Università degli Studi di Padova

Dipartimento di Fisica e Astronomia Galileo Galilei

Corso di Laurea Magistrale in Fisica

Tesi di Laurea Magistrale

The inflationary tensor power spectrum in an approximate quantum de Sitter state

Laureanda:
Agnese Tucci

Relatore:
Prof. Sabino Matarrese

Correlatore:
Prof. Stefan Hofmann

Anno Accademico 2014 – 2015

Contents

The aim of science is not to open the door to infinite wisdom, but to set a limit to infinite error.

B. Brecht
Life of Galileo

Introduction

The discovery of the cosmic microwave background (CMB) by Penzias and Wilson [44] confirmed the Hot Big Bang paradigm and established the CMB as a central tool. In recent years, observations of its temperature anisotropies have helped establish and refine the "standard" cosmological model now known as ΛCDM, under which our universe is understood to be spatially flat, dominated by cold dark matter, and with a cosmological constant (Λ) driving accelerated expansion at late times.

Inflationary cosmology extends the standard model by postulating an early period of nearly exponential expansion which sets the initial conditions for the subsequent Hot Big Bang. It was proposed and developed in the early 1980s to resolve mysteries for which the standard model offered no solution, including the flatness, smoothness, entropy and monopole problems (e.g. [2, 29, 37]; see [13] for a review). Inflation also explains the universe's primordial perturbations as originating in quantum fluctuations strechted by exponential expansion into seeds that eventually cause the formation of galaxies and clusters of galaxies [6, 42, 58], and thus to be correlated on superhorizon scales. The simplest models further predict these perturbations to be highly adiabatic and Gaussian and nearly scale-invariant (though typically slightly stronger on large scales). These qualities, though not necessarily unique to the inflationary paradigm, have all been confirmed by subsequent observations (e.g. [55,60]). Although highly successful, the inflationary paradigm represents a vast extrapolation from well-tested regimes in physics. It invokes quantum effects in highly curved spacetime at energies near 10^{16} GeV and timescales less tha 10^{-32} s. A definitive test of this paradigm would be of fundamental importance.

Gravitational waves (tensor perturbations) generated by inflation have the potential to provide such a definitive test. Inflation predicts that the quantizion of the gravitational field coupled to exponential expansion produces a primordial background of stochastic gravitational waves with a characteristic spectral shape, i.e., they have a nearly scale invariant primordial power spectrum [27, 57]. Though unlikely to be detectable in modern instruments, these gravitational waves would have imprinted an unique signature upon CMB. Gravitational waves induce local quadrupole anisotropies in the radiation field within the last-scattering surface, inducing polarization in the scattered light [48]. This polarization pattern will include a "curl" or B-mode component at degree angular scales that cannot be generated primordially by density perturbations. The amplitude of this signal depends upon the tensor-to-scalar ratio r, which itself is a function of the energy scale of inflation. The detection of B-mode polarization of the CMB at large angular scales would provide an unique confirmation of inflation and a probe of its energy scale [33, 52].

The gravitational wave background (GWB) also persists as a sea of relic gravitational radiation filling the universe today [51, 54, 62]. Direct detection of this relic radiation has received considerable attention over the past year or so, since it has been realized that space-based laser interfereometers operating in the frequency range 0.1 Hz $< f <$ 1 Hz might achieve the necessary sensitivity and foreground subtraction [53, 61].

The gravitational-wave spectrum generated by inflation carries important information about the conditions during inflation. But the spectrum also receives corrections, both large and small, from the subsequent evolution and matter content of the universe after inflation. In this thesis we identify various postinflationary physical effects, which modify the GWB, and show how they may be encoded in the gravitational-wave transfer function that relates the primordial tensor power spectrum to the gravitational-wave spectrum at a later point in cosmic history. It is necessary to properly understand and disentangle the postinflationary effects in order to optimally extract the inflationary information in the GWB. But these modifications are also interesting in their own right, since they offer a rare window onto the physical properties of the high-energy universe during the "primordial dark age" between the end of inflation and the electroweak phase transition.

The same features that make the inflationary GWB difficult to detect—namely its small amplitude and gravitational-strength coupling to matter—also make it a clean cosmological probe. First, because of their tiny amplitude, the gravitational waves obey linear equation of motion, so that their evolution may be predicted analytically with high precision. Second, because of their ultraweak interactions with matter, the gravitational waves have been free streaming since the end of inflation—in contrast to neutrinos (which began streming roughly a second later) and photons (which began streaming several hundred thousand year later). The gravitational waves carry unsullied information from the early universe, and subsequent modifications of the gravitational-wave spectrum are not washed out by thermal effects (since the gravitons are thermally decoupled).

The gravitational-wave spectrum near a given wave number k is primarily sensitive to two "moments" in cosmic history: (1) the moment when the mode "left the horizon" (i.e., became longer than the instantaneous Hubble radius during inflation), and (2) the moment when the moment "re-entered the horizon" (i.e., became shorter that the instantaneous Hubble radius once again, after the end of inflation). The first moment imprints information about inflation itself, while the second moment imprints information about postinflationary conditions. The CMB is sensitive to long-length modes that re-entered at relatively low tempertures (well after the Big Bang Nucleosynthesis (BBN)), corresponding to relatively well-understood physics. By contrast, laser interferometers are sensitive to shorter wavelengths that entered the horizon at high temperatures ($T \sim 10^7$ GeV), well above the electroweak phase transition. The physical conditions at such high energies, which are considerably beyond the reach of particle accelerators, are a mystery, so that any information about this epoch from the GWB would be very valuable.

In the usual model of inflation, the initial state is assumed to be the empty vacuum in the infinite past when all scales that have a finite linear size today have a size infinitely smaller than the Planck scale. Even though this does not make too much sense—after all, we have no idea of how the physics at these scales work—it is interesting that this naive approach seems to give sensible results. But if I fill spacetime entirely with a spacetime homogeneous source (energy density), I cannot have Minkowski not even asymptotically because there is a medium everywhere. A de Sitter source, no matter how small its strength is, will destabilize the spacetime.

How does the choice of vacuum reflect to the background metric? You stick to the original Minkowski definition of a perturbative vacuum and try to lift it to non-trivial geometries. The usual argument goes as follows: if you go to high-frequency limit, then you're which is arbitrary small and you expect that morally you're still in the tangent space, which would be Minkowski anyhow. If you think in a real world what is the true ground state of nature on a curved geometry (you take gravity with you), you have to deal with a difficult question.

Let's describe our path of line. We want to investigate if there is some physics hidden in a non-perturbative vacuum proposal that could boost the power spectrum (maybe in the B-mode polarization). There are two logical possibilities: either spacetime geometry, as well as Quantum Mechanics is fundamental, or it is not fundamental. We have no reason to know that spacetime geometry is fundamental: Minkowski geometry is already granted by special relativity (I don't need general relativity for Minkowski). Let's take one of these logical possibilities: spacetime geometry is not fundamental. **Pseudo-Riemannian geometry is not as fundamental as Hilbert geometry**. Nobody can prove that this point of view is wrong, it's not even unattractive by the way. Anything that is not fundamental has to have somehow a reflection in a Hilbert space because Quantum Mechanics is fundamental. We now set it as the only fundamental framework we have in physics and if that is true, everything we encounter in nature has to fit in this framework. Then the notion of spacetime geometry has to emerge and has to be anchored fundamentally in a Hilbert space. In other words, spacetime itself should have a quantum mechanical description.

What does it mean? First of all, it means that I need a quantum mechanical state for what I want to describe, which will be the non-perturbative vacuum state. That would mean also that to a given geometry would correspond a quantum mechanical state and in this state I want to evaluate my autocorrelation function, i.e., the power spectrum. This path might be unorthodox, but we're exploring a logical possibility. How could the state look like that corresponds to a certain geometry? The answer is the Auxiliary Current Description (ACD).

It's difficult to imagine how these macroscopic objects (like a table) fit in a Hilbert space. The table for instance should live in the bound state sector of the Hilbert space. What is the state of the table? I would give up because it has too many particles, but then I can think of what is essential for the state description. Usually in Quantum Mechanics the state description comes with two sort of lists: you can picture a state as a measuring device where you can write in and read out information. What would you want to write in and read out? There are the quantum numbers and the identifiers of the state that are connected to the spacetime isometries, e.g., electrical charge, spin, etc. But I can assign to it a momentum if I want. Why the momentum? Because out of the momentum I build a Casimir operator which gives the mass of the our object. For a free object the momentum doesn't change so it's a smart thing to use it. To sum up there are the two type of lists that you can store in a quantum state: the quantum numbers and the spacetime isometries. Then a very complicated object from that point of view is not so complicated: we just have to write a state with all the quantum numbers and the correct isometries.

This state needs not to be the true state in nature, it could be a trial state. Why? Because, first of all, there could be quantum numbers we didn't know about and the other thing is, that this description is purely kinematical. Of course there should be a dynamics and the true state in nature should be solution of the dynamical equation of motion. But the equation of motion for a bound state we don't know: we only know the equation of motion for the constituents (free or interacting). We construct now a kinematical state which is not yet a solution of the dynamical equation, but it has the right quantum numbers. Now we claim that this kinematical state has a non-trivial projection on the true physical state. But the true physical state could be extremely complicated and since there is a non-vanishing overlap, we can work with the kinematical state to learn something about the true state. The kinematical state is fully under our control and if we follow that path of line, then it's also clear that there is also a chance to describe geometry. We play the same game with the quantum numbers.

We want something like an empty (i.e., no stored intrinsic quantum numbers) memory configuration which we usually call a ground state. Then we want to create the kinematical state by operating with certain "read-in" devices on that ground state. The "read-in" device is like a current $\mathcal{J}(x)$. It's like a computer works: the current stores information into the memory and the memory configuration, the current generates, is what you're interested in. Now you've operation definition which is useful because we can relate any kinematical state to this specific ground state configuration.

The amazing thing is that is not so hard to think about currents: if it's really true that we anchor geometry in Hilbert space, we fix spacetime symmetries beforehand to avoid any problem. If isometries specify the quantum state of the geometry, then we try to construct a current that transforms invariantly under the appropriate Killing vectors. It's using a quantum operator $\mathcal{J}$ which isolates the right quantum numbers that we identify the spacetime. $\mathcal{J}$ has to be a composite operator, composite of the degrees of freedom you're investigating in the theory.

Suppose now we can do that with geometry: we build our theory on a kinematical description. How an observable (e.g. the power spectrum) looks in a geometry? This question becomes encoded in a quantum mechanical framework. I formulate the problem as a quantum one because fundamentally the geometry plays no role in this description. Based on that, we want to calculate the power spectrum and compare it to what we observe. We take the operator that measures the power spectrum and sandwich it with a state that is the kinematical representation of the geometry in which you want to evaluate that operator.

This can give rise to a radical different power spectrum. It changes the vacuum energy density in de Sitter.

CHAPTER 1

Cosmological perturbation theory

In this chapter, we first summarize basic facts of cosmological perturbation theory. [38, 43, 45] Then we concentrate on quantum fluctuations during inflation. [19, 36, 41]

The reason why inflation inevitably produces fluctuations is simple: the inflaton evolution $\phi(t)$ governs the energy density of the early universe $\rho(t)$ and hence controls the end of inflation. Essentially, ϕ plays the role of a local "clock" reading off the amount of inflationary expansion remaining. By uncertainity principle, arbitrarly precise timing is not possible in quantum mechanics. Instead, quantum-mechanical clocks necessarily have some variance, so the inflaton will have spatially varying fluctuations $\delta\phi(t, \mathbf{x}) \equiv \phi(t, \mathbf{x}) - \bar{\phi}(t)$. There will hence be local differences in the time when inflation ends, $\delta t(\mathbf{x})$, so that different regions of space inflate by different amounts. Moreover, these differences in the local expansion histories lead to differences in the local densities after inflation. In quantum theory, local fluctuations in $\delta\rho(t, \mathbf{x})$ are therefore unavoidable. The main purpose of this chapter is to compute this effect.

1.1 The perturbed universe

We consider perturbations to the homogeneous background and the stress-energy of the universe.

1.1.1 Metric perturbations

The most general first-order perturbation to a spatially flat Friedmann-Robertson-Walker (FRW) metric is

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + 2a(t)B_i\,\mathrm{d}x^i\,\mathrm{d}t + a^2(t)\left[(1-2\Psi)\delta_{ij} + 2E_{ij}\right]\mathrm{d}x^i\,\mathrm{d}x^j \tag{1.1.1}$$

where Φ is a 3-scalar called the lapse, B_i is a 3-vector called the shift, Ψ is a 3-scalar called the spatial curvature perturbation, and E_{ij} is a spatial shear 3-tensor which is symmetric and traceless, E^i_i slices and curves of constant spatial coordinates x^i but varying time t are called threads.

1.1.2 Matter perturbations

The energy-momentum tensor may be described by a density ρ, a pressure p, a 4-velocity u^μ (of the frame in which the 3-momentum density vanishes) and an anisotropic stress $\Pi^{\mu\nu}$.

Density and pressure perturbations are defined in an obvious way

$$\delta\rho(t, x^i) \equiv \rho(t, x^i) - \bar{\rho}(t), \quad \text{and} \quad \delta p(t, x^i) \equiv p(t, x^i) - \bar{p}(t). \tag{1.1.2}$$

Here, the background values have been denoted by overbars. The 4-velocity has only three indipendent components (after the metric is fixed) since it has to satisfy the constraint $g_{\mu\nu}u^\mu u^\nu = -1$. In the perturbed metric (1.1.1) the perturbed 4-velocity is

$$u_\mu \equiv (-1-\Phi, v_i), \quad \text{or} \quad u^\mu \equiv (1-\Phi, v^i + B^i). \tag{1.1.3}$$

Here, u_0 is chosen so that the constraint $u_\mu u^\mu = -1$ is satisfied to first order in all perturbations. Anisotropic stress vanishes in the unperturbed FRW universe, so $\Pi^{\mu\nu}$ is a first-order perturbation. Furthermore, $\Pi^{\mu\nu}$ is constrained by

$$\Pi^{\mu\nu}u_\nu = \Pi^\mu_\mu = 0. \tag{1.1.4}$$

The orthogonality with u_μ implies $\Pi^{00} = \Pi^{0j} = 0$, i.e. only the spatial components Π^{ij} are non-zero. The trace condition then implies $\Pi^i_i = 0$. Anisotropic stress is therefore a traceless symmetric 3-tensor.

Finally, with these definitions the perturbed stress-tensor

$$T^0_0 = -(\bar\rho + \delta\rho), \tag{1.1.5}$$

$$T^0_i = (\bar\rho + \bar p)v_i, \tag{1.1.6}$$

$$T^i_0 = -(\bar\rho + \bar p)\left(v^i + B^i\right), \tag{1.1.7}$$

$$T^i_j = \delta^i_j(\bar p + \delta p) + \Pi^i_j. \tag{1.1.8}$$

If there are several contributions to the energy-momentum tensor (e.g. photons, baryons, dark matter, etc.), they are added: $T_{\mu\nu} = \sum_I T^I_{\mu\nu}$. This implies

$$\delta\rho = \sum_I \delta\rho_I \tag{1.1.9}$$

$$\delta p = \sum_I \delta p_I, \tag{1.1.10}$$

$$(\bar\rho + \bar p)v^i = \sum_I (\bar\rho_I + \bar p_I)v^i_I, \tag{1.1.11}$$

$$\Pi^{ij} = \sum_I \Pi^{ij}_I. \tag{1.1.12}$$

Density, pressure and anisotropic stress perturbations simply add. However, velocities do not add, which motivates defining the 3-momentum density

$$\delta q^i \equiv (\bar\rho + \bar p)v^i, \tag{1.1.13}$$

such that

$$\delta q^i = \sum_I \delta q^i_I. \tag{1.1.14}$$

1.2 Scalars

1.2.1 Metric perturbations

Four scalar metric perturbations Φ, $B_{,i}$, $\Psi\delta_{ij}$ and $E_{,ij}$ may be constructed from 3-scalars, their derivatives and the background spatial metric, i.e.

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + 2a(t)B_{,i}\,\mathrm{d}x^i\,\mathrm{d}t + a^2(t)\left[(1-2\Psi)\delta_{ij} + 2E_{,ij}\right]\mathrm{d}x^i\,\mathrm{d}x^j, \tag{1.2.1}$$

Here, we have absorbed the $\nabla^2 E\delta_{ij}$ part of the helicity scalar E^S_{ij} [1] in $\Psi\delta_{ij}$.

The intrinsic Ricci scalar curvature of constant time hypersurfaces is

$$R_{(3)} = \frac{4}{a^2}\nabla^2\Psi. \tag{1.2.2}$$

This explains why Ψ is often referred to as the curvature perturbation.

There are two scalar gauge transformations

$$t \to t + \alpha, \tag{1.2.3}$$
$$x^i \to x^i + \delta^{ij}\beta_{,j}\,. \tag{1.2.4}$$

Under these coordinate transformations the scalar metric perturbations transform as

$$\Phi \to \Phi - \dot{\alpha}, \tag{1.2.5}$$
$$B \to B + a^{-1}\alpha - a\dot{\beta}, \tag{1.2.6}$$
$$E \to E - \beta, \tag{1.2.7}$$
$$\Psi \to \Psi + H\alpha. \tag{1.2.8}$$

Note that the combination $\dot{E} - B/a$ is indipendent of the spatial gauge and only depends on the temporal gauge. It is called the scalar potential for the anisotropic shear of the world lines orthogonal to constant time hypersurfaces. To extract physical results it is useful to define gauge-invariant combinations of the scalar metric perturbations. Two important gauge-invariant quantities were introduced by Bardeen [5]

$$\Phi_B \equiv \Phi - \frac{\mathrm{d}}{\mathrm{d}t}\left[a^2\left(\dot{E} - B/a\right)\right], \tag{1.2.9}$$
$$\Psi_B \equiv \Psi + a^2 H\left(\dot{E} - B/a\right) \tag{1.2.10}$$

1.2.2 Matter perturbations

Matter perturbations are also gauge-dependent, e.g. density and pressure perturbations transform as follows under temporal gauge transformations

$$\delta\rho \to -\dot{\bar{\rho}}\alpha, \qquad \delta p \to \delta p - \dot{\bar{p}}\alpha. \tag{1.2.11}$$

Adiabatic pressure perturbations are defined as

$$\delta p_{ad} \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\delta\rho. \tag{1.2.12}$$

The non-adiabatic, or entropic, part pf the pressure perturbations is then gauge-invariant

$$\delta p_{en} \equiv \delta p - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\delta\rho. \tag{1.2.13}$$

The scalar part of the 3-momentum density, $(\delta q)_{,i}$, transforms as

$$\delta q \to \delta q + (\dot{\bar{\rho}} + \dot{\bar{p}})\alpha. \tag{1.2.14}$$

[1] See SVT decompositon or Helmholtz's theorem [38]

We may then define the gauge-invariant comoving density perturbation

$$\delta\rho_m \equiv -3H\delta q \tag{1.2.15}$$

Finally, two important gauge-invarinat quantities are formed from combinations of matter and metric perturbations. The *curvature perturbation on uniform density hypersurfaces* is

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar\rho}}\delta\rho. \tag{1.2.16}$$

The comoving curvature perturbation is

$$\mathcal{R} = \Psi - \frac{H}{\bar\rho + \bar p}\delta q. \tag{1.2.17}$$

We will show that ζ and $\mathcal{R}$ are equal on superhorizon scales, where they become time-independent. The computation of the inflationary perturbation is most clearly phrased in terms of ζ and $\mathcal{R}$.

1.2.3 Einstein equations

To relate the metric and stress-energy perturbations [41,43], we consider the perturbed Einstein equations

$$\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu} \tag{1.2.18}$$

We work at linear order. This leads to the energy and momentum constraint equations

$$3H\left(\dot\Psi + H\Phi\right) + \frac{k^2}{a^2}\left[\Psi + H\left(a^2\dot E - aB\right)\right] = -4\pi\delta\rho \tag{1.2.19}$$

$$\dot\Psi + H\Psi = -4\pi G\delta q. \tag{1.2.20}$$

These can be combined into the gauge-invariant Poisson equation

$$\frac{k^2}{a^2}\Psi_B = -4\pi G\delta\rho_m. \tag{1.2.21}$$

The Einstein equation also yield two evolution equations

$$\ddot\Psi + 3H\dot\Psi + H\dot\Phi + \left(3H^2 + 2\dot H\right)\Phi = 4\pi G\left(\delta p - \frac{2}{3}k^2\delta\Pi\right) \tag{1.2.22}$$

$$\left(\partial_t + 3H\right)\left(\dot E - B/a\right) + \frac{\Psi - \Phi}{a^2} = 8\pi G\delta\Pi. \tag{1.2.23}$$

The last equation may be written as

$$\Psi_B - \Phi_B = 8\pi Ga^2\delta\Pi. \tag{1.2.24}$$

In the absence of anisotropic stress this implies, $\Psi_B = \Phi_B$. Energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, gives the continuity equation and the Euler equation

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) = \frac{k^2}{a^2}\delta q + (\bar\rho + \bar p)\left[3\dot\Psi + k^2\left(\dot E + B/a\right)\right], \tag{1.2.25}$$

$$\dot{\delta q} + 3H\delta q = -\delta p + \frac{2}{3}k^2\delta\Pi - (\bar\rho + \bar p)\,\Phi. \tag{1.2.26}$$

Expressed in terms of the curvature perturbation on uniform density hypersurfaces, ζ, (1.2.26) reads

$$\dot{\zeta} = -H\frac{\delta p_{en}}{\bar{\rho}+\bar{p}} - \Pi, \tag{1.2.27}$$

where δp_{en} is the non-adiabatic component of the pressure, and Π is the scalar shear along comoving worldlines

$$\begin{aligned} \frac{\Pi}{H} &\equiv -\frac{k^2}{3H}\left[\dot{E} - \frac{B}{a} + \frac{\delta q}{a^2(\bar{\rho}+\bar{p})}\right] = \\ &= -\frac{k^2}{3a^2H^2}\left[\zeta - \Psi_B\left(1 - \frac{2\bar{\rho}}{9(\bar{\rho}+\bar{p})}\frac{k^2}{a^2H^2}\right)\right]. \end{aligned} \tag{1.2.28}$$

For adiabatic perturbations, $\delta p_{en} = 0$ on superhorizon scales, $k/(aH) << 1$ (i.e. $\Pi/H \to 0$ for finite ζ and Ψ_B), the curvature perturbation ζ is constant.

1.2.4 Popular gauges

For reference we now give the Einstein equations and the conservation equations in various popular gauges [38]:

Synchronous gauge A popular gauge, especially for numerical implementation of the perturbation equations, is synchronous gauge. It is defined by

$$\Phi = B = 0. \tag{1.2.29}$$

The Einstein equations become

$$3H\dot{\Psi} + \frac{k^2}{a^2}\left[\Psi + Ha^2\dot{E}\right] = -4\pi G\delta\rho, \tag{1.2.30}$$

$$\dot{\Psi} = -4\pi G\delta q, \tag{1.2.31}$$

$$\ddot{\Psi} + 3H\dot{\psi} = 4\pi G\left(\delta p - \frac{2}{3}k^2\delta\Pi\right), \tag{1.2.32}$$

$$(\partial_t + 3H)\,\dot{E} + \frac{\Psi}{a^2} = 8\pi G\delta\Pi \tag{1.2.33}$$

The conservation equations are

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) = \frac{k^2}{a^2}\delta q + (\bar{\rho}+\bar{p})\left[3\dot{\Psi} + k^2\dot{E}\right], \tag{1.2.34}$$

$$\dot{\delta q} + 3H\delta q = -\delta p + \frac{2}{3}k^2\delta\Pi \tag{1.2.35}$$

Newtonian gauge The Newtonian gauge has its name because it reduces to Newtonian gravity in the samll-scale limit. It is popular for analytic work since it leads to algebraic relations between metric and stress-energy perturbations.

Newtonian gauge is defined by

$$B = E = 0, \tag{1.2.36}$$

and

$$\mathrm{d}s^2 - (1+2\Phi)\,\mathrm{d}t^2 + a^2(t)(1-2\Psi)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{1.2.37}$$

The Einstein equations are

$$3H(\dot{\psi} + H\Phi) + \frac{k^2}{a^2}\Psi = -4\pi G\delta\rho, \tag{1.2.38}$$

$$\dot{\Psi} + H\Phi = -4\pi G\delta q, \tag{1.2.39}$$

$$\ddot{\Psi} + 3H\dot{\Phi} + \left(3H^2 + 2\dot{H}\right)\Phi = 4\pi G\left(\delta p - \frac{2}{3}k^2\delta\Pi\right), \tag{1.2.40}$$

$$\frac{\Psi - \Phi}{a^2} = 8\pi G\delta\Pi. \tag{1.2.41}$$

The continuity equations are

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) = \frac{k^2}{a^2}\delta q + 3(\bar{\rho} + \bar{p})\dot{\Psi}, \tag{1.2.42}$$

$$\dot{\delta q} + 3H\delta q = -\delta p + \frac{2}{3}k^2\delta\Pi - (\bar{\rho} + \bar{p})\,\Phi. \tag{1.2.43}$$

Uniform density gauge The uniform density gauge is useful for describing the evolutions of perturbations on superhorizon scales. As its name suggests it is defined by

$$\delta\rho = 0. \tag{1.2.44}$$

In addition, it is convenient to take

$$E = 0, \qquad -\psi \equiv \zeta. \tag{1.2.45}$$

The Einstein equations are

$$3H\left(-\dot{\zeta} + H\Phi\right) - \frac{k^2}{a^2}[\zeta + aHB] = 0 \tag{1.2.46}$$

$$-\dot{\zeta} + H\Phi = -4\pi G\delta q, \tag{1.2.47}$$

$$-\ddot{\zeta} - 3H\dot{\zeta} + H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi = 4\pi G\left(\delta p - \frac{2}{3}k^2\delta\Pi\right), \tag{1.2.48}$$

$$(\partial_t + 3H)\frac{B}{a} + \frac{\zeta + \Phi}{a^2} = -8\pi G\delta\Pi. \tag{1.2.49}$$

The continuity equations are

$$3H\delta p = \frac{k^2}{a^2}\delta q + (\bar{\rho} + \bar{p})\left[-3\dot{\zeta} + k^2B/a\right], \tag{1.2.50}$$

$$\dot{\delta q} + 3H\delta q = -\delta p + \frac{2}{3}k^2\delta\Pi - (\bar{\rho} + \bar{p})\,\Phi. \tag{1.2.51}$$

Comoving gauge Comoving gauge is defined by the vanishing of the scalar momentum density,

$$\delta q = 0, \qquad E = 0. \tag{1.2.52}$$

It is also conventional to set $-\Psi \equiv \mathcal{R}$ in this gauge.

The Einstein equations are

$$3H\left(-\dot{\mathcal{R}}+H\Phi\right)-\frac{k^2}{a^2}[\mathcal{R}+aHB]=-4\pi G\delta\rho, \tag{1.2.53}$$

$$-\dot{\mathcal{R}}+H\Phi=0, \tag{1.2.54}$$

$$-\ddot{\mathcal{R}}-3H\dot{\mathcal{R}}+H\dot{\Phi}+(3h^2+2\dot{H})\Phi=4\pi G\left(\delta p-\frac{2}{3}k^2\delta\Pi\right), \tag{1.2.55}$$

$$(\partial_t+3H)\frac{B}{a}+\frac{\mathcal{R}+\Phi}{a^2}=-8\pi G\delta\Pi. \tag{1.2.56}$$

The continuity equations are

$$\dot{\delta\rho}+3H(\delta\rho+\delta p)=(\bar{\rho}+\bar{p})\left[-3\dot{\mathcal{R}}+k^2B/a\right], \tag{1.2.57}$$

$$0=-\delta p+\frac{2}{3}k^2\delta\Pi-(\bar{\rho}+\bar{p})\,\Phi. \tag{1.2.58}$$

Equations (1.2.58) and (1.2.55) may be combined into

$$\Phi=\frac{-\delta p+\frac{2}{3}\Pi}{\bar{\rho}+\bar{p}},\qquad kB=\frac{4\pi Ga^2\delta\rho-k^2\mathcal{R}}{aH}. \tag{1.2.59}$$

Spatially-flat gauge A convenient gauge for computing inflatioanry perturbation is spatially-flat gauge

$$\Psi=E=0. \tag{1.2.60}$$

During inflayion all scalar perturbations are then described by $\delta\phi$.

The Einstein equations are

$$3H^2\Pi+\frac{k^2}{a^2}[-aHB]=-4\pi G\delta\rho, \tag{1.2.61}$$

$$H\Phi=-4\pi G\delta q, \tag{1.2.62}$$

$$H\dot{\Phi}+\left(3H^2+2\dot{H}\right)\Phi=4\pi G\left(\delta p-\frac{2}{3}k^2\delta\Pi\right), \tag{1.2.63}$$

$$(\partial_t+3H)\frac{B}{a}+\frac{\Phi}{a^2}=-8\pi G\delta\Pi. \tag{1.2.64}$$

The continuity equations are

$$\dot{\delta\rho}+3H(\delta\rho+\delta p)=\frac{k^2}{a^2}\delta q+(\bar{\rho}+\bar{p})\,[k^2B/a], \tag{1.2.65}$$

$$\dot{\delta q}+3H\delta q=-\delta p+\frac{2}{3}k^2\delta\Pi-(\bar{\rho}+\bar{p})\,\Phi. \tag{1.2.66}$$

1.3 Vectors

1.3.1 Metric perturbations

Vector type metric perturbations are defined as

$$\mathrm{d}s^2=-\,\mathrm{d}t^2+2a(t)S_i\,\mathrm{d}x^i\,\mathrm{d}t+a^2(t)\left[\delta_{ij}+2F_{(i,j)}\right]\mathrm{d}x^i\,\mathrm{d}x^j, \tag{1.3.1}$$

where $S_{i,i} = F_{i,i} = 0$. The vector gauge transformation is

$$x^i \to x^i + \beta^i, \qquad \beta_{i,i} = 0. \tag{1.3.2}$$

They lead to the transformations

$$S_i \to S_i + a\dot{\beta}_i \tag{1.3.3}$$
$$F_i \to F_i - \beta_i. \tag{1.3.4}$$

The combination $\dot{F}_i + S_i/a$ is called the gauge-invariant vector shear perturbation.

1.3.2 Matter perturbations

We define the vector part of the anisotropic stress by

$$\delta\Pi_{ij} = \partial_{(i}\Pi_{j)}, \tag{1.3.5}$$

where Π_i is divergence-free, $\Pi_{i,i} = 0$.

1.3.3 Einstein equations

For vector perturbations there are only two Einstein equations,

$$\dot{\delta q}_i + 3H\delta q_i = k^2\delta\Pi_i, \tag{1.3.6}$$
$$k^2\left(\dot{F}_i + S_i/a\right) = 16\pi G\delta q_i. \tag{1.3.7}$$

In the absence of anisotropic stress ($\delta\Pi_i = 0$) the divergence-free momentum δq_i decays with the expansion of the universe; see Eq. (1.3.7). The shear perturbation $\dot{F}_i + S_i/a$ then vanishes by Eq. (1.3.7). Under most circumstances vector perturbations are therefore subdominant. In particular, vector perturbations aren't created by inflation.

1.4 Tensors

1.4.1 Metric perturbations

Tensor metric perturbations are defined as

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left[\delta_{ij} + h_{ij}\right]\mathrm{d}x^i\,\mathrm{d}x^j, \tag{1.4.1}$$

where $h_{ij,i} = h^i_i = 0$. Tensor perturbations are automatically gauge-invariant (at linear order). It is conventional to decompose tensor perturbations into eigenmodes of the spatial Laplacian, $\nabla^2 e_{ij} = -k^2 e_{ij}$, with comoving wavenumber k and scalar amplitude $h(t)$,

$$h_{ij} = h(t)e^{(+,\times)}_{ij}(\mathbf{x}). \tag{1.4.2}$$

Here, + and × denote the two possible polarization states.

1.4.2 Matter perturbations

Tensor perturbations are sourced by anisotropic stress Π_{ij}, with $\Pi_{ij,i} = \Pi^i_i = 0$. It is typically a good approximation to assume that the anisotropic stress is negligible, although a small amplitude is induced by neutrino free-streaming.

1.4.3 Einstein equations

For tensor perturbations there is only one Einstein equation. In the absence stress this is

$$\ddot{h} + 3H\dot{h} + \frac{k^2}{a^2}h = 0. \tag{1.4.3}$$

This is a wave equation describing the evolution of gravitational waves in an expanding universe. Gravitational waves are produced by inflation, but then decay with the expansion of the universe. However, at recombination their amplitude may still be large enough to leave distinctive signatures in B-modes of CMB polarization. We will analyze the tensor perturbations in more detail in the next chapter.

1.5 Classical perturbations

For concreteness, we will consider single-field slow-roll models of inflation

$$S = \int \mathrm{d}^4 x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}\mathcal{R} - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{1.5.1}$$

In this chapter we will study scalar fluctuations. For the scalar modes we have to be careful to identify the true physical degrees of freedom. A priori, we have 5 scalar modes: 4 metric perturbations— δg_{00}, δg_{ii}, $\delta g_{0i} \sim \partial_i B$ and $\delta g_{ij} \sim \partial_i\partial_j H$— and 1 scalar field perturbation $\delta\phi$. Gauge invariances associated with the invariance of (1.5.1) under scalar coordinate transformations —$t \to t + \epsilon_0$ and $x_i \to x_i + \partial_i\epsilon$— remove two modes. The Einstein constraint equations remove two more modes, so that we are left with 1 physical scalar mode. Deriving the quadratic action for this mode is the aim of this section.

1.5.1 Comoving gauge

We will work in a fixed gauge throughout. For a number of reasons it will be convenient to work in comoving gauge, defined by the vanishing of the momentum density, $\delta T_{0i} \equiv 0$, as we saw in (1.2.52). For slow-roll inflation this becomes

$$\delta\phi = 0. \tag{1.5.2}$$

In this gauge, perturbations are characterized purely by fluctations in the metric,

$$\delta g_{ij} = a^2(1 - 2\mathcal{R})\,\delta_{ij} + a^2 h_{ij} \tag{1.5.3}$$

where $\mathcal{R}$ is the comoving curvature perturbation.

1.5.2 Quadratic action

From now on we set $M_{Pl} \equiv 1$. Substituting δg_{00} and δg_{0i} into (1.5.1) and expanding in powers of $\mathcal{R}$, we find

$$S = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}\mathbf{x}\,a^3\frac{\dot{\phi}}{H^2}\left[\dot{\mathcal{R}}^2 - \frac{1}{a^2}(\nabla\mathcal{R})^2\right] + \ldots \tag{1.5.4}$$

The ellipses in (1.5.4) refer to terms that are higher order in $\mathcal{R}$. Being interested only in the quadratic action of $\mathcal{R}$ we will now drop these terms. We define the canonically-normalized Mukhanov variable

$$v \equiv z\mathcal{R}, \tag{1.5.5}$$

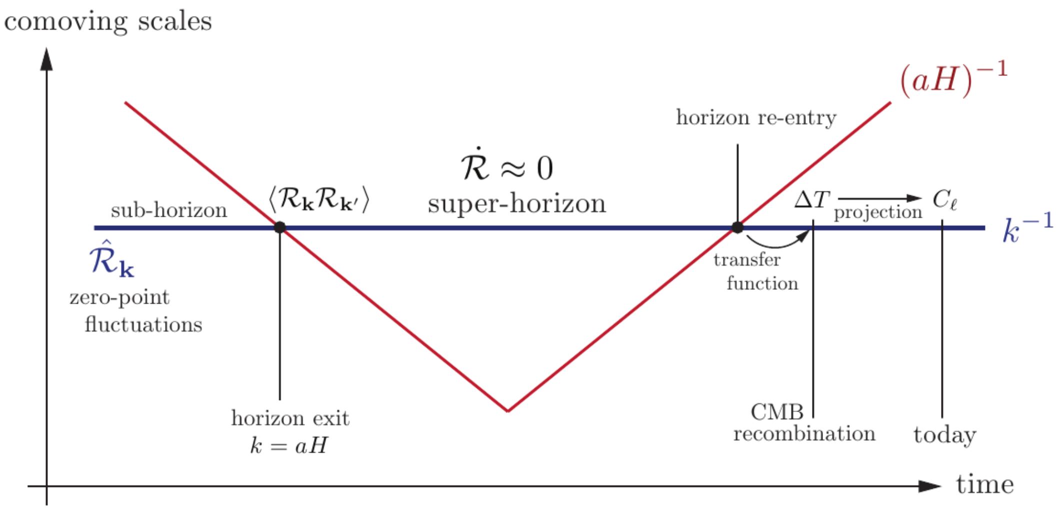


Figure 1.1: Curvature perturbations during and after inflation: The comoving horizon $(aH)^{-1}$ shrinks during inflation and grows in the subsequent FRW evolution. This implies that comoving scales k^{-1} exit the horizon at early times and re-enter the horizon at late times. While the curvature perturbations $\mathcal{R}$ are outside of the horizon they don't evolve, so our computation for the correlation function $\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle$ at horizon exit during the early de Sitter phase can be related directly to CMB observables at late times.

where

$$z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2 \varepsilon \tag{1.5.6}$$

where $\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{Pl}H^2}$. Switching to conformal time, we get

$$S = \frac{1}{2} \int \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z} v^2 \right]. \tag{1.5.7}$$

We recognize this as the action of an harmonic oscillator with time-dependent mass

$$S = \int \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu v \partial_\nu v - \frac{1}{2} m_{\text{eff}}^2(\tau) v^2 \right], \tag{1.5.8}$$

where

$$m_{\text{eff}}(\tau) \equiv -\frac{z''}{z} = -\frac{h}{a\dot{\phi}} \frac{\partial^2}{\partial \tau^2} \left(\frac{a\dot{\phi}}{H} \right). \tag{1.5.9}$$

Given a solution for the homogeneous background $a(t)$ and $\phi(t)$ one obtains $m_{\text{eff}}(\tau)$. The time-dependence of the effective mass accounts for the interaction of the scalar field $\mathcal{R}$ with the gravitational background.

1.5.3 Mukhanov-Sasaki equation

Varying the action S, we arrive at the classical equation of motion (the *Mukhanov-Sasaki equation*)

$$v_{\mathbf{k}}'' + \underbrace{\left(k^2 - \frac{z''}{z} \right)}_{\equiv \omega_k^2(\tau)} v_{\mathbf{k}} = 0, \tag{1.5.10}$$

where we defined the Fourier modes

$$v_{\mathbf{k}}(\tau) \equiv \int \frac{\mathrm{d}\mathbf{x}}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} v(\tau, \mathbf{x}). \tag{1.5.11}$$

In de Sitter space, $a = -(H\tau)^{-1}$, the effective frequency reduces to

$$\omega_k^2(\tau) = k^2 - \frac{2}{\tau^2} \quad \text{(de Sitter)}. \tag{1.5.12}$$

Let us study the solution of (1.5.10) in special limits:

Subhorizon At sufficiently early times, $-k\tau >> 1$, the comoving Hubble radius was larger than the wavelengths of all modes of interest. We say that the modes were subhorizon. In that case, $k^2 >> |z''/z|$, and we get

$$v''_{\mathbf{k}} + k^2 v_{\mathbf{k}} = 0. \tag{1.5.13}$$

This is the equation of motion of a massless scalar field in Minkowski space which has oscillating solutions:

$$v_{\mathbf{k}} \propto e^{\pm ik\tau} \tag{1.5.14}$$

Superhorizon As the comoving Hubble radius shrinks during inflation (see fig. 1.1), the modes eventually cross the Hubble radius (at $-k\tau = 1$) and become superhorizon thereafter. Superhorizon modes satisfy $k^2 << |z''/z|$, and we find instead

$$\frac{v''_{\mathbf{k}}}{v_{\mathbf{k}}} = \frac{z''}{z} \approx \frac{2}{\tau^2}. \tag{1.5.15}$$

This has the growing solution $v_{\mathbf{k}} \propto z \propto \tau^{-1}$ (and the decaying solution $v_{\mathbf{k}} \propto \tau^2$). This implies that $\mathcal{R}$ freezes on superhorizon scales

$$\mathcal{R}_{\mathbf{k}} = z^{-1} v_{\mathbf{k}} \propto \text{const} \tag{1.5.16}$$

1.5.4 Mode expansion

Since the frequency $\omega_k(\tau)$ in (1.5.10) depends only on $k \equiv |\mathbf{k}|$, the most general solution of (1.5.10) can be written as [2]

$$v_{\mathbf{k}} \equiv a^-_{\mathbf{k}} v_k(\tau) + a^+_{-\mathbf{k}} v^*_k(\tau). \tag{1.5.17}$$

Here, $v_k(\tau)$ and its complex conjugate $v^*_k(\tau)$ are two linearly indipendent solutions of (1.5.10). As indicated by dropping the vector notation $\mathbf{k}$ on the subscript, $v_k(\tau)$ and $v^*_k(\tau)$ are the same for all Fourier modes with $k \equiv |\mathbf{k}|$. The Wronskian of the mode functions is

$$W[v_k, v^*_k] \equiv v'_k v^*_k - v_k v'^*_k = 2i\Im(v'_k v^*_k). \tag{1.5.18}$$

From the equation of motion (1.5.10) it follows that $W[v_k, v^*_k]$ is time-independent. Furthermore, by rescaling the mode functions $v_k \to \lambda v_k$ (giving $W[v_k, v^*_k] \to |\lambda|^2 W[v_k, v^*_k]$) we can always normalize v_k such that

$$W[v_k, v^*_k] = v'_k v^*_k - v_k v'^*_k \equiv -i. \tag{1.5.19}$$

The reason for this particular choice of normalization will be clear momentarily.

[2] The $-\mathbf{k}$ on $a^+_{-\mathbf{k}}$ was chosen for later convenience.

The two time-independent integration constants $a^{\pm}_{\mathbf{k}}$ in (1.5.17) are

$$a^{-}_{\mathbf{k}} = \frac{v'^{*}_{k} v_{\mathbf{k}} - v^{*}_{k} v'_{\mathbf{k}}}{v'^{*}_{k} v_{k} - v^{*}_{k} v'_{k}} = \frac{W[v^{*}_{k}, v_{\mathbf{k}}]}{W[v^{*}_{k}, v_{k}]} \quad \text{and} \quad a^{+}_{\mathbf{k}} = \left(a^{-}_{\mathbf{k}}\right)^{*}, \tag{1.5.20}$$

where the relation between $a^{+}_{\mathbf{k}}$ and $a^{-}_{\mathbf{k}}$ follows from the reality of v. Note that the constants $a^{\pm}_{\mathbf{k}}$ may depend on the direction of the vector $\mathbf{k}$.

Finally, Fourier transforming (1.5.17) gives

$$\begin{aligned} v(\tau,\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \left[a^{-}_{\mathbf{k}} v_k(\tau) + a^{+}_{-\mathbf{k}} v^{*}_{k}(\tau)\right] e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \left[a^{-}_{\mathbf{k}} v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + a^{+}_{\mathbf{k}} v^{*}_{k}(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \end{aligned} \tag{1.5.21}$$

where the second line is manifestly real, since $a^{+}_{\mathbf{k}} = \left(a^{-}_{\mathbf{k}}\right)^{*}$.

