Università
degli Studi
di Padova

DEPARTMENT OF MATHEMATICS "TULLIO LEVI CIVITA"
Second level Degree in Mathematics

Master's Thesis

A mixed formulation for linear viscoelasticity

Candidate:
Francesco Fantin
Student ID 2017356

Thesis supervisor:
Prof. Federico Piazzon

Co supervisor:
Prof. Mario Putti

Academic year 2021-2022
23 September 2022

Abstract

In this work we address the problem of the derivation and the study of a mixed formulation of linear viscoelasticity, focusing on the case of the Kelvin-Voigt model. Namely, we construct an equivalent mathematical model in which the stress tensor appears as independent variable and its natural regularity is explicitly enforced.

In the introductory part of this work we review the derivation of the mixed formulation for static linear elasticity. Indeed, we aim to extend the duality techniques underlying such construction to the more general context of viscoelastic deformations.

This task is accomplished in the second part of the present work. Where, after studying existence, uniqueness, and fine regularity properties of the solution of the primal formulation of viscoelastic waves propagation, we manage to find a variational characterization of the solution of the primal problem that well fits in the above mentioned duality technique. We finally arrive to a mixed formulation of viscoelasticity.

In the last part of this thesis we present a mixed finite elements discretization which is conformal with respect to the functional spaces considered in our mixed (infinite dimensional) formulation. A simple two-dimensional test case is also implemented and presented as an example.

Contents

Introduction

Viscoelastic models

Elastic and viscoelastic models are used, in the framework of continuum mechanics, to describe the behaviour of certain materials, e.g. compressible fluids, gases, as well as most solids, in presence of a (possibly time dependent) load. Elastic models are conservative mechanical systems, indeed they are used for describing the mechanical response of the so-called purely elastic materials, i.e., media for which the internal friction can be neglected. In contrast, viscoelastic models are used when the friction is non negligible with respect to the time and spatial scale of the quantity of interest, leading to a dissipative system that, from a mathemtical point of view, can be casted in the framework of parabolic PDEs.

In the 17th century R. Hooke started the study of the modelization of elastic and viscous deformation in the seminal work [10], while the first treatment of this subject taking into account several modes of deformation is due to T. Kelvin in 1865, see e.g. [17]. The modern treatment of viscoelasticity relies upon the coupling of the Newton's laws

$$\rho \underline{u}_{tt} = \operatorname{div}(\underline{\underline{\sigma}}) + \underline{f},$$

with a stress-strain constitutive equation defining $\underline{\underline{\sigma}}$. Here $\underline{u}$ denotes the (Lagrangian) displacement vector field, $\underline{\underline{\sigma}}$ is the stress tensor, and $\underline{f}$ denotes the load. We warn the reader that hereafter we use the notation $\underline{\cdot}$ for vectors and $\underline{\underline{\cdot}}$ for tensors, and the subscripts $\cdot_t$, $\cdot_{tt}$ denote time derivatives. The stress-strain constitutive equation relates the symmetric gradients of the displacement, $\underline{\underline{\epsilon}}(\underline{u})$, and of the velocity, $\underline{\underline{\epsilon}}(\underline{u}_t)$, with the stress tensor $\underline{\underline{\sigma}}$.

In the literature there is a number of models of viscoelasticity obtained by different choices of the constitutive equation whose mathematical justification is given by possible combinations (in series and/or in parallel) of springs and dash-pots. Among this class of models we mention the Maxwell, the Kelvin-Voigt, and the Burgers models. Namely, the Maxwell constitutive equation (typically used for modelling fluids) is obtained by combining a spring and a dash-pot in series, the Kelvin-Voigt constitutive equation (typically used for modelling solids) is obtained by combining a spring and a dash-pot in parallel, and the Burgers models by combining the Kevin-Voigt and Maxwell in series, see e.g. [6, Ch. 1].

It is well-known that, if we attack the solution of the above models working with the displacement as unique unknown, some issues arise. This difficulties are even stronger when elastic or viscoelastic waves propagates in an heterogeneous or non isotropic media, a typical framework for many practical applications. Indeed, even if the stress tensor given by the exact solution of the infinite dimensional model (by plugging the displacement and velocity gradients in the constitutive equation) possesses L^2 divergence, the finite dimensional solutions obtained by classical discretization methods (e.g., spectral, finite elements, etc.) do not provide a stress tensor with the same regularity.

We remark here that the condition $\operatorname{div}(\underline{\underline{\sigma}}) \in L^2$ allows to trace the flux of the normal component of $\underline{\underline{\sigma}}$ across any rectifiable hypersurface, i.e., the total force acting on the hypersurface: in many applications this is the quantity of interest. Even worse, it can happen that the hypersurface along which we aim at tracing $\underline{\underline{\sigma}}$ separates two media characterized by very different constitutive equations. Moreover, from a numerical point of view, differentiating the displacement, in order to obtain the stress tensor, implies a loss of accuracy.

These reasons justify the search for a matematical modelization of the propagation of elastic and viscoelastic waves that entails the suitable regularity of the stress tensor $\underline{\underline{\sigma}}$.

For certain PDEs that present the above described difficulties, as e.g. Stokes and Poisson problems (see e.g. [4] and reference therein), it can be introduced the so-called mixed formulation, in which, roughly speaking, the equation is splitted into a system, that includes an additional variable. This procedure can be carried out in order to enforce the regularity of the additional variable or for treating it as the main quantity of interest.

In the last decades several authors studied different mixed formulation of viscoelastic models, see [1], [2], [13], and [14]. In the preliminary study that we made of the subject we could not find any approach that satisfies the following properties: a formal variational derivation of the mixed formulation is provided, the natural $\underline{\underline{H}}(\operatorname{div}, \Omega)_{sym}$ regularity of the stress tensor is enforced both in infinite and discrete version of the model, and a stable mixed finite element discretization of the model is provided.

Our study

In this work we focus on the Voigt viscoelastic model

$$\underline{\underline{\sigma}} = A_{el}(\underline{\underline{\epsilon}}(\underline{u})) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_t)),$$

where $A_{el}(\underline{\underline{\tau}}) = 2\mu_{el}\underline{\underline{\tau}} + \lambda_{el}\operatorname{tr}(\underline{\underline{\tau}})\underline{\underline{id}}$ and $A_{vis}(\underline{\underline{\tau}}) = 2\mu_{vis}\underline{\underline{\tau}} + \lambda_{vis}\operatorname{tr}(\underline{\underline{\tau}})\underline{\underline{id}}$. In this setting the Newton's law complemented with homogeneous Dirichlet conditions and initial conditions reads as

$$\begin{cases} \rho\underline{u}_{tt} - \operatorname{div}(A_{el}(\underline{\underline{\epsilon}}(\underline{u})) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_t)) - \underline{f} = 0 \\ \underline{u}_t|_{\partial\Omega}(\cdot, t) = 0 \\ \underline{u}|_{\partial\Omega}(\cdot, t) = 0 \\ \underline{u}_t(\cdot, 0) = \underline{u}_1(\cdot) \\ \underline{u}(\cdot, 0) = \underline{u}_0(\cdot) \end{cases} . \tag{1}$$

For this set of equations we address the problem of deriving a mixed formulation and mixed FEM discretization enjoying the three properties we have defined above.

This task is accomplished following four steps. First, we study existence, uniqueness, and regularity of solutions of (1) in the framework of second order evolution equation in Hilbert spaces, see [15]. In particular, we were able to verify all the properties that are needed to prove the regularity of the resulting stress tensor (see Corollary 2.3.1), provided natural conditions on the initial data hold.

As second step, we characterize the solution of (1) as the unique minimum of a variational principle. Unfortunately, due to the dissipative nature of the system, we could not construct an objective functional starting from Hamiltonian or Lagrangian mechanics, as it can be done for very simple systems as, e.g., a damped harmonic oscillator. In contrast, in order to construct such variational principle, we use the solution of (1) itself (see (2.101)) as a parameter in the objective functional. This procedure, that apparently is running in circles, would not concern the reader, since our final aim is only to construct a mixed formulation for the viscoelastic model. We arrive to the following

$$\inf_{\underline{v}\in\Gamma}\Bigg\{ \int_0^T \int_\Omega \rho \underline{\bar{u}}_{tt}\underline{u} + (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{\epsilon}}(u) + \mu_{el}|\underline{\underline{\epsilon}}(u)|^2 + \frac{1}{2}\lambda_{el}|\mathrm{tr}(\underline{\underline{\epsilon}}(u))|^2 \;\mathrm{dx} - \langle \underline{f}, \underline{u}\rangle_{L^2(\Omega)^2} \;\mathrm{dt} \Bigg\},$$

where $\Gamma = L^2(0,T;H^1_0(\Omega)^2)$.
In the next step we play with the duality theory of perturbation of variational problems (see [8]) and, starting from our variational principle, we construct its dual and mixed formulations. Namely, the latter reads

$$= \inf_{\underline{\underline{p}}\in Y}\sup_{\underline{u}\in\Gamma}\Bigg\{\int_0^T \int_\Omega \frac{1}{4\mu_{el}}|\underline{\underline{p}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\underline{\underline{p}}) - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho\underline{\bar{u}}_{tt}\underline{u} + \underline{f}(x,t)\underline{u} + \mathrm{div}(\underline{\underline{p}})\underline{u} \;\mathrm{dx}\;\mathrm{dt}\Bigg\},$$

where $W = L^2(0,T;\underline{\underline{H}}(\mathrm{div},\Omega)_{sym})$ and $\Lambda = L^2(0,T;L^2(\Omega)^2)$.
Such an approach has already been used for introducing the dual and the mixed formulations of several steady-state problems, such as linear elasticity and Poisson (see e.g. [4] [8]). We remark that in our problem we have to overcome the difficulty of working with an evolution equation, and the drawbacks arising from its dissipative nature.

In the last step of our construction, that can be thought as main result of this work, we prove that the solution of the mixed formulation (2.126) has an unique solution that coincides with the solution of (1), see Theorem 2.12.

The study of a stable finite element discretization scheme of our model has been only partially carried out. This is mainly due to the time limits and to the fact that the implementation and the testing of the scheme we designed are highly non-trivial. In the last chapter of this work we present a numerical approximation scheme where the spatial discretization is performed by a conformal Galerkin method using vectorial P^1_0 elements for the displacement and the velocity fields, and a subspace of the symmetric tensor discontinuous Galerkin elements of order 2 where the $H(\mathrm{div})$ regularity is preserved. Our preliminary results seem to show that this couple of spaces provide a stable discretization of our mixed formulation in the sense of Brezzi, Ladyženskaja and Babuška inf-sup condition, see [4, Sec. 4.2]. We plan to conclude this study in the next months.

Chapter 1

Linear Elasticity

In this chapter we want to introduce primal, dual and saddle point problems for linear elasticity and also the mixed problem given by Brezzi. First of all we will fix some notations, secondly we will analyse the primal problem and see that it admits an unique solution, then we will construct the dual problem starting form the primal and see its properties. In the end we will construct the saddle point problem and see its relation with the one given by Brezzi.

1.1 Description of the problem

Let $\Omega \subseteq \mathbb{R}^2$ a bounded Lipschitz domain, which represents the initial configuration of the body and we denote with $\partial\Omega$ its boundary. Our aim is to determinate the displacement $\underline{u} : \overline{\Omega} \to \mathbb{R}^2$, since, for simplicity, we consider the boundary $\partial\Omega$ fixed we have that $\underline{u}|_{\partial\Omega} = 0$. We also have a force $\underline{f} \in L^2(\Omega)^2$ acting on Ω and since $\partial\Omega$ is fixed the force g acting on the boundary is equal to 0. In order to introduce the problem it is convenient to give the definition of symmetric gradient $\underline{\underline{\epsilon}}(u)$ of a function $\underline{u} \in H_0^1(\Omega)^2$.

Remark 1.1.1. *In order to emphasize the fact that the symmetric tensor of a vector is a tensor, we have decided to omit the bar under the vector and write $\underline{\underline{\epsilon}}(u)$ instead of $\underline{\underline{\epsilon}}(\underline{u})$.*

We define the symmetric gradient as the linear map $\underline{\underline{\epsilon}} : H_0^1(\Omega)^2 \longrightarrow L^2(\Omega)^{2\times 2}_{sym}$ such that

$$\underline{\underline{\epsilon}}(u)_{ij} := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

From the previous definition can be easily seen that $\underline{\underline{\epsilon}}(u)$ is a symmetric tensor and its trace, where $\operatorname{tr}(\underline{\underline{\tau}}) = \sum_i \tau_{ii}$, is $\operatorname{div}(\underline{u})$. We define the double dot product of two tensors as

$$\underline{\underline{\tau}} : \underline{\underline{\sigma}} = \sum_{i,j} \tau_{ij}\sigma_{ij},$$

and in case we have $\underline{\underline{\tau}} : \underline{\underline{\tau}}$ we will write $\left|\underline{\underline{\tau}}\right|^2$. Now we are ready to introduce the starting problem.

1.2 Starting Problem

Our aim is to solve the following minimisation problem

$$\inf_{\underline{v}\in H_0^1(\Omega)^2}\left\{\int_\Omega \mu|\underline{\underline{\epsilon}}(v)|^2+\frac{\lambda}{2}|\text{div}(\underline{v})|^2-\underline{f}\cdot\underline{v}\ \text{dx}\right\}, \tag{1.1}$$

where we recall that $\underline{f}$ is a function in $L^2(\Omega)^2$.
First of all we can rewrite the previous problem in the following way:

$$\inf_{\underline{v}\in H_0^1(\Omega)^2}\left\{G(\underline{\underline{\epsilon}}(v))-F(\underline{v})\right\},$$

where $G: L^2(\Omega)^{2\times 2}_{sym}\longrightarrow\mathbb{R}$ is

$$G(\underline{\underline{p}})=\int_\Omega \mu|\underline{\underline{p}}|^2+\frac{\lambda}{2}|\text{tr}(\underline{\underline{p}})|^2\ \text{dx},$$

and $F: H_0^1(\Omega)^2\longrightarrow\mathbb{R}$ is

$$F(\underline{v})=\int_\Omega \underline{f}\cdot\underline{v}\ \text{dx},$$

a linear and continuous operator.
In order to show the existence and uniqueness of solution of problem (1.1), it is enough to prove that $G\circ\underline{\underline{\epsilon}}$ is a strictly convex function. We will show that $G\circ\underline{\underline{\epsilon}}$ is not only strictly convex but also strongly convex.

Proposition 1.2.1. *G is 2μ-convex function*

Proof. Let $\underline{\underline{p}},\underline{\underline{q}}\in L^2(\Omega)^{2\times 2}_{sym}$ and $\alpha\in[0,1]$, then:

$$\begin{aligned}G(\alpha\underline{\underline{p}}+(1-\alpha)\underline{\underline{q}}) &= \int_\Omega \mu|\alpha\underline{\underline{p}}+(1-\alpha)\underline{\underline{q}}|^2\ \text{dx}+\frac{\lambda}{2}|\alpha\,\text{tr}(\underline{\underline{p}})+(1-\alpha)\,\text{tr}(\underline{\underline{q}})|^2\ \text{dx}\\ &= \alpha G(\underline{\underline{p}})+(1-\alpha)G(\underline{\underline{q}})-\alpha(1-\alpha)\Big[\int_\Omega \mu|\underline{\underline{p}}-\underline{\underline{q}}|^2-\frac{\lambda}{2}|\text{tr}(\underline{\underline{p}}-\underline{\underline{q}})|^2\Big]\ \text{dx}\\ &\le \alpha G(\underline{\underline{p}})+(1-\alpha)G(\underline{\underline{q}})-\alpha(1-\alpha)\int_\Omega \mu|\underline{\underline{p}}-\underline{\underline{q}}|^2\ \text{dx}\\ &\le \alpha G(\underline{\underline{p}})+(1-\alpha)G(\underline{\underline{q}})-\frac{1}{2}(2\mu)\alpha(1-\alpha)\left\|\underline{\underline{p}}-\underline{\underline{q}}\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}}.\end{aligned}$$

□

Now we want to prove that $G\circ\underline{\underline{\epsilon}}$ is a strong convex function.

Proposition 1.2.2. *$G\circ\underline{\underline{\epsilon}}$ is $2\mu\mathcal{K}$-convex function where $\mathcal{K}$ is the constant of Korn.*

Proof. From Proposition 1.2.1 we have that

$$G(\alpha\underline{\underline{\epsilon}}(v)+(1-\alpha)\underline{\underline{\epsilon}}(w))\le\alpha G(\underline{\underline{\epsilon}}(v))+(1-\alpha)G(\underline{\underline{\epsilon}}(w))-\frac{1}{2}(2\mu)\alpha(1-\alpha)\left\|\underline{\underline{\epsilon}}(v)-\underline{\underline{\epsilon}}(w)\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}}. \tag{1.2}$$

According to Korn inequality A.5 we have that

$$\left\|\underline{\underline{\epsilon}}(v) - \underline{\underline{\epsilon}}(w)\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}} = \sum_{i,j=1}^{2} \int_\Omega \left|\underline{\underline{\epsilon}}_{ij}(v-w)\right|^2 \geq \mathcal{K}\|\underline{v} - \underline{w}\|^2_{H^1_0(\Omega)^2}. \tag{1.3}$$

So from (1.2) and (1.3) we obtain

$$G(\alpha\underline{\underline{\epsilon}}(v) + (1-\alpha)\underline{\underline{\epsilon}}(w)) \leq \alpha G(\underline{\underline{\epsilon}}(v)) + (1-\alpha)G(\underline{\underline{\epsilon}}(w)) - \frac{1}{2}(2\mu\mathcal{K})\alpha(1-\alpha)\|\underline{v} - \underline{w}\|^2_{H^1_0(\Omega)^2},$$

for all $\underline{v}, \underline{w} \in H^1_0(\Omega)^2$. □

The fact that $G \circ \underline{\underline{\epsilon}}$ is strictly convex and F is a linear and continuous functional implies that there exists a unique $\underline{u}$, solution of (1.1), that realize the minimum. Such function is characterized by the Euler-Lagrange equations, so we have that the minimum $\underline{u} \in H^1_0(\Omega)^2$ satisfies:

$$\int_\Omega 2\mu\underline{\underline{\epsilon}}(u)\underline{\underline{\epsilon}}(v) + \lambda \text{div}(\underline{u})\text{div}(\underline{v}) - \underline{f}\cdot\underline{v}\ \text{dx} = 0 \quad \forall \underline{v} \in H^1_0(\Omega)^2. \tag{1.4}$$

Now we recall the classical integration by parts formula between a smooth symmetric tensor $\underline{\underline{m}}$ and $\underline{\underline{\epsilon}}(v)$:

$$\int_\Omega \underline{\underline{m}} : \underline{\underline{\epsilon}}(v)\ \text{dx} = -\int_\Omega \text{div}(\underline{\underline{m}})\cdot\underline{v}\ \text{dx} + \int_{\partial\Omega} \underline{\underline{m}}\ \underline{n}\cdot\underline{v}\ \text{dx}.$$

Since in our case $\underline{v} \in H^1_0(\Omega)^2$ then the boundary term is equal to zero. The problem in applying such equation is that $\text{div}(\underline{\underline{\epsilon}}(\underline{u}))$ does not make sense for $\underline{u}$ in $H^1_0(\Omega)^2$, unless we consider $\text{div}(\underline{\underline{\epsilon}}(\underline{u}))$ as an element of $H^{-1}(\Omega)^2$ such that:

$$\langle \text{div}(\underline{\underline{\epsilon}}(u)), \underline{v}\rangle_{H^1_0(\Omega)^2} := -\int_\Omega \underline{\underline{\epsilon}}(\underline{u})\underline{\underline{\epsilon}}(v)\ \text{dx}. \tag{1.5}$$

Now recalling the classical integration by parts in higher dimension between a smooth scalar function $\underline{w}$ and $\underline{v}$:

$$\int_\Omega w\ \text{div}(\underline{v})\ \text{dx} = -\int_\Omega \nabla w\cdot\underline{v}\ \text{dx} + \int_{\partial\Omega} w\ \underline{v}\cdot\underline{n}\ \text{dx}.$$

As before we have that the boundary term is equal to zero because $\underline{v} \in H^1_0(\Omega)^2$ and we have to consider $\nabla(\text{div}(\underline{u}))$ as an element of $H^{-1}(\Omega)^2$ such that:

$$\langle \nabla\,\text{div}(\underline{u}), \underline{v}\rangle := -\int_\Omega \text{div}(\underline{u})\text{div}(\underline{v})\ \text{dx}.$$

As a consequence of the previous reasoning we have that equation (1.4), with a change of sign, becomes

$$2\mu\ \text{div}(\underline{\underline{\epsilon}}(u)) + \lambda\ \nabla\,\text{div}(\underline{u}) + \underline{f} = 0 \quad \text{in } H^{-1}(\Omega)^2,$$

then adding the boundary condition of $\underline{u}$ we obtain

$$\begin{cases} 2\mu\ \text{div}(\underline{\underline{\epsilon}}(u)) + \lambda\,\nabla(\text{div}(\underline{u})) + \underline{f} = 0 \\ u|_{\partial\Omega} = 0 \end{cases}. \tag{1.6}$$

From $\underline{u}$, solution of (1.1), we can define the stress tensor $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^D + p\ \underline{\underline{id}}$, where

$$\begin{cases} \underline{\underline{\sigma}}^D = 2\mu \underline{\underline{\epsilon}}(u)^D \\ p = (\lambda + \mu)\operatorname{tr}(\underline{\underline{\epsilon}}(u)) \end{cases} . \tag{1.7}$$

Recalling that $\operatorname{tr}(\underline{\underline{\epsilon}}(\underline{u})) = \operatorname{div}(\underline{u})$ and that for every tensor $\underline{\underline{\tau}}$ the following decomposition is unique

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}^D + \frac{1}{n}\operatorname{tr}(\underline{\underline{\tau}})\underline{\underline{id}},$$

where n is the the dimension of the space, substituting $\underline{\underline{\sigma}}$, define in (1.7), in the first equation of (1.6) we obtain

$$\operatorname{div}(\underline{\underline{\sigma}}) + \underline{f} = 0, \tag{1.8}$$

which expresses the equilibrium condition of continuum mechanics.

1.3 Dual problem

In order to compute the dualisation of the linear elasticity problem, define in (1.1), we will follow [8]. We will divide this section in three parts, in the first one we will introduce the dual problem and see its relation to the primal one and in the second one we will see the particular case in which the primal problem has the form $F(v) + G(\Lambda(v))$. In the last part we will compute explicitly the dualisation of the linear elasticity problem.

1.3.1 Introduction to the dual problem

We start by giving the definition of primal problem.

Definition 1.3.1 (Primal Problem)**.** *Let V be a topological vector space, V^* its dual and F a function from V to $\overline{\mathbb{R}}$, then the primal problem is*

$$\inf_{v\in V} F(v). \tag{$\mathcal{P}$}$$

If there exists a function $u \in V$ such that the infimum is realize we denote it with

$$F(u) = \inf_{v\in V} F(v) = \inf \mathcal{P}.$$

The problem $\mathcal{P}$ is said to be non-trivial if there exists an element $u_0 \in V$ such that

$$F(u_0) < +\infty.$$

In order to introduce the dual problem of primal problem $\mathcal{P}$ we have to perturb it. First of all we need Y and Y^* two Hausdorff topological vector spaces placed in duality, we define a function $\Phi : V \times Y \longrightarrow \overline{\mathbb{R}}$ such that

$$\Phi(u, 0) = F(u),$$

and for every $p \in Y$ we shall consider the minimisation problem

$$\inf_{v \in V} \Phi(u, p). \qquad (\mathcal{P}_p)$$

We call $\mathcal{P}_p$ the perturbed problems of $\mathcal{P}$ and in case $p = 0$ we return to the primal problem $\mathcal{P}_0 = \mathcal{P}$. The $\mathcal{P}_p$ is a family of problems and for a given primal problem $\mathcal{P}$ the perturbation is not unique, so we could choose for better perturbation than others, as we will do for the linear elasticity problem. Now we are able to define the dual problem of $\mathcal{P}$.

Definition 1.3.2 (Dual Problem). *Under the same assumption of Definition 1.3.1 of primal problem, the dual problem is*

$$\sup_{p^* \in Y^*} \big[-\Phi^*(0, p^*) \big], \qquad (\mathcal{P}^*)$$

where $\Phi^* : V^* \times Y^* \longrightarrow \overline{\mathbb{R}}$ is the Legendre transform of Φ. If there exists an element q^* such that the supremum is realized we denote it with

$$-\Phi^*(0, q^*) = \sup_{p^* \in Y^*} \big[-\Phi^*(0, p^*) \big] = \sup \mathcal{P}^*.$$

Now we want to see the relation between $\mathcal{P}$ and $\mathcal{P}^*$.

Proposition 1.3.1. *The following chain of inequalities hold:*

$$-\infty \leq \sup \mathcal{P}^* \leq \inf \mathcal{P} \leq +\infty.$$

Proof. Let $p^* \in Y^*$, by Definition A.1.8 we have

$$\Phi^*(0, p^*) = \sup_{v \in V p \in Y} \big[\langle p^*, p \rangle_Y - \Phi(v, p) \big].$$

Taking $p = 0$ in the previous equation we obtain

$$\Phi^*(0, p^*) \geq -\Phi(v, 0) \quad \forall v \in V. \tag{1.9}$$

By changing the sign of the previous equation and for the arbitrary of p^* we have

$$\sup \mathcal{P}^* \leq \inf \mathcal{P}$$

□

If both primal Problem $\mathcal{P}$ and dual Problem $\mathcal{P}^*$ are non-trivial we have that

$$-\infty < \sup \mathcal{P}^* \leq \inf \mathcal{P} < +\infty.$$

Our aim now is to see when the previous inequality is an equality, i.e. $\sup \mathcal{P}^* = \inf \mathcal{P}$. If we try to construct the dual problem of $\mathcal{P}$ in general we can not conclude anything unless we suppose that

$$\Phi \in \Gamma_0(V \times Y),$$

where $\Gamma_0(V \times Y)$ is the set of convex and l.s.c. (lower semi continuous) functions from $V \times Y$ into $\overline{\mathbb{R}}$ that can not be constantly equal to $+\infty$ and $-\infty$.

Assumption 1.3.1. *From now we shall assume that*

$$\Phi \in \Gamma_0(V \times Y).$$

For $p \in Y$ let us define

$$h(p) = \inf \mathcal{P}_p = \inf_{v \in V} \big[\Phi(v,p)\big]. \tag{1.10}$$

Proposition 1.3.2. *The function $h : Y \longrightarrow \overline{\mathbb{R}}$ is convex.*

Proof. Let $p, q \in Y$ and $\lambda \in]0,1[$. We have to show that

$$h(\lambda p + (1-\lambda)q) \leq \lambda h(p) + (1-\lambda)h(q).$$

Of course if $h(p)$ or $h(q)$ are equal to $+\infty$ the inequality is true. Let us assume that $h(p) < +\infty$ and $h(q) < \infty$, so for every $a > h(p)$ and $b > h(q)$ there exists $u, v \in V$ such that

$$\begin{aligned} h(p) &\leq \Phi(u,p) \leq a \\ h(q) &\leq \Phi(v,p) \leq b. \end{aligned}$$

Then:

$$\begin{aligned} h(\lambda p + (1-\lambda)q) &= \inf_{w \in V} \Phi(w, \lambda p + (1-\lambda)q) \\ &\leq \Phi(\lambda u + (1-\lambda)v, \lambda p + (1-\lambda)q) \\ &\leq \lambda \Phi(u,p) + (1-\lambda)\Phi(v,q) \\ &\leq \lambda a + (1-\lambda)b. \end{aligned}$$

Taking the limit for $a \to h(p)$ and $b \to h(q)$ we can conclude. □

Remark 1.3.1. *Even if the function $\Phi \in \Gamma_0(V \times Y)$, in general $h \notin \Gamma_0(Y)$.*

Proposition 1.3.3. *The following chain of equality holds:*

$$\sup \mathcal{P}^* = \sup_{p^* \in Y^*} \big[-h^*(p^*)\big] = h^{**}(0).$$

Proof. First we notice that for every $p^* \in Y^*$ we have:

$$\begin{aligned} h^*(p^*) &= \sup_{p \in Y} \big[\langle p^*, p\rangle_Y - h(p)\big] \\ &= \sup_{p \in Y} \big[\langle p^*, p\rangle_Y - \inf_{v \in V} \Phi(v,p)\big] \\ &= \sup_{p \in Y} \sup_{v \in V} \big[\langle p^*, p\rangle_Y - \Phi(v,p)\big] \\ &= \Phi^*(0, p^*). \end{aligned}$$

So, according to the Definition 1.3.2 of dual problem, we obtain

$$\sup \mathcal{P}^* = \sup_{p^* \in Y^*} \big[-\Phi^*(0,p^*) \big] = \sup_{p^* \in Y^*} \big[-h^*(p^*) \big] = h^{**}(0). \tag{1.11}$$

□

In order to have equivalence between primal and dual problem we need stronger assumption. We start by the following definition.

Definition 1.3.3. *The problem $\mathcal{P}$ is said to be normal if $h(0)$ is finite and h is l.s.c. at* 0.

These conditions imply that the primal problem is non trivial, because $h(0) = \inf \mathcal{P}$, and lower semi continuity at 0 is needed for $h^{**}(0) = h(0)$ even if $h \notin \Gamma_0(Y)$.
In the next Proposition we use some results we have reported in Appendix A.

Proposition 1.3.4. *The following conditions are equivalent to each other*

(i)	$\mathcal{P}$ *is normal,*
(ii)	$\mathcal{P}^*$ *is normal,*
(iii)	$\inf \mathcal{P} = \sup \mathcal{P}^*$ *and this number is finite.*

Proof. $(i) \Rightarrow (iii)$
We assume that $\mathcal{P}$ is normal, so we can define $\overline{h}$, which is the l.s.c. regularization of h ($\overline{h}$ is the largest l.s.c. minorant of h). From Proposition A.1.3 and Proposition A.1.4 we obtain

$$h^{**} \leq \overline{h} \leq h.$$

Now if we dualize the previous equation we obtain $h^* = h^{***} \geq \overline{h}^* \geq h^*$, so $h^* = \overline{h}^*$ and also $h^{**} = \overline{h}^{**}$.
By hypothesis we have that $h(0) = \overline{h}(0) \in \mathbb{R}$ and since h is convex also $\overline{h}$ is convex and can not take the value $-\infty$, because if it does $\overline{h}$ would be equal to $-\infty$ everywhere which is a contradiction according to Proposition A.1.1. So we have that $\overline{h} \in \Gamma_0(Y)$ and thus $\overline{h}^{**} = \overline{h}$. This means that $h(0) = \overline{h}(0) = \overline{h}^{**}(0) = h^{**}(0)$. According to (1.11) we obtain (iii).
$(iii) \Rightarrow (i)$
We have that $h(0)$ is finite and $h(0) = h^{**}(0)$, this implies that $h(0) = \overline{h}(0)$ so

$$h(0) = \overline{h}(0) \leq \liminf_{x \to 0} \overline{h}(x) \leq \liminf_{x \to 0} h(x),$$

so h is l.s.c. at 0. Thus, according to Definition 1.3.3, we have that $\mathcal{P}$ is normal.
$(ii) \Leftrightarrow (iii)$
The equivalence between (ii) and (iii) follows by the fact that $\mathcal{P}^{**} = \mathcal{P}$. Indeed we can consider the dual problem as a primal problem and $\Phi^*(v^*, p^*)$ as the function from which we construct its perturbed problem, as done for the primal problem $\mathcal{P}$ in $\mathcal{P}_p$, we obtain

$$\tilde{h}(v^*) = \inf_{p^* \in Y^*} \big[\Phi^*(v^*, p^*)\big],$$

which corresponds to the function h defined in (1.10) for the primal problem. According to $\mathcal{P}^*$

and since $V \cong V^{**}$ and $Y \cong Y^{**}$ we have

$$\begin{aligned}\tilde{h}^*(v) &= \sup_{v^* \in V^*} \left[\langle v^*, v \rangle_V - \tilde{h}(v^*) \right] \\ &= \sup_{v^* \in V^*} \left[\langle v^*, v \rangle_V - \inf_{p^* \in Y^*} \Phi^*(v^*, p^*) \right] \\ &= \sup_{v^* \in V^*} \sup_{p^* \in Y^*} \left[\langle v^*, v \rangle_V - \Phi^*(v^*, p^*) \right] \\ &= \Phi^{**}(v, 0).\end{aligned}$$

According to Assumption 1.3.1 and Proposition A.1.4 we have $\Phi = \Phi^{**}$, so we obtain

$$\tilde{h}^{**}(0) = \sup_{v \in V} \left[- \tilde{h}^*(v) \right] = \sup_{v \in V} \left[- \Phi^{**}(v, 0) \right] = \sup_{v \in V} \left[- \Phi(v, 0) \right] = - \inf_{v \in V} \left[F(v) \right] = - \inf \mathcal{P}.$$

Thus $\mathcal{P}^{**} = \mathcal{P}$ and since $(i) \Leftrightarrow (iii)$, we obtain $(ii) \Leftrightarrow (iii)$. □

Even if we are ready to introduce the explicit formulation of the dual problem for linear elasticity, it is convenient to us to go a bit further and see which conditions on $\mathcal{P}$ guarantee us existence and uniqueness of solution for the dual problem $\mathcal{P}^*$.

1.3.2 Existence of solution

In order to have at least one solution for the dual problem $\mathcal{P}^*$ we need stronger assumptions, so we start form a definition.

Definition 1.3.4. *Problem $\mathcal{P}$ is said to be stable if $h(0)$ is finite and h is subdifferentiable at 0.*

Requiring $\mathcal{P}$ to be stable is a stronger assumption than $\mathcal{P}$ to be normal because subdifferentiability implies l.s.c. In the following proof we recall same results that are collected in [8].

Proposition 1.3.5 (A Stability criterion). *Let us assume that $\Phi \in \Gamma_0(V \times Y)$ (it is enough only requiring Φ to be convex) , $\inf \mathcal{P}$ is finite and that there exists $u_0 \in V$ such that $p \mapsto \Phi(u_0, p)$ is finite and continuous at 0. Then problem $\mathcal{P}$ is stable.*

Proof. Since $\inf \mathcal{P}$ is finite we have that $h(0)$ is finite. The function $p \mapsto \Phi(u_0, p)$ is convex and continuous at 0, so there exists a neighbourhood $\mathcal{V}$ of 0 in Y such that is bounded form above:

$$\Phi(u_0, p) \leq M < +\infty \quad \forall p \in \mathcal{V}.$$

Since $h(p)$ is the infimum in V this means that even h is bounded by M in $\mathcal{V}$. applying Proposition A.1.2 we have that h is continuous at 0, and, according to Proposition A.1.6, we have that h is subdifferentiable at 0. □

This result is important because gives an easy way to see if problem $\mathcal{P}$ is stable. Before analyzing the consequence of stability for problem $\mathcal{P}$ we see a preliminary Lemma.

Lemma 1.3.1. *The set of solution of $\mathcal{P}^*$ is identical to $\partial h^{**}(0)$.*

Proof. If $q^* \in Y^*$ is a solution of $\mathcal{P}^*$, then

$$-h^*(q^*) = \sup_{p^* \in Y^*} \big[-h^*(p^*) \big] = \sup \mathcal{P}^* = h^{**}(0).$$

This is equivalent to say that $q^* \in \partial h^{**}(0)$. □

Proposition 1.3.6. *The following conditions are equivalent:*

(i) $\mathcal{P}$ *is stable,*

(ii) $\mathcal{P}$ *is normal and* $\mathcal{P}^*$ *has at least one soultion.*

Proof. If $\mathcal{P}$ is stable we have that $h(0)$ is finite and $\partial h(0)$ is non-empty, so from Proposition A.1.5 we have that $h(0) = h^{**}(0)$ which implies, from Proposition 1.3.4, that $\mathcal{P}$ is normal. Also from Proposition A.1.5 we have that $\partial h^{**}(0) = \partial h(0) \neq \emptyset$, so, using Lemma 1.3.1, $\mathcal{P}^*$ has at least one solution.
Conversely, if $\mathcal{P}$ is normal, using Proposition 1.3.4, we have that $h(0) = h^{**}(0)$ and it is finite, so from Proposition A.1.5 we have that $\partial h^{**}(0) = \partial h(0)$. Since $\mathcal{P}^*$ has at least one solution, according to Lemma 1.3.1, $\partial h^{**}(0) \neq \emptyset$. Thus $\mathcal{P}$ is stable. □

Now we study the particular case where the function F, that characterize the primal Problem 1.3.1 has the form

$$F(v) = J(v, \Lambda v),$$

where Λ is a linear operator from V to Y.
So $\mathcal{P}$ takes the form

$$\inf_{v \in V} J(v, \Lambda v). \tag{1.12}$$

We will denote with $\Lambda^* : Y^* \longrightarrow V^*$ the transpose (linear) operator.
Now we have to choose the perturbation needed for the construction of the dual problem. In order to do that we should introduce another functional space, as we did before, but since the function J takes value in Y (we suppose that Y and Y^* are two Hausdorff spaces) we use it instead. So the function Φ will be

$$\Phi(v, p) = J(v, \Lambda v + p),$$

which is clearly a good choice for us.
Now we want to develop the Legendre transform of $\Phi(0, p^*)$ and write it as a function of J^*:

$$\begin{aligned} \Phi^*(0, p^*) &= \sup_{v \in V p \in Y} \big[\langle p^*, p \rangle_Y - \Phi(v, p) \big] \\ &= \sup_{v \in V} \sup_{p \in Y} \big[\langle p^*, p \rangle_Y - J(v, \Lambda v + p) \big]. \end{aligned}$$

Let fix $q = \Lambda v + p$, so

$$\begin{aligned}\Phi^*(0,p^*) &= \sup_{v\in V q\in Y} \big[\langle p^*, -\Lambda u\rangle_Y + \langle p^*, q\rangle_Y - J(v,q)\big] \\ &= \sup_{v\in V} \sup_{q\in Y} \big[\langle -\Lambda^* p^*, u\rangle_V + \langle p^*, q\rangle_Y - J(v,q)\big] \\ &= J^*(-\Lambda^* p^*, p^*).\end{aligned}$$

So the dual problem $\mathcal{P}^*$ of (1.12) becomes:

$$\sup_{p^*\in Y^*} \big[- J^*(-\Lambda^* p^*, p^*)\big]. \tag{1.13}$$

Now we see a direct consequence of Proposition 1.3.5 for this special case in which $\Phi(v,p) = J(v, \Lambda v + p)$.

Theorem 1.1. *Let us assume that J is convex, $\inf \mathcal{P}$ is finite and that there exists $u_0 \in V$ such that $J(u_0, \Lambda u_0) < +\infty$ and the function $p \mapsto J(u_0, p)$ being continuous at Λu_0.*
Then $\mathcal{P}$ is stable, $\inf \mathcal{P} = \sup \mathcal{P}^*$ and $\mathcal{P}^*$ has at least one solution $\overline{p}^*$.

Proof. Clearly the hypothesis of this theorem imply the condition needed for Proposition 1.3.5 to be applied, then using Proposition 1.3.6 we can conclude. □

Now we want to study the special case in which $J(v, \Lambda v) = F(v) + G(\Lambda v)$.
In this case primal problem $\mathcal{P}$ becomes

$$\inf_{v\in V} \big[F(v) + G(\Lambda v)\big]. \tag{1.14}$$

Now we want to write $J^*(-\Lambda^* p^*, p^*)$ as a function of F^* and G^*:

$$\begin{aligned}J^*(u^*,p^*) &= \sup_{v\in V p\in Y} \big[\langle u^*, v\rangle_V + \langle p^*, p\rangle_Y - F(v) - G(p)\big] \\ &= \sup_{v\in V} \big[\langle u^*, v\rangle_V - F(v)\big] + \sup_{p\in Y} \big[\langle p^*, p\rangle_Y - G(p)\big] \\ &= F^*(u^*) + G^*(p^*),\end{aligned}$$

so from (1.13) we have that $\mathcal{P}^*$ is

$$\sup_{p^*\in Y^*} \big[- F^*(-\Lambda^* p^*) - G^*(p^*)\big]. \tag{1.15}$$

Now we want to see the relation between solutions of primal and dual problem in case $J(v, \Lambda v) = F(v) + G(\Lambda v)$.

Proposition 1.3.7. *Let $u \in V$ be a solution of $\mathcal{P}$, q^* be a solution of $\mathcal{P}^*$ and $\inf \mathcal{P} = \sup \mathcal{P}^*$. Then we have that*

$$\begin{aligned}-\Lambda^* q^* &\in \partial F(u) \\ q^* &\in \partial G(\Lambda u).\end{aligned}$$

Proof. Since $u \in V$ is a solution of $\mathcal{P}$ we have

$$F(u) + G(\Lambda u) = J(u, \Lambda u) = \inf_{v \in V} \big[J(v, \Lambda v)\big] = \inf \mathcal{P},$$

and from (1.13) and (1.15), since $q^* \in Y^*$ is a solution of $\mathcal{P}^*$, we have

$$-F^*(-\Lambda^* q^*) - G^*(q^*) = -J(-\Lambda^* q^*, q^*) = \sup_{p^* \in Y^*} \big[-J^*(-\Lambda^* p^*, p^*)\big] = \sup \mathcal{P}^*.$$

So we obtain

$$\begin{aligned} 0 &= J(u, \Lambda u) + J^*(-\Lambda^* q^*, q^*) \\ &= F(u) + G(\Lambda u) + F^*(-\Lambda^* q^*) + G^*(q^*) \\ &= [F(u) + F^*(-\Lambda^* q^*) - \langle -\Lambda^* q^*, u\rangle] \\ &+ [G(\Lambda u) + G^*(q^*) - \langle q^*, \Lambda u\rangle]. \end{aligned}$$

From the definition of Legendre transform we have

$$\begin{aligned} F^*(-\Lambda^* q^*) &= \sup_{v \in V}[\langle -\Lambda^* q^*, v\rangle_V + F(v)] \\ G^*(q^*) &= \sup_{p \in Y}[\langle q^*, p\rangle_Y + G(p)], \end{aligned}$$

so we can deduce that $-\Lambda^* q^*$ realizes the supremum for F^* and q^* realizes the supremum for G^*. This is equivalent to say that

$$\begin{aligned} -\Lambda^* q^* &\in \partial F(u) \\ q^* &\in \partial G(\Lambda u). \end{aligned}$$

□

Remark 1.3.2. *This last result is strictly related to the choice of the perturbation, if we had chosen $\Phi(v, p) = J(v, \Lambda v - p)$ we would have obtain*

$$\begin{aligned} \Lambda^* q^* &\in \partial F(u) \\ -q^* &\in \partial G(\Lambda u). \end{aligned}$$

Proposition 1.3.8. *If the primal Problem $\mathcal{P}$ is in the form (1.14), it is stable, it admits a unique solution u and G is differentiable at $G(\Lambda u)$, then the dual Problem $\mathcal{P}^*$ admits a unique solution.*

Proof. Since primal Problem $\mathcal{P}$ is stable then dual Problem $\mathcal{P}^*$ admits at least one solution. According to Proposition 1.3.7 we have that all the solutions of the dual problem must be in $\partial G(\Lambda u)$, but, according to Proposition A.1.7, we have that $\partial G(\Lambda u) = \{G'(\Lambda u)\}$, so $q^* = G'(\Lambda u)$. □

In this section we have introduced the dual problem $\mathcal{P}^*$ of primal problem $\mathcal{P}$. We have seen that if the primal is normal then we have that $\inf \mathcal{P} = \sup \mathcal{P}^*$ and, if we require $\mathcal{P}$ to be stable then we have that $\mathcal{P}^*$ has at least one solution. In the end we studied the case where

$J(v, \Lambda v) = F(v) + G(\Lambda v)$ and notice that under the hypothesis of Proposition 1.3.8 we have uniqueness of solution for $\mathcal{P}^*$. In the next subsection we want to apply these results to the linear elasticity problem.

1.3.3 Application to linear elasticity problem

According to the theory we have developed in the previous subsection now we are able to introduce the primal problem for the linear elasticity starting from the starting problem (1.1) that we have defined in Section 1.2.

Problem 1.3.1 (Primal formulation of linear elasticity). *Find* $\underline{u} \in H_0^1(\Omega)^2$ *such that*

$$\inf_{\underline{v} \in H_0^1(\Omega)^2} \left\{ \int_\Omega \mu |\underline{\underline{\epsilon}}(v)|^2 + \frac{\lambda}{2} |\mathrm{div}(\underline{v})|^2 - \underline{f} \cdot \underline{v} \, \mathrm{dx} \right\}. \tag{1.16}$$

From the analysis that we have done in Section 1.2 we have seen that Problem 1.3.1 admits a unique solution. In order to dualize (1.16) it is convenient to rewrite it in the form (1.14), where: $\Lambda = \underline{\underline{\epsilon}} : H_0^1(\Omega)^2 \longrightarrow L^2(\Omega)^{2\times 2}_{sym}$ is a linear operator, the function $F : H_0^1(\Omega)^2 \longrightarrow \mathbb{R}$ is

$$F(\underline{v}) = -\int_\Omega \underline{f} \cdot \underline{v} \, \mathrm{dx}, \tag{1.17}$$

a linear and continuous operator and $G : L^2(\Omega)^{2\times 2}_{sym} \longrightarrow \mathbb{R}$ is

$$G(\underline{\underline{p}}) = \int_\Omega \mu |\underline{\underline{p}}|^2 + \frac{\lambda}{2} |\mathrm{tr}(\underline{\underline{p}})|^2 \, \mathrm{dx}, \tag{1.18}$$

a 2μ-convex function. Now we can rewrite the primal problem (1.16) as follows

$$\inf_{\underline{v} \in H_0^1(\Omega)^2} \left\{ F(\underline{v}) + G(\underline{\underline{\epsilon}}(v)) \right\}.$$

In order to obtain $\inf \mathcal{P} = \sup \mathcal{P}^*$ and the existence of solutions for $\mathcal{P}^*$ we will show that Primal problem is stable.

