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**Study of spectral and polarimetric
properties
of soft X-rays from stellar-mass
black hole accretion discs**

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Introduction

Stellar-mass black holes represent the final stage of most massive stars. These compact objects can be found in binary systems, and they might attract matter from their companion star, giving origin to an accretion disc. The inner regions of these discs reach high temperatures, capable of emitting radiation in the X-ray band. Due to scattering effects, the emerging photons turns out to be linearly polarized.

The aim of this thesis is to discuss the main results obtained by Taverna et al. 2020[23], Taverna et al. 2021[22] and Marra et al. 2026[14], using the most commonly-accepted model. The study of General Relativity effects, combined with the polarization properties of the emerging photons, gives us a crucial way to determine fundamental properties of the black hole, such as its spin.

The theoretical understanding of the emerging (and returning) radiation of the black hole has progressively evolved over time, incorporating increasingly realistic details. Dovčiak et al. (2008)[9] developed a model focused solely on direct radiation. Later, Schnittman & Krolik (2009)[20] introduced for the first time the contribution of returning radiation. Taverna et al. (2020)[23] expanded the model by introducing deviations from the ideal Chandrasekhar profile for the intrinsic polarization[4], exploring the effect of finite optical depths in the disc atmosphere τ and implementing a realistic albedo profile. Finally, Taverna et al. (2021)[22] and Marra et al. (2026)[14] further refined the analysis by including the effects of photoelectric absorption. These studies investigated the impact of partially ionized atmospheres both in collisional ionization equilibrium (CIE) and in photoionization equilibrium (PIE) regimes for direct radiation.

Chapter 1 describes the basics of the birth of a black hole and the conditions for hydrostatic equilibrium. Chapter 2 introduces the fundamental concepts of General Relativity useful for this work. Chapter 3 addresses the theoretical accretion model, explaining the origin of X-ray polarization from the disc atmosphere. Chapter 4 discusses the numerical implementations of the theoretical model, separately analyzing the direct radiation, the returning radiation, the effects of a realistic albedo, and the impact of different plasma ionization mechanisms. Finally, Chapter 5 summarizes the main conclusions of the work.

Chapter 1

Birth of a Black Hole

Black holes are born as the final evolutive stage of a massive star. For most of their existence, stars remain in a delicate hydrostatic equilibrium. However, when they exhaust their fuel, the balance is permanently broken.

1.1 Hydrostatic equilibrium and Virial theorem

Assuming a spherically-symmetric structure with radius r , the condition of hydrostatic equilibrium is given by

$$\frac{dP}{dr} = -g(r)\rho(r) = \frac{-Gm(r)\rho(r)}{r^2}, \quad (1.1)$$

where $P(r)$ is the pressure, $\rho(r)$ is the density, $m(r)$ the mass of the cloud within the radius r and G is the universal gravitational constant.

Multiplying both sides of equation (1.1) by $4\pi r^3$ and, integrating over the radius across the entire star ($r \in (0, R]$) and doing some convenient rearrangements of the equation, one obtains the following relationship

$$\langle P \rangle = -\frac{1}{3} \frac{E_{GR}}{V}. \quad (1.2)$$

where the brackets signify the average value over the volume of the star.

Equation (1.2) expresses the so-called Virial theorem: it shows that for a system in which the gravitational potential energy density E_{GR}/V is stored, the necessary pressure to reach hydrostatic equilibrium is equal to 1/3 of this potential energy density [19].

Studying the same problem from the microscopic point of view and assuming that the star is composed of a gas containing N particles, it is possible to obtain a relation between the gravitational potential energy and the internal

energy (which, for simplicity, can be assumed to be simply the kinetic energy E_k of the N particles, which corresponds to neglect the non-translational degrees of freedom of the particles[19]). For non-relativistic and ultra-relativistic particles, the relation between the average pressure $\langle P \rangle$ and the kinetic energy density changes. In particular:

- For non-relativistic particles, the average pressure reads $\langle P \rangle = 2E_k/3V$ [19] and, substituting in equation (1.2), the expression of the Virial theorem is

$$2E_k + E_{GR} = 0. \quad (1.3)$$

This expression describes a stable hydrostatic equilibrium that characterizes the star for the vast majority of its life. From equation (1.3), the total energy of the system $E_{tot} = E_k + E_{GR}$ can be rewritten as:

$$E_{tot} = \frac{E_{GR}}{2} = -E_k.$$

The kinetic energy is strictly positive ($E_k > 0$), while the total energy is negative ($E_{tot} < 0$); hence the star is a bound system. Furthermore, this relation explains the mechanism of hydrostatic equilibrium: if the star undergoes a compression, its radius decreases, causing a decrease in the gravitational potential energy. Consequently, the total energy decreases as well, meaning the kinetic energy $E_k = -E_{tot}$ must increase. Higher kinetic energy leads to a rise in the star's internal temperature, allowing the star to resist further contraction and restore equilibrium. The exact opposite occurs if the star expands.

- For ultra-relativistic particles, one has $\langle P \rangle = E_{kv}/3V$ so that, substituting in equation (1.2), the expression of the Virial theorem becomes

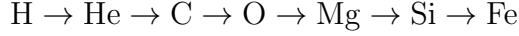
$$E_{tot} = E_k + E_{GR} = 0. \quad (1.4)$$

Under the assumption of no dissipative forces, equation (1.4) shows that the total energy must be zero; in other words, the system is unstable: a small perturbation can put the star in a bound or unbound state (respectively, $E_{tot} < 0$ or $E_{tot} > 0$).

After the star exhausts its fuel, it can no longer contrast gravity, leading the star to gravitational contractions and raising the temperature. The star reaches high energies so that the system is able to transition to an ultra relativistic limit.

1.2 Lane-Emden equation

In stars the heat and high density in the core allow them to synthesize heavier and heavier nuclei [19], following the chain



When the nuclear reactions reach the elements of the iron group (mass number $A = 56$) nuclear fusion becomes endothermic, and energy cannot be extracted by fusing elements heavier than iron. Hence, as soon as iron starts to be produced and builds up in the center of the structure, the balance between gravity and internal forces breaks, causing the star to collapse under its own weight. Particles (electrons and nuclei) are stored in a volume smaller and smaller. However, because they are fermions, they must obey Pauli's principle [5], i.e. two particles cannot occupy the same quantum state, but, while the first ones will occupy the lowest energy states, the subsequent ones will arrange themselves at progressively higher energies, thus becoming degenerate. This means that fermions in the star must have a minimum energy, which is called Fermi's energy [21], that depends on the mass of the particles. Because electrons have a mass much smaller than the nuclei, they will be the first to become degenerate (higher Fermi's energy). From equation (1.1), where

$$m(r) = \int_0^r 4\pi \tilde{r}^2 \rho d\tilde{r},$$

differentiating with respect to r both members, we have

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho. \quad (1.5)$$

To solve equation (1.5), we need an equation of state (i.e., a relation that gives P as a function of density and temperature). If the electrons are degenerate, the thermal energy does not have any influence on the pressure, which is exclusively provided by the degenerate gas pressure. In this way, we can use an adiabatic equation of state [21], which can be written as

$$P = K \rho^{1+\frac{1}{\eta}},$$

called the polytropic equation of state, where K is a constant and η is the polytropic index. In the case at hand, an additional assumption can be made, writing

$$\rho = \rho_c \phi^\eta$$

where $\phi = \phi(r)$, is a dimensionless function of the radius r and ρ_c is the central density of the structure. After some algebraic manipulation, we have

$$\frac{dP}{dr} = \frac{d}{dr} \left(K \rho^{1+\frac{1}{\eta}} \right) = K \rho_c^{1+\frac{1}{\eta}} (1 + \eta) \phi^\eta \frac{d\phi}{dr},$$

and substituting this into equation (1.5) gives

$$\frac{K(\eta + 1)\rho_c^{-1+\frac{1}{\eta}}}{4\pi G r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -\phi^\eta.$$

The constant term at the LHS of the equation has dimension of a squared length; defining

$$a^2 := \frac{K(\eta + 1)\rho_c^{-1+\frac{1}{\eta}}}{4\pi G r^2},$$

the equation becomes

$$\frac{a^2}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -\phi^\eta,$$

and again, defining $\xi := r/a$, we finally obtain

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\phi}{d\xi} \right) = -\phi^\eta. \quad (1.6)$$

Equation (1.6) is the Lane-Emden equation, a non-linear second order ODE which describes the distribution of density, temperature, and pressure at the center of a star sustained by the degenerate (electron) gas pressure in hydrostatic equilibrium[21].

From equation (1.6) it is possible to find the total mass $M(R)$ and, for fixed values of η , its dependence on the central density. The same can be done for R . By combining these dependencies, one obtains the mass-radius relation

$$R \propto M^{(1-\eta)/(3-\eta)},$$

and one can derive that, for ultra-relativistic particles ($\eta = 3$), the denominator in the exponent diverges. So even a small change in the mass translates into a very large change in the radius, making the structure unstable and breaking the hydrostatic equilibrium guaranteed by the pressure of the degenerate electrons.

1.3 Chandrasekhar limit

For ultra relativistic electrons it is possible to estimate the Chandrasekhar mass limit beyond which the spherically-symmetric structure cannot be maintained anymore in equilibrium by the pressure of degenerate electrons. Calculating $M(R)$ for $\eta = 3$ we obtain the Chandrasekhar mass

$$M_{Ch} = 4\pi \left(\frac{K}{\pi G} \right)^{\frac{3}{2}} \cdot 2.0182, \quad (1.7)$$

where the number 2.0182 is obtained using the Chandrasekhar's tables[4]. Finally, using the proper value of K for our case, we obtain

$$M_{Ch} \approx 1.44 M_{\odot}. \quad (1.8)$$

It should be noted that 1.44 is not a fixed value, but it depends on the star's composition. When the core exceeds the 1.44 solar masses, it collapses and in a fraction of a second the star implodes, causing the outer layers to bounce off the compact core and be expelled by a shockwave. At this point, the gravitational collapse can be stopped by the neutron degenerate gas pressure, giving birth to a neutron star. However, if the core exceeds $M \approx 3.2M_{\odot}$, we must underline that the core (not the star) continues its gravitational collapse until it falls below the limit where the escape velocity equals the speed of light, known as the event horizon.

That is why we talk about black holes (BHs): astrophysical objects so compact that neither light can escape.

1.4 A Newtonian problem

Consider a particle with mass m moving away radially from a spherically symmetric distribution of matter with radius R , uniform density ρ and total mass M . If the particle has a velocity v at a distance r from the center, then its total energy E in the absence of dissipative forces is given by

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r}.$$

We define the escape velocity v_e as the velocity at the surface of the distribution of matter ($r = R$) that enables the particle to escape up to arrive at infinity with zero velocity. By applying the conservation of energy between the surface and infinity, we obtain

$$\frac{1}{2}mv_e^2 - G\frac{Mm}{R} = 0.$$

Imposing that the escape velocity is equal to the speed of light ($v_e = c$), we have

$$R = \frac{2GM}{c^2}$$

Since $v_e = \sqrt{2GM/R}$, the escape velocity increases as the star's radius R decreases, meaning that a collapsing core will eventually reach a critical size where $v_e = c$. However, this theory has the problem that it mixes classical mechanics with the speed of light, which according to special relativity, cannot be reached by massive bodies. This problem will be solved in Chapter 2, introducing General Relativity.

