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Some Problems of Unlikely Intersections

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Abstract

The theory of unlikely intersection comes from the dimension theory. Indeed, if two subvarieties X and Y in a space of dimension n have a nonempty intersection, then we expect that $\dim X \cap Y \leq \dim X + \dim Y - n$. Thus, we say that the intersection is unlikely when this dimension exceeds the expected dimension, that is $\dim X \cap Y > \dim X + \dim Y - n$. Fixing X in the multiplicative algebraic group $\mathbf{G}_m^n(\overline{\mathbb{Q}})$, we are specifically interested in the intersection of X with the union of Y where Y runs through the algebraic subgroups of $\mathbf{G}_m^n$ restricted only by dimension. We shall prove that, if X is an irreducible subvariety of dimension at most $n-1$ in $\mathbf{G}_m^n$ and the considered algebraic subgroups have codimension at least $\dim X$, then this intersection has a bounded height by removing the *anomalous subvarieties* of X. Furthermore, we have the finiteness of such intersection by considering the algebraic subgroups of codimension at least $1 + \dim X$.

La théorie de l'intersection exceptionnelle ou l'intersection improbable provient de la théorie de la dimension. En effet, si deux sous-variétés X et Y dans un espace de dimension n ont une intersection non vide, alors nous attendons à ce que $\dim X \cap Y \leq \dim X + \dim Y - n$. Ainsi, on parle d'intersection exceptionnelle lorsque cette dimension dépasse la dimension attendue, i.e $\dim X \cap Y > \dim X + \dim Y - n$. En fixant X dans le groupe multiplicatif $\mathbf{G}_m^n(\overline{\mathbb{Q}})$, nous sommes intéressés à l'intersection de X avec l'union des Y où Y parcourt l'ensemble des sous-groupes algébriques de $\mathbf{G}_m^n$ de dimension fixée. Pour une variété irréductible X de dimension au plus $n-1$ dans le groupe multiplicatif $\mathbf{G}_m^n$, on prouvera que cette intersection a une hauteur bornée en enlevant certaines de ses intersections exceptionelles et en considérant les sous-groupes de codimenion au moins égale à $\dim X$. Nous retrouvons la finitude de cette intersection en considérant les sous-groupes algébriques de codimension au moins égale à $1 + \dim X$.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Njaka Harilala ANDRIAMANDRATOMANANA, July 15, 2020.

Contents

1. Introduction

Roughly speaking, the term "unlikely intersections" refers to varieties which we do not expect to intersect, due to natural dimensional reasons. For example if X and Y are varieties of dimensions r, s in a space of dimension n, we expect $X \cap Y$ to have dimension at most $r+s-n$ and in particular to be empty if $r+s<n$. If the contrary happens, i.e $X \cap Y$ has dimension strictly lager than $r+s-n$, then we say that X, Y have unlikely intersection.

More specifically, let X be fixed and let Y run through a denumerable set $\mathcal{Y}$ of algebraic varieties, chosen in advance independently of X, with a certain structure and such that $\dim X + \dim Y < n$ where $n = \dim$ (ambient space). Then we expect that only for a small subset of $Y \in \mathcal{Y}$, we shall have $X \cap Y \neq \emptyset$, unless there is a special structure relating X with Y which forces the contrary to happen. We shall express this by saying that X is a *special variety*. When X is not a special variety, the said expected smallness may be measured in terms of the union of the intersections $\cup_{Y \in \mathcal{Y}}(X \cap Y)$. So the natural questions to ask are: how is this set distributed in X? Is this set finite? Similarly, we study the analogous situations when $\dim Y \leq s$ for any fixed number s and where $\dim(X \cap Y) > \dim X + \dim Y - n$ for several $Y \in \mathcal{Y}$. It corresponds to the unlikely intersections or the unexpected intersections.

In this essay, we would like to introduce some known problems that can be viewed in this perspective. In our situation, we shall consider $\mathcal{Y}$ as the multiplicative algebraic group $G = \mathbf{G}_m^n(\overline{\mathbb{Q}})$ called *algebraic torus*, defined over the field of algebraic number $\overline{\mathbb{Q}}$. Particularly, we are interested in the intersection of a subvariety X in G with varying algebraic subgroups of G restricted only by dimension. Hence, Chapter 1 is dedicated to the study of algebraic subgroups of $\mathbf{G}_m^n$. We also recall there some results about the Weil Height. Using this height function, we may also previously ask if our intersection has a bounded height.

The real interest in this realm begins with the discovering of the existence of these unexpected intersections and its structure. This leads us to the concept of "anomalous variety" defined in Chapter 2. We will present there, in the same chapter, some structures of such anomalous varieties. After, we describe its consequences on the boundedness of the intersection of X with the union of the algebraic subgroups of G restricted by dimension. We will see that the finiteness of this intersection holds under slightly change of the assumption. The Chapter 3 will be dedicated for the proof of main results. We also present there some generalizations and other related problems.

