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The role of supersymmetry in near-extremal black hole thermodynamics

Thesis supervisor | **Candidate**

Dr. Davide Cassani | **Damiano Tietto**

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Abstract

In this work we study the effective Schwarzian theories arising in the description of near-extremal and near-BPS black holes in 4d, $\mathcal{N} = 2$ ungauged supergravity. The supergravity models we consider are such that the supersymmetry of the extremal solutions can be either preserved or broken by flipping the sign of some of the charges. We obtain the Schwarzian in two ways: by studying the (super)isometries of the extremal solutions and by performing the dimensional reduction to 2d of both the bosonic and fermionic sectors in the 4d supergravity. We then quantize exactly these generalized Schwarzian theories and extract the spectra. When supersymmetry is preserved, the spectrum has a degenerate ground state and a mass gap, in accordance with the semiclassical analysis. When supersymmetry is broken, the spectrum is continuous and has no extremal degeneracy, due to strong quantum effects modifying the Bekenstein–Hawking area law. The 2d gravitinos set apart the dynamics in the two cases, becoming respectively massless and massive.

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1 Introduction

The understanding of black holes is one of the main tools to gain deeper insights on the quantum behavior of gravity. In particular, one of the simplest setup to study is the one offered by extremal black holes: in any dimension, their near-horizon geometry is usually characterized by a more symmetric AdS_2 throat, in contrast to the 2d Rindler (Rind_2) throat of typical near-extremal black holes. By using the Euclidean gravitational path integral, one can analyze their thermodynamics, discovering a peculiar behavior: while extremal black holes have a temperature $T = 0$, their extremal entropy $\mathcal{S}_* = \mathrm{Area}/4G_N$ — as given by the Bekenstein–Hawking area law [1, 2] — remains finite, showing a huge degeneracy in the ground state of the underlying microscopic theory describing the black holes.

A paradoxical behavior also arises when analyzing the thermodynamics of near-extremal black holes. Their energy $\mathcal{E}$ above the extremal energy $\mathcal{E}_*$ in general scales like $\mathcal{E} - \mathcal{E}_* \sim T^2/\Delta\mathcal{E}_{\text{gap}}$. For $T \lesssim \Delta\mathcal{E}_{\text{gap}}$, the average energy of a Hawking quanta is greater than the mass gap above extremality and thus the near-extremal black hole cannot radiate, despite having a temperature $T \neq 0$; this signals a breakdown of the semiclassical description of the black holes [3, 4]. One way out of the problem is to assume that there actually is a gap in the spectrum of microstates, with no states between the extremal energy $\mathcal{E}_*$ and $\mathcal{E}_* + \Delta\mathcal{E}_{\text{gap}}$. This is supported by some string constructions of the microstates in supergravity [5, 6]. These constructions, however, require some amount of supersymmetry to work; hence the presence of the gap protecting the degeneracy of states at $T = 0$ might not be a general property of near-extremal black holes, but rather it might just be peculiar to supersymmetry.

In recent years, there has been accumulating evidence [7, 8, 9, 10] showing that the paradox of the mass gap is solved once one includes gravitational quantum corrections. These quantum corrections are encoded in generalized Schwarzian theories, which are particular kinds of effective 1d boundary theories describing the near-extremal dynamics. These theories feature a Schwarzian mode [11, 12] — related to the $SL(2,\mathbb{R})$ symmetry of the near-horizon AdS_2 — together with some additional gauge degrees of freedom (and possibly fermions). These theories are usually 1-loop exact thanks to fermionic localization [13]; therefore, one just needs to calculate 1-loop determinants in order to obtain the fully quantum corrected partition function. In general, it turns out that when considering near-extremal black holes in non-supersymmetric theories [7] the 1-loop determinants add a temperature dependent logarithmic contribution to the entropy, $\mathcal{S} \sim \log T + \dots$; when taking $T \to 0$, the entropy $\mathcal{S} \to -\infty$, signaling that there is just a single extremal microstate and no extremal degeneracy. The energy is also modified and behaves as $\mathcal{E} - \mathcal{E}_* \gtrsim T$ for $T \to 0$, and hence the black holes can always radiate and the thermodynamic description never breaks down. Vice-versa, when considering near-BPS black holes (i.e. near-extremal black holes close to an extremal solution that is also supersymmetric) in supersymmetric theories [8, 9], we still get an extremal degeneracy even after quantum corrections. However, analyzing the spectrum (or energy density of the states) of the theory shows exactly a gap of order $\Delta\mathcal{E}_{\text{gap}}$ with no states, thus solving the problem of the mass gap as expected from microscopic string constructions. Therefore, one expects the presence of a gap in the spectrum only when expanding around supersymmetric black holes, while around non-supersymmetric ones quantum corrections remove the extremal degeneracy that is found classically, leaving a single extremal microstate.

The goal of this work is to assess this conjectured behavior by analyzing the case of a supersymmetric theory in which we can have both supersymmetric and non-supersymmetric extremal black holes. One motivation for considering this setup is that sometimes non-supersymmetric states within a supersymmetric theory enjoy particularly nice properties, and we would like to figure out if this can happen for near-extremal black holes. In particular, we would like to understand if the fact that the Lagrangian is supersymmetric may lead to cancellations in the quantum corrections to the classical entropy, thus protecting the degeneracy of states at extremality, similarly to what happens when the solution is supersymmetric. We will focus on slowly rotating black holes in $\mathcal{N} = 2$ 4d ungauged supergravity with no hypermultiplets [14], working at fixed electric and magnetic charges. $\mathcal{N} = 2$ ungauged supergravity contains (in flat space language) one graviton multiplet (with one graviton, two gravitinos and one $U(1)$ gauge field) and n_V vector multiplets (each with one $U(1)$ gauge field, two gauginos and one scalar). The scalar manifold is a special Kähler manifold, with a holomorphic line bundle (as in usual $\mathcal{N} = 1$ supergravity) and a holomorphic symplectic bundle, which implements the electromagnetic duality among the $n_V + 1$ $U(1)$ gauge fields. One of the main features of black holes in $\mathcal{N} = 2$ ungauged supergravity — working at fixed electric and magnetic charges at infinity — is the $\mathcal{N} = 2$ attractor mechanism [15, 16, 17]. Let us consider extremal solutions with the values of the scalars fixed at infinity, $z^i = z^i_\infty$; as we move towards the horizon, the scalars will flow towards some value $z^i = z^i_0$. In extremal black holes, the horizon sits infinitely far away from spatial infinity; hence the scalars "forget" their value at infinity, and solutions with different z^i_∞ flow towards the same z^i_0, organized in basins of attractions. The fixed points of the flow are exactly the minima of some real functions W, denoted "superpotentials" as they determine the scalar potential[1]. Notice that different z^i_0 correspond to fixed points of different superpotentials W, and a given theory may admit different superpotentials.

In particular, in some models we can have, at the same time, some fixed points which lead to BPS black holes and some fixed points which instead belong to non-supersymmetric black holes [18, 19]; in the latter case, their W are often referred to as "fake-superpotentials". The scalars and the metric satisfy in both cases some first order flow equations. When dealing with supersymmetric attractors, this is the typical behavior of a BPS solution; while dealing with a non-supersymmetric attractor, instead, this is quite atypical and thus these solutions are referred to as "fake-BPS". We will review the attractor mechanism in section 2, mainly following [19]. Notice that these fake-BPS black holes can be quite similar to actual BPS black holes; in fact

[1] Despite the name, notice that these "superpotentials" are not related to the usual superpotentials appearing in supersymmetric theories.

we will review an example of a fake-BPS black hole which is obtained from an actual BPS black hole by just flipping the sign of one of the electric charges.

We now have to find the correct 1d effective Schwarzian theory which correctly captures the near-extremal dynamics. There are two main ways to do this: one can analyze the symmetry breaking pattern in the near-horizon, or alternatively one can perform a Kaluza–Klein dimensional reduction to 2d starting from the 4d supergravity, and later identify the 1d boundary theory. Let us begin with the former. If we take the near-horizon limit of the metric of a near-extremal black hole, we get $\mathrm{Rind}_2 \times S^2$; if we do the same for an extremal black hole, we get $\mathrm{AdS}_2 \times S^2$ instead. In particular, the latter — once Wick rotating to the Euclidean signature — is symmetric under $SL(2,\mathbb{R})$ isometries, which are not present in the former. The Schwarzian mode can then be interpreted as the "Goldstone boson" associated with this symmetry breaking. The procedure for obtaining the 1d effective theory can be summarized as follows:

1. identify the (super)isometries of the near-horizon extremal geometry;
2. choose the correct generalized Schwarzian theory which describes the symmetry breaking;
3. match the classical result of the Schwarzian with the one from the Euclidean path integral to fix the energy scale of the theory (and other unknown constants).

This approach, particularly the last point, is the typical top-down approach to construct an effective theory.

We will follow this approach in section 3, fixing the electric and magnetic charges while allowing the black holes to rotate, generalizing the results from [8]. The isometry group of the near-horizon geometry of both supersymmetric and non-supersymmetric attractors is $SL(2,\mathbb{R}) \times SU(2)$, where the two factors come respectively from the AdS_2 and S^2 components of the geometry (after Wick rotation). The $SL(2,\mathbb{R})$ is fully broken by turning on a small temperature T, which sends $\mathrm{AdS}_2 \to \mathrm{Rind}_2$; the $SU(2)$ is broken to $U(1)$ by turning on a small angular velocity Ω, which turns on — in the dimensionally reduced 2d theory — a background $SU(2)$ gauge field. Up to now, the analysis cannot distinguish between supersymmetric and non-supersymmetric attractors. When considering the latter, however, we have yet to consider supersymmetry, i.e. the presence of superisometries. We will show that the supersymmetric attractors possess an enlarged superisometry group $PSU(1,1|2)$, whose bosonic subgroup is exactly $SL(2,\mathbb{R}) \times SU(2) \subset PSU(1,1|2)$. This highlights how one cannot just analyze the bosonic part of the symmetries, but needs a fully fermionic analysis to distinguish the near-extremal dynamics around supersymmetric and non-supersymmetric attractors. The effective theory describing the $SL(2,\mathbb{R}) \times SU(2) \to \varnothing \times U(1)$ symmetry breaking is the one of a Schwarzian mode (associated to $SL(2,\mathbb{R})$) together with a particle moving on an $SU(2)$ group manifold (associated to $SU(2)$) [7]; the effective theory describing the $PSU(1,1|2) \to \varnothing \times U(1)$ symmetry breaking is instead the $\mathcal{N}=4$ super-Schwarzian, a supersymmetric generalization of the usual Schwarzian, whose bosonic part of the action is exactly the Schwarzian mode together with a particle moving on $SU(2)$ [8].

Although this approach yields sensible results, it does not explain in a clear way what really changes, within a supersymmetric theory, in going from a supersymmetric to a non-supersymmetric attractor (whose difference, as explained above, can be as small as a flip in the sign of an electric charge). For this reason, most of this work is dedicated to following the dimensional reduction approach. We first start from the original 4d supergravity theory and perform a Kaluza–Klein dimensional reduction to 2d. Since we are only interested in the lightest degrees of freedom above extremality, we neglect the infinite towers of massive fields that are generated, and keep only the massless ones (which is enough to guarantee a consistent truncation of the 4d theory). We perform the bosonic part of the dimensional reduction in section 4, following [7, 8]; we obtain a 2d dilaton gravity [20], coupled to some 2d scalars and an $SU(2)$ gauge fields, which arises when gauging the $SU(2)$ isometry of the angular part of the 4d metric.

Given once again the insight from [7, 8], we then separate the contribution of the spacetime into two regions: the near-horizon region (NHR) and the far-away region (FAR). The idea is that in the FAR region the curvature is negligible and we can thus employ the semiclassical approximation; this way the FAR region only contributes to the extremal energy and entropy of the black holes, without influencing the near-extremal dynamics. In the NHR region, instead, the curvature is not negligible and we need to take into account the quantum effects due to gravity. Imposing the near-extremality conditions, however, reduces the dilaton gravity to a far simpler (generalized) Jackiw–Teitelboim (JT) gravity [21, 22], which can be quantized in various ways [11, 23, 24, 8]. It is from JT gravity that we can eventually recover the generalized Schwarzian, which arises from its boundary terms.

This highlights however an important problem: in JT gravity, the whole near-extremal dynamics is determined by the boundary terms, and therefore choosing the correct boundary conditions for the NHR region is essential to get the correct result. These boundary conditions are determined when gluing together the NHR and FAR region in such a way to obtain a unique spacetime that is well defined. How to exactly do this procedure, however, is not entirely clear. We will show that the proposal of [7, 8, 10] of using the FAR region to propagate the boundary conditions from infinity using the equation of motions does not seem to yield a consistent result. We will therefore follow a different approach. We first show, extending the argument given in [24], that (the bosonic part of) the generalized JT gravity we obtain is equivalent — in the first order formulation — to an $SL(2,\mathbb{R}) \times SU(2)$ BF theory, which is a particular type of 2d topological gauge theory [25]. Then the Dirichlet boundary conditions for the metric — needed to recover the Schwarzian mode — become mixed boundary conditions, fixing a combination of the $SL(2,\mathbb{R})$ gauge field and the related Lagrange multiplier [24]; these boundary conditions are also equivalent to introducing a defect in the 2d bulk. In the supersymmetric case, the $SL(2,\mathbb{R})$ and $SU(2)$ gauge fields are not independent and become part of a unique $PSL(1,1|2)$ gauge field; to preserve supersymmetry, we must impose the same boundary conditions on (or introduce the same defect for) both the $SL(2,\mathbb{R})$ and $SU(2)$ components of the gauge field. In the non-supersymmetric case, since the bosonic part of the theory is the same as in the supersymmetric case, we also expect the same boundary conditions to apply; this is because, in principle, they should be obtained only from the gluing procedure between the NHR and FAR region. This way we obtain the boundary terms that exactly contain the Schwarzian boundary theory together with a particle moving on the group manifold $SU(2)$, as expected from the previous analysis of the bosonic symmetries.

As before, however, a bosonic analysis alone is not enough to distinguish between a supersymmetric and a non-supersymmetric attractor; while we do get the Schwarzian and the particle moving on $SU(2)$, there is no way to know — purely from the bosonic dimensional reduction — whether we should include the fermions appearing in the $\mathcal{N}=4$ super-Schwarzian or not. Therefore, in section 5, we will perform the dimensional reduction of the gravitinos and the gauginos at the quadratic level; we choose the dimensional reduction ansatz slightly modifying [26, 27]. The 2d gauginos — arising from the s-wave reduction of the 4d gauginos — have masses of the order of the Kaluza–Klein scales, and thus we will neglect their contribution, as we did for the other Kaluza–Klein modes. The gravitinos, instead, assume a central role. In the dimensional reduction, they get a mass term both from their 4d kinetic term and the interactions with the graviphoton field strength: the latter, in particular, can distinguish between supersymmetric and non-supersymmetric attractors. Around supersymmetric attractors, the two contributions cancel each other out and the gravitinos remain massless, becoming part of the $\mathcal{N}=4$ JT supergravity; $\mathcal{N}=4$ JT supergravity is equivalent to a $PSU(1,1|2)$ BF theory, which in turn is related to the $\mathcal{N}=4$ super-Schwarzian [8] (provided that we pick the correct boundary conditions as before). Around non-supersymmetric attractors, instead, the gravitinos gain a large mass, and therefore they will not be excited in the near-extremal limit. Hence we are just left with the bosonic part of the $\mathcal{N}=4$ JT supergravity, which in turn leads to a $SL(2,\mathbb{R})\times SU(2)$ BF theory and the boundary action of a Schwarzian together with a particle moving on the group manifold $SU(2)$. Hence we get the same result given by the symmetry considerations of section 3.

Finally, we analyze the thermodynamics of the two generalized Schwarzian theories in section 6, following [8, 28]. First, we show that in the path integrals defining the grand-canonical partition functions we are integrating over symplectic manifolds with a $U(1)$ symmetry; we can thus apply fermionic localization [13], granting that the partition functions are 1-loop exact. We then calculate the 1-loop determinants [8] and obtain the quantum corrected partition functions. At last, we extract the spectrum of the near-extremal black holes in the two cases. Around non-supersymmetric attractors, the gravitational quantum corrections remove the extremal degeneracy, and we are just left with a continuous spectrum for all the possible values of the total angular momentum $\mathcal{J}$ of the black holes. As for supersymmetric attractors, we get a purely continuous spectrum only for $\mathcal{J}=1/2,1,\ldots$; for $\mathcal{J}=0$, instead, we get a delta function at the extremal energy — signaling the presence of a degeneracy of the supersymmetric ground state — and a continuum of states starting after a mass gap $\Delta\mathcal{E}_{\rm gap}$ from extremality [28]. This confirms that the gravitational quantum corrections actually solve the mass gap problem and the semiclassical thermodynamic description never breaks down; in particular the problem is solved either by removing the extremal degeneracy and by raising the energy so that $\mathcal{E} - \mathcal{E}_* \gtrsim T$ (in the non-supersymmetric case) or by adding a mass gap (in the supersymmetric case). More importantly, this also shows that the presence of a mass gap — argued from string constructions of black holes [5, 6] — is an artifact of supersymmetry. This suggests that one has to be careful when trying to extend to non-supersymmetric black holes conclusions drawn using supersymmetry, with the dynamics differing already qualitatively. The semiclassical thermodynamics — obtained from the Euclidean path integral — is thus a good approximation of the near-extremal dynamics only when supersymmetry is preserved; when supersymmetry is broken, instead, the gravitational quantum corrections strongly modify the thermodynamics, violating the Bekenstein–Hawking area law.

To sum up and to compare with the symmetry approach, we also write here in a list the procedure we use to obtain the 1d effective theory following the dimensional reduction approach:

1. perform the Kaluza–Klein dimensional reduction from 4d to 2d, keeping only massless fields;
2. separate the spacetime into FAR and NHR regions, in order to obtain a generalized JT gravity in the NHR;
3. relate JT gravity to the appropriate $SL(2,\mathbb{R}) \times \dots$ BF theory;
4. pick the correct boundary conditions on the gauge field, eventually using supersymmetry to relate the gravitational $SL(2,\mathbb{R})$ components to the other (super-)groups;
5. calculate the boundary terms needed to make the variational problem well defined.

2 The black hole attractor mechanism

2.1 The $\mathcal{N} = 2$ supergravity action

Before finding the effective Schwarzian theories describing the near-extremal black hole dynamics, let us start by briefly reviewing first the $\mathcal{N} = 2$ ungauged supergravity action (with no hypermultiplets) and then the attractor mechanism for black holes in asymptotically flat space[2] [15, 16, 17, 18, 19, 14].

The bosonic part of the action for $\mathcal{N} = 2$ ungauged supergravity can be expressed as

$$S^{4\rm d} = S^{4\rm d}_{\rm grav} + S^{4\rm d}_{\rm scal} + S^{4\rm d}_{\rm EM} + S^{4\rm d}_{\rm ferm} + S^{4\rm d}_{\partial} \,, \tag{2.1}$$

[2] We will mainly follow the conventions of [14].

where[3] [14]:

$$
\begin{aligned}
S^{\rm 4d}_{\rm grav} =& \frac{1}{2\kappa^2}\int d^4X\,\sqrt{-G}\; R\;,\\
S^{\rm 4d}_{\rm scal} =& \int d^4X\,\sqrt{-G}\,\left(-g_{i\bar{j}}(z^i,\bar{z}^{\bar{i}})\partial_M z^i \partial^M \bar{z}^{\bar{j}}\right)\;,\\
S^{\rm 4d}_{\rm EM} =& \frac{1}{2\kappa^2}\int I_{IJ}(z^i,\bar{z}^{\bar{i}})F^I\wedge\star F^J + R_{IJ}(z^i,\bar{z}^{\bar{i}})F^I\wedge F^J\;,\\
S^{\rm 4d}_{\rm ferm} =& \frac{1}{\kappa^2}\int d^4X\,\sqrt{-G}\,\left(-\kappa^{-2}\overline{\psi}_{AM}\Gamma^{MNP}D_N\psi^A{}_P\right)\\
&-\frac{1}{4}\int d^4X\,\sqrt{-G}\,\left(g_{i\bar{j}}\bar{\xi}_A{}^i \not{D}\xi^{A\bar{j}} + \text{h.c.}\right)\\
&+\frac{1}{\kappa^2}\int d^4X\,\sqrt{-G}\,\left(F^{-I}_{MN}I_{IJ}X^J\overline{\psi}_A{}^M\psi_B{}^N\varepsilon^{AB} + \text{h.c.}\right)\\
&+\frac{1}{\kappa^2}\int d^4X\,\sqrt{-G}\,\left(F^{-I}_{MN}I_{IJ}\overline{\nabla}_{\bar{i}}\overline{X}^J\overline{\xi}^{A\bar{i}}\Gamma^M\psi^{BN}\varepsilon_{AB} + \text{h.c.}\right)\\
&+\frac{1}{2}\int d^4X\,\sqrt{-G}\,\left(g_{i\bar{j}}\overline{\psi}_{AM}\not{\partial} z^i\Gamma^M\xi^{A\bar{j}} + \text{h.c.}\right)\\
&+\text{(four-fermions terms)}\;;
\end{aligned}
\tag{2.2}
$$

$S^{\rm 4d}_{\partial}$ contains the necessary boundary terms, which depend on the boundary conditions chosen at infinity — i.e. the ensemble chosen for the black holes — such as the Gibbons–Hawking–York boundary term [29]. In flat space language, this action contains a graviton multiplet together with n_V vector multiplets. As for the bosonic sector, in $\mathcal{N}=2$ supergravity the n_V $U(1)$ gauge fields of the vector multiplets mix with the vector field in the graviton multiplet; together, they are described by the $U(1)$ field strengths $F^I = dA^I$, where $I = 0,\ldots,n_V$ and A^I are the corresponding $U(1)$ gauge potentials. The gauge fields also interact with the n_V complex scalars z^i and $\bar{z}^{\bar{i}}$, with $i = 1,\ldots,n_V$, through the gauge kinetic functions $R_{IJ}(z^i,\bar{z}^{\bar{i}})$ and $I_{IJ}(z^i,\bar{z}^{\bar{i}})$, which together compose the matrix $\mathcal{N}_{IJ} = R_{IJ} + iI_{IJ}$. Finally, the scalars form a non-linear sigma model due to the matrix $g_{i\bar{j}}(z^i,\bar{z}^{\bar{i}})$ appearing in their kinetic term. As for the fermionic sector, we have the gravitinos ψ_{AM} — where M are spacetime indices and $A = 1,2$ — and the gauginos $\xi_A{}^i$ and $\bar{\xi}_A{}^{i}$. The fermionic sector of the action contains their kinetic terms, as well as their interactions with the scalars, the vectors, the graviton and among themselves.

The main feature of $\mathcal{N}=2$ supergravity is that the scalar manifold is a special Kähler manifold [14, 30, 31]. Special Kähler manifolds are characterized by the presence of both a holomorphic line bundle (as for Kähler–Hodge manifold of $\mathcal{N}=1$ supergravity) and a holomorphic symplectic bundle, i.e. a vector bundle with structure group $Sp(2n_V+2,\mathbb{R})$. This is needed to implement the electromagnetic duality in the theory. In particular, taking the (anti-)self dual part[4] of the field strength F^I and their electromagnetic dual $G_I \coloneqq R_{IJ}F^J + I_{IJ}\star F^J$, we can

[3] In this work we will freely switch between $\kappa^2 = 8\pi G_N$ depending on convenience.

[4] We define the (anti-)self-dual part of a 2-form ω as $\omega^{\pm} = (\omega \mp i\star\omega)/2$.

compose the symplectic vectors (which are covariant with respect to the line bundle)

$$\mathcal{F}^{+} = \begin{pmatrix} F^{+I} \\ G^{+}{}_{I} \end{pmatrix}, \qquad \mathcal{F}^{-} = \begin{pmatrix} F^{-I} \\ G^{-}{}_{I} \end{pmatrix}, \tag{2.3}$$

whose upper and lower component are related by

$$G^{+}{}_{I} = \mathcal{N}_{IJ} F^{+J}, \qquad G^{-}{}_{I} = \overline{\mathcal{N}}_{IJ} F^{-J}. \tag{2.4}$$

The scalars z^i themselves are essentially just a choice of homogeneous coordinates on the (projective) special Kähler manifold. They parameterize the covariantly constant symplectic vectors

$$V = \begin{pmatrix} X^I(z^i, \bar{z}^{\bar{i}}) \\ F_I(z^i, \bar{z}^{\bar{i}}) \end{pmatrix} \qquad \text{and} \qquad \overline{V} = \begin{pmatrix} \overline{X}^I(z^i, \bar{z}^{\bar{i}}) \\ \overline{F}_I(z^i, \bar{z}^{\bar{i}}) \end{pmatrix}, \tag{2.5}$$

with

$$F_I = \mathcal{N}_{IJ} X^J \qquad \text{and} \qquad \overline{F}_I = \overline{\mathcal{N}}_{IJ} \overline{X}^J. \tag{2.6}$$

Additionally, the constraint $\langle V, \nabla_i V \rangle = 0$ must be imposed, where we denoted the symplectic product via the bracket $\langle . , . \rangle$. This can be solved if the matrix $\left(X^I, \nabla_i X^I\right)$ is invertible, by choosing $F_I = \partial_I F(X^J)$, where $F(X^J)$ is called the prepotential and is a homogeneous function of second degree in X^J. Choosing a specific model of $\mathcal{N} = 2$ supergravity then simply amounts to the choice of the prepotentials. Notice also that, while in general the matrix $\left(X^I, \nabla_i X^I\right)$ might not be invertible, it is always possible to make a choice of frame (through a symplectic transformation) such that the matrix becomes invertible, thus allowing for a prepotential [14].

2.2 The 1 dimensional effective potential

Let us now search for bosonic black hole solutions which are static and spherically symmetric, carrying both electric and magnetic charges. A generic ansatz for the metric that allows for both extremal and non-extremal black holes is given by [32]

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} \left(\frac{c^4 dr^2}{\sinh^4(cr)} + \frac{c^2}{\sinh^2(cr)} d\Omega^2_{S^2} \right), \tag{2.7}$$

where extremal black holes are recovered in the limit $c = (r_+ - r_-)/2 \to 0$:

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} \left(\frac{dr^2}{r^4} + \frac{1}{r^2} d\Omega^2_{S^2} \right). \tag{2.8}$$

To see this, let us change the variables from r to R in such a way that [14]:

$$\frac{c^2}{\sinh^2(cr)} = (R - R_+)(R - R_-); \tag{2.9}$$

we thus get the metric:

$$ds^2 = -e^{2U(R)}dt^2 + e^{-2U(R)}\left(dR^2 + (R - R_+)(R - R_-)\, d\Omega^2_{S^2}\right), \tag{2.10}$$

which, for $R \to \infty$, is the typical form of the metric for a charged black hole. We choose the parameter r to run from $-\infty$ to 0, where $r = -\infty$ is the location of the black hole outer horizon and $r = 0$ is spatial infinity. In order to recover flat space at infinity, we thus impose the condition $U(r = 0) = 0$.

Before proceeding with the gauge fields and the scalars ansatze, we note here that, throughout this work, we will freely Wick rotate between Lorentzian signature and Euclidean signature by using $t_E = -it$. For example, Wick rotation transforms the metric (2.7) to:

$$ds^2_{\mathrm{E}} = +e^{2U(r)}dt_E^2 + e^{-2U(r)}\left(\frac{c^4 dr^2}{\sinh^4(cr)} + \frac{c^2}{\sinh^2(cr)}d\Omega^2_{S^2}\right). \tag{2.11}$$

This Euclidean metric describes only the region outside of the horizon; the smoothness of this region — since the only physical singularity is inside the horizon — implies that the Euclidean time becomes compactified, with periodicity $t_E \sim t_E + \beta$. The constant $\beta = 1/T$ is the inverse of the Hawking temperature T of the black hole, which is related to the parameter c from (2.7) as follows [32]:

$$c = \frac{\text{Area}}{2G_N}T = 2\mathcal{S}T\,; \tag{2.12}$$

here the Bekenstein–Hawking area law is used to relate the area of the horizon to the (semiclassical) black hole entropy $\mathcal{S}$. With abuse of notation, in the rest of this work we will drop the subscript E from the Euclidean time t_E — which we will just denote with t — in order to ease the notation, leaving the interpretation up to the context. To avoid any confusion, we will highlight explicitly when we perform Wick rotations. As a rule of thumb, we will use Lorentzian signature when discussing the (super)isometries of the solutions and the Kaluza–Klein dimensional reduction from 4d to 2d, while we will use Euclidean signature when discussing the generalized Schwarzians and the (super-)BF theories.

Let us now go back to searching for black hole solutions. For the electromagnetic fields, we will assume that there are no charged objects outside the black hole horizon, given that all the other fields in the supergravity actions are neutral, i.e.

$$\begin{aligned} &dF^I = 0\,,\\ &dG_I \equiv d\left(R_{IJ}F^J + I_{IJ} \star F^J\right) = 0\,. \end{aligned} \tag{2.13}$$

This implies that the relations defining the charges,

$$
\begin{aligned}
p^I &= -\int_{S^2_\infty} F^I \,, \\
q_I &= -\int_{S^2_\infty} G_I \,,
\end{aligned}
\tag{2.14}
$$

usually valid only when integrating on a 2-sphere S^2_∞ at spatial infinity, can be imposed on a sphere S^2_r at any radius r outside the horizon. These conditions are satisfied under the assumptions of spherical symmetry by:

$$
4\pi F^I = -p^I \sin\theta d\theta \wedge d\phi + \left(I^{-1}\right)^{IJ} \left(q_J - R_{JK} p^K\right) \star \left(\sin\theta d\theta \wedge d\phi\right) \;; \tag{2.15}
$$

notice that $I_{IJ}(z^i, \overline{z}^{\bar{i}})$ and $R_{IJ}(z^i, \overline{z}^{\bar{i}})$ intrinsically depend on the radius r, given that they are functions of the scalar fields z^I. We also define the chemical potential for the electric charge as

$$
e^{\oint A^I} = e^{-\mu^I \beta} \,, \tag{2.16}
$$

where we integrate over the boundary of the spacetime at infinity along the periodic Euclidean time direction from 0 to β. This implies the relation

$$
4\pi \mu^I = -\int_{-\infty}^{0} dr\, e^{2U} \left(I^{-1}\right)^{IJ} \left(q_J - R_{JK} p^K\right) \;; \tag{2.17}
$$

the above expression relates the chemical potential μ^I to the values of the charges q_I that dominate the sum over different charges in the grand-canonical partition function. Finally, we will simply assume that the scalar fields are simple functions of the radial coordinate, i.e. $z^i = z^i(r)$.

Placing these ansatze into the equations of motion of the supergravity theory yields some second order differential equations that $U(r)$ and $z(r)$ must satisfy in order to obtain an actual black hole solution. These differential equations can alternatively be obtained as the equation of motions of the following 1-dimensional effective theory:

$$
S_{\mathrm{1d}} = -\frac{4\pi}{\kappa^2} \int dt \int dr\, U'^2 + \kappa^2 g_{i\bar{j}} z^{i\prime} \overline{z}^{\bar{j}\prime} + e^{2U} V_{\mathrm{BH}} - c^2 \,, \tag{2.18}
$$

where we define the black hole potential $V_{\rm BH}$ as [32]:

$$\begin{aligned} (4\pi)^2 V_{\rm BH} &= \frac{1}{2} Q^\top \mathcal{M} Q \,, \\ \mathcal{M} &= \begin{pmatrix} -(I + R I^{-1} R)_{IJ} & (R I^{-1})_I{}^J \\ (I^{-1} R)^I{}_J & -(I^{-1})^{IJ} \end{pmatrix} \,, \\ Q &= \begin{pmatrix} p^I \\ q_I \end{pmatrix} \,. \end{aligned} \tag{2.19}$$

In order to correctly reproduce the equations of motion of the original 4d theory, we also need to impose the following constraint,

$$U'^2 + \kappa^2 g_{i\bar{\jmath}} z^{i\prime} \bar{z}^{\bar{\jmath}\prime} - e^{2U} V_{\rm BH} - c^2 = 0 \,, \tag{2.20}$$

which can be seen as the conservation of energy in the 1d effective theory.

Notice that, to get the effective theory (2.18), it is not enough to simply plug the ansatz into the action. With no additional boundary term, we are fixing the gauge field at infinity, i.e. we are fixing the magnetic charges and the electric chemical potentials. To work in an ensemble with both types of charges fixed — and get the proper $V_{\rm BH}$ — we thus have to add the term $+\beta \mu^I q_I$, properly converted to real Minkowski time via

$$\beta \to i \int dt \,. \tag{2.21}$$

Notice that to get the 1d theory (2.18) we also ignore the contribution of a total derivative (proportional to U'') and we should in principle renormalize the theory by fixing a boundary near $r \sim 0$ and subtracting a diverging constant; this is akin to what usually happens while studying black holes thermodynamics, when one subtracts the contribution of flat space to the action by adding a boundary near spatial infinity [29] (or using holographic renormalization in the case of asymptotically AdS spacetimes).

2.3 Flow equations for black holes

To find actual black hole solutions, we now have to solve the classical equations of motion of the 1d theory; they are given by [18]:

$$\begin{aligned} U'' &= e^{2U} V_{\rm BH} \,, \\ z^{i\prime\prime} + \Gamma^i{}_{jk} z^{j\prime} z^{k\prime} &= e^{2U} \kappa^{-2} g^{i\bar{\jmath}} \partial_{\bar{\jmath}} V_{\rm BH} \,. \end{aligned} \tag{2.22}$$

Typically, one should solve these second order differential equations in full generality. However, when we are dealing with extremal black holes (and for particular types of $V_{\rm BH}$) it turns out that we can find BPS-like solutions that satisfy just first order differential equations instead.

