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"A comparative analysis of Resampling Efficiency and Black-Litterman portfolio optimization"

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Abstract

It is common knowledge that the traditional Mean-Variance (MV) approach presents several non-negligible criticalities such as unintuitive and highly concentrated portfolios, input sensitivity, and estimation error maximization. Black-Litterman (BL) model and Resampled Efficiency (RE) techniques are advanced methods that help to generate better allocations than the traditional "a la Markowitz" method. They both provide more diversified and performing portfolios.

On the one hand, the Black-Litterman model represents a well-known approach that overcomes this issue by assuming partial information on the expected returns. By blending a reference market distribution with subjective views on the market, the approach yields optimal portfolios that smoothly reflect those views. On the other, portfolio resampling, following a heuristic approach (there is no economic rationale derived from the optimizing behaviours of rational agents that supports this method) allows the portfolio manager to visualize the estimation error in traditional portfolio optimization methods. RE optimization was invented by Richard Michaud and Robert Michaud and is a U.S. patented procedure, with worldwide patents pending. It was originally described in Michaud (1998, Chapter 6). New Frontier Advisors, LLC (NFA) is an exclusive worldwide licensee.

Starting from a review of the literature about modern portfolio theory and a discussion of its limitations, the two approaches: BL and RE are introduced. The core of this work will focus on the implementation of Resampling techniques together with the Black-Litterman model for portfolio allocations, carrying out an empirical examination of the usefulness of those technologies. Both the approaches are tested over the month that follows the end of the sampling period. A full set of portfolio performance indicators is provided. Both RE and BL approaches overperformed the MV under three dimensions: holding period return, absolute performance indicators such as Sharpe ratio and Sortino ratio, and relative performance indicators such as Tracking Errors and Information ratio. A short conclusion will be made at the end.

1. The evolution of portfolio theory

Portfolio Theory appeared at the beginning of the 20th century playing a crucial role in the world of finance until the publication of Markowitz's article "Portfolio selection" (1952). In its very early stage, from the beginning of the 20th century until 1933, Portfolio Theory was mainly focused on the analysis of individual securities, it was centred on agents' individual skills and capabilities, and was based on subjective assessment, without any codified analytical basis and omitting the way in which securities included in the portfolio impacted on the overall.

Before Markowitz's article, portfolio construction was plainly based on the ”cherry-picking”, following a simple diversification principle derived from the law of large numbers introduced by Bernoulli in 1713. R. Hicks (1935) advocated a simple concept of diversification based on the law of large numbers. A more refined concept of diversification was given by John Burr Williams. Williams rightfully deserves to be recognized as the progenitor of the "Gordon growth formula," the Modigliani-Miller capital structure irrelevancy theorem, and the dividend discount model. Despite that, Williams had very little to say about the effects of risk on valuation because he believed that all risk could be diversified away. One last milestone before the "turning point" is represented by the work made by Dickson Hammond Leavens (Diversification of Investments, 1945). In his work, he illustrated the benefits that diversification might generate to a given portfolio when risks are independent introducing some vague notion of covariance, but most importantly, he recognized that full independence is very difficult to reach in capital markets and therefore there is a fraction of risk that cannot be diversified. What was missing prior to 1952 was a properly defined theory of investment that included the effects of diversification when different sources of risks are correlated with one another, distinguished between dominating and dominated portfolios, and analyzed risk-return trade-offs on the portfolio in a wider way.

In 1952, an article titled "Portfolio Selection" authored by Harry Markowitz was published in The Journal of Finance. It improved financial theory and investment practice enabling the optimization of the relationship between the expected return and the assumed risk and introducing the principle of diversification, coming to form the foundations of what is now referred to as the Modern Portfolio Theory. In addition to that, he re-elaborated the concept of diversification and its benefits. Yet, probably the most important aspect of Markowitz's work was to show that it is not a security's own risk that is important to an investor, but rather the contribution the security makes to the variance of his entire portfolio and that this was primarily a question of how that particular security covariate with all the other securities in his portfolio.

Markowitz's (1952) approach is now commonplace among institutional portfolio managers who use it as a key principle both to structure their portfolios and measure their performance. It has been generalized, refined and adopted in uncountable ways and is even being used to manage the portfolios of ordinary investors. Markowitz's mean-variance methodology, later expanded into a seminal book (Markowitz 1959), is the classic paradigm of modern finance for efficiently allocating capital among risky assets. In his 1959 book on portfolio selection (Markowitz, 1959), Markowitz provided an extended and detailed development of the 1952 mean-variance model of portfolio choice, purposely designed for access by readers with a modest quantitative background.

Markowitz's article on portfolio selection proposed expected return (defined by the mean) and variance of the return of the portfolio (as a whole) as criteria for portfolio selection. Both were considered as a possible hypothesis about actual behaviour and as an adage for how investors should act[1]. Markowitz distinguished between efficient and inefficient portfolios defining the so-called "efficient frontier", a term subsequently coined to define what Markowitz initially defined "set of efficient mean-variance combinations". He proposed means, variances, and covariances of securities to be estimated by a combination of statistical analysis and analyst’s judgment. From these estimates, the set of efficient mean-variance combinations can be derived and presented to the investor for choice of the desired risk-return combination. Markowitz in his article used geometrical analyses of three- and four-security examples to illustrate properties of efficient sets, He assumed nonnegative investments subject to a budget constraint. In particular, he showed that the set of efficient mean-variance combinations is piecewise parabolic.

There would be another interesting element that certainly deserves to be mentioned. A British economist, Andrew Donald Roy, in the same year, 1952, independently came up with the same equation relating portfolio variance of return to the variances of return of the

[1] It is worth to mentioning here that the theory of portfolio selection is a normative theory. A normative theory is one that describes a standard or norm assuming that decision-makers apply a set of rational criteria during the decision-making process, which is also considered rational.

In this work, $\tau = 0$, p=1, q=2

constituent securities (Roy, 1952). He developed a pretty similar mean-variance efficient set which included the covariances of returns among securities. There were two chief differences between Markowitz's and Roy's works:

I. whereas Markowitz left it up to the investor to choose where along with the efficient set he would invest, Roy advised choosing the single portfolio in the mean-variance efficient set that maximizes the excess return of the portfolio over a "disaster level" return the investor places a high priority on not falling below per unit of volatility of the portfolio.
II. Markowitz's required non-negative investments whereas Roy's allowed the amount invested in any security to be positive or negative

The principles of Modern Portfolio Theory changed the investment world forever and paved the way for the development of pooled funds, indexed-based funds, as well as exchange-traded funds. Portfolio theory concerns the design of optimal portfolios and its implication for asset pricing (H. K. Baker, G. Filbek, 2013) providing a framework within which to make sensible asset allocation decisions. The theory has undergone tremendous development since Markowitz (1952) " who laid out the initial mean-variance framework. At the very beginning of its introduction, the so-called Modern Portfolio Theory was not very successful. It generated relatively little interest among practitioners indeed, however, with time, the financial community began to appreciate its potential and now, 70 years later, the concepts and intuitions illustrated in the theory continue to be one of the pillars of portfolio managers' day-to-day activity. They are equipped with many more tools and concepts, but anyway, their financial models are based on those very same principles, constantly being re-engineered to incorporate all the new findings.

This, is probably, a result of its flexibility that allows the investment agent to consider elements such as various trading costs and investment policy constraints. The theory behind MPT is relatively straightforward but its implementation can get quite complicated anyway. MV optimization is useful as an asset management tool for many applications, including (Michaud, 1998):

I. Implementing investment objectives and constraints
II. Controlling the components of portfolio risk
III. Implementing the asset manager's investment philosophy, style, and market outlook

IV. Efficiently using active return information (Sharpe, 1985)

V. Conveniently and efficiently embedding new information into portfolios

Markowitz's Mean-Variance efficiency is also the basis for many important advances in financial theory. Some of the most important are the works of William Sharpe (1964), John Lintner (1965), and Jack Treynor (1961): the well-known Capital Asset Pricing Model, which addresses the formalization of the relationship that should exist between asset returns and risk and the fundamental dichotomy between systematic and diversifiable risk. They defined the role that portfolios play in determining the appropriate individual asset risk premium (i.e., the return in excess of the risk-free return expected by investors as compensation for the asset's risk). According to their theory, the "priced" risk of an individual security is affected by holding it in a well-diversified portfolio. The early research provided the insight that an asset's risk should be measured in relation to the remaining systematic or non-diversifiable risk, which should be the only risk that affects the asset's price (meaning: the market remunerates only the systematic risk: the beta).

The starting point of Modern Portfolio Theory is that of a rational investor who, at time *t*, decides what portfolio of investments to hold for a time horizon of T. The investor makes decisions on the gains and losses he will make at time $t+T$, without considering eventual gains and losses either during or after the period *T*. At time $t+T$, the investor will reconsider the situation and decide anew. The theory dictates that given estimates of the returns, volatilities, correlations of a set of investments and constraints on investment choices (e.g., maximum exposures and turnover constraints), it is possible to perform an optimization that results in the risk/return or mean-variance efficient frontier. This frontier is "efficient" because underlying every point on this frontier is a portfolio that results in the greatest possible expected return for that level of risk or, the other way round, results in the smallest possible risk for that level of expected return. The portfolios that lie on the frontier make up the set of efficient portfolios (Fabozzi et al, 2002). At its very basic kind, it provides a framework for building and selecting portfolios based on the expected return of the investment and the risk appetite of the investor. The figure below schematizes the Mean-Variance Portfolio optimization process.

1.1. The mean-variance optimization approach

The notation adopted in this, and the following sections, is based on Meucci, 2009, which provides an extremely clear and detailed description of the optimization problem. Consider a market of N securities. Denote P_t the prices at the generic time t of the N securities in the investor's market. At the time T when the investment is made, the investor can purchase α_n units of the generic n-th security[2]. The N-dimensional vector α represents the outcome of the allocation decision, which can be seen as a "black box" that processes two types of inputs: the information on the investor's profile P and the information i_T on the market available at the time the investment decision is made.

A generic allocation decision processes the information on the market and on the investor and outputs the amounts to invest in each security in the given market:

1

$$\alpha[\cdot]:[i_T, P] \longmapsto R^N$$

The investor evaluates the potential advantages of an allocation α based on his primary index of satisfaction S, provided that the allocation is feasible since the investor is bound by a set (vector) of investment constraints C that limit his feasible allocations. Therefore, the optimal allocation is the solution to the following maximization problem:

2

$$\alpha^{¿} \equiv \underset{\alpha \in C}{argmax} \{S(\alpha)\}$$

In general, it is not possible to determine the analytical solution to this problem (cfr Meucci, 2009) but the Mean-Variance provides with two-step approximation to the general allocation optimization problem defined above. In order to solve explicitly the general allocation problem it is necessary to determine the functional dependence of the index of satisfaction on all the moments and the dependence of each moment on the allocation.

The index of satisfaction is a function defined on the infinite-dimensional space of the moments of the distribution of the investors' objectives ψ:

[2] These units are specific to the security: for example in the case of equities the units are shares, in the case of futures the units are contracts, etc.

3

$$S(\alpha) \approx H\left(E\{\psi_\alpha\}, CM_2\{\psi_\alpha\}, CM_3\{\psi_\alpha\}, \cdots\right).$$

Where CM_k denotes the central moment of order k of a univariate distribution.

Suppose that it is possible focusing on the two first moments only and neglect all the higher moments, than, for a suitable bivariate function $\widetilde{H}$:

4

$$S(\alpha) \approx \widetilde{H}\left(E\{\psi_\alpha\}, Var\{\psi_\alpha\}\right)$$

Since all the indices of satisfaction S (cfr Meucci, 2009) are consistent with weak stochastic dominance, for a given level of variance of the objective, higher expected values of the objective are always appreciated, no matter the functional expression of $\widetilde{H}$. Hence, if for each target value of variance of the investor's objective, its maximum possible expected value is pursued, the solution to the general allocation problem is captured for sure.

In other words, the optimal allocation$\alpha^{¿}$ that solves the maximization problem (i.e. the Mean-Variance approach pioneered by Markowitz) must belong to the one-parameter family $\alpha(v)$ defined as follows:

5

$$\alpha(v) \equiv \underset{\substack{\alpha \in C \\ Var\{\psi_\alpha\} = v}}{argmax}\, E\{\psi_\alpha\}$$

Where v is the target variance, $v \geq 0$. Its solution is called the Mean-Variance efficient frontier. As a result, the general problem is reduced to a two-step recipe:

I. The computation of the mean-variance efficient frontier,
II. The following one-dimensional search to define the optimal allocation:

6

$$\alpha^{¿} \equiv \alpha(v^{¿}) \equiv \underset{v \geq 0}{argmax}\{S(\alpha(v))\}$$

According to the investor's objective ψ_α, the target variance v in the MV optimization can be interpreted as the riskiness of the solution $\alpha(v)$: for a given level of risk v, the investor seeks the allocation that maximizes the expected value of his objective. As the risk level v spans all the positive numbers, the one-parameter family of solutions $\alpha(v)$ describes a one-dimensional curve in the *N*-dimensional space of all possible allocations, and the optimal allocation $\alpha^{¿}$ must lie on this curve.

1.1.1. The set up

The investor's objective that appears in the MV problem is a linear function of the allocation and of the market vector (the set of stochastic returns):

7

$$\psi_\alpha \equiv \alpha^{'} M$$

The market vector M is a simple invertible affine transformation of the market returns at the investment horizon. If the investor focuses on the final wealth, as in the MV approach, the market vector reads:

$$M \equiv P_{T+\tau}$$

Within the MV framework prices are considered normally distributed, hence:

$$P_{T+\tau} \quad N(\mu, \Sigma)$$

Thus the objective (equation 7) is normally distributed:

$$\psi_\alpha \quad N(\mu_\alpha, \sigma_\alpha^2)$$

where,

$$\mu_\alpha = \mu^{'} \alpha, \sigma_\alpha^2 = \alpha^{'} \Sigma \alpha$$

The expected value and the covariance are respectively defined by:

8

$$E\{\psi_\alpha\} = \alpha^{'} E\{M\}$$

9

$$var\{\psi_\alpha\} = \alpha^{'} cov\{M\} \alpha$$

Hence, the MV efficient frontier can be re-expressed in the following form:

$$\alpha(v) \equiv \underset{\substack{\alpha \in C \\ \alpha' cov(M) \alpha = v}}{argmax} \; \alpha' E(M) \tag{10}$$

where $v \geq 0$.

In addition to no short selling constraints and the other subjective set of constraints C, the only inputs required to compute the MV efficient frontier are the expected values of the market vector $E(M)$ and the respective covariance matrix $cov(M)$. The mean-variance approach is often presented and solved in terms of the returns instead of the market vector. Expressing an allocation in terms of relative weights is more intuitive than expressing it in absolute terms. This comes at a cost. To present the formulation in terms of returns it is necessary to make two restrictive assumptions:

I. the investor's objective is final wealth, or equivalently that the market vector is represented by the prices of the securities at the investment horizon:

$$\psi_\alpha \equiv \alpha' P_{T+\tau} \tag{11}$$

II. the investor's initial capital is not null:

$$w_T \equiv \alpha' p_T \neq 0 \tag{12}$$

The linear return from the investment date T to the investment horizon τ of a security/portfolio that at time t trades at the price P_t is defined as follows:

$$L \equiv \frac{P_{T+\tau}}{P_T} - 1 \tag{13}$$

Now, considering the linear return on wealth:

14

$$L^{\psi_\alpha} \equiv \frac{\psi_\alpha}{w_T} - 1$$

it can be shown[3] that under the assumption I and II, the MV efficient frontier can be expressed equivalently in terms of the linear return on wealth as follows:

15

$$\alpha(v) \equiv \underset{\substack{\alpha \in C \\ var\{L^{\psi_\alpha}\} = v}}{argmax}\, E\{L^{\psi_\alpha}\}$$

where $v \geq 0$.

Considering now the relative weights w of a generic allocation:

16

$$w \equiv \frac{diag(p_T)}{\alpha' p_T} \alpha$$

Since the current prices p_T are known, the relative weights w are a scale-independent equivalent representation of the allocation α. Hence, the linear return on wealth can be expressed in terms of the linear returns of the securities in the market and the respective relative weights:

17

$$L^{\psi_\alpha} = w' L$$

As a result of the above properties for the derivation of the expected and the covariance matrix, it is possible to write the relative weights as follows:

18

$$w(v) \equiv \underset{\substack{w \in C \\ w' cov\{L\} w = v}}{argmax}\; w' E\{L\}$$

where $v \geq 0$.

[3] See Appendix 6.6, Meucci (2009)

Then, the efficient frontier expressed in terms of the allocation vector $\alpha(v)$ can be derived from equation 18 by inverting equation 16. The inputs necessary to solve equation 18 are the expected value of the horizon specific linear returns $E\{L\}$ and the respective covariance matrix $cov\{L\}$. Below the steps to compute these inputs necessary to solve equation 18

I. Detect the invariants $X_{t,\tilde{\tau}}$ behind the market relative to a suitable estimation horizon $\tilde{\tau}$.

II. Estimate the distribution of the invariants $X_{t,\tilde{\tau}}$

III. Project these invariants $X_{t,\tilde{\tau}}$ to the investment horizon, obtaining the distribution of $X_{T+\tau,\tilde{\tau}}$

IV. Map the distribution of the invariants $X_{T+\tau,\tilde{\tau}}$ into the distributionof the prices at the investment horizon of the securities $P_{T+\tau}$

V. Compute the expected value $\mathrm{E}\{P_{T+\tau}\}$ and the covariance matrix $cov\{P_{T+\tau}\}$ of the distribution of the market prices

VI. Compute the inputs for the optimization of equation 10 i.e. the expected value and the covariance matrix of the linear returns, from equation 13 using the affine equivariance of the expected value and of the covariance matrix respectively:

$$E\{L\} = diag(p_T)^{-1} E\{P_{T+\tau}\} - 1 \tag{19}$$

$$cov\{L\} = diag(p_T)^{-1} cov\{P_{T+\tau}\} diag(p_T)^{-1} \tag{20}$$

1.2. Structural pitfalls of classical Mean-Variance approach and possible alternatives

By virtue of what was stated in the previous section, it is not difficult to identify the reasons that motivate the popularity of the Mean-Variance theory. First, the paradigm proposed by Markowitz incorporates two crucial aspects for every asset manager: the contribution of diversification and the trade-off existing between risk and expected return.

Despite the incredible contribution that the model has made, both academics and practitioners have raised several objections to its efficiency as the appropriate framework for defining portfolio optimality. Except for a first period, in which economic literature ignored this methodology, from the mid-1960s the works dedicated to the confutation of the Mean-Variance approach have exponentially increased, fueling real debates between critics and proponents of Markowitz's theory. Criticisms and perplexities are mainly concerned to investors behaviours and how they are modelled within the optimization process:

I. the optimization based on the mean-variance principle ignores any preference toward statistical moments of order higher than the second,
II. Mean-Variance efficiency is not strictly consistent with expected utility maximization.
III. the model assumes that investors have a single time horizon,
IV. even in a two-dimensional context, the standard deviation (variance) is an overly simplified measure of risk, unable to discriminate between rewarding and penalizing phenomena.