1.6 Quantum origin of cosmological perturbations

Our task now is to quantize the field v. This is not much more complicated than quantizing the simple harmonic oscillator in quantum mechanics, except for a small subtelty in the vacuum choice arising from the time dependence of the oscillator frequencies $\omega_k(\tau)$. [8, 40]

1.6.1 Canonical quantization

The canonical quantization procedure proceeds in the standard way: the field v and its canonically conjugate momentum $\pi \equiv v'$ are promoted to quantum operators $\hat{v}$ and $\hat{\pi}$, which satisfy the standard equal-time commutation relations[3]

$$\left[\hat{v}(\tau,\mathbf{x}), \hat{\pi}(\tau,\mathbf{y})\right] = i\delta(\mathbf{x}-\mathbf{y}), \tag{1.6.1}$$

and

$$\left[\hat{v}(\tau,\mathbf{x}), \hat{v}(\tau,\mathbf{y})\right] = \left[\hat{\pi}(\tau,\mathbf{x}), \hat{\pi}(\tau,\mathbf{y})\right] = 0. \tag{1.6.2}$$

It follows from (1.5.10) that the commutation relation (1.6.1) holds at all times if it holds at any one time. The delta function is a signature of *locality*: modes at different points in space commute. The Hamiltonian is

$$\hat{H}(\tau) = \frac{1}{2} \int \mathrm{d}\mathbf{x} \left[\hat{\pi}^2 + (\nabla\hat{v})^2 + m_{\text{eff}}(\tau)\hat{v}^2\right]. \tag{1.6.3}$$

The constants of integration $a^{\pm}_{\mathbf{k}}$ in the mode expansion of v become operators $\hat{a}^{\pm}_{\mathbf{k}}$, so that the field operator $\hat{v}$ is expanded as

$$\hat{v}(\tau,\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \left[\hat{a}^{-}_{\mathbf{k}} v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^{+}_{\mathbf{k}} v^{*}_{k}(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{1.6.4}$$

Substituting (1.6.4) into (1.6.1) and (1.6.2) implies

$$\left[\hat{a}^{-}_{\mathbf{k}}, \hat{a}^{+}_{\mathbf{k}'}\right] = \delta(\mathbf{k}-\mathbf{k}') \quad \text{and} \quad \left[\hat{a}^{-}_{\mathbf{k}}, \hat{a}^{-}_{\mathbf{k}'}\right] = \left[\hat{a}^{+}_{\mathbf{k}}, \hat{a}^{+}_{\mathbf{k}'}\right] = 0. \tag{1.6.5}$$

[3] Here, we defined $\hbar = 1$

We realize that our normalization for the mode functions (1.5.19) was wisely chosen to make (1.6.5). The operators $\hat{a}^+_{\mathbf{k}}$ and $\hat{a}^-_{\mathbf{k}}$ may then be interpreted as creation and annihilation operators, respectively. As usual, quantum states in the Hilbert space are constructed by defining the vacuum state $|0\rangle$ via

$$\hat{a}^-_{\mathbf{k}}|0\rangle = 0, \tag{1.6.6}$$

and by producing excited states by repeated application of creation operators

$$\left|m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots\right\rangle = \frac{1}{\sqrt{m!n!\dots}}\left[\left(\hat{a}_{\mathbf{k}_1}\right)^m\left(\hat{a}_{\mathbf{k}_2}\right)^n\dots\right]|0\rangle \tag{1.6.7}$$

1.6.2 Non-Uniqueness of the vacuum

An unambiguous physical interpretation of the states in (1.6.6) and (1.6.7) arises only after the mode function $v_k(\tau)$ are selected.[4] However, the normalization (1.5.19) is not sufficient to completely fix the solutions $v_k(\tau)$ to the second-order ODE (1.5.10). An unambiguous definition of the vacuum still requires additonal physical input.

To illustrate this ambiguity explicitly, consider the following functions

$$u_k(\tau) = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau), \tag{1.6.8}$$

where α_k and β_k are complex constants. The functions $u_k(\tau)$ of course also satisfy the equation of motion (1.5.10). Moreover, they satisfy the normalization (1.5.19), i.e. $W[u_k, u_k^*] = -i$, if the coefficients α_k and β_k obey

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{1.6.9}$$

At this point there is therefore nothing that permits us to favor $v_k(\tau)$ over $u_k(\tau)$ in our choice of mode functions. In terms of $u_k(\tau)$ the expansion $\hat{v}$ takes the form

$$\hat{v}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}}\left[\hat{b}^-_{\mathbf{k}} u_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}^+_{\mathbf{k}} u_k^*(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{1.6.10}$$

where $\hat{b}^\pm_{\mathbf{k}}$ are alternative creation and annihilation operators satisfying (1.6.5). Comparing (1.6.10) to (1.6.4) leads to the *Bogoliubov transformation* [8] between $\hat{b}^\pm_{\mathbf{k}}$ operators and $\hat{a}^\pm_{\mathbf{k}}$ operators:

$$\hat{a}^-_{\mathbf{k}} = \alpha_k^* \hat{b}^-_{\mathbf{k}} + \beta_k \hat{b}^+_{-\mathbf{k}} \quad \text{and} \quad \hat{a}^+_{\mathbf{k}} = \alpha_k \hat{b}^+_{\mathbf{k}} + \beta_k^* \hat{b}^-_{-\mathbf{k}}. \tag{1.6.11}$$

Both sets of operators can be used to construct a basis of states in the Hilbert space:

$$\hat{a}^-_{\mathbf{k}}|0\rangle_a = 0 \qquad \hat{b}^-_{\mathbf{k}}|0\rangle_b = 0, \tag{1.6.12}$$

and

$$\left|m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots\right\rangle_a = \frac{1}{\sqrt{m!n!\dots}}\left[\left(\hat{a}_{\mathbf{k}_1}\right)^m\left(\hat{a}_{\mathbf{k}_2}\right)^n\dots\right]|0\rangle_a, \tag{1.6.13}$$

$$\left|m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots\right\rangle_b = \frac{1}{\sqrt{m!n!\dots}}\left[\left(\hat{b}_{\mathbf{k}_1}\right)^m\left(\hat{b}_{\mathbf{k}_2}\right)^n\dots\right]|0\rangle_b. \tag{1.6.14}$$

[4] Changing $v_k(\tau)$ while keeping $\hat{v}$ fixed, changes $\hat{a}^\pm_{\mathbf{k}}$ [cfr. (1.5.20)] and hence changes the vacuum $|0\rangle$ and the excited states $|m, n, \dots\rangle$.

It should be clear that the b-states are in general different form the a-states. In particular, the b-vacuum contains a-particles:

$$\begin{aligned} {}_b\langle 0|\hat{N}^{(a)}_{\mathbf{k}}|0\rangle_b &= {}_b\langle|\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}}|0\rangle_b = \\ &= {}_b\langle 0|\left(\alpha_k\hat{b}^+_{\mathbf{k}} + \beta^*_k\hat{b}^-_{-\mathbf{k}}\right)\left(\alpha^*_k\hat{b}^-_{\mathbf{k}} + \beta_k\hat{b}^+_{-\mathbf{k}}\right)|0\rangle_b = \\ &= |\beta_k|^2\,\delta(0). \end{aligned} \tag{1.6.15}$$

The divergent factor $\delta(0)$ arises because we are considering an infinite spatial volume, but the mean density of a-particles in the b-vacuum is finite (and typically not zero):

$$n \equiv \int \mathrm{d}\mathbf{k}\, n_{\mathbf{k}} = \int \mathrm{d}\mathbf{k}\, |\beta_k|^2\,. \tag{1.6.16}$$

1.6.3 Choice of the physical vacuum

Clearly, we are still missing some essential physical input to define the unique vacuum state.

Vacuum in Minkowski space

How do we usually do this? In a time-indipendent spacetime a preferable set of mode functions and thus an unambiguous physical vacuum can be defined by requiring that the expectation value of the Hamiltonian in the vacuum state is minimized. To illustrate this let us consider the Mukhanov-Sasaki equation in Minkowski space (i.e. the $a \equiv 0$ limit of (1.5.10)):

$$v''_k + k^2 v_k = 0. \tag{1.6.17}$$

We aim to find the mode functions v_k that minimize the expectation value of the Hamiltonian in the vacuum. We will therefore compute ${}_v\langle 0|\hat{H}|0\rangle_v$ for an arbitrary mode function v and then find the preferred function v that minimizes the result. In terms of our mode expansion, the Hamiltonian (1.6.3) becomes

$$\hat{H} = \frac{1}{2}\int \mathrm{d}\mathbf{k}\left[\hat{a}^-_{\mathbf{k}}\hat{a}^-_{-\mathbf{k}}F^*_k + \hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}}F_k + \left(2\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}} + \delta(0)\right)E_k\right] \tag{1.6.18}$$

where

$$E_k \equiv \left|v'_k\right|^2 + k^2\,|v_k|^2\,, \tag{1.6.19}$$

$$F_k \equiv {v'_k}^2 + k^2 v_k^2. \tag{1.6.20}$$

Since $\hat{a}^-_{\mathbf{k}}|0\rangle_v = 0$, we have

$${}_v\langle 0|\hat{H}|0\rangle_v = \frac{\delta(0)}{4}\int \mathrm{d}\mathbf{k}\, E_k. \tag{1.6.21}$$

Dividing out the uninteresting divergence , $\delta(0)$, we infer that the energy density in the vacuum state is

$$\varepsilon = \frac{1}{4}\int \mathrm{d}\mathbf{k}\, E_k \tag{1.6.22}$$

It is clear that this is minimized if each k-mode E_k is minimized separately. We therefore need to determine the v_k and v'_k that minimize the expression

$$E_k = \left|v'_k\right|^2 + k^2\,|v_k|^2\,. \tag{1.6.23}$$

We mustn't forget that the mode functions v_k satisfy the normalization (1.5.19),

$$v_k' v_k^* - v_k v_k'^* = -i. \tag{1.6.24}$$

using the parametrization $v_k = r_k e^{i\alpha_k}$, for real r_k and α_k, (1.6.24) becomes

$$r_k^2 \alpha_k' = -\frac{1}{2} \tag{1.6.25}$$

and (1.6.23) gives

$$\begin{aligned} E_k &= r_k'^2 + r_k^2 \alpha_k'^2 + k^2 r_k^2 = \\ &= r_k'^2 + \frac{1}{4r_k^2} + k^2 r_k^2. \end{aligned} \tag{1.6.26}$$

It is easily seen that (1.6.26) is minimized if $r_k' = 0$ and $r_k = \frac{1}{\sqrt{2k}}$. Integrating (1.6.25) gives $\alpha_k = -k\tau$ (up to an irrilevant constant that doesn't affect any observables; e.g. this constant phase factor drops out in the computation of the power spectrum) and hence

$$v_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{1.6.27}$$

This defines the preferred mode functions for fluctuations in Minkowski space. For these mode functions we find $E_k = k \equiv \omega_k$ and $F_k = 0$, so the Hamiltonian is

$$\hat{H} = \int \mathrm{d}\mathbf{k} \left[\hat{a}_{\mathbf{k}}^+ \hat{a}_{\mathbf{k}}^- + \frac{1}{2}\delta(0) \right]. \tag{1.6.28}$$

Hence, the Hamiltonian is diagonal in the eigenbasis of the occupation number operator $\hat{N}_{\mathbf{k}} \equiv \hat{a}_{\mathbf{k}}^+ \hat{a}_{\mathbf{k}}^-$.

Vacuum in time-dependent spacetimes

The vacuum prescription which we just applied to Minkowski space does not generalize straightforwardly to time-dependent spacetimes.

In this case the mode equation (1.5.10) involves time-dependent frequencies $\omega_k(\tau)$ and the "minimum-energy vacuum" depends on the time τ_0 at which it is defined. Repeating the above argument, one can nevertheless determine the vacuum which instantaneously minimizes the expectation value of the Hamiltonian at some time τ_0. One finds that the initial conditions

$$v_k(\tau_0) = \frac{1}{\sqrt{2\omega_k(\tau_0)}} e^{-i\omega_k(\tau_0)\tau_0}, \qquad v_k' = -i\omega_k(\tau_0) v_k(\tau_0) \tag{1.6.29}$$

select the preferred mode functions which determine the vacuum $|0\rangle_{\tau_0}$. However, since $\omega_k(\tau)$ changes with time, the mode functions satisfying (1.6.29) at $\tau = \tau_0$ will typically be different from the mode functions that satisfy the same conditions at a different time $\tau_1 \neq \tau_0$. This implies that $|0\rangle_{\tau_1} \neq |0\rangle_{\tau_0}$ and the state $|0\rangle_{\tau_0}$ is not the lowest-energy state at a later time τ_1.

Bunch-Davies vacuum

How do we resolve this ambiguity for the inflationary quasi-de Sitter spacetime?

From Fig. 1.1 we note that a sufficiently early times (large negative conformal time τ) all modes of cosmological interest were deep inside the horizon:

$$\frac{k}{aH} \sim |k\tau| >> 1 \qquad \text{(subhorizon)}. \tag{1.6.30}$$

This means that in the remote past all observable modes had time-indipendent frequencies; e.g. in perfect de Sitter space:

$$\omega_k^2 = k^2 - \frac{2}{\tau^2} \to k^2. \tag{1.6.31}$$

The corresponding modes are therefore not affected by gravity and behave just like in Minkowski space:

$$v_k'' + k^2 v_k = 0. \tag{1.6.32}$$

The two indipendent solutions of (1.6.32) are $v_k \propto e^{\pm ik\tau}$. As we have seen above only the positive frequency mode $v_k \propto e^{-ik\tau}$ is the the "minimal excitation state", cf. Eq. (1.6.27).

Given that at sufficiently early times all modes have time-indipendent frequencies, we can now avoid the ambiguity in defining the inital conditions for the mode functions that afflicts the treatment in more general time-dependent spacetimes. In practice, this means solving the Mukhanov-Sasaki equation with the (Minkowski) initial condition

$$\lim_{\tau\to-\infty} v_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{1.6.33}$$

This defines a preferable set of mode functions and a unique physical vacuum, the *Bunch-Davies vacuum*.

1.6.4 Quantum fluctuations in de Sitter

We are now ready to derive the correlation functions for quantum fluctuations in de Sitter space.

de Sitter mode functions

In de Sitter space the Mukhanov-Sasaki equation is:

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0. \tag{1.6.34}$$

The exact solution of (1.6.34) is

$$v_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right). \tag{1.6.35}$$

The initial condition (1.6.33) fixes $\beta = 0$, $\alpha = 1$. Hence, the unique mode function is

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right). \tag{1.6.36}$$

Since the mode function is completely fixed, the future evolution of the mode including its superhorizon dynamics is determined:

$$\lim_{k\tau\to 0} v_k(\tau) = \frac{1}{i\sqrt{2}} \cdot \frac{1}{k^{3/2}\tau}. \tag{1.6.37}$$

Zero-point fluctuations

Knowledge of the mode functions for canonically-normalized fields in de Sitter space allows us to compute the effect due to quantum zero-point fluctuations:

$$\begin{aligned}
\langle \hat{v}_{\mathbf{k}} \hat{v}_{\mathbf{k}'} \rangle &= \langle 0 | \, \hat{v}_{\mathbf{k}} \hat{v}_{\mathbf{k}'} \, | 0 \rangle = \\
&= \langle 0 | \left(\hat{a}^-_{\mathbf{k}} v_k + \hat{a}^+_{-\mathbf{k}} v_k^* \right) \left(\hat{a}^-_{\mathbf{k}'} v_{k'} + \hat{a}^+_{-\mathbf{k}'} v_{k'}^* \right) | 0 \rangle = \\
&= v_k v_k^* \, \langle 0 | \, \hat{a}^-_{\mathbf{k}} \hat{a}^+_{-\mathbf{k}'} \, | 0 \rangle = \\
&= v_k v_k^* \, \langle 0 | \left[\hat{a}^-_{\mathbf{k}}, \hat{a}^+_{-\mathbf{k}'} \right] | 0 \rangle = \\
&= |v_k|^2 \, \delta(\mathbf{k} + \mathbf{k}') = \\
&= P_v(k) \delta(\mathbf{k} + \mathbf{k}')
\end{aligned} \tag{1.6.38}$$

On superhorizon scales this approaches [cf. Eq. (1.6.37)]

$$P_v = \frac{1}{2k^3} \frac{1}{\tau^2} = \frac{1}{2k^3} (aH)^2. \tag{1.6.39}$$

All power spectra for fields in de Sitter space are simple rescalings of this power spectrum for the canonically-normalized field.

We define the (dimensionless) power spectrum as

$$\Delta_v^2(k) = \frac{k^3}{2\pi^2} P_v(k) \tag{1.6.40}$$

CHAPTER 2

Gravitational waves from inflation

One of the most robust and model-independent predictions of inflation is a stochastic background of gravitational waves with an amplitude given simply by the Hubble scale H during inflation. The simplicity of this prediction means that a measurement of primordial gravitational waves would give clean information about arguably the most important inflationary parameter, namely the energy scale of inflation. Most excitingly, inflationary gravitational waves lead to a unique signature in the polarization of the CMB. A large number of ground-based, balloon and satellite experiments are currently searching for this signal.

Near comoving wave-number k, the gravitational-wave background from inflation carries information about the physical conditions near two moments in cosmic history: the moment when k "left the horizon" during inflation, and the moment when it "re-entered the horizon" after inflation. The discussion about physical effects incorporated in a gravitational-wave transfer function is mostly based on the work of Steinhardt and Boyle [10].

2.1 Tensor perturbations

Tensor perturbations in a spatially flat FRW universe are described by a line element

$$\mathrm{d}s^2 = a^2\left[-\,\mathrm{d}\tau^2 + \left(\delta_{ij} + h_{ij}\right)\mathrm{d}x^i\,\mathrm{d}x^j\right] \tag{2.1.1}$$

where τ is the conformal time, x^i are comoving spatial coordinates, and h_{ij} is the gauge-invariant tensor metric perturbation. The perturbation h_{ij} is symmetric ($h_{ij} = h_{ji}$), traceless ($h_{ii} = 0$), and transverse ($h_{ij,j} = 0$) and therefore contains $6-3-1=2$ indipendent modes (corresponding to the "+" and "×" gravitational-wave polarizations).

One can think of $h_{ij}(\tau, \mathbf{x})$ as a quantum field in an unperturbed FRW background metric $\bar{g}_{\mu\nu} = \mathrm{diag}\{-a^2, a^2, a^2, a^2\}$. At quadratic order in h_{ij} (which is adequate, since h_{ij} is tiny), tensor perturbations are governed by the second-order action, which comes from the expansion of Einstein-Hilbert action (see [43])

$$S = \int \mathrm{d}\tau\,\mathrm{d}\mathbf{x}\sqrt{-\bar{g}}\left[\frac{-\bar{g}^{\mu\nu}}{64\pi G}\partial_\mu h_{ij}\partial_\nu h_{ij} + \frac{1}{2}\Pi_{ij}h_{ij}\right] \tag{2.1.2}$$

where $\bar{g}^{\mu\nu}$ and $\bar{g}$ are the inverse and the determinant of $\bar{g}_{\mu\nu}$, respectively, and G is the Newton's constant. The tensor part of the anisotropic stress Π_{ij} is given by $T_{ij} = pg_{ij} + a^2\Pi_{ij}$, or equivalently

$$\Pi_{ij} = T^i_j - p\delta_{ij} \qquad \text{p = unperturbed pressure} \tag{2.1.3}$$

along with the conditions $\Pi_{ii} = 0$ and $\partial_i \Pi_{ij} = 0$. It couples to h_{ij} like an external source in (2.1.2). By varying h_{ij} in (2.1.2):

$$\begin{aligned}
\delta S =& \int \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \quad \delta(\sqrt{-\bar{g}}) \left[\frac{-\bar{g}^{\mu\nu}}{64\pi G} \, \partial_\mu h_{ij} \partial_\nu h_{ij} + \frac{1}{2} \Pi_{ij} \, h_{ij} \right] + \\
&+ \sqrt{-\bar{g}} \left[\frac{-\delta \bar{g}^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \partial_\nu h_{ij} - \frac{\bar{g}^{\mu\nu}}{64\pi G} \delta \left(\partial_\mu h_{ij} \right) \partial_\nu h_{ij} - \frac{\bar{g}^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \delta \left(\partial_\nu h_{ij} \right) + \right. \\
&\left. + \frac{1}{2} \Pi_{ij} \delta h_{ij} + \frac{1}{2} \delta \Pi_{ij} h_{ij} \right] = \\
=& \int \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \quad \delta(\sqrt{-\bar{g}}) \left[\frac{-\bar{g}^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \partial_\nu h_{ij} + \frac{1}{2} \Pi_{ij} h_{ij} \right] + \\
&- \frac{\sqrt{-\bar{g}} \, \delta \bar{g}^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \partial_\nu h_{ij} - \partial_\mu \left(\frac{\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu}}{64\pi G} \partial_\nu h_{ij} \delta h_{ij} \right) + \\
&+ \frac{\partial_\mu \left(\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu} \partial_\nu h_{ij} \right)}{64\pi G} \delta h_{ij} - \partial_\nu \left(\frac{\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \delta h_{ij} \right) + \\
&+ \frac{\partial_\nu \left(\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu} \partial_\mu h_{ij} \right)}{64\pi G} \delta h_{ij} + \sqrt{-\bar{g}} \left[\frac{1}{2} \Pi_{ij} \, \delta h_{ij} + \frac{1}{2} \delta \Pi_{ij} \, h_{ij} \right] = 0
\end{aligned} \tag{2.1.4}$$

The boundary terms as always vanish:

$$\frac{\delta S}{\delta h_{ij}} = \frac{1}{64\pi G} \left[\partial_\mu \left(\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu} \partial_\nu h_{ij} \right) + \partial_\nu \left(\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu} \partial_\mu h_{ij} \right) \right] + \frac{\sqrt{-\bar{g}}}{2} \Pi_{ij} = 0 \tag{2.1.5}$$

Then,

$$\frac{1}{32\pi G} \partial_\mu \left(\sqrt{-\bar{g}} \, \bar{g}^{\mu\nu} \partial_\nu h_{ij} \right) = -\frac{\sqrt{-\bar{g}}}{2} \Pi_{ij} \tag{2.1.6}$$

Since $\sqrt{-\bar{g}} = a^4$ and substituting the inverse metric terms:

$$\partial_0 \left(\sqrt{-\bar{g}} \, \bar{g}^{00} \partial_0 h_{ij} \right) = -\partial_0 \left(a^2 \partial_0 h_{ij} \right) = -a^2 h''_{ij} - 2a'(\tau) a(\tau) h_{ij} \tag{2.1.7}$$

$$\partial_k \left(\sqrt{-\bar{g}} \, \bar{g}^{kl} \partial_l h_{ij} \right) = \partial_k \left(a^2 \partial_l h_{ij} \right) = a^2 \partial_k \partial_l h_{ij} \tag{2.1.8}$$

Hence we obtain the equation of motion

$$h''_{ij} + 2 \frac{a'(\tau)}{a(\tau)} h'_{ij} - \nabla^2 h_{ij} = 16\pi G a^2(\tau) \Pi_{ij}(\tau, \mathbf{x}) \tag{2.1.9}$$

where a ($'$) indicates a conformal time derivative $\mathrm{d}/\mathrm{d}\tau$. Next, it is convenient to Fourier transform as follows,

$$h_{ij}(\tau, \mathbf{x}) = \sum_r \sqrt{16\pi G} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \epsilon^r_{ij}(\mathbf{k}) h^r_{\mathbf{k}}(\tau) e^{i\mathbf{k}\mathbf{x}} \tag{2.1.10}$$

$$\Pi_{ij}(\tau, \mathbf{x}) = \sum_r \sqrt{16\pi G} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \epsilon^r_{ij}(\mathbf{k}) \Pi^r_{\mathbf{k}}(\tau) e^{i\mathbf{k}\mathbf{x}} \tag{2.1.11}$$

where $r =$ ("+" or "×") labels the polarization state, and the polarization tensors are symmetric $\left[\epsilon^r_{ij}(\mathbf{k}) = \epsilon^r_{ji}(\mathbf{k})\right]$, traceless $\left[\epsilon^r_{ii}(\mathbf{k}) = 0\right]$, and transverse $\left[k_i \epsilon^r_{ij}(\mathbf{k}) = 0\right]$. We also choose a circular-polarization basis in which $\epsilon^r_{ij}(\mathbf{k}) = \left(\epsilon^r_{ij}(-\mathbf{k})\right)^*$, and normalize the polarization basis as follows:

$$\sum_{i,j} \epsilon^r_{ij}(\mathbf{k}) \left(\epsilon^s_{ij}(\mathbf{k}) \right)^* = 2\delta^{rs}. \tag{2.1.12}$$

Substituting (2.1.10) into (2.1.2):

$$S = \int \mathrm{d}\tau\, \mathrm{d}\mathbf{x}\, a^4 \{ \frac{1}{4a^2} \sum_r \sum_s \int \frac{\mathrm{d}\mathbf{k}\ \mathrm{d}\mathbf{k'}}{8\pi^3} \epsilon^r_{ij}(\mathbf{k})\epsilon^s_{ij}(\mathbf{k'})\times$$
$$\times \left[h'^{\,r}_{\mathbf{k}}(\tau) h'^{\,s}_{\mathbf{k'}}\tau) + \mathbf{k}\,\mathbf{k'} h^r_{\mathbf{k}}(\tau) h^s_{\mathbf{k'}}(\tau) \right] e^{i\mathbf{kx}} e^{i\mathbf{k'x}} + \tag{2.1.13}$$
$$+ \sum_r \sum_s 8\pi G \int \frac{\mathrm{d}\mathbf{k}\ \mathrm{d}\mathbf{k'}}{8\pi^3} \epsilon^r_{ij}(\mathbf{k})\epsilon^s_{ij}(\mathbf{k'}) \Pi^r_{\mathbf{k}}(\tau) h^s_{\mathbf{k'}}(\tau) e^{i\mathbf{kx}} e^{i\mathbf{k'x}}\ \}$$

Since the domain of integration is symmetric, $\mathbf{k} \longrightarrow -\mathbf{k}$,

$$S = \int \mathrm{d}\tau\, \mathrm{d}\mathbf{x}\, a^2 \{ \sum_r \sum_s \int \frac{\mathrm{d}\mathbf{k}\ \mathrm{d}(-\mathbf{k})}{3\pi^3} \epsilon^r_{ij}(\mathbf{k})\epsilon^s_{ij}(-\mathbf{k})\times$$
$$\times \left[h'^{\,r}_{\mathbf{k}}(\tau) h'^{\,s}_{-\mathbf{k}}(\tau) - k^2 h^r_{\mathbf{k}}(\tau) h^s_{-\mathbf{k}}(\tau) \right] + \tag{2.1.14}$$
$$+ \sum_r \sum_s a^2 \frac{G}{\pi^2} \int \mathrm{d}\mathbf{k}\ \mathrm{d}(-\mathbf{k})\, \epsilon^r_{ij}(\mathbf{k})\epsilon^s_{ij}(-\mathbf{k}) \Pi^r_{\mathbf{k}}(\tau) h^s_{-\mathbf{k}}(\tau)\ \}$$

Using (2.1.12) then it yields

$$S = \sum_r \int \mathrm{d}\tau\, \mathrm{d}\mathbf{k} \frac{a^2}{2} \left[h'^{\,r}_{\mathbf{k}} h'^{\,r}_{-\mathbf{k}} - k^2 h^r_{\mathbf{k}} h^r_{-\mathbf{k}} + 32\pi G a^2 \Pi^r_{\mathbf{k}} h^r_{-\mathbf{k}} \right] \tag{2.1.15}$$

Now we can canonically quantize by promoting $h^r_{\mathbf{k}}$ and it conjugate momentum

$$\pi^r_{\mathbf{k}}(\tau) = a^2(\tau) h'^{\,r}_{-\mathbf{k}}(\tau) \tag{2.1.16}$$

to operators, $\hat{h}^r_{\mathbf{k}}$ and $\hat{\pi}^r_{\mathbf{k}}$, satisfying the equal-time commutation relations

$$\left[\hat{h}^r_{\mathbf{k}}(\tau), \hat{\pi}^s_{\mathbf{k'}}(\tau) \right] = i\delta^{rs}\delta^{(3)}(\mathbf{k}-\mathbf{k'}) \tag{2.1.17}$$
$$\left[\hat{h}^r_{\mathbf{k}}(\tau), \hat{h}^s_{\mathbf{k'}}(\tau) \right] = \left[\hat{\pi}^r_{\mathbf{k}}(\tau), \hat{\pi}^s_{\mathbf{k'}}(\tau) \right] = 0. \tag{2.1.18}$$

Since $\hat{h}_{ij}(\tau, \mathbf{x})$ is Hermitian, its Fourier components satisfy $\hat{h}^r_{\mathbf{k}} = \hat{h}^{r\dagger}_{-\mathbf{k}}$, and we write them as

$$\hat{h}^r_{\mathbf{k}}(\tau) = h_k(\tau)\hat{a}^r_{\mathbf{k}} + h^*_k(\tau)\hat{a}^{r\dagger}_{\mathbf{k}} \tag{2.1.19}$$

where the creation and annihilation operators, $\hat{a}^r_{\mathbf{k}}$ and $\hat{a}^{r\dagger}_{\mathbf{k}}$, satisfy standard commutation relations

$$\left[\hat{a}^r_{\mathbf{k}}(\tau), \hat{a}^{s\dagger}_{\mathbf{k'}}(\tau) \right] = \delta^{rs}\delta^{(3)}(\mathbf{k}-\mathbf{k'}) \tag{2.1.20}$$
$$\left[\hat{a}^r_{\mathbf{k}}(\tau), \hat{a}^s_{\mathbf{k'}}(\tau) \right] = \left[\hat{a}^{r\dagger}_{\mathbf{k}}(\tau), \hat{a}^{s\dagger}_{\mathbf{k'}}(\tau) \right] = 0. \tag{2.1.21}$$

while the (c-number) mode functions $h_k(\tau)$ and $h^*_k(\tau)$ are linearly indipendent solutions of the Fourier-transformed equation of motion

$$h''_k + 2\frac{a'(\tau)}{a(\tau)} h'_k + k^2 h_k = 16\pi G a^2(\tau) \Pi_k(\tau) \tag{2.1.22}$$

Equation (2.1.19) makes use of the fact that, by isotropy, the mode functions $h_k(\tau)$ will depend on the time (τ) and the wave numer $(k = |\mathbf{k}|)$, but not on the direction $(\hat{\mathbf{k}})$ or the polarization

(r). Note that consistency between the two sets of commutation relations, (2.1.17) and (2.1.20), requires that the mode functions satisfy the Wronskian normalization condition

$$h_k(\tau)h_k^{*\prime}(\tau) - h_k^*(\tau)h_k'(\tau) = \frac{i}{a^2(\tau)} \tag{2.1.23}$$

in the past. In particular, the standard initial condition for the mode function in the far past (when the mode k was still far inside the horizon during inflation),

$$h_k(\tau) \longrightarrow \frac{\exp(-ik\tau)}{a(\tau)\sqrt{2k}} \qquad (\text{as } \tau \longrightarrow -\infty) \tag{2.1.24}$$

satisfies (2.1.23) — but it is not the unique inital condition which does so. This is a manifestation of the well-known **vacuum ambiguity** that is responsible for particle production in cosmological spacetimes (see [8]).

In the early universe, the gravitational-wave background (GWB) is usually characterized by the tensor power spectrum $\Delta_h^2(k,\tau)$. With the formalism developed thus so far

$$\begin{aligned}\langle 0|\hat{h}_{ij}(\tau,\mathbf{x})\hat{h}_{ij}(\tau,\mathbf{x})|0\rangle &= \sum_r\sum_s 16\pi G\int \frac{\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k'}}{8\pi^3}\epsilon_{ij}^r(\mathbf{k})\epsilon_{ij}^s(\mathbf{k'})\times \\ \times\,\langle 0|\,[\,\left(h_k(\tau)\hat{a}_{\mathbf{k}}^r + h_k^*(\tau)\hat{a}_{-\mathbf{k}}^{r\dagger}\right)&\left(h_{k'}(\tau)\hat{a}_{\mathbf{k'}}^s + h_{k'}^*(\tau)\hat{a}_{-\mathbf{k'}}^{s\dagger}\right)\,]\,|0\rangle e^{i\mathbf{kx}}e^{i\mathbf{k'x}}\end{aligned} \tag{2.1.25}$$

The only non-vanishing term is:

$$\langle 0|h_k(\tau)\hat{a}_{\mathbf{k}}^r h_{k'}^*(\tau)\hat{a}_{-\mathbf{k'}}^{s\dagger}|0\rangle = h_k(\tau)h_{k'}^*(\tau)\langle 0|\hat{a}_{\mathbf{k}}^r\hat{a}_{\mathbf{k'}}^{s\dagger}|0\rangle \overset{(2.1.20)}{=} h_k(\tau)h_{-k}^*(\tau)\delta^{rs} \tag{2.1.26}$$

Then,

$$\begin{aligned}\langle 0|\hat{h_{ij}}(\tau,\mathbf{x})\hat{h_{ij}}(\tau,\mathbf{x})|0\rangle &= \sum_r \frac{4G}{\pi^2}\int \mathrm{d}\mathbf{k}\,\epsilon_{ij}^r(\mathbf{k})\epsilon_{ij}^r(-\mathbf{k})\left|h_k(\tau)\right|^2 = \\ &= \frac{8G}{\pi^2}4\pi\int_0^\infty \mathrm{d}k\,k^2\left|h_k(\tau)\right|^2 = \\ &= \int_0^\infty 64\pi G\frac{k^3}{2\pi^2}|h_k(\tau)|^2\frac{\mathrm{d}k}{k}\end{aligned} \tag{2.1.27}$$

so that the tensor power spectrum is given by

$$\Delta_h^2(k,\tau) \equiv \frac{\mathrm{d}\langle 0|\hat{h}_{ij}^2|0\rangle}{\mathrm{d}\ln k} = 64\pi G\frac{k^3}{2\pi^2}|h_k(\tau)|^2 \tag{2.1.28}$$

Since (2.1.28) defines the tensor power spectrum in terms of the full tensor perturbation h_{ij}, the normalization of the power spectrum is indipendent of the normalization (2.1.12) of the polarization basis.

Next, let us "derive" the slow-roll expression for the primordial tensor power spectrum. As long as k remains inside the Hubble horizon ($k \gg aH$) during inflation, the mode function $h_k^{(in)}(\tau)$ is given by (2.1.24) ; and once k is outside the horizon ($k \ll aH$), the mode function $h_k^{(out)}$ is indipendent of τ. Then, by simply matching $|h_k^{(in)}|$ to $|h_k^{(out)}|$ at the moment of horizon exit ($k = aH$), one finds $h_k^{(out)} = 1/(a_*\sqrt{2k})$, where an $(*)$ denotes that a quantity is evaluated at horizon ($k = a_* H_*$). Thus, the primordial tensor power spectrum is given by

$$\begin{aligned}\Delta_h^2(k) &\equiv 64\pi G \frac{k^3}{2\pi^2} |h_k^{(out)}|^2 \approx 64\pi G \frac{k^3}{2\pi^2} \frac{1}{a_*^2 2k} = \\ &\overset{a_* = \frac{k}{H_*}}{=} \frac{16G}{\pi} \frac{k^2}{a_*^2} = \frac{16G}{\pi} H_*^2 = \\ &= \frac{2}{M_{Pl}\pi^2} H_*^2 = 8\left(\frac{H_*}{2\pi M_{Pl}}\right)^2\end{aligned} \tag{2.1.29}$$

where in this equation $M_{Pl} = (8\pi G)^{-1/2}$ is the "reduced Planck mass". Note that the primordial power spectrum is time indipendent, since (by definition) it is evaluated when the mode k is far outside the horizon. Although our derivation of (2.1.29) may seem crude, it is well known that (2.1.29) provides a very good approximation to the inflationary tensor spectrum.

I

The same result can be obtained applying the formalism we introduced for scalar fluctuations in the previous chapter. In fact, in this case, our job is considerably simpler due to the fact that first-order tensor perturbations are gauge-invariant and don't backreact on the inflationary background. Inserting the metric in Eq. (2.1.2) then yields

$$S = \int \mathrm{d}\tau\, \mathrm{d}\mathbf{x} \left\{ \frac{M_{Pl}^2}{8} a^2 \left[(h'_{ij})^2 - (\nabla h_{ij})^2 \right] + \frac{a^4}{2} \Pi_{ij} h_{ij} \right\} \tag{2.1.30}$$

Here, we have introduce the explicity factor of M_{Pl}^2 to make h_{ij} manifestly dimensionless. Up to the normalization factor of $\frac{M_{Pl}}{2}$ this is the same action for a massless scalar field in an FRW universe. Substituting (2.1.10) (for convenience let's forget about the factor $\sqrt{16\pi G}$) into (2.1.30), we get again the Fourier representation of the action

$$S = \sum_r \int \mathrm{d}\tau\, \mathrm{d}\mathbf{k} \left\{ \frac{a^2}{4} M_{Pl}^2 \left[(h'^r_{\mathbf{k}})^2 - k^2 (h^r_{\mathbf{k}})^2 \right] + \frac{a^4}{2} \Pi^r_{\mathbf{k}} h^r_{-\mathbf{k}} \right\} \tag{2.1.31}$$

For canonically-normalized fields,

$$v^r_{\mathbf{k}} \equiv \frac{a}{2} M_{Pl} h^r_{\mathbf{k}}, \tag{2.1.32}$$

this reads (forgetting for the moment about the anisotropic stress Π_{ij})

$$S = \sum_r \int \mathrm{d}\tau\, \mathrm{d}\mathbf{k} \left[(v'^r_{\mathbf{k}})^2 - \underbrace{\left(k^2 - \frac{a''}{a}\right)}_{\equiv \omega_k^2(\tau)} (v^r_{\mathbf{k}})^2 \right]. \tag{2.1.33}$$

For a de Sitter background, we have

$$\frac{a''}{a} = \frac{2}{\tau^2}. \tag{2.1.34}$$

Eq. (2.1.33) should be recognized as essentially two copies of the action (1.5.7). Hence, we can jump straight to the result in Eq. (1.6.39):

$$P_v = \frac{1}{2k^3} (aH)^2. \tag{2.1.35}$$

Defining the tensor power spectrum P_t as the sum of the power spectra for each polarization mode of h_{ij}, we find

$$P_t = 2 \cdot P_h = 2 \cdot \left(\frac{2}{aM_{Pl}}\right)^2 \cdot P_v = \frac{4}{k^3}\frac{H^2}{M_{Pl}^2} \tag{2.1.36}$$

or the dimensionless spectrum

$$\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{Pl}^2}\bigg|_{k=aH} . \tag{2.1.37}$$

2.2 The tensor transfer function

Since cosmological tensor perturbations are tiny, they are well described by linear perturbation theory, so that each Fourier mode evolves independently. Thus, we see from Eq. (2.1.28) that the primordial tensor power spectrum — defined at some conformal τ_i shortly after the end of inflation, when all modes of interest have already left the horizon, but have not yet re-entered — is related to the tensor power spectrum at a later time τ by a multiplicative transfer function

$$\Delta_h^2(k,\tau) = T_h(k,\tau)\Delta_h^2(k,\tau_i) \tag{2.2.1}$$

where

$$T_h(k,\tau) = \left|\frac{h_k(\tau)}{h_k(\tau_i)}\right|^2 . \tag{2.2.2}$$

Note that we will not necessarily want to evaluate $T_h(k,\tau)$ at the present time ($\tau = \tau_0$), since different experiments probe the gravitational-wave spectrum at different redshift. For example, while laser interferometers measure T_h today, CMB experiments measure it near the redshift of recombination. As long as a mode remains outside the horizon ($k \ll aH$), the corresponding perturbations does not vary with the time $[h_k(\tau) = h_k(\tau_i)]$, so that the transfer function is very well approximated by $T_h(k,\tau) = 1$. [1] So we will focus on $T_h(k,\tau)$ for modes that have already re-enterd the horizon prior to time τ.

It is very convenient to split the transfer function (2.2.2) into three factors as follows:

$$T_h(k,\tau) = \left|\frac{\bar{h}_k(\tau)}{h_k(\tau_i)}\frac{\tilde{h}_k(\tau)}{\bar{h}_k(\tau)}\frac{h_k(\tau)}{\tilde{h}_k(\tau)}\right|^2 = C_1C_2C_3 \tag{2.2.3}$$

Here $h_k(\tau)$, $\tilde{h}_k(\tau)$, and $\bar{h}_k(\tau)$ represent three different solutions of the tensor mode equation (2.1.22). In particular, $h_k(\tau)$ is the true (exact) solution of (2.1.22); $\tilde{h}_k(\tau)$ is an approximate solution obtained by ignoring the tensor anisotropic stress Π_k on the right-hand side of (2.1.22); $\bar{h}_k(\tau)$ is an even cruder approximation obtained by first ignoring Π_k and then using the "thin-horizon" approximation to solve (2.1.22). (Briefly, the thin-horizon approximation treats horizon re-entry as a "sudden" or instantaneous event. In this approximation, $\bar{h}_k(\tau)$ is frozen outside the Hubble horizon, redshifts as $1/a(\tau)$ inside the Hubble horizon, and has a sharp transition between these two behaviours at the moment when the mode re-enters the Hubble horizon ($k = aH$). We will describe this approximation in more detail later.)

These three factors each represent a different physical effect. The first factor,

$$C_1 = \left|\frac{\bar{h}_k(\tau)}{h_k(\tau_i)}\right|^2 \tag{2.2.4}$$

[1] For a general proof, even in presence of anisotropic stress, see Appendix A.

represents the redshift suppression of the gravitational-wave amplitude after the mode k re-enters the horizon. The second factor,

$$C_2 = \left| \frac{\tilde{h}_k(\tau)}{\bar{h}_k(\tau)} \right|^2 \tag{2.2.5}$$

captures the behaviour of the background equation-of-state parameter $w(\tau) = p(\tau)/\rho(\tau)$ around the time horizon re-entry. The third factor,

$$C_3 = \left| \frac{h_k(\tau)}{\tilde{h}_k(\tau)} \right|^2 \tag{2.2.6}$$

measures the damping of the gravitational-wave spectrum due to tensor anisotropic stress. Note that C_1 by itself is $\ll 1$ and provides a rough approximation to the full transfer function T_h. The two other factors, C_2 and C_3, are typically of order unity, and are primarily sensitive to the physical condition near the time that the mode k re-entered the Hubble horizon.

2.2.1 The redshift-suppression factor, C_1

The mode function $h_k(\tau)$ behaves simply in two regimes: far outside the horizon ($k \ll aH$), and far inside the horizon ($k \gg aH$). Far outside, $h_k(\tau)$ is τ-indipendent (as we have seen). Far inside, after horizon re-entry, $h_k(\tau)$ oscillates with a decaying envelope

$$h_k(\tau) = \frac{A_k}{a(\tau)} \cos[k(\tau - \tau_k) + \phi_k], \tag{2.2.7}$$

as we shall see in the next two subsections, where A_k and ϕ_k are constants representing the amplitude and phase shift of the oscillation, and τ_k is the conformal time at the horizon re-entry ($k = aH$). These two simple regimes are separated by an intermediate period (horizon crossing) when $k \sim aH$.

