angles.

Appendix C

Spherical Bessel functions

Spherical Bessel functions are intensively used in the study of large scale anisotropies; for this reason in this Appendix we give a brief review of their main properties, while more details can be found in [85]. By definition spherical Bessel functions are functions $j_\ell(x)$ satisfying the differential equation

$$\frac{d^2 j_\ell(x)}{dx^2} + \frac{2}{x}\frac{dj_\ell(x)}{dx} + \left[1 - \frac{\ell(\ell+1)}{x^2}\right] j_\ell(x) = 0. \tag{C.1}$$

Just to give some easy example, since they are used within the main text, we show the two lowest orders:

$$j_0(x) = \frac{\sin x}{x}, \qquad j_1(x) = \frac{\sin x - x\cos x}{x^2}. \tag{C.2}$$

Spherical Bessel functions are related to the Legendre polynomial through the key relation

$$\int_{-1}^{1} \frac{d\mu}{2} P_\ell(\mu) e^{ix\mu} = \frac{j_\ell(x)}{(-i)^\ell}. \tag{C.3}$$

An important relation that allows to eliminate derivatives of the spherical Bessel function is:

$$\frac{dj_\ell(x)}{dx} = j_{\ell-1}(x) - \frac{l+1}{x} j_\ell(x). \tag{C.4}$$

Moreover, in the explicit computation of the leading order anisotropies, it often appears the integral of the squared spherical Bessel functions multiplied by a certain power of the integral variable. We computed this integral with Mathematica and get

$$\int_0^\infty dx\, x^{n-2} j_\ell^2(x) = 2^{n-4}\pi \frac{\Gamma\left(\ell - \frac{1}{2} + \frac{n}{2}\right)}{\Gamma\left(2 - \frac{n}{2}\right)}\, {}_2F_1\left[-1 + \frac{n}{2}, \ell - \frac{1}{2} + \frac{n}{2}; \ell + \frac{3}{2}; 1\right] \tag{C.5}$$

$$\text{if} \quad n \text{ odd} \vee 1 < \mathrm{Re}(n) < 4 \qquad \wedge \qquad \mathrm{Re}(2\ell + n) > 1, \tag{C.6}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function and $\Gamma(n)$ the Euler Gamma function. Moreover, in [104] it is shown that the hypergeometric function of the type ${}_2F_1(a, b; c; 1)$ admits the following explicit expression in terms of the Gamma functions:

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{C.7}$$

Therefore, if the above conditions (C.6) are verified, the integral eventually reads

$$\int_0^\infty dx\, x^{n-2} j_\ell^2(x) = 2^{n-4}\pi \frac{\Gamma\left(\ell + \frac{n}{2} - \frac{1}{2}\right)}{\Gamma\left(\ell + \frac{5}{2} - \frac{n}{2}\right)} \frac{\Gamma(3-n)}{\Gamma^2\left(2 - \frac{n}{2}\right)}, \tag{C.8}$$

that is the expression one can find in [70]. Other useful relations are the closure equation

$$\frac{2}{\pi}\int dx\, x^2 j_\ell(\alpha x) j_\ell(\alpha' x) = \frac{\delta(\alpha-\alpha')}{\alpha^2}, \tag{C.9}$$

and the recurrence relation, which one can infer from the same relation holding for the Legendre polynomials

$$(\ell+1)P_{\ell+1}(\mu) = (2\ell+1)\mu P_\ell(\mu) - \ell P_{\ell-1}(\mu). \tag{C.10}$$

Exploiting (C.3), one can easily verify that for the spherical Bessel functions:

$$\frac{j_\ell(x)}{x} = \frac{j_{\ell-1}(x) + j_{\ell+1}(x)}{2\ell+1}. \tag{C.11}$$

Appendix D

Second order linear differential equations with non constant coefficients

This Appendix focuses on the solution of second order differential equations with non constant coefficients with the reduction of order method. All the derivation follows the reference [71]. It is known that any homogeneous linear differential equation of second order, that is of the type

$$y^{''} + p(t)y' + q(t)y = 0, \tag{D.1}$$

admits as solution a linear combination of two linearly independent functions satisfying the above differential equation. In this Appendix we want to show how to solve (D.1) in the case the coefficients $p(t)$ and $q(t)$ are time-dependent. In this case we have no way to find analytically the solution in full generality. The best we can do is to find one of the two linearly independent solutions in the case the other one is already known.