In particular, if the black hole potential can be written as

$$V_{\rm BH} = W^2 + 4\kappa^{-2} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} W \tag{2.23}$$

for some real function $W = W(z^i, \bar{z}^{\bar{i}})$ — denoted "superpotential"[5] in analogy with the function that appears in supersymmetric theories — and if we set $c = 0$, the equations of motion and the constraint are reduced to first order differential equations[6]:

$$\begin{aligned} U' &= +e^U W \,, \\ z^{i\prime} &= +2e^U \kappa^{-2} g^{i\bar{j}} \partial_{\bar{j}} W \,. \end{aligned} \tag{2.24}$$

This is why these solutions are called BPS-like. Notice however that these solutions are not necessarily actual BPS solutions, i.e. supersymmetric solutions of the 4d supergravity satisfying the BPS bound $\mathcal{E}_* = |\mathcal{Z}_\infty|$, where $\mathcal{E}_*$ is the (extremal) mass of the solution, $\mathcal{Z}_\infty$ is the central charge evaluated at infinity,

$$\mathcal{Z}_\infty = -i\kappa^{-2} \int_{S^2_\infty} T^- = 2\kappa^{-2} \left(X^I_\infty q_I - F_{I\,\infty} p^I \right) \equiv 2\kappa^{-2} \langle V_\infty, Q \rangle \,, \tag{2.25}$$

and

$$T^- = -4 X^I I_{IJ} F^{-J} \tag{2.26}$$

is the bosonic and anti-self-dual part of the graviphoton field strength.

If we look at equations (2.24), we see that they are simply flow equations. In particular, the scalar fields flow from the values z^i_∞ — fixed by the boundary conditions at infinity — towards z^i_0 at the horizon, which are fixed points of the flow, i.e. $\partial_j W(z^i_0, \bar{z}^{\bar{i}}_0, p^I, q_I) = \partial_{\bar{j}} W(z^i_0, \bar{z}^{\bar{i}}_0, p^I, q_I) = 0$; in particular, given that the superpotential $W \geq 0$ always, these fixed points are minima of the superpotential [19]. Given that the black hole potential $V_{\rm BH}$ and thus the superpotential W are only functions of the charges and the scalar fields, we see that the fixed points of the flow $z^i_0 = z^i_0 \left(p^I, q_I \right)$ are just functions of the charges themselves, and are thus independent of the values z^i_∞ set by the boundary conditions at infinity. This can be intuitively understood as a consequence of the fact that, for extremal black holes, the horizon sits infinitely far away from spatial infinity; as the scalar fields flow moving towards the horizon, they "forget" the boundary conditions at infinity, and therefore they can only depend on the near-horizon properties of the black hole, such as the electric and magnetic charges. Notice also that there could be more than one fixed point for each superpotential, and that there could be different superpotentials with different fixed points that satisfy (2.23) for the same $V_{\rm BH}$. The moduli space of the z^i_∞ of the

[5] The analogy is only partial, as the standard superpotential that is found in supersymmetric theories is a holomorphic function of its variables.

[6] If we choose $r \in (0, +\infty)$ we should replace the $+$ sign at the beginning of both equations with a $-$ sign. In this work we choose $r \in (-\infty, 0)$ such that $\varepsilon_{tr\theta\phi} = +1$, leaving the orientation of the coordinates as the usual one in spherical coordinates.

possible extremal black holes will therefore be divided into multiple basins of attractions, each with its own fixed point [19].

2.4 Fake-superpotentials in $\mathcal{N} = 2$ supergravity

Up to now, the whole analysis still holds true for more general theories with the bosonic action identical to (2.1), but with coefficients $g_{i\bar{j}}$, I_{IJ} and R_{IJ} not related to each other by special Kähler geometry as in $\mathcal{N} = 2$ supergravity; let us now focus on the latter. In $\mathcal{N} = 2$ supergravity, we can always find a superpotential W that is given by:

$$W^{\mathcal{Z}} = G_N|\mathcal{Z}| \,. \tag{2.27}$$

It turns out that BPS-like solutions of (2.24) are actual BPS solutions [14]. In general, however, it might be possible to find different $\widetilde{W}$ that satisfy (2.23); the BPS-like solutions obtained this way are not supersymmetric, and thus these $\widetilde{W}$ are referred to as a fake-superpotentials.

One way to find other superpotentials is to notice that V_{BH} is left unchanged if we rotate $Q \to \Theta Q$, using a matrix Θ such that $\Theta^\top \mathcal{M} \Theta = \mathcal{M}$ [18]. If this matrix is a constant, we can define a "fake-central charge"

$$\mathcal{Z} = 2\kappa^{-2}\langle V, Q\rangle \to \widetilde{\mathcal{Z}} = 2\kappa^{-2}\langle V, \Theta Q\rangle \tag{2.28}$$

and get, from (2.27), the fake-superpotential

$$\widetilde{W} = G_N|\widetilde{\mathcal{Z}}| \,. \tag{2.29}$$

A useful trick to simplify the procedure of finding the appropriate matrix Θ is to rewrite (2.19) as [18]

$$\mathcal{M} = \mathcal{I} M \,, \qquad\qquad \mathcal{I} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{2.30}$$

The condition $\Theta^\top \mathcal{M} \Theta = \mathcal{M}$ then simply becomes $[\Theta, M] = 0$.

We also notice here, for future convenience (see section 3.2), that the mass of the extremal black holes is given by:

$$\mathcal{E}_* = \frac{\partial_r U_\infty}{G_N} = \frac{W(z^i_\infty, \bar{z}^{\bar{i}}_\infty)}{G_N} \equiv \frac{W_\infty}{G_N} \,; \tag{2.31}$$

the mass can either be calculated as a Komar mass or obtained by looking at the asymptotic form of the metric (2.8) near spatial infinity (since we are in flat space). For the supersymmetric black holes with superpotential (2.27), where $W_\infty = W^{\mathcal{Z}}(z^i_\infty, \bar{z}^{\bar{i}}_\infty)$ is the superpotential evaluated at infinity, the relation (2.31) correctly recovers the BPS relation $\mathcal{E}_* = |\mathcal{Z}_\infty|$. Notice also that, since the attractors of the flow are minima of the superpotential W, in each basin of attraction the solution with minimal mass (which we denote by $\mathcal{E}^0_*$) is the one given with constant scalars $z^i(x^\mu) = z_0^i$; for the "true" superpotential (2.27), this becomes $\mathcal{E}_*^0 = |\mathcal{Z}_0|$, while for fake superpotentials given by (2.29), this yields $\mathcal{E}_*^0 = |\widetilde{\mathcal{Z}}_0|$.

Let us now show one of the simplest theories which admit both BPS-like supersymmetric and non-supersymmetric black holes. We consider the $SU(1,1)/U(1)$ model of $\mathcal{N} = 2$ supergravity [19], with the prepotential:

$$F(X^I) = -i\kappa^{-2} X^0 X^1 \,. \tag{2.32}$$

This is a model with only one vector multiplet in addition to the supergravity multiplet. Choosing the normal coordinates $Z^0 = e^{-\kappa^2\mathcal{K}/2}X^0 = 1$ and $Z^1 = e^{-\kappa^2\mathcal{K}/2}X^1 = z = x + iy$ on the scalar manifold, we have the covariant symplectic vector

$$V = \begin{pmatrix} X^0 \\ X^1 \\ F_0 \\ F_1 \end{pmatrix} = \frac{1}{2\sqrt{x}} \begin{pmatrix} 1 \\ z \\ -i\kappa^{-2} z \\ -i\kappa^{-2} \end{pmatrix} . \tag{2.33}$$

Due to the simplicity of the theory, we can find the attractors by looking directly at the minima of the potential $V_{\rm BH}$ (without finding superpotentials first). The black hole potential is given by:

$$V_{\rm BH} = \frac{1}{32\pi^2\kappa^2 x}\left[\kappa^4 (q_0)^2 + (p^1)^2 + (x^2+y^2)\left(\kappa^4(q_1)^2 + (p^0)^2\right) + 2\kappa^2\left(p_0 q^0 + p_1 q^1\right) y\right] . \tag{2.34}$$

Its extrema are given by

$$\begin{aligned} z_0^+ &= \frac{\left(p^0p^1 + \kappa^2 q_0 q_1\right) + i\kappa^2\left(p^0 q_0 - p^1 q_1\right)}{(p^0)^2 + \kappa^4 (q_1)^2} \qquad && \text{if } p^0p^1 + \kappa^2 q_0 q_1 > 0\,, \\ z_0^- &= \frac{-\left(p^0p^1 + \kappa^2 q_0 q_1\right) + i\kappa^2\left(p^0 q_0 - p^1 q_1\right)}{(p^0)^2 + \kappa^4 (q_1)^2} \qquad && \text{if } p^0p^1 + \kappa^2 q_0 q_1 < 0\,; \end{aligned} \tag{2.35}$$

notice that the "if" conditions are needed in order to have $\mathrm{Re}(z) > 0$ and thus $I_{IJ} < 0$, so that we have the proper sings in the gauge kinetic functions of the gauge fields. In particular, we can pass from z_0^+ to z_0^- (and vice-versa) by simply flipping the sign of some charges, for example by sending $q_0 \to -q_0$ and $p^0 \to -p^0$. Notice also that, at both $z_0^\pm$, the value of the black hole potential is the same:

$$V_{\rm BH}|_{r\to-\infty} = \frac{\left|p^0p^1 + \kappa^2 q_0 q_1\right|}{16\pi^2\kappa^2} \,. \tag{2.36}$$

Let us now find the appropriate superpotentials yielding the two solutions $z_0^\pm$. For sure we have one superpotential — proportional to the central charge — given by (2.27). The central charge is

$$\mathcal{Z} = \frac{\kappa^{-2}}{\sqrt{x}}\left(q_0 + z q_1 + i\kappa^{-2}\left(p^1 + z p^0\right)\right) , \tag{2.37}$$

and its absolute value $|\mathcal{Z}|$ is extremized exactly by z_0^+. Therefore, the black extremal black holes in the z_0^+ basin of attraction are actual BPS black holes. In order to find a superpotential $\widetilde{W}$ with fixed point z_0^-, instead, we follow the procedure described in (2.28) and (2.29). Looking at the matrix (2.27),

$$\mathcal{M} = \begin{pmatrix} \kappa^{-2}\begin{pmatrix} \frac{x^2+y^2}{x} & 0 \\ 0 & \frac{1}{x} \end{pmatrix} & \begin{pmatrix} -\frac{y}{x} & 0 \\ 0 & \frac{y}{x} \end{pmatrix} \\ \begin{pmatrix} -\frac{y}{x} & 0 \\ 0 & \frac{y}{x} \end{pmatrix} & \kappa^2\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & \frac{x^2+y^2}{x} \end{pmatrix} \end{pmatrix}, \tag{2.38}$$

we find that the matrix

$$\Theta = \cos\eta \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + \sin\eta \begin{pmatrix} 0 & -i\kappa^2\sigma_2 \\ i\kappa^{-2}\sigma_2 & 0 \end{pmatrix} \tag{2.39}$$

satisfies $\Theta^\top \mathcal{M}\Theta = \Theta$. We thus get a fake-superpotential $\widetilde{W}$ as in (2.29) using the fake-central charge

$$\widetilde{\mathcal{Z}} = \frac{\kappa^{-2}}{\sqrt{x}} e^{i\eta}\left(-q_0 + zq_1 + i\kappa^{-2}\left(p^1 - zp^0\right)\right), \tag{2.40}$$

with $\widetilde{W}$ being extremized by z_0^-. These kind of black holes still satisfy BPS-like equations, but they are not supersymmetric solutions. We also have that the value of W_0 and $\widetilde{W}_0$, which are the superpotentials evaluated at the horizon, are exactly the same:

$$W_0^{\mathcal{Z}} = \widetilde{W}_0 = \sqrt{V_{\mathrm{BH}}}\Big|_{r\to-\infty} = \left(\frac{\left|p^0p^1 + \kappa^2 q_0 q_1\right|}{16\pi^2\kappa^2}\right)^{1/2}. \tag{2.41}$$

Thus we cannot distinguish the supersymmetric and the non-supersymmetric attractors from W_0 only, which — as we will show in the next section 3.1 — is the radius of the extremal AdS_2 throat. Vice-versa, the two solutions differ for their values of the central charge:

$$\begin{aligned} \mathcal{Z}(z_0^+) =& \frac{2i\sqrt{p^0p^1 + \kappa^2 q_0 q_1}\sqrt{(p^0)^2 + \kappa^4 (q_1)^2}}{\kappa^4\left(p^0 + i\kappa^2 q^1\right)}, \\ \mathcal{Z}(z_0^-) =& 0\,. \end{aligned} \tag{2.42}$$

This indicates that only quantities coupling directly to the central charge, such as the graviphoton field strength, behave differently for the two attractors, making the near-extremal dynamics differ in the two cases. Finally, notice that we can have both BPS-like supersymmetric and non-supersymmetric black holes even when we consider only electric charges; as we will see in section 4.1, this setup will be easier to analyze in the dimensional reduction, and thus we will fix $p^I = 0$ for simplicity.

3 Symmetries in the near-horizon

3.1 Bosonic symmetries

Let us now study the symmetries of the extremal solutions satisfying (2.24), focusing on the near-horizon limit of the solutions. One way to choose the correct generalized Schwarzian theory describing the extremal/near-extremal transition is by looking at the symmetry breaking pattern in the near-horizon limit. In particular, the extremal solution admits more (super)isometries than the near-extremal ones; we can then interpret the extremal/near-extremal transition as a sort of symmetry breaking, to be described by the means of an effective theory (in the top-down approach). We thus pick the generalized Schwarzian action by looking at the broken symmetries, interpreting the generalized Schwarzian modes as the “Goldstone bosons” of the symmetry breaking. Finally, we should in principle fix the energy scale of the effective theory, along with other unknown constants, by matching — at the classical level — the partition function of the generalized Schwarzians with the ones from the 4d supergravity; this is equivalent to comparing and matching various thermodynamic quantities such as the energy, the entropy and the specific heat of the black hole.

As we move towards the horizon $r \to -\infty$ of (2.8), the scalar fields approach the constant values z_0^i, set by the attractor mechanism; given that $\partial_i W = \partial_{\bar{i}} W = 0$, we will approximate the first equation of (2.24) as

$$U' = e^U W_0 + \dots\,, \tag{3.1}$$

where $W_0 = W(z_0^i, \bar{z}_0^{\bar{i}})$ is the superpotential evaluated at the horizon (i.e. at the fixed point of the attractor flow). Solving this equation yields:

$$e^{-U(r)}\Big|_{\text{NHR}} = -W_0 r + \dots\,; \tag{3.2}$$

plugging this approximate result into (2.8), we get:

$$ds^2 = W_0^2 \left(\frac{-d\tilde{t}^2 + dr^2}{r^2} + d\Omega_{S^2}^2 \right)\,, \tag{3.3}$$

where we rescaled $\widetilde{t} = t/W_0^2$. This is exactly the metric of $\text{AdS}_2 \times S^2$, with AdS radius W_0. Hence, after a Wick rotation, all the extremal solutions generated through the attractor mechanism possess an $SL(2,\mathbb{R}) \times SU(2)$ symmetry in the near-horizon, independently of whether they are supersymmetric or not. Notice the following delicate point: from the metric $\text{AdS}_2 \times S^2$ one would generically consider the isometry group to be $PSL(2,\mathbb{R}) \times SO(3)$, and not $SL(2,\mathbb{R}) \times SU(2)$. As pointed out in [28], however, in a theory with fermions one wants the spacetime to admit a spin structure. More heuristically, the antiperiodicity of fermions in Euclidean times forces us to consider the double cover of $PSL(2,\mathbb{R}) \times SO(3)$, that is $SL(2,\mathbb{R}) \times SU(2)$. This is also consistent with the results of section 3.2, where we analyze the superisometries of the extremal solutions, and of section 4.7, where we discuss the BF formulation of the generalized JT gravities: in the former case, the supersymmetric attractors possess a $PSU(1,1|2)$ superisometry group, which exactly contains the bosonic subgroup $SL(2,\mathbb{R}) \times SU(2)$; in the latter case, see in particular the comment at page 34 of [28].

Let us now discuss the symmetry breaking pattern in the extremal/near-extremal transition heuristically. The breaking of $SL(2,\mathbb{R})$ is caused by turning on a finite temperature $T > 0$ and moving away from extremality, modifying the near-horizon throat from an AdS_2 to a less symmetric Rind_2 one. The breaking of $SU(2)$ is instead caused by turning on a small angular velocity Ω (which we will take, without loss of generality, along the z axis) and considering slowly rotating black holes. Since giving an angular velocity along z still leaves the solutions symmetric under rotations along the z axis, the symmetry breaking pattern is $SU(2) \to U(1)$. Putting everything together, the extremal $\to$ near-extremal transition is described by a symmetry breaking with bosonic subgroup $SL(2,\mathbb{R}) \times SU(2) \to \varnothing \times U(1)$. Finally, notice that the analysis of the bosonic symmetries is completely independent on whether we chose $W^{\mathcal{Z}} = G_N|\mathcal{Z}|$ as in (2.27) or not, that is on whether the BPS-like solution is supersymmetric or not.

3.2 Fermionic symmetries

Let us now search for the supersymmetry transformations that leave the near-horizon solutions (3.3) invariant; once again, we will approximate $z^i(r) \sim z_0^i$. The supersymmetric transformation of the gravitinos is given by[7] [14]:

$$\delta\psi_M{}^A = \nabla_M \epsilon^A - \frac{1}{16}\Gamma^{NP} T^-_{NP} \Gamma_M \varepsilon^{AB} \epsilon_B = 0\,, \tag{3.4}$$

where $\nabla_M = \partial_M + \frac{1}{4}\omega_M{}^{NP}\Gamma_{NP}$ and the graviphoton field strength is

$$4\pi T^- = i\kappa^2 \mathcal{Z}\,(1 + i\star)\,(\sin\theta d\theta \wedge d\phi)\ . \tag{3.5}$$

To find the solutions, it is convenient to separate the $x^M = (t, r, \theta, \phi)$ coordinates into two sets: $x^\mu = (t, r)$ and $y^\alpha = (\theta, \phi)$. This way we can write the 4d Γ matrices in terms of 2d γ matrices on μ and α coordinates as[8]:

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}\,, \qquad\qquad \Gamma^\alpha = \gamma_3 \otimes \gamma^\alpha\,, \tag{3.6}$$

where $\gamma_3 = \gamma^0\gamma^1$ is the pseudo-chirality matrix in μ coordinates[9]. Next, in order to obtain a simpler spinor equation, we define the following Dirac spinor:

$$\epsilon := \epsilon^1 - ie^{i\arg\mathcal{Z}_0}\epsilon_2\,, \tag{3.7}$$

[7] We choose the matrix ε_{AB} such that $\varepsilon_{01} = \varepsilon^{01} = 1$.

[8] For the choice of the 4d Γ matrices and the 2d γ matrices, we will follow the conventions of [8]; see in particular Appendix B (at page 70).

[9] When acting on a 4d spinor, we will simply write $\gamma_3 \otimes \mathbb{1}$ as γ_3.

by combining together the left-handed spinor ϵ^1 and the right-handed spinor ϵ_2. We can then express (3.4) in terms of the spinor ϵ by taking the equation with $A = 1$, together with the charge conjugate of the equation with $A = 2$; in doing so we use the fact that the Killing spinors that parameterize the supersymmetry transformation are Majorana. Finally, by expanding

$$4\pi\Gamma^{NP}T^-_{NP} = -\frac{4\kappa^2}{W_0^2}\mathcal{Z}_0\gamma_3 P_L \tag{3.8}$$

using the ansatz (2.15), we get

$$\nabla_M\epsilon - \frac{i}{2}\frac{G_N|\mathcal{Z}_0|}{W_0^2}\gamma_3\Gamma_M\epsilon = 0. \tag{3.9}$$

Splitting into two 2d components the spinor $\epsilon = \epsilon_{\text{AdS}} \otimes \eta$, we obtain

$$\begin{aligned}\left(\nabla_\mu - \frac{i}{2W_0}\zeta\gamma_3\gamma_\mu\right)\epsilon_{\text{AdS}} = 0\,,\\ \left(\nabla_\alpha - \frac{i}{2W_0}\zeta\gamma_\alpha\right)\eta = 0\,,\end{aligned} \tag{3.10}$$

where we introduced the parameter

$$\zeta = \frac{G_N|\mathcal{Z}_0|}{W_0}\,. \tag{3.11}$$

Equation (3.10) is the standard form of the Killing spinor equation on a $\text{AdS}_2 \times S^2$ spacetime of radius W_0/ζ. The general solutions to these equations can be found in [33]; there are 8 free real parameters controlling the solutions, implying that — in the near-horizon — the solutions preserve the full $\mathcal{N} = 2$ supersymmetry. Had we not considered the near-horizon limit, we would have had to impose a supersymmetric projection condition such as $\epsilon^A \sim \varepsilon^{AB}\epsilon_B$; this would have halved the number of free parameters, making the full black hole solution only 1/2-BPS [18]. There is thus a symmetry enhancement in the near-horizon.

So far, the solutions coming from fake-superpotentials could still be supersymmetric, despite the mismatch between the AdS radii W_0 (coming from the metric) and W_0/ζ (coming from the supersymmetric Killing spinors). We have however yet to look at the supersymmetric transformation of the gauginos, once again in the $z^i \sim$ const. approximation [14]:

$$\delta\xi^i_A = -\frac{1}{2}G^{-\ i}_{NP}\Gamma^{NP}\varepsilon_{AB}\epsilon^B = 0\,. \tag{3.12}$$

Expanding $G^{-\ i}_{NP}$, we get:

$$\begin{aligned}\partial_{\bar{i}}|\mathcal{Z}_0|\epsilon^1 = 0\,,\\ \partial_i|\mathcal{Z}_0|\epsilon_2 = 0\,.\end{aligned} \tag{3.13}$$

Therefore, in order to get non trivial Killing spinors, we must have $\partial_i|\mathcal{Z}_0| = \partial_{\bar{i}}|\mathcal{Z}_0| = 0$; in turn, this implies that the only supersymmetric extremal solutions come from the "true" superpotential (2.27) and that fake-BPS solutions are indeed not supersymmetric. The ζ parameter introduced in (3.11) thus controls whether the solution is supersymmetric or not: we can have Killing spinors only for $\zeta = 1$, and in that case they are exactly the Killing spinors on an $\mathrm{AdS}_2 \times S^2$ spacetime of AdS radius $W_0^{\mathcal{Z}}$. It can be shown [8] that the Killing spinors obtained from (3.10) generate the superisometry group $PSU(1,1|2)$, whose bosonic part is $SL(2,\mathbb{R}) \times SU(2) \subset PSU(1,1|2)$ as expected from the analysis of section 3.1.

At last, let us discuss in more depth the meaning of the parameter ζ introduced in (3.11); it will in fact assume a central role in the dimensional reduction of the fermionic sector of the 4d supergravity (see section 5, and in particular 5.2). For supersymmetric solutions with superpotential (2.27), we simply have $\zeta = 1$. If the superpotential is obtained from a fake-central charge via (2.29), instead, we have

$$\zeta = \frac{|\mathcal{Z}_0|}{|\widetilde{\mathcal{Z}}_0|} = \frac{|\mathcal{Z}_0|}{\mathcal{E}_*^0} \,. \tag{3.14}$$

ζ is thus the inverse ratio between the mass $\mathcal{E}_*^0$ of the lightest extremal black hole in the chosen basin of attraction (see section 2.4) and the mass $|\mathcal{Z}_0|$ which the black hole would have if it satisfied the BPS bound (and hence if it were supersymmetric). Since the BPS bound implies $\mathcal{E}_*^0 \geq |\mathcal{Z}_0|$ — with $\mathcal{E}_*^0 = |\mathcal{Z}_0|$ for supersymmetric solutions only — ζ must take the values $0 \leq \zeta \leq 1$, with $\zeta = 1$ just for supersymmetric attractors. In other words, ζ measures how far are we from a true BPS black hole, and controls whether we are in the basin of attraction of a supersymmetric attractor or not.

3.3 Choice of the Schwarzian theory

The above analysis shows that the symmetry broken in the extremal/near-extremal black hole transition in the near-horizon region is $SL(2,\mathbb{R}) \times SU(2) \to \varnothing \times U(1)$ for non-supersymmetric attractors and $PSU(1,1|2) \to \varnothing \times U(1)$ for supersymmetric attractors. These symmetry breaking patterns have already been studied in [7, 10] and [8], respectively, for simpler types of black holes. We can thus already guess the Schwarzian theories that will describe the behavior of the black holes. We remark that we are working in an ensemble with fixed electric and magnetic charges, but fixed angular velocity Ω.

Let us start from the symmetry breaking of $SL(2,\mathbb{R}) \times SU(2)$ for a non-supersymmetric attractor. The symmetry breaking of $SL(2,\mathbb{R})$ is described by a pure Schwarzian theory [11, 12]:

$$S_{\mathrm{Sch}} = -\frac{1}{M_{SL(2)}} \int_0^\beta du \, \mathrm{Sch}\left(\tan \frac{\pi \tau(u)}{\beta}, u\right) ; \tag{3.15}$$

here u is the Euclidean time of the theory, $\tau(u)$ is a monotonically increasing function that parameterizes the boundary mode, $M_{SL(2)}$ is the energy scale associated to the Schwarzian mode — to be determined by matching with the on-shell supergravity action — and

$$\mathrm{Sch}\left(f(u),u\right)=\frac{f'''(u)}{f'(u)}-\frac{3}{2}\left(\frac{f''(u)}{f'(u)}\right)^2 \tag{3.16}$$

is the Schwarzian derivative[10]. The symmetry breaking of $SU(2)$ can instead be described using the action of a particle moving on the group manifold $SU(2)$ [7, 10]:

$$S_{SU(2)}=-\frac{1}{M_{SU(2)}}\int_0^\beta du\ \mathrm{Tr}\left[\left(g^{-1}(u)g'(u)\right)^2\right]\,, \tag{3.17}$$

where $g(u)\in SU(2)_{\mathrm{fund}}$ is an $SU(2)$ group element in the fundamental representation and $M_{SU(2)}$ is the energy scale associated with the $SU(2)$ symmetry breaking. Finally, by adopting the effective theory approach to add all the possible terms compatible with the symmetries of the system, we can add a constant term $\mathcal{E}_*$ to the Lagrangian; we will also add an overall constant $-\mathcal{S}_*$ to the action. These terms account for the (classical) mass and entropy of the extremal black hole; we need to add them by hand since the effective Schwarzians only describe the near-extremal excitations, and not the extremal black hole itself. We thus get the effective action:

$$\begin{aligned}S^{\mathrm{eff}}_{\text{non-SUSY}}=&-\mathcal{S}_*+\beta\mathcal{E}_*-\frac{1}{M_{SL(2)}}\int_0^\beta du\,\mathrm{Sch}\left(\tan\frac{\pi\tau(u)}{\beta},u\right)\\&-\frac{1}{M_{SU(2)}}\int_0^\beta du\ \mathrm{Tr}\left[\left(g^{-1}(u)g'(u)\right)^2\right]\,,\end{aligned} \tag{3.18}$$

where the constants $\mathcal{E}_*$, $\mathcal{S}_*$, $M_{SL(2)}$ and $M_{SU(2)}$ should be determined by matching. Note that there is no term form $\Omega\mathcal{J}_*$ — with $\mathcal{J}_*$ the extremal angular momentum — in contrary to what happens for the extremal energy ($+\beta\mathcal{E}_*$) and entropy ($-\mathcal{S}_*$); this is simply because the extremal solutions that we consider are static and not rotating, so $\mathcal{J}_*=0$.

Let us now focus on the symmetry breaking of $PSU(1,1|2)$. As shown in [8], it is described by the $\mathcal{N}=4$ super-Schwarzian theory; its action is based on the $\mathcal{N}=4$ super-Schwarzian, a supersymmetric generalization (with $SU(2)$ R-symmetry) of the usual Schwarzian, first described in [34, 35] (we give more details in section 6.1). The bosonic part of the action is given by:

$$S^{\mathrm{bos}}_{\mathrm{Sch},\mathcal{N}=4}=-\frac{1}{M_{PSU(1,1|2)}}\int_0^\beta du\,\mathrm{Sch}\left(\tan\frac{\pi\tau(u)}{\beta},u\right)+\mathrm{Tr}\left[\left(g^{-1}(u)g'(u)\right)^2\right]+(\text{fermions})\,, \tag{3.19}$$

where we introduced yet another energy scale $M_{PSU(1,1|2)}$. The fermionic degrees of freedom are given by the two Grassmanian functions $\eta^p(u)$ and $\overline{\eta}_p(u)$, where $p=1,2$. Adding the extremal

[10] We denote the derivative $f'(u)\equiv\partial_u f(u)$.

energy and entropy contribution as before, we get the effective theory:

$$
\begin{aligned}
S_{\text{SUSY}}^{\text{eff, bos}} &= -\,\mathcal{S}_* + \beta\mathcal{E}_* - \frac{1}{M_{PSU(1,1|2)}}\int_0^\beta du\,\text{Sch}_{\mathcal{N}=4}\left(\tau, g, \eta, \overline{\eta}\right) \\
&= -\,\mathcal{S}_* + \beta\mathcal{E}_* \\
&\quad - \frac{1}{M_{PSU(1,1|2)}}\int_0^\beta du\,\text{Sch}\left(\tan\frac{\pi\tau(u)}{\beta}, u\right) + \text{Tr}\left[\left(g^{-1}(u)g'(u)\right)^2\right] \\
&\quad + (\text{fermions})\ .
\end{aligned}
\tag{3.20}
$$

By comparing (3.20) with (3.18), we see that the bosonic part of the action is essentially the same, with the identification $M_{SL(2)} = M_{SU(2)} \equiv M_{PSU(1,1|2)}$. This is to be expected, given that the bosonic subgroup of symmetry is identical in the two cases. Notice also that the two energy scales $M_{SL(2)}$ and $M_{SU(2)}$ become related thanks to the supersymmetry of the super-Schwarzian theory; therefore, in the $PSU(1,1|2)$ case, there is apparently one less parameter to determine by matching.

However, it is important to note that the underlying supersymmetry of the original supergravity action (2.1) imposes the relation $M_{SL(2)} = M_{SU(2)}$ even for near-extremal black holes around non-supersymmetric attractors [10]. Although we will verify this directly in section 4.6, the reason can be understood without doing any calculations. In fact, not only the bosonic symmetries are the same, but if we look at (2.1) we see that the bosonic part of the action is not dependent on whether we consider supersymmetric attractors or not[11]; in particular, the only pieces depending on the graviphoton field strength T^- — and thus on the value of the central charge $\mathcal{Z}$ — are in the fermionic terms. Since the generalized Schwarzians actions are obtained via dimensional reduction of (2.1) to generalized JT gravities, the bosonic part of the JT gravities and thus of the effective Schwarzians will be independent on whether we consider supersymmetric attractors or not. In turn, since we must have a single energy scale $M_{SL(2)} = M_{SU(2)} \equiv M_{PSU(1,1|2)}$ for the supersymmetric attractors (due to supersymmetry), the relation $M_{SL(2)} = M_{SU(2)}$ will also be enforced in the non-supersymmetric case.

4 The bosonic 1d boundary theory

4.1 Dimensional reduction of bosons

In principle, all that remains is to match the effective theories (3.18) and (3.20) to the classical partition functions of the supergravity. Then, one computes 1-loop quantum corrections and extracts the quantum corrected spectra of states. A remarkable property of these theories is that they have been shown to be 1-loop exact thanks to fermionic localization in the Schwarzian path integral [13]. The overall qualitative features of the spectrum — such as the presence of

[11] While technically $\mathcal{N}_{IJ}$ (evaluated at the horizon) can be different for different attractors, it does not change significantly the results to distinguish between supersymmetric and non-supersymmetric attractors.

a gap or of a degenerate ground state — are however already set by the choice of generalized Schwarzian theory. In fact, the 1-loop corrections do not depend on the actual value of the mass scales, but they are simply related to how many bosonic and fermionic zero-modes one has to gauge fix when computing the path integral [13]. In particular — as we will review in section 6 — the non-supersymmetric Schwarzian (3.18) has a continuous spectrum for all the possible values of the angular momentum $\mathcal{J}$ and thus a vanishing extremal entropy [7, 10]; instead, the $\mathcal{N} = 4$ super-Schwarzian has a discrete part of the spectrum associated with the $\mathcal{J} = 0$ states, due to the presence of the fermionic zero-modes, with a non-zero extremal entropy [8, 28].

While this is sufficient for understanding the behavior of the black hole spectra, the approach of section 3 is not really transparent on what sets apart the two boundary theories we have identified, given that they should both be obtained from dimensional reduction of the same supergravity action (2.1). Since the difference between BPS and fake-BPS extremal black holes can be as simple as flipping the sign of some charges, we would like to understand what is the actual mechanism which differentiates between the two effective theories.