Proposition 1.3.9. *Problem 1.16 is stable. Moreover the dual problem of Problem 1.16 admits a unique solution.*

Proof. In order to verifies stability we will show that Problem 1.16 verifies the hypothesis of Theorem 1.1, i.e. we need to find $\underline{u}_0 \in V$ such that $J(\underline{u}_0, \underline{\underline{\epsilon}}(u_0)) < +\infty$ and the function $\underline{\underline{p}} \mapsto J(\underline{u}_0, \underline{\underline{p}})$ being continuous at $\underline{\underline{\epsilon}}(u_0)$. If we fix $\underline{u}_0 = 0$ we have that $\underline{\underline{\epsilon}}(0) = \underline{\underline{0}}$, and $J(\underline{0}, \underline{\underline{0}}) = F(\underline{0}) + G(\underline{\underline{0}}) = 0 < +\infty$. So it remains to verify that $p \mapsto J(\underline{0}, \underline{\underline{p}}) = G(\underline{\underline{p}})$ is continuous at $\underline{\underline{0}}$. In order to do that we will show that G is continuous for every $\underline{\underline{p}}$. Let $\underline{\underline{p}}_n \to \underline{\underline{p}}$ in $L^2(\Omega)^{2\times 2}_{sym}$ as $n \to \infty$, then it is enough to show

$$\lim_{n\to\infty} \int_\Omega \mu |\underline{\underline{p}}_n|^2 \, \mathrm{dx} = \int_\Omega \mu |\underline{\underline{p}}|^2 \, \mathrm{dx}, \tag{1.19}$$

and

$$\lim_{n\to\infty}\int_\Omega \frac{1}{2}\lambda|\mathrm{tr}(\underline{\underline{p}}_n)|^2\ \mathrm{dx} = \int_\Omega \frac{1}{2}\lambda|\mathrm{tr}(\underline{\underline{p}})|^2\ \mathrm{dx}. \tag{1.20}$$

Since $\int_\Omega \mu|\underline{\underline{p}}|^2\ \mathrm{dx\ dt} = \mu\left\|\underline{\underline{p}}\right\|_Y$ and the norm is continuous, then (1.19) is true. Since $|\mathrm{tr}(\cdot)| : Y \to \mathbb{R}$ can be seen as the composition of the projection $\pi : Y \to Y$ on the trace components, the sum of them, and the norm of Y and because all of them are continuous, we obtain (1.20). So form (1.19) and (1.20) we have the following

$$\lim_{n\to\infty} G(\underline{\underline{p}}_n) = G(\underline{\underline{p}}).$$

Thus Problem 1.3.1 is stable and due to Proposition 1.3.6, its dual problem has at least one solution.
Can be noticed that G is differentiable and the differential of G in $\underline{\underline{p}}$ evaluated in $\underline{\underline{q}}$ is

$$G'[\underline{\underline{p}}](\underline{\underline{q}}) = \int_\Omega 2\mu\underline{\underline{p}} : \underline{\underline{q}} + \lambda\,\mathrm{tr}(\underline{\underline{p}})\,\mathrm{tr}(\underline{\underline{q}})\ \mathrm{dx}.$$

Since G is differentiable and primal Problem 1.16 admits a unique solution, according to Proposition 1.3.8, we have that the dual problem of primal Problem 1.16 admits a unique solution. □

Before compute the dual problem explicitly let do some consideration about the spaces we are using. First of all $H_0^1(\Omega)$ is an Hilbert space and its dual is denoted with $H^{-1}(\Omega)$. We also recall that given two Banach spaces V and W the dual space of $V \times W$ con be identified with $V^* \times W^*$, so $(H_0^1(\Omega)^2)^*$ is $H^{-1}(\Omega)^2$.
Secondly there exists an isometry between $L^2(\Omega)^3$ and $L^2(\Omega)^{2\times 2}_{sym}$ such that

$$\underline{\varphi} = \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix} \longrightarrow \begin{bmatrix}\varphi_1 & \frac{1}{\sqrt{2}}\varphi_2\\ \frac{1}{\sqrt{2}}\varphi_2 & \varphi_3\end{bmatrix} = \underline{\underline{\xi}},$$

The factor $\frac{1}{\sqrt{2}}$ is needed for having an isometry:

$$\begin{aligned}\left\|\underline{\varphi}\right\|^2_{L^2(\Omega)^3} &= \|\varphi_1\|^2_{L^2(\Omega)} + \|\varphi_2\|^2_{L^2(\Omega)} + \|\varphi_3\|^2_{L^2(\Omega)}\\ &= \|\varphi_1\|^2_{L^2(\Omega)} + 2\left\|\frac{1}{\sqrt{2}}\varphi_2\right\|^2_{L^2(\Omega)} + \|\varphi_3\|^2_{L^2(\Omega)} = \left\|\underline{\underline{\xi}}\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}}.\end{aligned}$$

So we obtain that, because $L^2(\Omega)^3$ is an Hilbert space, even $L^2(\Omega)^{2\times 2}_{sym}$ is an Hilbert space and using the Riesz representation define in Theorem (A.1) we have that for every $p^* \in (L^2(\Omega)^{2\times 2}_{sym})^*$ there exists a unique element $\underline{\underline{r}} \in L^2(\Omega)^{2\times 2}_{sym}$ such that

$$\langle p^*, \underline{\underline{\varphi}}\rangle_{L^2(\Omega)^{2\times 2}_{sym}} = (\underline{\underline{r}}, \underline{\underline{\varphi}})_{L^2(\Omega)^{2\times 2}_{sym}} \quad \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym},$$

and we denote $\underline{\underline{r}}$ with $\mathcal{R}(p^*)$.

According to (1.15) we have that the dual problem have the following form:

$$\sup_{p^* \in L^2(\Omega)^{2\times 2}_{sym}} \Big\{ -F^*(-\underline{\underline{\epsilon}}^*(p^*)) - G^*(p^*) \Big\}. \tag{1.21}$$

So we have to develope F^* and G^* in order to get an explicit formulation. According to the Definition A.1.8 we have that $F^* : H^{-1}(\Omega)^2 \longrightarrow \overline{\mathbb{R}}$ is the Legendre transform of F, so

$$F^*(v^*) = \sup_{\underline{v} \in H^1_0(\Omega)^2} \big[\langle v^*, \underline{v} \rangle_{H^1_0(\Omega)^2} - F(\underline{v}) \big] = \sup_{\underline{v} \in H^1_0(\Omega)^2} \big[\langle v^*, \underline{v} \rangle_{H^1_0(\Omega)^2} + \int_\Omega \underline{f} \cdot \underline{v} \ \mathrm{dx} \big].$$

Now we need to calculate $F^*(-\underline{\underline{\epsilon}}^*(p^*))$:

$$\begin{aligned} F^*(-\underline{\underline{\epsilon}}^*(p^*)) &= \sup_{\underline{v} \in H^1_0(\Omega)^2} \big[\langle -\underline{\underline{\epsilon}}^*(p^*), \underline{v} \rangle_{H^1_0(\Omega)^2} + \int_\Omega \underline{f} \cdot \underline{v} \ \mathrm{dx} \big] \\ &= \sup_{\underline{v} \in H^1_0(\Omega)^2} \big[- (\mathcal{R}(p^*), \underline{\underline{\epsilon}}(v))_{L^2(\Omega)^{2\times 2}_{sym}} + \int_\Omega \underline{f} \cdot \underline{v} \ \mathrm{dx} \big] \\ &= \sup_{\underline{v} \in H^1_0(\Omega)^2} \big[- \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(v) + \int_\Omega \underline{f} \cdot \underline{v} \ \mathrm{dx} \big]. \end{aligned}$$

We can see that the value of $F^*(-\underline{\underline{\epsilon}}^*(p^*))$ can only be $+\infty$, if there exists at least one $\underline{v} \in H^1_0(\Omega)^2$ such that the argument we are maximizing is different form zero, or 0 if we have that

$$- \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(v) + \underline{f} \cdot \underline{v} \ \mathrm{dx} = 0 \quad \forall \underline{v} \in H^1_0(\Omega)^2.$$

According to (1.5) we can define $\mathrm{div}(\mathcal{R}(p^*))$ as an element of $H^{-1}(\Omega)^2$ such that

$$\langle \mathrm{div}(\mathcal{R}(p^*)), \underline{v} \rangle_{H^1_0(\Omega)^2} \coloneqq - \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(v) \ \mathrm{dx}. \tag{1.22}$$

Since $\mathcal{R}(p^*)$ is an element of $L^2(\Omega)^{2\times 2}_{sym}$ and $\underline{f}$ is in $L^2(\Omega)^2$ and we can conclude that if there exists a $\underline{\underline{q}}^* \in Y^*$ such that $F^*(-\underline{\underline{\epsilon}}^*(\underline{\underline{q}}^*)) = 0$, then $\mathcal{R}(\underline{\underline{q}}^*)$ belongs to $\underline{\underline{H}}(\mathrm{div}, \Omega)$ and $\mathrm{div}(\mathcal{R}(\underline{\underline{q}}^*)) + \underline{f} = 0$. So we have that

$$F^*(-\underline{\underline{\epsilon}}^*(p^*)) = \begin{cases} 0 & \text{if } \ \mathrm{div}(\mathcal{R}(p^*)) + \underline{f} = 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{1.23}$$

Now we can calculate $G^*(-p^*)$, where G^* is the Legendre transform of G:

$$\begin{aligned} G^*(p^*) &= \sup_{\underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym}} \big[\langle p^*, \underline{\underline{p}} \rangle_{L^2(\Omega)^{2\times 2}_{sym}} - G(\underline{\underline{p}}) \big] \\ &= \sup_{\underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym}} \big[\langle p^*, \underline{\underline{p}} \rangle_{L^2(\Omega)^{2\times 2}_{sym}} - \int_\Omega \mu |\underline{\underline{p}}|^2 - \frac{\lambda}{2} |\mathrm{tr}(\underline{\underline{p}})|^2 \ \mathrm{dx} \big]. \end{aligned} \tag{1.24}$$

Since G is strictly convex then for every fixed $p^* \in Y^*$ there exists a unique $\underline{\underline{q}} \in L^2(\Omega)^{2\times 2}_{sym}$ such that the supremum in the previous equation is realize. Now we compute the first variation respect to $\underline{\underline{p}}$ evaluated in $\underline{\underline{q}}$ of the argument inside the supremum in (1.24) and since $\underline{\underline{q}}$ realize the maximum we have

$$\langle p^*, \underline{\underline{\varphi}} \rangle_{L^2(\Omega)^{2\times 2}_{sym}} - \int_\Omega 2\mu \, \underline{\underline{q}} : \underline{\underline{\varphi}} - \lambda \, \mathrm{tr}(\underline{\underline{q}}) \, \mathrm{tr}(\underline{\underline{\varphi}}) \, \mathrm{dx} = 0 \quad \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym}. \tag{1.25}$$

If we take $\underline{\underline{\varphi}} = \underline{\underline{q}}$ we obtain that

$$\langle p^*, \underline{\underline{q}} \rangle_{L^2(\Omega)^{2\times 2}_{sym}} = \int_\Omega 2\mu |\underline{\underline{q}}|^2 - \lambda |\mathrm{tr}(\underline{\underline{q}})|^2 \, \mathrm{dx} = 2G(\underline{\underline{q}}). \tag{1.26}$$

Substituting (1.26) in (1.24) we obtain

$$G^*(p^*) = 2G(\underline{\underline{q}}) - G(\underline{\underline{q}}) = G(\underline{\underline{q}}). \tag{1.27}$$

Now we want to write $\underline{\underline{q}}$ as a function of p^*. According to Theorem A.1 we can rewrite (1.25) as follows.

$$(\mathcal{R}(p^*), \underline{\underline{\varphi}})_{L^2(\Omega)^{2\times 2}_{sym}} - \int_\Omega 2\mu \, \underline{\underline{q}} : \underline{\underline{\varphi}} - \lambda \, \mathrm{tr}(\underline{\underline{q}}) \underline{\underline{id}} : \underline{\underline{\varphi}} \, \mathrm{dx} = 0 \quad \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym},$$

from the arbitrariness of $\underline{\underline{\varphi}}$ in the previous equation we obtain

$$\mathcal{R}(p^*) = 2\mu \underline{\underline{q}} + \lambda \, \mathrm{tr}(\underline{\underline{q}}) \underline{\underline{id}} = 2\mu \underline{\underline{q}}^D + (\lambda + \mu) \, \mathrm{tr}(\underline{\underline{q}}) \underline{\underline{id}},$$

where we recall that the decomposition of a tensor into its deviatoric and trace components is orthogonal and unique. This means that we can define the function

$$\underline{\underline{q}} \longrightarrow 2\mu \underline{\underline{q}}^D + (\lambda + \mu) \, \mathrm{tr}(\underline{\underline{q}}) \underline{\underline{id}},$$

where the inverse can be easily seen to be

$$\underline{\underline{p}} \longrightarrow \frac{1}{2\mu} \underline{\underline{p}}^D + \frac{1}{4(\lambda + \mu)} \mathrm{tr}(\underline{\underline{p}}) \underline{\underline{id}}.$$

According to the previous equation we can rewrite (1.27):

$$\begin{aligned} G^*(p^*) &= G(\underline{\underline{q}}) \\ &= G\Big(\frac{1}{2\mu} \mathcal{R}(p^*)^D + \frac{1}{4(\lambda + \mu)} \mathrm{tr}(\mathcal{R}(p^*)) \underline{\underline{id}} \Big) \\ &= \int_\Omega \frac{1}{4\mu} |\mathcal{R}(p^*)^D|^2 + \frac{1}{8(\mu + \lambda)} |\mathrm{tr}(\mathcal{R}(p^*))|^2 \, \mathrm{dx}. \end{aligned} \tag{1.28}$$

Now we aim to write (1.21) explicitly. Since the only value of $F^*(-\underline{\epsilon}^*(p^*))$, defined in (1.23), are 0 and $+\infty$, and $\sup \mathcal{P}^*$ is finite, we can rewrite (1.21) as follows.

$$\sup_{\substack{p^* \in (L^2(\Omega)^{2\times 2}_{sym})^* \\ \mathrm{div}(\mathcal{R}(p^*)) + \underline{f} = 0}} \Big\{ -G^*(p^*) \Big\}.$$

Changing the sup into a inf, since we are looking for p^* and not the value of $\sup \mathcal{P}^*$ and using

(1.28), we have

$$\inf_{\substack{p^* \in (L^2(\Omega)^{2\times 2}_{sym})^* \\ \operatorname{div}(\mathcal{R}(p^*))+\underline{f}=0}} \left\{ G^*(p^*) \right\} = \inf_{\substack{p^* \in (L^2(\Omega)^{2\times 2}_{sym})^* \\ \operatorname{div}(\mathcal{R}(p^*))+\underline{f}=0}} \left\{ \int_\Omega \frac{1}{4\mu} |\mathcal{R}(p^*)^D|^2 + \frac{1}{8(\mu+\lambda)} |\operatorname{tr}(\mathcal{R}(p^*))|^2 \, \mathrm{dx} \right\}. \quad (1.29)$$

Since the functional define in (1.29) depends on p^* only through $\mathcal{R}$ and since $L^2(\Omega)^{2\times 2}_{sym}$ is an Hilbert space, so $\mathcal{R}$ is an isomorphism, we can replace Y^* with Y and $\mathcal{R}(p^*)$ with $\underline{\underline{p}}$ and we can introduce the following.

Problem 1.3.2 (Dual formulation of linear elasticity)**.** *Find* $\underline{\underline{q}} \in L^2(\Omega)^{2\times 2}_{sym}$ *such that*

$$\int_\Omega \frac{1}{4\mu} |\underline{\underline{q}}^D|^2 + \frac{1}{8(\mu+\lambda)} |\operatorname{tr}(\underline{\underline{q}})|^2 \, \mathrm{dx} = \inf_{\substack{\underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym} \\ \operatorname{div}(\underline{\underline{p}})+\underline{f}=0}} \left\{ \int_\Omega \frac{1}{4\mu} |\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)} |\operatorname{tr}(\underline{\underline{p}})|^2 \, \mathrm{dx} \right\}. \quad (1.30)$$

1.4 Saddle point Problem

In this section we want to introduce the saddle point problem for linear elasticity. We will proceed in the same way as the previous section: according to the theory developed in [8], we will introduce the saddle point problem and see its relation with primal and dual and then we will apply it to the linear elasticity problem.

1.4.1 Introduction to saddle point problem

The assumption on the spaces are the same of the previous section. We will see that the work here will be very similar to the one that we have done previously for duality. We start form two definitions.

Definition 1.4.1 (Lagrangian)**.** *Let* $L : V \times Y^* \longrightarrow \overline{\mathbb{R}}$ *such that*

$$-L(v, p^*) = \sup_{p \in Y} \left[\langle p^*, p \rangle_Y - \Phi(v, p) \right],$$

for $v \in V$ *and* $p^* \in Y^*$*.* $L(v, p^*)$ *will be called the Lagrangian of the primal Problem 1.3.1 relative to the given perturbation* Φ*.*

Definition 1.4.2 (Saddle point)**.** *The point* $(u, q^*) \in V \times Y$ *is called saddle point of* L *if*

$$L(u, p^*) \leq L(u, q^*) \leq L(v, q^*) \quad \forall v \in V, \quad \forall p^* \in Y^*. \quad (1.31)$$

We recall that $\Phi(v, p)$ is such that $\Phi(v, 0) = F(v) \;\; \forall v \in V$ and such perturbation it is not unique. Now we want to see the relation between $\mathcal{P}$ and $\mathcal{P}^*$ in terms of the Lagrangian. We start from noticing that

$$\begin{aligned}\Phi^*(v^*,p^*) &= \sup_{v\in V\, p\in Y}\big[\langle v^*,v\rangle_V + \langle p^*,p\rangle_Y - \Phi(v,p)\big]\\ &= \sup_{v\in V}\big[\langle v^*,v\rangle_V + \sup_{p\in Y}\big[\langle p^*,p\rangle_Y - \Phi(v,p)\big]\big]\\ &= \sup_{v\in V}\big[\langle v^*,v\rangle_V - L(v,p^*)\big].\end{aligned}$$

From the previous equation we can rewrite the dual problem $\mathcal{P}^*$ as

$$\sup_{p^*\in Y^*}\big[-\Phi^*(0,p^*)\big] = \sup_{p^*\in Y^*}\big[-\sup_{v\in V}\big[-L(v,p^*)\big]\big] = \sup_{p^*\in Y^*}\inf_{v\in V}\big[L(v,p^*)\big].$$

Now we want to rewrite the primal problem $\mathcal{P}$ as a function of L, in order to do that we need $\Phi\in\Gamma_0(V\times Y)$, an assumption that we have already made in Assumption 1.3.1, so the function $\Phi_v : p \longrightarrow \Phi(v,p)$ is $\Gamma_0(Y)$ and from Proposition A.1.4 we have $\Phi_v = \Phi_v^{**}$, that implies

$$\begin{aligned}\Phi(v,p) &= \Phi_v^{**}(p)\\ &= \sup_{p^*\in Y^*}\big[\langle p^*,p\rangle_Y - \Phi_v^*(p^*)\big]\\ &= \sup_{p^*\in Y^*}\big[\langle p^*,p\rangle_Y + L(v,p^*)\big],\end{aligned}$$

where, according to Definition 1.4.1, we have used

$$\Phi_v^*(p^*) = \sup_{p\in Y}\big[\langle p^*,p\rangle_Y - \Phi_v(p)\big] = \sup_{p\in Y}\big[\langle p^*,p\rangle_Y - \Phi(v,p)\big] = -L(v,p^*).$$

So primal Problem $\mathcal{P}$ can also be written in the following way:

$$\inf_{v\in V}\big[F(v)\big] = \inf_{v\in V}\big[\Phi(v,0)\big] = \inf_{v\in V}\sup_{p^*\in Y^*}\big[L(v,p^*)\big].$$

Now we want to see the relation between the solutions of primal problem $\mathcal{P}$, dual problem and $\mathcal{P}^*$ and the saddle points of the Lagrangian.

Proposition 1.4.1. *If $\Phi\in\Gamma_0(V\times Y)$, then the following conditions are equivalent:*

(i)	*(u,q^*) is a saddle point ol L,*
(ii)	*u is a solution of $\mathcal{P}, q^*$ is a solution of $\mathcal{P}^*$ and $\inf\mathcal{P} = \sup\mathcal{P}^*$.*

Proof. If (u,q^*) is a saddle point, then according to (1.31), we have

$$\inf\mathcal{P} = \inf_{v\in V} F(v) = \inf_{v\in V}\Phi(v,0) = \Phi(u,0) \tag{1.32}$$

$$\sup\mathcal{P}^* = \sup_{p^*\in Y^*} -\Phi^*(0,p^*) = -\Phi^*(0,q^*), \tag{1.33}$$

so u is a solution to $\mathcal{P}$ and q^* is a solution to $\mathcal{P}^*$. Then we can also notice that if (u,q^*) is a saddle point, then

$$\Phi(u,0) = \sup_{p^*\in Y^*} L(u,p^*) = L(u,q^*) = \inf_{v\in V} L(v,q^*) = -\Phi^*(0,q^*). \tag{1.34}$$

So according to (1.32), (1.33) and (1.34) we have

$$\inf \mathcal{P} = \Phi(u,0) = -\Phi^*(0,q^*) = \sup \mathcal{P}^*.$$

Conversely, if u is a solution of $\mathcal{P}$ and q^* is a solution of $\mathcal{P}^*$ we have

$$\Phi(u,0) = \sup_{p^*\in Y^*} L(u,p^*) \geq L(u,q^*)$$
$$-\Phi^*(0,q^*) = \inf_{v\in V} L(v,q^*) \leq L(u,q^*),$$

then using that $\inf \mathcal{P} = \sup \mathcal{P}^*$, we obtain $\Phi(u,0) = -\Phi^*(0,q^*)$, so we can conclude. □

Remark 1.4.1. *For our purpose it is important that $(ii) \Rightarrow (i)$, because it guarantees existence of solution for the saddle point problem since primal problem and dual problem of linear elasticity admit solutions. Since for linear elasticity we know that solutions of primal and dual are unique, form $(i) \Rightarrow (ii)$, we have that the saddle point is unique.*

Definition 1.4.3 (Saddle point problem)**.** *If the primal problem constructed from $\Phi(v,p)$ is stable, then the saddle point problem consists in finding $(u,q^*) \in V \times Y^*$ such that*

$$L(u,q^*) = \sup_{p^*\in Y^*} \inf_{v\in V} \big[L(v,p^*)\big] = \inf_{v\in V} \sup_{p^*\in Y^*} \big[L(v,p^*)\big] \tag{1.35}$$

1.4.2 Application to linear elasticity problem

We start from recalling that in our case $\Phi(\underline{v},\underline{\underline{p}}) = F(\underline{v}) + G(\underline{\underline{\epsilon}}(v) + \underline{\underline{p}})$, where F is define in (1.17) and G is define in (1.18), so the Lagrangian becomes

$$-L(\underline{v},\underline{\underline{q}}^*) = \sup_{\underline{\underline{p}}\in L^2(\Omega)^{2\times 2}_{sym}} \Big[\int_\Omega \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{p}}\,\mathrm{dx} - F(\underline{u}) - G(\underline{\underline{\epsilon}}(v) + \underline{\underline{p}})\Big], \tag{1.36}$$

where $\mathcal{R} : (L^2(\Omega)^{2\times 2}_{sym})^* \to L^2(\Omega)^{2\times 2}_{sym}$ is the Riesz operator that we have introduced in the previous section.

Now we want to calculate the Lagrangian explicitly. Since G is a 2μ-convex functional and F is linear and continuous, then there exists a unique $q \in L^2(\Omega)^{2\times 2}_{sym}$ such that the supremum is realized, so if we take the Euler-Lagrange equations of

$$\int_\Omega \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{p}}\,\mathrm{dx} - F(\underline{u}) - G(\underline{\underline{\epsilon}}(v) + \underline{\underline{p}}),$$

evaluated in $\underline{\underline{q}}$ we obtain

$$\int_\Omega \mathcal{R}(p^*) : \underline{\underline{\varphi}} - 2\mu(\underline{\underline{\epsilon}}(v) + \underline{\underline{q}}) : \underline{\underline{\varphi}} - \lambda(\mathrm{div}(\underline{v}) + \mathrm{tr}(\underline{\underline{q}}))\underline{\underline{id}} : \underline{\underline{\varphi}}\,\mathrm{dx} = 0 \quad \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym}. \tag{1.37}$$

Now we proceed in the same way as done for duality: first we will try to characterize $L(\underline{v}, \underline{\underline{q}}^*)$ as a function of $\underline{\underline{q}}$ and then we will write $\underline{\underline{q}}$ as a function of $\underline{v}$ and $\mathcal{R}(p^*)$. We start from noticing that if we take $\underline{\underline{\varphi}} = \underline{\underline{\epsilon}}(v) + \underline{\underline{q}}$ in (1.37) then we obtain

$$\int_\Omega \mathcal{R}(p^*) : \underline{\underline{q}} \text{ dx} = 2G(\underline{\underline{\epsilon}}(v) + \underline{\underline{q}}) - \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(\underline{v}) \text{ dx}.$$

Substituting the previous equation in (1.36) we obtain

$$\begin{aligned}
-L(\underline{v}, \underline{\underline{q}}^*) &= \sup_{\underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym}} \Big[\int_\Omega \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{p}} \text{ dx} - F(\underline{u}) - G(\underline{\underline{\epsilon}}(v) + \underline{\underline{p}}) \Big] \\
&= \int_\Omega \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{q}} \text{ dx} - F(\underline{v}) - G(\underline{\underline{\epsilon}}(v) + \underline{\underline{q}}) \\
&= G(\underline{\underline{\epsilon}}(v) + \underline{\underline{q}}) - F(\underline{v}) - \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(\underline{v}) \text{ dx}. \qquad (1.38)
\end{aligned}$$

From the arbitrariness of $\underline{\underline{\varphi}}$ in (1.37) we must have that

$$\mathcal{R}(\underline{\underline{q}}^*) - 2\mu(\underline{\underline{\epsilon}}(v) + \underline{\underline{q}}) - \lambda(\text{div}(\underline{v}) + \text{tr}(\underline{\underline{q}}))\underline{\underline{id}} = 0.$$

If one proceed in the same way as done for duality, i.e. write $\mathcal{R}(\underline{\underline{q}}^*)$ as a function of $\underline{\underline{q}}$, where $\underline{\underline{\epsilon}}(v)$ is fixed, one could see that this function is invertible because the decomposition in deviatoric and trace components is unique, and would obtain that

$$\underline{\underline{q}} = \frac{1}{2\mu}\mathcal{R}(\underline{\underline{q}}^*)^D + \frac{1}{4(\lambda+\mu)} \text{tr}(\mathcal{R}(\underline{\underline{q}}^*))\underline{\underline{id}} - \underline{\underline{\epsilon}}(v).$$

Substituting the previous result in (1.38) we obtain

$$\begin{aligned}
L(\underline{v}, \underline{\underline{q}}^*) &= -G(\frac{1}{2\mu}\mathcal{R}(\underline{\underline{q}}^*)^D + \frac{1}{4(\lambda+\mu)} \text{tr}(\mathcal{R}(\underline{\underline{q}}^*))\underline{\underline{id}}) + F(\underline{v}) + \int_\Omega \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{\epsilon}}(\underline{v}) \text{ dx} \\
&= \int_\Omega -\frac{1}{4\mu}|\mathcal{R}(\underline{\underline{q}}^*)^D|^2 - \frac{1}{8(\mu+\lambda)}|\text{tr}(\mathcal{R}(\underline{\underline{q}}^*))|^2 - \underline{f} \cdot \underline{v} + \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{\epsilon}}(\underline{v}) \text{ dx}.
\end{aligned}$$

So the right-hand side of (1.35) becomes

$$\sup_{\underline{\underline{q}}^* \in (L^2(\Omega)^{2\times 2}_{sym})^*} \inf_{\underline{v} \in H^1_0(\Omega)^2} \Big\{ \int_\Omega -\frac{1}{4\mu}|\mathcal{R}(\underline{\underline{q}}^*)^D|^2 - \frac{1}{8(\mu+\lambda)}|\text{tr}(\mathcal{R}(\underline{\underline{q}}^*))|^2 - \underline{f} \cdot \underline{v} + \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{\epsilon}}(\underline{v}) \text{ dx} \Big\}.$$

Since the aim of the saddle point Problem 1.4.3 consists in finding the couple $(\underline{u}, \underline{\underline{\sigma}})$ and not $L(\underline{u}, \underline{\underline{\sigma}})$, we can change the sign of the previous equation and we obtain

$$\inf_{\underline{\underline{q}}^* \in (L^2(\Omega)^{2\times 2}_{sym})^*} \sup_{\underline{v} \in H^1_0(\Omega)^2} \Big\{ \int_\Omega \frac{1}{4\mu}|\mathcal{R}(\underline{\underline{q}}^*)^D|^2 + \frac{1}{8(\mu+\lambda)}|\text{tr}(\mathcal{R}(\underline{\underline{q}}^*))|^2 + \underline{f} \cdot \underline{v} - \mathcal{R}(\underline{\underline{q}}^*) : \underline{\underline{\epsilon}}(\underline{v}) \text{ dx} \Big\}.$$

Now recalling that $L^2(\Omega)^{2\times 2}_{sym}$ is an Hilbert space, so $\mathcal{R}$ is surjective, then, as done for the dual Problem 1.3.2 we can replace Y^* with Y and $\mathcal{R}(p^*)$ with $\underline{\underline{p}}$. Now we are ready to introduce the following.

Problem 1.4.1 (Saddle point formulation of linear elasticity)**.** *Find* $(\underline{w}, \underline{\underline{\tau}}) \in H_0^1(\Omega)^2 \times L^2(\Omega)^{2\times 2}_{sym}$ *such that*

$$\int_\Omega \frac{1}{4\mu}|\underline{\underline{\tau}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{\tau}})|^2 + \underline{f}\cdot\underline{w} - \underline{\underline{\tau}} : \underline{\underline{\epsilon}}(w)\ \mathrm{dx}$$
$$= \inf_{\underline{\underline{p}}\in L^2(\Omega)^{2\times 2}_{sym}} \sup_{\underline{v}\in H_0^1(\Omega)^2} \left\{ \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{p}})|^2 + \underline{f}\cdot\underline{v} - \underline{\underline{p}} : \underline{\underline{\epsilon}}(v)\ \mathrm{dx} \right\}. \quad (1.39)$$

Proposition 1.4.2. *Problem 1.4.1 admits a unique solution* $(\underline{u}, \underline{\underline{\sigma}})$*, where* $\underline{u}$ *is the solution of Problem 1.3.1 and* $\underline{\underline{\sigma}} = \mathcal{R}(q^*)$*, where* q^* *is solution of Problem 1.3.2.*

Proof. Since Problem 1.3.1 and Problem 1.3.2 admit a unique solution, then, according to Proposition 1.4.1, also 1.4.1 admits a unique solution. □

1.4.3 Characterization of saddle point

In this part we want to characterize the unique solution of Problem 1.4.1 through Euler-Lagrange equations:

$$\begin{cases} \int_\Omega \underline{f}\cdot\underline{\varphi} - \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(\varphi)\ \mathrm{dx} = 0 & \forall \underline{\varphi} \in H_0^1(\Omega)^2 \\ \int_\Omega \frac{1}{2\mu}\underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D + \frac{1}{4(\mu+\lambda)}\mathrm{tr}(\underline{\underline{\sigma}})\,\mathrm{tr}(\underline{\underline{\varphi}}) + \underline{\underline{\varphi}} \cdot \underline{\underline{\epsilon}}(u)\ \mathrm{dx} = 0 & \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym} \end{cases}. \quad (1.40)$$

According to (1.22) the first equation is equivalent to $\mathrm{div}(\underline{\underline{\sigma}}) + \underline{f} = 0$ in $H^{-1}(\Omega)^2$. Noticing that $\underline{\underline{\varphi}} = \underline{\underline{\varphi}}^D + \frac{1}{2}\mathrm{tr}(\underline{\underline{\varphi}})\underline{\underline{id}}$ from the second equation we have that

$$\int_\Omega \left(\frac{1}{2\mu}\underline{\underline{\sigma}}^D - \underline{\underline{\epsilon}}^D(u)\right) : \underline{\underline{\varphi}}^D + \left(\frac{1}{4(\lambda+\mu)}\mathrm{tr}(\sigma) - \frac{1}{2}\mathrm{tr}(\underline{\underline{\epsilon}}(u))\right)\mathrm{tr}(\underline{\underline{\varphi}})\ \mathrm{dx} = 0 \ \ \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym}. \quad (1.41)$$

Considering $\underline{\underline{\varphi}} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ with $\varphi_{12} = \varphi_{21} = 0$, $\varphi_{11} = \varphi_{22} = \frac{\varphi}{2}$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$, we have that $\underline{\underline{\varphi}}^D = 0$ and from (1.41) we obtain

$$\int_\Omega \left(\frac{1}{4(\lambda+\mu)}\mathrm{tr}(\underline{\underline{\sigma}}) - \frac{1}{2}\mathrm{tr}(\underline{\underline{\epsilon}}(u))\right)\varphi = 0 \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega),$$

then using Lemma A.2.1 we obtain

$$\frac{1}{4(\lambda+\mu)}\mathrm{tr}(\underline{\underline{\sigma}}) = \frac{1}{2}\mathrm{tr}(\underline{\underline{\epsilon}}(u)). \quad (1.42)$$

From (1.41) and (1.42) we obtain

$$\int_\Omega \left(\frac{1}{2\mu}\underline{\underline{\sigma}}^D - \underline{\underline{\epsilon}}^D(u)\right) : \underline{\underline{\varphi}}^D\ \mathrm{dx} = 0 \quad \forall \underline{\underline{\varphi}} \in L^2(\Omega)^{2\times 2}_{sym}$$

If we proceed in the exact same way as done before but with $\varphi_{12} = \varphi_{21} = \frac{\varphi}{2}$ and $\varphi_{11} = \varphi_{22} = 0$, $\varphi_{12} = \varphi_{21} = \varphi_{22} = 0$ and $\varphi_{11} = \varphi$, $\varphi_{12} = \varphi_{21} = \varphi_{11} = 0$ and $\varphi_{22} = \varphi$ with $\varphi \in \mathcal{C}_c^\infty(\Omega)$ we get

$$\frac{1}{2\mu}\underline{\underline{\sigma}}^D - \underline{\underline{\epsilon}}^D(u) = 0.$$

So the saddle point is characterize by the following equations.

$$\begin{cases} \operatorname{div}(\underline{\underline{\sigma}}) + \underline{f} = 0 \\ \underline{\underline{\sigma}}^D = 2\mu\underline{\underline{\epsilon}}^D(u) \\ \operatorname{tr}(\underline{\underline{\sigma}}) = 2(\lambda+\mu)\operatorname{tr}(\underline{\underline{\epsilon}}(u)) \end{cases}. \tag{1.43}$$

According to Proposition 1.4.2 and (1.43) we can conclude that $\underline{\underline{\sigma}}$ is the stress tensor define in (1.7).

1.5 Mixed Problem Brezzi

In this section we want to introduce the mixed problem for linear elasticity developed by Brezzi in [4]. The difference form saddle point Problem 1.4.1 consists in changing the functional spaces in (1.39).

1.5.1 Mixed Problem formulation

First of all we recall that the solutions of primal Problem 1.3.1 and dual Problem 1.3.2 are unique and that the solution of saddle point Problem 1.4.1 is the combination of those two. From (1.43) we must have $\operatorname{div}(\underline{\underline{\sigma}}) + \underline{f} = 0$, so, since $\underline{f} \in L^2(\Omega)^2$ we can deduce that $\operatorname{div}(\underline{\underline{\sigma}}) \in L^2(\Omega)^2$, thus $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\operatorname{div}, \Omega)$. So in (1.39) we can restrict the domain of $\underline{\underline{p}}$ form $L^2(\Omega)^{2\times 2}_{sym}$ to $\underline{\underline{H}}(\operatorname{div}, \Omega)_{sym}$ and still have guarantee the uniqueness of solution which is the same of Problem 1.4.1. Thus we can rewrite (1.39) as

$$\inf_{\substack{\underline{\underline{p}} \in \underline{\underline{H}}(\operatorname{div},\Omega)_{sym} \\ \operatorname{div}(\underline{\underline{p}})+\underline{f}=0}} \left\{ \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\operatorname{tr}(\underline{\underline{p}})|^2 \,\mathrm{dx} \right\}. \tag{1.44}$$

Now we want to introduce the constrain $\operatorname{div}(\underline{\underline{p}}) + \underline{f} = 0$ in the equation, but since both $\operatorname{div}(\underline{\underline{p}})$ and $\underline{f}$ are in $L^2(\Omega)^2$, we can use as test space $L^2(\Omega)^2$ instead of $H^1_0(\Omega)^2$, so we obtain

$$\inf_{\underline{\underline{p}} \in \underline{\underline{H}}(\operatorname{div},\Omega)_{sym}} \sup_{\underline{v} \in L^2(\Omega)^2} \left\{ \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\operatorname{tr}(\underline{\underline{p}})|^2 + (\operatorname{div}(\underline{\underline{p}}) + \underline{f}) \cdot \underline{v} \,\mathrm{dx} \right\}. \tag{1.45}$$

The formulation above is the one given by Brezzi in [4]. The difference between (1.39) and (1.45) is that we have restrict the space where $\underline{\underline{p}}$ lives, but enlarge the space where $\underline{v}$ lives. Now we are ready to introduce the mixed problem.

Problem 1.5.1 (Mixed formulation of linear elasticity). *Find* $(\underline{w}, \underline{\underline{\tau}}) \in L^2(\Omega)^2 \times \underline{\underline{H}}(\operatorname{div}, \Omega)_{sym}$

such that

$$\int_\Omega \frac{1}{4\mu}|\underline{\underline{\tau}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{\tau}})|^2 + \underline{f}\cdot\underline{w} + \mathrm{div}(\underline{\underline{\tau}})\cdot\underline{w}\ \mathrm{dx}$$
$$= \inf_{\underline{\underline{p}}\in\underline{\underline{H}}(\mathrm{div},\Omega)_{sym}} \sup_{\underline{v}\in L^2(\Omega)^2} \left\{ \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{p}})|^2 + \underline{f}\cdot\underline{v} + \mathrm{div}(\underline{\underline{p}})\cdot\underline{v}\ \mathrm{dx} \right\}. \tag{1.46}$$

1.5.2 Existence and uniqueness of saddle point for the mixed problem

The aim of this subsection is to show that $(\underline{u}, \underline{\underline{\sigma}})$, unique saddle point of Problem 1.4.1, is also the unique saddle point of Problem 1.5.1, in the sense of Definition 1.4.2.

Proposition 1.5.1. *If $(\underline{w}, \underline{\underline{\tau}})$ is a saddle point of Problem 1.5.1 then $\underline{w} \in H^1_0(\Omega)^2$.*

Proof. Let $(\underline{w}, \underline{\underline{\tau}})$ be a saddle point of Problem 1.5.1, then if we take the Euler-Lagrange equations of the right-hand side of (1.46) evaluated in $(\underline{w}, \underline{\underline{\tau}})$ we obtain

$$\begin{cases} \int_\Omega \mathrm{div}(\underline{\underline{\tau}})\cdot\underline{\varphi} + \underline{f}\cdot\underline{\varphi}\ \mathrm{dx} = 0 & \forall \underline{\varphi}\in L^2(\Omega)^2 \\ \int_\Omega \frac{1}{2\mu}\underline{\underline{\tau}}^D : \underline{\underline{\varphi}}^D + \frac{1}{4(\mu+\lambda)}\mathrm{tr}(\underline{\underline{\tau}})\,\mathrm{tr}(\underline{\underline{\varphi}}) + \mathrm{div}(\underline{\underline{\varphi}})\cdot\underline{w}\ \mathrm{dx} = 0 & \forall \underline{\underline{\varphi}}\in \underline{\underline{H}}(\mathrm{div},\Omega)_{sym}. \end{cases} \tag{1.47}$$

We can rewrite the second equation of the previous system in the following way:

$$\int_\Omega \left(\frac{1}{2\mu}\underline{\underline{\tau}}^D + \frac{1}{4(\mu+\lambda)}\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{id}}\right) : \underline{\underline{\varphi}}\ \mathrm{dx} = -\int_\Omega \mathrm{div}(\underline{\underline{\varphi}})\cdot\underline{\tilde{w}}\ \mathrm{dx} \quad \forall \underline{\underline{\varphi}}\in \underline{\underline{H}}(\mathrm{div},\Omega)_{sym}. \tag{1.48}$$

Since $\mathcal{C}_c^\infty(\Omega)^{2\times 2}_{sym}$ are contained in $\underline{\underline{H}}(\mathrm{div},\Omega)_{sym}$, as [16] suggests, we obtain that $\underline{w}$ admits symmetric gradient $\underline{\underline{\epsilon}}(w)$ and

$$\underline{\underline{\epsilon}}(w) = \frac{1}{2\mu}\underline{\underline{\tau}}^D + \frac{1}{4(\mu+\lambda)}\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{id}}, \tag{1.49}$$

which implies that $\underline{w} \in \underline{H}(sym,\Omega)$, which is a Banach space. Now we want to show that if $\underline{w} \in \underline{H}(sym,\Omega)$, then $\underline{w} \in H^1(\Omega)^2$. Since $\mathcal{C}^\infty(\overline{\Omega})^2$ is dense in $\underline{H}(sym,\Omega)$, then we can take $\{\underline{\varphi}_n\}_{n\in\mathbb{N}} \in \mathcal{C}^\infty(\overline{\Omega})^2$ such that $\underline{\varphi}_n \to \underline{w}$ in $\underline{H}(sym,\Omega)$, that means:

$$\begin{aligned} &\underline{\varphi}_n \longrightarrow \underline{w} && \text{in } L^2(\Omega)^2 \\ &\underline{\underline{\epsilon}}(\varphi_n) \longrightarrow \underline{\underline{\epsilon}}(w) && \text{in } L^2(\Omega)^{2\times 2}_{sym}. \end{aligned}$$

Using Korn inequality (A.5) for $H^1(\Omega)^2$ we have that

$$\left\|\underline{\varphi}_n\right\|_{H^1(\Omega)^2} \le \mathcal{K}\big(\left\|(\underline{\varphi}_n)\right\|_{L^2(\Omega)^2} + \left\|\underline{\underline{\epsilon}}(\varphi_n)\right\|_{L^2(\Omega)^{2\times 2}_{sym}}\big),$$

which implies that $\{\underline{\varphi}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)^2$. Since this space is complete there exists an element $\underline{\varphi} \in H^1(\Omega)^2$ such that $\underline{\varphi}_n \to \underline{\varphi}$ in $H^1(\Omega)^2$. We can notice that for every

$\underline{u} \in H^1(\Omega)^2$ we have

$$\left\|\underline{\underline{\epsilon}}(u)\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}} \coloneqq \|\partial_{x_1} u_1\|^2_{L^2(\Omega)} + \|\partial_{x_2} u_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_{x_1} u_2 + \partial_{x_2} u_1\|^2_{L^2(\Omega)}$$
$$\leq \|\partial_{x_1} u_1\|^2_{L^2(\Omega)} + \|\partial_{x_2} u_2\|^2_{L^2(\Omega)} + \|\partial_{x_1} u_2\|^2_{L^2(\Omega)} + \|\partial_{x_2} u_1\|^2_{L^2(\Omega)} \eqqcolon \|\nabla(\underline{u})\|^2_{L^2(\Omega)^{2\times 2}},$$

so we can deduce that

$$\|\underline{u}\|_{\underline{H}(sym,\Omega)} \coloneqq \|\underline{u}\|^2_{L^2(\Omega)^2} + \left\|\underline{\underline{\epsilon}}(\underline{u})\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}} \leq \|\underline{u}\|^2_{L^2(\Omega)^2} + \|\nabla(\underline{u})\|^2_{L^2(\Omega)^{2\times 2}} \eqqcolon \|\underline{u}\|_{H^1(\Omega)^2}.$$

Thus $H^1(\Omega)^2$ immerses with continuity in $\underline{H}(sym,\Omega)$ and this implies that $\underline{\varphi} = \underline{w}$, so $\underline{w}$ is an element of $H^1(\Omega)^2$. Now we have to show that $\underline{w} \in H^1_0(\Omega)^2$. According to (1.48), (1.49) and since $\mathcal{C}^\infty(\overline{\Omega})^{2\times 2}_{sym}$ are dense in $\underline{\underline{H}}(\mathrm{div},\Omega)_{sym}$ we have:

$$0 = \int_\Omega \underline{\underline{\epsilon}}(\underline{w}) : \underline{\underline{\varphi}} \,\mathrm{dx} + \int_\Omega \mathrm{div}(\underline{\underline{\varphi}}) \cdot \underline{w} \,\mathrm{dx} = \int_{\partial\Omega} \mathrm{Tr}(\underline{w}) \cdot \underline{\varphi_\nu} \, d\mathcal{H}^1 \qquad \forall \underline{\underline{\varphi}} \in \mathcal{C}^\infty(\overline{\Omega})^{2\times 2}_{sym},$$

where $\underline{\varphi_{\nu}}_i = \sum_{j=1}^2 \varphi_{ij}\nu_j$, where $\underline{\nu}$ is the normal unit vector of $\partial\Omega$. Then according to Theorems A.2, Theorem A.3 and since $\underline{\underline{\varphi}}$ is arbitrary, we can conclude $\underline{w} \in H^1_0(\Omega)^2$. □

Proposition 1.5.2. *If $(\underline{w}, \underline{\underline{\tau}})$ is a saddle point of Problem 1.5.1, then it is also a saddle point of Problem 1.4.1.*

Proof. If $(\underline{w}, \underline{\underline{\tau}})$ is a saddle point of Problem 1.5.1, then if we define

$$F(\underline{v}, \underline{\underline{p}}) \coloneqq \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{p}})|^2 + \underline{f}\cdot\underline{v} + \mathrm{div}(\underline{\underline{p}})\cdot\underline{v} \,\mathrm{dx}, \tag{1.50}$$

according to Definition 1.4.2 of saddle point we have that

$$\forall \underline{v} \in L^2(\Omega)^2 \quad F(\underline{v}, \underline{\underline{\tau}}) \leq F(\underline{w}, \underline{\underline{\tau}}) \leq F(\underline{w}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in \underline{\underline{H}}(\mathrm{div},\Omega)_{sym}. \tag{1.51}$$

If $(\underline{u}, \underline{\underline{\sigma}})$ is a saddle point of Problem 1.4.1, then if we define

$$G(\underline{v}, \underline{\underline{p}}) \coloneqq \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{p}})|^2 + \underline{f}\cdot\underline{v} - \underline{\underline{p}} : \underline{\underline{\epsilon}}(v) \,\mathrm{dx}, \tag{1.52}$$

we have that

$$\forall \underline{v} \in H^1_0(\Omega)^2 \quad G(\underline{v}, \underline{\underline{\sigma}}) \leq G(\underline{u}, \underline{\underline{\sigma}}) \leq G(\underline{u}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym}. \tag{1.53}$$

Now we want to show that $(\underline{w}, \underline{\underline{\tau}})$, saddle point of Problem 1.5.1, is also a saddle point of Problem 1.4.1. According to Proposition 1.5.3 we have that $\underline{w} \in H^1_0(\Omega)^2$ and so $(\underline{w}, \underline{\underline{\tau}}) \in H^1_0(\Omega)^2 \times L^2(\Omega)^{2\times 2}_{sym}$. Thus $(\underline{w}, \underline{\underline{\tau}})$ is an admissible saddle point for Problem 1.4.1. If we show

$$\forall \underline{v} \in H^1_0(\Omega)^2 \quad G(\underline{v}, \underline{\underline{\tau}}) \leq G(\underline{w}, \underline{\underline{\tau}}) \leq G(\underline{w}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym},$$

then we obtain that $(\underline{w}, \underline{\underline{\tau}})$ is a saddle point of Problem 1.4.1. For every $\underline{v} \in H^1_0(\Omega)^2$ $\underline{\underline{p}} \in$

$\underline{\underline{H}}(\text{div}, \Omega)_{sym}$ if we integrate by parts (1.50) we obtain

$$F(\underline{v}, \underline{\underline{p}}) := \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\text{tr}(\underline{\underline{p}})|^2 + \underline{f} \cdot \underline{w} + \text{div}(\underline{\underline{p}}) \cdot \underline{v} \text{ dx}$$
$$= \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\text{tr}(\underline{\underline{p}})|^2 + \underline{f} \cdot \underline{w} - \underline{\underline{p}} : \underline{\underline{\epsilon}}(v) \text{ dx} =: G(\underline{v}, \underline{\underline{p}}). \quad (1.54)$$

Since $H_0^1(\Omega)^2 \subseteq L^2(\Omega)^2$ and according to Proposition 1.5.3 we have that $\underline{w} \in H_0^1(\Omega)^2$, from (1.54) and (1.51) we have that

$$G(\underline{w}, \underline{\underline{\tau}}) = F(\underline{w}, \underline{\underline{\tau}}) \geq F(\underline{v}, \underline{\underline{\tau}}) = G(\underline{v}, \underline{\underline{\tau}}) \quad \forall \underline{v} \in H_0^1(\Omega)^2.$$

According to Proposition 1.5.3 we have that $\underline{w} \in H_0^1(\Omega)^2$ and from (1.54) we can deduce that

$$G(\underline{w}, \underline{\underline{\tau}}) = F(\underline{w}, \underline{\underline{\tau}}) \leq F(\underline{w}, \underline{\underline{p}}) = G(\underline{w}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in \underline{\underline{H}}(\text{div}, \Omega)_{sym}.$$

Since $\underline{\underline{p}} \to G(\underline{w}, \underline{\underline{p}})$ is continuous in $L^2(\Omega)^{2\times 2}_{sym}$ and since $\mathcal{C}^\infty(\overline{\Omega})^{2\times 2}_{sym}$ is dense in $\underline{\underline{H}}(\text{div}, \Omega)_{sym}$, then $\underline{\underline{H}}(\text{div}, \Omega)_{sym}$ is dense in $L^2(\Omega)^{2\times 2}_{sym}$. So we obtain

$$G(\underline{w}, \underline{\underline{\tau}}) \leq G(\underline{w}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in L^2(\Omega)^{2\times 2}_{sym}.$$

Thus $(\underline{w}, \underline{\underline{\tau}})$ is a saddle point of Problem 1.4.1. □

Proposition 1.5.3. *If* $(\underline{u}, \underline{\underline{\sigma}})$ *the unique saddle point of Problem 1.4.1 is also a saddle point of Problem 1.5.1.*

Proof. According to what we have said in Subsection 1.4.3 we have that $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\text{div}, \Omega)_{sym}$, so $(\underline{u}, \underline{\underline{\sigma}})$ is an admissible saddle point for Problem 1.5.1. In order to prove the thesis we need to show that $(\underline{u}, \underline{\underline{\sigma}})$ satisfies (1.51). Since $(\underline{u}, \underline{\underline{\sigma}})$ is a saddle point for Problem 1.4.1 then it satisfies (1.53), and because $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\text{div}, \Omega)_{sym}$, due to (1.54) we have

$$F(\underline{u}, \underline{\underline{\sigma}}) = G(\underline{u}, \underline{\underline{\sigma}}),$$

so, since $\underline{\underline{H}}(\text{div}, \Omega)_{sym} \subseteq L^2(\Omega)^{2\times 2}_{sym}$, from (1.54) we obtain

$$F(\underline{u}, \underline{\underline{\sigma}}) = G(\underline{u}, \underline{\underline{\sigma}}) \leq G(\underline{u}, \underline{\underline{p}}) = F(\underline{u}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in \underline{\underline{H}}(\text{div}, \Omega)_{sym}.$$

From (1.53) and (1.54) we obtain

$$F(\underline{u}, \underline{\underline{\sigma}}) = G(\underline{u}, \underline{\underline{\sigma}}) \geq G(\underline{v}, \underline{\underline{\sigma}}) = F(\underline{v}, \underline{\underline{\sigma}}) \quad \forall \underline{v} \in H_0^1(\Omega)^2.$$

Since $\underline{v} \to F(\underline{v}, \underline{\underline{\sigma}})$ is continuous in $L^2(\Omega)^2$ and $H_0^1(\Omega)^2$ is dense in $L^2(\Omega)^2$ we obtain

$$F(\underline{u}, \underline{\underline{\sigma}}) = G(\underline{u}, \underline{\underline{\sigma}}) \geq G(\underline{v}, \underline{\underline{\sigma}}) = F(\underline{v}, \underline{\underline{\sigma}}) \quad \forall \underline{v} \in L^2(\Omega)^2.$$

Thus $(\underline{u}, \underline{\underline{\sigma}})$ is a saddle point of Problem 1.5.1. □

Theorem 1.2. *Problem 1.5.1 admits a unique saddle point which is the same one of Problem*

1.4.1.