1.2.1. Mean-Variance approximates investors' satisfaction

Recall from equation 3 that the investor's satisfaction depends on all the moments of the distribution of investor's objective. The mean-variance approach relies on the approximation, according to which, the investor's satisfaction is determined by the first two moments of the distribution of his objective (Equation 4). The "plain" Mean-Variance optimization ignores any preference toward statistical moments of a higher order than the second. This contributed to embracing the inclusion within the investor preference function of higher-order moments: the Skewness (i.e. the third moment) and the Kurtosis (i.e. the fourth moment). Among the numerous works addressing this issue it is worth mentioning Samuelson (1958), Arditti and Levy (1975), Lee (1977), Kraus and Litzemberger (1976), Kane (1982), Lai (1991), Konno, Shirakawa and Yamazaki (1993), Simaan (1993), Konno and Suzuki (1995), Chunhachinda, Dandapani et al. (1997), de Athayde and Flores (2004), Pomante (2008), Meucci (2009). As rightly both Pomante (2008) and Meucci (2009) point out, it should be noted that in the case of inclusion of the third and fourth statistical moments, the portfolio optimization must be developed on a four-dimensional plane, with the consequent impossibility of graphically representing the combinations of efficient portfolios. Furthermore, like the standard deviation, neither the Skewness nor the Kurtosis can be estimated as a weighted average of the measures of the individual assets, as these are influenced by the joint movements of the assets. A portfolio optimization that includes statistical moments higher than the second hides some pitfalls that are not negligible: the inclusion of the third and/or fourth moment increases the probability that there is no vector of weights capable of maximizing the expected return, and portfolio asymmetry/extreme even exposure at the same time is quite high.

The second problem concerns the ability of an average investor to translate, with rationality, his degree of preference for indicators such as asymmetry and the kurtosis of returns. It is undeniable that the limited rationality of the typical investor, and its scarce familiarity with the moments of a distribution, make the identification of these parameters very complex, also due to the impossibility of attributing to them a meaning that goes beyond the mere mathematical value.

1.2.2. Mean-Variance inconsistency with expected utility maximization

Utility functions to define portfolio optimality often divide practitioners from academics. MV efficient portfolios are often good approximations of maximum expected utility[4] and a practical framework for portfolio optimization. (Kroll, Levy, & Markowitz 1984; Levy & Markowitz, 1979; Markowitz, 1987, chapter 3).

Markowitz's efficiency is strictly consistent with expected utility maximization only under either of two conditions: normally distributed asset returns or quadratic utility, but the normal distribution assumption is unacceptable as a realistic hypothesis. Returns are neither strictly normal nor log-normal. Returns are not normal due to limited liability. Returns are not log-normal due to the possibility of an adverse event like market crashes. Additionally, the limitations of quadratic utility as a representation of investors' behaviour are well known and unacceptable (Cfr. Meucci 2009). This is because a quadratic function

[4] Note that the best-approximating quadratic function is simply some two-moment approximation of maximum expected utility that is a function of utility parameters (Michaud, 2008)

is not monotone increasing as a function of wealth, from some point on, the expected quadratic utility declines as a function of increasing wealth. Quadratic utility functions are primarily useful as approximations of expected utility maximization in some regions of the wealth spectrum (Michaud, 2008).

The lack of specificity of the investor's utility function is another important limitation of the utility function approach. Even small errors in estimating utility function parameters can bring the asset manager to a quite inefficient asset allocation. Perhaps differing in the value of only one or two parameters, yet represent a very wide, even contradictory, spectrum of risk-bearing and investment behaviour (Rubinstein, 1973). Note that the best-approximating quadratic function is simply some two-moment approximation of maximum expected utility that is a function of utility parameters.

1.2.3. Single time horizon

Modern Portfolio Theory framework, which is principally based on the mean-variance model, is fundamentally single-period in nature where the investor makes his portfolio decision at the beginning of a period and then waits until the end of the period when the rate of return of his portfolio materializes. He cannot make any intermediate changes to the composition of his portfolio (Mossin, 1968; Grauer and Hakansson, 1982). The model neglects the fact that the perception that investors have of the goodness of an investment is path-dependent since investors, discounting the effects of an early divestment, might also attribute value to the evolution over different timeframes of the principal itself. Starting from the example proposed by Mossin (1968), Pomante (2008) proposed a simplified case of a multiperiod model in which each investment decision implemented in the single sub-period depends on what has been achieved in previous periods and on the opportunities relating to future sub-periods only at the beginning of the last subperiod the construction of the portfolio can be set up according to the one-period Markowitz logic. A similar work has been performed also by Michaud and Monahan (1981).

An intuitive alternative is to consider the multiperiod distribution of the geometric mean of return instead of the simple average (Crf. Michaud, 1981, 2003, 2008) since Mean-Variance geometric mean investment objectives are often consistent with many institutional investment mandates. The geometric mean, or compound, return is the statistic of choice for summarizing portfolio return over multiple periods. In fact, assuming that an investor experiences a 100% return in one period and a 50% negative return in the next period. The two-period average return is 25%, but the two-period wealth is the same as at the beginning. Therefore, the true multiperiod return is 0%. Basing his thoughts on Hakansson (1971), Michaud (2008) stated that the mean and variance of N-period geometric mean return is a natural N-period generalization of Markowitz efficiency. Various approximations show that portfolios on the single-period MV efficient frontier are often good approximations of N-period geometric mean efficient portfolios. Consequently, N-period geometric mean MV efficiency is roughly a special case of MV efficiency in many cases of practical interest.

1.2.4. The standard deviation is an over-simplified measure of risk

In addition to the problem relating to the one-period investment decision framework and the two-dimensional preference space, there is a literary strand that considers the use of the standard deviation an over-simplified measure of risk, as investors, rather than being averse to the mere average distance from the mean, they should discriminate between positive volatility (favourable, with respect to a given threshold value) and negative volatility (adverse, with respect to a given threshold value) and therefore consider only the so-called lower partial moments. One obvious and intuitively appealing example of a "downside" risk measure discussed as early as Markowitz (1959), is the semi-variance or semi-standard deviation of return.

To satisfy an explicit request from asset managers and investors, who are increasingly comfortable with downside and extreme measures, such as the percentile, the Value at Risk, and the probability of shortfall, and are more averse to the manifestation of returns placed in the left-hand portion of the distribution is also expressed there are alternative measures of risk such as the VaR, the Conditional VaR.

Telser (1956) and Kataoka (1963), Harlow (1991), Nawrocki (1991), Feiring, Wong et al. (1994) implemented an optimization model aimed at minimizing downside risk measures, while Basak and Shapiro (2001), Cuoco, He and Issaenko (2001) and Alexander and Baptista (2002) adopted a maximization of expected utility subject to maintaining a critical level of Value-at-Risk (VaR). The pros and cons of various risk measures depend on the nature of the return distribution. In the event that the returns are distributed normally, the use of the full moment indicator or the partial risk indicator leads to the same result; this is because, under the Gaussian distribution assumption, the portfolios that minimize the variance are the same ones that minimize a partial moment. Hence, adopting a partial risk measure focused only on the left portion of the distribution would only make sense if the distribution of returns were not normal and therefore skewness and kurtosis phenomena appeared. (Cfr. Pomante Ugo, 2008). From this, it follows that, in the hypothesis in which the Mean-Variance maximization is used in the presence of investors exclusively adverse to the downside, the portfolios thus selected could be sub-optimal, since the portfolios capable of minimizing the variance could not be the same ones that minimize the partial third or fourth moment. It must also be said that the return distribution of an asset or portfolio depends on several factors: like the time horizon. On the one hand, the returns of diversified equity portfolios, equity indexes, and other assets are often approximately symmetric over periods of institutional interest, hence efficiency based on non-variance risk measures may be nearly equivalent to MV efficiency. On the other hand, the return distribution of diversified equity portfolios becomes increasingly asymmetric over long time horizons. Furthermore, some securities, such as options, swaps, hedge funds, and private equity, have return distributions that are unlikely to be symmetric. The return distributions of fixed-income and real estate indices are generally less symmetric than equity indices (Michaud, 2008).

R. Michaud in his book 2008, said that comparing the MV efficient frontier with a mean-semi-variance efficient frontier based on the same historic data[5] results in two virtually identical efficient frontiers. There is a small mismatch in the middle reflecting the fact that some equity indices have asymmetrically less downside risk.

1.2.5. Difficulty in estimating the inputs

The critical elements analyzed so far focus on the basic assumptions of the model, however, it is possible to identify a further source of perplexity related to the inability of

[5] Michaud used eight different asset class: Euro Bonds, US Bonds, Canada Equity, France Equity, Germany Equity, Japan Equity, UK Equity, US Equity.

the Mean-Variance framework to adapt to the operating practices of the subjects involved in the asset allocation process. Markowitz's optimization doesn't make any assumptions about the composition of efficient portfolios. The problem of restrictions on the composition of portfolios, which is quite an issue from an operational standpoint[6], has been tackled by many authors. Among them Sharpe (1974), Fisher (1975), Rosemberg and Rudd (1978, 1979), Dybvig (1984) Black and Litterman (1992), Michaud (1998).

From an operational point of view all the literature converges defining:

I. instability of the final output, and
II. the ambiguity of the final output,

as three of the most important and limitations of MV optimization. Small changes in input assumptions often imply large changes in the optimized portfolio and could suggest extreme and/or non-intuitive weights for some of the assets in the portfolio (Jorion, 1985 Black & Litterman,1990,1991,1992, Green & Hollifield, 1992, Best e Grauer (1991).

In many practical applications, an equally weighted portfolio may be substantially closer to the "true" Mean-Variance optimality than an optimized portfolio. Meaning: in many cases, equally-weighted portfolios outperform mean-variance portfolios (DeMiguel, Garlappi, & Uppal, 2009; Jobson & Korkie, 1981; Jorion, 1985). Moreover, the overuses of statistically estimated information exaggerate the impact of estimation errors. Estimation errors in the forecasts significantly impact the resulting portfolio weights. As a result, MV-optimized portfolios are likely to be "error maximized" (Brandt 1995, Michaud, 1998) and, often, have little, if any, reliable investment value.

Markowitz's Mean-Variance optimization provides the right way to invest given that the risk-return estimates are correct. Both instability and ambiguity of the optimized asset allocation can be therefore connected to estimation errors.

In recognition of these concerns, the original approach proposed by Markowitz only serves as a starting point and the classical mean-variance framework is often extended in several different directions for portfolio management in practice indeed.

[6] Because beyond their personal preferences, asset managers must also be compliant with regulatory, corporate, and investment policy constraints

2. Alternative approaches for portfolio selection

The previous section summarizes the major weaknesses of the mean-variance model, noting that the most illustrated shortcoming in implementing the model is the sensitivity of the portfolio constructed to changes in inputs. Perhaps the simplest and most immediate of the heuristic techniques is the equally weighted strategy. This strategy is characterized by high diversification. Including all the investable universe in the portfolio, ensures the agent has selected those assets that would be chosen if he would have had the correct input parameters. However, the use of this strategy implies an a priori renunciation of selecting the construction of a truly effective solution and that, in fact, the outcome of the strategy is left to luck, not to the skill of the asset manager. Meucci (2005) made a point here:

"Just like the hands of a broken watch, which happen to correctly indicate the time only twice a day, the prior allocation is only good in those markets, if any, where the optimal allocation happens to be close to the prior allocation"

Imposing weight constraints within quadratic programming to overcome estimation error is another of the most widely used Heuristic Techniques (Frost and Savarino, 1988, Jagannathan and Ma, 2003). The effect that the imposition of constraints produces on the composition of efficient portfolios is intuitive: by restricting the numerical range within which the weights of the asset classes can vary, the portfolios obtained are more diversified than those that would be obtained with the unconstrained[7] optimizations.

Portfolio managers are required to mitigate these types of problems by engineering techniques capable of re-proposing the original model more consistently with the real operational context. Practitioners, as well as academic financial literature, allocated a considerable amount of resources to the estimation error issue, developing various exotic statistical and investment proposals for improving optimization inputs. Starting from a classic Mean-Variance optimization, there are two paths (not necessarily alternatives) that can be followed in order to minimize the distance that separates the risk-return

[7] The unconstrained expression identifies the optimizations that have the sole constraint of non-negativity of the weights.

combinations of true efficient portfolios compared to the combinations of portfolios considered erroneously optimal (Pomante, 2008):

I. act on optimization, trying to get closer to the composition of truly efficient portfolios (Resampled Efficiency).
II. operate on estimated parameters, reducing the error (Bayesian techniques);

II.1. Michaud Resampled Efficiency

Michaud Resampled Efficiency [RE] technology introduces Monte Carlo resampling and bootstrapping methods into MV optimization to reflect the uncertainty in investment information more realistically.

Consider an allocation α. The market prices $P^{\theta}_{T+\tau}$ and the allocation α determine the investor's objective ψ [8], which in turn determines the investor's satisfaction S:

$$\left(\alpha, P^{\theta}_{T+\tau}\right) \longmapsto \psi^{\theta}_{\alpha} \longmapsto S_{\theta}(\alpha) \tag{21}$$

A chain similar to equation 21 holds for the investor's constraints ensuing from the investor's multiple secondary objectives[9]:

$$\left(\alpha, P^{\theta}_{T+\tau}\right) \longmapsto \widetilde{\psi}^{\theta}_{\alpha} \longmapsto \widetilde{S}_{\theta}(\alpha) \longmapsto C_{\theta} \tag{22}$$

Hence, the optimal allocation function that, for each value of the input parameters θ, maximizes the investor's satisfaction given his investment constraints is the following:

$$\alpha(\theta) \equiv \underset{\alpha \in C_{\theta}}{argmax} \left\{ S_{\theta}(\alpha) \right\} \tag{23}$$

---

[8] Notice that, in all its specifications, the objective is a linear function of the allocation and of a market vector: $\psi_{\alpha} = \alpha' M$ and the distribution of M can be easily computed from the distribution of the security prices $P_{T+\tau}$ at the investment horizon and vice versa (cfr Meucci, 2009, appendix 2.4).

[9] When an index of the satisfaction is estimable, the satisfaction associated with the allocation α is a function of any of the equivalent representations of the distribution of the objective ψ_{α}(cfr Meucci, 2009, sez 5.3)

Since the true value θ^t of the market parameters is not known, the truly optimal allocation cannot be implemented. Furthermore, as already discussed, the optimal allocation function is extremely sensitive to the input parameters θ: a slightly wrong input can give rise to a very large opportunity cost.

Unlike the Black-Litterman approach (see section 3.3), where the estimation error problem is addressed by smoothing the estimate of the input parameters before the optimization, the resampling technique tackles the above issue by averaging the outputs of a set of optimizations.

The assumptions of the original resampling recipe are the following:

I. the agent's objective is based on the Mean-variance formulations in terms of linear returns and relative weights,

II. the market consists of equity-like securities for which the linear returns are market invariants,

III. the investment horizon and the estimation interval coincide,

IV. the investment constraints are such that the dual formulation is correct,

V. the constraints do not depend on unknown market parameters.

Under the above assumptions the Mean-Variance problem expressed by Equation 18 can be written in its dual formulation as follows:

$$w^{(i)} \equiv \underset{\substack{w \in C \\ w'\mu \geq e^{(i)}}}{argmin}\, w' \Sigma w, i=1,\ldots,I \tag{24}$$

Where:

μ: vector of expected linear returns of the assets

Σ: matrix of the covariances of linear returns of the assets

$\{e^{(1)},\ldots,e^{(I)}\}$: the significative grid of target expected values

C: the set of investment constraints

$i=1,\ldots,I$ represents the number of portfolios composing the efficient frontier.

The resampled efficiency engineered by Michaud (1998) defines the efficient portfolio weights through the following steps:

I. Estimate the inputs ${}_0\hat{\mu}$, and ${}_0\hat{\Sigma}$ of the mean-variance framework from the analysis of the observed time series i_T of the past linear returns[10]:

$$i_T \equiv \{I_1, \ldots, I_T\}$$

II. Consider the time series i_T as the realization of a set of market invariants, i.e. independent and identically distributed returns:

$$I_T \equiv \{L_1, \ldots, L_T\}$$

III. Make assumptions on the distribution generating the returns, for instance assuming normality, and set the estimated parameters (sample mean) as the true parameters (population mean) that determine the distribution of the returns:

$$L_t \quad N\left({}_0\hat{\mu}, {}_0\hat{\Sigma}\right)$$

IV. Resample a large number Q of Monte Carlo scenarios of realizations of the returns [step II] from the adopted distribution [step III]:

$${}_q i_T \equiv \{{}_q I_1, \ldots, {}_q I_T\}, q = 1, \ldots, Q$$

V. Estimate the inputs ${}_q\hat{\mu}$ and ${}_q\hat{\Sigma}$ of the mean-variance framework from the resampled time series [step IV]: as in Step I.

VI. Compute the global minimum-variance portfolio from each of the resampled inputs:

$${}_q w_{MV} = \underset{w \in C}{argmin}\, w' {}_q\hat{\Sigma} w, q = 1, \ldots, Q$$

VII. Compute the respective estimated expected value in each scenario:

$${}_q \underline{e} \equiv {}_q w_{MV} \, {}_q\hat{\mu}, q = 1, \ldots, Q$$

VIII. Compute the maximum estimated expected value in each scenario:

---

[10] By means of additional constraints it is possible to include the investor's multiple objectives in the allocation problem. Indeed, the multiple objectives are accounted for by imposing that the respective index of satisfaction $\tilde{S}$ exceed a minimum acceptable threshold $\tilde{s}$ (cfr Meucci, 2009, sez 6.1).

$$\overline{{}_q e} \equiv max\left\{ {}_q\hat{\mu}' \delta^{(1)}, \ldots, {}_q\hat{\mu}' \delta^{(N)} \right\}, q=1,\ldots,Q$$

where δ is the canonical basis[11]

IX. For each scenario q determine a grid $\left\{ {}_q e^{(1)}, \ldots, {}_q e^{(I)} \right\}$ of equally-spaced target expected values

$$\begin{bmatrix} {}_q e^{(1)} \\ \vdots \\ {}_q e^{(i)} \\ \vdots \\ {}_q e^{(I)} \end{bmatrix} \equiv \begin{bmatrix} {}_q\underline{e} \\ \vdots \\ {}_q\underline{e} + \frac{{}_q\overline{e} - {}_q\underline{e}}{I-1}(i-1) \\ \vdots \\ {}_q\overline{e} \end{bmatrix}$$

X. Solve the mean-variance dual problem [Equation 25] for all the Monte Carlo scenarios $q=1,\ldots,Q$ and all the target expected values $i=1,\ldots,I$:

25

$${}_q w^{(i)} \equiv \underset{\substack{w \in C \\ w' {}_q\hat{\mu} \geq {}_q e^{(i)}}}{argmin}\, w' {}_q\hat{\Sigma} w$$

XI. Define the resampled e□cient frontier as the average of the above allocations, possibly rejecting some outliers:

$$w_{\Re}^{(i)} \equiv \frac{1}{Q} \sum_{q=1}^{Q} {}_q w^{(i)}, i=1,\ldots,I$$

where "RE" stands for "resampled"

XII. Compute the efficient allocations from the respective relative weights:

$$\alpha_{\Re}^{(i)} \equiv w_T\, diag\left(p_T\right)^{-1} w_{\Re}^{(i)}, i=1,\ldots,I$$

where w_T, adopting the same notation of the previous section is the initial budget.