We will therefore proceed with the Kaluza–Klein dimensional reduction of (2.1). We follow the conventions of[12] [8], with minor modifications; more details can be found for similar cases in [26, 27]. Notice that, since we are interested just in the lowest energy excitations above extremality, we will only consider the Kaluza–Klein modes that remain massless, ignoring the infinite tower of massive particles that will be generated. This can be justified because when going in the near-horizon region (which is the region we are actually interested in), the Kaluza–Klein modes have a mass of order $m_{KK} \sim 1/W_0$ [7]. As we will see in section 4.6, the energy scale associated with the effective Schwarzian boundary modes and thus the order of magnitude of the temperatures T we are interested in is $T \sim \Phi_{\text{ren}}^{-1} = G_N/W_0^3$; since $T \ll m_{KK}$ for macroscopic black holes with radius much larger than Planck length — which satisfy $G_N \ll W_0^2$ — the Kaluza–Klein modes are too heavy to be excited in the near-extremal limit we consider. It is also important to note that, in general, these massive modes contribute to the value of the extremal entropy and energy [7], but do not give rise to other temperature dependent terms in their 1-loop determinants (which are the main focus of this work).

As in section 3.2, we once again split the 4d coordinates $x^M = (t, r, \theta, \phi)$ into the two sets $x^\mu = (t, r)$ and $y^\alpha = (\theta, \phi)$. The ansatz for the metric in the Kaluza–Klein reduction is given by [8]:

$$\begin{aligned} ds^2 =& G_{MN} dX^M dX^N \\ =& \frac{W_\infty}{\chi^{1/2}} g_{\mu\nu} dx^\mu dx^\nu + \chi h_{\alpha\beta} \left(dy^\alpha + T_i{}^\alpha B^i{}_\mu dx^\mu \right) \left(dy^\beta + T_j{}^\beta B^j{}_\nu dx^\nu \right) ; \end{aligned} \tag{4.1}$$

here $W_\infty = W(z^i_\infty, \bar{z}^{\bar{i}}_\infty)$ is just a constant, $g_{\mu\nu}(x^\mu)$ is an arbitrary 2d metric and $h_{\alpha\beta} dy^\alpha dy^\beta = d\Omega^2_{S^2}$ is the metric of a unit sphere. The field $\chi(x^\mu)$ is the dilaton; its normalization has been

[12] In particular, see the appendix B of [8].

chosen such that it has no kinetic term in the resulting action. Finally $B^i{}_\mu(x^\mu)$ is the $SU(2)$ gauge fields transforming in the adjoint of $SU(2)$; together with the Killing vectors $T_i{}^\alpha$ (see [26] for their explicit form) it gauges the $SU(2)$ isometry of the extremal solution. We denote the frame indices of the 2d metrics $g_{\mu\nu}$ and $h_{\alpha\beta}$ with $(m, n, \dots)$ and $(a, b, \dots)$ respectively, while we use $\hat{M} = (\hat{m}, \hat{a})$ for the frame indices of the 4d metric G_{MN}; the vierbein and the zweibeins are given by

$$
\begin{aligned}
G_{MN} dX^M dX^N &= \eta_{\widehat{M}\widehat{N}} e^{\widehat{M}} e^{\widehat{N}} = \eta_{mn} e^{\widehat{m}} e^{\widehat{n}} + \delta_{ab} e^{\widehat{a}} e^{\widehat{b}} \ , \\
g_{\mu\nu} dx^\mu dx^\nu &= \eta_{mn} e^m e^n \ , \\
h_{\alpha\beta} dy^\alpha dy^\beta &= \delta_{ab} e^a e^b \ ,
\end{aligned}
\tag{4.2}
$$

where

$$
e^{\widehat{m}} = W_\infty^{1/2} \chi^{-1/4} e^m \, , \qquad\qquad e^{\widehat{a}} = \chi^{1/2} \left(e^a + e^a{}_\alpha T_i{}^\alpha B^i{}_\mu dx^\mu \right) \, . \tag{4.3}
$$

The explicit expression for the spin connection ω is obtained by solving the torsionless condition:

$$
de^{\widehat{M}} + \omega^{\widehat{M}}{}_{\widehat{N}} e^{\widehat{N}} = 0 \, . \tag{4.4}
$$

Calling ω_g and ω_h the spin connections of $g_{\mu\nu}$ and $h_{\alpha\beta}$ respectively, we get:

$$
\begin{aligned}
\omega^{\widehat{m}}{}_{\widehat{n}} &= \omega_{(g)}{}^m{}_n - \frac{1}{2} \partial_{[n} \log \chi e^{m]} - \frac{1}{2} T_{ai} H^{im}{}_n \left(e^a + T_i{}^a B^i \right) \, , \\
\omega^{\widehat{a}}{}_{\widehat{m}} &= \frac{1}{2} W_\infty^{-1/2} \chi^{3/4} \left[T_i{}^a H^i{}_{mn} e^n + \partial_m \log \chi \left(e^a + T_i^a B^i \right) \right] \, , \\
\omega^{\widehat{a}}{}_{\widehat{b}} &= \omega_{(h)}{}^a{}_b + \nabla_b T_i{}^a B^i \, ,
\end{aligned}
\tag{4.5}
$$

where $\nabla_b = e_b{}^\beta \nabla_\beta$ is the gravitational covariant derivative and $H^i = dB^i + \frac{1}{2} \varepsilon^i{}_{jk} B^j \wedge B^k$ is the $SU(2)$ field strength associated to the gauge field B. Evaluating the curvature, plugging the result in $S^{\rm 4d}_{\rm grav}$ given by (2.2) and integrating over the sphere S^2 yields [8]:

$$
S^{\rm 2d}_{\rm grav} = \frac{2\pi}{\kappa^2} \int d^2x \sqrt{-g} \left(\chi R + \frac{2W_\infty}{\chi^{1/2}} \right) + \frac{2}{3} \frac{\chi^{5/2}}{W_\infty} \, {\rm Tr} \left(H \wedge \star H \right) \, ; \tag{4.6}
$$

here we write $B = B^i T_i$ and $H = dB - B \wedge B$, with T_i generators in the fundamental of $SU(2)$ normalized such that $[T_i, T_j] = -\varepsilon_{ijk} T_k$ and ${\rm Tr}\,(T_i T_j) = -\delta_{ij}/2$. This is the action of a 2d dilaton gravity [20], coupled with a 2d $SU(2)$ gauge field.

As for the scalar field, if we were interested in the full Kaluza–Klein expansion, we should in principle assume an expansion in spherical harmonics such as

$$
z^i(X^M) = \sum_l z^i_l(x^\mu) Y^l(y^\alpha) \, ; \tag{4.7}
$$

after integrating spherical harmonics we get a tower of scalars of mass $m_l^2 \sim l(l+1)/\chi$ [7].

Therefore, taking into account only the massless modes, we simply need to replace $z^i = z^i(x^\mu)$ into $S^{\rm 4d}_{\rm scal}$ from (2.2), obtaining:

$$S^{\rm 2d}_{\rm scal} = \frac{2\pi}{\kappa^2}\int d^2x\,\sqrt{-g}\left(-2\chi\kappa^2 g_{i\bar{j}}\partial_\mu z^i\partial^\mu\bar{z}^{\bar{j}}\right)\,. \tag{4.8}$$

Finally, we have to expand the $U(1)$ gauge fields. The full expansion in spherical harmonics can be found in [36]; once again, since we are only interested in the massless terms, we will take:

$$\begin{aligned} A^I_\pm(X^M) &= a^I(x^\mu) - \frac{p^I}{4\pi}\left(\cos\theta\mp 1\right)d\phi\,,\\ F^I(X^M) &= f^I(x^\mu) + \frac{p^I}{4\pi}\sin\theta d\theta\wedge d\phi\,. \end{aligned} \tag{4.9}$$

Notice that we defined the 2d field strength $f^I = da^I$ and that we specified the potential $A^I_\pm(X^M)$ into two different charts in order to properly account for the magnetic charge of the black hole. Notice also that by fixing p^I we are implicitly assuming to work in an ensemble with fixed magnetic charges and thus fixed holonomy of the gauge field at infinity (and, equivalently, fixed electric chemical potentials). Plugging (4.9) into $S^{\rm 4d}_{\rm EM}$ from (2.2), we get

$$\begin{aligned} S^{\rm 2d}_{\rm EM} =& \frac{2\pi}{\kappa^2}\int \frac{\chi^{3/2}}{W_\infty}I_{IJ}f^I\wedge\star f^J + \frac{1}{2\pi}R_{IJ}p^I f^J + \frac{1}{16\pi^2}\frac{W_\infty}{\chi^{3/2}}I_{IJ}p^Ip^J\star 1\\ &+\frac{2\pi}{\kappa^2}\int\frac{1}{24\pi^2}I_{IJ}p^Ip^JB^i{}_\mu B_i{}^\mu\,\star 1\\ &+\frac{2\pi}{\kappa^2}\int\frac{1}{96\pi^2}\frac{\chi^{3/2}}{W_\infty}I_{IJ}p^Ip^J\left[\left(B_i{}^\mu B^i{}_\mu\right)^2-\left(B^i{}_\mu B^{j\mu}\right)^2\right]\star 1\,. \end{aligned} \tag{4.10}$$

This action is quite complicated: apart from the usual kinetic term of the 2d $U(1)$ gauge fields (coupled to the scalars and the dilaton), we also get some non-trivial quadratic and quartic terms for the isometry gauge field B, due to the presence of the magnetic charges p^I in the angular part of (4.9). To simplify the treatment of the 2d theory, we will turn off all the magnetic charges, setting $p^I = 0$, thus getting the far simpler action:

$$S^{\rm 2d}_{\rm EM} = \frac{2\pi}{\kappa^2}\int\frac{\chi^{3/2}}{W_\infty}I_{IJ}f^I\wedge\star f^J\,. \tag{4.11}$$

This action simply describes 2d $U(1)$ gauge fields coupled to the scalars and the dilaton. As shown in section 2.4, it is still possible to have non-supersymmetric BPS-like black holes even without magnetic charges, so this setup is still general enough to investigate the mechanism that generates the different effective Schwarzians for supersymmetric and non-supersymmetric attractors. This way we are also free to chose whether to work with fixed electric chemical potential or fixed electric charges, provided we add the correct boundary terms to have a well defined variational principle.

At last, we conclude this section by combining together all the relevant contributions into a single 2d action:

$$\begin{aligned} S^{2\mathrm{d}} = \frac{2\pi}{\kappa^2} \int \star 1 \left(\chi R + \frac{2W_\infty}{\chi^{1/2}} - 2\chi\kappa^2 g_{i\bar{j}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} \right) \\ + \frac{2}{3} \frac{\chi^{5/2}}{W_\infty} \mathrm{Tr}\,(H \wedge \star H) + \frac{\chi^{3/2}}{W_\infty} I_{IJ} f^I \wedge \star f^J + S_\partial^{2\mathrm{d}} \,, \end{aligned} \tag{4.12}$$

where $S_\partial^{2\mathrm{d}}$ represents additional boundary terms dependent on the choice of ensemble for the black holes. In the near-AdS limit — which arises when considering the near-horizon limit of the near-extremal black holes — we can linearize the dilaton around a constant value,

$$\chi(x^\mu) \approx \chi_0 + 2G_N \Phi(x^\mu) + \dots \,, \tag{4.13}$$

thus obtaining the usual JT gravity [21, 22] coupled to additional scalars and gauge fields.

4.2 Boundary conditions for the metric

Even though (4.12) is simpler than the original 4d action (2.1), the theory as it is is far too complicated to be directly quantized. What one can do then is to separate the contribution of the action into two pieces: one coming from the near-horizon region (NHR) — where we consider the JT gravity limit of (4.12) at the quantum level — and the other coming from far away from the black hole (FAR), in which the curvature is small and thus we assume that the classical approximation is valid [7]. In principle, one cannot simply separate the two contributions: in the gravitational path integral, one should only set the boundary conditions at infinity, allowing the metric inside to fluctuate freely; this might also include different topologies and perhaps complex metrics [37, 38]. Fixing the metric at its classical value in the FAR region thus might not capture some of the quantum corrections due to gravity; however, given that the curvature is small away from the horizon, one expects this approximation to hold relatively well. Notice also that one could also expand around fluctuations of the FAR metric, computing perturbative loop corrections due to the gravitons in the FAR region.

In the NHR region, however, the classical curvature is not negligible and thus we have to consider the full quantum gravity. Fortunately, when expanding the dilaton as in (4.13), the action (4.12) simply becomes a generalization of JT gravity, which one can then quantize much more easily following different approaches [11, 23, 24, 8]. In the simplest approach, one must first go from the generalized JT gravity description (4.12) to the corresponding generalized Schwarzian; for example, the pure Schwarzian mode arises from the Gibbons–Hawking–York boundary term of pure JT gravity. Thus understanding the correct boundary conditions to impose at the boundary between the NHR and FAR regions (for the fields in the NHR action) is essential to obtain the correct description of the near-extremal black holes. Another way to see the importance of the boundary conditions is to rewrite the generalized JT gravity (4.12) as a BF theory (see sections 4.7 and 5.3) plus additional bulk contributions. A BF theory is a 2d topological gauge theory; its whole dynamics is therefore determined by the boundary conditions of the NHR region, highlighting once again how they encode the near-extremal degrees of freedom. In particular, as we will show in section 5.3, the bosonic fields that will be included into the BF description are essentially $g_{\mu\nu}, \Phi, a^I$ and B; in the supersymmetric case, there are also the additional fermions λ and Ψ, which are respectively the dilatinos and the gravitinos of the 2d theory. Therefore, we will be mainly concerned by their boundary conditions, without caring too much about the scalars and the gauginos.

Now, choosing the correct boundary conditions for the NHR region is not simple and there is no clear way to do so. Let us start from the metric $g_{\mu\nu}$ and the dilaton Φ. Ideally, we want to glue together the NHR and FAR spacetimes along their boundary [7]; given that the FAR region has a fixed spacetime, we want this boundary to be fixed. Hence we must impose the Dirichlet boundary conditions $\delta h_{uu}|_{r_\partial} = 0$ (with h_{uu} the intrinsic boundary metric) and $\delta\chi|_{r_\partial} = 0$, along a boundary set at a fixed distance $r = r_\partial$ close to the horizon in the FAR region, i.e. $r_\partial \sim -\infty$.

There is also another reason to justify these Dirichlet boundary conditions. At the classical level, when we calculate the entropy of the black holes from the Euclidean path integral, the extremal entropy essentially comes from the contribution of the horizon, i.e. the end of the "cigar" shaped spacetime (typical of black hole geometries). We therefore want the JT gravity obtained from the action (4.12) to produce exactly the extremal entropy $\mathcal{S}_* = \text{Area}/4G_N = \pi W_0^2/G_N$. As we will show in section 4.5, the extremal entropy contribution comes from rewriting — after expanding the dilaton — the 2d Einstein–Hilbert action in terms of the Euler characteristic of the 2d spacetime, using the Gauss–Bonnet theorem [39, 31]. In manifolds with a boundary, one also has to include the usual Gibbons–Hawking–York boundary proportional to the extrinsic curvature K to apply the Gauss–Bonnet theorem. Therefore one must have Dirichlet boundary conditions both for $g_{\mu\nu}$ and Φ. Since Φ is just a scalar field, we also assume that the z^i will themselves satisfy Dirichlet boundary conditions at the boundary, even though, as said before, we do not particularly care about them.

Let us now explicitly calculate how to join the FAR and NHR regions along their boundaries. The 4d metric in the FAR is given by the classical solution (2.8); comparison to (4.1) implies:

$$\begin{aligned} \chi &= \frac{e^{-2U}}{r^2}\,, \\ g_{\mu\nu}dx^\mu dx^\nu &= -\frac{e^{-U}}{W_\infty r}\left(-e^{2U}dt^2 + e^{-2U}\frac{dr^2}{r^4}\right)\,, \\ \sqrt{-g} &= -\frac{e^{-U}}{W_\infty r^3}\,. \end{aligned} \tag{4.14}$$

In order to perform the matching we expand near the boundary at $r = r_\partial$ the quantities above. In principle, to obtain an explicit expression for $U(r)$ in the FAR region, we should solve (2.24); doing so in full generality requires an explicit expression of W as a function of the z^i, which requires us to specify a particular model of $\mathcal{N} = 2$ supergravity (i.e. we need to pick the prepotential $F(X^I)$). To keep the discussion more general, we will employ the following approximation: we consider only solutions whose values of the fields at infinity z^i_∞ are already "close enough" to the values z^i_0 of the attractor. Let us quantify what this means. If we expand around $z^i(x^\mu) = z^i_0 + \delta z^i(x^\mu)$ and use the fact that z^i_0 are fixed points of the superpotentials W, we can expand the first line of (2.24) as:

$$U' = e^U \left(W_0 + \frac{1}{2} \partial^{(2)}_{(i)(j)} W_0 \delta z^{(i)} \delta z^{(j)} + \dots \right) , \tag{4.15}$$

where we use the notation $(i) = i, \bar{\imath}$ for the indices of the scalars to make the notation more compact. If we fix z^i_∞ such that

$$\frac{1}{2} \partial^{(2)}_{(i)(j)} W_0 \left(z^{(i)}_\infty - z^{(i)}_0 \right) \left(z^{(j)}_\infty - z^{(j)}_0 \right) \ll W_0 \, , \tag{4.16}$$

the second (and higher) order terms of (4.15) become negligible, and thus we can approximate

$$U' \approx e^U W_0 \tag{4.17}$$

throughout the whole solution (and not just in the NHR region, as in (3.1)). This equation is then solved by

$$e^{-U(r)} = -W_0 r + 1 \, , \tag{4.18}$$

which gives us an explicit expression of $U(r)$, independent from the specific supergravity model. Expanding near the boundary at $r = r_\partial$ thus yields[13]:

$$\begin{aligned} h_{uu} = g_{tt}|_{r_\partial} &\approx -\frac{1}{W_\infty W_0 r_\partial^2} + \dots \, , \\ \chi|_{r_\partial} &\approx W_0^2 - 2\frac{W_0}{r_\partial} + \dots \, . \end{aligned} \tag{4.19}$$

Notice also that, for constant radius surfaces of radius r_∂, the intrinsic boundary metric $h_{uu} du^2$ coincides with $g_{tt} dt^2$. In the NHR region, instead, we have a generalized JT gravity. The linearized dilaton acts as a Lagrange multiplier, enforcing a (near-)AdS_2 metric:

$$ds^2_{\mathrm{NHR}} = \ell^2 \frac{-dt^2 + dr^2}{r^2} \, , \tag{4.20}$$

where the AdS radius ℓ will be determined in section 4.5. As the dilaton moves towards the boundary, its value diverges in the near-AdS case [11]; to avoid this divergence, we cut the AdS_2 spacetime by adding a boundary, which acts as an IR cutoff. The boundary is chosen such that

[13] u is the intrinsic boundary time coordinate that parameterizes the boundary in both the FAR and NHR region.

— with Euclidean boundary time $u \in (0, \beta)$ — we have:

$$
\begin{aligned}
h_{uu} &= \frac{\ell^2}{\varepsilon^2}\,, \\
\chi|_{r_\partial} &= \chi_0 + 2G_N \frac{\Phi_{\rm ren}}{\varepsilon} + \ldots\,;
\end{aligned}
\tag{4.21}
$$

here we introduced the renormalized dilaton $\Phi_{\rm ren} = \varepsilon\ \Phi|_{r_\partial}$, which remains finite near the boundary. Finally, comparing (4.19) and (4.21) yields the matching conditions between the FAR and NHR regions:

$$
\begin{aligned}
\chi_0 &= W_0^2\,, \\
G_N \Phi_{\rm ren} &= W_0^{3/2} W_\infty^{1/2} \ell\,, \\
r_\partial &= \frac{\varepsilon}{W_0^{1/2} W_\infty^{1/2} \ell}\,.
\end{aligned}
\tag{4.22}
$$

As for the other scalars, we will assume that, due to the attractor mechanism in the FAR region, the fields at the boundary will be essentially at their fixed points, that is $z^i\big|_{r_\partial} \approx z_0^i$.

Now that we found the correct boundary conditions for the metric and the dilaton of the reduced action (4.12), let us briefly discuss the boundary terms that one needs to add to the action in the NHR region in order to obtain the correct variations. Notice that in principle we should do the same also for the FAR region action, in order to get the complete treatment of the black hole thermodynamics; however, since the contribution of the FAR region is just a shift in the extremal energy that does not modify the spectrum of the black hole (as we will briefly explain in section 4.3), we will just focus on the NHR region. When gluing together the NHR and FAR region, we impose Dirichlet boundary conditions fixing the boundary as a circle of fixed length:

$$
h_{uu}|_{r_\partial} = \frac{\ell^2}{\varepsilon^2} \implies \delta h_{uu}|_{r_\partial} = 0\,;
\tag{4.23}
$$

we thus need to add the usual Gibbons–Hawking–York boundary term to (4.12),

$$
S^{\rm 2d}_{\partial,{\rm grav}} = \frac{2\pi}{\kappa^2} \int du\, \sqrt{-h} \chi K\,,
\tag{4.24}
$$

where K is the extrinsic curvature of the boundary. For what concerns the dilaton χ and the scalars z^i, we also chose the Dirichlet boundary conditions:

$$
\begin{aligned}
\chi|_{r_\partial} &= W_0^2 + \frac{2G_N \Phi_{\rm ren}}{\varepsilon} \implies \delta\chi|_{r_\partial} = 0\,, \\
z^i\big|_{r_\partial} &= z_0^i \implies \delta z^i\big|_{r_\partial} = 0\,,
\end{aligned}
\tag{4.25}
$$

where z_0^i are the fixed points of the black hole attractor flow determined by which basin of attraction the scalar at infinity z_∞^i belong to. Therefore, we do not need to add boundary terms for the scalars in order to have a well defined variational problem.

4.3 Boundary conditions for the gauge fields

Picking the boundary conditions for the gauge fields is instead far trickier. In particular, the situation is rather problematic given the plethora of boundary conditions that we can already put on them at infinity. An approach to tackle this problem has been proposed by [7, 10, 8]: the idea is to set the boundary conditions at infinity and then "propagate" them to the NHR/FAR boundary using the equation of motions. To argue why this could be the case, let us try to translate this idea in terms of path integrals. The partition function of the original theory can be written heuristically as

$$\mathcal{Z} = \int_{\Psi(0)=\Psi_\infty} \mathcal{D}\Psi e^{iS^{4\mathrm{d}}(\Psi)}\,, \tag{4.26}$$

where we denote collectively all the fields with Ψ and the measure $\mathcal{D}\Psi$ includes the appropriate quotient over the gauge and (super-)diffeomorphisms groups. What we want to highlight is the fact that, since spacetime itself is dynamical, we can just set the boundary conditions at spatial infinity — collectively denoted as $\Psi(r=0)=\Psi_\infty$ — and thus we cannot in principle distinguish between the FAR and NHR region. However, if we ignore possible non-perturbative effects away from the horizon, we can expand perturbatively around the classical metric (3.3) in the FAR region. This allows us to identify a FAR region going from spatial infinity ($r=0$) down to a fixed arbitrary radius $r=r_\partial$, where the background spacetime is fixed and gravitons propagate. The remaining NHR region, instead, still contains a dynamic spacetime, and is joined with the FAR region at its boundary. In terms of path integrals, this can be expressed as:

$$\mathcal{Z} = \int \mathcal{D}\Psi_\partial \int_{\Psi^{\mathrm{NHR}}(r_\partial)=\Psi_\partial} \mathcal{D}\Psi^{\mathrm{NHR}} e^{iS^{4\mathrm{d}}\left(\Psi^{\mathrm{NHR}}\right)} \int_{\Psi^{\mathrm{FAR}}(0)=\Psi^{\mathrm{FAR}}_\infty}^{\Psi^{\mathrm{FAR}}(r_\partial)=\Psi_\partial} \mathcal{D}\Psi^{\mathrm{FAR}} e^{iS^{4\mathrm{d}}\left(\Psi^{\mathrm{FAR}}\right)}\,. \tag{4.27}$$

Now we employ the classical approximation in the FAR region. This amounts to simply solving the equation of motions of Ψ^{FAR} while imposing the boundary conditions $\Psi^{\mathrm{FAR}}(0)=\Psi^{\mathrm{FAR}}_\infty$, thus evaluating the action $S^{4\mathrm{d}}\left(\Psi^{\mathrm{FAR}}\right)$ on-shell. There is however another consequence due to the presence of the other boundary at $r=r_\partial$: the classical equations of motion will constrain the possible values of the field Ψ_∂ at the boundary between the NHR and FAR region to a subset $\Psi_\partial^{\mathrm{constr}}$. In formula this becomes:

$$\mathcal{Z} = e^{iS^{4\mathrm{d}}\left(\Psi^{\mathrm{FAR}}_{\mathrm{eom}}\right)} \int_{\Psi_\partial^{\mathrm{constr}}} \mathcal{D}\Psi_\partial \int_{\Psi^{\mathrm{NHR}}(r_\partial)=\Psi_\partial} \mathcal{D}\Psi^{\mathrm{NHR}} e^{iS^{4\mathrm{d}}\left(\Psi^{\mathrm{NHR}}\right)}\,. \tag{4.28}$$

The role of the FAR region is thus twofold: first, it provides a contribution $\exp\left[iS^{4\mathrm{d}}\left(\Psi^{\mathrm{FAR}}_{\mathrm{eom}}\right)\right]$ to the partition function; second, most importantly, it propagates the boundary conditions at infinity to the new boundary at $r=r_\partial$, altering the dynamic of the NHR modes. This is particularly important for the gauge fields, given the various possible choices for their boundary conditions already at infinity. Finally, notice that the contribution to the action of the FAR region just amounts to a shift in the extremal energy of the black hole, and thus cannot influence the near-extremal dynamics and in particular the presence of a mass gap. In fact, the classical black hole solutions we are interested in are static, and thus the integral in time of the action factors out; after Wick rotation, we therefore get an overall β factor multiplying the spatial integrals which are β independent. This means that

$$e^{-S^{4\mathrm{d}}\left(\Psi^{\mathrm{FAR}}_{\mathrm{eom}}\right)} = e^{-\beta\cdot\mathrm{const}}\,, \tag{4.29}$$

i.e. the on-shell action only changes the extremal energy. While we justified this procedure heuristically in 4d, [7, 8, 10] propose to apply it directly using the 2d action (4.12) (in place of the original 4d action (2.1)).

Let us now attempt to propagate the boundary conditions of the gauge fields, starting from the $U(1)$ gauge fields. The equations of motion given by (4.12) are:

$$d\left(\chi^{3/2} I_{IJ} \star f^I\right) = 0\,. \tag{4.30}$$

If we look for electrically charged solutions, these equations are solved by:

$$\begin{aligned} a^I &= \left({c_{(0)}}^I + \int_0^r dr'\, e^{2U}\left(I^{-1}\right)^{IJ} {c_{(1)}}_J\right) dt\,,\\ f^I &= -e^{2U}\left(I^{-1}\right)^{IJ} {c_{(1)}}_J dt\wedge dr\,; \end{aligned} \tag{4.31}$$

${c_{(0)}}^I$ and ${c_{(1)}}_J$ are two constants that must be determined by imposing the boundary conditions at infinity. Notice also that $a|_{r=0} = {c_{(0)}}^I$; in fact, the integral in the first line vanishes, provided that $\left(I^{-1}\right)^{IJ}$ reaches a finite value as $r\to 0$. Let us now impose the boundary conditions at infinity ($r\to 0$). If we fix the field strength at infinity — $\delta f^I|_\infty = 0$ — by specifying electric charges via (2.14), we have:

$${c_{(0)}}^I \ \text{ undetermined}\,, \qquad\qquad {c_{(1)}}_I = \frac{q_I}{4\pi}\,; \tag{4.32}$$

this means that fixing f^I at infinity also completely fixes f^I at the boundary between the NHR and FAR region as

$$4\pi\, f^I\big|_{r_\partial} = -\left[e^{2U}\left(I^{-1}\right)^{IJ}\right]\Big|_{r_\partial} q_J dt\wedge dr\,, \tag{4.33}$$

where in the second line we expanded for $r\to r_\partial$. Something different happens instead if we want to fix the holonomy along the imaginary time boundary at infinity. Using (2.16) we get:

$${c_{(0)}}^I = -i\left(\mu^I + \frac{2\pi i}{\beta} n^I\right) \equiv -i\mu^I_n \ , \qquad\qquad {c_{(1)}}_I \text{ undetermined}\,, \tag{4.34}$$

with $n^I\in\mathbb{Z}^I$ a vector of integers characterizing solutions differing from large gauge transformations. The constant ${c_{(1)}}_I$ is left undetermined, and we can thus express it in terms of f^I and then plug it back into the expression (4.31) of a^I. Looking at the boundary $r = r_\partial$, we now have — fixing the holonomy at infinity — that:

$$\oint_{r=r_\partial} a^I - \int_{r\in(r_\partial,0)} f^I = -i\mu_n^I \beta = \oint_{r=0} a^I \,. \tag{4.35}$$

Fixing the holonomies of the gauge fields at infinity thus does not fix the holonomies at the boundary between NHR and FAR, but rather fixes a particular combination of the electromagnetic potentials and the field strengths.

Let us now work out the correct boundary terms for (4.12), which are needed to obtain the correct variational problem. Let us start from fixing the charges q_I at infinity: expanding around $r \to r_\partial$ equation (4.33), we have

$$4\pi \star f^I\big|_{r_\partial} \approx -\frac{W_\infty}{W_0^3}\left(I_0^{-1}\right)^{IJ} q_J + \dots \implies \delta f^I\big|_{r_\partial} = 0\,. \tag{4.36}$$

The correct boundary term one needs to add is then:

$$S_{\partial,\mathrm{EM}}^{2\mathrm{d},\delta f} = -\frac{4\pi}{\kappa^2}\int_\partial \frac{\chi^{3/2}}{W_\infty} I_{IJ} a^I \wedge \star f^J = \frac{q_I}{\kappa^2}\int_\partial a^I = \frac{q_I}{\kappa^2}\int f^I \,, \tag{4.37}$$

where in the last equivalence we used Stoke's theorem. If we instead fix the chemical potentials μ^I at infinity, we must impose (4.35) at the boundary between FAR and NHR; expanding around $r \to r_\partial$ — using (3.2) and $\chi^2 = W_0^2 + \dots$ — we have:

$$a^I\big|_{r_\partial} + \frac{W_0^2}{W_\infty} \star f^I\big|_{r_\partial} dt \approx -i\mu_n^I dt + \dots \implies \left[\delta a^I + \frac{W_0^2}{W_\infty}\star \delta f^I dt\right]\Bigg|_{r_\partial} = 0\,. \tag{4.38}$$

These mixed boundary conditions are a hybrid between fixing the holonomy (which requires no additional boundary term) and fixing the field strength (see (4.37)). The correct boundary term one needs to add to the action is:

$$S_{\partial,\mathrm{EM}}^{2\mathrm{d},\delta a} = \frac{2\pi}{\kappa^2}\int_\partial du\, \frac{W_0^5}{W_\infty} I_{0IJ} \star f^I \star f^J \,. \tag{4.39}$$

Finally, we will repeat the same steps for the $SU(2)$ gauge field; since we will be mainly interested in fixing the angular velocity of the black holes, we will focus directly on the case of fixed holonomy at infinity. The equations of motion for B are

$$D\left(\chi^{5/2}\star H\right) = d\left(\chi^{5/2}\star H\right) - \chi^{5/2}[B, \star H] = 0\,. \tag{4.40}$$

Searching for black holes solutions rotating along the polar axis[14], we find:

$$\begin{aligned} B &= \left(\tilde{c}_{(0)} + \int_0^r dr'\, e^{4U} r'^2 \tilde{c}_{(1)}\right) dt\, T_3\,, \\ H &= -e^{4U} r^2 \tilde{c}_{(1)} dt \wedge dr\, T_3\,. \end{aligned} \tag{4.41}$$

Fixing the holonomy of B at infinity, i.e. the angular velocity, via

$$\mathcal{P}\left(e^{\oint B}\right) = e^{-2i\Omega\beta T_3}\,, \tag{4.42}$$

gives

$$\tilde{c}_{(0)} = 2\left(\Omega + \frac{2\pi i}{\beta} n\right) \equiv 2\Omega_n \quad , \qquad\qquad \tilde{c}_{(1)} \text{ undetermined}\,. \tag{4.43}$$

Fixing the holonomy at infinity thus implies the following matching condition:

$$\oint_{r=r_\partial} B - \int_{r\in(r_\partial,0)} H = 2\Omega_n \beta T_3 \;=\oint_{r=0} B; \tag{4.44}$$

notice that this condition is exactly the same as (4.35), the one for $U(1)$ gauge fields, despite the differences in the action (4.12) between the two types of gauge fields and the presence of the scalars in the $U(1)$ gauge kinetic functions. Expanding (4.44) around $r \to r_\partial$ yields:

$$B|_{r_\partial} + \frac{W_0^2}{3W_\infty} \star H|_{r_\partial} dt \approx 2\Omega_n dt\, T_3 + \ldots \implies \left[\delta B + \frac{W_0^2}{3W_\infty} \star \delta H dt\right]\Bigg|_{r_\partial} = 0\,; \tag{4.45}$$

the boundary term needed for the correct variational principle is thus:

$$S^{2\mathrm{d},\delta B}_{\partial,SU(2)} = \frac{4\pi}{9\kappa^2} \int_\partial du\, \frac{W_0^7}{W_\infty^2} \operatorname{Tr}\left[(\star H)^2\right]\,. \tag{4.46}$$

Although this procedure of "propagating" the boundary conditions seems fine, it provides quite problematic results. In fact, if we continue with the analysis of the theory (4.12) using the steps of the next sections (4.4, 4.5 and 4.6), what we get with these boundary conditions is not the bosonic part of the $\mathcal{N} = 4$ super-Schwarzian, despite what is claimed in [8, 10][15]. While this is in stark contrast with the analysis purely based on symmetry breaking considerations of section 3.3, in principle this might just signal that these symmetry considerations are not enough to capture the near-extremal black hole dynamics. There is another reason, however, to believe that the propagated boundary conditions (4.44) are wrong. The generalized Schwarzian

[14] We always have the freedom to rotate (in 4d) or gauge transform (in 2d) to get black holes rotating along the polar axis.