The mathematical object that we use to probe the linear independence of the two solutions (the we will call y_1 and y_2) is the Wronskian $W(y_1, y_2)(t)$, that is the determinant of the coefficient matrix built from the system

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2, \\ y' &= C_1 y_1' + C_2 y_2', \\ W(y_1, y_2)(t) &\equiv \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'. \end{aligned} \tag{D.2}$$

Therefore, if y_1 and y_2 are two linear solutions of (D.1), and if $W(y_1, y_2)(t) \neq 0$ for t belonging to an interval I, then their linear combination $y = C_1 y_1 + C_2 y_2$ forms a general solution of (D.1) within the interval I. In the following we will benefit from the *Abel's theorem*, that states:

Theorem 1. ***Abel's Theorem***

If y_1 and y_2 are any two solutions of the equation

$$y^{''} + p(t)y' + q(t)y = 0,$$

with $p(t)$ and $q(t)$ continuous functions on the interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by:

$$W(y_1, y_2)(t) = Ce^{-\int dt p(t)}, \tag{D.3}$$

where C is a time-independent constant that depends on the two solutions y_1 and y_2. Further, $W(y_1, y_2)(t)$ is either zero for all $t \in I$ (if $C = 0$), or else is never zero for all $t \in I$ (if $C \neq 0$).

At this point we are able to find the desired solution to (D.1). Let's suppose now to have found the first solution y_1. Assume further that there exists a second solution of the form $y_2 = y_1\nu(t)$, with $\nu(t)$ some differentiable function. The Wronskian then reads:

$$W(y_1, y_2) = y_1 y_1' \nu(t) + {y_1}^2 \nu'(t) - y_1' y_1 \nu(t) = {y_1}^2 \nu'(t). \tag{D.4}$$

However the same Wronskian can be evaluated exploiting the Abel's theorem as (D.3). Therefore the two expression must coincide. By equating them:

$${y_1}^2 \nu'(t) = Ce^{-\int dt p(t)}, \tag{D.5}$$

with $C \neq 0$ to ensure the linear independence of the two solutions. Now we can integrate this expression to obtain

$$\nu(t) = C \int dt \, \frac{e^{-\int dt p(t)}}{{y_1}^2} + \tilde{C}, \tag{D.6}$$

with $\tilde{C}$ an integration constant that we can safely set to zero since, in the final result $y_2 = y_1\nu(t)$, it would bring an additional term that is proportional to y_1, that is linearly dependent. Therefore, at the end of the day, the desired solution for the differential equation (D.1) is

$$y = C_1 y_1 + C_2 y_1 \int dt \, \frac{e^{-\int dt p(t)}}{{y_1}^2}. \tag{D.7}$$

Appendix E

Wigner 3-j Symbol

This chapter of the Appendix is meant to give a basic review on the properties of the Wigner 3-j symbol which is intensively used in the computation of the bispectrum in chapter (5). In quantum mechanics the Clebsh–Gordan coefficients describe the addition of two eigenstate of the angular momentum operator. Let us denote as $|\ell\, m\rangle$ a quantum state where ℓ is the eigenvalue of the angular momentum operator $\mathbf{L}^2$ such that $\mathbf{L}^2|\ell, m\rangle = \ell(\ell+1)|\ell, m\rangle$, and m the eigenvalue relative to the z-component of the angular momentum $L_z|\ell\, m\rangle = m|\ell\, m\rangle$. Then, given two initial states $|\ell_1\, m_1\rangle$ and $|\ell_2\, m_2\rangle$, a final state $|\ell_3\, m_3\rangle$ which is the sum of the initial ones can be decomposed as [91]

$$|\ell_3\, m_3\rangle = \sum_{m_1, m_2} |\ell_1, m_1, \ell_2\, m_2\rangle\langle\ell_1, m_1, \ell_2\, m_2|\ell_3\, m_3\rangle, \tag{E.1}$$

where the last contracted term represents indeed the Clebsh-Gordan coefficients. As shown in the Appendix of [92], the Wigner 3-j symbols

