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A stochastic control perspective of multi-curve term structures under the benchmark approach

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Chapter 1

Introduction

In recent years, both market analysts and academic researchers have had to face new challenges due to the dramatic turbulences in the financial markets. The *interest rate* (or *fixed income*) *market* has been deeply affected by the credit crisis that exploded in 2007 and led to a paradigm shift in interest rate theory. An increase in credit and liquidity risk in the interbank system has given rise to significant *spreads* between interbank (Libor/Euribor) and risk-free rates, prompting mathematical finance experts to explore new modeling perspectives for the interest rate market. Given the need to provide a framework for spread term structures, several authors have proposed martingale models based on the assumption that the fixed income market is *free of arbitrage*. As in the pre-crisis environment, these models can be divided into three fundamental classes: the short-rate models, the Heath-Jarrow-Morton (HJM) framework, and the Libor market models. The exhaustive monograph of Z. Grbac and W.J. Runggaldier (2015) summarizes all these approaches, which typically have in common the fact of presupposing the existence of an *equivalent martingale measure* (EMM). Focusing on *spot spreads*, Backwell et al. (2019) developed an arbitrage-free approach which links spreads to *roll-over risk*, providing an economic explanation which interprets the spot spread as a *term premium* paid by the borrower of a Libor loan to avoid roll-over risk.

While martingale models for spread curves have been carefully studied to address post-crisis challenges, relatively little attention has been paid to models that do not ensure the existence of equivalent martingale measures. As explained in many textbooks (e.g. Björk T., *Arbitrage theory in continuos time*), a market model provides an equivalent martingale measure if and only if it meets the *No Free Lunch With Vanishing Risk* (NFLVR) condition, which is the cornerstone of the classical no-arbitrage theory and it allows to apply the *risk-neutral* approach to the pricing of contingent claims. Since financial phenomena cannot always be analyzed with standard no-arbitrage theory based on NFLVR, several authors have studied instances of market where an equivalent martingale measure may fail to exist. In particular, C. Fontana and W.J. Runggaldier (2013) studied a general class of diffusion-based financial models without relying on the existence of an EMM and they shown that a fair valuation of contingent claims is still possible, provided that only limited arbitrage opportunities are permitted. In such a context, the risk-neutral pricing is replaced by the so-called *benchmark approach*. The first purpose of the present work is to provide a roll-over risk formulation for spreads under the benchmark approach and use it to express the *spot and forward spreads* as solutions of partial differential equations (PDEs) with terminal conditions. To this end, we will need to introduce a Markov structure depending on a multifactor process.

The representation via PDEs makes it possible to link the pricing of interest rate derivatives to dynamic stochastic optimization. Stochastic control theory studies how to optimize performance criteria subject to stochastic dynamics and it finds wide application in finance, mainly for portfolio optimization. An organic discussion on stochastic control techniques and their applications can be found in H. Pham (2009). In the work of A. Gombani and W.J. Runggaldier (2013) it is shown that multifactor term structures free of arbitrage can be represented as solutions of suitable stochastic control problems, providing an alternative approach to pricing of bond derivatives. The second aim of the thesis is to extedend their analysis and to derive a stochastic control perspective of bonds and spreads under the benchmark approach.

Within the vast panorama of stochastic control, a special class of problems explicitly characterizes investors' risk attitude through a *risk-sensitivity parameter*. They are called *risk-sensitive* control problems and they were extensively investigated by M. Davis and S. Lleo (2011), due to their application to dynamic benchmarked asset management. As far as we know, risk-sensitive approach has however not been applied in the context of spread modeling. The last and most significant aim of the thesis is to provide a risk-sensitive representation of spreads, accompanied by a comprehensive economic interpretation and a possible relation with the roll-over risk approach.

The thesis is structured as follows. In chater 2 we provide an overview on the benchmark approach. In particular, Sect. 2.1 introduces the general setting, which consists of a diffusion-based market model including risky assets and a savings account, and it recalls the basic concepts of *self-financing investment strategy* and *portfolio process*. In Sect. 2.2 we discuss different notions of no-arbitrage that are weaker than the traditional NFLVR condition and we introduce the concept of *viable market*, which closely related to the *market price of risk*. In Sect. 2.3 we show that in viable markets it is possible to evaluate contingent claims without martingale measures, due to the *numéraire property* of the *growth-optimal portfolio* (GOP). In chapter 3 we adapt the roll-over risk formulation of spreads to the benchmark approach.

More specifically, in Sec. 3.1 we recall some basics from the standard interest rate theory and we enlarge the diffusion-based market model presented in Sect. 2.1 by adding *zero-coupon bonds* and *forward rate agreements*. Sect. 3.2 exposes the paradigm shift in the interest rate market due to the crisis and it presents the roll-over risk approach. In Sect. 3.3 we introduce a multifactor Markov structure which makes it possible to derive a PDE representation for bonds and spreads. We propose some analitical results for the special case of linear dynamics and exponential quadratic structures. Chapter 4 is dedicated to the stochastic control derivations. In Sect. 4.1 we apply the classical stochastic control approach to bond prices and spreads. In Sec. 4.2 we deal with a risk-sensitive asset allocation problem, deriving a risk-sensitive portfolio process used to define a new spread formulation. In turn, this allows to express the spot spread as solution of a risk-sensitive control problem. We conclude by determining endogenous conditions of equivalence between the risk-sensitive representation and the roll-over risk approach..

Chapter 2

An introduction to the benchmark approach

The concept of *equivalent (local) martingale measure* and its relation to the *NFLVR* condition are considered as the foundation of modern mathematical finance. Indeed, the existence of martingale measures makes it possible to determine fair values of contingent claims, giving rise to a coherent and functioning pricing theory for derivative instruments presented in many textbooks; see for instance Björk (2020). In recent years, there has been an increase in anomalies in the financial markets and special attention is being paid to market models which may not provide an equivalent martingale measure or, in other words, allow for some limited arbitrage opportunities. In this chapeter we show that it is still possible to solve the fundamental problems of portfolio optimization and contingent claim valuation in absence of martingale measures, as long as the model satisfies a minimal condition. We thus propose one of the possible approaches for evaluating contingent claims without martingale measures: the *real-world pricing* based on the *benchmark approach*. To this end, we introduce the fundamental notion of *growth-optimal portfolio*, which often occurs in the subsequent chapters.

2.1 The general framework

We base our analysis of financial markets not admitting martingale measures mainly on the work of C. Fontana and W.J. Runggaldier (2013). For a fixed time horizon $\mathcal{T} \in (0, \infty)$, let $W = (W_t)_{0 \le t \le \mathcal{T}}$ be an $\mathbb{R}^d$-valued Wiener process defined on a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le \mathcal{T}}$ denote a complete and right-continuos filtration. More precisely, we suppose $\mathbb{F} = \mathbb{F}^W$ in order to use the martingale representation theorem (see [18, Theorem 4.15, ch. 3]). We consider a financial market composed of N risky assets $S^1, \ldots, S^N$, with $N \le d$, and a savings account S^0. For every $i = 1, \ldots, N$, we fix a diffusion dynamics for the processes

$S^i = (S^i_t)_{0\le t\le \mathcal{T}}$

$$dS^i_t = S^i_t \mu^i_t dt + S^i_t \sum_{j=1}^{N} \sigma^{i,j}_t dW^j_t, \quad S^i_0 = s^i \in \mathbb{R}_+ \tag{2.1}$$

If we denote by $D(S_t)$ the $N \times N$-matrix $\mathrm{diag}(S^1, \ldots, S^N)$, we can rewrite equation (2.1) more compactly as

$$dS_t = D(S_t) \cdot \mu_t dt + D(S_t) \cdot \sigma_t \cdot dW_t$$

where the processes $\mu = (\mu^i_t)_{0\le t\le \mathcal{T}}$ and $\sigma = (\sigma^{i,j}_t)_{0\le t\le \mathcal{T}}$, for $i = 1, \ldots, N$, $j = 1, \ldots, d$, satisfy the minimal conditions in order to have a meaningful definition of S^i_t in terms of ordinary and stochastic integrals (see [11, sec. 4.2]). We define the savings account process $S^0 = (S^0_t)_{0\le t\le \mathcal{T}}$ as solution of the differential equation

$$dS^0_t = S^0_t r_t dt, \quad S^0_0 = 1 \tag{2.2}$$

where the *short interest rate* process $r = (r_t)_{0\le t\le \mathcal{T}}$ is a real-valued progressively measurable process such that $\int_0^{\mathcal{T}} |r_t|\, dt < \infty$.

In order to describe the activity of trading in the financial market, we introduce formally the concepts of *trading strategy* and *portfolio process.*

Definition 2.1.1. A trading strategy (or *portfolio strategy*) is any $\mathbb{R}^{N+1}$-valued progressively measurable process $h = (h_t)_{0\le t\le \mathcal{T}}$. The portfolio process corresponding to h is a real-valued progressively measurable process $V^h = (V^h_t)_{0\le t\le \mathcal{T}}$ given by

$$V^h_t :\, = \sum_{i=0}^{N} h^i_t S^i_t$$

If we cosider positive portfolio processes, it is convenient to describe a trading strategy h in relative terms, through the *portfolio weights* defined by

$$\pi^i_t :\, = \frac{h^i_t S^i_t}{V^h_t}, \quad i = 0, \ldots, N$$

Therefore, in particular, we have

$$\pi^0_t + \pi'_t \cdot \mathbf{1} = 1 \tag{2.3}$$

where $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^N$ and $\pi_t = (\pi^1_t, \ldots, \pi^N_t)'$. Thanks to (2.3), from now on we will indicate by π a trading strategy and by V^π the associated positive portfolio process.

Definition 2.1.2. A trading strategy π is said *self-financing* if the corresponding portfolio process $V^\pi = (V_t^\pi)_{0\le t\le\mathcal{T}}$ satisfies

$$dV_t^\pi = V_t^\pi \sum_{i=0}^{N} \pi_t^i \frac{dS_t^i}{S^i}, \quad V_0^\pi = v \in \mathbb{R}^+ \tag{2.4}$$

Remark 2.1.3. We can set the initial wealth $v = 1$ without loss of generality (see [11, sec. 4.2]).

Remark 2.1.4. From an economic point of view, a portfolio strategy is a choice of allocating an amount of money available at time t. The self-financing condition is equivalent to assume that there is no exogenous inflow or withdrawal of money; in other words, the value of the portfolio depends only on how much is invested in the assets already in the portfolio.

If we insert equation (2.1) and condition (2.3) into (2.4), we get

$$\frac{dV_t^\pi}{V_t^\pi} = \pi_t^0 \frac{dS_t^0}{S_t^0} + \sum_{i=1}^{N} \pi_t^i \frac{dS_t^i}{S_t^i} = r_t dt + \pi_t^{'} \cdot (\mu_t - r_t \mathbf{1})dt + \pi_t^{'} \cdot \sigma_t \cdot dW_t \tag{2.5}$$

Definition 2.1.5. An *admissible trading strategy* is any $\mathbb{R}^N$-valued progressively measurable process $\pi = (\pi_t)_{0\le t\le\mathcal{T}}$ such that $\int_0^{\mathcal{T}} \|\sigma_t^{'} \cdot \pi_t\|^2 \, dt < \infty$ $\mathbb{P}$-a.s and $\int_0^{\mathcal{T}} |\pi_t^{'} \cdot (\mu_t - r_t \mathbf{1})| \, dt < \infty$ $\mathbb{P}$-a.s. We denote by $\mathcal{A}$ the set of all admissible strategies.

If $\pi \in \mathcal{A}$, then the solution of (2.5) is well-defined in terms of ordinary and stochastic intergrals.

2.2 Viable markets

One of the main topics in modern finance is the pricing of *contingent claims*. In financial literature they are also known as *derivative instruments*, while the assets S are called *underlyings*. From a mathematical point of view, a contingent claim with date of maturity (or exercise date) $\mathcal{T}$ is any random variable $\mathcal{X} \in \mathcal{F}_{\mathcal{T}}$. A contigent claim $\mathcal{X}$ is called a *simple claim* if it can be expressed as $\mathcal{X} = \phi(S_{\mathcal{T}})$ for a given fuction $\phi \colon \mathbb{R}^N \to \mathbb{R}$. The price (or value) of a contintingent claim at time $t \in [0, \mathcal{T})$ is often not given a priori. In order to provide a rigorous pricing technique, standard financial theory requires the market satisfies the conditions of *no-arbitrage* and *No Unbounded Profit with Bounded Risk.*

Definition 2.2.1. An *arbitrage opportunity* on a financial market is a self-financing portfolio strategy $\pi \in \mathcal{A}$ such that

$$\mathbb{P}(V_{\mathcal{T}}^\pi \ge V_0^\pi) = 1$$
$$\mathbb{P}(V_{\mathcal{T}}^\pi > V_0^\pi) > 0$$

A financial market is said *arbitrage free* if it satisfies the *no-arbitrage* (NA) condition, i.e. if it admits no arbitrage opportunities.

Definition 2.2.2. A financial market is said to satisfy the condition of *No Unbounded Profit with Bounded Risk* (NUPBR) if the set $\{ V^{\pi}_{\mathcal{T}} \colon \pi \in \mathcal{A} \}$ is bounded in probability

A market is said to satisfy the *No Free Lunch with Vanishing Risk* (NFLVR) condition if both NUPBR and NA condition hold (see [11, sec. 4.3]).

It is well known that the NFLVR condition is related to the notion of *equivalent martigale measure.*

Definition 2.2.3. A probability measure $\mathbb{Q}$ on $\mathcal{F}$ is called equivalent (local) martingale measure (E(L)MM) for the market model (2.1), the *numéraire* S^0, and the time interval $[0, \mathcal{T}]$, if it is equivalent[1] to $\mathbb{P}$ and, for any choice of $\pi \in \mathcal{A}$, the process $\frac{V^{\pi}}{S^0}$ is a (local) martingale with respect to the filtration $\mathbb{F}$.

The following result is known as *First Fundamental Theorem of Asset Pricing* (FFTAP).

Theorem 2.2.4. *A market model satisfies the NFLVR condition if and only if there exists an equivalent (local) martingale measure* $\mathbb{Q}$.

See [4, sec. 11.3] for a sketch of the proof provided by Delbaen and Schachermayer (1998).

The existence of martingale measures allows to formulate a consistent pricing theory. However, arbitrage opportunities are often admitted in real markets. Practical reasons, therefore, lead us to provide a characterization of the market that is weaker than NFLVR but which guarantees a still functioning pricing procedure. In other words, we aim to define market models which respect a minimum condition such that fair prices are still possible for contingent claims, though an ELMM may not exist. Since such a condition must be written as a no-arbitrage condition, we have to carefully answer the question of which types of arbitrage opportunities must be avoided in order to reach our purpose. We start by giving the following definition.

Definition 2.2.5. A trading strategy $\pi \in \mathcal{A}$ is said to yield an *increasing profit* if the corresponding portfolio process V^{π} satisfies the following two conditions:

(a) V^{π} is $\mathbb{P}$-a.s. increasing, in the sense that

$$\mathbb{P}(V^{\pi}_s \leq V^{\pi}_t \,\forall\, s, t \in [0, \mathcal{T}] \,\text{with}\, s \leq t) = 1;$$

[1] We recall that two probabilty measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if for every $A \in \mathcal{F}$ we have that $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$.

(b) $\mathbb{P}(V_{\mathcal{T}}^{\pi} > V_0^{\pi}) > 0$.

The notion of increasing profit represents the strongest arbitrage opportunity in our analysis. The following lemma provides a useful characterization of increasing profit.

Lemma 2.2.6. *There exists an increasing profit if and only if there exists a trading strategy $\pi \in \mathcal{A}$ satisfying the following two conditions:*

(a) $\pi_t^{'} \cdot \sigma_t = 0\, \mathbb{P} \otimes l$*-a.e., where we denote by l the Lebesgue measure on* $[0, \mathcal{T}]$*;*

(b) $\pi_t^{'} \cdot (\mu_t - r_t \mathbf{1}) \neq 0$ *on some subset of* $\Omega \times [0, \mathcal{T}]$ *of positive* $\mathbb{P} \otimes l$*-measure.*

A proof can be found in [11, Lemma 4.3.2]. The previous lemma makes it possible to derive the following result.

Proposition 2.2.7. *There are no increasing profits if and only if there exists an $\mathbb{R}^d$-valued progressively measurable process $\gamma = (\gamma_t)_{0 \leq t \leq \mathcal{T}}$ which satisfies the condition*

$$\sigma_t \cdot \gamma_t = \mu_t - r_t \mathbf{1} \quad \mathbb{P} \otimes l - a.e. \tag{2.6}$$

Proof. If equation (2.6) is satisfied for some progressively measurable process $\gamma = (\gamma_t)_{0 \leq t \leq \mathcal{T}}$, then there cannot exist a trading strategy $\pi \in \mathcal{A}$ satisfying condions $(a) - (b)$ of Lemma 2.2.6. Indeed, if such a strategy exists, (a) and (2.6) give

$$\pi_t^{'} \cdot (\mu_t - r_t \mathbf{1}) = \pi_t^{'} \cdot \sigma_t \cdot \gamma_t = 0 \quad \mathbb{P} \otimes l - \text{a.e.}$$

so condition (b) fails. The equivalence result of Lemma 2.2.6 implies that there are no increasing profits.

On the other hand, suppose that there exists no trading strategy in $\mathcal{A}$ yielding an increasing profit. Let $\mathrm{Im}(\sigma_t)$ and $\ker(\sigma_t^{'})$ be the image space of σ_t and the kernel of $\sigma_t^{'}$ respectively, for every $t \in [0, \mathcal{T}]$. We denote by $\mathcal{P}_{\ker(\sigma_t^{'})}$ the othogonal projection on $\ker(\sigma_t^{'})$. We define a process $p = (p_t)_{0 \leq t \leq \mathcal{T}}$ by

$$p_t :\, = \mathcal{P}_{\ker(\sigma_t^{'})}(\mu_t - r_t \mathbf{1})$$

We define then the trading strategy $\hat{\pi} = (\hat{\pi}_t)_{0 \leq t \leq \mathcal{T}}$ by

$$\hat{\pi} :\, = \begin{cases} \frac{p_t}{\|p_t\|} & \text{if } p_t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

It can be shown that $\hat{\pi}$ is progressively measurable (see [11, Proposition 4.3.4]) and it is clear that $\hat{\pi} \in \mathcal{A}$. Moreover, since $p_t \in \ker(\sigma_t^{'})$ by definition, then $\hat{\pi}_t \in \ker(\sigma_t^{'})$, meaning that $\hat{\pi}$ satisfies condition (a) of Lemma 2.2.6.

Since there are no increasing profits, Lemma 2.2.6 implies that condition (b) must not hold. Therefore, we have $\mathbb{P} \otimes l$-a.e.

$$0 = \hat{\pi}_t^{'} \cdot (\mu_t - r_t\mathbf{1}) = \frac{p_t^{'}}{\|p_t\|} \cdot (\mu_t - r_t\mathbf{1})\mathbb{1}_{p_t \neq 0} = \|p_t\| \tag{2.7}$$

where the last equality uses the fact that $\mu_t - r_t\mathbf{1} - p_t \in \ker(\sigma_t^{'})^{\perp}$ for all $t \in [0, \mathcal{T}]$, with the superscript $\perp$ denoting the orthogonal complement[2]. The identity (2.7) implies that $p_t = 0\,\mathbb{P} \otimes l$-a.e., meaning that $\mu_t - r_t\mathbf{1} \in \ker(\sigma_t^{'})^{\perp} = \mathrm{Im}(\sigma_t)\,\mathbb{P} \otimes l$-a.e. The definition of $\mathrm{Im}(\sigma_t)$ leads to conclude that there exists some process $\gamma = (\gamma_t)_{0 \le t \le \mathcal{T}}$ such that

$$\sigma_t \cdot \gamma_t = \mu_t - r_t\mathbf{1} \quad \mathbb{P} \otimes l - \text{a.e.}$$

See again [11] to get the progressive measurability of γ. □

In order to eliminate increasing profits from the diffusion-based financial market described in Sect. 2.1, one has to assume that equation (2.6) admits solution. This is guarateed, for instance, by the following standing assumption.

Assumption 2.2.8. For all $t \in [0, \mathcal{T}]$, the $(N \times d)$-matrix σ_t has $\mathbb{P}$-a.s. full rank.

From a financial point of view, the assumption above corresponds to avoid the existence of redundant assets in the finanacial market, i.e. there does not exist a non-trivial linear combination of $(S^1, \dots, S^N)$ which is locally riskless. If Assumption 2.2.8 holds, then (2.6) has solution for every choice of the processes μ, r and σ. However, such a solution may not be unique and, therefore, we are interested in characterizing the minimum norm process which solves (2.6).

Definition 2.2.9 (The market price of risk). If equation (2.6) holds for some process γ, the $\mathbb{R}^d$-valued progressively measurable *market price of risk* process $\theta = (\theta_t)_{0 \le t \le \mathcal{T}}$ is the minimum norm solution process of (2.6) and it is defined as

$$\theta_t : = \sigma_t^{+} \cdot (\mu_t - r_t\mathbf{1}) \tag{2.8}$$

where we denote by σ_t^{+} the *Moore-Penrose pseudoinverse* of the matrix σ_t (see [13, ch. 3, sect. 7])

[2] More precisely, we have

$$\frac{p_t^{'}}{\|p_t\|} \cdot (\mu_t - r_t\mathbf{1}) = \frac{p_t^{'}}{\|p_t\|} \cdot (\mu_t - r_t\mathbf{1} - p_t) + \frac{p_t^{'}}{\|p_t\|} \cdot p_t = \frac{\|p_t\|^2}{\|p_t\|}$$

From an economic perspective, θ_t measures the excess return $\mu_t - r_t\mathbf{1}$ of the risky assets (with respect to the savings account) in terms of their volatility. Under Assumption 2.2.8 the market price of risk $\theta = (\theta_t)_{0\le t\le\mathcal{T}}$ can be written as

$$\theta_t = \sigma_t^{'} \cdot (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot (\mu_t - r_t\mathbf{1}) \tag{2.9}$$

The assumption below will be crucial.

Assumption 2.2.10. The market price of risk process θ belongs to $L^2_{loc}(W)$, meaning that $\int_0^{\mathcal{T}} \|\theta_t\|^2\,dt < \infty$ $\mathbb{P}$-a.s.

Many of the following results rely on the key relation existing between Assumption 2.2.10 and no-arbitrage conditions. We have just seen that the existence of incresing profits can be avoided by assuming that equation (2.6) has solution. However, the concept of increasing profit represents an almost pathological notion of arbitrage opportunity. Hence, we would like to characterize a market without martingale measures through a stronger and more economically meaningful no-arbitrage condition. To this effect, let us give the following definition.