As said above, we start with the definition of algebraic group $\mathbf{G}_m^n$ and the study of its algebraic subgroups.

1.1 Algebraic tori

Let K be an algebraically closed field of characteristic 0. We define $\mathbf{G}_m$ to be multiplicative group $K^\times$. We shall also denote it by $\mathbf{G}_m^1$. For a positive integer $n \geq 1$, we define $\mathbf{G}_m^n$ as the direct product

$$\underbrace{\mathbf{G}_m^1 \times \cdots \times \mathbf{G}_m^1}_{n \text{ times}}.$$

We endow it with the obvious multiplication $(x_1, \cdots, x_n) \cdot (y_1, \cdots, y_n) = (x_1 y_1, \cdots, x_n y_n)$. The identity element is denoted $\mathbf{1}_n = (1, \cdots, 1)$. It is an algebraic group with ring of regular functions $K[G] = K[T_1, \cdots, T_n, T_1^{-1}, \cdots, T_n^{-1}]$. As a variety, we identity $\mathbf{G}_m^n$ as the Zariski open subset $x_1 \cdots x_n \neq 0$ of the affine space $\mathbb{A}^n$. It is irreducible as $\mathbf{G}_m^1$ is.

1.1.1 Notation. If $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ and $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbf{G}_m^n$, we set $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. If $a \in \mathbb{Z}$, we may simply write $\mathbf{x}^a$ for $\mathbf{x}^{\mathbf{a}} = x_1^a \cdots x_n^a$. From now, we set $G = \mathbf{G}_m^n$.

1.1.2 Definition. We say that $\mathbf{x} \in G$ is a *torsion point* if it has a finite order in G i.e the coordinates of $\mathbf{x}$ are roots of unity.

1.1.3 Definition. A *coset* of G is simply a translate of subgroup of G. A *torsion coset* of G is a translate of a subgroup of G by a torsion point. A torsion point is a torsion coset by taking H as the trivial subgroup.

An algebraic subgroup of G is *a subtorus (or simply a torus)* if it is irreducible as an algebraic variety. For instance, $\mathbf{G}_m^n$ is a subtorus of itself. So we call it also n*-dimensional torus*. We will show that a subtorus of dimension r in G is isomorphic to $\mathbf{G}_m^r$.

1.1.4 Proposition. Any algebraic subgroup of G is a Zariski closed subgroup.

Proof. Let H be a subgroup of G. Its Zariski closure $\overline{H}$ is a subgroup of G. Indeed, since the inversion map is a isomorphism, it is clear that $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$. Similarly, the translation by $x \in H$ is a isomorphism, so $x\overline{H} = \overline{xH} = H$ i.e $H\overline{H} \subseteq H$. Moreover, if $x \in \overline{H}$, then $Hx \subseteq \overline{H}$. It implies that $\overline{H}x = \overline{Hx} \subset \overline{H}$. Hence $\overline{H} \cdot \overline{H} \subset \overline{H}$. This gives the group structure on H. Now, we endow $\overline{H}$ with the reduced structure, so $\overline{H}$ is an algebraic group (since the ground field is algebraically closed). So we can replace H by its closure and we can suppose that H is dense in G. The canonical morphism $H \hookrightarrow G$ then has a dense image. By Chevalley's theorem [24, p. 19, Chap 1], its image contains a dense open subset U of G i.e $U \subseteq H \subseteq G$. Let $x \in G$, then xU is open and so there exists $y \in xU \cap U$.This implies that $xy^{-1} \in U$, so $x \in yU \subset U \subseteq H$. $\square$

1.1.5 Remark. Let Λ be a subgroup of $\mathbb{Z}^n$. This subgroup determines an algebraic subgroup

$$H_\Lambda := \{\mathbf{x} \in G \mid \mathbf{x}^{\mathbf{a}} = 1,\ \forall\ \mathbf{a} \in \Lambda\}$$

of G. We will prove that the natural map $\Lambda \mapsto H_\Lambda$ is a bijection between subgroups of $\mathbb{Z}^n$ and algebraic subgroups of G. This map is contravariant i.e if $\Lambda \subset \Lambda' \subset \mathbb{Z}^n$, then $H_{\Lambda'} \subset H_\Lambda$.