1.5 Types of Black Holes

- **Stellar-mass black holes.** These are black holes that form as the final product of the evolution of stars with an initial mass greater than $20\text{--}25 M_\odot$ (where the core mass exceeds the previously mentioned limit of $\approx 3.2 M_\odot$). Their typical mass ranges between 3 and $10 M_\odot$, although they can reach higher values. There is an upper limit to the mass of stellar black holes, as the mass of stars is bounded from above by the threshold at which radiation pressure disrupts the entire stellar structure, corresponding to $\approx 100 M_\odot$.
- **Supermassive black holes.** These are black holes located at the center of many galaxies, including the Milky Way, whose mass has been estimated by observing the motion of closely orbiting stars. Their masses range from $10^6 M_\odot$ to over $10^9 M_\odot$ for older and more distant galaxies.
- **Intermediate-mass black holes.** These objects have masses ranging between 100 and $10^5 M_\odot$; recently, first through the detection of ultra-luminous X-ray sources (ULXs) and then through the observation of gravitational waves from black hole–black hole mergers, it has become possible to observe this class of objects as well.

Chapter 2

Elements of General Relativity

In Classical Mechanics, gravity is regarded as action at distance. The revolution of the General Relativity mainly consists in changing this point of view, by associating gravity with the curvature of the spacetime. In this chapter we briefly discuss some of the main mathematical aspects of this theory.

2.1 Metric Tensors

Let M be a n -differentiable manifold and $p \in M$ a point. Let $V := T_p M$ be the vector space tangent to the manifold at p and let $V^* = T_p^* M$ be its dual space.

A tensor of type (r, s) at p is defined as a multilinear map

$$T: \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \longrightarrow \mathbb{R}.$$

The metric tensor $\vec{g}$ is a $(0,2)$ tensor that, having two arbitrary vectors $\vec{A}$ and $\vec{B}$ as arguments, associates to them a real number that is the inner product $\vec{A} \cdot \vec{B}$:

$$g(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}.$$

The components of the metric tensor are obtained by replacing $\vec{A}$ and $\vec{B}$ with the basis vectors

$$g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta) = \partial_\alpha \cdot \partial_\beta. \quad (2.1)$$

It allows us to compute the scalar product of two vectors in any space and whatever coordinates we use

$$\vec{A} \cdot \vec{B} = g(\vec{A}, \vec{B}) = g(A^\alpha \partial_\alpha, B^\beta \partial_\beta) = A^\alpha B^\beta g(\partial_\alpha, \partial_\beta) = A^\alpha B^\beta g_{\alpha\beta}. \quad (2.2)$$

Hence, it allows to compute the distance between two points,

$$ds^2 = g(\vec{ds}, \vec{ds}) = g_{\alpha\beta} dx^\alpha dx^\beta.$$

Other properties are: it maps one-forms into vectors or vice versa, and it allows to raise and lower indices.

2.2 Affine Connection

Let us consider a vector field $\vec{V} = V^\mu \vec{e}_{(\mu)}$. The derivative of $\vec{V}$ is

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_{(\alpha)} + V^\alpha \frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} \quad (2.3)$$

The first term on the RHS is a linear combination of the basis vectors. The second term, instead, arises because in a curved spacetime the basis vectors change while one moves on the surface. To solve this, one introduces the Christoffel symbols or affine connections, $\Gamma_{\alpha\beta}^\mu$ to quantify this spatial variation, expressing the derivative of a basis vector as a linear combination of the basis itself:

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \Gamma_{\alpha\beta}^\mu \vec{e}_{(\mu)} \quad (2.4)$$

where the indices α and β indicate respectively the basis vector that we are differentiating, and the coordinate with respect to which the differentiation is performed. Thus, coming back to the derivative of $\vec{V}$, and introducing the notation $V^\alpha_{;\beta} = \frac{\partial V^\alpha}{\partial x^\beta}$, we can define the covariant derivative for the components of a vector:

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + V^\mu \Gamma_{\beta\mu}^\alpha \quad (2.5)$$

In the end, this allows us to define the covariant derivative for a vector:

$$\nabla \vec{V} = [V^\alpha_{;\beta} \vec{e}_{(\alpha)}] \otimes d\vec{x}^\beta \quad (2.6)$$

that is a (1,1) tensor.

We want to underline that if the Christoffel symbols are symmetric, i.e: $\Gamma_{\beta\mu}^\gamma = \Gamma_{\mu\beta}^\gamma$ and the covariant derivative of the metric tensor is zero [7], then we have

$$\Gamma_{\beta\mu}^\gamma = \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \quad (2.7)$$

This expression allows to compute the affine connection in terms of the components of the metric.

2.3 Parallel transport

Parallel transport means that for each infinitesimal displacement, the displaced vector must be parallel to the original one, and all the subsequent displacements must have the same length.

The crucial difference between flat and curved spaces is that, in the latter case, the results of parallel transporting a vector from one point to another depends on the path between the points.

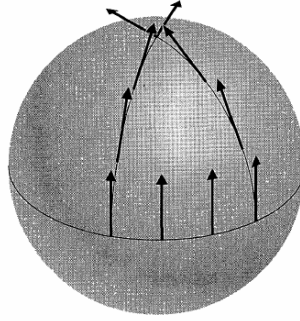


Figure 2.1: Parallel transport on a two-sphere[3].

Now consider a curve parameter λ and a vector field $\vec{V}$ defined at every point of the curve. Let $\vec{U} \rightarrow \{\frac{\partial x^\alpha}{\partial \lambda}\}$ be the vector tangent to the curve. At every point of the curve we can choose a locally inertial frame $\{\xi^\alpha\}$. In this frame we move $\vec{V}$ along the curve of a segment $d\lambda$, parallel to itself and keeping its length unchanged, i.e.

$$\frac{dV^\alpha}{d\lambda} = \frac{\partial V^\alpha}{\partial \xi^\beta} \frac{d\xi^\beta}{d\lambda} = U^\beta V^\alpha{}_{,\beta} = 0. \quad (2.8)$$

In this way, equation (2.8) is the frame invariant definition of parallel transport of $\vec{V}$ along the curve identified by the tangent vector $\vec{U}$.

If we want to write it in a frame independent form we shall write

$$\nabla_{\vec{U}} \vec{V} = 0, \quad (2.9)$$

which means that the covariant derivative along the direction of the vector $\vec{U}$ is zero.

2.3.1 Geodesic equation

With the parallel transport defined, the next logical step is to discuss geodesics. A geodesic is the curved-space generalization of the notion of a straight line in Euclidean space. It is commonly known that, in Euclidean geometry, a

straight line is the path of shortest distance between two points. An equivalent definition for a straight line is a path that parallel transports its own tangent vector. The tangent vector to a path $x^\mu(\lambda)$ is $dx^\mu/d\lambda$. The condition that it is parallel transported imposes that it satisfies the so called geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \quad (2.10)$$

2.3.2 Curvature tensors

From equation (2.5), it is possible to define

$$R^\lambda_{\mu\nu\kappa} = - \left[\frac{\partial}{\partial x^\kappa} \Gamma_{\mu\nu}^\lambda - \frac{\partial}{\partial x^\nu} \Gamma_{\mu\kappa}^\lambda + \Gamma_{\kappa\eta}^\lambda \Gamma_{\mu\nu}^\eta - \Gamma_{\nu\eta}^\lambda \Gamma_{\mu\kappa}^\eta \right], \quad (2.11)$$

that is the (symmetric) curvature tensor or Riemann Tensor.

An important property of the Riemann tensor is that it gives us the commutator of the covariant derivatives

$$[\nabla_\alpha, \nabla_\beta] V^\mu = R^\mu_{\nu\alpha\beta} V^\nu. \quad (2.12)$$

Since this is a tensor equation, equation (2.12) should be valid in any frame. It implies also that in curved spacetime covariant derivatives do not commute and therefore the order in which they appear is important.

We mention another important tensor equation, the Bianchi identity

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0. \quad (2.13)$$

2.4 Einstein's equations

The concepts of space and time are no longer separated in general relativity: they belong to a four-dimensional manifold, called spacetime.

The way in which the space time is warped (e.g. due to the presence of matter) is described by the Einstein field equations,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (2.14)$$

The equality (2.14) tells us not only how matter warps spacetime, but also how spacetime dictates the motion of matter. Solving equation (2.14) means determining the geometry of the spacetime, i.e. its metric.

In equation (2.14), $T_{\mu\nu}$ is the stress-energy tensor, G and c are respectively

the Newton's constant and the speed of light, and $G_{\mu\nu}$ is the Einstein's tensor defined in terms of the Ricci's tensor $R_{\mu\nu}$ and Ricci's scalar R

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}.$$

Note that in vacuum $T_{\mu\nu} = 0$, hence calculating the trace of equation (2.14) (i.e. $g^{\mu\nu}G_{\mu\nu} = 0$) we have

$$R_{\mu\nu} = 0. \quad (2.15)$$

2.4.1 Symmetries for the Einstein's equations

Let us consider a vector field $\vec{\xi}(x^\mu)$ defined at every point x^α of a spacetime region. $\vec{\xi}$ identifies a symmetry if an infinitesimal translation along $\vec{\xi}$ leaves the line element ds^2 unchanged, i.e.

$$\delta(ds^2) = \delta(g_{\alpha\beta}dx^\alpha dx^\beta) = 0. \quad (2.16)$$

A vector field satisfying this property is called a Killing vector field. After some computation, it is possible to obtain that a solution of Einstein's equations is invariant under translation along $\vec{\xi}$, if and only if

$$g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi_{,\alpha}^\delta + g_{\alpha\delta}\xi_{,\beta}^\delta = 0. \quad (2.17)$$

To find the Killing vector field of a given metric $g_{\alpha\beta}$ we need to solve equation (2.17), which is a system of differential equations for the components of $\vec{\xi}$. This is equivalent to solve the Killing equation

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \quad (2.18)$$

If equation (2.18) has no solutions, the spacetime has no symmetries.

2.5 The Schwarzschild solution

Now we can analyze two fundamental solutions for Einstein's equations, which describe uncharged black holes.

The most obvious application of a theory of gravity is to a spherically symmetric and static (i.e. time independent) gravitational field. In this section, we examine the simple case of vacuum solutions, i.e. $R_{\mu\nu} = 0$.

Without giving more details (that can be found in [21], [7]), it follows that the Schwarzschild solution (written in spherical polar coordinates) is

$$ds^2 = - \left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.19)$$

which describes the spacetime outside a non-rotating BH. In equation (2.19) M is the mass of the BH measured by an observer orbiting the mass source, also known as the Keplerian mass [3].

Note that, when $r \rightarrow \infty$ the metric reduces to that of a flat spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2);$$

therefore, we say the metric is asymptotically flat. Furthermore, from Birkhoff's theorem[3], it can be shown that equation (2.19) is the unique static vacuum solution with a spherical symmetry.