Definition 2.2.11. An $\mathcal{F}$-measurable random variable ξ is called an *arbitrage of the first kind* if $\xi \ge 0$ $\mathbb{P}$-a.s., $\mathbb{P}(\xi > 0) > 0$, and for all $v \in (0,\infty)$ there exists a trading strategy $\pi^v \in \mathcal{A}$ such that $V_{\mathcal{T}}^{v,\pi^v} \ge \xi$ $\mathbb{P}$-a.s., where V^{v,π^v} is the portfolio process corresponding to π^v with initial wealth $V_0^{v,\pi^v} = v$. We say that the financial market is *viable* if there are no arbitrages of the first kind.

We denote by NA_1 the viability condition. It can be proved that if there exists a trading strategy yielding an increasing profit, then there exists an arbitrage of the first kind (see [11, Proposition 4.3.11]). Below we explain the difference between the conditions of viability and NFLVR, that is why the viability condition is not sufficient to guarantee the existence of martigale measures. First of all, let us give the following result.

Proposition 2.2.12. *The NA_1 and NUPBR conditions are equivalent.*

A proof can be found in [15, Proposition 1.2]. The NA_1 condition is also related to the concept of *martingale deflator*.

Definition 2.2.13. A *martingale deflator* is a real-valued, non-negative and adapted process $D = (D_t)_{0\le t\le\mathcal{T}}$ with $D_0 = 1$, $D_{\mathcal{T}} > 0$ $\mathbb{P}$-a.s., and such that the process $D\bar{V}^\pi = (D\bar{V}^\pi)_{0\le t\le\mathcal{T}}$ is a local martingale for every $\pi \in \mathcal{A}$, where we denote by $\bar{V}^\pi$ the discounted portfolio process

$$\bar{V}^\pi :\, = \left(\frac{V_t^\pi}{S_t^0}\right)_{0\le t\le\mathcal{T}} \tag{2.10}$$

In particular, taking $\pi = 0$, a martingale deflator is a non-negative local martingale and, therefore, a supermartingale. We denote by $\mathcal{D}$ the set of all martingale deflators.

Proposition 2.2.14. *There cannot exist arbitrages of the first kind if and only if $\mathcal{D} \neq \emptyset$.*

We refer to [11, Proposition 4.4.16] and [16, Theorem 1.1] for the proof. Now we can state the following theorem, the second part of which follows from [17, Proposition 3.19].

Theorem 2.2.15. *The following are equivalent:*

(a) $\mathcal{D} \neq \emptyset$;

(b) the NA_1 *condition holds;*

(c) the NUPBR condition holds.

Moreover, for every concave and strictly increasing utility function

$$U\colon [0,\infty] \to \mathbb{R}$$

the expected utility maximization problem of finding a strategy $\pi^* \in \mathcal{A}$ *such that*

$$\mathbb{E}[U(V_{\mathcal{T}}^{\pi^*})] = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(V_{\mathcal{T}}^{\pi})] \tag{2.11}$$

either does not have a solution or has infinitely many solutions when any of conditions (a)-(b) fails.

In view of the second part of the theorem above, the NA_1 condition can be seen as the minimal no-arbitrage condition in order to be able to meaningfully solve portfolio optimisation problems.

If Assumpion 2.2.10 holds, the market model described in Sect. 2.1 has a natural martingale deflator $Z = (Z_t)_{0\le t\le\mathcal{T}}$ given by

$$Z_t\colon = \exp\left(-\int_0^t \theta_s \cdot dW_s - \frac{1}{2}\int_0^t \|\theta_s\|^2\, ds\right) \tag{2.12}$$

It is clear that Z is a positive process and applying the Itô's formula we get that Z is solution of the following SDE:

$$dZ_t = -Z_t\theta_t \cdot dW_t, \quad Z_0 = 1 \tag{2.13}$$

Therefore, Z is a positive local martingale and it can be easily shown that $Z\bar{V}^{\pi}$ is a local martingale for every $\pi \in \mathcal{A}$ (see [11, Proposition 4.3.9]), where $\bar{V}^{\pi}$ is the discounted portfolio process defined by (2.10). Assumption 2.2.10 guarantees that Z is well-defined in terms of ordinary and stochastic integrals, meaning that Assumption 2.2.10 is a sufficient condition to make the market (S^0, S) viable. Since the converse result can be proved (see [11, Corollary 4.3.19]), we have the following statement.

Proposition 2.2.16. *The financial market* (S^0, S) *is viable, i.e. it respects the* NA_1 *condition, if and only if Assumpion 2.2.10 holds.*

Let us now stress the crucial point. The well known Girsanov Theorem states that if the martingale deflator Z is a true martingale, then the process $W^{\mathbb{Q}} = (W_t^{\mathbb{Q}})_{0 \le t \le \mathcal{T}}$ defined by

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s \, ds \tag{2.14}$$

is an $\mathbb{R}^d$-valued Brownian motion with respect to an equivalent (local) martingale measure $\mathbb{Q}$. To get the martingality of Z it is necessary that the process θ satisfies stronger conditions than Assumption 2.2.10, e.g. the Novikov criterion (see [18, Sect. 3.5]). Therefore, by Proposition 2.2.16, the viability condition is not sufficient to provide the existence of ELMMs. From the arbitrage point of view, viable markets satisfy the NUPBR condition (by Proposition 2.2.12) but admit some arbitrage opportunity, in the sense of Definition 2.2.1. This implies that the NFLVR condition may fail, as well as the existence of martingale measures. Fortunately, in viable markets pricing contingent claims is still possible. In the next section we present an approach based on the direct use of the original real-world probability measure $\mathbb{P}$.

We close this section with a simple technical result that turns out useful in the following.

Lemma 2.2.17. *Suppose that Assumption 2.2.10 holds. An* $\mathbb{R}^N$*-valued progressively measurable process* $\pi = (\pi_t)_{0 \le t \le \mathcal{T}}$ *belongs to* $\mathcal{A}$ *if and only if* $\int_0^{\mathcal{T}} \|\sigma_t' \cdot \pi_t\|^2 \, dt < \infty$ $\mathbb{P}$*-a.s.*

Proof. We only need to show that $\int_0^{\mathcal{T}} \|\sigma_t' \cdot \pi_t\|^2 \, dt < \infty$ $\mathbb{P}$-a.s. and Assumpion 2.2.10 together imply that $\int_0^{\mathcal{T}} |\pi_t' \cdot (\mu_t - r_t \mathbf{1})| \, dt < \infty$ $\mathbb{P}$-a.s. This follows from the Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_0^{\mathcal{T}} |\pi_t' \cdot (\mu_t - r_t \mathbf{1})| \, dt &= \int_0^{\mathcal{T}} |\pi_t' \cdot \sigma_t \cdot \theta_t| \, dt \\
&\le \left(\int_0^{\mathcal{T}} \|\sigma_t' \cdot \pi_t\|^2 \, dt \right)^{\frac{1}{2}} \left(\int_0^{\mathcal{T}} \|\theta_t\|^2 \, dt \right)^{\frac{1}{2}} < \infty
\end{aligned}$$ □

2.3 The growth-optimal portfolio and the benchmark approach

Before explaining how contingent claims can be evaluated in viable markets, we have to define the fundamental notion of *growth-optimal portfolio*, that is the portfolio process with maximum growth rate.

Definition 2.3.1. For a trading strategy $\pi \in \mathcal{A}$, we call *growth rate* process the process $g^\pi = (g_t^\pi)_{0\leq t\leq \mathcal{T}}$ appearing in the drift term of the SDE satisfied by the process $\log(V^\pi) = (\log(V_t^\pi))_{0\leq t\leq \mathcal{T}}$, i.e. the term g_t^π in the SDE

$$d\log(V_t^\pi) = g_t^\pi dt + \pi_t^{'} \cdot \sigma_t \cdot dW_t \tag{2.15}$$

A trading strategy $\pi^* \in \mathcal{A}$ and its corresponding portfolio process V^* are said *growth optimal* if $g_t^{\pi^*} \geq g_t^\pi$ $\mathbb{P}$-a.s. for all $t \in [0, \mathcal{T}]$ and for any trading strategy $\pi \in \mathcal{A}$.

The following theorem gives an explicit description of the growth optimal strategy $\pi^* \in \mathcal{A}$.

Theorem 2.3.2. *Suppose that Assumptions 2.2.8 and 2.2.10 hold. Then there exist a unique growth-optimal strategy $\pi^* \in \mathcal{A}$, in the sense of Definition 2.3.1, explicity given by*

$$\pi_t^* = (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot \sigma_t \cdot \theta_t \tag{2.16}$$

where the process $\theta = (\theta_t)_{0\leq t\leq \mathcal{T}}$ is the market price of risk introduced in Definition 2.2.9. The corresponding growth optimal portfolio (GOP) process $V^* = (V_t^*)_{0\leq t\leq \mathcal{T}}$ satisfies the following dynamics:

$$\frac{dV_t^*}{V_t^*} = (r_t + \|\theta_t\|^2)dt + \theta^{'} \cdot dW_t \tag{2.17}$$

Proof. Let $\pi \in \mathcal{A}$ be a trading strategy. A simple application of Itô's formula gives that

$$d\log(V_t^\pi) = g_t^\pi dt + \pi_t^{'} \cdot \sigma_t \cdot dW_t$$

where the growth rate is given by

$$g_t^\pi = r_t + \pi_t^{'} \cdot (\mu_t - r_t\mathbf{1}) - \frac{1}{2}\pi_t^{'} \cdot \sigma_t \cdot \sigma_t^{'} \cdot \pi_t, \quad \forall t \in [0, \mathcal{T}]$$

By differentiating g_t^π w.r.t. π_t we get the following first-order condition:

$$\sigma_t \cdot \sigma_t^{'} \cdot \pi_t = \mu_t - r_t\mathbf{1} \tag{2.18}$$

There exists a growth optimal strategy if the linear system above admits solution. Since σ_t is assumed to have full rank $\mathbb{P}$-a.s. (Assumption 2.2.8), the matrix $\sigma_t \cdot \sigma_t^{'}$ is $\mathbb{P}$-a.s. positive definite for all $t \in [0, \mathcal{T}]$ and, therefore, it is invertible. Thus, by using Definition 2.2.9, we get the a unique optimiser π_t^* given by

$$\pi_t^* = (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot (\mu_t - r_t\mathbf{1}) = (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot \sigma_t \cdot \theta_t, \quad \forall t \in [0, \mathcal{T}]$$

We now need to verify that $\pi^* \in \mathcal{A}$. Due to Lemma 2.2.17, it suffices to check that $\int_0^{\mathcal{T}} \|\sigma_t^{'} \cdot \pi_t\|^2 \, dt < \infty \, \mathbb{P}$-a.s. To show this, it is enough to notice that

$$\int_0^{\mathcal{T}} \|\sigma_t^{'} \cdot \pi_t^*\|^2 \, dt = \int_0^{\mathcal{T}} \|\sigma_t^{'} \cdot (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot (\mu_t - r_t \mathbf{1})\|^2 \, dt$$
$$= \int_0^{\mathcal{T}} \|\theta_t\|^2 \, dt$$

due to equation (2.9). Finally, by inserting π_t^* into the stochastic dynamics of V^π given by (2.5) we get

$$\begin{aligned} \frac{dV_t^*}{V_t^*} &= r_t dt + (\pi_t^*)^{'} \cdot (\mu_t - r_t \mathbf{1}) dt + (\pi_t^*) \cdot \sigma_t \cdot dW_t \\ &= (r_t + \theta_t^{'} \cdot \sigma_t^{'} \cdot (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot \sigma_t \cdot \theta_t) dt + \theta_t^{'} \cdot \sigma'_t \cdot (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot \sigma_t \cdot dW_t \\ &= (r_t + \|\theta_t\|^2) dt + \theta_t^{'} \cdot dW_t \end{aligned}$$

□

Remark 2.3.3. If Assumption 2.2.8 holds, then expression (2.16) makes sense and there exists a unique optimal strategy. Actually, we could relax such assumption without losing the existence of a growth optimal strategy. Indeed, the first-order condition given by (2.18) corresponds to the *normal equations system* associated to the linear system $\sigma_t^{'} \cdot \pi_t = \theta_t$ (see [13, ch. 3, sect. 7]) and, therefore, it has solution if $\theta_t \in \mathrm{Im}(\sigma_t^{'})$, which is given by Defininition 2.2.9. In such a more general case, many optimal strategies may exist, but all of them lead to the same stochastic dynamics for the GOP, that is (2.17), as proved by the computation below:

$$\frac{dV_t^*}{V_t^*} = r_t dt + (\pi_t^*)^{'} \cdot \sigma_t \cdot \theta_t dt + (\pi_t^*) \cdot \sigma_t \cdot dW_t = (r_t + \|\theta_t\|^2) dt + \theta^{'} \cdot dW_t$$

where $\pi^* = (\pi_t^*)_{0 \le t \le \mathcal{T}}$ is an admible strategy satisfying (2.18). See also [8, sect. 3].

Besides maximizing the growth rate, the GOP enjoys several other optimality properties. In particular, we recall a well known result (see [11, sect. 4.6.3] and [17, Proposition 2.19]).

Theorem 2.3.4. *Assume that* $\mathbb{E}[\log(V_{\mathcal{T}}^\pi)] < \infty$ *for all* $\pi \in \mathcal{A}$. *Then, the GOP is the solution to the expected log-utility maximization problem*

$$\mathbb{E}[\log(V_{\mathcal{T}}^*)] = \max_{\pi \in \mathcal{A}} \mathbb{E}[\log(V_{\mathcal{T}}^\pi)] \tag{2.19}$$

Proof. From (2.15) we deduce that

$$\mathbb{E}[\log(V_{\mathcal{T}}^\pi)] = \mathbb{E}\left[\int_0^{\mathcal{T}} g_t^\pi \, dt\right]$$

We can thus define a function $J\colon \mathcal{A} \to \mathbb{R}$ such that

$$J(\pi)\colon = \mathbb{E}[\log(V^{\pi}_{\mathcal{T}})] = \int_{\Omega} \int_{0}^{\mathcal{T}} g^{\pi}_t(\omega)\, dt\, d\mathbb{P}(\omega) \tag{2.20}$$

The optimization problem (2.19) is equivalent to determine the maximum of J. Since we have no constraints, we can maximize $g^{\pi}_t(\omega)$ w.r.t π for each $\omega \in \Omega$ and $t \in [0, \mathcal{T}]$. We can thus impose the first-order condition by simply differetiating g^{π}_t w.r.t. π and we get again equation (2.18). □

Notice that, by Theorem 2.2.15, the viability condition is sufficient to ensure the existence of the GOP as solution to an expected log-utility maximization problem. Since the converse can be proved (see [11, Corollary 4.4.9]), we can state the following.

Corollary 2.3.5. *The financial market (S^0, S) is viable, i.e. it respects the NA_1 condition, if and only if the GOP exists.*

The GOP plays an important role in defining the *real-world price*, which is at the core of the so-called *benchmark approach* (see [11, sect. 4.6.1]) to the valuation of contingent claims. The feature which makes the GOP so important for our purpose is the numéraire property.

Definition 2.3.6. Let $\tilde{\pi} \in \mathcal{A}$ be an admissible strategy. The portfolio process $V^{\tilde{\pi}} = (V^{\tilde{\pi}}_t)_{0 \le t \le \mathcal{T}}$ is called *numéraire portfolio* (NP) if all admissible porfolio processes $V^{\pi} = (V^{\pi}_t)_{0 \le t \le \mathcal{T}}$, when denominated in units of V^{π}, are local martingales, i.e. if the *benchmarked portfolio process*

$$\hat{V}^{\pi}\colon = \left(\frac{V^{\pi}_t}{V^{\tilde{\pi}}_t}\right)_{0 \le t \le \mathcal{T}} \tag{2.21}$$

is a local martingale for all $\pi \in \mathcal{A}$

Proposition 2.3.7. *The GOP has the numéraire property*

Proof. The result above follows easily from the following lemma.

Lemma 2.3.8. *Suppose that Assumption 2.2.10 holds. Then the discounted GOP process $\bar{V}^* = \left(\frac{V^*_t}{S^0_t}\right)_{0 \le t \le \mathcal{T}}$ satisfies the folowing relation*

$$\bar{V}^*_t = \frac{1}{Z_t} \tag{2.22}$$

where the process $Z = (Z_t)_{0 \le t \le \mathcal{T}}$ is the martingale deflator defined by (2.12).

Proof. Assumption 2.2.10 ensures that the process Z is $\mathbb{P}$-a.s. strictly positive and well defined as a martingale deflator. Furthermore, due to Theorem

2.3.2, the GOP exists and its dynamics is given by (2.17). Now we just need to apply the stochastic differentiation by parts

$$d\bar{V}_t^* = d\left(\frac{V_t^*}{S_t^0}\right) = dV_t^* \frac{1}{S_t^0} + V_t^* d\left(\frac{1}{S_t^0}\right) = \bar{V}_t^*(\|\theta_t\|^2 dt + \theta' \cdot dW_t)$$

The SDE above is satisfied by

$$\bar{V}^* = \exp\left(\int_0^{\mathcal{T}} \theta_s \cdot dW_s + \frac{1}{2}\int_0^{\mathcal{T}} \|\theta_s\|^2 \, ds\right) = \frac{1}{Z_t}$$

where the last equality follows from (2.12). □

Let us now consider the process $\hat{V}^\pi = \left(\frac{V_t^\pi}{V_t^*}\right)_{0 \le t \le \mathcal{T}}$ for any $\pi \in \mathcal{A}$. Passing to the discounted quantities, we have

$$\hat{V}_t^\pi = \frac{V_t^\pi}{V_t^*} = \frac{\bar{V}_t^\pi}{\bar{V}_t^*} = \bar{V}_t^\pi Z_t$$

and we can conclude because Z is a martingale deflator. □

Since it can be shown that the numéraire portfolio is unique (see [11, Proposition 4.4.7], we can state the following.

Theorem 2.3.9. *The GOP coincides with the NP.*

We can finally illustrate the application of the growth optimal portfolio to the pricing of derivative instruments in absence of equivalent martingale measures. Due to Proposition 2.3.7, all portfolio processes V^π, $\pi \in \mathcal{A}$, are local martingales when denominated in units of the GOP V^*. This means that, if we express all price processes in terms of the GOP, then the original probability measure $\mathbb{P}$ becomes a local martingale measure and it can be used to evaluate replicable contingent claims.

Definition 2.3.10. Let $\mathcal{X}$ be a positive $\mathcal{F}$-measurable contingent claim such that $\mathbb{E}\left[\frac{Z_{\mathcal{T}}\mathcal{X}}{S_{\mathcal{T}}^0}\right] < \infty$. We say that $\mathcal{X}$ is *replicable* if there exists a couple $(v^{\mathcal{X}}, \pi^{\mathcal{X}}) \in \mathbb{R}_+ \times \mathcal{A}$ such that $V_{\mathcal{T}}^{v^{\mathcal{X}},\pi^{\mathcal{X}}} = \mathcal{X}$ $\mathbb{P}$-a.s., where we denote by $V^{v^{\mathcal{X}},\pi^{\mathcal{X}}}$ the portfolio process corresponding to $\pi^{\mathcal{X}}$ with initial value $V_0^{v^{\mathcal{X}},\pi^{\mathcal{X}}} = v^{\mathcal{X}}$.

Definition 2.3.11. A trading strategy $\pi \in \mathcal{A}$ and the associated portfolio process V^π are said to be *fair* if the benchmarked portfolio process $\hat{V}^\pi = \frac{V^\pi}{V^*}$ is a true martingale. We denote by $\mathcal{A}^F$ the set of all fair trading strategies in $\mathcal{A}$. A replicable contingent claim $\mathcal{X}$ is said *fairly priced* if the replicating strategy $\pi^{\mathcal{X}}$ belongs to $\mathcal{A}^F$, and we define its *real-world price* as

$$\Pi_t^{\mathcal{X}} := V_t^* \mathbb{E}\left[\frac{\mathcal{X}}{V_{\mathcal{T}}^*}\middle|\mathcal{F}_t\right] \tag{2.23}$$

for every $t \in [0, \mathcal{T}]$.

Remark 2.3.12. The terminology real-world price is used to indicate that, unlike the traditional setting, all contingent claims are valued under the original real-world probability measure $\mathbb{P}$ and not under an equivalent risk-neutral measure. This makes it possible to extend the valuation of contingent claims to financial markets for which no ELMM may exist. The concept of real-world price gives rise to the so-called *benchmark approach* to the pricing of contingent claims in view of the fact that the GOP plays the role of the natural numéraire portfolio. Such approach is possible as long as the GOP exists, meaninig that, by Corolloary 2.3.5, if the market is viable. This observation suggests that the NA_1 condition can be seen as the minimal and natural condition guaranteeing that the problems of pricing and portfolio optimization make sense. However, note that every benchmarked portfolio process is a local martingale but not necessarily a true martingale. This amounts to saying that there may exist unfair portfolios, namely portfolios for which the corresponding benchmarked value process is a strict local martingale. This means that not every contingent claim can be replicated by a fair strategy, since the market (S^0, S) is not *complete* a priori. There exist conditions which provide the completeness of viable markets [11, sect. 4.5] but, to make our analysis more general, we do not impose them.

We conclude this section characterizing when the GOP is invariant with respect to an extension of the financial market with additional assets.

Theorem 2.3.13. *Suppose that the market (S^0, S) is viable and denote by V^* its GOP. Consider an additional asset Σ defined by*

$$d\Sigma_t = \Sigma_t \alpha_t dt + \Sigma_t \beta_t' \cdot dW_t$$

where $\alpha = (\alpha_t)_{0 \le t \le \mathcal{T}}$ is a real-valued process, $\beta = (\beta_t)_{0 \le t \le \mathcal{T}}$ is an $\mathbb{R}^d$-valued process. We assume that the $(N+1) \times d$-matrix

$$\begin{pmatrix} \sigma_t \\ \beta_t' \end{pmatrix} \tag{2.24}$$

has $\mathbb{P}$-a.s. full rank for every $t \in [0, \mathcal{T}]$. If the process $\hat{\Sigma} = \frac{\Sigma}{V^*}$ is a local martingale, then V^* is GOP also for the market (S^0, S, Σ).