Proof. From Proposition 1.5.3 we have guarantee that there exists a saddle point for Problem 1.5.1, then from Proposition 1.5.2 we have that all the saddle points of Problem 1.5.1 are also saddle points of Problem 1.4.1. Since Problem 1.4.1 admits a unique saddle point we have that Problem 1.5.1 must have a unique saddle points that coincide to the one of Problem 1.4.1. □

Chapter 2

Dynamic linear elasticity and viscoelasticity

This chapter aims at the derivation and the analysis of a mixed formulation for the evolution equation of linear elasticity and viscoelasticity. It is divided into four sections in each of which we will analyse a different topic: elastic waves, linear elasticity, elastic waves with dissipation and linear viscoelasticity. In each section, following the theory developed in the Sections 1.3 and 1.4, we will present mixed formulations similar to the one obtained in the previous chapter when we have introduced the Problem 1.4.1. Only for elastic waves with dissipation and linear viscoelasticity we will be able to introduce another mixed formulation similar to the one introduced with Problem 1.5.1 and this will be possible thanks to the dissipative term that will allow the solution of the primal to have a greater regularity.

2.1 One dimensional elastic waves

2.1.1 Starting problem

The aim of this section is to introduce a mixed formulation for one dimensional elastic waves. We will begin by stating the starting problem and then will show that it admits a unique solution. After that, according to the theory developed in Section 1.3 and Section 1.4, we will introduce a mixed formulation for elastic waves. Finally we will show that the mixed problem has an unique solution that coincides with the one of the starting problem.

Remark 2.1.1. *One of the main differences in working in one dimension respect to higher ones is that the gradient and the divergence operator coincide, since we want to make the transition to multidimensional easier we will represent differently this two operators. We will denote with* $\cdot'$ *the spatial derivative and with* $\frac{\mathrm{d}}{\mathrm{dx}}$ *the divergence operator.*

Assumption 2.1.1. *Let* $\Omega \subset \mathbb{R}$ *be a bounded and Lipschitz domain,* $T > 0$*,* $u_0 \in H_0^1(\Omega)$*,* $u_1 \in L^2(\Omega)$*,* $f \in L^2(0, T; L^2(\Omega)^*)$ *and* $\rho, k \in L^\infty(\Omega)$ *such that* $0 < A \leq k, \rho \leq B < \infty$ *a.e. in* Ω*.*

For k and Ω satisfying Assumption 2.1.1, we introduce the symmetric bilinear form a :

$H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$:

$$a(u,v) := \int_\Omega k u' v' \text{ dx}. \tag{2.1}$$

Note that by classical Poincaré Inequality a is in particular positive definite.

We consider the following.

Problem 2.1.1 (Starting Problem one dimensional elastic waves)**.** *Find u such that*

$$u \in L^2(0,T;H^1_0(\Omega)) \quad u_t \in L^2(0,T;L^2(\Omega)) \quad \rho u_{tt} \in L^2(0,T;H^{-1}(\Omega)),$$

and satisfies

$$\begin{cases} \int_0^T \langle \rho u_{tt}, \varphi\rangle_{H^1_0(\Omega)} + a(u,\varphi) \text{ dt} = \int_0^T \langle f, \varphi\rangle_{L^2(\Omega)} \text{ dt} & \forall \varphi \in L^2(0,T;H^1_0(\Omega)) \\ u(\cdot,0) = u_0 & \textit{in } \Omega \\ u_t(\cdot,0) = u_1 & \textit{in } \Omega. \end{cases} \tag{2.2}$$

Following the same ideas of [9, Thm. 1, 2, and 3 in Sec. 7.2], we obtain the following result.

Theorem 2.1 (Existence and uniqueness of solution)**.** *Let $f \in L^2(0,T;L^2(\Omega)^*)$, $u_0 \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$. There exists a unique function u such that*

$$u \in L^2(0,T;H^1_0(\Omega)) \quad u_t \in L^2(0,T;L^2(\Omega)) \quad \rho u_{tt} \in L^2(0,T;H^{-1}(\Omega)),$$

and satisfies

$$\begin{cases} \int_0^T \langle \rho u_{tt}, \varphi\rangle_{H^1_0(\Omega)} + a(u,\varphi) \text{ dt} = \int_0^T \langle f, \varphi\rangle_{L^2(\Omega)} \text{ dt} & \forall \varphi \in L^2(0,T;H^1_0(\Omega)) \\ u(\cdot,0) = u_0 \\ u_t(\cdot,0) = u_1. \end{cases} \tag{2.3}$$

Sketch of the proof. The result can be proved following the proof of Theorem 2.3 with minor changes. □

Definition 2.1.1. *We will denote by $\bar{u}$ the unique solution of Problem 2.1.1.*

2.1.2 Construction of mixed formulation

In order to apply the theory that we have developed in Section 1.3 and Section 1.4 we have to give a variational characterization of $\bar{u}$.

Let us introduce the functional

$$\mathcal{S}(u) = \int_0^T \langle \rho \bar{u}_{tt}, u\rangle_{H^1_0(\Omega)} + \frac{1}{2} a(u,u) - \langle f, u\rangle_{L^2(\Omega)} \text{ dt}, \tag{2.4}$$

acting on

$$\Gamma = \left\{ u \;\middle|\; u(x,t) \in L^2(0,T;H^1_0(\Omega)) \right\}.$$

Remark 2.1.2. *We warn the reader that the functional $\mathcal{S}$ depends on the function $\bar{u}$, the solution of Problem 2.1.1. In the following we will consider the minimisation of the functional $\mathcal{S}$ and recover $\bar{u}$ itself as solution. This procedure, that apparently is running in circles, would not concern the reader. Indeed, we remark that our final aim is to construct a saddle point formulation of Problem 2.1.1, and we can safely use its unique solution as a parameter in the intermediate steps. We will use this type of construction a number of times in the next sections, the present remark holding true for all the considered instances.*

Problem 2.1.2 (Primal formulation of one dimensional elastic waves)**.** *Find $u \in \Gamma$ such that*

$$\mathcal{S}(u) = \inf_{v\in\Gamma}\{\mathcal{S}(v)\}. \tag{2.5}$$

Proposition 2.1.1. *The function $\bar{u}$ is the unique minimiser of Problem 2.1.2.*

Proof. Since the solutions of Problem 2.1.1 live in a subspace of Γ, then we have that $\bar{u} \in \Gamma$. Computing the first variation of (2.4) evaluated in $\bar{u}$, we obtain

$$d\mathcal{S}(\bar{u}).y = \int_0^T \langle \rho\bar{u}_{tt}, y\rangle_{H_0^1(\Omega)} + a(\bar{u}, y) - \langle f, y\rangle_{L^2(\Omega)} \text{ dt} = 0 \quad \forall \underline{y} \in \Gamma,$$

where we have used that $\bar{u}$ satisfies the first equation of (2.2). Thus $\bar{u}$ is a critical point of $\mathcal{S}$. Now we have to prove that $\mathcal{S}$ is strongly convex in Γ, if that happens, then $\bar{u}$ is the unique minimiser. This is equivalent to prove that the second variation of $\mathcal{S}$ is positive define for all $u \in \Gamma$:

$$\begin{aligned} d^2\mathcal{S}(u).(y,y) = a(y,y) = \int_0^T \int_\Omega k|y'|^2 \text{ dx dt} \\ \geq \int_0^T A\|y\|^2_{H_0^1(\Omega)} \text{ dt} \geq A\|y\|^2_{L^2(0,T;H_0^1(\Omega))} \qquad \forall y \in \Gamma. \end{aligned}$$

□

In order to introduce the dual and the saddle point formulation for Problem 2.1.2 it is convenient to rewrite (2.5) as follows:

$$\mathcal{S}(\bar{u}) = \inf_{v\in\Gamma}\{\mathcal{S}(v)\} = \inf_{v\in\Gamma}\big\{F(v) + G(v')\big\},$$

where $F : \Gamma \to \mathbb{R}$ is

$$F(v) = \int_0^T \langle \rho\bar{u}_{tt}, v\rangle_{H_0^1(\Omega)^2} - \langle f, v\rangle_{L^2(\Omega)} \text{ dt}, \tag{2.6}$$

and $G : Y := L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ is

$$G(p) = \int_0^T \int_\Omega k|p|^2 \text{ dx dt}. \tag{2.7}$$

Now we introduce the dual problem and show that it admits a unique solution. Starting from Problem 2.1.2, if we define $\Phi(u,p) = F(u) + G(u' + p)$, then, according to Definition 1.3.2, we have the following.

Problem 2.1.3 (Dual formulation of one dimensional elastic waves)**.** *Find* $q^* \in Y^*$ *such that*

$$-\Phi^*(0,q^*) = \sup_{p^*\in Y^*}\big[-\Phi^*(0,p^*)\big] = \sup_{p^*\in Y^*}\big[-F^*(-\Lambda^* p^*) - G^*(p^*)\big], \tag{2.8}$$

where Λ^* *is* $\frac{\mathrm{d}}{\mathrm{dx}}$*, the adjoint of* $\cdot'$*, according to Definition A.1.8* $F^* : \Gamma^* \to \mathbb{R}$ *is*

$$F^*(-\frac{\mathrm{d}}{\mathrm{dx}}p^*) = \begin{cases} 0 & \text{if } -\frac{\mathrm{d}}{\mathrm{dx}}p^* - \rho\bar{u}_{tt} + f = 0 \\ +\infty & \text{otherwise} \end{cases},$$

and $G^* : Y^* \to \mathbb{R}$ *is*

$$G^*(p^*) = \int_0^T \int_\Omega \frac{1}{4k}|\mathcal{R}(p^*)|^2 \ \mathrm{dx}\ \mathrm{dt}$$

where $\mathcal{R}$ *is the Riesz operator such that* $\mathcal{R} : Y^* \to Y$*.*

Proposition 2.1.2. *Problem 2.1.2 is stable, in the sense of Definition 1.3.4. Moreover Problem 2.1.3 admits a unique solution.*

Proof. In order to prove stability we verify the hypothesis of Theorem 1.1. Since F, defined in (2.6), is linear and G, defined in (2.7), is strictly convex, then $J(u,p) = F(u) + G(p)$ is convex in $\Gamma \times Y$:

$$\begin{aligned} J(\alpha(u,p) + (1-\alpha)(v,q)) &= F(\alpha u + (1-\alpha)v) + G(\alpha p + (1-\alpha)q) \\ &= \alpha F(u) + (1-\alpha)F(v) + G(\alpha p + (1-\alpha)q) \\ &\le \alpha F(u) + (1-\alpha)F(v) + \alpha G(p) + (1-\alpha)G(q) \\ &\le \alpha J(u,p) + (1-\alpha)J(v,q), \end{aligned}$$

for all $\alpha \in [0,1]$, $u, v \in \Gamma$ and $p, q \in Y$.
Clearly for $u_0 = 0$ we have that $F(0) = 0$ and $G(0') = 0$, so we need to verify that $p \to F(u_0) + G(p)$ is continuos at 0. Let $p_n \to p$ in Y as $n \to \infty$, then, since $k \in L^\infty(\Omega)$ and $\|p\|_Y^2 := \int_0^T \int_\Omega |p|^2 \ \mathrm{dx}\ \mathrm{dt}$ we have

$$\lim_{n\to\infty} F(u_0) + G(p_n) = \lim_{n\to\infty} \int_0^T \int_\Omega k|p_n|^2 \ \mathrm{dx}\ \mathrm{dt} = \int_0^T \int_\Omega k|p|^2 \ \mathrm{dx}\ \mathrm{dt} = F(u_0) + G(p).$$

So, Problem 2.1.2 is stable and due to Proposition 1.3.6, Problem 2.1.3 has at least one solution. Can be noticed that G is differentiable and the differential of G in p evaluated in q is

$$G'[p](q) = \int_0^T \int_\Omega 2kpq \ \mathrm{dx}\ \mathrm{dt}.$$

Since G is differentiable and the primal Problem 2.1.2 admits a unique solution, according to Proposition 1.3.7 and Proposition A.1.7, the dual Problem 2.1.3 admits a unique solution. □

Before introducing the saddle point problem, according to the theory developed in Section 1.4, we need to study the Lagrangian defined in Definition 1.4.1:

$$-L(u,p^*) = \sup_{p\in Y}\big[\langle p^*, p\rangle_Y - F(u) - G(u' + p)\big]. \tag{2.9}$$

Since F, defined in (2.6), is linear and G, defined in (2.7) is strictly convex we have that the argument of the supremum in (2.9) is a strictly concave function. Thus there exists a unique $q \in Y$ such that

$$\langle p^*, q\rangle_Y - F(u) - G(u' + q) = \sup_{p\in Y} \big[\langle p^*, p\rangle_Y - F(u) - G(u' + p)\big].$$

Therefore the first variation respect to p evaluated in q of $\langle p^*, p\rangle_Y - F(u) - G(u'+p)$ must vanish, that is

$$\int_0^T \int_\Omega \mathcal{R}(p^*)\varphi - 2k(q + u')\varphi \text{ dx dt} = 0 \quad \forall \varphi \in Y, \tag{2.10}$$

where $\mathcal{R} : Y^* \to Y$ is the Riesz operator defined in Theorem A.1 such that

$$\langle p^*, \varphi\rangle_Y := \int_0^T \int_\Omega \mathcal{R}(p^*)\varphi \text{ dx dt} \quad \forall \varphi \in Y.$$

Taking $\varphi = q + u'$ in (2.10) we obtain

$$\langle p^*, q\rangle_Y = 2G(u' + q) - \int_0^T \int_\Omega \mathcal{R}(p^*)u' \text{ dx dt}.$$

Then from (2.9) and the previous equation we can deduce that

$$\begin{aligned} -L(u, p^*) &= \sup_{p\in Y} \big[\langle p^*, p\rangle_Y - F(u) - G(u' + p)\big] \\ &= \langle p^*, q\rangle_Y - F(u) - G(u' + q) \\ &= G(u' + q) - F(u) - \int_0^T \int_\Omega \mathcal{R}(p^*)u' \text{ dx dt}. \end{aligned} \tag{2.11}$$

Since (2.10) is true for all φ in Y, then we have

$$\mathcal{R}(p^*) = 2k(q + u'),$$

and we can deduce that

$$q = \frac{1}{2k}\mathcal{R}(p^*) - u'. \tag{2.12}$$

Substituting (2.12) in (2.11) we obtain

$$-L(u, p^*) = \int_0^T \int_\Omega \frac{1}{4k}|\mathcal{R}(p^*)|^2 - \mathcal{R}(p^*)u' \text{ dx} - \langle \rho\bar{u}_{tt}, u\rangle_{H_0^1(\Omega)} + \langle f, u\rangle_{L^2(\Omega)} \text{ dt}. \tag{2.13}$$

Since the right-hand side of the last equation depends on p^* only through the Riesz operator $\mathcal{R}$

and Y is an Hilbert space, we can replace Y^* with Y and $\mathcal{R}(p^*)$ with p, and we obtain

$$\inf_{p^*\in Y^*}\sup_{u\in\Gamma}\left\{\int_0^T\int_\Omega \frac{1}{4k}|\mathcal{R}(p^*)|^2-\mathcal{R}(p^*)u'\ \mathrm{dx}-\langle\rho\bar{u}_{tt},u\rangle_{H_0^1(\Omega)}+\langle f,u\rangle_{L^2(\Omega)}\ \mathrm{dt}\right\}$$
$$=\inf_{p\in Y}\sup_{u\in\Gamma}\left\{\int_0^T\int_\Omega \frac{1}{4k}|p|^2-pu'\ \mathrm{dx}-\langle\rho\bar{u}_{tt},u\rangle_{H_0^1(\Omega)}+\langle f,u\rangle_{L^2(\Omega)}\ \mathrm{dt}\right\}.$$

Thus, from the previous equation and according to Definition 1.4.3 of saddle point problem, we arrive to consider the following.

Problem 2.1.4 (Saddle point formulation of one dimensional elastic waves)**.** *Find $(w,\tau)\in\Gamma\times Y$ such that*

$$\int_0^T\int_\Omega \frac{1}{4k}|\tau|^2-\tau w'\ \mathrm{dx}-\langle\rho\bar{u}_{tt},w\rangle_{H_0^1(\Omega)}+\langle f,w\rangle_{L^2(\Omega)}\ \mathrm{dt}$$
$$=\inf_{p\in Y}\sup_{u\in\Gamma}\left\{\int_0^T\int_\Omega \frac{1}{4k}|p|^2-pu'\ \mathrm{dx}-\langle\rho\bar{u}_{tt},u\rangle_{H_0^1(\Omega)}+\langle f,u\rangle_{L^2(\Omega)}\ \mathrm{dt}\right\}. \quad (2.14)$$

Proposition 2.1.3. *Problem 2.1.4 admits a unique solution $(\bar{u},\sigma)$ where $\bar{u}$ is the unique solution of Problem 2.1.2 and $\sigma=\mathcal{R}(q^*)$, where q^* is the unique solution of Problem 2.1.3.*

Proof. According to Proposition 1.4.1 we have that the solutions of Problem 2.1.4 are the product of the solutions of Problem 2.1.2 and Problem 2.1.3. Since Problem 2.1.2 and Problem 2.1.3 admit unique solution $\bar{u}$ and $\sigma=\mathcal{R}(q^*)$, where q^* is the unique solution of Problem 2.1.3, then we have that Problem 2.1.4 admits a unique solution that is $(\bar{u},\sigma)$. □

If we compute the Euler-Lagrange equation of (2.14) evaluated in $(\bar{u},\sigma)$ we obtain

$$\begin{cases}\int_0^T\int_\Omega \sigma y'\ \mathrm{dx}+\langle\rho\bar{u}_{tt},y\rangle_{H_0^1(\Omega)}-\langle f,y\rangle_{L^2(\Omega)}\ \mathrm{dt}=0 & \forall y\in L^2(0,T;H_0^1(\Omega))\\ \int_0^T\int_\Omega \sigma\varphi-2k\varphi\bar{u}'\ \mathrm{dx}\ \mathrm{dt}=0 & \forall\varphi\in L^2(0,T;L^2(\Omega))\end{cases} \quad (2.15)$$

Remark 2.1.3. *We can define $\frac{\mathrm{d}}{\mathrm{dx}}\sigma\in L^2(0,T;H^{-1}(\Omega))$ as follows*

$$\langle\frac{\mathrm{d}}{\mathrm{dx}}\sigma,y\rangle_{H_0^1(\Omega)}\coloneqq\int_\Omega\sigma y'\ \mathrm{dx}\quad\forall y\in H_0^1(\Omega).$$

The main difference from the non evolutive case, that we have analysed in Subsection 1.4.2 and Subsection 1.5.1, is that even if we require $f\in L^2(0,T;L^2(\Omega))$ we can not state that $\frac{\mathrm{d}}{\mathrm{dx}}\sigma\in L^2(0,T;L^2(\Omega))$ because $\rho\bar{u}_{tt}$ is in $L^2(0,T;H^{-1}(\Omega))$ and not in $L^2(0,T;L^2(\Omega))$. We will see, in Section 2.3, that adding a viscosity term to the first equation of (2.2) we will gain enough regularity to $\bar{u}$ to have $\sigma\in L^2(0,T;H(\mathrm{div},\Omega))$.

2.1.3 Mixed formulation Elastic waves

Now we are interested in finding the couple (u,σ) solution of the system (2.15), with the addition of the initial conditions for u. We introduce the following.

Problem 2.1.5 (Mixed PDE formulation in $L^2(0,T;L^2(\Omega))$ of one dimensional elastic waves)**.** *Find* (u,σ) *such that* $u \in L^2(0,T;H^1_0(\Omega))$, $u_t \in L^2(0,T;L^2(\Omega))$, $\rho u_{tt} \in L^2(0,T;H^{-1}(\Omega))$, $\sigma \in L^2(0,T;L^2(\Omega))$ *and satisfies*

$$\begin{cases} \int_0^T \int_\Omega \sigma y' \text{ dx} + \langle \rho u_{tt}, y\rangle_{H^1_0(\Omega)} - \langle f, y\rangle_{L^2(\Omega)} \text{ dt} = 0 & \forall y \in L^2(0,T;H^1_0(\Omega)) \\ \int_0^T \int_\Omega \sigma\varphi - 2k\varphi u' \text{ dx dt} = 0 & \forall \varphi \in L^2(0,T;L^2(\Omega)) \\ u(\cdot,0) = u_0 & \textit{in } \Omega \\ u_t(\cdot,0) = u_1 & \textit{in } \Omega. \end{cases} \tag{2.16}$$

Proposition 2.1.4. *For every* (u,σ) *solution of Problem 2.1.5,* u *must satisfy the following*

$$\int_0^T \langle \rho u_{tt}, y\rangle_{H^1_0(\Omega)} + a(u,y) \text{ dt} = \int_0^T \langle f, y\rangle_{L^2(\Omega)} \text{ dt} \quad \forall y \in L^2(0,T;H^1_0(\Omega)), \tag{2.17}$$

where $a(\cdot,\cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ *is defined in* (2.1)*.*

Proof. Since (u,σ) is solution of Problem 2.1.5, then it satisfies (2.16). From the second equation of (2.16) and since $\sigma \in L^2(0,T;L^2(\Omega))$ we obtain

$$\sigma = 2k\bar{u}'.$$

Substitute σ is the first equation of (2.16) and according to the definition of the operator $a(\cdot,\cdot)$ we obtain

$$\int_0^T \langle \rho u_{tt}, y\rangle_{H^1_0(\Omega)^2} + a(u,y) \text{ dt} = \int_0^T \langle f, y\rangle_{L^2(\Omega)} \text{ dt} \quad \forall y \in L^2(0,T;H^1_0(\Omega)^2),$$

that is exactly (2.17). □

Now we state the main result of this section.

Theorem 2.2. *The function* $\bar{u}$ *is the unique solution of Problem 2.2 if and only if the couple* $(\bar{u},\sigma)$ *is the unique solution of Problem 2.1.5, where*

$$\sigma = 2k\bar{u}'. \tag{2.18}$$

Proof. (⇒)
From the previous subsection we have obtained that $(\bar{u},\sigma)$ satisfies (2.15). Form the second equation of (2.15) and since $\sigma \in L^2(0,T;L^2(\Omega))$ we have that σ is defined as in (2.18). Finally, since $\bar{u}$ is the solution of Problem 2.2, then it satisfies the initial conditions.
(⇐)
Let (u,σ) be solution of Problem 2.1.5. According to Proposition 2.1.4 we have that u satisfies the following system.

$$\begin{cases} \int_0^T \langle \rho u_{tt}, y\rangle_{H^1_0(\Omega)} + a(u,y) \text{ dt} = \int_0^T \langle f, y\rangle_{L^2(\Omega)} \text{ dt} & \forall y \in L^2(0,T;H^1_0(\Omega)) \\ u(\cdot,0) = u_0 & \text{in } \Omega \\ u_t(\cdot,0) = u_1 & \text{in } \Omega. \end{cases}$$

Thus u is solution of Problem 2.2.
Since Problem 2.2 admits a unique solution, then $(\bar{u}, \sigma)$ is the unique solution of Problem 2.1.5.

□

2.2 Linear Elasticity

The aim of this section is to introduce a mixed formulation for linear elasticity. We will start from presenting the starting problem and then following the same ideas of [7] and [9], we will prove that it admits a unique solution. Then, using the theory we have developed in Sections 1.3 and 1.4, we will construct a mixed formulation. Finally we will show that the mixed formulation has a unique solution that coincide with the starting problem.

2.2.1 Starting problem

Assumption 2.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and Lipschitz domain, $T > 0$, $\underline{u}_0 \in H_0^1(\Omega)^2$, $\underline{u}_1 \in L^2(\Omega)^2$, $\underline{f} \in L^2(0,T;(L^2(\Omega)^2)^*)$, $\rho \in L^\infty(\Omega)$ such that $0 < A \leq \rho \leq B < \infty$ a.e. in Ω, $\mu > 0$ and $\lambda \geq 0$.*

Satisfying Assumption 2.2.1 now we introduce the symmetric bilinear form $a : H_0^1(\Omega)^2 \times H_0^1(\Omega)^2 \to \mathbb{R}$:

$$a(\underline{u}, \underline{v}) := \int_\Omega 2\mu \underline{\underline{\epsilon}}(u) : \underline{\underline{\epsilon}}(v) + \lambda \mathrm{div}(\underline{u})\mathrm{div}(\underline{v}) \ \mathrm{dx}. \tag{2.19}$$

We now consider the following.

Problem 2.2.1 (Starting Problem Linear Elasticity)**.** *Find $\underline{u}$ such that*

$$\underline{u} \in L^2(0,T;H_0^1(\Omega)^2) \quad \underline{u}_t \in L^2(0,T;L^2(\Omega)^2) \quad \rho\underline{u}_{tt} \in L^2(0,T;H^{-1}(\Omega)^2),$$

and satisfies

$$\begin{cases} \int_0^T \langle \rho\underline{u}_{tt}, \underline{\varphi}\rangle_{H_0^1(\Omega)^2} + a(\underline{u},\underline{\varphi}) \ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi}\rangle_{L^2(\Omega)} \ \mathrm{dt} & \forall \underline{\varphi} \in L^2(0,T;H_0^1(\Omega)^2) \\ \underline{u}(\cdot,0) = \underline{u}_0 & \text{in } \Omega \\ \underline{u}_t(\cdot,0) = \underline{u}_1 & \text{in } \Omega. \end{cases} \tag{2.20}$$

Following the same ideas of [7, Thm. 4.1 in Ch. 3] and [9, Thm. 1, 2, and 3 in Sec. 7.2], we obtain the following.

Theorem 2.3 (Existence and uniqueness of solution)**.** *Let $\underline{f} \in L^2(0,T;(L^2(\Omega)^2)^*)$, $\underline{u}_0 \in H_0^1(\Omega)^2$ and $\underline{u}_1 \in L^2(\Omega)^2$. There exists a unique function $\underline{u}$ such that*

$$\underline{u} \in L^2(0,T;H_0^1(\Omega)^2) \quad \underline{u}_t \in L^2(0,T;L^2(\Omega)^2) \quad \rho\underline{u}_{tt} \in L^2(0,T;H^{-1}(\Omega)^2),$$

and satisfies

$$\begin{cases} \int_0^T \langle \rho\underline{u}_{tt}, \underline{\varphi}\rangle_{H_0^1(\Omega)^2} + a(\underline{u},\underline{\varphi}) \ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi}\rangle_{L^2(\Omega)} \ \mathrm{dt} & \forall \underline{\varphi} \in L^2(0,T;H_0^1(\Omega)^2) \\ \underline{u}(0) = \underline{u}_0 & \\ \underline{u}_t(0) = \underline{u}_1. & \end{cases} \tag{2.21}$$

Proof. **Construction of approximate solution.** Since $H_0^1(\Omega)^2$ and $L^2(\Omega)^2$ are separable, then there exists functions w_k, such that $\{w_k\}_{k=1}^\infty$ is an orthogonal basis of $H_0^1(\Omega)^2$, $\{w_k\}_{k=1}^\infty$ is

an orthonormal basis of $L^2(\Omega)^2$ and $w_k > 0$ for all k. As Evans suggests in [9] it is enough to consider the normalize eigenfunctions of the Laplacian operator in $H^1_0(\Omega)^2$.
For every fixed integer m we write

$$\underline{u}_m(t) := \sum_{k=1}^{m} d^k_m(t)\underline{w}_k, \tag{2.22}$$

where $\underline{w}_k := (w_k, 0)$ or $\underline{w}_k := (0, w_k)$, and we consider the problem of finding the coefficients $d^k_m(t)$, such that

$$(\rho\underline{u}_{m\,tt}, \underline{w}_s)_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{w}_s) = \langle \underline{f}, \underline{w}_s \rangle_{L^2(\Omega)} \quad \text{for } 0 \le t \le T, \tag{2.23}$$

for $s = 1, 2, \ldots, m$ and

$$d^k_m(0) = (\underline{u}_0, \underline{w}_k)_{L^2(\Omega)^2} \tag{2.24}$$

$$d^k_{m\,t}(0) = (\underline{u}_1, \underline{w}_k)_{L^2(\Omega)^2}, \tag{2.25}$$

for $k = 1, 2, \ldots, m$.
Since $w_k > 0$ for all k and $0 < A \le \rho \le B < \infty$ a.e. in Ω, we have the following

$$A(\underline{w}_k, \underline{w}_s)_{L^2(\Omega)^2} \le (\rho\underline{w}_k, \underline{w}_s)_{L^2(\Omega)^2} := \int_\Omega \rho \underline{w}_k \underline{w}_s \text{ dx} \le B(\underline{w}_k, \underline{w}_s)_{L^2(\Omega)^2}, \tag{2.26}$$

that implies

$$(\rho\underline{u}_m(t), \underline{w}_s)_{L^2(\Omega)^2} = \sum_{k=1}^{m} d^k_{m\,tt}(t) \int_\Omega \rho|\underline{w}_k|^2 \text{ dx} = \sum_{k=1}^{m} C_k d^k_{m\,tt}(t).$$

Denoting with $f^k(t) := \langle \underline{f}(t), \underline{w}^k \rangle_{L^2(\Omega)^2}$ and $b^{kl} := a(\underline{w}_k, \underline{w}_l)$ we can rewrite (2.23) as a linear system of ODE

$$C_k d^k_{m\,tt}(t) + \sum_{l=1}^{m} b^{kl} d^l_m(t) = f^k(t) \quad \text{for } 0 \le t \le T, \tag{2.27}$$

for $k = 1, 2, \ldots, m$ and $0 \le t \le T$. According to the standard theory for ordinary differential equations, there exists a unique function $d_m(t) = (d^1_m(t), \ldots, d^m_m(t))$ satisfying (2.27) for $0 \le t \le T$, (2.24) and (2.25). So, if we define $\underline{u}_{0m} = \sum_{k=1}^m d^k_m(0)\underline{w}_k$ and $\underline{u}_{1m} = \sum_{k=1}^m d^k_{m\,t}(0)\underline{w}_k$, we have that for every $m = 1, 2, \ldots$, there exists a unique function $\underline{u}_m(t)$, of the form (2.22), that satisfies (2.23) and

$$\underline{u}_m(0) = \underline{u}_{0m} \tag{2.28}$$

$$u_{m\,t}(0) = \underline{u}_{1m}. \tag{2.29}$$

A priori estimates. Now we want to prove the following:

$$\max_{0 \le t \le T}(\|\underline{u}_m\|^2_{H^1_0(\Omega)^2} + \|\underline{u}_{m\,t}\|^2_{L^2(\Omega)^2}) + \|\rho\underline{u}_{m\,tt}\|^2_{L^2(0,T;H^{-1}(\Omega)^2)}$$
$$\le C(\|\underline{f}\|^2_{L^2(0,T;(L^2(\Omega)^2)^*)} + \|\underline{u}_1\|^2_{L^2(\Omega)^2} + \|\underline{u}_0\|^2_{H^1_0(\Omega)^2}). \tag{2.30}$$

Taking (2.23), multiply it by d^k_{mt} and then sum for $k = 1, 2, \dots, m$, we obtain

$$(\rho \underline{u}_{mtt}, \underline{u}_{mt})_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{u}_{mt}) = \langle \underline{f}, \underline{u}_{mt} \rangle_{L^2(\Omega)} .$$

Observing that

$$(\rho \underline{u}_{mtt}, \underline{u}_{mt})_{L^2(\Omega)^2} = (\sqrt{\rho}\underline{u}_{mtt}, \sqrt{\rho}\underline{u}_{mt})_{L^2(\Omega)^2} = \frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\rho}\underline{u}_{mt}\|^2_{L^2(\Omega)^2}\big),$$

and, since $a(\cdot,\cdot)$ is a symmetric operator, we have

$$a(\underline{u}_m, \underline{u}_{mt}) = \int_\Omega 2\mu \underline{\underline{\epsilon}}(\underline{u}_m) : \underline{\underline{\epsilon}}(\underline{u}_{mt}) + \lambda \mathrm{div}(\underline{u}_m)\mathrm{div}(\underline{u}_{mt}) \text{ dx} = \frac{1}{2}\frac{d}{dt}\big(a(\underline{u}_m, \underline{u}_m)\big),$$

recalling that $2ab \le a^2 + b^2$, (2.26) and $a(\underline{u}, \underline{u}) \ge \mathcal{K}\mu \|\underline{u}\|^2_{H^1_0(\Omega)^2}$ for all $\underline{u} \in H^1_0(\Omega)^2$, we obtain the following

$$\begin{aligned}
\frac{d}{dt}\big(\|\sqrt{\rho}\underline{u}_{mt}\|^2_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{u}_m)\big)\big) &= 2\big[(\rho \underline{u}_{mtt}, \underline{u}_{mt})_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{u}_{mt})\big] \\
&= 2\big[\, \langle \underline{f}, \underline{u}_{mt} \rangle_{L^2(\Omega)^2} - a(\underline{u}_m, \underline{u}_{mt}) + a(\underline{u}_m, \underline{u}_{mt})\big] \\
&\le 2\big[\|f\|_{(L^2(\Omega)^2)^*} \|\underline{u}_{mt}\|_{L^2(\Omega)^2}\big] \\
&\le \|f\|^2_{(L^2(\Omega)^2)^*} + \|\underline{u}_{mt}\|^2_{L^2(\Omega)^2} \\
&\le C\big[\|f\|^2_{(L^2(\Omega)^2)^*} + \big\|\sqrt{\rho}\underline{u}_{mt}\big\|^2_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{u}_m)\big]. \qquad (2.31)
\end{aligned}$$

If we define

$$\begin{aligned}
\nu(t) &:= \|\sqrt{\rho}\underline{u}_{mt}(t)\|^2_{L^2(\Omega)^2} + a(\underline{u}_m(t), \underline{u}_m(t)) \\
\xi(t) &:= \big\|\underline{f}(t)\big\|^2_{(L^2(\Omega)^2)^*},
\end{aligned}$$

we can rewrite (2.31) as follows

$$\nu_t(t) \le C\nu(t) + C\xi(t) \quad \text{for } 0 \le t \le T,$$

and according to Proposition A.2.6 we obtain

$$\nu(t) \le e^{Ct}\big(\nu(0) + C\int_0^t \xi(s) \text{ ds }\big) \quad \text{for } 0 \le t \le T. \qquad (2.32)$$

Due to (2.26) we can notice that

$$\nu(0) = \|\sqrt{\rho}\underline{u}_{mt}(0)\|^2_{L^2(\Omega)^2} + a(\underline{u}_m(0), \underline{u}_m(0)) \le C\big(\|\underline{u}_1\|^2_{L^2(\Omega)} + \|\underline{u}_0\|^2_{H^1_0(\Omega)^2}\big), \qquad (2.33)$$

then from (2.32) and (2.33) we obtain

$$\|\sqrt{\rho}\underline{u}_{mt}(t)\|^2_{H^1_0(\Omega)^2} + a(\underline{u}_m(t), \underline{u}_m(t)) \le C(\|\underline{u}_1\|^2_{L^2(\Omega)^2} + \|\underline{u}_0\|^2_{H^1_0(\Omega)^2} + \big\|\underline{f}\big\|^2_{L^2(0,T;(L^2(\Omega)^2)^*)}). \qquad (2.34)$$

Since for all $\underline{u} \in H_0^1(\Omega)^2$ we have

$$a(\underline{u}, \underline{u}) \geq 2\mu\mathcal{K}\|\underline{u}\|^2_{H_0^1(\Omega)^2},$$

and

$$\|\sqrt{\rho}\underline{u}_{mt}(t)\|^2_{L^2(\Omega)^2} = (\sqrt{\rho}\underline{u}_{mt}(t), \sqrt{\rho}\underline{u}_{mt}(t))_{L^2(\Omega)^2} = \\ (\rho\underline{u}_{mt}(t), \underline{u}_{mt}(t))_{L^2(\Omega)^2} \geq A(\underline{u}_{mt}(t), \underline{u}_{mt}(t))_{L^2(\Omega)^2} = A\|\underline{u}_{mt}(t)\|^2_{L^2(\Omega)^2},$$

from (2.34) we obtain

$$\max_{0\leq t\leq T}(\|\underline{u}_m(t)\|^2_{H_0^1(\Omega)^2} + \|\underline{u}_{mt}(t)\|^2_{L^2(\Omega)^2}) \leq \\ C(\|\underline{f}\|^2_{L^2(0,T;(L^2(\Omega)^2)^*)} + \|\underline{u}_1\|^2_{L^2(\Omega)^2} + \|\underline{u}_0\|^2_{H_0^1(\Omega)^2}). \quad (2.35)$$

Now we fix $\underline{v} \in H_0^1(\Omega)^2$ such that $\|\underline{v}\|_{H_0^1(\Omega)^2} \leq 1$ and $\underline{v} = \underline{v}_1 + \underline{v}_2$, where $\underline{v}_1 \in span\{\underline{w}_k\}_{k=1}^m$ and $(\underline{v}_2, \underline{w}_k)_{L^2(\Omega)^2} = 0$ for every $k = 1, 2, \ldots, m$, then we obtain

$$\langle\rho\underline{u}_{mtt}, \underline{v}\rangle_{H_0^1(\Omega)^2} \coloneqq (\rho\underline{u}_{mtt}, \underline{v})_{L^2(\Omega)^2} = (\rho\underline{u}_{mtt}, \underline{v}_1)_{L^2(\Omega)^2} = (\underline{f}, \underline{v}_1) - a(\underline{u}_m, \underline{v}_1),$$

Since $\|\underline{v}\|_{H_0^1(\Omega)^2} \leq 1$, we have

$$\|\rho\underline{u}_{mtt}\|_{H^{-1}(\Omega)^2} \coloneqq \sup_{\underline{v}\in H_0^1(\Omega)^2\setminus\{0\}} \frac{|\langle\rho\underline{u}_{mtt}, \underline{v}\rangle_{H_0^1(\Omega)^2}|}{\|\underline{v}\|_{H_0^1(\Omega)^2}} \leq C\big(\|\underline{f}\|_{(L^2(\Omega)^2)^*} + \|\underline{u}_m\|_{H_0^1(\Omega)^2}\big),$$

from the previous equation and (2.35) we can deduce the following

$$\begin{aligned} \|\rho\underline{u}_{mtt}\|^2_{L^2(0,T;H^{-1}(\Omega)^2)} &= \int_0^T \|\rho\underline{u}_{mtt}\|^2_{H^{-1}(\Omega)^2}\, \mathrm{dt} \\ &\leq \int_0^T C\big(\|\underline{f}\|^2_{(L^2(\Omega)^2)^*} + \|\underline{u}_m\|^2_{H_0^1(\Omega)^2}\big) \\ &\leq C\big(\|\underline{f}\|^2_{(L^2(\Omega)^2)^*} + \|\underline{u}_1\|^2_{L^2(\Omega)^2} + \|\underline{u}_0\|^2_{H_0^1(\Omega)^2}\big). \end{aligned} \quad (2.36)$$

Thus from (2.35) and (2.36) we obtain (2.30).

Existence of solution. Since (2.30) is independent of m we can deduced that $\{\underline{u}_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;H_0^1(\Omega)^2)$, $\{\underline{u}_{mt}\}_{m=1}^\infty$ is bounded in $L^2(0,T;L^2(\Omega)^2)$ and $\{\rho\underline{u}_{mtt}\}_{m=1}^\infty$ is bounded in $L^2(0,T;H^{-1}(\Omega)^2)$. According to Theorem A.4, we have that there exists a subsequence $\{\underline{u}_{m_l}\}_{l=1}^\infty$, $\underline{u} \in L^2(0,T;H_0^1(\Omega)^2)$, $\underline{u}_t \in L^2(0,T;L^2(\Omega)^2)$ and $\rho\underline{u}_{tt} \in L^2(0,T;H^{-1}(\Omega)^2)$ such that

$$\begin{cases} \underline{u}_{m_l} \rightharpoonup \underline{u} & \text{weakly in } L^2(0,T;H_0^1(\Omega)^2) \\ \underline{u}_{m_l t} \rightharpoonup \underline{u}_t & \text{weakly in } L^2(0,T;L^2(\Omega)^2) \\ \rho\underline{u}_{m_l tt} \rightharpoonup \rho\underline{u}_{tt} & \text{weakly in } L^2(0,T;H^{-1}(\Omega)^2). \end{cases} \quad (2.37)$$

From now on we identify $\{\underline{u}_m\}_{m=1}^\infty$ with its subsequence $\{\underline{u}_{m_l}\}_{l=1}^\infty$.