In the thin-horizon approximation, we neglect this intermediate regime. That is, we assume that $\bar{h}_k(\tau) = h_k(\tau_i)$ when $k < aH$; and that $\bar{h}_k(\tau)$ is given by Eq. (2.2.7) for $k > aH$; and that the outside amplitude is connected to the inside envelope via the matching condition $h_k(\tau_i) = A_k/a(\tau_k)$. Ignoring the phase shift ϕ_k, which is really an asymptotic quantity, this matching condition simply imposes continuity of the inside and outside amplitudes at $k = aH$. Combining the matching condition with Eq. (2.2.4), we see that

$$\begin{aligned} C_1 &= \left| \frac{\frac{A_k}{a(\tau)} \cos[k(\tau - \tau_k) + \phi_k]}{\frac{A_k}{a(\tau_k)}} \right|^2 = \\ &= \left| \frac{a(\tau_k)}{a(\tau)} \cos[k(\tau - \tau_k) + \phi_k] \right|^2 = \\ &= \left(\frac{1+z}{1+z_k} \right)^2 \cos^2[k(\tau - \tau_k) + \phi_k] \end{aligned} \tag{2.2.8}$$

where $1 + z = a_0/a(\tau)$ is the redshift at which the spectrum is to be probed, and $1 + z_k = a_0/a_k$ is the redshift at which the mode re-entered the Hubble horizon ($k = aH$).

The relic GWB from inflation is often treated as "quasistationary" process (which means that its statistical properties only vary on cosmological time scales—much longer than the time scales in a terrestrial experiment). But the $\cos^2[\ldots]$ factor in Eq. (2.2.8) implies that the background is actually highly *nonstationary*—its power spectrum oscillates as a function of both wave number k and time τ. This $\cos^2[\dots]$ factor represents a genuine feature, and is *not* a spurious by-product of our thin-horizon approximation. It was found by Grishchuk [28] that the expectation values of operators which measure the amplitude of gravitational radiation have a non trivial time dependance. A state which initially has a vacuum-state occupation numbers, at later times corresponds to a squeezed state with occupation numbers which differ from those of the later-time vacuum state. Thus, in an expanding universe gravitons can be produced. Physically the gravitational GWB consists of gravitational *standing waves* with random *spatial* phases, and coherent *temporal* phases. All modes $\vec{k}$ at fixed wave number $k = |\vec{k}|$ re-enter the Hubble horizon simultaneously, and subsequently oscillate in phase with one another—even until the present day. In other words, the modes are synchronized by inflation [28]. Thus, at a fixed wave number, $T_h(k,\tau)$ is sinusoidal in τ, with oscillation frequency k, and a phase shift ϕ_k (computed in the next subsection). Alternatively, at fixed time, $T_h(k,\tau)$ oscillates rapidly in wave number.

In the remainder of this subsection, we derive an accurate expression for $(1+z_k)$, the horizon-crossing redshift. To start, let us write the background energy density ρ as a sum of several components. The ith component has energy density ρ_i, pressure p_i, equation-of-state parameter $w_i \equiv p_i/\rho_i$, and obeys the continuity equation

$$\mathrm{d}\rho_i = -3\rho_i(1+\frac{p_i}{\rho_i})\,\mathrm{dln}\,a \quad \Rightarrow \quad \frac{\mathrm{d}\rho_i}{\rho_i} = 3\left(1+w_i(z)\right)\frac{\mathrm{d}z}{1+z} \tag{2.2.9}$$

since

$$\mathrm{dln}\,a = \frac{\mathrm{d}a(\tau)}{a(\tau)} = \frac{\frac{a_0}{a^2(\tau)}\,\mathrm{d}a(\tau)}{\frac{a_0}{a(\tau)}} = -\frac{\mathrm{d}\left(\frac{a_0}{a(\tau)}-1\right)}{\frac{a_0}{a(\tau)}} = -\frac{\mathrm{d}z}{1+z}. \tag{2.2.10}$$

Integrating this equation yields

$$\rho_i(z)/\rho_i^{(0)} = (1+z)^3\exp\left[3\int_0^z\frac{w_i(\tilde{z})}{1+\tilde{z}}\,\mathrm{d}\tilde{z}\right], \tag{2.2.11}$$

where $\rho_i^{(0)}$ is the present value. Then the Friedmann equation

$$H^2(z) = \frac{8\pi G}{3}\sum_i\rho_i(z) \tag{2.2.12}$$

may be rewritten as (using $\frac{a^2}{a_0^2} = (1+z)^{-2}$)

$$\begin{aligned}\frac{a^2H^2}{a_0^2H_0^2} &= \frac{8\pi G}{3H_0^2}\sum_i\left\{\rho_i^{(0)}(1+z)\exp\left[3\int_0^z\frac{w_i(\tilde{z})}{1+\tilde{z}}\,\mathrm{d}\tilde{z}\right]\right\}\\ &= \sum_i\Omega_i^{(0)}(1+z)\exp\left[3\int_0^z\frac{w_i(\tilde{z})}{1+\tilde{z}}\,\mathrm{d}\tilde{z}\right]\end{aligned} \tag{2.2.13}$$

where H_0 is the present Hubble rate, and the density parameter $\Omega_i^{(0)} \equiv \rho_i^{(0)}/\rho_{cr}^{(0)}$ represents the ith component's present energy density in units of the present critical density $\rho_{cr}^{(0)} = 3H_0^2/8\pi G$. Hence z_k is obtained by solving the equation

$$\left(\frac{k}{k_0}\right)^2 = F(z_k), \tag{2.2.14}$$

where

$$F(z_k) = \sum_i \Omega_i^{(0)}(1+z_k)\exp\left[3\int_0^{z_k}\frac{w_i(z)}{1+z}\,\mathrm{d}z\right] \tag{2.2.15}$$

ans $k_0 = a_0H_0 = h \times 3.24 \times 10^{-18}$ Hz is today's horizon wave number.

Before solving this equation properly, let us pause to extract a few familiar approximate scalings from our formalism. Since the primordial inflationary power spectrum $\Delta_h^2(k,\tau_i)$ is roughly scale invariant [$\propto (k/k_0)^0$], the current power spectrum $\Delta_h^2(k,\tau_0)$ is roughly $\propto C_1$, and hence $\propto (1+z_k)^{-2}$. For modes that re-enter the horizon during radiation domination, when $w_r = 1/3$ term dominates the sum (2.2.15), we solve (2.2.14) to find $(1+z_k) \propto (k/k_0)$. For modes that re-enter during matter domination, when $w_m = 0$ term dominates the sum (2.2.15), we find $(1+z_k) \propto (k/k_0)^2$.

For a more proper analysis, consider a universe with 4 components: matter ($w_m = 0$), curvature ($w_K = -1/3$), dark energy ($w_{de}(z)$), and radiation ($w_r(z) = 1/3 + \delta w_r(z)$). Note that, although one often assumes $w_r = 1/3$ during radiation domination, we have allowed for corrections $\delta w_r(z)$ due to early universe effects discussed later. Then we can write

$$F(z) = \hat{F}(z) + \delta F(z) \tag{2.2.16}$$

where

$$\hat{F}(z) = \Omega_r^{(0)}(1+z)^2 + \Omega_m^{(0)}(1+z) + \Omega_K^{(0)} \tag{2.2.17}$$

and

$$\begin{aligned}\delta F(z) =&\Omega_{de}^{(0)}(1+z)\exp\left[3\int_0^{z}\frac{w_{de}(\tilde{z})}{1+\tilde{z}}\,\mathrm{d}\tilde{z}\right]+\\ &+\Omega_r^{(0)}(1+z)^2\left\{\exp\left[3\int_0^{z}\frac{\delta w_r(\tilde{z})}{1+\tilde{z}}\,\mathrm{d}\tilde{z}\right]-1\right\}.\end{aligned} \tag{2.2.18}$$

Here $\hat{F}$ represents an universe with spatial curvature, matter and "standard" ($w_r = 1/3$) radiation; and δF contains the modifications due to dark energy (w_{de}) and equation-of-state corrections (δw_r).

If we neglect these modifications [by setting $\Omega_{de}^{(0)} = 0 = \delta w_r(z)$ so that $\delta F = 0$], Eq. (2.2.14)

$$\Omega_r^{(0)}(1+\hat{z}_k)^2 + \Omega_m^{(0)}(1+\hat{z}_k) + \left[\Omega_K^{(0)} - \left(\frac{k}{k_0}\right)^2\right] = 0 \tag{2.2.19}$$

has the exact solution

$$1+\hat{z}_k \equiv \frac{1+z_{eq}}{2}\left[-1+\sqrt{1+\frac{4\left[(k/k_0)^2-\Omega_K^{(0)}\right]}{(1+z_{eq})\Omega_m^{(0)}}}\right], \tag{2.2.20}$$

where $1+z_{eq} \equiv \Omega_m^{(0)}/\Omega_r^{(0)}$ is the redshift of matter-radiation equality. Then, including both modifications, the solution becomes

$$(1+z_k) = (1+\hat{z}_k) + \delta z_k, \tag{2.2.21}$$

where $\hat{z}_k$ is defined by (2.2.20). By Taylor expanding $F(z_k)$ around $\hat{z}_k$ (to 2nd order in δz_k)

$$\begin{aligned}&F(z_k) \cong F(\hat{z}_k) + F'(\hat{z}_k)\delta z_k + \frac{F''(\hat{z}_k)}{2}\delta z_k^2 = \hat{F}(\hat{z}_k)\\ &\Rightarrow \frac{F''(\hat{z}_k)}{2}\delta z_k^2 + F'(\hat{z}_k)\delta z_k + \left[F(\hat{z}_k) - \hat{F}(\hat{z}_k)\right] = 0\end{aligned} \tag{2.2.22}$$

and then solving the equation $F(\hat{z}_k) = \hat{F}(\hat{z}_k)$ for δz_k

$$\delta z_k = \frac{F'(\hat{z}_k)}{F''(\hat{z}_k)}\left[-1 + \sqrt{1 - 2\frac{F''(\hat{z}_k)\delta F(\hat{z}_k)}{[F'(\hat{z}_k)]^2}}\right] \tag{2.2.23}$$

where $\delta F(\hat{z}_k) = F(\hat{z}_k) - \hat{F}(\hat{z}_k)$. It is extremely accurate for a wide range of $\Omega_{de}^{(0)}$, $w_{de}(z)$, and $\delta w_r(z)$. Indeed the simpler 1st-order expression

$$\delta z_k = -\frac{\delta F(\hat{z}_k)}{F'(\hat{z}_k)} \tag{2.2.24}$$

is often sufficiently accurate.

2.2.2 The horizon-crossing factor, C_2

In the previous subsection, we treated horizon re-entry as a sudden event that occurs when $k = aH$. In reality, the "outside" behaviour (h_k = constant) only holds when $k \ll aH$, and the "inside" behaviour ($h_k \propto a^{-1}\cos[k\tau + \text{phase}]$) only holds when $k \gg aH$. In between, when $k \sim aH$, neither behaviour holds—i.e., the horizon has a non-zero "thickness".

The behaviour of the background equation of state $w(\tau) = p(\tau)/\rho(\tau)$ during the period of horizon re-entry is imprinted in the factor C_2. For example, let us compute C_2 for a mode k that re-enters the Hubble horizon when $w(\tau)$ is varying slowly relative to the instantaneous Hubble rate. Then we can write

$$a = a_0\left(\frac{\tau}{\tau_0}\right)^{\alpha} \quad \text{with} \quad \alpha = \frac{2}{1+3w} \tag{2.2.25}$$

so that the equation of motion for $\tilde{h}_k$

$$\tilde{h}_k'' + 2\left(\frac{a'}{a}\right)\tilde{h}_k' + k^2\tilde{h}_k = 0 \tag{2.2.26}$$

has the general solution

$$\tilde{h}_k(\tau) = c_1\tau^{\frac{1}{2}-\alpha}J_{\alpha-\frac{1}{2}(k\tau)} + c_2\tau^{\frac{1}{2}-\alpha}Y_{\alpha-\frac{1}{2}(k\tau)} \tag{2.2.27}$$

where we have absorbed the constant τ_0 in α. Since we are interested in $k\tau_i \ll 1$, $Y_{\alpha-\frac{1}{2}} \to -\infty$ and we set $c_2 = 0$. If we evaluate $\tilde{h}_k(\tau)$ at the time τ_i, we get

$$\begin{aligned}\tilde{h}_k(\tau_i) &= c_1\tau_i^{\frac{1}{2}-\alpha}\frac{1}{\Gamma\left(\alpha+1/2\right)}\left(\frac{k\tau_i}{2}\right)^{\alpha-\frac{1}{2}} = \\ &= c_1\frac{1}{\Gamma\left(\alpha+1/2\right)}\left(\frac{k}{2}\right)^{\alpha-\frac{1}{2}}\end{aligned} \tag{2.2.28}$$

where we have used the asymptotic form of the Bessel function of first kind: $J_m(z) \sim \frac{1}{\Gamma(m+1/2)}\left(\frac{z}{2}\right)^m$. Hence,

$$\tilde{h}_k = \tilde{h}_k(\tau_i)\Gamma\left(\alpha+\frac{1}{2}\right)\left[\frac{k\tau}{2}\right]^{1/2-\alpha} J_{\alpha-1/2}(k\tau) \tag{2.2.29}$$

where we have used $\tilde{h}_k' = 0$. (Early in the radiation era, the relevant modes were far outside the horizon, and the corresponding mode functions were τ-indipendent). We have neglected the spatial curvature, K, because the two conditions $K \ll a_0^2 H_0^2$ (current observations indicate that the spatial curvature is small) and $k > a_0 H_0$ (we are only interested in modes that are already inside the horizon) imply that K produces a negligible correction to the equation of motion for h_k. Once the modes are well inside the horizon ($k\tau \gg 1$), we can use the asymptotic Bessel formula

$$J_{\alpha-1/2}(k\tau) \to \sqrt{\frac{2}{\pi k\tau}} \cos\left(k\tau - \alpha\frac{\pi}{2}\right) \tag{2.2.30}$$

to find

$$\frac{\tilde{h}_k^2(\tau)}{h_k^2(\tau_i)} = \frac{\Gamma^2(\alpha+1/2)}{\pi}\left[\frac{k\tau}{2}\right]^{-2\alpha} \cos^2\left(k\tau - \alpha\frac{\pi}{2}\right). \tag{2.2.31}$$

On the other hand, since a mode k re-enters the horizon ($k = aH = a'$) at time $\tau_k = \alpha/k$, we can rewrite Eq. (2.2.8) for C_1 as

$$\begin{aligned} C_1 &= \left[\frac{\tau}{\tau_k}\right]^{-2\alpha} \cos^2[k(\tau-\tau_k)+\phi_k] \\ &= \left[\frac{k\tau}{\alpha}\right]^{-2\alpha} \cos^2(k\tau - \alpha + \phi_k) \end{aligned} \tag{2.2.32}$$

which comes from

$$\left(\frac{1+z}{1+z_k}\right)^2 = \left(\frac{a_k}{a(\tau)}\right)^2 \overset{(2.2.25)}{=} \left(\frac{\tau_k}{\tau}\right)^{2\alpha} \overset{\tau_k=\frac{\alpha}{k}}{=} \left(\frac{k\tau}{\alpha}\right)^{-2\alpha} \tag{2.2.33}$$

Comparing Eqs. (2.2.4), (2.2.5), (2.2.31), and (2.2.32), we see that the phase shift ϕ_k in Eq. (2.2.8) is given by

$$\phi_k = \left[1 - \frac{\pi}{2}\right]\alpha, \tag{2.2.34}$$

and that C_2, dividing (2.2.31) by (2.2.32), is given by

$$C_2(k) = \frac{\Gamma^2(\alpha+1/2)}{\pi}\left[\frac{2}{\alpha}\right]^{2\alpha} \tag{2.2.35}$$

where α should be evaluated at horizon re-entry ($k = aH$). In particular, note that

$$w = 0 \Rightarrow C_2(k) = \frac{9}{16} \quad \text{and} \quad \phi_k = 2 - \pi \tag{2.2.36}$$

$$w = \frac{1}{3} \Rightarrow C_2(k) = 1 \quad \text{and} \quad \phi_k = 1 - \frac{\pi}{2}. \tag{2.2.37}$$

2.2.3 The anisotropic-stress damping factor, C_3

In this subsection, we will include the effects of the anistropic-stress term Π_k on the right-hand side of the tensor mode equation (2.1.22). A non-negligible tensor anisotropic stress Π_k is most naturally generated by relativistic particles free-streaming along geodesics that are perturbed by the presence of the tensor metric perturbations h_k. In this situation, Weinberg [64] has shown that the tensor mode equation (2.1.22) may be rewritten as a fairly simple integrodifferential equation for h_k—see Eq. (18) in [64].

Let us focus on a particularly interesting case: a radiation-dominated universe in which the free-streaming particles constitute a nearly constant fraction f of the background (critical) energy density. (Physically, if the free-streaming particles are stable, or long-lived relative to the instantaneous Hubble time at re-entry, then f will indeed be nearly constant, as required). In this case, following an approach that is essentially identical to the one outlined in [18] (see the appendix B for the explicit calculation), we write the solution in the form

$$h_k(\tau) = h_k(\tau_i) \sum_{n=0}^{\infty} a_n j_n(k\tau) \tag{2.2.38}$$

where $j_n(k\tau)$ are spherical Bessel functions, and find the first five non vanishing coefficients to be given by

$$a_0 = 1, \tag{2.2.39}$$

$$a_2 = \frac{10f}{(15+4f)}, \tag{2.2.40}$$

$$a_4 = \frac{18f(3f+5)}{(15+4f)(50+4f)}, \tag{2.2.41}$$

$$a_6 = \frac{\frac{130}{7}f(14f^2+92f+35)}{(15+4f)(50+4f)(105+4f)}, \tag{2.2.42}$$

$$a_8 = \frac{\frac{85}{343}f(4802f^3+78266f^2+161525f-29400)}{(15+4f)(50+4f)(105+4f)(180+4f)}. \tag{2.2.43}$$

The odd coefficients all vanish: $a_{2n+1} = 0$. Keeping these first five nonvanishing terms yields a solution for $h_k(\tau)$ that is accurate to within 0.1% for all values $0 < f < 1$. Next, as observed in [18], we can use the asymptotic expression

$$j_{2n}(k\tau) \to (-1)^n \frac{\sin(k\tau)}{k\tau} \quad \text{as} \quad k\tau \to \infty \tag{2.2.44}$$

along with the $f = 0$ solution $\tilde{h}_k(\tau) = h_k(\tau_i) j_0(k\tau)$ to infere that the tensor anisotropic stress Π_k induces no additional phase shift in h_k, so that our earlier expression (2.2.34) for ϕ_k is unchanged.[2] In this way, one also see that Π_k damps the tensor power spectrum by the asymptotic factor

$$C_3 = |A|^2 \tag{2.2.45}$$

where

$$A = \sum_{n=0}^{\infty} (-1)^n a_{2n}. \tag{2.2.46}$$

For example, keeping the first 4 terms in this sum, we find an approximate expression for A:

$$A = \frac{-\frac{10}{7}(98f^3-589f^2+9380f-55125)}{(15+4f)(50+4f)(105+4f)}, \tag{2.2.47}$$

which is accurate to within 1% for all values $0 < f < 1$. If we keep the first 5 terms in the sum, we find an even better approximation for A:

$$A = \frac{15(14406f^4-55770f^3+3152975f^2-48118000f+34135000)}{343(15+4f)(50+4f)(105+4f)(180+4f)} \tag{2.2.48}$$

which is accurate to within 0.1% for all values $0 < f < 1$. These calculations improve on the accuracy of previous calculations [7] [49]

[2] See [7] for a complementary explanation of this null result, based on causality.

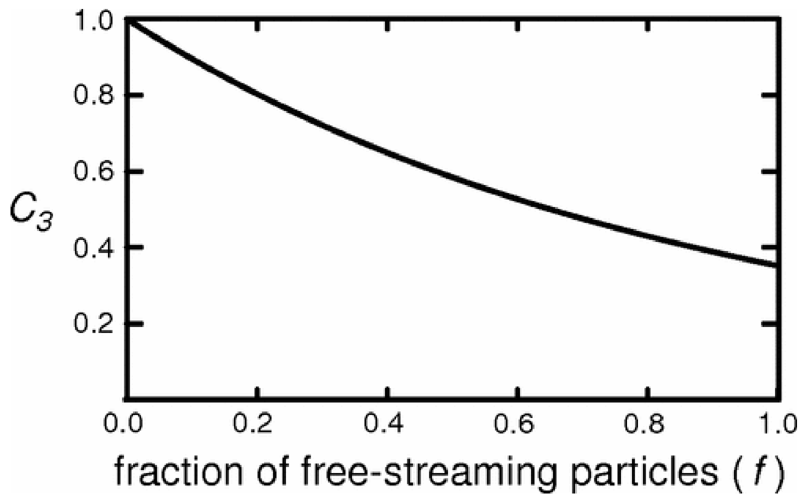


Figure 2.1: C_3 is the transfer function factor that accounts for the damping of the tensor power spectrum due to the tensor anisotropic stress. The factor depends on the fraction f of the background (critical) energy density contained in the free-streaming relativistic particles. The figure plots this dependence for $0 < f < 1$

The exact dependance of C_3 on f is shown in Fig. 2.1. Note that, as f varies between 0 and 1, the damping factor C_3 varies between 1.0 and 0.35. In particular, if we substitute $f = 0.4052$, corresponding to 3 standard neutrino species, the damping factor agrees with the results of the numerical integrations [49, 64]. When the modes probed by the CMB re-enter the horizon, the temperature is relatively low (corresponding to atomic-physics energies), so we are fairly confident that neutrinos are the only free-streaming relativistic particles. But when the modes probed by laser interferometers re-enter the horizon, the temperature is much-higher (above the electroweak phase transition, $T \sim 10^7$ GeV) so that the physics (and, in particular, the instantaneous free-streaming fraction f) is much more uncertain. Thus, laser interferometers offer the possibility of learning about the free-streaming fraction f in the very early universe at temperatures between the inflationary and electroweak symmetry breaking scales.

Finally, although Weinberg and subsequent authors have concentrated on the tensor anisotropic stress due to single fermionic species (the neutrino), it is straightforward to generalize the analysis to include a combination of species which (i) may each decouple at a different time and temperature, and (ii) may be an arbitrary mixture of bosons and fermions. We find that, as long as all of these free-streaming species decouple well before the modes of interest re-enter the horizon, then all of the results presented in this section are completely unchanged. In other words, in order to determine the behaviour of the tensor mode function, one only needs to know one number—the total fraction f of the critical density contained in the free-streaming particles—even if the particles are a mixture of fermionic and bosonic species with different temperature and decoupling times.

2.2.4 Equation-of-state corrections, δw_r

In this subsection, we consider various physical effects that cause the equation of state $w_r(z)$ to deviate from 1/3 during the radiation-dominated epoch, and the corresponding modifications that these effects induce in the GWB transfer function. The deviations

$$\delta w_r(z) = w_r(z) - 1/3 \qquad (2.2.49)$$

primarily modify the transfer function through the redshift factor $(1 + z_k)$ that appears C_1 [see Eqs. (2.2.23) and (2.2.24)]; through the horizon-crossing factor $C_2(k)$ [see Eq. (2.2.35)]; and through the phase shift ϕ_k [see Eq. (2.2.34)]. We consider here six physical effects which can produce these kinds of modifications of the transfer function.

First, deviations can be caused by mass thresholds in the early universe. Suppose that all particle species are described by equilibrium distribution functions. Then we can write ρ and p as

$$\rho = \frac{1}{2\pi^2}\sum_i g_i T_i^4 \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{\exp[u - y_i] \pm 1} u^2 \, \mathrm{d}u, \tag{2.2.50}$$

$$p = \frac{1}{6\pi^2}\sum_i g_i T_i^4 \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{3/2}}{\exp[u - y_i] \pm 1} \, \mathrm{d}u, \tag{2.2.51}$$

where the ith species (with mass m_i, and g_i internal degrees of freedom) is described by temperature T_i and chemical potential μ_i, and we have defined the dimensionless quantities $x_i \equiv m_i/T_i$ and $y_i \equiv \mu_i/T_i$ [35]. In the denominator, the + and − signs are for fermions and bosons, respectively. Then the deviation is given by the exact expression

$$\delta w_r = \sum_i \delta w_r^{(i)} \tag{2.2.52}$$

where

$$\begin{aligned}
\delta w_r^{(i)} &= \frac{p^{(i)}}{\rho^{(i)}} - \frac{1}{3} = \\
&= \frac{1}{3}\left\{ \frac{\int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{3/2}}{\exp[u - y_i] \pm 1} \mathrm{d}u}{\int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{\exp[u - y_i] \pm 1} u^2 \, \mathrm{d}u} - 1 \right\} = \\
\text{(integrating by parts)} \qquad &= -\frac{1}{3} \frac{x_i^2 \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{\exp[u - y_i] \pm 1} \mathrm{d}u}{\int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{\exp[u - y_i] \pm 1} u^2 \, \mathrm{d}u} \\
&= -\frac{5}{\pi^4} \frac{g_i}{g_{*\rho}} \frac{T_i^4}{T^4} f(x_i, y_i)
\end{aligned} \tag{2.2.53}$$

represents the contribution from the ith species,

$$g_{*\rho} \equiv \sum_i g_i \frac{T_i^4}{T^4} \frac{15}{\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2} u^2}{\exp[u - y_i] \pm 1} \mathrm{d}u \tag{2.2.54}$$

represents the effective number of relativistic degrees of freedom, T is conventionally chosen to be the photon temperature, and we have defined the function

$$f(x, y) \equiv x^2 \int_x^{\infty} \frac{(u^2 - x^2)^{1/2}}{\exp[u - y] \pm 1} \mathrm{d}u. \tag{2.2.55}$$

For fixed y_i, note that $f(x_i, y_i)$ vanishes as x_i goes to 0 or ∞; and inbetween it has fairly broad peak, with maximum located at x_i^{peak}, and a peak value $f_i^{peak} = f(x_i^{peak}, y_i)$. In particular, when $y_i = 0$, then the ordered pair (x_i^{peak}, f_i^{peak}) is $(2.303, 1.196)$ for bosons and $(2.454, 1.125)$ for fermions. This makes sense: we expect $\delta w_r^{(i)}$ to vanish when $x_i \ll 1$ (since the species is relativistic) and when $x_i \gg 1$ (since the species is nonrelativistic, and makes a negligible contribution to the energy density). In between, when $x_i \sim x_{i,peak}$, the ith species is cold enough to exhibit nonrelativistic behaviour, yet hot enough to contribute non-negligibly to the energy density.

Using the above equations, we can compute $\delta w_r(z)$ once we know $T_i(z)$ and $\mu_i(z)$. But let us estimate the size of the effect. As a species becomes nonrelativistic, it produces maximum equation-of-state deviation

$$\delta w_r^{(i)} = -\frac{5 f_i^{peak}}{\pi^4} \frac{g_i}{g_{*\rho}} \frac{T_i^4}{T^4} \tag{2.2.56}$$

in the background equation of state. Furthermore, if N_S different species (with the same temperature and similar masses) become nonrelativistic at the same time, then (roughly speaking) the effect is multiplied by N_S (since their $\delta w_r^{(i)}$'s add). Ultimately, the fractional correction $\delta w_r / w_r$ is model dependent, but it can conceivably be as large as a few percent.

Second, deviations can be produced by a trace anomaly in the early universe. During the radiation-dominated epoch, the universe is dominated by highly relativistic particles whose masses may be neglected. Thus, each species is governed by a classical action that is conformally invariant at the classical level, leading to the usual conclusion that the stress-energy tensor is traceless and $w_r = 1/3$. But conformal invariance is broken at the quantum level by interactions among the particles, so that $T_\mu^\mu \neq 0$. For example, for a quark-gluon plasma governed by $\mathrm{SU}(N_c)$ gauge theory, with N_f flavours, and gauge-coupling g, the equation-of-state correction is given [up to $\mathcal{O}(g^5)$ corrections] by [17,31]

$$\delta w_r = \frac{5}{18\pi^2} \frac{g^4}{(4\pi)^2} \frac{(N_c + \frac{5}{4} N_f)(\frac{2}{3} N_f - \frac{11}{3} N_c)}{2 + \frac{7}{2}[N_c N_f / (N_c^2 - 1)]} \tag{2.2.57}$$

Note that this effects can be non-negligible: for large gauge groups (i.e large N_c) in the early universe (prior to the electroweak phase transition), the equation of state w_r may easily be reduced from 1/3 by several percent, or more.

Third, deviations can be produced if the early universe behaves like a slightly imperfect fluid. The stress-energy tensor for an imperfect fluid contains (in addition to the usual perfect-fluid terms) three extra terms whose coefficients (χ, η and ζ) represent heat conduction, shear viscosity, and bulk viscosity (see [63], Chapter 2.11). Of these dissipative effects, only the bulk viscosity term

$$\Delta T^{\mu\nu} = -\zeta (g^{\mu\nu} + u^\mu u^\nu) u^\lambda_{;\lambda} \tag{2.2.58}$$

can contribute to the background evolution in an FRW universe (see [63], Chapters 15.10-15.11). This term modifies the continuity equation

$$\dot{\rho} = -3H(\rho + p) + 9\zeta H^2 = -3H\rho \left[1 + w - \frac{8\pi G \zeta}{H} \right] \tag{2.2.59}$$

so that, as far as gravitational waves are concerned, the effective equation is corrected by

$$\delta w_r = -\frac{8\pi G \zeta}{H}. \tag{2.2.60}$$

Whereas the three effects discussed thus far produce small corrections to the equation of state, it is worth mentioning three other effects that can produce much larger deviations. The first example is a massive particle species that decouples from the thermal plasma before its abundance becomes negligible. Since its energy density falls as a^{-3} (more slowly than the radiation density, which falls as a^{-4}), it can come to dominate the energy density of the universe before it decays (if uts lifetime τ_{decay} is sufficiently long). In this case, w drops to zero when the particle becomes dominant, and rises back to $w = 1/3$ over a time scale given by the decay lifetime τ_{decay}. Second, extra-dimensional physics typically modifies the effective 4-dimensional Friedmann equation. Such modifications which, from the standpoint of the GWB, can in some case look like corrections to the effective radiation equation of state, are primarly constrained by the requirement that the Friedmann equation becomes sufficiently similar to the ordinary 4-dimensional Friedmann equation (with ordinary matter) by the time of Big Bang Nucleosynthesis. Third, due to the weak coupling of the inflaton, the temperature at the start of the radiation-dominted epoch (the reheat temperature) can be much lower than the energy scale at the end of inflation. In this case, laser-interferometer scales might actually re-enter the horizon during the reheating epoch (before the start of radiation domination), when the equation of state was probably quite different from $w = 1/3$. The actual equation of state depends on the details of the reheating process, but a commonly discussed vakue is $w = 0$, or some value in the range $0 < w < 1/3$ [46]. If $w = 0$ during reheating, the corresponding modification of the GWB might be similar to the modification due to the long-lived massive relic discussed above.

Note that the first of these six processes can be expressed as a modification of g_*, and the effect on the transfer function can be computed using the methods discussed in Ref. [54]. However, the other five cannot.

2.2.5 Results

Figure 2.2 illustrates some of the effects discussed in the previous subsections. In this figure, the solid black curve represents the present-day energy spectrum, $\Omega_{gw}(f, \tau_0)$[3], generated by a particular inflationary model—namely, a quandratic potential $V(\phi) = (1/2)m^2\phi^2$. The red dotted curve illustrates the damping due to tensor anisotropic stress from free-streaming neutrinos. Boyle and Steinhardt have assumed that the free-streaming fraction is $f = 0.4052$, which is the f value for three standard neutrino species which decouple around the time of Big Bang Nucleosynthesis. The green dot-dashed curve represents the damping due to tensor anisotropic stress from various particle species (X particles) which begin free-streaming before the scales detected by NASA's "Big Bang Observer" (BBO) and the Japanese "Deci-hertz Interferometer Gravitational Wave Observatory" (DECIGO) re-enter the horizon and then decay after the scales re-enter, but prior to the electroweak phase transition. As an example, Boyle and Steinhardt have assumed that the free-streaming fraction $f = 0.5$. Finally the blue dashed curve represents damping due to a trace anomaly that is present above the electroweak scale. For illustration, they have assumed that this anomaly, through Eq. (2.2.57), reduces the equation of state $w_r = 1/3$ by $\delta w_r = -0.02$. This reduction may be achieved by various combinations of the number of colours N_c, the number of flavours N_f, and the gauge coupling g; but the point is that Boyle and Steinhardt have not chosen an unreasonable large value for δw_r, given the large gauge groups that are often theorized to be present at high energies.

The key point conveyed by Fig. 2.2 is that there a variety of plausible postinflationary effects that can produce rather large modifications of the gravitational-wave spectrum on laser-interferometer scales, without modifying the spectrum on CMB scales. This is tantalizing, since the modifications on laser-interferometer scales reflect the primordial dark age between the end of inflation and the electroweak phase transition, at energies beyond the reach of terrestrial accelerators.

[3]The energy spectrum is related to the tensor power spectrum via: $\Omega_{gw}(k,\tau) = \frac{1}{12}\frac{k^2\Delta_h^2(k,\tau)}{a^2(\tau)H^2(\tau)}$

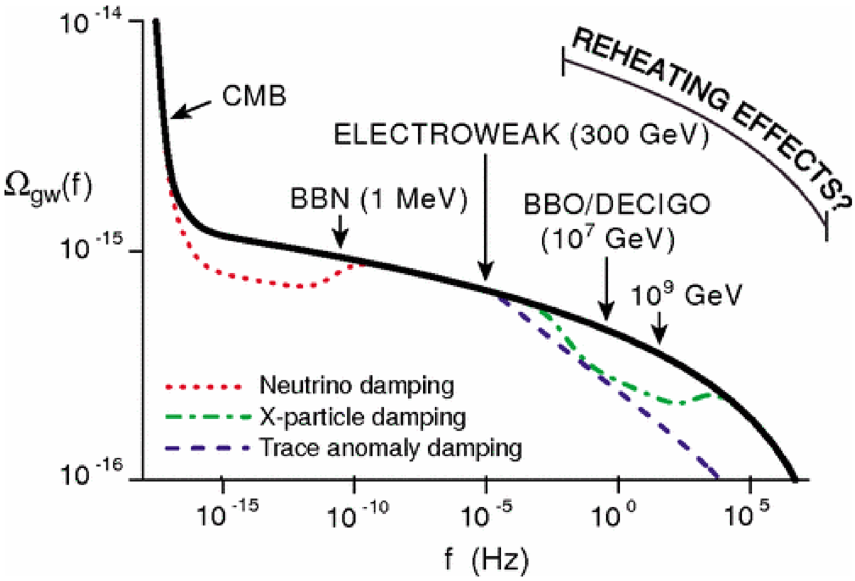


Figure 2.2: The black solid curve represents the present-day gravitational wave-energy spectrum, $\Omega_{gw}(f, \tau_0)$, for the inflationary model $V(\phi) = (1/2)m^2\phi^2$. The red dotted curve shows the damping due to (three ordinary massless species of) free-streaming neutrinos. The green dotted dashed curve shows the damoing effect which arises if free-streaming particles make up 50% of the background energy density at the time τ_{BBO} when the modes probed by BBO/DECIGO re-enter the horizon. As shown in the figure, the particles begin free-streaming sometime before before τ_{BBO}, and decay sometime after τ_{BBO}, but prior to the electroweak symmetry breaking. FInally, the blue dashed curve shows the effect of a conformal anomaly in the early universe that slightly reduces the equation of state from $w = 0.33$ to $w = 0.31$ above the electroweak phase transition. The spectrum will also be modified on comoving scales that re-enter the horizon during the reheating epoch after inflation; but the range of scales affected by reheating is unknown. Finally, note that the correlated BBO interferometer proposal claims a sensitivity that extends beyond the botton of the figure (down to roughly $\Omega_{gw} \sim 10^{-17}$) in the frequency range from 10^{-1} Hz to 10^0 Hz.

Chapter 3

Beyond Bunch-Davies vacuum

In the standard inflationary scenario the modes of the inflaton field can be carried back in time to eras when they start out with a linear size much smaller than the Planck scale. For small scales the expansion of the universe can be ignored and an unique vacuum can be chosen for the inflaton quantum fluctuations. This is the Bunch-Davies vacuum. But the construction ignores the Planck scale and the natural expectation physics beyond is very different from physics at low energies, and not possible to describe using a quantum field theory.

Our ignorance of the high energy physics is encoded in a cut-off imposed on the theory at the Planck scale, i.e. in the choice of the initial conditions for the field modes when that start out at planckian size. Contrary to the standard scenario, since the Planck scale is not infinitely smaller than the inflationary Hubble scale, the initial conditions are imposed in a situation where the time dependence of the background can't be ignored. There is no unique natural vacuum: various ways of choosing the vacuum (minimal uncertainity, adiabatic to all orders, etc.) now give different results. A conservative approach is then to investigate the span of possibilities and its effects.

In [15] (and more recently in [22]) initial conditions (for a particular mode) were imposed when the wavelength was comparable to some fundamental length scale in the theory.

3.1 Inflation and trans-Planckian physics

Let us consider a real massless scalar field ϕ. It is described by the action S

$$S = \frac{1}{2}\int \mathrm{d}^4 x\sqrt{-g}\,\partial^\mu\phi\partial_\mu\phi \tag{3.1.1}$$

on an inflating (de Sitter) background with metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\,\mathrm{d}\mathbf{x}^2, \tag{3.1.2}$$

where the scale factor is given by $a(t) = e^{Ht}$. The equation for a scalar field $\phi(t,\mathbf{x})$ [1] in this background is given by

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi = 0. \tag{3.1.3}$$

In terms of the conformal time $\tau = -\frac{1}{aH}$, and the rescaled field $f = a\phi$, we find

$$y''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)y_{\mathbf{k}} = 0 \tag{3.1.4}$$

[1] Considering FRW spacetime, consistency with its symmetries requires that the background value of the inflaton only depends on time, $\phi = \phi(t)$

in Fourier space (this is the Mukhanov-Sasaki equation (1.5.10) derived in Chapter 1, where we've chosen a different variable f). Note that we have $k = ap$, where p is the physical momentum which is redshifting away with the expansion (k is fixed). We will also need the conjugate momentum to $f_{\mathbf{k}}$ which is given by

$$\pi_{\mathbf{k}} = y'_{\mathbf{k}} - \frac{a'}{a} y_{\mathbf{k}}. \tag{3.1.5}$$

When quantizing the system it turns out that the Heisenberg picture is the most convenient one to use. [47] In terms of time dependent oscillators we can write

$$y_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}} \left(a_{\mathbf{k}}(\tau) + a^{\dagger}_{-\mathbf{k}}(\tau) \right) \tag{3.1.6}$$

$$\pi_{\mathbf{k}}(\tau) = -i\sqrt{\frac{k}{2}} \left(a_{\mathbf{k}}(\tau) - a^{\dagger}_{-\mathbf{k}}(\tau) \right). \tag{3.1.7}$$

The oscillators can be conveniently expressed in terms of their values at some fixed time τ_0,

$$a_{\mathbf{k}}(\tau) = u_k(\tau) a_{\mathbf{k}}(\tau_0) + v_k(\tau) a^{\dagger}_{-\mathbf{k}} \tag{3.1.8}$$

$$a^{\dagger}_{-\mathbf{k}}(\tau) = u^*_k(\tau) a_{-\mathbf{k}}(\tau_0)^{\dagger} + v^*_k(\tau) a_{\mathbf{k}}, \tag{3.1.9}$$

which is nothing but the Bogoliubov transformations which describe the mixing of the creation and annihilation operators as time goes by. Plugging this back into the expressions for $y_{\mathbf{k}}(\tau)$ (see Eq. (1.5.17)) and $\pi_{\mathbf{k}}(\tau)$ we find

$$y_{\mathbf{k}}(\tau) = f_k(\tau) a_{\mathbf{k}}(\tau_0) + f^*_k(\tau) a_{-\mathbf{k}}(\tau_0)^{\dagger} \tag{3.1.10}$$

$$\pi_{\mathbf{k}}(\tau) = -i \left(g_k(\tau) a_{\mathbf{k}}(\tau_0) - g^*_k(\tau) a_{-\mathbf{k}}(\tau_0)^{\dagger} \right) \tag{3.1.11}$$

where

$$f_k(\tau) = \frac{1}{\sqrt{2k}} \left(u_k(\tau) + v^*_k(\tau) \right) \tag{3.1.12}$$

$$g_k(\tau) = \sqrt{\frac{k}{2}} \left(u_k(\tau) - v^*_k(\tau) \right). \tag{3.1.13}$$

$f_k(\tau)$ is a solution of the mode equation (3.1.4). We are now in the position to start discussing the **choice of the vacuum**. A reasonable candidate for a vacuum is

$$a_{\mathbf{k}}(\tau_0) |0, \tau_0\rangle = 0. \tag{3.1.14}$$

In general this corresponds to a class of different vacua depending on the parameter τ_0. At this initial time it follows from Eq. (3.1.9) that $v_k(\tau_0) = 0$, and the relation between the field and its conjugate momentum is particularly simple:

$$\pi_{\mathbf{k}}(\tau_0) = iky_{\mathbf{k}}(\tau_0). \tag{3.1.15}$$

The choice of the vacuum has a simple physical interpretation. Following [47] it is easy to show that it corresponds to a state which minimizes the uncertainty at $\tau = \tau_0$. Using $\langle y_{\mathbf{k}} \rangle = \langle \pi_{\mathbf{k}} \rangle = 0$ it follows from the definiton $\Delta\Phi \equiv \Phi - \langle\Phi\rangle$ that

$$\begin{aligned}
\langle \Delta y_{\mathbf{k}} \Delta y_{\mathbf{k}'} \rangle &= \langle y_{\mathbf{k}} y_{\mathbf{k}'} \rangle = \\
&= \langle 0| \left[f_k(\tau) a_{\mathbf{k}}(\tau_0) + f^*_k(\tau) a_{-\mathbf{k}}(\tau_0)^{\dagger} \right] \left[f_{k'}(\tau) a_{\mathbf{k}'}(\tau_0) + f^*_{k'}(\tau) a_{-\mathbf{k}'}(\tau_0)^{\dagger} \right] |0\rangle = \\
&\overset{(3.1.14)}{=} f_k(\tau) f^*_{k'}(\tau) \langle 0| a_{\mathbf{k}}(\tau_0) a_{-\mathbf{k}'}(\tau_0)^{\dagger} |0\rangle = \\
&= f_k(\tau) f^*_{k'}(\tau) \langle 0| \left[a_{\mathbf{k}}(\tau_0), a_{-\mathbf{k}'}(\tau_0)^{\dagger} \right] |0\rangle = \\
&= |f_k|^2 \delta^{(3)}(\mathbf{k} - \mathbf{k}')
\end{aligned} \tag{3.1.16}$$

thus,

$$\langle \Delta y_{\mathbf{k}} \Delta y_{\mathbf{k}'} \rangle = |f_k|^2 \, \delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{3.1.17}$$

$$\langle \Delta \pi_{\mathbf{k}} \Delta \pi_{\mathbf{k}'} \rangle = |g_k|^2 \, \delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{3.1.18}$$

where

$$|f_k|^2 \, |g_k|^2 = \frac{1}{4} \left(1 + \left| uv - u^* v^* \right|^2 \right) \tag{3.1.19}$$

and we have further adopted the notation $\left\langle \Phi(\mathbf{k}, \tau) \Phi^\dagger(\mathbf{k}', \tau) \right\rangle \equiv \Phi^2(k) \delta^{(3)}(\mathbf{k} - \mathbf{k}')$, where the quantity $\Phi^2(k)$, the power spectrum of the quantity Φ, depends only on k if the state is invariant under spatial translations and rotations. The latter expression is indeed minimized at $\tau = \tau_0$ where $v_k(\tau_0) = 0$.