2.5.1 Singularities in the Schwarzschild metrics

From equation (2.19), the metric coefficient g_{rr} becomes infinite at $r = 0$ and $r = 2GM/c^2$. We define $r = R_s = 2GM/c^2$ as the Schwarzschild radius, which seems to indicate some pathologies of the metric tensor. Actually, the metric coefficients are coordinate dependent quantities, hence it is possible to have a "coordinate singularity", which is only a matter of the chosen coordinate system and not a physical problem of the spacetime.

A precise criterion to determine whether a metric divergence indicates a real spacetime pathology is to study where the curvature tensor diverges[3]. As seen in section (2.3.2) the curvature is a tensor, and obviously its components are coordinate dependent. Nevertheless, from the Riemann tensor some useful scalar quantities can be constructed, which are coordinate independent[3], such as the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$. In our case, direct calculation on the Schwarzschild metric[3] reveals that

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2 M^2}{r^6}, \quad (2.20)$$

and this is enough to say that $r = 0$ represents an actual spacetime singularity. For the divergence at $r = 2GM/c^2$ the case is different. Taking for simplicity the geometrical units $G = c = 1$ and introducing the change of coordinates[21]

$$r' = r \quad \theta' = \theta \quad \phi' = \phi \quad t' = t - 2M \log \left(1 - \frac{r'}{2M}\right),$$

we have

$$dr' = dr \quad d\theta' = d\theta \quad d\phi' = d\phi$$

$$dt = dt' - 2M \frac{dr'}{1 - r'/2M} \left(-\frac{1}{2M} \right) = dt' + 2M \frac{dr'}{2M - r'}.$$

Eventually, substituting in equation (2.19), we obtain

$$ds^2 = - \left(1 - \frac{2M}{r'} \right) dt'^2 + \frac{4M}{r'} dt' dr' + \left(1 + \frac{2M}{r'} \right) dr'^2 + r'^2 (d\theta'^2 + \sin^2 \theta' d\phi'^2). \quad (2.21)$$

Although an off-diagonal term $dt' dr'$ came out, the divergence at $r = 2M$ has vanished, so it is merely apparent.

In particular, in eq. (2.21), considering $d\theta = d\phi = 0$ (radial motion) we have

$$ds^2 = - \left(1 - \frac{2M}{r'} \right) dt'^2 + \frac{4M}{r'} dr' dt' + \left(1 + \frac{2M}{r'} \right) dr'^2, \quad (2.22)$$

For a massive particle, its trajectory must be a timelike trajectory, which, adopting as metric signature $(-, +, +, +)$, means $ds^2 < 0$. Analyzing equation (2.22), we notice that the third term on the RHS is always positive. Therefore, to satisfy the condition $ds^2 < 0$:

- Outside the Schwarzschild radius ($r' > 2M$): the first term $-(1 - 2M/r')$ is negative. This is enough to obtain $ds^2 < 0$ regardless of the sign of the second term. Thus, dr' can be either positive or negative, meaning the particle can move radially both towards the infinity or towards $r = 0$.
- Inside the Schwarzschild radius ($r' < 2M$): the first term becomes positive. Since the third term is also positive, hence to obtain $ds^2 < 0$ the second term $\frac{4M}{r'} \frac{dr'}{dt'}$ must be negative. This requires $dr' < 0$. Consequently, a massive particle can only move radially inwards, inevitably approaching the singularity at $r = 0$.

Similarly, for a massless particle, like photons, the trajectory is constrained to be a null curve, i.e. $ds^2 = 0$. Applying the same logic to equation (2.22) to obtain zero, the photon can move in both directions outside $r' = 2M$, but it is forced to move inwards ($dr' < 0$) once inside $r' < 2M$. Therefore, we can think of the surface at $r = R_s$ as a semipermeable membrane: it can be crossed by photons in one direction, from the outer to the inner region, while once inside, it is impossible to escape. This defines the Schwarzschild radius as the event horizon in a static, spherically-symmetric space-time.

2.5.2 Geodesics and the Innermost Stable Circular Orbit (ISCO)

We now want to study the motion of a free particle in the Schwarzschild metric, that is, a particle subject to no forces other than gravity (which is not treated as a force in General Relativity). While a complete study would require solving the full geodesic equations in the Schwarzschild spacetime, we can focus on the most relevant properties of geodesic motion by analyzing only the radial motion and exploiting the constants of motion.

Let us consider a particle of mass m moving in the equatorial plane, the Lagrangian of this free particle is

$$\mathcal{L} = \frac{1}{2} \left[- \left(1 - \frac{2m}{r} \right) \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2m}{r} \right)} + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right]. \quad (2.23)$$

where ($G = c = 1$). $\mathcal{L}$ does not depend on t and ϕ , hence these act as cyclic coordinates. The symmetries of the metric associated with this coordinates give two conserved quantities, which correspond to the respective covariant components of the 4-momentum, p_t and p_ϕ . These are used to define the specific energy $E = -p_t/m$ and the specific angular momentum $L = p_\phi/m$. For a massive particle, the radial equation of motion can be written in the form of a one dimensional classical mechanics problem

$$\left(\frac{dr}{dt} \right)^2 = E^2 - V_{eff}(r), \quad (2.24)$$

where $V_{eff}(r)$ is the effective potential, defined as

$$V_{eff}(r) = \left(1 - \frac{2M}{r} \right) \left(1 + \frac{L^2}{r^2} \right). \quad (2.25)$$

By analyzing equation (2.24), one can identify different types of orbits:

- For $E^2 \geq 1$, the trajectories are unbound open orbits, analogous to Newtonian hyperbolic orbits.
- For $V_{\min} < E^2 < 1$, the particle is in a bound orbit, but due to relativistic effects, they result rosette-like trajectories with a secular precession of the periastron.
- For $E^2 = V_{\max}$ and $E^2 = V_{\min}$, we have respectively unstable and stable circular orbits.

Differentiating equation (2.25) leads to two possible roots solutions:

$$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M}. \quad (2.26)$$

The root r_+ corresponds to a minimum of the potential, representing a stable circular orbit, while the root r_- is a local maximum, representing an unstable circular orbit.

These two roots are real and distinct only if $L^2 > 12M^2$. When $L^2 = 12M^2$, the two roots merge into a single point

$$r_{ISCO} = 6M. \quad (2.27)$$

Hence, there are no stable circular orbits for $r < 6M$. We define r_{ISCO} as radius of the Innermost Stable Circular Orbit (ISCO). Matter in an accretion disc will spiral inward through stable circular orbits, losing energy and angular momentum due to friction, until it reaches the ISCO. Once it crosses $r = 6M$, it undergoes a rapid radial fall into the black hole.

2.6 Kerr solution

The Kerr's solution describes the spacetime outside a rotating BH. Since the mass distribution is rotating, it is convenient to abandon the spherical symmetry approximation and consider an axisymmetric coordinate system, choosing the rotation axis of the BH as the preferred reference axis. In these conditions, one can use the Boyer-Lindquist coordinate system

$$\begin{cases} x = \sqrt{r^2 + a^2} \sin \theta \cos \phi \\ y = \sqrt{r^2 + a^2} \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad (2.28)$$

with $a = J/M$ the specific angular momentum[21]. Hence the line element of the Kerr's metric is

$$ds^2 = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{2A}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{aA}{\Sigma} \right) \sin^2 \theta d\phi^2 \quad (2.29)$$

where

$$\begin{aligned} \Sigma &:= r^2 + a^2 \cos^2 \theta \\ \Delta &:= r^2 - 2Mr + a^2 \\ A &:= 2Mar \sin^2 \theta \end{aligned} \quad (2.30)$$

2.6.1 Singularities in the Kerr metric

As for the Schwarzschild solution, the Kerr solution also converges to a flat space for $r \rightarrow \infty$; moreover for $a \rightarrow 0$ the solution converges to the external Schwarzschild one.

From the calculation of the scalar invariant $R^{\mu\nu\gamma\sigma}R_{\mu\nu\gamma\sigma}$ [7], it comes out that equation (2.29) diverges for $\Sigma \rightarrow 0$, indicating that $\Sigma = 0$ is an intrinsic singularity. This occurs when

$$x^2 + y^2 = a^2, \quad z = 0. \quad (2.31)$$

In the equatorial plane ($z = 0$) the singularity turns out to be shaped as a ring.

The event horizon in Kerr is obtained by studying the divergence of the component g_{11} of the metric. It is yielded by solving

$$\Delta = 0,$$

hence

$$r_h = M \pm \sqrt{M^2 - a^2}. \quad (2.32)$$

We are only interested in the positive solution, the outer event horizon, because the negative one is covered by the external horizon (all inside of the external horizon is invisible to an external observer).

Instead the element g_{00} is zero for

$$r = M \pm \sqrt{M^2 - a^2 \cos^2 \theta}. \quad (2.33)$$

As for g_{11} , we're only interested in the positive solution, which is called ergosphere.

From equation (2.32), the condition for the horizon to exist is $M^2 - a^2 \geq 0$, which yields

$$\frac{a}{M} \leq 1. \quad (2.34)$$

Furthermore, since $\cos^2 \theta \leq 1$, the condition for the existence of the ergosphere ($M^2 - a^2 \cos^2 \theta \geq 0$) is automatically satisfied whenever the horizon exists.

Hence the Kerr solution describes a rotating BH if equation (2.34) holds.

In addition, while in the non-rotating case the ISCO is located at $r_{ISCO} = 6M$, in the Kerr spacetime the radius depends on the spin a . In fact, the radius decreases as the spin increases, eventually reaching $r = M$ for a maximally rotating BH.

Chapter 3

X-ray Polarization from BH Accretion Discs

The main objective of this thesis is to discuss what information can be obtained by studying the radiation coming from an accretion disc around a BH. Chapter 1 described the origin of BHs, while Chapter 2 established the mathematical framework of the General Relativity. By means of these concepts, in this last chapter we aim to show how the strong gravity leaves an observable imprint on the emitted X-ray of the accretion disc.

3.1 Polarization from the disc atmosphere

3.1.1 Polarization

The study of time-dependent electric and magnetic fields revealed the existence of electromagnetic waves which constitute a complete set of solutions for Maxwell's equations in vacuum ($\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field)

$$\begin{cases} \nabla \cdot \vec{E} = 0, & \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \\ \nabla \cdot \vec{B} = 0, & \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0. \end{cases} \quad (3.1)$$

In three dimensions, the set of monochromatic plane waves is described by

$$f_k(t, \vec{x}) = e^{i(\omega t - \vec{k} \cdot \vec{x})}, \quad (3.2)$$

and the general solution is

$$f(t, \bar{x}) = \text{Re} \int f_0(\bar{k}) e^{i(\omega t - \bar{k} \cdot \bar{x})} d^3 k, \quad (3.3)$$

where $\bar{k}$ is the wave vector, which points in the direction of propagation of the wave.