Now we fix an integer N and consider the function $\underline{\varphi} \in \mathcal{C}^1([0,T]; H_0^1(\Omega)^2)$ such that

$$\underline{\varphi}(t) := \sum_{i=1}^{N} d^k(t)\underline{w}_k, \tag{2.38}$$

where $\{d^k\}_{k=1}^N$ are smooth functions, i.e. $\{d^k\}_{k=1}^N \subseteq \mathcal{C}^\infty([0,T])$. Taking $m \leq N$, multiply (2.23) by $d^k(t)$ and sum $k = 1, 2, \dots N$ we obtain

$$(\rho \underline{u}_{m\,tt}, \underline{\varphi}(t))_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{\varphi}(t)) = \langle \underline{f}, \underline{\varphi}(t) \rangle_{L^2(\Omega)} \quad \text{for } 0 \leq t \leq T.$$

Integrating the previous equation between $[0,T]$ with respect to the variable t, we obtain

$$\int_0^T \langle \rho \underline{u}_{m\,tt}, \underline{\varphi} \rangle_{H_0^1(\Omega)^2} + a(\underline{u}_m, \underline{\varphi})\ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi} \rangle_{L^2(\Omega)}\ \mathrm{dt}. \tag{2.39}$$

According to (2.37) if we pass to the limit in the previous equation, we obtain

$$\int_0^T \langle \rho \underline{u}_{tt}, \underline{\varphi} \rangle_{H_0^1(\Omega)^2} + a(\underline{u}, \underline{\varphi})\ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi} \rangle_{L^2(\Omega)}\ \mathrm{dt}.$$

Since the functions defined in (2.38) are dense in $L^2(0,T; H_0^1(\Omega)^2)$ we can deduce that

$$\int_0^T \langle \rho \underline{u}_{tt}, \underline{\varphi} \rangle_{H_0^1(\Omega)^2} + a(\underline{u}, \underline{\varphi})\ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi} \rangle_{L^2(\Omega)}\ \mathrm{dt} \quad \forall \underline{\varphi} \in L^2(0,T; H_0^1(\Omega)^2). \tag{2.40}$$

In order to verify that $\underline{u}$ satisfy (2.21) we need to check that $\underline{u}(0) = \underline{u}_0$ and $\underline{u}_t(0) = \underline{u}_1$. If we take $\underline{\varphi} \in \mathcal{C}^2([0,T]; H_0^1(\Omega)^2)$ such that $\underline{\varphi}(T) = 0$ and $\underline{\varphi}_t(T) = 0$ and integrating by parts equation (2.40) respect to the variable t, we obtain

$$\begin{aligned}\int_0^T (\rho \underline{u}, \underline{\varphi}_{tt})_{L^2(\Omega)^2} + a(\underline{u}, \underline{\varphi})\ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi} \rangle_{L^2(\Omega)}\ \mathrm{dt}& \\ + (\rho \underline{u}(0), \underline{\varphi}_t(0))_{L^2(\Omega)^2} + (\rho \underline{u}_t(0), \underline{\varphi}(0))_{L^2(\Omega)^2}&.\end{aligned} \tag{2.41}$$

Starting from equation (2.39), integrating twice by parts respect to the variable t, we obtain

$$\begin{aligned}\int_0^T (\rho \underline{u}_m, \underline{\varphi}_{tt})_{L^2(\Omega)^2} + a(\underline{u}_m, \underline{\varphi})\ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi} \rangle_{L^2(\Omega)}\ \mathrm{dt}& \\ + (\rho \underline{u}_m(0), \underline{\varphi}_t(0))_{L^2(\Omega)^2} + (\rho \underline{u}_m(0)_t \underline{\varphi}(0))_{L^2(\Omega)^2}&,\end{aligned}$$

from (2.28), (2.29) and (2.37) if we pass to the limit of the previous equation we obtain

$$\begin{aligned}\int_0^T (\rho \underline{u}, \underline{\varphi}_{tt})_{L^2(\Omega)^2} + a(\underline{u}, \underline{\varphi})\ \mathrm{dt} = \int_0^T \langle \underline{f}, \underline{\varphi} \rangle_{L^2(\Omega)}\ \mathrm{dt}& \\ + (\rho \underline{u}_0, \underline{\varphi}_t(0))_{L^2(\Omega)^2} + (\rho \underline{u}_1, \underline{\varphi}(0))_{L^2(\Omega)^2}&.\end{aligned} \tag{2.42}$$

Confronting (2.41) and (2.42) we obtain that

$$(\rho\underline{u}(0),\underline{\varphi}_t(0))_{L^2(\Omega)^2}+(\rho\underline{u}_t(0),\underline{\varphi}(0))_{L^2(\Omega)^2}=(\rho\underline{u}(0),\underline{\varphi}_t(0))_{L^2(\Omega)^2}+(\rho\underline{u}_t(0),\underline{\varphi}(0))_{L^2(\Omega)^2},$$

form the arbitrary of $\underline{\varphi}(0)$ and $\underline{\varphi}_t(0)$, implied by the arbitrary of $\underline{\varphi}\in\mathcal{C}^2([0,T];H_0^1(\Omega)^2)$, and since $0<A\le\rho\le B<+\infty$ a.e. in Ω, we can deduce that

$$\underline{u}(0)=\underline{u}_0$$
$$\underline{u}_t(0)=\underline{u}_1,$$

so $\underline{u}$ is a solution of (2.21).

Uniqueness of solution Now we want to show that there exists a unique solution of (2.21), in order to do that we are going to prove that if $\underline{u}$ is solution of (2.21) with the choice of $\underline{u}_0=\underline{u}_1=\underline{f}=0$, then $\underline{u}=0$. Let $\underline{\varphi}\in\mathcal{C}^1([0,T];(L^2(\Omega)^2)^*)$ and according to what we have prove until now there exists a function $\underline{w}$ solution of (2.21) with the choice $\underline{u}_0=\underline{u}_1=0$ and $\underline{f}=\underline{\varphi}$. Now we consider the function $\underline{\tilde{w}}(T-t)=\underline{w}(t)$, so we have that

$$\begin{aligned}
&\underline{\tilde{w}}\in L^2(0,T;H_0^1(\Omega)^2)\quad \underline{\tilde{w}}_t\in L^2(0,T;L^2(\Omega)^2)\quad \rho\underline{\tilde{w}}_{tt}\in L^2(0,T;H^{-1}(\Omega)^2)\\
&\int_0^T\langle\rho\underline{\tilde{w}}_{tt},\underline{\varphi}\rangle_{H_0^1(\Omega)^2}+a(\underline{\tilde{w}},\underline{\varphi})\ \mathrm{dt}=\int_0^T\langle\underline{f},\underline{\varphi}\rangle_{L^2(\Omega)}\ \mathrm{dt}\quad\forall\underline{\varphi}\in L^2(0,T;H_0^1(\Omega)^2)\\
&\underline{\tilde{w}}(T)=0\\
&\underline{\tilde{w}}_t(T)=0.
\end{aligned}\tag{2.43}$$

Since $\underline{\tilde{w}}(T)=\underline{\tilde{w}}_t(T)=\underline{u}(0)=\underline{u}_t(0)=0$, then we have

$$\begin{aligned}
\int_0^T\langle\rho\underline{u}_{tt},\underline{\tilde{w}}\rangle_{H_0^1(\Omega)^2}\ \mathrm{dt}&=-\int_0^T(\rho\underline{u}_t,\underline{\tilde{w}}_t)_{L^2(\Omega)^2}\ \mathrm{dt}\\
&\quad+(\rho\underline{u}_t(T),\underline{\tilde{w}}(T))_{L^2(\Omega)^2}-(\rho\underline{u}_t(0),\underline{\tilde{w}}(0))_{L^2(\Omega)^2}\\
&=\int_0^T\langle\rho\underline{\tilde{w}}_{tt},\underline{u}\rangle_{H_0^1(\Omega)^2}\ \mathrm{dt}\\
&\quad+\langle\rho\underline{\tilde{w}}_{tt}(T),\underline{u}(T)\rangle_{L^2(\Omega)^2}-\langle\rho\underline{\tilde{w}}_{tt}(0),\underline{u}(0)\rangle_{L^2(\Omega)^2}\\
&=\int_0^T\langle\rho\underline{\tilde{w}}_{tt},\underline{u}\rangle_{H_0^1(\Omega)^2}\ \mathrm{dt},
\end{aligned}\tag{2.44}$$

and, because $\underline{u}$ is solution of (2.21), with $\underline{f}=0$ and $\underline{\tilde{w}}$ satisfies (2.43) we have that

$$\int_0^T\langle\rho\underline{\tilde{w}}_{tt},\underline{u}\rangle_{H_0^1(\Omega)^2}\ \mathrm{dt}=\int_0^T\langle\underline{\varphi},\underline{u}\rangle_{L^2(\Omega)^2}-a(\underline{\tilde{w}},\underline{u})\ \mathrm{dt}\tag{2.45}$$

$$\int_0^T\langle\rho\underline{u}_{tt},\underline{\tilde{w}}\rangle_{H_0^1(\Omega)^2}\ \mathrm{dt}=\int_0^T-a(\underline{u},\underline{\tilde{w}})\ \mathrm{dt}.\tag{2.46}$$

From (2.44), (2.45), (2.46) and because $a(\cdot,\cdot)$ is a symmetric operator we obtain

$$\int_0^T\langle\underline{\varphi},\underline{u}\rangle_{L^2(\Omega)^2}=0\quad\forall\underline{\varphi}\in\mathcal{C}^1([0,T];L^2(\Omega)^2).$$

Since $\mathcal{C}^1([0,T];(L^2(\Omega)^2)^*)$ is dense in $L^2(0,T;(L^2(\Omega)^2)^*)$, then we can deduce that $\underline{u}=0$. $\square$

Definition 2.2.1. *We will denote $\underline{\bar{u}}$ the unique solution of Problem 2.2.1.*

2.2.2 Construction of mixed formulation

In order to apply the duality theory that we have seen in Section 1.3 and Section 1.4, we need to characterize $\underline{\bar{u}}$ as the unique minimum of a certain variational principle. Let us start by introducing the functional

$$\mathcal{S}(\underline{u}) = \int_0^T \langle \rho \underline{\bar{u}}_{tt}, \underline{u} \rangle_{H_0^1(\Omega)^2} + \frac{1}{2} a(\underline{u},\underline{u}) - \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)} \text{ dt}, \tag{2.47}$$

acting on the Hilbert space

$$\Gamma = \left\{ \underline{u}(x,t) \;\middle|\; \underline{u}(x,t) \in L^2(0,T;H_0^1(\Omega)^2) \right\},$$

and the variational problem related to $\mathcal{S}$.

Problem 2.2.2 (Primal formulation of linear elasticity)**.** *Find $\underline{u} \in \Gamma$ such that*

$$\mathcal{S}(\underline{u}) = \inf_{\underline{v} \in \Gamma} \{\mathcal{S}(\underline{v})\} \tag{2.48}$$

As suggested in Remark 2.1.2 the presence of $\underline{\bar{u}}$ in (2.47) is not a problem for the sake of our purpose.

Proposition 2.2.1. *The function $\underline{\bar{u}}$ is the unique minimum of Problem 2.2.2.*

Proof. Clearly we have that $\underline{\bar{u}} \in \Gamma$, if we compute the first variation of (2.47) evaluated in $\underline{\bar{u}}$, we obtain

$$d\mathcal{S}(\underline{\bar{u}}).\underline{y} = \int_0^T \langle \rho \underline{\bar{u}}_{tt}, \underline{\varphi} \rangle_{H_0^1(\Omega)^2} + a(\underline{u},\underline{\varphi}) - \langle \underline{f}, \underline{y} \rangle_{L^2(\Omega)} \text{ dt} = 0 \quad \forall \underline{y} \in \Gamma,$$

where the last equality is a consequence of the first equation of (2.20). Thus $\underline{\bar{u}}$ is a critical point of $\mathcal{S}$. In order to prove that it is the unique minimum of $\mathcal{S}$ we will show that the action $\mathcal{S}$ is strongly convex in Γ. This is equivalent to prove that the second variation of $\mathcal{S}$ is positive define for all $\underline{u} \in L^2(0,T;H_0^1(\Omega)^2)$:

$$\begin{aligned}
d^2\mathcal{S}(\underline{u}).(\underline{y},\underline{y}) &= \int_0^T a(\underline{y},\underline{y}) \text{ dt} \\
&= \int_0^T \int_\Omega 2\mu|\underline{\epsilon}(y)|^2 + \lambda|\text{tr}(\underline{\epsilon}(y))|^2 \text{ dx dt} \\
&\geq \int_0^T \int_\Omega 2\mu|\underline{\epsilon}(y)|^2 \text{ dx dt} \\
&\geq \int_0^T 2\mu\mathcal{K}\|\underline{y}\|^2_{H_0^1(\Omega)^2} \text{ dt} \geq 2\mu\mathcal{K}\|\underline{y}\|^2_{L^2(0,T;H_0^1(\Omega)^2)^2} \quad \forall \underline{y} \in L^2(0,T;H_0^1(\Omega)^2),
\end{aligned}$$

where we have used Korn Inequality (A.4). $\square$

In order to introduce the dual and the saddle point formulation for Problem 2.2.2 it is convenient to rewrite (2.48) as follows:

$$\mathcal{S}(\underline{u}) = \inf_{\underline{v}\in\Gamma}\{\mathcal{S}(\underline{v})\} = \inf_{\underline{v}\in\Gamma}\{F(\underline{v}) + G(\underline{\underline{\epsilon}}(v))\},$$

where $F : \Gamma \to \mathbb{R}$ is

$$F(\underline{v}) = \int_0^T \langle\rho\underline{\bar{u}}_{tt}, \underline{v}\rangle_{H_0^1(\Omega)^2} - \langle\underline{f}, \underline{v}\rangle_{L^2(\Omega)} \text{ dt}, \tag{2.49}$$

and $G : Y \coloneqq L^2(0, T; L^2(\Omega)^{2\times 2}_{sym}) \to \mathbb{R}$ is

$$G(\underline{\underline{p}}) = \int_0^T \int_\Omega \mu|\underline{\underline{p}}|^2 + \frac{1}{2}\lambda|\text{tr}(\underline{\underline{p}})|^2 \text{ dx dt}. \tag{2.50}$$

Now we need to introduce the dual problem and show that it admits a unique solution. Starting from Problem 2.2.2, according to Definition 1.3.2 with the choice of $\Phi(\underline{u}, \underline{\underline{p}}) = F(\underline{u}) + G(\underline{\underline{\epsilon}}(u) + \underline{\underline{p}})$, we obtain the following.

Problem 2.2.3 (Dual formulation of linear elasticity)**.** *Find $q^* \in Y^*$ such that*

$$-\Phi^*(0, q^*) = \sup_{p^*\in Y^*}\big[-\Phi^*(0, p^*)\big] = \sup_{p^*\in Y^*}\big[-F^*(-\Lambda^* p^*) - G^*(p^*)\big], \tag{2.51}$$

where Λ^* is $\underline{\underline{\epsilon}}^*$, the adjoint of $\underline{\underline{\epsilon}}$, according to Definition A.1.8 $F^* : \Gamma^* \to \mathbb{R}$ is

$$F^*(-\underline{\underline{\epsilon}}^*(p^*)) = \begin{cases} 0 & \text{if } -\underline{\underline{\epsilon}}^*(p^*) - \rho\underline{\bar{u}}_{tt} + \underline{f} = 0 \\ +\infty & \text{otherwise} \end{cases},$$

and $G^* : Y^* \to \mathbb{R}$ is

$$G^*(p^*) = \int_0^T \int_\Omega \frac{1}{4\mu}|\mathcal{R}(p^*)^D|^2 + \frac{1}{8(\mu+\lambda)}|\text{tr}(\mathcal{R}(p^*))|^2 \text{ dx dt},$$

where $\mathcal{R}$ is the Riesz operator such that $\mathcal{R} : Y^* \to Y$.

Proposition 2.2.2. *Problem 2.2.2 is stable, in the sense of Definition 1.3.4. Moreover Problem 2.2.3 admits a unique solution.*

Proof. In order to prove stability we verify the hypothesis of Theorem 1.1. Since F, defined in (2.49), is linear and G, defined in (2.50), is strictly convex, then $J(\underline{u}, \underline{\underline{p}}) = F(\underline{u}) + G(\underline{\underline{p}})$ is convex in $\Gamma \times Y$:

$$\begin{aligned} J(\alpha(\underline{u}, \underline{\underline{p}}) + (1-\alpha)(\underline{v}, \underline{\underline{q}})) &= F(\alpha\underline{u} + (1-\alpha)\underline{v}) + G(\alpha\underline{\underline{p}} + (1-\alpha)\underline{\underline{q}}) \\ &= \alpha F(\underline{u}) + (1-\alpha)F(\underline{v}) + G(\alpha\underline{\underline{p}} + (1-\alpha)\underline{\underline{q}}) \\ &\leq \alpha F(\underline{u}) + (1-\alpha)F(\underline{v}) + \alpha G(\underline{\underline{p}}) + (1-\alpha)G(\underline{\underline{q}}) \\ &\leq \alpha J(\underline{u}, \underline{\underline{p}}) + (1-\alpha)J(\underline{v}, \underline{\underline{q}}), \end{aligned}$$

for $\alpha \in [0, 1]$, $\underline{u}, \underline{v} \in \Gamma$ and $\underline{\underline{p}}, \underline{\underline{q}} \in Y$.

Now we need to find a value $\underline{u}_0$ such that $F(\underline{u}_0)+G(\underline{\underline{\epsilon}}(\underline{u}_0))<+\infty$ and verify that the functional $\underline{\underline{p}} \to F(\underline{u}_0)+G(\underline{\underline{p}})$ is continuos at $\underline{\underline{\epsilon}}(\underline{u}_0)$. If we take $\underline{u}_0=\underline{0}$ then $F(\underline{0})=0$ and, since $\underline{\underline{\epsilon}}(\underline{0})=\underline{\underline{0}}$, then $G(\underline{\underline{\epsilon}}(\underline{0}))=G(\underline{\underline{0}})=0$, so the first hypothesis is verified. Now we need to show that $\underline{\underline{p}} \to F(\underline{u}_0)+G(\underline{\underline{p}})$ is continuos at $\underline{\underline{\epsilon}}(\underline{u}_0)$. Let $\underline{\underline{p}}_n \to \underline{\underline{p}}$ in Y as $n\to\infty$, then it is enough to show

$$\lim_{n\to\infty}\int_0^T\int_\Omega \mu|\underline{\underline{p}}_n|^2 \text{ dx dt} = \int_0^T\int_\Omega \mu|\underline{\underline{p}}|^2 \text{ dx dt} \tag{2.52}$$

and

$$\lim_{n\to\infty}\int_0^T\int_\Omega \frac{1}{2}\lambda|\text{tr}(\underline{\underline{p}}_n)|^2 \text{ dx dt} = \int_0^T\int_\Omega \frac{1}{2}\lambda|\text{tr}(\underline{\underline{p}})|^2 \text{ dx dt}. \tag{2.53}$$

Since $\int_0^T\int_\Omega \mu|\underline{\underline{p}}|^2 \text{ dx dt} = \mu\left\|\underline{\underline{p}}\right\|_Y^2$ and the norm is continuous, then (2.52) is true. Since $|\text{tr}(\cdot)|: Y \to \mathbb{R}$ can be seen as the composition of the projection $\pi: Y \to Y$ on the trace components, the sum of them, and the norm of Y and because all of them are continuous, we obtain (2.53). So form (2.52) and (2.53) we have the following

$$\begin{aligned}\lim_{n\to\infty} F(\underline{u}_0)+G(\underline{\underline{p}}_n) &= \lim_{n\to\infty}\int_0^T\int_\Omega \mu|\underline{\underline{p}}_n|^2+\frac{1}{2}\lambda|\text{tr}(\underline{\underline{p}}_n)|^2 \text{ dx dt} \\ &= \int_0^T\int_\Omega \mu|\underline{\underline{p}}|^2+\frac{1}{2}\lambda|\text{tr}(\underline{\underline{p}})|^2 \text{ dx dt} = F(\underline{u}_0)+G(\underline{\underline{p}}).\end{aligned}$$

Thus Problem 2.2.2 is stable and due to Proposition 1.3.6 we have that Problem 2.2.3 has at least one solution. Can be noticed that G is differentiable and the differential of G in $\underline{\underline{p}}$ evaluated in $\underline{\underline{q}}$ is

$$G'[\underline{\underline{p}}](\underline{\underline{q}}) = \int_0^T\int_\Omega 2\mu\underline{\underline{p}}:\underline{\underline{q}}+\lambda\,\text{tr}(\underline{\underline{p}})\,\text{tr}(\underline{\underline{q}}) \text{ dx dt}.$$

Since G is differentiable and primal Problem 2.2.2 admits a unique solution, according to Proposition 1.3.7 and Proposition A.1.7, we have that the dual Problem 2.2.3 admits a unique solution. □

In order to write the saddle point formulation of Problem 2.2.2, using the theory developed in Section 1.4, we need to study the Lagrangian, defined in Definition 1.4.1, of Problem 2.2.1:

$$-L(\underline{u},p^*) = \sup_{\underline{\underline{p}}\in Y}\left[\langle p^*,\underline{\underline{p}}\rangle - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u)+\underline{\underline{p}})\right]. \tag{2.54}$$

Since $\langle p^*,\underline{\underline{p}}\rangle - F(\underline{u})$ is linear in $\underline{\underline{p}}$ and G is strictly convex, so $-G$ is strictly concave, we have that the argument of the supremum in the previous equation is strictly concave, so there exists a unique $\underline{\underline{q}}\in Y$ such that the supremum is realized. Computing the first variation respect to $\underline{\underline{p}}$ evaluated in $\underline{\underline{q}}$ of $\langle p^*,\underline{\underline{p}}\rangle - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u)+\underline{\underline{p}})$ we obtain

$$\int_0^T\int_\Omega \mathcal{R}(p^*):\underline{\underline{\varphi}} - 2\mu(\underline{\underline{q}}+\underline{\underline{\epsilon}}(u)):\underline{\underline{\varphi}} - \lambda(\text{tr}(\underline{\underline{q}})\underline{\underline{id}}+\text{tr}(\underline{\underline{\epsilon}}(u))\underline{\underline{id}}):\underline{\underline{\varphi}} \text{ dx dt} = 0 \qquad \forall\underline{\underline{\varphi}}\in Y, \tag{2.55}$$

where $\mathcal{R}: Y^* \to Y$ is the Riesz operator defined in Theorem A.1. If we take $\underline{\underline{\varphi}} = \underline{\underline{q}}+\underline{\underline{\epsilon}}(u)$ in

(2.55) we obtain

$$\langle p^*, \underline{\underline{q}} \rangle = \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\underline{q}} \text{ dx dt} = 2G(\underline{\underline{\epsilon}}(u) + q) - \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(u) \text{ dx dt}.$$

Substituting the previous equation in (2.54) we have

$$\begin{aligned} -L(\underline{u}, p^*) &= \sup_{\underline{\underline{p}} \in Y} \left[\langle p^*, \underline{\underline{p}} \rangle - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u) + \underline{\underline{p}}) \right] \\ &= \langle p^*, \underline{\underline{q}} \rangle - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u) + \underline{\underline{q}}) \\ &= G(\underline{\underline{\epsilon}}(u) + q) - F(u) - \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(u) \text{ dx dt}. \end{aligned} \tag{2.56}$$

From the arbitrariness of $\underline{\underline{\varphi}} \in Y$ in (2.55) we obtain

$$\mathcal{R}(p^*) = 2\mu(\underline{\underline{q}} + \underline{\underline{\epsilon}}(u)) + \lambda(\text{tr}(\underline{\underline{q}})\underline{\underline{id}} + \text{tr}(\underline{\underline{\epsilon}}(u))\underline{\underline{id}}),$$

and we can deduce that

$$\begin{aligned} \underline{\underline{q}}^D &= \frac{1}{2\mu} \mathcal{R}(p^*)^D - \underline{\underline{\epsilon}}(u)^D \\ \text{tr}(\underline{\underline{q}}) &= \frac{1}{2\mu + 2\lambda} \text{tr}(\mathcal{R}(p^*)) - \text{tr}(\underline{\underline{\epsilon}}(u)) \\ \underline{\underline{q}} = \underline{\underline{q}}^D + \frac{1}{2}\text{tr}(\underline{\underline{q}})\underline{\underline{id}} &= \frac{1}{2\mu} \mathcal{R}(p^*)^D + \frac{1}{4(\mu + \lambda)} \text{tr}(\mathcal{R}(p^*)) - \underline{\underline{\epsilon}}(u). \end{aligned} \tag{2.57}$$

Substituting (2.57) in (2.56) we obtain

$$\begin{aligned} - L(\underline{u}, p^*) = \int_0^T \int_\Omega \frac{1}{4\mu} |\mathcal{R}(p^*)^D|^2 + \frac{1}{8(\mu + \lambda)} |\text{tr}(\mathcal{R}(p^*))|^2 - \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(u) \text{ dx} \\ - \langle \rho \underline{\bar{u}}_{tt}, \underline{u} \rangle_{H_0^1(\Omega)^2} + \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)^2} \text{ dt}. \end{aligned} \tag{2.58}$$

Notice that the right-hand side of the last equation depends on $\underline{\underline{p}}^*$ only through the Riesz operator $\mathcal{R}$. Since Y is an Hilbert space, so $\mathcal{R}$ is an isomorphism, we can replace Y^* with Y and $\mathcal{R}(p^*)$ with $\underline{\underline{p}}$, and we obtain

$$\begin{aligned} &\inf_{p^* \in Y^*} \sup_{\underline{u} \in \Gamma} \bigg\{ \int_0^T \int_\Omega \frac{1}{4\mu} |\mathcal{R}(p^*)^D|^2 + \frac{1}{8(\mu + \lambda)} |\text{tr}(\mathcal{R}(p^*))|^2 - \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(u) \text{ dx} \\ &\qquad - \langle \rho \underline{\bar{u}}_{tt}, \underline{u} \rangle_{H_0^1(\Omega)^2} + \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)^2} \text{ dt} \bigg\} \\ &= \inf_{\underline{\underline{p}} \in Y} \sup_{\underline{u} \in \Gamma} \bigg\{ \int_0^T \int_\Omega \frac{1}{4\mu} |\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu + \lambda)} |\text{tr}(\underline{\underline{p}})|^2 - \underline{\underline{p}} : \underline{\underline{\epsilon}}(u) \text{ dx} - \langle \rho \underline{\bar{u}}_{tt}, \underline{u} \rangle_{H_0^1(\Omega)^2} + \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)^2} \text{ dt} \bigg\}. \end{aligned}$$

Thus, from the previous equation and according to Definition 1.4.3 of saddle point problem, we arrive to consider the following.

Problem 2.2.4 (Saddle point formulation of linear elasticity). *Find* $(\underline{w}, \underline{\underline{\tau}}) \in \Gamma \times Y$ *such that*

$$\begin{aligned}&\int_0^T \int_\Omega \frac{1}{4\mu}|\underline{\underline{\tau}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{\tau}})|^2 - \underline{\underline{\tau}} : \underline{\underline{\epsilon}}(w) \ \mathrm{dx} - \langle \rho \underline{\bar{u}}_{tt}, \underline{w} \rangle_{H_0^1(\Omega)^2} + \langle \underline{f}, \underline{w} \rangle_{L^2(\Omega)^2} \ \mathrm{dt} \\ &= \inf_{\underline{\underline{p}} \in Y} \sup_{\underline{u} \in \Gamma} \left\{ \int_0^T \int_\Omega \frac{1}{4\mu}|\underline{\underline{p}}^D|^2 + \frac{1}{8(\mu+\lambda)}|\mathrm{tr}(\underline{\underline{p}})|^2 - \underline{\underline{p}} : \underline{\underline{\epsilon}}(u) \ \mathrm{dx} - \langle \rho \underline{\bar{u}}_{tt}, \underline{u} \rangle_{H_0^1(\Omega)^2} + \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)} \ \mathrm{dt} \right\}.\end{aligned} \tag{2.59}$$

Proposition 2.2.3. *Problem 2.2.4 admits a unique solution* $(\underline{\bar{u}}, \underline{\underline{\sigma}})$ *where* $\underline{\bar{u}}$ *is the unique solution of Problem 2.2.2 and* $\underline{\underline{\sigma}} = \mathcal{R}(q^*)$*, where* q^* *is the unique solution of Problem 2.2.3.*

Proof. According to Proposition 1.4.1, since Problem 2.2.2 and Problem 2.2.3 admit unique solution, respectively due to Proposition 2.2.1 and Proposition 2.2.2, we have guarantee that Problem 2.2.4 admits a unique solution, which is the composition of the solutions of Problem 2.2.2 and Problem 2.2.3. Recalling that we have replace Y^* with Y, if we denote $\underline{\underline{\sigma}} = \mathcal{R}(q^*)$, where q^* is the unique solution of Problem 2.2.3, we have that the unique saddle point of Problem 2.2.4 is $(\underline{\bar{u}}, \underline{\underline{\sigma}})$. □

Computing the Euler-Lagrange equations of (2.59) we obtain

$$\begin{cases} \int_0^T \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(y) \ \mathrm{dx} + \langle \rho \underline{\bar{u}}_{tt}, \underline{y} \rangle_{H_0^1(\Omega)^2} - \langle \underline{f}, \underline{y} \rangle_{L^2(\Omega)} \ \mathrm{dt} = 0 & \forall \underline{y} \in L^2(0,T; H_0^1(\Omega)^2) \\ \int_0^T \int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D - 2\mu \underline{\underline{\epsilon}}(\bar{u})^D : \underline{\underline{\varphi}}^D \ \mathrm{dx \ dt} = 0 & \forall \underline{\underline{\varphi}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym}) \\ \int_0^T \int_\Omega \mathrm{tr}(\underline{\underline{\sigma}}) \,\mathrm{tr}(\underline{\underline{\varphi}}) - 2(\mu+\lambda) \,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u})) \,\mathrm{tr}(\underline{\underline{\varphi}}) \ \mathrm{dx \ dt} = 0 & \forall \underline{\underline{\varphi}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym}) \end{cases} \tag{2.60}$$

Remark 2.2.1. *From the first equation of the previous system since* $\rho \underline{u}_{tt}$ *is in* $L^2(0,T; H^{-1}(\Omega)^2)$ *and not in* $L^2(0,T; L^2(\Omega)^2)$ *even if we require* $\underline{f} \in L^2(0,T; L^2(\Omega)^2)$ *we can not deduce that* $\mathrm{div}(\sigma) \in L^2(0,T; L^2(\Omega)^2)$*. We will see in Section 2.4 that adding a viscosity term to the first equation of* (2.20) *we will gain enough regularity to* $\underline{\bar{u}}$ *to have* $\underline{\underline{\sigma}} \in L^2(0,T; \underline{\underline{H}}(\mathrm{div}, \Omega))$*.*

2.2.3 Mixed formulation

Now we are interested in finding the couple $(\underline{u}, \underline{\underline{\sigma}})$ solution of the system (2.60), with the addition of the initial conditions for $\underline{u}$, i.e. $\underline{u}(\cdot, 0) = \underline{u}_0$ and $\underline{u}_t(\cdot, 0) = \underline{u}_1$. We introduce the following.

Problem 2.2.5 (Mixed PDE formulation in $L^2(0,T; L^2(\Omega)^{2\times 2}_{sym})$ of linear elasticity). *Find the couple* $(\underline{u}, \underline{\underline{\sigma}})$ *such that* $\underline{u} \in L^2(0,T; H_0^1(\Omega)^2)$*,* $\underline{u}_t \in L^2(0,T, L^2(\Omega)^2)$*,* $\rho \underline{u}_{tt} \in L^2(0,T, H^{-1}(\Omega)^2)$*,* $\underline{\underline{\sigma}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym})$ *and satisfies*

$$\begin{cases} \int_0^T \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(y) \ \mathrm{dx} + \langle \rho \underline{u}_{tt}, \underline{y} \rangle_{H_0^1(\Omega)^2} - \langle \underline{f}, \underline{y} \rangle_{L^2(\Omega)} \ \mathrm{dt} = 0 & \forall \underline{y} \in L^2(0,T; H_0^1(\Omega)^2) \\ \int_0^T \int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D - 2\mu \underline{\underline{\epsilon}}(u)^D : \underline{\underline{\varphi}}^D \ \mathrm{dx \ dt} = 0 & \forall \underline{\underline{\varphi}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym}) \\ \int_0^T \int_\Omega \mathrm{tr}(\underline{\underline{\sigma}}) \,\mathrm{tr}(\underline{\underline{\varphi}}) - 2(\mu+\lambda) \,\mathrm{tr}(\underline{\underline{\epsilon}}(u)) \,\mathrm{tr}(\underline{\underline{\varphi}}) \ \mathrm{dx \ dt} = 0 & \forall \underline{\underline{\varphi}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym}) \\ \underline{u}(\cdot, 0) = \underline{u}_0 & \text{in } \Omega \\ \underline{u}_t(\cdot, 0) = \underline{u}_1 & \text{in } \Omega. \end{cases} \tag{2.61}$$

Proposition 2.2.4. *For every $(\underline{u}, \underline{\underline{\sigma}})$ solution of Problem 2.2.5, $\underline{\underline{\sigma}}$ is unique determined by $\underline{u}$ and*

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^D + \frac{1}{2}\operatorname{tr}(\underline{\underline{\sigma}})\underline{\underline{id}} = 2\mu\underline{\underline{\epsilon}}(u) + \lambda \operatorname{div}(u)\underline{\underline{id}} \tag{2.62}$$

Proof. Since $(\underline{u}, \underline{\underline{\sigma}})$ is a solution, then $\underline{\underline{\sigma}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym})$. From the second and third equations of (2.61) we can deduce that

$$\begin{aligned}\underline{\underline{\sigma}}^D &= 2\mu\underline{\underline{\epsilon}}(u)^D \\ \operatorname{tr}(\underline{\underline{\sigma}}) &= (2\mu + 2\lambda)\operatorname{tr}(\underline{\underline{\epsilon}}(u)).\end{aligned}$$

Thus we have (2.62). □

Proposition 2.2.5. *For every $(\underline{u}, \underline{\underline{\sigma}})$ solution of Problem 2.2.5, $\underline{u}$ must satisfy the following*

$$\int_0^T \langle \rho \underline{u}_{tt}, \underline{y}\rangle_{H_0^1(\Omega)^2} + a(\underline{u}, \underline{y}) \text{ dt} = \int_0^T \langle \underline{f}, \underline{y}\rangle_{L^2(\Omega)^2} \text{ dt} \quad \forall \underline{y} \in L^2(0,T; H_0^1(\Omega)^2) \tag{2.63}$$

where $a(\cdot,\cdot) : H_0^1(\Omega)^2 \times H_0^1(\Omega)^2 \to \mathbb{R}$ is defined in (2.19).

Proof. A direct consequence of Proposition 2.2.4 is that we can substitute $\underline{\underline{\sigma}}$ defines in (2.62) in the first equation of (2.61) and recalling the definition of $a(\cdot,\cdot)$ we obtain

$$\int_0^T \langle \rho \underline{u}_{tt}, \underline{y}\rangle_{H_0^1(\Omega)^2} + a(\underline{u}, \underline{y}) \text{ dt} = \int_0^T \langle \underline{f}, \underline{y}\rangle_{L^2(\Omega)^2} \text{ dt} \quad \forall \underline{y} \in L^2(0,T; H_0^1(\Omega)^2),$$

that is exactly (2.63). □

Now we state the main result of this section.

Theorem 2.4. *The function $\underline{\bar{u}}$ is the unique solution of Problem 2.20 if and only if the $(\underline{\bar{u}}, \underline{\underline{\sigma}})$ is the unique solution of Problem 2.2.5, where $\underline{\underline{\sigma}}$ is defined in (2.62).*

Proof. (⇒)
From the construction we have done in the previous section we have that $(\underline{\bar{u}}, \underline{\underline{\sigma}})$ satisfies (2.60) and $\underline{\underline{\sigma}} = \mathcal{R}(q^*)$, where q^* is the unique solution of Problem 2.2.3. Since $\underline{\underline{\sigma}} \in L^2(0,T; L^2(\Omega)^{2\times 2}_{sym})$ from the second and third equations of (2.60) we obtain that $\underline{\underline{\sigma}}$ is defined as in (2.62). Finally, since $\underline{\bar{u}}$ is the unique solution of Problem 2.20, then it satisfies the initial conditions.
(⇐)
Let $(\underline{u}, \underline{\underline{\sigma}})$ be solution of Problem 2.2.5, from Proposition 2.2.5 and, since $(\underline{u}, \underline{\underline{\sigma}})$ is solution of Problem 2.2.5, we have that

$$\begin{cases} \int_0^T \langle \rho \underline{u}_{tt}, \underline{y}\rangle_{H_0^1(\Omega)^2} + a(\underline{u}, \underline{y}) \text{ dt} = \int_0^T \langle \underline{f}, \underline{y}\rangle_{L^2(\Omega)} \text{ dt} & \forall \underline{y} \in L^2(0,T; H_0^1(\Omega)^2) \\ \underline{u}(\cdot,0) = \underline{u}_0 & \text{in } \Omega \\ \underline{u}_t(\cdot,0) = \underline{u}_1 & \text{in } \Omega. \end{cases}$$

Thus $\underline{u}$ is solution of Problem 2.20.
Since there exists a unique solution of Problem 2.20, then $(\underline{\bar{u}}, \underline{\underline{\sigma}})$ is the unique solution of Problem 2.2.5. □

2.3 One dimensional elastic waves with dissipation

In this section we aim to introduce a mixed formulation for one dimensional elastic waves with dissipation. As we have announced in Remark 2.1.3 in Section 2.1 if we require stronger condition on the forcing function and the velocity initial date, respect to the condition we have used in the conservative case, we obtain that $\sigma \in L^2(0,T;H(\mathrm{div},\Omega))$.
We will start form presenting the starting problem and using a result from [15] we will show that it has a unique solution. Then, proceeding as in Section 2.1, according to the theory we have developed in the previous chapter in Sections 1.3 and 1.4 we will introduce a variational principle through which we will construct a mixed formulation. After noticing that under our assumption $\underline{\underline{\sigma}}$ is in $L^2(0,T;H(\mathrm{div},\Omega))$ we will be able to introduce a new mixed formulation. Finally we will show that the solution of the new mixed formulation coincides with the solution of the starting problem.

2.3.1 Starting problem

Assumption 2.3.1. *Let $\Omega \subset \mathbb{R}$ bounded , $T > 0$, $u_0 \in H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f : [0,T] \to L^2(\Omega)^*$ Hölder continuous and $\rho, k, \gamma \in L^\infty(\Omega)$ such that $0 < A \le \rho, k, \gamma \le B < \infty$ a.e. in Ω.*

We recall Remark 2.1.1 according to which even if we are work in one dimension we will denote with $\cdot'$ the spatial derivative and with $\frac{\mathrm{d}}{\mathrm{dx}}$ the divergence operator. We consider the following.

Problem 2.3.1 (Starting Problem one dimensional elastic waves with dissipation)**.** *Find u such that*

$$u \in \mathcal{C}^1((0,T];H_0^1(\Omega)) \cap \mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^2((0,T];L^2(\Omega)) \cap \mathcal{C}^1([0,T];H_0^1(\Omega)) \tag{2.64}$$

and satisfies

$$\begin{cases} \rho u_{tt} = 2\frac{\mathrm{d}}{\mathrm{dx}}(ku') + 2\frac{\mathrm{d}}{\mathrm{dx}}(\gamma u_t') + f & \text{in } \Omega \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega \\ u_t(\cdot,0) = u_1 & \text{in } \Omega. \end{cases} \tag{2.65}$$

Problem 2.3.1 can be cast within the framework of second order evolution equation in Hilbert spaces. Indeed we can provide existence ad uniqueness of solution of Problem 2.3.1 using [15, Thm. 2.2 in Ch. 6]. For the reader's convenience we first recall this result.

Theorem 2.5 (Theorem 2.2 in Chapter 6 of [15])**.** *Let $\mathcal{A}$ and $\mathcal{C}$ be the Riesz maps of the Hilbert spaces V and W and V is dense and continuously embedded in W. Let $\mathcal{B}$ a linear operator form V to V^* and assume that there exists constants $C > 0$ and $\delta > 0$ such that $\mathcal{B} + \delta\mathcal{C}$ satisfies*

$$(\mathcal{B} + \delta\mathcal{C})[u](u) \ge C\|u\|_V^2 \quad \forall u \in V,$$

Then for every Hölder continuous $f : [0,\infty) \to W^*$, $u_0 \in V$ and $u_1 \in W$, there is a unique solution $u(t)$ of Problem A.2.1 on $t > 0$ with $u(0) = u_0$ and $u_t(0) = u_1$.

A direct consequence of the previous theorem is the most important result of this subsection.

Theorem 2.6 (Existence and uniqueness of solution)**.** *There exists a unique solution of Problem 2.3.1.*

Proof. In order to show that there exists a unique solution to the Problem 2.3.1 we will verify that (2.65) satisfies the hypothesis of Theorem 2.5. We start form setting $V = H_0^1(\Omega)$ and $W = L^2(\Omega)$. We have to show that $\mathcal{A} = -\frac{\mathrm{d}}{\mathrm{dx}}(k\cdot') : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a Riesz map, i.e. we need to check that

$$(u, v)_{H_0^1(\Omega)} := \mathcal{A}[v](u) = \int_\Omega ku'v' \ \mathrm{dx} \quad \text{for } u, v \in H_0^1(\Omega) \tag{2.66}$$

is a scalar product (i.e. symmetric, linear and positive define). Clearly $(\cdot,\cdot)_{H_0^1(\Omega)}$ is symmetric and linear. Since $0 < A \leq k \leq B < \infty$ we have that

$$(u, u)_{H_0^1(\Omega)} := \int_\Omega k|u'|^2 \ \mathrm{dx} \geq A \int_\Omega |u'|^2 \ \mathrm{dx} = A\|u\|^2_{H_0^1} \quad \forall u \in H_0^1(\Omega), \tag{2.67}$$

where $\|\cdot\|_{H_0^1(\Omega)^2}$ is the usual norm in $H_0^1(\Omega)$, so it is positive define. So the bilinear form defined in (2.66) is a scalar product, where $\mathcal{A}$ is its Riesz map.

Now we have to show that $\mathcal{C} = \rho : L^2(\Omega) \to (L^2(\Omega))^*$ is a Riesz map, i.e. we need to check that

$$(u, v)_{L^2(\Omega)} := \mathcal{C}[v](u) = \int_\Omega \rho uv \ \mathrm{dx} \quad \text{for } u, v \in H_0^1(\Omega), \tag{2.68}$$

is a scalar product. Clearly it is symmetric and linear and since $0 < A \leq \rho \leq B < \infty$, we have that

$$(u, u)_{L^2(\Omega)} := \int_\Omega \rho|u|^2 \ \mathrm{dx} \geq A\|u\|^2_{L^2(\Omega)} \quad \forall u \in L^2(\Omega), \tag{2.69}$$

where $\|\cdot\|_{L^2(\Omega)^2}$, so is the usual norm in $L^2(\Omega)^2$ it is positive define. Thus the bilinear form defined in (2.68) is a scalar product where $\mathcal{C}$ is its Riesz map.

Since $0 < A \leq k, \rho \leq B < \infty$, we have that

$$(u, u)_{H_0^1(\Omega)} := \int_\Omega k|u'|^2 \ \mathrm{dx} \leq B \int_\Omega |u'|^2 \ \mathrm{dx} = B\|u\|^2_{H_0^1} \quad \forall u \in H_0^1(\Omega), \tag{2.70}$$

and

$$(u, u)_{L^2(\Omega)} := \int_\Omega \rho|u|^2 \ \mathrm{dx} \leq B\|u\|^2_{L^2(\Omega)} \quad \forall u \in L^2(\Omega). \tag{2.71}$$

So from (2.67) and (2.70) we can deduce that the scalar product defined in (2.66) induces a norm that is equivalent to the usual one in $H_0^1(\Omega)$ and from (2.69) and (2.71) we can deduce that the scalar product defined in (2.68) induces a norm that is equivalent to the usual one in $L^2(\Omega)$. Since with the usual norms $H_0^1(\Omega)^2$ is dense and immersed with continuity in $L^2(\Omega)^2$ it is also true for the norms induced by $\mathcal{C}$ and $\mathcal{A}$.

Now it remains to show that there exists a constant $C > 0$ and $\delta > 0$ such that $\mathcal{B} + \delta\mathcal{C}$ satisfies

$$(\mathcal{B} + \delta\mathcal{C})[u](u) \geq C\|u\|^2_{H_0^1(\Omega)} \quad \forall u \in H_0^1(\Omega). \tag{2.72}$$

where $\mathcal{B} = -2\frac{\mathrm{d}}{\mathrm{dx}}(\gamma\cdot') : H_0^1(\Omega) \to H^{-1}(\Omega)$.

Since $0 < A \leq \gamma \leq B < \infty$, we obtain

$$(\mathcal{B} + \delta\mathcal{C})[u](u) = \int_\Omega \gamma|u'|^2 + \delta\rho|u|^2 \mathrm{dx} \geq \int_\Omega \gamma|u'|^2 \mathrm{dx} \geq A\|u\|^2_{H^1_0(\Omega)}.$$

Thus (2.72) is satisfy for all $\delta > 0$. Since all hypothesis of Theorem 2.5 are verified we have that there exists a unique solution of Problem 2.3.1. □

The regularity of the solution obtained by applying Theorem 2.5 is not enough for us. Indeed, according to [15, Cor. 3.2 in Ch. 4], we are able to sharpen the above regularity result and obtain the regularity we need for our purpose, in the case $(u_0, u_1) \in D(A)$, where

$$A = \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{C}^{-1} \end{bmatrix} \begin{bmatrix} 0 & -\mathcal{A} \\ \mathcal{A} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \mathcal{C}^{-1}\mathcal{A} & \mathcal{C}^{-1}\mathcal{B} \end{bmatrix}$$

and

$$D(A) = \left\{ (x_1, x_2) \in H^1_0(\Omega) \times H^1_0(\Omega) \mid \mathcal{A}(x_1) + \mathcal{B}(x_2) \in L^2(\Omega)^* \right\}.$$

Since $\mathcal{A} = -2\frac{\mathrm{d}}{\mathrm{dx}}(k\cdot')$ and $\mathcal{B} = -2\frac{\mathrm{d}}{\mathrm{dx}}(\gamma\cdot')$ we have that $(x_1, x_2) \in D(A)$ if and only if $(x_1, x_2) \in H^1_0(\Omega) \times H^1_0(\Omega)$ and

$$2\frac{\mathrm{d}}{\mathrm{dx}}(kx_1') + 2\frac{\mathrm{d}}{\mathrm{dx}}(\gamma x_2') \in L^2(\Omega)^*.$$

So we have the following.