Proof. We can apply the stochastic differetiation by parts to $\hat{\Sigma}$:

$$d\hat{\Sigma}_t = d\left(\frac{\Sigma_t}{V_t^*}\right) = \frac{d\Sigma_t}{V_t^*} + \Sigma_t \left(\frac{1}{V_t^*}\right) + d\left\langle \Sigma, \frac{1}{V^*}\right\rangle_t$$

Due to Theorem 2.3.2 and Itô's formula, we get

$$d\left(\frac{1}{V_t^*}\right) = -\frac{1}{V_t^*}(r_t dt + \theta_t' \cdot dW_t)$$

Therefore, by knowing the dynamics of Σ and $\frac{1}{V^*}$, we have

$$\begin{aligned} d\hat{\Sigma}_t &= \hat{\Sigma}_t(\alpha_t dt + \beta_t^{'} \cdot dW_t) - \hat{\Sigma}_t(r_t dt + \theta_t^{'} \cdot dW_t) - \hat{\Sigma}_t\theta_t^{'} \cdot \beta_t dt \\ &= \hat{\Sigma}_t[(\alpha_t - r_t - \theta_t^{'} \cdot \beta_t)dt + (\beta_t - \theta_t)^{'} \cdot dW_t] \end{aligned}$$

Since $\hat{\Sigma}$ is a local martingale by hypothesis, we get the following condition on the drift of Σ:

$$\alpha_t = r_t + \theta_t^{'} \cdot \beta_t \quad \forall t \in [0, \mathcal{T}] \tag{2.25}$$

We denote by $\hat{\pi} = (\pi, \pi^\Sigma)$ a genereric admissible strategy in the extended market, where $\pi^\Sigma = (\pi_t^\Sigma)_{0 \le t \le \mathcal{T}}$ is a real-valued progressively measurable process representing the portion of wealth invested in the new asset Σ. The portfolio process $V^{\hat{\pi}} = (V^{\hat{\pi}})_{0 \le t \le \mathcal{T}}$ satisfies the following dynamics, adapted from (2.5):

$$\frac{dV_t^{\hat{\pi}}}{V_t^{\hat{\pi}}} = [r_t + \pi_t^{'} \cdot (\mu_t - r_t\mathbf{1}) + \pi_t^\Sigma(\alpha_t - r_t)]dt + (\pi_t^{'} \cdot \sigma_t + \pi_t^\Sigma \beta_t^{'}) \cdot dW_t \tag{2.26}$$

Through the application of Itô's formula we have

$$d\log(V_t^{\hat{\pi}}) = g_t^{\pi,\pi^\Sigma} dt + (\pi_t^{'} \cdot \sigma_t + \pi_t^\Sigma \beta_t^{'}) \cdot dW_t$$

where the process g^{π,π^Σ} is the growth rate process in the extended market and it is given by

$$g_t^{\pi,\pi^\Sigma} = r_t + \pi_t^{'} \cdot (\mu_t - r_t\mathbf{1}) + \pi_t^\Sigma(\alpha_t - r_t) - \frac{1}{2}\pi_t^{'} \cdot \sigma_t \cdot \sigma_t^{'} \cdot \pi_t - \frac{1}{2}(\pi_t^\Sigma)^2 \beta_t^{'} \cdot \beta_t - \pi_t^\Sigma \pi_t^{'} \cdot \sigma_t \cdot \beta_t$$

Due to the first-order condition, there exists a growth optimal strategy $\hat{\pi}^*$ if the following system admits solution:

$$\begin{cases} \mu_t - r_t\mathbf{1} - \sigma_t \cdot \sigma_t^{'} \cdot \pi_t - \pi_t^\Sigma \sigma_t \cdot \beta_t = 0 \\ \alpha_t - r_t - \pi_t^\Sigma \beta_t^{'} \cdot \beta_t - \pi_t^{'} \cdot \sigma_t \cdot \beta_t = 0 \end{cases}$$

Substituting the drift condition (2.25) into the second line gives

$$\begin{cases} \mu_t - r_t\mathbf{1} - \sigma_t \cdot \sigma_t^{'} \cdot \pi_t - \pi_t^\Sigma \sigma_t \cdot \beta_t = 0 \\ (\theta_t^{'} - \pi_t^\Sigma \beta_t^{'} - \pi_t^{'} \cdot \sigma_t) \cdot \beta_t = 0 \end{cases}$$

Since $\beta_t \neq 0$ $\mathbb{P}$-a.s., the second line becomes

$$\sigma_t^{'} \cdot \pi_t = \theta_t - \pi_t^\Sigma \beta_t \quad \forall t \in [0, \mathcal{T}] \tag{2.27}$$

and by inserting it into the first one we get

$$\begin{cases} \sigma_t \cdot \theta_t = \mu_t - r_t\mathbf{1} \\ \sigma_t^{'} \cdot \pi_t = \theta_t - \pi_t^\Sigma \beta_t \end{cases}$$

Since the first line holds by definition of θ, we have that an optimal strategy is given by any choice of π and π^Σ satisfying condition (2.27). In particular, if we insert $\pi^\Sigma = 0$ into equation (2.27) we have

$$\begin{aligned}\sigma_t^{'} \cdot \pi_t &= \theta_t \\ \sigma_t \cdot \sigma_t^{'} \cdot \pi_t &= \sigma_t \cdot \theta_t \\ \pi_t &= (\sigma_t \cdot \sigma_t^{'})^{-1} \cdot \sigma_t \cdot \theta_t\end{aligned}$$

Therefore, a possible choice is given by $(\pi^*, 0)$ where π^* is the growth optimal strategy in the original market (S^0, S). Since the matrix (2.24) has $\mathbb{P}$-a.s. full rank, then the quadratic form definining g^{π,π^Σ} is positive definite with respect to π and π^Σ, implying that $\hat{\pi}^* = (\pi^*, 0)$ is the only solution of the growth rate maximization problem in the extended market. Hence, we obtain that V^* is GOP also for the extended market. □

Remark 2.3.14. By induction, we can consider m assets $\Sigma = (\Sigma^1, \ldots, \Sigma^m)$, $m \leq d - N$, with diffusion dynamics given by

$$d\Sigma_t = D(\Sigma_t) \cdot \alpha_t dt + D(\Sigma_t) \cdot \eta_t \cdot dW_t$$

where $\alpha = (\alpha_t)_{0 \leq t \leq \mathcal{T}}$ is an $\mathbb{R}^m$-valued process and $\eta = (\eta_t)_{0 \leq t \leq \mathcal{T}}$ is an $\mathbb{R}^{m \times d}$-valued process. Suppose that the $(N + m) \times d$-matrix

$$\begin{pmatrix} \sigma_t \\ \eta_t \end{pmatrix} \tag{2.28}$$

has $\mathbb{P}$-a.s. full rank for every $t \in [0, \mathcal{T}]$. If the processes $\frac{\Sigma^i}{V^*}$, $i = 1, \ldots, m$ are local martingales, then V^* is the GOP of the market (S^0, S, Σ).

We can justify the result above with an economically significant intuition. Since the new assets are local martingales when denominated in units of the original GOP, they they do not contribute to the price of risk and, therefore, they cannot affect the dynamics of the GOP in the extended market. From an economic point of view, we can thus state that extending the market with assets which are somehow fair has no influence on the choice by investors of allocating optimally a given wealth at a given time by creating a growth-optimal portfolio.

Remark 2.3.15. If we relax the assumption concerning the full rank of the diffusion matrix given by (2.24), Theorem 2.3.13 still holds but we cannot use the uniqueness of the growth optimal strategy to conclude the proof. In order to obtain the same result, let $\hat{\pi} = (\pi, \pi^\Sigma)$ be a possible optimal strategy satisfying condition (2.27) and let us compute the dynamics of the corresponding portfolio starting from (2.26):

$$
\begin{aligned}
\frac{dV_t^{\hat{\pi}}}{V_t^{\hat{\pi}}} &= [r_t + \pi_t^{'} \cdot (\mu_t - r_t\mathbf{1}) + \pi_t^{\Sigma}(\alpha_t - r_t)]dt + (\pi_t^{'} \cdot \sigma_t + \pi_t^{\Sigma}\beta_t^{'}) \cdot dW_t \\
&= (r_t + \pi_t^{'} \cdot \sigma_t \cdot \theta_t + \pi_t^{\Sigma}\theta_t^{'} \cdot \beta_t)dt + \theta_t^{'} \cdot dW_t \\
&= (r_t + \theta^{'} \cdot \theta - \pi_t^{\Sigma}\beta_t^{'} \cdot \theta_t + \pi_t^{\Sigma}\theta_t^{'} \cdot \beta_t)dt + \theta_t^{'} \cdot dW_t \\
&= (r_t + \|\theta_t\|^2)dt + \theta_t^{'} \cdot dW_t
\end{aligned}
$$

Therefore, any optimal strategy $\hat{\pi}$ leads to the GOP V^* of the original market. In such a more general setting we can have infinitely many optimal strategies, but the corresponding GOP is still the same. In this case we can add an infinite number of new assets by induction, as long as they are local martingales when denominated in units of V^*.

Chapter 3

Multi-curve term structures under the benchmark approach

In the previous chapter we have shown one way to deal with diffusion-based markets which may not provide a martingale measure to evaluate contingent claims. Now, one of our main purposes is to apply what we have seen in chapter 2 to markets which involve *interest rate derivatives.* As explained in Sect. 3.2, the credit crisis that started in 2007 has had an impact on interest rate market and irreversibly changed the way it works in practice, offering new modeling challenges. In this chapter we propose a possible way to deal with interest rate derivatives in a post-crisis framework, combining it with the benchmark approach. The main objects in our analysis are *zero coupon bonds* and *Libor-OIS spreads*, and we aim at representing them as solutions of partial differential equations.

3.1 Extending viable markets with term structures

In this section we extend the financial market (S^0, S) by adding the simple examples of interest rate derivatives, that is *zero coupon bonds* and *forward rate agreemets.* Before that, we recall some basics from standard interest rate theory by referring to Björk (2020).

Definition 3.1.1. A *zero coupon bond* (ZCB) with maturity $T \geq 0$, also called T-bond, is a contract which guarantees the holder one dollar to be paid on the date T. We denote by $P(t, T)$ the price at time $t \in [0, T]$ of a T-bond and we have $P(T, T) = 1$ for every $T \geq 0$.

Zero coupon bonds deliver a deterministic payment at an agreed time of maturity, but their value at any time before maturity depends on the stochastic fluctuation of the short interest rate. Indeed, we recall that if there exists an equivalent martingale measure $\mathbb{Q}$ with numéraire S^0, that is the savings account associated to $r = (r_t)_{0 \leq t \leq \mathcal{T}}$, then the price at time t of a T-bond satisfies the following relation:

$$P(t,T) = S_t^0 \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{S_T^0} \Big| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(- \int_t^T r_s \, ds \right) \Big| \mathcal{F}_t \right] \tag{3.1}$$

as shown for instance in [4, sect. 3, ch. 20]. Besides ZCBs, the other crucial objects in our analysis are forward rate agreements.

Definition 3.1.2. A *forward rate agrement* (FRA) with starting date $T \geq 0$, notional equal to one dollar, and tenor $\delta > 0$ is a contract which binds the owner to exchange at time $T + \delta$ a payment of one dollar at a fixed rate R for a payment of one dollar at a floating rate, typically a *spot Libor rate* $L(T, T+\delta)$ over the interval $[T, T+\delta]$. The price at time $t \in [0, T]$ of a FRA is denoted by $P^{FRA}(t; T, T+\delta, R)$ and its payoff is given by

$$P^{FRA}(T+\delta; T, T+\delta, R) = \delta(L(T, T+\delta) - R) \tag{3.2}$$

The holder of a forward rate agreement receives a cashflow based on a floating rate in exchange for a payment based on a fixed rate. Such exchange occurs at maturity $T + \delta$, but the payoff of a FRA is known at time T. In the arbitrage-free setting, the price of a FRA at time t under an equivalent martingale measure $\mathbb{Q}$ is given by

$$P^{FRA}(t; T, T+\delta, R) = S_t^0 \mathbb{E}^{\mathbb{Q}} \left[\frac{\delta(L(T, T+\delta) - R)}{S_{T+\delta}^0} \Big| \mathcal{F}_t \right] \tag{3.3}$$

It is often covenient to evaluate forward rate agreements with respect to the $(T + \delta)$*-forward measure.*

Definition 3.1.3. For a fixed $T \geq 0$, the T*-forward measure* $\mathbb{Q}^T$ is defined as the equivalent martingale measure for the numéraire process $P(t, T)$.

A proof of the following result can be found in [4, Proposition 15.11]

Proposition 3.1.4. *The density process of* $\mathbb{Q}^T$ *with respect to* $\mathbb{Q}$ *is given by*

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\Big|_t = \frac{P(t,T)}{S_t^0 P(0,T)} \quad \forall t \in [0, T] \tag{3.4}$$

If we choose $P(t, T+\delta)$ as numéraire, an application of Bayes' theorem (see [4, Proposition B.41]) to expression (3.3) leads to write the price of a FRA at time t as follows:

$$P^{FRA}(t; T, T+\delta, R) = P(t, T+\delta)\delta \mathbb{E}^{T+\delta}[L(T, T+\delta) - R | \mathcal{F}_t] \tag{3.5}$$

where $\mathbb{E}^{T+\delta}$ denotes the expectation under $\mathbb{Q}^{T+\delta}$. We recall another well-known definition (compare [5, Appendix A]).

Definition 3.1.5. The *forward Libor rate* $L(t;T,T+\delta)$ contracted at time $t \in [0,T]$ for the interval $[T,T+\delta]$ is that value of R which makes the price of a FRA at time t equal to zero.

Due to equation (3.5), in the arbitrage-free framework the forward Libor rate can be written as

$$L(t;T,T+\delta) = \mathbb{E}^{T+\delta}[L(T,T+\delta)|\mathcal{F}_t] \tag{3.6}$$

In particular, $L(T;T,T+\delta) = L(T,T+\delta)$. In other words, if there exist a martingale measure $\mathbb{Q}$, then $(L(t;T,T+\delta))_{0\leq t\leq T}$ is a $\mathbb{Q}^{T+\delta}$-martingale for all $T \geq 0$.

Let us now go back to the diffusion-based setting described in Sect. 2.1. We recall that (S^0, S) represents a finacial market composed of N risky assets $S = (S^1,\ldots,S^N)$ and a savings account S^0. We have shown that, if Assumption 2.2.10 holds, the financial market (S^0, S) is viable and, therefore, there exists a growth optimal portfolio V^*. The definition below sets up a framework which includes both risky assets modeled as in (2.1) and interest rate derivatives.

Definition 3.1.6. We call *extented financial market* the original market (S^0, S) enlarged with the following traded assets:

- zero-coupon bonds for all maturities $T \in [0,\mathcal{T}]$;
- forward rate agreements for all inception dates $T \in [0,\mathcal{T})$ and for all tenors $\delta_i > 0$, $i \in \{1,\ldots,k\}$, such that $T + \delta_i \leq \mathcal{T}$.

We need the following assumtion.

Assumption 3.1.7. All the ZCBs and FRAs added to the market (S^0, S) are fairly priced by the growth optimal portfolio V^*, meaning that, by Definition 2.3.11, they are true $\mathbb{P}$-martingales when discounted by V^*.

In view of Definition 2.3.11, the assumption above implies the following expressions for pricing ZCBs and FRAs in the extended market:

$$P(t,T) = V_t^*\mathbb{E}\left[\frac{1}{V_T^*}\Big|\mathcal{F}_t\right] \tag{3.7}$$

$$P^{FRA}(t;T,T+\delta_i,R) = V_t^*\mathbb{E}\left[\frac{\delta_i(L(T,T+\delta_i)-R)}{V_{T+\delta_i}^*}\Big|\mathcal{F}_t\right] \tag{3.8}$$

In order to simplify the notation, from now on we consider the case in which only FRAs related to a single tenor $\delta > 0$ are negotiated. In view of Definition 3.1.5, another consequence of Assumption 3.1.6 is the following expression for Libor rates:

$$L(t;T,T+\delta) = \frac{V_t^*}{P(t,T+\delta)}\mathbb{E}\left[\frac{L(T,T+\delta)}{V_{T+\delta}^*}\Big|\mathcal{F}_t\right] \tag{3.9}$$

Thanks to Theorem 2.3.13 we are able to state what follows.

Corollary 3.1.8. *If Assumption 3.1.7 holds, then the extended financial market defined in 3.1.6 is viable and its growth optimal portofolio is still the process* $V^* = (V_t^*)_{0\leq t\leq \mathcal{T}}$.

To complete the picture, let us dip the extended market described above into a post-crisis setting by considering Libor rates as affected by the *interbank risk* (see [9]). The next section provides an overview on the post-crisis approach based mainly on the work of Z. Grbac and W. J. Runggaldier (2015) and proposes an interpretation of the interbank risk as roll-over risk borrowed from A. Backwell et al. (2019).

3.2 Multi-curve spreads and roll-over risk approach

Before the start of the global crisis in 2007, Libor rates associated to different tenors were simply related by no-arbitrage arguments (see [4, ch. 19, sect. 2]), since both market practitioners and academic researchers used to consider negligible the *interbank risk*. As explained by Filipović and Trolle (2013), such risk can be separated into two risky components: a credit and a liquidity risk. The *credit* or *default risk* is the possibility of a counterparty failing to fulfill its obligations in a financial contract. Typically, it refers to the risk that a lender may not receive the owed notional and interest due to the default of the corresponding borrower. The *liquidity* or *funding risk* is defined as the risk of not being able to fund a position in a financial contract due to the lack of liquidity in the market; it can occur when a financial agent cannot convert an asset into cash without considerable losses. Before the last financial breakdown, the global interbank system was believed virtually not affected by credit and liquidity risk. For such reason, Libor rates were usually considered as simple compounded rates.

Definition 3.2.1. The *simply compounded spot rate* $F(T, T+\delta)$ over an interval $[T, T+\delta]$ is the rate given by

$$F(T, T+\delta) \colon = \frac{1}{\delta}\left(\frac{1}{P(T, T+\delta)} - 1\right) \tag{3.10}$$

The *simply compounded forward rate* at time $t \in [0, T]$ for the future interval $[T, T+\delta]$ is defined as

$$F(t; T, T+\delta) \colon = \frac{1}{\delta}\left(\frac{P(t, T)}{P(t, T+\delta)} - 1\right) \tag{3.11}$$

In particular, $F(T; T, T+\delta) = F(T, T+\delta)$.

Since the simply compounded rate is defined in terms of ZCBs, it is also called *risk-free rate*. It is well known that the simply compounded rates are usually considered as underlyings of *overnight indexed swaps* (OIS), contracts which bind the holder to exchange a stream of fixed rate payments for a stream of floating rate payments linked to a compounded overnight rate. *Overnight rates* are the rates at which banks lend funds to each other at the end of the day in the overnight market. Since the interbank risk is negligible in the overnight market, compounded overnight rates are assumed risk-free and they coincide with simply compounded rates. For this reason, the overnight indexed swaps can be thought as linear combinations of FRAs exchanging a fixed rate R for a floating rate $F(T_{i-1}, T_i)$, over a tenor structure $\{\, 0 \le T_0 < T_1 \cdots < T_k \le \mathcal{T} \,\}$:

$$P^{OIS}(T_k; T_0, T_k, R) = \sum_{i=1}^{k} \delta_i (F(T_{i-1}, T_i) - R) \tag{3.12}$$

where $\delta_i = T_i - T_{i-1}$. The simply compounded rates are thus often called *OIS rates* in the literature.

In the pre-crisis setting, the following relation was assumed to hold for every $T \ge 0$ and $\delta > 0$:

$$L(T, T+\delta) = F(T, T+\delta) \tag{3.13}$$

In other words, Libor rates were considered risk-free rates. Due to equation (3.6), in an arbitrage-free framework the forward Libor rates could be written as

$$L(t; T, T+\delta) = \mathbb{E}^{T+\delta}[L(T, T+\delta)|\mathcal{F}_t] = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right) = F(t; T, T+\delta)$$

and we are able to express the price of a FRA in terms of ZCBs:

$$\begin{aligned} P^{FRA}(t; T, T+\delta, R) &= P(t, T+\delta)\delta\mathbb{E}^{T+\delta}[L(T, T+\delta) - R|\mathcal{F}_t] \\ &= P(t, T+\delta)\left(\frac{P(t,T)}{P(t,T+\delta)} - 1 - \delta R\right) \end{aligned}$$

More generally, up to the crisis the prices of all Libor derivatives could be defined as output of linear (FRAs, floating rate bonds, and swaps) or non-linear (swaptions, caps, and floors) functions of ZCBs, which allowed to characterize the whole Libor market by simply modelling the ZCB curve.

During the crisis, significant *spreads* have emerged between interbank and risk-free rates, as consequence of an increase in credit and liquidity risk within the interbank system. Therefore, in the post-crisis market the Libor rate can no longer be considered immune from various interbank risks and the connection to the zero coupon bonds, which are assumed risk-free, is lost:

$$L(T, T+\delta) \neq \frac{1}{\delta}\left(\frac{1}{P(T,T+\delta)} - 1\right)$$

In other words, after the crisis the Libor curve must be modeled separately. In view of this, such framework is named *multi-curve* in the literature (see for instance [1], [5], and [12]). By adapting the multiplicative convention proposed in [5], we can now give the following definition.

Definition 3.2.2. The *spot spread* $S(T, T+\delta)$ over an interval $[T, T+\delta]$ is the quantity defined by

$$S(T, T+\delta) := \frac{1+\delta L(T, T+\delta)}{1+\delta F(T, T+\delta)} = [1+\delta L(T, T+\delta)] P(T, T+\delta) \quad (3.14)$$

The *forward spread* at time $t \in [0, T]$ for the future interval $[T, T+\delta]$ is given by

$$S(t; T, T+\delta) := \frac{1+\delta L(t; T, T+\delta)}{1+\delta F(t; T, T+\delta)} \quad (3.15)$$

In particular $S(T; T, T+\delta) = S(T, T+\delta)$.

Remark 3.2.3. Since $L(t; T, T+\delta) > F(t; T, T+\delta)$ for all $t \in [0, T]$, the spread between Libor and risk-free rates is typically greater than one and it increases with the tenor length $\delta > 0$. Indeed, the longer the interval $[T, T+\delta]$, the greater the risk to grant a Libor loan, which becomes more expensive. On the other hand, when δ is close to zero the risk affecting Libor rates can be considered negligible and identity (3.13) still holds. We thus obtain the following terminal condition for spot spreads:

$$S(T, T) = 1 \quad (3.16)$$

In view of Definition 3.2.2, instead of dealing directly with the Libor curve, we can provide a model for the spread curve. Moreover, as consequence of Assumption 3.1.7, we are able to establish a relation between spot spread and forward spread (compare [5, Lemma 3.11]).