From equation (3.1), one can derive the expressions for the electric field and magnetic field relative to a generic electromagnetic monochromatic wave

$$\begin{cases} \bar{E}_{\bar{k}}(t, \bar{x}) = \text{Re}[\bar{E}_0 e^{i(\omega t - \bar{k} \cdot \bar{x})}] \\ \bar{B}_{\bar{k}}(t, \bar{x}) = \text{Re}[\bar{B}_0 e^{i(\omega t - \bar{k} \cdot \bar{x})}]. \end{cases} \quad (3.4)$$

where ω corresponds to the wave angular frequency, and $|\bar{k}| = k$ corresponds to the wave number. Therefore, the following relations hold:

$$\omega = kv, \quad \lambda\nu = v, \quad k = \frac{2\pi}{\lambda} \quad (3.5)$$

where ν corresponds to the wave frequency, v to the propagation velocity, and λ to the wavelength.

If $\bar{E}$ and $\bar{B}$ solve equations (3.1), then they solve the wave equation in three dimensions,

$$\square \bar{E} = 0 = \square \bar{B}, \quad \square := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \bar{\nabla}. \quad (3.6)$$

Hence, in the xy plane the components of $\bar{E}$ can be written as

$$\begin{cases} E_x = E_{0x} \cos(\omega t - kz) \\ E_y = E_{0y} \cos(\omega t - kz + \phi), \end{cases} \quad (3.7)$$

where ϕ is the initial phase.

We point out that, if we define $\bar{n} = \frac{\bar{k}}{k}$, then $\bar{E}$, $\bar{B}$ and $\bar{n}$ are always orthogonal to each other, and they are linked by the relation

$$\bar{B}(t, \bar{x}) = \bar{n} \times \bar{E}(t, \bar{x}). \quad (3.8)$$

Polarization refers to the direction properties of the electric field vector $\bar{E}$, which in turn are determined by the properties of the complex vector $\bar{E}_0$. For a generic $\bar{E}_0$ and $\bar{x}$ fixed, as t changes, $\bar{E}$ describes an ellipse. If $\bar{x}$ changes along the direction $\bar{n}$, for a t fixed, $\bar{E}$ describes a helix with an elliptical cross-section. In the end, if $\phi = 0, \pi$, the polarization is called linear and the electric field holds the same direction during propagation.

3.1.2 Stokes parameters

The Stokes parameters are a set of values, obtained as suited combinations of the electric field oscillation amplitudes along the x and y directions, that describe polarization state of electromagnetic field radiation. We can define them as[18]

$$\begin{cases} I = E_{0x}^2 + E_{0y}^2 \\ Q = E_{0x}^2 - E_{0y}^2 \\ U = 2E_{0x}E_{0y} \cos \phi \\ V = 2E_{0x}E_{0y} \sin \phi, \end{cases}$$

where I is the total intensity of the beam, Q and U describe the linear polarization, and V describe the circular polarization.

Through the Stokes parameters, it is possible to define specific polarization observables. In particular, the linear polarization degree P and the polarization angle χ are defined as:

$$P = \frac{\sqrt{Q^2 + U^2}}{I}, \quad \chi = \frac{1}{2} \arctan \left(\frac{U}{Q} \right) \quad (3.9)$$

Physically, P represents the fraction of the total collected radiation that is linearly polarized. The angle χ indicates the polarization direction, which is the net orientation of the oscillating electric field (summing the electric fields of all detected photons) with respect to a chosen reference direction in the plane of the sky. In this work, we will focus exclusively on linear polarization (hence neglecting V). This choice is motivated by the fact that the dominant process responsible for producing polarization in the radiation emitted by accretion discs is electron-photon scattering, which generates primarily linear polarization. Furthermore, from an observational standpoint, accretion discs around stellar-mass BHs emit mainly in the X-ray band, and modern X-ray polarimetry technologies are currently sensitive only to linear polarization[15].

3.2 The accretion disc model

A stellar mass BH is characterized by its mass M and its dimensionless angular momentum a , while the surrounding spacetime is generally described by the Kerr metric[23]. In our galaxy, binary star systems are actually much more common than single stars[19]. The companion star can survive the supernova explosion of the massive primary star, remaining gravitationally bound to the newly formed compact object. Later in its evolution, when the companion becomes a red giant, it expands its outer layers. Once the star expands, a mass transfer mechanism known as Roche-Lobe overflow is triggered. The enormous

gravitational field of the BH pulls the matter of the companion star, and the outward transfer of angular momentum leads to the formation of an accretion disc. According to the standard Shakura-Sunayev model[16], this disc can be described as a geometrically thin and optically thick structure. As the matter spirals (with Keplerian velocity) into the BH, internal friction between particles heats up the disc, causing its surface layers to emit thermal radiation as a local isotropic blackbody distribution.

Assuming no torque at the inner edge of the disc, which is taken coincident with the ISCO. According to the Shakura-Sunyaev model, the temperature decreases with the radius as $r^{-3/4}$; nevertheless, to account the general relativistic effects, we must use the Novikov-Thorne profile[17], where the radial dependence is far more complex and it is given by the function $f(\xi, a)$. The temperature profile of the accretion disc is:

$$T(M, r, a) = 741 f_{\text{col}} \left(\frac{M}{M_{\odot}} \right)^{-1/2} \left(\frac{\dot{M}}{M_{\odot} \text{ yr}^{-1}} \right)^{1/4} \times [f(\xi, a)]^{1/4} \text{ keV}, \quad (3.10)$$

with $\dot{M}$ the mass accretion rate, $\xi = (r/r_g)^{1/2}$, r the radial coordinate and $r_g = GM/c^2$ the gravitational radius. Furthermore, $f(\xi, a)$ is a relativistic correction factor and f_{col} is a constant used to account for the effects of scatterings inside the disc. The resulting radial dependence of the temperature is illustrated in Figure 3.1. Given the composition of the disc, the emerging radiation is expected to be polarized; hence, we assume a Chandrasekhar profile (Figure 3.2) for the photon polarization state. In fact, photons originating from the disc midplane interact with the disc material primarily through scattering off charged particles. This process tends to linearly polarize the scattered radiation: this is due to the fact that the electric field of the scattered photons oscillates perpendicular to the scattering plane, while remaining constrained to be perpendicular to the propagation direction. Therefore, radiation emitted by the accretion disc is expected to be linearly polarized, with a degree of polarization depending on the observer's inclination. Since the scattering cross section increases significantly as the particle mass decreases, scattering is far more efficient on electrons than on heavier constituents like protons or neutral atoms. Consequently, the basic assumption is that radiation passes through a surface layer (the disc atmosphere) where the dominant photon-matter interaction is electron scattering. This process is described by the optical depth $\tau = \int n \sigma dz$, where n is the number density of particles per unit of geometric thickness dz , and σ is the scattering cross section. When the optical depth is much larger than one, every photon will eventually be scattered by electrons. Analyzing the symmetry of the system, we can have two polarization states:

parallel or perpendicular to the projection of the disc symmetry axis in the plane of the sky.

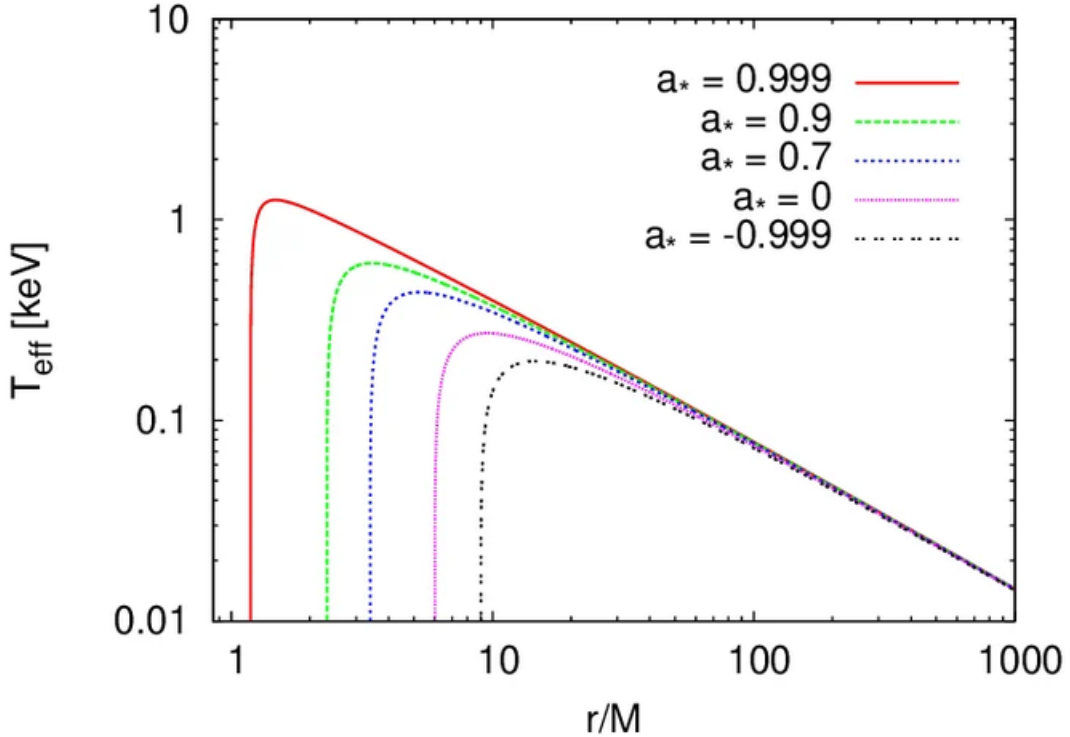


Figure 3.1: Radial profile of the effective temperature T_{eff} in Novikov-Thorne discs around Kerr black holes with $M = 10 M_{\odot}$, $\dot{M} = 10^{18} \text{ g s}^{-1}$, and different values of the black hole spin parameter a_* (figure taken from [2]).

Since we are considering radiation propagating in the vicinity of a BH, general relativistic effects must be taken into account. Due to the strong gravitational field, photons are redshifted and undergo ray-bending as they follow null geodesics. Consequently, an observer at infinity does not merely receive photons traveling in straight lines, as the trajectories of photons emitted closer to the BH are strongly deflected. Radiation arriving directly from the disc to the observer without further interactions with the disc material is referred to as direct radiation. Conversely, depending on their emission point and initial direction, some photons are bent to such an extent that they return to strike the disc on the opposite side. Then, these photons can be reflected or reprocessed by the disc matter before eventually escaping, forming the returning radiation component. Furthermore, general relativity also influences the polarization state. Due to the strong gravity, the photon polarization plane is parallel transported along its null geodesic [23]. In the BH space-time, this geometrical effect causes a rotation of the polarization vector with respect to its

initial direction at the emission point, an effect known as gravitational Faraday rotation. Because this rotation is directly tied to the spacetime metric, measuring it gives us a direct way to extrapolate physical parameters of the BH, such as its spin.