Corollary 2.3.1. *If $u_1 \in H^1_0(\Omega)$ and u_0, u_1 are such that*

$$2\frac{\mathrm{d}}{\mathrm{dx}}(ku_0') + 2\frac{\mathrm{d}}{\mathrm{dx}}(\gamma u_1') \in L^2(\Omega)^*.$$

Then we have that $\bar{u}$, unique solution of Problem 2.3.1, satisfies

$$\bar{u} \in \mathcal{C}^1([0, T]; H^1_0(\Omega)) \cap \mathcal{C}^2([0, T]; L^2(\Omega)).$$

For the purpose of our work from now on we can assume the following.

Assumption 2.3.2. *The function $u_1 \in H^1_0(\Omega)$ and*

$$2\frac{\mathrm{d}}{\mathrm{dx}}(ku_0') + 2\frac{\mathrm{d}}{\mathrm{dx}}(\gamma u_1') \in L^2(\Omega)^*.$$

Definition 2.3.1. *We will denote by*

$$\bar{u} \in \mathcal{C}^1([0, T]; H^1_0(\Omega)) \cap \mathcal{C}^2([0, T]; L^2(\Omega)),$$

the solution of Problem 2.3.1 provided by Theorem 2.5 and Corollary 2.3.1.

2.3.2 Construction of mixed formulation

In order to derive our mixed formulation we proceed as follows. First we characterize $\bar{u}$ as the unique minimum of a variational problem, then according to the theory we have developed in Section 1.3 and Section 1.4 we write its dual and saddle point formulation. We start form introducing the functional

$$\mathcal{S}(u) = \int_0^T \int_\Omega \rho \bar{u}_{tt} u + |ku'|^2 + 2\gamma \bar{u}_t' u' \text{ dx} - \langle f, u \rangle_{L^2(\Omega)} \text{ dt}, \tag{2.73}$$

acting on

$$\Gamma = \left\{ u \;\middle|\; u(x,t) \in L^2(0,T; H_0^1(\Omega)) \right\},$$

and now we are ready to consider the following problem.

Problem 2.3.2 (Primal formulation of one dimensional elastic waves)**.** *Find $u \in \Gamma$ such that*

$$\mathcal{S}(u) = \inf_{v \in \Gamma} \{\mathcal{S}(v)\} \tag{2.74}$$

Proposition 2.3.1. *The function $\bar{u}$ is the unique minimum of Problem 2.3.2.*

Proof. Since $\bar{u} \in \Gamma$, then we can compute the first variation of (2.73) evaluated in $\bar{u}$ and we obtain

$$\begin{aligned} d\mathcal{S}(\bar{u}).y &= \int_0^T \rho \bar{u}_{tt} y + 2k\bar{u}' y' + 2\gamma \bar{u}_t' y' \text{ dx} - \langle f, y \rangle_{L^2(\Omega)} \text{ dt} \\ &= \int_0^T \left\langle \rho \bar{u}_{tt} + \frac{\text{d}}{\text{dx}}(2k\bar{u}') - \frac{\text{d}}{\text{dx}}(2\gamma \bar{u}_t') - f, y \right\rangle_{L^2(\Omega)} \text{ dt} = 0 \quad \forall y \in \Gamma, \end{aligned}$$

where we have used that $\bar{u}$ satisfies the first equation of (2.65). Thus $\bar{u}$ is a critical point of $\mathcal{S}$. In order to prove that $\bar{u}$ is the unique minimum of $\mathcal{S}$ in Γ we will show that $\mathcal{S}$ is strongly convex in Γ. This is equivalent to prove that the second variation of $\mathcal{S}$ is positive define for all $u \in \Gamma$:

$$d^2\mathcal{S}(u).(y,y) = \int_0^T \int_\Omega 2k|y'|^2 \text{ dx dt} \geq \int_0^T 2A\|y\|^2_{H_0^1(\Omega)} \text{ dt} \geq 2A\|y\|^2_{L^2(0,T;H_0^1(\Omega))} \quad \forall y \in \Gamma.$$

□

In order to introduce the dual and the saddle point formulation for Problem 2.3.2 it is convenient to rewrite (2.74) as follows:

$$\mathcal{S}(u) = \inf_{v \in \Gamma} \{\mathcal{S}(v)\} = \inf_{v \in \Gamma} \{F(v) + G(v')\},$$

where $F : \Gamma \to \mathbb{R}$ is

$$F(v) = \int_0^T \int_\Omega \rho \bar{u}_{tt} v \text{ dx} - \langle f, v \rangle_{L^2(\Omega)} \text{ dt}, \tag{2.75}$$

and $G : Y := L^2(0,T; L^2(\Omega)) \to \mathbb{R}$ is

$$G(p) = \int_0^T \int_\Omega k|p|^2 + 2\gamma \bar{u}_t' p \text{ dx dt}. \tag{2.76}$$

Now we check that the dual problem admits a unique solution. Starting from Problem 2.3.2, if we define $\Phi(u, p) = F(u) + G(u' + p)$, then, according to Definition 1.3.2, we can consider the following.

Problem 2.3.3 (Dual formulation of one dimensional elastic waves with dissipation)**.** *Find $q^* \in Y^*$ such that*

$$-\Phi^*(0, q^*) = \sup_{p^* \in Y^*} \big[-\Phi^*(0, p^*)\big] = \sup_{p^* \in Y^*} \big[-F^*(-\Lambda^* p^*) - G^*(p^*)\big], \tag{2.77}$$

where Λ^* is $\frac{\mathrm{d}}{\mathrm{dx}}$, the adjoint of $\cdot'$, according to Definition A.1.8, $F^* : \Gamma^* \to \mathbb{R}$ is

$$F^*(-\frac{\mathrm{d}}{\mathrm{dx}}p^*) = \begin{cases} 0 & \textit{if } -\frac{\mathrm{d}}{\mathrm{dx}}p^* - \rho\bar{u}_{tt} + f = 0 \\ +\infty & \textit{otherwise}, \end{cases}$$

and $G^* : Y^* \to \mathbb{R}$ is

$$G^*(p^*) = \int_0^T \int_\Omega \frac{1}{4k} |\mathcal{R}(p^*) - 2\gamma\bar{u}_t'|^2 \ \mathrm{dx}\ \mathrm{dt}$$

where $\mathcal{R}$ is the Riesz operator such that $\mathcal{R} : Y^* \to Y$.

Proposition 2.3.2. *Problem 2.3.2 is stable, in the sense of Definition 1.3.4. Moreover Problem 2.3.3 admits a unique solution.*

Proof. In order to prove stability we verify the hypothesis of Theorem 1.1. Since F, defined in (2.75), is linear and G, defined in (2.76), is strictly convex, because is sum of a strictly convex functional

$$\int_0^T \int_\Omega k|p|^2 \ \mathrm{dx}\ \mathrm{dt},$$

and a linear one

$$\int_0^T \int_\Omega 2\gamma\bar{u}_t' p \ \mathrm{dx}\ \mathrm{dt},$$

then $J(u, p) = F(u) + G(p)$ is convex in $\Gamma \times Y$:

$$\begin{aligned} J(\alpha(u, p) + (1-\alpha)(v, q)) &= F(\alpha u + (1-\alpha)v) + G(\alpha p + (1-\alpha)q) \\ &= \alpha F(u) + (1-\alpha)F(v) + G(\alpha p + (1-\alpha)q) \\ &\leq \alpha F(u) + (1-\alpha)F(v) + \alpha G(p) + (1-\alpha)G(q) \\ &\leq \alpha J(u, p) + (1-\alpha)J(v, q), \end{aligned}$$

for $\alpha \in [0, 1]$, $u, v \in \Gamma$ and $p, q \in Y$.
Now we need to verify that there exists u_0 such that $F(u_0)+G(u_0') < +\infty$ and $p \to F(u_0)+G(p')$ is continuous in u_0'. Taking $u_0 = 0$ we have that $F(0) = 0$ and, since $0' = 0$, $G(0') = 0$, so we need to verify that $p \to F(u_0) + G(p)$ is continuos at 0. Let $p_n \to p$ in Y as $n \to \infty$, then since $k \in L^\infty(\Omega)$ and $\|p\|_Y^2 \coloneqq \int_0^T \int_\Omega |p|^2 \ \mathrm{dx}\ \mathrm{dt}$ we have

$$\lim_{n\to\infty} \int_0^T \int_\Omega k|p_n|^2 \ \mathrm{dx}\ \mathrm{dt} = \int_0^T \int_\Omega k|p|^2 \ \mathrm{dx}\ \mathrm{dt},$$

and since $\gamma \in L^\infty(\Omega)$ and $\bar{u}'_t \in L^2(0,T;L^2(\Omega))$ we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_\Omega 2\gamma\bar{u}'_t p_n \text{ dx dt} &= \lim_{n\to\infty}\int_0^T \langle 2\gamma\bar{u}'_t, p_n\rangle_{L^2(\Omega)} \text{ dt}\\ &= \lim_{n\to\infty}\langle 2\gamma\bar{u}'_t, p_n\rangle_Y\\ &= \langle 2\gamma\bar{u}'_t, p\rangle_Y\\ &= \int_0^T\int_\Omega 2\gamma\bar{u}'_t p \text{ dx dt}.\end{aligned}$$

So Problem 2.3.2 is stable and due to Proposition 1.3.6, Problem 2.3.3 has at least one solution. Can be noticed that G is differentiable and the differential of G in p evaluated in q is

$$G^{'}[p](q) = \int_0^T\int_\Omega 2kpq + 2\gamma\bar{u}'_t q \text{ dx dt}.$$

Since G is differentiable and primal Problem 2.3.2 admits a unique solution, according to Proposition 1.3.7 and Proposition A.1.7, we have that the dual Problem 2.3.3 admits a unique solution. □

Before introducing the saddle point problem, according to the theory in Section 1.4, we need to study the Lagrangian defined in Definition 1.4.1:

$$-L(\underline{u}, p^*) = \sup_{\underline{\underline{p}}\in Y}\left[\langle p^*, \underline{\underline{p}}\rangle - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u) + \underline{\underline{p}})\right]. \tag{2.78}$$

Since F, defined in (2.75), is linear and G, defined in (2.76) is strictly convex we have that the argument inside the supremum in (2.78) is a strictly concave function. Thus there exists a unique $q \in Y$ such that

$$\langle p^*, q\rangle_Y - F(u) - G(u' + q) = \sup_{p\in Y}\left[\langle p^*, p\rangle_Y - F(u) - G(u' + p)\right].$$

Computing the first variation respect to p evaluated in q of $\langle p^*, p\rangle_Y - F(u) - G(u' + p)$ we obtain

$$\int_0^T\int_\Omega \mathcal{R}(p^*)\varphi - 2k(q + u')\varphi - 2\gamma\bar{u}'_t\varphi \text{ dx dt} = 0 \quad \forall\varphi \in Y, \tag{2.79}$$

where $\mathcal{R} : Y^* \to Y$ is the Riesz operator defined in Theorem A.1 such that

$$\langle p^*, \varphi\rangle_Y := \int_0^T\int_\Omega \mathcal{R}(p^*)\varphi \quad \forall\varphi \in Y.$$

Taking $\varphi = q + u'$ in (2.79) we obtain

$$\langle p^*, q\rangle_Y = 2G(u' + q) - \int_0^T\int_\Omega \mathcal{R}(p^*)u' - 2\gamma\bar{u}'_t(u' + q) \text{ dx dt}. \tag{2.80}$$

Now we introduce the functional $\tilde{G} : Y \to \mathbb{R}$:

$$\tilde{G}(p) = \int_0^T \int_\Omega k|p|^2 \text{ dx dt},$$

which is the functional G without the dissipative term. Then from (2.78) and (2.80) we can deduce that

$$\begin{aligned}
-L(u,p^*) &= \sup_{p\in Y} \left[\langle p^*, p\rangle - F(u) - G(u'+p)\right] \\
&= \langle p^*, q\rangle - F(u) - G(u'+q) \\
&= G(u'+q) - F(u) - \int_0^T \int_\Omega \mathcal{R}(p^*)u' - 2\gamma\bar{u}_t'(u'+q) \text{ dx dt} \\
&= \tilde{G}(u'+q) - F(u) - \int_0^T \int_\Omega \mathcal{R}(p^*)u' \text{ dx dt}. \qquad (2.81)
\end{aligned}$$

Since (2.79) is true for all φ in Y, then we have

$$\mathcal{R}(p^*) = 2k(q+u') + 2\gamma\bar{u}_t',$$

and we can deduce that

$$q = \frac{1}{2k}(\mathcal{R}(p^*) - 2\gamma\bar{u}_t') - u'. \qquad (2.82)$$

Substituting (2.82) in (2.81) we obtain

$$-L(u,p^*) = \int_0^T \int_\Omega \frac{1}{4k}|\mathcal{R}(p^*) - 2\gamma\bar{u}_t'|^2 - \mathcal{R}(p^*)u' - \rho\bar{u}_{tt}u \text{ dx } + \langle f,u\rangle_{L^2(\Omega)} \text{ dt}. \qquad (2.83)$$

Since the right-hand side of the last equation depends on p^* only through the Riesz operator $\mathcal{R}$ and Y is an Hilbert space, we can replace Y^* with Y and $\mathcal{R}(p^*)$ with p, and we obtain

$$\begin{aligned}
&\inf_{p^*\in Y^*} \sup_{u\in\Gamma} \left\{ \int_0^T \int_\Omega \frac{1}{4k}|\mathcal{R}(p^*) - 2\gamma\bar{u}_t'|^2 - \mathcal{R}(p^*)u' \text{ dx} - \rho\bar{u}_{tt}u \text{ dx} + \langle f,u\rangle_{L^2(\Omega)} \text{ dt} \right\} \\
&\qquad = \inf_{p\in Y} \sup_{u\in\Gamma} \left\{ \int_0^T \int_\Omega \frac{1}{4k}|p - 2\gamma\bar{u}_t'|^2 - pu' \text{ dx} - \rho\bar{u}_{tt}u \text{ dx } + \langle f,u\rangle_{L^2(\Omega)} \text{ dt}. \right\}.
\end{aligned}$$

Thus, from the previous equation and according to Definition 1.4.3 of saddle point problem, we arrive to consider the following problem.

Problem 2.3.4 (Saddle point formulation of one dimensional elastic waves with dissipation)**.** *Find* $(w,\tau) \in \Gamma \times Y$ *such that*

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{1}{4k}|\tau - 2\gamma\bar{u}_t'|^2 - \tau w' \text{ dx} - \rho\bar{u}_{tt}w \text{ dx } + \langle f,w\rangle_{L^2(\Omega)} \text{ dt} \\
&\qquad = \inf_{p\in Y} \sup_{u\in\Gamma} \left\{ \int_0^T \int_\Omega \frac{1}{4k}|p - 2\gamma\bar{u}_t'|^2 - pu' \text{ dx} - \rho\bar{u}_{tt}u \text{ dx } + \langle f,u\rangle_{L^2(\Omega)} \text{ dt}. \right\}. \qquad (2.84)
\end{aligned}$$

Proposition 2.3.3. *Problem 2.84 admits a unique solution* $(\bar{u}, \sigma)$ *where* $\bar{u}$ *is the unique solution of Problem 2.3.2 and* $\sigma = \mathcal{R}(q^*)$*, where* q^* *is the unique solution of Problem 2.3.3.*

Proof. According to Proposition 1.4.1 we have that the solutions of Problem 2.3.4 are the product of the solutions of Problem 2.3.2 and Problem 2.3.3. Since Problem 2.3.2 and Problem 2.3.3 admit unique solution $\bar{u}$ and $\sigma = \mathcal{R}(q^*)$, where q^* is the unique solution of Problem 2.3.3, then we have that the saddle point of Problem 2.3.4 admits a unique solution that is $(\bar{u}, \sigma)$. □

If we compute the Euler-Lagrange equations of (2.84) evaluated in $(\bar{u}, \sigma)$ we obtain

$$\begin{cases} \int_0^T \int_\Omega \rho \bar{u}_{tt} y + \sigma y' \text{ dx} - \langle f, y \rangle_{L^2(\Omega)} \text{ dt} = 0 & \forall y \in L^2(0,T; H_0^1(\Omega)) \\ \int_0^T \int_\Omega \sigma\varphi - (2k\bar{u}' + 2\gamma \bar{u}_t')\varphi \text{ dx dt} = 0 & \forall \varphi \in L^2(0,T; L^2(\Omega)) \end{cases} \tag{2.85}$$

According to what we have noticed in Remark 2.1.3, for the purpose of our work form now on we can assume the following.

Assumption 2.3.3. $f : [0,T] \to L^2(\Omega)$ *Hölder continuous.*

A direct consequence of Assumption 2.3.3 is that we can no longer consider

$$\langle f(\cdot, t), u \rangle_{L^2(\Omega)} \quad \forall u \in L^2(\Omega),$$

but

$$(f(\cdot, t), u)_{L^2(\Omega)} := \int_\Omega f(\cdot, t) u \text{ dx} \quad \forall u \in L^2(\Omega).$$

In the first equation of (2.85) since for every $y \in \Gamma$ we have that $y(\cdot, t) \in H_0^1(\Omega)$ for a.e. $t \in [0,T]$, we can define $\frac{\mathrm{d}}{\mathrm{dx}}(\sigma)$ as an element of $L^2(0,T; H^{-1}(\Omega)^2)$ as follows

$$\langle \frac{\mathrm{d}}{\mathrm{dx}}(\sigma), y \rangle_{L^2(0,T;H_0^1(\Omega)^2)} := \int_0^T \langle \frac{\mathrm{d}}{\mathrm{dx}}(\sigma), y \rangle_{H_0^1(\Omega)^2} \text{ dt} = -\int_0^T \int_\Omega \sigma y' \text{ dx dt} \quad \forall y \in H_0^1(\Omega).$$

Now we can rewrite the first equation of (2.85) as follows

$$\rho \bar{u}_{tt} - \frac{\mathrm{d}}{\mathrm{dx}}(\sigma) - f(x,t) = 0 \quad \text{ in } L^2(0,T; L^2(\Omega)^*).$$

Since $\bar{u} \in \mathcal{C}^2([0,T]; L^2(\Omega))$ and, according to Assumption 2.3.3, $f : [0,T] \to L^2(\Omega)$ is Hölder continuous, then we have that $\frac{\mathrm{d}}{\mathrm{dx}}(\sigma) \in L^2(0,T; L^2(\Omega))$, that implies $\sigma \in L^2(0,T; H(\mathrm{div}, \Omega))$.

Remark 2.3.1. *Since we are working with one dimension in space, i.e.* $\Omega \subset \mathbb{R}$*, we have that* $L^2(0,T; H(\mathrm{div}, \Omega)) = L^2(0,T; H^1(\Omega))$ *because* $\frac{\mathrm{d}}{\mathrm{dx}}(\tau) = \tau' = \partial_x \tau$*. If we pass in higher dimension the last chain of equality does not hold, and we can not identify* $L^2(0,T; H(\mathrm{div}, \Omega)^{n\times n})$ *with* $L^2(0,T; H^1(\Omega)^{n\times n})$ *for* $n > 1$*, so we will distinguish* $L^2(0,T; H(\mathrm{div}, \Omega))$ *with* $L^2(0,T; H^1(\Omega))$ *even if they coincide in one dimension.*

Since $\sigma \in L^2(0,T; H(\mathrm{div}, \Omega))$ we can restrict Y to $L^2(0,T; H(\mathrm{div}, \Omega))$ in Problem 2.3.4, if we do that we can integrate by parts $\int_0^T \int_\Omega \sigma y' \text{ dx dt}$ in (2.84) and we obtain

$$\int_0^T \int_\Omega \frac{1}{4k} |p - 2\gamma \bar{u}_t'|^2 + \frac{\mathrm{d}}{\mathrm{dx}}(p) u - \rho \bar{u}_{tt} u + f u \text{ dx dt}.$$

After doing that we no longer need u to be in $\Gamma = L^2(0,T;H_0^1(\Omega))$, because the new functional does not involve space derivative of u, so we can replace Γ with the largest space $L^2(0,T;L^2(\Omega))$. If we make these changes is no longer guarantee that $(\bar{u},\sigma)$ is a saddle point, indeed we do not even know if there exists one for the new problem. Now we are interested in studying the following.

Problem 2.3.5 (Mixed formulation of one dimensional elastic waves with dissipation)**.** *Find* $(w,\tau)\in\Lambda\times W$ *such that*

$$\int_0^T\int_\Omega \frac{1}{4k}|\tau-2\gamma\bar{u}_t'|^2+\frac{\mathrm{d}}{\mathrm{dx}}(\tau)w-\rho\bar{u}_{tt}w+fw\ \mathrm{dx\ dt}$$
$$=\inf_{p\in Y}\sup_{u\in\Gamma}\left\{\int_0^T\int_\Omega \frac{1}{4k}|p-2\gamma\bar{u}_t'|^2+\frac{\mathrm{d}}{\mathrm{dx}}(p)u-\rho\bar{u}_{tt}u+fu\ \mathrm{dx\ dt}\right\}, \quad (2.86)$$

where $W=L^2(0,T;H(\mathrm{div},\Omega))$ *and* $\Lambda=L^2(0,T;L^2(\Omega))$.

Proposition 2.3.4. *If* (w,τ) *is a saddle point of Problem 2.3.5, in sense of Definition 1.4.2, then* $w\in L^2(0,T;H_0^1(\Omega))$.

Proof. Taking the first variation of

$$\int_0^T\int_\Omega \frac{1}{4k}|p-2\gamma\bar{u}_t'|^2+\frac{\mathrm{d}}{\mathrm{dx}}(p)u-\rho\bar{u}_{tt}u+fu\ \mathrm{dx\ dt},$$

respect to the variable p evaluated in (w,τ), we obtain

$$\int_0^T\int_\Omega \frac{\mathrm{d}}{\mathrm{dx}}(\varphi)w\ \mathrm{dx\ dt}=-\int_0^T\int_\Omega\frac{1}{2k}(\tau-2\gamma\bar{u}_t')\varphi\ \mathrm{dx\ dt}\quad \forall\varphi\in W. \quad (2.87)$$

Since $\bar{u}\in\mathcal{C}^1([0,T];H_0^1(\Omega))$ and $L^2(0,T;\mathcal{C}_c^\infty(\Omega))\subset W$, we obtain that we can define

$$w'=\frac{1}{2k}(\tau-2\gamma\bar{u}_t').$$

So we can deduce that $\underline{w}\in L^2(0,T;H^1(\Omega)^2)$. From (2.87), since $L^2(0,T;\mathcal{C}^\infty(\overline{\Omega}))\subset W$, according to Theorem A.3, we obtain that $\mathrm{Tr}(w)=0$ for a.e. $t\in[0,T]$ and that is equivalent to stating that $w\in L^2(0,T;H_0^1(\Omega))$. □

Proposition 2.3.5. *Problem 2.3.5 admits a unique saddle point which is the solution of Problem 2.3.4.*

Proof. Since Problem 2.3.4 admits a unique solution, then the thesis is equivalent to prove that (w,τ) is saddle point of (2.86) if and only if is a saddle point of (2.84). We call $F_1(v,p)$ the functional in (2.84) and $F_2(v,p)$ the functional in (2.86), we can see that if $p\in W$ and $v\in\Gamma$ then we have $F_1(v,p)=F_2(v,p)$ and, due to Proposition 2.3.4, we can deduce that all saddle points of (2.84) and (2.86) satisfy such condition.
Let (w,τ) be a saddle point of (2.84), then

$$\forall v\in\Gamma\quad F_1(v,\tau)\le F_1(w,\tau)\le F_1(w,p)\quad\forall p\in Y, \quad (2.88)$$

and we want to show that

$$\forall v \in \Lambda \quad F_2(v,\tau) \leq F_2(w,\tau) \leq F_2(w,p) \quad \forall p \in W. \tag{2.89}$$

Since $W \subset Y$ and if $p \in W$ we have $F_1(w,p) = F_2(w,p)$, then the right inequality is satisfy. The left one is given by the denseness of Γ in Λ and the continuity of $v \to F_2(v,\tau)$ in Λ. Indeed due to the denseness we have that for every $v \in \Lambda$ there exists $\{v_n\}_{n\in\mathbb{N}} \in \Gamma$ such that $v_n \to v$ in Λ as $n \to \infty$, recalling that if $u \in \Gamma$ then $F_1(u,\tau) = F_2(u,\tau)$, we obtain

$$F_2(w,\sigma) = F_1(w,\sigma) \geq \lim_{n\to\infty} F_1(v_n,\sigma) = \lim_{n\to\infty} F_2(v_n,\sigma) = F_2(v,\sigma).$$

Now we do the converse, let (w,τ) be a saddle point of (2.86), so it satisfies (2.89), we want to prove (2.88). The left side follow from $\Gamma \subset \Lambda$ and $F_1(v,\tau) = F_2(v,\tau)$ for all $v \in \Gamma$. The right side is given by the denseness of Y in W, the continuity of $p \to F_1(w,p)$ in W and that $F_1(w,p) = F_2(w,p)$ for all $p \in W$. Indeed for every $p \in W$ there exists $\{p_n\}_{n\in\mathbb{N}} \in Y$ such that $p_n \to p$ in Y as $n \to \infty$, then we obtain

$$F_1(w,\tau) = F_2(w,\tau) \leq \lim_{n\to\infty} F_2(w,p_n) = \lim_{n\to\infty} F_1(w,p_n) = F_2(w,p).$$

□

From the previous Proposition we have that $(\bar{u},\sigma)$, unique saddle point of Problem 2.3.4, is also the unique solution of Problem 2.3.5, then if we calculate the Euler-Lagrange equations of (2.86), evaluated in $(\bar{u},\sigma)$, we obtain:

$$\begin{cases} \int_0^T \int_\Omega \rho \bar{u}_{tt} y - \frac{\mathrm{d}}{\mathrm{dx}}(\sigma) y - f y \ \mathrm{dx}\ \mathrm{dt} = 0 & \forall y \in \Lambda \\ \int_0^T \int_\Omega \sigma\varphi - (2k\bar{u}' + 2\gamma\bar{u}_t')\varphi \ \mathrm{dx}\ \mathrm{dt} = 0 & \forall \varphi \in W \end{cases} \tag{2.90}$$

2.3.3 Mixed formulation 1

Now we are interested in finding the functions (u,σ) solution of system (2.85) with the addition of initial conditions for u. We introduce the mixed problem:

Problem 2.3.6 (Mixed PDE formualtion in $L^2(0,T;L^2(\Omega))$ of one dimensional elastic waves)**.** *Find the couple (u,σ) such that $u \in H^1(0,T;H_0^1(\Omega))$, $u_{tt} \in L^2(0,T;L^2(\Omega))$, $\sigma \in L^2(0,T;L^2(\Omega))$ and satisfies:*

$$\begin{cases} \int_0^T \int_\Omega \rho u_{tt} y + \sigma y' - f y \ \mathrm{dx}\ \mathrm{dt} = 0 & \forall y \in L^2(0,T;H_0^1(\Omega)) \\ \int_0^T \int_\Omega \sigma\varphi - (2ku' + 2\gamma u_t')\varphi \ \mathrm{dx}\ \mathrm{dt} = 0 & \forall \varphi \in L^2(0,T;L^2(\Omega)) \\ u(\cdot,0) = u_0 & \mathit{in}\ \Omega \\ u_t(\cdot,0) = u_1 & \mathit{in}\ \Omega. \end{cases} \tag{2.91}$$

Theorem 2.7. *The function u is the unique solution of Problem 2.3.1 if and only if (u,σ) is the unique solution of Problem 2.3.6 where σ is define (2.92).*

Proof. The proof is rather long and we decided to divided it in Proposition 2.3.6 and Proposition 2.3.7. □

Proposition 2.3.6. *For every (u, σ) solution of Problem 2.3.6, σ is uniquely determined by u and*

$$\sigma = 2ku' + 2\gamma u_t' \tag{2.92}$$

Proof. Since $\sigma \in L^2(0,T;L^2(\Omega))$ and satisfies the second equation of (2.91) we can deduce (2.92). □

Proposition 2.3.7. *The function u is solution of Problem 2.3.1 if and only if there exists σ such that (u, σ) is solution of the mixed Problem 2.3.6.*

Proof. $(\Rightarrow)$
Let $\bar{u}$ be solution of Problem 2.3.1, then Corollary 2.3.1 implies that $\bar{u}$ is in $H^1(0,T;H_0^1(\Omega))$ and $\bar{u}_{tt} \in L^2(0,T;L^2(\Omega))$. From the construction we have done in Section 2.4.2 we have that $(\bar{u}, \sigma)$, where $\sigma \in L^2(0,T;L^2(\Omega))$, is solution of (2.85), so it is also solution of Problem 2.3.6.
$(\Leftarrow)$
Now we proceed with the opposite, let u_m be the first component of a solution of Problem 2.3.6. We want to prove that $u_m = \bar{u}$. Let start by considering the following.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}(u_{mt} - \bar{u}_t)\|^2_{L^2(\Omega)} &= \langle\sqrt{\rho}(u_{mtt} - \bar{u}_{tt}), \sqrt{\rho}(u_{mt} - \bar{u}_t)\rangle_{L^2(\Omega)} \\ &= (\sqrt{\rho}(u_{mtt} - \bar{u}_{tt}), \sqrt{\rho}(u_{mt} - \bar{u}_t))_{L^2(\Omega)} \\ &= (\rho(u_{mtt} - \bar{u}_{tt}), u_{mt} - \bar{u}_t)_{L^2(\Omega)} \\ &= \langle\rho(u_{mtt} - \bar{u}_{tt}), u_{mt} - \bar{u}_t\rangle_{L^2(\Omega)}\,,\end{aligned} \tag{2.93}$$

where we have used that $\rho \geq A > 0$. Since $u_{mt} - \bar{u}_t \in L^2(0,T;H_0^1(\Omega))$ and recalling that u_m satisfies the first equation of (2.91), and $\bar{u}$ satisfies the first equation of (2.65), from (2.93) we obtain

$$\begin{aligned}&\langle\rho(u_{mtt} - \bar{u}_{tt}), u_{mt} - \bar{u}_t\rangle_{L^2(\Omega)} = \\ &\qquad - \langle\sigma, u_{mt}' - \bar{u}_t'\rangle_{L^2(\Omega)} - \langle\frac{\mathrm{d}}{\mathrm{dx}}(2k\bar{u}' + 2\gamma\bar{u}_t'), u_{mt} - \bar{u}_t\rangle_{L^2(\Omega)}\,.\end{aligned} \tag{2.94}$$

Since $u_{mt} - \bar{u}_t \in L^2(0,T;H_0^1(\Omega))$ we have that $u_{mt}' - \bar{u}_t' \in L^2(0,T;L^2(\Omega))$, so according to the second equation of (2.91) we have

$$-\langle\sigma, u_{mt}' - \bar{u}_t'\rangle_{L^2(\Omega)} = -\langle 2ku_m' + 2\gamma u_{mt}', u_{mt}' - \bar{u}_t'\rangle_{L^2(\Omega)}\,,$$

and since $u_{mt}' - \bar{u}_t' \in L^2(0,T;L^2(\Omega))$ we have

$$\langle\frac{\mathrm{d}}{\mathrm{dx}}(2k\bar{u}' + 2\gamma\bar{u}_t'), u_{mt} - \bar{u}_t\rangle_{L^2(\Omega)} = -\langle 2k\bar{u}' + 2\gamma\bar{u}_t', u_{mt}' - \bar{u}_t'\rangle_{L^2(\Omega)}\,,$$

Denoting $\underline{w} = \underline{u}_m - \underline{\bar{u}}$ and $\underline{w}_t = \underline{u}_{mt} - \underline{\bar{u}}_t$, we can rewrite (2.94) as follows

$$\begin{aligned}
-\langle \sigma, w_t' \rangle_{L^2(\Omega)} + \langle 2k\bar{u}' + 2\gamma \bar{u}_t', w_t' \rangle_{L^2(\Omega)} &= -\langle 2kw' + 2\gamma w_t', w_t' \rangle_{L^2(\Omega)} \\
&= -\frac{d}{dt} \left\| \sqrt{2k} w' \right\|^2_{L^2(\Omega)} - \langle 2\gamma w_t', w_t' \rangle_{L^2(\Omega)} \\
&\leq -\langle 2\gamma w_t', w_t' \rangle_{L^2(\Omega)} \\
&\leq -2A \|w_t\|^2_{H^1_0(\Omega)},
\end{aligned} \tag{2.95}$$

where we have used that $0 < A \leq k \leq B < \infty$ and $k \in L^\infty(\Omega)$.
Since $\rho \geq A > 0$, then from (2.93) and (2.95) we can conclude that

$$\frac{d}{dt} \|u_{mt} - \bar{u}_t\|^2_{L^2(\Omega)} = 0.$$

From the previous equation and because u_m and $\bar{u}$ satisfy the same initial condition, we have $u_m = \bar{u}$. □

2.3.4 Mixed formulation 2

Now we aim to find (u, σ) solution of system (2.90) with the addition of initial conditions for u. We introduce the mixed problem:

Problem 2.3.7 (Mixed PDE formulation in $L^2(0,T; H(\text{div}, \Omega))$ of one dimensional elastic waves)**.** *Find (u, σ) such that $u \in H^1(0,T; H^1_0(\Omega))$, $u_{tt} \in L^2(0,T; L^2(\Omega))$, $\sigma \in L^2(0,T; H(\text{div}, \Omega))$ and satisfies:*

$$\begin{cases}
\int_0^T \int_\Omega \rho u_{tt} y - \frac{\mathrm{d}}{\mathrm{dx}}(\sigma) y - f y \, \mathrm{dx} \, \mathrm{dt} = 0 & \forall y \in \Lambda \\
\int_0^T \int_\Omega \sigma \varphi - (2ku' + 2\gamma u_t')\varphi \, \mathrm{dx} \, \mathrm{dt} = 0 & \forall \varphi \in W \\
u(\cdot, 0) = u_0 & \text{in } \Omega \\
u_t(\cdot, 0) = u_1 & \text{in } \Omega,
\end{cases} \tag{2.96}$$

where $W = L^2(0,T; H(\text{div}, \Omega))$ and $\Lambda = L^2(0,T; L^2(\Omega))$.

Proposition 2.3.8. *The couple (u, σ) is solution of Problem 2.3.6 if and only if is solution of Problem 2.3.7.*

Proof. $(\Rightarrow)$
Let (u, σ) be solution of Problem 2.3.6, so $u = \bar{u}$ and σ is define in (2.92). We have already notice, at the end of the Subsection 2.3.2, that $\sigma \in L^2(0,T; H(\text{div}, \Omega))$, so if (u, σ) satisfies (2.96), then it is a solution of Problem 2.3.7. Since $\rho \bar{u}_{tt}, f, \frac{\mathrm{d}}{\mathrm{dx}}(\sigma) \in L^2(0,T; L^2(\Omega))$ and u satisfies the first equation of (2.91), then, because $L^2(0,T; H^1_0(\Omega))$ is dense in Λ, for every $y \in \Lambda$ there exists

$\{y_n\}_{n\in\mathbb{N}} \subset L^2(0,T;H_0^1(\Omega))$ such that $y_n \to y$ in Λ as $n \to \infty$, then

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \int_0^T \int_\Omega \rho u_{tt} y_n + \sigma y_n' - f y_n \text{ dx dt} \\
&= \lim_{n\to\infty} \int_0^T \int_\Omega \rho u_{tt} y_n - \frac{\text{d}}{\text{dx}}(\sigma) y_n - f y_n \text{ dx dt} \\
&= \lim_{n\to\infty} (\rho u_{tt} - \frac{\text{d}}{\text{dx}}(\sigma) - f, y_n)_\Lambda \\
&= \lim_{n\to\infty} \langle \rho u_{tt} - \frac{\text{d}}{\text{dx}}(\sigma) - f, y_n \rangle_\Lambda \\
&= \langle \rho u_{tt} - \frac{\text{d}}{\text{dx}}(\sigma) - f, y \rangle_\Lambda \\
&= \int_0^T \int_\Omega \rho u_{tt} y - \frac{\text{d}}{\text{dx}}(\sigma) y - f y \text{ dx dt.}
\end{aligned}
$$

Thus the first equation of (2.96) is satisfy. Second equation of (2.96) is satisfy because $W \subset Y$. The initial conditions, because are the same, are clearly satisfy.

$(\Leftarrow)$

Let (u, σ) be solution of Problem 2.3.7, we only need to verify that (u, σ) satisfies (2.91). Since $H_0^1(\Omega) \subset \Lambda$ we can restrict the test functions in the first equation of (2.96) to $L^2(0, T, H_0^1(\Omega))$, if we do that according to the integration by parts we have

$$
\int_0^T \int_\Omega \rho u_{tt} y + \sigma y' - f y \text{ dx dt} = \int_0^T \int_\Omega \rho u_{tt} y - \frac{\text{d}}{\text{dx}}(\sigma) y - f y \text{ dx dt} = 0
$$

Thus the first equation of (2.91) is satisfy.

From the second equation of (2.96), $u \in H^1(0,T;H_0^1(\Omega))$ and $\sigma \in L^2(0,T;H(\text{div},\Omega))$ then, because $H(\text{div}(\Omega))$ is a dense subset of $L^2(\Omega)$, for every $\varphi \in L^2(0,T;L^2(\Omega))$ there exists $\{\varphi_n\}_{n\in\mathbb{N}} \subset L^2(0,T;H(\text{div},\Omega))$ such that $\varphi_n \to \varphi$ in $L^2(0,T;L^2(\Omega))$ as $n \to \infty$, then

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \int_0^T \int_\Omega \sigma \varphi_n - (2ku' + 2\gamma u_t')\varphi_n \text{ dx dt} \\
&= \lim_{n\to\infty} (\sigma - (2ku' + 2\gamma u_t'), \varphi_n)_{L^2(0,T;L^2(\Omega))} \\
&= \lim_{n\to\infty} \langle \sigma - (2ku' + 2\gamma u_t'), \varphi_n \rangle_{L^2(0,T;L^2(\Omega))} \\
&= \langle \sigma - (2ku' + 2\gamma u_t'), \varphi \rangle_{L^2(0,T;L^2(\Omega))} \\
&= \int_\Omega \sigma \varphi - (2ku' + 2\gamma u_t')\varphi \text{ dx dt.}
\end{aligned}
$$

Thus the second equation of (2.91) is satisfy. Since the initial conditions are the same, they are fulfil. □

Finally we can state the main result.

Theorem 2.8. *The function u is the unique solution of Problem 2.3.1 if and only if (u, σ) is the unique solution of the Problem 2.3.7 where σ is define* (2.92).

Proof. A direct consequence of Proposition 2.3.8 and Theorem 2.7. □

2.4 Viscoelasticity

In this section we will derive a mixed formulation for viscoelasticity, where the stress tensor $\underline{\underline{\sigma}}$ will be in $L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$. In order to do that we need stronger assumption on the forcing function and the velocity initial data respect to the condition we have used in the conservative case, see Section 2.2.
We will start form presenting the problem, then we will define the starting problem and cite a result form [15] that guarantee existence and uniqueness of solution. We will construct a mixed formulation following the same steps we have taken in the previous chapter in Sections 1.3 and 1.4. After noticing that under our assumption $\underline{\underline{\sigma}}$ is $L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$, we will introduce a new mixed formulation, the most important one. Finally we will show that the solution of the last mixed formulation and starting problem coincide.

2.4.1 Starting problem

Assumption 2.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $T > 0$, $\underline{u}_0 \in H_0^1(\Omega)^2$, $\underline{u}_1 \in L^2(\Omega)^2$, $\underline{f} : [0,T] \to (L^2(\Omega)^2)^*$ Hölder continuous, $\rho \in L^\infty(\Omega)$ such that $0 < A \le \rho \le B < \infty$ a.e. in Ω, $\mu_{el} > 0$, $\mu_{vis} > 0$, $\lambda_{el} \ge 0$, and $\lambda_{vis} \ge 0$.*

We consider the following.

Problem 2.4.1 (Starting problem of linear viscoelasticity)**.** *Find*

$$\underline{u} \in \mathcal{C}^1((0,T];H_0^1(\Omega)^2) \cap \mathcal{C}^0([0,T];H_0^1(\Omega)^2) \cap \mathcal{C}^2((0,T];L^2(\Omega)^2) \cap \mathcal{C}^1([0,T];H_0^1(\Omega)^2)$$

such that

$$\begin{cases} \rho \underline{u}_{tt} = 2\mu_{el}\text{div}(\underline{\underline{\epsilon}}(u)) + \lambda_{el}\text{div}(\text{tr}(\underline{\underline{\epsilon}}(u))\underline{\underline{id}}) \\ \qquad\qquad + 2\mu_{vis}\text{div}(\underline{\underline{\epsilon}}(u_t)) + \lambda_{vis}\text{div}(\text{tr}(\underline{\underline{\epsilon}}(u_t))\underline{\underline{id}}) + \underline{f} & \text{in } \Omega \times (0,T] \\ \underline{u}(\cdot,0) = \underline{u}_0 & \text{in } \Omega \\ \underline{u}_t(\cdot,0) = \underline{u}_1 & \text{in } \Omega. \end{cases} \tag{2.97}$$

Problem 2.4.1 can be cast within within the framework of second order evolution equation in Hilbert spaces. Indeed, we can provide existence ad uniqueness of solution of Problem 2.4.1 using [15, Thm. 2.2 in Ch. 6]. For the reader's convenience we first recall this result.

Theorem 2.9 (Theorem 2.2 in Chapter 6 of [15])**.** *Let $\mathcal{A}$ and $\mathcal{C}$ be the Riesz maps of the Hilbert spaces V and W and V is dense and continuously embedded in W. Let $\mathcal{B}$ a linear form V to V^* and assume that there exists constants $C > 0$ and $\delta > 0$ such that $\mathcal{B} + \delta\mathcal{C}$ satisfies*

$$(\mathcal{B} + \delta\mathcal{C})[u](u) \ge C\|u\|_V^2 \quad \forall u \in V,$$

Then for every Hölder continuous $f : [0,\infty) \to W^*$, $u_0 \in V$ and $u_1 \in W$, there is a unique solution $u(t)$ of Problem A.2.1 on $t > 0$ with $u(0) = u_0$ and $u_t(0) = u_1$.

Now we are ready to state and prove the main result of this subsection.

Theorem 2.10 (Existence and uniqueness of solution)**.** *There exists a unique solution of Problem 2.4.1.*

Proof. We want to use Theorem 2.9 in order to prove existence and uniqueness of solution to the Problem 2.4.1, so we need to verify its hypothesis. We set $V = H_0^1(\Omega)^2$ and $W = L^2(\Omega)^2$. We have to show that $\mathcal{A} = -\text{div}(2\mu_{el}\underline{\epsilon}(\cdot) + \lambda_{el}\text{div}(\cdot)\underline{\underline{id}}) : H_0^1(\Omega)^2 \to H^{-1}(\Omega)^2$ is a Riesz map, i.e. we need to check that

$$(\underline{u}, \underline{v})_{H_0^1(\Omega)^2} \coloneqq \mathcal{A}[\underline{v}](\underline{u}) = \int_\Omega (2\mu_{el}\underline{\epsilon}(u) + \lambda_{el}\text{div}(\underline{u})\underline{\underline{id}}) : \underline{\epsilon}(v) \text{ dx} \quad \text{for } \underline{u}, \underline{v} \in H_0^1(\Omega)^2, \tag{2.98}$$

is a scalar product (i.e. symmetric, linear and positive define). From the previous equation we can deduce that

$$(\underline{u}, \underline{v})_{H_0^1(\Omega)^2} \coloneqq \int_\Omega 2\mu_{el}\underline{\epsilon}(u) : \underline{\epsilon}(v) + \lambda_{el}\text{div}(\underline{u})\text{div}(\underline{v}) \text{ dx} \quad \text{for } \underline{u}, \underline{v} \in H_0^1(\Omega)^2,$$

then clearly $(\cdot, \cdot)_{H_0^1(\Omega)^2}$ is symmetric and linear. Since $0 < \mu_{el} < \infty$, using Korn inequality (A.4), we have that

$$(\underline{u}, \underline{u})_{H_0^1(\Omega)^2} \geq \int_\Omega 2\mu_{el}|\underline{\epsilon}(u)|^2 \text{ dx} \geq 2\mu_{vis}\mathcal{K}\|u\|^2_{H_0^1(\Omega)^2} \quad \forall \underline{u} \in H_0^1(\Omega)^2,$$

where $\|\cdot\|_{H_0^1(\Omega)^2}$ is the usual norm in $H_0^1(\Omega)^2$ and $\mathcal{K}$ is the constant of Korn. So it is positive define and we can deduce that $(\cdot, \cdot)_{H_0^1(\Omega)^2}$ is a scalar product, where $\mathcal{A}$ is its Riesz map.
Now we have to show that $\mathcal{C} = \rho : L^2(\Omega)^2 \to (L^2(\Omega)^2)^*$ is a Riesz map, i.e. we need to check that

$$(\underline{u}, \underline{v})_{L^2(\Omega)} \coloneqq \mathcal{C}[\underline{v}](\underline{u}) = \int_\Omega \rho\underline{u}\underline{v} \text{ dx} \quad \text{for } \underline{u}, \underline{v} \in L^2(\Omega)^2, \tag{2.99}$$

is a scalar product (i.e. symmetric, linear and positive define). From the previous equation we can clearly see that $(\cdot, \cdot)_{L^2(\Omega)^2}$ is symmetric and linear. Since $0 < A \leq \rho < B \leq \infty$ we have that

$$(\underline{u}, \underline{u})_{L^2(\Omega)} \coloneqq \int_\Omega \rho|\underline{u}|^2 \text{ dx} \geq A\|\underline{u}\|^2_{L^2(\Omega)^2} \quad \forall \underline{u} \in L^2(\Omega)^2,$$

where $\|\cdot\|_{L^2(\Omega)^2}$ is the usual norm in $L^2(\Omega)^2$. So it is positive define and we can deduce that $(\cdot, \cdot)_{L^2(\Omega)}$ is a scalar product, where $\mathcal{C}$ is its Riesz map.
Since we have that

$$(\underline{u}, \underline{u})_{H_0^1(\Omega)^2} \leq 2(\mu_{el} + \lambda_{el}) \int_\Omega |\underline{\epsilon}(u)|^2 \text{ dx} \leq 2(\mu_{el} + \lambda_{el})\|u\|^2_{H_0^1(\Omega)^2} \quad \forall \underline{u} \in H_0^1(\Omega)^2,$$

and

$$(\underline{u}, \underline{u})_{L^2(\Omega)} \leq B\int_\Omega |\underline{u}|^2 \text{ dx} \leq B\|\underline{u}\|^2_{L^2(\Omega)^2} \quad \forall \underline{u} \in L^2(\Omega)^2,$$

we can see that $(\cdot, \cdot)_{L^2(\Omega)}$, define in (2.99), and $(\cdot, \cdot)_{H_0^1(\Omega)^2}$, define in (2.98), induces norms that are equivalent to the usual one in $H_0^1(\Omega)^2$ and $L^2(\Omega)^2$. Since, with the usual norms, $H_0^1(\Omega)^2$ is dense and immersed with continuity in $L^2(\Omega)^2$ it is also true for the norms induced by $\mathcal{C}$ and $\mathcal{A}$.