XIII. Choose the optimal allocation according to the preferences.

[11] This can be done by means of the sample estimators for example

This model has been the subject of debate (Scherer 2002) because even though portfolio resampling is a thoughtful heuristic, some features make it difficult to interpret by the inexperienced. On the one hand, by factoring in estimation error in a Bayesian-like framework, RE optimization tries to avoid unreliable and self-defeating principles of design and management that follow from in-sample parameter certainty MV portfolio optimization analytics. On the other, unlike the Bayesian and the Black-Litterman approaches, where the estimation error is tackled by smoothing the estimate of the input parameters before the optimization, the resampling technique averages the outputs of a set of optimizations.

The resampling technique is very innovative. It displays several advantages but also a few drawbacks such as reducing the sensitivity to the market parameters. Still Markowitz and Usmen (2003) expressed concern that RE optimization illustrated in Michaud (1998) may require revision of expected utility axioms. Michaud (2003) clarified the issue (the misunderstanding) demonstrating that RE optimization is fundamentally based on expected utility considerations. Note that resampling deals with sampling error only. In theory, sampling error in means that arises from not having enough data can be cured by lengthening the observation period (in the case of variance, increasing the frequency of observations would help). Because the involved distributions are likely to be nonstationary, however (i.e., the mean and covariance tend to vary over time), enlarging the data set in this way is not always appropriate. Scherer (2002) dealt with this trade-off.

Kohli (2005) conducts an empirical study based on stock market data from 2011 to 2013 and find “no conclusive advantage or disadvantage of using resampling as a technique to obtaining better returns,” but “resampled portfolios do seem to offer higher stability and lower transaction costs.” Scherer (2006) runs a new Monte Carlo simulation, which reveals that the James–Stein shrinkage estimator outperforms portfolio resampling.

Frahm (2013), by contrast, tried to justify the application of Resampled EfficiencyTM and proved that portfolio resampling has a strong foundation in the classic theory of rational behaviour. Every noise trader could do better by applying the Michaud procedure. By contrast, a signal trader who has enough prediction power and risk-management skills should refrain from portfolio resampling. The crucial point is that in most simulation studies, investors are considered noise traders. This explains why portfolio resampling performs well in simulation studies but could be mediocre in real life.

2.1.1. Peculiarities of the model: The Resampled Efficient Frontier™ Maximum Return Point

The RE technique is characterized by a peculiarity that is a source of perplexity: by increasing the risk of a specific asset class and leaving the remaining parameters unchanged, the weight of the latter is bound to increase (see Scherer, 2004 and Pomante, 2008). It is disheartening for an asset manager to know that the increase in the standard deviation of the returns of an asset class, a "symptom" of a deterioration in quality, is received by resampling with an increase in the weight of the market itself. However, this limit is destined to disappear if applied to optimizations not subject to the hypothesis of the impossibility of carrying out short selling since in the absence of short selling, the greater weight that this market assumes in the positive projections (highly positive simulated returns) is not offset by the negative weights assumed in the unfavourable (highly negative simulated returns) simulations (Pomante, 2008).

Furthermore, the RE technique tends to violate one of the fundamental rules of mean-variance optimization: the resampled frontier may have a negative slope, thus negating the investor risk aversion principle.

A statistical approach to portfolio optimality leads to some significant differences from classical MV optimization. One significant difference is that the Resampled Frontier [RF] curve may have, at some point, a negative first derivative. This means that the maximum risk portfolio on the RF may not be also the maximum return portfolio. Michaud (1998) distinguished the two points on the frontier calling this the latter “maximum return point” [MRP] of the RF.

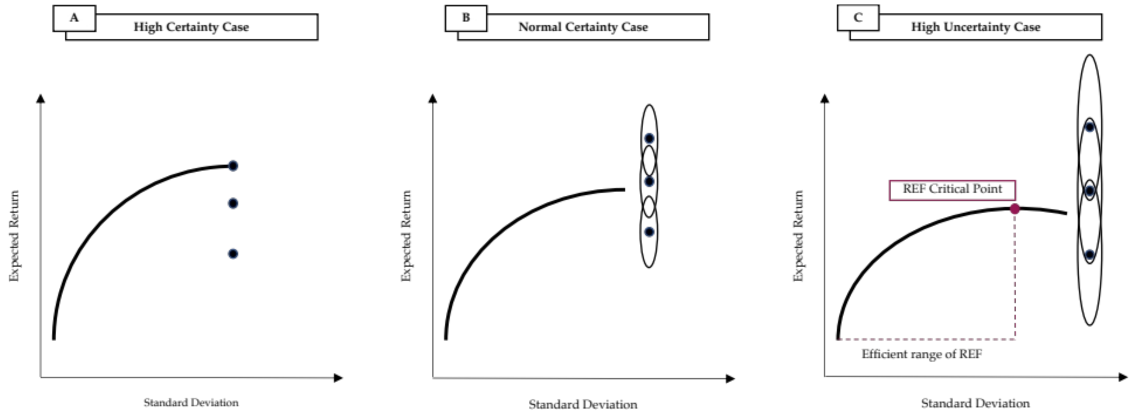

Figure 1: Certainty Levels and the REF Maximum Return Point (from Michaud, 2008a)

The possible existence of an MRP is a quite important and very useful property of RF portfolio optimality. Figure[12] 4 illustrates how the RF MRP may arise. In each of the above panel (A,B, and C) there are the plots of the efficient frontier and three high risk assets (top-right of the plots). The uncertainty of information is indicated by the elliptical confidence area around each of them. Panel "A" reflects the Markowitz case, where risk and return are point estimates and it is assumed high certainty of information. Here, the maximum return portfolio includes only the maximum return asset. Panel "B" exhibits a moderate amount of uncertainty in the return distribution of the three assets. Since the true maximum return asset can no longer be known with certainty, the Resampled Efficient Frontier [REF] maximum return portfolio includes a significant amount of the middle asset (although not as much as the top asset) and has a lower expected return than the Markowitz maximum return portfolio (Panel "A"). Panel "C" depicts a high level of uncertainty in the expected return estimates. High uncertainty of information implies small statistical distinction between the three assets and the RF includes significant allocations in all three assets. In this case, an MRP may emerge where the RF has a downward-sloping inefficient

[12] A basis for a vector space is a set of linearly independent elements of that space that can generate all the other vectors by means of linear combinations. The number of these elements is the dimension of that vector space. In the case of the Euclidean space R^N , this number is N. Therefore, a basis is a set of vectors $e^{(n)}, n=1,\ldots,N$, such that, for suitable scalars, any vector v of R^N can be expressed as a linear combination: $v=\sum_{n=1}^{N} w_n e^{(n)}$. The canonical basis is the following set of vectors (cfr Meucci, 2009):

$$\begin{bmatrix} \delta^{(1)} \\ \vdots \\ \delta^{(N)} \end{bmatrix} \equiv \begin{bmatrix} (1,0,\ldots,0)' \\ \vdots \\ (0,0,\ldots,1)' \end{bmatrix}$$

segment. Any risk beyond the MRP is not optimal indeed and is not, by definition, part of the REF.

II.2. Bayesian estimation

Bayesian approach has its roots in works of Zellner and Chetty (1965), Mao and Sarndal (1966), Kalymon (1971), Barry (1974), Barry and Winkler (1975, 1976), Klein and Bawa (1976, 1977), Brown (1978), Bawa, Brown and Klein (1979). Acting directly on the input parameters could alleviate many of the unpleasant inconveniences involved in a la Markowitz portfolio optimization. Bayesian estimation in a simulation study often reflects a superior level of risk-return estimation than can be achieved in investment practice (cfr. Robert, 1994). Markowitz and Usmen (2003) addressed the issue of the relative importance of Bayesian estimation versus RE optimization.

Bayesian statistics offers an operational advantage of no small importance: the final estimators can be directly influenced by asset managers' market views, thus favouring the construction of inputs and therefore portfolios which are much more consistent with the expectations of the asset managers.

Indeed, the Bayesian estimation process allows the portfolio manager to figure out the posterior distribution of the market parameters. This distribution explicitly acknowledges that an estimate cannot be a single number. Furthermore, the posterior distribution includes within a sound statistical framework both the investor's experience (the prior knowledge), and the information from the market. Bayesian allocations rely on Bayes-Stein shrinkage estimators of the market parameters, providing a mechanism that mixes the positive features of the prior allocation and the sample-based allocation. The estimate of the market is shrunk (indirectly, through the market parameters) towards the investor's prior in a self-adjusting way and the overall opportunity cost is reduced.

The "classical" estimator is a function that processes current information i_T and outputs an S-dimensional vector $\hat{\theta}$. Information consists of a time series of T past observations of the market invariants[13]:

---

[13] This figure is a.

$$i_T \equiv \{x_1, \cdots, x_T\}$$

The output $\hat{\theta}$ is a number which is supposed to be close to the true, unknown parameter θ^t:

$$i_T \longmapsto \hat{\theta}$$

The Bayesian estimation is different both in terms of "input" and "output". First of all, in a Bayesian context, an estimator does not yield a number $\hat{\theta}$**.** Instead, it yields a random variable θ, which can take values within a given range Θ. The distribution of θ is called the posterior distribution. It can be represented for instance in terms of its probability density function $f_{po}(\theta)$. The true, unknown parameter θ^t is assumed to be hidden most likely in the proximity of those values where the posterior distribution is more peaked. The possibility that θ^t might lie in some other region of the range Θ is acknowledged as well.

Secondly, in a Bayesian context, an estimator does not depend only on backwards-looking historical information i_T. Indeed, the investor typically has some prior knowledge of the unknown value θ^t based on his experience e_C, where C denotes the level of confidence in his experience:

$$i_T, e_C \longmapsto f_{po}(\theta)$$

The figure below, taken from Meucci, 2009, provides a graphical interpretation of Bayesian approach to parameter estimation.

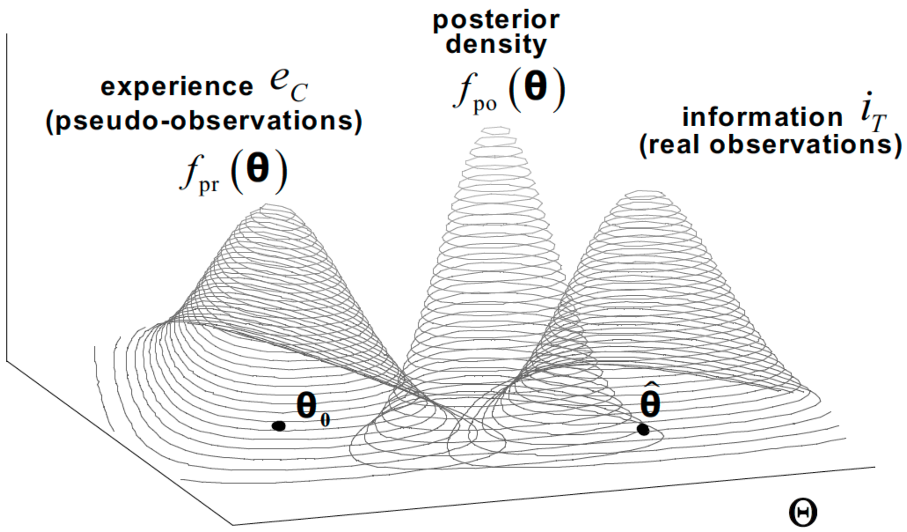

Figure 2: Bayesian approach to parameter estimation

On the one hand, the purely classical estimator based on historical information i_T gives rise to a distribution of the market parameters θ that reaches its peak around the classical parameter estimate $\hat{\theta}$. The larger the number of observations T in the time series, the higher the concentration of the historical distribution around the parameter $\hat{\theta}$.

On the other hand, the investor equates his personal experience e_C to a number C of pseudo-observations located in a "prior" value θ_0. These observations give rise to a distribution of the market parameters θ which is called the prior distribution, whose probability density function is denotet as $f_{po}(\theta)$. The larger the number of these pseudo-observations, the higher the investor's confidence in his own experience and thus the more concentrated the prior distribution around θ_0. Therefore, the Bayesian posterior provides a way to blend the above two distributions into a third distribution, i.e. a spectrum of values and respective probabilities for the parameters θ. In particular, when the confidence in the investor's experience is large, the posterior becomes peaked around the prior value θ_0. When the number of observations (the level of confidence on its prior knowledge) C in the time series is large, the posterior becomes peaked around the classical estimate $\hat{\theta}$.

The literature on Bayesian-style portfolio optimization models is extremely rich. Consequently, preferring to remain adherent to the arguments that are intended to be treated in this work, it is considered appropriate to provide an overview of the literature related to the Black-Litterman model.

II.3. Black-Litterman allocation

Fischer Black and Robert Litterman revolutionize portfolio management in 1990 with the creation of the Black-Litterman Global Asset Allocation Model. The model has been quickly adopted for optimal portfolio allocation across international equity, fixed income and currency markets. One of these was the Black-Litterman Global Asset Allocation Model.

While the more famous Black-Scholes Option Pricing Model was developed before Fischer Black's move to Goldman Sachs's Trading and Arbitrage Division, innovations such as the Black-Litterman Global Asset Allocation model and the Black-Derman-Toy (a model for pricing fixed income derivatives) were developed while he worked at the firm.

In 1986 Fischer Black brought Robert Litterman, a PhD in economics from the University of Minnesota, onto the Goldman Sachs Fixed Income Research team. Three years later, the two were tasked with creating an asset allocation model to help clients diversify their global bond portfolios. The goal was to create a quantitative and disciplined approach to structuring international bond portfolios in a manner consistent with the portfolio manager's unique view of markets. The model could also be used reversely engineered, evaluating an existing portfolio to quantify the investor's implied viewpoint on various markets as reflected in current holdings. The user of the model needed only to determine where his or her assumptions regarding returns diverged from the market view and the degree of confidence in these contrarian assumptions . With this input, Black-Litterman would calculate the appropriate asset allocation.

Developed in 1990, the Black-Litterman model became a prized asset of the recently formed Goldman Sachs Asset Management (GSAM) Division, established in 1988. Fischer Black himself moved to GSAM in 1990 and the model, initially used exclusively for bonds, was extended to equities in 1991.

The Black-Litterman model was published in the Journal of Fixed Income in September 1991. In the article, Black and Litterman outlined the value of their model: saying that this new asset allocation approach lets investors derive portfolios that appear balanced and reflect their views, without resorting to arbitrary constraints on portfolio composition. The model also lets portfolio managers incorporate their market views in a manner that approximates the way they actually think about their outlook. Further studies about the Black-Litterman approach have been made by He and Litterman (1999, 2002) providing more detail on the workings of the model, but not quite a complete set of formulas.

Bevan and Winkelmann (1998) provide details on how they use Black-Litterman as part of their broader Asset Allocation process at Goldman Sachs, including some calibrations of the model which they perform. This is useful information for anybody planning on building Black-Litterman into an ongoing asset allocation process. Drobetz (2001) provides further description of the Black-Litterman model including a good discussion of how to interpret the confidence in the estimates. Fusai and Meucci (2003) introduced yet another non-Bayesian variant of the model which removed the parameter τ altogether. Meucci (2005) followed up on this paper and coined the phrase, "Beyond Black-Litterman".

Meucci (2006) provides a method to use non-normal views in Black-Litterman. Meucci (2008) extends this method to any model parameter and allows for both analysis of the full distribution as well as scenario analysis.

Sahamkhadam et al (2020) extended the Black-Litterman approach to incorporate tail dependency in portfolio optimization and estimate the posterior joint distribution of returns using vine copulas.

II.3.1. The Black-Litterman set-up

The Black-Litterman optimization makes use of Bayes' rule directly shrinking the market towards the investor's prior views[14].

Consider the optimal allocation function defined by Equation 23 that, for each value of the input parameters θ, maximizes the investor's satisfaction given his investment constraints:

$$\alpha(\theta) \equiv \underset{\alpha \in C_\theta}{argmax} \{S_\theta(\alpha)\}$$

As stated before, since the true value θ^t of the market parameters is not known, the truly optimal allocation cannot be implemented. Furthermore, the allocation function is

[14] The invariants are random variables that refer to a specific estimation-horizon $\tilde{\tau}$ and are independent and identically distributed (i.i.d.) across time (cfr Meucci, 2009, sez 4).

extremely sensitive to the input parameters θ. In the general case the market can be described by a generic distribution and the investor can express views on any function of the market.

Consider a market represented by the multivariate random variable X. This could be the set of any variable that directly or indirectly fully determines the market (market invariants, market prices, asset returns…). Assume that it is possible to determine the distribution of this random variable, as represented for instance by the probability density function f_X, by means of a reliable model/estimation technique (e.g.: general equilibrium arguments, nonparametric estimators, or maximum likelihood estimators).

To smooth out the estimation error risk that affects f_X, the portfolio manager, according to his knowledge of the prior distribution "tweaks" the outcome expressing his view V as a conditional distribution $V \vee x$. He assesses that the outcome of the market is V, a random variable that, depending on the market, would be larger or smaller than the value X predicted by the "standard" model. The conditional distribution is modelled, as represented for instance by the probability density function $f_{V \vee x}$, according to the portfolio manager's level of confidence in his own view. His opinion might regard a specific area of the market, therefore the view refers to a generic multivariate function $g(X)$ on the market and, as a consequence, the conditional model for the view becomes $f_{V \vee g(X)}$.