[15] While [8] does claim that one gets the bosonic part of the $\mathcal{N} = 4$ super-Schwarzian, there are many typos and inconsistencies in the paper; in particular the boundary conditions for B are different in different formulas (see page 15 versus equation (4.48) at page 51) and there are some missing 2 factors in the treatment of the B fields (see equations (4.35) and (4.37) from pages 46/47). Therefore we will proceed differently in our discussion of the boundary conditions, even though this point requires certainly further investigation.

that one obtains is in fact of the form:

$$S = -\frac{1}{M_{PSU(1,1|2)}} \int_0^\beta du \, \mathrm{Sch}\left(\tan\frac{\pi\tau(u)}{\beta}, u\right) - \frac{1}{2}\,\mathrm{Tr}\left[\left(g^{-1}(u)g'(u)\right)^2\right] , \tag{4.47}$$

which, as we already said, is different from the $\mathcal{N} = 4$ super-Schwarzian action (3.19). The issue with this action is that, apart to the factor of $1/2$ in front of the trace, we also get an additional $-$ sign; this gives the wrong sign to the kinetic term of the particle moving in $SU(2)$, making the action unphysical. The origin of this paradoxical behavior is unclear to us; one possible source for the problem might be the fact that, while we are working in the 2d theory, the separation in NHR and FAR region should be analyzed in the full 4d supergravity, in which the B field becomes itself part of the metric. Notice also that we could check the validity of the procedure of propagating the boundary condition by applying it to the metric $g_{\mu\nu}$ and then checking whether or not we recover (4.22) (even though we will not do so in this work).

Therefore, we must find another way to pick the boundary conditions for the gauge fields. As we will show in section 5.3, we argue that the boundary conditions for the B field — once fixing the holonomy at infinity — should be:

$$\begin{gathered} B|_{r_\partial} - \frac{W_0^2}{6W_\infty} \star H|_{r_\partial}\, dt \approx 2\Omega_n dt\, T_3 + \ldots \implies \left[\delta B - \frac{W_0^2}{6W_\infty} \star \delta H dt\right]\Big|_{r_\partial} = 0\,, \\ S^{2\mathrm{d},\delta B}_{\partial,SU(2)} = -\frac{2\pi}{9\kappa^2}\int_\partial du\, \frac{W_0^7}{W_\infty^2}\,\mathrm{Tr}\left[(\star H)^2\right] ; \end{gathered} \tag{4.48}$$

they differ from (4.45) only for the factor of $-1/2$ in front of $\star H$, needed to recover the bosonic part of the $\mathcal{N} = 4$ super-Schwarzian. In the following, we will just assume that (4.48) holds true, leaving the explanation for this (as of now arbitrary) choice to section 5.3.

Finally, the problems arising in the treatment of the B field at fixed holonomy suggests that the propagation for the $U(1)$ gauge fields — namely (4.35) and (4.33) — might also be incorrect. In particular, from the B field results, we can argue that the procedure is somewhat problematic, but at least provide the right kind of mixed boundary conditions, although off by few numerical factors. We are substantially interested in just fixing the charges of the black holes, rather than the electric potentials; since the boundary condition $\delta f^I = 0$ is independent on eventual proportionality constants, we might think that (4.33) still holds true. We also point out that, since there are no charged objects in the FAR region, all the charges measured at the near horizon must be the same as the one measured at infinity. This is exactly what the boundary conditions (4.33) implies, and thus we will assume in the following that they remain valid, even though a more accurate analysis of the boundary conditions of the gauge fields is needed to better validate the results.

4.4 Simplifying the action

We now have to find a way to quantize the action (4.12). Before doing so, however, we will further simplify (4.12) in order to get a more manageable problem. First off, since we are interested in computing the 1-loop quantum corrections, it is convenient to linearly expand the scalars and the gauge fields around their classical values. As for the scalars, we will expand them as

$$z^i(x^\mu) \to z_0^i + z^i(x^\mu)\,, \tag{4.49}$$

where we will only keep terms up to order $O\left((z^i)^2\right)$ in the action. Notice that we avoid calling the fluctuations δz^i — and simply use z^i instead — in order to make the notation less cluttered; we will do the same for the other fields we expand. As for the $SU(2)$ gauge field, since we are expanding around extremal configurations which are not rotating, the classical values of the field is $B = 0$; the expansion is thus simply:

$$\begin{aligned} &B(x^\mu) \to 0 + B(x^\mu)\,, \\ &H(x^\mu) \to 0 + H(x^\mu) = 0 + (dB - B \wedge B) = dB\,. \end{aligned} \tag{4.50}$$

Finally, for the $U(1)$ gauge fields, we will focus for simplicity on the case where we fix the electric charges of the black holes. In particular, spherical symmetry — together with (2.14) — dictates

$$\star f^I\big|_{\text{eom}} = -W_\infty \chi^{-3/2} I_0^{-1\,IJ} \frac{q_J}{4\pi} \tag{4.51}$$

throughout the classical solution. Therefore we expand:

$$\begin{aligned} &a^I \to a^I_{\text{eom}} + a^I\,, \\ &\star f^I \to \star d\left(a^I_{\text{eom}} + a^I\right) = -W_\infty \chi^{-3/2} \left(I_0^{-1}\right)^{IJ} \frac{q_J}{4\pi} + \star f^I\,, \end{aligned} \tag{4.52}$$

where we left the classical value of the gauge fields a^I_{eom} implicit. Finally, notice that we did not use the explicit expression of the metric $g_{\mu\nu}$ and of the dilaton χ to expand the scalars and the gauge fields; therefore, we can safely leave them in their fully non-linear expression, making it easier to recognize and treat JT gravity. Plugging the expansions back into the respective actions, and using the fact that the attractor solution is not only a fixed point of W but also of V_{BH} (i.e. the classical equation of motion), we get the action for the gauge fields and the scalars (where we include the appropriate boundary terms):

$$\begin{aligned}
S = &-\frac{4\pi}{\kappa^2}\int \star 1 \left(W_\infty W_0^2 \chi^{-3/2} + g_{0 i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}}\right) \\
&+\frac{2\pi}{\kappa^2}\int W_\infty^{-1}\chi^{3/2} I_{0IJ} f^I \wedge \star f^J - \frac{4\pi}{\kappa^2}\int \frac{q_I}{4\pi} {I_0^{-1}}^{IJ} \partial_{(i)} I_{0JK} z^{(i)} f^K \\
&-\frac{2\pi}{\kappa^2}\int \star 1\, W_\infty \chi^{-3/2} M^2_{(i)(j)} z^{(i)} z^{(j)} \\
&-\frac{8\pi}{3\kappa^2 W_\infty}\int \chi^{5/2}\,\mathrm{Tr}\,(H\wedge\star H) - \frac{2\pi}{9\kappa^2}\int_\partial du\, \frac{W_0^7}{W_\infty^2}\,\mathrm{Tr}\left[(\star H)^2\right] ,
\end{aligned} \tag{4.53}$$

where the index (i) runs over $(i) = i, \bar{i}$, $z^{(i)} = (z^i, \bar{z}^{\bar{i}})$ and we defined the scalar mass matrix

$$M^2_{(i)(j)} \coloneqq \frac{1}{2}\left(\partial^{(2)}_{(i)(j)} I_{0IJ}\right)\left({I_0^{-1}}^{IK}\frac{q_K}{4\pi}\right)\left({I_0^{-1}}^{JL}\frac{q_L}{4\pi}\right) . \tag{4.54}$$

First off, on the first line we see that the expansion generates a term proportional to

$$W_0^2 = V_{\mathrm{BH}}|_{r\to-\infty} = -\frac{1}{2}\left(I_0^{-1}\right)^{IJ}\frac{q_I}{4\pi}\frac{q_J}{4\pi} ; \tag{4.55}$$

this term essentially acts as a cosmological constant due to the fixed charges of the black hole, and it is the dominant term that fixes the near-AdS background. Then we have, apart from the usual kinetic terms for the scalars and the $U(1)$ gauge fields, a mass term ($\sim z^{(i)} z^{(j)}$) for the scalars proportional to the charges $\sim {q_I}^2$ and an interaction term between the scalars and the gauge fields ($\sim z^{(i)} f^K$); both of them are due to the presence of the scalars in the gauge kinetic functions of the action (4.10). The interaction term appears to greatly increase the complexity of the dynamics, coupling the gauge fields and the scalars fluctuations together. We can however work around this problem by defining a new composite "$U(1)$ field strength":

$$\widetilde{f}^I = f^I - W_\infty \chi^{-3/2}\frac{q_K}{4\pi}\partial_{(i)}\left(I_0^{-1}\right)^{KI} z^{(i)} \star 1 . \tag{4.56}$$

In terms of this new field, we can rewrite the second and third line of (4.53) as:

$$\begin{aligned}
&+\frac{2\pi}{\kappa^2}\int W_\infty^{-1}\chi^{3/2} I_{0IJ}\widetilde{f}^I \wedge \star \widetilde{f}^J \\
&-\frac{2\pi}{\kappa^2}\int \star 1 W_\infty \chi^{-3/2}\widetilde{M}^2_{(i)(j)} z^{(i)} z^{(j)} ,
\end{aligned} \tag{4.57}$$

with the new scalar mass matrix

$$\widetilde{M}^2_{(i)(j)} = \partial^{(2)}_{(i)(j)}\, V_{\mathrm{BH}}|_{r\to-\infty} . \tag{4.58}$$

While this may seem just a way to rewrite the action, this allows us to essentially reabsorbe the $\sim z^{(i)} f^I$ interaction, leaving only the additional mass term for the scalars. The contribution of the $SU(2)$ gauge field, instead, is left unchanged[16]. Considering the B field as a fluctuation will however be important in section 5.1, where it will simplify the dimensional reduction of the fermions. Finally, in terms of the fluctuations around the classical solutions, we can express the boundary conditions (4.33), (4.44) and (4.25) as[17]:

$$\begin{aligned} z^i\big|_{r\partial} &= 0\,, \\ \widetilde{f}^I\Big|_{r\partial} &= 0\,, \\ B|_{r\partial} &= \frac{W_0^2}{6W_\infty}\,\star H|_{r\partial}\,du + 2\Omega_n du\,T_3\,. \end{aligned} \tag{4.59}$$

Next, to further simplify the treatment of the gauge fields in the action (4.53), we will employ a typical trick used to deal with 2d gauge theories [40]. What one can do is to rewrite the action in terms of the Lie-algebra-valued scalars $\star\widetilde{f}^I$ and $\star H$ in place of the field strength $\widetilde{f}^I$ and H. To do so, we introduce the Lie-algebra-valued Lagrange multipliers φ^I and b respectively, related to $\star\widetilde{f}^I$ and $\star H$ by:

$$\begin{aligned} \star\widetilde{f}^I &= 2G_N W_\infty \chi^{-3/2}\varphi^I\,, \\ \star H &= 3G_N W_\infty \chi^{-5/2} b\,. \end{aligned} \tag{4.60}$$

In terms of these Lagrange multipliers, we can rewrite the contributions of the gauge fields to the action (together with their boundary conditions) as:

$$\begin{aligned} S =& \frac{2\pi}{\kappa^2}\int W_\infty^{-1}\chi^{3/2} I_{0IJ}\widetilde{f}^I\wedge\star\widetilde{f}^J \\ =& \int I_{0IJ}\varphi^I\widetilde{f}^J + G_N W_\infty\int \star 1\,\chi^{-3/2} I_{0IJ}\varphi^I\varphi^J\,, \\ \varphi^I\big|_{r\partial} =& 0\,, \end{aligned} \tag{4.61}$$

and

$$\begin{aligned} S =& \frac{4\pi}{3\kappa^2 W_\infty}\int \chi^{5/2}\,\mathrm{Tr}\,(H\wedge\star H) - \frac{2\pi}{9\kappa^2}\int_\partial du\,\frac{W_0^7}{W_\infty^2}\,\mathrm{Tr}\left[(\star H)^2\right] \\ =& \int \mathrm{Tr}\,[bH] + \frac{3}{2}G_N W_\infty\int \star 1\,\chi^{-5/2}\,\mathrm{Tr}\left[b^2\right] - \frac{G_N}{4W_0^3}\int_\partial du\;\mathrm{Tr}\left[b^2\right]\,, \\ B|_{r\partial} =& \frac{G_N}{2W_0^3}\,b|_{r\partial}\,du + 2\Omega_n du\,T_3\,. \end{aligned} \tag{4.62}$$

[16] In principle we should replace $H = dB - B\wedge B \to dB$ in the action if we expand up to second order, but essentially this does not simplify further our treatment.

[17] Notice that we deliberately confuse the boundary time u (which coincide for both the NHR and FAR region) with the time t in the FAR region, since the two match at the boundary.

4.5 Extracting the boundary modes

We are now finally ready to combine all the pieces back together and extract the boundary modes that describe the near-extremal dynamics. To do so, we merge the purely gravitational terms of the 2d action (4.12) with the action (4.53) written in terms of the Lagrange multipliers φ^I and b. In order to recover JT gravity, we linearize the dilaton in all the terms of the action as in (4.13). The bosonic part of the final 2d action dictating the near-extremal dynamics is:

$$\begin{aligned} S^{2\mathrm{d}}_{\mathrm{bos}} =& \frac{W_0^2}{G_N}\pi\chi(\mathcal{M}) + \int \star 1\,\frac{\Phi}{2}\left(R + \frac{2}{\ell^2}\left(1+\Delta_{\mathrm{bos}}\right)\right) + \int_\partial du\,\sqrt{-h}\;\Phi|_{r_\partial} K \\ &+ \int \star 1\left(-4\pi W_0^2 g_{0i\bar{j}}\partial^\mu z^i \partial_\mu \bar{z}^{\bar{j}} - \frac{1}{4G_N\ell^2}\widetilde{M}^2_{(i)(j)} z^{(i)} z^{(j)}\right) \\ &+ \int I_{0IJ}\varphi^I \widetilde{f}^J + \int \star 1\,\frac{G_N}{\ell^2} I_{0IJ}\varphi^I\varphi^J \\ &+ \int \mathrm{Tr}\,[bH] + \frac{3}{2}G_N W_\infty \int \star 1\,\chi^{-5/2}\,\mathrm{Tr}\left[b^2\right] - \frac{G_N}{4W_0^3}\int_\partial du\;\mathrm{Tr}\left[b^2\right]\,. \end{aligned} \tag{4.63}$$

Here we defined the AdS radius[18]

$$\ell = \frac{W_0^{3/2}}{W_\infty^{1/2}}\,, \tag{4.64}$$

and the dynamical correction to the AdS radius

$$\begin{aligned} \Delta_{\mathrm{bos}} =& -\frac{6G_N}{W_0^2} I_{0IJ}\varphi^I\varphi^J + \frac{3G_N}{W_0^2} I_{0IJ}\varphi^I \frac{q_K}{4\pi}\partial_{(i)}\left(I_0^{-1}\right)^{KI} z^{(i)} - \frac{15G_N}{W_0^4}\,\mathrm{Tr}\left[b^2\right] \\ &- 16\pi G_N \ell^2 g_{0i\bar{j}}\partial^\mu z^i \partial_\mu \bar{z}^{\bar{j}} + \frac{6}{W_0^2}\widetilde{M}^2_{(i)(j)} z^{(i)} z^{(j)}\,; \end{aligned} \tag{4.65}$$

we also have that

$$2\pi\chi(\mathcal{M}) = \frac{1}{2}\int d^2x\,\sqrt{-g}\,R + \int_\partial du\,\sqrt{-h}\,K \tag{4.66}$$

is the Euler characteristic of the 2d spacetime [39, 41, 31]. Starting from the top, we have the term proportional to $\chi(\mathcal{M}) = 1$ (for a disk) that provides the classical extremal entropy $\mathcal{S}_*$ of the black hole:

$$\mathcal{S}_* = \pi\frac{W_0^2}{G_N} = \frac{1}{4G_N}\left(4\pi W_0^2\right) = \frac{\text{Area}}{4G_N}\,. \tag{4.67}$$

Next, we have the bulk term proportional to the dilaton Φ; the dilaton acts as a Lagrange multiplier, forcing the spacetime to be (near-)AdS. The dominant contribution to the AdS

[18] The mismatch between this AdS radius $\ell = W_0^{3/2}/W_\infty^{1/2}$ and $\ell = W_0$ (from (3.3)) is just a consequence of the different normalizations of the 2d metrics (along t and r) of (4.1) and (3.3). This actually gives us a way to check if our calculations make sense. In particular, we expect the near-extremal dynamics in the NHR to be dependent only on the properties of the near-horizon, that is on the charges of the black hole and the values of the scalars at the attractor of the flow, and thus on W_0 only. While $\ell = W_0^{3/2}/W_\infty^{1/2}$ depends also on W_∞, we will show in section 4.6 that the whole near-extremal dynamics depends only on the length scale Φ_{ren}. Calculating Φ_{ren} using (4.22), we see that the dependence on W_∞ cancels, since $\Phi_{\mathrm{ren}} = W_0^3/G_N$, leaving all the dependence only on W_0 (as expected) and giving us a simple check that our calculations make sense.

radius is given by the classical part of the $U(1)$ gauge fields, i.e. from the fixed electric charges of the black hole; the corrections contained in Δ_{bos} instead come from all the fluctuations around the background values of the other fields. Next, we have the Gibbons–Hawking–York boundary term $\sim \Phi K$, which contains the Schwarzian mode dominating the gravitational dynamics in the near-extremal limit. Finally, we have the contributions of n_V free massive scalars, n_V+1 $U(1)$ gauge fields and an $SU(2)$ gauge field (together with its boundary term).

To further simplify (4.63), we will adopt one more approximation, namely the macroscopic or large mass (or charge) limit [8]. Since our original supergravity theory only holds down to length scales of the order of the Planck length $\ell_P \sim \sqrt{G_N}$, we can only trust it when analyzing extremal black holes whose horizon is much larger than the Planck length, i.e. in the limit

$$W_0, W_\infty, \ell \gg \sqrt{G_N}\,. \tag{4.68}$$

In this limit, we have two major simplifications in the action (4.63). First, the dynamical corrections to the AdS radius become suppressed: $\Delta_{\text{bos}}/\ell^2 \ll 1$; approximating $\Delta_{\text{bos}} \approx 0$, we see that Φ now forces properly the spacetime to be locally AdS, with a fixed AdS radius ℓ. Notice that, apparently, the contributions of the scalars is not suppressed; however, after we rescale $g_{0i\bar{j}} \to g_{0i\bar{j}}/W_0^2$ and $\widetilde{M}^2_{(i)(j)} \to G_N \ell^2 \widetilde{M}^2_{(i)(j)}$ to get properly normalized kinetic and mass terms for the $z^{(i)}$, we see that the scalars contributions to Δ_{bos} are in fact suppressed in the macroscopic limit. The second simplification is that the bulk terms proportional to $\varphi^I \varphi^J$ and b^2 are also suppressed[19] with respect to the terms $\varphi^I \widetilde{f}^J$ and bH. With these approximations, the action (4.63) can be divided[20] into a purely topological term $S^{\text{top}}_{\text{bos}}$ — which contains JT gravity together with the gauge fields — and a more usual bulk contribution $S^{\text{bulk}}_{\text{bos}}$:

$$\begin{aligned}
S^{\text{2d}}_{\text{bos}} =& S^{\text{top}}_{\text{bos}} + S^{\text{bulk}}_{\text{bos}}\,,\\
S^{\text{top}}_{\text{bos}} =& \int \star 1\, \frac{\Phi}{2}\left(R + \frac{2}{\ell^2}\right) + \int_\partial du\,\sqrt{-h}\ \Phi|_{r_\partial}\, K\\
&+ \int I_{0IJ}\varphi^I \widetilde{f}^J + \int \operatorname{Tr}\left[bH\right] - \frac{G_N}{4W_0^3}\int_\partial du\ \operatorname{Tr}\left[b^2\right]\,,\\
S^{\text{bulk}}_{\text{bos}} =& \int \star 1\left(-4\pi W_0^2 g_{0i\bar{j}}\partial^\mu z^i \partial_\mu \bar{z}^{\bar{j}} - \frac{1}{4G_N\ell^2}\widetilde{M}^2_{(i)(j)} z^{(i)} z^{(j)}\right)\,.
\end{aligned} \tag{4.69}$$

However, the presence of scalars — whether massive or massless — can only modify the extremal entropy and energy of the black holes, without actually modifying the other properties of the spectrum. As explained in section (4.3) of [7] (see also [42]), the 1-loop determinant of the scalars can be computed using the Gelfand-Yaglom method [43]. The result of this calculation is that the only temperature dependent contribution of the scalars 1-loop determinant to the partition

[19] We will however keep the boundary term proportional to b^2 since it is the only non-zero term in the action and it is not dominated by any other term.

[20] From now on we will forget the term proportional to $\chi(\mathcal{M})$, since it only contributes to the extremal entropy and it does not influence the qualitative features of the spectrum.

function is independent of the masses of the scalars[21]; therefore, one can then calculate it using massless scalars. Massless scalars on a Poincaré disk can then be treated as a 2d Conformal Field Theory (CFT). The end result is that — after appropriate renormalization — they only contribute to the extremal entropy (i.e. they provide just a constant term), and thus do not modify the temperature dependence of the partition function [7]; for this reason, the whole near-extremal dynamic is determined just by $S^{\text{top}}_{\text{bos}}$. Finally, since there are no boundary terms related to $\widetilde{f}^I$ in the action, integrating out φ^I eliminates the bulk contribution without leaving any boundary term coming from the $U(1)$ gauge fields; the $\widetilde{f}^I$ do not influence the spectrum and thus we will ignore them in the following analysis.

4.6 Analysis of the bosonic boundary modes

Let us now finally analyze the remaining contribution to the topological 2d action $S^{\text{top}}_{\text{bos}}$, which, after Wick rotating, becomes:

$$\begin{aligned} S^{\text{top}}_{\text{bos}} = &- \int \star 1 \, \frac{\Phi}{2} \left(R + \frac{2}{\ell^2} \right) - \int_\partial du \, \sqrt{-h} \;\, \Phi|_{r_\partial} K \\ &- i \int \text{Tr}\,[bH] + \frac{G_N}{4W_0^3} \int_\partial du \;\, \text{Tr}\left[b^2\right] \; . \end{aligned} \tag{4.70}$$

The most straightforward way to show that $S^{\text{top}}_{\text{bos}}$ is topological is to integrate out the Lagrange multipliers Φ and b. This way, all the 2d bulk integrals are set to 0 and we are left with just the boundary terms:

$$S = -\int_0^\beta du \, \sqrt{-h} \, \left(\Phi|_{r_\partial} K - \frac{G_N}{4W_0^3} \, \text{Tr}\left[b^2\right] \right) \; . \tag{4.71}$$

We will now cast the above action as (the bosonic part of) the generalized Schwarzian theory which controls the dynamics of near-extremal black holes.

Starting from the Gibbons–Hawking–York boundary term, we follow the usual procedure for obtaining the Schwarzian action from JT gravity [11, 12]. We parameterize the boundary of AdS, with metric

$$ds^2_{\text{AdS}} = g_{\mu\nu} dx^\mu dx^\nu = \ell^2 \frac{dt^2 + dr^2}{r^2} \, , \tag{4.72}$$

using

$$ds^2_\partial = \ell^2 \frac{r'(u)^2 + t'(u)^2}{r(u)^2} du^2 \, ; \tag{4.73}$$

here $t(u)$ is a monotonic function that is always growing, with $t' > 0$ always. Let us now impose the boundary condition (4.22):

$$ds^2_\partial = \frac{\ell^2}{\epsilon^2} du^2 \, . \tag{4.74}$$

[21] Dependence on the scalar masses only enters as a shift in extremal entropy, which we do not care about.

If we make the change of variable

$$t(u) = \tan\frac{\pi\tau(u)}{\beta}\,, \tag{4.75}$$

where $\tau(u)$ is thus an element of Diff(S^1), we have

$$K = 1 + \frac{\varepsilon^2}{\ell}\text{Sch}\left(\tan\frac{\pi\tau(u)}{\beta}, u\right) + \dots\,. \tag{4.76}$$

Notice that the $K = 1 + \dots$ gives a divergent contribution proportional to β when inserted into the action (4.71); this divergent contribution can however be simply removed by a local counterterm, and thus we are free to ignore it. Plugging everything back into the Gibbons–Hawking–York boundary term yields the effective boundary action

$$S = -\Phi_{\text{ren}}\int_0^\beta du\,\text{Sch}\,(t(u), u) = -\Phi_{\text{ren}}\int_0^\beta du\,\text{Sch}\left(\tan\frac{\pi\tau(u)}{\beta}, u\right)\,, \tag{4.77}$$

with the renormalized dilaton

$$\Phi_{\text{ren}} = \frac{W_0^3}{G_N} \tag{4.78}$$

controlling the energy scale associated to the Schwarzian effective theory.

As for the $SU(2)$ component of the boundary term (4.71), we can rewrite it in terms of the B field by using (4.62) (after Wick rotation) as:

$$S = \frac{G_N}{4W_0^3}\int_0^\beta du\ \text{Tr}\left[b^2\right] = -\Phi_{\text{ren}}\int_0^\beta du\ \text{Tr}\left[\left(B_u + 2i\Omega_n T_3\right)^2\right]\,. \tag{4.79}$$

This is useful because from (4.70) we see that integrating out b forces $H = 0$, i.e. the $SU(2)$ gauge field becomes pure gauge. Near the boundary we thus have

$$B_u = -\widetilde{g}^{-1}(u)\partial_u\widetilde{g}(u)\,, \tag{4.80}$$

where $\widetilde{g}(u)$ maps u to an $SU(2)$ matrix in the fundamental representation, implying:

$$S = -\Phi_{\text{ren}}\int_0^\beta du\ \text{Tr}\left[\left(B_u + 2i\Omega_n T_3\right)^2\right] = -\Phi_{\text{ren}}\int_0^\beta du\ \text{Tr}\left[\left(\widetilde{g}^{-1}(u)\widetilde{g}'(u) - 2i\Omega_n T_3\right)^2\right]\,. \tag{4.81}$$

Finally, defining $g(u) \coloneqq \exp\left(2i\Omega_n T_3 u\right)\widetilde{g}(u)$, we can rewrite the above action as

$$S = -\Phi_{\text{ren}}\int_0^\beta du\ \text{Tr}\left[\left(g^{-1}(u)g'(u)\right)^2\right]\,, \tag{4.82}$$

where $g(u)$ now has the periodicity condition:

$$g(\beta) = e^{-2i\Omega_n T_3\beta}g(0)\,. \tag{4.83}$$

We have therefore managed to rewrite the boundary action (5.21) as the following generalized Schwarzian theory:

$$S_{\text{eff}}^{\text{bos}} = -\Phi_{\text{ren}} \int_0^\beta du \left(\text{Sch}\left(\tan \frac{\pi\tau}{\beta}, u \right) + \text{Tr}\left[\left(g^{-1} g' \right)^2 \right] \right) ; \tag{4.84}$$

in particular, since we started from the bosonic part of the original 4d supergravity, we should interpret this as the bosonic part of the effective generalized Schwarzian theory which describes the near-extremal dynamics. The action above is exactly the action of a Schwarzian boundary mode together with a particle moving on a $SU(2)$ group manifold (with a non-zero holonomy), which together describe the symmetry breaking of the bosonic isometry group $SL(2,\mathbb{R}) \times SU(2) \to \varnothing \times U(1)$, as explained in section 3.3. Notice that the action (4.84) does not distinguish between supersymmetric and non-supersymmetric attractors; this is to be expected, since the bosonic symmetry breaking pattern is the same in both for the supersymmetric and non-supersymmetric case. This means, however, that in order to check whether the supersymmetric and non-supersymmetric spectra differ we must also perform the dimensional reduction of the fermionic sector of the supergravity (2.1). We also see that, as argued once again in section 3.3, the energy scales associated to the $SL(2,\mathbb{R})$ and $SU(2)$ symmetry breaking are related to each other as expected from the supersymmetric $\mathcal{N}=4$ super-Schwarzian, even in the non-supersymmetric case; in particular they are related to the renormalized value of the dilaton at the boundary:

$$M_{PSU(1,1|2)} = M_{SL(2)} = M_{SU(2)} = \frac{1}{\Phi_{\text{ren}}} = \frac{G_N}{W_0^3} . \tag{4.85}$$

Finally, notice that the $U(1)$ gauge fields have no impact on the near-extremal dynamics, as long as we fix the electric charge of the black hole. Had we fixed the chemical potentials μ^I instead, there would be two main differences with respect to the previous analysis. First, there would be another additional boundary term[22]:

$$S \sim \frac{G_N}{W_0} \int_0^\beta du \, I_{0IJ} \varphi^I \varphi^J . \tag{4.86}$$

After integrating out the φ^I and imposing the pure gauge condition $f^I = 0$, this would result in a new boundary term[23]:

$$S \sim -\frac{W_0}{G_N} \int_0^\beta du \, I_{0IJ} \alpha^{I\prime}(u) \alpha^{J\prime}(u) , \tag{4.87}$$

where the functions $\alpha^I(u)$ — analogously to the $g(u)$ — satisfy $a_u^I = i\partial_u \alpha^I(u)$. These boundary modes describe $n_V + 1$ $U(1)$ symmetry breakings; notice that the energy scale of these symmetry

[22] Modulo the additional complication of understanding the correct mixed boundary conditions for the $U(1)$ gauge fields, since we might encounter a similar problem to that of the $SU(2)$ gauge field of section 4.3.

[23] Note that technically to obtain the following boundary term we also expanded the chemical potential μ^I around its background value.

breakings is not given simply by G_N/W_0, since we also have the explicit appearance of I_{0IJ} in the action. A similar $U(1)$ boundary mode, describing the breaking of only one $U(1)$, arises — without the dependence on I_{0IJ} — in the bosonic part of the $\mathcal{N}=2$ super-Schwarzian [44]; for its application to the description of near-BPS black holes, see [9]. The second difference with respect to the action (4.84) is however quite problematic: changing boundary conditions flips the sign of the cosmological constant $+2/\ell^2 \to -2/\ell^2$ in the term proportional to Φ. Therefore the spacetime is not AdS anymore, it becomes dS instead; this makes the interpretation of the JT gravity somewhat troublesome, since heuristically one identifies the AdS_2 background of JT gravity with the near-horizon AdS_2 throat of the extremal black hole. A possible interpretation of this peculiar behavior is that when we calculate the partition function and we fix the holonomy, already at the classical level we are summing over all the possible values of the charges q_I. This way, we are including in the sum also black holes that do not respect the macroscopic limit (and thus Δ_{bos} cannot be neglected), and even black holes whose horizon is comparable to the Planck length $\ell_P \sim \sqrt{G_N}$; hence the supergravity description breaks down. When we instead fix the charges and consider the macroscopic limit, we are sure that the black holes that we sum over in the partition function have a horizon which is much greater than ℓ_P, and we are thus in a regime where the supergravity still holds. Notice that a somewhat similar problem seems to arise in[24] [7]; in that instance, when one consider the partition function for fixed $U(1)$ chemical potentials, the sum over the possible charges diverges due to the breakdown of the $(q_I)^2 \gg G_N$ approximation. The problem there is then avoided considering only black holes in an asymptotically AdS background with a finite value of the AdS radius, which cannot be sent to infinity to recover the flat space limit (or otherwise one gets a divergent result).

4.7 Boundary action as a bosonic BF theory

We will now reanalyze the action (4.70) from a different point of view: we will show how the generalized JT gravity can be rewritten as a BF theory, a 2d topological gauge theory [25]. This will become much more useful once we perform the dimensional reduction of the fermions in order to understand the fermionic part of the generalized Schwarzian theory, shedding also some light on the (as of now) arbitrary choice of the boundary conditions (4.62) for the $SU(2)$ gauge field.

A 2d BF theory is a purely topological gauge theory of the form (in the Euclidean) [40, 41]

$$S^{\mathrm{BF}} = -i \int \mathrm{Tr}\,[\mathcal{B}\mathcal{F}] + S_\partial\,; \tag{4.88}$$

here $\mathcal{F} = d\mathcal{A} - \mathcal{A}\wedge\mathcal{A}$ is the field strength of a gauge theory with group $\mathcal{G}$ (with generators $\mathcal{T}_i$), $\mathcal{B}$ is a Lie-algebra-valued Lagrange multiplier and S_∂ represent a generic boundary term depending on the chosen boundary conditions. Integrating over the Lagrange multiplier $\mathcal{B}$ forces the constraint $\mathcal{F}=0$, making the gauge field pure gauge. It also sets the bulk term to 0; the

[24] In particular, see the beginning of page 27 of [7].

whole dynamics thus depends only on the boundary term S_∂ and the theory becomes topological. Notice that we can define the trace starting from the quadratic Casimir C_2 — $C_2 = g^{AB}\mathcal{T}_A\mathcal{T}_B$ for some metric g^{AB} — and defining $\mathrm{Tr}\,[\mathcal{T}_A\mathcal{T}_B] := \left(g^{AB}\right)^{-1}$ [8]. Notice also that the BF theory (4.88) can be easily generalized for supergroups, as we will do in section 5.3.