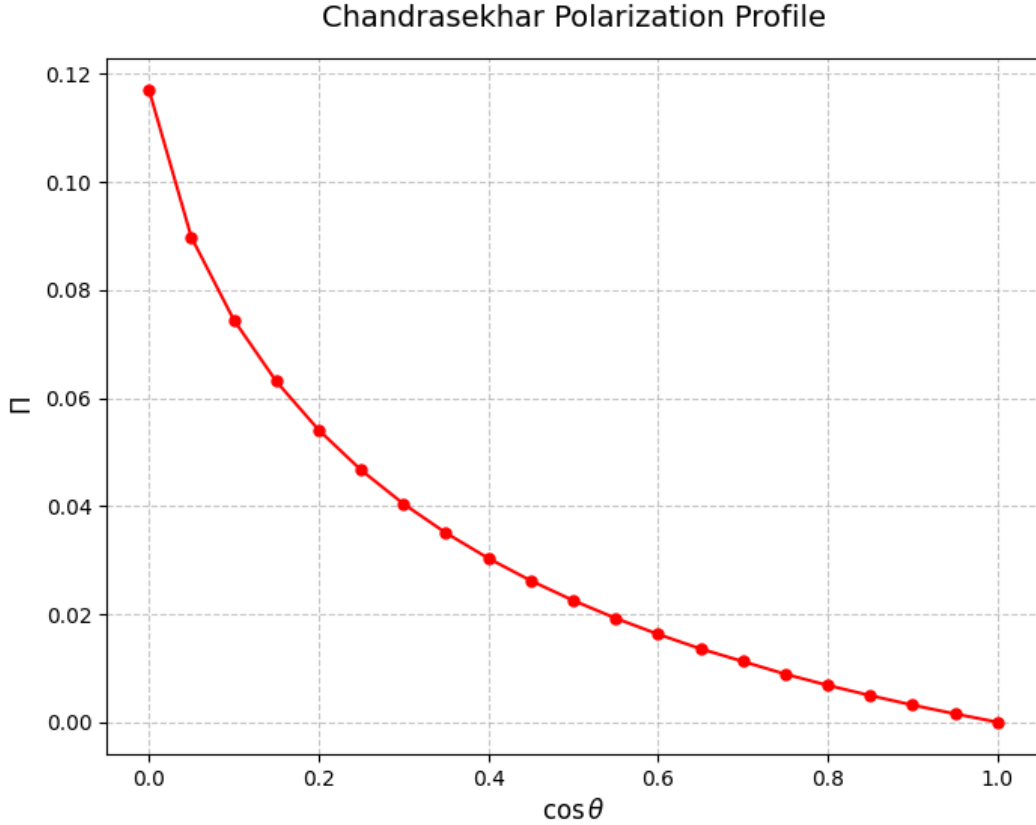


Figure 3.2: Chandrasekhar profile. Π is the polarization degree, while $\cos \theta$ is the cosine of the angle between the line of sight and the normal to the disc.

To measure this variation in the polarization direction, we can exploit an additional constant of motion present in both the Kerr and Schwarzschild metrics: the Walker-Penrose constant[6, 12]. The deviation angle Ψ can be computed as:

$$\tan \Psi = \frac{Y}{X} \quad (3.11)$$

where the terms X and Y are given by[23]:

$$X = -(\alpha - a \sin \theta_{obs})\kappa_1 - \beta\kappa_2 \quad (3.12)$$

$$Y = (\alpha - a \sin \theta_{obs})\kappa_2 - \beta\kappa_1 \quad (3.13)$$

In these equations, κ_1 and κ_2 are the components of the Walker-Penrose constant, α and β are the impact parameters, and θ_{obs} is the observer's inclination. It is crucial to note that the BH spin parameter a explicitly appears in both X and Y . While a portion of the deviation Ψ is intrinsically present due to the mere existence of the BH, an additional contribution arises only if the BH is rotating ($a \neq 0$). This spin dependent effect can be understood as a frame-dragging phenomenon that pulls the polarization direction along with the rotation of the BH.

Chapter 4

Numerical Simulations

Knowing the theoretical models governing accretion discs and X-ray polarization in the previous chapter, we now see their numerical evaluation. This chapter shows the numerical implementation of the theoretical framework described in Chapter 3 and discusses the resulting simulations. To perform these calculations, we rely on the numerical approach described in [23]. It evaluates the spectral and polarization properties of the radiation emitted by stellar-mass BHs in the soft state (i.e., for sources from which the collected radiation is observed to be purely thermal).

4.1 Numerical Implementations

Our models are evaluated using the KYN package[8], which provides a set of different emission models built-in into XSPEC[1]. The code exploits a fully relativistic, ray-tracing technique based on an observer-to-emitter approach. Since the photons emitted from the disc are assumed to follow an isotropic blackbody distribution with a Novikov-Thorne temperature profile, the reference model is KYNBB.

To properly account for strong gravity effects, [23] expanded the original KYN model by adding a specific routine to include the contributions of returning photons to spectra and polarization observables. This was achieved using the C++ code SELFIRR (based on the ray-tracing SIM5 package), which computes all the possible null geodesics connecting two different points on the disc surface. Consequently, the total observed radiation is computed by evaluating and summing two distinct components, which are detailed in the following subsections.

4.1.1 Direct radiation

To simulate the photon emission which arrives directly to the observer from the disc, the surface of the latter is modeled as an $N_r \times N_\phi$ grid using polar coordinates (r, ϕ) , where r is the radial distance from the BH and ϕ is the azimuth. It is also defined θ_{obs} as the observer's inclination with respect to the disc normal[23]. The integrated photon flux over an energy interval ΔE is computed as

$$\Delta f_{\text{obs}} = \int_{r_{\text{in}}}^{r_{\text{out}}} r dr \int_{\phi'}^{\phi' + \Delta\phi'} d\phi \int_{E/g}^{(E+\Delta E)/g} G f_{\text{loc}} dE_{\text{loc}}, \quad (4.1)$$

where r_{in} and r_{out} are respectively the inner and outer radii of the disc, and f_{loc} is the local, energy dependent photon number flux. In this expression, g is the redshift factor, which accounts for the gravitational redshift, and G is the transfer function that accounts for the transport of photons along the null geodesics[23]. Note that r_{in} can be chosen to be smaller, larger, or equal to the ISCO (in our next simulation it will be always $r_{\text{in}} = r_{\text{ISCO}}$). The Stokes parameters at the observer can be calculated as

$$\begin{cases} i_{\text{obs}} = \int dS i_{\text{loc}}^{\text{dir}}(r, \phi) G \\ q_{\text{obs}} = \int dS [q_{\text{loc}}^{\text{dir}}(r, \phi) \cos(2\Psi) - u_{\text{loc}}^{\text{dir}}(r, \phi) \sin(2\Psi)] G \\ u_{\text{obs}} = \int dS [u_{\text{loc}}^{\text{dir}}(r, \phi) \cos(2\Psi) + q_{\text{loc}}^{\text{dir}}(r, \phi) \sin(2\Psi)] G. \end{cases} \quad (4.2)$$

Clearly $dS = r dr d\phi$, while $G = g^2 l \cos \theta$ is the transfer function (accounting for general relativistic effects in the photon trajectory from the disc to the observer), with $g = E_{\text{obs}}/E_{\text{loc}}$ is the energy shift and l is the lensing factor. At this point, the polarization observable P and χ can be obtained substituting the Stokes parameters computed through equations (4.2) inside equations (3.9).

4.1.2 Returning radiation

To account for returning radiation, the code computes all the possible null geodesics that connect two points on the disc surface, placed at opposite sides with respect to the BH. To do this, the disc is first divided into a number of incidence patches, each one characterized only by the radial distance r_i from the black hole of its center. Exploiting axis-symmetry, the same calculation is performed at every azimuthal direction, covering the entire disk. For each

radial patch, a number of different directions, along which returning photons are considered to collide with the disc, are sampled through a grid of polar angles (θ_i, ϕ_i) , referred to the disc normal direction. Hence, for every triplet $(r_i, \theta_i$ and $\phi_i)$ the geodesic equations are solved to find the radial distance r_e of the photon emission point and the polar angles (θ_e, ϕ_e) characterizing the emission direction. When the photons hit the disc, they are assumed to be reflected; the reflected Stokes parameters $(\bar{i}_{\text{refl}}, \bar{q}_{\text{refl}}, \bar{u}_{\text{refl}})$ can be computed using Chandrasekhar's formulae for diffuse reflection[4]. Then, the contributions from all possible incidence directions are summed together, obtaining the Stokes parameters for returning photons,

$$\begin{cases} i_{\text{loc}}^{\text{ret}}(\bar{r}_i) = \sum_{\bar{\theta}_i, \bar{\varphi}_i} \bar{i}_{\text{refl}}(\bar{r}_i, \bar{\theta}_i, \bar{\varphi}_i) \bar{g}^2(\bar{r}_i, \bar{\theta}_i, \bar{\varphi}_i) \bar{\mu}_i \Delta \bar{\Omega}_i \\ q_{\text{loc}}^{\text{ret}}(\bar{r}_i) = \sum_{\bar{\theta}_i, \bar{\varphi}_i} \bar{q}_{\text{refl}}(\bar{r}_i, \bar{\theta}_i, \bar{\varphi}_i) \bar{g}^2(\bar{r}_i, \bar{\theta}_i, \bar{\varphi}_i) \bar{\mu}_i \Delta \bar{\Omega}_i \\ u_{\text{loc}}^{\text{ret}}(\bar{r}_i) = \sum_{\bar{\theta}_i, \bar{\varphi}_i} \bar{u}_{\text{refl}}(\bar{r}_i, \bar{\theta}_i, \bar{\varphi}_i) \bar{g}^2(\bar{r}_i, \bar{\theta}_i, \bar{\varphi}_i) \bar{\mu}_i \Delta \bar{\Omega}_i, \end{cases} \quad (4.3)$$

where $\Delta \bar{\Omega}_i$ is the solid angle subtended by the emission cone with angular extension $\Delta \theta_i, \Delta \phi_i$ and $\bar{\mu}_i = \cos \bar{\theta}_i$. In the end, since the total radiation reaching the observer at infinity is the sum of the direct photons and those reflected by the disc, the total local Stokes parameters are obtained by summing these two contributions:

$$\begin{cases} i_{\text{loc}}(r, \phi) = i_{\text{loc}}^{\text{dir}}(r, \phi) + i_{\text{loc}}^{\text{ret}}(r) \\ q_{\text{loc}}(r, \phi) = q_{\text{loc}}^{\text{dir}}(r, \phi) + q_{\text{loc}}^{\text{ret}}(r) \\ u_{\text{loc}}(r, \phi) = u_{\text{loc}}^{\text{dir}}(r, \phi) + u_{\text{loc}}^{\text{ret}}(r). \end{cases} \quad (4.4)$$

4.2 Determining the BH Spin: The Ideal 100 per cent Albedo Case

As previously discussed, a fraction of the photons emitted is bent by the strong gravitational field of the BH and forced to hit the accretion disc surface again, generating returning radiation. At the surface, these photons interact with the electrons in the atmosphere, experiencing multiple scatterings similarly to Section 3.2. The fate of the returning photon depends on these scatterings. Clearly, if the first scattering event occurs at the disc surface, redirecting the photon away from the disc, we have reflection. Alternatively, it can penetrate the disc before encountering an electron: at this point, if it scatters, it could be

(again) redirected far from the disc. Alternatively, at the first scattering, the photon can be bounced into the disc, where it will undergo other scatterings. Of course, some photons will eventually escape the disc, but a fraction of them will inevitably be trapped or absorbed in the disc.