Once the model has been set up, the portfolio manager will produce a specific number v: his prediction on V. At this point the distribution of the market conditioned on his opinion $X \vee v$ can be computed. The representation of this distribution in terms of its probability density function follows from Bayes' rule, that in this context is reframed as follows:

$$f_{X \vee v}(x \vee v) = \frac{f_{V \vee g(X)}(v \vee x) f_X(x)}{\int f_{V \vee g(X)}(v \vee x) f_X(x) dx} \tag{26}$$

Black and Litterman (1990, 1992) computed and discussed the analytical solution to equation 26 in a general case. Firstly, the "official" model for the *N*-dimensional market vector X was assumed to follow a normal distribution[15]:

[15] See, He and Litterman (2002) for an interpretation in terms of shrinkage of market parameters.

$$X \sim N(\mu, \Sigma) \quad (27)$$

Second, the investor's area of expertise is a linear function of the market:

$$g(X) \equiv P x \quad (28)$$

where P, called the "pick" matrix is a *KxN* matrix. Each of its K rows is an *N*-dimensional vector that corresponds to one view and defines the linear combination of the market involved in that view. The above identity (equation 29) is very "flexible", in the sense that the investor does not necessarily need to express views on all the market variables. In addition, the views do not necessarily need to be expressed in absolute terms for each market variable considered, as any linear combination of the market constitutes a potential view.

Third, the conditional distribution of the investor's views given the outcome of the market is assumed normal:

$$V \vee P x \sim N(P x, \Omega) \quad (29)$$

Where Ω is a matrix (which is symmetric and positive) that denotes the portfolio manager's confidence in his own opinion. Ω can be defined (Meucci, 2009) as follows:

$$\Omega \equiv \left(\frac{1}{c} - 1\right) P \Sigma P' \quad (30)$$

where c is a positive scalar.

This reflects an "empirical Bayesian" approach: the agent gives, relatively speaking, more leeway to those combinations that are more volatile according to the official market model (equation 27). The scalar c tweaks the absolute confidence in the investor's views:

XIV. If $c \longrightarrow 0$, the scatter matrix Ω is infinite, the portfolio manager's is not confident in his own at all and, therefore his views have no impact.

xv. If $c \longrightarrow 1$ on the contrary, the scatter matrix Ω is null, which means the portfolio manager's confidence in his own views is total.

Fourth, the portfolio manager expresses his opinion on his area of expertise. This will turn into a specific value v of the views V.

By means of Bayes' rule (equation 26) the distribution of the market, conditioned on the investor's views, can be computed:

31

$$X \vee v \quad N\left(\mu_{BL}, \Sigma_{BL}\right)$$

Where, the expected returns values are:

32

$$\mu_{BL}(v, \Omega) \equiv \mu + \Sigma P^{'}\left(P \Sigma P^{'} + \Omega\right)^{-1} P \Sigma$$

and the covariance matrix is:

33

$$\Sigma_{BL}(\Omega) \equiv \Sigma - \Sigma P^{'}\left(P \Sigma P^{'} + \Omega\right)^{-1} P \Sigma$$

Notice that the value of the views v, does not affect the expression of the covariance. This is a peculiarity of the normal setting (cfr Meucci 2009). The expression of the Black-Litterman market distribution can be used to determine the optimal asset allocation that includes the investor's views. At this point, the Black-Litterman allocation α_{BL} is defined as the optimal allocation function [equation 23] computed using the market "adjusted" by the view [equation 26]:

34

$$\alpha_{BL}[v] \equiv \underset{\alpha \in C_v}{argmax}\left\{S_v(\alpha)\right\}$$

3. Empirical analysis

Beyond the theoretical profile, the topic of greatest interest is analysing the behaviour of the models on an operational level. The purpose of this paragraph is to show a practical implementation case. Assume an asset manager wants to build long-only portfolios by combining multiple equity asset classes. The investment time horizon is monthly. There are ten asset classes that he considers potentially interesting, and a market index is associated with each of them:

1. MSCI ACWI AERODEFENSE
2. MSCI ACWI BEVERAGES
3. MSCI ACWI BIOTEC
4. MSCI ACWI ENERGY
5. MSCI ACWI FINANCIALS
6. MSCI ACWI HEALTHCARE
7. MSCI ACWI INDUSTRIALS
8. MSCI ACWI IT
9. MSCI ACWI CHEMICALS
10. MSCI ACWI PHARMA

Assuming that the returns of the indices are serially independent and identically distributed (i.i.d.), the inputs are estimated by considering a sample of 120 months (Apr. 2012 – Apr 2022)[16] historical returns. All the asset classes are in dollars, this to avoid the discussion of currencies[17]. The optimized portfolios are constrained to have non-negative weights and sum to one. The risk free rate is set to zero. Figure 3 and 4 summarizes the set of sample estimates: average monthly returns, standard deviations and correlations.

[16] The original paper assumed the market was represented by the linear returns on a set of securities and the parameters (μ, Σ) satisfy a general equilibrium model.

[17] Source: Refinitiv EIKON [Thompson Reuters]

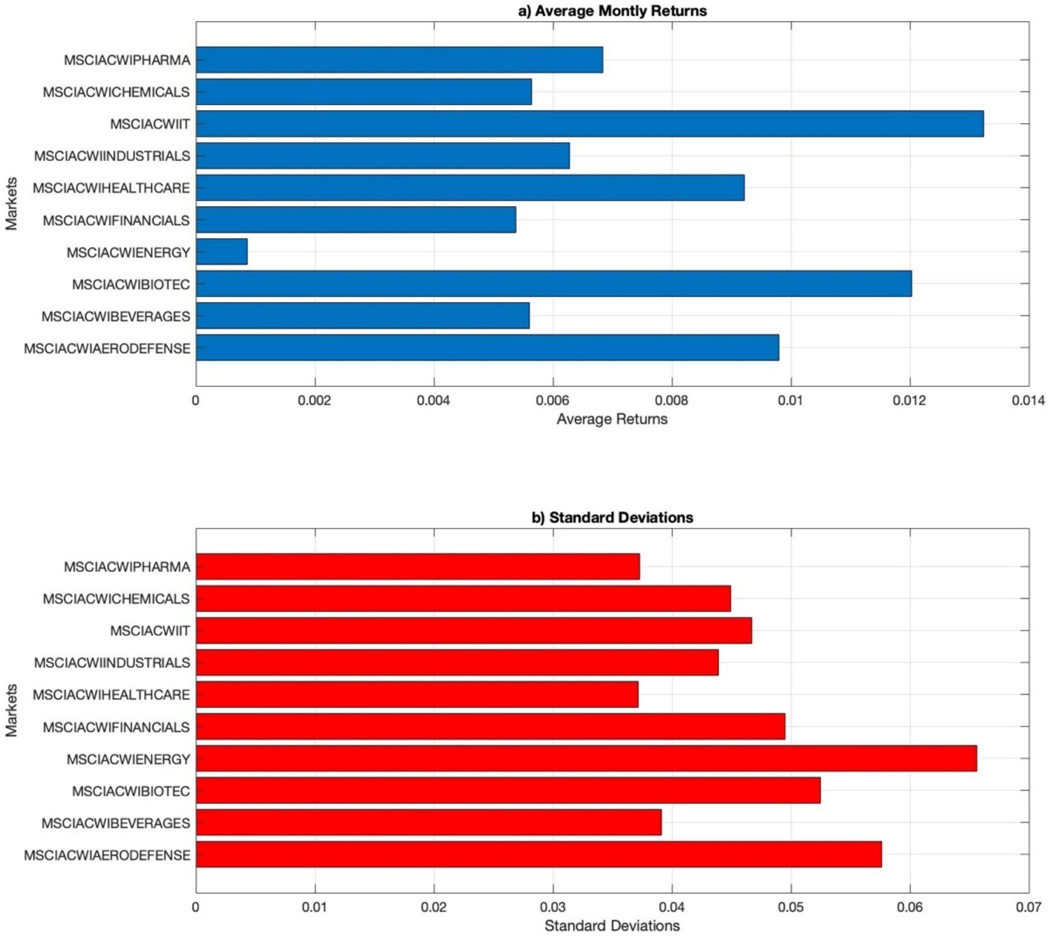

Figure 3 Assets moments: average returns and standard deviations

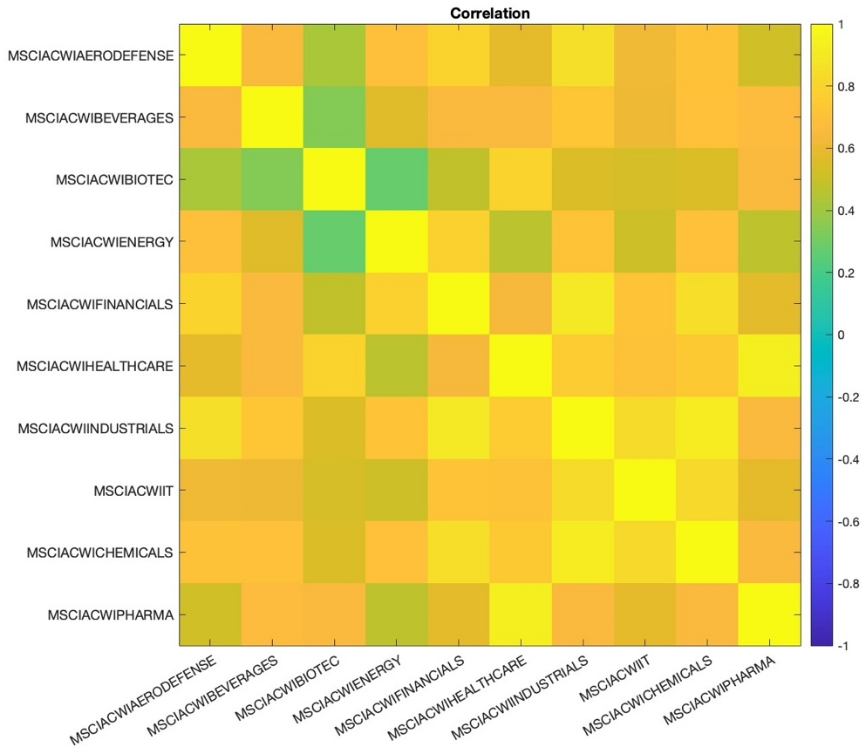

Figure 4 Assets correlation

3.1. The classical MV optimization

The application of the classical MV optimization model allows identifying the return-risk combinations that on the risk-return plan draw the set of points representing efficient portfolios: Figure 5. The task of showing the composition map of the hundred optimal portfolios is entrusted to Figure 6, which displays the allocations from minimum risk (on the left-hand side of the charts) to the maximum return (on the right-hand side of the charts). Each colour represents a particular asset class. A vertical slice of the chart illustrates the weights of each asset class in the portfolio at each level of risk. The leftmost points of the efficient frontier, those characterized by the lowest risk (whose composition is shown in the left part of Figure 6), are obtained by combining approximately 40% of MSCI BEVERAGES, 30% of MSCI PHARMA, 20% of MSCI HEALTHCARE, 5% OF MSCI IT, and 5% of MSCI BIOTEC. The rightmost points of the efficient frontier assume quite extreme exposures, combining only the two assets that are expected to generate the highest returns: MSCI IT (≈ 75% of the overall) and MSCI BIOTEC. MSCI IT is present in almost all efficient portfolios because of its extremely high expected return and moderate volatility (Figure 3a). The last portfolio on the far right of the Efficient Frontier, i.e. the maximum return portfolio, is actually a 100% bet on the IT market.

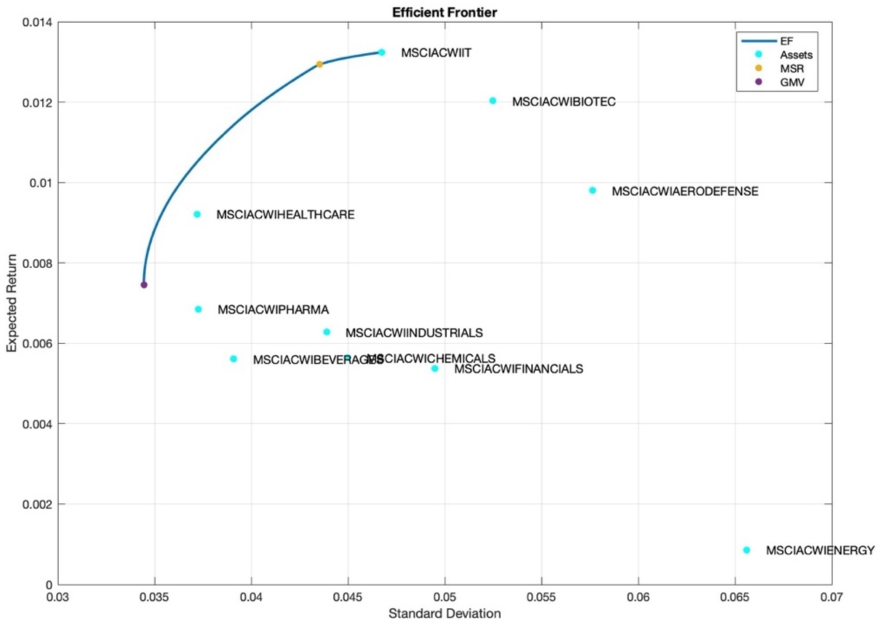

Figure 5 MV efficient Frontier

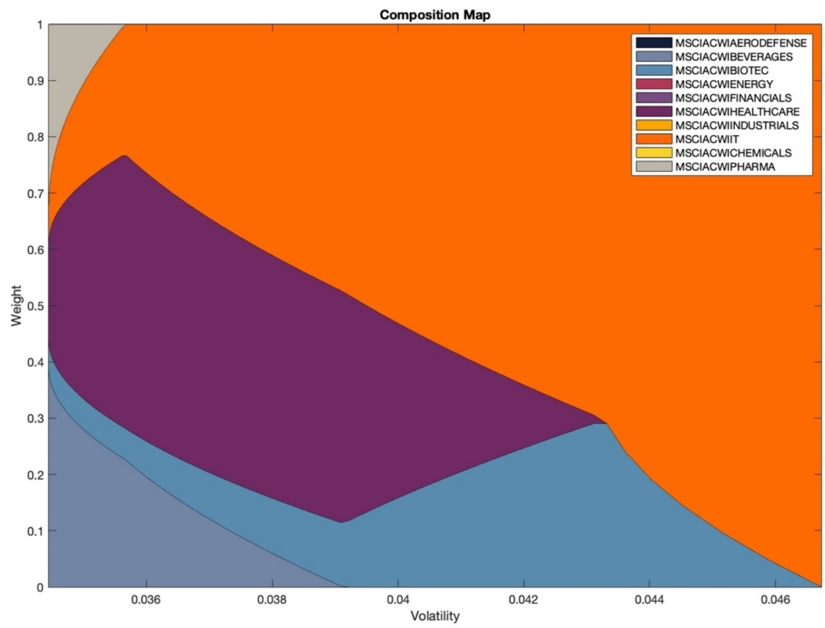

Figure 6 MV Efficient frontier composition map

Figure 7 shows the weights of the MSR (or Tangency) and the GMV portfolios. As expected, 50% of the asset classes: MSCI ACWI AERODEFENCE, MSCI ACWI ENERGY, MSCI ACWI FINANCIALS, MSCI ACWI INDUSTRIALS and MSCI ACWI CHEMICALS are never selected. With reference to the reasonableness of the allocation, beyond the need to maximize the expected return given a level of risk, optimal portfolios must have composition requirements that make them appear reasonable, showing a convincing diversified allocation. Looking at the composition of the MSR and MV portfolios (Figure 7), the allocation is extremely concentrated.

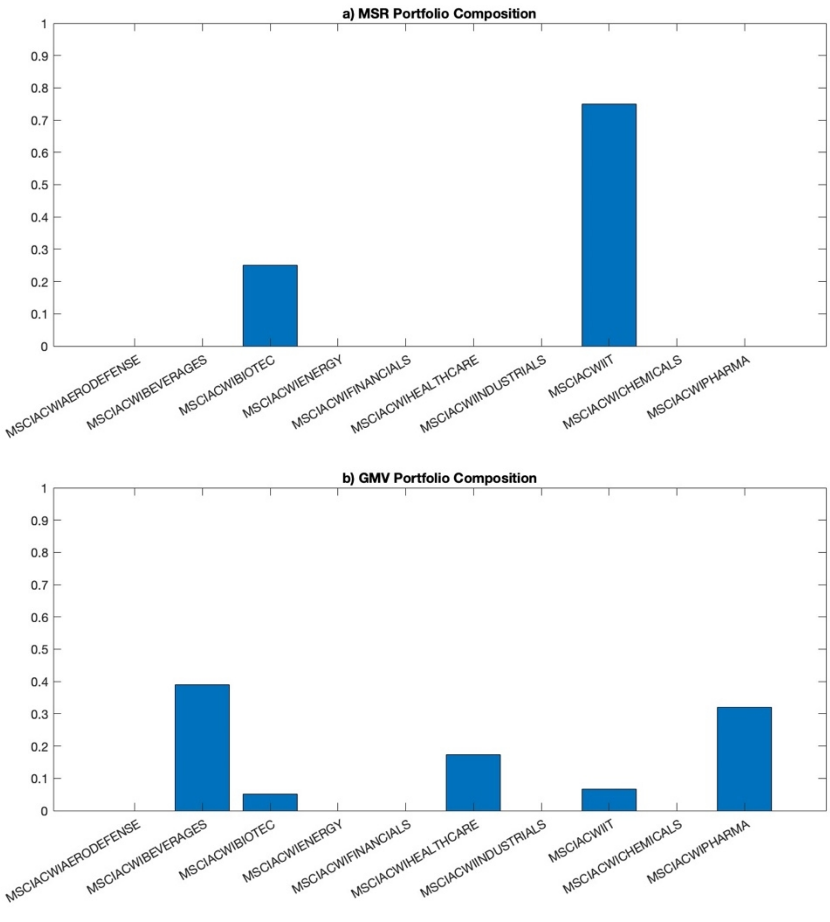

Figure 7 Mean-Variance MSR and GMV portfolios weights

3.2. Visualization of the input estimation pitfall

The simplest approach to estimating the input is to rely on historical data. Figure 8 shows four efficient frontiers where one is constructed using the last 10 (Figure 8a), 5 (Figure 8b), 2 (Figure 8c), and 1 (Figure 8d) years of monthly returns. The optimized portfolios are constrained to have non-negative weights and sum to one as in the previous section.

The estimated frontiers show a considerable discrepancy, and this is no surprise because the four frontiers use different input values in portfolio construction using the mean-variance method.