We will now show that the vacuum defined in this way can be referred to as the zeroth order adiabatic vacuum.

3.1.1 The role of the adiabatic vacuum

As discussed in Chapter 1, in a time-dependent background the notion of a vacuum is a tricky issue. One possibility is to use the adiabatic vacuum, where the solution of the wave equation is, formally, assumed to be of WKB form. Often the exact solution is expanded to some finite order in the adiabatic parameter that determines the slowness of the process. The idea is to approximate the field equation, at some moment in time, with their time indipendent counterparts (possibly with some corrections to some finite order) and define positive and negative energy using solutions to these approximative equations.

The adiabatic vacuum prescription relies on the WKB approximation for the solution of the Mukhanov-Sasaki equation for the Fourier modes

$$v_k'' + \omega_k^2(\tau) v_k = 0, \quad \omega_k(\tau) \equiv \sqrt{k^2 + m_{eff}^2(\tau)} \tag{3.1.20}$$

in the case of slowly varying $\omega_k^2(\tau)$. Substituting the Ansatz

$$v_k(\tau) = \frac{1}{\sqrt{W_k(\tau)}} \exp\left[i \int_{\tau_0}^{\tau} W_k(\tau) \, \mathrm{d}\tau \right] \tag{3.1.21}$$

into (3.1.20) we find that the function $W_k(\tau)$ must obey the nonlinear equation

$$W_k^2 = \omega_k^2 - \frac{1}{2} \left[\frac{W_k''}{W_k} - \frac{3}{2} \left(\frac{W_k''}{W_k} \right)^2 \right]. \tag{3.1.22}$$

Let us consider the case when ω_k is a slowly varying function of time. More precisely, we assume that ω_k and all its derivatives change substantially, i.e. $\frac{\Delta \omega_k}{\omega_k} \sim \mathcal{O}(1)$, only during time intervals $T \gg \frac{1}{\omega_k}$. In this case, Eq. (3.1.22) can be used as a recurrence relation which allows us to find a particular solution for W_k in the form of asymptotic series in the powers of small parameter $(\omega_k T)^{-1}$. For example, to zeroth order in $(\omega_k T)^{-1}$ we have

$${}^{(0)}W_k = \omega_k, \tag{3.1.23}$$

while to second order

$${}^{(2)}W_k = \omega_k \left(1 - \frac{1}{4} \frac{\omega_k''}{\omega_k^3} + \frac{3}{8} \frac{\omega_k'^{\,2}}{\omega_k^4} \right). \tag{3.1.24}$$

In principle one could find ${}^{(N)}W_k$ to an arbitrary order N. However the series obtained is asymptotic, and so the accuracy of the approximation reaches at a particular N and subsequently becomes worse as N grows. Substituting ${}^{(N)}W_k$ in (3.1.21) we obtain an approximate WKB solution $v_k^{(N)}(\tau)$ of the mode equation (3.1.20) to adiabatic order N.Then the mode function $v_k(\tau)$ determining the *adiabatic vacuum of order N at a particular time* τ_0 are definied by the requirement that the exact solution $v_k(\tau)$ of equation (3.1.20) satisfies the following initial conditions,

$$v_k(\tau_0) = v_k^{(N)}(\tau_0), \quad v_k'(\tau_0) = v_k'^{(N)}(\tau_0). \tag{3.1.25}$$

What one should remember is that the adiabatic vacuum (to some finite order in the adiabatic parameter) is not unique but depends on what moment in time one uses for its definition. In de Sitter space, however, it happens that the finite order adiabatic vacuum obtained in the infinite past actually corresponds to an exact solution of the exact field equations, and therefore in some sense is distinguished. After all, when the modes are small enough they do not care about the expansion of the universe.

Which vacuum should we choose? There is no unique choice: one viable alternative is the minimum uncertainty vacuum and we will argue that it agrees with the adiabatic vacuum only to zeroth order. In fact, it is only at zeroth order, where the expansion of the universe can be ignored , that ambiguities in the definition of the vacuum are removed. It is important to observe that these distinctions between various vacua only become important since we insist on imposing the choice of vacua at a finite time corresponding to some specific finite wavelength because we don't have any knowlwdge of physics beyond the Planck scale.

In the zeroth order adiabatc approximation, the solution of a mode equation of the form

$$y_{\mathbf{k}} + \left(k^2 - C(\tau)\right) y_{\mathbf{k}} = 0, \tag{3.1.26}$$

is given by

$$y_{\mathbf{k}} = \frac{1}{\sqrt{2\omega}} e^{\pm i\omega\tau}, \tag{3.1.27}$$

where

$$\omega = \sqrt{k^2 - C(\tau)}. \tag{3.1.28}$$

For the approximation to make sense we must have an ω that varies slowly enough (i.e., adiabatically). A necessary condition for this to be the case is that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ln C \ll \omega, \tag{3.1.29}$$

which for us [where $C(\tau) = 2/\tau^2$] typically leads to

$$k\tau \gg 1. \tag{3.1.30}$$

With the help of this the zeroth order solution simply degenerates into

$$y_{\mathbf{k}} = \frac{1}{\sqrt{2k}} e^{\pm ik\tau}, \tag{3.1.31}$$

and one finds a conjugate momentum given by

$$\pi_{\mathbf{k}} = iky_{\mathbf{k}}. \tag{3.1.32}$$

This is precisely what our choice in the previous subsection led to, and we can therefore refer to the vacuum that we will analyze as the zeroth order adiabatic vacuum. A finite order adiabatic mode is in general not an exact solution, but the vacuum that it corresponds to is nevertheless an honest proposal for a vacuum. One should view Eq. (3.1.32) as initial conditions with a subsequent time evolution given by the exact solution.

3.1.2 Gaussianity of zeroth order adiabatic vacuum

We defined our vacuum at some time τ_0 as

$$\forall \mathbf{k}: \quad a_{\mathbf{k}}(\tau_0)|0,\tau_0\rangle = 0. \tag{3.1.33}$$

We want to show that this state corresponds to a Gaussian state and time evolution preserves its Gaussianity. Indeed, it follows from (3.1.11) that in the Heisenberg representation, the time-indipendent state $|0,\tau_0\rangle_H$ is an eigenstate of the operator $y_{\mathbf{k}} + i\gamma_k^{-1}(\tau)\pi_{\mathbf{k}}$, namely

$$\left\{y_{\mathbf{k}}(\tau) + i\gamma_k^{-1}(\tau)\pi_{\mathbf{k}}(\tau)\right\}|0,\tau_0\rangle_H = 0, \tag{3.1.34}$$

where the operators $y_{\mathbf{k}}(\tau)$, $\pi_{\mathbf{k}}(\tau)$ as well as the function γ_k depend on time,

$$\gamma_k = k\frac{u_k^* - v_k}{u_k^* + v_k} = \frac{1}{2|f_k|^2} - i\frac{F_k}{|f_k|^2} \tag{3.1.35}$$

$$F(k) = \Im u_k v_k = \Im f_k^* g_k \tag{3.1.36}$$

On the other hand, in the Schrödinger representation the time-evolved state $|0,\tau\rangle_S \equiv S|0,\tau_0\rangle$, where S is the S-matrix, satisfies the equation

$$S a_{\mathbf{k}}(\tau_0) S^{-1}|0,\tau\rangle_S = 0. \tag{3.1.37}$$

or, equivalently,

$$\left\{y_{\mathbf{k}}(\tau_0) + i\gamma_k^{-1}(\tau)\pi_{\mathbf{k}}(\tau_0)\right\}|0,\tau\rangle_S = 0. \tag{3.1.38}$$

Note the similar structure of Eq. (3.1.34) and (3.1.38). In the coordinate Schrödinger representation, $\pi_{\mathbf{k}}(\tau_0) = -i\frac{\partial}{\partial y_{-\mathbf{k}}(\tau_0)}$. Then

$$\left\{y_{\mathbf{k}}(\tau_0) + \gamma_k^{-1}(\tau)\frac{\partial}{\partial y_{-\mathbf{k}}(\tau_0)}\right\}\Psi = 0 \tag{3.1.39}$$

$$\frac{\gamma_k^{-1}(\tau)\frac{\partial\Psi}{\partial y_{-\mathbf{k}}(\tau_0)}}{\Psi} = -y_{\mathbf{k}}(\tau_0) \tag{3.1.40}$$

Solving the differential equation, we get

$$\Psi = \exp\left(-\frac{y_{\mathbf{k}}(\tau_0)^2 + C}{2\gamma_k^{-1}(\tau)}\right) \tag{3.1.41}$$

The costant C can be set to zero. Hence the state $|0,\tau_0\rangle_S$ has a Gaussian wave functional in this representation, consisting of the product of

$$\begin{aligned}\Psi\left[y_{\mathbf{k}}(\tau_0), y_{-\mathbf{k}}(\tau_0)\right] &= N_k \exp\left(-\frac{y_{\mathbf{k}}(\tau_0)y_{-\mathbf{k}}(\tau_0)}{2|f_k|^2}\left\{1 - i2F(k)\right\}\right) = \\ &= N_k \exp\left(-\frac{\left|y_{\mathbf{k}}(\tau_0)\right|^2}{2|f_k|^2}\left\{1 - i2F(k)\right\}\right)\end{aligned} \tag{3.1.42}$$

for each pair $\mathbf{k}$, $-\mathbf{k}$ where N_k is the normalization coefficient.

3.1.3 Imposing the initial conditions

Let us now consider the standard treatement of fluctuations in inflation. In this case we have (1.6.36)

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \left(1 - \frac{i}{k\tau}\right) \tag{3.1.43}$$

and

$$g_k = \sqrt{\frac{k}{2}} e^{-ik\tau}. \tag{3.1.44}$$

The logic behind the choice is that the mode at early times (when $\tau \to -\infty$) is of positive frequency and corresponds to what one would naturally think of as the vacuum. It is nothing but the state obeying (3.1.14) for $\tau_0 \to -\infty$ and is therefore the zeroth order adiabatic vacuum of the infinite past. Note that the zeroth order adiabatic vacuum in this case is actually an exact solution (for $\tau \to -\infty$). For later times (when $\tau \to 0$ and the second term of $f_{\mathbf{k}}$ becomes important) we see how the initial vacuum leads to particle creation thereby providing the fluctuation spectrum.

But what if the initial conditions are chosen differently? As we've seen in Chapter 1, Eq. (1.6.35), in general we could have

$$f_k = A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) \tag{3.1.45}$$

$$g_k = A_k \sqrt{\frac{k}{2}} e^{-ik\tau} - B_k \sqrt{\frac{k}{2}} e^{ik\tau}, \tag{3.1.46}$$

with a nonzero B_k. If we then work backwards, we can calculate what this corresponds to in terms of u_k and v_k.

$$u_k = \frac{1}{2} \left(\sqrt{2k} f_k(\tau) + \sqrt{\frac{2}{k}} g_k(\tau) \right) \tag{3.1.47}$$

$$v_k^* = \frac{1}{2} \left(\sqrt{2k} f_k(\tau) - \sqrt{\frac{2}{k}} g_k(\tau) \right) \tag{3.1.48}$$

Substituting (3.1.46),

$$\begin{aligned} u_k &= \frac{\sqrt{2k}}{2} \left(A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) \right) + \frac{1}{2} \sqrt{\frac{2}{k}} \left(A_k \sqrt{\frac{k}{2}} e^{-ik\tau} - B_k \sqrt{\frac{k}{2}} e^{ik\tau} \right) = \\ &= \frac{1}{2} \left(A_k e^{-ik\tau} \left(1 - \frac{i}{k\tau}\right) + B_k e^{ik\tau} \left(1 + \frac{i}{k\tau}\right) + A_k e^{-ik\tau} - B_k e^{ik\tau} \right) \end{aligned} \tag{3.1.49}$$

and similarly for v_k^*. The result is

$$u_k = \frac{1}{2} \left(A_k e^{-ik\tau} \left(2 - \frac{i}{k\tau}\right) + B_k e^{ik\tau} \frac{i}{k\tau} \right) \tag{3.1.50}$$

$$v_k^* = \frac{1}{2} \left(B_k e^{ik\tau} \left(2 + \frac{i}{k\tau}\right) - A_k e^{-ik\tau} \frac{i}{k\tau} \right). \tag{3.1.51}$$

At this point we should also remember that

$$|u_k|^2 - |v_k|^2 = 1 \tag{3.1.52}$$

from which we find

$$|A_k|^2 - |B_k|^2 = 1. \tag{3.1.53}$$

As we have seen, the choice of the vacuum that we make requires that we put $v_k^*(\tau_0) = 0$ at some initial moment τ_0. This implies that

$$B_k = \frac{ie^{-2ik\tau_0}}{2k\tau_0 + i} A_k \tag{3.1.54}$$

from which we conclude that

$$\begin{aligned} |A_k|^2 &= 1 + |B_k|^2 = \\ &= 1 + \frac{ie^{-2ik\tau_0}}{2k\tau_0 + i} \frac{-ie^{-2ik\tau_0}}{2k\tau_0 - i} |A_k|^2 = \\ &= 1 + \frac{1}{4k^2\tau_0^2 + 1} |A_k|^2 \\ \Rightarrow |A_k|^2 &\left(1 - \frac{1}{4k^2\tau_0^2 + 1}\right) = 1 \\ |A_k|^2 &= \frac{1}{1 - \frac{1}{4k^2\tau_0^2 + 1}} \end{aligned} \tag{3.1.55}$$

Hence,

$$|A_k|^2 = \frac{1}{1 - |\alpha_k|^2}, \tag{3.1.56}$$

where

$$\alpha_k = \frac{i}{2k\tau_0 + i}. \tag{3.1.57}$$

We next move to the calculation of the dimensionless power spectrum, i.e., the variance of inflaton fluctuations due to quantum zero-point fluctuations, given by

$$\begin{aligned} \Delta_\phi^2(k,\tau) &= a^{-2}\Delta_f^2(k,\tau) = \frac{k^3}{2\pi^2 a^2} |f_k(\tau)|^2 = \\ &= \frac{k^3}{2\pi^2 a^2} \left| A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) \right|^2 = \\ &= \frac{k^2}{4\pi^2 a^2} \left[|A_k|^2 \left(1 + \frac{1}{k^2\tau^2}\right) + |B_k|^2 \left(1 + \frac{1}{k^2\tau^2}\right) + A_k B_k^* e^{-2ik\tau} \left(1 - \frac{i}{k\tau}\right)^2 + \right. \\ &\left. + B_k A_k^* e^{2ik\tau} \left(1 + \frac{i}{k\tau}\right)^2 \right] \end{aligned} \tag{3.1.58}$$

For $\tau \to 0$, $e^{\pm 2ik\tau} \to 1$ and we can neglect the terms $\sim \frac{1}{k^2\tau^2}$ coming from the square of the parenthesis

$$\left(1 - \frac{i}{k\tau}\right)^2 = 1 - \frac{2i}{k\tau} + \dots \tag{3.1.59}$$

$$\left(1 + \frac{i}{k\tau}\right)^2 = 1 + \frac{2i}{k\tau} + \dots \tag{3.1.60}$$

Then,

$$
\begin{aligned}
\Delta_\phi^2(k,\tau) &\sim \frac{1}{4\pi^2\tau^2a^2}\left(|A_k|^2+|B_k|^2-A_k^*B_k-A_kB_k^*\right)= \\
&=\left(\frac{H}{2\pi}\right)^2\left(\frac{1}{1-|\alpha_k|^2}+\frac{1}{4k^2\tau_0^2+1}\frac{1}{1-|\alpha_k|^2}-\frac{ie^{-2ik\tau_0}}{2h\tau_0+i}\frac{1}{1-|\alpha_k|^2}+\frac{ie^{2ik\tau_0}}{2h\tau_0-i}\frac{1}{1-|\alpha_k|^2}\right)= \\
&=\left(\frac{H}{2\pi}\right)^2\left(1+|\alpha_k|^2-\alpha_ke^{-2ik\tau_0}-\alpha_k^*e^{2ik\tau_0}\right)\frac{1}{1-|\alpha_k|^2},
\end{aligned}
\tag{3.1.61}
$$

where we have used $\tau=-1/aH$ in the prefactor and considered the leading term at late times when $\tau\to 0$. If we impose the initial condition at $\tau_0\to-\infty$ we get $\alpha=0$ and recover the standard result $\Delta_\phi^2(k,\tau)=\left(\frac{H}{2\pi}\right)^2$.

Now, for a given k we choose a finite τ_0 such that the physical momentum corresponding to k is given by some fixed scale Λ. Λ is the energy scale of new physics, e.g., the Planck scale or the string scale. From

$$
k=ap=-\frac{p}{\tau H} \tag{3.1.62}
$$

with $p=\Lambda$ we find

$$
\tau_0=-\frac{\Lambda}{Hk}. \tag{3.1.63}
$$

it is important to note that τ_0 depends on k. Using (3.1.57)

$$
\alpha_k=\frac{i}{i-2\frac{\Lambda}{H}} \tag{3.1.64}
$$

and then

$$
\begin{aligned}
\Delta_\phi^2(k,\tau) &=\left(\frac{H}{2\pi}\right)^2\left(1+\frac{1}{1+4\frac{\Lambda^2}{H^2}}-\frac{i}{i-2\frac{\Lambda}{H}}e^{2i\frac{\Lambda}{H}}-\frac{i}{i+2\frac{\Lambda}{H}}e^{-2i\frac{\Lambda}{H}}\right)\frac{1}{1-\frac{1}{1+4\Lambda^2/h^2}}= \\
&=\left(\frac{H}{2\pi}\right)^2\left[\frac{1+4\frac{\Lambda^2}{H^2}+1+i\left(i+2\frac{\Lambda}{H}\right)e^{2i\frac{\Lambda}{H}}+i\left(i-2\frac{\Lambda}{H}\right)e^{-2i\frac{\Lambda}{H}}}{1+4\frac{\Lambda^2}{H^2}}\right]\frac{1+4\frac{\Lambda^2}{H^2}}{4\frac{\Lambda^2}{H^2}}= \\
&=\frac{2+4\frac{\Lambda^2}{H^2}+i2\frac{\Lambda}{H}\left(e^{2i\frac{\Lambda}{H}}-e^{-2i\frac{\Lambda}{H}}\right)-\left(e^{2i\frac{\Lambda}{H}}+e^{-2i\frac{\Lambda}{H}}\right)}{4\frac{\Lambda^2}{H^2}}.
\end{aligned}
\tag{3.1.65}
$$

If we assume $\Lambda/H\gg 1$ we get

$$
\Delta_\phi^2(k,\tau)=\left(\frac{H}{2\pi}\right)^2\left(1-\frac{H}{\Lambda}\sin\left(\frac{2\Lambda}{H}\right)\right). \tag{3.1.66}
$$

In conclusion, the size of the correction ($\sim H/\Lambda=|1/(k\tau_0)|$) is precisely what is to be expected from a higher order correction to the zeroth order adiabatic vacuum. If the vacuum is imposed in the infinite past, the vacuum is exact, but if it is imposed at a later time it is natural to expect nonvanishing corrections.

From Chapter 2 we know that the tensor power spectrum is just a rescaling of the scalar field power spectrum:

$$\Delta_h^2(k,\tau) = \frac{8}{M_{Pl}^2}\Delta_\phi^2(k,\tau) = \\ = 2\left(\frac{H}{\pi M_{Pl}}\right)^2\left(1 - \frac{H}{\Lambda}\sin\left(\frac{2\Lambda}{H}\right)\right). \tag{3.1.67}$$

So far no real consensus has been reached in the literature, and there are at least two competing estimates of the size of the corrections to the standard scale invariant spectrum. In, e.g., Refs. [34] the corrections are argued to be of size $(H/\Lambda)^2$, while in, e.g., Refs. [21, 23], one is dealing with substantially larger corrections of our order of magnitude (as we have seen this can be expected on quite general grounds). In [32] it was argued, using a low energy effective field theory, that local physics imply that the effects cannot be larger than $(H/\Lambda)^2$. This conclusion has been criticized in [11] , where it was pointed out that trans-Planckian physics can be effectively provide the low energy theory with an excited vacuum, therby circumventing the arguments of [32].

3.2 On the consistency of de Sitter α-vacua

The vacua selected in [15] and used in the section above correspond to an one-parameter subfamily of the two-parameter family of de Sitter invariant vacua, also called α-vacua , introduced in [3, 12, 25, 39] and more recently discussed in [9, 56] in the context of de Sitter holography. Formally, the α-vacua are realized as squeezed states over the Bunch-Davies vacuum. This is only a formal correspondence because $|\alpha\rangle$ is a non-normalizable excitation in the Fock space constructed over $|0\rangle$, i.e., each of the α-vacua is the de Sitter invariant ground state of a different Hilbert space. α is a superselection parameter. The simplified approach discussed in [15] and above essentialy amounts to an investigation of the physics of α-vacua and the main purpose of [14] is to translate some of the observations made in [15] into a language appropriate for holographic studies.

Understanding whether interacting field theory in any of the $|\alpha \neq 0\rangle$-vacua physics makes sense as a consistent theory of physics is still a matter of debate. [16].

In [4, 24] non renormalization problems were pointed out: the loop amplitudes seem to be ill defined, but none of these problems seems to be relevant to the issue of transplanckian physics in cosmology. In [14] it's suggested that a planckian input/cutoff would give well defined answers.

Another issue that has been brought up in the same articles is the long distance behaviour of the theory, e.g., casuality problems. Vacuum-dependent Green functions are expected to be non-zero outside the light cone, as for the Bunch-Davies vacuum. In [14] it's stressed that an unorthodox behaviour of these Green functions doesn't imply that the theory is inconsistent. The Green functions which are important for causality are the commutator and the retarded Green function, which are indipendent of the choice of the vacuum and always vanish outside the light cone.

In [32] it is argued that α can't vary with a changing H because local physics can't know how H changes ad consequently α must remain constant. But even the Bunch-Davies modes have a dependence on H, which is generated through the expansion of the universe while the mode is in the trans-Planckian regime. A further dependence on H through α isn't qualitatively much different [14].

Even if the α-vacua are not thermal, in [32] they presented an argument based on holography and complementarity to show that all perturbations will be inevitably thermalized. But taking a point of view based on Robertson-Walker coordinates, Danielsson claimed that we can follow the modes through the de Sitter horizon and this is the appropriate way to discuss fluctuations generated from inflation.

Chapter 4

A new approach

In this chapter, we will follow the proposal of Dvali and Gomez [20] and Hofmann and Rug [30] that black holes are graviton bound states and a Fock vacuum does not support a bound state. So we will need to construct a new state basis for a Black Hole (BH). We will use the results of Appendix C to express operators and the Fock basis in terms of fields. The advantage will be that we can use fields on any kind of vacuum state. Since we can't hope to know the real BH state at this point we will borrow the idea of non-zero overlaps with it from QCD. In this way we will be able to model a state from our fields using a new type of current $\mathcal{J} = \phi^N$. With the help of this tool we will be able to calculate expectation value of observables using Wick's theorem. We will just focus on one of the two observable considered in [30] since we need it as an example to introduce the new language: the particle density operator $n(\mathbf{k})$ and its $d^3 k$ integral to produce a constituent number $\mathcal{N}_C$ to count the effective number of particles inside the BH.

We are not going to deal with gravitons, which would have to be rank 2 tensors. This would be technically much more demanding, but should be possible. For the present work we will indeed work with scalar fields. Harmonic (or de Donder) gauge, $\Gamma^{\alpha}_{\beta\gamma} g^{\beta\gamma} = 0$, which one uses frequently when calculating Einstein field equations's solutions in the first order approximation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ leads, up to constant factors, to the same field equations and propagator structure as the one for scalar fields: $\Box h^{\mu}_{\mu} = 0$ and $\langle 0| T\left[h^{\mu}_{\mu}(x) h^{\alpha}_{\alpha}(y)\right] |0\rangle \sim \frac{1}{\Box}\delta^{(4)}(x-y)$. So we can consider our scalar field to be the trace of the graviton. Clearly, this means that we are not taking into account vector and tensor degrees of freedom. Nonetheless, we gain a technically far less challenging analysis of large bound states in return.

4.1 Auxiliary current description: introduction

In standard QFT we deal with Fock states. However, we are set out to consider BH as bound state. This kind of state cannot be created just by free particles. Therefore, we will now work with a quantum state $|\psi\rangle$ that is a bound state. Sadly, we do not know the full quantum theory of gravity that would allow us to write down a state that describes the real world BH. What we can do, though, is construct a state from a theory we know ad let it have a non-vanishing overlap with the real world BH state $|\mathcal{B}\rangle$. Overlap constants are a common tools in QCD [50]. Here we shall also introduce a type of current that will allow us to create a state from the vacuum, as well as choose the overlap to be a, so far unknown, (dimensionful) constant which will encode the structural information for gravity, that allows us to have a bound state. Thereby, we will have a way of writing down the BH state in terms of the known scalar field.

We start with the BH state and expand it in terms of a basis $\left\{|\mathcal{L}\rangle := |K, \mathcal{Q}\rangle\right\}$ for the BH's total momentum K and a set of quantum numbers $\mathcal{Q} := \{Q\}$ that identifies the bound state's various internal symmetries — e.g., total spin, total angular momentum, mass, etc. Additionally, let us assume that the BH is in a specific state with respect to the internal symmetry operators, $\mathcal{Q} = \mathcal{Q}_B$, but it is not necessarily in a momentum eigenstate. So the expansion reads

$$\begin{aligned}|\mathcal{B}\rangle &= \int \mathrm{d}^4 K\, \tilde{\mathcal{B}}(\mathcal{L}_B)|\mathcal{L}_B\rangle = \\ &= \int \mathrm{d}^3 K\, \mathcal{B}(\mathcal{L}_B)|\mathcal{L}_B\rangle,\end{aligned} \tag{4.1.1}$$

where the second step consisted of taking the BH state to be on-shell, i.e. such that $K^2 = M^2$, with M the BH's mass. In the above expression, then, $\tilde{\mathcal{B}}(K, \mathcal{Q}_B) = \delta(K^0 - E)\mathcal{B}(\mathbf{K}, \mathcal{Q}_B)$ (useful also in the next chapter). With these results, though, we have no way of making uselful QFT calculations with the bound state. To relate the state to something we can work with, we shall introduce the auxiliary current description for the BH state.

Let $\mathcal{J}$ be a source that creates a state in the non-perturbative vacuum $|\Omega\rangle$ that has the exact same quantum numbers as the BH state—from hereon we shall call it Auxiliary Current, or AC. We can fix this statement with the following expression:

$$\langle\mathcal{L}|\mathcal{J}(0)|\Omega\rangle \equiv \Gamma_B\delta(\mathcal{Q}, \mathcal{Q}_B). \tag{4.1.2}$$

The independence of the RHS from the momentum K means that AC is compatible with any total momentum for the BH and, by defining Γ_B over the basis, this statement is more general and applies to the BH state which can be seen inserting a $\mathbb{1}$:

$$\begin{aligned}\langle\mathcal{B}|\mathcal{J}(x)|\Omega\rangle &= \sum_{\mathcal{Q}}\int \mathrm{d}^4 K\, \tilde{\mathcal{B}}^*(\mathcal{L})\langle\mathcal{L}|\mathcal{J}(x)|\Omega\rangle = \\ &= \sum_{\mathcal{Q}}\int \mathrm{d}^4 K\, \tilde{\mathcal{B}}^*(\mathcal{L})e^{-iKx}\langle\mathcal{L}|\mathcal{J}(0)|\Omega\rangle = \\ &= \Gamma_B\sum_{\mathcal{Q}}\int \mathrm{d}^4 K\, e^{-iKx}\tilde{\mathcal{B}}^*(\mathcal{L}_B).\end{aligned} \tag{4.1.3}$$

What we can do with (4.1.2) now is to express the basis kets $\{|\mathcal{L}\rangle\}$ in terms of the AC. So we write down the state that is created from the current at spacetime point x and insert a $\mathbb{1}$ again:

$$\begin{aligned}\mathcal{J}(x)|\Omega\rangle &= \sum_{\mathcal{Q}}\int \mathrm{d}^4 K\, |\mathcal{L}\rangle\langle\mathcal{L}|\mathcal{J}(x)|\Omega\rangle = \\ &= \sum_{\mathcal{Q}}\int \mathrm{d}^4 K\, e^{-iKx}\langle\mathcal{L}|\mathcal{J}(0)|\Omega\rangle|\mathcal{L}\rangle = \\ &= \Gamma_B\sum_{\mathcal{Q}}\int \mathrm{d}^4 K\, e^{-iKx}|\mathcal{L}_B\rangle,\end{aligned} \tag{4.1.4}$$

and to solve for the momentum kets we transform the above result into momentum space:

$$\begin{aligned}\int \frac{\mathrm{d}^4 x}{(2\pi)^4}\, e^{iPx}\mathcal{J}(x)|\Omega\rangle &= \Gamma_B\int \frac{\mathrm{d}^4 x}{(2\pi)^4}\int \mathrm{d}^4 K\, e^{i(P-K)x}|K, \mathcal{Q}_B\rangle = \\ &= \Gamma_B|P, \mathcal{Q}_B\rangle,\end{aligned} \tag{4.1.5}$$

so that

$$|P, \mathcal{Q}_B\rangle = \tilde{\Gamma}_B^{-1}\int \mathrm{d}^4 x\, e^{iPx}\mathcal{J}(x)|\Omega\rangle \tag{4.1.6}$$

where the modified overlap constant absorbs the $(2\pi)^4$ factor. We now insert it in (4.1.1) and get the final result for the BH state:

$$|\mathcal{B}\rangle = \tilde{\Gamma}_B^{-1} \int \mathrm{d}^3 P \, \mathcal{B}(\mathbf{P}, \mathcal{Q}_B) \int \mathrm{d}^4 x \, e^{iPx} \mathcal{J}(x) |\Omega\rangle. \tag{4.1.7}$$

This is the expansion for the BH state we are going to work with for the rest of the chapter. Note that the BH's wavefunction dependence on the momentum is reduced to the 3-momentum $\mathbf{P}$ because we took it to be on-shell right from the beginning—this also means that the Minkowski product in the exponential has the momentum's zero component the on-shell dispersion relation. But one is, of course, also free to use the full $\mathrm{d}^4 P$ measure and use the $\tilde{\mathcal{B}}(P, \mathcal{Q}_B)$ as it was in the inital expansion instead.

Now, as mentioned above, $\mathcal{J}$ is supposed to create a state which has the same quantum numbers as the BH. Let's consider the free scalar field case in Appendix C, where the momentum basis states are created by products of the scalar field operator. But now we will take all fields to be at the same spacetime point. This doesn't come unmotivated at all: first of all, it resembles the treatment of hadrons [50]. Furthermore, it's also physically intuitive: we want to spawn a bound state of particles at a given spacetime point. However due to quantum fluctuations, especially the ones resulting from non-perturbative effects, the position of the individual constituents is never fixed over time. Introducing an extra distribution of each individual constituent would complicate our initial problem a lot; especially because our goal is finding a bound state description for a BH that takes its macroscopic attributes from input parameters, like mass and Schwarzschild radius. So we reduce the problem to work with centre-of-mass quantities, e.g., the momentum P which represents the overall momentum of the BH, so in that expansion x would be the position the bound state is created at. But, what is important to remember, is that the AC creates a state on the right vacuum and with the right quantum numbers (ant thus right symmetries) necessary for a non-vanishing overlap with $|\mathcal{L}_B\rangle$. As this gives us a certain amount of freedom for the form of the current, we choose it such that it is the simplest to work with. The vacuum expectation value of observables, e.g., the particle number density, should not be affected by the exact expression of the AC. Therefore, let us use the current

$$\mathcal{J}(x) = \phi^N(x), \tag{4.1.8}$$

where $N \in \mathbb{N}$. It creates a state from N field operators.

4.2 Application: constituent number of a black hole

The goal of this section is to calculate the constituent number for a BH state (4.1.7). We make the following assumptions about the BH:

- its mass M_B is much larger than the energy of the individual constituent fields
- N is large

The large M makes sense if we consider the BHs we have "observed" so far in the universe compared to gravitons. The large N assumption doesn't necessarily mean $N \to \infty$. The forthcoming calculations will be valid for all $N \geq 3$. The details of $N = 2$ can be looked up in [30]. Anything larger, but finite, will produce $\frac{1}{N}$ corrections that are vital for getting rid of the famous BH information paradox.

As we've already said, unlike in the case of free particles created on a perturbative vacuum $|0\rangle$, the current $\mathcal{J}$ creates a bound state on a non-perturbative vacuum $|\Omega\rangle$. In the light of purely non-perturbative effects and interactions, one can expect the number of physical bound state constituents to end up being different form N.

So to properly distinguish the number of measured constituents and the number of graviton fields, let us denote the integrated number density as the constituent number

$$\mathcal{N}_C \equiv \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_k} \langle \mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle, \tag{4.2.1}$$

while N remains the number describing the fields building up the AC (4.1.8). It would then mean: for the non-perturbative case we do not necessarily expect $\mathcal{N}_C \sim N$ anymore.

Now let's start calculating the particle density of $|\mathcal{B}\rangle$, where we first insert the expansion (4.1.7):

$$\langle \mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle = \tilde{\Gamma}_B^{-2} \int \mathrm{d}^3 P' \,\mathrm{d}^3 P \,\mathcal{B}^*(\mathbf{P}', \mathcal{Q}_B)\mathcal{B}(\mathbf{P}, \mathcal{Q}_B) \int \mathrm{d}^4 z \,\mathrm{d}^4 u \, e^{iP'z - iPu} \langle \Omega|\mathcal{J}(z) n(\mathbf{k}) \mathcal{J}(u)|\Omega\rangle. \tag{4.2.2}$$

Plugging in for the density operator (C.0.23), we get for the expectation value on the RHS:

$$\langle \Omega|\mathcal{J}(z) n(\mathbf{k}) \mathcal{J}(u)|\Omega\rangle = \frac{(2E_k)^2}{2} \int \mathrm{d}^3 x \,\mathrm{d}^3 y \, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \langle \Omega| T\left[\mathcal{J}(z)\phi(0,\mathbf{x})\phi(0,\mathbf{y})\mathcal{J}(u)\right]|\Omega\rangle. \tag{4.2.3}$$

The operator product can be made time ordered because, just as in the Fock case, all involved operators are commuting. Our job now is making use of the Wick's theorem to solve time ordered product. The expectation value has a few important features, though, which aren't present in the case of the perturbative vacuum expectation value (C.0.14):

- since the currents' fields are at one spacetime point, we can't exclude diagrams like (C.0.18) on the basis of disconnectedness from the measurement device. But there will be disconencted diagrams that we can skip.
- The vacuum expectation value is evaluated on a non-perturbative vacuum. We will have to consider all normal ordered terms from Wick's theorem, i.e., not only then ones where all operators are contracted.

For the last point, let's keep in mind that for general operators subjected to normal ordering we have:

$$\langle 0| : AB : |0\rangle = 0, \tag{4.2.4}$$

$$\langle \Omega| : AB : |\Omega\rangle \neq 0 \tag{4.2.5}$$

and since the last term does not vanish in our present case, and it's in fact the vital one for our results, we shall denote it simply as:

$$\langle \Omega| : AB : |\Omega\rangle \equiv \langle AB\rangle \tag{4.2.6}$$

There will be three types of diagrams resulting from the time ordered product according to Wick's theorem:

- purely perturbative diagrams
- mixed diagrams
- purely non-perturbative diagrams

Now, one would naturally think that it will be necessary to take into account every single diagram type listed above, but in the case of a heavy bound state most of the diagrams will turn out to have a vanishing contribution and only one type will contribute. We shall now discuss and show what types of diagrams will vanish and which will survive.

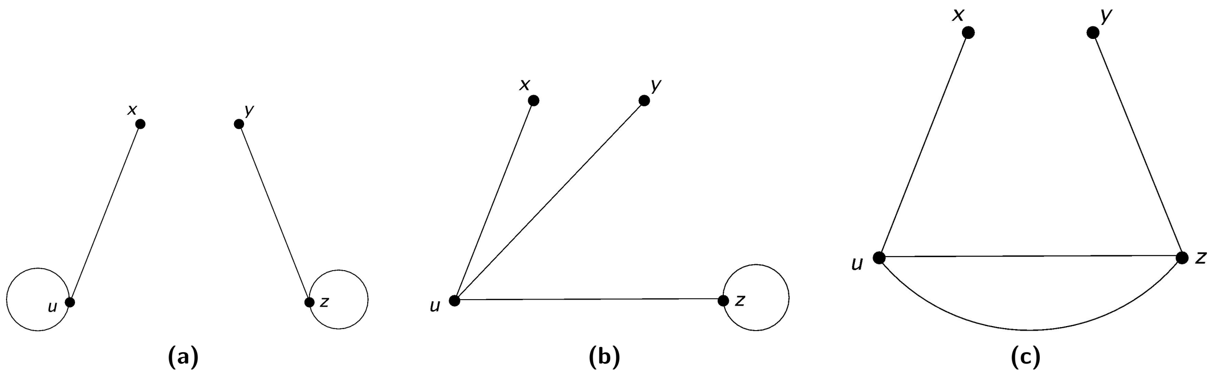


Figure 4.1: Purely perturbative diagrams for $N = 3$ number density measurement.

4.2.1 Purely perturbative diagrams

Purely perturbative terms are Wick terms with the maximum number of contractions, that is with all fields contracted. For simplicity, let us start with a particular value of fields: $N = 3$, so that for us the object of interst will be

$$\langle\Omega| T[\phi^3(z)\phi(x)\phi(y)\phi^3(u)]|\Omega\rangle_{pp}. \tag{4.2.7}$$

The diagrams resulting from all possible contractions are shown in Fig. 4.1 —modulo the diagrams with the same structure at the ACs spacetime points but with x and y interchanged; but in order to better focus on the techniques and important qualitative results of the purely perturbative diagrams, we disregard the alternative diagrams for now. What's special about the first two diagrams in Fig. 4.1 is that they contain 1-point loops from the contraction

$$\overbrace{\phi(z)\phi(z)} = \overbrace{\phi(u)\phi(u)} = S(0). \tag{4.2.8}$$

Expressions like

$$S(0) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4 \frac{1}{p^2 - m^2}} \tag{4.2.9}$$

need to be renormalized for physicality, but prior to this one needs to regularize this loop integral. At this point, let us make the following claim: we can take the mass m, which would be the mass of the constituent fields in the propagator, to be negligible in presence of the BH's overall mass M_B. [1] As a result of the claim we will get that such 1-point loops will have zero contribution to the amplitude. We can see it if we explicitly write out the regularization process. Let's consider a $4 \to D = 4 - 2\epsilon$ ($\epsilon > 0$) spacetime and keep mass in the integral. Now we apply dimensional regularization to the integral:

$$\lim_{m\to 0} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 - m^2}. \tag{4.2.10}$$

The result for the loop integral with dimensional regularization is:

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 - m^2} = -\frac{im^2}{16\pi^2} \left(\frac{1}{\pi m^2}\right)^{\epsilon} \Gamma(\epsilon - 1) \approx \frac{im^2}{16\pi^2} \left(\frac{1}{\epsilon} + 1 - \gamma - \ln(\pi m^2) + \mathcal{O}(\epsilon)\right), \tag{4.2.11}$$

[1] Since we consider our constituents to be gravitons, we get anyway 1-loop diagrams for massless particles.

where the integration was performed via Wick rotation in the p^0 coordinate and the result is approximated for a small, non-zero ϵ. The result for $m \to 0$ is therefore

$$\lim_{m\to 0} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 - m^2}\bigg|_{\epsilon>0} = 0. \tag{4.2.12}$$

So all diagrams involving 1-point loops $S(0)$ for approximately massless constitutents vanish. This means that the first two diagrams in Fig. 4.1 don't contribute to the overall particle number.

It's important to note that this result is neither limited to the $N = 3$ case nor to the purely perturbative diagrams. It applies to all diagrams containing 1-point loops, i.e., for all N and will extremely reduce the number of non-vanishing diagrams in the case of mixed diagrams. We should remember this result also for the next chapter.

What's left now is the last diagram in Fig. 4.1. The contractions give

$$\begin{aligned}
&\langle\Omega|\, T[\phi(z)\phi(z)\phi(z)\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)]|\Omega\rangle_{pp} + \text{permutations} = \\
&= f_N iS(z-x) iS(y-u) iS(z-u) iS(z-u) = \\
&= f_N \int \frac{\sigma(q_1,q_2,q_3,q_4)}{q_1^2 q_2^2 q_3^2 q_4^2} e^{-iq_1(z-x)} e^{-iq_2(y-u)} e^{-iq_3(z-u)} e^{-iq_4(z-u)} = \\
&= f_N \int \frac{\sigma(q_1,q_2,q_3,q_4)}{q_1^2 q_2^2 q_3^2 q_4^2} e^{-i(q_1+q_3+q_4)z} e^{i(q_2+q_3+q_4)u} e^{iq_1 x} e^{-iq_2 y}
\end{aligned} \tag{4.2.13}$$

where f_N stands for a dimensionless permutational factor, taking into account contractions that lead to the same expression and we expanded the free scalar propagator S in the Fourier scalar modes.