Lemma 3.2.4. *Suppose that Assumption 3.1.7 holds. For every $T > 0$, the stochastic process $(S(t; T, T+\delta))_{0 \le t \le T}$ is a $\mathbb{Q}^T$-martingale, where $\mathbb{Q}^T$ denotes the T-forward measure whose density process is given by*

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{P}}\right|_t = \frac{P(t, T)}{V_t^* P(0, T)} = \frac{\hat{P}(t, T)}{P(0, T)} \quad \forall t \in [0, T] \quad (3.17)$$

Proof. Thanks to Assumption 3.1.7, the price of a FRA at time t is given by

$$P^{FRA}(t; T, T+\delta, R) = V_t^* \mathbb{E}\left[\delta \frac{L(T, T+\delta) - R}{V_{T+\delta}^*} \Big| \mathcal{F}_t\right]$$

Due to Definition 3.1.5, replacing the fixed rate R by $L(t; T, T+\delta)$ makes null the quantity above

$$\mathbb{E}\left[\frac{V_t^*}{V_{T+\delta}^*} \delta [L(T, T+\delta) - L(t; T, T+\delta)] \Big| \mathcal{F}_t\right] = 0$$

In view of equation (3.7) and Definition 3.2.1, the tower property of the conditional expectation provides

$$\begin{aligned}[1+\delta L(t;T,T+\delta)]P(t,T+\delta) &= \mathbb{E}\left[\frac{V_t^*}{V_{T+\delta}^*}[1+\delta L(T,T+\delta)]\Big|\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\frac{V_t^*}{V_T^*}P(T,T+\delta)[1+\delta L(T,T+\delta)]\Big|\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\frac{V_t^*}{V_T^*}\frac{1+\delta L(T,T+\delta)}{1+\delta F(T,T+\delta)}\Big|\mathcal{F}_t\right]\end{aligned}$$

As consequence of the defining property of forward spread, specified by (3.15), we obtain

$$S(t;T,T+\delta) = \frac{1}{P(t,T)}\mathbb{E}\left[\frac{V_t^*}{V_T^*}S(T,T+\delta)\Big|\mathcal{F}_t\right] \tag{3.18}$$

Since the process V^* can be considered as the numéraire associated to the real-world measure $\mathbb{P}$, an application of the change of numéraire formula (see [4, Proposition 15.3]) provides (3.17) as density process of $\mathbb{Q}^T$ with respect to $\mathbb{P}$. By starting from expression (3.18) we have

$$S(t;T,T+\delta) = \frac{V_t^* P(0,T)}{P(t,T)}\mathbb{E}\left[\frac{S(T,T+\delta)}{V_T^* P(0,T)}\Big|\mathcal{F}_t\right] = \mathbb{E}^T\left[S(T,T+\delta)\Big|\mathcal{F}_t\right]$$

where the last equality follows from an application of Bayes' theorem. □

The problem now is to characterize the spot spread. There exists a large variety of modeling possibilities; in what follows we adopt the formulation advaced by A. Backwell et al. in [1], which consists of interpreting the interbank risk as roll-over risk. The *roll-over risk* is a risk related to the refinancing of debt and it is faced by countries and companies when a loan or another debt obligation is close to maturity and it may require a higher interest rate to be converted, or rolled over, into new debt. In other terms, the roll-over risk emerged when a financial entity faces the istantaneous funding cost

$$\bar{r}_t :\, = r_t + \zeta_t$$

where $\zeta = (\zeta_t)_{0\le t\le \mathcal{T}}$ is a non-negative adapted process w.r.t. $\mathbb{F}$ which represents the *roll-over-risk spread*. To link the roll-over risk to the Libor risk, we introduce the non-negative adapted processes $\varphi = (\varphi_t)_{0\le t\le \mathcal{T}}$, the *funding-liquidity spread*, and $\lambda = (\lambda)_{0\le t\le \mathcal{T}}$, the *credit-default spread*, and we assume that

$$\zeta_t = \varphi_t + \lambda_t$$

for every $t \in [0, \mathcal{T}]$.

Remark 3.2.5. Using the *intensity approach* to credit risk described in [3, ch. 3], we suppose that λ corresponds to the *default intensity* associated to a random default time τ satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for any $t \in [0, \mathcal{T}]$. We denote by $\mathbb{H} = (\mathcal{H}_t)_{0 \le t \le \mathcal{T}}$ the filtration generated by the right-continuous *default indicator* process $\mathbb{1}_{\{\tau \le t\}}$. The intensity approach provides thus two kinds of information: the information related to asset prices and other economic factors, denoted by $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le \mathcal{T}}$, that in our setting coincides with the Brownian filtration, and the information about the occurence of the default time. For each $t \in [0, \mathcal{T}]$, the total information available at time t is captured by the σ-algebra $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. In this framework, the credit-default spread is defined by

$$\lambda_t = \lim_{h \to 0} \frac{\mathbb{P}(\tau \le t + h | \mathcal{G}_t, \tau > t)}{h} \tag{3.19}$$

See [3] for further details.

We can now give the following definition.

Definition 3.2.6. A *defaultable zero coupon bond* with zero recovery and maturity date $T \ge 0$ provides the holder with one dollar at time T if $\tau > T$ and with zero dollars otherwise. We denote by $P^d(t, T)$ the price at time $t \in [0, T]$ of a defaultable ZCB and we have $P^d(T, T) = \mathbb{1}_{\{\tau > T\}}$ for every $T \ge 0$.

By enlarging the market described in Definition 3.1.6 with defaultable zero coupon bonds $P^d(t, T)$ for all $T \in [0, \mathcal{T}]$, and assuming them fairly priced, we get the following expression for $P^d(t, T)$:

$$P^d(t, T) = V_t^* \mathbb{E}\left[\frac{1}{V_T^*} \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right] \tag{3.20}$$

We recall a useful result from [3, Corollary 3.1.1].

Lemma 3.2.7. *For all $T \in [0, \mathcal{T}]$, let X be a $\mathcal{F}_T$-measurable and $\mathbb{P}$-integrable random variable. Then, for every $t \le T$,*

$$\mathbb{E}[X \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[X \exp\left(- \int_t^T \lambda_s \, ds \right) \middle| \mathcal{F}_t \right]$$

An application of the lemma above gives

$$P^d(t, T) = \mathbb{1}_{\{\tau > t\}} V_t^* \mathbb{E}\left[\frac{1}{V_T^*} \exp\left(- \int_t^T \lambda_s \, ds \right) \middle| \mathcal{F}_t \right]$$

Along the line of [1], we are now in the position to define the process $(A(t, T))_{t \in [0, T]}$ of the value at time t of the *roll-over-risk-adjusted borrowing account* for every $T \in [0, \mathcal{T}]$. In other words, $A(t, T)$ represents the value at time t of a loan of one dollar continuously rolled-over until time T. If $A(t,T)$ is fairly priced by the GOP V^*, we obtain

$$\begin{aligned} A(t,T) :&= V_t^* \mathbb{E}\left[\frac{1}{V_T^*}\exp\left(\int_t^T \bar{r}_s\, ds\right)\mathbb{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] \\ &= V_t^* \mathbb{E}\left[\frac{1}{V_T^*}\exp\left(\int_t^T (r_s+\varphi_s+\lambda_s)\, ds\right)\mathbb{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} V_t^* \mathbb{E}\left[\frac{1}{V_T^*}\exp\left(\int_t^T (r_s+\varphi_s)\, ds\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} \frac{V_t^*}{\tilde{S}_t^0} \mathbb{E}\left[\frac{\tilde{S}_T^0}{V_T^*}\Big|\mathcal{F}_t\right] \end{aligned} \tag{3.21}$$

where the second equality relies again on Lemma 3.2.7 and the third one follows from defining a fundingaccount $\tilde{S} = (\tilde{S}_t)_{0\le t\le \mathcal{T}}$ based on the roll-over rate process $\tilde{r} = (\tilde{r}_t)_{0\le t\le \mathcal{T}}$ given by $\tilde{r}_t = r_t + \varphi_t$. Notice that, due to the cancellation of the default intensity, the present value of the cost of a continuosly rolled borrowing over the period $[t,T]$ only depends on the funding-liquidity spread. As in [1, sect. 2, ch. 1], in equilibrium the following quantities must be equal:

- the value at T of borrowing one dollar and rolling-over the loan until time $T+\delta$, which is given by $A(T,T+\delta)$;
- the value at T of borrowing one dollar at the term rate $L(T,T+\delta)$ over the interval $[T,T+\delta]$, that is $[1+\delta L(T,T+\delta)]P^d(T,T+\delta)$.

Since $A(T,T+\delta) = [1+\delta L(T,T+\delta)]P^d(T,T+\delta)$, conditionally on the event $\{\tau > T\}$ one gets

$$L(T,T+\delta) = \frac{1}{\delta}\left(\frac{A(T,T+\delta)}{P^d(T,T+\delta)} - 1\right) \tag{3.22}$$

and the spot spread between Libor and risk-free rates is given by

$$S(T,T+\delta) = \frac{1+\delta L(T,T+\delta)}{1+\delta F(T,T+\delta)} = \frac{A(T,T+\delta)}{P^d(T,T+\delta)}P(T,T+\delta) \tag{3.23}$$

The spot spread formulation based on the roll-over risk approach offers an interesting economic explanation. Indeed, the Libor-OIS spread can be seen as a premium paid by the borrower at Libor to avoid funding and credit risk over the period of the Libor loan; see [1] for more details.

Remark 3.2.8. Another popular model for the post-crisis interest rates market, adopted for instance in [12, ch. 2], consists of assuming that the Libor rates satisfy an equation similar to (3.10), but replacing the bond prices $P(t,T)$ by fictitious ones $\bar{P}(t,T)$:

$$L(T,T+\delta) = \frac{1}{\delta}\left(\frac{1}{\bar{P}(T,T+\delta)} - 1\right) \tag{3.24}$$

Such bonds are called *risky*[3] since they are supposed to be affected by the same risk factors of the Libor rates. Indeed, if we assume that $\bar{P}(t,T)$ are fairly priced by the GOP V^* for all $T \in [0,\mathcal{T}]$, we can adapt the formulation presented in [12, equation (2.35)] and write the price at time t of a T-risky-bond as follows:

$$\bar{P}(t,T) = V_t^* \mathbb{E}\left[\frac{1}{V_T^*} \exp\left(-\int_t^T (\varphi_s + \lambda_s)\, ds\right) \Big| \mathcal{F}_t\right] \tag{3.25}$$

The risky bonds are not traded assets, but we can still assume $\bar{P}(T,T) = 1$ and, due to (3.15), we can write the spot spread as

$$S(T,T+\delta) = S(T;T,T+\delta) = \frac{P(T,T+\delta)}{\bar{P}(T,T+\delta)} \tag{3.26}$$

for every $T \in [0,\mathcal{T})$. This formulation is equivalent to the roll-over risk approach if the defining property of T-risky-bonds given by (3.25) is replaced by

$$\bar{P}(t,T) = \frac{P^d(t,T)}{A(t,T)}, \quad t < \tau$$

Both approaches provide a definition of $\bar{P}(t,T)$ depending on the processes φ and λ.

In our analysis, we deal with a specific instance of the roll-over risk framework, in which the market defined in 3.1.6 is assumed credit risk free, leaving only the funding-liquidity component of roll-over risk. Such choice is justified by the fact that the credit risk is often mitigated by collatelatizations.[4] In absence of credit risk, the argument above can be repeated replacing the defaultable bonds by the risk-free ones and, therefore, one get the following expressions for Libor rates and spreads, for all $T \in [0,\mathcal{T})$:

$$L(T,T+\delta) = \frac{1}{\delta}\left(\frac{A(T,T+\delta)}{P(T,T+\delta)} - 1\right) \tag{3.27}$$

$$S(T,T+\delta) = A(T,T+\delta) = \frac{V_T^*}{\tilde{S}_T^0} \mathbb{E}\left[\frac{\tilde{S}_{T+\delta}^0}{V_{T+\delta}^*}\Big| \mathcal{F}_T\right] \tag{3.28}$$

Since $A(t,t) = 1$ for all $t \in [0,\mathcal{T}]$ by construction, equation (3.28) gives the $\mathbb{P}$-martingality of the process $\left(S(t,T)\frac{\tilde{S}_t^0}{V_t^*}\right)_{0 \le t \le T}$ for every $T \in [0,\mathcal{T}]$, where we denote by $S(t,T)$ the spot spread over the interval $[t,T]$. Due to Lemma

[3]The artificial risky bonds are only introduced here as an explanatory tool and they are not be considered in the following sections of the thesis.

[4]In finance the term *collateral* refers to assets or cash posted by a borrower to a lender in order to secure a loan. See [1, ch. 5] or [12, ch. 1, sect 2].

3.2.4, the roll-over risk approach provides the following characterization for multiplicative forward spreads:

$$
\begin{aligned}
S(t;T,T+\delta) &= \frac{1}{P(t,T)}\mathbb{E}\left[\frac{V_t^*}{V_T^*}S(T,T+\delta)\middle|\mathcal{F}_t\right] \\
&= \frac{V_t^*}{P(t,T)}\mathbb{E}\left[\frac{1}{V_{T+\delta}^*}\frac{\tilde{S}^0_{T+\delta}}{\tilde{S}^0_T}\middle|\mathcal{F}_t\right] \\
&= \frac{P(t,T+\delta)}{P(t,T)}\mathbb{E}^{T+\delta}\left[\frac{\tilde{S}^0_{T+\delta}}{\tilde{S}^0_T}\middle|\mathcal{F}_t\right]
\end{aligned}
\tag{3.29}
$$

Moreover, since under Assumption 3.1.7 the forward Libor rate is that rate which makes the fair value at time t of a FRA equal to zero, by inserting equation (3.27) into expression (3.9) we obtain

$$
L(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{V_t^*}{P(t,T+\delta)}\mathbb{E}\left[\frac{A(T,T+\delta)}{V_T^*}\middle|\mathcal{F}_t\right]-1\right)
\tag{3.30}
$$

and due to (3.28) the forward Libor rate can be written as

$$
\begin{aligned}
L(t;T,T+\delta) = &\frac{1}{\delta}\left(\frac{S(t,T+\delta)}{P(t,T+\delta)}\mathbb{E}\left[\frac{\tilde{S}^0_t}{\tilde{S}^0_T}\middle|\mathcal{F}_t\right]-1\right) \\
&+\frac{1}{\delta}\frac{V_t^*}{P(t,T+\delta)}\operatorname{Cov}\left(\frac{1}{\tilde{S}^0_T},\frac{\tilde{S}^0_{T+\delta}}{V_{T+\delta}^*}\middle|\mathcal{F}_t\right)
\end{aligned}
\tag{3.31}
$$

In the next section we endow the market defined in 3.1.6 with a Markovian stucture in order to characterize the bonds and spreads as solutions of specific partial differential equations which will be used in chapter 4 as starting point to derive a stochastic control representation under the benchmark approach.

3.3 Diffusion-based Markov models for multi-curve term structures

To define a Markov model for all the assets we are considering, we assume that the whole economy is driven by a single $\mathbb{R}^n$-valued Markov process $X = (X_t)_{0\le t\le \mathcal{T}}$ satisfying

$$
dX_t = f(t,X_t)dt + g(t,X_t)dW_t, \quad X_0 = x_0
\tag{3.32}
$$

where $f\colon \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}_+\times\mathbb{R}^n \to \mathcal{M}(n,d)$ are functions for which the following condition holds.

Assumption 3.3.1. There exists $M>0$ such that, for every $t\in[0,\mathcal{T}]$ and $x\in\mathbb{R}^n$,

$$
\|f(t,x)\| \le M(1+\|x\|), \quad |||g(t,x)||| \le M
$$

where we denote by $|||\cdot|||$ the euclidean matrix norm.

Under the assumption above, equation (3.32) has a unique strong solution. Let us set $n > N$ in order to consider the stochastic differentials $\frac{dS_t^i}{S_t^i}$ defined by (2.1) as the first N components of dX_t.

Assumption 3.3.2 (Diffusion-based Markov structure). The stochastic dynamics of (S^0, S) defined in Sect. 2.1 can be written as

$$dS_t = D(S_t) \cdot \mu(t, X_t)dt + D(S_t) \cdot \sigma(t, X_t) \cdot dW_t \tag{3.33}$$

$$dS_t^0 = S_t^0 r(t, X_t)dt \tag{3.34}$$

where μ, σ and r are functions of time and the factor process $X = (X_t)_{0 \leq t \leq \mathcal{T}}$ which is assumed to contain the assets $S = (S^1, \dots, S^N)$ as first N components and to satisfy (3.32). The value at time t of $P(t,T)$ and $S(t,T)$ is defined by

$$P(t,T) =: p^T(t, X_t) \tag{3.35}$$

$$S(t;T) =: s^T(t, X_t) \tag{3.36}$$

for some functions $p^T, s^T : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belonging to class $\mathcal{C}^{1,2}$. Finally, we assume that there exists $M > 0$ such that, for every $t \in [0, \mathcal{T}]$ and $x \in \mathbb{R}^n$,

$$\|r(t,x)\|, \|\varphi(t,x)\| \leq M(1 + \|x\|^2)$$

where $\varphi_t =: \varphi(t, X_t)$ is the funding-liquidity spread.

We recall from [4, ch. 6] the following definition.

Definition 3.3.3. A portfolio process $V^\pi = (V_t^\pi)_{0 \leq t \leq \mathcal{T}}$ associated to an admissible strategy π is said to be *Markovian* if there exists a function $v^\pi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $v^\pi \in \mathcal{C}^{1,2}$, such that

$$V_t^\pi = v^\pi(t, X_t) \quad \forall t \in [0, \mathcal{T}] \tag{3.37}$$

The following result provides a characterization for any function v^π defining a Markovian porfolio process.

Proposition 3.3.4. *Let $v \colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be a function belonging to class $\mathcal{C}^{1,2}$. The stochastic process $V = (V_t)_{0 \leq t \leq \mathcal{T}}$ given by $V_t = v(t, X_t)$ for every $t \in [0, \mathcal{T}]$ is an admissible portfolio process if and only if the function v satisfies the following partial differential equation:*

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x) &+ \nabla_x v(t,x) \cdot (f(t,x) - g(t,x) \cdot \theta(t,x)) \\ &+ \frac{1}{2} \operatorname{tr}(g^{'}(t,x) \cdot \nabla_{xx}^2 v(t,x) \cdot g(t,x)) \\ &- v(t,x) r(t,x) = 0 \end{aligned} \tag{3.38}$$

where θ is a deterministic function such that $\theta(t, X_t) \colon = \theta_t$ for all $t \in [0, \mathcal{T}]$, that is it defines the market price of risk process.

Proof. Suppose that there exists an admissible strategy $\pi = (\pi_t)_{0 \le t \le \mathcal{T}}$ such that its corresponding portfolio process is $V^\pi = (V_t^\pi)_{0 \le t \le \mathcal{T}}$ given by

$$V_t^\pi = v(t, X_t) \quad \forall t \in [0, \mathcal{T}]$$

By applying the Itô's formula we get

$$dV_t^\pi = \left(\frac{\partial v}{\partial t} + \nabla_x v \cdot f + \frac{1}{2} \operatorname{tr}(g' \cdot \nabla_{xx}^2 v \cdot g) \right) dt + \nabla_x v \cdot g \cdot dW_t$$

where we have suppressed the dependence on (t, X_t) for brevity of notation. Since V^π is an admissible portfolio process, the above expression must be equal to (2.5). Matching the deterministic and stochastic parts gives

$$\begin{cases} \frac{\partial v}{\partial t} + \nabla_x v \cdot f + \frac{1}{2} \operatorname{tr}(g^{'} \cdot \nabla_{xx}^2 v \cdot g) = (r + \pi^{'} \cdot \sigma \cdot \theta) v \\ \nabla_x v \cdot g = (\pi^{'} \cdot \sigma) v \end{cases}$$

and inserting the second line into the first one leads to equation (3.38).

Conversely, let us now assume that v satisfies equation (3.38). Another application of Itô's formula provides the following:

$$\begin{aligned} dv(t, X_t) =& (v(t, X_t) r(t, X_t) + \nabla_x v(t, X_t) \cdot g(t, X_t) \cdot \theta(t, X_t)) dt \\ &+ \nabla_x v(t, X_t) \cdot g(t, X_t) \cdot dW_t \end{aligned}$$

where we have used equation (3.38) to get the drift of $v(t, X_t)$. The process $V = v(\cdot, X)$ defines a portfolio process if there exists an admissible strategy $\pi = (\pi_t)_{0 \le t \le \mathcal{T}}$ satisfying

$$\sigma_t^{'} \cdot \pi_t = g^{'}(t, X_t) \cdot \frac{\nabla_x^{'} v(t, X_t)}{v(t, X_t)} \tag{3.39}$$

Since σ_t has $\mathbb{P}$-a.s. full rank for every t (Assumptions 2.2.8), then the equation above admits the following solution:

$$\pi_t = \left(\sigma(t, X_t) \cdot \sigma^{'}(t, X_t) \right)^{-1} \cdot \sigma(t, X_t) \cdot g^{'}(t, X_t) \cdot \frac{\nabla_x^{'} v(t, X_t)}{v(t, X_t)}$$

for every $t \in [0, \mathcal{T}]$. Such a process is progressively measurable and we have

$$\int_0^{\mathcal{T}} \|\sigma_t^{'} \cdot \pi_t\|^2 \, dt \le \int_0^{\mathcal{T}} |||g(t, X_t)|||^2 \left\| \frac{\nabla_x v(t, X_t)}{v(t, X_t)} \right\|^2 dt < \infty$$

since $v \in \mathcal{C}^{1,2}$ and $|||g(t, x)||| \le M$ for every $(t, x) \in [0, \mathcal{T}] \times \mathbb{R}^n$ (according to Assumption 3.3.1). Due to Lemma 2.2.17, the process π defined above is an admissible strategy. □

For our purposes, an interesting implication of the previous result concerns the growth optimal portfolio.

Corollary 3.3.5. *Let $V^* = (V^*_t)_{0\le t\le \mathcal{T}}$ be an admissible portfolio process. Let us assume that there exists a function $v^*\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $v^* \in \mathcal{C}^{1,2}$, such that $V^*_t = v^*(t, X_t)$. The process V^* is the GOP of the market (S^0, S) if and only if the function v^* is related to the market price of risk $\theta = (\theta_t)_{0\le t\le \mathcal{T}}$ as follows:*

$$\nabla_x v^*(t, X_t) \cdot g(t, X_t) = v^*(t, X_t)\theta^{'}_t \tag{3.40}$$

for all $t \in [0, \mathcal{T}]$.

Proof. If V^* is the GOP of the market (S^0, S), then it satisfies the stochastic dynamics given by equation (2.17). An application of Itô's formula gives

$$dV^*_t = \left(\frac{\partial v^*}{\partial t} + \nabla_x v^* \cdot f + \frac{1}{2}\operatorname{tr}(g' \cdot \nabla^2_{xx} v^* \cdot g)\right) dt + \nabla_x v^* \cdot g \cdot dW_t$$

and we get equation (3.40) by comparing the stochastic component of (2.17) with $\nabla_x v^* \cdot g$.

On the other hand, due to Proposition 3.3.4, the function v^* solves equation (3.38) and, therefore, by applying the Itô's formula we have

$$\begin{aligned} dv^*(t, X_t) =&(v^*(t, X_t) r(t, X_t) + \nabla_x v^*(t, X_t) \cdot g(t, X_t) \cdot \theta(t, X_t))dt \\ &+ \nabla_x v^*(t, X_t) \cdot g(t, X_t) \cdot dW_t \end{aligned}$$

By using (3.40), we get the stochastic dynamics (2.17) which defines the GOP of the market (S^0, S). □

From now on we assume the following.