If we look at the second line of (4.70), we see that the $SU(2)$ contribution is already expressed as a BF theory. Therefore we just need to rewrite the first line of $S^{\text{top}}_{\text{bos}}$ as a BF theory, i.e. the gravitational contribution; we will do so following [24]. The BF theory which is equivalent to the gravitational part of $S^{\text{top}}_{\text{bos}}$ is based on the non-compact gauge group $SL(2,\mathbb{R})$; notice that the equivalence requires passing from the second order to the first order formulation of the JT gravity. $SL(2,\mathbb{R})$ is generated by the three generators L_0, L_{+1} and L_{-1} satisfying:

$$[L_m, L_n] = (m-n)\,L_{m+n}\,, \qquad\qquad C_2^{SL(2)} = 2L_0^2 - \{L_{+1}, L_{-1}\}\,. \tag{4.89}$$

It is convenient to redefine the basis of generators as

$$P_0 := L_0\,, \qquad P_1 := -\frac{L_{+1}-L_{-1}}{2}\,, \qquad J := \frac{L_{+1}+L_{-1}}{2}\,, \tag{4.90}$$

such that

$$\begin{gathered}[P_a, P_b] = \varepsilon_{ab} J\,, \qquad [P_a, J] = \varepsilon_{ab} P_b\,,\\ C_2^{SL(2)} = 2P_aP_a - 2J^2\,,\end{gathered} \tag{4.91}$$

where $a = 0, 1$ and we sum over repeated indices. To recover the gravitational description, it is sufficient to expand the $SL(2,\mathbb{R})$ gauge field $\mathcal{A}_{SL(2)}$ and Lagrange multiplier $\mathcal{B}_{SL(2)}$ as:

$$\mathcal{A}_{SL(2)}(x) = \frac{1}{\ell}e^a(x)P_a - \omega(x)J \qquad \mathcal{B}_{SL(2)}(x) = 2\ell\phi^a(x)P_a - 2\phi(x)J\,, \tag{4.92}$$

where e^a are the 2d zweibein and $\omega \equiv \omega^1{}_2 = -\omega^2{}_1$ is the 2d spin connection. Plugging (4.92) into the BF action (4.88), we get:

$$S^{\text{BF}}_{SL(2)} = -i\int -\phi\left(d\omega + \frac{1}{\ell^2}e^0\wedge e^1\right) + \phi^a\left(de^a + \varepsilon_{ab}\,\omega\wedge e^b\right)\,. \tag{4.93}$$

Integrating out ϕ^a forces the torsionelss condition needed to recover the second order formulation of the JT gravity. Integrating out ϕ instead imposes the spacetime to be AdS, as it happens for JT gravity; this can be seen using the relation $d\omega = R\,e^1\wedge e^2/2$ valid for 2d manifolds [24], which implies:

$$d\omega + \frac{1}{\ell^2}e^0\wedge e^1 = \star 1\,\frac{1}{2}\left(R + \frac{2}{\ell^2}\right)\,. \tag{4.94}$$

Notice in particular that, to recover JT gravity, the dilaton is related to the Lagrange multiplier by $\phi = i\Phi$.

We still need to express the Gibbons–Hawking–York boundary term as a function of $\mathcal{A}_{SL(2)}$

and $\mathcal{B}_{SL(2)}$. As pointed out in [24], this boundary term is equivalent to introducing a line defect in the BF theory; this is in turn equivalent to imposing mixed boundary conditions which relate the gauge field and its field strength. The Schwarzian action can be obtained from the $SL(2,\mathbb{R})$ BF theory provided that we add the contribution of a line defect[25] of the form:

$$S_\partial = \int_0^\beta du\ \mathrm{Tr}\left[\mathcal{B}^2_{SL(2)}\right] . \tag{4.95}$$

To see this, one partially integrates out the $\mathcal{A}_{SL(2)}$ gauge field along the boundary, solving the equations of motion along the u direction:

$$\begin{aligned} D_u\left(\mathcal{B}_{SL(2)}\right) =&\partial_u \mathcal{B}_{SL(2)} - \left[\mathcal{A}^{SL(2)}_u, \mathcal{B}_{SL(2)}\right] = 0 \\ =&\partial_u \mathcal{B}_{SL(2)} - t'(u)\left[\mathcal{A}^{SL(2)}_t, \mathcal{B}_{SL(2)}\right] = 0\,. \end{aligned} \tag{4.96}$$

Here $t(u)$ is exactly the boundary parameterization (4.75); ${e^a}_t$ and ω_t in $\mathcal{A}^{SL(2)}_u$ are constant along the boundary[26], and can calculated from (4.73). To solve these equations, it is convenient to combine $\phi^\pm := \phi^0 \pm i\phi^1$ and ${e^\pm}_t := {e^0}_t \pm i{e^1}_t$; this way we can express (4.96) as:

$$\begin{aligned} \partial_u \phi^\pm \pm t'\left(-i\phi^\pm \omega_t + \frac{i}{\ell^2}\phi {e^\pm}_t\right) = 0\,, \\ \partial_u \phi + \frac{it'}{2}\left(\phi^- {e^+}_t - \phi^+ {e^-}_t\right) = 0\,. \end{aligned} \tag{4.97}$$

The equations for $\partial_u\phi^-$ and $\partial_u \phi$ are solved by:

$$\begin{aligned} \phi^- &= \Lambda t' {e^-}_t\,, \\ \phi &= \Lambda\left(\omega_t t' - i\frac{t''}{t'}\right)\,, \\ \phi_+ &= \frac{\Lambda}{\ell^2}\left({e^+}_t t' + \frac{2\ell^2}{{e^-}_t t'^3}\left(t''^2 - t't''' - i\omega_t t'^2 t''\right)\right)\,, \end{aligned} \tag{4.98}$$

where Λ is a constant. Plugging them into the equation for $\partial_u\phi^+$ yields the additional constraint:

$$t'''' - 4\frac{t''t'''}{t'} + 3\frac{t''^3}{t'^2} = 0\,, \tag{4.99}$$

which is exactly the equation of motion for a Schwarzian theory. To recover the Schwarzian action, we plug the partially integrated out fields (4.98) into the line defect contribution (4.95), obtaining:

$$S_\partial = \int du\ \mathrm{Tr}\left[\mathcal{B}^2_{SL(2)}\right] = -4\Lambda^2 \int du\, \mathrm{Sch}\left(t,u\right)\,. \tag{4.100}$$

[25] We choose the defect of length β and really close the boundary of the spacetime. Notice that it can always be moved near the boundary thanks to the topological properties of the theory away from the defect.

[26] Even if they were not, they can be set this way — without loss of generality — owing once again to the topological nature of the BF theory [24].

To find out the value of Λ, let us relate the Lagrange multiplier ϕ with the dilaton Φ. If we compare (4.93) and (4.70), we naively obtain the relation $\phi = i\Phi$; however, looking at (4.70), we see that in order for Φ to enforce an AdS spacetime once integrated out, it must be integrated over imaginary values — instead of real values – when we are working in the Euclidean. Therefore, we should fix its value at the boundary in the Euclidean not to $\Phi_{\text{ren}}/\varepsilon$, but rather to[27] $-i\Phi_{\text{ren}}/\varepsilon$. By using the relations $t' \approx 1$ and $t'' \approx 0$ — obtained from (4.73) — and applying them to (4.98), we obtain the following relation:

$$\phi \approx \frac{\Lambda}{\varepsilon} = i\Phi \qquad \Longrightarrow \qquad \Lambda = \Phi_{\text{ren}} \,. \tag{4.101}$$

By rescaling (4.100), we can see that the line defect makes the dynamics equivalent to the one the Schwarzian mode coming from the Gibbons–Hawking–York boundary term:

$$S_\partial = \frac{1}{4\Phi_{\text{ren}}} \int_0^\beta du \ \text{Tr}\left[\mathcal{B}^2_{SL(2)}\right] = -\Phi_{\text{ren}} \int_0^\beta du \, \text{Sch}\left(t(u), u\right) \,. \tag{4.102}$$

In summary, we have that the BF action

$$S^{\text{BF}}_{SL(2)} = -i \int \text{Tr}\left[\mathcal{B}_{SL(2)}\mathcal{F}_{SL(2)}\right] + \frac{1}{4\Phi_{\text{ren}}} \int_0^\beta du \ \text{Tr}\left[\mathcal{B}^2_{SL(2)}\right] \tag{4.103}$$

is equivalent — boundary terms included — to the one of the JT gravity in (4.70). Notice also that while we introduced the boundary term (4.102) as a line defect contribution, it is essentially the same as modifying the boundary conditions for the $\mathcal{A}_{SL(2)}$ and $\mathcal{B}_{SL(2)}$ fields. In particular, varying (4.103), we see that the variational problem is well defined with the following boundary conditions:

$$2i\Phi_{\text{ren}} \, \delta\mathcal{A}_{SL(2)}\big|_\partial - \delta\mathcal{B}_{SL(2)} du\big|_\partial = 0 \,. \tag{4.104}$$

Finally, let us put together the gravitational and $SU(2)$ contributions, in order to express the whole (4.70) as a single $SL(2,\mathbb{R}) \times SU(2)$ BF theory. We can write the complete algebra of the $SL(2,\mathbb{R}) \times SU(2)$ BF theory, with its Casimir, as:

$$\begin{gathered} [P_a, P_b] = \varepsilon_{ab} J \,, \qquad [P_a, J] = \varepsilon_{ab} P_b \,, \qquad [T_i, T_j] = -\varepsilon_{ijk} T_k \\ C_2^{SL(2)\times SU(2)} = 2P_a P_a - 2J^2 - 2T_i T_i \,. \end{gathered} \tag{4.105}$$

The choice of the gauge field and Lagrange multiplier is:

$$\mathcal{A}_{SL(2)\times SU(2)} = \frac{1}{\ell} e^a P_a - \omega J + B^i T_i \qquad \mathcal{B}_{SL(2)\times SU(2)} = 2\ell\phi^a P_a - 2\phi J + b^i T_i \,. \tag{4.106}$$

[27] The sign in front of the i is not important.

Plugging everything in (4.88) yields:

$$S^{\mathrm{BF}}_{SL(2)\times SU(2)} = -i\int -\phi\left(d\omega + \frac{1}{\ell^2}e^0\wedge e^1\right) + \phi^a\left(de^a + \varepsilon_{ab}\,\omega\wedge e^b\right) + \mathrm{Tr}\,[bH] + S_\partial\,; \tag{4.107}$$

here we deliberately confuse the generators T_i of the $SU(2)$ algebra with their representatives $T_i = i\sigma_i/2$ in the fundamental representation, such that $H = H^iT_i$, $b = b^iT_i$ and $\mathrm{Tr}[T_iT_j] = -\delta_{ij}/2$. To find S_∂, we use the boundary conditions (4.104) and (4.48); the latter can be rewritten in terms of the BF theory fields B and b as:

$$2i\Phi_{\mathrm{ren}}\,\delta B|_\partial - \delta b\,du|_\partial = 0\,. \tag{4.108}$$

Notice that the conditions above closely resemble (4.104); this is not by chance, as we will explain in section 5.3. Putting everything back together, we can finally rewrite the action (4.70) — along with the correct boundary terms that reproduces the generalized Schwarzian (4.84) — as:

$$\begin{aligned} &S^{\mathrm{BF}}_{SL(2)\times SU(2)} = -i\int \mathrm{Tr}\left[\mathcal{B}_{SL(2)\times SU(2)}\mathcal{F}_{SL(2)\times SU(2)}\right] + \frac{1}{4\Phi_{\mathrm{ren}}}\int_0^\beta du\;\mathrm{Tr}\left[\mathcal{B}^2_{SL(2)\times SU(2)}\right]\,,\\ &2i\Phi_{\mathrm{ren}}\,\delta\mathcal{A}_{SL(2)\times SU(2)}\big|_\partial - \delta\mathcal{B}_{SL(2)\times SU(2)}du\big|_\partial = 0\,. \end{aligned} \tag{4.109}$$

5 The complete 1d boundary theory

5.1 Dimensional reduction of the fermions

As we have shown above in section 4, studying only the dimensional reduction of the bosons is not enough to distinguish between the behavior of the supersymmetric and non-supersymmetric attractors. Therefore we will now perform the dimensional reduction of the fermionic terms of the original supergravity action (2.1). Given the complexity of performing a complete dimensional reduction of the fermionic terms, we will proceed as in section 4.4: we focus only on the terms that are quadratic in the fluctuations of the fermions above their background. Since the classical solution is fully bosonic — i.e. the fermionic background is vanishing — we only need to consider terms that are quadratic in the fermions, replacing all the other (bosonic) fields with their classical values. This kills the interactions the fermions have with the scalars and $SU(2)$ gauge field, as well as the four-fermions terms.

Let us now start from the dimensional reduction of the gravitinos contributions. Their 4d kinetic term, at the quadratic level, is given by the first line of $S^{\mathrm{4d}}_{\mathrm{ferm}}$ from (2.2):

$$e^{-1}\mathcal{L} = -\kappa^{-2}\overline{\psi}_{AM}\Gamma^{MNP}\nabla_N\psi^A{}_P\,. \tag{5.1}$$

In order to gauge the superisometry of the $\mathrm{AdS}_2\times S^2$ solutions, it is convenient to combine the two spinors ψ^1 (left-handed) and ψ_2 (right-handed) into a single Dirac spinor, similarly to what is done in (3.7) (in section 3.2) in order to solve the Killing spinor equation. More explicitly, following [26, 27], we decompose:

$$\begin{aligned} \psi^1{}_m + ie^{i\arg \mathcal{Z}_0}\psi_{2m} &= \sqrt{\frac{2}{\ell}}\chi^{1/4}\Psi^k{}_m \otimes \eta_k \,, \\ \psi^1{}_a + ie^{i\arg \mathcal{Z}_0}\psi_{2a} &= \sqrt{\frac{2}{\ell}}\frac{G_N}{W_0^2}\chi^{1/4}\lambda^k \otimes \gamma_{\widehat{a}}\eta_k \,. \end{aligned} \tag{5.2}$$

The 2d spinors $\Psi^k{}_m(x^\mu)$ and $\lambda^k(x^\mu)$ are respectively the 2d gravitinos and dilatinos. The index $k = 1, 2$ — which is summed over[28] — labels the Killing spinors η_k of the sphere S^2:

$$\eta_1(\theta,\phi) = e^{\frac{i}{2}\theta\sigma_1}e^{\frac{1}{2}\phi\sigma_{13}}\begin{pmatrix}1\\ i\end{pmatrix}\,, \qquad \eta_2(\theta,\phi) = e^{\frac{i}{2}\theta\sigma_1}e^{\frac{1}{2}\phi\sigma_{13}}\begin{pmatrix}1\\ -i\end{pmatrix}\,; \tag{5.3}$$

they satisfy the Killing spinor equations

$$\partial_\theta\eta_k = \frac{i}{2}\sigma_1\eta_k \,, \qquad \left(\partial_\phi - \frac{1}{2}\cos\theta\sigma_{13}\right)\eta_k = \frac{i}{2}\sin\theta\sigma_3\eta_k \,, \tag{5.4}$$

which are exactly the angular part of (3.10). As a consequence, any spinor of the form $(\cdots)^k \otimes \eta_k$ satisfies[29]:

$$e_a{}^\alpha\nabla_\alpha(\cdots)^k\otimes\eta_k = \left(\frac{i}{2}\Gamma_{\widehat{a}}\gamma_3 + \frac{1}{4}W_\infty^{-1/2}\chi^{3/4}\partial_m\log\chi\Gamma_{\widehat{a}}\Gamma^{\widehat{m}}\right)(\cdots)^k\otimes\eta_k \,. \tag{5.5}$$

Notice that the $\chi^{1/4}$ in (5.2) have been chosen such that the factors of $\partial_m \log\chi$ — coming from the spin connection (4.5) — cancel; the other numerical prefactors have been chosen future convenience, in order to obtain a final action which is as similar as possible to $\mathcal{N} = 4$ JT supergravity. Notice also that while the two radial components are independent, the two spherical components of the 4d gravitinos are related to each other.

We can now plug (5.2) into the Lagrangian (5.1). Defining the 1-form $\Psi_k := \Psi_{k\mu}dx^\mu$, such that

$$\overline{\Psi}_k \wedge \Psi^{k'} = \overline{\Psi}_{km}\Psi^{k'}{}_n e^m \wedge e^n = \overline{\Psi}_{km}\Psi^{k'}{}_n\varepsilon_{mn}\sqrt{-g}\,d^2x \,, \tag{5.6}$$

and integrating over θ and ϕ using

$$\int_{S^2} d\Omega\, \eta_k{}^\dagger\eta^{k'} = 8\pi\delta_k^{k'} \,, \tag{5.7}$$

[28] The position of the index k — downstairs or upstairs — is not important and we will move it freely; we will keep k upstairs for spinors and downstairs for their conjugates for consistency. When two of such indices are contracted, we will sum over them as usual.

[29] One might expect also a term proportional to B from the spin connection (4.5); however, it is negligible in the small B approximation at the order we are working at (and would provide in (5.19) the correct gravitinos-B interaction that appears in (5.35)).

yields the following contribution to the 2d action:

$$
\begin{aligned}
S_\psi^{\rm kin} &= \int \frac{2}{\ell}\frac{W_0^2}{G_N}\frac{\chi}{W_0^2} i\overline{\Psi}_k \wedge \Psi^k \\
&+ \int -\left[2\frac{\chi}{W_0^2}\overline{\lambda}_k \left(\nabla_{(g)} - \frac{1}{2\ell}\left(\frac{\chi}{W_0^2}\right)^{-3/4}\gamma_3 \not{e}\wedge\right)\Psi^k + \text{h.c.}\right] ,
\end{aligned}
\tag{5.8}
$$

where

$$
\nabla_{(g)} = d - \frac{1}{2}\omega_{(g)}\gamma_3\wedge \tag{5.9}
$$

is the gravitational covariant exterior derivative and $\not{e} := \gamma^{\widehat{m}} e_m$.

Next, we have the interaction term between the gravitinos and the graviphoton field strength:

$$
\begin{aligned}
e^{-1}\mathcal{L} &= -\kappa^{-2} F_{\mu\nu}^{-I} I_{IJ} X^J \left(\overline{(\psi^{1\mu})^c}\psi_2{}^\nu - \overline{\psi_2{}^\mu}\left(\psi^{1\nu}\right)^c\right) + \text{h.c.} \\
&= \frac{1}{4\kappa^2} T^{-\mu\nu}\left(\overline{\left(\psi^1{}_\mu\right)^c}\psi_{2\nu} - \overline{\psi_{2\mu}}\left(\psi^1{}_\nu\right)^c\right) + \text{h.c.} \\
&= \frac{1}{4\kappa^2}\left(T^{-\mu\nu}\overline{\left(\psi^1{}_\mu\right)^c}\psi_{2\nu} - T^{+\mu\nu}\overline{(\psi_{2\mu})^c}\psi^1{}_\nu\right) + \text{h.c.} .
\end{aligned}
\tag{5.10}
$$

Notice that this piece depends explicitly on the graviphoton field strength T^-, and thus can distinguish whether we are working in the near-horizon of a BPS or fake-BPS black hole. As explained before, we can simply replace $T_-^{\mu\nu}$ with its classical value (3.5):

$$
\begin{aligned}
4\pi T^- &= i\kappa^2 \mathcal{Z}\left(1 + i\star\right)\left(\sin\theta d\theta \wedge d\phi\right) , \\
T^{-\widehat{m}\widehat{n}} &= -\frac{\kappa^2}{4\pi}\mathcal{Z}\chi^{-1}\varepsilon^{\widehat{m}\widehat{n}} = \frac{\kappa^2}{4\pi}\mathcal{Z}\chi^{-1}\varepsilon_{mn} ,
\end{aligned}
\tag{5.11}
$$

with the central charge $\mathcal{Z}_0 = 2\kappa^{-2} q_I X_0{}^I$. Plugging in the dimensional reduction ansatz (5.2) and integrating over θ and ϕ produces the following contribution to the 2d action:

$$
S_\psi^{\rm EM} = \int -\zeta \frac{2}{\ell}\frac{W_0^2}{G_N}\left(\frac{\chi}{W_0^2}\right)^{1/2} i\overline{\Psi}_k \wedge \Psi_k . \tag{5.12}
$$

We see that the parameter ζ (introduced in (3.11)), which controls whether the attractor is supersymmetric ($\zeta = 1$) or not ($0 \le \zeta < 1$), appears explicitly in the action; this was to be expected, given the explicit appearance of the graviphoton field strength in the interaction.

Next, let us focus on the gauginos $\xi_A{}^i$; we choose the dimensional reduction ansatz:

$$
\begin{aligned}
\xi^{1\bar{i}} &= \sqrt{\frac{\ell}{2}} W_0^{1/4}\chi^{-3/8}\Xi^{k\bar{i}} \otimes \eta_k , \\
ie^{i\arg\mathcal{Z}_0}\xi_2{}^i &= \sqrt{\frac{\ell}{2}} W_0^{1/4}\chi^{-3/8}\Xi^{ki} \otimes \eta_k ,
\end{aligned}
\tag{5.13}
$$

where $\Xi^{k\bar{i}}(x^\mu)$ and $\Xi^{ki}(x^\mu)$ are 2d spinors and we will use once again the notation $(i) = i, \bar{\imath}$ to collectively denote both components. Notice that once again the factor $\chi^{-3/8}$ is needed to cancel the $\partial_m \log \chi$ factors coming from the spin connection. The dimensional reduction of the kinetic term of the gauginos,

$$e^{-1}\mathcal{L} = -\frac{1}{4} g_{0i\bar{j}} \overline{\xi}_A{}^i \nabla\!\!\!\!/\, \xi^{A\bar{j}} + \text{h.c.}\,, \tag{5.14}$$

yields[30]:

$$\begin{aligned} S_\xi^{\text{kin}} = &-\pi W_0^2 \int \star 1\, g_{0(i)(j)} \overline{\Xi}_k{}^{(i)} \nabla\!\!\!\!/_{(g)} \Xi^{k(i)} \\ &- \pi W_0^2 \int \star 1 \left(\frac{\chi}{W_0^2}\right)^{-3/4} g_{0(i)(j)} \overline{\Xi}_k{}^{(i)} \frac{i}{\ell} \gamma_3 \Xi^{k(i)}\,, \end{aligned} \tag{5.15}$$

where $\nabla\!\!\!\!/_{(g)} = \gamma^{\widehat{m}} \nabla_{(g)m}$ is the (slashed) 2d gravitational covariant derivative. Finally, we are left with the interaction term between gauginos, gravitinos and the graviphoton field strength:

$$e^{-1}\mathcal{L} = \kappa^{-2} F^{-I\,MN} I_{0IJ} \overline{\nabla}_{\bar{i}} \overline{X}^J \overline{\xi}^{A\bar{i}} \Gamma^M \psi^{BN} \varepsilon_{AB} + \text{h.c.}\,. \tag{5.16}$$

By using the holomorphicity of the central charge — $\overline{\nabla}_{\bar{i}} \mathcal{Z} = 0$ — and plugging in the ansatz for the dimensional reduction, we get the interaction terms:

$$\begin{aligned} S_\xi^{\text{EM}} = &\int i\partial_{(i)} |\mathcal{Z}_0| \frac{W_0}{\ell} \left(\frac{\chi}{W_0^2}\right)^{-3/8} \overline{\Xi}_k{}^{(i)} e\!\!\!/ \wedge \Psi^k + \text{h.c.} \\ &+ \int \star 1\, i\partial_{(i)} |\mathcal{Z}_0| \frac{G_N}{W_0 \ell} \left(\frac{\chi}{W_0^2}\right)^{-9/8} \overline{\Xi}_k{}^{(i)} \lambda^k + \text{h.c.}\,. \end{aligned} \tag{5.17}$$

We notice that once again, as expected, these terms distinguish between supersymmetric and non-supersymmetric attractors. In particular, supersymmetric attractors are fixed points of the attractor flow satisfying $\partial_i |\mathcal{Z}_0| = \partial_{\bar{i}} |\mathcal{Z}_0| = 0$, while this is not true for the non-supersymmetric case.

5.2 Simplifying the action

At last, we can finally put together the four contributions (5.8), (5.12), (5.15) and (5.17); linearizing the dilaton around (4.13), we get the following fermionic terms to add to the 2d

[30] $\overline{\Xi}_k{}^{(i)}$ denotes the Dirac conjugate of the spinor, as for the gravitinos $\overline{\Psi}_k$.

bosonic action[31] (4.63):

$$\begin{aligned}
S^{\text{2d}}_{\text{fer}} = &\int \star 1\, \frac{\Phi}{2}\left(-\frac{4}{\ell}\,(2-\zeta)\, i \star \left(\overline{\Psi}_k \wedge \Psi^k\right) + \frac{2}{\ell^2}\Delta_{\text{fer}}\right) \\
&+ \int \frac{2}{\ell}\frac{W_0^2}{G_N}\,(1-\zeta)\, i\overline{\Psi}_k \wedge \Psi^k + \left(-2\overline{\lambda}_k \mathcal{D}\Psi^k + \text{h.c.}\right) \\
&+ \int \star 1\, \left(-\pi W_0^2 g_{0(i)(j)} \overline{\Xi_k}^{(i)} \left(\nabla\!\!\!\!/ + \frac{i}{\ell}\gamma_3\right) \Xi_k{}^{(j)}\right) \\
&+ \int i\frac{W_0}{\ell}\partial_{(i)}|\mathcal{Z}_0| \overline{\Xi_k}^{(i)} \left(e\!\!\!/ \wedge \Psi^k + \frac{G_N}{W_0^2}\lambda^k \star 1\right) + \text{h.c.}\,,
\end{aligned} \tag{5.18}$$

where Δ_{fer} are the fermionic dynamical corrections to the AdS radius (which we will not write down explicitly) and

$$\mathcal{D} = d - \frac{1}{2}\gamma_3 \omega \wedge - \frac{1}{2\ell}\gamma_3 e\!\!\!/ \wedge \tag{5.19}$$

is the $PSU(1,1|2)$ gauge covariant derivative. Before discussing the different behavior of the fermions for supersymmetric and non-supersymmetric attractors, let us impose the macroscopic limit, so that the dynamical fermionic corrections to the AdS radius Δ_{ferm} become negligible. Notice that the macroscopic limit also makes the interaction term $\sim \overline{\Xi_k}^{(i)}\lambda^k$ small; however, we still keep it in the action as it makes it easier to identify the role of the dilatinos in the near-extremal dynamics. Putting together (4.63) and (5.18), we get the total action that determines, at the quadratic level, the dynamics of the near-extremal black holes:

$$\begin{aligned}
S^{\text{2d}} =& S^{\text{top}} + S^{\text{bulk}}\,, \\
S^{\text{top}} =& \int \star 1\, \frac{\Phi}{2}\left(R + \frac{2}{\ell^2} - \frac{4}{\ell}(2-\zeta) i \star \left(\overline{\Psi}_k \wedge \Psi^k\right)\right) \\
&- 2\int \left(\overline{\lambda}_k \mathcal{D}\Psi^k + \text{h.c.}\right) + \int I_{0IJ}\varphi^I \widetilde{f}^J + \int \text{Tr}\,[bH] + S^{\text{top}}_{\partial}\,, \\
S^{\text{bulk}} =& \int \star 1\, \left(-4\pi W_0^2 g_{0i\bar{j}} \partial^\mu z^i \partial_\mu \bar{z}^{\bar{j}} - \frac{1}{4G_N \ell^2}\widetilde{M}^2_{(i)(j)} z^{(i)} z^{(j)}\right) \\
&+ \int \star 1\, \left(-\pi W_0^2 g_{0(i)(j)} \overline{\Xi_k}^{(i)} \left(\nabla\!\!\!\!/ + \frac{i}{\ell}\gamma_3\right) \Xi^{k(j)}\right) \\
&+ \frac{2W_0^2}{G_N \ell}(1-\zeta)\int i\overline{\Psi}_k \wedge \Psi^k \\
&+ \partial_{(i)}|\mathcal{Z}_0| \int \overline{\Xi_k}^{(i)} \left(e\!\!\!/ \wedge \Psi^k + \frac{2G_N}{W_0 \ell}\lambda^k \star 1\right) + \text{h.c.}\,;
\end{aligned} \tag{5.20}$$

where $S^{\text{top}}_{\partial}$ is the boundary term for the action S^{top}, which is given by

$$S^{\text{top}}_{\partial} = \int_{\partial} du \sqrt{-h}\ \Phi|_{r_\partial} K - \frac{G_N}{4W_0^3}\int_{\partial} du\ \text{Tr}\left[b^2\right] + (\text{fermions})\,. \tag{5.21}$$

[31] From now on, we drop the subscript (g) on the spin connection $\omega_{(g)}$ and the covariant derivative $\nabla_{(g)}$.

As we did for (4.69), we split the contributions to the 2d action into two pieces: $S^{\rm top}$ is a fully topological theory, since integrating out the Lagrange multipliers Φ, λ_k, φ^I and b leaves only the boundary terms; $S^{\rm bulk}$ is instead a usual 2d action, which essentially contributes only to the extremal entropy and energy of the system, as it was for the bulk term in (4.69). Looking at (5.20), we see that the fermionic terms do actually distinguish between the supersymmetric and non-supersymmetric attractors. In particular, what really differentiates between the attractors are the last two lines of $S^{\rm bulk}$ in (5.20): both terms vanish when the attractor is supersymmetric — given that $\zeta = 1$ and $\partial_i|\mathcal{Z}_0| = \partial_{\bar{i}}|\mathcal{Z}_0| = 0$ — and become relevant only for non-supersymmetric attractors. The main fermions that are responsible for the different behaviors in the two cases are the gravitinos Ψ_k and the dilatinos λ^k (which are the corresponding Lagrange multipliers). In fact, while all the other fields can be separated[32] into "topological" fields and "bulk" fields, this is not the case for Ψ^k and λ^k, given that they appear in both $S^{\rm top}$ and $S^{\rm bulk}$.

$$
\begin{array}{ccccc}
\textbf{Topological} & & & & \textbf{Bulk} \\
g_{\mu\nu}, \Phi, a^I, \varphi^I, B, b & \xleftarrow{\text{SUSY}} & \Psi^k, \lambda^k & \xrightarrow{\text{non-SUSY}} & z^{(i)}, \Xi^{k(i)}
\end{array}
$$

The actual roles of Ψ^k and λ_k are determined by whether or not we are expanding around a supersymmetric attractor. For a supersymmetric attractor, the last two lines of $S^{\rm bulk}$ in (5.20) become 0, and thus the gravitinos and the dilatinos are fully "topological" fields; the final action which determines the dynamics thus becomes:

$$
\begin{aligned}
S^{\rm 2d}_{\rm SUSY} =& S^{\rm top}_{\rm SUSY} + S^{\rm bulk}_{\rm SUSY}\,, \\
S^{\rm top}_{\rm SUSY} =& \int \star 1\, \frac{\Phi}{2}\left(R + \frac{2}{\ell^2} - \frac{4}{\ell} i \star \left(\overline{\Psi}_k \wedge \Psi^k\right)\right) \\
& - 2\int \left(\overline{\lambda}_k \mathcal{D}\Psi^k + \text{h.c.}\right) + \int I_{0IJ}\varphi^I \widetilde{f}^J + \int \mathrm{Tr}\,[bH] + S^{\rm top}_{\partial,\rm SUSY}\,, \\
S^{\rm bulk}_{\rm SUSY} =& \int \star 1\, \left(-4\pi W_0^2 g_{0i\bar{j}} \partial^\mu z^i \partial_\mu \bar{z}^{\bar{j}} - \frac{1}{4G_N\ell^2} M^2_{(i)(j)} z^{(i)} z^{(j)}\right) \\
& + \int \star 1\, \left(-\pi W_0^2 g_{0(i)(j)} \overline{\Xi}_k{}^{(i)} \left(\nabla\!\!\!\!/ + \frac{i}{\ell}\gamma_3\right) \Xi^{k(j)}\right) .
\end{aligned}
\tag{5.22}
$$

Notice that the $\sim \overline{\Xi}_k{}^{(i)} \gamma_3 \Xi_k{}^{(j)}$ term is actually just the mass term for fermions with a mass matrix $\pi g_{0(i)(j)}/\ell$ (once normalizing the kinetic term by sending $\Xi_k{}^{(j)} \to \Xi_k{}^{(j)}/W_0^2$). To see this, one simply needs to redefine the 2d γ matrices as

$$
\gamma^t \to i\gamma'^r\,, \qquad\qquad \gamma^r \to i\gamma'^t\,, \tag{5.23}
$$

[32] Technically, as highlighted in section 4.4, the scalars z^i do actually appear in the "topological" action in $\widetilde{f}^I$ thanks to the gauge kinetic term. Since their main contribution is however still in the "bulk" term, we will consider them as "bulk" fields.

obtaining:[33]:

$$-\frac{i}{\ell}\overline{\Xi}_k{}^{(i)}\gamma_3\Xi^{k(j)} + \text{h.c.} \to -\frac{1}{\ell}\overline{\Xi}'_k{}^{(i)}\Xi^{k(j)} + \text{h.c.}\,. \tag{5.24}$$

These masses are of the order of the Kaluza–Klein masses (see section 4.1) which we have excluded in our analysis of the near-extremal (i.e. lowest energy) excitations, and therefore we will ignore the gauginos contribution to the partition function.