For the next step, we shall introduce a trick where we make use of our assumption that the BH's mass M_B is large: we insert the P momentum exponentials from (4.2.2) in the above expression and then substitute q_1 and q_2 in such a way that it will get the BH momenta into the denominators and, using the large M_B limit, get rid of some q-momenta squares in the denominator:

$$f_N \int \frac{\sigma(q_1,q_2,q_3,q_4)}{q_1^2 q_2^2 q_3^2 q_4^2} e^{iq_1 x} e^{-iq_2 y} e^{i(P'-q_1-q_3-q_4)z} e^{-i(P-q_2+q_3+q_4)u}. \tag{4.2.14}$$

So, introducing the variables

$$\tilde{q}_1 := P' - q_1 - q_3 - q_4, \tag{4.2.15}$$

$$\tilde{q}_2 := P - q_2 - q_3 - q_4 \tag{4.2.16}$$

and substitute q_1 and q_2 everywhere by the new variables:

$$f_N \int \frac{\sigma(\tilde{q}_1,\tilde{q}_2,q_3,q_4)}{(P'-\tilde{q}_1-q_3-q_4)^2 (P-\tilde{q}_2-q_3-q_4)^2 q_3^2 q_4^2} e^{i(P'-\tilde{q}_1-q_3-q_4)x} e^{-i(P-\tilde{q}_2-q_3-q_4)y} e^{i\tilde{q}_1 z} e^{-i\tilde{q}_2 u}. \tag{4.2.17}$$

The reason why we would like this is: we can use our first assumption for the BH, i.e., its mass is much larger than the individual energies of the consitutents. The result for Minkowski squares is then:

$$(P-q)^2 = P^2 - 2Pq + q^2 \approx M_B^2 + \mathcal{O}(Pq, q^2). \tag{4.2.18}$$

This should also hold for P'. Therefore, we can approximate the above integral with:

$$(4.2.17) \approx \frac{f_N}{M_B^4} \int \frac{\sigma(\tilde{q}_1,\tilde{q}_2,q_3,q_4)}{q_3^2 q_4^2} e^{i(P'-\tilde{q}_1-q_3-q_4)x} e^{-i(P-\tilde{q}_2-q_3-q_4)y} e^{i\tilde{q}_1 z} e^{-i\tilde{q}_2 u}. \tag{4.2.19}$$

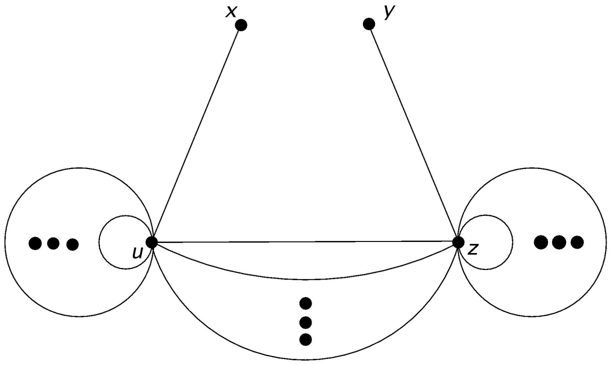


Figure 4.2: A general N diagram for $\mathcal{J} = \phi^N$ containing perturbative lines only, with l 1-point loops at each current's spacetime point and $N-2l-2$ 2-point loops at the centre ($N-2l-1$ propagators five $N-2l-2$ loops).

So far we managed to simplify the momentum integrals for q_1 and q_2 to the point where the momentum variables appear only in the exponentials, we can reorder the exponentials and get Dirac deltas for spacetime coordinates:

$$(4.2.17) \approx \frac{f_N}{M_B^4}\int \frac{\sigma(q_3,q_4)}{q_3^2 q_4^2} e^{i(P'-q_3-q_4)x} e^{-i(P-q_3-q_4)y} \delta^{(4)}(z-u)\delta^{(4)}(u-y). \tag{4.2.20}$$

If we perform the trick a second time for, e.g., q_3 we are going to be left with a 1-point loop again. So shift

$$\tilde{q}_3 := P - q_3 - q_4 \tag{4.2.21}$$

and get:

$$\begin{aligned}(4.2.17) \approx & \frac{f_N}{M_B^6}\int \frac{\sigma(\tilde{q}_3,q_4)}{q_4^2} e^{i(P'-P+\tilde{q}_3)x} e^{-i\tilde{q}_3 y} \delta^{(4)}(z-u)\delta^{(4)}(u-y) = \\ =& \frac{f_N}{M_B^6}\int \sigma(\tilde{q}_3) S(0) e^{i(P'-P+\tilde{q}_3)x} e^{-i\tilde{q}_3 y} \delta^{(4)}(z-u)\delta^{(4)}(u-y) = \\ =& 0\end{aligned} \tag{4.2.22}$$

the last equation follows from our analysis of $S(0)$, which vanishes when regularized, as we have seen (4.2.12). Therefore, these diagrams do not contribute to (4.2.2). The two important results for $N=3$ are: 1-point loops and 2-point loops give a factor of 0. These diagrams in fact vanish for any $N>3$. Let us show this statement for the case of an arbitrary number of 1-loops and 2-loops. So let's take our current $\mathcal{J} = \phi^N$ and evaluate diagrams with l 1-point loops at each side and $(N-2l-2)$ 2-point loops (this is the number so that the diagrams only involve perturbative lines; the number means the point z and u are connected via $(N-2l-1)$ lines)—see Fig. 4.2. However, any $l>0$ will instantly produce a factor $S(0)$, so any diagrams involving 1-point loops instantly drop out. So let's consider the case $l=0$, as depicted in Fig. 4.3, (4.2.14) for general N is

$$f_N \int \frac{\sigma(q_1,q_2,\ldots,q_{N+1})}{q_1^2 q_2^2 \cdots{}^2 q_{N+1}^2} e^{iq_1 x} e^{-iq_2 y} e^{i(P'-q_1-\sum_{k=3}^{N+1} q_k)z} e^{-i(P-q_2-\sum_{k=3}^{N+1} q_k)u}. \tag{4.2.23}$$

A substitution of q_1 and q_2 just as before

$$\tilde{q}_1 := P' - q_1 - \sum_{k=3}^{N+1} q_k, \tag{4.2.24}$$

$$\tilde{q}_2 := P - q_2 - \sum_{k=3}^{N+1} q_k \tag{4.2.25}$$

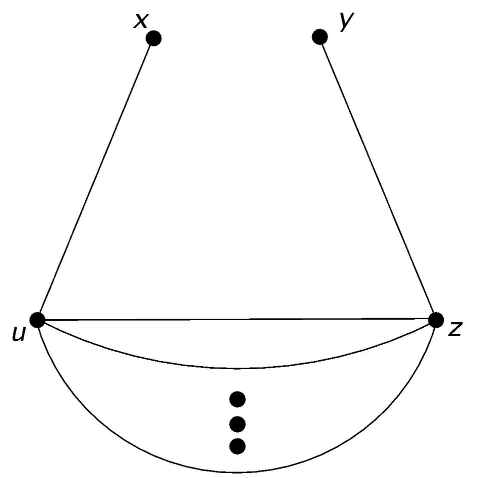


Figure 4.3: The purely perturbative diagram for $N-2$ central loops only.

allows us to approximate the integral to

$$(4.2.23) \approx \frac{f_N}{M_B^4} \int \frac{\sigma(q_3,\dots,q_{N+1})}{q_3^2\cdots q_{N+1}^2} e^{i(P'-\sum_{k=3}^{N+1} q_k)x} e^{-i(P-\sum_{k=3}^{N+1} q_k)y} \delta^{(4)}(z-x)\delta^{(4)}(u-y). \quad (4.2.26)$$

Finally substituing q_3 as before

$$\tilde{q}_3 := P - \sum_{k=3}^{N+1} q_k, \quad (4.2.27)$$

yields

$$\begin{aligned}(4.2.23) &\approx \frac{f_N}{M_B^6} \int \frac{\sigma(\tilde{q}_3,\dots,q_{N+1})}{q_4^2\cdots q_{N+1}^2} e^{i(P'-P+\tilde{q}_3)x} e^{-i\tilde{q}_3 y} \delta^{(4)}(z-x)\delta^{(4)}(u-y) = \\ &= 0.\end{aligned} \quad (4.2.28)$$

This includes a product of several $S(0)$ and thus gives 0. So our result for purely perturbative diagrams with current content of $N \geq 3$ is that they simply all vanish in the large M_B limit: $P^2 = P'^2 = M_B^2 \gg Pq, q^2$.

In [30] it is shown that $N = 2$ case does have a non-vanishing perturbative contribution. However, as we have shown, this contribution is really limited to the small N case.

4.2.2 Mixed diagrams

Once again, before we deal with general N case, let us do a concrete calculation for the case of $N = 3$ to focus on the tricks that have to be used in order to produce concrete results for otherwise complicated diagrams

$$\langle\Omega| T\left[\phi^3(z)\phi(x)\phi(y)\phi^3(u)\right] |\Omega\rangle_{mix}. \quad (4.2.29)$$

There are of course a lot of possible diagrams we can construct from the time ordered product that contain both perturbative and non-perturbative factors, but taking into account what we found out in the previous analysis of purely perturbative diagrams, we know that 1-point loops and 2-point loops give 0 contribution in our large M_B approximation. So we already know that the diagram in Fig 4.3, for $N = 3$, gives no contribution to the expectation value. Another constraint is given by perturbative connectedness of the points x to u, y to z and u to z, i.e.,

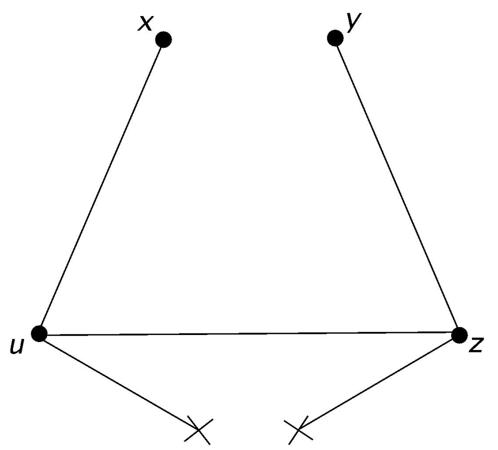


Figure 4.4: The only physical diagram contributing to $N = 3$ BH state particle number density measurement.

these points need to have at least one propagator connecting them. This is not so clear at this point, but we will see from the previous result about purely non-perturbative diagrams and the next one that the mixed diagrams with three perturbative lines will give, on the light cone, the dominating contribution of all possible mixed diagrams. This leaves us with the following non-vanishing, dominating diagrams:

$$\langle\Omega|T[\phi(z)\phi(z)\phi(z)\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)]|\Omega\rangle_{mix}, \tag{4.2.30}$$

which is shown in Fig. 4.4. The number of possible contraction like this are $[3^2 \cdot (3-1)^2]$ for each of them. As in the free case, though, there are also the diagrams of the type

$$\langle\Omega|T[\phi(z)\phi(z)\phi(z)\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)]|\Omega\rangle_{mix}, \tag{4.2.31}$$

and here we shall consider them, because we will get a non-vanishing result, which will be of interest for us quantitavely. The calculation will show, though, that these two expressions above are actually the same, so the latter contractions don't lead to different diagrams, but just give an additional factor of 2 for the amplitude. But for sake of accuracy, we shall treat these two diagrams separately until we know for sure that they lead to the same result. Because they will appear in the general N case as well, we shall now introduce a shorthand notation for the two-fold contribution so that we don't have to write everything twice. The result of the above contractions is (including the combinatorial factor for the number of equal diagrams):

$$\begin{aligned} &3^2 \cdot (3-1)^2 \left\{ iS(z-x)\, iS(y-u)\, iS(z-u) + iS(z-y)\, iS(x-u)\, iS(z-u) \right\} \langle\phi(z)\phi(u)\rangle \equiv \\ &\equiv 3^2 \cdot (3-1)^2 iS(z - \left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\})\, iS(\left\{ \begin{smallmatrix} y \\ x \end{smallmatrix} \right\} - u)\, iS(z-u) \langle\phi(z)\phi(u)\rangle. \end{aligned} \tag{4.2.32}$$

As next step, let us expand the free scalar propagator S in the Fourier modes and plug in the P-momentum exponentials of (4.2.2):

$$\begin{aligned} &36 i^3 e^{iP'z - iPu} S(z - \left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\}) S(\left\{ \begin{smallmatrix} y \\ x \end{smallmatrix} \right\} - u) S(z-u) \langle\phi(z)\phi(u)\rangle = \\ &= 36 i^3 \int \frac{\sigma(q_1, q_2, q_3)}{q_1^2 q_2^2 q_3^2} e^{i(P' - q_1 - q_3)z} e^{-i(P - q_2 - q_3)u} e^{iq_1 \left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\}} e^{-iq_2 \left\{ \begin{smallmatrix} y \\ x \end{smallmatrix} \right\}} \langle\phi(z)\phi(u)\rangle. \end{aligned} \tag{4.2.33}$$

Here we will perform the previous trick again: substituting the momenta q_1 and q_2 such that they are shifted by the total BH momenta P and P':

$$\tilde{q}_1 := P' - q_1 - q_3, \tag{4.2.34}$$
$$\tilde{q}_2 := P - q_2 - q_3 \tag{4.2.35}$$

Just as before, this momentum translation of the two q will have the effect of bringing the momenta P into the Minkowski square in the denominator:

$$(4.2.33) = 36i^3 \int \frac{\sigma(\tilde{q}_1, \tilde{q}_2, q_3)}{(P' - \tilde{q}_1 - q_3)^2 (P - \tilde{q}_2 - q_3)^2 q_3^2} e^{i\tilde{q}_1 z} e^{-i\tilde{q}_2 u} e^{i(P' - \tilde{q}_1 - q_3)\left\{\substack{x\\y}\right\}} e^{-i(P - \tilde{q}_2 - q_3)\left\{\substack{y\\x}\right\}} \langle\phi(z)\phi(u)\rangle. \tag{4.2.36}$$

Here we apply again the large BH mass approximation (4.2.18) and thus get rid of $\tilde{q}_1$ and $\tilde{q}_2$ in the denominator. Consequently, we rearrange the exponentials again and can replace the integrals with tilde variables with Dirac deltas:

$$\begin{aligned}(4.2.33) &= \frac{36i^3}{M_B^4} \int \frac{\sigma(q_3)}{q_3^2} e^{i(P' - q_3)\left\{\substack{x\\y}\right\}} e^{-i(P - q_3)\left\{\substack{y\\x}\right\}} \delta^{(4)}(z - \left\{\substack{x\\y}\right\}) \delta^{(4)}(u - \left\{\substack{y\\x}\right\}) \langle\phi(z)\phi(u)\rangle = \\ &= \frac{36i^3}{M_B^4} e^{iP'\left\{\substack{x\\y}\right\} - iP\left\{\substack{y\\x}\right\}} S(\left\{\substack{y\\x}\right\} - \left\{\substack{x\\y}\right\}) \delta^{(4)}(z - \left\{\substack{x\\y}\right\}) \delta^{(4)}(u - \left\{\substack{y\\x}\right\}) \langle\phi(z)\phi(u)\rangle\end{aligned} \tag{4.2.37}$$

Having the delta functions there, we can now easily perform the z and u integration from (4.2.2) and get:

$$\begin{aligned}(4.2.33) &= \frac{36i^3}{M_B^4} e^{iP'\left\{\substack{x\\y}\right\} - iP\left\{\substack{y\\x}\right\}} S(\left\{\substack{y\\x}\right\} - \left\{\substack{x\\y}\right\}) \langle\phi(\left\{\substack{x\\y}\right\})\phi(\left\{\substack{y\\x}\right\})\rangle = \\ &= \frac{36i^3}{M_B^4} e^{iP'\left\{\substack{x\\y}\right\} - iP\left\{\substack{y\\x}\right\}} S(x - y) \langle\phi(\left\{\substack{x\\y}\right\})\phi(\left\{\substack{y\\x}\right\})\rangle.\end{aligned} \tag{4.2.38}$$

There we used for the last line that the free scalar propagator is symmetric in its argument. The overall expression for the density (4.2.2) modulo the constant overlap, P and P' integration and wavefunctions is then:

$$\begin{aligned}&\int \mathrm{d}^4 z\, \mathrm{d}^4 u\, e^{iP'z - iPu} \frac{(2E_k)^2}{2} \int \mathrm{d}^3 x\, \mathrm{d}^3 y\, e^{-ik(x-y)} \langle\Omega|\mathcal{J}(z)\phi(x)\phi(y)\mathcal{J}(u)|\Omega\rangle \approx \\ &\frac{36i^3}{M_B^4} \frac{(2E_k^2)}{2} \int \mathrm{d}^3 x\, \mathrm{d}^3 y e^{-ik(x-y)} e^{iP'\left\{\substack{x\\y}\right\} - iP\left\{\substack{y\\x}\right\}} S(x - y) \langle\phi(\left\{\substack{x\\y}\right\})\phi(\left\{\substack{y\\x}\right\})\rangle.\end{aligned} \tag{4.2.39}$$

Now before we proceed, let us shift coordinates of the condenstate on the RHS using Poincaré transformations (like in Appendix C, Eq. (C.0.29)) and also rewrite the exponential involving the BH momenta:

$$(4.2.39) = \frac{36i^3}{M_B^4} \frac{(2E_k^2)}{2} \int \mathrm{d}^3 x\, \mathrm{d}^3 y e^{-ik(x-y)} e^{i\left\{\substack{P'\\-P}\right\}x - i\left\{\substack{P\\-P'}\right\}y} S(x - y) \langle\phi(\left\{\substack{x-y\\y-x}\right\})\phi(0)\rangle. \tag{4.2.40}$$

The integrand has a dominating contribution for light-like distances, [2] which allows us to

[2] The Green function in configuration space is

$$S(x - y) = \begin{cases} \frac{m}{8\pi\sqrt{(x-y)^2 - i\epsilon}} H_1^{(2)}(m\sqrt{(x-y)^2 - i\epsilon}) & \text{if } m \neq 0 \\ \frac{i}{4\pi^2} \frac{1}{(x-y)^2 - i\epsilon} & \text{if } m = 0 \end{cases}$$

where $H_1^{(2)}$ is an Hankel function. [26, 65]

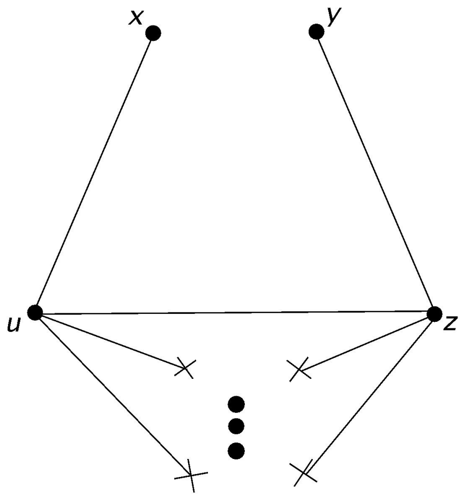


Figure 4.5: The only physical diagram contributing to N, $\mathcal{J} = \phi^N$, particle number density measurement.

approximate the condensate with

$$\langle \phi(\pm r)\phi(0)\rangle = \sum_{n=0}^{\infty} c_n r^{2n} \langle \partial^{2n}\phi(0)\phi(0)\rangle \approx \langle \phi(0)\phi(0)\rangle + \mathcal{O}(r^2) =: \langle \phi^2\rangle + \mathcal{O}(r^2); \tag{4.2.41}$$

the reason the first equality stems from the request that the vacuum is Lorentz invariant (more details on the Taylor expansion and the factor c_n see [50]. For convenience we can also rewrite it substituting x and y to one variable by defining the distance $r := x - y$. Hence

$$(4.2.39) \approx \frac{36i^3}{M_B^4}\frac{(2E_k^2)}{2}\langle\phi^2\rangle \int \mathrm{d}^3 r\, \mathrm{d}^3 y\, e^{-ikr} e^{i\left\{ {P' \atop -P} \right\} r} e^{i\left(\left\{ {P' \atop -P} \right\} - \left\{ {P \atop -P'} \right\}\right) y} S(r), \tag{4.2.42}$$

which gives us a Dirac delta for the BH momenta:

$$(4.2.39) = \frac{36i^3}{M_B^4}\frac{(2E_k^2(2\pi)^3)}{2}\langle\phi^2\rangle \int \mathrm{d}^3 r\, e^{-ikr} e^{i\left\{ {P' \atop -P} \right\} r} \delta^{(3)}(\mathbf{P}' - \mathbf{P}) S(r). \tag{4.2.43}$$

With the only assumption that we are dealing with a large mass $P^0 \approx P'^0 \approx M_B$, we managed to reduce the expectation value (4.2.2) to

$$\langle \mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle_{mix} \approx \frac{36i^3(2E_k)^2(2\pi)^3}{2\tilde{\Gamma}_B^2 M_B^4}\langle\phi^2\rangle \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P}, \mathcal{Q}_B)\right|^2 \int \mathrm{d}^3 r\, e^{-ikr} e^{i\left\{ {P' \atop -P} \right\} r} S(r). \tag{4.2.44}$$

Notice that our x and y integration was still over a 3-volume, while the expressions in the exponentials remain 4-products. This means we aren't necessarily considering $y^0 = x^0 = 0$. In the Fock case we could fix the time variable to our convenience because our choice wouldn't alter the result of the number density (see Appendix C). We shall specify later which values for the time variables we choose, for now let them just be non-zero. This will help us to calculate the consitutent number in a rather short way. For N constituent fields all combinatorial elements, i.e., any expressions involving N, are only constant factors in front of the integrals, and thus, do not affect the dependence on the 3-momentum variable $\mathbf{k}$ in the density expectation value (4.2.2).

Let's deal with the case for general N

$$\langle \Omega | T\left[\phi^N(z)\phi(x)\phi(y)\phi^N(u)\right] |\Omega\rangle_{mix}. \tag{4.2.45}$$

Just as before we can exclude a great deal of diagrams involving perturbative loops that will give vanishing contribution in the large M_B limit. One can see from the calculation of the amplitude of the diagrams involving central loops, Fig. 4.3 , that, even in the presence of condensate terms, $\langle\phi^n(z)\phi^n(u)\rangle$, one will still end up with an integral of the form (4.2.28)—the only difference will be the number of q momenta we integrate over, but as long as there are four or more qs, there will be factors of $S(0)$ which give vanishing contribution. Since "four or more qs" means four or more propagators in the diagrams, the only non-vanishing diagrams will be the ones with three propagators, i.e., the type of diagrams shown in Fig. 4.5. The non-vanishing diagrams will be akin to (4.2.30),

$$\langle\Omega|T[\phi(z)\ldots\phi(z)\ldots\phi(z)\ldots\phi(z)\phi(x)\phi(y)\phi(u)\ldots\phi(u)\ldots\phi(u)\ldots\phi(u)]|\Omega\rangle_{mix} \tag{4.2.46}$$

and akin to (4.2.31),

$$\langle\Omega|T[\phi(z)\ldots\phi(z)\ldots\phi(z)\ldots\phi(z)\phi(x)\phi(y)\phi(u)\ldots\phi(u)\ldots\phi(u)\ldots\phi(u)]|\Omega\rangle_{mix}. \tag{4.2.47}$$

The number of possible contractions like that are, in both cases, $N^2(N-1)^2$: N for each contraction with a field operator from the measurement device, this leaves us with $N-1$ operators at point z and $N-1$ operators at point u; now pick one $\phi(z)$ out and contract it with one of the $N-1$ $\phi(u)$, but since we have $N-1$ $\phi(z)$ this contraction has multiplicity $(N-1)^2$. The Wick term resulting from these $2N^2(N-1)^2$ is

$$N^2(N-1)^2 iS(z-\left\{\begin{smallmatrix}x\\ y\end{smallmatrix}\right\})\, iS(\left\{\begin{smallmatrix}y\\ x\end{smallmatrix}\right\}-u)\, iS(z-u)\langle\phi^{N-2}(z)\phi^{N-2}(u)\rangle. \tag{4.2.48}$$

The way to proceed from here is the same as for $N=3$: expand the propagators in Fourier modes, shift (i.e., substitute) two of their momenta by the BH momenta P, use the assumption that the BH's mass is much larger than the individual constituent energies and work in the rest-frame. Then we manage to reduce the three propagators to one propagator and two 4-deltas:

$$\frac{N^2(N-1)^2 i^3}{M_B^4} e^{iP'\left\{\begin{smallmatrix}x\\ y\end{smallmatrix}\right\}-iP\left\{\begin{smallmatrix}y\\ x\end{smallmatrix}\right\}} iS(\left\{\begin{smallmatrix}y\\ x\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}x\\ y\end{smallmatrix}\right\})\delta^{(4)}(z-\left\{\begin{smallmatrix}x\\ y\end{smallmatrix}\right\}))\delta^{(4)}(u-\left\{\begin{smallmatrix}y\\ x\end{smallmatrix}\right\})\langle\phi^{N-2}(z)\phi^{N-2}(u)\rangle. \tag{4.2.49}$$

Now we can perform the z and u integrations and an expression equivalent to (4.2.40),

$$\frac{N^2(N-1)^2 i^3}{M_B^4}\frac{(2E_k)^2}{2}\int \bar{\mathrm{d}}^3 x\, \bar{\mathrm{d}}^3 y\, e^{-ik(x-y)} e^{i\left\{\begin{smallmatrix}P'\\ -P\end{smallmatrix}\right\}x-i\left\{\begin{smallmatrix}P\\ -P'\end{smallmatrix}\right\}y} S(x-y)\langle\phi^{N-2}(\left\{\begin{smallmatrix}x-y\\ y-x\end{smallmatrix}\right\})\phi^{N-2}(0)\rangle. \tag{4.2.50}$$

The condensate can now be "Taylored" just as before around the light cone, where we can neglect the higher order $\mathcal{O}^2$ terms due to the remaining propagator giving the main contribution near the light cone. The contribution from the condensate is then a constant number, $\langle\phi^{N-2}(0)\phi^{N-2}(0)\rangle$, which we can move in front of the integral so that the density as a function of $\mathbf{k}$ only depends on N via constant pre-factors:

$$\begin{aligned}\langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle_{mix} \approx &\frac{N^2(N-1)^2 i^3(2\pi)^3}{2\tilde{\Gamma}_B^2 M_B^2}\langle\phi^{2(N-2)}\rangle(2E_k)^2\times\\ &\times\int \bar{\mathrm{d}}^3 P\left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2\int \bar{\mathrm{d}}^3 r\, e^{-ikr}e^{i\left\{\begin{smallmatrix}P\\ -P\end{smallmatrix}\right\}r}S(r).\end{aligned} \tag{4.2.51}$$

This is the non-vanishing contribution to the number density of $|\mathcal{B}\rangle$ coming from the diagrams incorporating both perturbative and non-perturbative lines between spacetime locations of the ACs.

4.2.3 Purely non-perturbative diagrams

Only the final terms from Wick's theorem are left: the ones that involve no contractions between the $\mathcal{J}$'s individual fields, i.e., produce no $S(z-u)$, and consist of a condensate only. For the $N=3$ case it would be:

$$\langle\Omega|T[\phi(z)\phi(z)\phi(z)\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)]|\Omega\rangle_{pnp} \tag{4.2.52}$$

and the one with the roles of x and y switched. The result here is, after performing q_1 and q_2 substitutions and approximating the propagators with $1/M_B^2$ each, and going on with the calculation to the point of (4.2.40),

$$\frac{3^2 i^2}{M_B^4}\frac{(2E_k)^2}{2}\int \mathrm{d}^3 x\, \mathrm{d}^3 y\, e^{-ik(x-y)} e^{i\left\{ {P' \atop -P} \right\}x - i\left\{ {P \atop -P'} \right\}y}\langle\phi(\left\{ {x-y \atop y-x} \right\})\phi(0)\rangle. \tag{4.2.53}$$

But notice that, compared to Eq. (4.2.40), a propagator $S(x-y)$ is missing. This is because our contractions gave us a propagator less, so its absence is not going to be changed by considering higher Ns. Therefore it is missing for all N,

$$\approx \frac{N^2 i^2}{M_B^4}\frac{(2E_k)^2}{2}\int \mathrm{d}^3 x\, \mathrm{d}^3 y\, e^{-ik(x-y)} e^{i\left\{ {P' \atop -P} \right\}x - i\left\{ {P \atop -P'} \right\}y}\langle\phi^{N-1}(\left\{ {x-y \atop y-x} \right\})\phi^{N-1}(0)\rangle \tag{4.2.54}$$

This contribution, on its own, is non-vanishing, but when taking into account the other non-vanishing contribution to the number density, (4.2.50), we notice that the latter contributes with a light cone singularity at $(x-y)^2 = 0$ (at the lowest order expansion of the field condensate). Therefore, when we sum up all the non-vanishing contributions,

$$\langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle \approx \langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle_{mix} + \langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle_{pnp}, \tag{4.2.55}$$

we see that the largest contribution will come at the light cone, which is only present in the mixed part. Contributions from $(x-y)^2 \neq 0$ will be much smaller compared to that. So if we neglect them, only the mixed part matters. Therefore,

$$\langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle \approx \langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle_{mix} \tag{4.2.56}$$

4.2.4 Constituent number

To calculate the actual constituent number from the density expectation value, we use our definition (4.2.1)

$$\begin{aligned}\mathcal{N}_C &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_k}\langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle = \\ &= \frac{N^2(N-1)^2 i^3 (2\pi)^3}{2\tilde{\Gamma}_B^2 M_B^2}\langle\phi^{2(N-2)}\rangle\int \mathrm{d}^3 P\, \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \times \\ &\times \int \frac{\mathrm{d}^3 k}{(2\pi)^3} 2E_k \int \mathrm{d}^3 r\, e^{-ikr} e^{i\left\{ {P \atop -P} \right\}r} S(r).\end{aligned} \tag{4.2.57}$$

Before solving these integrals let us give first a couple of comments about $\mathrm{d}^3 r$ integral: recalling that the behaviour of the free, massless, scalar propagator is $S(r) \sim 1/r^2$, we can notice that the r integral looks almost like the Fourier transform of the scalar propagator: $\frac{1}{(k\pm P)^2 - m^2}$. In that case the integration measure would be $\mathrm{d}^4 r$. To make up for our measure, let us rewrite the above expression in the following way: we first insert a 1 with a 4-dimensional Dirac delta replacing the momentum $\mathbf{k}$ with another 4-momentum variable

$$\begin{aligned}\mathcal{N}_C &= A_N \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3}(1)2E_k \int \mathrm{d}^3 r\, e^{-ikr} e^{i\left\{ {P \atop -P} \right\} r} S(r) = \\ &= A_N \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \mathrm{d}^4 q\, \delta^{(4)}(q-k)\, 2E_q \int \mathrm{d}^3 r\, e^{-ikr} e^{i\left\{ {P \atop -P} \right\} r} S(r) = \\ &= A_N \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \mathrm{d}^4 q \int \sigma(\rho) e^{-iq\rho} 2E_q \int \mathrm{d}^3 r\, e^{-ik(r-\rho)} e^{i\left\{ {P \atop -P} \right\} r} S(r) = \\ &= A_N \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \mathrm{d}^4 q \int \sigma(\rho) e^{-iq\rho} 2E_q \int \mathrm{d}^3 r\, e^{-i\mathbf{k}(\mathbf{r}-\rho)} e^{i\left\{ {P \atop -P} \right\} r} S(r)\Big|_{r^0=\rho^0}\end{aligned} \tag{4.2.58}$$

where $A_N := \frac{N^2(N-1)^2 i^3 (2\pi)^3}{2\bar{\Gamma}_B^2 M_B^4}\langle \phi^{2(N-2)}\rangle$ is just a shorthand for all constant factors to shorten the expression, and the third equality was just writing the delta in the Fourier representation.

Let's have a closer look at what happened in the last line. As was remarked below (4.2.51), the particle is time indipendent in the Fock case and so we are free to evaluate its r^0 variable at any time we want. For our present problem we want to get rid of the $\mathrm{d}^3 k$ integral, which, by the state of the expression in third line, is only present in the second exponential function. But there it is a Minkowski product, not the usual $\mathbb{R}^3$ scalar product. We can reduce to it if the time component is zero, which it is the case if r^0 in chosen appropriately

$$k(r-\rho) = E_k(r^0-\rho^0) - \mathbf{k}(\mathbf{r}-\rho) \stackrel{!}{=} E_k(\rho^0-\rho^0) - \mathbf{k}(\mathbf{r}-\rho) = -\mathbf{k}(\mathbf{r}-\rho) \tag{4.2.59}$$

that is $r^0 = \rho^0$. Note that, while ρ^0 is certainly a running variable inside the $\mathrm{d}^4\rho$ integral, the number density remains invariant under infinitesimal changes of r^0 (which was proven by showing that its derivative with respect to time is 0); so if we think of the integral as a sum, the individual summands containing the density can all have different r^0s without changing the overall sum.

So now we know that the $\mathbf{k}$ integral is a 3-dimensional Dirac delta,

$$(4.2.58) = A_N \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \int \mathrm{d}^4 q \int \sigma(\rho) e^{-iq\rho} 2E_q \int \mathrm{d}^3 r\, \delta^{(3)}(\mathbf{k}(\mathbf{r}-\rho)) e^{i\left\{ {P \atop -P} \right\} r} S(r)\Big|_{r^0=\rho^0}, \tag{4.2.60}$$

and we can also now perform the trivial $\mathbf{r}$ integration:

$$(4.2.58) = A_N \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right|^2 \int \mathrm{d}^4 q\, 2E_q \int \sigma(\rho) e^{-i\left(q-\left\{ {P \atop -P} \right\}\right)\rho} S(\rho). \tag{4.2.61}$$

To sum up, we managed to replace all 3-dimensional integrals with 4-dimensional ones thanks to the constancy of $\langle\mathcal{B}|n(\mathbf{k})|\mathcal{B}\rangle$ with respect to time.

$$(4.2.58) = \frac{iA_N}{4\pi^2} \int \mathrm{d}^3 P \left[\mathcal{B}(\mathbf{P},\mathcal{Q}_B)\right]^2 \int \mathrm{d}^4 q\, 2E_q \int \sigma(\rho) e^{-i\left(q-\left\{ {P \atop -P} \right\}\right)\rho} \frac{1}{\rho^2 - i\epsilon} \tag{4.2.62}$$

Notice that, indeed, the $\mathrm{d}^4\rho$ integral now is the inverse Fourier transform of the massless scalar Green's function, which is the Feynman propagator (for vanishing mass) in momentum space with ϵ at its position

$$\int \sigma(\rho)\, e^{i\left(q-\left\{ {P \atop -P} \right\}\right)\rho} \frac{1}{\rho^2 - i\epsilon} = \frac{1}{\left|q - \left\{ {P \atop -P} \right\}\right|^2 + i\epsilon}. \tag{4.2.63}$$

Recall, however, our large BH mass approximation. With the q just representing the individual constituents' momenta, having the BH momenta P in the denominator we can approximante the previous expression with

$$\frac{1}{\left[q - \left\{ \begin{smallmatrix} P \\ -P \end{smallmatrix} \right\}\right]^2 + i\epsilon} \approx \frac{2}{M_B^2}, \tag{4.2.64}$$

where the 2 came from the definition of the curly brackets as a shorthand notation for a sum: $\frac{1}{\left[q - \left\{ \begin{smallmatrix} P \\ -P \end{smallmatrix} \right\}\right]^2 + i\epsilon} \equiv \frac{1}{[q-P]^2 + i\epsilon} + \frac{1}{[q+P]^2 + i\epsilon}$.

So, altogether, we managed to reduce the integrals for the consitutent number down to

$$\mathcal{N}_C = \frac{iA_N}{4\pi^2 M_B^2} \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P}, \mathcal{Q}_B)\right|^2 \int \mathrm{d}^4 q \, 2E_q. \tag{4.2.65}$$

next we will have to deal with the $\mathrm{d}^4 q$ integal. It obviously diverges like $|q|^5$. As a solution to this mathematical problem, let us impose a physical constraint to this constituent momentum variable: let us restrain the integration 4-volume to $[0, M_B] \times [0, M_B] \times S^2(M_B)$. This makes sense if we model our BH to be a bound state of constituents ϕ and only these can contribute to the overall energy, so every constituent should be bounded from above by the overall energy, which is the BH's mass in the centre of mass frame. If we actually consider the integral with those boundaries, we get

$$\begin{aligned} \int_{[0,M_B]\times[0,M_B]\times S^2(M_B)} \mathrm{d}^4 q \, 2E_q &\approx 2 \int_0^{M_B} \mathrm{d}q^0 \int_{[0,M_B]\times S^2(M_B)} \mathrm{d}^3 q |\mathbf{q}| = \\ &= 2M_B \Omega(S^2) \int_0^{M_B} \mathrm{d}|\mathbf{q}||\mathbf{q}|^3 = \\ &= 8\pi M_B \int_0^{M_B} |\mathbf{q}||\mathbf{q}|^3 = \\ &= 8\pi M_B \frac{1}{4} M_B^4. \end{aligned} \tag{4.2.66}$$

This give for the constituent number:

$$\begin{aligned} \mathcal{N}_C &= \frac{2\pi i A_N M_B^5}{4\pi^2 M_B^2} \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P}, \mathcal{Q}_B)\right|^2 = \\ &= \frac{i^4 N^2 (N-1)^2 (2\pi)^3 M_B^3}{2\pi 2 \tilde{\Gamma}_B^2 M_B^4} \langle \phi^{2(N-2)} \rangle \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P}, \mathcal{Q}_B)\right|^2 = \\ &= N^2 (N-1)^2 \frac{(2\pi)^2 \langle \phi^{2(N-2)} \rangle}{2\tilde{\Gamma}_B^2 M_B} \int \mathrm{d}^3 P \left|\mathcal{B}(\mathbf{P}, \mathcal{Q}_B)\right|^2. \end{aligned} \tag{4.2.67}$$

Let's give a few comments about this result.

Firstly, a thorough dimensional analysis of all quantities involved reveals that $\mathcal{N}_C$ has mass dimension m^0, just as one would expect from a particle number.

Secondly, it scales with N; combining the definition of the overlap Γ_B (4.1.2), plugging it in the expression of the momentum eigenstates (4.1.6) and performing the calculations involving Wick's theorem will reveal how the overlap constant scales with N. The result is $\tilde{\Gamma}_B \sim (N/M_B)^2 \langle \phi^{2(N-1)} \rangle$ so that

$$\mathcal{N}_C \sim N^2 (N-1)^2 \overset{N \gg 1}{\approx} N^4 + \mathcal{O}(N^3). \tag{4.2.68}$$

This result is different from the one for free, non-interacting particles, where it scaled linear with N (see Appendix C, Eq. (C.0.5)). In that case we act with a "current" $\sim \phi^N(x)$ on a perturbative vacuum $|0\rangle$ and get $\mathcal{N}_C = N$. The different scaling here stems from the fact that our auxiliary currents are acting on a non-perturbative vacuum. This setting leads to extra non-vanishing terms in Wick's theorem because the normal ordered terms don't have to vanish anymore and result in field condensates.

For $N > 2$ and with fields assumed to be on the light-cone, we received neither purely perturbative nor purely non-perturbative contributions and, in fact, only the mixed case with a minimal number of perturbative lines contribute so that the diagrams remained effectively connected.

Thirdly, the constant factors $\mathcal{O}(m^0, N^0)$, numbers like 2, 2π, etc., depend on the conventions one uses. For example, for the Lorentz invariant momentum integration measure we used $\frac{\mathrm{d}^3 k}{(2\pi)^3 2E_k}$. However, there exists also the convention $\frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2E_k}}$, which Hofmann and Rug use in [30]. So, their slightly different result for $\mathcal{N}_C$, which differs namely in those factors, can be assigned to different conventions. The relevant N and M_B scaling is identical.

Fourthly, when we compare our calculation process to that in [30] we see that they worked with the constituent distribution, which in essence is a Fourier transform of the particle density expectation value (4.2.51),

$$\mathcal{D}(r) := \int \mathrm{d}^3 k e^{ikr} (2E_{\mathbf{k}})^{-2} \langle \mathcal{B} | n(\mathbf{k}) | \mathcal{B} \rangle; \tag{4.2.69}$$

r represents an external scale of the observable—compare to Appendix C, Eq. (C.0.29) This is the alternative density one can work with. To get the constituent number one then has to integrate over the scale r over all spacetime. In the the double scaling limit $N/M_B = const$, this distribution is constant. So the graviton constituent of a quantum BH bound state are evenly distributed.

Chapter 5

The primordial power spectrum

5.1 Auxiliary current description: generalization

In this section we want to generalize the auxiliary current description originally developed for black holes to arbitrary solutions of Einstein's field equations. In particular, it should be possible to describe classical backgrounds with small fluctuations in terms of quantum bound states in an appropriate Hilbert space.

Knowing the exact quantum state $|\mathcal{G}\rangle$ is definitely not possible. It is possible, however, to store kinematical data in a quantum state, in such a way that its overlap with $|\mathcal{G}\rangle$ is non-vanishing. This can only be the case if this state has the same quantum numbers as $|\mathcal{G}\rangle$. One can achieve this by acting with an appropriate auxiliary current, $\mathcal{J}(x)$, on a non-perturbative vacuum $|\Omega\rangle$ supporting the creation of quantum bound states and possible fluctuations. As we will discuss in detail below, inclusion of fluctuations effectively leads to a factorization of $|\Omega\rangle$ into a purely perturbative ($|0\rangle$) and non-perturbative ($|\tilde{\Omega}\rangle$) ground state.

Since in our picture the "would-be" classical geometry should be understood as bound state of the (weakly interacting) elementary degrees of freedom of the underlying quantum field theory, $\mathcal{J}(x)$ must provide the correct field content.

Furthermore $\mathcal{J}(x)$ has to respect the isometries of a given background. In general the associated symmetries will be broken softly by small fluctuations. In order to have a complete quantum description we also need to incorporate these fluctuations in the construction of $\mathcal{J}(x)$. This implies that the state $\mathcal{J}(x)|\Omega\rangle$ is no longer invariant under the action of the symmetry group of the background. The isometries, however, must be recovered in the case of vanishing fluctuations. Since we consider only small fluctuations which should not destabilize the background, this idea can be understood as a realization of the mean-field idea within the auxiliary current description.

Any degree of freedom we want in the universe we should include in principle, but this is too much. The advantage of ACD is: we have a true state in nature (horrific complex with all Standard Model fields), but we have a toy/model state which has the right isometries. Any current that can create a state with the right isometries is a nice current because we assume that the state we produce, by the operation of the current, has a non-trivial overlap with the true state. We have a quantum mechanical treament: we can never switch Quantum Mechanics off (that's what tells us harmonic oscillator: we cannot think of any regime where the oscillator doesn't give rise to fluctuations unless the mass of the degree of freedom that is oscillating becomes infinite) and then fluctuations are always there. We never think of classical GR solutions as being realized perfectly in nature.

The quantum bound states are everywhere so, e.g., in de Sitter spacetime, which is a perfect symmetry group, we have no more notion of asymptotic Minkowski as in the case of Bunch-

Davies vacuum definition. Since we are working with bound states, we don't have a concept of momentum (or mass) either. We cannot make any measurement that would globally define a mass for our background field. Incorporating fluctuations we can define again a good Casimir operator, the momentum.

Let us now explain how $|\mathcal{G}\rangle$ can be expressed in terms of $\mathcal{J}(x)|\Omega\rangle$ and how to compute variables in this state. First of all, we can think of $|\mathcal{G}\rangle$ as a quantum superposition of states representing classical geometries:

$$|\mathcal{G}\rangle = \sum_i \alpha_i |\mathcal{G}_i\rangle. \tag{5.1.1}$$

Here $|\mathcal{G}_0\rangle$ corresponds to the classical background solution (e.g., pure de Sitter) and the other $|\mathcal{G}_i\rangle$ correspond to fluctuations around where i counts the order of fluctuation. The different basis states are weighted with coefficients α_i. Notice that in order to realize a mean-field description in this framework, we should assume that $|\alpha_0| \simeq 1$ while all the other α_i are close to zero. Furthermore, we will see, that $\langle \mathcal{G}_i|\mathcal{G}_{\mathcal{J}}\rangle = \delta_{i\mathcal{J}}$.