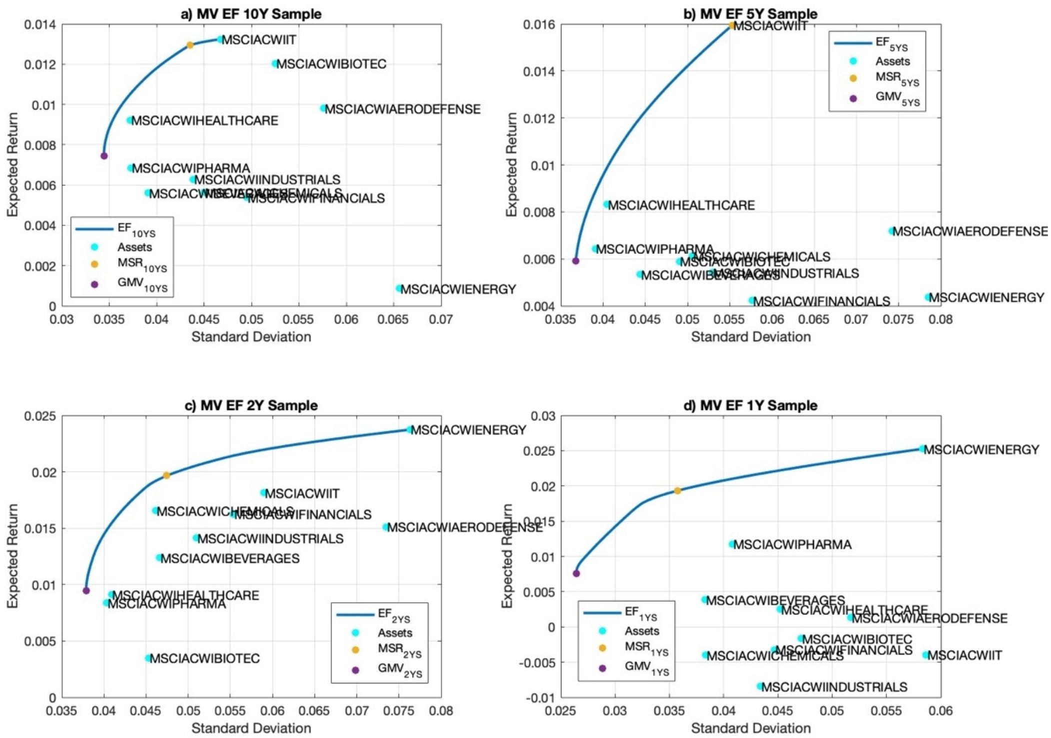

Figure 8: MV Efficient frontiers with four different time windows

Figure 9 shows the weights of the Maximum Sharpe Ratio (MSR) and the Global Minimum Variance (GMV) Portfolios of the four samples, while figure 10 maps the composition of the four efficient frontiers. Figures 8 and 9 clearly show how the optimal portfolios for the four cases differ. Especially when investors are interested in tangency portfolios (Figure 8).

The effect of input estimation on the allocation's performance is quite relevant. When longer time series are considered (Figures 8a and 8b), the MSR portfolios are expected to generate a monthly return of approximately 1,3% and 1,6 for the 10Y and 5Y samples respectively, with 4.3% and 5.5% volatility. Considering shorter time windows (fig. 8c and fig. 8d), returns are much higher: around 2%, whit volatility between 3.6 and 4.7 percentage points.

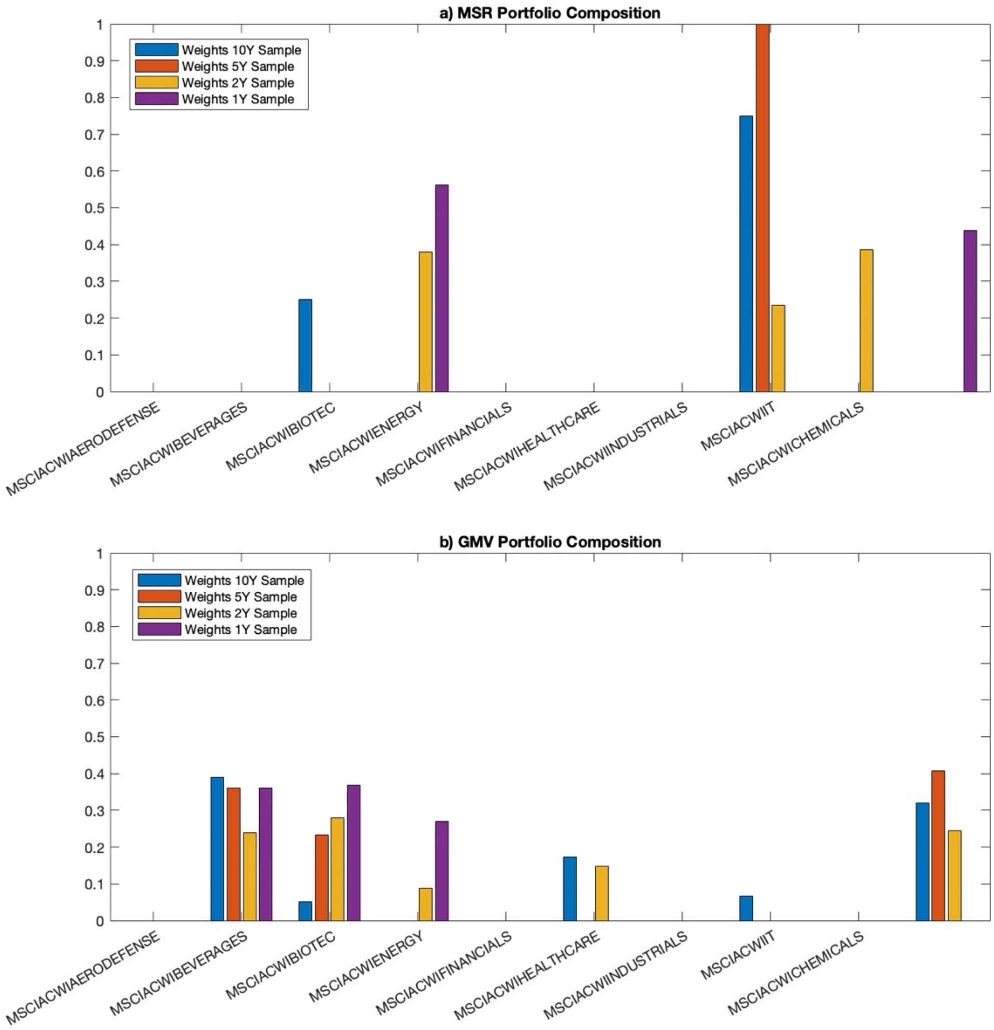

Figure 9: Mean-Variance MSR and GMV Portfolio Weights

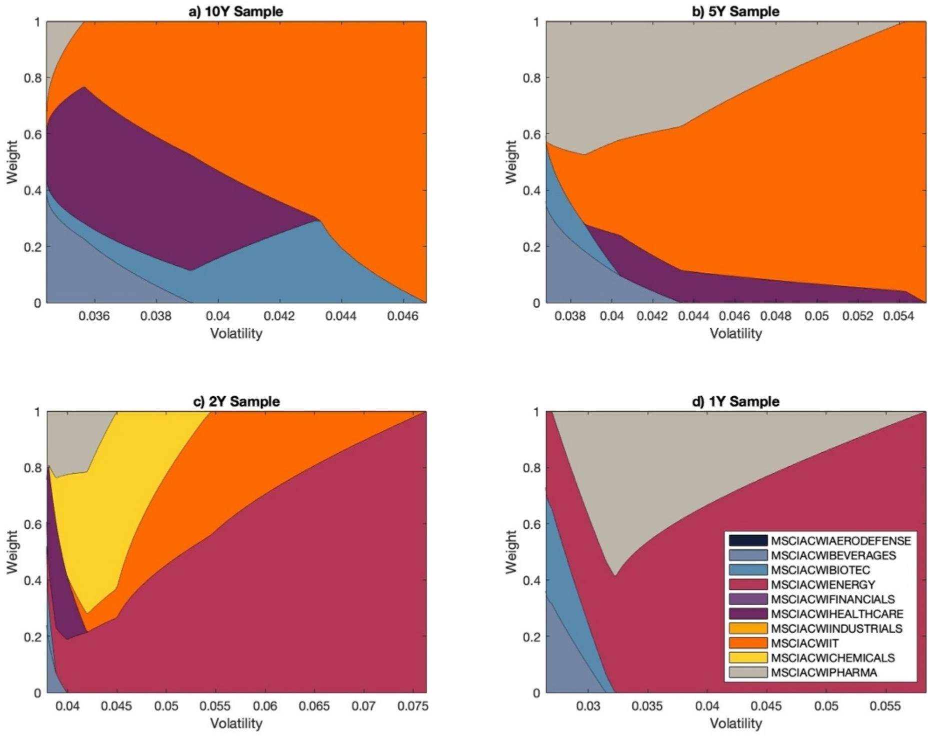

Figure 10 MV Efficient Frontier Composition Map. Different samples: 10Y, 5Y, 3Y, 1Y

Portfolios' composition changes dramatically as well. While 10Y and 5Y samples consider (in terms of return) MSCI ACWI IT the most performing asset class, both 2Y and 1Y samples "bet" on MSCI ACWI ENERGY. Using historical measures raises the problem of establishing how much to go back in the process of calculating the inputs. Nobody knows whether an allocation based on one year of data is less/or reliable than an allocation based on ten years of data. There is no optimal length of time series: short time series have the peculiarity of better reflecting more recent history, while long time series are instead statistically more significant (often only a long series is able to catch surprises).

The visualization of the effect of the input estimation problem can also be approached differently. Figure 11 exhibits 1000 statistically equivalent MV EF (fig. 11a) and the "original" MV EF (fig. 11b). Each simulated EF in the left-hand panel, as well as the original EF in the right-hand panel, consist of 100 portfolios from lowest to the highest return. Each simulated EF is defined to be consistent with the uncertainty in the original data set, i.e., the 10Y sample data adopted in the previous sections. The simulated returns assume a multivariate normal distribution, hence the correlation between asset prices is maintained. The optimized portfolios are constrained to non-negative weights and sum to one.

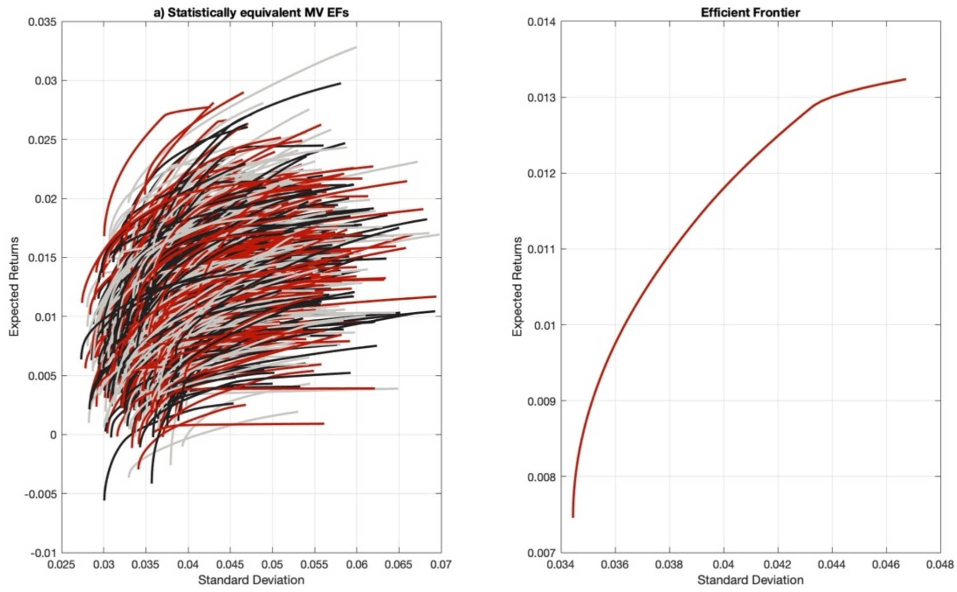

Figure 11 Simulated and original MV Efficient frontiers

For the efficient frontier in Figure 11, the data resampling simulation proceeds as follows:

I. Monte Carlo simulation of 10 years of correlated monthly returns based on the data in Figures 3 and 4 for the nine asset classes.
II. Computation of optimization inputs parameters from the simulated return data.
III. Computation of efficient frontier portfolios that satisfy the long-only constraint
IV. Repeat steps I through III 1000 times. By definition, each simulated efficient frontier is statistically equivalent to the efficient frontier in Exhibit 2.5.

By definition, each simulated Efficient Frontier of Figure 11a is statistically equivalent to the Efficient Frontiers in Figure 11b, Figure8. The simulations clearly show that estimation error in both the risk and return dimensions strongly affects MV optimality ambiguity. The dispersion of the simulated frontiers is enormous. Some simulated frontiers have roughly half the range of risk of the original MV efficient frontier while others have significantly more risk. The range of returns among the simulated frontiers is even more impressive

Note the range of risk and return displayed in the simulated efficient frontiers compared to the original frontier. While the original frontier returns range from 0.75% to 1.35%, the simulated efficient frontier returns range from -0.5% to 3.3%.

3.3. A comparison between MV and RE

In Figure 11a, every simulated MV EF is the optimal way to invest for a given set of inputs. However, the inputs are highly uncertain. From a practical perspective, the instability of MV efficiency with estimation error demonstrated in Figure 11a may indicate little investment value. In reality, the variation suggests a statistical route for transforming MV optimization into a more investment useful procedure (Michaud, 2007). Resampled Efficiency (RE) defines the optimal portfolio weights by averaging the weights of each simulated optimal portfolio. For example:

1. Since all simulated EF are equally likely, the optimal GMV RE portfolio (the leftmost portfolio on the EF) is the average of the portfolio weights of all the 1000 simulated GMV MV portfolios.
2. Equivalently, since all simulated EF are equally likely, the optimal MR RE portfolio (the rightmost portfolio on the EF) is the average of the portfolio weights of all the 1000 simulated MR MV portfolios.

Figure 12 displays a comparison between the Markowitz-like EF and the RE optimized frontier (REF) based on the 10Y sample data adopted in the previous sections[18]. The optimized portfolios are, as usual, constrained to have non-negative weights and sum to one. The risk-free rate is set to zero.

[18] The currency theme is well explained in the Litterman (2003), Black and Litterman (1991, 1992), Black (1989a, 1989b), Grinold (1996), Meese and Crownover (1999), and Grinold and Meese (2000).

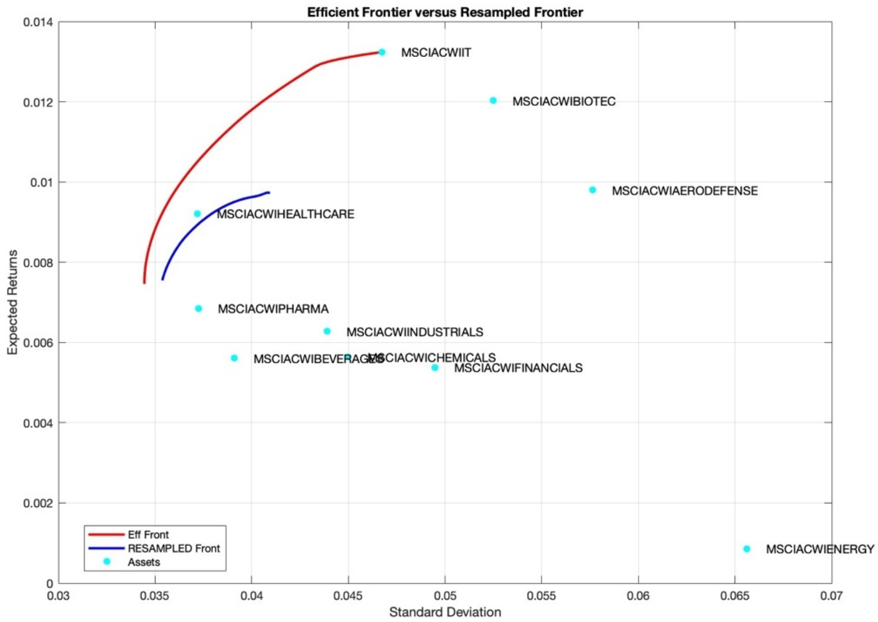

Figure 12 MV Efficient Frontier vs Resampled Efficient Frontier

Figure 12 illustrates that REF plots below the MV frontier. Superficially this may suggest that the RE optimization may be inferior as an investment framework. Let the next section investigate it.

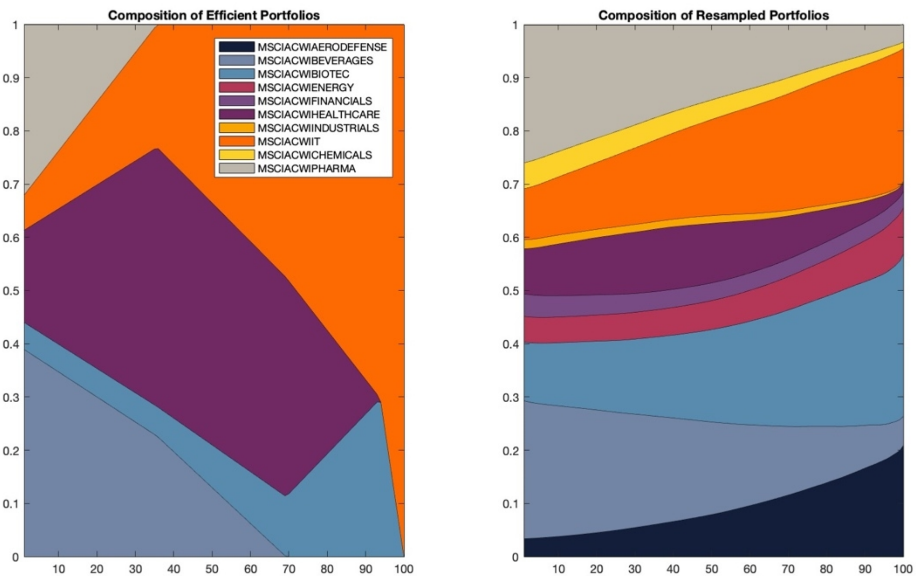

Figure 13 MV and RE Frontier Portfolio Composition Maps

Figure 13 goes provides a portfolio composition map of the MV and RE optimal asset allocations of Figure 12. The left-hand panel is the composition map for MV efficiency; the right-hand panel represents RE optimality.

The composition map for the REF illustrates very different and more diversified portfolios. REF optimality includes all assets. There is a smooth transition from one risk level to another.

The greater degree of diversification of resampled solutions compared to efficient ones is evident. Furthermore, the RE technique is capable of discriminating between asset classes, especially favouring the diversification of the riskier ones. This phenomenon is quite welcome since most of the problem of estimation error is attributable to the concentration in high-risk asset classes. As Figure 13 indicates, while the MR (Maximum Return) MV optimal portfolio represents a 100% bet in the MSCI ACWI IT, the MR RE portfolio is very well diversified and much less risky (as figure 12 exhibits, the MR RE delivers 4.1% volatility against the 4.7% vol. delivered by the MR MV), more acceptable, investment[19].

3.4. A comparison between MV and Black-Litterman

The Black-Litterman model, as discussed above, is an asset allocation approach that allows investment analysts to incorporate subjective views (based on investment analyst estimates) into market equilibrium returns. By blending analyst views and equilibrium returns instead of relying only on historical asset returns.