If we consider a non-supersymmetric attractor instead, the last two lines of (5.20) become relevant and, in particular, the gravitinos acquire a mass through the term $\sim \overline{\Psi}_k \wedge \Psi^k$, since $\zeta < 1$. In particular, exploiting once again the same γ matrices redefinition (5.23), the mass term can be rewritten as

$$\frac{2W_0^2}{G_N\ell}(1-\zeta)\int i\overline{\Psi}_k \wedge \Psi^k = \frac{2W_0^2}{G_N\ell}(1-\zeta)\int \star 1\, \overline{\Psi}'_{k\mu}\gamma'^{\mu\nu}\Psi^k{}_\nu\,, \tag{5.25}$$

which is the standard form of the mass term [14] for Rarita–Schwinger fields of mass

$$m_\Psi = \frac{2W_0^2}{G_N\ell}(1-\zeta)\,. \tag{5.26}$$

For large black holes, the gravitinos mass becomes much greater[34] than the energy scale of the Schwarzian energy scale, i.e. $m_\Psi \gg \Phi_{\text{ren}}^{-1}$; we therefore conclude that the gravitinos degrees of freedom cannot be excited at the energy scale that dominates the near-extremal limit and thus we will consider their contribution negligible (together with their corresponding Lagrange multipliers, the dilatinos). The final action which determines the dynamics is therefore[35]:

$$\begin{aligned}
S^{\text{2d}}_{\text{non-SUSY}} =& S^{\text{top}}_{\text{non-SUSY}} + S^{\text{bulk}}_{\text{non-SUSY}}\,,\\
S^{\text{top}}_{\text{non-SUSY}} =& \int \star 1\,\frac{\Phi}{2}\left(R+\frac{2}{\ell^2}\right) + \int I_{0IJ}\varphi^I\widetilde{f}^J + \int \operatorname{Tr}[bH] + S^{\text{top}}_{\partial,\text{bos}}\,,\\
S^{\text{bulk}}_{\text{non-SUSY}} =& \int \star 1\left(-4\pi W_0^2 g_{0i\bar{j}}\partial^\mu z^i\partial_\mu\bar{z}^{\bar{j}} - \frac{1}{4G_N\ell^2}\widetilde{M}^2_{(i)(j)}z^{(i)}z^{(j)}\right)\\
&+\int \star 1\left(-\pi W_0^2 g_{0(i)(j)}\overline{\Xi}_k{}^{(i)}\left(\not{\nabla}+\frac{i}{\ell}\gamma_3\right)\Xi^{k(j)}\right)\,,
\end{aligned} \tag{5.27}$$

where $S^{\text{top}}_{\partial,\text{bos}}$ is given by (5.21) and $S^{\text{bulk}}_{\text{non-SUSY}} = S^{\text{bulk}}_{\text{SUSY}} \equiv S^{\text{bulk}}$. The dilatinos and gravitinos are thus ignored like all the other massive Kaluza–Klein modes that would belong to the "bulk".

Finally, as in section 4.5, we point out that S^{bulk} only contributes to the extremal energy

[33] For $\overline{\Xi}'_k{}^{(i)}$ we mean the Dirac conjugate of the spinor using γ'^t in place of γ^t.

[34] One might worry in principle that, if ζ is close enough to 1, the mass of the gravitinos is comparable (and not much greater) than the energy scale Φ_{ren}^{-1} of the near-extremal dynamics; this however can be avoided for large enough black holes. Notice also that if we consider for example the theory described at the end of section 2.4, we have $\zeta = 0$ for the non-BPS attractors (see (2.42)); hence, at least for this example, there is no such problem. We will therefore assume $m_\Psi \gg \Phi_{\text{ren}}^{-1}$ always.

[35] As explained before, we also ignore the $\sim \overline{\Xi}_k{}^{(i)}\lambda^k$ interaction, which is negligible in the macroscopic limit.

and entropy of the black holes and does not influence the spectrum. The scalar contribution is in fact the same as in section 4.5, while the gauginos have a mass of the same order of the Kaluza–Klein modes and can thus be neglected.

5.3 Boundary action as a BF theory

Let us now analyze the two terms $S^{\text{top}}_{\text{SUSY}}$ and $S^{\text{top}}_{\text{non-SUSY}}$. In the non-supersymmetric case, we see that $S^{\text{top}}_{\text{non-SUSY}}$ is the same as $S^{\text{top}}_{\text{bos}}$ from (4.69); as we already showed in section 4.7, this is the action of a $SL(2,\mathbb{R})\times SU(2)$ BF theory, whose contribution to the partition function is topological. In the supersymmetric case, we will once again follow the steps of 4.7, generalizing the BF theory to a super-BF theory (as done in [8]).

A super-BF theory is a theory of the form:

$$S^{\text{sBF}} = -i\int \operatorname{Str}\left[\mathcal{B}\mathcal{F}\right] + S_\partial\,; \tag{5.28}$$

here $\mathcal{F} = d\mathcal{A} - \mathcal{A}\wedge\mathcal{A}$ and $\mathcal{B}$ are respectively the field strength and the Lie-algebra-valued Lagrange multiplier of a gauge theory with supergroup (instead of a group) $\mathcal{G}$. Str denotes the supertrace over the generators $\mathcal{T}_A$; writing the Casimir as $C_2 = g^{AB}\mathcal{T}_A\mathcal{T}_B$, we define the supertrace as $\operatorname{Str}\left[\mathcal{T}_A\mathcal{T}_B\right] \coloneqq (-1)^{|\mathcal{T}_A|}\left(g^{AB}\right)^{-1}$, where $|\mathcal{T}_A| = 0,1$ for bosonic and fermionic generators respectively [8]. Once again, integrating over $\mathcal{B}$ forces $\mathcal{F}=0$, setting the bulk term to 0 and making the dynamics determined only by the boundary term S_∂. Focusing on the case at hand, we can recover $S^{\text{top}}_{\text{SUSY}}$ by picking the supergroup $PSU(1,1|2)$ [8], as hinted by the superisometries of the extremal solution analyzed in section 3.2. The algebra $\mathfrak{psu}(1,1|2)$ is given by [8]:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}\,, & [T_i, T_j] &= -\varepsilon_{ijk}T_k\,,\\
[L_m, {G_p}^\alpha] &= \left(\frac{m}{2}-\alpha\right){G_p}^{\alpha+m}\,, & \left[L_m, {\overline{G}^p}_\alpha\right] &= \left(\frac{m}{2}-\alpha\right){\overline{G}^p}_{\alpha+m}\,,\\
[T_i, {G_p}^\alpha] &= -\frac{i}{2}{(\sigma_i)_p}^q {G_q}^\alpha\,, & \left[T_i, {\overline{G}^p}_\alpha\right] &= \frac{i}{2}{(\sigma_i^*)^p}_q {\overline{G}^q}_\alpha\,,\\
\{{G_p}^\alpha, {G^q}_\beta\} &= 2{\delta_p}^q L_{\alpha+\beta} - 2(\alpha-\beta){(\sigma_i)^q}_p T_i\,.
\end{aligned} \tag{5.29}$$

Here L_0, L_{+1} and L_{-1} are the bosonic generators of the $\mathfrak{sl}(2,\mathbb{R})$ sub-algebra, T_i are the bosonic generators of the $\mathfrak{su}(2)$ sub-algebra and ${G_p}^\alpha$ and ${\overline{G}^q}_\beta$ are the fermionic generators of $\mathfrak{psu}(1,1|2)$, with $p,q=1,2$ and $\alpha,\beta = +1/2,-1/2$.

In order to simplify the calculations, we first redefine the generators of $SL(2,\mathbb{R})$ as in (4.90). Then, when dealing with the fermionic quantities ${G_p}^\alpha$ and ${\overline{G}^q}_\beta$, it is convenient not to write explicitly the p and α indices, in order to ease the notation; we will always contract the α indices from "south-west" to "north-east", while we will always contract the p indices from "north-west" to "south-east". As an example, if we have the two fermionic quantities ${\psi_p}^\alpha$ and ${\overline{\chi}^p}_\alpha$, we will write $\overline{\chi}\psi \coloneqq {\overline{\chi}^p}_\alpha{\psi_p}^\alpha$. We choose the indices of the Pauli matrices such that $\sigma_i = {(\sigma_i)_p}^q$. Finally, we introduce the following 2d Euclidean gamma matrices with indices $\gamma \coloneqq (\gamma)^{\alpha}{}_{\beta}$:

$$\gamma^0 = \sigma_1 \,, \qquad \gamma^1 = -\sigma_3 \,, \qquad \gamma_3 = \gamma^0\gamma^1 = i\sigma_2 \,. \tag{5.30}$$

This allows us to rewrite the algebra (5.29) in a more concise form, making the calculations more manageable:

$$\begin{aligned}
&[P_a, J] = \varepsilon_{ab} P_b \,, && [P_a, P_b] = \varepsilon_{ab} J \,, && [J, G] = -\frac{1}{2}\gamma_3 G \,, \\
&[P_a, G] = \frac{1}{2}\varepsilon_{ab}\gamma_b G \,, && [P_a, \overline{G}] = \frac{1}{2}\varepsilon_{ab}\overline{G}\gamma_b^\top \,, && [J, \overline{G}] = -\frac{1}{2}\overline{G}\gamma_3^\top \,, \\
&[T_i, T_j] = -\varepsilon_{ijk} T_k && [T_i, G] = \frac{i}{2}\sigma_i G && [T_i, \overline{G}] = \overline{G}\left(\frac{i}{2}\sigma_i\right)^\dagger \,, \\
&\qquad \left\{G, \overline{G}\right\} = 2\left(J + \gamma^a P_a - i\gamma_3\sigma_i^\top T_i\right) \,.
\end{aligned} \tag{5.31}$$

The $PSU(1,1|2)$ gauge field $\mathcal{A}_{PSU(1,1|2)}$ and Lagrange multiplier $\mathcal{B}_{PSU(1,1|2)}$ are then chosen as follows:

$$\begin{aligned}
\mathcal{A}_{PSU(1,1|2)} =& \frac{1}{\ell} e^a P_a - \omega J + B^i T_i + \frac{1}{\sqrt{\ell}}\left(\Psi G - \overline{G}\Psi\right) \,, \\
\mathcal{B}_{PSU(1,1|2)} =& 2\ell\phi^a P_a - 2\phi J + b^i T_i + \sqrt{\ell}\left(\overline{\lambda}\gamma_3 G - \overline{G}\gamma_3\lambda\right) \,.
\end{aligned} \tag{5.32}$$

Using the Casimir [8]

$$C_2^{PSU(1,1|2)} = 2P_aP_a - 2J^2 - 2T_iT_i - \frac{1}{2}\overline{G}\gamma_3 G + \frac{1}{2}G\gamma_3^\top\overline{G} \,, \tag{5.33}$$

we get the following $PSU(1,1|2)$ super-BF theory:

$$\begin{aligned}
S^{\mathrm{BF}}_{PSU(1,1|2)} =& -i\int -\phi\left(d\omega + \frac{1}{\ell^2}e^0\wedge e^1 - \frac{2}{\ell}\overline{\Psi}\wedge\Psi\right) \\
& - i\int \phi^a\left(de^a + \varepsilon_{ab}\,\omega\wedge e^b + 2\overline{\Psi}\wedge\gamma^a\Psi\right) \\
& - i\int \mathrm{Tr}\left[bH\right] + \frac{2}{\ell}\overline{\Psi}\wedge\gamma_3 b^\top\Psi + \left(-2\lambda\mathcal{D}\Psi + \text{h.c.}\right) \,,
\end{aligned} \tag{5.34}$$

where the $PSU(1,1|2)$ covariant exterior derivative

$$\mathcal{D} = d - \frac{1}{2}\gamma_3\omega\wedge - \frac{1}{2\ell}\gamma_3 \not{e}\wedge - B\wedge \tag{5.35}$$

generalizes the expression (5.19), obtained while considering terms like $\sim \overline{\Psi}\Psi B$ negligible. Notice that once again we are not distinguishing between the generators of $SU(2)$ and their representative in the fundamental representation, as in (4.107). After a Wick rotation, a redefinition of the gamma matrices and while neglecting the terms of the third order in the fluctuations ($\sim \overline{\Psi}\Psi B, \overline{\Psi}\Psi b$), the action (5.34) is exactly equal to the bulk of the action $S^{\mathrm{top}}_{\mathrm{SUSY}}$ obtained via dimensional reduction. As in section 4.7, using (4.94) one can see that the super-JT gravity dilaton Φ is related to ϕ by $\phi = i\Phi$. Notice that the terms $\sim \overline{\Psi}\Psi B, \overline{\Psi}\Psi b$ can also be obtained from the dimensional reduction, provided that we do not ignore the terms dependent on B and H in the 4d spin connection (4.5). Notice also that, as expected, the bosonic part of the $PSU(1,1|2)$ super-BF theory is exactly the $SL(2,\mathbb{R}) \times SU(2)$ BF theory.

We are once again left to find the appropriate boundary terms that generate the dynamics; let us start from the $PSU(1,1|2)$ BF theory arising from supersymmetric attractors. We know that, in order to glue correctly the NHR and FAR region (and get the Gauss–Bonnet term generating the extremal entropy), we must impose Dirichlet boundary conditions on the metric $g_{\mu\nu}$ and the dilaton Φ; these in turn generate, via the Gibbons–Hawking–York boundary term, the usual Schwarzian action. In the $SL(2,\mathbb{R})$ BF theory formulation of JT gravity, the Schwarzian mode arises from the presence of a defect, which is equivalent to imposing the mixed boundary conditions (4.104). In the $PSU(1,1|2)$ theory we thus have to impose the same boundary conditions at least for the $SL(2,\mathbb{R})$ component of the $PSU(1,1|2)$ gauge field. By applying supersymmetry transformations to (4.104) — which in BF language are just gauge transformations along the fermionic directions of $PSU(1,1|2)$ — we can send the $SL(2,\mathbb{R})$ components $\mathcal{A}_{SL(2)}$ and $\mathcal{B}_{SL(2)}$ respectively to $\mathcal{A}_{SU(2)} \equiv B$ and $\mathcal{B}_{SU(2)} \equiv b$, as well as to the other fermionic components of the fields $\mathcal{A}_{PSU(1,1|2)}$. Therefore, for consistency, we must extend the boundary conditions (4.104) to:

$$2i\Phi_{\text{ren}}\, \delta\mathcal{A}_{PSU(1,1|2)}\big|_{\partial} - \delta\mathcal{B}_{PSU(1,1|2)} du\big|_{\partial} = 0\,. \tag{5.36}$$

Alternatively, this means that we must introduce a defect in the super-BF theory which acts in the same way as (4.95) but along all of the components of the $PSU(1,1|2)$ gauge field.

Although these boundary conditions were obtained via supersymmetric considerations, in principle they come just from the gluing of the NHR and FAR region to get a consistent spacetime. Therefore, the bosonic part of (5.36) should still hold true when we consider just the bosonic part of the $PSU(1,1|2)$ BF theory, i.e. the $SL(2,\mathbb{R}) \times SU(2)$ BF theory describing black holes close to the non-supersymmetric attractors. This justifies our choice of boundary conditions (4.108), which in turn justifies the arbitrary choice of boundary conditions (4.48) (needed to recover the effective action of a particle moving in an $SU(2)$ group manifold).

To sum up, we can write the effective theory describing the dynamics of near-extremal black holes around supersymmetric attractors as:

$$\begin{aligned} &S^{\text{BF}}_{PSU(1,1|2)} = -i\int \operatorname{Tr}\left[\mathcal{B}_{PSU(1,1|2)}\mathcal{F}_{PSU(1,1|2)}\right] + \frac{1}{4\Phi_{\text{ren}}}\int_0^\beta du\, \operatorname{Tr}\left[\mathcal{B}^2_{PSU(1,1|2)}\right]\,, \\ &2i\Phi_{\text{ren}}\, \delta\mathcal{A}_{PSU(1,1|2)}\big|_{\partial} - \delta\mathcal{B}_{PSU(1,1|2)} du\big|_{\partial} = 0\,. \end{aligned} \tag{5.37}$$

The $PSU(1,1|2)$ BF theory with the boundary conditions above has been shown in [8] to be equivalent to a $\mathcal{N} = 4$ super-Schwarzian (3.20) with energy scale $M_{PSU(1,1|2)} = \Phi_{\text{ren}}^{-1}$, as expected from the symmetry considerations of section 3.3. This was done by studying and matching the transformation of the BF theory and the $\mathcal{N}=4$ Schwarzian under gauge transformations and super-diffeomorphisms respectively. As for the theory around non-supersymmetric attractors, the fermions are not present and thus we only get an $SL(2,\mathbb{R})\times SU(2)$ bosonic BF theory:

$$\begin{aligned}
&S^{\mathrm{BF}}_{SL(2)\times SU(2)} = -i\int \mathrm{Tr}\left[\mathcal{B}_{SL(2)\times SU(2)}\mathcal{F}_{SL(2)\times SU(2)}\right] + \frac{1}{4\Phi_{\mathrm{ren}}}\int_0^\beta du\ \mathrm{Tr}\left[\mathcal{B}^2_{SL(2)\times SU(2)}\right] , \\
&2i\Phi_{\mathrm{ren}}\ \delta\mathcal{A}_{SL(2)\times SU(2)}\big|_\partial - \delta\mathcal{B}_{SL(2)\times SU(2)}du\big|_\partial = 0 .
\end{aligned} \tag{5.38}$$

As shown in section 4.7, this is equivalent to the action of a Schwarzian together with a particle on an $SU(2)$ group manifold (3.18), with energy scales $M_{SL(2)} = M_{SU(2)} = \Phi^{-1}_{\mathrm{ren}}$.

At last, we point out that we obtained these (super)-BF theories by assuming, in the matching between the NHR and FAR boundary (4.22), that the approximation (3.1) is valid. Despite this, the resulting effective Schwarzian theories coincide with those obtained by symmetry considerations in section 3, which were obtained without assuming the approximation (3.1). The associated energy scale Φ^{-1}_{ren} depends only on W_0, that is only on the properties of the attractors (electric charges and values of the scalars at the attractor), and not on other properties of the extremal solutions at infinity. Therefore, it seems reasonable to assume that the mechanism that differentiates between the attractors — the gravitinos becoming massless/massive — still remains what sets apart the boundary theories near supersymmetric/non-supersymmetric attractors. Finally, the symmetry considerations of section 3 hold even in the presence of non-zero magnetic charges. Since $W_0 = \sqrt{V_{\mathrm{BH}}}\big|_{r\to-\infty}$ characterizes the attractors also in the presence of magnetic charges, we argue that the generalized Schwarzians we obtained are still describing correctly the near-extremal excitations, with the same energy scale Φ^{-1}_{ren}.

6 Analysis of the Schwarzian theories

6.1 The path integral measure

We are now ready to determine the spectra of the black holes from their corresponding generalized Schwarzian theories. The grand-canonical partition functions for the black holes will be determined by evaluating the corresponding path integrals at 1-loop, expanding around the classical saddles of the generalized Schwarzian theories; after an inverse Laplace transform, we will then isolate the density of states $\rho(\mathcal{E},\mathcal{J})$ of the black holes. Notice that one can also calculate the grand-canonical partition functions in other ways. For example, one can evaluate directly the partition functions of the (super-)BF theories on a disk; this approach has been followed in [24] for a simple Schwarzian mode. Alternatively, for the case of the $\mathcal{N}=4$ super-Schwarzian, [8] shows — on the basis of [45] — how one can obtain the same results by canonical quantization of the phase space of theory, namely $\mathrm{SDiff}(S^{1|4})/PSU(1,1|2)$.

We will now briefly focus on the path integral measure for the Schwarzian theories. Understanding over what space we are integrating over has a twofold importance. First, when one calculates the 1-loop determinants for Schwarzian theories, one usually gets some zero modes; these can be then gauge fixed away only by knowing what gauge freedom one has, which in turn comes from the properties of the space we are integrating over. Second, if we manage to show that we are integrating over a symplectic manifold, the path integral turns out to be 1-loop exact. This is a consequence of the Duistermaat–Heckman theorem, which applies fermionic localization to symplectic (super-)manifolds with a $U(1)$ symmetry, which in our case is due to the periodicity $u \to u+\beta$ of Euclidean time [13].

To start, let us sum up the results of section 3, 4 and 5. The dynamics of near-extremal black holes around (static) non-supersymmetric attractors is given — up to extremal energy and entropy shifts — by the action of a Schwarzian mode[36] and of a particle moving on an $SU(2)$ group manifold:

$$S_{SL(2)\times SU(2)} = -\Phi_{\rm ren} \int_0^\beta du \left(\mathrm{Sch}\left(\tan\frac{\pi\tau}{\beta}, u\right) + \mathrm{Tr}\left[\left(g^{-1}g'\right)^2\right]\right) . \tag{6.1}$$

Here $\tau(u)$ is an element of $\tau(u) \in \mathrm{Diff}(S^1)$, i.e. a map with $\tau(\beta)-\tau(0)=\beta$ and $\tau'(u)>0$, $g(u)$ maps the time u to a group element of $SU(2)$ and $\Phi_{\rm ren}$ sets the energy scale of the theory. If we expand around supersymmetric attractors instead, we get the action of an $\mathcal{N}=4$ super-Schwarzian, whose bosonic part is exactly given by (6.1):

$$\begin{aligned} S_{PSU(1,1|2)} &= -\,\Phi_{\rm ren}\int_0^\beta du\, \mathrm{Sch}_{\mathcal{N}=4}\left(\tau, g, \eta, \overline{\eta}\right) \\ &= -\,\Phi_{\rm ren}\int_0^\beta du\, \left(\mathrm{Sch}\left(\tan\frac{\pi\tau}{\beta}, u\right) + \mathrm{Tr}\left[\left(g^{-1}g'\right)^2\right]\right) + (\text{fermions})\,, \end{aligned} \tag{6.2}$$

where $\eta^p(u)$ and $\overline{\eta}_p(u)$ $(p=1,2)$ are fermionic modes. The $\mathcal{N}=4$ super-Schwarzian was first described in [34, 35]; it arises from the anomalous part of the transformation of the super stress-energy tensor under $\mathcal{N}=4$ $SU(2)$-extended superconformal transformations in 2d SCFTs. The corresponding Schwarzian action can then be obtained by integrating the anomalous part of the transformations in superspace. Notice that the $\mathcal{N}=4$ super-Schwarzian is invariant under $PSU(1,1|2)$ transformations (i.e. global superconformal transformations), similarly to how the Schwarzian is invariant under $SL(2,\mathbb{R})$ transformations (i.e. global conformal transformations); its bosonic part also coincides with the action of a Schwarzian plus a particle moving in an $SU(2)$ group manifold, as expected from the fact that $PSU(1,1|2)$ has $SL(2,\mathbb{R})\times SU(2)$ as its bosonic subgroup. We will not explicitly write down the fully non-linear fermionic terms in (6.2) (see [48] for details), because for the 1-loop calculations one just needs the quadratic expansion around the classical saddles (with $\eta=\overline{\eta}=0$).

[36] Note that the Schwarzian mode itself can be written in terms of a particle moving on an $SL(2,\mathbb{R})$ group manifold [46]. The Schwarzian is also equivalent to a particle moving on the hyperbolic plane, in the presence of a large magnetic (Euclidean) or electric (Lorentzian) field [47, 24]; this equivalence is particularly useful when discussing the bulk dual to partially entangled states in the SYK model [47].

Let us now focus on the non-supersymmetric attractors. The Schwarzian part of the action is described by the function $\tau(u) \in \mathrm{Diff}(S^1)$. The action is however invariant under an $SL(2,\mathbb{R})$ transformation sending

$$\mathrm{Sch}\left(t(u),u\right) \to \mathrm{Sch}\left(\frac{at(u)+b}{ct(u)+d},u\right) . \tag{6.3}$$

In JT gravity, this invariance is due to the possibility of having equivalent choices of the IR cutoffs thanks to the invariance of the Poincaré half-plane (that is, Euclidean AdS_2) metric (4.73) under diffeomorphisms given by $SL(2,\mathbb{R})$ transformations. Hence, the phase space of the Schwarzian mode — i.e. the space over which we will integrate in the path integral — is $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$. This quotient in turn is exactly the one needed to eliminate the zero modes arising in the 1-loop expansion of the Schwarzian action. We also have that $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ is a symplectic manifold with a $U(1)$ symmetry; one way to see this [13] is to show that $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ can be constructed as the coadjoint orbit of the Virasoro group, which guarantees both the conditions above[37]. As a consequence, the Duistermaat–Heckman theorem implies that the path integral for the Schwarzian mode is 1-loop exact[13].

As for the $SU(2)$ component of (6.1), the function $g(u)$ is a function that maps each value of u to an element of $SU(2)$. Its boundary conditions are not periodic, in order to account for the angular velocity of the black hole (see (4.83)):

$$g(\beta) = e^{-2i\Omega T_3 \beta} g(0) . \tag{6.4}$$

We can however make a change of variable such that:

$$\begin{aligned} g(u) =& e^{-2i\Omega T_3 u}\widetilde{g}(u) , \\ \widetilde{g}(\beta) =& \widetilde{g}(0) . \end{aligned} \tag{6.5}$$

This way, $\widetilde{g}$ maps S^1 to a loop in $SU(2)$, that is $\widetilde{g} \in \mathrm{loop}(SU(2))$. Additionally, $\mathrm{Tr}\left[\left(g^{-1}g'\right)^2\right]$ is invariant under global $SU(2)$ transformations, which is just a consequence of the spherical symmetry of the extremal solutions; we thus need to quotient $\mathrm{loop}(SU(2))$ by $SU(2)$. As explained in [49], $\mathrm{loop}(SU(2))/SU(2)$ is Kähler and thus possesses a symplectic form, so that we can apply the Duistermaat–Heckman theorem. We will therefore integrate over $\mathrm{loop}(SU(2))/SU(2)$. Notice that this is also expected since the boundary gauge transformations of an $SU(2)$ BF theory are parameterized by $\mathrm{loop}(SU(2))/SU(2)$ [8]. Therefore, we have that the partition function

$$\mathcal{Z}_{SL(2)\times SU(2)} = \int \mathcal{D}\tau \left[\frac{\mathrm{Diff}(S^1)}{SL(2,\mathbb{R})}\right] \mathcal{D}g \left[\frac{\mathrm{loop}(SU(2))}{SU(2)}\right] e^{-S_{SL(2)\times SU(2)}(\tau,g)} \tag{6.6}$$

is 1-loop exact. Finally, notice also that, as pointed out once again in [13], we can get both the actions of the Schwarzian and the particle moving on a group manifold and their corresponding symplectic manifolds by considering coadjoint orbits of an $SU(2)$ Virasoro–Kac–Moody algebra

[37] This construction of the symplectic space also provides the Schwarzian action directly.

(instead of considering the $SL(2,\mathbb{R})$ and $SU(2)$ components separately).

A similar reasoning also holds for the case of a supersymmetric attractor. The functions τ, g, η and $\overline{\eta}$ parameterizes the superdiffeomorphisms SDiff($S^{1|4}$) [34, 35], with the super-Schwarzian being invariant under superconformal $PSU(1,1|2)$ transformations. We will thus integrate over the quotient SDiff($S^{1|4}$)$/PSU(1,1|2)$. Notice also that one can also directly build the $\mathcal{N}=4$ super-Schwarzian theory exactly through the coadjoint orbit method [50] of an $\mathcal{N}=4$ super-Virasoro algebra, showing that SDiff($S^{1|4}$)$/PSU(1,1|2)$ is a symplectic manifold with a $U(1)$ symmetry and the Duistermaat–Heckman theorem applies. We therefore get the 1-loop exact partition function:

$$\mathcal{Z}_{PSU(1,1|2)} = \int \mathcal{D}\tau\mathcal{D}g\mathcal{D}\eta\mathcal{D}\overline{\eta}\left[\frac{\text{Diff}(S^{1|4})}{PSU(1,1|2)}\right] e^{-S_{PSU(1,1|2)}(\tau,g,\eta,\overline{\eta})}\,. \tag{6.7}$$

Finally, we write here the boundary conditions for η and $\overline{\eta}$ (rewriting also the one for τ and g for convenience) [8]:

$$\tau(\beta)=\tau(0)\,,\qquad g(\beta)=e^{-2i\Omega T_3\beta}g(0)\,,\qquad \eta(\beta)=-e^{-2i\Omega T_3\beta}\eta(0)\,. \tag{6.8}$$

As always, the fermions are chosen to be antiperiodic in Euclidean time, hence the $-$ sign; additionally, we also have the exponential factor due to the $SU(2)$ holonomy.

6.2 The classical contribution

We will now calculate the partition functions in two steps, mainly following [8, 28]: first — in this section — we evaluate the contribution of the classical saddles; then — in section 6.3 — we expand the action to second order in the fluctuations and compute the 1-loop quantum corrections. Since the classical bosonic saddles are the same for both generalized Schwarzians, we will treat only the non-supersymmetric theory in this section, with everything still true for the supersymmetric case with $\eta=\overline{\eta}=0$. The equation of motion of $S_{SL(2)\times SU(2)}$, from (6.1), are given by

$$\begin{aligned}\frac{\tau''''}{\tau'^2}-4\frac{\tau'''\tau''}{\tau'^3}+3\frac{\tau''^3}{\tau'^4}+\frac{4\pi^2}{\beta^2}\tau''=0\,,\\ g''-g'g^\dagger g'=0\,,\end{aligned} \tag{6.9}$$

which can be rewritten in the more convenient form:

$$\left[\text{Sch}\left(\tan\frac{\pi\tau}{\beta},u\right)\right]'=0\,,\qquad \left(g^\dagger g'\right)'=0\,. \tag{6.10}$$

The classical solutions, once enforcing the boundary conditions (6.8), are labeled by an integer n, and represent the various saddles with angular velocities $\Omega_n = \Omega + 2\pi i n/\beta$:

$$\tau(u) = u\,, \qquad\qquad g(u) = e^{-2i\Omega_n T_3 u}\,. \tag{6.11}$$

Notice that these solutions are not invariant under a generic $SL(2,\mathbb{R}) \times SU(2)$ transformation, despite the invariance of the generalized Schwarzian actions under such a transformation. The only transformations leaving (6.11) unchanged are $U(1)$ rotations along T_3 of $g(u)$, which correspond to rotations of the original 4d black hole along its axis of rotation. Hence the classical saddles (6.11) minimizing the action break the $SL(2,\mathbb{R}) \times SU(2)/PSU(1,1|2)$ symmetry of the generalized Schwarzians down to just a $U(1) \in SU(2)$. The generalized Schwarzians thus implements correctly the symmetry breaking pattern of the near-horizon geometry, described in section 3, with the generalized Schwarzian modes interpreted as "Goldstone bosons" of the symmetry breaking.

Plugging (6.11) into the action yields the classical partition function:

$$\begin{aligned} \mathcal{Z}^{\mathrm{cl}}(\beta,\Omega) :=& \mathcal{Z}^{\mathrm{cl}}_{SL(2)\times SU(2)} = \mathcal{Z}^{\mathrm{cl}}_{PSU(1,1|2)} \\ =& e^{\mathcal{S}_* - \beta\mathcal{E}_*} \sum_{n\in\mathbb{Z}} e^{\Phi_{\mathrm{ren}} \frac{2\pi^2}{\beta}\left(1 + \frac{\beta^2}{\pi^2}\left(\Omega + \frac{2\pi i}{\beta} n\right)^2\right)}\,. \end{aligned} \tag{6.12}$$

In the equation above we added back the (classical) extremal energy $\mathcal{E}_*$ and entropy $\mathcal{S}_*$, which cannot be obtained from the generalized Schwarzian theory. These terms arise from the joint contribution of the Gauss–Bonnet term of JT gravity, the action of the FAR region, the scalars $z^{(i)}$ and the gauginos $\Xi^{(i)}$ and all the other massive Kaluza–Klein modes that we have ignored in the dimensional reduction (possibly including logarithmic corrections [51, 52]). However, notice that both terms do not influence the qualitative features of the spectrum: the $\mathcal{S}_*$ term acts only as a proportionality constant, while the $\beta\mathcal{E}_*$ terms simply shifts the energies of the spectrum.

We will now briefly analyze the classical thermodynamics of the black holes, for later comparison with the quantum corrected ones (see section 6.4). For simplicity, we will just study the properties of the thermodynamic potentials, without extracting the spectrum of the states (as we will do later for the quantum corrected partition functions). By using the relation $\mathcal{Z}^{\mathrm{cl}}(\beta,\Omega) = \exp(-\beta\mathcal{G})$ — where $\mathcal{G}$ is the grand-canonical free energy — we obtain (via Legendre transforms) the following thermodynamic quantities:

$$\begin{aligned} \mathcal{G} &= \mathcal{E}_* - T\mathcal{S}_* - 2\pi^2\Phi_{\mathrm{ren}}T^2 - 2\Phi_{\mathrm{ren}}\Omega^2 + \dots\,, \\ \mathcal{S} &= \mathcal{S}_* + 4\pi^2\Phi_{\mathrm{ren}}T + \dots\,, \\ \mathcal{E} &= \mathcal{E}_* + 2\pi^2\Phi_{\mathrm{ren}}T^2 + 2\Phi_{\mathrm{ren}}\Omega^2 + \dots\,. \end{aligned} \tag{6.13}$$

This is the typical expansion of the thermodynamic potentials for near-extremal black holes. In particular we notice that there is an extremal entropy $\mathcal{S}_*$; one therefore expect a degeneracy of the ground state, i.e. there should be many different microstates corresponding to the extremal black hole. We also see that — as usual for classical extremal black holes — we have a breakdown of the thermodynamic for small temperatures. In particular, the black holes radiate Hawking quanta with an average energy $\mathcal{E}_\gamma \sim T$. When $T \lesssim 1/\left(2\pi^2\Phi_{\text{ren}}\right)$, the energy above extremality of the black hole is less than the energy needed to emit a Hawking quantum: $\mathcal{E} - \mathcal{E}_* \sim \mathcal{E}_\gamma$; the near-extremal black hole thus cannot radiate and it cannot behave as a thermodynamic object, signaling a breakdown in the semiclassical thermodynamic description. One possible workaround is to assume that there is indeed a gap of the order

$$\Delta\mathcal{E}_{\text{gap}} \sim \frac{1}{2\pi^2\Phi_{\text{ren}}} \tag{6.14}$$

in the energy spectrum of the black holes, with no microstates with energies between $\mathcal{E}_*$ and $\mathcal{E}_* + \Delta\mathcal{E}_{\text{gap}}$. Another possibility is instead that, once accounting for the quantum effects of gravity, the energy $\mathcal{E}$ from (6.13) is modified in such a way that $\mathcal{E} \gtrsim T$ down to extremality, i.e. the thermodynamic description never breaks down and the black holes are always well-behaved. As we will show in section 6.4, these two possibilities are realized, for supersymmetric and non-supersymmetric attractors respectively.