Assumption 3.3.6 (The GOP as Markovian porfolio process). There exists a function $v^*\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $v^* \in \mathcal{C}^{1,2}$, such that

$$V^*_t = v^*(t, X_t) \tag{3.41}$$

for all $t \in [0, \mathcal{T}]$

We are now in condition to give a characterization for multi-curve term structures under the benchmark approach as solutions of terminal value problems. First of all, let us denote by $\hat{P}(t, T) = \frac{P(t,T)}{V^*_t}$ the benchmarked price of a T-bond. Due to Assumptions 3.3.2 and 3.3.6, there exists a function $\hat{p}^T \in \mathcal{C}^{1,2}$ such that

$$\hat{P}(t, T) = \hat{p}^T(t, X_t) = \frac{p^T(t, X_t)}{v^*(t, X_t)} \tag{3.42}$$

for every $T \in [0, \mathcal{T}]$. Remembering that Assumption 3.1.7 gives the $\mathbb{P}$-martingale property of $(\hat{P}(t, T))_{0\le t\le \mathcal{T}}$ and combining it with an application of Itô's formula, we get that the function $\hat{p}^T$ must solve the following PDE:

$$\frac{\partial}{\partial t}\hat{p}^T(t, x) + \nabla_x \hat{p}^T(t, x) \cdot f(t, x) + \frac{1}{2}\operatorname{tr}\left(g^{'}(t, x) \cdot \nabla^2_{xx}\hat{p}^T(t, x) \cdot g(t, x)\right) = 0 \tag{3.43}$$

with boundary condition given by

$$\hat{p}^T(T,x) = \frac{p^T(T,x)}{v^*(T,x)} = \frac{1}{v^*(T,x)} \quad \forall x \in \mathbb{R}^n \tag{3.44}$$

By adopting the roll-over risk formulation described in Sect. 3.2, a similar representation can be provided for spot spreads $S(t,T)$ for all $T \in [0,\mathcal{T}]$.

Proposition 3.3.7. *Suppose that the roll-over-risk-adjusted borrowing account related to $\tilde{r} = (\tilde{r}_t)_{0\le t\le \mathcal{T}}$ is fairly priced. Assume that the market is not affected by credit risk. Recalling that φ represents the funding-liquidity risk, if Assumptions 3.3.1, 3.3.2 and 3.3.6 hold, then the only function $s^T \in \mathcal{C}^{1,2}$ such that $S(t,T) = s^T(t,X_t)$ solves the following partial differential equation*

$$\begin{aligned}\frac{\partial}{\partial t}s^T(t,x) + \nabla_x s^T(t,x)\cdot\left(f(t,x) - g(t,x)\cdot g^{'}(t,x)\cdot\frac{\nabla_x^{'}v^*(t,x)}{v^*(t,x)}\right)\\ + \frac{1}{2}\,\mathrm{tr}\left(g^{'}(t,x)\cdot\nabla^2_{xx}s^T(t,x)\cdot g(t,x)\right) + \varphi(t,x)s^T(t,x) = 0\end{aligned} \tag{3.45}$$

with terminal condition $s^T(T,x) = 1$ for every $x \in \mathbb{R}^n$.

Proof. In the absence of credit risk, $S(t,T)$ coincides with the roll-over-risk-adjusted borrowing account $A(t,T)$ which is assumed fairly priced, meaning that

$$S(t,T) = \frac{V_t^*}{\tilde{S}_t^0}\mathbb{E}\left[\frac{\tilde{S}_T^0}{V_T^*}\middle|\mathcal{F}_t\right] \tag{3.46}$$

Therefore, the process $\left(S(t,T)\frac{\tilde{S}_t^0}{V_t^*}\right)_{0\le t\le T}$ is a $\mathbb{P}$-martingale. An application of the stochastic differentiation by parts provides

$$d\left(S(t,T)\frac{\tilde{S}_t^0}{V_t^*}\right) = dS(t,T)\frac{\tilde{S}_t^0}{V_t^*} + S(t,T)d\left(\frac{\tilde{S}_t^0}{V_t^*}\right) + d\left\langle S(\cdot,T),\frac{\tilde{S}^0}{V^*}\right\rangle_t$$

Consequently to the definition of $\tilde{S}^0$ as funding account associated to $\tilde{r}$ and the GOP dynamics formula (2.17), the stochastic differential of $\frac{\tilde{S}^0}{V^*}$ is given by

$$d\left(\frac{\tilde{S}_t^0}{V_t^*}\right) = \frac{\tilde{S}_t^0}{V_t^*}\left[(\tilde{r}_t - r_t)dt - \theta_t^{'}\cdot dW_t\right] = \frac{\tilde{S}_t^0}{V_t^*}\left(\varphi_t dt - \theta_t^{'}\cdot dW_t\right)$$

Substituting the last expression into the first equation leads to

$$\begin{aligned} d\left(S(t,T)\frac{\tilde{S}^0_t}{V^*_t}\right) &= \frac{\tilde{S}^0_t}{V^*_t}\left(\frac{\partial}{\partial t}s^T(t,X_t) + \nabla_x s^T(t,X_t)\cdot f(t,X_t)\right)dt \\ &+ \frac{1}{2}\frac{\tilde{S}^0_t}{V^*_t}\operatorname{tr}\left(g^{'}(t,X_t)\cdot\nabla^2_{xx}s^T(t,X_t)\cdot g(t,X_t)\right)dt \\ &+ \frac{\tilde{S}^0_t}{V^*_t}[\varphi_t s^T(t,X_t) - \nabla_x s^T(t,X_t)\cdot g(t,X_t)\cdot\theta_t]dt \\ &+ (\dots)\cdot dW_t \end{aligned}$$

By setting the drift equal to zero in view of the $\mathbb{P}$-martingality of $S(\cdot,T)\frac{\tilde{S}^0}{V^*}$ one get

$$\frac{\partial}{\partial t}s^T + \nabla_x s^T\cdot(f - g\cdot\theta) + \frac{1}{2}\operatorname{tr}(g^{'}\cdot\nabla^2_{xx}s^T\cdot g) + \varphi s^T = 0 \tag{3.47}$$

Due to Assumption 3.3.6 and Corollary 3.3.5, the market price of risk can be replaced by the expression deriving from (3.40) and we obtain equation (3.45) with boundary condition given by $s^T(T,x) = S(T,T) = 1$. The existence and uniqueness of solution are guaranteed by the hypotheses of the Proposition. □

Remark 3.3.8. The result obtained confirms the relation between term premia and spot spreads provided by the roll-over risk interpretation. Since we have excluded the presence of credit risk, the spot spread can be can be seen as a term premium paid by the holder of a Libor loan to hedge against the roll-over risk, in addition to the premium due to the market price of risk. In this setting, the value $\varphi_t = \varphi(t,X_t)$ of the funding-liquidity spread determines the instantaneous rate of increase of the Libor loan or, equivalently, the loan corresponding to the roll-over-risk-adjusted borrowing account.

In order to derive an analogous result for forward spreads, let us now assume that the value at time t of $S(t;T,T+\delta)$ is given by

$$S(t;T,T+\delta) =: s^{T,\delta}(t,X_t) \tag{3.48}$$

for some function $s^{T,\delta}\colon \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}_+$ belonging to class $\mathcal{C}^{1,2}$. We recall that the value at time $t = T$ of $S(t;T,T+\delta)$ coincides with $S(T,T+\delta)$, so that

$$s^{T,\delta}(T,X_T) = S(T,T+\delta) \tag{3.49}$$

Corollary 3.3.9. *Suppose that the hypotheses of Proposition 3.3.7 hold. Let $s^{T+\delta}$ be solution of the boundary value problem given in Proposition 3.3.7. Then the only function $s^{T,\delta}\in\mathcal{C}^{1,2}$ defined by (3.48) satisfies the following* *partial differential equation:*

$$\frac{\partial}{\partial t}s^{T,\delta}(t,x) + \nabla_x s^{T,\delta}(t,x)\cdot\left(f(t,x) + g(t,x)\cdot g^{'}(t,x)\cdot\frac{\nabla^{'}_x\hat{p}^T(t,x)}{\hat{p}^T(t,x)}\right) + \frac{1}{2}\operatorname{tr}\left(g^{'}(t,x)\cdot\nabla^2_{xx}s^{T,\delta}(t,x)\cdot g(t,x)\right) = 0 \tag{3.50}$$

with terminal condition $s^{T,\delta}(T,x) = s^{T+\delta}(T,x)$ *for every* $x\in\mathbb{R}^n$*.*

Proof. From Lemma 3.2.4 we deduce that $(\hat{P}(t,T)S(t;T,T+\delta))_{0\le t\le T}$ is a $\mathbb{P}$-martingale, as consequence of Bayes' rule. In view of Assumption 3.3.2, we can apply the stochastic differentiation by parts to $\hat{p}^T(t,X_t)s^{T,\delta}(t,X_t)$:

$$d(\hat{p}^T(t,X_t)s^{T,\delta}(t,X_t)) = d\hat{p}^T(t,X_t)s^{T,\delta}(t,X_t) + \hat{p}^T(t,X_t)ds^{T,\delta}(t,X_t) + d\langle\hat{p}^T(\cdot,X),s^{T,\delta}(\cdot,X)\rangle_t$$

Since $\hat{p}^T(t,X_t)$ is a $\mathbb{P}$-martingale, the first term has finite variation and we get

$$\begin{aligned} d(\hat{p}^T(t,X_t)s^{T,\delta}(t,X_t)) = {} & \hat{p}^T(t,X_t)\left(\frac{\partial}{\partial t}s^{T,\delta}(t,X_t) + \nabla_x s^{T,\delta}(t,X_t)\cdot f_t\right)dt \\ & + \frac{1}{2}\hat{p}^T(t,X_t)\operatorname{tr}\left(g^{'}_t\cdot\nabla^2_{xx}s^{T,\delta}(t,X_t)\cdot g_t\right)dt \\ & + \nabla_x s^{T,\delta}(t,X_t)\cdot g_t\cdot g^{'}_t\cdot\nabla^{'}_x\hat{p}^T(t,X_t)dt \\ & + (\dots)\cdot dW_t \end{aligned}$$

where $f_t = f(t,X_t)$ and $g_t = g(t,X_t)$. Due to the $\mathbb{P}$-martingality of the process $(\hat{p}^T(t,X_t)s^{T,\delta}(t,X_t))_{0\le t\le T}$, we can set the drift equal to zero obtaining equation (3.50). The boundary condition is provided by (3.49) and Assumption 3.3.2, while existence and uniqueness of solution are guaranteed by Assumptions 3.3.1, 3.3.2, and 3.3.6. □

Remark 3.3.10. As consequence of (3.42), equation (3.50) can be written as

$$\frac{\partial}{\partial t}s^{T,\delta} + \nabla_x s^{T,\delta}\cdot\left[f + g\cdot g^{'}\cdot\left(\frac{\nabla^{'}_x p^T}{p^T} - \frac{\nabla^{'}_x v^*}{v^*}\right)\right] + \frac{1}{2}\operatorname{tr}(g^{'}\cdot\nabla^2_{xx}s^{T,\delta}\cdot g) = 0$$

where we have suppressed the dependence on (t,x) for brevity of notation. We enphasize that equation (3.50) suffices to provide a Markovian picture for forward spreads and the terminal condition given by the spot spread could be provided by exogenous factors. In our setting, the roll-over-risk approch introduces an endogenous perspective of spot spred and we are thus able to reduce the terminal condition of equation (3.50) to a further boundary value problem, given by Proposition 3.3.7.

3.3.1 Example: linear dynamics and exponential quadratic structures

An interesting case for its analytical tractability is given by exponential quadratic term structures driven by a factor process with linear dynamics. Let us assume that equation (3.32) is of the form

$$dX_t = (F(t)X_t + H(t))dt + G(t) \cdot dW_t \tag{3.51}$$

where, for all $t \in [0, \mathcal{T}]$, $H(t)$ is an n-dimensional column vector, while $F(t)$ and $G(t)$ are matrices of dimension $n \times n$ and $n \times d$.

Lemma 3.3.11. *Suppose that the short rate $r_t = r(t, X_t)$ is given by*

$$r(t, X_t) = a(t) + b(t) \cdot X_t + X_t^{'} \cdot c(t) \cdot X_t \tag{3.52}$$

where, for all $t \in [0, \mathcal{T}]$, $a(t)$ is a scalar quantity, $b(t)$ is an n-dimensional row vector, and $c(t)$ is a non-negative symmetric $(n \times n)$-matrix. Let C^*, B^*, A^* be functions satisfying the following system of ordinary differential equations:

$$\frac{\partial}{\partial t} C^*(t) + 2C^*(t) \cdot F(t) - 2C^*(t) \cdot G(t) \cdot G^{'}(t) \cdot C^*(t) - c(t) = 0 \tag{3.53}$$

$$\begin{aligned}\frac{\partial}{\partial t} B^*(t) + B^*(t) \cdot F(t) + 2H^{'}(t) \cdot C^*(t) - 2B^*(t) \cdot G(t) \cdot G^{'}(t) \cdot C^*(t) \\ - b(t) = 0\end{aligned} \tag{3.54}$$

$$\begin{aligned}\frac{\partial}{\partial t} A^*(t) + B^*(t) \cdot H(t) - \frac{1}{2} B^*(t) \cdot G(t) \cdot G^{'}(t) \cdot (B^*)^{'}(t) \\ + \mathrm{tr}(G^{'}(t) \cdot C^*(t) \cdot G^{'}(t)) - a(t) = 0\end{aligned} \tag{3.55}$$

with boundary conditions given by

$$C^*(\mathcal{T}) = \phi_3 \qquad B^*(\mathcal{T}) = \phi_2 \qquad A^*(\mathcal{T}) = \phi_1 \tag{3.56}$$

where $\phi_1 \in \mathbb{R}$, $\phi_2 \in \mathbb{R}^n$ is a row vector, and $\phi_3 \in \mathrm{Sym}(n)$ is a positive semi-definite matrix. Then the process defined by

$$v^*(t, X_t) = \exp(A^*(t) + B^*(t) \cdot X_t + X_t^{'} \cdot C^*(t) \cdot X_t) \tag{3.57}$$

for some function $v^* \in \mathcal{C}^{1,2}$ such that

$$v^*(\mathcal{T}, x) = \exp(\phi_1 + \phi_2 \cdot x + x^{'} \cdot \phi_3 \cdot x) \tag{3.58}$$

defines a Markovian model for the growth optimal portfolio.

Proof. Due to Proposition 3.3.4 and Corollary 3.3.5, a fuction $v^* \in \mathcal{C}^{1,2}$ defines a Markovian GOP if and only if it satisfyies the following partial differential equation:

$$\frac{\partial v^*}{\partial t}(t,x) + \nabla_x v^*(t,x) \cdot \left(f(t,x) - g(t,x) \cdot g'(t,x) \cdot \frac{\nabla'_x v^*(t,x)}{v^*(t,x)} \right) + \frac{1}{2}\operatorname{tr}(g'(t,x) \cdot \nabla^2_{xx} v^*(t,x) \cdot g(t,x)) - v^*(t,x) r(t,x) = 0 \tag{3.59}$$

By (3.51) and the hypotheses on $r(t,x)$, Assumption 3.3.1 and the part of Assumption 3.3.2 concerning the short rate are satisfied and, given a boundary condition on $v^*(t,x)$, equation (3.59) has a unique solution. Suppose that v^* is of the form (3.57) and its terminal value is given by (3.58). Then, denoting by a subscript the partial derivatives w.r.t. t,

$$\begin{aligned}
\frac{\partial v^*}{\partial t}(t,x) &= v^*(t,x)[A^*_t(t) + B^*_t(t) \cdot x + x' \cdot C^*_t(t) \cdot x] \\
\nabla_x v^*(t,x) &= v^*(t,x)[B^*(t) + 2x' \cdot C^*(t)] \\
\nabla^2_{xx} v^*(t,x) &= v^*(t,x)\{[B^*(t) + 2x' \cdot C^*(t)]' \cdot [B^*(t) + 2x' \cdot C^*(t)] + 2C^*(t)\}
\end{aligned}$$

Inserting the expressions above and the linear stucture of f and g into equation (3.59) and dividing by $v^*(t,x)$ we obtain

$$\begin{aligned}
A^*_t + B^*_t \cdot x + x' \cdot C^*_t \cdot x + B^* \cdot F \cdot x + B^* \cdot H + 2x' \cdot C^* \cdot F \cdot x \\
+ 2x' \cdot C^* \cdot H - \frac{1}{2}(B^* + 2x' \cdot C^*) \cdot G \cdot G' \cdot (B^* + 2x' \cdot C^*)' \\
+ \operatorname{tr}(G' \cdot C^* \cdot G) - a - b \cdot x - x' \cdot c \cdot x = 0
\end{aligned}$$

Since the latter relation has to hold for all possible values of $x \in \mathbb{R}^n$, we can set the second-order, first-order, and constant terms equal to zero thereby getting the desired result. The boundary conditions follow from imposing (3.58) as terminal value of $v^*(t,x)$. □

Remark 3.3.12. Since the differetial system (3.53)-(3.56) is equivalent to equation (3.59) with terminal condition given by (3.58), under the hypotheses of Lemma 3.3.11 it admits a unique solution. The matrix equation (3.53) is a *Riccati equation* and there exist powerful algorithms for solving it numerically. For instance, equation (3.53) can be reduced to a linear differential equation that can be easily solved by numerical methods. Given $C^*(t)$, the other two equations are linear and explicit solutions in terms of line integrals can be exhibited. See [14, Appendix B] for more details.

A similar result can be derived for the benchmarked price of a T-bond.

Proposition 3.3.13. *Let v^* be a function satisfying the conditions of Lemma 3.3.11. For all $T \in [0, \mathcal{T}]$ let C^T, B^T, A^T be functions satisfying the following system of ordinary differential equations:*

$$\frac{\partial}{\partial t} C^T(t) + 2C^T(t) \cdot F(t) - 2C^T(t) \cdot G(t) \cdot G^{'}(t) \cdot C^T(t) = 0 \tag{3.60}$$

$$\frac{\partial}{\partial t} B^T(t) + B^T(t) \cdot F(t) + 2H^{'}(t) \cdot C^T(t) - 2B^T(t) \cdot G(t) \cdot G^{'}(t) \cdot C^T(t) = 0 \tag{3.61}$$

$$\begin{aligned} \frac{\partial}{\partial t} A^T(t) + B^T(t) \cdot H(t) - \frac{1}{2} B^T(t) \cdot G(t) \cdot G^{'}(t) \cdot (B^T)^{'}(t) \\ + \operatorname{tr}(G^{'}(t) \cdot C^T(t) \cdot G^{'}(t)) = 0 \end{aligned} \tag{3.62}$$

with boundary conditions given by

$$C^T(T) = C^*(T) \qquad B^T(T) = B^*(T) \qquad A^T(T) = A^*(T) \tag{3.63}$$

where C^*, B^*, A^* are determined by (3.53)-(3.56). Then

$$\hat{p}^T(t, X_t) = \exp(-A^T(t) - B^T(t) \cdot X_t - X_t^{'} \cdot C^T(t) \cdot X_t) \tag{3.64}$$

defines the benchmarked price of a fairly priced T-bond.

Proof. Under the assumptions of the Proposition the benchmarked price of a T-bond is uniquely determined by the boundary value problem (3.43)-(3.44). If $\hat{p}^T(t, x)$ is of the form (3.64), a simple computation leads to

$$\begin{aligned} \frac{\partial \hat{p}^T}{\partial t}(t, x) &= -\hat{p}^T(t, x)[A_t^T(t) + B_t^T(t) \cdot x + x^{'} \cdot C_t^T(t) \cdot x] \\ \nabla_x \hat{p}^T(t, x) &= -\hat{p}^T(t, x)[B^T(t) + 2x^{'} \cdot C^T(t)] \\ \nabla_{xx}^2 \hat{p}^T(t, x) &= \hat{p}^T(t, x)\{[B^T(t) + 2x^{'} \cdot C^T(t)]^{'} \cdot [B^T(t) + 2x^{'} \cdot C^T(t)] \\ &\quad - 2C^T(t)\} \end{aligned}$$

Substituting these expressions and the linear stucture of f and g into equation (3.43) and dividing by $-\hat{p}^T(t, x)$ we get

$$\begin{aligned} A_t^T &+ B_t^T \cdot x + x^{'} \cdot C_t^T \cdot x + B^T \cdot F \cdot x + B^T \cdot H + 2x^{'} \cdot C^T \cdot F \cdot x \\ &+ 2x^{'} \cdot C^T \cdot H - \frac{1}{2}(B^T + 2x^{'} \cdot C^T) \cdot G \cdot G^{'} \cdot (B^T + 2x^{'} \cdot C^T)^{'} \\ &+ \operatorname{tr}(G^{'} \cdot C^T \cdot G) = 0 \end{aligned}$$

By setting the second-order, first-order, and constant terms equal to zero we get equations (3.60)-(3.62). The terminal conditions follow from

$$\hat{p}^T(T, x) = \frac{1}{v^*(T, x)} \quad \forall x \in \mathbb{R}^n$$

where v^* is assumed to be of the form (3.57) and it defines the GOP, in view of Lemma 3.3.11. □

While the conditions above ensure that the bond price processes $P(\cdot, T)$ are martingales when discounted by the GOP, consistently with Assumption 3.1.7, the initial condition, that is, the observed bond prices $P^{OBS}(0,T)$ at time $t = 0$, is not necessarily matched. Similarly to [14], this can be easily obtained by imposing a condition on $A^*(t)$. To this end, we recall the following definition from [4, ch. 19].

Definition 3.3.14. The *istantaneous forward rate* with maturitity T, contranced at time t, is defined as

$$f(t,T) \colon = -\frac{\partial}{\partial T} \log P(t,T) \tag{3.65}$$

Analogously, the observed forward rate is given by

$$f^{OBS}(0,T) = -\frac{\partial}{\partial T} \log P^{OBS}(0,T) \tag{3.66}$$

where $P^{OBS}(0,T)$ denotes the initial data provided by the market for the price of a T-bond for each value of $T \in [0, \mathcal{T}]$. Since $V_0^* = 1$, to impose the initial condition we can set

$$\hat{P}(0,T) = P^{OBS}(0,T) \tag{3.67}$$

for all $T \in [0, \mathcal{T}]$.