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{E.2}$$

are defined in relation to the Clebsh-Gordan coefficients by

$$\langle\ell_1, m_1, \ell_2\, m_2|\ell_3\, m_3\rangle = (-1)^{-\ell_1+\ell_2-m_3}\sqrt{2\ell_3+1}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{E.3}$$

This means that the Wigner symbols in (E.3) describe the coupling of two angular momenta $\mathbf{L}_1$ and $\mathbf{L}_2$ such that $\mathbf{L}_3 = \mathbf{L}_1 + \mathbf{L}_2$. In the same way the symbol (E.2) describes three angular momenta forming a triangle $\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 = 0$. Since the triangle condition must hold for any of the three spatial direction, projecting the relation between the angular momenta on the z-axis it immediately derives that $m_1 + m_2 + m_3 = 0$. Moreover, any side of the triangle must satisfy the geometrical condition $|L_i - L_j| \leq L_k \leq L_i + L_j$ with $i, j, k = 1, 2, 3)$. All in all the triangle condition demands the following two relations

$$\begin{aligned} m_1 + m_2 + m_3 &= 0, \\ |\ell_i - \ell_j| \leq \ell_k &\leq \ell_i + \ell_j. \end{aligned} \tag{E.4}$$

If one of these conditions are not satisfied, then the Wigner 3-j symbols automatically vanish. To find explicit expression for the symbols is a difficult task, and one should often refers to the Clebsch-Gordan tables. However for many practical purposes we can exploit some symmetry properties which simplify the computations

Symmetries The Wigner 3-j symbols are invariant under even permutations, while they change sign for odd permutation if $\ell_{tot} = \ell_1 + \ell_2 + \ell_3 = 0$, i.e.

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} \ell_3 & \ell_1 & \ell_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} \ell_2 & \ell_3 & \ell_1 \\ m_2 & m_3 & m_1 \end{pmatrix}, \tag{E.5}$$

$$(-1)^{\ell_{tot}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} \ell_2 & \ell_1 & \ell_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_3 & \ell_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \\ = \begin{pmatrix} \ell_3 & \ell_2 & \ell_1 \\ m_3 & m_2 & m_1 \end{pmatrix}. \tag{E.6}$$

If $\ell_{tot} =$ odd, the phase changes even under the transformation $m_i \to -m_i$:

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_{tot}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{E.7}$$

Moreover, if the z components of the angular moments are all vanishing, then the Wigner symbol

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{E.8}$$

is invariant under any kind of permutation of the momenta ℓ_i and non vanishing only if $\ell_{tot} =$ even.

Orthogonality In accordance with the orthonormality of the angular momentum eigenstates and the unitarity of the Clebsh–Gordan coefficients, we have the following orthonormality conditions for the Wigner 3-j symbols:

$$\sum_{\ell_3 m_3} (2\ell_3 + 1) \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}, \tag{E.9}$$

$$\sum_{m_1 m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{\ell_3 \ell_3'} \delta_{m_3 m_3'}}{2\ell_3 + 1}, \tag{E.10}$$

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1. \tag{E.11}$$

Rotation matrix The intrinsic meaning of the Wigner 3-j symbols to describe the coupling of two angular momenta can be naturally extended to the coupling of two rotation operators. Indeed the angular momenta enter in the definition of a rotation operator $D(\alpha, \beta, \gamma)$ with α, β and γ Euler angles, in the following way

$$D(\alpha, \beta, \gamma) = e^{-i\alpha L_x} e^{-i\beta L_y} e^{-i\gamma L_z}. \tag{E.12}$$

Opening this matrix as $D^{(\ell)}_{m'm} = \langle \ell, m' | D | \ell, m \rangle$, one can show that

$$D^{(\ell_1)}_{m_1' m_1} D^{(\ell_2)}_{m_2' m_2} = \sum_{\ell_3} (2\ell_3 + 1) \sum_{m_3 m_3'} D^{(\ell_3)*}_{m_3' m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1' & m_2' & m_3' \end{pmatrix}. \tag{E.13}$$