Now it remains to show that there exists a constant $C > 0$ and $\delta > 0$ such that $\mathcal{B} + \delta\mathcal{C}$ satisfies

$$(\mathcal{B} + \delta\mathcal{C})[\underline{u}](\underline{u}) \geq C\|\underline{u}\|^2_{H^1_0(\Omega)^2} \quad \forall u \in H^1_0(\Omega)^2. \tag{2.100}$$

where $\mathcal{B} = -\mathrm{div}(2\mu_{vis}\underline{\underline{\epsilon}}(\cdot) + \lambda_{vis}\mathrm{div}(\cdot)\underline{\underline{id}}) : H^1_0(\Omega)^2 \to H^{-1}(\Omega)^2$.
Since $\mu_{vis} > 0$, using Korn inequality (A.4) we obtain

$$(\mathcal{B} + \delta\mathcal{C})[\underline{u}](\underline{u}) = \int_\Omega 2\mu_{vis}|\underline{\underline{\epsilon}}(u)|^2 + \lambda_{vis}|\mathrm{div}(\underline{u})|^2 + \delta\rho|\underline{u}|^2\mathrm{dx} \geq \int_\Omega 2\mu_{vis}|\underline{\underline{\epsilon}}(u)|^2\mathrm{dx}$$
$$\geq 2\mu_{vis}\left\|\underline{\underline{\epsilon}}(u)\right\|^2_{L^2(\Omega)^2} \geq 2\mu_{vis}\mathcal{K}\|\underline{u}\|^2_{H^1_0(\Omega)^2} \quad \forall \underline{u} \in H^1_0(\Omega)^2,$$

where $\mathcal{K}$ is the constant of Korn. Thus (2.100) is satisfy for all $\delta > 0$. Since all hypothesis of Theorem 2.9 are verified we have that there exists a unique solution of the Problem 2.4.1. □

The regularity of the solution obtained by a direct application of Theorem 2.9 is not sufficient for our purposes, which include duality technique that require stronger time regularity of solution at $t = 0$. According to [15, Cor. 3.2 in Ch. 4] we are able to sharpen the above regularity result in the case $(\underline{u}_0, \underline{u}_1) \in D(A)$, where

$$A = \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{C}^{-1} \end{bmatrix} \begin{bmatrix} 0 & -\mathcal{A} \\ \mathcal{A} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \mathcal{C}^{-1}\mathcal{A} & \mathcal{C}^{-1}\mathcal{B} \end{bmatrix}$$

and

$$D(A) = \left\{ (\underline{x}_1, \underline{x}_2) \in H^1_0(\Omega)^2 \times H^1_0(\Omega)^2 \mid \mathcal{A}(\underline{x}_1) + \mathcal{B}(\underline{x}_2) \in (L^2(\Omega)^2)^* \right\}.$$

Since $\mathcal{A} = -\mathrm{div}(2\mu_{el}\underline{\underline{\epsilon}}(\cdot) + \lambda_{el}\mathrm{div}(\cdot)\underline{\underline{id}})$ and $\mathcal{B} = -\mathrm{div}(2\mu_{vis}\underline{\underline{\epsilon}}(\cdot) + \lambda_{vis}\mathrm{div}(\cdot)\underline{\underline{id}})$ we have that $(\underline{x}_1, \underline{x}_2) \in D(A)$ if and only if $(\underline{x}_1, \underline{x}_2) \in H^1_0(\Omega)^2 \times H^1_0(\Omega)^2$ and

$$\mathrm{div}(2\mu_{el}\underline{\underline{\epsilon}}(\underline{x}_1) + \lambda_{el}\mathrm{div}(\underline{x}_1)\underline{\underline{id}} + 2\mu_{vis}\underline{\underline{\epsilon}}(\underline{x}_2) + \lambda_{vis}\mathrm{div}(\underline{x}_2)\underline{\underline{id}}) \in (L^2(\Omega)^2)^*.$$

So we have the following.

Corollary 2.4.1. *If $\underline{u}_1 \in H^1_0(\Omega)^2$ and $\underline{u}_0, \underline{u}_1$ are such that*

$$\mathrm{div}(2\mu_{el}\underline{\underline{\epsilon}}(\underline{u}_0) + \lambda_{el}\mathrm{div}(\underline{u}_0)\underline{\underline{id}} + 2\mu_{vis}\underline{\underline{\epsilon}}(\underline{u}_1) + \lambda_{vis}\mathrm{div}(\underline{u}_1)\underline{\underline{id}}) \in (L^2(\Omega)^2)^*.$$

Then we have that $\underline{\bar{u}}$, unique solution of Problem 2.4.1, satisfies

$$\underline{\bar{u}} \in \mathcal{C}^1([0,T]; H^1_0(\Omega)^2) \cap \mathcal{C}^2([0,T]; L^2(\Omega)^2).$$

For the purpose of our work from now on we can assume the following.

Assumption 2.4.2. *The function $\underline{u}_1 \in H^1_0(\Omega)^2$ and*

$$\mathrm{div}(2\mu_{el}\underline{\underline{\epsilon}}(\underline{u}_0) + \lambda_{el}\mathrm{div}(\underline{u}_0)\underline{\underline{id}} + 2\mu_{vis}\underline{\underline{\epsilon}}(\underline{u}_1) + \lambda_{vis}\mathrm{div}(\underline{u}_1)\underline{\underline{id}}) \in (L^2(\Omega)^2)^*.$$

Definition 2.4.1. *We will denote by*

$$\underline{\bar{u}} \in \mathcal{C}^1([0,T]; H^1_0(\Omega)^2) \cap \mathcal{C}^2([0,T]; L^2(\Omega)^2),$$

the solution of Problem 2.4.1 provided by Theorem 2.9 and Corollary 2.4.1.

2.4.2 Construction of mixed formulation

In order to derive our mixed formulation we proceed as follows. First we characterize $\underline{\bar{u}}$ as the unique minimum of a variational problem, then we apply the theory developed in Sections 1.3 and 1.4 in order to write its saddle point formulation. Let us start by introducing the functional

$$\mathcal{S}(\underline{u}) = \int_0^T \int_\Omega \rho \underline{\bar{u}}_{tt}\underline{u} + (2\mu_{vis}\underline{\epsilon}(\bar{u}_t) + \lambda_{vis}\text{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\epsilon}(u) + \mu_{el}|\underline{\epsilon}(u)|^2 + \frac{1}{2}\lambda_{el}|\text{tr}(\underline{\epsilon}(u))|^2 \text{ dx} - \langle \underline{f}, \underline{u}\rangle_{L^2(\Omega)^2} \text{ dt}, \quad (2.101)$$

acting on the Hilbert space

$$\Gamma = \left\{ \ \underline{u}(x,t) \ \middle| \ \underline{u}(x,t) \in L^2(0,T; H_0^1(\Omega)^2) \ \right\},$$

then we consider the following.

Problem 2.4.2 (Primal formulation of linear viscoelasticity)**.** *Find $\underline{u} \in \Gamma$ such that*

$$\mathcal{S}(\underline{u}) = \inf_{\underline{v}\in\Gamma}\{\mathcal{S}(\underline{v})\}. \quad (2.102)$$

Proposition 2.4.1. *Problem 2.4.2 has the unique solution $\underline{\bar{u}}$.*

Proof. If we compute the first variation of $\mathcal{S}$ at $\underline{\bar{u}}$ we obtain

$$\begin{aligned} d\mathcal{S}(\underline{\bar{u}}).(\underline{y}) &= \int_0^T \int_\Omega \rho\underline{\bar{u}}_{tt}\underline{y} + (2\mu_{vis}\underline{\epsilon}(\bar{u}_t) + \lambda_{vis}\text{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\epsilon}(y) + 2\mu_{el}\underline{\epsilon}(\bar{u}) : \underline{\epsilon}(y) \\ &\quad + \lambda_{el}\,\text{tr}(\underline{\epsilon}(\bar{u}))\underline{\underline{id}} : \underline{\epsilon}(y) \text{ dx} - \langle \underline{f}, \underline{y}\rangle_{L^2(\Omega)^2} \text{ dt} \\ &= \int_0^T \langle \rho\underline{\bar{u}}_{tt} - 2\mu_{el}\text{div}(\underline{\epsilon}(\bar{u})) - \lambda_{el}\text{div}(\text{tr}(\underline{\epsilon}(\bar{u}))\underline{\underline{id}}) \\ &\quad - 2\mu_{vis}\text{div}(\underline{\epsilon}(\bar{u}_t)) - \lambda_{vis}\text{div}(\text{tr}(\underline{\epsilon}(\bar{u}_t))\underline{\underline{id}}) - \underline{f}, \underline{y}\rangle_{L^2(\Omega)^2} \text{ dt} = 0 \quad \forall \underline{y} \in \Gamma, \end{aligned}$$

where the last equation is implied by the fact that $\underline{\bar{u}}$ is solution of Problem 2.4.1, so it satisfies the first equation of (2.97), that implies

$$\langle \rho\underline{\bar{u}}_{tt} - 2\mu_{el}\text{div}(\underline{\epsilon}(\bar{u})) - \lambda_{el}\text{div}(\text{tr}(\underline{\epsilon}(\bar{u}))\underline{\underline{id}}) - 2\mu_{vis}\text{div}(\underline{\epsilon}(\bar{u}_t)) - \lambda_{vis}\text{div}(\text{tr}(\underline{\epsilon}(\bar{u}_t))\underline{\underline{id}}) - \underline{f}, \cdot\ \rangle_{L^2(\Omega)^2} = 0.$$

Hence $\underline{\bar{u}}$ is a critical point of $\mathcal{S}$.
Now we want to show that the functional $\mathcal{S}$ is strictly convex, in order to do that we verify that

the second variation of $\mathcal{S}$ is positive definite at any $\underline{v} \in \Gamma$:

$$d^2\mathcal{S}(\underline{v}).(\underline{y},\underline{y}) = \int_0^T \int_\Omega 2\mu_{el}|\underline{\underline{\epsilon}}(y)|^2 + \lambda_{el}|\mathrm{tr}(\underline{\underline{\epsilon}}(y))|^2 \text{ dx dt} \geq \int_0^T \int_\Omega 2\mu_{el}|\underline{\underline{\epsilon}}(y)|^2 \text{ dx dt}$$
$$\geq 2\mu_{el}\mathcal{K}\int_0^T \|\underline{y}\|^2_{H^1_0(\Omega)^2} \text{ dt} = 2\mu_{el}\mathcal{K}\|\underline{y}\|^2_{L^2(0,T;H^1_0(\Omega)^2)} \quad \forall \underline{y} \in \Gamma,$$

where we used the Korn inequality (A.4). The functional $\mathcal{S}$ is strictly convex and it admits a unique critical point, which is indeed its unique minimum. Thus $\bar{\underline{u}}$ is the unique minimiser of $\mathcal{S}$ in Γ. □

Before introducing the saddle point problem using the theory that we have developed in Section 1.4, first we need to check that the dual problem, constructed from Problem 2.4.2, is stable. In order to that it is convenient to rewrite the functional $\mathcal{S}$ in the form

$$F(\underline{u}) + G(\underline{\underline{\epsilon}}(u)),$$

where $F : \Gamma \to \mathbb{R}$ is

$$F(\underline{v}) = \int_0^T \int_\Omega \rho\bar{\underline{u}}_{tt}\underline{u} \text{ dx} - \langle \underline{f}, \underline{u}\rangle_{L^2(\Omega)^2} \text{ dt}, \tag{2.103}$$

and $G : Y := L^2(0,T;L^2(\Omega)^{2\times 2}_{sym}) \to \mathbb{R}$ is

$$G(\underline{\underline{p}}) = \int_0^T \int_\Omega \mu_{el}|\underline{\underline{p}}|^2 + \frac{1}{2}\lambda_{el}|\mathrm{tr}(\underline{\underline{p}})|^2 + (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\bar{\underline{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}} \text{ dx dt}. \tag{2.104}$$

With the choice of $\Phi(\underline{u},\underline{\underline{p}}) = F(\underline{u}) + G(\underline{\underline{\epsilon}}(u) + \underline{\underline{p}})$, we can construct the dual problem of Problem 2.4.2 according to Definition 1.3.2, obtaining the following.

Problem 2.4.3 (Dual formulation of linear viscoelasticity)**.** *Find $q^* \in Y^*$ such that*

$$-\Phi^*(0,q^*) = \sup_{p^*\in Y^*}\big[-\Phi^*(0,p^*)\big] = \sup_{p^*\in Y^*}\big[-F^*(-\Lambda^*p^*) - G^*(p^*)\big], \tag{2.105}$$

where Λ^* is the adjoint of $\underline{\underline{\epsilon}}$ and, according to Definition A.1.8, $F^* : \Gamma^* \to \mathbb{R}$ is

$$F^*(-\underline{\underline{\epsilon}}^*(p^*)) = \begin{cases} 0 & \text{if } -\underline{\underline{\epsilon}}^*(p^*) - \rho\bar{\underline{u}}_{tt} + \underline{f} = 0 \\ +\infty & \text{otherwise} \end{cases}$$

and $G^* : Y^* \to \mathbb{R}$ is

$$G^*(p^*) = \int_0^T \int_\Omega \frac{1}{4\mu_{el}}|\mathcal{R}(p^*)^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2$$
$$+ \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\mathcal{R}(p^*)) - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 \text{ dx dt},$$

where $\mathcal{R}$ is the Riesz operator such that $\mathcal{R} : Y^* \to Y$.

Proposition 2.4.2. *Problem 2.4.2 is stable in the sense of Definition 1.3.4. Moreover Problem 2.4.3 admits a unique solution.*

Proof. In order to prove stability we verify the hypothesis of Theorem 1.1. Since we have that F, defined in (2.103), is linear and G, defined in (2.104), is strictly convex, because is sum of a strictly convex function

$$\int_0^T \int_\Omega \mu_{el} |\underline{\underline{p}}|^2 + \frac{1}{2}\lambda_{el} |\mathrm{tr}(\underline{\underline{p}})|^2 \text{ dx dt},$$

and a linear one

$$\int_0^T \int_\Omega (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}} \text{ dx dt},$$

then $J(\underline{u}, \underline{\underline{p}}) = F(\underline{u}) + G(\underline{\underline{p}})$ is convex in $\Gamma \times Y$:

$$\begin{aligned} J(\alpha(\underline{u}, \underline{\underline{p}}) + (1-\alpha)(\underline{v}, \underline{\underline{q}})) &= F(\alpha\underline{u} + (1-\alpha)\underline{v}) + G(\alpha\underline{\underline{p}} + (1-\alpha)\underline{\underline{q}}) \\ &= \alpha F(\underline{u}) + (1-\alpha)F(\underline{v}) + G(\alpha\underline{\underline{p}} + (1-\alpha)\underline{\underline{q}}) \\ &\leq \alpha F(\underline{u}) + (1-\alpha)F(\underline{v}) + \alpha G(\underline{\underline{p}}) + (1-\alpha)G(\underline{\underline{q}}) \\ &\leq \alpha J(\underline{u}, \underline{\underline{p}}) + (1-\alpha)J(\underline{v}, \underline{\underline{q}}), \end{aligned}$$

for $\alpha \in [0,1]$, $\underline{u}, \underline{v} \in \Gamma$ and $\underline{\underline{p}}, \underline{\underline{q}} \in Y$.
Now we have to find a value $\underline{u}_0$ such that $F(\underline{u}_0) + G(\underline{\underline{\epsilon}}(u_0)) < +\infty$ and verify that the functional $\underline{\underline{p}} \to F(\underline{u}_0) + G(\underline{\underline{p}})$ is continuos at $\underline{\underline{\epsilon}}(u_0)$. If we take $\underline{u}_0 = \underline{0}$ then $F(\underline{0}) = 0$ and, since $\underline{\underline{\epsilon}}(\underline{0}) = \underline{\underline{0}}$, then $G(\underline{\underline{\epsilon}}(\underline{0})) = G(\underline{\underline{0}}) = 0$, so the first hypothesis is verified. Now we need to show that $\underline{\underline{p}} \to F(\underline{u}_0) + G(\underline{\underline{p}})$ is continuos at $\underline{\underline{\epsilon}}(u_0)$. Let $\underline{\underline{p}}_n \to \underline{\underline{p}}$ in Y as $n \to \infty$, then it is enough to show

$$\lim_{n\to\infty} \int_0^T \int_\Omega \mu_{el} |\underline{\underline{p}}_n|^2 \text{ dx dt} = \int_0^T \int_\Omega \mu_{el} |\underline{\underline{p}}|^2 \text{ dx dt}, \tag{2.106}$$

and

$$\lim_{n\to\infty} \int_0^T \int_\Omega \frac{1}{2}\lambda_{el} |\mathrm{tr}(\underline{\underline{p}}_n)|^2 \text{ dx dt} = \int_0^T \int_\Omega \frac{1}{2}\lambda_{el} |\mathrm{tr}(\underline{\underline{p}})|^2 \text{ dx dt}, \tag{2.107}$$

and

$$\begin{aligned} \lim_{n\to\infty} \int_0^T \int_\Omega &(2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}}_n \text{ dx dt} \\ &= \int_0^T \int_\Omega (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}} \text{ dx dt}. \end{aligned} \tag{2.108}$$

Since $\int_0^T \int_\Omega \mu_{el} |\underline{\underline{p}}|^2 \text{ dx dt} = \mu_{el} \left\| \underline{\underline{p}} \right\|_Y^2$ and the norm is continuous, then (2.106) is true. Since $|\mathrm{tr}(\cdot)| : Y \to \mathbb{R}$ can be seen as the composition of the projection $\pi : Y \to Y$ on the trace components, the sum of them, and the norm of Y, and because of them are continuous, then (2.107) holds. Since

$$\int_0^T \int_\Omega (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}}_n \text{ dx dt} = \left\langle (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}), \underline{\underline{p}}_n \right\rangle_Y,$$

is a linear and continuous operator, then also (2.108) is true. So form (2.106), (2.107), and

(2.108) we have the following

$$\begin{aligned}\lim_{n\to\infty} F(\underline{u}_0) + G(\underline{\underline{p}}_n) &= \lim_{n\to\infty} \int_0^T \int_\Omega \mu_{el}|\underline{\underline{p}}_n|^2 + \frac{1}{2}\lambda_{el}|\mathrm{tr}(\underline{\underline{p}}_n)|^2 \\ &\qquad + (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}}_n \text{ dx dt} \\ &= \int_0^T \int_\Omega \mu_{el}|\underline{\underline{p}}|^2 + \frac{1}{2}\lambda_{el}|\mathrm{tr}(\underline{\underline{p}})|^2 + (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{p}} \text{ dx dt} \\ &= F(\underline{u}_0) + G(\underline{\underline{p}}).\end{aligned}$$

Thus Problem 2.4.1 is stable and due to Proposition 1.3.6, Problem 2.4.3 has at least one solution. Can be noticed that G is differentiable and the differential of G in $\underline{\underline{p}}$ evaluated in $\underline{\underline{q}}$ is

$$G'[\underline{\underline{p}}](\underline{\underline{q}}) = \int_0^T \int_\Omega 2\mu_{el}\underline{\underline{p}} : \underline{\underline{q}} + \lambda_{el}\,\mathrm{tr}(\underline{\underline{p}})\,\mathrm{tr}(\underline{\underline{q}}) + (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{q}} \text{ dx dt}.$$

Since G is differentiable and primal Problem 2.4.1 admits a unique solution, according to Proposition 1.3.7 and Proposition A.1.7, we have that the dual Problem 2.4.3 admits a unique solution. □

In order to write the saddle point problem that we have introduce in Section 1.4 we need to introduce the Lagrangian 1.4.1:

$$-L(\underline{u}, p^*) = \sup_{\underline{\underline{p}}\in Y} \left[\langle p^*, \underline{\underline{p}}\rangle_{Y^*} - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u) + \underline{\underline{p}})\right]. \tag{2.109}$$

Since G is strictly convex, because is sum of positive quadratic terms and linear terms, and $\langle p^*, \underline{\underline{p}}\rangle - F(\underline{u})$ is linear in $\underline{\underline{p}}$, we have that the argument inside the supremum of the Lagrangian is strictly concave, so there exists a unique $\underline{\underline{q}} \in Y$ that realizes the supremum. If we compute the first variation in Y of

$$\langle p^*, \underline{\underline{p}}\rangle_{Y^*} - F(\underline{u}) - G(\underline{\underline{\epsilon}}(u) + \underline{\underline{p}}),$$

evaluated in $\underline{\underline{q}}$, we obtain:

$$\begin{aligned}\int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\varphi}} - 2\mu_{el}(\underline{\underline{q}} + \underline{\underline{\epsilon}}(u)) : \underline{\underline{\varphi}} - \lambda_{el}(\mathrm{tr}(\underline{\underline{q}}) + \mathrm{tr}(\underline{\underline{\epsilon}}(u))\underline{\underline{id}}) : \underline{\underline{\varphi}} \\ - (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : \underline{\underline{\varphi}} \text{ dx dt} = 0 \quad \forall \underline{\underline{\varphi}} \in Y,\end{aligned} \tag{2.110}$$

where $\mathcal{R} : Y^* \to Y$ is the Riesz operator define in Theorem A.1. If we take $\underline{\underline{\varphi}} = \underline{\underline{q}} + \underline{\underline{\epsilon}}(u)$ in (2.110) we obtain

$$\begin{aligned}\langle p^*, \underline{\underline{q}}\rangle = \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\underline{q}} \text{ dx dt} = 2G(\underline{\underline{\epsilon}}(u) + q) - \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\underline{\epsilon}}(u) - (2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t) \\ + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : (\underline{\underline{q}} + \underline{\underline{\epsilon}}(u)) \text{ dx dt}.\end{aligned}$$

Substituting the previous equation in (2.109) we obtain

$$\begin{aligned}
-L(\underline{u},p^*) &= \sup_{\underline{\underline{p}}\in Y} \left[\langle p^*, \underline{\underline{p}}\rangle_{Y^*} - F(\underline{u}) - G(\underline{\epsilon}(u) + \underline{\underline{p}})\right] \\
&= \langle p^*, \underline{\underline{q}}\rangle_{Y^*} - F(\underline{u}) - G(\underline{\epsilon}(u) + \underline{\underline{q}}) \\
&= G(\underline{\epsilon}(u) + q) - F(u) - \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\epsilon}(u) - (2\mu_{vis}\underline{\epsilon}(\bar{u}_t) \\
&\quad + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}}) : (\underline{\epsilon}(u) + \underline{\underline{q}}) \text{ dx dt}.
\end{aligned}$$

It can be seen that the dissipative term disappears from $-L(\underline{u},p^*)$, so if we introduce $\tilde{G} : Y \to \mathbb{R}$ such that

$$\tilde{G}(\underline{\underline{p}}) = \int_0^T \int_\Omega \mu_{el}|\underline{\underline{p}}|^2 + \frac{1}{2}\lambda_{el}|\mathrm{tr}(\underline{\underline{p}})|^2 \text{ dx dt},$$

then we have

$$-L(\underline{u},p^*) = \tilde{G}(\underline{\epsilon}(u) + q) - F(u) - \int_0^T \int_\Omega \mathcal{R}(p^*) : \underline{\epsilon}(u) \text{ dx dt}. \tag{2.111}$$

From the arbitrariness of $\underline{\underline{\varphi}} \in Y$ in (2.110) we obtain

$$\mathcal{R}(p^*) = 2\mu_{el}(\underline{\underline{q}} + \underline{\epsilon}(u)) + \lambda_{el}(\mathrm{tr}(\underline{\underline{q}}) + \mathrm{tr}(\underline{\epsilon}(u))\underline{\underline{id}}) + 2\mu_{vis}\underline{\epsilon}(\bar{u}_t) + \lambda_{vis}\mathrm{div}(\underline{\bar{u}}_t)\underline{\underline{id}},$$

and we can deduce that

$$\begin{aligned}
\underline{\underline{q}}^D &= \frac{1}{2\mu_{el}}\left[\mathcal{R}(p^*)^D + 2\mu_{vis}\underline{\epsilon}(\bar{u}_t)^D\right] - \underline{\epsilon}(u)^D \\
\mathrm{tr}(\underline{\underline{q}}) &= \frac{1}{2\mu_{el} + 2\lambda_{el}}\left[\mathrm{tr}(\mathcal{R}(p^*)) + 2\mu_{vis}\,\mathrm{tr}(\underline{\epsilon}(\bar{u}_t)) + 2\lambda_{vis}\,\mathrm{tr}(\underline{\epsilon}(\bar{u}_t))\right] - \mathrm{tr}(\underline{\epsilon}(u)),
\end{aligned}$$

from which we have

$$\begin{aligned}
\underline{\underline{q}} = \underline{\underline{q}}^D + \frac{1}{2}\,\mathrm{tr}(\underline{\underline{q}})\underline{\underline{id}} &= \frac{1}{2\mu_{el}}\left[\mathcal{R}(p^*)^D + 2\mu_{vis}\underline{\epsilon}(\bar{u}_t)^D\right] \\
&\quad + \frac{1}{4(\mu_{el}+\lambda_{el})}\left[\mathrm{tr}(\mathcal{R}(p^*)) + 2\mu_{vis}\,\mathrm{tr}(\underline{\epsilon}(\bar{u}_t)) + 2\lambda_{vis}\,\mathrm{tr}(\underline{\epsilon}(\bar{u}_t))\right] - \underline{\epsilon}(u).
\end{aligned} \tag{2.112}$$

Substituting (2.112) in (2.111), we obtain

$$\begin{aligned}
-L(\underline{u},p^*) = \int_0^T \int_\Omega \frac{1}{4\mu_{el}}|\mathcal{R}(p^*)^D - 2\mu_{vis}\underline{\epsilon}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\mathcal{R}(p^*)) \\
- 2(\mu_{vis} + \lambda_{vis})\,\mathrm{tr}(\underline{\epsilon}(\bar{u}_t))|^2 - \rho\underline{\bar{u}}_{tt}\underline{u} - \mathcal{R}(p^*) : \underline{\epsilon}(u) \text{ dx} + \langle \underline{f}, \underline{u}\rangle_{L^2(\Omega)^2} \text{ dt}.
\end{aligned}$$

Notice that the right-hand side of the last equation depends on $\underline{\underline{p}}^*$ only through the Riesz operator $\mathcal{R}$. Since Y is an Hilbert space, so $\mathcal{R}$ is an isomorphism, we can replace Y^* with Y and $\mathcal{R}(p^*)$

with $\underline{\underline{p}}$, and we obtain

$$\inf_{\underline{\underline{p}}^*\in Y^*}\sup_{\underline{u}\in\Gamma}\bigg\{\int_0^T\int_\Omega \frac{1}{4\mu_{el}}|\mathcal{R}(p^*)^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\mathcal{R}(p^*)) - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho\underline{\bar{u}}_{tt}\underline{u} - \mathcal{R}(p^*):\underline{\underline{\epsilon}}(u)\ \mathrm{dx} + \langle \underline{f},\underline{u}\rangle_{L^2(\Omega)^2}\ \mathrm{dt}.\bigg\}$$

$$= \inf_{\underline{\underline{p}}\in Y}\sup_{\underline{u}\in\Gamma}\bigg\{\int_0^T\int_\Omega \frac{1}{4\mu_{el}}|\underline{\underline{p}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\underline{\underline{p}}) - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho\underline{\bar{u}}_{tt}\underline{u} - \underline{\underline{p}}:\underline{\underline{\epsilon}}(u)\ \mathrm{dx} + \langle \underline{f},\underline{u}\rangle_{L^2(\Omega)^2}\ \mathrm{dt}\bigg\}.$$

From the previous equation and recalling the Definition 1.4.3 of saddle point problem, we can consider the following.

Problem 2.4.4 (Saddle point formulation of linear viscoelasticity)**.** *Find* $(\underline{w},\underline{\underline{\tau}})\in\Gamma\times Y$ *such that*

$$\int_0^T\int_\Omega \frac{1}{4\mu_{el}}|\underline{\underline{\tau}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\underline{\underline{\tau}}) - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho\underline{\bar{u}}_{tt}\underline{w} - \underline{\underline{\tau}}:\underline{\underline{\epsilon}}(w)\ \mathrm{dx} + \langle \underline{f},\underline{w}\rangle_{L^2(\Omega)^2}\ \mathrm{dt}$$
$$= \inf_{\underline{\underline{p}}\in Y}\sup_{\underline{u}\in\Gamma}\bigg\{\int_0^T\int_\Omega \frac{1}{4\mu_{el}}|\underline{\underline{p}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\mathrm{tr}(\underline{\underline{p}}) - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho\underline{\bar{u}}_{tt}\underline{u} - \underline{\underline{p}}:\underline{\underline{\epsilon}}(u)\ \mathrm{dx} + \langle \underline{f},\underline{u}\rangle_{L^2(\Omega)^2}\ \mathrm{dt}\bigg\}. \tag{2.113}$$

Proposition 2.4.3. *Problem 2.4.4 admits a unique solution* $(\underline{\bar{u}},\underline{\underline{\sigma}})$ *where* $\underline{\bar{u}}$ *is the unique solution of Problem 2.4.2 and* $\underline{\underline{\sigma}}=\mathcal{R}(q^*)$*, where* q^* *is the unique solution of Problem 2.4.3.*

Proof. Recalling that Problems 2.4.2 and 2.4.3 admit a unique solution, then, due to Proposition 1.4.1, we have guarantee that Problem 2.4.4 admits a unique solution $(\underline{\bar{u}},\underline{\underline{\sigma}})$, where $\underline{\bar{u}}$ is the unique solution of Problem 2.4.2 and $\underline{\underline{\sigma}}=\mathcal{R}(q^*)$, where q^* is the unique solution of the Problem 2.4.3, since $\mathcal{R}$ is an isometry we have that $\underline{\underline{\sigma}}$ is unique. □

If we compute the Euler-Lagrange equations of (2.113) evaluated in $(\underline{\bar{u}},\underline{\underline{\sigma}})$, we obtain:

$$\begin{cases} \int_0^T\int_\Omega \rho\underline{\bar{u}}_{tt}\underline{y} + \underline{\underline{\sigma}}:\underline{\underline{\epsilon}}(y)\ \mathrm{dx} - \langle \underline{f},\underline{y}\rangle_{L^2(\Omega)^2}\ \mathrm{dt} = 0 & \forall \underline{y}\in\Gamma \\ \int_0^T\int_\Omega \underline{\underline{\sigma}}^D:\underline{\underline{\varphi}}^D - 2\mu_{el}\underline{\underline{\epsilon}}(\bar{u})^D:\underline{\underline{\varphi}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D:\underline{\underline{\varphi}}^D\ \mathrm{dx\ dt} = 0 & \forall\underline{\underline{\varphi}}\in Y \\ \int_0^T\int_\Omega \mathrm{tr}(\underline{\underline{\sigma}})\,\mathrm{tr}(\underline{\underline{\varphi}}) - (2\mu_{el}+2\lambda_{el})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}))\,\mathrm{tr}(\underline{\underline{\varphi}}) & \\ \quad - (2\mu_{vis}+2\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))\,\mathrm{tr}(\underline{\underline{\varphi}})\ \mathrm{dx\ dt} = 0 & \forall\underline{\underline{\varphi}}\in Y \end{cases} \tag{2.114}$$

For the purpose of our work form now on we can assume the following.

Assumption 2.4.3. $\underline{f}:[0,T]\to L^2(\Omega)^2$ *Hölder continuous.*

A direct consequence of the previous assumption is that we can no longer consider

$$\langle \underline{f}(\cdot,t), \underline{u} \rangle_{L^2(\Omega)^2} \quad \forall \underline{u} \in L^2(\Omega)^2,$$

but

$$(\underline{f}(\cdot,t), \underline{u})_{L^2(\Omega)^2} \coloneqq \int_\Omega \underline{f}(\cdot,t)\underline{u} \text{ dx} \quad \forall \underline{u} \in L^2(\Omega)^2.$$

In the first equation of (2.114) since $\underline{y} \in \Gamma$ then for a.e. $t \in [0,T]$ we have that $\underline{y}(\cdot,t) \in H_0^1(\Omega)^2$, so we can define $\text{div}(\underline{\underline{\sigma}}) \in L^2(0,T;H^{-1}(\Omega)^2)$ as follows

$$\langle \text{div}(\underline{\underline{\sigma}}), \underline{y} \rangle_{L^2(0,T;H_0^1(\Omega)^2)} \coloneqq \int_0^T \langle \text{div}(\underline{\underline{\sigma}}), \underline{y} \rangle_{H_0^1(\Omega)^2} \text{ dt} = -\int_0^T \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(y) \text{ dx dt}. \tag{2.115}$$

From the first equation we can deduce that

$$\rho \underline{\bar{u}}_{tt} - \text{div}(\underline{\underline{\sigma}}) - \underline{f} = 0 \quad \text{ in } L^2(0,T;(L^2(\Omega)^2)^*),$$

recalling that $\underline{\bar{u}} \in \mathcal{C}^2([0,T];L^2(\Omega)^2)$ and $\underline{f} : [0,T] \to L^2(\Omega)^2$ is Hölder continuous, then we have that $\text{div}(\underline{\underline{\sigma}}) \in L^2(0,T;L^2(\Omega)^2)$, that implies $\underline{\underline{\sigma}} \in L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$.
Since $\underline{\underline{\sigma}} \in L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$ we are interested in finding the saddle point of Problem 2.4.4 we can restrict Y to $L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$, if we do that, according to the integration by parts, we can rewrite

$$\int_0^T \int_\Omega \frac{1}{4\mu_{el}} |\underline{\underline{p}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\text{tr}(\underline{\underline{p}}) - 2(\mu_{vis}+\lambda_{vis})\,\text{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2$$
$$- \rho \underline{\bar{u}}_{tt}\underline{u} - \underline{\underline{p}} : \underline{\underline{\epsilon}}(u) \text{ dx} + \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)^2} \text{ dt},$$

with

$$\int_0^T \int_\Omega \frac{1}{4\mu_{el}} |\underline{\underline{p}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})}|\text{tr}(\underline{\underline{p}}) - 2(\mu_{vis}+\lambda_{vis})\,\text{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2$$
$$- \rho \underline{\bar{u}}_{tt}\underline{u} + \text{div}(\underline{\underline{p}})u \text{ dx} + \langle \underline{f}, \underline{u} \rangle_{L^2(\Omega)^2} \text{ dt}.$$

After doing that we no longer need $\underline{u}$ to be in $\Gamma = L^2(0,T;H_0^1(\Omega)^2)$, because the new functional does not involve space derivative of $\underline{u}$, so we can replace Γ with the largest space $L^2(0,T;L^2(\Omega)^2)$. If do that we have no longer guarantee that $(\underline{\bar{u}}, \underline{\underline{\sigma}})$ is a saddle point, indeed we do not even know if there exists one for the new problem. Now we are interested in studying the following.

Problem 2.4.5 (Mixed formulation of linear viscoelasticity). *Find $(\underline{w}, \underline{\underline{\tau}}) \in \Lambda \times W$ such that*

$$\begin{aligned}\int_0^T \int_\Omega \frac{1}{4\mu_{el}} |\underline{\underline{\tau}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})} |\mathrm{tr}(\underline{\underline{\tau}}) \\ - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho \underline{\bar{u}}_{tt}\underline{w} + \underline{f}(x,t)\underline{w} + \mathrm{div}(\underline{\underline{\tau}}) : \underline{w} \text{ dx dt} \\ = \inf_{\underline{\underline{p}}\in Y} \sup_{\underline{u}\in\Gamma} \Big\{ \int_0^T \int_\Omega \frac{1}{4\mu_{el}} |\underline{\underline{p}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\bar{u}_t)^D|^2 + \frac{1}{8(\mu_{el}+\lambda_{el})} |\mathrm{tr}(\underline{\underline{p}}) \\ - 2(\mu_{vis}+\lambda_{vis})\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))|^2 - \rho \underline{\bar{u}}_{tt}\underline{u} + \underline{f}(x,t)\underline{u} + \mathrm{div}(\underline{\underline{p}})\underline{u} \text{ dx dt} \Big\},\end{aligned} \tag{2.116}$$

where $W = L^2(0,T;\underline{\underline{H}}(\mathrm{div},\Omega)_{sym})$ and $\Lambda = L^2(0,T;L^2(\Omega)^2)$.

Proposition 2.4.4. *If $(\underline{w}, \underline{\underline{\tau}})$ is a saddle point of Problem 2.4.5, in sense of Definition 1.4.2, then $\underline{w} \in L^2(0,T;H_0^1(\Omega)^2)$.*

Proof. If we take the first variation of (2.116) respect to the variable $\underline{\underline{p}}$ evaluated in $(\underline{w}, \underline{\underline{\tau}})$, we obtain that for every $\underline{\underline{\varphi}} \in W$:

$$\begin{aligned}\int_0^T \int_\Omega \mathrm{div}(\underline{\underline{\varphi}})\underline{w} \text{ dx dt} = -\int_0^T \int_\Omega \underline{\underline{\varphi}} : \Big[\frac{1}{2\mu_{el}}\big(\underline{\underline{\sigma}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\underline{\bar{u}}_t)^D\big) \\ + \frac{1}{4(\mu_{el}+\lambda_{el})}\big(\mathrm{tr}(\underline{\underline{\sigma}}) - 2\mu_{vis}\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t)) - 2\lambda_{vis}\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))\big)\Big] \text{ dx dt},\end{aligned} \tag{2.117}$$

Since $\underline{\bar{u}} \in \mathcal{C}^1([0,T];H_0^1(\Omega)^2)$ and $L^2(0,T;\mathcal{C}_c^\infty(\Omega)^{2\times 2}_{sym}) \subset W$, according to what we have seen in Appendix A about $\underline{H}(sym,\Omega)$, we obtain that $\underline{w}$ admit symmetric gradient:

$$\underline{\underline{\epsilon}}(w) = \frac{1}{2\mu_{el}}\big(\underline{\underline{\sigma}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(\underline{\bar{u}}_t)^D\big) + \frac{1}{4(\mu_{el}+\lambda_{el})}\big(\mathrm{tr}(\underline{\underline{\sigma}}) - 2\mu_{vis}\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t)) - 2\lambda_{vis}\,\mathrm{tr}(\underline{\underline{\epsilon}}(\bar{u}_t))\big).$$

Since $\underline{w} \in L^2(0,T;\underline{H}(sym,\Omega))$, then there exists $\{\underline{\varphi}_n\}_{n\in\mathbb{N}} \subset L^2(0,T;\mathcal{C}^\infty(\overline{\Omega})^2)$ such that

$$\begin{aligned}&\underline{\varphi}_n \longrightarrow \underline{w} && \text{in } L^2(0,T;L^2(\Omega)^2) \\ &\underline{\underline{\epsilon}}(\varphi_n) \longrightarrow \underline{\underline{\epsilon}}(w) && \text{in } L^2(0,T;L^2(\Omega)^{2\times 2}_{sym}).\end{aligned}$$

From Korn inequality (A.5) we have

$$\int_0^T \left\|\underline{\varphi}_n\right\|^2_{H^1(\Omega)^2} \text{dt} \le \int_0^T \mathcal{K}^2\Big(\left\|(\underline{\varphi}_n)\right\|_{L^2(\Omega)^2} + \left\|\underline{\underline{\epsilon}}(\varphi_n)\right\|_{L^2(\Omega)^{2\times 2}_{sym}}\Big)^2 \text{ dt},$$

so $\{\underline{\varphi}_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L^2(0,T;H^1(\Omega)^2)$. Since $L^2(0,T;H^1(\Omega)^2)$ is an Hilbert space, then it is complete, so there exists $\underline{\varphi} \in L^2(0,T;H^1(\Omega)^2)$ such that $\{\underline{\varphi}_n\}_{n\in\mathbb{N}}$ converges to it. As $L^2(0,T;H^1(\Omega)^2) \hookrightarrow L^2(0,T;\underline{H}(sym,\Omega))$, i.e. $L^2(0,T;H^1(\Omega)^2)$ immerses with continuity in $L^2(0,T;\underline{H}(sym,\Omega))$, because $\left\|\underline{\underline{\epsilon}}(\varphi)\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}} \le \left\|\nabla\,\underline{\underline{\varphi}}\right\|^2_{L^2(\Omega)^{2\times 2}_{sym}}$, we must have $\underline{\varphi} = \underline{w}$, so $\underline{w} \in L^2(0,T;H^1(\Omega)^2)$. From (2.117), since $L^2(0,T;\mathcal{C}^\infty(\overline{\Omega})^{2\times 2}_{sym}) \subset W$, according to Theorem A.3, we obtain that $\mathrm{Tr}(\underline{w}) = 0$ for a.e. $t \in [0,T]$ and that is equivalent to stating that $\underline{w} \in L^2(0,T;H_0^1(\Omega)^2)$. ∎

Proposition 2.4.5. *Problem 2.4.5 admits a unique saddle point which is the solution of Problem 2.4.4.*

Proof. Since Problem 2.4.4 admits a unique solution, then the thesis is equivalent to prove that $(\underline{w}, \underline{\underline{\tau}})$ is saddle point of (2.116) if and only if is a saddle point of (2.113). We call $F_1(\underline{v}, \underline{\underline{p}})$ the functional in (2.113) and $F_2(\underline{v}, \underline{\underline{p}})$ the functional in (2.116), we can see that if $\underline{\underline{p}} \in W$ and $\underline{v} \in \Gamma$ then we have $F_1(\underline{v}, \underline{\underline{p}}) = F_2(\underline{v}, \underline{\underline{p}})$ and, due to Proposition 2.4.4, we can deduce that all saddle points of (2.113) and (2.116) satisfy such condition.
Let $(\underline{w}, \underline{\underline{\tau}})$ be a saddle point of (2.113), then

$$\forall \underline{v} \in \Gamma \quad F_1(\underline{v}, \underline{\underline{\tau}}) \leq F_1(\underline{w}, \underline{\underline{\tau}}) \leq F_1(\underline{w}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in Y, \tag{2.118}$$

and we want to show that

$$\forall \underline{v} \in \Lambda \quad F_2(\underline{v}, \underline{\underline{\tau}}) \leq F_2(\underline{w}, \underline{\underline{\tau}}) \leq F_2(\underline{w}, \underline{\underline{p}}) \quad \forall \underline{\underline{p}} \in W. \tag{2.119}$$

Since $W \subset Y$ and if $\underline{\underline{p}} \in W$ we have $F_1(\underline{w}, \underline{\underline{p}}) = F_2(\underline{w}, \underline{\underline{p}})$, then the right inequality is satisfy. The left one is given by the denseness of Γ in Λ and the continuity of $\underline{v} \to F_2(\underline{v}, \underline{\underline{\tau}})$ in Λ. Indeed due to the denseness we have that for every $\underline{v} \in \Lambda$ there exists $\{\underline{v}_n\}_{n\in\mathbb{N}} \in \Gamma$ such that $\underline{v}_n \longrightarrow \underline{v}$ in Λ as $n \to \infty$, recalling that if $\underline{u} \in \Gamma$ then $F_1(\underline{u}, \underline{\underline{\tau}}) = F_2(\underline{u}, \underline{\underline{\tau}})$, we obtain

$$F_2(\underline{w}, \underline{\underline{\sigma}}) = F_1(\underline{w}, \underline{\underline{\sigma}}) \geq \lim_{n\to\infty} F_1(\underline{v}_n, \underline{\underline{\sigma}}) = \lim_{n\to\infty} F_2(\underline{v}_n, \underline{\underline{\sigma}}) = F_2(\underline{v}, \underline{\underline{\sigma}}).$$

Now we do the converse, let $(\underline{w}, \underline{\underline{\tau}})$ be a saddle point of (2.116), so it satisfies (2.119), we want to prove (2.118). The left side follow from $\Gamma \subset \Lambda$ and $F_1(\underline{v}, \underline{\underline{\tau}}) = F_2(\underline{v}, \underline{\underline{\tau}})$ for all $\underline{v} \in \Gamma$. The right side is given by the denseness of Y in W, the continuity of $\underline{\underline{p}} \to F_1(\underline{w}, \underline{\underline{p}})$ in W and that $F_1(\underline{w}, \underline{\underline{p}}) = F_2(\underline{w}, \underline{\underline{p}})$ for all $\underline{\underline{p}} \in W$. Indeed for every $\underline{\underline{p}} \in W$ there exists $\{\underline{\underline{p}}_n\}_{n\in\mathbb{N}} \in Y$ such that $\underline{\underline{p}}_n \longrightarrow \underline{\underline{p}}$ in Y as $n \to \infty$, then we obtain

$$F_1(\underline{w}, \underline{\underline{\tau}}) = F_2(\underline{w}, \underline{\underline{\tau}}) \leq \lim_{n\to\infty} F_2(\underline{w}, \underline{\underline{p}}_n) = \lim_{n\to\infty} F_1(\underline{w}, \underline{\underline{p}}_n) = F_2(\underline{w}, \underline{\underline{p}}).$$

□

From Proposition 2.4.5 we have that $(\underline{\bar{u}}, \underline{\underline{\sigma}})$, unique saddle point of Problem 2.4.4, is also the unique saddle point of Problem 2.4.5, then from the Euler-Lagrange equations of (2.116), evaluated in $(\underline{\bar{u}}, \underline{\underline{\sigma}})$, we obtain:

$$\begin{cases} \int_0^T \int_\Omega \rho \underline{\bar{u}}_{tt} \underline{y} - \operatorname{div}(\underline{\underline{\sigma}}) \underline{y} - \underline{f} \underline{y} \, \mathrm{dx}\, \mathrm{dt} = 0 & \forall \underline{y} \in \Lambda \\ \int_0^T \int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D - 2\mu_{el} \underline{\underline{\epsilon}}(\underline{\bar{u}})^D : \underline{\underline{\varphi}}^D - 2\mu_{vis} \underline{\underline{\epsilon}}(\underline{\bar{u}}_t)^D : \underline{\underline{\varphi}}^D \, \mathrm{dx}\, \mathrm{dt} = 0 & \forall \underline{\underline{\varphi}} \in W \\ \int_0^T \int_\Omega \operatorname{tr}(\underline{\underline{\sigma}}) \operatorname{tr}(\underline{\underline{\varphi}}) - (2\mu_{el} + 2\lambda_{el}) \operatorname{tr}(\underline{\underline{\epsilon}}(\underline{\bar{u}})) \operatorname{tr}(\underline{\underline{\varphi}}) & \\ \qquad - (2\mu_{vis} + 2\lambda_{vis}) \operatorname{tr}(\underline{\underline{\epsilon}}(\underline{\bar{u}}_t)) \operatorname{tr}(\underline{\underline{\varphi}}) \, \mathrm{dx}\, \mathrm{dt} = 0 & \forall \underline{\underline{\varphi}} \in W \end{cases} \tag{2.120}$$

2.4.3 Mixed formulation 1

Now we are interested in finding the functions $(\underline{u}, \underline{\underline{\sigma}})$ solution of system (2.114) with the addition of initial conditions for $\underline{u}$. We introduce the mixed problem:

Problem 2.4.6 (Mixed PDE formulation in $L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$ of linear viscoelasticity)**.** *Find the couple* $(\underline{u}, \underline{\underline{\sigma}})$ *such that* $\underline{u} \in H^1(0,T;H^1_0(\Omega)^2)$, $\underline{u}_{tt} \in L^2(0,T;L^2(\Omega)^2)$, $\underline{\underline{\sigma}} \in L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$ *and satisfies:*

$$\begin{cases} \int_0^T \int_\Omega \rho \underline{u}_{tt}\underline{y} + \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(y) - \underline{f}\underline{y} \text{ dx dt} = 0 & \forall \underline{y} \in \Gamma \\ \int_0^T \int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D - 2\mu_{el}\underline{\underline{\epsilon}}(u)^D : \underline{\underline{\varphi}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D : \underline{\underline{\varphi}}^D \text{ dx dt} = 0 & \forall \underline{\underline{\varphi}} \in Y \\ \int_0^T \int_\Omega \operatorname{tr}(\underline{\underline{\sigma}}) \operatorname{tr}(\underline{\underline{\varphi}}) - (2\mu_{el} + 2\lambda_{el}) \operatorname{tr}(\underline{\underline{\epsilon}}(u)) \operatorname{tr}(\underline{\underline{\varphi}}) & \\ \qquad - (2\mu_{vis} + 2\lambda_{vis}) \operatorname{tr}(\underline{\underline{\epsilon}}(u_t)) \operatorname{tr}(\underline{\underline{\varphi}}) \text{ dx dt} = 0 & \forall \underline{\underline{\varphi}} \in Y \\ \underline{u}(\cdot, 0) = \underline{u}_0 & \textit{in } \Omega \\ \underline{u}_t(\cdot, 0) = \underline{u}_1 & \textit{in } \Omega, \end{cases} \tag{2.121}$$

where $Y = L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$ *and* $\Gamma = L^2(0,T;H^1_0(\Omega)^2)$.