The 100 per cent albedo[20] means that all the returning photons are eventually reflected, so this is the simplest case to start our study. Figure 4.1 reports the energy-dependent flux (Stokes parameter I), polarization degree and polarization angle, simulated by KYNBBRR for three different values of the BH spin ($a = 0, 0.9$ and 0.998) assuming 100% albedo. Looking at the top row, one can see that the observed spectrum is dominated by the direct radiation contribution at lower energies. In fact, lower energy photons predominantly originate from the colder regions of the disc, which are also the farthest from the BH. Given that, at large distances, BH gravity is relatively weak, very few photons are bent enough to return and strike the disc again, meaning that the contribution from these regions consists essentially of direct radiation. Conversely, at higher energies, deviations from the direct radiation spectrum become visible, particularly for rotating BHs ($a = 0.9$ and $a = 0.998$). High-energy photons are emitted in the hottest regions close to the BH, where they are subject to strong gravitational ray-bending and can easily return to the disc to be reflected. Furthermore, as the spin rises, the ISCO shrinks towards the BH (from $6r_g$ for a Schwarzschild BH down to r_g for an extreme Kerr BH). Consequently, the inner regions of a Kerr disc extend closer to the event horizon which, according to the temperature profile (3.10), are hotter than those of a Schwarzschild disc, enhancing the returning radiation component at high energies. From the middle row of Figure 4.1, the direct radiation exhibits a relatively low polarization degree (below 10% in all cases), which is consistent with the expectations from the Chandrasekhar profile. On the other hand, the returning radiation component is more polarized ($> 10\%$) because it is affected by the reflection process, rather than by the simple electron scattering assumed in Chandrasekhar's approximation. Much in the same way of the spectrum, also the total polarization degree tends to follow the direct curve at low energies and approaches the returning curve as energy increases. In all three cases, a distinct minimum appears before the curve rises. To understand this minimum, we need to observe the polarization angle (bottom row in Figure 4.1). For the direct radiation, the angle stays around $\sim 90^\circ$ from low to high energies for BHs with $a = 0, 0.9$, while it suddenly transitions from 90° to 0° for $a = 0.998$. The 90° value implies a polarization parallel to the disc plane, consistent with Chandrasekhar's model. Small deviations from 90° occur due to general relativistic effects: for $a = 0$, the polarization plane is rotated only via parallel transport; for rotating BHs ($a = 0.9, 0.998$), the frame dragging effect of the Kerr metric induces an additional rotation of the

polarization plane known as gravitational Faraday rotation.

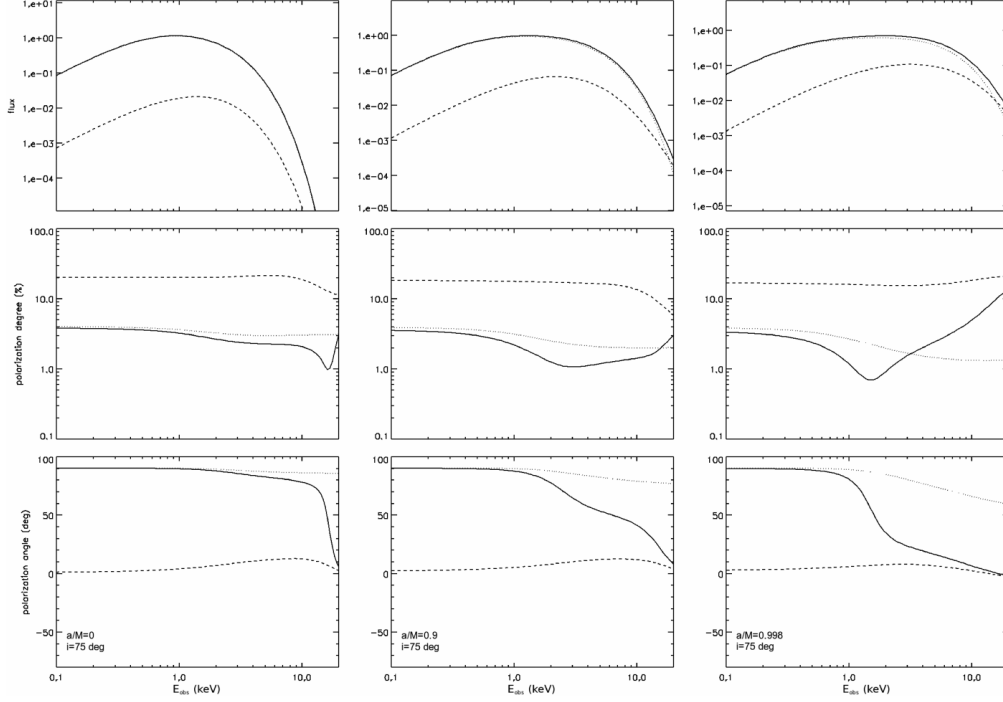


Figure 4.1: Spectrum (top row), polarization degree (middle row), and polarization angle (bottom row) plotted as functions of the photon energy at the observer for a BH mass $M = 10 M_{\odot}$ and spin $a = 0$ (left-hand column), 0.9 (middle column), and 0.998 (right-hand column). The inclination angle i between the observer’s line of sight and the disc symmetry axis is taken as 75° , while the BH mass accretion rate is chosen in such a way that the accretion luminosity is equal to 10 per cent of the Eddington limit. In particular, $\dot{M} = 2.45$ (for $a = 0$), 0.90 (for $a = 0.9$), and 0.35 (for $a = 0.998$) in units of 10^{18} g s^{-1} . In each plot, the contributions of direct and returning photons alone are marked by dotted and dashed lines, respectively, while the joint contribution (direct + returning radiation) is marked by solid lines[23].

The returning radiation, instead, is polarized at roughly 0° (perpendicular to the disc plane, or aligned with the rotation axis), confirming that reflection dictates its polarization state and that polarization can only be parallel or perpendicular to the disc. Now, observing the total angle trend, we note a transition from the direct (90°) to the returning (0°) regime exactly at the energy where the minimum occurs in the corresponding polarization degree plot. Changing the angle so abruptly means that, when we sum the electric field vectors at infinity, we are adding components with highly different direc-

tions. This vectorial sum tends to partially cancel out, perfectly explaining the observed dip in the total polarization degree. Thus, in general, one can see that the higher the spin, the closer the ISCO is to the event horizon, and consequently, the lower the energy at which this transition becomes evident. In fact, the transition energy drops from $\gtrsim 10$ keV for $a = 0$, to $\gtrsim 2$ keV for $a = 0.998$. Because the exact energy at which the polarization degree drops to its minimum and the angle changes depends directly on the BH spin, observing these particular features in the energy patterns would provide a direct way to measure the BH spin.

4.3 The Influence of changing the Optical Depth

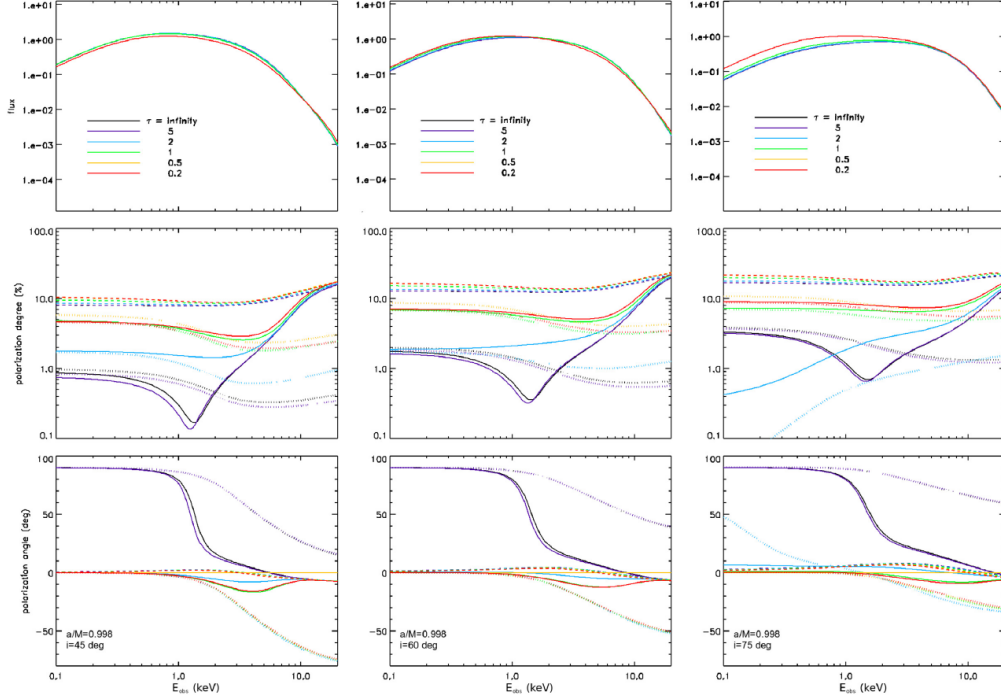


Figure 4.2: Spectrum (top row), polarization degree (middle row), and polarization angle (bottom row) plotted as functions of the photon energy at the observer for $a = 0.998$ (the results for different values of a are qualitatively similar and are not shown) and $i = 45^\circ$ (left-hand column), 60° (middle column), and 75° (right-hand column), and for different value of the atmosphere optical depth: $\tau \rightarrow \infty$ (black), 5 (purple), 2 (cyan), 1 (green), 0.5 (orange), and 0.2 (red). Details as in Figure 3.1 (for the sake of convenience, the separate direct and returning radiation contributions are shown in the polarization observable plots only).[23].

In the previous results, an optical depth $\tau \rightarrow \infty$ for the disc atmosphere was assumed, following the Chandrasekhar profile. Now we analyze what happens for different optical depths of the disc atmosphere. Since we are no longer within the limits of Chandrasekhar's analytical approximation, the intrinsic polarization pattern is evaluated by numerically simulating the radiative transfer within a layer of matter of a fixed optical depth. This has been done using the spectro-polarimetric Monte Carlo code Stokes, developed by [11][13]. The results of these simulations are reported in Figure 4.2.

First, we observe that, while the flux is similar across different depths, at lower energies a slightly higher flux is observed for small values of τ . This is expected because a small τ means the radiation undergoes very few scatterings; therefore, the photons emitted in the observer's direction reach us directly without being deflected away. At higher energies, however, there is practically no difference. Second, for $\tau \lesssim 2$ the polarization angle is $\sim 0^\circ$, hence the polarization is parallel to the disc symmetry axis, even in the case of direct radiation. This marks a substantial difference from Chandrasekhar's model, which is only recovered as $\tau \rightarrow \infty$; for $\tau \lesssim 2$, we are entirely outside its approximations. This behavior can be further confirmed by the fact that a transition occurs at $\tau \sim 2$ in the orientation of the polarization vectors for direct photons [23].