[19] The simulated MV efficient frontier procedure may have a number of variations. For example, the return distribution assumption can be changed, or historical data may be bootstrapped. In many cases in practice the number of simulated returns for computing the simulated MV efficient frontiers is not associated with a historical return data set and must be assumed (Michaud, 2007).

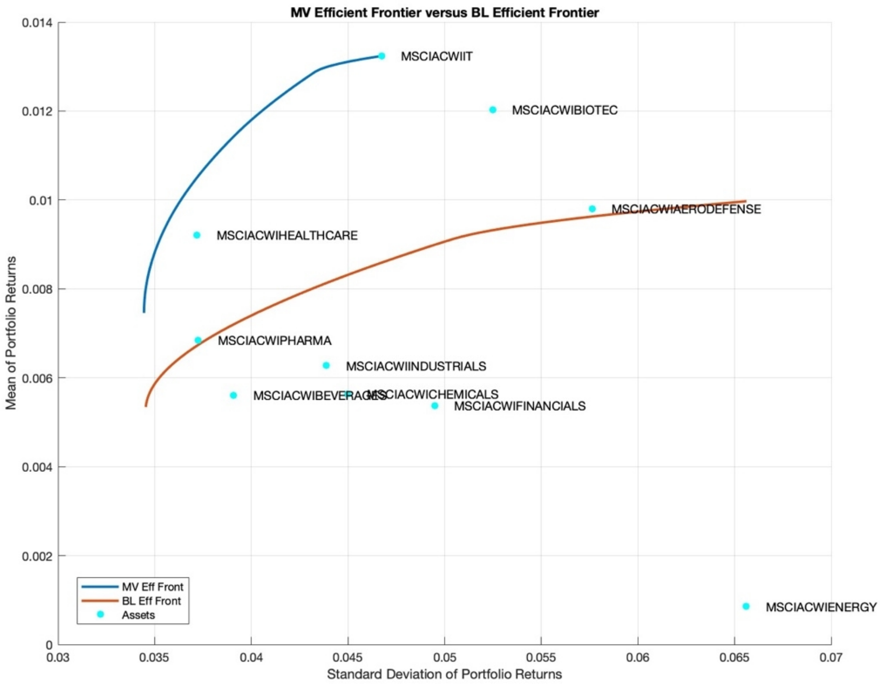

Figure 14 MV vs BL Efficient Frontier

The Universe of Assets is defined at the beginning of the section. The optimization is, as always, based on the same 10Y sample and constrained to have non-negative weights and sum to one. The risk-free rate is set to zero. This example assumes two strong dependent views and one strong independent view[20]:

I. MSCI ACWI ENERGY is going to have a 0.1% monthly return with uncertainty 1e-6;
II. MSCI ACWI AERODEFENCE is going to outperform MSCI ACWI HEALTHCARE by 0.2% monthly return with uncertainty 1e-5;
III. MSCI ACWI FINANCIALS is going to outperform MSCI ACWI IT by 0.5% monthly return with uncertainty 1e-5.

The implied returns are calculated by reverse optimization. The general formulation of portfolio optimization is given by the Markowitz optimization problem. To find the market portfolio, each asset is regressed against the benchmark: the index MSCI ACWI. The imposed constraints are fully invested and long-only. As regards the scalar that defines the

[20] Notice that, statistically thinking, each asset weight in the maximum return RE portfolio is equal to the probability that it is truly the maximum return asset (Michaud, 2007).

degree of confidence in the prior belief of the expected return, this example uses 1/n (where n is the number of the asset, i.e. 10).

Comparing the BL Blended Expected Return to the Prior Belief of Expected Return, the expected return from the Black-Litterman (3rd column of Table 1) model is indeed a mixture of both prior belief and investor views. For example, as shown in Table 1, the prior belief assumes return for MSCI ACWI FINANCIALS similar to MSCI ACWI IT: 0.751% against 0.699%. In the blended expected return, MSCI ACWI FINANCIALS has a higher return than MSCI ACWI IT by approximately 20 basis points 0.893% against 0.612%. This difference is due to the imposed strong view that MSCI ACWI FINANCIALS outperforms MSCI ACWI IT by 0.2%. MSCI ACWI ENERGY, according to the strong absolute view, has a BL Blended Expected Return of 0.997%.

Asset Name	Prior Belief of Expected Return	BL Blended Expected Return
MSCIACWIAERODEFENSE	0.768%	**0.816%**
MSCIACWIBEVERAGES	0.503%	**0.536%**
MSCIACWIBIOTEC	0.534%	**0.562%**
MSCIACWIENERGY	0.805%	**0.997%**
MSCIACWIFINANCIALS	0.751%	**0.893%**
MSCIACWIHEALTHCARE	0.514%	**0.543%**
MSCIACWIINDUSTRIALS	0.707%	**0.745%**
MSCIACWIIT	0.699%	**0.612%**
MSCIACWICHEMICALS	0.703%	**0.746%**
MSCIACWIPHARMA	0.461%	**0.507%**

Table 1 Prior Belief vs BL Blended Expected Return

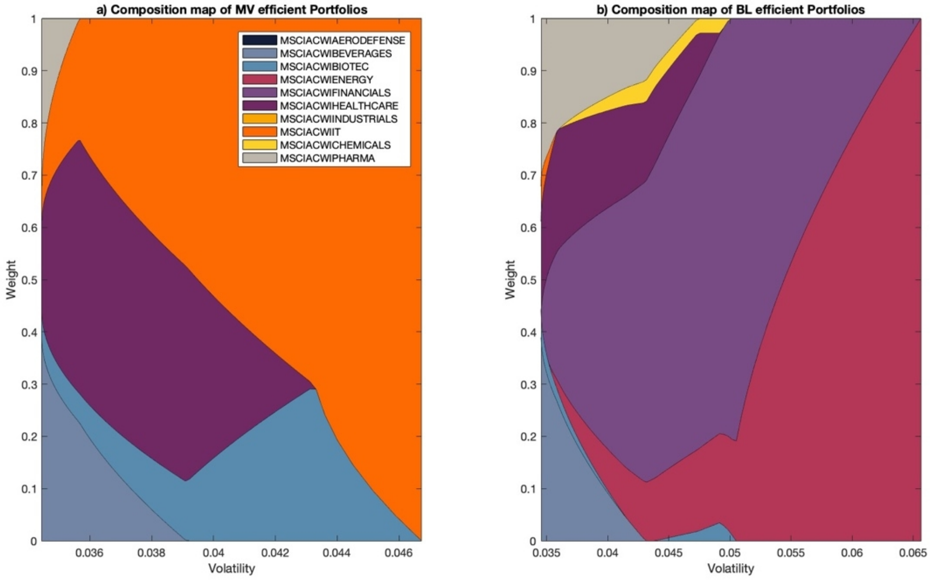

Figure 15 MV and BL Frontier Portfolio Composition Maps

The allocation from the BL model is slightly more diversified compared to the MV (Figure 15), especially in the MSR portfolio. See Figure 16. Looking at the MSR portfolio, BL optimization has generated more diversified portfolios and therefore it seemed to be effective in overcoming the problem of the reasonableness of the allocation. In the left end of the EF, the problem related to extreme allocations and corner portfolios persists.

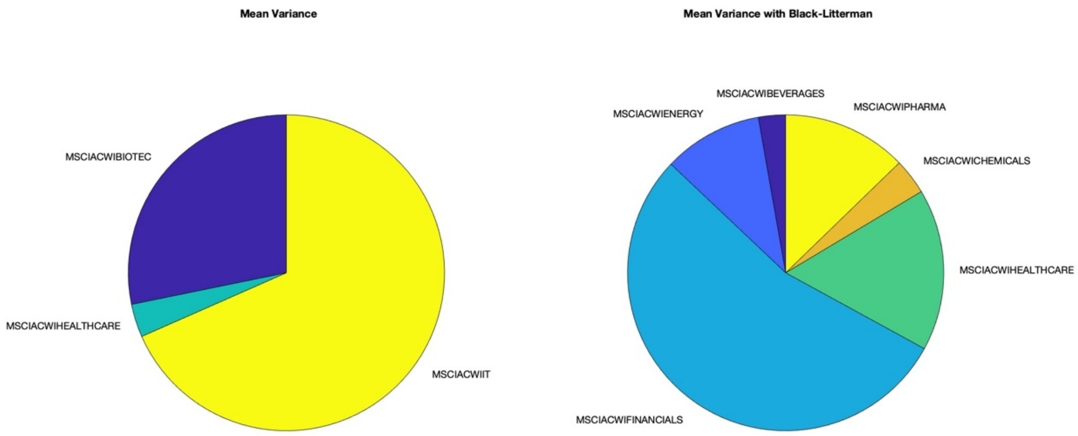

Figure 16 MV vs BL Maximum Sharpe ratio portfolio composition

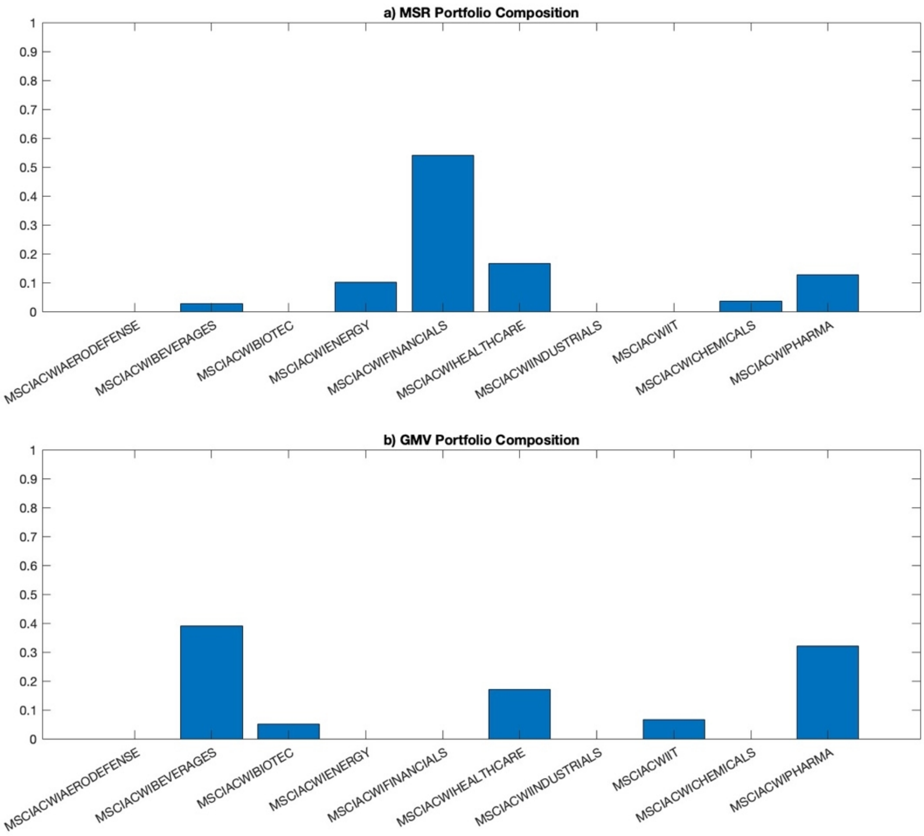

Figure 17 Black-Litterman MSR and GMV portfolios weights

3.5. Portfolio Performance

The main purpose of this section is to present how Michaud's and Black-Litterman optimization frameworks perform compared to the conventional Mean-Variance approach. In order to do this, the performance of the following portfolios has been evaluated over the month which followed the end of the sample period adopted to run the optimizations (2/05/2022 - 02/06/2022)[21]:

1. Benchmark [MSCI ACWI][22]
2. GMV MV [Mean-Variance]

[21] All views reflect the outlook of the writer and have been defined ex-ante with respect to the valuation period

[22] dd/mm/yyyy

3. GMV RE [Resampling Efficiency]
4. GMV BL [Black-Litterman]
5. MSR MV
6. MR [Maximum Return] RE
7. MSR BL

It must be emphasized that the fact that a single month is taken into consideration in this analysis represents a great limitation of this report. Who writes preferred not to implement a backtest logic as the asset allocation is derived based on the performance of the assets themselves. This could have compromised the unbiasedness of the performance analysis. Considering the single month, there is a risk that certain results are purely coincidental. However, due to a lack of time, it was impossible to adopt a longer estimation time after both the estimation period of the moments and the definition of the views in the BL model.

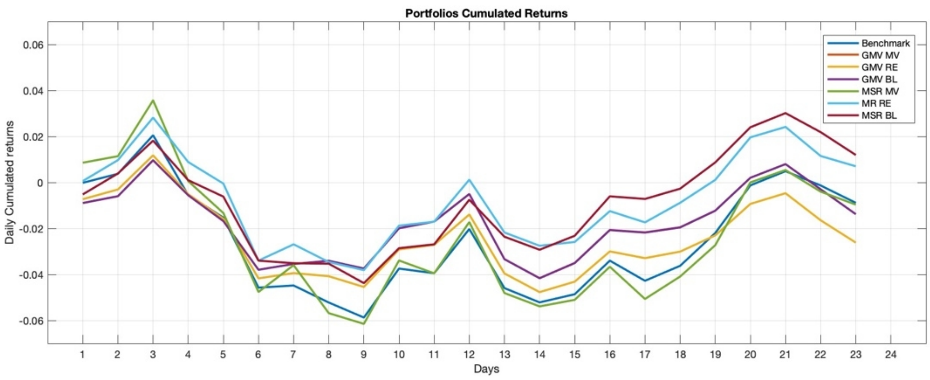

Figure 16 Relevant Portfolios Daily Cumulated Returns

The simplest way to measure the return of an investment is to calculate the percentage change of its total value during the investment horizon, also known as the holding period. Figure 16 exhibits the performance of each portfolio over the tested period. The only two portfolios that provided a positive gross monthly return are the MR RE, and the MSR BL (Table 2).

Portfolio	Holding Period Return
Benchmark	-0.887%
GMV MV	-1.381%
GMV RE	-2.616%
GMV BL	-1.379%
MSR MV	-0.965%
MR RE	**0.702%**
MSR BL	**1.194%**

Table 2 Holding period return

In this analysis, the MSR portfolio in the Michaud framework is not analyzed due to the way the model is developed in MATLAB. While in the MV and BL cases the definition of the efficient frontier occurs through the use of specific commands ("p = Portfolio" first, and "p.estimateFrontier" then), in the case of the Michaud model, the weights of the 100 portfolios that make up the RE frontier are extrapolated (taking the average value) by a three-dimensional array (in the code: "STORE_ WTS") of size 100x10x1000. Where 100 are the efficient portfolios composing the efficient frontier, 10 are the asset classes that make up the investable universe and 1000 are the number of Monte Carlo simulations. The vectors relating to risks and rewards are then calculated accordingly (see code on page 69) and plotted in the risk-return area.

There are no functions in MATLAB that allow the user to retrieve the Tangency portfolio from this process. The GMV RE portfolio, on the other hand, can be easily defined as it is the average of the portfolio weights of all the 1000 simulated GMV MV portfolios and is the first portfolio from the left for both the frontiers (MV and RE).

Despite this, considering that the strength of the methodology invented by Michaud is to provide a well-diversified MR portfolio, it makes sense to evaluate its performance. The same does not apply to MV and BL methods that, considering the MR portfolio as a 100% bet on the asset with a higher expected return, don't have any optimization logic.

3.5.1. Absolute performance indicators

Even though the holding period return characterizes a fundamental aspect of performance, to appraise the risk-adjusted performance of each strategy the following indicators are applied (Table 3):

1. Sharpe Ratio (Sh): which measures the excess return of an asset per unit of volatility;
2. Sortino Ratio (So): which measures the excess return of an asset per unit of downside volatility;
3. The Treynor ratio (Tr): measures the excess return of an asset per unit of systematic risk;
4. Value-at-Risk (VaR): which, in a horizon equivalent to the frequency of the returns (daily in this case), the VaR (α) is the maximum loss that can be suffered with probability 1 – α[23] (with a probability α the loss will be larger than the Value-at-Risk);
5. Expected Shortfall (ES): that is is a risk measure sensitive to the shape of the tail of the distribution of returns on a portfolio. It is calculated by averaging all of the returns in the distribution that are worse than the VAR of the portfolio at a given level of confidence;
6. Calmar Ratio (Cal): which is an index used to measure the return in relation to the drawdown risk[24]: it allows the investor to compare the potential gain and the possibility of loss of a given investment;
7. Sterling Ratio (Ste): which is the return per unit of extreme risks where those are set to the (absolute value of) the average of the k largest Drawdowns[25];

---

[23] The movement of a market index is a representation of how the market behaves, and market indices are usually benchmarks for evaluating portfolio performance. Thus, a composite index it is also used as the main benchmark (Index).

[24] In this work, α=0.05

[25] The Drawdown monitors the losses and is recovered in a recursive way (see MATLAB code). In general, the Drawdown sequence is graphically analyzed to identify the largest losses, and the time to recover from losses (a better strategy has smaller losses and a quick recovery from the minimums)

8. Farinelli-Tibiletti ratio (FT): which are ratios of average gains to average losses with respect to a target τ, each raised by a power index: p (upside power) and q(downside power)[26].

Portfolio	Sh	So	Tr	VaR	ES	Cal	Ste	FT
Benchmark	-0.01869	-0.02774	-0.00028	0.01004	0.00905	-0.00360	-0.00396	0.52866
GMV MV	-0.04305	-0.06574	-0.00069	0.02218	0.01865	-0.01047	-0.01138	0.49935
GMV RE	-0.08739	-0.12848	-0.00136	0.04130	0.04094	-0.01835	-0.01965	0.45633
GMV BL	-0.04294	-0.06564	-0.00069	0.02212	0.01862	-0.01043	-0.01134	0.49961
MSR MV	-0.01292	-0.02067	-0.00020	0.00720	0.00708	-0.00263	-0.00282	0.55221
MR RE	0.02841	0.04072	0.00044	0.01499	0.01191	0.00619	0.00680	0.56612
MSR BL	0.04762	0.07367	0.00074	0.02839	0.02106	0.00970	0.01119	0.61941

Table 3 Absolute performance indicators

When comparing many performance measures results are, in general, not concordant. Consequently, there is no one portfolio strategy that is ultimately better than others. Likewise, no one measure of performance can fully represent the performance of portfolios. In this work, to appraise the overall performance, a composite index (CI) which compares/ranks strategies across performance measures is adopted (Table 4). The scoring method is simple. For each indicator, a value ranging from 1 (best) to 7 (worst) is assigned to each strategy. The minimum score i.e. 7 is given by the fact that 7 different strategies are being considered. The CI simply adds up the scores received in each indicator. Assuming that the indicators are equally important, the strategy that, from a general point of view, has performed best, is the one with the lowest value. For example: strategy A receives a 1 for each of the 8 indicators, strategy B receives a 2 for each of the 8 indicators. The CI of strategy A is 8, the CI of strategy B is 16. Result: Strategy A performed better than strategy B.