Corollary 3.3.15. *Suppose that $X_0 = 0$. Let $\hat{P}(t,T)$ be the benchmarked price of a T-bond satisfying the conditions of Proposition 3.3.13 and let $P^{OBS}(0,T)$ be its observed price. Then condition (3.67) holds if and only if the deterministic coefficient $A^*(t)$ of $v^*(t, X_t)$ in (3.57) satisfies*

$$\begin{aligned} A^*(t) = \int_0^t & f^{OBS}(0,s) - B^t(s) \cdot H(s) + \frac{1}{2} B^t(s) \cdot G(s) \cdot G^{'}(s) \cdot (B^t)^{'}(s) \\ & - \operatorname{tr}(G^{'}(s) \cdot C^t(s) \cdot G^{'}(s)) \, ds \end{aligned} \tag{3.68}$$

Proof. It easy to show that

$$f(t,T) = -\frac{\partial}{\partial T} \log P(t,T) = -\frac{\partial}{\partial T} \log(\hat{P}(t,T) V_t^*) = -\frac{\partial}{\partial T} \log \hat{P}(t,T)$$

therefore the condition $\hat{P}(0,T) = P^{OBS}(0,T)$ is equivalent to

$$f^{OBS}(0,T) = f(0,T) = -\frac{\partial}{\partial T} \log \hat{P}(0,T) = -\frac{\partial}{\partial T} \log \hat{p}^T(0, X_0) = \frac{\partial}{\partial T} A^T(0)$$

We can now integrate equation (3.62)

$$\begin{aligned} A^T(t) = A^T(0) - \int_0^t & B^T(s) \cdot H(s) - \frac{1}{2} B^T(s) \cdot G(s) \cdot G^{'}(s) \cdot (B^T)^{'}(s) \\ & + \operatorname{tr}(G^{'}(s) \cdot C^T(s) \cdot G^{'}(s)) \, ds \end{aligned}$$

By choosing $t = T$ and imposing the initial condition

$$A^T(0) = \int_0^T f^{OBS}(0,s)\,ds$$

we obtain

$$\begin{aligned} A^t(t) = \int_0^t & f^{OBS}(0,s) - B^t(s) \cdot H(s) + \frac{1}{2} B^t(s) \cdot G(s) \cdot G^{'}(s) \cdot (B^t)^{'}(s) \\ & - \operatorname{tr}(G^{'}(s) \cdot C^t(s) \cdot G^{'}(s))\,ds \end{aligned}$$

We can conclude in view of the terminal condition $A^t(t) = A^*(t)$. □

Analogous results can be obtained for the spot spread $S(t,T)$ under the roll-over risk approach.

Proposition 3.3.16. *Let v^* be a function satisfying the conditions of Lemma 3.3.11. Assume that the funding-liquidity spread $\varphi_t = \varphi(t, X_t)$ is given by*

$$\varphi(t, X_t) = \alpha(t) + \beta(t) \cdot X_t + X_t^{'} \cdot \gamma(t) \cdot X_t \tag{3.69}$$

For all $T \in [0, \mathcal{T}]$ let R^T, Q^T, U^T be functions of time satisfying the following system of ordinary differential equations:

$$\begin{aligned} \frac{\partial}{\partial t} R^T(t) + 2R^T(t) \cdot F(t) - 2R^T(t) \cdot G(t) \cdot G^{'}(t) \cdot (2C^*(t) - R^T(t)) \\ + \gamma(t) = 0 \end{aligned} \tag{3.70}$$

$$\begin{aligned} \frac{\partial}{\partial t} Q^T(t) + Q^T(t) \cdot F(t) + 2H^{'}(t) \cdot R^T(t) - 2B^*(t) \cdot G(t) \cdot G^{'}(t) \cdot R^T(t) \\ - 2Q^T(t) \cdot G(t) \cdot G^{'}(t) \cdot (C^*(t) - R^T(t)) + \beta(t) = 0 \end{aligned} \tag{3.71}$$

$$\begin{aligned} \frac{\partial}{\partial t} U^T(t) + Q^T(t) \cdot H(t) - Q^T(t) \cdot G(t) \cdot G^{'}(t) \cdot \left(B^*(t) - \frac{1}{2} Q^T(t) \right)^{'} \\ + \operatorname{tr}(G^{'}(t) \cdot R^T(t) \cdot G^{'}(t)) + \alpha(t) = 0 \end{aligned} \tag{3.72}$$

with boundary conditions given by

$$R^T(T) = 0 \qquad Q^T(T) = 0 \qquad U^T(T) = 0 \tag{3.73}$$

Then

$$s^T(t, X_t) = \exp(U^T(t) + Q^T(t) \cdot X_t + X_t^{'} \cdot R^T(t) \cdot X_t) \tag{3.74}$$

defines a spot spread as a fairly priced roll-over-risk-adjusted borrowing account with roll-over rate $\tilde{r} = r + \varphi$.

Proof. If $s^T(t, X_t)$ represents a fairly priced roll-over-risk-adjusted borrowing account with roll-over rate $\tilde{r} = r + \varphi$, then the function s^T is solution of the boundary value problem provided by Proposition 3.3.7. In view of Lemma 3.3.11, Assumption 3.3.6 hold. Since Assumptions 3.3.1 and 3.3.2 are satisfied by the hypotheses of the Proposition, s^T is uniquely determined and, assuming it of the form (3.74), it has the following partial derivatives:

$$
\begin{aligned}
\frac{\partial s^T}{\partial t}(t,x) &= s^T(t,x)[U_t^T(t) + Q_t^T(t) \cdot x + x' \cdot R_t^T(t) \cdot x] \\
\nabla_x s^T(t,x) &= s^T(t,x)[Q^T(t) + 2x' \cdot R^T(t)] \\
\nabla^2_{xx} s^T(t,x) &= s^T(t,x)\{[Q^T(t) + 2x' \cdot R^T(t)]' \cdot [Q^T(t) + 2x' \cdot R^T(t)] \\
&\quad + 2R^T(t)\}
\end{aligned}
$$

By inserting all the corresponding expressions into equation (3.45) we get

$$
\begin{aligned}
&U_t^T + Q_t^T \cdot x + x' \cdot R_t^T \cdot x + Q^T \cdot F \cdot x + Q^T \cdot H + 2x' \cdot R^T \cdot F \cdot x \\
&\quad + 2x' \cdot R^T \cdot H - (Q^T + 2x' \cdot R^T) \cdot G \cdot G' \cdot (B^* + 2x' \cdot C^*)' \\
&\quad + \operatorname{tr}(G' \cdot R^T \cdot G) + \frac{1}{2}(Q^T + 2x' \cdot R^T) \cdot G \cdot G' \cdot (Q^T + 2x' \cdot R^T)' \\
&\quad + \alpha + \beta \cdot x + x' \cdot \gamma \cdot x = 0
\end{aligned}
$$

We conclude again by setting the terms of different order equal to zero and deriving the boundary conditions from $s^T(T, x) = 1$ for all $x \in \mathbb{R}^n$. □

Again, to complete the picture we have to match the initial condition given by

$$S^{OBS}(0,T) = [1 + TL^{OBS}(0,T)]P^{OBS}(0,T)$$

where $L^{OBS}(0,T)$ is the initial data provided by the market for the value of a Libor rate with maturity $T \in [0, \mathcal{T}]$. Similarly to [5], we define an analogous of the forward rate for spot spreads.

Definition 3.3.17. The *istantaneous forward spread rate*[5] corresponding to $S(t,T)$ is defined by

$$\psi(t,T) \colon = \frac{\partial}{\partial T} \log S(t,T) \tag{3.75}$$

Corollary 3.3.18. *Suppose that $X_0 = 0$. Let $S(t,T)$ be a roll-over-risk-adjusted borrowing satisfying the conditions of Proposition 3.3.16 and let*

[5] Actually, in practice spot spreads are available for finite tenors and, therefore, giving a financial explanation of instantaneous forward spread rate is not so easy. We invite the reader to see [5], where an HJM framework for multi-curve term structures is provided. In our analysis, the instantaneous forward spread rate is introduced as theoretical tool for making it easier to match the initial condition.

$S^{OBS}(0,T)$ be the observed value at initial time of the corresponding spot spread. Then

$$S(0,T) = S^{OBS}(0,T)$$

if and only if the deterministic coefficient $\alpha(t)$ of $\varphi(t, X_t)$ in (3.69) satisfies

$$\alpha(t) = \psi^{OBS}(0,t) - \frac{\partial}{\partial t}\int_0^t L(s,t)\,ds \tag{3.76}$$

where

$$\begin{aligned} L(t,T) :&= Q^T(t)\cdot H(t) - Q^T(t)\cdot G(t)\cdot G^{'}(t)\cdot\left(B^*(t) - \frac{1}{2}Q^T(t)\right)^{'} \\ &+ \operatorname{tr}(G^{'}(t)\cdot R^T(t)\cdot G^{'}(t)) \end{aligned}$$

Proof. Setting the initial condition $S(0,T) = S^{OBS}(0,T)$ leads to

$$\psi^{OBS}(0,T) = \psi(0,T) = \frac{\partial}{\partial T}\log S(0,T) = \frac{\partial}{\partial T}\log s^T(0,X_0) = \frac{\partial}{\partial T}U^T(0)$$

We can now integrate equation (3.72)

$$\begin{aligned} U^T(t) = U^T(0) - \int_0^t & Q^T(s)\cdot H(s) + \operatorname{tr}(G^{'}(s)\cdot R^T(s)\cdot G^{'}(s)) + \alpha(s) \\ & - Q^T(s)\cdot G(s)\cdot G^{'}(s)\cdot\left(B^*(s) - \frac{1}{2}Q^T(s)\right)^{'} ds \end{aligned}$$

By choosing $t = T$ and using the terminal condition $U^t(t) = 0$ we get

$$\begin{aligned} 0 = U^t(0) - \int_0^t & Q^t(s)\cdot H(s) + \operatorname{tr}(G^{'}(s)\cdot R^t(s)\cdot G^{'}(s)) + \alpha(s) \\ & - Q^t(s)\cdot G(s)\cdot G^{'}(s)\cdot\left(B^*(s) - \frac{1}{2}Q^t(s)\right)^{'} ds \end{aligned}$$

The desired result is obtained by differentiating both members of the equation above w.r.t. t. □

We conclude the section with a similar formulation of forward spreads.

Corollary 3.3.19. *Let $\hat{p}^T$ and $s^{T+\delta}$ be functions satisfying the hypotheses of Propositions 3.3.13 and 3.3.16. Let $C^{T,\delta}$, $B^{T,\delta}$, $A^{T,\delta}$ be functions satisfying* *the following system of ordinary differential equations:*

$$\frac{\partial}{\partial t} C^{T,\delta}(t) + 2C^{T,\delta}(t) \cdot F(t) + 2C^{T,\delta}(t) \cdot G(t) \cdot G^{'}(t) \cdot (2C^{T}(t) + C^{T,\delta}(t)) = 0 \tag{3.77}$$

$$\frac{\partial}{\partial t} B^{T,\delta}(t) + B^{T,\delta}(t) \cdot F(t) + 2H^{'}(t) \cdot C^{T,\delta}(t) - 2B^{T}(t) \cdot G(t) \cdot G^{'}(t) \cdot C^{T,\delta}(t) + 2B^{T,\delta}(t) \cdot G(t) \cdot G^{'}(t) \cdot (C^{T}(t) + C^{T,\delta}(t)) = 0 \tag{3.78}$$

$$\frac{\partial}{\partial t} A^{T,\delta}(t) + B^{T,\delta}(t) \cdot H(t) + B^{T,\delta}(t) \cdot G(t) \cdot G^{'}(t) \cdot \left(B^{T}(t) + \frac{1}{2} B^{T,\delta}(t) \right)^{'} + \mathrm{tr}(G^{'}(t) \cdot C^{T,\delta}(t) \cdot G^{'}(t)) = 0 \tag{3.79}$$

with boundary conditions given by

$$C^{T,\delta}(T) = R^{T+\delta}(T) \qquad B^{T,\delta}(T) = Q^{T+\delta}(T) \qquad A^{T,\delta}(T) = U^{T+\delta}(T) \tag{3.80}$$

Then

$$s^{T,\delta}(t, X_t) = \exp(A^{T,\delta}(t) + B^{T,\delta}(t) \cdot X_t + X_t^{'} \cdot C^{T,\delta}(t) \cdot X_t) \tag{3.81}$$

defines the forward spread at time $t \leq T$ related to a Libor rate that is underlying of a fairly priced FRA.

Proof. In view of Propositions 3.3.13 and 3.3.16, the functions $\hat{p}^T$ and $s^{T+\delta}$ determine respectively the benchmarked price of a T-bond and a spot spread with maturity $T+\delta$. Since Assumptions 3.3.1, 3.3.2, and 3.3.6 are satisfied by the hypotheses of the Proposition, $s^{T,\delta}$ is uniquely determined by the terminal value problem given by Corollary 3.3.9. Now, the result can be easily obtained by repeating the same scheme of the previous propositions. □

Chapter 4

Benchmark approach and stochastic control derivations

As shown in A. Gombani and W.J. Runggaldier (2013), the bond prices can be represented as solutions of suitable stochastic control problems. In the arbitrage-free setting, this result can be achieved by applying a *log-transform* to the bond price function and obtaining from the *term stuctures equation* a non-linear PDE which can be rewritten in the form of a Hamilton-Jacobi-Bellman equation (see [14, sect. 1, ch. 3]). In this chapter we aim at generalizing such a result and providing a stochastic control representation for bonds and spreads under the benchmark approach. Moreover, we propose a modeling framework for spot spreads that takes into account the Libor risk through a risk sensitivity parameter. In particular, we postulate that the value of spot spreads depends on a portfolio process whose corresponding strategy is the optimal solution of a risk-sensitive asset allocation problem of the type treated by M. Davis and S. Lleo (2011). In turn, this formulation leads to express the spot spread as solution of a risk-sensitive stochastic control problem. In the end, we show that there are endogenous conditions which link the risk-sensitive approach with the roll-over risk approach to spot spread modeling.

Before introducing the main topics we recall some general notions from stochastic control theory and *dynamic programming* (see [4, ch. 25]). Let us consider $\mathcal{T} \in [0, \infty)$ as time horizon. Given the deterministic functions

$$
\begin{aligned}
u &\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^k \\
F &\colon \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \\
\Phi &\colon \mathbb{R}^n \to \mathbb{R}
\end{aligned}
$$

a *stochastic control problem* consists of maximizing (or minimizing) the quantity

$$
\mathbb{E}_{t,x}\left[\int_t^{\mathcal{T}} F(s, X_s^u, u(s, X_s^u))\, ds + \Phi(X_{\mathcal{T}}^u)\right] \tag{4.1}
$$

where $X^u = (X^u_t)_{0\le t\le \mathcal{T}}$ is an $\mathbb{R}^n$-valued process satisfying the *controlled* stochastic differential equation

$$dX^u_t = h(t, X^u_t, u(t, X^u_t))dt + q(t, X^u_t, u(t, X^u_t)) \cdot dW_t \qquad (4.2)$$

for some functions h and q. We call *optimal value function* the value of the expectation (4.1) corresponding to the *optimal control.* The process X^u is viewed as a *state process* that can be partially controlled by choosing the function u, which is named *feedback control law.*[6]

Definition 4.0.1. We call *admissible* a feedback control law for which equation (4.2) has a unique weak solution and the expectation given by (4.1) is finite. We denote by $\mathcal{U}$ the set of all admissible control laws of the feedback type.

A proof of the following fundamental theorem can be find in [4, Theorem 25.7].

Theorem 4.0.2 (Verification Theorem). *Suppose that we have two functions $V(t,x)$ and $u^*(t,x)$, such that*

- *V is sufficiently integrable and solves the* ***Hamilton-Jacobi-Bellman equation***

  $$\frac{\partial}{\partial t}V(t,x) + \sup_{u\in\mathcal{U}}\{\, \mathcal{L}^u V(t,x) + F(t,x,u)\,\} = 0, \qquad V(\mathcal{T},x) = \Phi(x)$$

  where

  $$\begin{aligned}\mathcal{L}^u V(t,x) :&= \nabla_x V(t,x)\cdot h(t,x,u)\\ &+ \frac{1}{2}\,\mathrm{tr}\left(q'(t,x,u)\cdot \nabla^2_{xx} V(t,x)\cdot q(t,x,u)\right)\end{aligned}$$

- *the function $u^* \in \mathcal{U}$;*

- *for each fixed (t,x), the supremum in the expression*

  $$\sup_{u\in\mathcal{U}}\{\, \mathcal{L}^u V(t,x) + F(t,x,u)\,\}$$

  is reached by the choice $u = u^*$.

Then V and u^* are respectively the optimal value function and the optimal control of the stochastic control problem

$$\begin{cases} dX^u_t = h(t, X^u_t, u(t, X^u_t))dt + q(t, X^u_t, u(t, X^u_t)) \cdot dW_t \\ \max\limits_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^{\mathcal{T}} F(s, X^u_s, u(s, X^u_s))\, ds + \Phi(X^u_{\mathcal{T}})\right] \end{cases}$$

In case of a minimization problem an analogous result holds with the obvius modification consisting of replacing the supremum for the infimum.

[6] In a general stochastic control problem the state process X^u depends on a k-dimensional control process $u = (u_t)_{0\le t\le\mathcal{T}}$. A control process whose value u_t depends on the observed value of the state process X^u at time t is called feedback or closed loop control.

4.1 Bond prices and spreads as optimal values of stochastic control problems

We recall that, if Assumptions 3.3.1, 3.3.2, and 3.3.6 hold, the only function $\hat{p}^T \in \mathcal{C}^{1,2}$ such that

$$\hat{p}^T(t, X_t) = \hat{P}(t,T) = \frac{P(t,T)}{V_t^*}$$

is solution of the boundary value problem

$$\frac{\partial}{\partial t}\hat{p}^T + \nabla_x \hat{p}^T \cdot f + \frac{1}{2}\operatorname{tr}\left(g^{'} \cdot \nabla^2_{xx}\hat{p}^T \cdot g\right) = 0, \qquad \hat{p}^T(T,x) = \frac{1}{v^*(T,x)} \tag{4.3}$$

Proposition 4.1.1. *Suppose that Assumptions 3.3.1, 3.3.2, and 3.3.6 hold. The stochastic control problem*

$$\begin{cases} dX_t^u = [f(t,X_t^u) + g(t,X_t^u) \cdot u(t,X_t^u)]dt + g(t,X_t^u) \cdot dW_t \\ w^T(t,x) = \min\limits_{u \in \mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^T \frac{1}{2} u^{'}(s,X_s^u) \cdot u(s,X_s^u)\, ds + \log v^*(T,X_T^u)\right] \end{cases} \tag{4.4}$$

has an optimal value function given by

$$w^T(t,x) = -\log \hat{p}^T(t,x) \tag{4.5}$$

and the optimal control is

$$u^*(t,x) = -g^{'}(t,x) \cdot \nabla_x^{'} w^T(t,x) = g^{'}(t,x)\frac{\nabla_x^{'}\hat{p}^T(t,x)}{\hat{p}^T(t,x)} \tag{4.6}$$

Proof. Due to the hypotheses of the Proposition, equation (4.3) admits a unique solution. By applying the logarithmic transform (4.5) and computing the partial derivatives of $\hat{p}^T$ we obtain from (4.3) the following PDE after division by $\exp(-w^T(t,x))$

$$\begin{aligned} \frac{\partial}{\partial t} w^T(t,x) + \nabla_x w^T(t,x) \cdot f(t,x) + \frac{1}{2}\operatorname{tr}\left(g^{'}(t,x) \cdot \nabla^2_{xx} w^T(t,x) \cdot g(t,x)\right) \\ - \frac{1}{2}\nabla_x w^T(t,x) \cdot g(t,x) \cdot g^{'}(t,x) \cdot \nabla_x^{'} w^T(t,x) = 0 \end{aligned} \tag{4.7}$$

with terminal condition $w^T(T,x) = \log v^*(T,x)$. Consider now the HJB equation

$$\begin{aligned} &\frac{\partial}{\partial t} w^T + \inf_{u \in \mathcal{U}}\left\{\nabla_x w^T \cdot (f + g \cdot u) + \frac{1}{2}\operatorname{tr}\left(g^{'} \cdot \nabla^2_{xx} w^T \cdot g\right) + \frac{1}{2}u^{'} \cdot u\right\} = 0 \\ &w^T(T,x) = \log v^*(T,x) \end{aligned} \tag{4.8}$$

The expression inside the brackets is a convex function of u and by imposing the first order condition we get that the infimum is attained by

$$u^*(t,x) = -g^{'}(t,x) \cdot \nabla_x^{'} w^T(t,x)$$

If we insert u^* into equation (4.8) we obtain (4.7), that has

$$w^T(t,x) = -\log \hat{p}^T(t,x)$$

as unique solution. The proposition now follows by noting that (4.8) is the HJB equation associated to the stochastic control problem (4.4). We conclude by observing that, due to Assumption 3.3.1 and the smoothness of $\hat{p}^T \in \mathcal{C}^{1,2}$, the control law u^* is admissible. □

Remark 4.1.2. The state process X^u results from an alteration of the factor process X which changes the drift through the feedback while keeping the same measure. In the stochastic control methodology, the feedback control plays thus the same role of the Girsanov kernel in the traditional martingale approach (see [4, ch. 12]), where a change of drift is implicit in the change of measure. In the control approach an auxilary state process X^u is created: it is driven by the same Brownian motion of the original process but its trajectories are different, as consequence of the feedback action on the drift.

Remark 4.1.3. From an economic poit of view, the present formulation reflects the perspective of the issuer of the bond, who aims at minimizing the yield of the bond, subject to the constraint of a given dynamics in the first line of (4.4) and with a quadratic penalty function for the control law. The main difference with respect to the arbitrage-free framework presented in [14] is the terminal value of w^T depending on the GOP, as consequence of the benchmark approach.