Spherical harmonics Other useful relations come from the composition of spherical harmonics. We just limit ourselves to report some of the properties shown in [93].

$$\int d^2\hat{n}\, Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n}) Y_{\ell_3 m_3}(\hat{n}) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{E.14}$$

From the relation between the spherical harmonics and the Legendre polynomials

$$Y_{\ell 0}(\hat{n}) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta), \tag{E.15}$$

in the case where all the m's are vanihing, the relation (E.14) simplifies in terms of the latter as

$$\int_{-1}^{1} \frac{dx}{2} P_{\ell_1}(x) P_{\ell_2}(x) P_{\ell_3}(x) = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{E.16}$$

Appendix F

Primordial vector power spectrum

The equation of motion for the Goldstone boson σ_i^T are readily derived from the action (2.92). It is useful to redefine the vector field in order to obtain a canonically normalized action

$$\hat{\sigma}_T^i = \alpha\sqrt{2\left(-\bar{F}_Z + \frac{\bar{F}_{Y^2}}{2}\right)}\,\sigma_T^i. \tag{F.1}$$

This way the vector sector of the quadratic action for Goldstone bosons becomes

$$S^{(V)} = \int d^4x\, a^3 \frac{1}{2}\left[\dot{\hat{\sigma}}_T^i \dot{\hat{\sigma}}_{i,T} + c_v^2 \frac{\partial_j \hat{\sigma}_T^i \partial_j \hat{\sigma}_{i,T}}{a^2}\right], \tag{F.2}$$

with the definition

$$c_v^2 = \frac{2\left(\bar{F}_Z + 2\frac{\alpha^2}{a^2}\bar{F}_{ZZ}\right)}{-\bar{F}_Z + \frac{\bar{F}_{Y2}}{2}}. \tag{F.3}$$

This is the action for a free particle as it was already studied in the scalar case. In second quantization vector fields decompose in Fourier modes in the following way:

$$\hat{\sigma}_T^i = -i\int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=\pm 1} \epsilon_\lambda^i(\mathbf{k})\, \hat{\sigma}_T(k,t) e^{\mathbf{k}\cdot\mathbf{x}}. \tag{F.4}$$

where ϵ_λ^i are two polarization vectors. In terms of the Fourier modes, the Euler-Lagrange equations are

$$\ddot{\hat{\sigma}}_T + 3H\dot{\hat{\sigma}}_T - c_v^2\frac{\nabla^2}{a^2}\pi = 0. \tag{F.5}$$

This is the same equation (2.101) encountered for scalar modes. Therefore we can immediately write the solution for the σ_T^i Goldstone boson as

$$|\hat{\sigma}_{k,T}| \simeq \frac{H}{\sqrt{2c_v^3 k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}, \tag{F.6}$$

with $3/2 - \nu = -\epsilon$, and the primordial power spectrum

$$\mathcal{P}_\sigma = \frac{H_k^2}{16\pi^2 c_v \alpha^2\left(\bar{F}_Z + 2\frac{\alpha^2}{a^2}\bar{F}_{ZZ}\right)}. \tag{F.7}$$

However, since in section (4) we will deal with measurable quantities, it is useful to work with gauge invariant quantities, as shown above for the scalar case. As before, we firstly pass to the unitary gauge, where all the Goldstone bosons are set to zero. This configuration is achieved through the gauge transformation $x^\mu \to x^\mu - \xi^\mu$ with $\xi^0 = \pi$ and $\xi^i = \sigma^i$. In the high energy limit $k/a \ll H$, as said above, the gravitational metric perturbations decouple, such that the metric in the $\pi - \sigma$ language is unperturbed. Then, after a gauge transformation