[26] In this work, k=5

Portfolio	Composite	Sh	So	Tr	VaR	ES	Cal	Ste	FT
Benchmark	28	4	4	4	2	2	4	4	4
GMV MV	46	6	6	6	5	5	6	6	6
GMV RE	56	7	7	7	7	7	7	7	7
GMV BL	38	5	5	5	4	4	5	5	5
MSR MV	20	3	3	3	1	1	3	3	3
MR RE	18	2	2	2	3	3	2	2	2
MSR BL	18	1	1	1	6	6	1	1	1

Table 4 Ranking absolute performance indicators

The portfolios with a higher risk component (MSR and MR) are those that overall generated the best performance from all points of view. In particular, the MR RE and MSR portfolios are the ones that are best positioned in the composite index (Table 4). Constructing portfolios from BL and RE optimization leads to investment with the overall highest Sharpe and Sortino ratios. MSR BL has the highest Sharpe ratio (0.04762) and Sortino ratio (0.07367). The high return of RE and MV portfolios comes at a price; when it comes to the VaR and ES the best portfolio is the MSR MV.

3.5.2. Relative performance indicators

Most performance metrics present the absolute performance of portfolios. Sometimes, relative performance is of greater interest. In fact, most portfolio managers regularly assess their performance relative to a benchmark (Table 5). The relative performance indicators adopted in this analysis are the following:

1. Tracking Errors (TE): which measures the mean of the difference between the returns of the portfolio and its benchmark;
2. Tracking Error Volatility (TEV): which measures the variance of the difference between the returns of the portfolio and its benchmark;
3. Semi TEV: which measures the downside deviations of the difference between the returns of the portfolio and its benchmark;
4. Information Ratio (IR): which is TE over TEV;
5. Semi IR: which is TE over semiTEV;

Portfolio	TE	TEV	semi TEV	IR	semi IR
GMV MV	-0.00025	0.00003272	0.00287	-7.68561	-0.08753
GMV RE	-0.00080	0.00002186	0.00253	-36.54191	-0.31548
GMV BL	-0.00025	0.00003249	0.00286	-7.71538	-0.08755
MSR MV	0.00003	0.00003455	0.00385	0.95936	0.00862
MR RE	0.00068	0.00001202	0.00173	56.48784	0.39257
MSR BL	0.00087	0.00002236	0.00210	38.89486	0.41396

Table 5 Relative performance indicators

The variance underlying these calculations is the empirical variance calculated on the daily returns that make up the month of observation.

MSR MV has a very low tracking error compared to the other portfolios but both RE MR and MSR BL were able to exploit the upside volatility delivering a positive average excess return over the benchmark with a lower TEV (Table 5). The ability to exploit the upside benefit of the volatility is reflected also in having dominant IR and semiIR.

Again, to provide a more intuitive picture of the overall relative performances, a composite index of ranks is created (Table 6). In this case, the MS RE is, overall, the best portfolio. It is worth mentioning the high IR of the MS RE: 56.48784. The MSR BL is the second-best: 38.89486 (Table 5).

Portfolio	Composite	TE	TEV	semi TEV	IR	semi IR
GMV MV	25	5	5	5	4	6
GMV RE	22	6	2	3	6	5
GMV BL	21	4	4	4	5	4
MSR MV	19	3	6	6	3	1
MR RE	8	2	1	1	1	3
MSR BL	10	1	3	2	2	2

Table 6 Ranking relative performance indicators

4. Conclusions

Markowitz, in 1952, revolutionised the financial sector with his Portfolio Selection work, creating a scientific method for researching and managing an investment portfolio. The theory has shown some problems in its practical implementation:

1. The estimation risk related to the model parameters;
2. The instability of the mean-variance solutions;
3. The operative choice of the target returns;
4. The absence of investor views.

The RE and BL methods are two alternative and more sophisticated tools that can be used to determine the optimal weight of a portfolio. Both the methods enhance portfolio performance in terms of better diversification (RE in particular) and in terms of risk-adjusted returns.

The RE method can overcome the problem of sensitivity to input changes that characterize the MV method. The problem is solved by generating data using Monte Carlo simulation. The rationale behind this approach consists in limiting the extreme sensitivity of the optimal allocation function to the market parameters by averaging several sample-based allocations in different scenarios. The Black-Litterman is an effective approach that can, but does not necessarily need to, rely on the contingent historical information available when the investment decision is made. The model's great turning point is the inclusion of investor views, combined with historical estimated returns, generating a new set of expected returns, capable of improving and stabilising the portfolio's performance as seen above. Yet, a very pleasant point of the BL model is that of being able to compensate for the limiting effect of considering the standard deviation (or the variance) as a risk measure, as it can exploit the potential benefit of the upside volatility by means of the views of the asset manager. Clearly, this peculiarity can also prove to be a double-edged sword as the model will prove to be much more performing, the better are the skills of the asset manager who implements it in determining how the market will evolve. In light of that, while the Michaud model is also suitable for an "uninformed" asset manager, the effectiveness of the BL model is strictly dependent on the capabilities of the asset manager.

From the point of view of rationality in the composition of portfolios, it is evident that both the logic adopted by BL and the logic devised by Michaud, represent an evolution of the Mean-Variance method. It is worth mentioning that while MR MV is a 100% bet on MSCI ACWI IT, and the MR BL is a 100% bet on MSCI ACWI ENERGY, the RE method is the only one of the three methods that can offer a well-diversified MR portfolio.

As regards the performance of portfolios, both the Michaud model and the BL model outperform the MV model. In particular, the MSR portfolio of the BL model is the portfolio that, among those analyzed, can offer a greater holding period: 1.194%. The MR portfolio of the Michaud method follows with a holding period equal to 0.702%. The MV method MSR generates a negative return of -0.965%. Looking at the absolute performance indices, the MR portfolio of the Michaud method and the MSR portfolio of the BL method are the two portfolios that have performed better with a CI of 18 for both. The relative performance indicators instead indicate the MR portfolio developed according to the Michaud logic as the best portfolio with a CI of 8.

As already explained above, at the beginning of section 3.5, the performance analysis carried out in this paper presents a major limitation due to the shortness of the analysis period: only one month. Having more time available, it would certainly have been more exhaustive to extend the analysis period to at least 12 months, making a rebalancing of portfolios on a monthly basis. A performance evaluation through the simulation of n possible scenarios is not a viable path as it would make the BL model lose its forward-looking peculiarity.

Within this work, it was decided not to apply the logic of the RE method to the Black-Litterman model as this would nullify the benefit generated by the insertion of the views; but an interesting enrichment of this analysis could be given by inserting weight constraints to the MV method and then resampling the resulting efficient frontier.

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Annex: MATLAB CODE

Mean Variance

```
clear
clc

%% UNCONSTRAINED MEAN-VARIANCE PORTFOLIO OPTIMIZATION - ESTIMATION ERROR

% loading data
A1=readtable("Assets10yS.xlsx");
A2=readtable("Assets5yS.xlsx");
A3=readtable("Assets2yS.xlsx");
A4=readtable("Assets1yS.xlsx");
LABELS=A1.Properties.VariableNames(2:end)';

% Computing monthly returns
mret1=tick2ret(A1{:,2:end});
%mret1=mret1*100;
mret2=tick2ret(A2{:,2:end});
%mret2=mret2*100;
mret3=tick2ret(A3{:,2:end});
%mret3=mret3*100;
mret4=tick2ret(A4{:,2:end});
%mret4=mret4*100;

%storing a vector which defines colors
COL=['#131E3A','#7285A5',"#598BAF","#B43757",'#784B84','#702963', ...
'#F9A602', '#FD6A02','#FCD12A','#BDB7AB'];

%Risk free rate (monthly)
Rf=0.0;

MON_RET=mean(mret1);
STD=std(mret1);
CORRELATIONS=corr(mret1);


figure(1);
subplot(2,1,1);
barh(MON_RET);
title('a) Average Montly Returns');
ylabel('Markets');
xlabel('Average Returns');
set(gca,'YTickLabel',LABELS);
grid on;
subplot(2,1,2);
barh(STD,'r');
```

```
title('b) Standard Deviations');
ylabel('Markets');
xlabel('Standard Deviations');
set(gca,'YTickLabel',LABELS);
grid on;

figure(2);
plot(mret1(:,:));
grid on;
title('Volatility');
ylabel('Monthly return');
xlabel('Time');
colororder(COL);
LEGEND= legend(LABELS,'Location','SouthOutside');

figure(3);
clrLim = [-1,1];
diamLim = [0.3, 1];
imagesc(CORRELATIONS)
colormap(gca,'parula');
colorbar();
caxis(clrLim);
set(gca,'Xtick',1:10,'XTickLabel',LABELS);
set(gca,'Ytick',1:10,'YTickLabel',LABELS);
axis equal
axis tight
title('Correlation');


%% MV OPT

% Creating Portfolio
p1=Portfolio("AssetList",LABELS,'RiskFreeRate',Rf);
p1=estimateAssetMoments(p1,mret1);
p1=setDefaultConstraints(p1);
p2=Portfolio("AssetList",LABELS,'RiskFreeRate',Rf);
p2=estimateAssetMoments(p2,mret2);
p2=setDefaultConstraints(p2);
p3=Portfolio("AssetList",LABELS,'RiskFreeRate',Rf);
p3=estimateAssetMoments(p3,mret3);
p3=setDefaultConstraints(p3);
p4=Portfolio("AssetList",LABELS,'RiskFreeRate',Rf);
p4=estimateAssetMoments(p4,mret4);
p4=setDefaultConstraints(p4);

pwgt1=p1.estimateFrontier(100);
pwgt2=p2.estimateFrontier(100);
pwgt3=p3.estimateFrontier(100);
pwgt4=p4.estimateFrontier(100);

% MAX Sharpe (or Tangency Portfolio)
wMSR1=estimateMaxSharpeRatio(p1);
[riskMSR1, retMSR1]=estimatePortMoments(p1,wMSR1);
wMSR1=round(wMSR1,6);
wMSR2=estimateMaxSharpeRatio(p2);
[riskMSR2, retMSR2]=estimatePortMoments(p2,wMSR2);
wMSR2=round(wMSR2,6);
wMSR3=estimateMaxSharpeRatio(p3);
[riskMSR3, retMSR3]=estimatePortMoments(p3,wMSR3);
wMSR3=round(wMSR3,6);
```

```
wMSR4=estimateMaxSharpeRatio(p4);
[riskMSR4, retMSR4]=estimatePortMoments(p4,wMSR4);
wMSR4=round(wMSR4,6);

% Minimum Variance Portfolio
wGMV1=p1.estimateFrontierLimits('Min');
[riskGMV1, retGMV1]=estimatePortMoments(p1,wGMV1);
wGMV1=round(wGMV1,6);
wGMV2=p2.estimateFrontierLimits('Min');
[riskGMV2, retGMV2]=estimatePortMoments(p2,wGMV2);
wGMV2=round(wGMV2,6);
wGMV3=p3.estimateFrontierLimits('Min');
[riskGMV3, retGMV3]=estimatePortMoments(p3,wGMV3);
wGMV3=round(wGMV3,6);
wGMV4=p4.estimateFrontierLimits('Min');
[riskGMV4, retGMV4]=estimatePortMoments(p4,wGMV4);
wGMV4=round(wGMV4,6);


figure(4);
[EF1risk,EF1ret]=p1.plotFrontier(100);
hold on
[mu1, sigma1] = getAssetMoments(p1);
scatter(sqrt(diag(sigma1)), mu1,'oc','filled');
scatter(riskMSR1,retMSR1,'filled','o');
scatter(riskGMV1,retGMV1,'filled','o');
legend('EF','Assets','MSR','GMV',Location='best')
ylabel('Expected Return')
xlabel('Standard Deviation')
text(sqrt(diag(sigma1))+0.001,mu1,LABELS)%,'FontSize',7); % Label ticker
names
hold off


figure(5)
area(EF1risk,pwgt1')
xlabel ('Volatility'); ylabel ('Weight');
colororder(COL);ylim([0 1]); xlim([min(EF1risk) max(EF1risk)]);
title('Composition Map') %Efficient Portfolio EP
legend(LABELS);


figure(6)
subplot(2,1,1);
bar([wMSR1]);
set(gca,'Xtick',1:10,'XTickLabel',LABELS);
title('a) MSR Portfolio Composition');
ylim([0 1])
subplot(2,1,2)
bar([wGMV1]);
set(gca,'Xtick',1:10,'XTickLabel',LABELS);
title('b) GMV Portfolio Composition');
ylim([0 1])

%% Different samples analysis
% Data Visualization: Plot 4 different frontiers with different time
windows

figure(7);
hold on;
```

```
[EF1risk,EF1ret]=p1.plotFrontier(100);
[EF2risk,EF2ret]=p2.plotFrontier(100);
[EF3risk,EF3ret]=p3.plotFrontier(100);
[EF4risk,EF4ret]=p4.plotFrontier(100);

scatter(riskMSR1,retMSR1,'filled','o');
scatter(riskMSR2,retMSR2,'filled','o');
scatter(riskMSR3,retMSR3,'filled','o');
scatter(riskMSR4,retMSR4,'filled','o');
scatter(riskGMV1,retGMV1,'filled','o');
scatter(riskGMV2,retGMV2,'filled','o');
scatter(riskGMV3,retGMV3,'filled','o');
scatter(riskGMV4,retGMV4,'filled','o');

legend('EF 10Y Sample','EF 5Y Sample','EF 2Y Sample','EF 1Y
Sample','MSR_1_0_Y_S','MSR_5_Y_S','MSR_2_Y_S','MSR_1_Y_S','GMV_1_0_Y_S','
GMV_5_Y_S','GMV_2_Y_S','GMV_1_Y_S',Location='best')

hold off

% Data Visualization: plot bar graphs with weights

figure(8)
subplot(2,1,1)
bar([wMSR1,wMSR2,wMSR3,wMSR4]);
set(gca,'Xtick',1:9,'XTickLabel',LABELS);
legend('Weights 10Y Sample','Weights 5Y Sample','Weights 2Y
Sample','Weights 1Y Sample',Location='best');
title('a) MSR Portfolio Composition')
ylim([0 1])
subplot(2,1,2)
bar([wGMV1,wGMV2,wGMV3,wGMV4]);
set(gca,'Xtick',1:9,'XTickLabel',LABELS);
legend('Weights 10Y Sample','Weights 5Y Sample','Weights 2Y
Sample','Weights 1Y Sample',Location='best');
title('b) GMV Portfolio Composition')
ylim([0 1])


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Optional

figure(9)
subplot(2,2,1)
[EF1risk,EF1ret]=p1.plotFrontier(100);
hold on
[mu1, sigma1] = getAssetMoments(p1);
scatter(sqrt(diag(sigma1)), mu1,'oc','filled');
scatter(riskMSR1,retMSR1,'filled','o');
scatter(riskGMV1,retGMV1,'filled','o');
legend('EF_1_0_Y_S','Assets','MSR_1_0_Y_S','GMV_1_0_Y_S',Location='best')
ylabel('Expected Return')
xlabel('Standard Deviation')
text(sqrt(diag(sigma1))+0.0001,mu1,LABELS)%,'FontSize',7); % Label ticker
names
title('a) MV EF 10Y Sample') %Efficient Portfolio EP
hold off

subplot(2,2,2)
[EF2risk,EF2ret]=p2.plotFrontier(100);
hold on
```

```
[mu2, sigma2] = getAssetMoments(p2);
scatter(sqrt(diag(sigma2)), mu2,'oc','filled')
scatter(riskMSR2,retMSR2,'filled','o');
scatter(riskGMV2,retGMV2,'filled','o');
legend('EF_5_Y_S','Assets','MSR_5_Y_S','GMV_5_Y_S',Location='best')
ylabel('Expected Return')
xlabel('Standard Deviation')
text(sqrt(diag(sigma2))+0.0001,mu2,LABELS)%,'FontSize',7); % Label ticker
names
title('b) MV EF 5Y Sample') %Efficient Portfolio EP
hold off

subplot(2,2,3)
[EF3risk,EF3ret]=p3.plotFrontier(100);
hold on
[mu3, sigma3] = getAssetMoments(p3);
scatter(sqrt(diag(sigma3)), mu3,'oc','filled');
scatter(riskMSR3,retMSR3,'filled','o');
scatter(riskGMV3,retGMV3,'filled','o');
legend('EF_2_Y_S','Assets','MSR_2_Y_S','GMV_2_Y_S',Location='best')
ylabel('Expected Return')
xlabel('Standard Deviation')
text(sqrt(diag(sigma3))+0.0001,mu3,LABELS)%,'FontSize',7); % Label ticker
names
title('c) MV EF 2Y Sample') %Efficient Portfolio EP
hold off

subplot(2,2,4)
[EF4risk,EF4ret]=p4.plotFrontier(100);
hold on
[mu4, sigma4] = getAssetMoments(p4);
scatter(sqrt(diag(sigma4)), mu4,'oc','filled');
scatter(riskMSR4,retMSR4,'filled','o');
scatter(riskGMV4,retGMV4,'filled','o');
legend('EF_1_Y_S','Assets','MSR_1_Y_S','GMV_1_Y_S',Location='best')
ylabel('Expected Return')
xlabel('Standard Deviation')
text(sqrt(diag(sigma4))+0.0001,mu4,LABELS)%,'FontSize',7); % Label ticker
names
title('d) MV EF 1Y Sample') %Efficient Portfolio EP
hold off

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Data Visualization: plot EF Composition maps

figure(10)
subplot(2,2,1)
area(EF1risk,pwgt1')
xlabel ('Volatility'); ylabel ('Weight');
colororder(COL);ylim([0 1]); xlim([min(EF1risk) max(EF1risk)]);
title('a) 10Y Sample') %Efficient Portfolio EP

subplot(2,2,2)
area(EF2risk,pwgt2')
xlabel ('Volatility'); ylabel ('Weight');
colororder(COL);ylim([0 1]); xlim([min(EF2risk) max(EF2risk)]);
title('b) 5Y Sample') %Efficient Portfolio EP

subplot(2,2,3)
```

```
area(EF3risk,pwgt3')
xlabel ('Volatility'); ylabel ('Weight');
colororder(COL);ylim([0 1]); xlim([min(EF3risk) max(EF3risk)]);
title('c) 2Y Sample') %Efficient Portfolio EP

subplot(2,2,4)
area(EF4risk,pwgt4')
xlabel ('Volatility'); ylabel ('Weight');
colororder(COL);ylim([0 1]); xlim([min(EF4risk) max(EF4risk)]);
title('d) 1Y Sample') %Efficient Portfolio EP

legend(LABELS);


save Sez4.mat
```