Let us now explain how $|\mathcal{G}\rangle$ can be expressed in terms of $\mathcal{J}(x)$. Let us define $|\mathcal{L}(\mathcal{G})\rangle = \int \mathrm{d}^4 x F_{\mathcal{L}}(x)\mathcal{J}(x)|\Omega\rangle$, where $\mathcal{L}(\mathcal{G})$ is a state of quantum numbers compatible with $\mathcal{G}$ and $F_{\mathcal{L}}$ is a weight function. Inserting a complete set of such states we can write

$$|\mathcal{G}\rangle = \sum\!\!\!\!\!\!\int_{\mathcal{L}(\mathcal{G})} \mathcal{G}(\mathcal{L}) \int \mathrm{d}^4 x F_{\mathcal{L}}(x)\mathcal{J}(x)|\Omega\rangle. \tag{5.1.2}$$

Here $\mathcal{G}(\mathcal{L})$ is the wavefunction of $|\mathcal{L}(\mathcal{G})\rangle$ in the basis $|\mathcal{L}\rangle$, which then represents the geometry and carries the quantum numbers provided by the current $\mathcal{J}(x)$. Decomposing the current in terms of a background and fluctuations,

$$\mathcal{J}(x) = \mathcal{J}_0(x) + \delta\mathcal{J}(x) \equiv \sum_{j=0}^{max(j)} \delta_{(j)}\mathcal{J}(x) \tag{5.1.3}$$

leads to

$$|\mathcal{G}\rangle = \sum_{j=0}^{max(j)} \sum\!\!\!\!\!\!\int_{\mathcal{L}(\mathcal{G})} \mathcal{G}(\mathcal{L}) \int \mathrm{d}^4 x F_{\mathcal{L}_j}(x)\delta_{(j)}\mathcal{J}(x)|\Omega\rangle. \tag{5.1.4}$$

A few comments concerning this equation are in order. As we already discussed, $|\mathcal{G}\rangle$ should be understood as quantum superposition. Equation (5.1.4) gives an explicit realization of this idea. The different basis states are represented by different wavefunctions $\mathcal{G}(\mathcal{L})$ and currents $\delta_{(i)}\mathcal{J}(x)$. For the zeroth order holds $\delta_{(0)}\mathcal{J}(x) = \mathcal{J}_0(x)$ where $\mathcal{J}_0(x)$ only consists of background fields. Therefore, $\mathcal{J}_0(x)$ should be invariant under background isometries. This induces $\mathcal{J}_0(x) = \mathcal{J}_0(\tilde{x})$. The $\delta_{(i)}\mathcal{J}(x)$ correspond to geometries containing i fluctuating fields while all other fields provide an effective background geometry for these fluctuations. Furthermore, the functions $F_{\mathcal{L}_j}(x)$ display the realtive weights of different quantum geometries constituting $|\mathcal{G}\rangle$.

Notice that the background $|\mathcal{G}_0\rangle$ can be interpreted as a non-perturbative condensation process of fields with proper quantum numbers in the current $\mathcal{J}_0(x)$ on Minkowski spacetime. The fluctuations, however, are perturbative in nature. Thus they can be classified according to the Casimir operator on flat spacetime. Effectively, this tells us that all background fields create states in the bound state spectrum of the theory, i.e., they act on a non-perturbative vacuum state while fluctuations act on the perturbative vacuum. This explicitly realizes the factorization idea of QCD in the quantum description of general relativity.

Now one can make an Ansatz for current consisting of M different types of fields. Therefore, we split $\delta_{(j)}\mathcal{J}(x)$ in a background and a fluctuating part where the index represents the number of fields in the current.

$$\delta_{(j)}\mathcal{J}(x) = \mathcal{J}^b_{\sum_{l=1}^M N_l - j_l}\delta_{(j)}\tilde{\mathcal{J}}. \tag{5.1.5}$$

Here $N_l - j_l$ is the number of background fields of type l and $\mathcal{J}_l$ the corresponding number of fluctuations with $j = \sum_{l=1}^{M} j_l$. Now it is easy to see that $\langle \mathcal{G}_i | \mathcal{G}_j \rangle = \delta_{ij}$. Using the auxiliary current representation, terms of the form

$$\langle \Omega | \delta_{(i)} \mathcal{J}(x) \delta_{(\mathcal{J})} \mathcal{J}(x) | \Omega \rangle = \langle \Omega \, | \mathcal{J}^b_{\sum_{l=1}^{M} N_l - i_l} \delta_{(i)} \tilde{\mathcal{J}} \mathcal{J}^b_{\sum_{l=1}^{M} N_l - \mathcal{J}_l} \delta_{(\mathcal{J})} \tilde{\mathcal{J}} | \Omega \rangle \tag{5.1.6}$$

contribute. If $i_l \neq j_l$, because of the factorization property, we need to evaluate the overlap of two different effective backgrounds. Since the quantum numbers of these two states are different, their overlap must vanish. Thus $i_l = j_l \, \forall l$ has to hold. From this follows automatically $j = i$. Notice that this argument can also be applied for any observable $\mathcal{O}(x_1, x_2, \ldots, x_L)$ consisting of fluctuations only. In practice this tells us that only "diagonal" elements contribute to $\langle \mathcal{G} | \mathcal{O}(x_1, x_2, \ldots, x_L) | \mathcal{G} \rangle$. Schematically, $\langle \mathcal{G} | \mathcal{O}(x_1, x_2, \ldots, x_L) | \mathcal{G} \rangle = \sum_i |\alpha_i|^2 \langle \mathcal{G}_i | \mathcal{O}(x_1, x_2, \ldots, x_L) | \mathcal{G}_i \rangle$.

5.1.1 De Sitter spacetime

What are the isometries of the de Sitter? What are the generators of those isometries? I would strict to the intrisic description in 4 dimensions ($SO(1,3)$). I first try with one species (a scalar field) current: $\mathcal{J}(x) = \phi^N(x)$. We just demand that the isometries of de Sitter would be implemented in that current in an algebraic fashion. But this algebraic fashion traslate to a statement about the spacetime dependence of the fields which constitue the current. We will see that this form reduces to a differential equation for a single $\phi(x)$ and that will tell us on what the background field could possibly depend. The solution of this differential equation, the condition we get for every Killing vector, we'll tell us something. Because the Schwarzschild's metric is static ($SO(3)$), the background field can only depend on the distance r. The intersting thing with de Sitter: if I have a de Sitter spacetime, distances between points vary in time but we still have a timelike Killing vector; it just tells you a priori that the timelike KV can't be just ∂_t (in the Schwarzschild case, locally you can check e.g. energy conservation). In addition we have a Killing vector which tells us that we have homogeneity in space that combines to a state description which is effectively time-indipendent. One of de Sitter isometry is spatial homogeneity, which will tell us that ∂_x acting on the state woud be zero as well, which means effectively that this KV collapses to ∂_t. Then, in principle, ϕ is not allowed to depend on anything: isometries force us to say that ϕ is spacetime-indipendent. What can I do then? This gives us an idea how the current construction needs to look like.

Let G be a Lie group (e.g., $SO(1,3)$). If we want our state $|\mathcal{G}\rangle$ to be invariant under the action of G

$$g \in G \Rightarrow g|\mathcal{G}\rangle = |\mathcal{G}\rangle \tag{5.1.7}$$

we have to require that the variation of the state vanishes

$$\delta_g |\mathcal{G}\rangle = 0 \Rightarrow \delta_g \mathcal{J}(x) = 0 \tag{5.1.8}$$

since $\langle \mathcal{G} | \mathcal{J}(x) | \Omega \rangle \neq 0$. Differential geometry tells us that the derivative in the direction of the vector field g is given by

$$\delta_g \mathcal{J}(x) = [g, \mathcal{J}]. \tag{5.1.9}$$

From the commutator $[g, \mathcal{J}] = 0$ we get the differential equation our current $\mathcal{J}(x)$ has to satisfy. For convenience we will consider as g the generators of G.

De Sitter spacetime is the maximally symmetric so it has $\frac{D(D+1)}{2}$ Killing vectors (in the gruop theory language, the generators of $SO(1,3)$) where D is the dimension of the space. So we have 6 Killing vectors of the form

$$T_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}. \tag{5.1.10}$$

Considering $x^0 = ct$, the de Sitter Killing vectors are

$$T_{01} = ct\partial_x + \frac{x}{c}\partial_t = -T_{10} \tag{5.1.11}$$

$$T_{02} = ct\partial_y + \frac{y}{c}\partial_t = -T_{20} \tag{5.1.12}$$

$$T_{03} = ct\partial_z + \frac{z}{c}\partial_t = -T_{30} \tag{5.1.13}$$

$$T_{12} = -x\partial_y + y\partial_x = -T_{21} \tag{5.1.14}$$

$$T_{13} = -x\partial_z + z\partial_x = -T_{31} \tag{5.1.15}$$

$$T_{23} = -y\partial_z + z\partial_y = -T_{32} \tag{5.1.16}$$

We have to solve the commutator $[T_{\mu\nu}, \mathcal{J}(x)] = 0$ where $\mathcal{J}(x) = \phi^N(x)$ (x is a 4-dimensional coordinate). We get a set of differential equations

$$\left.\begin{aligned} N\phi^{N-1}(x)\left(ct\frac{\partial\phi}{\partial x} + \frac{x}{c}\frac{\partial\phi}{\partial t}\right) = 0 \\ N\phi^{N-1}(x)\left(ct\frac{\partial\phi}{\partial y} + \frac{y}{c}\frac{\partial\phi}{\partial t}\right) = 0 \\ N\phi^{N-1}(x)\left(ct\frac{\partial\phi}{\partial z} + \frac{z}{c}\frac{\partial\phi}{\partial t}\right) = 0 \end{aligned}\right\} \left(ct\frac{\partial}{\partial x^i} - \frac{x_i}{c}\frac{\partial}{\partial t}\right)\phi(x) = 0 \quad i = 1,2,3 \tag{5.1.17}$$

and

$$\left.\begin{aligned} N\phi^{N-1}(x)\left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial x}\right) = 0 \\ N\phi^{N-1}(x)\left(x\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial x}\right) = 0 \\ N\phi^{N-1}(x)\left(z\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial z}\right) = 0 \end{aligned}\right\} \left(x_i\frac{\partial}{\partial x^{\mathcal{J}}} - x_{\mathcal{J}}\frac{\partial}{\partial x^i}\right)\phi(x) = 0 \quad i \neq \mathcal{J},\, i,\mathcal{J} = 1,2,3 \tag{5.1.18}$$

Because of spatial homogeneity $\partial_x\phi = \partial_y\phi = \partial_z\phi = 0$ so from the first set of differtial equations above we $\partial_t\phi = 0$. Then $\mathcal{J}_0(x) = \mathcal{J}_0$, i.e., our background current is a constant.

The spacetime we'r working with is effectively static, this means it's not made up of dynamical degrees of freedom. How can you resolve that with an ACD? From the quantum mechanical point of view, there is nothing like exact de Sitter because having no classical solution is realized in nature (you cannot freeze the quantum fluctuations). We have to allow for quantum flucutations. What does it mean to be approximately close to de Sitter? We need a criteria. We can take the conditions we have for the current construction right now (namely commutation with de Sitter generators). Now we perturbe this equation to a certain order. The current now consists of a lot of fields such that they respect de Sitter isometries and we add a certain amount of fields that don't obey the isometries. But we demand that the equation $[\mathcal{J}, g] = 0$ still holds and this gives us a backreaction relation for how $\delta\phi$ and $\delta\psi$ are connected. That would be the qualification what means to be close. $[\delta\mathcal{J}, g] \approx 0$ would be even fulfilled at the perturbative level.

5.2 The de Sitter power spectrum

I want to calculate the power spectrum of the inflaton field. In our new language two ways are possible:

1. we try to come up with a bi-local operator which is associated with the power spectrum
2. I consider the bi-local fluctuations already emerging from the auxiliary currents.

Shall we then assign to the power spectrum an operator and view it as a measurement of that operator or should we say that the power spectrum arises because of the fluctuations in the currents? To some extent the power spectrum is very well measured from an operator $\phi(x)\phi(y)$ which I evaluate in the Bunch-Davies vacuum. In the first option I insert a measurement device in an approximate de Sitter background. The first order calculation should give the standard result. In the second one it seems that, contrary to the original calculation, the result doesn't depend on any scale because we integrate over 2 out of the 4 coordinate we have in the diagram (see below).

5.2.1 A first attempt

We have an operator, $P_{\delta\phi}(x,y) = \delta\phi(x)\delta\phi(y)$ where $\delta\phi$ is the inflaton fluctuation, and we can think of it as a measuring device. What does this operator do? At the first order, it connects two endpsoint, x and y. But after the first term is done, the only thing it can do is connecting to other stuff. The minimal connection to other stuff is via at least two additional points like an open square diagram. The other diagrams just enhance the connection between z and u. Basically, what is measuring the power spectrum is the open square and the rest is just quanta that don't directly partecipate in the power spectrum: they just correct it.

We have a measurement device operator that when applied to a state gives us the power spectrum of a certain fluctuation type in that state. The measurement device is at two spacetime points and it consists of two fields, which means you can connect it to two other fluctuations.

$$\langle \mathcal{G}| P_{\delta\phi} |\mathcal{G}\rangle = \big(\langle \mathcal{G}_0| + \delta \langle \mathcal{G}|\big) P_{\delta\phi}(x,y) \big(|\mathcal{G}_0\rangle + \delta |\mathcal{G}\rangle\big) \tag{5.2.1}$$

We have an observable, we put it in the universe so you have your measurement device: you measure 2 quanta in the universe but there are loads of other quanta floating around. Now suppose we would have many more fluctuations: they would not display in our measurement process. As we proceed, we cannot have an inflaton fluctuation without causing the background, i.e. the scalar graviton fluctuation $\delta\psi$, to fluctuate as well. Even if we couldn't forget about backreaction in nature, let's ignore the graviton contributions to the auxiliary current (D.0.2): $(\psi + \delta\psi)^{N_\psi} \longrightarrow 1$. Now let's expand the interpolating current to the second order in the fluctuations:

$$\begin{aligned} \mathcal{J}(x) &= \mathcal{J}_0 + \delta\mathcal{J}(x) = (\phi + \delta\phi)^{N_\phi} = \\ &\approx \phi^{N_\phi} + N_\phi \phi^{N_\phi - 1}\delta\phi + N_\phi(N_\phi - 1)\phi^{N_\phi - 2}\delta\phi^2 + \dots = \\ &= \mathcal{J}_0 + \delta_{(1)}\mathcal{J}(x) + \delta_{(2)}\mathcal{J}(x). \end{aligned} \tag{5.2.2}$$

Using the definition (5.1.4), the state we work with is

$$\begin{aligned} |\mathcal{G}\rangle = &\sum\!\!\!\!\!\!\!\int_{\mathcal{L}_0} \mathcal{G}_0(\mathcal{L}_0) \int \sigma(z) F_{\mathcal{L}_0}(z) \phi^{N_\phi} |\Omega\rangle + \\ &+ N_\phi \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1} \mathcal{G}_1(\mathcal{L}_1) \int \sigma(z) F_{\mathcal{L}_1}(z) \phi^{N_\phi - 1}\delta\phi |\Omega\rangle + \\ &+ N_\phi(N_\phi - 1) \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_2} \mathcal{G}_2(\mathcal{L}_2) \int \sigma(z) F_{\mathcal{L}_2}(z) \phi^{N_\phi - 2}\delta\phi^2 |\Omega\rangle . \end{aligned} \tag{5.2.3}$$

We know that for $\langle\Omega|\,\phi^{N_\phi-k}\phi^{N_\phi-m}\,|\Omega\rangle \neq 0$ we need $k=m$, so we get

$$\begin{aligned}\langle\mathcal{G}|\,P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2 = N_\phi^2(N_\phi^2-1)^2 &\oint_{\mathcal{L}'_2}\oint_{\mathcal{L}_2}\mathcal{G}_2^*(\mathcal{L}'_2)\mathcal{G}_2(\mathcal{L}_2)\\ &\times\int\sigma(z)\sigma(u)F^*_{\mathcal{L}'_2}(u)F_{\mathcal{L}_2}(z)C(2,2)\times\\ &\times\langle 0|\,\delta\phi^2(u)P_{\delta\phi}(x,y)\delta\phi^2(z)\,|0\rangle\end{aligned}\tag{5.2.4}$$

where $C(2,2)=\left\langle\tilde{\Omega}\middle|\,\phi^{N_\phi-2}(u)\,\phi^{N_\phi-2}(z)\,\middle|\tilde{\Omega}\right\rangle$. The other three contributions would give disconnected diagrams so we don't consider them.

There is a hierarchy: those terms which have the least amount of fluctuations should be the dominant contributions. The open square (Fig. 5.1) is leading. But in what sense? We have no coupling constant here! Which term would be more important? If we let $N_\phi\longrightarrow\infty$, the background becomes arbitrary important and the system becomes more and more classical in a sense. The background field opertors are going to end up in the condensates and the condensates are the classicalized version of our background, i.e., the higher the condensates are populated the more classical our background is. We are penalized by something out of the condensate promoting it to fluctuation and letting it propagate. On one hand, it's a physical input, on the other hand, it's a technicality.

Naively speaking, you start out with a fixed number of fields operator N_ϕ and then you split the field $\phi(x)$ in a background operator plus fluctuation. We have a general fields operator and we split it up in two pieces: $\mathcal{J}(x)=\phi^{(0)}(x)+\phi^{(1)}(x)$. $\phi^{(0)}(x)$ respects the isometries and it is a good background because $\phi^{(1)}(x)$ is only a small violation of the isometries (but the quantum fluctuations don't destabilize the spacetime). If it's not small, we can't sit in de Sitter. Now we have the split meaning that in every term of our expansion we get a certain number of background field operators and a certain number of fluctuations such that the total number of fields will be always the same. Let's call $\phi^{(0)}\equiv\phi(x)$ and $\phi^{(1)}\equiv\delta\phi$. From the computational point of view: we have a mean field split

$$\begin{aligned}(\phi+\delta\phi)^{N_\phi} &= \phi^{N_\phi}+N_\phi\delta\phi\,\phi^{N_\phi-1}+\cdots=\\ &\approx\phi^{N_\phi}\left(1+\frac{N_\phi\delta\phi}{\langle\phi\rangle}\right).\end{aligned}\tag{5.2.5}$$

In the mean-field split, you want the background to be the dominant part and just have small perturbations. But the N_ϕ appears here. We want the second term to be small compared to 1. Then $\langle\phi\rangle\gg N_\phi\delta\phi$ ($\delta\phi$ is a typical energy). In the physical point of view: $\langle\phi\rangle$ is fixed to have a mean-field approximation. In the limit $N_\phi\longrightarrow\infty$ (semiclassical limit) the background must become rigid so we can neglect the backreaction. The background contribution becomes completely dominant and this tells us that all the fluctuations decouple in that limit: $\delta\phi\longrightarrow 0$. Whatever you put $N_\phi\longrightarrow\infty$, also the importance of the background increases infinitely and we have a rescaling $\delta\phi\sim\frac{1}{N_\phi}$.

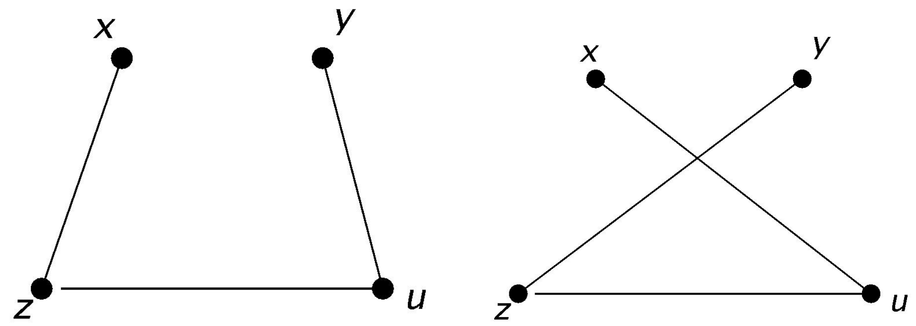
Furthermore, we demand $|\mathcal{G}\rangle\approx|\mathcal{G}_0\rangle=\sum_{i=0}^N\alpha_i\,|\mathcal{G}_i\rangle$ (Eq. (5.1.1)). Thus $|\alpha_0| >> |\alpha_1| >> \cdots >> |\alpha_N|$.

$$\int\sigma(x)F^*_{\mathcal{L}_m}(x)F_{\mathcal{L}_0}(x)\overset{m,n\neq 0}{<<}\int\sigma(x)\left|F_{\mathcal{L}_0}(x)\right|^2\tag{5.2.6}$$

$$\int\sigma(x)\left|F_{\mathcal{L}_m}(x)\right|^2\overset{m,n\neq 0}{<<}\int\sigma(x)\left|F_{\mathcal{L}_0}(x)\right|^2\tag{5.2.7}$$

Let's compute $\langle 0|\,\delta\phi^2(u)P_{\delta\phi}(x,y)\delta\phi^2(z)\,|0\rangle\equiv\xi_{\delta\phi}(u,x,y,z)$, which explicitly is

$$\xi_{\delta\phi}=2\left\langle\delta\phi(u)\delta\phi(z)\right\rangle\Big[\left\langle\delta\phi(u)\delta\phi(x)\right\rangle\left\langle\delta\phi(y)\delta\phi(z)\right\rangle+\left\langle\delta\phi(u)\delta\phi(y)\right\rangle\left\langle\delta\phi(x)\delta\phi(z)\right\rangle\Big]\tag{5.2.8}$$



(a) (b)

Figure 5.1: The only physical diagrams where all the lines are inflatons since we neglected backreaction

From the results in the Appendix D:

$$\left\langle 0 \middle| \delta\phi(u)\delta\phi(z) \middle| 0 \right\rangle = \int \frac{d^3 k}{(2\pi)^3 2\sqrt{|\mathbf{k}|^2 + m_\phi^2}} \, e^{ik(u-z)}\Big|_{os} \tag{5.2.9}$$

where "os" means on shell, i.e., the dispersion relations are fixed $p^0 = \omega(\mathbf{p})$ and $q^0 = \omega(\mathbf{q})$. Then we get

$$\begin{aligned} \xi_{\delta\phi}(u,x,y,z) = \int \frac{d^3 k}{(2\pi)^3\, 2\sqrt{|\mathbf{k}|^2 + m_\phi^2}} \, e^{ik(u-z)}\Big|_{os} \Bigg\{ \int \frac{d^3 p}{(2\pi)^3\, 2\sqrt{|\mathbf{p}|^2 + m_\phi^2}} e^{ip(u-x)} \times \\ \times \int \frac{d^3 p'}{(2\pi)^3\, 2\sqrt{|\mathbf{p}'|^2 + m_\phi^2}} e^{ip'(y-z)} + \int \frac{d^3 p}{(2\pi)^3\, 2\sqrt{|\mathbf{p}|^2 + m_\phi^2}} e^{ip(u-y)} \times \\ \times \int \frac{d^3 p'}{(2\pi)^3\, 2\sqrt{|\mathbf{p}'|^2 + m_\phi^2}} e^{ip'(x-z)} \Bigg\}\Bigg|_{os} \end{aligned} \tag{5.2.10}$$

The whole expression gives

$$\times$$

$$\times \left\{ e^{iu(k+p)}\, e^{-iz(k+p')} \left[e^{-ipx} e^{ip'y} + e^{-ipy} e^{ip'x} \right] \right\}\Big|_{os} \tag{5.2.11}$$

We can apply the Operator Product Expansion (OPE) to $C(2,2)$ then, by equations of motion, higher order is suppressed: we have the fields at one point and we can shift them by unitary translation (see previous chapter). The lowest order term is dominant and we can think at $C(2,2)$ as local. Suppose we can write

$$F_{\mathcal{L}_k} = \bar{F}_{\mathcal{L}_k} e^{iP_k x} \Rightarrow \sum\!\!\!\!\!\!\int_{\mathcal{L}_k} \to \int \bar{\sigma}(P_k) \quad \text{and} \quad \mathcal{G}_k(\mathcal{L}_k) \to \mathcal{G}_k(P_k)$$

thus

$$\begin{aligned}\langle\mathcal{G}|\,P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2 =& N_\phi^2(N_\phi-1)^2C(2,2)\left[\int\frac{\mathrm{d}^3 k}{(2\pi)^3\,2\sqrt{|\mathbf{k}|^2+m_\phi^2}}\right]\int\bar\sigma(P_2')\bar\sigma(P_2)\mathcal{G}_2^*(P_2')\mathcal{G}_2(P_2)\times\\ &\times\int\frac{\mathrm{d}^3 p}{(2\pi)^3\,2\sqrt{|\mathbf{p}|^2+m_\phi^2}}\frac{\mathrm{d}^3 p'}{(2\pi)^3\,2\sqrt{|\mathbf{p'}|^2+m_\phi^2}}\left(e^{-ixp}e^{iyp'}+e^{-iyp}e^{ixp'}\right)\Big|_{os}\times\\ &\times\underbrace{\int\sigma(z)\sigma(u)\bar F^*_{\mathcal{L}'_2}\bar F_{\mathcal{L}_2}\,e^{iu(-P_2'+k+p)}e^{-iz(-P_2+k+p')}}_{=\tilde F^*_{P_2'}(k+p-P_2')\tilde F_{P_2}(k+p'-P_2)}\Bigg|_{os}\,.\end{aligned} \tag{5.2.12}$$

Now our unknown quantities are the functions $\bar F_{\mathcal{L}_2}$ and the wavefunctions $\mathcal{G}_2(P_2)$. The wavefunction should be able to modify the power spectrum. If $\bar F_{\mathcal{L}_2}$ were 1, then the wavefunctions would give any statistical weight on the actual observable we calculated. To have a non-trivial dependence we should keep $\bar F_{\mathcal{L}_2}$ general.

Why did we introduce $F_{\mathcal{L}_k}(x)$? The current is an operator. In some sense I can understand the functional $\mathcal{J}(x)$ depending on $F_{\mathcal{L}_k}(x)$, like in QFT we have creation and annihilation operators: usually in physics they depend on $\mathbf{p}$, but they is not really good because their algebra is defined as $\left[\hat a_{\mathbf{p}},\hat a^\dagger_{\mathbf{p'}}\right]=\delta(\mathbf{p}-\mathbf{p'})$. But we can also introduce creation and annihilation operators which depend on functions and those are clean objects because they don't give the Dirac delta. Then you introduce an operator $a^\dagger(f)$, which is basically $a^\dagger(f)=\int\mathrm{d}^3p\,a^\dagger(\mathbf{p})f(\mathbf{p})$. The meaning is that you create a particle in a state described by the function f: the process of creation does not only move the counter up, but it also tells you what the wavefunction of the state which has been created. This is the idea we had in mind when we introduced $F_{\mathcal{L}_k}(x)$, i.e., creation of a state represented by the wavefunction $F_{\mathcal{L}_k}(x)$.

For pure perturbations the exponential in $F_{\mathcal{L}_2}(x)$ makes sense because this would be the momentum of the particle. As soon as I have some fluctuations in my current, then I can associate a plane wave with it and I would assume that I can write it as e^{iP_2x}. It's basically a measure of the frequency of the particle with respect to Minkowski spacetime (even though you have bound state) and you can define the Casimir operator of the particle which is P^2. In a pure de Sitter state without any perturbations the notion of P^2 is gone. We associate P_2 to perturbations $\delta\phi$ in de Sitter within the current. The plane waves have mass dimension if they are a function of P_2: we need to compensate with a mass scale (see the next subsection for more details). This is encoded in the $\bar F_{\mathcal{L}_2}$. [1] In the end $\bar F^*_{\mathcal{L}'_2}$ and $\bar F_{\mathcal{L}_2}$ are constant functions of

$$\int\sigma(z)\sigma(u)e^{iu(-P_2'+k+p)}e^{-iz(-P_2+k+p')}=\delta^4(-P_2'+k+p)\delta^4(P_2-k-p') \tag{5.2.13}$$

Suppose we wouldn't have the wavefunction $\mathcal{G}_2$, then the result we might get is the power spectrum in Minkowski spacetime and we have to see how the wavefunctions, that are actually telling us that we are in the de Sitter, transform the power spectrum in Minkowski spacetime

[1] Even if we set it to 1, we can reabsorb this dependence in the definition of the condensate.

to the one in de Sitter. Inserting the previous result in (5.2.12):

$$\begin{aligned}\langle\mathcal{G}|\, P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2 = N_\phi^2(N_\phi-1)^2C(2,2)\left[\int\frac{\mathrm{d}^3 k}{(2\pi)^3\,2\sqrt{|\mathbf{k}|^2+m_\phi^2}}\right]\int\bar{\sigma}(P_2')\bar{\sigma}(P_2)\mathcal{G}_2^*(P_2')\mathcal{G}_2(P_2)\times\\ \times\int\frac{\mathrm{d}^3 p}{(2\pi)^3\,2\sqrt{|\mathbf{p}|^2+m_\phi^2}}\frac{\mathrm{d}^3 p'}{(2\pi)^3\,2\sqrt{|\mathbf{p'}|^2+m_\phi^2}}\left(e^{-ixp}e^{iyp'}+e^{-iyp}e^{ixp'}\right)\Big|_{os}\times\\ \times\,\delta^4(-P_2'+k+p)\delta^4(P_2-k-p')\Big|_{os}\end{aligned} \tag{5.2.14}$$

Now let's trasform the $\mathrm{d}^3 p$ and $\mathrm{d}^3 p'$ integrals in 4-integrals:

$$\int \mathrm{d}^3 p = \int \mathrm{d}^4 p\,\theta(p^0)\delta(p^2=p_0^2-\mathbf{p}^2) \tag{5.2.15}$$

and the same for p'. Then we can get rid of these two integrals:

$$\begin{aligned}&\int\frac{\mathrm{d}^4 p}{(2\pi)^3\,2\sqrt{|\mathbf{p}|^2+m_\phi^2}}\theta(p^0)\delta(p^2)\int\frac{\mathrm{d}^4 p'}{(2\pi)^3\,2\sqrt{|\mathbf{p'}|^2+m_\phi^2}}\theta(p'^0)\delta(p'^2)\times\\ &\times left.\delta^4(-P_2'+k+p)\delta^4(P_2-k-p')\left(e^{-ixp}e^{iyp'}+e^{-iyp}e^{ixp'}\right)_{os}=\\ &=\frac{\theta(P_2'^0-k^0)\delta((P_2'-k^0)^2)}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2'}-\mathbf{k}\right|^2+m_\phi^2}}\frac{\theta(P_2^0-k^0)\delta((P_2-k^0)^2)}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2}-\mathbf{k}\right|^2+m_\phi^2}}\left(e^{-ix(P_2'-k)}e^{iy(P_2-k)}+e^{-iy(P_2'-k)}e^{ix(P_2'-k)}\right)\Big|_{os}\end{aligned} \tag{5.2.16}$$

Thus

$$\begin{aligned}\langle\mathcal{G}|\, P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2 =&N_\phi^2(N_\phi-1)^2C(2,2)\left[\int\frac{\mathrm{d}^3 k}{(2\pi)^3\,2\sqrt{|\mathbf{k}|^2+m_\phi^2}}\right]\int\bar{\sigma}(P_2')\bar{\sigma}(P_2)\;\mathcal{G}_2^*(P_2')\mathcal{G}_2(P_2)\Big|_{os}\times\\ &\times\frac{\theta(P_2'^0-k^0)\delta((P_2'-k^0)^2)}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2'}-\mathbf{k}\right|^2+m_\phi^2}}\frac{\theta(P_2^0-k^0)\delta((P_2-k^0)^2)}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2}-\mathbf{k}\right|^2+m_\phi^2}}\times\\ &\times\left(e^{-ix(P_2'-k)}e^{iy(P_2-k)}+e^{-iy(P_2'-k)}e^{ix(P_2'-k)}\right)\Big|_{os}=\\ =&N_\phi^2(N_\phi-1)^2C(2,2)\left[\int\frac{\mathrm{d}^3 k}{(2\pi)^3\,2\sqrt{|\mathbf{k}|^2+m_\phi^2}}\right]\int\mathrm{d}^3 P_2'\,\mathrm{d}^3 P_2\mathcal{G}_2^*(\mathbf{P}_2')\mathcal{G}_2(\mathbf{P}_2)\times\\ &\times\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2'}-\mathbf{k}\right|^2+m_\phi^2}}\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2}-\mathbf{k}\right|^2+m_\phi^2}}\times\\ &\times\left(e^{-ix(P_2'-k)}e^{iy(P_2-k)}+e^{-iy(P_2'-k)}e^{ix(P_2'-k)}\right)\Big|_{os}\end{aligned} \tag{5.2.17}$$

Let's rearrange this expression

$$\begin{aligned}\langle\mathcal{G}|\,P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2 =& N_\phi^2(N_\phi-1)^2C(2,2)\left[\int\frac{\mathrm{d}\Omega\,\mathrm{d}k\,k^3}{(2\pi)^3\,2k\,\sqrt{|\mathbf{k}|^2+m_\phi^2}}\right]\int\frac{\mathrm{d}^3P_2'}{(2\pi)^3}\frac{\mathrm{d}^3P_2}{(2\pi)^3}\mathcal{G}_2^*(\mathbf{P}_2')\mathcal{G}_2(\mathbf{P}_2)\times\\&\times\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2'}-\mathbf{k}\right|^2+m_\phi^2}}\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2}-\mathbf{k}\right|^2+m_\phi^2}}\times\\&\times\left(e^{ik(x-y)}e^{-ixP_2'}e^{iyP_2}+e^{-ik(x-y)}e^{-iyP_2'}e^{ixP_2}\right)\Big|_{os}.\end{aligned}\tag{5.2.18}$$

The previous equation has then the form

$$\langle\mathcal{G}|\,P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2\sim\int\mathrm{d}\ln k\,\frac{\sin(k|\mathbf{x}-\mathbf{y}|)}{k|\mathbf{x}-\mathbf{y}|}P_{\delta\phi}^M(k)g(k;x,y)\tag{5.2.19}$$

where $P_{\delta\phi}^M$ is the power spectrum in Minkowski spacetime and

$$\begin{aligned}g(k;x,y)=&\int\frac{\mathrm{d}^3P_2'}{(2\pi)^3}\frac{\mathrm{d}^3P_2}{(2\pi)^3}\mathcal{G}_2^*(\mathbf{P}_2')\mathcal{G}_2(\mathbf{P}_2)\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2'}-\mathbf{k}\right|^2+m_\phi^2}}\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{P_2}-\mathbf{k}\right|^2+m_\phi^2}}\times\\&\times\left(e^{-ixP_2'}e^{iyP_2}+e^{-iyP_2'}e^{ixP_2'}\right)\Big|_{os}.\end{aligned}$$

Let's shift the $\mathbf{k}$ dependence in the wavefunction:

$$\begin{aligned}\langle\mathcal{G}|\,P_{\delta\phi}(x,y)\,|\mathcal{G}\rangle_2 =& N_\phi^2(N_\phi-1)^2C(2,2)\left[\int\frac{\mathrm{d}\Omega\,\mathrm{d}k\,k^3}{(2\pi)^3\,2k\,\sqrt{|\mathbf{k}|^2+m_\phi^2}}\right]\int\frac{\mathrm{d}^3Q_2'}{(2\pi)^3}\frac{\mathrm{d}^3Q_2}{(2\pi)^3}\mathcal{G}_2^*(\mathbf{Q}_2'+\mathbf{k})\mathcal{G}_2(\mathbf{Q}_2+\mathbf{k})\times\\&\times\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{Q_2}'\right|^2+m_\phi^2}}\frac{1}{(2\pi)^3\,2\sqrt{\left|\mathbf{Q}_2\right|^2+m_\phi^2}}\times\\&\times\left(e^{-ixQ_2'}e^{iyQ_2}+e^{-iyQ_2'}e^{ixQ_2}\right)\Big|_{os}\end{aligned}\tag{5.2.20}$$

where $\mathbf{Q}_2=\mathbf{P}_2-\mathbf{k}$ and $\mathbf{Q}_2'=\mathbf{P}_2'-\mathbf{k}$. In this way I decouple those integrals from the one over $\mathbf{k}$.

In my actual calculation I found a modification of the orthodox Minkowski spacetime. From my calculation we learn that those diagrams we calculate in Minkowski spacetime in our new language don't correspond to the diagrams you would calculate in standard QFT, but that means that even the Minkowski result will get modified and we want to understand how much it gets modified and how we can reproduce from the modified Minkowski calculation the de Sitter power spectrum. We have to note the difference between these two results.

Can the modified Minkowski result be converted to another spacetime geometry? A priori we don't know how the wavefunctions look like. Due to the construction we made, the power spectrum in Minkowski spacetime gets convoluted with the wavefunction. Now the question is: given that the diagramatic is still in Minkowski spacetime, how do the wavefunctions need to look like such that when we build the convolution with the wavefunction and the modified Minkowski power spectrum we would get out the de Sitter power spectrum? The wavefunctions have to depend on the curvature scale (on the Hubble radius) and then on H. This result assumes that we load the information about the geometry at every spacetime point with equal probability. Can we get an idea about $\mathcal{G}_2(P_2)$ that reproduces a reasonable power spectrum? The Minkowski power spectrum goes like k^3, while the de Sitter power spectrum goes like k^2 and then evaluated at the curvature scale. That's basically what we have to achieve.

5.2.2 A second attempt

As in the previous subsection, we forget about graviton backreaction, but this time we expand the current until the first order

$$\mathcal{J}(x) = \phi^N(x)\psi^M(x), \quad \phi(x) = \phi^{(0)}(\hat{x}) + \phi^{(1)}(x), \quad \psi(x) \stackrel{!}{=} \psi^{(0)}(\hat{x}) \tag{5.2.21}$$

where $\hat{x}$ points out that the background fields don't depend on any spacetime point since we sit in a pure de Sitter background.

$$\begin{aligned} \mathcal{J}(x) = \mathcal{J}^{(0)}(\hat{x}) + \mathcal{J}^{(1)}(x) + \ldots, \quad & \mathcal{J}^{(0)}(\hat{x}) = \phi^{(0)N}(\hat{x})\psi^{(0)M}(\hat{x}), \\ & \mathcal{J}^{(1)}(x) = N\phi^{(0)N-1}(\hat{x})\psi^{(0)M}(\hat{x})\phi^{(1)}(x) \end{aligned} \tag{5.2.22}$$

Introduce $\theta(A,B;\hat{x}) \coloneqq \phi^{(0)N}(\hat{x})\psi^{(0)M}(\hat{x})$. Then

$$\mathcal{J}(x) = \theta(N,M;\hat{x}) + N\theta(N-1,M;\hat{x})\phi^{(1)}(x) + \ldots \tag{5.2.23}$$

In de Sitter, $\theta(\hat{x}) \neq f(x)$ we drop the $\hat{x}$-dependence and "renormalize"

$$\mathcal{J}(x) \to \mathcal{J}(x) + \theta(N,M): \quad \mathcal{J}(x) = N\theta(N-1,M)\phi^{(1)}(x) + \ldots \tag{5.2.24}$$

The approximate $\mathcal{G}$-quantum bound state is given by

$$|\mathcal{G},\phi^{(1)}(x)[F_{\mathcal{L}_1}]\rangle = \sum\!\!\!\!\!\!\int_{\mathcal{L}_1} \mathcal{G}(\mathcal{L}_1)N\theta(N-1,M)\phi^{(1)}[F_{\mathcal{L}_1}]|\Omega\rangle \tag{5.2.25}$$

where with $\phi^{(1)}(x)[F_{\mathcal{L}_1}]$ I mean $\phi^{(1)}(x)[F_{\mathcal{L}_1}] = \int \sigma(x)F_{\mathcal{L}_1}(x)\phi(x)$. Thus

$$\begin{aligned} \langle\mathcal{G},\phi^{(1)}[F_{\mathcal{L}_1'}]|\mathcal{G},\phi^{(1)}[F_{\mathcal{L}_1}]\rangle =& N^2 \sum\!\!\!\!\!\!\int_{\mathcal{L}_1} \sum\!\!\!\!\!\!\int_{\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1)\langle\Omega|\theta^2(N-1,M)\phi^{(1)2}[F_{\mathcal{L}_1}]|\Omega\rangle = \\ &= N^2\langle\Omega|\theta^2(N-1,M)|\Omega\rangle \sum\!\!\!\!\!\!\int_{\mathcal{L}_1} \sum\!\!\!\!\!\!\int_{\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1)\langle 0|\phi^{(1)2}[F_{\mathcal{L}_1}]|0\rangle \end{aligned} \tag{5.2.26}$$

where we drop the normal order term in $\phi^{(1)2}[F_{\mathcal{L}_1}] \coloneqq \phi^{(1)2}[F_{\mathcal{L}_1}] : +\langle 0|\phi^{(1)2}[F_{\mathcal{L}_1}]|0\rangle$. Now we need to calculate $\langle 0|\phi^{(1)2}[F_{\mathcal{L}_1}]|0\rangle$

$$\langle 0|\phi^{(1)2}[F_{\mathcal{L}_1}]|0\rangle = \int \sigma(x,y)F^*_{\mathcal{L}_1'}(x)F_{\mathcal{L}_1}(y)\langle 0|\phi(x)\phi(y)|0\rangle \tag{5.2.27}$$

where we $\phi(x) \equiv \delta\phi(x)$ of the previous is our usual inflaton field.

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \int \omega(\mathbf{p},\mathbf{q})e^{ipx}e^{-iqy}\langle 0|a(\mathbf{p})a^\dagger(\mathbf{q})|0\rangle = \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}e^{ip(x-y)} = \\ &= \int \frac{\mathrm{d}\Omega\,\mathrm{d}|\mathbf{p}||\mathbf{p}|^2}{(2\pi)^3}\frac{1}{2p^0(|\mathbf{p}|)}e^{-ip_0(|\mathbf{p}|)(x-y)^0)}e^{i|\mathbf{p}||\mathbf{x}-\mathbf{y}|cos(\alpha)} = \\ &= (2\pi)^{-2}\int_0^\infty \mathrm{d}|\mathbf{p}||\mathbf{p}|^2\frac{e^{-ip_0(|\mathbf{p}|)(x-y)^0)}}{2p^0(|\mathbf{p}|)}\frac{2i\sin(|\mathbf{p}||\mathbf{x}-\mathbf{y}|)}{i|\mathbf{p}||\mathbf{x}-\mathbf{y}|} \end{aligned} \tag{5.2.28}$$

where $\omega(\mathbf{p}) := \frac{d^3 p}{(2\pi)^{3/2}(2p^0)^{1/2}}$ and α is the angle between $(\mathbf{p}, \mathbf{x}-\mathbf{y})$.