Theorem 2.11. *The function* $\underline{u}$ *is the unique solution of Problem 2.4.1 if and only if* $(\underline{u}, \underline{\underline{\sigma}})$ *is the unique solution of Problem 2.4.6 where* $\underline{\underline{\sigma}}$ *is define* (2.122).

Proof. The proof is rather long and we decided to divided it in Proposition 2.4.6 and Proposition 2.4.7. □

Proposition 2.4.6. *For every* $(\underline{u}, \underline{\underline{\sigma}})$ *solution of Problem 2.4.6,* $\underline{\underline{\sigma}}$ *is unique determined by* $\underline{u}$ *and*

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^D + \frac{1}{2}\operatorname{tr}(\underline{\underline{\sigma}})\underline{\underline{id}} = 2\mu_{el}\underline{\underline{\epsilon}}(u) + \lambda_{el}\operatorname{div}(u)\underline{\underline{id}} + 2\mu_{vis}\underline{\underline{\epsilon}}(u_t) + \lambda_{vis}\operatorname{div}(u_t)\underline{\underline{id}}. \tag{2.122}$$

Proof. Since $(\underline{u}, \underline{\underline{\sigma}})$ is a solution, then $\underline{\underline{\sigma}} \in L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$. From the second and third equations of (2.121) we can deduce that

$$\begin{aligned} \underline{\underline{\sigma}}^D &= 2\mu_{el}\underline{\underline{\epsilon}}(u)^D + 2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D \\ \operatorname{tr}(\underline{\underline{\sigma}}) &= (2\mu_{el} + 2\lambda_{el}) \operatorname{tr}(\underline{\underline{\epsilon}}(u)) + (2\mu_{vis} + 2\lambda_{vis}) \operatorname{tr}(\underline{\underline{\epsilon}}(u_t)) \end{aligned}$$

□

Proposition 2.4.7. *The function* $\underline{u}$ *is solution of Problem 2.4.1 if and only if there exists* $\underline{\underline{\sigma}}$ *such that* $(\underline{u}, \underline{\underline{\sigma}})$ *is solution of the mixed Problem 2.4.6.*

Proof. ($\Rightarrow$)
Let $\bar{\underline{u}}$ be solution of Problem 2.4.1, then Corollary 2.4.1 implies that $\bar{\underline{u}}$ is in $H^1(0,T;H^1_0(\Omega)^2)$ and $\bar{\underline{u}}_{tt} \in L^2(0,T;L^2(\Omega)^2)$. From the construction we have done in Section 2.4.2 we have that $(\bar{\underline{u}}, \underline{\underline{\sigma}})$, where $\bar{\underline{u}}$ is the unique solution of Problem 2.4.2, that coincides with the one of Problem 2.4.1 by construction, and $\underline{\underline{\sigma}} = \mathcal{R}(q^*)$, where q^* is the unique solution of Problem 2.4.3 satisfies (2.114), so it is also a solution of Problem 2.4.6 since $\bar{\underline{u}}$ satisfies the initial conditions of Problem 2.4.6.

($\Leftarrow$)
Now we proceed with the opposite, let $\underline{u}_m$ be the first component of a solution of Problem 2.4.6. We want to prove that $\underline{u}_m = \underline{\bar{u}}$. Let start by considering the following.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}(\underline{u}_{mt} - \underline{\bar{u}}_t)\|^2_{L^2(\Omega)^2} &= \langle\sqrt{\rho}(\underline{u}_{mtt} - \underline{\bar{u}}_{tt}), \sqrt{\rho}(\underline{u}_{mt} - \underline{\bar{u}}_t)\rangle_{L^2(\Omega)^2} \\ &= (\sqrt{\rho}(\underline{u}_{mtt} - \underline{\bar{u}}_{tt}), \sqrt{\rho}(\underline{u}_{mt} - \underline{\bar{u}}_t))_{L^2(\Omega)^2} \\ &= (\rho(\underline{u}_{mtt} - \underline{\bar{u}}_{tt}), \underline{u}_{mt} - \underline{\bar{u}}_t)_{L^2(\Omega)^2} \\ &= \langle\rho(\underline{u}_{mtt} - \underline{\bar{u}}_{tt}), \underline{u}_{mt} - \underline{\bar{u}}_t\rangle_{L^2(\Omega)^2}\,, \end{aligned} \tag{2.123}$$

where we have used $\rho \geq A > 0$. Since $\underline{u}_{mt} - \underline{\bar{u}}_t \in L^2(0,T;H^1_0(\Omega)^2)$ and recalling that $\underline{u}_m$ satisfies the first equation of (2.121), and $\underline{\bar{u}}$ satisfies the first equation of (2.97), from (2.123) we obtain

$$\begin{aligned}\langle\rho(\underline{u}_{mtt} - \underline{\bar{u}}_{tt}), \underline{u}_{mt} - \underline{\bar{u}}_t\rangle_{L^2(\Omega)^2} =& -\langle\underline{\underline{\sigma}}, \underline{\underline{\epsilon}}(u_{mt}) - \underline{\underline{\epsilon}}(\bar{u}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}} \\ &- \langle\mathrm{div}(A_{el}(\underline{\underline{\epsilon}}(\bar{u})) + A_{vis}(\underline{\underline{\epsilon}}(\bar{u}_t))), \underline{u}_{mt} - \underline{\bar{u}}_t\rangle_{L^2(\Omega)^2}\,, \end{aligned} \tag{2.124}$$

where $A_{el}(\underline{\underline{\epsilon}}(u)) = 2\mu_{el}\underline{\underline{\epsilon}}(u) + \lambda_{el}\,\mathrm{tr}(\underline{\underline{\epsilon}}(u))\underline{\underline{id}}$ and $A_{vis}(\underline{\underline{\epsilon}}(u)) = 2\mu_{vis}\underline{\underline{\epsilon}}(u) + \lambda_{vis}\,\mathrm{tr}(\underline{\underline{\epsilon}}(u))\underline{\underline{id}}$.
Since $\underline{u}_{mt} - \underline{\bar{u}}_t \in L^2(0,T;H^1_0(\Omega)^2)$ we have that $\underline{\underline{\epsilon}}(u_{mt}) - \underline{\underline{\epsilon}}(\bar{u}_t) \in L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$, so according to the second and third equations of (2.121), we obtain

$$-\langle\underline{\underline{\sigma}}, \underline{\underline{\epsilon}}(u_{mt}) - \underline{\underline{\epsilon}}(\bar{u}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}} = -\langle A_{el}(\underline{\underline{\epsilon}}(u_m)) + A_{vis}(\underline{\underline{\epsilon}}(u_{mt})), \underline{\underline{\epsilon}}(u_{mt}) - \underline{\underline{\epsilon}}(\bar{u}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}}\,,$$

and, since $\underline{\underline{\epsilon}}(\underline{u}_{mt}) - \underline{\underline{\epsilon}}(\underline{\bar{u}}_t) \in L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$, we have

$$\langle\mathrm{div}(A_{el}(\underline{\underline{\epsilon}}(\bar{u})) + A_{vis}(\underline{\underline{\epsilon}}(\bar{u}_t))), \underline{u}_{mt} - \underline{\bar{u}}_t\rangle_{L^2(\Omega)^2} = -\langle A_{el}(\underline{\underline{\epsilon}}(\bar{u})) + A_{vis}(\underline{\underline{\epsilon}}(\bar{u}_t)), \underline{\underline{\epsilon}}(\underline{w}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}}\,.$$

Denoting $\underline{w} = \underline{u}_m - \underline{\bar{u}}$ and $\underline{w}_t = \underline{u}_{mt} - \underline{\bar{u}}_t$, we can rewrite (2.124) as follows

$$\begin{aligned}&-\langle\underline{\underline{\sigma}}, \underline{\underline{\epsilon}}(\underline{w}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}} + \langle A_{el}(\underline{\underline{\epsilon}}(\bar{u})) + A_{vis}(\underline{\underline{\epsilon}}(\bar{u}_t)), \underline{\underline{\epsilon}}(\underline{w}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}} \\ &= -\langle A_{el}(\underline{\underline{\epsilon}}(w)) + A_{vis}(\underline{\underline{\epsilon}}(w_t)), \underline{\underline{\epsilon}}(\underline{w}_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}} \\ &= -\left(\mu_{el}\frac{d}{dt}\|(\underline{\underline{\epsilon}}(w))\|^2_{L^2(\Omega)^{2\times 2}_{sym}} + \frac{\lambda_{el}}{2}\frac{d}{dt}\|\mathrm{tr}((\underline{\underline{\epsilon}}(w)))\|^2_{L^2(\Omega)^{2\times 2}_{sym}} + \langle A_{vis}(\underline{\underline{\epsilon}}(w_t)), \underline{\underline{\epsilon}}(w_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}}\right) \\ &\geq -\langle A_{vis}(\underline{\underline{\epsilon}}(w_t)), \underline{\underline{\epsilon}}(w_t)\rangle_{L^2(\Omega)^{2\times 2}_{sym}} = -\left(\int_0^T\int_\Omega 2\mu_{vis}\|(\underline{\underline{\epsilon}}(w_t))\|^2_{L^2(\Omega)^{2\times 2}_{sym}} + \lambda_{vis}|\mathrm{tr}(\underline{\underline{\epsilon}}(w_t))^2|\,\mathrm{dx}\,\mathrm{dt}\right) \\ &\geq -2\mu_{vis}\mathcal{K}\|\underline{w_t}\|^2_{H^1_0(\Omega)^2}, \end{aligned} \tag{2.125}$$

where $\mathcal{K}$ is the constant of Korn. Since $\rho, \mu_{vis} \geq A > 0$, then from (2.123) and (2.125) we can conclude that

$$\frac{d}{dt}\|\underline{u}_{mt} - \underline{\bar{u}}_t\|^2_{L^2(\Omega)^2} = 0.$$

From the previous equation and because $\underline{u}_m$ and $\underline{\bar{u}}$ satisfy the same initial condition, we have $\underline{u}_m = \underline{\bar{u}}$. □

2.4.4 Mixed formulation 2

Now we aim to find $(\underline{u}, \underline{\underline{\sigma}})$ solution of system (2.120) with the addition of initial conditions for $\underline{u}$. We introduce the mixed problem:

Problem 2.4.7 (Mixed formulation in $L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$ of linear viscoelasticity)**.** *Find the couple* $(\underline{u}, \underline{\underline{\sigma}})$ *such that* $\underline{u} \in H^1(0,T;H_0^1(\Omega)^2)$, $\underline{u}_{tt} \in L^2(0,T;L^2(\Omega)^2)$, $\underline{\underline{\sigma}} \in L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$ *and satisfies:*

$$\begin{cases} \int_0^T \int_\Omega \rho \underline{u}_{tt} \underline{y} - \text{div}(\underline{\underline{\sigma}}) \underline{y} - \underline{f}\underline{y} \text{ dx dt} = 0 & \forall \underline{y} \in \Lambda \\ \int_0^T \int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D - 2\mu_{el} \underline{\underline{\epsilon}}(u)^D : \underline{\underline{\varphi}}^D - 2\mu_{vis} \underline{\underline{\epsilon}}(u_t)^D : \underline{\underline{\varphi}}^D \text{ dx dt} = 0 & \forall \underline{\underline{\varphi}} \in W \\ \int_0^T \int_\Omega \text{tr}(\underline{\underline{\sigma}}) \,\text{tr}(\underline{\underline{\varphi}}) - (2\mu_{el} + 2\lambda_{el}) \,\text{tr}(\underline{\underline{\epsilon}}(u)) \,\text{tr}(\underline{\underline{\varphi}}) & \\ \qquad - (2\mu_{vis} + 2\lambda_{vis}) \,\text{tr}(\underline{\underline{\epsilon}}(u_t)) \,\text{tr}(\underline{\underline{\varphi}}) \text{ dx dt} = 0 & \forall \underline{\underline{\varphi}} \in W \\ \underline{u}(\cdot,0) = \underline{u}_0 & \textit{in } \Omega \\ \underline{u}_t(\cdot,0) = \underline{u}_1 & \textit{in } \Omega, \end{cases} \tag{2.126}$$

where $W = L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$ *and* $\Lambda = L^2(0,T;L^2(\Omega)^2)$.

Proposition 2.4.8. *The couple* $(\underline{u}, \underline{\underline{\sigma}})$ *is solution of Problem 2.4.6 if and only if is solution of Problem 2.4.7.*

Proof. $(\Rightarrow)$
Let $(\underline{u}, \underline{\underline{\sigma}})$ be solution of Problem 2.4.6, so $\underline{u} = \bar{u}$ and $\underline{\underline{\sigma}}$ is define in (2.122). We have already noticed, at the end of Section 2.4.2, that $\underline{\underline{\sigma}} \in L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$, so if $(\underline{u}, \underline{\underline{\sigma}})$ satisfies (2.126), then it is solution of Problem 2.4.7. Since $\rho\underline{\bar{u}}_{tt}, \underline{f}, \text{div}(\underline{\underline{\sigma}}) \in L^2(0,T;(L^2(\Omega)^2))$ and $\underline{u}$ satisfies the first equation of (2.121), then, because $L^2(0,T;H_0^1(\Omega)^2)$ is dense in Λ, for every $\underline{y}$ in Λ there exists $\{\underline{y}_n\}_{n\in\mathbb{N}}$ such that $\underline{y_n} \longrightarrow y$ in Λ as $n \to \infty$, then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_0^T \int_\Omega \rho \underline{u}_{tt} \underline{y}_n + \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(y_n) - \underline{f}\underline{y} \text{ dx dt} \\ &= \lim_{n\to\infty} \int_0^T \int_\Omega \rho \underline{u}_{tt} \underline{y}_n - \text{div}(\underline{\underline{\sigma}}) \underline{y}_n - \underline{f}\underline{y} \text{ dx dt} \\ &= \lim_{n\to\infty} (\rho \underline{u}_{tt} - \text{div}(\underline{\underline{\sigma}}) - \underline{f}, \underline{y}_n)_\Lambda \\ &= (\rho \underline{u}_{tt} - \text{div}(\underline{\underline{\sigma}}) - \underline{f}, \underline{y})_\Lambda \\ &= \int_0^T \int_\Omega \rho \underline{u}_{tt} \underline{y} - \text{div}(\underline{\underline{\sigma}}) \underline{y} - \underline{f}\underline{y} \text{ dx dt}. \end{aligned}$$

Thus the first equation of (2.126) is satisfy.
Second and third equations of (2.126) are satisfy because $W \subset Y$ and the initial conditions, because are the same, are clearly satisfy.
$(\Leftarrow)$
Let $(\underline{u}, \underline{\underline{\sigma}})$ be solution of Problem 2.4.7, we only need to verify that $(\underline{u}, \underline{\underline{\sigma}})$ satisfies (2.121). The first equation of (2.121) is satisfy because $H_0^1(\Omega)^2 \subset \Lambda$. Since $(\underline{u}, \underline{\underline{\sigma}})$ satisfies the second and third equations of (2.126), $\underline{u} \in H^1(0,T;H_0^1(\Omega)^2)$ and $\underline{\underline{\sigma}} \in L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$ then, because $\underline{\underline{H}}(\text{div},\Omega)_{sym}$ is a dense subset of $L^2(\Omega)^{2\times 2}_{sym}$, for every $\underline{\underline{\varphi}}$ in $L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$ there

exists $\{\underline{\underline{\varphi}}_n\}_{n\in\mathbb{N}}$ in $L^2(0,T;\underline{\underline{H}}(\text{div},\Omega)_{sym})$, such that $\underline{\underline{\varphi}}_n \longrightarrow \underline{\underline{\varphi}}$ in $L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})$ as $n\to\infty$, then

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\int_0^T\int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}_n{}^D - 2\mu_{el}\underline{\underline{\epsilon}}(u)^D : \underline{\underline{\varphi}}_n{}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D : \underline{\underline{\varphi}}_n{}^D \text{ dx dt}\\
&= \lim_{n\to\infty}(\underline{\underline{\sigma}}^D - 2\mu_{el}\underline{\underline{\epsilon}}(u)^D - -2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D, \underline{\underline{\varphi}}_n{}^D)_{L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})}\\
&= \lim_{n\to\infty}\langle\underline{\underline{\sigma}}^D - 2\mu_{el}\underline{\underline{\epsilon}}(u)^D - -2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D, \underline{\underline{\varphi}}_n{}^D\rangle_{L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})}\\
&= \langle\underline{\underline{\sigma}}^D - 2\mu_{el}\underline{\underline{\epsilon}}(u)^D - -2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D, \underline{\underline{\varphi}}^D\rangle_{L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})}\\
&= \int_0^T\int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\varphi}}^D - 2\mu_{el}\underline{\underline{\epsilon}}(u)^D : \underline{\underline{\varphi}}^D - 2\mu_{vis}\underline{\underline{\epsilon}}(u_t)^D : \underline{\underline{\varphi}}^D \text{ dx dt},
\end{aligned}
$$

and

$$
\begin{aligned}
0 = \lim_{n\to\infty}&\int_0^T\int_\Omega \operatorname{tr}(\underline{\underline{\sigma}})\operatorname{tr}(\underline{\underline{\varphi}}_n) - (2\mu_{el}+2\lambda_{el})\operatorname{tr}(\underline{\underline{\epsilon}}(\bar{u}))\operatorname{tr}(\underline{\underline{\varphi}}_n)\\
&\qquad - (2\mu_{vis}+2\lambda_{vis})\operatorname{tr}(\underline{\underline{\epsilon}}(u_t))\operatorname{tr}(\underline{\underline{\varphi}}_n) \text{ dx dt}\\
= \lim_{n\to\infty}&(\operatorname{tr}(\underline{\underline{\sigma}}) - (2\mu_{el}+2\lambda_{el})\operatorname{tr}(\underline{\underline{\epsilon}}(\bar{u})) - (2\mu_{vis}+2\lambda_{vis})\operatorname{tr}(\underline{\underline{\epsilon}}(u_t)), \operatorname{tr}(\underline{\underline{\varphi}}_n))_{L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})}\\
= \lim_{n\to\infty}&\langle\operatorname{tr}(\underline{\underline{\sigma}}) - (2\mu_{el}+2\lambda_{el})\operatorname{tr}(\underline{\underline{\epsilon}}(\bar{u})) - (2\mu_{vis}+2\lambda_{vis})\operatorname{tr}(\underline{\underline{\epsilon}}(u_t)), \operatorname{tr}(\underline{\underline{\varphi}}_n)\rangle_{L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})}\\
= \langle&\operatorname{tr}(\underline{\underline{\sigma}}) - (2\mu_{el}+2\lambda_{el})\operatorname{tr}(\underline{\underline{\epsilon}}(\bar{u})) - (2\mu_{vis}+2\lambda_{vis})\operatorname{tr}(\underline{\underline{\epsilon}}(u_t)), \operatorname{tr}(\underline{\underline{\varphi}})\rangle_{L^2(0,T;L^2(\Omega)^{2\times 2}_{sym})}\\
= \int_0^T&\int_\Omega \operatorname{tr}(\underline{\underline{\sigma}})\operatorname{tr}(\underline{\underline{\varphi}}) - (2\mu_{el}+2\lambda_{el})\operatorname{tr}(\underline{\underline{\epsilon}}(\bar{u}))\operatorname{tr}(\underline{\underline{\varphi}}) - (2\mu_{vis}+2\lambda_{vis})\operatorname{tr}(\underline{\underline{\epsilon}}(u_t))\operatorname{tr}(\underline{\underline{\varphi}}) \text{ dx dt}
\end{aligned}
$$

Thus the second and third equations of (2.121) are satisfy. Since the initial conditions are the same, they are fulfil. □

Finally we can state the main result.

Theorem 2.12. *The function $\underline{u}$ is the unique solution of Problem 2.4.1 if and only if $(\underline{u}, \underline{\underline{\sigma}})$ is the unique solution of the Problem 2.4.7 where $\underline{\underline{\sigma}}$ is define* (2.122) *.*

Proof. A direct consequence of Proposition 2.4.8 and Theorem 2.11. □

Chapter 3

Mixed FEM discretization

The aim of this chapter is to construct a numerical method that solves equation (2.97), since we have seen that (2.97) is equivalent to (2.126), we want to use the second equation and its advantages. We will try different approaches to solve it: in the first two, even if $\underline{\underline{\sigma}}$ is the quantity of interest that we want to find, it is convenient, due to the size of the matrices we have to work with, to implement a method with $\underline{u}$ and $\underline{u}_t$ as explicit variables and then, in a second moment derive $\underline{\underline{\sigma}}$ as a function of these two. We will also introduce another method in which $\underline{\underline{\sigma}}$ is an explicit variable as $\underline{u}$ and $\underline{u}_t$.

3.1 Finite element spaces

Since the aim of this section is to introduce the finite-dimensional subspaces that we will use to approximate the solution of (2.126), we will start from presenting a partition of Ω upon which we will build our discretization. Then we will choose appropriate finite-dimensional subspaces, the discretization, since we want that our method is conform, i.e. these subspaces must be contained in the infinite dimensional spaces $(H_0^1(\Omega)^2$ and $\underline{\underline{H}}(\mathrm{div}, \Omega)_{sym})$ we are trying to approximate.

As domain discretization we use the triangulation, i.e. we divide the domain into triangles T and we write $\mathcal{T}_h = \cup\{T\}$, where $\mathcal{T}_h$ is the triangulation of Ω and T is a single triangle, also called element. As the step h increase the triangulation $\mathcal{T}_h$ becomes uniformly thinner in the domain. Now that we have fixed the triangulation we are ready to do the same for the finite-dimensional subspaces. Since $\underline{u}$ and $\underline{u}_t$ live in $H_0^1(\Omega)^2$ we approximate them with the classical P1-Galerkin, that, for sake of completeness, we introduce. For the moments let us focus on the scalar case, if we define $\mathcal{P}_k(T)$, the set of polynomial of degree k in T, then we can introduce the following

$$\mathcal{S}_h = \{ \ v \in \mathcal{C}^0(\overline{\Omega}) \mid v|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_h \ \},$$

i.e. the piecewise-continuous function that restricted to every triangle are $\mathcal{P}_1(T)$. Since we have to approximate $H_0^1(\Omega)$, we need to find a subspace $\mathcal{J}_h$ of $\mathcal{S}_h$ such that $\mathcal{J}_h \subset H_0^1(\Omega)$. In order to do that we define $\{\Phi_i\}_{i=1}^N$ such that $\mathcal{J}_h = \mathrm{span}\{\Phi_i\}$. The functions $\{\Phi_i\}$ that we choose are as many as the internal nodes of the triangulation $\mathcal{T}_h$ and every function values 1 on an internal node x_i, each function on a different node, and 0 on the others. Since each element T contains or not the point x_i and $\Phi_i|_T$ is a $\mathcal{P}_1(T)$:

$$\Phi_i|_T = a_0 + a_1 x + a_2 y.$$

From the conditions we have decided we can deduce that Φ_i is 0 in every element that does not contain the point x_i. Since the three conditions of the value of the polynomial at the vertices of the triangle determinate uniquely the value of Φ_i in such element and because they all are zero we have that $\Phi_i|_T = 0$. If the element T contains the point x_i, then clearly $\Phi_i|_T \neq 0$, since $\Phi_i|_T$ is a $\mathcal{P}_1(T)$ and $\Phi_i(x_i) = 1$. So we can notice that each Φ_i does have support only in the elements that are near x_i. This choice for $\{\Phi_i\}_{i=1}^N$ has many advantages, one is that it usually leads to sparse matrices since $\text{supp}\{\Phi_i\} \cap \text{supp}\{\Phi_j\} \neq \emptyset$ only when x_i and x_j, internal nodes in which the two functions value 1, have an edge that connects them. Another advantage is that when we want to evaluate a function v of $\mathcal{J}_h$, i.e.

$$v(x) = \sum_{i=1}^{n} v_i \Phi_i(x),$$

in a internal point. Indeed, we have

$$v(x_j) = \sum_{i=1}^{n} v_i \Phi_i(x_j) = v_j \Phi_j(x_j) = v_j.$$

Since our aim is to create a basis for $\mathcal{V}_h$ we choose the basis spanned by $\{(\Phi_i, 0), (0, \Phi_i)\}_{i=1}^N$, where the functions Φ_i are the one that we have defined above.
Now we have to define the finite-dimensional subspace that approximate $\underline{\underline{H}}(\text{div}, \Omega)_{sym}$ conformely. In order to do that we decide to introduce $\mathcal{Q}_h$, the space of discontinuous Galerkin symmetric tensor that is often used as discretization for $L^2(\Omega)^{2\times 2}_{sym}$, and then impose that belongs to $\underline{\underline{H}}(\text{div}, \Omega)_{sym}$ through the intersection, i.e. $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$. As before, in order to introduce this space, it is convenient to study first the simpler formulation of the scalar case. We focus on the construction of $\mathcal{A}_h$, the scalar discontinuous Galerkin of degree 2 (DG2-Galerkin) :

$$\mathcal{A}_2 = \{v \mid v|_T \in \mathcal{P}_2(T) \ \forall T \in \mathcal{T}_h\},$$

i.e. the function that restricted to every triangle are $\mathcal{P}_2(T)$. Since a polynomial of degree 2 in $\mathbb{R}^2$ has 6 degrees of freedom, we need to fix 6 conditions to define it uniquely: 3 conditions are the values of the vertices, that are the same of the $\mathcal{P}_1$, and the other 3 are those at the middle points of the edges. So the basis of $\mathcal{A}_2$ is the set of functions τ_i that have value only on one element and, in that element, on one vertex they value 1 and on the others 0 and 0 on all the middle points of the edges, and the ones that, in the same element, are 0 on all the vertices and 1 on one middle point of an edge and 0 on the others. Since on every element we there are 6 basis function, then, we can deduce that the dimension of $\mathcal{A}_2$ is 6 times the number of elements. Since we want to introduce a basis for $\mathcal{Q}_h$, the space of discontinuous Galerkin symmetric tensors, then we choose the one spanned by

$$\left\{ \begin{bmatrix} \tau_i & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \tau_i \\ \tau_i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & \tau_i \end{bmatrix} \right\},$$

where the τ_i are defined above. So for the dimension of $\mathcal{Q}_h$ is 18 times the number of elements. Now we need to check that the choice of $\mathcal{P}_2$ is the right one, because, since we have to take the intersection of $\mathcal{Q}_h$ with $\underline{\underline{H}}(\text{div}, \Omega)_{sym}$, we need to check that $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym} \neq \langle 0 \rangle$. In order to do that we count the degrees of freedom that such condition needs and verify that the $\dim(\mathcal{Q}_h)$ is strictly greater of such number. Let $\underline{\underline{\sigma}}_i \in \mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$, then if we take two elements in $\mathcal{T}_h$, T_1 and T_2, with an edge in common, L_{12}, we need that $\underline{\underline{\sigma}}$ must satisfy

$$\underline{\underline{\sigma}}_1 \cdot n_1 = \underline{\underline{\sigma}}_2 \cdot n_2 \quad \text{in } L_{12}, \tag{3.1}$$

where $\underline{\underline{\sigma}}_1$ and $\underline{\underline{\sigma}}_2$ represent $\underline{\underline{\sigma}}$ restricted to T_1 and T_2, and n_1 and n_2 are the two normal components relative to the elements T_1 and T_2 on the edge L_{12}. Since $\underline{\underline{\sigma}} \cdot n_i$ is a polynomial of degree 2 we have that (3.1) is equivalent to impose that $\underline{\underline{\sigma}}_1(\xi_k) \cdot n_1 = \underline{\underline{\sigma}}_2(\xi_k) \cdot n_2$ are equal in 3 points (degree of the polynomial + 1), so we have that every edge require 6 degrees of freedom. Since we require such condition only for the internal edges we have that the number of conditions needed is 6 times the number of internal edges. So, the dimension of $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$ is, if we take the classical triangular mesh, $(n-1)(3n-1)$, where n is the number of points in an edge. Since $n > 1$, then working with the DG2-Galerkin symmetric tensors represent good choice for us.

3.2 PDE formulation

Even if $\underline{\underline{\sigma}}$ is the quantity of interest and trying to introduce a numerical mixed formulation in which it is an unknown, and not calculate it separately as a function of $\underline{u}$ and $\underline{u}_t$, is the main purpose of this thesis, we have that if we want to implement a method with such property the dimension of the matrices we obtain is too large. This is due to the fact that the dimension of $\underline{\underline{\sigma}}_h$ is 9 times the number of elements, instead of the dimension of $\underline{u}_h$ that is "only" 2 times the number of internal nodes. So, we need to rewrite (2.126), in a form such that the variables are $\underline{u}$ and $\underline{v}$ are the unknown, because we want to write it as a first order ODE, and, at the same time, we need to taking in account that $\underline{\underline{\sigma}}$ is in $\underline{\underline{H}}(\text{div}, \Omega)_{sym}$. So we will substitute $\underline{\underline{\sigma}}$ in the first equation and introduce the relation between $\underline{u}_t$ and $\underline{v}$ using a positive define operator.
Since the solution of (2.126) and (2.97) coincide and the solution of (2.97) is strong in time we have that also the solution of (2.126) is strong in time. This means that for every t in $[0, T]$ the following holds.

$$\begin{cases} \int_\Omega \rho \underline{u}_{tt}(\cdot,t)\underline{y} - \text{div}(\underline{\underline{\sigma}}(\cdot,t))\underline{y} - \underline{f}(\cdot,t)\underline{y} \,\text{dx} = 0 & \forall \underline{y} \in L^2(\Omega) \\ \int_\Omega \underline{\underline{\sigma}}(\cdot,t) : \underline{\underline{\varphi}} - (A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h(\cdot,t))) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h(\cdot,t)))) : \underline{\underline{\varphi}} = 0 & \forall \underline{\underline{\varphi}} \in \underline{\underline{H}}(\text{div}, \Omega)_{sym}, \end{cases}$$

where $A_{el}(\underline{\underline{\tau}}) = 2\mu_{el}\underline{\underline{\tau}} + \lambda_{el}\,\text{tr}(\underline{\underline{\tau}})\underline{\underline{id}}$ and $A_{vis}(\underline{\underline{\tau}}) = 2\mu_{vis}\underline{\underline{\tau}} + \lambda_{vis}\,\text{tr}(\underline{\underline{\tau}})\underline{\underline{id}}$. So, if we rewrite the above equations using the discretizations we have introduced before, we obtain:

$$\begin{cases} \int_\Omega \rho \underline{v_h}_t \underline{y_h} - \text{div}(\underline{\underline{\sigma_h}})\underline{y_h} - \underline{f_h}\underline{y_h} \,\text{dx} = 0 \ \ \forall \underline{y_h} \in \mathcal{V}_h \\ \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\tau_h}} - (A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h)) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h))) : \underline{\underline{\tau_h}} = 0 \qquad \forall \underline{\underline{\tau_h}} \in \mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}. \end{cases}$$

Since $\underline{\underline{\sigma_h}}$ is in $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$ and the elements $\mathcal{V}_h$ are continuous in Ω and $\mathcal{P}_1(T)$ restricted to every T, we can integrate by parts $\int_\Omega \text{div}(\underline{\underline{\sigma}}(\cdot,t))\underline{y}\ \text{dx}$ in the first equation and we obtain

$$\begin{cases} \int_\Omega \rho \underline{v_h}_t \underline{y_h} + \underline{\underline{\sigma_h}} : \underline{\underline{\epsilon}}(\underline{y_h}) - \underline{f_h}\underline{y_h}\ \text{dx} = 0 \quad \forall \underline{y_h} \in \mathcal{V}_h \\ \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\tau_h}} - (A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h)) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h))) : \underline{\underline{\tau_h}} = 0 \qquad \forall \underline{\underline{\tau_h}} \in \mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}. \end{cases} \tag{3.2}$$

From now on, in order to have simpler formulas to work with, we take $\underline{f_h} = 0$.

3.3 Construction of the matrices

In this section we will introduce the main matrices we will use later to attack (3.2) with different approach. Since we have several summations we can take the following assumption.

Remark 3.3.1. *From now on we use the Einstein convention and we write*

$$\alpha_i \beta_i \coloneqq \sum_{i=1}^{N} \alpha_i \beta_i.$$

We start from fixing some notation, from now on we will denote with $\underline{\underline{p}}$ the elements of $\mathcal{Q}_h$, $\underline{\underline{\tau}}$ an orthonormal basis of $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$, respect to the usual scalar product of $L^2(\Omega)^{2\times 2}_{sym}$, and $\underline{\varphi}$ the elements of $\mathcal{V}_h$. Since $\underline{u}_h$ and $\underline{v}_h$ are elements of $\mathcal{V}_h$, we can rewrite them using the basis spanned by the $\{\underline{\varphi_i}\}_{i=1}^N$ and we obtain

$$\underline{u_h} = \sum_{i=1}^{N} \underline{u_h}_i \underline{\varphi}_i$$

$$\underline{v_h} = \sum_{i=1}^{N} \underline{v_h}_i \underline{\varphi}_i.$$

Now we are ready to introduce the main matrices through which we will write equation (3.2):

$$A_{ij} = \int_\Omega A_{vis} \underline{\underline{\epsilon}}(\varphi_j) : \underline{\underline{p}}_i\ \text{dx}$$

$$B_{ij} = \int_\Omega A_{el} \underline{\underline{\epsilon}}(\varphi_j) : \underline{\underline{p}}_i\ \text{dx}$$

$$P_{ij} = \int_\Omega \underline{\underline{\epsilon}}(\varphi_j) : \underline{\underline{p}}_i\ \text{dx}$$

$$R_{ij} = \int_\Omega \rho \varphi_i \cdot \varphi_j\ \text{dx}.$$

The last matrix that we need is the one that permits to write the $\{\underline{\underline{\tau_i}}\}_{i=1}^R$ as a combination of $\{\underline{\underline{p_i}}\}_{i=1}^M$, i.e.

$$\underline{\underline{\tau}}_i = \sum_{j=1}^{M} \alpha_{ij} \underline{\underline{p}}_j\ \text{dx} = N_{ij} \underline{\underline{p}}_j.$$

Since $\underline{\underline{\sigma}}_h$ is in $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$, according to the second equation of (3.2) we have

$$\underline{\underline{\sigma}}_h = (A_{el}(\underline{\epsilon}(\underline{u}_h)) + A_{vis}(\underline{\epsilon}(\underline{v}_h))) \quad \text{in } \mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}.$$

Now we rewrite the previous equation through the orthonormal basis of $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$ spanned by $\{\underline{\underline{\tau_i}}\}_{i=1}^{R}$ and we obtain

$$\underline{\underline{\sigma}}_h = \langle \underline{\underline{\sigma}}_h, \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}} = \langle A_{el}(\underline{\epsilon}(\underline{u}_h)), \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}} + \langle A_{vis}(\underline{\epsilon}(\underline{v}_h)), \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}}.$$

Since we can represent $\underline{\underline{\tau_i}}$ as a composition of $\underline{\underline{p_i}}$ through the matrix N we can rewrite the previous equation as follows.

$$\begin{aligned} \underline{\underline{\sigma}}_h &= \langle A_{el}(\underline{\epsilon}(\underline{u}_h)), \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}} + \langle A_{vis}\underline{\epsilon}(\underline{v}_h), \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}} \\ &= \langle A_{el}(\underline{\epsilon}(\underline{u}_h)), \underline{\underline{p_l}} \rangle \, \alpha_{li} \underline{\underline{\tau_i}} + \langle A_{vis}(\underline{\epsilon}(\underline{v}_h)), \underline{\underline{p_l}} \rangle \, \alpha_{li} \underline{\underline{\tau_i}}. \end{aligned} \tag{3.3}$$

Substituting $\underline{\underline{\sigma}}_h$, define in (3.3), in the second component of the first equation of (3.2) we obtain

$$\begin{aligned} \int_\Omega \underline{\underline{\sigma}}_h : \underline{\epsilon}(\underline{\underline{y_h}}) \text{ dx} = \int_\Omega & \langle A_{el}(\underline{\epsilon}(\underline{u}_h)), \underline{\underline{p_l}} \rangle \, \alpha_{li} \underline{\underline{\tau_i}} : \underline{\epsilon}(\underline{\underline{y_h}}) \text{ dx} \\ &+ \int_\Omega \langle A_{vis}(\underline{\epsilon}(\underline{v}_h)), \underline{\underline{p_l}} \rangle \, \alpha_{li} \underline{\underline{\tau_i}} : \underline{\epsilon}(\underline{\underline{y_h}}) \text{ dx}. \end{aligned} \tag{3.4}$$

Since the $\{\underline{\underline{\tau_i}}\}_{i=1}^{R}$ are an orthonormal basis of $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$ respects to the scalar product of $L^2(\Omega)^{2\times 2}_{sym}$, then we can complete it to a an orthonormal one of $\mathcal{Q}_h$ with $\{\underline{\underline{\tilde{\tau}_i}}\}_{i=1}^{M-R}$, and we obtain

$$\begin{aligned} \int_\Omega \underline{\underline{\tau_i}} : \underline{\epsilon}(\underline{y_h}) \text{ dx} = \int_\Omega \underline{\underline{\tau_i}} : \langle \underline{\epsilon}(\underline{y_h}), \underline{\underline{\tau_j}} \rangle \, \underline{\underline{\tau_j}} \text{ dx} = \int_\Omega & \underline{\underline{\tau_i}} : \langle \underline{\epsilon}(\underline{y_h}), \underline{\underline{p_p}} \rangle \, \alpha_{pj} \underline{\underline{\tau_j}} \text{ dx} \\ &+ \int_\Omega \underline{\underline{\tau_i}} : \langle \underline{\epsilon}(\underline{y_h}), \underline{\underline{p_p}} \rangle \, \alpha_{pj} \underline{\underline{\tilde{\tau}_j}} \text{ dx} = \langle \underline{\epsilon}(\underline{y_h}), \underline{\underline{p_p}} \rangle \, \alpha_{pj}, \end{aligned} \tag{3.5}$$

where we have used that $\int_\Omega \underline{\underline{\tau_i}} : \underline{\underline{\tilde{\tau}_j}} \text{ dx} = 0$ for all $j \in \{1, 2, \dots, M-R\}$.
According to (3.4) and (3.5) we obtain

$$\begin{aligned} \int_\Omega \langle A_{el}(\underline{\epsilon}(\underline{u}_h)), \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}} : \underline{\epsilon}(\underline{\underline{y_h}}) \text{ dx} &= \langle A_{el}(\underline{\epsilon}(\underline{u}_h)), \underline{\underline{p_l}} \rangle \, \alpha_{lj}\alpha_{pj} \, \langle \underline{\epsilon}(\underline{y_h}), \underline{\underline{p_p}} \rangle \\ \int_\Omega \langle A_{vis}(\underline{\epsilon}(\underline{v}_h)), \underline{\underline{\tau_i}} \rangle \, \underline{\underline{\tau_i}} : \underline{\epsilon}(\underline{\underline{y_h}}) \text{ dx} &= \langle A_{vis}(\underline{\epsilon}(\underline{v}_h)), \underline{\underline{p_l}} \rangle \, \alpha_{lj}\alpha_{pj} \, \langle \underline{\epsilon}(\underline{y_h}), \underline{\underline{p_p}} \rangle \, . \end{aligned}$$

Using the matrix R to define the left side of the first equation of (3.2), we can rewrite it as follows.

$$R\underline{v}_{ht} = -P'N'NA\underline{v}_h - P'N'NB\underline{u}_h.$$

The last thing we need to do is to incorporate in the system that we want to introduce the relation between $\underline{v_h}$ and $\underline{u_{ht}}$, in order to do that we impose

$$BNN'P\underline{v}_h = BNN'P\underline{u_t}_h. \tag{3.6}$$

We can notice that $BNN'P$ is not a symmetric matrix since we are projecting into $\mathcal{Q}_h \cap \underline{\underline{H}}(\text{div}, \Omega)_{sym}$ both B and P. This in not a problem for us since the important relation that we need is that if $BNN'P\underline{u}_h = 0$, then $\underline{u}_h = 0$, so if (3.6) is satisfies then $\underline{v}_h = \underline{u_t}_h$. We have not proved this condition, but experimentally we have seen that the eigenvalues of $BNN'P$ are all strictly positive, so it is a positive definite form and, as a consequence, the above condition is verify.
Now we are ready to introduce the system we are going to solve:

$$\begin{cases} R\underline{v}_{ht} = -P'N'NA\underline{v}_h - P'N'NB\underline{u}_h \\ BNN'P\underline{u_t}_h = BNN'P\underline{v}_h. \end{cases}$$

If we define $\mathcal{A} = P'N'NA$ and $\mathcal{B} = P'N'NB$, we can rewrite it as follows.

$$\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix} \begin{bmatrix} v_h \\ u_h \end{bmatrix}_t = \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix} \begin{bmatrix} v_h \\ u_h \end{bmatrix}. \tag{3.7}$$

Since a method to solve this problem has not already been defined, we have faced it with different approaches and seen the pros and cons of each one.

3.4 Time integration

In this section we will describe three different approaches we have used to resolve (2.126). In the first two we implement (3.7), firstly trying to written the solution as an exponential one, and secondly using the implicit Euler method, which is more stable. The final approach we propose is to implement a system in which $\underline{\underline{\sigma}}$ is also an unknown.