On the other hand, for $\tau \gtrsim 5$, as in the scenario discussed in section 4.2, the polarization starts perpendicular to the disc axis, and around $\gtrsim 1$ keV it begins a transition that, for $\gtrsim 10$ keV, takes the angle to $\sim 0^\circ$. Furthermore, the polarization degree (middle column) shows that for $\tau \lesssim 2$, the fraction of polarized radiation significantly increases compared to the optically thick cases.

4.4 The Realistic Albedo Case: Tracing Absorption Effects

Up to this point, we have explored the ideal scenario of Chandrasekhar polarization with a 100% albedo, and then we tested different optical depth. The next step is to overcome the 100% albedo approximation (while temporarily maintaining the Chandrasekhar profile), before finally combining all these effects in the next section. The assumption of a 100% albedo is physically justifiable only if the atmospheric layer is completely ionized. In such a state, there is an abundance of free electrons capable of scattering photons (unlike neutral atoms, which are too massive to scatter them efficiently), resulting in a very high probability that incoming photons are reflected back. However, ionizing matter requires extremely high temperatures to strip electrons from

their orbitals via atomic collisions.

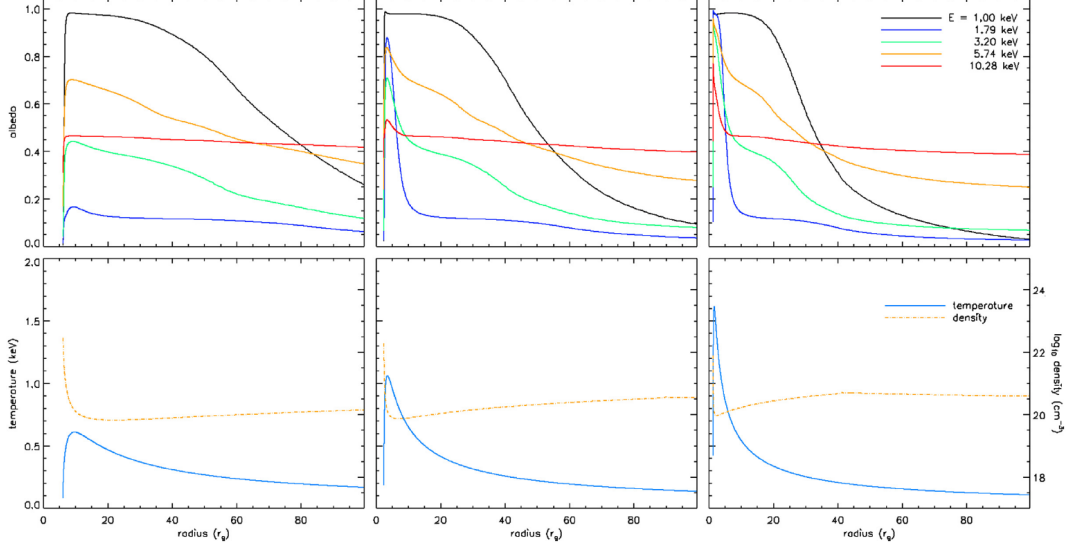


Figure 4.3: Top row: albedo profile as a function of the radial distance r , obtained for different local energies in the 1–10 keV range (see the plot legend for the colour code), in the cases of $a = 0$ (left), 0.9 (centre), and 0.998 (right). Bottom row: temperature (blue, solid line) and density (orange, dashed line) radial profiles, respectively, for $a = 0$ and $\dot{M} = 24.5 \times 10^{17} \text{ g s}^{-1}$ (left), $a = 0.9$ and $\dot{M} = 9 \times 10^{17} \text{ g s}^{-1}$ (middle), and $a = 0.998$ and $\dot{M} = 3.5 \times 10^{17} \text{ g s}^{-1}$ (right). In all the plots, a BH mass of $M = 10 M_\odot$ has been assumed[23].

As we discussed in section 3.2, the accretion disc is not uniformly heated: it is much hotter in the inner regions close to the BH and colder in the outer regions. Consequently, the plasma is likely to be highly ionized (albedo $\approx 100\%$) near the BH, whereas neutral atoms become more prevalent farther out, leading to increased absorption and a lower albedo ($< 100\%$). To obtain a physically realistic radial profile of the albedo, these microphysical collisional processes have been numerically evaluated using the photo-ionization code CLOUDY[10] and the radial dependent albedo profile is reported in Figure 4.3. To implement this realistic albedo into the physical model, the returning radiation contribution is multiplied by the local albedo value: if the albedo is 1, all returning radiation is assumed to be reflected to infinity; if it is 0, the returning radiation is entirely absorbed. The effects of this implementation are shown in Figure 4.4, which compares the ideal 100% albedo prescription with the more realistic one.

In the top row of Figure 4.4, although the returning radiation varies, the total flux remains unchanged. This can be explained by the fact that the ratio

between returning and direct photons is very small, around $\approx 10^{-2} - 10^{-3}$ over almost the entire energy range considered.

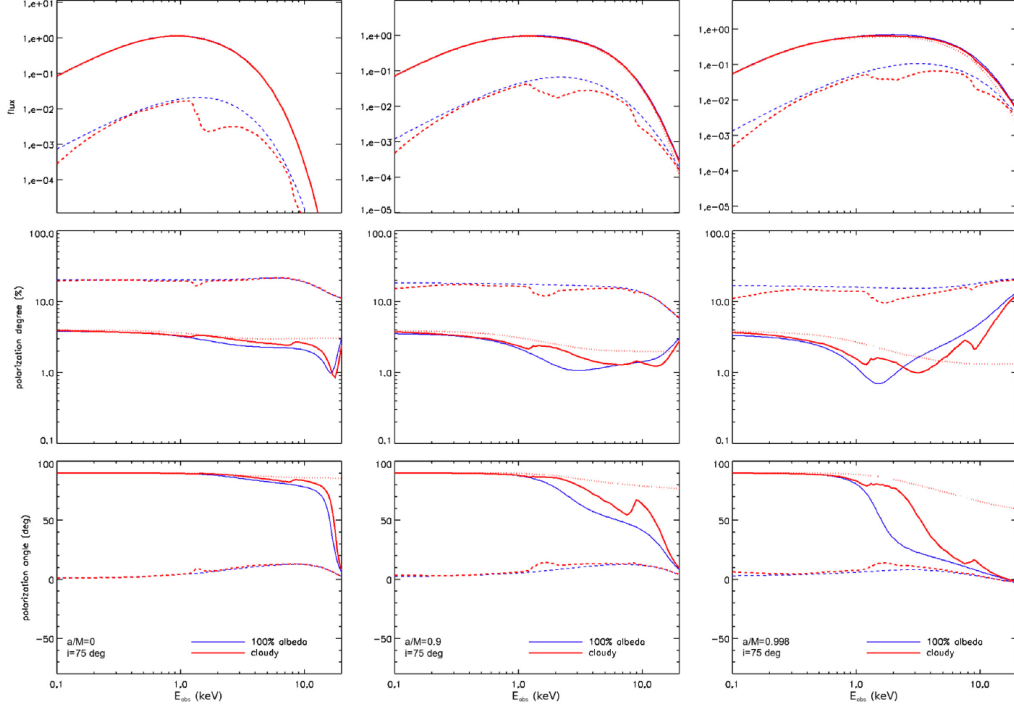


Figure 4.4: Spectrum (top row), polarization degree (middle row), and polarization angle (bottom row) behaviours, plotted as functions of the photon energy at the observer, for a 100 per cent albedo prescription (blue lines) and assuming the albedo profile obtained from CLOUDY[10] (red lines). The values of the other parameters and line styles are taken as in Fig.4.1[23].

The differences between the two cases are more evident in the polarization degree and angle, particularly at high spin values. In fact, in the bottom row for $a = 0.998$, the swing of the more realistic model results shifted toward higher energies with respect to the ideal case.

Furthermore, when assuming a disc albedo below 100%, both the minimum of the polarization degree and the transition of the polarization angle shift toward higher energies compared to the ideal case. As a result, the energy at which the polarization degree reaches its minimum can no longer be used to infer the spin independently. An estimate of the disc albedo is therefore also required to determine the spin.

4.5 The Impact of different Plasma Ionization Mechanisms

To obtain a more self consistent physical model, we apply the ionization profile to the disc atmosphere through which the radiation propagates. In the previous sections, the emission was assumed to be purely thermal and described by a blackbody spectrum. In a partially ionized plasma, photons are affected by both electron scattering and photoelectric absorption, so that the ionization state must be determined consistently with the radiation field.

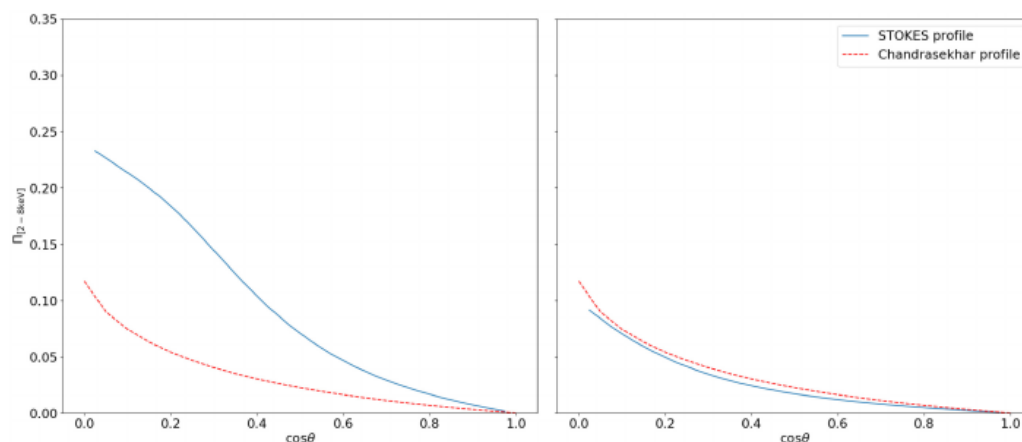


Figure 4.5: Polarization degree plotted as a function of the cosine of the inclination angle θ for the two cases of $a = 0$ (left-hand-panel) and $a = 0.998$ (right-hand-panel). Stokes parameters have been summed over the radial distance from the central BH and energy integrated between 2 and 8 keV. Blue solid lines mark the behaviour predicted by STOKES[11][13], while the red dashed lines that calculated according the formulae by Chandrasekhar[16][22].

Proper simulations in these conditions have been performed by [22], where solar abundances are assumed in the disc, while dust contribution is neglected. Focus is given to direct radiation only, while returning radiation is not considered. Two limits for the ionization of the disc atmosphere are considered: Collisional Ionization Equilibrium (CIE), where ionization is determined solely by collision between atoms (due to their thermal motion), and the Photo-Ionization Equilibrium (PIE), which includes ionization by photon absorption, via the photoelectric effect. For the collisional ionization case (Figure 4.6), slowly rotating BHs produce a higher PD (polarization degree) than expected in the pure scattering limit. For rapidly rotating BHs, the innermost stable circular orbit lies closer to the event horizon, resulting in higher temperatures in the

inner disc. The atmosphere is consequently driven toward complete ionization, reducing the contribution of absorption and bringing the polarization closer to the ideal Chandrasekhar result. The polarization differs substantially when the local emission is computed for a plasma in photoionization equilibrium (Figure 4.6). In this case, photoelectric absorption has a smaller effect on the emergent radiation, particularly for non rotating BHs.