Figure 11

```
clear
clc

load Sez4.mat

%% data set up

mu = mu1;
C = sigma1;
Nports = 100;

p_RE = Portfolio;
p_RE = setAssetMoments(p_RE, mu, C);
p_RE = setDefaultConstraints(p_RE);

PortWts = estimateFrontier(p_RE, Nports);
[EffRisk, EffReturn] = estimatePortMoments(p_RE, PortWts);

% keeping asset correlation

A = chol(C);

%%%%%%%%%%%%%%%%%%%%%%%%%%

B =1000;

for j = 1:B
    dbstop if error

    Gamma2 = randn(size(mu,1),Nports);
    R = A'*Gamma2;
    sim_mu = mu'+(mean(R'));
    sim_C = cov(R');

    p_RE = Portfolio;
```

```
    p_RE = setAssetMoments(p_RE, sim_mu, sim_C);
    p_RE = setDefaultConstraints(p_RE);

    PortWts = estimateFrontier(p_RE, Nports);
    [PortRisk, PortReturn,] = estimatePortMoments(p_RE, PortWts);

figure(1)
   subplot(1,2,1)
    [EF_RErisk,EF_REret]=p_RE.plotFrontier(100);
    xlabel('Standard Deviation')
    ylabel('Expected Returns')
    grid("on")
    colororder(['#C21807';'#222021';'#C7C6C1'])
    title('a) Statistically equivalent MV EFs')
    hold on
end

hold off

subplot(1,2,2)
[EF1risk,EF1ret]=p1.plotFrontier(100);
xlabel('Standard Deviation')
ylabel('Expected Returns')
grid("on")

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Michaud RE

```
clear
clc

load Sez4.mat

EXP_RET = mu1;
COV = sigma1;

ASSET=10;
SIZE=11;
SIM= 1000;

% MV eff front.
[RISK2,ROR2,WTS2]=portopt(EXP_RET,COV,100);

STORE_WTS=zeros(100,ASSET,SIM);

% Run the technology

for i = 1:SIM
i
SIM_RET= mvnrnd(EXP_RET, COV,SIZE);
EXP_RET_SIM=mean(SIM_RET);
COV_SIM=cov(SIM_RET);
```

```
[RISK,ROR,WTS]=portopt(EXP_RET_SIM,COV_SIM,100);

if i<100
figure(1)
area(WTS)
xlabel ('Efficinet Portfolios'); ylabel ('Weight');
colororder(COL);
ylim([0 1]);
legend(LABELS,'Location','EastOutside');
xlim([1 100]);
end

STORE_WTS(:,:,i)= WTS;
end

RESAPL_WEIGHTS=mean(STORE_WTS,3);

EXP_RET_RESAMPL= RESAPL_WEIGHTS*EXP_RET;

RISK_RESAMPL = zeros(100,1);

for i = 1 :100
RISK_RESAMPL(i,1) = sqrt(RESAPL_WEIGHTS(i,:)*COV*RESAPL_WEIGHTS(i,:)');
end

%% Plot frontiers

figure(2)
plot (RISK2,ROR2,'R','LineWidth',2)
hold on
plot (RISK_RESAMPL,EXP_RET_RESAMPL,'B','LineWidth',2)
scatter(sqrt(diag(sigma1)), mu1,'oc','filled');
hold off
title('Efficient Frontier versus Resampled Frontier')
legend('Eff Front','RESAMPLED Front','Assets',Location='best')
text(sqrt(diag(sigma1))+0.001,mu1,LABELS)
xlabel('Standard Deviation')
ylabel('Expected Returns')
grid("on")

figure(3)
subplot(1,2,1)
area(WTS2)
legend(LABELS,'Location','best')
title('Composition of Efficient Portfolios')
ylim([0 1]);
xlim([1 100]);
colororder(COL);
subplot(1,2,2)
area(RESAPL_WEIGHTS)
title('Composition of Resampled Portfolios')
ylim([0 1]);
xlim([1 100]);
colororder(COL);


save Michaud_RE.mat
```

Black Litterman

```
clear
clc

load Sez4.mat

% define data taking 10y sample
r_assets=mret1;
A_Bench=readtable("Bench10yS.xlsx");
r_Bench=tick2ret(A_Bench{:,2:end});
r_mkt=mean(r_Bench);
var_mkt=var(r_Bench);
Sigma=sigma1;

assetNames = LABELS';
numAssets = size(mret1, 2);

% total 3 views: 2 absolute, 1 relatives
v = 3;
P = zeros(v, numAssets);
q = zeros(v, 1);
Omega = zeros(v);

% View 1
P(1, assetNames == "MSCIACWIENERGY") = 1;
q(1) = 0.01;
Omega(1, 1) = 1e-6;

% View 2
P(2, assetNames == "MSCIACWIAERODEFENSE") = 1;
P(2, assetNames == "MSCIACWIHEALTHCARE") = -1;
q(2) = 0.002;
Omega(2, 2) = 1e-5;

% View 3
P(3, assetNames == "MSCIACWIFINANCIALS") = 1;
P(3, assetNames == "MSCIACWIIT") = -1;
q(3) = 0.005;
Omega(3, 3) = 1e-5;


viewtable = array2table([P q diag(Omega)], 'VariableNames', [assetNames
"View_Return" "View_Uncertainty"]);


tau = 1/size(r_assets, 1);
C = tau*Sigma;

% Find the market portfolio.

numAssets = size(r_assets,2);
LB = zeros(1,numAssets);
Aeq = ones(1,numAssets);
Beq = 1;
```

```
opts = optimoptions('lsqlin','Algorithm','interior-point',
'Display',"off");
wtsMarket = lsqlin(r_assets, r_Bench, [], [], Aeq, Beq, LB, [], [],
opts);

% Find δ
shpr = mean(r_Bench)/std(r_Bench);
delta = shpr/sqrt(wtsMarket'*Sigma*wtsMarket);

% Compute Implied expected rert
PI = delta*Sigma*wtsMarket;


% Compute the Estimated Mean Return and Covariance
mu_bl = (P'*(Omega\P) + inv(C)) \ ( C\PI + P'*(Omega\q));
cov_mu = inv(P'*(Omega\P) + inv(C));

table (assetNames', PI, mu_bl, 'VariableNames',
["Asset_Name","Prior_Belief_of_Expected_Return",
"Black_Litterman_Blended_Expected_Return"])


% Portfolio Optimization and Results
port = Portfolio('NumAssets', numAssets, 'lb', 0, 'budget', 1, 'Name',
'Mean Variance');
port = setAssetMoments(port, mu1, Sigma);
wts = estimateMaxSharpeRatio(port);


portBL = Portfolio('NumAssets', numAssets, 'lb', 0, 'budget', 1, 'Name',
'Mean Variance with Black-Litterman');
portBL = setAssetMoments(portBL, mu_bl, Sigma + cov_mu);
pwgtBL=portBL.estimateFrontier(100);

% MAX Sharpe (or Tangency Portfolio)
wMSRBL=estimateMaxSharpeRatio(portBL);
[riskMSRBL, retMSRBL]=estimatePortMoments(portBL,wMSR1);
wMSRBL=round(wMSRBL,6);

% Minimum Variance Portfolio
wGMVBL=portBL.estimateFrontierLimits('Min');
[riskGMVBL, retGMVBL]=estimatePortMoments(portBL,wGMV1);
wGMVBL=round(wGMVBL,6);


figure(1)
hold on;
[EF1risk,EF1ret]=p1.plotFrontier(100);
[EFBLrisk,EFBLret]=portBL.plotFrontier(100);
scatter(sqrt(diag(sigma1)), mu1,'oc','filled');
hold off
title('MV Efficient Frontier versus BL Efficient Frontier')
legend('MV Eff Front','BL Eff Front','Assets',Location='best')
text(sqrt(diag(sigma1))+0.001,mu1,LABELS)


figure(2)
ax1 = subplot(1,2,1);
idx = wts>0.001;
pie(ax1, wts(idx), assetNames(idx));
```

```
colororder(['#FD6A02';'#702963';"#598BAF"]);
title(ax1, port.Name ,'Position', [-0.05, 1.6, 0]);

ax2 = subplot(1,2,2);
idx_BL = wMSRBL>0.001;
pie(ax2, wMSRBL(idx_BL), assetNames(idx_BL));
colororder(['#BDB7AB';'#FCD12A';"#598BAF";'#702963';'#784B84';"#B43757";'
#7285A5']);
title(ax2, portBL.Name ,'Position', [-0.05, 1.6, 0]);


figure(3)
subplot(1,2,1)
area(EF1risk,pwgt1')
xlabel ('Volatility'); ylabel ('Weight');
colororder(COL);ylim([0 1]); xlim([min(EF1risk) max(EF1risk)]);
title('a) Composition map of MV efficient Portfolios')
legend(LABELS,'Location','best')

subplot(1,2,2)
area(EFBLrisk,pwgtBL')
xlabel ('Volatility'); ylabel ('Weight');
title('b) Composition map of BL efficient Portfolios')
colororder(COL);ylim([0 1]); xlim([min(EFBLrisk) max(EFBLrisk)]);


figure(4)
subplot(2,1,1);
bar([wMSRBL]);
set(gca,'Xtick',1:10,'XTickLabel',LABELS);
title('a) MSR Portfolio Composition');
ylim([0 1])
subplot(2,1,2)
bar([wGMVBL]);
set(gca,'Xtick',1:10,'XTickLabel',LABELS);
title('b) GMV Portfolio Composition');
ylim([0 1])


figure(5)
plot (RISK2,ROR2,'R','LineWidth',2)
hold on
plot (RISK_RESAMPL,EXP_RET_RESAMPL,'B','LineWidth',2)
[EFBLrisk,EFBLret]=portBL.plotFrontier(100);
scatter(sqrt(diag(sigma1)), mu1,'oc','filled');
hold off
title('MV Efficient Frontier, Resampled Frontier, BL Efficient Frontier')
legend('MV Eff Front','RE Front','BL Eff Front','Assets',Location='best')
text(sqrt(diag(sigma1))+0.001,mu1,LABELS)
xlabel('Standard Deviation')
ylabel('Expected Returns')
grid("on")


save BL.mat
```

Performance

```
clear
clc

load Sez4.mat
load BL.mat
load Michaud_RE.mat


 %% data setup

ASSET=10;
SIZE=23;
Assets_ret = readmatrix('Performance.xlsx','Range','B2');
Benchmark_ret = readmatrix('PerformanceBench.xlsx','Range','B2');
Rf=0
RF=ones(23,7).*Rf;

pretGMV_MV= zeros(12,1);
pretGMV_RE= zeros(12,1);
pretGMV_BL= zeros(12,1);
pretMSR_MV= zeros(12,1);
pretMR_RE= zeros(12,1);
pretMSR_BL= zeros(12,1);


for i=1:SIZE
    pretGMV_MV(i,:)= wGMV1' * Assets_ret(i,:)';
    pretGMV_RE(i,:)= (RESAPL_WEIGHTS(1,:)) * Assets_ret(i,:)';
    pretGMV_BL(i,:)= wGMVBL' * Assets_ret(i,:)';
    pretMSR_MV(i,:)= wMSR1' * Assets_ret(i,:)';
    pretMR_RE(i,:)= (RESAPL_WEIGHTS(end,:)) * Assets_ret(i,:)';
    pretMSR_BL(i,:)= wMSRBL' * Assets_ret(i,:)';
end

allret=[Benchmark_ret pretGMV_MV pretGMV_RE pretGMV_BL pretMSR_MV
pretMR_RE pretMSR_BL];

%% PERFORMANCE MEASURES

%% Cum ret

Cum_allret=cumprod(allret)-1;

figure(1)
for i=1:7
    plot(Cum_allret(:,i),LineWidth=2);
    hold on
end
xlabel('Days')
ylabel('Daily Cumulated returns')
grid("on")
title ('Portfolios Cumulated Returns')
legend({'Benchmark';'GMV MV';'GMV RE';'GMV BL';...
    'MSR MV';'MR RE';'MSR BL'});
```

```
ylim ([-0.07 +.07]);
set(gca,'Xtick',1:24);

hold off

allret=allret-1;

%% Absolute performance measures
%---------------------------------------------------------------------

% Sharpe
pm1=mean(allret)'./sqrt(var(allret))';

% Sortino
s2=zeros(size(allret,2),1);
for j=1:size(allret,2)
    % compute semi-standard deviation
    s2(j)=sqrt(var(allret(allret(:,j)<0,j)));
end
pm2=mean(allret)'./ s2;

% Treynor
s3=zeros(size(allret,2),1);
for j=1:size(allret,2)
    s3(j)= (((allret(:,1)-mean(allret(:,1)))')*(allret(:,j)-
mean(allret(:,j))))...
        /((allret(:,1)-mean(allret(:,1)))'*(allret(:,1)-
mean(allret(:,1))));
end
pm3=mean(allret)'./ s3;

%--------------------------------------------------------------------
% Value-at-Risk
alpha=0.05;
s4=quantile(allret,alpha);
pm4=mean(allret)'./ abs(s4)';
adjust=[-1;-1;-1;-1;-1;1;1];
pm4=pm4.*adjust;

% Expected Shortfall
alpha=0.05;
s5=zeros(size(allret,2),1);
for j=1:size(allret,2)
    % compute conditional mean
    s5(j)=mean(allret(allret(:,j)<quantile(allret(:,j),alpha),j));
end
pm5=mean(allret)'./ abs(s5);
pm5=pm5.*adjust;

%-----------------------------------------------------------------------
-
% DrawDown sequence
DD=zeros(size(allret,1),size(allret,2));
for i=1:size(allret,2)
    DD(1,i)=min(allret(1,i),0);
    for j=2:size(allret,1)
        DD(j,i)=min(0,(1+DD(j-1,i))*(1+allret(j,i))-1);
    end
end
s6=max(abs(DD))';
```

```
% Calmar ratio
pm6=mean(allret)'./s6;

% Sterling ratio
k=5;
s7=zeros(size(allret,2),1);
for j=1:size(allret,2)
    % average of the largest DD
    [sDDj,~]=sort(abs(DD(:,j)),'descend');
    s7(j)=mean(sDDj(1:k));
end
pm7=mean(allret)'./s7;


%-----------------------------------------------------------------------
% Farinelli-Tibiletti
P=1;    % upside power
Q=2;    % downside power
Tau=0;  % threshold
% compute upside and downside partial moments
u8=zeros(size(allret,2),1);
s8=zeros(size(allret,2),1);
for j=1:size(allret,2)
    % compute partial moments
    u8(j)=(mean((abs(allret(:,j)-Tau).*(allret(:,j)>=Tau)).^P)).^(1/P);
    s8(j)=(mean((abs(allret(:,j)-Tau).*(allret(:,j)<Tau)).^Q)).^(1/Q);
end
pm8=u8./s8;


%------------------------------------------------------------------------

% summarizing results
allPM=[pm1 pm2 pm3 pm4 pm5 pm6 pm7 pm8];
Tab3=table({'Benchmark';'GMV MV';'GMV RE';'GMV BL';...
    'MSR MV';'MR RE';'MSR
BL'},allPM(:,1),allPM(:,2),allPM(:,3),allPM(:,4),allPM(:,5),allPM(:,6),al
lPM(:,7),allPM(:,8),...
    'VariableNames',{'Strategy' 'Sh' 'So' 'Tr' 'VaR' 'ES' 'Cal' 'Ste'
'FT'});
Tab3

% Ranks
[~,pm1r]=sort(pm1,'descend');
pm1r(pm1r)=1:length(pm1r);
[~,pm2r]=sort(pm2,'descend');
pm2r(pm2r)=1:length(pm1r);
[~,pm3r]=sort(pm3,'descend');
pm3r(pm3r)=1:length(pm1r);
[~,pm4r]=sort(pm4,'ascend');
pm4r(pm4r)=1:length(pm1r);
[~,pm5r]=sort(pm5,'ascend');
pm5r(pm5r)=1:length(pm1r);
[~,pm6r]=sort(pm6,'descend');
pm6r(pm6r)=1:length(pm1r);
[~,pm7r]=sort(pm7,'descend');
pm7r(pm7r)=1:length(pm1r);
[~,pm8r]=sort(pm8,'descend');
pm8r(pm8r)=1:length(pm1r);
allPMr=[pm1r pm2r pm3r pm4r pm5r pm6r pm7r pm8r];
```

```
% compute a composite index
CIpm=allPMr*ones(size(allPMr,2),1);

% summary table
Tab4=table({'Benchmark';'GMV MV';'GMV RE';'GMV BL';...
    'MSR MV';'MR RE';'MSR BL'},CIpm,
allPMr(:,1),allPMr(:,2),allPMr(:,3),allPMr(:,4),allPMr(:,5),...
    allPMr(:,6),allPMr(:,7),allPMr(:,8),...
    'VariableNames',{'Strategy' 'CI' 'Sh' 'So' 'Tr' 'VaR' 'ES' 'Cal'
'Ste' 'FT'});
Tab4

%% Relative performance measures

% Compute tracking errors
allTE=allret(:,2:end)-(allret(:,1)*ones(1,size(allret,2)-1));
TE=mean(allTE);
TEV=var(allTE);
SemiTEV=zeros(size(allTE,2),1);
for j=1:size(allTE,2)
    % compute semi-standard deviation
    SemiTEV(j)=sqrt(var(allTE(allTE(:,j)<0,j)));
end
IR=TE'./TEV';
SemiIR=TE'./SemiTEV;

Tab5=table({'GMV MV';'GMV RE';'GMV BL';...
    'MSR MV';'MR RE';'MSR BL'},TE',TEV',SemiTEV,IR,SemiIR,...
    'VariableNames',{'Portfolio' 'TE' 'TEV' 'SemiTEV' 'IR' 'SemiIR'});
Tab5

% Rank the different scores
[~,pm1t]=sort(TE','descend');
pm1t(pm1t)=1:length(pm1t);
[~,pm2t]=sort(TEV','ascend');
pm2t(pm2t)=1:length(pm1t);
[~,pm3t]=sort(SemiTEV,'ascend');
pm3t(pm3t)=1:length(pm1t);
[~,pm4t]=sort(IR,'descend');
pm4t(pm4t)=1:length(pm1t);
[~,pm5t]=sort(SemiIR,'descend');
allPMt=[pm1t pm2t pm3t pm4t pm5t];

% Compute a composite index
CIpmt=allPMt*ones(size(allPMt,2),1);

Tab6=table({'GMV MV';'GMV RE';'GMV BL';...
    'MSR MV';'MR RE';'MSR
BL'},CIpmt,allPMt(:,1),allPMt(:,2),allPMt(:,3),allPMt(:,4),allPMt(:,5),..
.
    'VariableNames',{'Portfolio' 'CI' 'TE' 'TEV' 'SemiTEV' 'IR'
'SemiIR'});
Tab6

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```