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathscr{L}_\xi\, g^{(0)}_{\mu\nu} + o(\xi^2), \tag{F.8}$$

and the new metric components, in conformal time, are explicitly[1]

$$\begin{aligned}
\tilde{g}_{00} &= -a^2(1+2\dot{\pi}),\\
\tilde{g}_{0i} &= a^2\left[\sigma_i' + 2\mathcal{H}\sigma_i - \partial_i\pi\right] = a^2\left[2\mathcal{H}\sigma_{i,T} + \sigma'_{i,T} + \partial_i\left(\frac{2\mathcal{H}\sigma_L}{\sqrt{-\nabla^2}} + \frac{\sigma_L'}{\sqrt{-\nabla^2}} - \pi\right)\right],\\
\tilde{g}_{ij} &= a^2\left[(1+2\mathcal{H}\pi)\delta_{ij} + \partial_i\sigma_j + \partial_j\sigma_i\right] =\\
&= a^2\left[(1+2\mathcal{H}\pi)\delta_{ij} + \frac{1}{2}\partial_{(i,}\sigma_{j),T} + 2\partial_i\partial_j\frac{\sigma_L}{\sqrt{-\nabla^2}}\right].
\end{aligned} \tag{F.9}$$

which share the same form of the general linearly perturbed metric (2.23). Applying another gauge transformation and computing explicitly the Lie derivative of the metric as above, one can show (see ref. [123]) that there is only one possible vector gauge invariant quantity we can build out of the metric perturbations. This is found as the combination

$$V_i \equiv \omega_i^\perp - \chi_i^{\perp\,\prime} = 2\mathcal{H}\sigma_{i,T} + \sigma'_{i,T} - \sigma'_{i,T} = 2\mathcal{H}\sigma_{i,T}, \tag{F.10}$$

which represents the vector contribution to the extrinsic curvature. This is a very nice result, since it allows to relate the primordial power spectra of the gauge invariant variable V_i with the one of the vector transverse Goldstone mode $\sigma_{i,T}$, which was already studied above. Hence, remembering (F.7),

$$\mathcal{P}_V = 4\mathcal{H}^2\mathcal{P}_\sigma = \frac{H_k^4}{4\pi^2 c_v\left(\frac{\alpha^2}{a^2}\bar{F}_Z + 2\frac{\alpha^4}{a^4}\bar{F}_{ZZ}\right)}. \tag{F.11}$$

[1]Notice that if the contravariant vector $\xi^\mu = (\pi, \sigma^i)$, the covariant vector must be taken with the contraction with the unperturbed FLRW metric, that is $\xi_\mu = g_{\mu\nu}\xi^\nu = (-a^2\pi, a^2\sigma_i)$, having defined $\sigma_i \equiv \delta_{ij}\sigma^j$.

Appendix G

Modifications to the *hi_ class* code

In the case the reader may be interested in verifying our results by itself, we dedicate this chapter of the Appendix to illustrate the necessary modifications to apply to the source code of the program. The code itself is indeed meant for to purpose of studying CMB anisotropy, and some modifications are in order if one wants to focus on the background of gravitational waves. The first thing to notice is that gravitons have negligible interactions with the thermal plasma, such that they obey a collisionless Boltzmann equation, contrary to the case of the CMB photons. Moreover, while CMB was created on the last scattering surface, the SGWB was originated in primordial epochs during Iflation, and since their generation they freely stream until today.

Modification to the time integration range

By default, hi_class is developed in such a way that the source function is integrated from an early reference initial time ($\tau_{ini} \sim 200$Mpc) until today. This is because for very early times the optical depth $\tau \to \inf$ and suppresses the propagation of the temperature perturbations in the plasma. This is not the case for gravitons, since they do not undergo any collision process. We then would like firstly to push the initial time of integration to much smaller values. The information about the initial sampling time are contained in the file "*source/perturbations.c*" at the line

```
tau_lower = pth->tau_ini;
```

The simplest way to modify the initial sampling time is to replace this line with something as

```
tau_ini = 1;
```

remembering that both the CLASS and hi_class codes counts the time variables in units of Mpc. Actually this time value is really far from the time at which Iflation is expected to end. However, no cosmologically events relevant for the evolution of perturbations, like phase transition from matter to radiation dominance, are expected to take place before that time; hence this value should work correctly for our purposes[1]. As a confirmation, we verified that none variation in the spectra can be appreciate in varying the initial time from $\tau_{ini} = 1.0$ to $\tau_{ini} = 0.01$, and we fairly assume this holds for even smaller values. A more elegant way to shift the initial sampling time as back as possible is to identify τ_{lower}, which is the stating sampling time, with the earliest time tabulated by the code, that is