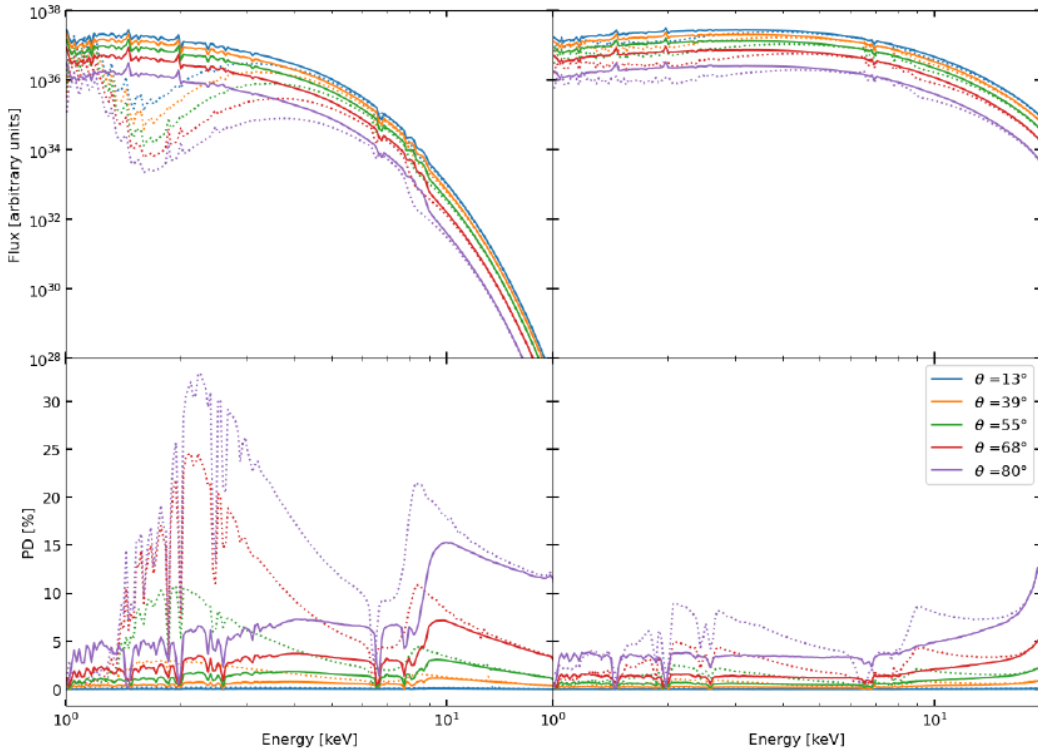


Figure 4.6: Spectra (top row) and PD (bottom row) of the disc radiation at the emission at different inclinations with respect to the disc normal. A BH mass of $M = 10 M_\odot$ and accretion rate of $0.1 \dot{M}_{\text{Edd}}$ were set, with a hardening factor of $f_{\text{col}} = 1.8$ and a stop column density of $N_{\text{H}} = 10^{24} \text{ cm}^{-2}$. The disc medium is modeled in either the CIE (dotted lines) or the PIE (solid lines) regimes[14].

The deep absorption features present in the spectra obtained in CIE are therefore strongly suppressed, and the polarization degree is correspondingly lower. Its energy dependence also approaches the flat profile expected for a pure-scattering atmosphere. These results show that the microphysical state of the disc atmosphere can significantly modify the intrinsic polarization before the radiation escapes from the BH.

4.6 Observational Signatures at infinity: coupling Microphysics and General Relativity effects

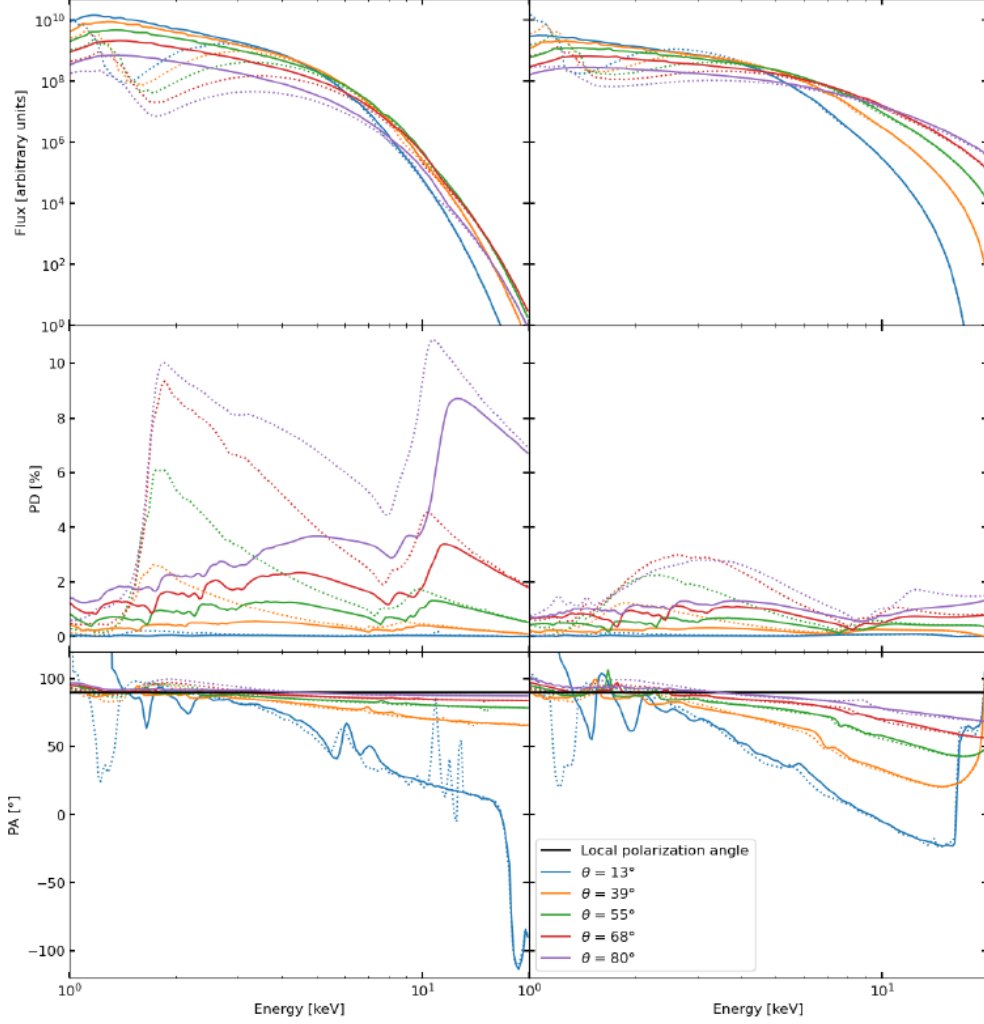


Figure 4.7: Spectra (top row), PD (middle row), and PA (bottom row) profiles of the direct radiation observed at infinity. Results are shown for two BH spin values, $a = 0$ (left column) and $a = 0.998$ (right column), and for both the CIE (dotted lines) and the PIE (solid lines) ionization regimes. The BH mass, accretion rate, hardening factor, and stopping column density, as well as the color code, are the same as in Figure 4.6. The horizontal black lines in the bottom panels indicate a reference PA of 90° , corresponding to a local polarization perpendicular to the disc symmetry axis[14].

While the previous sections detailed how the local microphysics of the disc (optical depth, albedo, and ionization state) generate the intrinsic polarization properties, we now couple these models with the strong gravity effects (geodesic propagation and parallel transport) experienced by these photons to obtain the observational signatures at infinity. To do this, we assumed the scenario of a partially ionized surface layer coming from underlying blackbody emitting disc. This last step is fundamental to correctly extract the BH information, like its spin.

Starting from the top row of Figure 4.7, considering only the direct radiation, we note that the flux curves are smoother than in section 4.5. This is due to the combined effect of gravitational redshift and relativistic Doppler shifts, which smear out the sharp absorption features.

In the middle row, a depolarization can be observed caused by the rotation of the polarization vector in the strong gravitational field. It is more evident for the maximally rotating BH, where we have a drop of PD $\lesssim 2\%$ at the observer (notice that this holds for both CIE and PIE). Furthermore, in the bottom row, at the disc surface the photons are polarized at PA = 90° with respect to the disc axis. However, the strong gravitational field forces the photon polarization plane to rotate, causing an important drop at high energies (closer to the BH center) which is more pronounced in the maximally rotating case. It is important also to notice that PA rotation is not dependent on the specific ionization regime of the disc atmosphere, meaning that both CIE and PIE regimes yield a similar PA behaviour[14].

These results can also be compared with observations from the IXPE satellite. In particular, the polarization peak around 2 keV predicted by the collisional ionization equilibrium (CIE) model has not been observed so far in accreting black hole systems. Observations of soft state sources such as 4U 1630-47 and LMC X-3 instead show an increasing polarization degree across the 2-8 keV energy range. This behaviour is consistent with the results obtained in the photoionization equilibrium (PIE) regime and in models with high optical depth. A more complete description of the spectro-polarimetric properties would therefore also require the effects of returning radiation to be taken into account.

Chapter 5

Conclusions

The aim of this thesis was to discuss how to determine the spin of BHs by studying the polarization properties of the radiation emitted from their accretion discs. In the pure scattering limit with albedo 100%, there is a direct way to estimate the spin using the minimum of the polarization degree and the polarization angle swing, both due to the transition between the direct and returning radiation dominated regimes. The rotation of the polarization plane, with the contribution from returning radiation, produces specific changes in the polarization degree and angle that depend on the spin.

Obviously, this relation becomes less simple when we move toward a realistic description. Departing from Chandrasekhar's approximation, assuming a finite optical depth in the disc atmosphere can substantially modify the polarization, and a realistic, temperature-dependent albedo profile on the disc surface introduces a dependence of the polarization features on photon energy. With polarization transitions moving towards higher energies. As a result, the energy at which the polarization degree reaches a minimum cannot be used as a reliable measure of the spin without an independent constraint on the disc albedo profile (to remove this degeneracy).

The ionization state of the disc atmosphere also has important effects on the polarization properties of the radiation emitted from the disc. In fact, when collisional ionization is considered, a non-negligible fraction of neutral atoms can survive in the atmosphere, leading to stronger photoelectric absorption. Furthermore, the additional absorption tends to increase the polarization degree. On the other hand, including photoionization produces a highly ionized atmosphere, which weakens the contribution of deep absorption features in the spectra and makes the resulting polarization closer to that of the pure scattering case.

Hence, these results indicate that the disc microphysics and relativistic effects cannot be considered separately when interpreting X-ray polarization from accreting black holes. The polarization observed at infinity carries information

about both the interaction of radiation with the disc atmosphere and its propagation through curved spacetime. Ultimately, the discussed results tell us that interpreting current polarimetric observations of stellar-mass BH accretion discs is far from being trivial. Accounting for these degeneracies is crucial, and fundamental to determine important parameters, such as the BH spin.

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