6.3 1-loop quantum corrections

We will now compute the quantum corrections to the partition functions. This requires expanding the action around the classical saddles and calculating the 1-loop determinants, properly regularized and with the zero modes gauge fixed away. We will first analyze the 1-loop contributions of the $SL(2,\mathbb{R})$ and $SU(2)$ modes, which are the same both for supersymmetric and non-supersymmetric attractors.

The classical solutions for τ and g are given by (6.11); we thus introduce the fluctuations:

$$\tau(u) = u + \delta\tau(u)\,, \qquad g(u) = e^{-2i\Omega_n T_3 u} e^{\delta g_1(u) T_1} e^{\delta g_2(u) T_2} e^{\delta g_3(u) T_3}\,. \tag{6.15}$$

Plugging (6.15) into the action (6.1) yields:

$$\begin{aligned} \delta S_{SL(2)\times SU(2)} &= \delta S_{SL(2)} + \delta S_{SU(2)}\,, \\ \delta S_{SL(2)} &= \frac{\Phi_{\text{ren}}}{2}\int_0^\beta du \left(\delta\tau''^2 - \frac{4\pi^2}{\beta^2}\delta\tau'^2\right)\,, \\ \delta S_{SU(2)} &= \frac{\Phi_{\text{ren}}}{2}\int_0^\beta du \left(\delta g_1'^2 + \delta g_2'^2 + \delta g_3'^2 - 2i\Omega_n\left(\delta g_2 \delta g_1' - \delta g_2' \delta g_1\right)\right)\,, \end{aligned} \tag{6.16}$$

To calculate the 1-loop determinants, we first rescale $u \to \beta u$ in order to eliminate the dependence on β of the extremum of integration; notice that this also require scaling $\delta\tau \to \beta\delta\tau$ in order to keep τ an element of Diff(S^1). Focusing first on the $\delta S_{SL(2)}$ component, the rescaled action and corresponding 1-loop determinant are given by:

$$\begin{aligned}\delta S_{SL(2)} &= \frac{1}{2}\frac{\Phi_{\text{ren}}}{\beta}\int_0^1 du\,\left(\delta\tau''^2 - 4\pi^2\delta\tau'^2\right)\,,\\ \det{}_{SL(2)} &= \det\left[\frac{\Phi_{\text{ren}}}{\beta}\left(\frac{d^4}{du^4}+4\pi^2\frac{d^2}{du^2}\right)\right]\,.\end{aligned} \tag{6.17}$$

We can evaluate the determinant as the product of all the eigenvalues λ_τ of the equation:

$$\frac{\Phi_{\text{ren}}}{\beta}\left(\frac{d^4}{du^4}+4\pi^2\frac{d^2}{du^2}\right)\delta\tau(u) = \lambda_\tau\delta\tau(u)\,. \tag{6.18}$$

We can obtain the eigenvalues λ_τ by means of Fourier series, which amounts to replacing $d/du \to 2\pi i m$ with $m \in \mathbb{Z}$. The eigenvalues are thus:

$$\lambda_\tau = \frac{\Phi_{\text{ren}}}{\beta}(2\pi)^4 m^2\left(m^2-1\right)\,, \qquad\qquad m\in\mathbb{Z}\,. \tag{6.19}$$

We see the appearance of three zero-modes, for values of $m = 0, \pm 1$; these are exactly the modes corresponding to infinitesimal $SL(2,\mathbb{R})$ transformations of $\tau(u) = u$. They can therefore be gauge fixed away, exhausting all the $SL(2,\mathbb{R})$ gauge freedom, so that the determinant is well defined:

$$\det{}_{SL(2)} = \prod_{m\neq 0,\pm 1}\frac{\Phi_{\text{ren}}}{\beta}(2\pi)^4 m^2\left(m^2-1\right)\,. \tag{6.20}$$

Finally, to obtain a finite result, we need to regularize the infinite product. This can be achieved through ζ-function regularization [53, 54]; in particular, we can use the formulas:

$$\prod_{m=1}^{\infty} x^{-2} = x^{-2\zeta(0)} = x\,, \qquad\qquad \prod_{m=1}^{\infty} m^y = e^{-y\zeta'(0)} = (2\pi)^{y/2}\,. \tag{6.21}$$

As we are only interested in the dependence on β and Φ_{ren}, we neglect the other terms in the product, which — after regularization — would provide only a shift in extremal entropy of the black hole. We thus obtain the 1-loop determinant:

$$\det{}_{SL(2)} = \left(\frac{\beta}{\Phi_{\text{ren}}}\right)^3\,. \tag{6.22}$$

Going back to the $SU(2)$ component instead, we have the action (after rescaling):

$$\delta S_{SU(2)} = \frac{1}{2}\frac{\Phi_{\text{ren}}}{\beta}\int_0^1 du\,\left(\delta g_1'^2+\delta g_2'^2+\delta g_3'^2 - 2i\beta\Omega_n\left(\delta g_2\delta g_1' - \delta g_2'\delta g_1\right)\right)\,. \tag{6.23}$$

To calculate the 1-loop determinant $\det_{SU(2)}$, it is convenient to diagonalize the kinetic terms by introducing the combinations $\delta g_\pm = \delta g_1 \pm i\delta g_2$; this way the kinetic term is diagonal and we can express the 1-loop determinant as:

$$\begin{aligned}\det{}_{SU(2)} = &\det\left[\frac{\Phi_{\rm ren}}{\beta}\left(-\frac{d^2}{du^2}\right)\right]\cdot \\ &\cdot \det\left[\frac{\Phi_{\rm ren}}{\beta}\left(-\frac{d^2}{du^2}-2\beta\Omega_n\frac{d}{du}\right)\right]\det\left[\frac{\Phi_{\rm ren}}{\beta}\left(-\frac{d^2}{du^2}+2\beta\Omega_n\frac{d}{du}\right)\right] .\end{aligned} \tag{6.24}$$

The three sets of eigenvalues $\lambda_3, \lambda_\pm$ of the respective three determinants are:

$$\lambda_3 = \frac{\Phi_{\rm ren}}{\beta}4\pi^2 m^2\,, \qquad\qquad \lambda_\pm = \frac{\Phi_{\rm ren}}{\beta}4\pi^2 m^2\left(1\mp\frac{i\beta\Omega_n}{\pi m}\right)\,, \tag{6.25}$$

where $m \in \mathbb{Z}$. We see that once again we have zero-modes for $m = 0$; these are exactly the modes which correspond to global $SU(2)$ rotations of $g(u)$ and which can be gauge fixed away, using all the $SU(2)$ gauge freedom. The determinant is therefore:

$$\det{}_{SU(2)} = \prod_{m\neq 0}\left(\frac{\Phi_{\rm ren}}{\beta}\right)^3\left(4\pi^2 m^6\right)\left(1-\frac{(i\beta\Omega_n)^2}{\pi^2 m^2}\right)\,. \tag{6.26}$$

In order to get a finite expression, we will once again employ ζ-function regularization, together with the expression:

$$\sin(x) = x\prod_{m=1}^{\infty}\left(1-\frac{x^2}{\pi^2 m^2}\right)\,. \tag{6.27}$$

Up to an overall constant — once again shifting the extremal entropy and regularization dependent — we get:

$$\det{}_{SU(2)} = \left(\frac{\beta}{\Phi_{\rm ren}}\right)^3\left(\frac{\sin\left(2\pi\alpha\right)}{\alpha+n}\right)^2\,, \tag{6.28}$$

where we introduced the parameter α defined by:

$$2\pi i\alpha \coloneqq \beta\Omega \qquad \Longrightarrow \qquad 2\pi i\left(\alpha+n\right) = \beta\Omega_n\,. \tag{6.29}$$

We are now just left to evaluate the 1-loop quantum corrections related to the fermionic modes, present only in the case of supersymmetric attractors. This requires expanding the $\mathcal{N} = 4$ super-Schwarzian to second order in the fermionic terms. This procedure has been carried out in [8, 48]; here we will simply quote the results, referring the reader to section 3.2 of [8] for more details. The quadratic part of the action yields:

$$\delta S_\eta = \Phi_{\rm ren}\int_0^\beta du\,\delta\overline{\eta}\left(2\partial_u^3+4\Omega_n\partial_u^2+\frac{2\pi^2}{\beta^2}\left(1+\frac{\beta^2\Omega_n^2}{\pi^2}\right)\,\partial_u\right)\delta\eta\,. \tag{6.30}$$

We then further rescale the action by sending $u \to \beta u$ and $\eta \to \sqrt{\beta}\eta$; the 1-loop determinant is thus:

$$\det_{\text{fer}} = \left(\det \left[\frac{\Phi_{\text{ren}}}{\beta} \left(2\frac{d^3}{du^3} + 4\Omega_n \beta \frac{d^2}{du^2} + 2\pi^2 \left(1 + \frac{\beta^2 \Omega_n^2}{\pi^2} \right) \frac{d}{du} \right) \right] \right)^2 , \tag{6.31}$$

where the square comes from the fact that we have two sets of modes, for $p = 1, 2$. Notice that this time we should impose antiperiodic boundary conditions on the spinor, i.e. we should take half-integers values for $m \in (2\mathbb{Z}+1)/2$ in the Fourier series sending $d/du \to 2\pi i m$. The 1-loop determinants (up to a proportionality constant[38]) turns out to be [8]:

$$\det_{\text{fer}} = \left(\frac{\beta}{\Phi_{\text{ren}}} \right)^4 \frac{2}{\pi^3} \frac{\cos(\pi\alpha)^2}{(1 - 4(\alpha+n)^2)^2} . \tag{6.32}$$

Notice that this requires gauge fixing the zero-modes arising from the $m = \pm 1/2$ eigenvalues using the residual (fermionic) gauge freedom that one has from the $PSU(1,1|2)$ quotient. Notice also that once again one can use ζ-function regularization to obtain a finite result, together with the formula:

$$\prod_{m = \ldots, -\frac{5}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \ldots} \left(1 - \frac{\alpha+n}{m} \right) = \frac{\cos(\pi\alpha)}{1 - 4(\alpha+n)^2} . \tag{6.33}$$

We can now finally obtain the 1-loop quantum corrected partition functions, which thanks to fermionic localization are 1-loop exact. In the case of non-supersymmetric attractors, we get:

$$\begin{aligned} \mathcal{Z}^{\text{1-loop}}_{SL(2)\times SU(2)}(\beta,\alpha) =& \mathcal{Z}^{\text{cl}} \cdot \left(\det_{SL(2)} \right)^{-1/2} \left(\det_{SU(2)} \right)^{-1/2} \\ =& e^{\mathcal{S}_* - \beta \mathcal{E}_*} \sum_{n\in\mathbb{Z}} \left(\frac{\Phi_{\text{ren}}}{\beta} \right)^3 \left(\frac{\alpha+n}{\sin(2\pi\alpha)} \right) e^{\Phi_{\text{ren}} \frac{2\pi^2}{\beta} \left(1 - 4(\alpha+n)^2 \right)} . \end{aligned} \tag{6.34}$$

As for supersymmetric attractors, instead, we have:

$$\begin{aligned} \mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\beta,\alpha) =& \mathcal{Z}^{\text{cl}} \cdot \left(\det_{SL(2)} \right)^{-1/2} \left(\det_{SU(2)} \right)^{-1/2} \left(\det_{\text{fer}} \right) \\ =& e^{\mathcal{S}_* - \beta \mathcal{E}_*} \sum_{n\in\mathbb{Z}} \frac{\beta}{\Phi_{\text{ren}}} \frac{2}{\pi^3} \frac{(\alpha+n)\cot(\pi\alpha)}{(1-4(\alpha+n)^2)^2} e^{\Phi_{\text{ren}} \frac{2\pi^2}{\beta} \left(1 - 4(\alpha+n)^2 \right)} . \end{aligned} \tag{6.35}$$

The main difference between the partition functions is the different scaling in Φ_{ren}/β. As pointed out in [13], the origin of the prefactors $(\Phi_{\text{ren}}/\beta)^{\#}$ lies in the gauge fixing of the zero-modes; for each bosonic zero-mode that is gauge fixed away, one gets a factor $(\Phi_{\text{ren}}/\beta)^{1/2}$ in front of the classical partition function, while for each fermionic mode one gets a factor $(\Phi_{\text{ren}}/\beta)^{-1/2}$. In the case of $SL(2,\mathbb{R}) \times SU(2)$, one gauge fixes 6 bosonic modes, 3 coming from the $SL(2,\mathbb{R})$ invariance of the Schwarzian derivative and 3 from the $SU(2)$ invariance of the trace under $SU(2)$ rotations; thus we get the prefactor $(\Phi_{\text{ren}}/\beta)^3$. As for $PSU(1,1|2)$, we have a total of 6 bosonic generators and 8 fermionic generators in $PSU(1,1|2)$ which leave the $\mathcal{N} = 4$ super-Schwarzian invariant; we thus get the prefactor β/Φ_{ren}. However, we still need to perform the 1-loop determinant in

[38] This proportionality constant is chosen such that the extremal entropy is exactly $\mathcal{S}_*$, see the end of this section.

order to get the correct dependence on α.

Notice also that the prefactor $e^{\mathcal{S}_*}$ in front of $\mathcal{Z}^{\rm cl}$ would be multiplied by an arbitrary normalization constant which is dependent on the regularization scheme of the 1-loop determinants; since this can only shift by a finite amount the extremal entropy and cannot influence the qualitative features of the spectrum, we will not care about it and simply redefine $\mathcal{S}_*$ in such a way that it remains the extremal entropy. This highlights the fact that the Schwarzian action does not describe the extremal black holes themselves; rather, the Schwarzian is just an effective description of their near-extremal excitations. Nevertheless, super-Schwarzians have also been used in recent attempts [55, 56] to reproduce the microstate counting of BPS black holes using the gravitational path integral. In particular, the super-Schwarzians arise when regularizing gravitational zero-modes by turning on a small temperature; for discussion about this approach and its validity, see [55, 56, 57, 58].

Finally, before extracting the density of states $\rho(\mathcal{E},\mathcal{J})$, let us briefly discuss what we can already learn from (6.34) and (6.35). In particular, the extremal microstates can be counted by sending $\beta\to\infty$, allowing us to see how the 1-loop corrections (dependent on β) influence the extremal entropy of the black holes. Let us start from the non-supersymmetric case. Sending $\beta\to\infty$ in (6.34) is a bit tricky: in fact, we would like to replace

$$e^{\Phi_{\rm ren}\frac{2\pi^2}{\beta}\left(1-4(\alpha+n)^2\right)} \xrightarrow{\beta\to\infty} 1+\dots\,, \tag{6.36}$$

since we cannot perform the sum over n analytically when the exponential is present; however, this leads us to divergent sums over n, since the terms with $|n|\gtrsim\sqrt{\beta/\Phi_{\rm ren}}$ are not suppressed anymore by the exponential. Hence we will regularize the sum by summing over just $|n|\lesssim\sqrt{\beta/\Phi_{\rm ren}}$, i.e. over the terms that does not receive an exponential suppression. This way we have:

$$e^{-\mathcal{S}_*+\beta\mathcal{E}_*}\mathcal{Z}^{\text{1-loop}}_{SL(2)\times SU(2)}(\beta,\alpha)\approx{\sum_{n\in\mathbb{Z}}}'\left(\frac{\Phi_{\rm ren}}{\beta}\right)^3\left(\frac{\alpha+n}{\sin(2\pi\alpha)}\right)\sim\left(\frac{\beta}{\Phi_{\rm ren}}\right)^{-2}\xrightarrow{\beta\to\infty}0\,, \tag{6.37}$$

where the $'$ over the summation $\sum'_{n\in\mathbb{Z}}$ highlights that we are summing over $|n|\lesssim\sqrt{\beta/\Phi_{\rm ren}}$. Notice that adding more terms from the Taylor series of the exponential in (6.36) does not modify the result, since these additional terms are further suppressed as $\beta\to\infty$. Hence (6.37) signals that there is only a single microstate of extremal energy, in contrast to what one expects from the classical analysis of section 6.2. In the supersymmetric case, instead, we can use (6.36) directly, since the sum over n in (6.35) does not diverge as $n\to\infty$. The expression we get is

$$e^{-\mathcal{S}_*+\beta\mathcal{E}_*}\mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\beta,\alpha)\approx\sum_{n\in\mathbb{Z}}\frac{\beta}{\Phi_{\rm ren}}\frac{2}{\pi^3}\frac{(\alpha+n)\cot(\pi\alpha)}{(1-4(\alpha+n)^2)^2}\,, \tag{6.38}$$

which apparently diverges as $\beta \to \infty$. However, if we perform the sum over n, we have

$$\sum_{n\in\mathbb{Z}} \frac{(\alpha+n)}{(1-4(\alpha+n)^2)^2} = 0\,, \tag{6.39}$$

so we do not actually get an infinite degeneracy at extremality. This suggests that we have to Taylor expand further the exponential (6.36) as follows [28]:

$$e^{\Phi_{\text{ren}}\frac{2\pi^2}{\beta}\left(1-4(\alpha+n)^2\right)} \xrightarrow{\beta\to\infty} 1 + \frac{2\pi^2\Phi_{\text{ren}}}{\beta}\left(1-4(\alpha+n)^2\right) + \dots\,. \tag{6.40}$$

By plugging (6.40) in (6.35), we get:

$$\begin{aligned}\mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\beta,\alpha) &\approx e^{\mathcal{S}_*-\beta\mathcal{E}_*}\sum_{n\in\mathbb{Z}}\frac{2\beta}{\pi^3\Phi_{\text{ren}}}\frac{(\alpha+n)\cot(\pi\alpha)}{(1-4(\alpha+n)^2)^2}\left(1+\frac{2\pi^2\Phi_{\text{ren}}}{\beta}\left(1-4(\alpha+n)^2\right)\right)\\ &\approx e^{\mathcal{S}_*-\beta\mathcal{E}_*}\sum_{n\in\mathbb{Z}}\frac{4}{\pi}\frac{(\alpha+n)\cot(\pi\alpha)}{(1-4(\alpha+n)^2)}\,.\end{aligned} \tag{6.41}$$

This sum over n is now divergent, once again because we are still considering values of $|n| \gtrsim \sqrt{\beta/\Phi_{\text{ren}}}$ which would be suppressed by the exponential. We can proceed as before, summing over just $|n| \lesssim \sqrt{\beta/\Phi_{\text{ren}}}$ to regularize the sum. This way we get [8, 28]:

$$\sum_{n\in\mathbb{Z}}{}' \frac{(\alpha+n)}{(1-4(\alpha+n)^2)} = \sum_{-\sqrt{\beta/\Phi_{\text{ren}}}}^{+\sqrt{\beta/\Phi_{\text{ren}}}} \frac{(\alpha+n)}{(1-4(\alpha+n)^2)} \xrightarrow{\beta\to\infty} \frac{\pi}{4}\tan(\pi\alpha)\,. \tag{6.42}$$

The supersymmetric partition function in the $\beta \to \infty$ limits thus becomes:

$$e^{-\mathcal{S}_*+\beta\mathcal{E}_*}\mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\beta,\alpha) \xrightarrow{\beta\to\infty} 1\,; \tag{6.43}$$

hence, in the supersymmetric case, the extremal entropy remains finite and there is a degeneracy in the ground states of the theory. Notice once again that considering additional terms in the Taylor series (6.40) would not change the result, since they would be further suppressed in β.

6.4 Spectra of the black holes

We are now finally ready to calculate the quantum corrected spectra of the black holes. The partition functions (6.34) and (6.35) are partition functions for black holes in a "grand-canonical" ensemble with fixed temperature T and angular velocity Ω (along the z axis), but fixed electric and magnetic charges; we can thus write them as[39]:

$$\mathcal{Z}(\beta,\Omega) = \text{Tr}\left[e^{-\beta\hat{H}+2\beta\Omega\hat{J}_3}\right]\,, \tag{6.44}$$

[39] The 2 factor in $2\beta\Omega\hat{J}_3$ is due to our normalization of the angular velocity in (4.42).

where $\hat{H}$ is the Hamiltonian operator and $\hat{J}_3$ is the angular momentum operator along the z axis. From the grand-canonical partition function $\mathcal{Z}(\beta,\Omega)$ we are interested in extracting the microcanonical partition function $\rho(\mathcal{E},j_3)$, i.e. the energy density of states (or spectrum) of the black holes for fixed energy $\mathcal{E}$ and angular momentum j_3.

The most direct approach one can take is to express $\rho(\mathcal{E},j_3)$ directly in terms of $\mathcal{Z}(\beta,\Omega)$. 4d spherical symmetry (i.e. 2d $SU(2)$ gauge symmetry) implies that the Hamiltonian operator $\hat{H}$ can only be a function of the total angular momentum operator $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, and not of the single components $\hat{J}_i$ separately. Given that $\left[\hat{J}^2, \hat{J}_3\right] = 0$ and thus $\left[H, \hat{J}_3\right] = 0$, we can expand (6.44) as:

$$
\begin{aligned}
\mathcal{Z}(\beta,\alpha) = \mathrm{Tr}\left[e^{-\beta\hat{H}+4\pi i\alpha\hat{J}_3}\right] &= \int_0^\infty d\mathcal{E}\, e^{-\beta\mathcal{E}} \int_{-\infty}^{+\infty} dj_3\, e^{4\pi i\alpha j_3} \underset{\mathcal{E},j_3\text{ fixed}}{\mathrm{Tr}}\left[\mathbb{1}\right] \\
&= \int_0^\infty d\mathcal{E}\, e^{-\beta\mathcal{E}} \int_{-\infty}^{+\infty} dj_3\, e^{4\pi i\alpha j_3}\rho(\mathcal{E},j_3)\,,
\end{aligned}
\tag{6.45}
$$

where we assumed that the zero-point energy is such that $\hat{H}\left|\Omega\right\rangle = 0$ (with $\left|\Omega\right\rangle$ the lowest energy states). The grand-canonical partition function is therefore the Laplace transform with respect to $\mathcal{E}$ and the Fourier transform with respect to j_3 of the spectrum $\rho(\mathcal{E},j_3)$ of the black holes, i.e. of the probability density of finding a black hole with energy $\mathcal{E}$ and angular momentum j_3. We can get $\rho(\mathcal{E},j_3)$ back by performing an inverse Laplace transform and an inverse Fourier transform. Notice in particular that introducing α instead of Ω allows us to perform the inverse transforms in any order; this would have not been possible had we kept Ω instead of α, since we would have had the appearance of $\beta\Omega$ in the coefficient of the Fourier transform, forcing us to perform first the inverse Fourier transform (rather than the Laplace transform).

The problem with the above approach is that performing both the inverse Laplace transform and inverse Fourier transform can get quite cumbersome. We can try to circumvent this difficulty by guessing that the spectrum $\rho(\mathcal{E},j_3)$ is organized as a sum over $SU(2)$ representations[40]:

$$
\rho(\mathcal{E},j_3) = \sum_{\mathcal{J}=\mathbb{N}/2}\left[\rho(\mathcal{E},\mathcal{J})\sum_{j_3'=-\mathcal{J}}^{+\mathcal{J}}\delta\left(j_3 - j_3'\right)\right]\,. \tag{6.46}
$$

Here $\rho(\mathcal{E},\mathcal{J})$ is the probability density of finding a black hole with energy $\mathcal{E}$ and total angular momentum $\hat{J}^2 = \mathcal{J}(\mathcal{J}+1)$. This allows us to rewrite (6.45) as:

$$
\mathcal{Z}(\beta,\alpha) = \int_0^\infty d\mathcal{E}\, e^{-\beta\mathcal{E}} \sum_{\mathcal{J}\in\mathbb{N}/2}\chi_{\mathcal{J}}(\alpha)\rho(\mathcal{E},\mathcal{J})\,, \tag{6.47}
$$

[40] We consider $\mathcal{J} = \mathbb{N}/2$ for both supersymmetric and non-supersymmetric attractors; this is due to the fact that we have fermions in our theory, even around non-supersymmetric attractors. Had we considered a 4d gravity theory without fermions, we would have just kept $\mathcal{J} = \mathbb{N}$, as done for example in [7].

where the $SU(2)$ character is defined as:

$$\chi_{\mathcal{J}}(\alpha) = \sum_{j_3=-\mathcal{J}}^{+\mathcal{J}} e^{4\pi i\alpha m} = \frac{\sin\left((2\mathcal{J}+1)2\pi\alpha\right)}{\sin\left(2\pi\alpha\right)}\,. \tag{6.48}$$

This way, we can obtain $\rho(\mathcal{E},\mathcal{J})$ by first performing an inverse Laplace transform and then identifying the $\chi_{\mathcal{J}}(\alpha)$ and what factors multiply them, instead of performing the inverse Fourier transform. The choice of the expansion (6.46) is up to know just an educated guess, which will turn out to be correct; as shown at the end of this section, however, such an expansion in $SU(2)$ representations can be justified from the point of view of an $SU(2)$ BF theory on a disk.

Let us now proceed by starting from the $SL(2,\mathbb{R})\times SU(2)$ generalized Schwarzian. Performing the inverse Laplace transform of (6.34) yields:

$$\begin{aligned}\mathcal{Z}^{\text{1-loop}}_{SL(2)\times SU(2)}(\mathcal{E},\alpha) =& e^{\mathcal{S}_*}\frac{\Phi^2_{\text{ren}}}{2\pi^2}\frac{E}{\sin 2\pi\alpha}\theta(E)\cdot \\ &\cdot\sum_{n=-\infty}^{+\infty}\frac{\alpha+n}{1-4(\alpha+n)^2}I_2\left(\sqrt{8\pi^2\Phi_{\text{ren}}E}\sqrt{1-4(\alpha+n)^2}\right)\,,\end{aligned} \tag{6.49}$$

where we defined the energy above extremality $E \coloneqq \mathcal{E}-\mathcal{E}_*$ to write a more concise expression. Here θ is the Heaviside theta and I_2 is the second modified Bessel function of the first kind. Next, we rewrite the expression above by the means of the relation (3.42) of [8], obtained by applying the Poisson summation formula to the second line of (6.49):

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty}\frac{\alpha+n}{1-4(\alpha+n)^2}I_2\left(\sqrt{8\pi^2\Phi_{\text{ren}}E}\sqrt{1-4(\alpha+n)^2}\right) = \\ &\sum_{m=1}^{+\infty}\frac{m}{8\pi\Phi_{\text{ren}}E}\sin(2\pi m\alpha)\sinh\left(\pi\sqrt{8\Phi_{\text{ren}}E-m^2}\right)\theta\left(E-\frac{m^2}{8\Phi_{\text{ren}}}\right)\,.\end{aligned} \tag{6.50}$$

In particular, notice the presence of the θ function: it arises in the computation of the Fourier transform with respect to n, once expanding for asymptotically large values the Bessel function I_2 and closing the complex contour. Plugging back into (6.49) yields:

$$\begin{aligned}\mathcal{Z}^{\text{1-loop}}_{SL(2)\times SU(2)}(\mathcal{E},\alpha) =& e^{\mathcal{S}_*}\frac{\Phi_{\text{ren}}}{16\pi^3}\cdot \\ &\cdot\sum_{m=1}^{+\infty}m\frac{\sin(2\pi m\alpha)}{\sin(2\pi\alpha)}\sinh\left(\pi\sqrt{8\Phi_{\text{ren}}E-m^2}\right)\theta\left(E-\frac{m^2}{8\Phi_{\text{ren}}}\right)\,.\end{aligned} \tag{6.51}$$

We now notice that we can rewrite the coefficient $\sin(2\pi m\alpha)/\sin(2\pi\alpha)$ in terms of the characters $\chi_{\mathcal{J}}(\alpha)$ as:

$$\frac{\sin(2\pi m\alpha)}{\sin(2\pi\alpha)} = \chi_{\mathcal{J}}(\alpha)\,, \qquad\qquad \mathcal{J} = \frac{m-1}{2}\,. \tag{6.52}$$

Comparison with (6.47) allows us to extract the spectrum for near-extremal black holes around the non-supersymmetric attractors[41]:

$$
\begin{aligned}
\rho_{\text{non-SUSY}}(\mathcal{E},\mathcal{J}) =& e^{\mathcal{S}_*}\frac{\Phi_{\text{ren}}}{16\pi^3}\left(2\mathcal{J}+1\right)\rho_{\text{c}}\left(\mathcal{E},\mathcal{J}+\frac{1}{2}\right)\,, \\
\rho_{\text{c}}\left(\mathcal{E},\mathcal{J}\right) \coloneqq& \sinh\left[2\pi\sqrt{2\Phi_{\text{ren}}}\sqrt{\mathcal{E}-\left(\mathcal{E}_*+\frac{\mathcal{J}^2}{2\Phi_{\text{ren}}}\right)}\right]\theta\left[\mathcal{E}-\left(\mathcal{E}_*+\frac{\mathcal{J}^2}{2\Phi_{\text{ren}}}\right)\right]\,,
\end{aligned}
\tag{6.53}
$$

with $\mathcal{J}\in\mathbb{N}/2$. Note that when going from (6.51) to (6.53) we should further redefine $\mathcal{E}_*\to\mathcal{E}_*-1/8\Phi_{\text{ren}}$, such that the ground state energy is still $\mathcal{E}_*$; however, we will not do so in order to make the comparison with the spectrum around supersymmetric attractors (6.60) clearer. We can see that, after adding the gravitational quantum corrections, the spectrum has no discrete components. This implies that there is no extremal degeneracy in the spectrum — there is just a single extremal microstate — and there is no mass gap, in agreement with the analysis in the $\beta\to\infty$ limit of the partition function in section 6.3. This is in contrast with the classical result, which predicts an extremal entropy of Area$/4G_N$ — i.e. the presence of a huge degeneracy of extremal microstates — and possibly a mass gap (6.14).