At $x^0 - y^0 = 0$,

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int_0^\infty \frac{d|\mathbf{p}|}{|\mathbf{p}|}\frac{\sin(|\mathbf{p}||\mathbf{x}-\mathbf{y}|)}{|\mathbf{p}||\mathbf{x}-\mathbf{y}|}\frac{|\mathbf{p}|^3}{2\pi^2}\frac{1}{2p^0(|\mathbf{p}|)} \tag{5.2.29}$$

That is the usual result for the primordial power spectrum in Minkowski spacetime.

$$\begin{aligned}
\mathcal{A} :=& \langle \mathcal{G}, \phi^{(1)}[F_{\mathcal{L}_1'}] | \mathcal{G}, \phi^{(1)}[F_{\mathcal{L}_1}]\rangle = \\
=& N^2 \langle \Omega|\theta^2(N-1,M)|\Omega\rangle \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1,\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1) \int \sigma(x,y) F^*_{\mathcal{L}_1'}(x) F_{\mathcal{L}_1}(y) \langle 0|\phi(x)\phi(y)|0\rangle = \\
=& N^2 \langle \Omega|\theta^2(N-1,M)|\Omega\rangle \int \omega(\mathbf{p},\mathbf{q}) \langle 0|a(\mathbf{p})a^\dagger(\mathbf{q})|0\rangle \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1,\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1) \times \\
& \times \int \sigma(x,y) e^{ipx} e^{-iqy} F^*_{\mathcal{L}_1'}(x) F_{\mathcal{L}_1}(y) \Big|_{os} = \\
=& N^2 \langle \Omega|\theta^2(N-1,M)|\Omega\rangle \int \omega(\mathbf{p},\mathbf{q}) \langle 0|a(\mathbf{p})a^\dagger(\mathbf{q})|0\rangle \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1,\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1) \tilde{F}^*_{\mathcal{L}_1'}(p) \tilde{F}_{\mathcal{L}_1}(q) \Big|_{os} = \\
=& N^2 \langle \Omega|\theta^2(N-1,M)|\Omega\rangle \int \frac{d^3 p}{(2\pi)^3 2p^0(|\mathbf{p}|)} \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1,\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1) \tilde{F}^*_1(p) \tilde{F}_1(p) \Big|_{os} = \\
=& N^2 \langle \Omega|\theta^2(N-1,M)|\Omega\rangle \int \frac{d\Omega}{4\pi} \int_0^\infty \frac{d|\mathbf{p}|}{|\mathbf{p}|} \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1,\mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1')\mathcal{G}(\mathcal{L}_1) \tilde{F}^*_{\mathcal{L}_1'}(p) \tilde{F}_{\mathcal{L}_1}(p) \Big|_{os} \frac{|\mathbf{p}|^3}{2\pi^2}\frac{1}{2p^0(|\mathbf{p}|)}
\end{aligned} \tag{5.2.30}$$

Now we want to infere the dimensions of our unknown functions. We know that

$$\langle \mathcal{G}|\mathcal{G}\rangle = 1 \tag{5.2.31}$$

to be a physical state and so $[|\mathcal{G}\rangle] = 0$. The state $|\mathcal{G}\rangle$ can be expressed as (5.1.2). Let us examine every component of this expansion. In momentum representation

$$\begin{aligned}
\langle \mathcal{G}|\mathcal{G}\rangle &= \int d^4 P \, \langle \mathcal{G}|P\rangle\langle P|\mathcal{G}\rangle = \\
&= \int d^4 P \, |\mathcal{G}(P)|^2 = 1
\end{aligned} \tag{5.2.32}$$

Then $[\mathcal{G}(P)] = -2$. Since $|\Omega\rangle$ is normalized, the state is dimensionless as $|\mathcal{G}\rangle$. If we assume the current to be $\mathcal{J}(x) \sim \phi^N(x)$, the mass dimension of a scalar (canonically normalized) field is 1. Then $[\mathcal{J}(x)] = N$. The only unknow dimension is the one of $F_{\mathcal{L}}(x)$ (knowing that $[d^4 x] = -4$ and $[d^4 p] = 4$)

$$\begin{aligned}
&\left[\int d^4 x \, \mathcal{J}(x) \int d^4 P \, \mathcal{G}(P)\right] = 2 + N - 4 = N - 2 \\
&\Rightarrow [F_{\mathcal{L}}(x)] = 2 - N.
\end{aligned} \tag{5.2.33}$$

So we can parametrize $F_{\mathcal{L}_1}(x)$ as $f(H, m_\phi)e^{iP_1 x}$ where P_1 is the momentum of the perturbation and $f(H, m_\phi)$ is a constant function, which has dimension $2-N$, depending on the characteristic scales of our de Sitter space: Hubble scale and the mass of the inflaton. Eq. (5.2.30) becomes

$$\begin{aligned}
\mathcal{A} :=& \langle \mathcal{G}, \phi^{(1)}[F_{\mathcal{L}_1'}] | \mathcal{G}, \phi^{(1)}[F_{\mathcal{L}_1}] \rangle = \\
=& N^2 \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \sum\!\!\!\!\!\!\!\int_{\mathcal{L}_1, \mathcal{L}_1'} \mathcal{G}^*(\mathcal{L}_1') \mathcal{G}(\mathcal{L}_1) \int \sigma(x,y) F^*_{\mathcal{L}_1'}(x) F_{\mathcal{L}_1}(y) \langle 0 | \phi(x) \phi(y) | 0 \rangle = \\
=& N^2 f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \omega(\mathbf{p}, \mathbf{q}) \langle 0 | a(\mathbf{p}) a^\dagger(\mathbf{q}) | 0 \rangle \int \mathrm{d}^4 P_1 \, \mathrm{d}^4 P_1' \, \mathcal{G}^*(P_1') \mathcal{G}(P_1) \times \\
& \times \int \sigma(x,y) e^{i(p - P_1')x} e^{-i(q - P_1)y} \Big|_{os} = \\
=& N^2 f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \omega(\mathbf{p}, \mathbf{q}) \langle 0 | a(\mathbf{p}) a^\dagger(\mathbf{q}) | 0 \rangle \int \mathrm{d}^4 P_1 \, \mathrm{d}^4 P_1' \, \mathcal{G}^*(P_1') \mathcal{G}(P_1) \times \\
& \times \delta^{(4)}(p - P_1') \delta^{(4)}(q - P_1) \Big|_{os} = \\
=& N^2 f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2 p^0(|\mathbf{p}|)} \int \mathrm{d}^4 P_1 \, \mathrm{d}^4 P_1' \, \mathcal{G}^*(P_1') \mathcal{G}(P_1) \times \\
& \times \delta^{(4)}(p - P_1') \delta^{(4)}(p - P_1) \Big|_{os} = \\
=& N^2 f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \frac{\mathrm{d}\Omega}{4\pi} \int_0^\infty \frac{\mathrm{d}|\mathbf{p}|}{|\mathbf{p}|} \, \mathcal{G}^*(p) \mathcal{G}(p) \Big|_{os} \frac{|\mathbf{p}|^3}{2\pi^2} \frac{1}{2 p^0(|\mathbf{p}|)}
\end{aligned} \tag{5.2.34}$$

The wavefunctions $\mathcal{G}(p)$ should reproduce the de Sitter power spectrum on superhorizon scales. We look then for a function which has the right dimensionality and which has an extremely compact support (peaked around $p = H$) such in order to get the standard de Sitter power spectrum:

$$\mathcal{G}(p) = \frac{1}{\pi m^2} \exp\left[-\frac{1}{2} \left(\frac{|\mathbf{p}| - H}{m} \right)^2 \right] \tag{5.2.35}$$

Hence

$$\mathcal{A} = \frac{N^2}{\pi^2 m_\phi^4} f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \frac{\mathrm{d}\Omega}{4\pi} \int_0^\infty \frac{\mathrm{d}|\mathbf{p}|}{|\mathbf{p}|} \exp\left[-\left(\frac{|\mathbf{p}| - H}{m} \right)^2 \right] \frac{|\mathbf{p}|^3}{2\pi^2} \frac{1}{2 p^0(|\mathbf{p}|)} \tag{5.2.36}$$

The Gaussian main contribution selects the values around $p = H$. We know that $m_\phi \ll H$ so we can expand the denominator $p^0 = \sqrt{|\mathbf{p}|^2 + m_\phi^2}$

$$\begin{aligned}
p^0 =& H \sqrt{1 + \left(\frac{m_\phi}{H} \right)^2} = \\
\approx& H \left(1 + \frac{1}{2} \left(\frac{m_\phi}{H} \right)^2 + \mathcal{O}\left(\left(\frac{m_\phi}{H} \right)^3 \right) \right)
\end{aligned} \tag{5.2.37}$$

Our final result is then

$$\begin{aligned}
\mathcal{A} \approx& \frac{N^2}{\pi^2 m_\phi^4} f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \frac{\mathrm{d}\Omega}{4\pi} \int_0^\infty \frac{\mathrm{d}|\mathbf{p}|}{|\mathbf{p}|} \frac{H^2}{2\pi^2} \frac{1}{2\left(1 + \frac{1}{2} \left(\frac{m_\phi}{H} \right)^2 \right)} = \\
\approx& \frac{N^2}{\pi^2 m_\phi^4} f^2(H, m_\phi) \langle \Omega | \theta^2(N-1, M) | \Omega \rangle \int \frac{\mathrm{d}\Omega}{4\pi} \int_0^\infty \frac{\mathrm{d}|\mathbf{p}|}{|\mathbf{p}|} \frac{H^2}{2\pi^2} \frac{1}{2} \left(1 - \frac{1}{2} \left(\frac{m_\phi}{H} \right)^2 \right)
\end{aligned} \tag{5.2.38}$$

Appendix A

Constant tensor mode

This appendix will prove that in all cases there is a tensor mode whose amplitude remains constant outside the horizon, even where some particles may have mean free times comparable to the Hubble time. The argument is based on the observation that for zero wave number the Newtonian gauge field equations and the dynamical equations for matter and radiation as well as the condition $k = 0$ are invariant under coordinate transformations that are not symmetries of the unperturbed metric. The most general such transformations are

$$x^0 \to x^0 + \epsilon(t), \tag{A.0.1}$$

$$x^i \to \left(\delta_{ij} - \frac{1}{2}\omega_{ij}\right) x^j, \tag{A.0.2}$$

where $H \equiv \frac{\dot{a}}{a}$, $\epsilon(t)$ is an arbitrary function of time and $\omega_{ij} = \omega_{ji}$ is an arbitrary constant matrix. under this conditions we have such a theorem: since the metric satisfies the field equations both before and after the transformation, the change in the metric under these transformations must also satisfy the field equations. The coordinate transformations (A.0.2) give

$$\begin{aligned} \tilde{g}_{00}(\tilde{x}^\mu) &= \frac{\partial x^\rho}{\partial \tilde{x}^0}\frac{\partial x^\sigma}{\partial \tilde{x}^0} g_{\rho\sigma}(x^\mu) = \\ &= \frac{\partial x^0}{\partial \tilde{x}^0}\frac{\partial x^0}{\partial \tilde{x}^0} g_{00}(x^\mu) = \\ &\approx g_{00}(x^\mu)\left(1 - \dot{\epsilon}(t)\right)^2 = \\ &\approx g_{00}(x^\mu) - 2\dot{\epsilon}(t) g_{00}(x^\mu) \quad \text{(linear order)} \end{aligned} \tag{A.0.3}$$

and

$$\begin{aligned} \tilde{g}_{ij}(\tilde{x}^\mu) &= \frac{\partial x^\rho}{\partial \tilde{x}^i}\frac{\partial x^\sigma}{\partial \tilde{x}^j} g_{\rho\sigma}(x^\mu) = \\ &= \frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j} g_{kl}(x^\mu) = \\ &\approx \left(\delta_{ik} + \frac{1}{2}\omega_{ik}\right)\left(\delta_{jl} + \frac{1}{2}\omega_{jl}\right) a^2(t)\delta_{kl} = \\ &\approx a^2(t)\left(\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{ik}\omega_{jl} + \frac{1}{2}\delta_{jl}\omega_{ik}\right)\delta_{kl} = \quad \text{(linear order)} \\ &= a^2(t)\left(\delta_{ik}\delta_{jk} + \frac{1}{2}\delta_{ik}\omega_{jk} + \frac{1}{2}\delta_{jk}\omega_{ik}\right) = \\ &= a^2(t)\left(\delta_{ij} + \frac{1}{2}\omega_{ji} + \frac{1}{2}\omega_{ij}\right) \overset{\omega_{ij}=\omega_{ji}}{=} a^2(t)\left(\delta_{ij} + \omega_{ij}\right). \end{aligned} \tag{A.0.4}$$

We have to expand also the argument of because an infinitesimal transformations is given by the Lie derivative of $g_{\mu\nu}$:

$$\begin{aligned} a^2(\tilde{t}) \equiv a^2\left(\tilde{x^0}\right) &= a^2\left(x^0 + \epsilon(t)\right) \\ &\approx a^2(t) + 2a(t)\dot{a}(t)\epsilon(t) = \\ &= a^2(t) + 2H(t)a^2(t)\epsilon(t). \end{aligned} \tag{A.0.5}$$

The change is simply

$$\delta g_{00} = \dot{\epsilon}(t), \tag{A.0.6}$$

$$\delta g_{i0} = 0, \tag{A.0.7}$$

$$\delta g_{ij} = a^2(t)\left[-H(t)\epsilon(t)\delta_{ij} + \omega_{ij}\right]. \tag{A.0.8}$$

This means that for zero wave number we always have a solution with scalar modes

$$\Psi = H\epsilon - \frac{\omega_{ii}}{3}, \qquad \Phi = -\dot{\epsilon} \tag{A.0.9}$$

and a tensor mode

$$h_{ij} = \omega_{ij} - \frac{1}{3}\delta_{ij}\omega_{kk}. \tag{A.0.10}$$

These are just gauge modes for zero wave number, but if they can be extended to non-zero wave number, they become physical modes, since (A.0.2) are not symmetries of the field equations except for zero wave number. For the scalar modes there are field equations that disappear in the limit of zero wave number, so that the conditions $\Phi = \Psi - 8\pi G\Pi_S$ and $\delta u = \epsilon$ (where Π_s is the scalar part of the anisotropic inertia and δu is the perturbation to the velocity potential) must be imposed on the solutions (A.0.9) for them to have an extension to non-zero wave number. It follows then that the zero wave number scalar modes that become physical for non-zero wave number satisfy

$$\dot{\epsilon} = -H\epsilon + \frac{\omega_{kk}}{3} - 8\pi G\Pi_S, \tag{A.0.11}$$

$$\delta u = \epsilon. \tag{A.0.12}$$

Then for zero wave number the quantity $\mathcal{R} \equiv -\Psi + H\delta u$ has the time-indipendent value

$$\mathcal{R} = \frac{\omega_{kk}}{3}. \tag{A.0.13}$$

For tensor modes there are no field equations that disappear for zero wave number, so the solution h_{ij} =const automatically has an extension to a physical mode for a non-zero wave number. The above theorem shows that this result applies even when some particle's mean free time is comparable with the Hubble time, in which case neither the hydrodynamic nor the free-streaming approximation are applicable.

The solution with $\dot{h}_{ij} = 0$ for zero wave number is not the only solution, but the other solutions decay rapidly after horizon exit. There is noanisotropic inertia in scalar field theories and in absence on anisotropic inertia

$$\ddot{h}_{ij} + \left(\frac{3\dot{a}}{a}\right)\dot{h}_{ij} - \left(\frac{\nabla^2}{a^2}\right)h_{ij} = 16\pi G\Pi_{ij} \tag{A.0.14}$$

for zero wave number has two solutions, one with h_{ij} a constant, and the other with $h_{ij} \propto a^{-3}$, for which h_{ij} rapidly becomes a constant.

Appendix B

Solution to Weinberg's gravitational waves equation

In [64] Weinberg gives an integrodifferential equation for the propagation of cosmological gravitational waves. In particular he writes an equation for the perturbation to the metric $h_{ij}(t, \mathbf{x})$ and then defines $\chi(u)$ as

$$h_{ij}(u) = h_{ij}(0)\chi(u), \tag{B.0.1}$$

where u is the conformal time multiplied by the wvenumber

$$u = k\int^{t} \frac{\mathrm{d}t'}{a(t')}. \tag{B.0.2}$$

$\chi(u)$ satisfies an integrodifferential equation ehich for short wavelengths (wavelengths which entered the horizon while the universe was still radiation dominated) is given by [64]

$$u^2\chi''(u) + 2u\chi'(u) + u^2\chi(u) = -24f_\nu(0)\int_0^u \mathrm{d}U(u-U)\chi'(U). \tag{B.0.3}$$

The fraction of the energy density in neutrinos is $f_\nu(0) = 0.40523$ and the kernel K will be discussed in detail below. The initial conditions are

$$\chi(0) = 1, \quad \chi'(0) = 0. \tag{B.0.4}$$

In the absence of free-streaming neutrinos the right-hand-side of (B.0.3) is zero and $\chi(u) = \frac{\sin u}{u}$. The suppression of these modes is due to the presence of the neutrinos where the solutions of (B.0.3) approaches, for $u >> 1$,

$$\chi(u) \longrightarrow A\frac{\sin(u+\delta)}{u}, \tag{B.0.5}$$

and the value of A^2 is the quantitative measure of that suppression. Our aim is to provide an analytic solution od (B.0.3) and (B.0.4).

A solution to Eq. (B.0.3) is a series of spherical Bessel functions [59]

$$\chi(u) = \sum_{n=0}^{\infty} a_n j_n(u). \tag{B.0.6}$$

Inserting Eq. (B.0.6) in the left-hand-side of Eq. (B.0.3) and using the differential equation for spherical Bessel functions leaves

$$\sum_{n=0}^{\infty} n(n+1)a_n j_n(u). \tag{B.0.7}$$

The RHS of Eq. (B.0.3) requires more work. The kernel is itslef a sum of spherical Bessel functions

$$\begin{aligned} K(u) &= \frac{1}{16}\int_{-1}^{1} \mathrm{d}x \left(1-x^2\right) e^{ixu} = \\ &= -\frac{\sin u}{u^3} - \frac{3\cos u}{u^4} + \frac{3\sin u}{u^5} = \\ &= \frac{1}{15}\left(j_0(u) + \frac{10}{27} j_2(u) + \frac{3}{7} j_4(u)\right), \end{aligned} \tag{B.0.8}$$

and the derivative of $\chi(u)$ is given by

$$\begin{aligned} \chi'(u) &= \sum_{n=0}^{\infty} a_n j_n' = \\ &= \sum_{n=0}^{\infty} a_n \frac{\left[n j_{n-1}(u) - (n+1) j_{n+1}(u)\right]}{(2n+1)}. \end{aligned} \tag{B.0.9}$$

So the RHS of Eq. (B.0.3) is $-CI(u)0$ with $C = \frac{24}{15} f_\nu(0) = 0.648368$ and $I(u)$ is given by

$$I(u) = \sum_{m=0,2,4} d_m \sum_{n=0}^{\infty} \frac{a_n}{(2n+1)} I_{n,m}(u) \tag{B.0.10}$$

where

$$I_{n,m}(u) = \int_0^u \mathrm{d}U j_m(u-U)\left[n j_{n-1}(U) - (n+1) j_{n+1}(U)\right]. \tag{B.0.11}$$

The d_m are given by Eq. (B.0.8) where we have factored out a 1/15. Let's evaluate $I_{n,m}$: first, use the fact that the Fourier transform of a Legendre polynomial is a spherical Bessel function (Eq. 10.1.14 in [1])

$$j_n(x) = \frac{(-i)^n}{2} \int_{-1}^{1} \mathrm{d}s\, e^{ixs} P_n(s) \tag{B.0.12}$$

to replace both Bessel functions in Eq. (B.0.11).

$$\begin{aligned} I_{m,n}(u) = &\int_0^u \mathrm{d}U \left\{ \int_{-1}^{1} \frac{(-i)^m}{2} \mathrm{d}s\, e^{i(u-U)s} P_m(s) \left[n \int_{-1}^{1} \frac{(-i)^{n-1}}{2} \mathrm{d}t\, e^{iUt} P_{n-1}(t) \right.\right. + \\ &\left.\left. -(n+1) \int_{-1}^{1} \frac{(-i)^{n+1}}{2} \mathrm{d}t\, e^{iUt} P_{n+1}(t) \right] \right\} = \\ = &\int_0^u \mathrm{d}U \left\{ \int_{-1}^{1} \frac{(-i)^m}{2} \mathrm{d}s\, e^{i(u-U)s} P_m(s) \int_{-1}^{1} \mathrm{d}t\, e^{iUt} \left[\frac{(-i)^{n-1}}{2} n P_{n-1}(t) \right.\right. + \\ &\left.\left. -\frac{(-i)^{n+1}}{2} (n+1) P_{n+1}(t) \right] \right\} \end{aligned} \tag{B.0.13}$$

This makes the integral over U trivial and we have

$$\begin{aligned} \int_0^u \mathrm{d}U\, e^{-iUs} e^{iUt} &= \int_0^u \mathrm{d}U\, e^{iU(t-s)} = \\ &= \frac{e^{iu(t-s)} - 1}{i(t-s)} = (-i)\frac{e^{iu(t-s)} - 1}{t-s} = \\ &= (-i)\frac{e^{itu} - e^{isu}}{e^{isu}(t-s)}, \end{aligned} \tag{B.0.14}$$

Hence, we get

$$I_{n,m}(u) = \frac{(-i)^{n+m}}{4} \int_{-1}^{1} \mathrm{d}s \int_{-1}^{1} \mathrm{d}t \, \frac{e^{itu} - e^{isu}}{t - s} P_m(s) \left[n P_{n-1}(t) + (n+1) P_{n+1}(t) \right]. \tag{B.0.15}$$

Now use the definition od the Legendre function of the second kind (8.8.3 in [1]),

$$Q_n(z) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}x (z - x)^{-1} P_n(x), \tag{B.0.16}$$

to evaluate the integral in Eq. (B.0.15) over the variable that does not appear in the exponent (s in the first term, t in the second) to obtain

$$I_{n,m}(u) = \frac{(-i)^{n+m}}{2} \int_{-1}^{1} \mathrm{d}t \, e^{itu} \left\{ Q_m(t) \left[n P_{n-1}(t) + (n+1) P_{n+1}(t) \right] + P_m(t) \left[n Q_{n-1}(t) + (n+1) Q_{n+1}(t) \right] \right\}. \tag{B.0.17}$$

Next, by replacing the remaining exponential with the familiar expression from quantum mechanical scattering,

$$e^{itu} = \sum_l (2l+1) i^l j_l(u) P_l(t), \tag{B.0.18}$$

the expressione for $I_{n,m}(u)$ becomes

$$I_{n,m}(u) = \sum_l \frac{(2l+1)}{2} (-i)^{n+m-l} j_l(u) \int_{-1}^{1} \mathrm{d}t \, P_l(t) \left\{ Q_m(t) \left[n P_{n-1}(t) + (n+1) P_{n+1}(t) \right] + \right. \\ \left. + P_m(t) \left[n Q_{n-1}(t) + (n+1) Q_{n+1}(t) \right] \right\}. \tag{B.0.19}$$

Eq. (B.0.19) can be simplified using (8.6.19 in [1]),

$$Q_m(x) = \frac{1}{2} P_m(x) \ln \frac{1+x}{1-x} - W_{m-1}(x), \tag{B.0.20}$$

where

$$W_{m-1}(x) = \sum_{k=0}^{\frac{m-1}{2}} \frac{2m - 4k - 1}{(2k+1)(m-k)} P_{m-2k-1}(x), \tag{B.0.21}$$

and the formula

$$P_l(x) P_m(x) = \sum_{L=|l-m|}^{l+m} \left| \langle l, 0, m, 0 | L, 0 \rangle \right|^2 P_L(x) \tag{B.0.22}$$

to express $P_l(x)P_m(x)$ in terms of $P_L(x)$'s and, with the aid of Eq. (B.0.20), $P_l(x)Q_m(x)$ in terms of $Q_L(x)$'s as

$$P_l(x) Q_m(x) = \sum_{L=|l-m|}^{l+m} \left[\left| \langle l, 0, m, 0 | L, 0 \rangle \right|^2 \left(Q_L(x) + W_{L-1}(x) \right) \right] - P_l(x) W_{m-1}(x). \tag{B.0.23}$$

Finally, the terms in Eq. (B.0.19) invlving the products of $P_n(x)$'s and $Q_m(x)$'s cancel using (8.14.10 in [1]),

$$\int_{-1}^{1} \mathrm{d}x \left(Q_L(x) P_{m\pm1}(x) + P_l(x) Q_{m\pm1} \right) = 0, \tag{B.0.24}$$

and $I_{n,m}(u)$ reduces to

$$I_{n,m}(u) = \sum_{l} \frac{(2l+1)}{2}(-i)^{n+m-l} j_l(u) \Bigg\{ \int_{-1}^{1} \mathrm{d}t \Bigg(\sum_{L=|l-m|}^{l+m} \Big[\big|\langle l,0,m,0|L,0\rangle\big|^2 \big(Q_L(t) + W_{L-1}(t)\big)\Big] +$$

$$- P_l(t) W_{m-1}(t) \Bigg) \big[nP_{n-1}(t) + (n+1)P_{n+1}(t)\big] + \int_{-1}^{1} \mathrm{d}t \Bigg(\sum_{L=|l-m|}^{l+m} \big|\langle l,0,m,0|L,0\rangle\big|^2 P_L(x) \Bigg) \times$$

$$\times \big[nQ_{n-1}(t) + (n+1)Q_{n+1}(t)\big] \Bigg\} =$$

$$= \sum_{l} \frac{(2l+1)}{2}(-i)^{n+m-l} j_l(u) \Bigg\{ \int_{-1}^{1} \mathrm{d}t \sum_{L=|l-m|}^{l+m} \big|\langle l,0,m,0|L,0\rangle\big|^2 W_{L-1} \big[nP_{n-1}(t) + (n+1)P_{n+1}(t)\big] +$$

$$- \int_{-1}^{1} \mathrm{d}t P_l(t) W_{m-1}(t) \big[nP_{n-1}(t) + (n+1)P_{n+1}(t)\big] \Bigg\}. \tag{B.0.25}$$

The contributions to the coefficient od each $j_l(u)$ in Eq. (B.0.25) can be straightforwardly evaluated; those in the sum by directly using orthogonality and the remaining terms by expresing the product of two P_l's as a sum of P_l's and again using orthogonality. The orthogonality of the Legendre functions means that the l which is summed over in Eq. (B.0.25) can only take on the values $n+2k$ where $k = 0, 1, 2, \ldots$ so we replace

$$\sum_{m=0,2,4} \frac{d_m}{2n+1} I_{n,m}(u) \tag{B.0.26}$$

with

$$\sum_{k=0}^{\infty} c_{n,k} j_{n+2k}(u), \tag{B.0.27}$$

i.e., the sum over l in $I_{n,m}(u)$ is replaced by a sum over k and each $c_{n,k}$ is the sum of the contributions from the three terms in the kernel, $m = 0, 2, 4$. Setting Eq. (B.0.7) equal to $-CI(u)$ we have

$$sum_{n=0}^{\infty} n(n+1) a_n j_n(u) = -C \sum_{n,k=0}^{\infty} a_n c_{n,k} j_{n+2k}(u) \tag{B.0.28}$$

where the $c_{n,k}$are known numbers and we can find the expansion coefficients, a_n, recursively bu equating the coefficients of each rder Bessel function in Eq. (B.0.28).

The coefficients of $j_1(u)$in Eq. (B.0.28) give

$$2a - 1 = -Cc_{1,0} a_1, \tag{B.0.29}$$

where $c_{n,0}$ is equal to $1 - \delta_{n,0}$ so $c_{1,0}$ is 1. The only solution of this equation is $a_1 = 0$. This ensures that the second of the intali conditions, Eq. (B.0.4), is satisfied. The equality of the coefficient of $j_3(u)$ shows that a_3 is proportional to a_1. Similarly a_5 is a linear combination of a_1 and a_3, a_7 a linear combination of a_1, a_3 and a_5, ... Thus the coefficients of all the odd order Bessel functions in Eq. (B.0.6) are zero. There is no mixing between the coefficients of the odd order Bessel functions and those of even order because the Clebsch-Gordan coefficients

$\langle a, 0, b, 0 | c, 0 \rangle$ are zero if $a + b + c$ is an odd number.

n	$c_{n,0}$	$c_{n,2}$	$c_{n,4}$	$c_{n,6}$	$c_{n,8}$
0	0	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{13}{60}$	$\frac{17}{735}$
2	1	$-\frac{3}{4}$	$-\frac{13}{21}$	$-\frac{17}{168}$	$\frac{1}{84}$
4	1	$-\frac{143}{210}$	$-\frac{221}{420}$	$-\frac{41}{504}$	$\frac{425}{45738}$
6	1	$-\frac{255}{392}$	$-\frac{17}{35}$	$-\frac{93575}{1280664}$	
8	1	$-\frac{19}{30}$	$-\frac{25}{54}$		

Thus the only non-zero a_n in Eq. (B.0.6) are those with an even n. a_0 doesn't appear in Eq. (B.0.28) but is determined by the first of the initial consitions of Eq. (B.0.4), which fixes it to be unity. The $c_{n,k}$ necarry to find $a_2, \ldots, a_8$ are shown in Table reftab. The equations for these a_n can be read off from Eq. (B.0.28). Using $a_0 = 1$, we have

$$a_2 = -C \frac{c_{0,2}}{6 + C} \tag{B.0.30}$$

$$a_4 = -C \frac{c_{0,4} + a_2 c_{2,2}}{20 + C} \tag{B.0.31}$$

$$\vdots \tag{B.0.32}$$

$$a_{2n} = -\frac{C \sum_{k=0}^{n-1} a_{2k} c_{2k,2n-2k}}{2n(2n+1) + C} \tag{B.0.33}$$

where $C = 1.6f$, which we used in Chapter 2. For large argument, all of the even order Bessel functions go as $\pm \frac{\sin x}{x}$ so the A in Eq. (B.0.5) is

$$A = \sum_{n=0}^{5} (-1)^n a_{2n} \tag{B.0.34}$$

Since there no odd order Bessel functions in the expansion, the hase δ in Eq. (B.0.5) is zero.

Appendix C

Particle density operator

In this appendix, our goal will be to represent the particle number density operator in terms of the fields instead of creation and annihilation operators. The next step will be verifying that the number operator observable is giving the same result both with the fields language and the creation/annihilation operator language, that is

$$\langle\psi|\hat{N}(k)|\psi\rangle \sim N\left|\psi(k)\right|^2, \quad \text{for}\,|\psi\rangle\,\text{an}\,N\text{-particle state.} \tag{C.0.1}$$

Our object of interest, expressed in terms of creation and annihilation operators, shall be:

$$\hat{n}(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}). \tag{C.0.2}$$

To check that this is indeed a number density operator we can apply it to a simple N- free particle state $|\psi\rangle$. The expectation value gives the number density with respect to the measure $\frac{\mathrm{d}^3 k}{(2\pi)^3 2E_k}$. It can be written as

$$\langle\psi|a^\dagger(\mathbf{k})a(\mathbf{k})|\psi\rangle = \frac{N}{N!}\int\prod_{j=1}^{N-1}\frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_{p_j}}\left|\psi(\mathbf{p}_1,\ldots,\mathbf{p}_{N-1},\mathbf{k})\right|^2, \tag{C.0.3}$$

which is to be understood as an expansion of the physical state $|\psi\rangle$ in terms of the Fock spae basis $\left\{|\mathbf{p}_1,\ldots,\mathbf{p}_N\rangle\right\}$ and the identity operator

$$\mathbb{1} = \frac{1}{N!}\int\prod_{j=1}^{N}\frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_{p_j}}|\mathbf{p}_1,\ldots,\mathbf{p}_N\rangle\langle\mathbf{p}_1,\ldots,\mathbf{p}_N|, \tag{C.0.4}$$

so, (C.0.3), integrated with respect to that measure, gives

$$\langle\psi|\int\frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_{p_j}}a^\dagger(\mathbf{k})a(\mathbf{k})|\psi\rangle = N\langle\psi|\psi\rangle. \tag{C.0.5}$$

The main goal of this appendix is to have both, (C.0.2) and the Fock basis be constructed out of the corresponding field, like

$$n(\mathbf{k}) \sim f(\mathbf{k})\phi\phi, \qquad |\mathbf{p}_1,\ldots,\mathbf{p}_N\rangle \sim g(\mathbf{p}_1,\ldots,\mathbf{p}_N)\phi^N|0\rangle \tag{C.0.6}$$

and have Lorentz invariance intact, density operator remain hermitian, and thus retain its status of a physical observable.

We begin by taking (C.0.2) and simply plugging in [1]

$$a_{\mathbf{p}} = 2E \int \mathrm{d}^3 x \, e^{i\mathbf{px}} \phi(t, \mathbf{x}). \tag{C.0.7}$$

For shortness reasons let us set the time to $t = 0$ in the previous equation. Thus we obtain

$$\begin{aligned} a^\dagger(\mathbf{k}) a(\mathbf{k}) &= \left(2E \int \mathrm{d}^3 x \, e^{i\mathbf{px}} \phi(0, \mathbf{x})\right)^\dagger 2E \int \mathrm{d}^3 y \, e^{i\mathbf{py}} \phi(0, \mathbf{y}) = \\ &= (2E_k)^2 \int \mathrm{d}^3 x \, \mathrm{d}^3 y \, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \phi(0, \mathbf{x}) \phi(0, \mathbf{y}), \end{aligned} \tag{C.0.8}$$

where the hermitian conjugate doesn't have any effect on ϕ because we took it as a real scalar field. Next come the Fock basis states. Here we also just plug in (C.0.7) and taking the hermitian conjugate:

$$|p\rangle = 2E \int \mathrm{d}^3 x \, e^{i\mathbf{px}} \phi(0, \mathbf{x}|0\rangle. \tag{C.0.9}$$

Furthermore, in these integrals, Eq. (C.0.8) as well as Eq. (C.0.9), are Lorentz invariant. [2]

Now, in order to calculate expectation values we would like to use a method that is the most efficient if one is dealing with fields. For inspiration, let's think of standard treatment of product of fields, namely perturbative S-matrix calculations: *Wick's theorem.* But to use it we don not only need to somehow introduce a time ordering operator T, but also make everything depend not on 3-vector but on 4-vector spacetime points.

Let's insert a 1 using Dirac delta and Heaviside theta functions into (C.0.8):

$$\begin{aligned} &\int \mathrm{d}^3 x \, \mathrm{d}^3 y \, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left[\int_{-\infty}^{\infty} \mathrm{d}x^0 \int_{-\infty}^{\infty} \mathrm{d}y^0 \, \delta(x^0)\delta(y^0) \left(\theta(x^0 - y^0) + \theta(y^0 - x^0)\right)\right] \phi(0, \mathbf{x})\phi(0, \mathbf{y}) = \\ &= \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \delta(x^0)\delta(y^0) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left[\theta(x^0 - y^0) + \theta(y^0 - x^0)\right] \phi(0, \mathbf{x})\phi(0, \mathbf{y}) = \\ &= \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \delta(x^0)\delta(y^0) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left[\theta(x^0 - y^0)\phi(0, \mathbf{x})\phi(0, \mathbf{y}) + \theta(y^0 - x^0)\phi(0, \mathbf{x})\phi(0, \mathbf{y})\right] = \\ &= \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \delta(x^0)\delta(y^0) e^{-ik(x-y)} \left[\theta(x^0 - y^0)\phi(x^0, \mathbf{x})\phi(y^0, \mathbf{y}) + \theta(y^0 - x^0)\phi(x^0, \mathbf{x})\phi(y^0, \mathbf{y})\right] = \\ &= \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \delta(x^0)\delta(y^0) e^{-ik(x-y)} T[\phi(x)\phi(y)]. \end{aligned} \tag{C.0.10}$$

where we used that bosonic fields commute at equal times and the Minkowski product in the exponent as $k(x - y) = E_k(x^0 - y^0) - \mathbf{k}(\mathbf{x} - \mathbf{y})$. So all in all we have

$$n(\mathbf{k}) = (2E_k)^2 \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \delta(x^0)\delta(y^0) e^{-ik(x-y)} T[\phi(x)\phi(y)]. \tag{C.0.11}$$

Eq. (C.0.7) with a Minkowski measure integral becomes

$$a_p = 2E_p \int \mathrm{d}^4 x \, \delta(x^0) e^{ipx} \phi(x), \tag{C.0.12}$$

[1] The expression of the annihilation opertator in terms of the respective scalar field can be obtained with a bit of manipulation using the conjugate momentun and the equation of motion for a free scalar field and then introducing the symplectic scalar product: $\left\langle e^{-ipx}, \phi(t, \mathbf{x})\right\rangle := i \int \mathrm{d}^3 x \left[(e^{-ipx})^* \overleftrightarrow{\partial} \phi(t, \mathbf{x})\right]$.

[2] To show it we just to insert a 1 with $1 = \int_{-\infty}^{\infty} \mathrm{d}x^0 \delta(x^0 - t)$ together with a timelike 4-vector $n := (1, 0, 0, 0)$

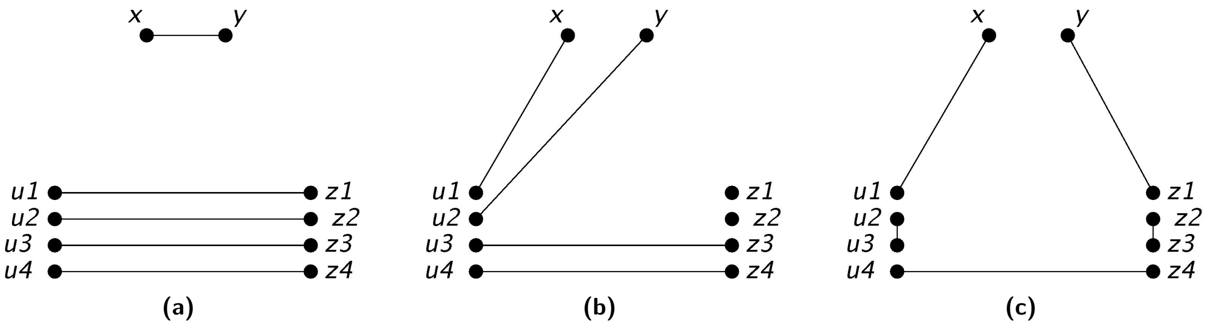


Figure C.1: Disconnected diagrams for the case $N = 4$; they feature contractions that lead to contributions which are irrelevant for particle number measurements. Spacetime points assign the field postion for the Fock basis state: $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\rangle \sim \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)|0\rangle$ (and respectively for u_i).

and we can now start calculating expectation values using Wick contractions. Let's consider a 1-free-particle state $|\psi\rangle_1 := |\psi\rangle$ and expand it in the Hilbert space basis

$$|\psi\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \psi(\mathbf{p})|\mathbf{p}\rangle, \tag{C.0.13}$$

and calculate the matrix elements of (C.0.11):

$$\begin{aligned}
\langle \mathbf{p}'|n(\mathbf{k})|\mathbf{p}\rangle =& (2E_k)^2 \int \mathrm{d}^4 x\, \mathrm{d}^4 y\, \delta(x^0)\delta(y^0) e^{-ik(x-y)} \langle \mathbf{p}'|T[\phi(x)\phi(y)]|\mathbf{p}\rangle = \\
=& (2E_k)^2 \int \mathrm{d}^4 x\, \mathrm{d}^4 y\, \delta(x^0)\delta(y^0) e^{-ik(x-y)} \langle 0|a_{\mathbf{p}'} T[\phi(x)\phi(y)] a^\dagger_{\mathbf{p}}|0\rangle = \\
=& (2E_k)^2 \int \mathrm{d}^4 x\, \mathrm{d}^4 y\, \delta(x^0)\delta(y^0) e^{-ik(x-y)} \langle 0|T\left[a_{\mathbf{p}'}\phi(x)\phi(y)a^\dagger_{\mathbf{p}}\right]|0\rangle = \\
=& (2E_k)^2 (2E_{p'})(2E_p) \int \mathrm{d}^4 x\, \mathrm{d}^4 y\, \mathrm{d}^4 z\, \mathrm{d}^4 u\, \delta(x^0)\delta(y^0)\delta(z^0)\delta(u^0) e^{-ik(x-y)} e^{-ipz+ip'u} \cdot \\
& \cdot \langle 0|T\left[\phi(u)\phi(x)\phi(y)\phi(z)\right]|0\rangle.
\end{aligned} \tag{C.0.14}$$

In the vacuum expectation value, we now use Wick's theorem and perform all physical contractions that give a non-vanishing contribution. Before we continue, let's point out which contractions we refer to: since the expectation value of the particle density operator is supposed to be a measurement of the incoming particle, terms containing $\overbrace{\phi(x)\phi(y)}$ or, if $N \geq 2$, contractions between fields that generate the "to-be-measured" state, would either be interactions that have nothing to do with density measurement, or would just count internal interactions of state-particles between each other before or after the measurement, respectively. Picturing the contractions as Feynman diagrams both would be disconnected diagrams. Therefore, we are actually interested in

$$n_{phys}(\mathbf{k}) := n(\mathbf{k}) - \text{disconnected diagrams}, \tag{C.0.15}$$

we just won't it as *phys* from hereon anymore. Practically, this means that the contraction rule for us is: **skip all terms containing contractions of the form**

$$\langle 0|T[\phi(u_N)\dots\phi(u_1)\overbrace{\phi(x)\phi(y)}\phi(z_1)\dots\phi(z_N)]|0\rangle, \tag{C.0.16}$$

$$\langle 0|T[\phi(u_N)\dots\phi(u_1)\phi(x)\phi(y)\phi(z_1)\dots\overbrace{\phi(z_l)\dots\phi(z_m)}\dots\phi(z_N)]|0\rangle, \quad \forall\, 1 \leq l, m \leq N, \tag{C.0.17}$$

$$\langle 0|T[\phi(u_N)\dots\overbrace{\phi(u_i)\dots\phi(u_j)}\dots\phi(u_1)\phi(x)\phi(y)\phi(z_1)\dots\phi(z_N)]|0\rangle, \quad \forall\, 1 \leq i, j \leq N, \tag{C.0.18}$$

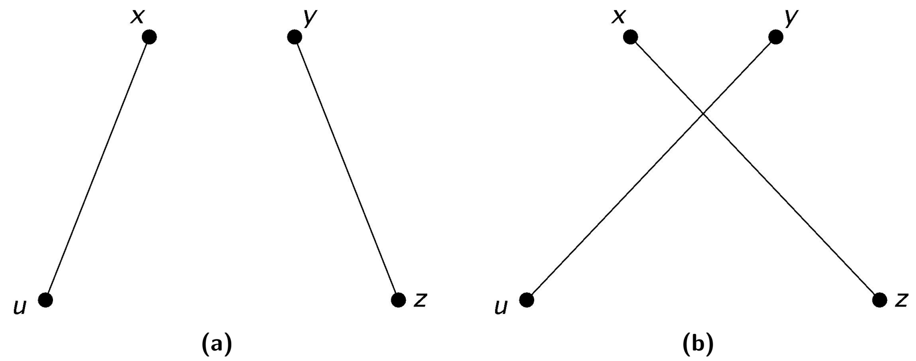


Figure C.2: The only physical diagrams contributing to $N = 1$ state particle number density measurement.