3.4.1 Semi-discrete approach

We have seen in Corollary 2.4.1, in Chapter 2, that if we require $\text{div}(\mathcal{A}_{vis}(\underline{\underline{\epsilon}}(\underline{v}_0) + \mathcal{A}_{el}(\underline{\underline{\epsilon}}(\underline{u}_0)) \in (L^2(\Omega)^2)^*$, then we have that the solution gains enough regularity to be a classical one. So, we can represent it as an exponential solution. In order to do that it is convenient to diagonalize the matrix

$$\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix}^{-1} \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix}.$$

According to [3, Proposition 2.12] we must have that every eigenvalue λ of the previous matrix, with non zero imaginary component, satisfies $\mathfrak{Re}(\lambda) \leq -\frac{1}{2}\lambda_{max}(\mathcal{A})$, where $\mathfrak{Re}(\lambda)$ is the real part of λ. Since $\mathcal{A}$ is positive define, then $\mathfrak{Re}(\lambda)$ must be strictly less then zero. Instead, also according to the same Proposition, for the eigenvalues with null imaginary component, we should have that $\mathfrak{Re}(\lambda) \leq 0$. So all the eigenvalues of the previous matrix must have $\mathfrak{Re}(\lambda)$ strictly less then 0. However, from a numerical point of view, we have found that the real part of some complex eigenvalues with small modulus change sign and becomes positive (see 3.1). This is caused by the fact that the coefficients μ_{el},μ_{vis}, λ_{el}, and λ_{vis} are not heterogeneous since they jump from two different regions of the domain.
Since we have found that for severely ill conditioned problem, as ours, this approach is not good

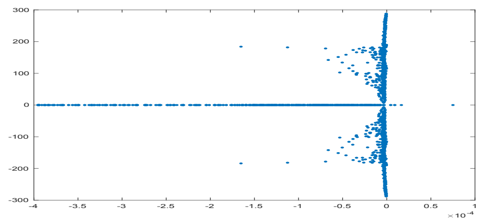

Figure 3.1: Eigenvalues when the problem is severely ill conditioned

for the above reason, we have tried to attack it using the implicit Euler method which is more stable for ill problems.

3.4.2 Implicit Euler

In this subsection we introduce two different formulations using the implicit Euler method. In the first one we try the implementation of (3.7) and in the second one we try to implement a formulation in which also $\underline{\underline{\sigma}}_h$ is a variable.
The advantage of solving system (3.7), instead of introducing $\underline{\underline{\sigma}}_h$, is that we have to work with matrices with less possible dimension since we do not have $\underline{\underline{\sigma}}_h$ as a variable. We remark that we have used the property that $\underline{\underline{\sigma}}_h$ is in $\underline{\underline{H}}(\mathrm{div}, \Omega)_{sym}$, since the matrix N represents the projection of the elements of $\mathcal{Q}_h$ into $\mathcal{Q}_h \cap \underline{\underline{H}}(\mathrm{div}, \Omega)_{sym}$.
We start form inverting the matrix on the left side in (3.7), this is possible since it is positive define, and we obtain

$$\begin{bmatrix} v_h \\ u_h \end{bmatrix}_t = \begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix}^{-1} \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix} \begin{bmatrix} v_h \\ u_h \end{bmatrix}.$$

If Δt is the time step we choose for the integration we have that the implicit Euler formulation of the previous equation is

$$\begin{bmatrix} v_h \\ u_h \end{bmatrix}_{k+1} = \begin{bmatrix} v_h \\ u_h \end{bmatrix}_k + \Delta\mathrm{t} \begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix}^{-1} \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix} \begin{bmatrix} v_h \\ u_h \end{bmatrix}_{k+1}.$$

Now we multiply both side by $\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix}$ and we obtain

$$\left(\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix} - \Delta\mathrm{t} \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix} \right) \begin{bmatrix} v_h \\ u_h \end{bmatrix}_{k+1} = \begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix} \begin{bmatrix} v_h \\ u_h \end{bmatrix}_k$$

In order to solve the previous system we have two choices that depends on the number time steps we have to take. If we have to take a number of steps greater than 2 times the number of internal nodes, that are the columns of $\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix}$, then it is convenient to pre-calculate

$$\mathcal{M}_1 = \left(\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix} - \Delta \mathrm{t} \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix},$$

and then solve at every time step $\begin{bmatrix} v_h \\ u_h \end{bmatrix}_{k+1} = \mathcal{M}_1 \begin{bmatrix} v_h \\ u_h \end{bmatrix}_k$. If the steps we take are less then 2 times the number of internal nodes, then it is convenient to calculate at every step $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}_k = \begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix} \begin{bmatrix} v_h \\ u_h \end{bmatrix}_k$ and pre-calculate $\mathcal{M}_2 = \left(\begin{bmatrix} R & 0 \\ 0 & \mathcal{B}' \end{bmatrix} - \Delta \mathrm{t} \begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{B}' & 0 \end{bmatrix} \right)^{-1}$, so, in this second case, the system becomes $\begin{bmatrix} v_h \\ u_h \end{bmatrix}_{k+1} = \mathcal{M}_2 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}_k$.
The problem related to this approach is that in both these cases the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are not sparse, so we have tried another method.

Now we introduce the second Implicit Euler Method where $\underline{\underline{\sigma}}_h$ enter in the equation as a variable, so starting from (3.2) and (3.3), we obtain

$$\begin{cases} \rho \underline{v_t}_h = \mathrm{div}(\underline{\underline{\sigma_h}}) & \text{in } \mathcal{V}_h \\ \mathcal{A}_{el}(\underline{\underline{\epsilon}}(\underline{u_t}_h)) = \mathcal{A}_{el}(\underline{\underline{\epsilon}}(\underline{v}_h)) & \text{in } \mathcal{V}_h \\ \underline{\underline{\sigma}}_h = A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h)) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h)) & \text{in } \mathcal{Q}_h \cap \underline{\underline{H}}(\mathrm{div}, \Omega)_{sym} \end{cases} \tag{3.8}$$

Since the elements of $\mathcal{V}_h$ are $\mathcal{P}_1$ on each element and $\underline{\underline{\sigma_h}}$ is in $\mathcal{Q}_h \cap \underline{\underline{H}}(\mathrm{div}, \Omega)_{sym}$, then we rewrite the first equation as follows.

$$\begin{aligned} \int_\Omega \rho \underline{v_t}_h \underline{\varphi}_j \, \mathrm{dx} &= \int_\Omega \mathrm{div}(\underline{\underline{\sigma}}_h) \underline{\varphi}_j \, \mathrm{dx} \\ &= -\int_\Omega \underline{\underline{\sigma}}_h : \underline{\underline{\epsilon}}(\underline{\varphi}_j) \, \mathrm{dx} \\ &= -\int_\Omega \underline{\underline{p_l}}_h \underline{\underline{\tau}}_k \alpha_{lk} : \underline{\underline{\epsilon}}(\underline{\varphi}_j) \, \mathrm{dx} = \underline{\underline{\sigma_k}}_h \alpha_{lk} \left(-\int_\Omega \underline{\underline{p}}_l : \underline{\underline{\epsilon}}(\underline{\varphi}_j) \, \mathrm{dx} \right) \end{aligned}$$

So the first equation can be written as follows:

$$R \underline{v_t}_h = -P' N' \underline{\underline{\sigma}}_h.$$

In order to write the second equation it is convenient to introduce the following symmetric matrix:

$$E_{ij} = \int_\Omega A_{el}(\underline{\underline{\epsilon}}(\varphi_i)) : \underline{\underline{\epsilon}}(\varphi_j) \, \mathrm{dx}.$$

So, the second equation can be written, using the matrix E, as follows:

$$E\underline{u_t}_h = E\underline{v}_h.$$

Remark 3.4.1. *We could have written the second equation introducing an orthonormal basis of $\mathcal{Q}_h$ and a matrix L that allow us to write this basis as a combination of $\{\underline{\underline{p}}\}_{i=1}^{M}$. Then, using the matrices B and P we should have obtained*

$$P'L'LB\underline{u_t}_h = P'L'LB\underline{v}_h.$$

Such approach is not convenient since calculating the product $P'L'LB$ represents a numerical cost that can be avoided by the introduction of E.

Now we rewrite the third equation of (3.8):

$$\begin{aligned}\int_\Omega \underline{\underline{\sigma_h}} : \underline{\underline{\tau}}_k \,\mathrm{dx} &= \int_\Omega A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h)) + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h)) : \underline{\underline{\tau}}_k \,\mathrm{dx}\\ &= \int_\Omega A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h)) : \underline{\underline{\tau}}_k + A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h)) : \underline{\underline{\tau}}_k \,\mathrm{dx}\\ &= \left(\int_\Omega A_{el}(\underline{\underline{\epsilon}}(\underline{u}_h)) : \underline{\underline{p}}_l \,\mathrm{dx}\right) \alpha_{lk} + \left(\int_\Omega A_{vis}(\underline{\underline{\epsilon}}(\underline{v}_h)) : \underline{\underline{p}}_s \,\mathrm{dx}\right) \alpha_{lk}.\end{aligned}$$

So the third equation can be written as follows:

$$N'N\underline{\underline{\sigma_h}} = N'B\underline{u_h} + N'A\underline{v_h}.$$

Now we can rewrite (3.8):

$$\begin{bmatrix} R & 0 & 0\\ 0 & P'L'LB & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}_t = \begin{bmatrix} 0 & 0 & -P'N'\\ P'L'LB & 0 & 0\\ -N'A & -N'B & N'N\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}.$$

According to the implicit Euler method we obtain

$$\begin{bmatrix} R & 0 & 0\\ 0 & P'L'LB & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}_{k+1} = \begin{bmatrix} R & 0 & 0\\ 0 & P'L'LB & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}_k + \Delta\mathrm{t}\begin{bmatrix} 0 & 0 & -P'N'\\ P'L'LB & 0 & 0\\ -N'A & -N'B & N'N\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}_{k+1},$$

this is equivalent to

$$\begin{bmatrix} R & 0 & \Delta\mathrm{t}P'N'\\ -\Delta\mathrm{t}P'L'LB & P'L'LB & 0\\ N'A & N'B & -N'N\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}_{k+1} = \begin{bmatrix} R & 0 & 0\\ 0 & P'L'LB & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} v_h\\ u_h\\ \sigma_h\end{bmatrix}_k,$$

where we have multiplicate for Δt the third equation.

3.5 A simple 2D test case

In this last section we introduce the setting of the example that we have implemented using the last approach we have introduced before: Implicit Euler where $\underline{\underline{\sigma}}$ is an unknown. As domain we have chosen $\Omega = [-1, 1] \times [0, 1]$, where $\Omega_1 = [-1, 0] \times [0, 1]$ represents the first material and $\Omega_2 =]0, 1] \times [0, 1]$ the second one. The parameters are as follows:

$$\begin{aligned} &\mu_{el} = 10^{-2}, 2 * 10^{-2} && \mu_{vis} = 10^{-5}, 10^{-4} \\ &\lambda_{el} = 5 * 10^{-2}, 4 * 10^{-2} && \lambda_{vis} = 2 * 10^{-4}, 10^{-3}, \\ &\rho = 1, 2 \end{aligned}$$

where the first number of each parameter is for Ω_1 and the second one for Ω_2. As time step we have taken $\Delta\mathrm{t} = 0.25$ For simplicity as forcing function we take $\underline{f} = 0$, as we have done in the previous section. In order to have $\underline{\underline{\sigma}}_0$ in $\underline{\underline{H}}(\mathrm{div}, \Omega)_{sym}$ we need that the initial data satisfies Corollary 2.4.1, so we take $\underline{u}_0$ and $\underline{v}_0$ compactly supported in Ω_1:

$$\begin{aligned} &\underline{u}_0 = \{\left(\sin(\frac{\pi}{2} + \frac{\pi}{0.6}(y - \frac{1}{2})) \sin(2\pi \frac{x}{0.4}) \right) \chi_{[-1,-0.6]\times[0.2,0.8]}, 0\} \\ &\underline{v}_0 = \{0, 0\}. \end{aligned}$$

As mesh we have used the usual uniform triangulation, see Figure 3.2.

We have reported different frames of the evolution of the viscoelastic wave subjected to the conditions described above. Indeed, we can notice that at the initial time, see Figure 3.3, the deformation, due to the initial conditions, acts only on Ω_1. From Figure 3.4 can be noticed, when looking at the displacement in $[-0.2, 0] \times [0, 1]$, the "rebound" of the wave when it hits the second material, i.e. Ω_2. Also from Figure 3.4, and even more from Figure 3.5, is visible the formation of vortex due to the Dirichlet boundary conditions. For completeness we have also put Figure 3.6 and Figure 3.7 that reported the evolution of the wave at $t = 15$ and $t = 20$. From all the figures mentioned above it can be noticed how the magnitude of the velocity and the displacement decay over time.

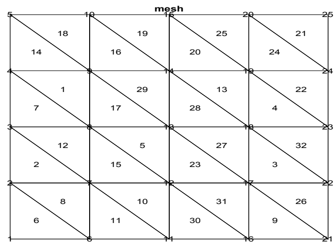

Figure 3.2: Mesh with $n = 5$

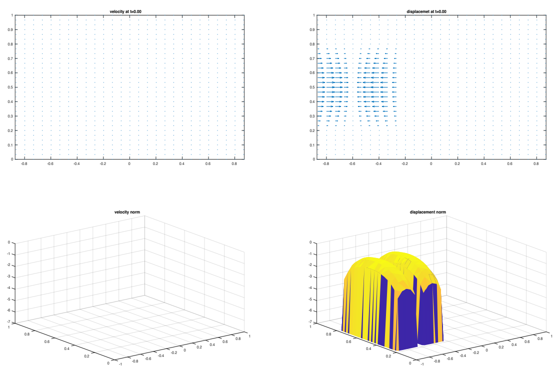

Figure 3.3: Evolution of the wave at $t = 0$ in the setting described in Section 3.5

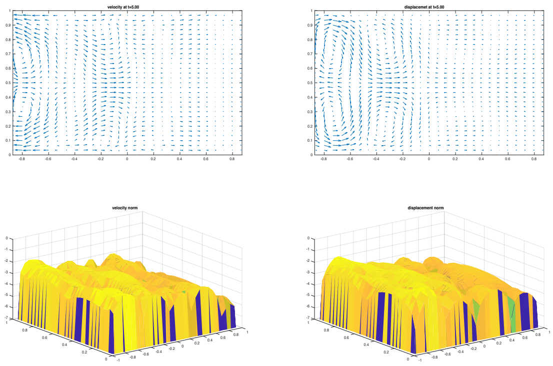

Figure 3.4: Evolution of the wave at time $t = 5$ in the setting described in Section 3.5

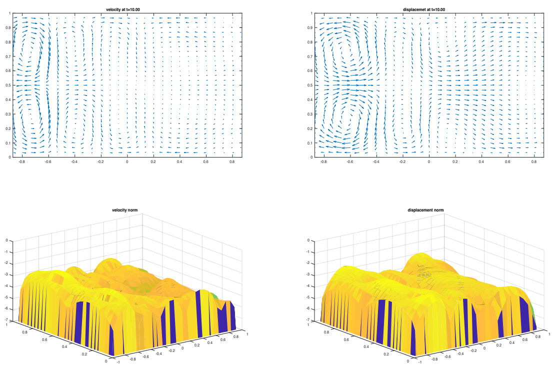

Figure 3.5: Evolution of the wave at time $t = 10$ in the setting described in Section 3.5

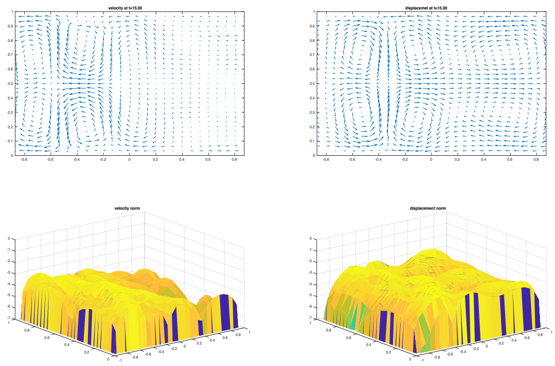

Figure 3.6: Evolution of the wave at time $t = 15$ in the setting described in Section 3.5

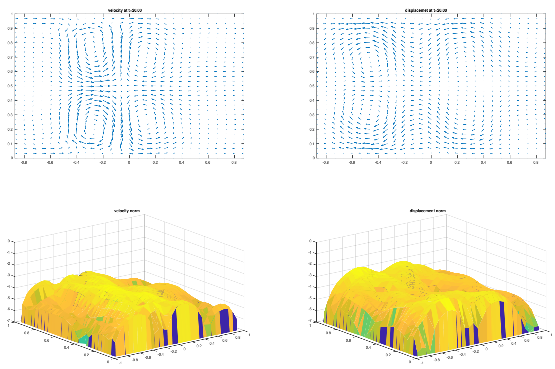

Figure 3.7: Evolution of the wave at time $t = 20$ in the setting described in Section 3.5

Appendix A

Tools

In this chapter we want to fix some notations that we need later or recall results that we are not going to prove here but are needed in some proofs.

A.1 Convex Analysis

All the results that we are going to take in this section came from [8] where a full coverage of the subject can be found.

Definition A.1.1 (Convex function)**.** *Let $\mathcal{A}$ be a convex subspace of V, and F a mapping of $\mathcal{A}$ into $\overline{\mathbb{R}}$. F is said to be convex if, for every u and v in $\mathcal{A}$, we have:*

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(v) + (1-\lambda)F(v) \quad \forall \lambda \in [0,1] \tag{A.1}$$

whenever the right-hand side is defined.

Definition A.1.2 (Strictly convex function)**.** *Let $\mathcal{A}$ be a convex subspace of V, and F a mapping of $\mathcal{A}$ into $\overline{\mathbb{R}}$. F is said to be strictly convex if it is convex and the strict inequality holds in A.1, $\forall u, v \in \mathcal{A}, u \neq v$ and $\forall \lambda \in]0,1[$*

The next definition is not in the [8] but can be found in every book of analysis.

Definition A.1.3 (Strong convex function)**.** *Let $\mathcal{A}$ be a convex subspace of V where V is a normed space, and F a mapping of $\mathcal{A}$ into $\overline{\mathbb{R}}$. F is said to be μ-convex if for every u and v in $\mathcal{A}$, we have:*

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(v) + (1-\lambda)F(v) - \frac{1}{2}\lambda(1-\lambda)\mu\|u-v\|_V^2 \quad \forall \lambda \in [0,1] \tag{A.2}$$

whenever the right-hand side is defined.

Definition A.1.4 (Lower semi-continuous function)**.** *Let V be a real l.c.s. (locally convex space). A function $F : V \longrightarrow \overline{\mathbb{R}}$ is said to be l.s.c. (lower semi-continuous) on V if*

$$\forall u \in V, \quad \liminf_{v \to u} F(v) \geq F(u).$$

Definition A.1.5 (lower semi-contineous regularization)**.** *Let V be a real l.c.s. (locally convex space) and $F : V \longrightarrow \overline{\mathbb{R}}$ a function, we define $\overline{F} : V \longrightarrow \overline{\mathbb{R}}$ the largest l.s.c. minorant of F and we call it the l.s.c. regularization of F and*

$$\forall u \in V, \quad \overline{F}(u) = \liminf_{v \to u} F(v).$$

Proposition A.1.1. *If $F : V \longrightarrow \overline{\mathbb{R}}$ is a lower semi-continuous convex function and assumes the value $-\infty$, it can not take any finite value.*

Proposition A.1.2. *Let $F : V \longrightarrow \overline{\mathbb{R}}$ be a convex function. The following statements are equivalent to each other:*
(i) there exists a non-empty open set θ on which F is not everywhere equal to $-\infty$ and is bounded above by a constant $a < +\infty$
(ii) F is a proper function, and it is continuous over the interior of its effective domain, which is non-empty.

Definition A.1.6 ($\Gamma(V)$ and $\Gamma_0(V)$)**.** *Let V be a locacly convex space, $\Gamma(V)$ is the set of lower semi-contnuous function from V into $\overline{\mathbb{R}}$, and if F takes the value $-\infty$ then F is identically equal to $-\infty$.*
We define $\Gamma_0(V)$ as $\Gamma(V)$ with out the function F that are equal to $+\infty$ and $-\infty$.

Definition A.1.7 (Γ-regularization)**.** *Let V be a real l.c.s. (locally convex space) and $F : V \longrightarrow \overline{\mathbb{R}}$ a function, we define G the largest minorant of F in $\Gamma(V)$ and we call it the Γ-regularizationof F.*

Proposition A.1.3. *Let $F : V \longrightarrow \overline{\mathbb{R}}$, and G be its Γ-regularization, then*
(i) $G \leq \overline{F} \leq F$;
(ii) if F is convex and admits a continuous affine minorant, $\overline{F} = G$.

Definition A.1.8 (Legendre Transform)**.** *Let V and V^* be two vector spaces placed in duality, let $F : V \longrightarrow \overline{\mathbb{R}}$ be a function. We define the Legendre transform of F from V^* to $\overline{\mathbb{R}}$ as*

$$F^*(u^*) = \sup_{v \in V} \big[\langle v^*, v \rangle - F(v) \big].$$

and $F^* \in \Gamma(V^*)$.

Proposition A.1.4. *Let F be a function of V into $\mathbb{R}$. Then its bipolar F^{**} is none other its Γ-regularization. In particular, if $F \in \Gamma(V)$, then $F^{**} = F$.*

Definition A.1.9 (subdifferentiability)**.** *A function F of V into $\overline{\mathbb{R}}$ is said to be subdifferentiable at the point $u \in V$ if it has a continuous affine minorant which is exact at u. The slope $u^* \in V^*$ of such a minorant is called a subgradient of F at u, and the set of subgradients at u is called the subdifferential at u and is denoted $\partial F(u)$.*

A much more easier characterization is

$$u^* \in \partial F(u) \iff F(u) \text{ is finite and } F^*(u^*) = \langle u^*, u \rangle - F(u).$$

We have also the following results

Proposition A.1.5. *Let F be a function from V to $\overline{\mathbb{R}}$, then*
(i) if $\partial F(u) \neq \emptyset$, then $F(u) = F^{**}(u)$,
(ii) if $F(u) = F^{**}(u)$, then $\partial F(u) = \partial F^{**}(u)$.

Proposition A.1.6. *Let F be a convex function of V into $\overline{\mathbb{R}}$, finite and continuous at the point $u \in V$. Then $\partial F(u) \neq \emptyset$*

Proposition A.1.7. *Let F be a convex function of V into $\overline{\mathbb{R}}$. If F is Gateaux-differentiable at $u \in V$, it is subdifferentiable at u and $\partial F(u) = \{F'(u)\}$.*

A.2 Functional analysis

The results in this section came from [5] and [16] where, as before, a full coverage of the subject can be found.

Definition A.2.1 (Banach space)**.** *$(V, \|\cdot\|_V)$ is a Banach space if it is a complete normed space where V is a vector space over a field $\mathbb{K}$, for us will be $\mathbb{R}$, and $\|\cdot\|_V : V \longrightarrow \mathbb{R}$ is a norm on V.*

Definition A.2.2 (Hilbert space)**.** *V is a Hilbert space if $(V, \|\cdot\|)$ is a Banach space where the norm $\|\cdot\|$ is the one induced by the scalar product $(\cdot,\cdot)_V : V \times V \longrightarrow \mathbb{R}$.*

Proposition A.2.1. *If V and X are two Banach spaces (Hilbert spaces), then $V \times X$ is still a Banach space (Hilbert space) with the norm $\|(v,x)\|_{V\times X} = \|v\|_V + \|x\|_X$ for all $v \in V$ and for all $x \in X$ (with the scalar product $((v,x),(w,y))_{V\times X} = (v,w)_V + (x,y)_X$ for all $v, w \in V$ and for all $x, y \in X$).*

Here we recall few spaces that we are using later, from now on we consider $\Omega \subseteq \mathbb{R}^n$.

1) $L^2(\Omega)$ is a Hilbert space with

$$(f,g) = \int_\Omega fg \; dx.$$

2) $H^1(\Omega)$ is a Hilbert space with

$$(f,g) = \int_\Omega f \; g \; dx + \int_\Omega Df \cdot Dg \; dx,$$

where Df and Dg denote the weak derivatives of f and g.
3) $H_0^1(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}$,which is the closure of test function $\mathcal{C}_c^\infty(\Omega)$ respect to the norm $\|\cdot\|_{H^1}$ in $H^1(\Omega)$, is a Hilbert space with

$$(f,g) = \int_\Omega Df \cdot Dg \; dx.$$

4) $\underline{H}(div, \Omega)$ is a Hilbert space with

$$(\underline{f}, \underline{g}) = \int_\Omega \underline{f} \cdot \underline{g} \; dx + \int_\Omega div(\underline{f}) \; div(\underline{g}) \; dx.$$

The elements of $\underline{H}(div, \Omega)$ are the elements of $L^2(\Omega)^n$ with $div(\underline{f}) \in L^2(\Omega)$.

The space $\underline{H}(div,\Omega)$ has to be seen as an intermediate space between $L^2(\Omega)^n$ and $H^1(\Omega)^n$ where we are not requiring that all the weak derivative exists and $D^\alpha f_j \in L^2(\Omega)$ for all $\alpha, j \in \{1,\dots,n\}$ but only that $div(\underline{f})$ exists and is in $L^2(\Omega)$. In the exact same way as it is done for Sobolev spaces we require that $div(\underline{f})$, in order to exists, must work in a good way with the elements of $\mathcal{C}_c^\infty(\Omega)$, which means

$$\int_\Omega div(\underline{f})\ \varphi\ dx = -\int_\Omega \underline{f}\cdot\nabla\,\varphi\ dx.$$

So, for example, for a fixed function $\underline{f} \in L^2(\Omega)^n$ if there exists a function $g \in L^2(\Omega)$ such that

$$\int_\Omega g\ \varphi\ dx = -\int_\Omega \underline{f}\cdot\nabla\,\varphi\ dx \qquad \forall\varphi \in \mathcal{C}_c^\infty(\Omega),$$

then we can say that $\underline{f} \in \underline{H}(div,\Omega)$ and $div(\underline{f}) = g$.
We recall that the extension of the div operator to the matrices of function is equal to apply the standard divergence to the lines of the matrix getting a vector of the same size of the number lines of the matrix, so if $\underline{\underline{\sigma}}$ is a matrix with n lines and m columns, then

$$div(\underline{\underline{\sigma}})_i = \sum_{j=1}^m \frac{\partial\sigma_{ij}}{\partial x_j}.$$

we will denote $div(\underline{\underline{\sigma}}) = \{div(\underline{\underline{\sigma}})_1, div(\underline{\underline{\sigma}})_2, \dots div(\underline{\underline{\sigma}})_n\}$.
5) $\underline{\underline{H}}(div,\Omega)$, where $\underline{\underline{\sigma}} \in \underline{\underline{H}}(div,\Omega)$ if $\sigma_{ij} \in L^2(\Omega)$ for all $i,j \in \{1,\dots,n\}$ and $div(\underline{\underline{\sigma}}) \in L^2(\Omega)^n$, is an Hilbert space with

$$(\underline{\underline{\sigma}},\underline{\underline{\tau}}) = \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\tau}}\ dx + \int_\Omega div(\underline{\underline{\sigma}})\cdot div(\underline{\underline{\tau}})\ dx.$$

As done for the space $\underline{H}(div,\Omega)$, we require that for $div(\underline{\underline{\sigma}})$, for $\underline{\underline{\sigma}} \in \underline{\underline{H}}(div,\Omega)$, in order to exist it must satisfy

$$\int_\Omega div(\underline{\underline{\sigma}})\cdot\underline{\varphi}\ dx = -\int_\Omega \underline{\underline{\sigma}} : \nabla\,\underline{\varphi}\ dx \qquad \forall\underline{\varphi} \in \mathcal{C}_c^\infty(\Omega)^n.$$

where with $\nabla\,\underline{\varphi}$ we denote the Jacobian.
6)$\underline{\underline{H}}(div,\Omega)_{sym}$, which is the subspace of $\underline{\underline{H}}(div,\Omega)$ where $\sigma_{ij} = \sigma_{ji}$ for all $i,j \in \{1,\dots,n\}$, it is also an Hilbert space with the same scalar product as above.
We have that for a given function $\underline{u} \in H_0^1(\Omega)^n$ we can define the strain tensor $\underline{\underline{\epsilon}}(\underline{u})$ through the linear map $\underline{\underline{\epsilon}} : H_0^1(\Omega)^n \longrightarrow L^2(\Omega)^{n\times n}_{sym}$ where

$$\epsilon(\underline{u})_{ij} \doteq \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

The relation between the strain tensor $\underline{\underline{\epsilon}}$ and $\underline{\underline{H}}(div,\Omega)_{sym}$ is that the condition for the existence of $div(\underline{\underline{\sigma}})$ becomes

$$\int_\Omega div(\underline{\underline{\sigma}})\cdot\underline{\varphi}\ dx = -\int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(\underline{\varphi})\ dx \qquad \forall\underline{\varphi} \in \mathcal{C}_c^\infty(\Omega)^n.$$

So, as we have done before, for a fixed tensor $\underline{\underline{\sigma}} \in L^2(\Omega)^{2\times 2}_{sym}$ if there exists a function $\underline{g} \in L^(\Omega)^n$ such that

$$\int_\Omega \underline{g} \cdot \underline{\varphi} \; dx = -\int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(\underline{\varphi}) \; dx \qquad \forall \underline{\varphi} \in \mathcal{C}_c^\infty(\Omega)^n. \tag{A.3}$$

then we can say that $\underline{\underline{\sigma}} \in \underline{\underline{H}}(div, \Omega)_{sym}$ and $div(\underline{\underline{\sigma}}) = \underline{g}$.
7)$\underline{H}(sym, \Omega)$ where $\underline{u}$ is in $\underline{H}(sym, \Omega)$ if $\underline{u} \in L^2(\Omega)^n$, $\underline{\underline{\epsilon}}(\underline{u}) \in L^2(\Omega)^{n\times n}_{sym}$ and $\underline{\underline{\epsilon}}(\underline{u})$ in order to exists must operate in a good way with the elements of $\mathcal{C}_c^\infty(\Omega)^{n\times n}_{sym}$:

$$\int_\Omega \underline{\underline{\epsilon}}(\underline{u}) : \underline{\underline{\varphi}} \; dx = -\int_\Omega \underline{u} \cdot div(\underline{\underline{\varphi}}) \; dx \qquad \forall \underline{\underline{\varphi}} \in \mathcal{C}_c^\infty(\Omega)^{n\times n}_{sym}.$$

The norm of this space is the following one:

$$\|\underline{v}\|^2_{\underline{H}(sym,\Omega)} = \|\underline{v}\|^2_{L^2(\Omega)^n} + \left\|\underline{\underline{\epsilon}}(\underline{v})\right\|^2_{L^2(\Omega)^{n\times n}}.$$

This space in not only a Banach space, but it is also an Hilbert space with scalar product:

$$(\underline{u}, \underline{v}) = \int_\Omega \underline{u} \cdot \underline{v} \; dx + \int_\Omega \underline{\underline{\epsilon}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{v}) \; dx.$$

Now we recall few inequality that are needed later. The first one is the classical Poincaré inequality, even if in [5] (Corollary 9.19 pag.290) the statement is for a general p we fix $p = 2$ for simplicity. The second one is the Korn inequality that can be found in [12] (Theorem 2.2 at pag.14) that will allow us to estimate function from above e show strong convexity.

Proposition A.2.2 (Poincaré inequality)**.** *Let Ω be a bounded open set. Then there exists a constant $\mathcal{C}$, depending on Ω, such that*

$$\|u\|_{L^2(\Omega)} \leq \mathcal{C}\|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H^1_0(\Omega)$$

In particular, the expression $\|\nabla u\|_{L^2(\Omega)}$ is a norm on $H^1_0(\Omega)$ and it is equivalent to the norm $H^1(\Omega)$; on $H^1_0(\Omega)$ the expression $\sum_{i,j=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \, dx$ is a scalar product that induces the norm $\|\nabla u\|_{L^2(\Omega)}$ and it is equivalent to the norm $\|u\|_{H^1}$.

Proposition A.2.3 (Korn inequality 1)**.** *Let Ω a bounded domain in $\mathbb{R}$. Then every vector valued function $\underline{v} \in H^1_0(\Omega)^n$ satisfies the inequality*

$$\sum_{i,j=1}^n \int_\Omega |\epsilon_{ij}(\underline{v})|^2 \geq \mathcal{K}\|\underline{v}\|^2_{H^1_0(\Omega)^n}. \tag{A.4}$$

where $\mathcal{K} > 0$ is the constant of Korn and it depens only on Ω.

Proposition A.2.4 (Korn inequality 2)**.** *Let Ω a bounded domain in $\mathbb{R}$. Then every vector valued function $\underline{v} \in H^1(\Omega)^n$ satisfies the inequality*

$$\|\underline{v}\|_{H^1(\Omega)^n} \leq \mathcal{K}\big(\|\underline{v}\|_{L^2(\Omega)^n} + \left\|\underline{\underline{\epsilon}}(\underline{v})\right\|_{L^2(\Omega)^{n\times n}}\big). \tag{A.5}$$

where $\mathcal{K} > 0$ is the constant of Korn and it depens only on Ω.

Now we want to see the dual space of the previous spaces and recall the most important result that are used later. A full coverage of this subject can be found in [5]

Definition A.2.3 (Dual space)**.** *Let V be a vector space over $\mathbb{R}$, then we denote with V^* its dual space which is the space of all continuous linear functional on V. The norm on V^* is*

$$\|v^*\|_{V^*} = \sup_{v^* \in V^*} \frac{|\langle v^*, v\rangle|}{\|v\|_V}$$

If V is an Hilbert space, then we have the following theorem (Theorem 5.5 pag.135) that relate an element of V^* with an unique element of V.

Theorem A.1 (Riesz–Fréchet representation theorem)**.** *Given any $v^* \in V^*$ then there exists an unique $v \in V$ such that*

$$\langle v^*, w\rangle = (v, w) \quad \forall w \in V.$$

Moreover,

$$\|v^*\|_{V^*} = \|v\|_V.$$

So we can define a map $\mathcal{R} : V^* \longrightarrow V$ such that $\mathcal{R}(v^*) = v$, this map is a bijection and it is an isometry.

So all the dual space of the previous spaces are isomorphic to themself. Sometimes we will denote the dual space of $H^1_0(\Omega)$ with $H^{-1}(\Omega)$. Now we recall a useful result that we need if we will work with product of Hilbert spaces.

Proposition A.2.5. *Let us consider $H \times V$ where H and V are two Hilbert spaces. Then $(H \times V)^* \cong H^* \times V^*$*

So, for example: $(H^1_0(\Omega)^2)^* = (H^1_0(\Omega) \times H^1_0(\Omega)^* \cong H^{-1}(\Omega) \times H^{-1}(\Omega)$.

Now we recall the fundamental lemma of calculus of variation:

Lemma A.2.1 (Fundamental lemma of calculus of variations)**.** *Let $\Omega \subset \mathbb{R}^n$ and $f \in L^1(\Omega)$, such that*

$$\int_\Omega f\varphi = 0 \quad \forall \varphi \in \mathcal{C}^\infty_c(\Omega).$$

Then $f = 0$ a.e. in Ω

We recall that if Ω is bounded then $L^p(\Omega) \subset L^q(\Omega)$ if $p > q \geq 1$, so $L^2(\Omega) \subset L^1(\Omega)$.

We present a version of Trace theorem that came from [11] (Theorem 18.1 pag. 592) and one important result (Theorem 18.7 pag.595):

Theorem A.2 (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open set whose boundary $\partial\Omega$ is Lipschitz continuous, let $1 \leq p \leq \infty$. There exists a unique linear operator*

$$Tr : W^{1,p}(\Omega) \to L^p_{loc}(\partial\Omega)$$

such that

(i) $\mathrm{Tr}(u) = u$ *on $\partial\Omega$ for all $u \in W^{1,p}(\Omega) \cap \mathbf{C}(\overline{\Omega})$*

(ii) for all $u \in W^{1,p}(\Omega)$, all $\psi \in \mathbf{C}^1_c(\mathbb{R}^n)$, and all i = 1,...,N, the following formula holds

$$\int_\Omega u \frac{\partial \psi}{\partial x_i}\ dx = -\int_\Omega \psi \frac{\partial u}{\partial x_i}\ dx + \int_{\partial\Omega} \mathrm{Tr}(u)\psi\nu_i\ d\mathcal{H}^{n-1}$$

where ν_i is the normal component.

If we have a function $\underline{v} \in H^1(\Omega)^n$, with $\mathrm{Tr}(\underline{v})$ we indicate the vector where $\mathrm{Tr}(\underline{v})_i = \mathrm{Tr}(v_i)$.

Theorem A.3 (Trace and $W^{1,p}_0(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ whose boundary $\partial\Omega$ is Lipschitz continuous, let $1 \leq p \leq \infty$ and let $u \in W^{1,p}(\Omega)$. Then $\mathrm{Tr}(u) = 0$ if and only if $u \in W^{1,p}_0(\Omega)$.*

Now we introduce a problem, that we will analyse in chapter 3, and then state a theorem that guarantee us existence and uniqueness of solution for it. The following result came form [15], where a full study of the argument can be found.
Let V and W two Hilbert spaces such that V is a dens subspace of W for which the injection is continuous. Let $\mathcal{A}$,$\mathcal{C}$ and $\mathcal{B}$ are a linear operators respectively form V to V^*, W to W^* and $D(B)$, subspace of V, to V^*. Let $u_0 \in V$, $u_1 \in W$ and $f \in \mathcal{C}([0,\infty), W')$ then our aim si to find $\bar{u} \in \mathcal{C}^1((0,T],V) \cap \mathcal{C}^0([0,T],V) \cap \mathcal{C}^2(0,T],W) \cap \mathcal{C}^1([0,T],W)$ such that $u(0) = u_0$ and $u_t(0) = u_1$ and

$$\mathcal{C}u_{tt}(t) + \mathcal{B}u_t(t) + \mathcal{A}u(t) = f(t).$$

for all $t > 0$, where $D(B)$ is

$$D(B) = \big\{x \in V \| \lim_{h\to 0^+} \frac{(S(h) - id)x}{h} = D^+(S(0)x) \text{ exists in V } \big\}.$$

Before recalling Theorem 2.2 at pag. 148 we need the following definition:

Definition A.2.4 (Hölder continuous)**.** *Let I an interval in $\mathbb{R}$, H a Banach space and $0 < \alpha \leq 1$, then $f : I \to H$ is an Hölder continuous function if*

$$\|f(x) - f(y)\|_H \leq |x - y|^\alpha \quad \forall x, y, \in I.$$

Problem A.2.1. *Let V and W be Hilbert spaces with V a dense subspace of W for which the injection is continuous. Thus, we identify $W^* \leq V^*$ by duality. Let $\mathcal{A}$ be a continuous linear operator from V to V^* and $\mathcal{C}$ be a continuous linear operator from W to W^* be given. Suppose $D(\mathcal{B}) \leq V$ and $\mathcal{B} : D(\mathcal{B}) \to V^*$ is linear. If $u_0 \in V$, $u_1 \in W$ and $f \in \mathcal{C}((0,\infty); W^*)$ are given, we consider the problem of finding $u \in \mathcal{C}^1((0,T]; H^1_0(\Omega)) \cap \mathcal{C}^0([0,T]; H^1_0(\Omega)) \cap \mathcal{C}^2((0,T]; L^2(\Omega)) \cap \mathcal{C}^1([0,T]; H^1_0(\Omega))$ such that $u(0) = u_0$ and $u_t(0) = u_1$ and*

$$\mathcal{C}u_{tt}(t) + \mathcal{B}u_t(t) + \mathcal{A}u(t) = f(t),$$

for all $t > 0$.

From [9] we have

Proposition A.2.6 (Gronwall's inequality)**.** *Let $\nu(t) : [0,T] \to \mathbb{R}$ be a non negative, absolute continuous function on $[0,T]$, such that*

$$\nu_t(t) \leq \Phi(t)\nu(t) + \psi(t),$$

where $\Phi(t)$ *and* $\psi(t)$ *are non negative, locally integrable function on* $[0,T]$. *Then*

$$\nu(t) \leq e^{\int_0^T \Phi(s)\ \mathrm{ds}}\big[\nu(0) + \int_0^t \psi(s)\ \mathrm{ds}\big] \tag{A.6}$$

From [5] we have

Theorem A.4 (Banach–Alaoglu–Bourbaki)**.** *Let* V *be a Banach space,* V^* *its dual and let*

$$B_{V^*} = \{v^* \in V^* | \|v^*\|_{V^*} \leq 1\}$$

is compact in the weak topology $\sigma(V^*, V)$.

Nomenclature

Constants

$\mathcal{K}$	The Korn constant
μ, λ	The Lamé coefficients
ρ	The mass density

Objects

a	Scalar
$\underline{v}$	Vector
$\underline{v} \cdot \underline{w}$	The dot product
$\underline{vw}$	Alternative notation for the dot product. We use this notation after the first chapter in order to have shorter notation
$\underline{\underline{\tau}}$	Tensor
$\mathrm{tr}(\underline{\underline{\sigma}})$	The trace of tensor $\underline{\underline{\sigma}}$: $\mathrm{tr}(\underline{\underline{\tau}}) = \sum_i \tau_{ii}$
$\underline{\underline{id}}$	The identical tensor with 1 on the diagonal and 0 outside it
$\underline{\underline{\sigma}}^D$	The deviatoric component of the tensor $\underline{\underline{\sigma}}$, i.e. $\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - \frac{1}{n}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{id}}$ where n is the dimension of the space
$\underline{\underline{\tau}} : \underline{\underline{\sigma}}$	The duble dot product between two tensor $\underline{\underline{\tau}} : \underline{\underline{\sigma}} = \sum_{i,j} \tau_{ij}\sigma_{ij}$

Spaces

Ω	The domain
$\partial\Omega$	The boundary of Ω
$\overline{V}$	The closure of V
V^*	The dual space of the vector space V
$\mathbb{N}$	The natural number
$\mathbb{R}$	The real number

$\mathcal{C}(\Omega)$ The set of continuous functions in Ω

$\mathcal{C}_c^\infty(\Omega)$ The set of test functions in Ω

Functional operators

$(u, v)_V$ The scalar product between u and v in V

$\langle v^*, v\rangle_V$ The dual pairing between v^* and v, where $v^* \in V^*$ and $v \in V$

v_{tt} The second time derivative of the function u

v_t The first time derivative of the function u

$\mathrm{div}(\underline{v})$ The divergence of $\underline{v}$

$\nabla \underline{v}$ The Gradient of $\underline{v}$

Λ^* The transpose oprator of the linear operator Λ

$\|v\|_V$ The norm of v in the space V

$\partial_x f$ The partial derivative of f respects to x

$\mathcal{R}$ The Riesz oprator

$\underline{\underline{\epsilon}}(u)$ The symmetric tensor of u, usually we denote with $\underline{\underline{\epsilon}}(u)$ the symmetric tensor of the vector $\underline{u}$, in order to emphasize that the object we obtain is a tensor and to mantain a cleaner notation we write $\underline{\underline{\epsilon}}(u)$ instead of $\underline{\underline{\epsilon}}(\underline{u})$

$\mathrm{Tr}(\underline{f})$ The trace of the function $\underline{f}$

$\mathrm{tr}(\underline{\underline{\sigma}})$ The trace of σ: $\mathrm{tr}(\underline{\underline{\tau}}) = \sum_i \tau_{ii}$

$v_n \rightharpoonup v$ The weakly convergence of $\{v_n\}$ to v

Convex analysis tools

$\mathcal{P}$ The primal probelm

$\mathcal{P}^*$ The dual probelm of primal problem $\mathcal{P}$

Φ^* The Legendre transform of Φ

$\Gamma_0(V)$ The set of lower and semi continuous function from V to $\overline{\mathbb{R}}$ that can not be constantly equal to $\pm\infty$

$\partial\Phi(x)$ The subdifferential of Φ in x

L The Lagrangian

Bibliography

[1] Douglas Arnold, Richard Falk, and Ragnar Winther. "Mixed finite element methods for linear elasticity with weakly imposed symmetry". In: *Mathematics of Computation* 76.260 (2007), pages 1699–1723 (cited on page 2).

[2] Eliane Bécache, Abdelaâziz Ezziani, and Patrick Joly. "A mixed finite element approach for viscoelastic wave propagation". In: *Computational Geosciences* 8.3 (2005), pages 255–299 (cited on page 2).

[3] Michele Benzi and Valeria Simoncini. "On the eigenvalues of a class of saddle point matrices". In: *Numerische Mathematik* 103.2 (2006), pages 173–196 (cited on page 86).

[4] Daniele Boffi, Franco Brezzi, Michel Fortin, et al. *Mixed finite element methods and applications*. Volume 44. Springer, 2013 (cited on pages 2, 3, 25).

[5] Haïm Brézis. *Functional analysis*. Masson, 1983 (cited on pages 97, 99, 100, 102).

[6] Richard Christensen. *Theory of viscoelasticity: an introduction*. Elsevier, 2012 (cited on page 1).

[7] Georges Duvant and Jacques Louis Lions. *Inequalities in mechanics and physics*. Volume 219. Springer Science & Business Media, 2012 (cited on page 39).

[8] Ivar Ekeland and Roger Temam. *Convex analysis and variational problems*. SIAM, 1999 (cited on pages 3, 8, 12, 20, 95).

[9] Lawrence C Evans. *Partial differential equations*. Volume 19. American Mathematical Soc., 2010 (cited on pages 32, 39, 40, 101).

[10] ROBERT HOOKE. "Lectures de Potentia Restitutiva, or of Spring Explaining the Power of Springing Bodies. London 1678. Facsimile reproduction in RT Gun-ther: Early". In: *Science in Oxford* 8 (1931), pages 331–356 (cited on page 1).

[11] Giovanni Leoni. *A first course in Sobolev spaces*. American Mathematical Soc., 2017 (cited on page 100).

[12] Olga Arsenievna Oleĭnik, AS Shamaev, and GA Yosifian. *Mathematical problems in elasticity and homogenization*. Elsevier, 2009 (cited on page 99).

[13] Marie E Rognes. "Mixed finite element methods with applications to viscoelasticity and gels". In: (2009) (cited on page 2).

[14] Marie Rognes and Ragnar Winther. "Mixed finite element methods for linear viscoelasticity using weak symmetry". In: *Mathematical Models Methods in Applied Sciences - M3AS* 20 (June 2010). DOI: 10.1142/S0218202510004490 (cited on page 2).

[15] Ralph E Showalter. *Hilbert space methods in partial differential equations*. Courier Corporation, 2010 (cited on pages 3, 51, 53, 64, 66, 101).

[16] Luc Tartar. *An introduction to Sobolev spaces and interpolation spaces*. Volume 3. Springer Science & Business Media, 2007 (cited on pages 26, 97).

[17] Isaac Todhunter. *A History of the Theory of Elasticity and of the Strength of Materials: From Galilei to the Present Time*. Edited by KarlEditor Pearson. Volume 2. Cambridge Library Collection - Mathematics. Cambridge University Press, 2014. DOI: 10.1017/CBO9781107280083 (cited on page 1).