Focusing now on supersymmetric attractors, the analysis of the spectrum of the $\mathcal{N}=4$ has already been performed in [8, 28]. The inverse Laplace transform of (6.35) is given by:

$$
\begin{aligned}
\mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\mathcal{E},\alpha) =& e^{\mathcal{S}_*}\frac{4\cot(\pi\alpha)}{\pi}\sum_{n=-\infty}^{+\infty}\frac{\alpha+n}{1-4(\alpha+n)^2}\cdot \\
&\cdot\left[\delta(E)+\frac{1}{2\pi^2\Phi_{\text{ren}}}\partial_E\delta(E)+\frac{1}{E}I_2\left(\sqrt{8\pi^2\Phi_{\text{ren}}E}\sqrt{1-4(\alpha+n)^2}\right)\right]\,.
\end{aligned}
\tag{6.54}
$$

The inverse Laplace transform contains both a continuum and a discrete part. The discrete part — which contributes only at extremality — seems rather problematic due to the presence of the $\partial_E\delta(E)$, which arises from the β multiplying the exponential in (6.35). As discussed already in the previous section 6.3, this apparent paradoxical behavior at extremality is due to the fact that we have yet to perform the sum over n, and we freely exchanged the sum and the inverse Laplace transform without any care. As shown in section 6.3, we actually have just an extremal

[41] In [8], the spectrum of an $SL(2,\mathbb{R})\times SU(2)$ generalized Schwarzian appears slightly modified, with $\mathcal{J}^2$ replaced by $\mathcal{J}(\mathcal{J}+1)$ in ρ_{c}; one might expect this, since in the BF formulation we get the Casimir of $SU(2)$, $C_2\sim\mathcal{J}(\mathcal{J}+1)$, in the expression of the energy. Given that we sum over $\mathcal{J}+1/2$ and not over $\mathcal{J}$, we have $\mathcal{J}(\mathcal{J}+1)=(\mathcal{J}+1/2)^2-1/4$; their spectrum is therefore equivalent to our result up to a shift in extremal energy of $1/8\Phi_{\text{ren}}$ (which as always we do not care about).

entropy $\mathcal{S}_*$ when taking $\beta \to \infty$; hence we will just replace (6.54) with[42]:

$$\begin{aligned}\mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\mathcal{E},\alpha) =& e^{\mathcal{S}_*}\delta(E) + e^{\mathcal{S}_*}\frac{4\cot(\pi\alpha)}{\pi}\sum_{n=-\infty}^{+\infty}\frac{\alpha+n}{1-4(\alpha+n)^2}\cdot \\ &\cdot\left[\frac{1}{E}I_2\left(\sqrt{8\pi^2\Phi_{\text{ren}}E}\sqrt{1-4(\alpha+n)^2}\right)\right]\,.\end{aligned} \tag{6.55}$$

We can once again simplify the term with the Bessel function I_2 in the above expression using (6.50), obtaining:

$$\begin{aligned}\mathcal{Z}^{\text{1-loop}}_{PSU(1,1|2)}(\mathcal{E},\alpha) =& e^{\mathcal{S}_*}\delta(E) + e^{\mathcal{S}_*}\frac{1}{2\pi^2\Phi_{\text{ren}}E^2}\cdot \\ &\cdot\sum_{m=1}^{+\infty} m\frac{\sin(2\pi m\alpha)}{\tan(2\pi\alpha)}\sinh\left(\pi\sqrt{8\Phi_{\text{ren}}E-m^2}\right)\theta\left(E-\frac{m^2}{8\Phi_{\text{ren}}}\right)\,.\end{aligned} \tag{6.56}$$

We are just left to reconstruct the coefficients in terms of the characteristics $\chi_{\mathcal{J}}(\alpha)$. For the $\delta(E)$, we simply notice that $1=\chi_0(\alpha)$: hence the $\delta(E)$ is the contribution of the ground state of total angular momentum $\mathcal{J}=0$. For the $\sin(2\pi m\alpha)/\tan(\pi\alpha)$, we recognize[43] [8]:

$$\frac{\sin(2\pi m\alpha)}{\tan(\pi\alpha)} = \left(\chi_{\mathcal{J}}(\alpha)+2\chi_{\mathcal{J}-1/2}(\alpha)+\chi_{\mathcal{J}-1}(\alpha)\right)\,, \qquad \mathcal{J}=\frac{m}{2}\,, \quad m\geq\frac{1}{2}\,. \tag{6.57}$$

Notice that the appearance of the combination $\chi_{\mathcal{J}}(\alpha)+2\chi_{\mathcal{J}-1/2}(\alpha)+\chi_{\mathcal{J}-1}(\alpha)$ is to be expected. The $\mathcal{N}=4$ super-Schwarzian is a supersymmetric quantum mechanics and thus we expect the spectrum to be organized in $\mathcal{N}=4$ supermultiplets [8]. In particular, we have 4 fermionic generators of the super-translations:

$$\{Q_p,\overline{Q}^q\} = 2\delta_p^q\hat{H}\,, \qquad \{Q_p,Q_q\} = \{\overline{Q}^p,\overline{Q}^q\} = 0\,; \tag{6.58}$$

when acting on a state $|\mathcal{J}\rangle$ of angular momentum $\mathcal{J}$, two of them will act as an $SU(2)$ doublet of lowering operators (sending $\mathcal{J}\to\mathcal{J}-1/2$) and two will act as an $SU(2)$ doublet of raising operator (sending $\mathcal{J}\to\mathcal{J}+1/2$), while leaving the energy $\mathcal{E}$ of the state unchanged. They also annihilate the ground state, defined as the state $|\Omega\rangle$ such that $\hat{H}|\Omega\rangle=0$. The spectrum of the $PSU(1,1|2)$ theory will thus be organized into $\mathcal{N}=4$ supermultiplets, containing both bosonic and fermionic states of the same energy. They can be constructed by starting with a state of maximum spin $\mathcal{J}$ and acting with the two lowering operators, obtaining the following supermultiplets:

$$\begin{aligned}&\mathbf{J} = \mathcal{J}\oplus 2\left(\mathcal{J}-\frac{1}{2}\right)\oplus\mathcal{J}-1\,, \qquad \mathbf{J}\geq 1\,,\\ &\frac{\mathbf{1}}{\mathbf{2}} = \frac{1}{2}\oplus 0\,.\end{aligned} \tag{6.59}$$

[42] See [28] for a more in depth explanation and a slightly different derivation of the same result.

[43] Notice that $\chi_{-1/2}(\alpha)=0$ [28], so that for $\mathcal{J}=1/2$ we get only $\chi_{1/2}(\alpha)+2\chi_0(\alpha)$.

Additionally, the fact that all the supercharges Q and $\overline{Q}$ annihilate the ground state implies that the ground state is by itself a full BPS state, i.e. it is its own supermultiplet (thus explaining why we get just the combination $\chi_0(\alpha)\delta(E)$). Finally, we get the following spectrum for near-extremal black holes around supersymmetric attractors:

$$\begin{aligned}\rho_{\text{SUSY}}(\mathcal{E},\mathcal{J}) =& e^{\mathcal{S}_*}\delta_{\mathcal{J},0}\delta(\mathcal{E}-\mathcal{E}_*) + \frac{e^{\mathcal{S}_*}}{\pi^2\Phi_{\text{ren}}(\mathcal{E}-\mathcal{E}_*)^2}\cdot \\ &\cdot\left[\mathcal{J}\rho_{\text{c}}(\mathcal{E},\mathcal{J}) + 2\left(\mathcal{J}+\frac{1}{2}\right)\rho_{\text{c}}\left(\mathcal{E},\mathcal{J}+\frac{1}{2}\right) + (\mathcal{J}+1)\,\rho_{\text{c}}\left(\mathcal{E},\mathcal{J}+1\right)\right],\end{aligned} \tag{6.60}$$

where $\mathcal{J}\in\mathbb{N}/2$ and where we used once again the function $\rho_{\text{c}}(\mathcal{E},\mathcal{J})$ defined in (6.53). We see that (6.60) is quite different from (6.53). The main difference is the presence of the $\delta(\mathcal{E}-\mathcal{E}_*)$, signaling that in the supersymmetric case we do indeed have a degeneracy of the ground state, similarly to the classical analysis. We then have a continuum of states of growing energies, organized by multiplets; they are separated by a gap $\Delta\mathcal{E}_{\text{gap}}$ with no states from the lowest energy discrete states at $\mathcal{E}=\mathcal{E}_*$, with:

$$\Delta\mathcal{E}_{\text{gap}} = \frac{1}{8\Phi_{\text{ren}}} = \frac{1}{8}\frac{G_N}{W_0^3}\,. \tag{6.61}$$

This is exactly the mass gap (6.14) that was argued directly from the classical thermodynamics. Notice that the gap is determined by the value of the superpotential at the horizon $W_0 = W(z_0^i,\bar{z}_0^{\bar{i}})$, which in turn is just a function of the electric and magnetic charges of the black hole (q_I and p^I) and of the values of the scalars z_0^i at the attractor of the flow; in particular, these are the quantities that characterize the near-horizon of the extremal black holes, and there is no dependence on quantities set at spatial infinity. Once again, we see that the problem of the mass gap, arising in the classical regime and signaling the breakdown of the thermodynamic description, is actually solved by the gravitational quantum corrections. Notice that some examples of black holes microstates counting in string theory suggest exactly the presence of a mass gap and of a degeneracy in the ground state [5, 6]. These qualitative features of the spectrum thus appear to be a consequence of supersymmetry, rather than general properties of generic near-extremal black holes.

For comparison between non-supersymmetric and supersymmetric attractors, we plot in Figure 1[44] the behavior of $\rho_{\text{non-SUSY}}$ and ρ_{SUSY}. In particular, notice the presence of the $\delta(\mathcal{E}-\mathcal{E}_*)$ in the supersymmetric $\mathcal{J}=0$ spectrum (together with the mass gap $\Delta\mathcal{E}_{\text{gap}} = 1/8\Phi_{\text{ren}}$), which is not present in the non-supersymmetric one. One can also notice a kink of $\rho_{\text{SUSY}}(\mathcal{E},1/2)$ for the supersymmetric spectrum at $\mathcal{E}-\mathcal{E}_* = 1/8\Phi_{\text{ren}}$; this is due to the contributions coming from different $\mathcal{N}=4$ supermultiplets to the same $\mathcal{J}$ (and happens for all the $\mathcal{J}$ in the supersymmetric case, even if it is not evident in the plot for $\mathcal{J}=0,1,3/2$).

[44] Due to the different multiplicative factors of ρ_{c} in (6.53) and (6.60), we have rescaled the y axis differently so that one can better compare the qualitative features of the two spectra.

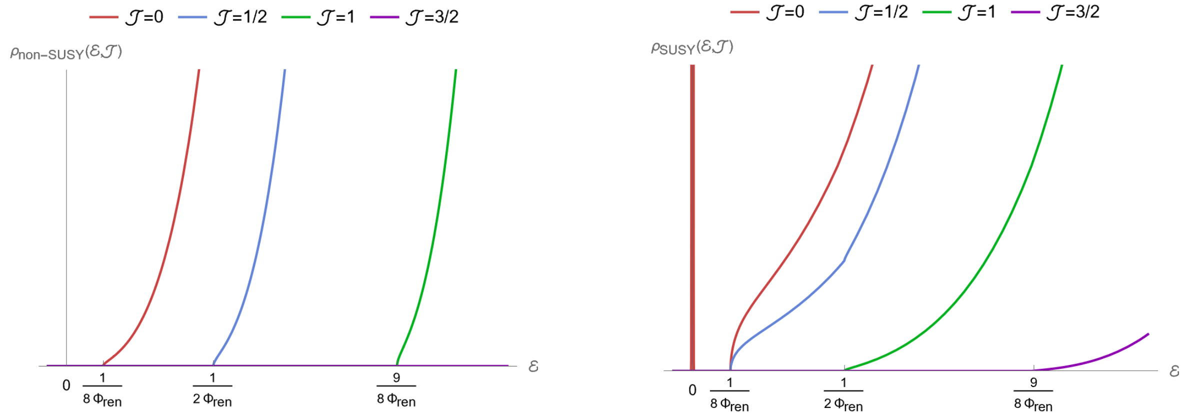

Figure 1: Energy density of states (or spectrum) $\rho(\mathcal{E},\mathcal{J})$ for near-extremal black holes around non-supersymmetric (on the left) and supersymmetric (on the right) attractors, as a function of the black hole energy $\mathcal{E}$; we set the extremal energy $\mathcal{E}_* = 0$ and plotted the spectra for values of total angular momentum $\mathcal{J} = 0, 1/2, 1, 3/2$.

At last, let us briefly discuss why the spectrum is organized as a sum over $SU(2)$ representations — weighted by their characters — by starting directly from the BF theories. Let us start by considering a pure $SU(2)$ BF theory, which arises around non-supersymmetric attractors (see section 4.7). Since we want to calculate the partition function on a disk, we will proceed with radial quantization, using r as the radius and t as our (periodic) angular coordinate. To quantize the gauge theory, we will follow [40]. To avoid dealing with ghosts, we can pick the Coulomb gauge:

$$D_0 H_{01} = \partial_0 H_{01} - [B_0, H_{01}] = 0\,. \tag{6.62}$$

In canonical quantization, this equation must be imposed as a constraint on the wavefunction (or more precisely wavefunctional) $\Psi\left[B^i{}_0(x)\right]$; as an operator equation, it becomes:

$$\left[\partial_0 \frac{\delta}{\delta B^i{}_0(x)} + \varepsilon_{ijk} B^j{}_0(x) \frac{\delta}{\delta B^k{}_0(x)}\right] \Psi = 0\,. \tag{6.63}$$

The above equation is solved by wavefunctions of the form:

$$\Psi\left[B^i{}_0(x)\right] = \Psi\left[\mathcal{P} e^{\oint_0^\beta dt\, B_0}\right] = \Psi\left[e^{-2i\Omega\beta T_3}\right]\,, \tag{6.64}$$

i.e. the wavefunction is just a function of the holonomy of the $SU(2)$ gauge field (4.42). In other words, we have that the Hilbert space of states is exactly the space of L^2 functions on $SU(2)$. Using the Peter-Weyl theorem, $L^2(SU(2))$ can be decomposed into matrix elements of the unitary irreducible representations of $SU(2)$, and a natural basis for such a space is provided by the characters in the unitary irreducible representations $\chi_{\mathcal{J}}\left(e^{\oint_0^\beta dt\, B_0}\right)$ [40]. This allows us to expand the trace over the Hilbert space — used to define the partition functions — as a sum over representations.

The partition function of the BF theory on the disk can be obtained by gluing together the contribution of a cylinder and of an infinitesimal cap [40]. The Hamiltonian of the theory is given just by (minus) the boundary term (or defect) (4.79) and is simply a function of Tr $[b^2]$. The b fields are proportional to the conjugate momenta to the B_0 (up to an $-2i$ factor); in canonical quantization, they act as functional derivatives on the wavefunctions and thus — in the chosen basis of irreducible representations — they act simply as the generators of $SU(2)$:

$$b_i = 2\frac{\delta}{\delta B^i{}_0} \qquad \Longrightarrow \qquad b_i\,\chi_{\mathcal{J}}\left(\mathcal{P}e^{\oint_0^\beta dt\,B_0}\right) = 2\chi_{\mathcal{J}}\left(T_i\,\mathcal{P}e^{\oint_0^\beta dt\,B_0}\right)\,. \tag{6.65}$$

Since $\sum_i T_i T^i = C_2(\mathcal{J}) = \mathcal{J}(\mathcal{J}+1)$, the Hamiltonian is diagonalized in the chosen basis and depends only on $\mathcal{J}$. It can be shown [41] that the grand-canonical partition function of an $SU(2)$ BF theory on a disk is of the form:

$$\mathcal{Z}(\beta,\alpha) = \sum_{\mathcal{J}\in\mathbb{N}/2} (2\mathcal{J}+1)\,\chi_{\mathcal{J}}(\alpha)e^{-\beta h(\mathcal{J})}\,, \tag{6.66}$$

where $h(\mathcal{J})$ is some function closely related to the Hamiltonian (which we will not determine explicitly here). This matches exactly the expansion (6.47), and also predicts the factor $2\mathcal{J}+1$ that appears in (6.53).

As for the $PSU(1,1|2)$ BF theory, things get more complicated since $PSU(1,1|2)$ is non-compact. As shown in [24], however, similar relations as the one used above still hold for non-compact groups, once appropriately generalized[45]. Heuristically, we expect the partition function of the $PSU(1,1|2)$ super-BF theory on a disk to behave similarly to the partition function on a disk of a generic BF theory with a compact gauge group, that is to be organized as a sum over $PSU(1,1|2)$ representations like above:

$$\mathcal{Z}(\beta,\alpha) = \sum_R \dim(R)\chi_R\,(\beta,\alpha)\,e^{-\beta\tilde{h}(C_2(R))}\,; \tag{6.67}$$

here R is a generic $PSU(1,1|2)$ representation, χ_R is the character of such a representation and $\tilde{h}(C_2(R))$ is a function of the Casimir of the representation. Notice that while we write $\sum_R$ for the purpose of our discussion (as for usual compact gauge groups), for non-compact groups one should consider an integral over continuous representations, rather than a sum over discrete ones [24]. As showed in (6.59), each $\mathcal{N}=4$ supermultiplet (i.e. irreducible representation of $PSU(1,1|2)$) contains four $SU(2)$ irreducible representations. Therefore, we expect to be able to rewrite the partition function — isolating the $SU(2)$ contribution — using (6.59) as [8, 28]

$$\mathcal{Z}(\beta,\alpha) = \sum_{\mathcal{J}\in\mathbb{N}/2} 8\mathcal{J}\left[\chi_{\mathcal{J}}(\alpha) + 2\chi_{\mathcal{J}-1/2}(\alpha) + \chi_{\mathcal{J}-1}(\alpha)\right](\ldots)\,; \tag{6.68}$$

[45] In particular, [24] generalized the above procedure to an $SL(2,\mathbb{R})$ theory, describing yet another way to quantize the Schwarzian.

in particular, notice the $8\mathcal{J}$ prefactor, coming from the sum of the dimensions of the four $SU(2)$ representations:

$$8\mathcal{J} = (2\mathcal{J}+1) + 2\left(2\left(\mathcal{J}-\frac{1}{2}\right)+1\right) + (2(\mathcal{J}-1)+1) \ . \tag{6.69}$$

The above expressions match exactly (6.57), together with the factor $m = 2\mathcal{J} = 8\mathcal{J}/4$ present in (6.56).

To sum up, we calculated the quantum corrected 1-loop exact partition functions of the generalized Schwarzian theories; then we extracted the spectra $\rho(\mathcal{E}, \mathcal{J})$ (plotted in Figure 1), i.e. the probability of finding a black hole with energy $\mathcal{E}$ and total angular momentum $\mathcal{J}$. The quantum corrections calculated using the Schwarzian theories profoundly modify the thermodynamics of the black holes. In particular, the thermodynamics around supersymmetric attractors is still similar to the classical thermodynamics: we have a degeneracy in the ground state — as predicted by the Bekenstein–Hawking area law — together with a mass gap where no state is present. In contrast, the spectrum around non-supersymmetric attractors strongly deviates from the classical prediction: the entropy $\mathcal{S}$ goes to $-\infty$ as $T \to 0$, and there is just a single non-degenerate ground state. The behavior around supersymmetric attractors supports what has been argued in [5, 6] from string construction of supersymmetric extremal black holes; the behavior around non-supersymmetric attractors is instead quite different, suggesting that the presence of an extremal degeneracy and of a mass gap is a consequence of supersymmetry, rather than a general property of black holes.

7 Discussion and outlook

7.1 Results of this work

Before discussing some open questions, let us briefly summarize the results of this work. We obtained the 1d effective theories describing near-extremal and near-BPS black holes in $\mathcal{N}=2$ ungauged supergravity (without hypermultiplets), in a 4d asymptotically flat background. We focused both on BPS and on so called "fake-BPS" static black holes — which we reviewed in section 2 — in the context of the $\mathcal{N}=2$ black hole attractor mechanism. The latter are extremal black holes whose geometry satisfies first order differential equations, despite not being supersymmetric. They can be obtained via very simple changes from BPS ones, as simple as flipping the sign of a charge. In particular, we worked at fixed values of the electric charges and with no magnetic charges, looking at near-extremal black holes which are slowly rotating. We then used the 1d effective theories to calculate the quantum corrected (and 1-loop exact) partition function and energy density of states; notice that we neglected all the corrections to the extremal entropy and energy of the black holes, focusing only on the qualitative features of the spectra. These effective theories are generalizations of the usual Schwarzian action appearing in the description of near-extremal black holes in Einstein–Maxwell theory, including additional gauge and fermionic degrees of freedom.

We followed two different approaches to obtain the generalized Schwarzians. First — in section 3 — we studied the symmetry breaking pattern in the near-horizon of the extremal black holes, guessing the 1d theory in the spirit of effective field theories. Then — in sections 4 and 5 — we performed the Kaluza–Klein dimensional reduction from the original 4d supergravity to a 2d dilaton gravity; in the near-extremal limit, the latter becomes a generalized JT gravity, itself related to a generalized Schwarzian. The near-extremal dynamics around non-supersymmetric attractors is described by the action of a Schwarzian mode and a particle moving on $SU(2)$, realizing the $SL(2,\mathbb{R})\times SU(2) \to \varnothing \times U(1)$ symmetry breaking; as for supersymmetric attractors, the dynamics is determined by the $\mathcal{N} = 4$ super-Schwarzian, realizing the $PSU(1,1|2) \to \varnothing \times U(1)$ symmetry breaking.

In both approaches, we emphasized that we cannot focus only on the bosonic side of the theories, but we need also an analysis of the fermionic side to correctly differentiate between the near-extremal dynamics around supersymmetric and non-supersymmetric attractors. This can be understood directly from the original 4d action (2.1): the bosonic part of the supergravity action, in fact, does not distinguish between supersymmetric and non-supersymmetric attractors (while the kinetic $\mathcal{N}_{IJ}$ might change, it does not alter the calculations of the effective theories).

Near-BPS and near-extremal dynamics can only be distinguished by the quantities that appear directly in the supersymmetric transformations, such as the graviphoton field strength T^- given in (2.26); T^- however appears explicitly only in the fermionic part of the 4d supergravity action. To distinguish the two cases we introduced in (3.11) the parameter ζ, which measures how "far" — in terms of distance from the BPS bound on the mass — the extremal solution is from a truly BPS one; we have $\zeta = 1$ for supersymmetric attractors, while the BPS bound on the black hole mass implies $0 \leq \zeta < 1$ for non-supersymmetric attractors. In the analysis of the symmetries, what sets supersymmetric and non-supersymmetric attractors apart are exactly the superisometries (i.e. the supersymmetries preserved by the solution), which are only present for $\zeta = 1$. In the dimensional reduction approach, the two attractors can be distinguished only when considering the gravitinos and the gauginos quadratic contributions. In particular, the gravitinos play a central role. From their 4d kinetic term and the interaction with the graviphoton field strength, they receive a mass term proportional to $(1-\zeta)$. Therefore they become heavy and thus negligible — for small energies above extremality — for the non-supersymmetric attractors, while they remain massless — supplementing the fermionic part of $\mathcal{N} = 4$ JT supergravity — in the case of supersymmetric attractors.

In the dimensional reduction approach, we found particularly useful to relate the generalized JT gravity to a 2d BF theory, a topological gauge theory. This highlights the topological nature of the reduced 2d theory, simplifying the process of relating it to a generalized Schwarzian. This, together with supersymmetry, has also been fundamental in imposing the correct mixed boundary conditions for the 2d action, needed to properly recover the correct generalized Schwarzians from the JT gravities; we will discuss more in depth this delicate (and still not fully understood) point later in section 7.3.

Finally, in section 6, we analyzed the partition functions of the effective Schwarzians and extracted the energy density of the states in the two cases. The partition functions are actually 1-loop exact: since we are integrating over (super-)symplectic manifolds with an $U(1)$ symmetry (given by Euclidean time translations) in the path integrals, we can apply the Duistermaat–Heckman theorem [13], localizing the path integrals. The 1-loop determinants all have zero-modes, due to the invariance of the generalized Schwarzian under "generalized" $SL(2,\mathbb{R})$ transformations; in the path integral, however, we quotient exactly over these symmetries of the generalized Schwarzians, and thus the zero-modes can be gauge fixed away. Extracting the spectrum, we see that the problem of the mass gap and the breakdown of the semiclassical thermodynamic limit we have discussed in section 1 is solved once we add the gravitational quantum corrections. The solution is different depending on whether we are around a supersymmetric attractor or not: in the former case, we have a degenerate ground state separated by a mass gap from the other continuous states (hence there are no non-extremal black holes that cannot emit Hawking quanta); in the latter case, the quantum corrections remove the extremal degeneracy and raise the energy such that black holes can always radiate (and the thermodynamic description never breaks down). This shows that the mass gap, first conjectured from black holes constructions in string theory, is closely related to supersymmetry. This suggests that, in general, we have to be careful when extending considerations coming from supersymmetry to non-supersymmetric black holes, already at the qualitative level. We also see that the semiclassical prediction — from the Euclidean path integral — is still a good approximation when supersymmetry is preserved; vice-versa, when supersymmetry is broken, the quantum corrections dramatically alter the semiclassical thermodynamics, and in particular the Bekenstein–Hawking area law. These results agree with the analysis of [7, 8, 9, 10], where near-extremal and near-BPS black holes were studied separately in different theories.

Finally, we notice that this is not the only example where quantum corrections coming from the gravitational side solve some of the paradoxes arising in the semiclassical treatment of black holes. In fact, a similar situation has been highlighted in [59] regarding the information paradox. In particular, in a simplified 2d setup, it has been shown that the Page curve can be recovered just from the gravitational path integral, once replica wormholes are included in the entanglement entropy calculation.

7.2 Near-CFT$_1$ duals, SYK and matrix models

We now discuss some possible directions of future work. For starters, an interesting idea is to study the Near-AdS$_2$/Near-CFT$_1$ correspondence in the cases of JT gravity coupled to an $SU(2)$ gauge field and of $\mathcal{N}=4$ JT supergravity, searching for holographic models describing the near-extremal and near-BPS dynamics respectively. As for $\mathcal{N}=0,1,2$ JT (super)gravities, it has been shown that the $\mathcal{N}=0,1,2$ super-Schwarzians can be obtained in slightly different ways: as the low-energy, strong-coupling conformal limit of SYK-like models [60, 44] and as double-scaled single-cut matrix models [23, 61, 28].

The SYK model is a 1d model of N Majorana fermions, interacting via q-particle couplings picked randomly from a Gaussian distribution; one can then perform a quenched or annealed average over the couplings to evaluate the thermodynamic potentials and the correlation functions [62, 63, 64]. In the large-N, strong coupling regime the SYK develops reparameterization invariance, typical of a "1d CFT"; the effective action describing the low-energy limit of the model is exactly the Schwarzian action [11, 12]. While currently there is no clear way to relate directly the SYK model to a bulk gravitational description, it is believed that the SYK model may be a good candidate for a holographic UV completion of JT gravity; for some recent progress in relating the boundary SYK to the gravitational bulk, see [24]. There have been various generalizations of the SYK model in the last decade. For example, the SYK model has been generalized in the cases of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry [44], and the connection with the $\mathcal{N} = 1, 2$ super-Schwarzians has been established. The latter in particular arises from $\mathcal{N} = 2$ JT supergravity, which in turn can appear in the dimensional reduction of near-BPS higher dimensional black holes (see for example near-BPS black holes in 5d supergravity [9]). Therefore, it could be interesting to formulate a $\mathcal{N} = 4$ generalization of the SYK model and study its relation to the $\mathcal{N} = 4$ super-Schwarzian. There are also generalizations of the SYK model which instead add invariance under some global symmetry group, without including supersymmetry [65, 66, 67]; hence one might hope that a similar setup with $SU(2)$ symmetry would contain the Schwarzian mode together with the particle moving on $SU(2)$ in its low energy, conformal limit.

Another interesting question would be to understand the role of the gravitinos in the boundary dual, or in other words what is the mechanism in the boundary that leads either to the $\mathcal{N} = 4$ super-Schwarzian or just to its bosonic part, depending on whether we expand around supersymmetric or non-supersymmetric attractors. Since the gravitino sources the supercurrent of the dual field theory, a massless/massive gravitino should correspond to a conserved/broken supercurrent, respectively.

As for matrix models, instead, it was first shown in [23] that JT gravity is equivalent to double-scaled matrix model. A matrix model is a 1d quantum mechanical system, whose Hamiltonian is a random $N \times N$ matrix; one then averages over all the possible matrices weighted by some potential. In the large-N limit, the model reproduces the partition functions of JT gravity with various boundary conditions. In particular, as highlighted in [23], one gets from the matrix model the full sum over different topologies that arises when evaluating the non-perturbative corrections to the JT gravity partition function. This correspondence has also been expanded to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ JT supergravity, in [61] and (very recently) in [28] respectively. One can therefore pose the same questions as for the generalized SYK models: can we formulate a $\mathcal{N} = 4$ generalization of the matrix models (and its correspondence with the sum over topologies of $\mathcal{N} = 4$ JT supergravity) and what mechanism differentiate between supersymmetric and non-supersymmetric attractors directly in the matrix model? Some step towards answering the former are also discussed in [28].

7.3 Propagation of the boundary conditions

A delicate point that remains to be clarified is how to pick the boundary conditions of the NHR region. These boundary conditions determine the 1d effective boundary theory, which in turn determines the near-extremal dynamics and the black hole spectrum. Understanding how we should choose the boundary conditions is therefore at the core of understanding the behavior of black holes near-extremality in any dimension. While in principle one can pick "by hand" the appropriate boundary condition based on the symmetry considerations of section 3, we would like to better understand the underlying physical mechanism.

Regarding the gravitational boundary conditions, one typically assumes Dirichlet boundary conditions for the metric and the dilaton [12]; the idea is that NHR and FAR region should be joined at their boundary, which must therefore coincide. The Dirichlet boundary conditions forces us to add the Gibbons–Hawking–York boundary term, which is also essential to recover the classical extremal entropy of the black holes. While this choice reproduces correctly the Schwarzian mode (the "Goldstone boson" of the broken $SL(2,\mathbb{R})$ symmetry), it is not particularly clear how it should be generalized for other fields. Another question is whether it is correct to do this at the level of the 2d action, or whether we should already separate the NHR and FAR region in the original 4d theory. This is particularly problematic since the 2d $SU(2)$ gauge field is itself part of the 4d metric; therefore, if we simply applied Dirichlet boundary conditions for the 4d metric, we would get the Dirichlet boundary conditions not only for the dilaton and the 2d metric, but also for the $SU(2)$ gauge field (which would make the $SU(2)$ contribution trivial).

A procedure for finding the boundary conditions of other fields has been proposed in [7, 8, 10]. The proposal is to "propagate" the boundary conditions at infinity up to the NHR boundary by using the equation of motion. As we discussed in more details in section 4.3, we find that this procedure does not yield consistent boundary conditions (let alone reproduce the boundary conditions needed for the $\mathcal{N}=4$ super-Schwarzian). This contradiction with the results of [7, 8, 10] is unclear to us at the moment. Notice that another way to check the validity of the claim of [7, 8, 10] could be to repeat the propagation for the metric and the dilaton; if propagating the boundary conditions is a correct procedure, we should expect to recover the Dirichlet boundary conditions. Once again, we could also ask what changes if we do this procedure in 4d (and not in the dimensionally reduced 2d theory).

Finally, as we explained in section 5.3, we decided to pick the boundary conditions by using insights from the supersymmetric theory. The Dirichlet boundary conditions for the metric and the dilaton are converted to mixed boundary conditions in the first order BF formulation of JT gravity for the $SL(2,\mathbb{R})$ gauge field $\mathcal{A}_{SL(2)}$. Then, in the $PSU(1,1|2)$ BF theory, supersymmetry imposes the same boundary conditions on both the $SU(2)$ gauge field $\mathcal{A}_{SU(2)} \equiv B$ and the other fermionic components. Finally, since the $SL(2,\mathbb{R}) \times SU(2)$ BF theory is just the bosonic part of the $PSU(1,1|2)$ BF theory, we also impose the same boundary conditions for the gauge field in that case. The problem with this approach is that it is not easily generalizable to other fields or to other boundary conditions at infinity (such as fixing the field strength). For example, the electromagnetic $U(1)$ gauge fields do not combine in a unique BF theory with the gravitational $SL(2,\mathbb{R})$ components, and thus we cannot use supersymmetry to relate the boundary conditions of the two fields. One possible workaround could be to consider the $U(1)$ gauge fields as the gauged isometry coming from the Kaluza–Klein reduction of a higher dimensional theory. This way, we could get some black holes with a larger set of near-horizon supersymmetries in this higher dimensional theory, with superisometry group $\mathcal{G}$ containing the subgroup $SL(2,\mathbb{R}) \times U(1)^{n_V+1} \subset \mathcal{G}$. The resulting super-BF theory describing the near-horizon dynamics will therefore be based on $\mathcal{G}$. This way, we can use the enhanced set of supersymmetric transformations to relate the boundary conditions for the $SL(2,\mathbb{R})$ gauge field $\mathcal{A}_{SL(2)}$ to the boundary conditions for the $U(1)$ gauge fields $\mathcal{A}_{U(1)} \equiv a^I$, similarly to how we used the $PSU(1,1|2)$ supersymmetries to obtain the $\mathcal{A}_{SU(2)} \equiv B$ boundary conditions as explained above (5.36) in the $PSU(1,1|2)$ super-BF theory. This way, since we expect the same boundary conditions to hold for the bosonic sector even when no supersymmetry is present, we have also obtained the boundary conditions for the $U(1)$ gauge fields even when no supersymmetry is present. The main problem with this workaround is that it does not tell us what precisely is the meaning of gluing the NHR and FAR region together; another drawback is that it still cannot be applied if we do not fix the holonomy of the $U(1)$ gauge fields at infinity (but choose some other boundary conditions instead).

7.4 Different kinds of backgrounds

In this work we used two different approaches to find the 1d effective theory describing the near-extremal black holes: first, we studied the symmetry of the extremal solutions; second, we performed a Kaluza–Klein dimensional reduction from 4d to 2d. Both these approaches are quite general, and thus an interesting direction is to apply them to different black hole solutions (or maybe even to different spacetimes altogether). These black holes solutions may differ for the number of dimensions, for their matter content, for their preserved supersymmetries and for the value of their cosmological constant.

For example, black holes in an AdS background have been studied in [8, 9]; depending on the preserved supersymmetries, they admit a near-BPS limit described by $\mathcal{N} = 2$ [9] and $\mathcal{N} = 4$ [8] super-Schwarzians. More exotic black holes have also been studied: for example, [68] focuses on Einstein–Maxwell theory in presence of a positive cosmological constant. Depending on how the near-extremal limit is taken, one can get three different near-horizon geometries: the usual $\mathrm{AdS}_2 \times S^2$ and the two unusual $\mathrm{dS}_2 \times S^2$ and $\mathrm{Mink}_2 \times S^2$. The $\mathrm{AdS}_2 \times S^2$ arises when the inner and outer horizons of the black holes coincide (as usual for typical extremal black holes), and is related to the Schwarzian action. The $\mathrm{dS}_2 \times S^2$ arises when the outer horizon coincides with the cosmological dS horizon. As is often the case for de Sitter spaces, dS JT gravity is harder to make sense of than regular (AdS) JT gravity; some recent developments are described in [42, 69, 70], highlighting also a connection to complex SYK models. Finally, the $\mathrm{Mink}_2 \times S^2$ arises from a particular limit where the three horizons coincide; it appears to be connected to the so called Callan–Giddings–Harvey–Strominger (CGHS) model, a 2d dilaton gravity which admits a flat background [69]. In turn, the CGHS model is related to the action of a Schwarzian mode together with a particle moving on a $U(1)$ group manifold [69]. Quantization of this theory has been performed in [71]; similarly to the cases of other non-supersymmetric Schwarzians, quantum corrections strongly modify the spectrum at low temperature, removing the extremal degeneracy and leaving only a single extremal microstate. This is to be expected from the relation between this generalized Schwarzian and the near-extremal black holes in dS; the latter in fact are non-supersymmetric, and thus from the result of this work we would expect that their entropy deviates from the semiclassical Bekenstein–Hawking area law due to growing quantum corrections as $T \to 0$.

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