[1]Pay attention that too small values of the initial time may force the code to run for too much time

```
tau_lower = pba->tau_table[0];
```

The value of the first element of the array is controlled by the parameter "*start_sources_at_tau_c_over_tau_h*" which can be arbitrary set inside the file *input.c*. In particular the lower the value of this parameter the lower the initial sampling time for the source the sources, and the above situation can be recovered setting

```
class_precision_parameter(start_sources_at_tau_c_over_tau_h,double,0.00008);
```

One problem that may arise in shifting the initial sampling time concerns the sampling rate. Indeed we must teach the program to decrease the interval between two subsequent samples in order to have a faithful representation of earlier times. This can be done adding the lines

```
if(tau<=30){timescale_source = tau*pow(10,-1);}
    tau = tau + ppr->perturb_sampling_stepsize*timescale_source;
    counter++;
```

inside the sampling cycle. These lines have the effect to decrease the sampling rate as

$$\tau_{i+1} = \tau_i + sampling\_stepsize * timescale\_source;$$

Modification of the source function

The temperature angular power spectrum in the multipole space are numerically evaluated by the CLASS/hi_class codes through

$$\mathcal{C}_\ell^{TT} = 4\pi \int \frac{dk}{k} \left(\Theta_\ell(\tau_0, k)\right)^2 \mathcal{P}(k), \tag{G.1}$$

with Θ_ℓ the temperature fluctuations of the photon population and $\mathcal{P}(k)$ the dimensionless power spectrum of primordial perturbations, summed over scalar and tensor modes[2]. After an integration by parts, the temperature fluctuations sourced by scalar and tensor perturbations are respectively

$$\begin{aligned}\Theta_\ell^S(\tau_0, k) &= \int_{\tau_{ini}}^{\tau_0} d\tau \left\{ S_T^0(\tau, k) j_\ell(k(\tau - \tau_0)) + S_T^1(\tau, k) \frac{dj_\ell}{dx}(k(\tau - \tau_0)) + \right. \\ &\quad \left. + S_T^2(\tau, k) \frac{1}{2} \left[3 \frac{d^2 j_\ell}{dx^2}(k(\tau - \tau_0)) + j_\ell(k(\tau - \tau_0)) \right] \right\}, \qquad \text{(G.2)} \\ \Theta_\ell^T(\tau_0, k) &= \int_{\tau_{ini}}^{\tau_0} d\tau\, S_T(\tau, k) \sqrt{\frac{3(\ell+2)!}{8(\ell-2)!}} \frac{j_\ell(k(\tau_0 - \tau)}{(k(\tau_0 - \tau))^2}, \qquad \text{(G.3)}\end{aligned}$$

with[3]

$$S_{T,S}^0 = g\left(\frac{\delta_g}{4} + \Phi\right) + e^{-K} 2\Phi' + g'\theta_b + g\theta_b', \qquad S_{T,S}^1 = e^{-K} K(\Psi - \Phi),$$

$$S_{T,S}^2 = \frac{g}{8}(G_0 + G_2 + F_2),$$

$$S_{T,T}^2 = -g\left(\frac{F_{\gamma 0}^{(2)}}{10} + \frac{F_{\gamma 2}^{(2)}}{7} + \frac{3F_{\gamma 4}^{(2)}}{70} - \frac{3G_{\gamma 0}^{(2)}}{5} + \frac{6G_{\gamma 2}^{(2)}}{7} - \frac{3G_{\gamma 4}^{(2)}}{70}\right) - e^{-K}\chi', \tag{G.4}$$

[2]Vector modes are neglected by the code

[3]Our choice is to work in the Newtonian gauge.

where $g(\tau) \equiv -\dot{K}e^{-K}$ is the visibility function, k the optical depth, δ_g and θ_b fluctuations in the fluid density, and F, G, F_γ, G_γ quantities related to polarization modes. Looking back to our solutions, these equations have to be compared with

$$S^0_{T,S} = v^{-2}\Phi\delta(\tau - \tau_{ini}) + (1 + v^{-2})\Phi', \qquad S^1_{T,S} = 0, \qquad S^2_{T,S} = 0,$$
$$S^2_{T,T} = -\chi'. \tag{G.5}$$