An analogous result can be provided for spot spreads under the roll-over-risk approach. We recall that, under the conditions expressed by 3.3.1, 3.3.2 and 3.3.6, if there exists a function $s^T \in \mathcal{C}^{1,2}$ such that $s^T(t, X_t) = S(t,T)$, where $S(t,T)$ is given by (3.46), then s^T is uniquely determined by the following terminal value problem:

$$\begin{aligned} &\frac{\partial}{\partial t} s^T + \nabla_x s^T \cdot \left(f - g \cdot g^{'} \cdot \frac{\nabla_x^{'} v^*}{v^*} \right) + \frac{1}{2} \operatorname{tr}\left(g^{'} \cdot \nabla_{xx}^2 s^T \cdot g \right) + \varphi s^T = 0 \\ &s^T(T,x) = 1 \end{aligned} \tag{4.9}$$

Proposition 4.1.4. *Suppose that Assumptions 3.3.1, 3.3.2 and 3.3.6 hold. The stochastic control problem*

$$z^T(t,x) = \max_{u \in \mathcal{U}} \mathbb{E}_{t,x} \left[\int_t^T \varphi(s, X_s^u) - \frac{1}{2} u^{'}(s, X_s^u) \cdot u(s, X_s^u) \, ds \right] \tag{4.10}$$

subject to the dynamics

$$dX_t^u = \left[f(t, X_t^u) - g(t, X_t^u) \cdot g^{'}(t, X_t^u) \cdot \frac{\nabla_x^{'} v^*(t, X_t^u)}{v^*(t, X_t^u)} + g(t, X_t^u) \cdot u(t, X_t^u) \right] dt + g(t, X_t^u) \cdot dW_t \tag{4.11}$$

has an optimal value function given by

$$z^T(t, x) = \log s^T(t, x) \tag{4.12}$$

and the optimal control is

$$u^*(t, x) = g^{'}(t, x) \cdot \nabla_x^{'} z^T(t, x) = g^{'}(t, x) \frac{\nabla_x^{'} s^T(t, x)}{s^T(t, x)} \tag{4.13}$$

Proof. If we apply the log-transform (4.12) to the function s^T, we obtain from (4.9) the following PDE

$$\frac{\partial}{\partial t} z^T + \nabla_x z^T \cdot \left(f - g \cdot g^{'} \cdot \frac{\nabla_x v^*}{v^*} \right) + \frac{1}{2} \operatorname{tr} \left(g^{'} \cdot \nabla_{xx}^2 z^T \cdot g \right) + \frac{1}{2} \nabla_x z^T \cdot g \cdot g^{'} \cdot \nabla_x^{'} z^T + \varphi = 0 \tag{4.14}$$

with terminal condition $z^T(T, x) = 0$. Consider now the HJB equation

$$\frac{\partial}{\partial t} z^T + \sup_{u \in \mathcal{U}} \left\{ \nabla_x z^T \cdot \left(f - g \cdot g^{'} \cdot \frac{\nabla_x^{'} v^*}{v^*} + g \cdot u \right) + \frac{1}{2} \operatorname{tr} \left(g^{'} \cdot \nabla_{xx}^2 z^T \cdot g \right) - \frac{1}{2} u^{'} \cdot u + \varphi \right\} = 0 \tag{4.15}$$

The expression inside the brackets is a concave function of u and by imposing the first order condition we get that the supremun is reached by

$$u^*(t, x) = g^{'}(t, x) \cdot \nabla_x^{'} z^T(t, x)$$

Due to Assumption 3.3.1 and the smoothness of $s^T \in \mathcal{C}^{1,2}$, the control law u^* is admissible and we can easily see that equation (4.14) corresponds to (4.15) when $u = u^*$ and with $z^T(T, x) = 0$ as boundary condition. The hypotheses of the Proposition ensure that equation (4.14) admits $z^T(t, x) = \log s^T(t, x)$ as unique solution. The statement now follows by noting that (4.15) is the HJB equation associated to the stochastic control problem (4.10)-(4.11). □

Remark 4.1.5. Again, an economic explanation can be offered. The stochastic control problem (4.10)-(4.11) reflects the point of view of the lender who aims at maximizing the yield of the loan corresponding to the roll-over-risk adjusted borrowing account. It can be interpreted also as the problem of maximizing the yield of a Libor loan. Indeed, we recall that

$$S(t,T) = [1 + (T - t)L(t,T)]P(t,T)$$

If $S(t,T)$ coincides with the roll-over-risk-adjusted borrowing account, the expression above means that, in equilibrium, the value of the continuosly rolled-over loan must be equal to that of the term loan (see Sect. 3.2).

Finally, let us recall that, if Assumptions 3.3.1, 3.3.2, and 3.3.6 hold, the unique function $s^{T,\delta} \in \mathcal{C}^{1,2}$ such that

$$s^{T,\delta}(t, X_t) = S(t; T, T + \delta) = \frac{1 + \delta L(t; T, T + \delta)}{1 + \delta F(t; T, T + \delta)}$$

must solves the boundary value problem

$$\begin{aligned} &\frac{\partial}{\partial t} s^{T,\delta} + \nabla_x s^{T,\delta} \cdot \left(f + g \cdot g^{'} \cdot \frac{\nabla_x^{'} \hat{p}^T}{\hat{p}^T} \right) + \frac{1}{2} \operatorname{tr}\left(g^{'} \cdot \nabla_{xx}^2 s^{T,\delta} \cdot g \right) = 0 \\ &s^{T,\delta}(T, x) = s^{T+\delta}(T, x) \end{aligned} \tag{4.16}$$

where $s^{T+\delta}$ is uniquely determined by a PDE of the form (4.9).

Corollary 4.1.6. *Suppose that Assumptions 3.3.1, 3.3.2 and 3.3.6 hold. Let w^T and $z^{T+\delta}$ be the optimal value functions of the stochastic control problems (4.4) and (4.10)-(4.11). The stochastic control problem*

$$z^{T,\delta}(t,x) = \max_{u \in \mathcal{U}} \mathbb{E}_{t,x} \left[- \int_t^T \frac{1}{2} u^{'}(s, X_s^u) \cdot u(s, X_s^u)\, ds + z^{T+\delta}(T, X_T^u) \right] \tag{4.17}$$

subject to the dynamics

$$\begin{aligned} dX_t^u = \Big[& f(t, X_t^u) - g(t, X_t^u) \cdot g^{'}(t, X_t^u) \cdot \nabla_x^{'} w^T(t, X_t^u) \\ & + g(t, X_t^u) \cdot u(t, X_t^u) \Big] dt + g(t, X_t^u) \cdot dW_t \end{aligned} \tag{4.18}$$

has an optimal value function given by

$$z^{T,\delta}(t,x) = \log s^{T,\delta}(t,x) \tag{4.19}$$

and the optimal control is

$$u^*(t,x) = g^{'}(t,x) \cdot \nabla_x^{'} z^{T,\delta}(t,x) = g^{'}(t,x) \frac{\nabla_x^{'} s^{T,\delta}(t,x)}{s^{T,\delta}(t,x)} \tag{4.20}$$

Proof. In view of 4.1.1 and 4.1.4, the optimal value functions w^T and $z^{T+\delta}$ are given by

$$w^T(t,x) = -\log \hat{p}^T(t,x), \quad z^{T+\delta}(t,x) = \log s^{T+\delta}(t,x)$$

The result now follows by repeating the same scheme of the previous propositions. □

In Gombani and Runggaldier (2013), a stochastic control formulation is provided also for the forward price of a T-bond and it is used for evaluating bond derivatives instead of the classical forward measure approach (see [14][ch. 4] for details). It is natural to ask whether these results can be extended to the approach being analyzed, but this is beyond our purpose and is left as a suggestion for the reader.

4.1.1 Example: linear dynamics and exponential quadratic structures

The class of models described in Subsect. 3.3.1 is particularly suitable in a stochastic control setting since allows to reduce the stochastic control problems above to *Linear-Quadratic-Gaussian* (LQG) problems (see [4, ch. 25]). Let us therefore assume that the factor process X satisfies equation (3.51). Suppose that the function v^* is of the form

$$v^*(t,x) = \exp(A^*(t) + B^*(t)\cdot x + x^{'}\cdot C^*(t)\cdot x)$$

where C^*, B^*, A^* are determined by (3.53)-(3.56). Due to Lemma 3.3.11, v^* defines a Markovian model for the GOP. The first equation in (4.4) becomes[7]

$$dX_t^u = [F(t)\cdot X_t^u + H(t) + G(t)\cdot u_t]dt + G(t)\cdot dW_t \tag{4.21}$$

while the second line can be written as

$$\min_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^T \frac{1}{2}u_s^{'}\cdot u_s\, ds + A^*(T) + B^*(T)\cdot X_T^u + (X_T^u)^{'}\cdot C^*(T)\cdot X_T^u\right] \tag{4.22}$$

If we call $w^T(t,x)$ the optimal value function, the associated HJB equation is

$$\begin{aligned}\frac{\partial}{\partial t}w^T + \inf_{u\in\mathcal{U}}\Big\{\nabla_x w^T\cdot(F\cdot x + H + G\cdot u) + \frac{1}{2}\operatorname{tr}\left(G^{'}\cdot\nabla_{xx}^2 w^T\cdot G\right)\\ + \frac{1}{2}u^{'}\cdot u\Big\} = 0\end{aligned} \tag{4.23}$$

with terminal condition

$$w^T(T,x) = A^*(T) + B^*(T)\cdot x + x^{'}\cdot C^*(T)\cdot x$$

[7] We set $u_t := u(t, X_t^u)$ to simplify the notation.

Making the usual Ansatz

$$w^T(t,x) = A^T(t) + B^T(t) \cdot x + x^{'} \cdot C^T(t) \cdot x$$

we obtain that the optimal control is given by

$$u^*(t,x) = -g^{'}(t,x) \cdot \nabla^{'}_x w^T(t,x) = -G^{'}(t) \cdot (B^T(t) + x^{'} \cdot C^T(t))^{'}$$

Now, it is easy to see that, by inserting u^* into equation (4.23), we get that C^T, B^T, A^T satisfy the same differential system of Proposition 3.3.13.

To provide a similar model for spot spreads as roll-over-risk-adjusted borrowing accounts, we need to assume that the funding-liquidity spread is given by

$$\varphi(t, X_t) = \alpha(t) + \beta(t) \cdot X_t + X^{'}_t \cdot \gamma(t) \cdot X_t$$

Then, the stochastic control problem (4.10) can be expressed as

$$\max_{u \in \mathcal{U}} \mathbb{E}_{t,x} \left[\int_t^T \alpha(s) + \beta(s) \cdot X^u_s + (X^u_s)^{'} \cdot \gamma(s) \cdot X^u_s - \frac{1}{2} u^{'}_s \cdot u_s \, ds \right] \tag{4.24}$$

given the dynamics

$$\begin{aligned} dX^u_t = \Big[& F(t) \cdot X^u_t + H(t) - G(t) \cdot G^{'}(t) \cdot \Big(B^*(t) + 2(X^u_t)^{'} \cdot C^*(t) \Big)^{'} \\ & + G(t) \cdot u_t \Big] dt + G(t) \cdot dW_t \end{aligned} \tag{4.25}$$

Equation (4.15) becomes

$$\begin{aligned} \frac{\partial}{\partial t} z^T + \sup_{u \in \mathcal{U}} \Big\{ & \nabla_x z^T \cdot \Big(F \cdot x + H - G \cdot G^{'} \cdot \big(B^* + 2x^{'} \cdot C^* \big)^{'} + G \cdot u \Big) \\ & + \frac{1}{2} \operatorname{tr} \Big(G^{'} \cdot \nabla^2_{xx} z^T \cdot G \Big) - \frac{1}{2} u^{'} \cdot u + \alpha + \beta \cdot x + x^{'} \cdot \gamma \cdot x \Big\} = 0 \end{aligned} \tag{4.26}$$

with boundary condition $z^T(T,x) = 0$. In view of the usual Ansatz

$$z^T(t,x) = U^T(t) + Q^T(t) \cdot x + x^{'} \cdot R^T(t) \cdot x$$

we obtain the following optimal control

$$u^*(t,x) = g^{'}(t,x) \cdot \nabla^{'}_x z^T(t,x) = G^{'}(t) \cdot (Q^T(t) + x^{'} \cdot R^T(t))^{'}$$

and if we substitute it into equation (4.26), we get that R^T, Q^T, U^T satisfy the differential system (3.70)-(3.73).

An analogous result can be obtained for the stochastic control problem of Corollary 4.1.6. If w^T and $z^{T+\delta}$ are solutions of (4.23) and (4.26), then the problem (4.17) becomes

$$\begin{aligned} \max_{u \in \mathcal{U}} \mathbb{E}_{t,x} \Big[& U^{T+\delta}(T) + Q^{T+\delta}(T) \cdot X^u_T + (X^u_T)^{'} \cdot R^{T+\delta}(T) \cdot X^u_T \\ & - \int_t^T \frac{1}{2} u^{'}_s \cdot u_s \, ds \Big] \end{aligned} \tag{4.27}$$

subject to the dynamics

$$dX_t^u = \Big[F(t)\cdot X_t^u + H(t) + G(t)\cdot G^{'}(t)\cdot \Big(B^T(t) + 2(X_t^u)^{'}\cdot C^T(t)\Big)^{'} + G(t)\cdot u_t\Big]dt + G(t)\cdot dW_t \tag{4.28}$$

If we call $z^{T,\delta}(t,x)$ the optimal value function, the corresponding HJB equation is

$$\frac{\partial}{\partial t}z^{T,\delta} + \sup_{u\in\mathcal{U}}\Big\{\nabla_x z^{T,\delta}\cdot\Big(F\cdot x + H + G\cdot G^{'}\cdot\big(B^T + 2x^{'}\cdot C^T\big)^{'} + G\cdot u\Big) + \frac{1}{2}\operatorname{tr}\Big(G^{'}\cdot\nabla_{xx}^2 z^{T,\delta}\cdot G\Big) - \frac{1}{2}u^{'}\cdot u\Big\} = 0 \tag{4.29}$$

with terminal condition

$$z^{T,\delta}(T,x) = U^{T+\delta}(T) + Q^{T+\delta}(T)\cdot x + x^{'}\cdot R^{T+\delta}(T)\cdot x$$

The usual Ansatz

$$z^{T,\delta}(t,x) = A^{T,\delta}(t) + B^{T,\delta}(t)\cdot x + x^{'}\cdot C^{T,\delta}(t)\cdot x$$

leads to the following optimal control

$$u^*(t,x) = G^{'}(t)\cdot(B^{T,\delta}(t) + x^{'}\cdot C^{T,\delta}(t))^{'}$$

and we obtain the same differential system of Corollary 3.3.19 for $C^{T,\delta}$, $B^{T,\delta}$, $A^{T,\delta}$.

4.2 A risk-sensitive formulation for spreads

In this section we present an alternative formulation for spreads which is partially inspired by the roll-over-risk approach. Our idea is to represent the risk affecting the Libor market through a risk-sensitivity parameter related to a risk-sensitive portfolio optimization. To do this, let us leave aside the interest rate derivatives for a moment and go back to the diffusion-based market model presented in Sect. 2.1. We recall that S^0 is the savings account whereas $S = (S^1,\ldots,S^N)$ is an $\mathbb{R}^N$-valued stochastic process representing N risky assets. We can now introduce the following.

Definition 4.2.1. For every $\eta\in\mathbb{R}\setminus\{-1,0\}$, we call *risk-sensitive portfolio* the solution to the expected power utility optimization problem of finding an element $\pi^\eta\in\mathcal{A}$ such that

$$\mathbb{E}\left[\left(V_T^{\pi^\eta}\right)^{-\eta}\right] = \begin{cases}\max\limits_{\pi\in\mathcal{A}}\mathbb{E}[(V_T^\pi)^{-\eta}] & \text{if } \eta > -1\\ \min\limits_{\pi\in\mathcal{A}}\mathbb{E}[(V_T^\pi)^{-\eta}] & \text{if } \eta < -1\end{cases} \tag{4.30}$$

Remark 4.2.2. Due to Assumption 2.2.10 and Proposition 2.2.16, the financial market (S^0, S) is viable. In view of Theorem 2.2.15, the optimization problem (4.30) makes sense and the portfolio process $V^{\pi^\eta} = \left(V_t^{\pi^\eta}\right)_{0\le t\le \mathcal{T}}$ is well-defined. We emphasize that V^{π^η} coexists with V^* and plays an auxilary role: the growth optimal portfolio allows to determine the fair value of a contingent claim in absence of martingale measures; the risk-sensitive portfolio (RSP) is used to provide a definition of $S(t,T)$ alternatively to the roll-over-risk approach.

Remark 4.2.3. The risk sensitivity parameter η represents the risk attidute of the economic agents and its value changes the nature of problem (4.30). The case $\eta > -1$ leads to a maximization over a concave function, while the case $\eta < -1$ leads to a minimization over a convex function (compare [7, ch. 3]). In economic terms, we say that for $\eta > -1$ there is *risk aversion*, whereas for $\eta < -1$ there is *risk seeking behaviour*. If we define a stochastic process $W^\pi = (W_t^\pi)_{0\le t\le\mathcal{T}}$ such that

$$W_t^\pi :\, = \log V_t^\pi$$

the optimization problem (4.30) can be reformulated as

$$\mathbb{E}\left[\exp\left(-\eta W_{\mathcal{T}}^{\pi^\eta}\right)\right] = \begin{cases} \max\limits_{\pi\in\mathcal{A}} \mathbb{E}[\exp(-\eta W_{\mathcal{T}}^\pi)] & \text{if } \eta > -1 \\ \min\limits_{\pi\in\mathcal{A}} \mathbb{E}[\exp(-\eta W_{\mathcal{T}}^\pi)] & \text{if } \eta < -1 \end{cases}$$

Therefore, the risk-sensitive asset management is equivalent to optimizing the *hyperbolic absolute risk aversion* (HARA) utility of a log portfolio. If we consider portfolio processes discounted by a benchmark asset, the risk sensitive criterion becomes optimizining the HARA utility gained from outperforming the benchmark on a monetary basis (see [6] and [7] for details).

Remark 4.2.4. The optimization problem (4.30) is equivalent to maximizing (or minimizing) the *certainty-equivalent expectation* of $W^\pi = (W_t^\pi)_{0\le t\le\mathcal{T}}$ defined above with respect to the HARA utility function. For any utility function U and any random variable X, the certainty-equivalent expectation is given by

$$\mathcal{E}(X) = U^{-1}\mathbb{E}[U(X)]$$

If $U(X) = \exp(-\eta X)$, we can define the function $\mathcal{J}^\eta \colon \mathcal{A} \to \mathbb{R}$ as

$$\mathcal{J}^\eta(\pi) :\, = \mathcal{E}(W_{\mathcal{T}}^\pi) = -\frac{1}{\eta}\log \mathbb{E}[\exp(-\eta W_{\mathcal{T}}^\pi)]$$

Since the logarithm is monotone, problem (4.30) can be reduced to maximize (or minimize) $J^\eta(\pi)$ with respect to $\pi \in \mathcal{A}$. Such a formulation allows to clarify the role of the risk-sensitivity parameter η. Indeed, notice that by performing the limit for $\eta \to 0$ of $\mathcal{J}^\eta(\pi)$ via De l'Hôpital rule we obtain

$$\lim_{\eta\to 0}\mathcal{J}^\eta(\pi) = \lim_{\eta\to 0} -\mathbb{E}\left[\frac{-W_{\mathcal{T}}^\pi \exp(-\eta W_{\mathcal{T}}^\pi)}{\exp(-\eta W_{\mathcal{T}}^\pi)}\right] = \mathbb{E}[W_{\mathcal{T}}^\pi]$$

The Taylor expansion of the risk-sensitive criterion J around $\eta = 0$ is

$$\mathcal{J}^\eta(\pi) = \mathbb{E}[W_\mathcal{T}^\pi] - \frac{\eta}{2}\,\mathrm{Var}(W_\mathcal{T}^\pi) + o(\eta^2)$$

Thus, the risk-sensitive optimization problem is amenable to a dynamic asset allocation model with *mean-variance* criterion for portfolio log returns (compare [7]). Notice that η determines the weight of the variance in the optimization. When η is close to zero the important term to maximize becomes $\mathbb{E}[W_\mathcal{T}^\pi]$. Since $W_\mathcal{T}^\pi = \log V_\mathcal{T}^\pi$, this means that for $\eta \to 0$ the RSP solves a log utility maximization problem and, therefore, it coincides with the GOP (see also Remark 4.2.6).

There exists a vast literature on risk-sensitive control problems and their applications; for further information the reader is invited to see Bensoussan (1992), Fleming (2006), and Davis and Lleo (2011). Now, we aim at characterizing the RSP and its corresponding strategy, as done for the GOP in Theorem 2.3.2. Solving (4.30) is less straightforward than dealing with the log utility case and, in a general framework, we should apply the martingale method to optimal investment in incomplete markets (see [19, Sect.6.7]). However, the diffusion-based Markov structure introduced in Assumption 3.3.2 makes it possible to attack problem (4.30) with dynamic programming and, therefore, to reduce it to an HJB equation. To this end, we recall that, in view of Assumption 3.3.2, the general portfolio dynamics equation (2.5) can be written as

$$\frac{dV_t^\pi}{V_t^\pi} = r(t,X_t)dt + \pi_t^{'}\cdot(\mu(t,X_t) - r(t,X_t)\mathbf{1})dt + \pi_t^{'}\cdot\sigma(t,X_t)\cdot dW_t \quad (4.31)$$

where the factor process $X = (X_t)_{0\le t\le\mathcal{T}}$ satisfies equation (3.32). In what follows, we suppose that Assumptions 3.3.1 and 3.3.2 hold in order to have existence and uniqueness of solution for the partial differential equations below.