Diagramatically these contractions can be pictures as shown in Fig. C.1 Therefore, excluding those ones, our case of $N = 1$ leaves us with

$$\phi(u)\phi(x)\phi(y)\phi(z) + \phi(u)\phi(x)\phi(y)\phi(z) \tag{C.0.19}$$

for the physical ones (see Fig. C.2). Now we go back to the matrix elements: continuing (C.0.14)

$$\begin{aligned}
\langle \mathbf{p}'|n(\mathbf{k})|\mathbf{p}\rangle =&(2E_k)^2(2E_{p'})(2E_p)\int \mathrm{d}^4 x\,\mathrm{d}^4 y\,\mathrm{d}^4 z\,\mathrm{d}^4 u\,\delta(x^0)\delta(y^0)\delta(z^0)\delta(u^0)e^{-ik(x-y)}e^{-ipz+ip'u}\cdot\\
&\cdot\left[iS(u,x)S(y,z)+iS(u,y)S(x,z)\right]=\\
=&-(2E_k)^2(2E_{p'})(2E_p)\int \mathrm{d}^4 x\,\mathrm{d}^4 y\,\mathrm{d}^4 z\,\mathrm{d}^4 u\,\delta(x^0)\delta(y^0)\delta(z^0)\delta(u^0)e^{-ik(x-y)}e^{-ipz+ip'u}\cdot\\
&\cdot\left[\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{d}^4 q'}{(2\pi)^4}\frac{e^{-iq(u-x)}}{q^2-m^2+i\epsilon}\frac{e^{-iq'(y-z)}}{q'^2-m^2+i\epsilon}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{d}^4 q'}{(2\pi)^4}\frac{e^{-iq(u-y)}}{q^2-m^2+i\epsilon}\frac{e^{-iq'(x-z)}}{q'^2-m^2+i\epsilon}\right]=\\
=&-(2E_k)^2(2E_{p'})(2E_p)\int \mathrm{d}^3 x\,\mathrm{d}^3 y\,\mathrm{d}^3 z\,\mathrm{d}^3 u\,\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{d}^4 q'}{(2\pi)^4}\cdot\\
&\cdot\left[\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}e^{i\mathbf{q}(\mathbf{u}-\mathbf{x})}}{q^2-m^2+i\epsilon}\frac{e^{i\mathbf{p}\mathbf{z}-i\mathbf{p}'\mathbf{u}}e^{i\mathbf{q}'(\mathbf{y}-\mathbf{z})}}{q'^2-m^2+i\epsilon}+\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}e^{i\mathbf{q}(\mathbf{u}-\mathbf{y})}}{q^2-m^2+i\epsilon}\frac{e^{i\mathbf{p}\mathbf{z}-i\mathbf{p}'\mathbf{u}}e^{i\mathbf{q}'(\mathbf{y}-\mathbf{z})}}{q'^2-m^2+i\epsilon}\right]=
\end{aligned}$$

$$
\begin{aligned}
=&-(2E_k)^2(2E_{p'})(2E_p)\int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{d}^3z\,\mathrm{d}^3u\,\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4q'}{(2\pi)^4}\cdot\\
&\cdot\left[\frac{e^{i(\mathbf{k}-\mathbf{q})\mathbf{x}}e^{i(\mathbf{q}'-\mathbf{k})\mathbf{y}}}{q^2-m^2+i\epsilon}\frac{e^{i(\mathbf{p}-\mathbf{q}'\mathbf{z}}e^{i(\mathbf{q}-\mathbf{p}')\mathbf{u}}}{q'^2-m^2+i\epsilon}+\frac{e^{i(\mathbf{k}+\mathbf{q}')\mathbf{x}}e^{-i(\mathbf{k}+\mathbf{q})\mathbf{y}}}{q^2-m^2+i\epsilon}\frac{e^{i(\mathbf{p}-\mathbf{q})'\mathbf{z}}e^{i(\mathbf{q}-\mathbf{p}')\mathbf{u}}}{q'^2-m^2+i\epsilon}\right]=\\
=&-(2E_k)^2(2E_{p'})(2E_p)\int \mathrm{d}^4q\,\mathrm{d}^4q'\left[\frac{\delta^{(3)}(\mathbf{k}-\mathbf{q})\delta^{(3)}(\mathbf{q}'-\mathbf{k})}{(q^0)^2-\mathbf{q}^2-m^2+i\epsilon}\frac{\delta^{(3)}(\mathbf{p}-\mathbf{q}')\delta^{(3)}(\mathbf{q}-\mathbf{p}')}{(q'^0)^2-\mathbf{q}'^2-m^2+i\epsilon}+\right.\\
&\left.+\frac{\delta^{(3)}(\mathbf{k}+\mathbf{q}')\delta^{(3)}(\mathbf{k}+\mathbf{q})}{(q^0)^2-\mathbf{q}^2-m^2+i\epsilon}\frac{\delta^{(3)}(\mathbf{p}-\mathbf{q}')\delta^{(3)}(\mathbf{q}-\mathbf{p}')}{(q'^0)^2-\mathbf{q}'^2-m^2+i\epsilon}\right]=\\
=&-(2E_k)^2(2E_{p'})(2E_p)(2\pi)^4\int \mathrm{d}q^0\,\mathrm{d}q'^0\left[\frac{1}{(q^0)^2-\mathbf{k}^2-m^2+i\epsilon}\frac{\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta^{(3)}(\mathbf{k}-\mathbf{p}')}{(q'^0)^2-\mathbf{k}^2-m^2+i\epsilon}+\right.\\
&\left.+\frac{1}{(q^0)^2-(-\mathbf{k})^2-m^2+i\epsilon}\frac{\delta^{(3)}(\mathbf{p}+\mathbf{k})\delta^{(3)}(-\mathbf{k}-\mathbf{p}')}{(q'^0)^2-(-\mathbf{k})^2-m^2+i\epsilon}\right]=\\
=&-(2E_k)^2(2E_{p'})(2E_p)(2\pi)^4\left[\frac{\pi}{\sqrt{-E_k^2+i\epsilon}}\frac{\pi\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta^{(3)}(\mathbf{k}-\mathbf{p}')}{\sqrt{-E_k^2+i\epsilon}}+\right.\\
&\left.+\frac{\pi}{(\sqrt{-E_k^2+i\epsilon}}\frac{\pi\delta^{(3)}(\mathbf{p}+\mathbf{k})\delta^{(3)}(-\mathbf{k}-\mathbf{p}')}{\sqrt{-E_k^2+i\epsilon}}\right]=\\
\xrightarrow{\epsilon\to 0}&(2\pi)^2(2E_{p'})(2E_p)(2\pi)^4\left[\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta^{(3)}(\mathbf{k}-\mathbf{p}')+\delta^{(3)}(\mathbf{p}+\mathbf{k})\delta^{(3)}(-\mathbf{k}-\mathbf{p}')\right],
\end{aligned}
$$

where, for the $\mathrm{d}q^0$ integration we used

$$
\int_{-\infty}^{\infty}\mathrm{d}q\,\frac{1}{q^2-E^2+i\epsilon}=\frac{\pi}{\sqrt{-E^2+i\epsilon}}. \tag{C.0.20}
$$

Finally we can get the expectation value of $n(\mathbf{k})$ for the state (C.0.13):

$$
\begin{aligned}
\langle\psi|n(\mathbf{k})|\psi\rangle=&\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E_{p'}}\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\,\psi^*(\mathbf{p}')\psi(\mathbf{p})\langle\mathbf{p}'|n(\mathbf{k})|\mathbf{p}\rangle=\\
=&\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E_{p'}}\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\,\psi^*(\mathbf{p}')\psi(\mathbf{p})\cdot\\
&\cdot(2\pi)^2(2E_{p'})(2E_p)(2\pi)^4\left[\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta^{(3)}(\mathbf{k}-\mathbf{p}')+\delta^{(3)}(\mathbf{p}+\mathbf{k})\delta^{(3)}(-\mathbf{k}-\mathbf{p}')\right]=\\
=&\psi^*(\mathbf{k})\psi(\mathbf{k})+\psi^*(-\mathbf{k})\psi(-\mathbf{k})
\end{aligned} \tag{C.0.21}
$$

Let's compare this result to (C.0.3) for $N=1$. With our field method we got an extra summand with $-\mathbf{k}$., where this second summand comes from the second possibility of contraction in (C.0.19). When dealing with the annihilation/creation operator version the second contraction term would only give a vanishing contribution, as it would involve the commutators of annihilation-annihilation/creation-creation operators:

$$
\contraction{}{a_{\mathbf{p}'}}{}{a^\dagger_{\mathbf{k}}}a_{\mathbf{p}'}a^\dagger_{\mathbf{k}}\contraction{}{a_{\mathbf{k}}}{}{a^\dagger_{\mathbf{p}}}a_{\mathbf{k}}a^\dagger_{\mathbf{p}}+\contraction{}{a_{\mathbf{p}'}}{a^\dagger_{\mathbf{k}}}{a_{\mathbf{k}}}\contraction[2ex]{a_{\mathbf{p}'}}{a^\dagger_{\mathbf{k}}}{a_{\mathbf{k}}}{a^\dagger_{\mathbf{p}}}a_{\mathbf{p}'}a^\dagger_{\mathbf{k}}a_{\mathbf{k}}a^\dagger_{\mathbf{p}} \tag{C.0.22}
$$

With fields a contraction is not equal to the commutator, therefore it gives a non-vanishing contribution. If we integrate the particle number (C.0.21) with measure $\frac{\mathrm{d}^3 k}{(2\pi)^3 2E_k}$, however, the second term will give the same contribution of because the measure is invariant under sign-flip transformations. So our new density counts twice as many particles. Overall, as the possibility to contract with either $\phi(x)$ or $\phi(y)$ exists for any number of fields operator that come the "to-be-measured" particle state, tha same thing will happen for all N, i.e. we will get twice as many particles for all N. So the proper number density operator in the field form should be divided by two. Therefore, at the end of our fine tuning for the operator we get:

$$n(\mathbf{k}) \mapsto \frac{(2E_k)^2}{2} \int \mathrm{d}^3 x \, \mathrm{d}^3 y \, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \phi(0,\mathbf{x})\phi(0,\mathbf{y}). \tag{C.0.23}$$

We shall give the result for the N-particle-generalized stare $|\psi\rangle$ now. Just as in $N = 1$ in (C.0.13), we expand the state in its wavefunction

$$|\psi\rangle = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_{p_i}} \psi(\mathbf{p}_1, \ldots, \mathbf{p}_N)|\mathbf{p}_1, \ldots, \mathbf{p}_N\rangle. \tag{C.0.24}$$

Using the contraction rules stated above, performing the calculation exctly as presented for $N = 1$ and remembering that wavefunctions for bosons are symmetric, we have

$$\langle\psi|n(\mathbf{k})|\psi\rangle = \frac{N}{2N!} \int \prod_{i=1}^{N-1} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_{p_i}} \left(\left|\psi(\mathbf{p}_1, \ldots, \mathbf{p}_{N-1}, \mathbf{k})\right|^2 + \left|\psi(\mathbf{p}_1, \ldots, \mathbf{p}_{N-1}, -\mathbf{k})\right|^2 \right) \tag{C.0.25}$$

for a N-particle state. Then, the resulting particle number, i.e., the integrated particle density expectation expectation value, is indeed equal to the right hand side (RHS) of (C.0.5)

$$\langle\psi|n(\mathbf{k})|\psi\rangle = N\langle\psi|\psi\rangle. \tag{C.0.26}$$

We have one last remark. One can write the fields' x, y dependence for the number density

$$\langle\mathbf{p}'|n(\mathbf{k})|\mathbf{p}\rangle = \frac{(2E_k)^2}{2} \int \mathrm{d}^3 x \, \mathrm{d}^3 y \, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} e^{-i(\mathbf{p}'-\mathbf{p})\mathbf{y}} \langle\mathbf{p}'|\phi(0,\mathbf{x}-\mathbf{y})\phi(0,\mathbf{0})|\mathbf{p}\rangle. \tag{C.0.27}$$

For the proof we will only consider the $N = 1$ case, as the other cases can be reduced to the same basic argument. The trick is using Poincaré transformation and inserting a $\mathbb{1}$:

$$\begin{aligned}
\langle\mathbf{p}'|\phi(0,\mathbf{x})\phi(0,\mathbf{y})|\mathbf{p}\rangle &= \langle\mathbf{p}'|e^{-i\mathbf{P}\mathbf{y}} e^{i\mathbf{P}\mathbf{y}} \phi(0,\mathbf{x}) e^{-i\mathbf{P}\mathbf{y}} \phi(0,\mathbf{0}) e^{i\mathbf{P}\mathbf{y}}|\mathbf{p}\rangle = \\
&= \langle\mathbf{p}'|e^{-i\mathbf{p}'\mathbf{y}} \phi(0,\mathbf{x}-\mathbf{y})\phi(0,\mathbf{0}) e^{i\mathbf{p}\mathbf{y}}|\mathbf{p}\rangle = \\
&= e^{-i\mathbf{p}'\mathbf{y}} e^{i\mathbf{p}\mathbf{y}} \langle\mathbf{p}'|\phi(0,\mathbf{x}-\mathbf{y})\phi(0,\mathbf{0})|\mathbf{p}\rangle = \\
&= e^{-i(\mathbf{p}'-\mathbf{p})\mathbf{y}} \langle\mathbf{p}'|\phi(0,\mathbf{x}-\mathbf{y})\phi(0,\mathbf{0})|\mathbf{p}\rangle
\end{aligned} \tag{C.0.28}$$

where $\mathbf{P}$ are the Poincaré generators and $\mathbf{p}$, $\mathbf{p}'$ their eigenvalues. Now we can shift the x integration variable such that $\mathbf{x} - \mathbf{y} \mapsto \mathbf{r}$, so that the integral (C.0.27) becomes

$$\begin{aligned}
\langle\mathbf{p}'|n(\mathbf{k})|\mathbf{p}\rangle &= \frac{(2E_k)^2}{2} \int \mathrm{d}^3 r \, \mathrm{d}^3 y \, e^{i\mathbf{k}\mathbf{r}} e^{-i(\mathbf{p}'-\mathbf{p})\mathbf{y}} \langle\mathbf{p}'|\phi(0,\mathbf{r})\phi(0,\mathbf{0})|\mathbf{p}\rangle \sim \\
&\sim \delta^{(3)}(\mathbf{p}'-\mathbf{p}) \frac{(2E_k)^2}{2} \int \mathrm{d}^3 r \, e^{i\mathbf{k}\mathbf{r}} \langle\mathbf{p}|\phi(0,\mathbf{r})\phi(0,\mathbf{0})|\mathbf{p}\rangle.
\end{aligned} \tag{C.0.29}$$

Thus, we have rewritten the bilocal observable $n(\mathbf{k})$ described by two spacetime points to a bilocal observable described by some scale $\mathbf{r}$.

Appendix D

General calculation

$$\langle \mathcal{G}| P_{\delta\phi} |\mathcal{G}\rangle = \big(\langle \mathcal{G}_0| + \delta \langle \mathcal{G}|\big) P_{\delta\phi}(x,y) \big(|\mathcal{G}_0\rangle + \delta |\mathcal{G}\rangle\big) \tag{D.0.1}$$

Interpolating current:

$$J(x) = J_0 + \delta J(x) = \sum_{k,l=0}^{N_\phi,N_\psi} \binom{N_\phi}{k}\binom{N_\psi}{l} \phi^{N_\phi-k}\psi^{N_\psi-l}\delta\phi^k\delta\psi^l \tag{D.0.2}$$

$$|\mathcal{G}\rangle = \sum_{k,l=0}^{N_\phi,N_\psi} \binom{N_\phi}{k}\binom{N_\psi}{l} \sum\!\!\!\!\!\!\int_{\mathcal{L}_{kl}} \mathcal{G}_{kl}(\mathcal{L}_{kl}) \int \sigma(z) F_{\mathcal{L}_{kl}}(z)\phi^{N_\phi-k}\psi^{N_\psi-l}\delta\phi^k\delta\psi^l |\Omega\rangle \tag{D.0.3}$$

where $\sigma(z) = \int \mathrm{d}^4 z$.

$$\begin{aligned}\langle \mathcal{G}| P_{\delta\phi}(x,y) |\mathcal{G}\rangle = &\sum_{k,l=0}^{N_\phi,N_\psi}\sum_{m,n=0}^{N_\phi,N_\psi} \binom{N_\phi}{k}\binom{N_\psi}{l}\binom{N_\phi}{m}\binom{N_\psi}{n} \sum\!\!\!\!\!\!\int_{\mathcal{L}_{mn}} \sum\!\!\!\!\!\!\int_{\mathcal{L}_{kl}} \mathcal{G}^*_{mn}(\mathcal{L}_{mn})\mathcal{G}_{kl}(\mathcal{L}_{kl}) \\ &\int \sigma(z)\sigma(u) F^*_{\mathcal{L}_{mn}}(u) F_{\mathcal{L}_{kl}}(z) C(k,l,m,n)\times \\ &\times \langle 0| \delta\phi^m(u)\delta\psi^n(u) P_{\delta\phi}(x,y)\delta\phi^k(z)\delta\psi^l(z) |0\rangle\end{aligned} \tag{D.0.4}$$

where $C(k,l,m,n) = \langle \Omega| \phi^{N_\phi-k}(z)\phi^{N_\phi-m}(u)\psi^{N_\psi-l}(z)\psi^{N_\psi-n}(u) |\Omega\rangle$.

Furthermore, we demand $|\mathcal{G}\rangle \approx |\mathcal{G}_0\rangle = \sum_{i=0}^{N} \alpha_i |\mathcal{G}_i\rangle$ $\Big(\langle \mathcal{G}||\mathcal{G}\rangle = 1 = \sum_{i=0} |\alpha_i|^2\Big)$. Thus $|\alpha_0| >> |\alpha_1| >> \cdots >> |\alpha_N|$.

$$\int \sigma(x) F^*_{\mathcal{L}_{mn}}(x) F_{\mathcal{L}_{00}}(x) \overset{m,n\neq 0}{<<} \int \sigma(x) \left|F_{\mathcal{L}_{00}}(x)\right|^2 \tag{D.0.5}$$

$$\int \sigma(x) \left|F_{\mathcal{L}_{mn}}(x)\right|^2 \overset{m,n\neq 0}{<<} \int \sigma(x) \left|F_{\mathcal{L}_{00}}(x)\right|^2 \tag{D.0.6}$$

Let's compute

$$\langle 0| \delta\phi^m(u)\delta\psi^n(u) P_{\delta\phi}(x,y)\delta\phi^k(z)\delta\psi^l(z) |0\rangle \equiv \xi^{mnkl}_{\delta\phi}(u,x,y,z) \tag{D.0.7}$$

where $P_{\delta\phi}(x,y) = \delta\phi(x)\delta\phi(y)$.

Note: for $\langle \Omega| \phi^{N_\phi-k}\phi^{N_\phi-m}\psi^{N_\psi-l}\psi^{N_\psi-n} |\Omega\rangle \neq 0$ we need $k=m$, $l=n$.

$$\xi^{kl}_{\delta\phi}(u,x,y,z) = \langle 0| \left(\delta\phi^m\delta\psi^n\right)(u)\delta\phi(x)\delta\phi(y)\left(\delta\phi^k\delta\psi^l\right)(z) |0\rangle \tag{D.0.8}$$

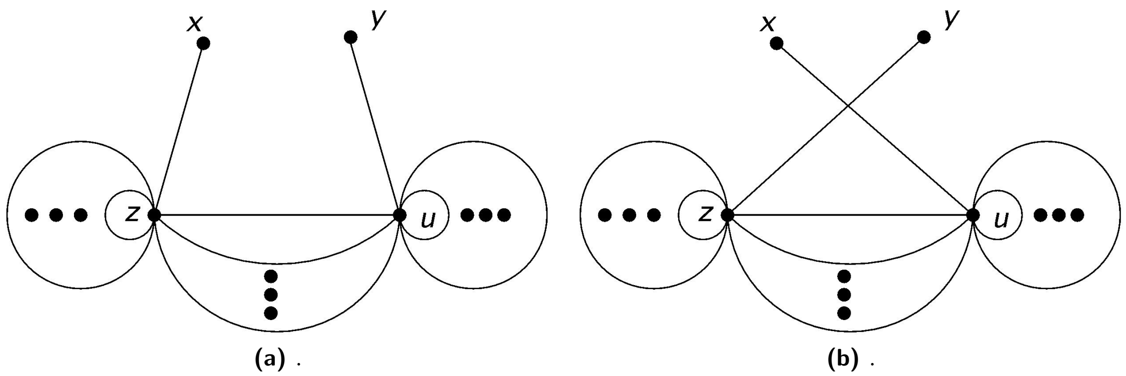


Figure D.1: Diagrams

$\Delta^{\delta\psi}(0)$ contributions vanish while $\Delta^{\delta\phi}(0) = \frac{m^2}{(4\pi)^2}\log\left(\frac{m^2}{\mu^2}\right)$

Combinatoric factors: assume that there are $r\,\delta\phi$ self-loops ($\Delta^{\delta\phi}(0)$) at x and y, then $\binom{l-1}{2r}\binom{l-1}{2r}$. Furthermore we need to connect one $\delta\phi$ at x and y with the power spectrum operator which gives

$$\binom{l}{1}\binom{l}{1}\left[\left\langle\delta\phi(u)\delta\phi(x)\right\rangle\left\langle\delta\phi(y)\delta\phi(z)\right\rangle+\left\langle\delta\phi(u)\delta\phi(y)\right\rangle\left\langle\delta\phi(x)\delta\phi(z)\right\rangle\right].$$

Then at x and y there are $(l-1-2r)\,\delta\phi$s and $k\,\delta\psi$s left.

We need to connect these with each other:

$$\left(\frac{[2(l-1-2r)]!}{(l-1-2r)!\,2^{l-1-2r}}-w(l-1-2r)\right)\left(\frac{(2k)!}{k!\,2^k}\right)$$

$$l^2\left[\frac{(l-1)!}{(l-1-2r)!(2r)!}w(2r)\right]^2\left[w(2(l-1-2r))-w(l-1-2r)\right]w(2k)\equiv f(k,l,r) \tag{D.0.9}$$

$$\xi^{kl}_{\delta\phi}(u,x,y,z)=\sum_{r=0}^{\frac{1}{2}(l-1)} f(k,l,r)\left\langle\delta\psi(u)\delta\psi(z)\right\rangle^k\left\langle T\left(\delta\phi(x)\delta\phi(x)\right)\right\rangle^{2r}\left\langle\delta\phi(u)\delta\phi(z)\right\rangle^{l-1-2r}$$
$$\left[\left\langle\delta\phi(u)\delta\phi(x)\right\rangle\left\langle\delta\phi(y)\delta\phi(z)\right\rangle+\left\langle\delta\phi(u)\delta\phi(y)\right\rangle\left\langle\delta\phi(x)\delta\phi(z)\right\rangle\right] \tag{D.0.10}$$

Thus, we need to evaluate the following correlators:

1. $\left\langle\delta\psi(u)\delta\psi(z)\right\rangle$
2. $\left\langle\delta\phi(u)\delta\phi(z)\right\rangle$
3. $\left\langle T\left(\delta\phi(x)\delta\phi(x)\right)\right\rangle=\frac{m^2}{(4\pi)^2}\log\left(\frac{m^2}{\mu^2}\right)$

Expansion:

$$\delta\psi(x)=\int\omega(\mathbf{k})\left\{e^{ikx}a(\mathbf{k})+e^{-ikx}a^\dagger(\mathbf{k})\right\}=\delta\phi|_{m=0} \tag{D.0.11}$$

where

$$\omega(k)=\frac{\mathrm{d}^3 k}{(2\pi)^{3/2}\sqrt{2k_0(\mathbf{k})}} \tag{D.0.12}$$

$$\left\langle 0 \left| \delta\phi(u)\delta\phi(z) \right| 0 \right\rangle = \int \omega(\mathbf{k},\mathbf{p}) \left\langle 0 \left| e^{iku} e^{-ipz} a(\mathbf{k}) a^\dagger(\mathbf{p}) \right| 0 \right\rangle = \\ = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0|} e^{ik(u-z)} \tag{D.0.13}$$

$$\begin{aligned} \xi^{kl}_{\delta\phi}(u,x,y,z) = &\sum_{r=0}^{\frac{1}{2}(l-1)} f(k,l,r) \left\{ \prod_{n=1}^{k} \int \frac{\mathrm{d}^3 k_n}{(2\pi)^3\, 2\,|k_n|} \right\} e^{i(u-z)\sum_{n=1}^{k} k_n} \times \\ &\times \left\{ \prod_{m=1}^{l-1-2r} \int \frac{\mathrm{d}^3 p_m}{(2\pi)^3\, 2\sqrt{|p_m|^2 + m_\phi^2}} \right\} e^{i(u-z)\sum_{m=1}^{l-1-2r} p_m} \left[\frac{m^2}{(4\pi)^2} \log\left(\frac{m^2}{\mu^2}\right) \right]^{2r} \times \\ &\times \left\{ \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\sqrt{|\vec{k}|^2 + m_\phi^2}} e^{ik(u-x)} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3\, 2\sqrt{|\vec{k}'|^2 + m_\phi^2}} e^{ik'(y-z)} + \right. \\ &\left. + \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\sqrt{|\vec{k}|^2 + m_\phi^2}} e^{ik(u-y)} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3\, 2\sqrt{|\vec{k}'|^2 + m_\phi^2}} e^{ik(x-z)} \right\} \end{aligned} \tag{D.0.14}$$

$$\begin{aligned} \langle \mathcal{G} | \, P_{\delta\phi}(x,y) \, | \mathcal{G} \rangle = &\sum_{k,l=0}^{N_\phi, N_\psi} \sum_{r=0}^{\frac{1}{2}(l-1)} \binom{N_\phi}{k}^2 \binom{N_\psi}{l}^2 f(k,l,r) \sum\!\!\!\!\!\!\int_{\mathcal{L}'_{kl}} \sum\!\!\!\!\!\!\int_{\mathcal{L}_{kl}} \mathcal{G}^*_{kl}(\mathcal{L}'_{kl}) \mathcal{G}_{kl}(\mathcal{L}_{kl}) \times \\ &\times \int \sigma(z)\sigma(u) F^*_{\mathcal{L}'_{kl}}(u) F_{\mathcal{L}_{kl}}(z) C(k,l,k,l) \left[\frac{m^2}{(4\pi)^2} \log\left(\frac{m^2}{\mu^2}\right) \right]^{2r} \times \\ &\times \left[\prod_{n=1}^{k} \prod_{m=1}^{l-1-2r} \int \frac{\mathrm{d}^3 k_n}{(2\pi)^3\, 2\,|k_n|} \frac{\mathrm{d}^3 p_m}{(2\pi)^3\, 2\sqrt{|p_m|^2 + m_\phi^2}} \right] \times \\ &\times \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\sqrt{|\vec{k}|^2 + m_\phi^2}} \frac{\mathrm{d}^3 k'}{(2\pi)^3\, 2\sqrt{|\vec{k}'|^2 + m_\phi^2}} \times \\ &\times \left[e^{iu\left(\sum_{n=1}^{k} k_n + \sum_{m=1}^{l-1-2r} p_m + k\right)} e^{-iz\left(\sum_{n=1}^{k} k_n + \sum_{m=1}^{l-1-2r} p_m + k'\right)} \left(e^{-ixk} e^{iyk'} + e^{-iyk} e^{ixk'} \right) \right] \end{aligned} \tag{D.0.15}$$

Suppose you can write $F_{\mathcal{L}_{kl}} = \bar{F}_{\mathcal{L}_{kl}} e^{iP_{kl}x} \Rightarrow \sum\!\!\!\!\!\!\int_{\mathcal{L}_{kl}} \to \int \bar{\sigma}(P_{kl})$ and $\mathcal{G}_{kl}(\mathcal{L}_{kl}) \to \mathcal{G}_{kl}(P_{kl})$ where $\bar{\sigma}(P_{kl}) = \int \frac{\mathrm{d}^4 P_{kl}}{(2\pi)^4}$.

$$\langle \mathcal{G}| P_{\delta\phi}(x,y) |\mathcal{G}\rangle = \sum_{k,l=0}^{N_\phi,N_\psi} \sum_{r=0}^{\frac{1}{2}(l-1)} \binom{N_\phi}{k}^2 \binom{N_\psi}{l}^2 f(k,l,r) C(k,l,k,l) \left[\frac{m^2}{(4\pi)^2}\log\left(\frac{m^2}{\mu^2}\right)\right]^{2r} \times$$
$$\times \left[\prod_{n=1}^{k}\prod_{m=1}^{l-1-2r} \int \frac{d^3 k_n}{(2\pi)^3\, 2\,|k_n|} \frac{d^3 p_m}{(2\pi)^3\, 2\sqrt{|p_m|^2 + m_\phi^2}}\right] \int \bar\sigma(P'_{kl})\bar\sigma(P_{kl})\mathcal{G}^*_{kl}(P'_{kl})\mathcal{G}_{kl}(P_{kl}) \times$$
$$\times \int \frac{d^3 k}{(2\pi)^3\, 2\sqrt{|\vec{k}|^2 + m_\phi^2}} \frac{d^3 k'}{(2\pi)^3\, 2\sqrt{|\vec{k}'|^2 + m_\phi^2}} \left(e^{-ixk}e^{iyk'} + e^{-iyk}e^{ixk'}\right) \times$$
$$\times \underbrace{\int \sigma(z)\sigma(u) F^*_{P'_{kl}}(u) F_{P_{kl}}(z) e^{iu\left(-P'_{kl} + \sum_{n=1}^{k} k_n + \sum_{m=1}^{l-1-2r} p_m + k\right)} e^{-iz\left(-P_{kl} + \sum_{n=1}^{k} k_n + \sum_{m=1}^{l-1-2r} p_m + k'\right)}}_{\tilde F^*_{P'_{kl}}(k + \sum_n k_n + \sum_m p_m - P'_{kl}) \tilde F_{P_{kl}}(k' + \sum_n k_n + \sum_m p_m - P_{kl})} \tag{D.0.16}$$

where in the last line we have precisely the Fourier transform of $F^*_{P'_{kl}}$ and $F_{P_{kl}}$.

Define

$$\prod_{n=1}^{k}\prod_{m=1}^{l-1-2r} \int \frac{d^3 k_n}{(2\pi)^3\, 2\,|k_n|} \frac{d^3 p_m}{(2\pi)^3\, 2\sqrt{|p_m|^2 + m_\phi^2}} \times$$
$$\times \tilde F^*_{P'_{kl}}(k + \sum_n k_n + \sum_m p_m - P'_{kl}) \tilde F_{P_{kl}}(k' + \sum_n k_n + \sum_m p_m - P_{kl}) \equiv \mathcal{W}(k, k', P_{kl}, P'_{kl})$$

$$\langle \mathcal{G}| P_{\delta\phi}(x,y) |\mathcal{G}\rangle = \sum_{k,l=0}^{N_\phi,N_\psi} \sum_{r=0}^{\frac{1}{2}(l-1)} \binom{N_\phi}{k}^2 \binom{N_\psi}{l}^2 f(k,l,r) C(k,l,k,l) \left[\frac{m^2}{(4\pi)^2}\log\left(\frac{m^2}{\mu^2}\right)\right]^{2r} \times$$
$$\times \int \bar\sigma(P'_{kl})\bar\sigma(P_{kl})\mathcal{G}^*_{kl}(P'_{kl})\mathcal{G}_{kl}(P_{kl}) \int \frac{d^3 k}{(2\pi)^3\, 2\sqrt{|\vec{k}|^2 + m_\phi^2}} \frac{d^3 k'}{(2\pi)^3\, 2\sqrt{|\vec{k}'|^2 + m_\phi^2}} \times \tag{D.0.17}$$
$$\times \left(e^{-ixk}e^{iyk'} + e^{-iyk}e^{ixk'}\right) \mathcal{W}_{m_\phi}(k, k', P_{kl}, P'_{kl}) C(k,l,k,l)$$

Conclusions

In this thesis we presented the calculation of the inflationary power spectrum in an approximate quantum de Sitter state.

First of all we recall cosmological perturbation theory and quantum fluctuations from inflation. That leads to the definition of the power spectrum, which is the central object in our work. We discussed the problem of choosing the physical vacuum in a curved spacetime too and we will try to give a new and general solution in the second part of the thesis. After reviewing the basics of inflationary gravitational waves, we presented a new calculation of the gravitational wave-transfer function which includes effects not consider in previous calculations [7, 54] and we separeted the transfer function into three factors, each with a distinct physical meaning.

The first factor accounts for the redshift suppression of the gravitational-wave amplitude after horizon re-entry. Among other things, this factor accomodates a dark energy component with a time-varying equation-of-state parameter $w(z)$. The second factor captures the behaviour of the background equation-of-state parameter w near the time of horizon re-entry. The third factor accounts for the damping of tensor modes due to tensor anisotropic stress from free-streaming relativistic particles in the early universe. We stressed that it is also necessary to consider this damping effect on laser-interferometer scales, which re-entered when free-streaming particles were an unknown function f. We also observe that Weinberg's analysis [64], which originally focused on the damping on CMB scales due to a single fermionic species (the neutrino), extends in a simple way way to the more general case of a mixture of free-streaming bosons and fermions with different temperatures and decoupling times.

We identified six physical effects which can modify the relic GWB by causing the equation-of-state parameter w to deviate from its standard value ($w = 1/3$) during the radiation-dominated epoch. Furthermore, although it is often treated as a stationary random process, the inflationary GWB is actually highly non-stationary (as emphasized in Grishchuk [28]). Thus, our transfer function keeps track of the coherent phase that it contains.

Starting from Chapter 3 we leave aside postinflationary effects that produce modifications of the gravitational-wave spectrum and we focus on the choice of the vacuum. At first, we studied the possible influence of trans-Planckian physics on the fluctuation spectrum predicted by inflation. We have made use of a natural initial condition: we required that the modes are created in a state of minimized uncertainity. If this is imposed in the infinite past, there is no difference between this choice and the usual choice of an adiabatic vacuum. But contrary to the standard treatment we have imposed the initial condition not in the infinite past, but at a mode dependent time determined by when a particular mode reaches a size of the order of the fundamental scale (e.g., the Planck scale). As a consequence our analysis agrees with the standard choice inly to zeroth order in an adiabatic expansion with corrections at first order. This should be viewed as a conservative approach appropriate for estimating how well the fluctuation spectrum can be predicted without any knowledge of high energy physics. We were indeed working with the α-vacua, which are formally squeezed states over the Bunch-Davies vacuum, and then they aren't the most general ground state for de Sitter space.

I have then switched to the presentation of the innovative formalism, which was first introduced in [30]. Since the power spectrum is a bi-local operator, I chose as an example the calculation that leads to the constituent number of a Black Hole (BH) to explain the conceptual and technical details of the new framework. We applied the bi-local particle number operator to the BH state $|\mathcal{B}\rangle$. Its proposal and construction involved the postulation of a non-perturbative, gravitational vacuum $|\Omega\rangle$ and an Auxiliary Current $\mathcal{J}$ that, acting on this vacuum, creates a quantum bound state with a non-vanishing overlap with the above BH state. Taking a specific current, we were able to perform explicit, quantum theoretical, calculations with it. Namely, we calculated the expectation value of the above particle number operator.

At this point the original contribution of my work starts. The Auxiliary Current Description (ACD) is then generalized to arbitrary spacetimes so that I can construct an approximate quantum de Sitter state too. Finally I had all the necessary tools to calculate the inflationary power spectrum. I used two different ways to calculate it: first, constructing a power spectrum operator; second, considering the power spectrum which emerges naturally from the current expansion. I managed to reproduce the standard behaviuor of the power spectrum present in the literature. In Chapter 5 I showed the calculation until the second order in the current expansion, while in Appendix D the general calculation is reported.

The results presented in this thesis are part of an ongoing project. In our opinion the results coming from the two ways I calculated the power spectrum should match. But then we still have some problems to solve. In the first case, my calculation seems to give already the second order correction to the standard power spectrum. So we have still to understand if it's possible a contraction within the fluctuations composing the operator. In the second one, I lose the dependence on the spacetime points, x and y. Then we have to think of a redefinition of our state in such a way that we still have the spacetime dependence (in the current culculation we integrated over x and y). Finally, the calculation of the primordial power spectrum gives us an idea about the wavefunctions $\mathcal{G}(\mathcal{L})$ and, if we can reproduce with no doubts the standard result, would be then a crucial test for this framework.

Bibliography

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions*. 1964.

[2] A. Albrecht and P.J. Steinhardt. Cosmology for grand unified theories with radiatively induced symmetry breaking. 1982.

[3] B. Allen. Vacuum states in de sitter space. 1985.

[4] T. Banks and L. Mannelli. De sitter vacua, renormalization and locality. 2003.

[5] J.M. Bardeen. Gauge-invariant cosmological perturbations. 1989.

[6] J.M. Bardeen, P.J. Steinhardt, and M.S. Turner. Spontaneous creation of almost scale-free density perturbations in an inflationary universe. 1983.

[7] S. Bashinsky. Coupled evolution of primordial gravity waves and gravity neutrinos, 2008.

[8] N.D. Birrell and P.C.W. Davies. *Quantum Fields in Curved Space*. 1982.

[9] R. Bousso, A. Maloney, and A. Strominger. Conformal vacua and entropy in de sitter space. 2002.

[10] L.A. Boyle and P.J. Steinhardt. Probing the early universe with inflationary gravitational waves. 2008.

[11] R.H. Brandenberger and J. Martin. On signatures of short distance physics in the cosmic microwave background. 2002.

[12] N.A. Chernikov and E.A. Tagirov. Quantum theory of scalar field in de sitter space-time. 1968.

[13] Planck Collaboration. Constraints on inflation. 2013.

[14] U.H. Danielsson. Inflation, holography and the choice of vacuum in de sitter. 2002.

[15] U.H. Danielsson. Note on inflation and tras-planckian physics. 2002.

[16] U.H. Danielsson. On the consistency of de sitter vacua. 2002.

[17] H. Davoudiasl, R. Kitano, G.D. Kribs, H. Murayama, and P.J. Steinhardt. Gravitational baryogenesis. 2004.

[18] D.A. Dicus and W.W. Repko. Comment on ”damping of tensor modes in cosmology”. 2005.

[19] S. Dodelson. *Modern cosmology*. 2003.

[20] G. Dvali and C. Gomez. Black hole's quantum n-portrait, 2011.

[21] R. Easther, B.R. Greene, W.H. Kinney, and G. Shiu. Inflation as a probe of short distance physics. 2001.

[22] R. Easther, B.R. Greene, W.H. Kinney, and G. Shiu. A generic estimate of trans-planckian modifications to the primordial power spectrum in inflation. 2002.

[23] R. Easther, B.R. Greene, W.H. Kinney, and G. Shiu. Imprints of short distance physics on inflationary cosmology. 2003.

[24] M.B. Einhorn and F. Larsen. Interacting quantum field theory in de sitter vacua. 2003.

[25] R. FLoreanini, C.T. Hill, and R. Jackiw. Functional representation for the isometris of de sitter space. 1987.

[26] W. Greiner and J. Reinhardt. *Quantum electrodynamics*. 2009.

[27] L. P. Grischuk. Amplification of gravitational waves in an isotropic universe. 1975.

[28] L.P. Grishchuk and Y.V. Sidorov. Squeezed quantum states of relic gravitons and primordial density fluctuations. 1990.

[29] A.H. Guth. Inflationary universe: a possible solution to the horizon and flatness problems. 1981.

[30] S. Hofmann and T. Rug. A quantum bound state description of a black hole, 2014.

[31] K. Kajantie, M. Laine, K. Rummukainene, and Y. Schröder. Pressure of hot qcd up to $g^6 \ln(1/g)$. 2003.

[32] N. Kaloper, M. Kleban, A. Lawrence, and S. Shenker. Signatures of short distance physics in the cosmic microwave background. 2002.

[33] M. Kamionkowski, A. Kosowsky, and A. Stebbins. A probe of primordial gravity waves and vorticity. 1997.

[34] A. Kempf and J.C. Niemeyer. Perturbation spectrum in inflation with a cutoff. 2001.

[35] E.W. Kolb and M.S. Turner. *The early universe*. 1990.

[36] A.R. Liddle and D.H. Lyth. *Cosmological Inflation and Large-Scale Structure*. 2000.

[37] A.D. Linde. A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. 1982.

[38] S. Matarrese. Cosmological perturbations in general relativity, 2013.

[39] E. Mottola. Particle creation in de sitter space. 1985.

[40] V. Mukhanov and S. Winitzki. *Introduction to Quantum Effects in Gravity*. 2007.

[41] V.F. Mukhanov. *Physical foundations of cosmology*. 2005.

[42] V.F. Mukhanov and G.F. Chibisov. Quantum fluctuations and a nonsingular universe. 1981.

[43] V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger. Theory of cosmological perturbations. 1992.

[44] A.A. Penzias and R.W. Wilson. A measurement of excess antenna temperature at 4080 mc/s. 1965.

[45] P. Peter and J.P. Uzan. *Primordial Cosmology.* 2009.

[46] D. Podolsky, G.N. Felder, L. Kofman, and M. Peloso. Equation of state and beginning of thermalization after preheating. 2006.

[47] D. Polarski and A.A. Starobinsky. Semiclassicality and decoherence of cosmological perturbations. 1996.

[48] A.G. Polnarev. 1985.

[49] J.R. Pritchard and M. Kamionkowski. Cosmic microwave background fluctuations from gravitational waves: an analytic approach. 2005.

[50] A. V. Radyushkin. *Introduction into QCD sum rule approach.* 2000.

[51] V. Sahni. Energy density of relic gravity waves from inflation. 1990.

[52] U. Seljak and M. Zaldarriaga. Signature of gravity waves in polarization of the microwave background. 1997.

[53] N. Seto, S. Kawamura, and T. Nakamura. Possibility of direct measurement of the acceleration of the universe using 0.1 hz band laser interferometer gravitational wave antenna in space. 2001.

[54] T.L. Smith, M. Kamionkowski, and A. Cooray. Direct detection of the inflationary gravitational-wave background. 2006.

[55] D.N. Spergel and M. Zaldarriaga. Cosmic microwave background polarization as a direct test of inflation. 1997.

[56] M. Spradlin and A. Volovich. Vacuum states and s-matric in ds/cft. 2002.

[57] A.A. Starobinsky. Spectrum of relic gravitational radiation and the early state of the universe. 1979.

[58] A.A. Starobinsky. Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations. 1982.

[59] J. M. Stewart. Perturbations in an expanding universe of free particles. 1972.

[60] H.V. Peiris et al. First year wilkinson microwave anisotropy probe (wmap) observations: Implications for inflation. 2003.

[61] E.S. Phinney *et al.* The big bang observer. 2004.

[62] M.S. Turner. Detectability of inflation-produced gravitational waves. 1997.

[63] S. Weinberg. *Gravitation and Cosmology*. 1972.

[64] S. Weinberg. Damping of tensor modes in cosmology. 2004.

[65] H.H. Zhang, K.Z. Feng, S.W. Qiu, A. Zhao, and X.S. Li. On analytic formulas of feynman propagators in position space. 2010.