Then, by comparison, we learn that the CMB sourced functions can recover the case for SGWB if we set $K = \delta_g = \theta_b = 0$ and $g = \delta(\tau - \tau_{ini})$, neglecting any polarization contribution which does not enter in our analysis. First of all we define the new variables

```
double grav_mass;
double grav_velocity;
```

in the file "*thermodynamics.h*". Then we implemented the above identifications in "*perturbations.c*". The most tricky step is how to reproduce a delta function peaked at the initial time. We inserted then, in a very rough approximation, a strongly peaked function with the shape of a power law to reproduce the same behavior. All in all the code lines we modified on the code read

```
if (ppt->gauge == newtonian) {
      double v2=pth->grav_velocity*pth->grav_velocity;
      pvecthermo[pth->index_th_exp_m_kappa] = 1;
      pvecthermo[pth->index_th_g] = pow(0.1*pow(2*3.14,0.5),-1)
      *exp(-pow(tau-1-5*0.1,2)      *pow(2*0.1*0.1,-1));
        _set_source_(ppt->index_tp_t0) =
          ppt->switch_sw * /*pth->factor */ pvecthermo[pth->index_th_g] *
           (0*delta_g / 4. + pvecmetric[ppw->index_mt_psi])/v2
          + switch_isw * (pvecthermo[pth->index_th_g] *
          (y[ppw->pv->index_pt_phi]-pvecmetric[ppw->index_mt_psi])
                          + pvecthermo[pth->index_th_exp_m_kappa]
                          * (1+1/v2) * pvecmetric[ppw->index_mt_phi_prime])
          + ppt->switch_dop /k/k * 0*(pvecthermo[pth->index_th_g]
           * dy[ppw->pv->index_pt_theta_b]
                                    + pvecthermo[pth->index_th_dg]
                                    * y[ppw->pv->index_pt_theta_b]);

        _set_source_(ppt->index_tp_t1) = switch_isw
        * pvecthermo[pth->index_th_exp_m_kappa] * k
        * (pvecmetric[ppw->index_mt_psi]-y[ppw->pv->index_pt_phi]);

        _set_source_(ppt->index_tp_t2) = 0*ppt->switch_pol
        * pvecthermo[pth->index_th_g] * P;
      }
```

for the scalar source function, and

```
 /* tensor temperature */
    if (ppt->has_source_t == _TRUE_) {
      _set_source_(ppt->index_tp_t2) = - y[ppw->pv->index_pt_gwdot]
    }
```

```
    /* tensor polarization */
    if (ppt->has_source_p == _TRUE_) {

      /* Note that the correct formula for the polarization source
         should have a minus sign, as shown in Hu & White. We put a
         plus sign to comply with the 'historical convention'
         established in CMBFAST and CAMB. */

      _set_source_(ppt->index_tp_p) =0;
    }
  }
```

for tensors, where *_set_source_(ppt->index_tp_t0)* and *_set_source_(ppt->index_tp_t2)* denotes the contributions to S_T^0 and S_T^2 respectively, while the other quantities can be easily recognized.

Modification of the Einstein equation

In order to account for the late graviton mass effects also the Einstein field equations must be modified according to

$$\chi^{''}(\eta, k) + 2\mathcal{H}\chi'(\eta, k) + (k^2 + m^2a^2)\chi(\eta, k) = 0. \tag{G.6}$$

This is simply performed by writing, in the file "*perturbations.c*

```
 /** - for tensor modes */

  if (_tensors_) {

    /* single einstein equation for tensor perturbations */
    if (pba->has_smg == _FALSE_) {
    double m2 = pth->grav_mass * pth->grav_mass;
      ppw->pvecmetric[ppw->index_mt_gw_prime_prime] =
       -2.*a_prime_over_a*y[ppw->pv->index_pt_gwdot]-(k2+2.*pba->K +
       m2*a2)*y[ppw->pv->index_pt_gw]+ppw->gw_source;
    }
```

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