Theorem 4.2.5. *Suppose that Assumptions 2.2.8 and 2.2.10 hold. Let us define the function* $\Xi\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d$ *given by*

$$\Xi(t,x)\colon = g^{'}(t,x)\cdot\nabla_x^{'}\phi(t,x) \quad (4.32)$$

where $\phi(t,x) \in \mathcal{C}^{1,2}$ *is the optimal value function of the following stochastic control problem*

$$\begin{cases} \max\limits_{u\in\mathcal{U}}\mathbb{E}_{t,x}\left[-\int_t^\mathcal{T} \frac{1}{2}\frac{u_s^{'}\cdot u_s}{\eta+1} + \eta\left(r_s + \frac{\|\theta_s\|^2}{\eta+1}\right)ds\right] & \text{if } \eta > -1 \\ \min\limits_{u\in\mathcal{U}}\mathbb{E}_{t,x}\left[-\int_t^\mathcal{T} \frac{1}{2}\frac{u_s^{'}\cdot u_s}{\eta+1} + \eta\left(r_s + \frac{\|\theta_s\|^2}{\eta+1}\right)ds\right] & \text{if } \eta < -1 \end{cases} \quad (4.33)$$

subject to dynamics

$$
\begin{aligned}
dX^{\eta,u} = &\left(f(t, X_t^{\eta,u}) - \frac{\eta}{\eta+1} g(t, X_t^{\eta,u}) \cdot \theta_t + \frac{1}{\eta+1} g(t, X_t^{\eta,u}) \cdot u_t \right) dt \\
&+ g(t, X_t^{\eta,u}) \cdot dW_t
\end{aligned} \tag{4.34}
$$

Then, there exist a unique strategy $\pi^\eta \in \mathcal{A}$ solving problem (4.30) and it is given by

$$
\pi_t^\eta = \frac{(\sigma_t \cdot \sigma_t^{'})^{-1}\sigma_t}{\eta+1} \cdot (\theta_t + \Xi_t) = \frac{\pi_t^*}{\eta+1} + \frac{(\sigma_t \cdot \sigma_t^{'})^{-1}\sigma_t}{\eta+1} \cdot \Xi_t \tag{4.35}
$$

where π^* is the growth optimal strategy introduced in Definition 2.3.1 and $\Xi = (\Xi_t)_{0 \le t \le \mathcal{T}}$ is the stochastic process defined by $\Xi_t := \Xi(t, X_t)$. The corresponding risk-sensitive portfolio process $V^{\pi^\eta} = (V_t^{\pi^\eta})_{0 \le t \le \mathcal{T}}$ satisfies the following dynamics:

$$
\frac{dV_t^{\pi^\eta}}{V_t^{\pi^\eta}} = \left(r_t + \frac{\|\theta_t + \Xi_t\|^2}{\eta+1} \right) dt + \frac{(\theta_t + \Xi_t)^{'}}{\eta+1} \cdot dW_t \tag{4.36}
$$

Proof. We can consider only the case $\eta > -1$. We embed problem (4.30) into a larger class of portfolio optmization problems defined by

$$
P(t,x,v) = \max_{\pi \in \mathcal{A}} \mathbb{E}_{t,x,v}[(V_\mathcal{T}^\pi)^{-\eta}] = \max_{\pi \in \mathcal{A}} \mathbb{E}[(V_\mathcal{T}^\pi)^{-\eta} | X_t = x,\, V_t^\pi = v] \tag{4.37}
$$

where the process $V^\pi = (V_t^\pi)_{0 \le t \le \mathcal{T}}$ satisfies equation (4.31) and its value at time t is given by $V_t^\pi = v \in \mathbb{R}_+$. In other terms, our original problem in equivalent to determining $P(0, x_0, 1)$. Since the state process related to problem (4.37) is the $\mathbb{R}^{n+1}$-valued process (X, V^π), the corresponding HJB equation is

$$
\begin{aligned}
\frac{\partial}{\partial t} P(t,x,v) + \sup_{\pi \in \mathcal{A}} \Big\{ & v \left[r(t,x) + \pi^{'}(t,x,v) \cdot (\mu(t,x) - r(t,x)\mathbf{1}) \right] \frac{\partial}{\partial v} P(t,x,v) \\
&+ \nabla_x P(t,x,v) \cdot f(t,x) + \frac{1}{2} \operatorname{tr} \left(g^{'}(t,x) \cdot \nabla_{xx}^2 P(t,x,v) \cdot g(t,x) \right) \\
&+ v\pi^{'}(t,x,v) \cdot \sigma(t,x) \cdot g^{'}(t,x) \cdot \nabla_x^{'} \frac{\partial}{\partial v} P(t,x,v) \\
&+ \frac{1}{2} v^2 \|\sigma^{'}(t,x) \cdot \pi(t,x,v)\|^2 \frac{\partial^2}{\partial v^2} P(t,v) \Big\} = 0
\end{aligned} \tag{4.38}
$$

together with the terminal condition

$$
P(\mathcal{T}, x, v) = v^{-\eta}, \quad x \in \mathbb{R}^n,\, v \in \mathbb{R}_+ \tag{4.39}
$$

Similarly to [20, Sect. 3.6.1], we look for a candidate solution of the form

$$
P(t,x,v) = v^{-\eta} \exp \phi(t,x)
$$

for some function $\phi\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belonging to class $\mathcal{C}^{1,2}$ such that $\phi(\mathcal{T}, x) = 0$ for every $x \in \mathbb{R}^n$. By replacing the derivatives into equation (4.38) and dividing by $v^{-\eta} \exp \phi(t,x)$ we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t,x) - \eta \sup_{\pi\in\mathcal{A}}\Big\{ & \Big[r(t,x) + \pi^{'}(t,x,v)\cdot(\mu(t,x) - r(t,x)\mathbf{1})\Big] \\
& - \frac{\nabla_x \phi(t,x)}{\eta}\cdot f(t,x) - \frac{1}{2\eta}\operatorname{tr}\Big(g^{'}(t,x)\cdot\nabla^2_{xx}\phi(t,x)\cdot g(t,x)\Big) \\
& - \frac{1}{2\eta}\nabla_x\phi(t,x)\cdot g(t,x)\cdot g^{'}(t,x)\cdot\nabla^{'}_x\phi(t,x) \\
& + \pi^{'}(t,x,v)\cdot\sigma(t,x)\cdot g^{'}(t,x)\cdot\nabla^{'}_x\phi(t,x) \\
& - \frac{1}{2}(\eta+1)\|\sigma^{'}(t,x)\cdot\pi(t,x,v)\|^2\phi(t,x)\Big\} = 0
\end{aligned} \tag{4.40}$$

For $\eta > -1$, the expression inside the brackets is a concave function of π and by imposing the first order condition we get that $\pi^\eta_t = \pi^\eta(t, X_t)$ must sastify the following equation

$$\sigma_t\cdot\sigma^{'}_t\cdot\pi^\eta_t = \frac{\mu_t - r_t\mathbf{1}}{\eta+1} + \frac{\sigma_t\cdot g^{'}(t,X_t)\cdot\nabla^{'}_x\phi(t,X_t)}{\eta+1}$$

Due to Assumption 2.2.8, the matrix $\sigma_t\cdot\sigma^{'}_t$ is $\mathbb{P}$-a.s. invertible and we obtain

$$\pi^\eta_t = \frac{(\sigma_t\cdot\sigma^{'}_t)^{-1}\sigma_t}{\eta+1}\cdot\Big(\theta_t + g^{'}(t,X_t)\cdot\nabla^{'}_x\phi(t,X_t)\Big)$$

which corresponds to equation (4.35), after setting

$$\Xi_t = \Xi(t,X_t) = g^{'}(t,X_t)\cdot\nabla^{'}_x\phi(t,X_t)$$

The stochastic dynamics of the RSP can be now easily obtained by inserting π^η_t into equation (4.31), similarly to what was done to deduce the GOP dynamics equation (2.17). If we subsititute π^η into (4.40) instead, we deduce that ϕ must satisfy the partial differential equation

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t,x) + \nabla_x\phi(t,x)\cdot & \Big(f(t,x) - \frac{\eta}{\eta+1}g(t,x)\cdot\theta(t,x)\Big) \\
& + \frac{1}{2}\frac{1}{\eta+1}\nabla_x\phi(t,x)\cdot g(t,x)\cdot g^{'}(t,x)\cdot\nabla^{'}_x\phi(t,x) \\
& + \frac{1}{2}\operatorname{tr}\Big(g^{'}(t,x)\cdot\nabla^2_{xx}\phi(t,x)\cdot g(t,x)\Big) \\
& - \eta\Big(r(t,x) + \frac{\|\theta(t,x)\|^2}{\eta+1}\Big) = 0
\end{aligned} \tag{4.41}$$

with terminal condition $\phi(\mathcal{T}, x) = 0$. Consider now the HJB equation

$$\frac{\partial}{\partial t}\phi + \sup_{u\in\mathcal{U}}\Big\{\nabla_x\phi\cdot\left(f - \frac{\eta}{\eta+1}g\cdot\theta + \frac{1}{\eta+1}\cdot g\cdot u\right) + \frac{1}{2}\operatorname{tr}\left(g^{'}\cdot\nabla^2_{xx}\phi\cdot g\right)$$
$$-\frac{1}{2}\frac{u^{'}\cdot u}{\eta+1} - \eta\left(r + \frac{\|\theta\|^2}{\eta+1}\right)\Big\} = 0 \tag{4.42}$$

For $\eta > -1$, the expression inside the brackets is a concave function of u and by imposing the first order condition we get that the infimum is attained by

$$u^*(t,x) = g^{'}(t,x)\cdot\nabla^{'}_x\phi(t,x)$$

If we insert u^* into equation (4.42) we obtain (4.41) and, therefore, we can conclude by observing that (4.42) is the HJB equation associated to the stochastic control problem (4.33)-(4.34) for $\eta > -1$. □

Remark 4.2.6. Our analysis is similar to that proposed in [6] and [7], with the difference that we have used the standard optimality criterion instead of the certainty-equivalent expectation criterion. In H. Pham (2009), Sect. 3.6.1, an analogous result is proved for the special case of a single risky asset with the Black-Scholes dynamics. Here we have instead chosen to deal with the more general situation introduced by Assumption 3.3.2 and we obtain a more complex representation for the optimal asset allocation of problem (4.30), which can be expressed as a portfolio of investments composed of two *mutual funds*. The first is the growth optimal or log-utility portfolio; the second is a correction portfolio related to the comovements of the risky assets $S = (S^1, \dots, S^N)$ with the factor process X.

We can now consider the exdended market introduced in Definition 3.1.6. We recall that, for all $T \in [0, \mathcal{T})$, the forward Libor-OIS spread is defined by

$$S(t;T,T+\delta) = \frac{1+\delta L(t;T,T+\delta)}{1+\delta F(t;T,T+\delta)}$$

and for $t = T$ we get the spot version

$$S(T,T+\delta) = [1+\delta L(T,T+\delta)]P(T,T+\delta)$$

Due to Lemma 3.2.4, $S(\cdot;T,T+\delta)$ is a martingale with respect to the T-forward measure, meaning that

$$S(t;T,T+\delta) = \mathbb{E}^T\left[S(T,T+\delta)\middle|\mathcal{F}_t\right] = \frac{1}{P(t,T)}\mathbb{E}\left[\frac{V^*_t}{V^*_T}S(T,T+\delta)\middle|\mathcal{F}_t\right] \tag{4.43}$$

We make the following modeling assumption.

Assumption 4.2.7. For all $T \in [0, \mathcal{T}]$, the spot spread $S(t, T)$ is given by

$$S(t,T) = V_t^{\pi^\eta} \mathbb{E}\left[\frac{1}{V_T^{\pi^\eta}} \Big| \mathcal{F}_t\right] \tag{4.44}$$

In other words, the process $\left(\frac{S(t,T)}{V_t^{\pi^\eta}}\right)_{0 \le t \le T}$ is a $\mathbb{P}$-martingale.

Remark 4.2.8. From equation (4.43) and Assumption 4.2.7 we can derive the following risk-sensitive representation for forward spreads:

$$S(t; T, T+\delta) = \frac{V_t^*}{P(t,T)} \mathbb{E}\left[\frac{1}{V_T^*} \frac{V_T^{\pi^\eta}}{V_{T+\delta}^{\pi^\eta}} \Big| \mathcal{F}_t\right] = \mathbb{E}^T\left[\frac{V_T^{\pi^\eta}}{V_{T+\delta}^{\pi^\eta}} \Big| \mathcal{F}_t\right] \tag{4.45}$$

Remark 4.2.9. The financial interpretation of Assumption 4.2.7 is similar to the economic explation provided by the roll-over-risk approach and the spot spread is still seen as a term premium. However, while the roll-over-risk formulation reflects only the point of view of the lender of a Libor loan, the risk sensitive representation makes it possible to express the point of view of both the lender and the borrower by choosing the value of the risk sensitivity parameter η (see Remark 4.2.11). An interesting example is $\eta = 0$ since in this case the quantity $S(t, T)$ coincides with the value at time t of a T-bond and the Libor-OIS spread at time $t \in [0, T]$ is given by

$$S(t; T, T+\delta) = \mathbb{E}^T\left[P(T, T+\delta) \Big| \mathcal{F}_t\right] = \frac{P(t, T+\delta)}{P(t,T)}$$

From a financial point of view, when $\eta = 0$ the borrower at spot Libor $L(T, T+\delta)$ can be seen as an issuer of a $(T+\delta)$-bond. In other words, in this particular case receiving a Libor loan is equivalent to borrowing at a risk-free rate plus a risk premium given by the price of bond with maturity $T + \delta$.

We are now in condition to provide a new stochastic control formulation for spot spreads.

Theorem 4.2.10. *Suppose that Assumptions 3.3.1, 3.3.2, and 4.2.7 hold. Let $s^T(t,x)$ be a function belonging to class $\mathcal{C}^{1,2}$ such that*

$$s^T(t, X_t) = S(t,T)$$

Assume that $s^T(t,x)$ satisfies the following condition

$$g^{'}(t,x) \cdot \frac{\nabla_x^{'} s^T(t,x)}{s^T(t,x)\eta} = -\frac{\theta(t,x) + \Xi(t,x)}{\eta + 1} \tag{4.46}$$

where the function $\theta(t,x)$ defines the market price of risk and $\Xi(t,x)$ is the function introduced in Theorem 4.2.5. Then, s^T is the optimal value function *of the stochastic control problem*

$$\begin{cases} \max\limits_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\exp\left(-\int_t^T \eta\frac{1}{2}u^{'}(s,X_s^u)\cdot u(s,X_s^u)+r(s,X_s^u)\,ds\right)\right] & \text{if } \eta>0 \\ \min\limits_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\exp\left(-\int_t^T \eta\frac{1}{2}u^{'}(s,X_s^u)\cdot u(s,X_s^u)+r(s,X_s^u)\,ds\right)\right] & \text{if } \eta<0 \end{cases} \tag{4.47}$$

subject to the dynamics

$$dX_t^u = \left[f(t,X_t^u) - g(t,X_t^u)\cdot u(t,X_t^u)\right]dt + g(t,X_t^u)\cdot dW_t \tag{4.48}$$

and the optimal control is

$$u^*(t,x) = \frac{\theta(t,x)+\Xi(t,x)}{\eta+1} \tag{4.49}$$

Proof. We consider only the case $\eta > 0$. First of all, let us apply the stochastic differentiation by parts to $\frac{S(t,T)}{V_t^{\pi^\eta}}$:

$$d\left(\frac{S(t,T)}{V_t^{\pi^\eta}}\right) = \frac{dS(t,T)}{V_t^{\pi^\eta}} + S(t,T)d\left(\frac{1}{V_t^{\pi^\eta}}\right) + d\left\langle S(\cdot,T),\frac{1}{V^{\pi^\eta}}\right\rangle_t$$

Due to Theorem 4.2.5, an application of Ito's formula provides the following

$$d\left(\frac{1}{V_t^{\pi^\eta}}\right) = -\frac{1}{V_t^{\pi^\eta}}\left[\left(r_t+\eta\frac{\|\theta_t+\Xi_t\|^2}{(\eta+1)^2}\right)dt + \frac{(\theta_t+\Xi_t)^{'}}{\eta+1}\cdot dW_t\right]$$

Inserting this into the line above and assuming that $S(t,T) = s^T(t,X_t)$ for some function $s^T \in \mathcal{C}^{1,2}$ we obtain

$$\begin{aligned} d\left(s^T(t,X_t)\frac{1}{V_t^{\pi^\eta}}\right) &= \frac{1}{V_t^{\pi^\eta}}\left(\frac{\partial}{\partial t}s^T(t,X_t) + \nabla_x s^T(t,X_t)\cdot f(t,X_t)\right)dt \\ &+ \frac{1}{2}\frac{1}{V_t^{\pi^\eta}}\operatorname{tr}\left(g^{'}(t,X_t)\cdot\nabla^2_{xx}s^T(t,X_t)\cdot g(t,X_t)\right)dt \\ &- \frac{1}{V_t^{\pi^\eta}}\left(\nabla_x s^T(t,X_t)\cdot g(t,X_t)\cdot\frac{\theta_t+\Xi_t}{\eta+1}\right)dt \\ &- \frac{1}{V_t^{\pi^\eta}}\left[s^T(t,X_t)\left(r_t+\eta\frac{\|\theta_t+\Xi_t\|^2}{(\eta+1)^2}\right)\right]dt \\ &+ (\dots)\cdot dW_t \end{aligned}$$

Due to Assumption 4.2.7, we can set the drift equal to zero and we get the following PDE

$$\begin{aligned} \frac{\partial}{\partial t}s^T(t,x) &+ \nabla_x s^T(t,x)\cdot\left(f(t,x) - g(t,x)\cdot\frac{\theta(t,x)+\Xi(t,x)}{\eta+1}\right) \\ &+ \frac{1}{2}\operatorname{tr}(g^{'}(t,x)\cdot\nabla^2_{xx}s^T(t,x)\cdot g(t,x)) \\ &- s^T(t,x)\left(\eta\frac{\|\theta(t,x)+\Xi(t,x)\|^2}{(\eta+1)^2} + r(t,x)\right) = 0 \end{aligned} \tag{4.50}$$

with terminal condition $s^T(T,x) = 1$. Due to the hypotheses of the Theorem, equation (4.50) admits a unique solution. Consider now the HJB equation

$$\begin{aligned}\frac{\partial}{\partial t}s^T + \sup_{u\in\mathcal{U}}\Big\{\nabla_x s^T \cdot (f - g\cdot u) + \frac{1}{2}\operatorname{tr}\left(g^{'}\cdot\nabla^2_{xx}s^T\cdot g\right) \\ - s^T\left(\eta\frac{1}{2}u^{'}\cdot u + r\right)\Big\} = 0\end{aligned} \tag{4.51}$$

For $\eta > 0$, the expression inside the brackets is a concave function of u and by imposig the first oder condition we get that the supremun is reached by

$$u^*(t,x) = -g^{'}(t,x)\cdot\frac{\nabla^{'}_x s^T(t,x)}{s^T(t,x)\eta} = \frac{\theta(t,x)+\Xi(t,x)}{\eta+1}$$

where the last equalitity follows from condition (4.46). Due to Assumption 2.2.10, the control law u^* is admissible and we can easily see that equation (4.50) corresponds to (4.51) when $u = u^*$ and with $s^T(T,x) = 1$ as boundary condition. We conclude by noting that (4.51) is the HJB equation associated to the stochastic control problem (4.47)-(4.48) for $\eta > 0$. □

Remark 4.2.11. Unlike the stochastic control representation for spot spreads deriving from the roll-over-risk approach (see Proposition 4.1.4), the present formulation offers two interpretations depending on the choice of the risk-sensitivity parameter. If $\eta > 0$, it reflects the point of view of the lender of a Libor loan who aims at maximizing the running gain; if $\eta > 0$ the perspective of the borrower is represented and the control problem becomes minimizing the running cost of a term loan.

Finally, by recalling that, under the the roll-over risk approach, the only function $s^T \in \mathcal{C}^{1,2}$ such that $s^T(t,X_t) = S(t,T)$ solves the boudary value problems of Proposition 3.3.7, we can conclude with the following statement.

Corollary 4.2.12. *Suppose that Assumptions 3.3.1, 3.3.2 and 3.3.6 hold. We recall that $\varphi = (\varphi_t)_{0\le t\le\mathcal{T}}$, $\varphi_t = \varphi(t,X_t)$, is the funding-liquidity spread. Then, the risk-sensitive representation of spot spreads is equivalent to the roll-over risk approach if and only if*

$$\Xi(t,x) = \eta g^{'}(t,x)\cdot\frac{\nabla^{'}_x v^*(t,x)}{v^*(t,x)} \tag{4.52}$$

$$\varphi(t,x) = \frac{\eta}{\eta+1}\left\|g^{'}(t,x)\cdot\frac{\nabla^{'}_x v^*(t,x)}{v^*(t,x)}\right\|^2 + r(t,x) \tag{4.53}$$

Proof. Due to Corollary 3.3.5, which links the market price of risk to the

GOP, equation (4.50) can be written as

$$\frac{\partial}{\partial t}s^T + \nabla_x s^T \cdot \left[f - \frac{g}{\eta+1} \cdot \left(g^{'} \cdot \frac{\nabla_x^{'} v^*}{v^*} + \Xi \right) \right] + \frac{1}{2}\operatorname{tr}(g^{'} \cdot \nabla_{xx}^2 s^T \cdot g)$$
$$- s^T \left(\frac{\eta}{(\eta+1)^2} \left\| g^{'} \cdot \frac{\nabla_x^{'} v^*}{v^*} + \Xi \right\|^2 + r \right) = 0$$

where we have suppressed the dependence on (t, x) for brevity of notation. The risk-sensitive representation is equivalent to the roll-over risk approach if the equation above coincides with (3.45). Comparing the coefficients of the partial derivatives, we obtain

$$\frac{1}{\eta+1}\left(g^{'}(t,x) \cdot \frac{\nabla_x^{'} v^*(t,x)}{v^*(t,x)} + \Xi(t,x) \right) = g^{'}(t,x) \cdot \frac{\nabla_x^{'} v^*(t,x)}{v^*(t,x)}$$
$$\varphi(t,x) = \frac{\eta}{(\eta+1)^2} \left\| g^{'}(t,x) \cdot \frac{\nabla_x^{'} v^*(t,x)}{v^*(t,x)} + \Xi(t,x) \right\|^2 + r(t,x)$$

that leads to

$$\Xi(t,x) = \eta g^{'}(t,x) \cdot \frac{\nabla_x^{'} v^*(t,x)}{v^*(t,x)}$$
$$\varphi(t,x) = \frac{\eta}{(\eta+1)^2} \left\| g^{'}(t,x) \cdot \frac{\nabla_x^{'} v^*(t,x)}{v^*(t,x)} + \Xi(t,x) \right\|^2 + r(t,x)$$

By inserting the first line into the second we deduce (4.52)-(4.53). □

Chapter 5

Conclusions

In conclusion, let us summarize the results obtained.

(i) Due to the benchmark approach, the pricing of contigent claims can be carried out without assuming the existence of an E(L)MM, as long as the market is viable, that is it does not allow for arbitrage of the first kind. In view of Theorem 2.2.16, the viability of the financial market is ensured by a square-integrability property of the market price of risk process (Assumption 2.2.10). We have obtained an expression for the value of a contingent claim depending on the growth-optimal portfolio and we have applied it to bonds and forward rate agreements, assuming them fairly priced (Assumption 3.1.7). In order to provide a spread representation under the benchmark approach, we have adapted the roll-over risk formulation presented in [1], which leads to interpreted the spot spread as a roll-over-risk adjusted borrowing account whose value at present time depends on the GOP and the fundig-liquidity spread. After introducing a multifactor Markov structure, we have been able to represent bonds and spreads as solutions of partial differential equations. For linear-quadratic models, this representation led us to ordinary differential systems.

(ii) The terminal value problems obtained have been the starting point to provide a stochastic control perspective of bonds and spreads under the benchmark approach. As explained in [14] and Remark 4.1.2, a feedback approach resulting from a stochastic control methodology determines the present values of term structures as solutions of stochatic control problems subject to an auxilary dynamics, which is generated by an alteration of the original factor process due to the action of a feedback control on its drift. We have accompanied our analysis with exhaustive economic explanations. In particular, the stochastic control representation of spot spreads reflects the point of view of the lender of a Libor loan, who aim at maximizing the running gain.

(iii) Motivated by the results of Sect. 4.1, in Sect. 4.2 we have experimented a new spread representation, based on the idea of connecting the spot spread to a portfolio process whose corresponding strategy solves a risk-sensitive portfolio optimization problem. We have addressed this problem with dynamic programming and we have determined an optimal asset allocation that is a linear combination of the growth-optimal strategy and a correction portfolio. This result has been used to derive a risk-sentive stochastic control representation of spot spreads capable to reflect the perspective of both the lender and the borrower of a Libor loan. We have concluded our analysis by showing that the risk-sensitive spread representation is equivalent to the roll-over risk approach, provided an endogenous condition which connect the funding-liquidity spread to the risk-sensitive parameter.

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