Every subgroup $\Lambda \subset \mathbb{Z}^n$ of finite rank is torsion free, hence is free. So it is a lattice in $\mathbb{R}\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ of rank $\dim_{\mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$. We define the *saturation* as

$$\overline{\Lambda} = \mathbb{Q}\Lambda \cap \mathbb{Z}^n = \{\lambda \in \mathbb{Z}^n \mid \exists k \in \mathbb{N},\ k\lambda \in \Lambda\}.$$

It is a subgroup of $\mathbb{Z}^n$ and we clearly have $\Lambda \subset \overline{\Lambda}$.

1.1.6 Proposition. Let $\Lambda \subset \mathbb{Z}^n$ be a subgroup of rank r. Then H_Λ is isomorphic to $F \times H_{\overline{\Lambda}}$, where F is a finite group of order $[\overline{\Lambda} : \Lambda]$. Moreover, $H_{\overline{\Lambda}} \cong \mathbf{G}_m^{n-r}$. In particular, H_Λ is the finite union of the torsion cosets $\epsilon H_{\overline{\Lambda}}$ with $\epsilon \in F$ and therefore it has codimension r.

Proof. Let $(\mathbf{a}_1, \cdots, \mathbf{a}_r)$ be a basis of $\overline{\Lambda}$. We can complete it to get a basis $(\mathbf{a}_1, \cdots, \mathbf{a}_n)$ of $\mathbb{Z}^n$. Let us denote by A the matrix formed by the $\mathbf{a}_i$'s as rows. Then $A \in \mathbf{GL}_n(\mathbb{Z})$ and we define φ_A the map

$$\begin{array}{rcl} \varphi_A : G & \longrightarrow & G \\ \mathbf{x} & \mapsto & (\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_n}). \end{array}$$

Then we have $\varphi_A(H_{\overline{\Lambda}}) = \mathbf{1}_r \times \mathbf{G}_m^{n-r}$. This map φ_A is an isomorphism with inverse $\varphi_{A^{-1}}$ since $\varphi_A \circ \varphi_B = \varphi_{AB}$ for any $B \in \mathbf{GL}_n(\mathbb{Z})$. As $H_{\overline{\Lambda}} \cong \mathbf{G}_m^{n-r}$ and up to a change of coordinates and a projection, we may assume that $r = n$, thus $\overline{\Lambda} = \mathbb{Z}^n$ and $H_{\overline{\Lambda}} = \{1\}$. Since $\Lambda \subset \overline{\Lambda}$, then there exist $\alpha_1, \cdots, \alpha_r \in \mathbb{N}$ and a basis $(\mathbf{a}_1, \cdots, \mathbf{a}_r)$ of $\mathbb{Z}^r$ such that $(\alpha_1 \mathbf{a}_1, \cdots, \alpha_r \mathbf{a}_r)$ is a basis of Λ. Up to change of coordinates, we may assume that $\mathbf{a}_i = e_i$ are the canonical basis. Then we have $\Lambda =< (\alpha_1, \cdots, 0), \cdots, (0, \cdots, \alpha_r) >$ and so

$$H_\Lambda = \{x \in G \mid x_i^{\alpha_i} = \mathbf{1}, \quad \forall i = 1, \cdots, r\} = \ker[\alpha_1] \times \cdots \times \ker[\alpha_r],$$

where $[\alpha_i] : \mathbf{G}_m^1 \longrightarrow \mathbf{G}_m^1$ is the map defined by $g \mapsto g^{\alpha_i}$. This gives further the cardinality of H_Λ

$$|H_\Lambda| = \prod_{i=1}^r |\ker[\alpha_i]| = \prod_{i=1}^r \alpha_i = [\mathbb{Z}^r : \Lambda].$$

The other assertions follow from the isomorphism $H_\Lambda \cong F \times H_{\overline{\Lambda}}$. □

1.1.7 Corollary. H_Λ is a subtorus if and only if $H_\Lambda \cong \mathbf{G}_m^{n-r}$, where $r = \text{rank}(\Lambda)$, or equivalently the subgroup Λ is *saturated* (i.e $\Lambda = \overline{\Lambda}$).

Proof. By the previous theorem, we have $H_\Lambda = \cup_{\epsilon \in F} \epsilon H_{\overline{\Lambda}}$. If H_λ is a torus, then H_Λ is irreducible and so $|F| = 1$. This means that $H_\Lambda = \mathbf{G}_m^{n-r}$. Conversely, we just remark that $\mathbf{G}_m^{n-r}$ is irreducible. □

1.1.8 Proposition. Let V be a Zariski closed subvariety of G defined by the equations

$$f_i(\mathbf{x}) = \sum_{\lambda \in I} a_{i,\lambda} \mathbf{x}^\lambda = 0, \text{ for some } a_{i,\lambda} \in K \text{ and for any } i = 1, \cdots, l,$$

for some $l \in \mathbb{N}$ and where $I \subset \mathbb{Z}^n$ the set of the of exponents appearing in the monomials in f_i.

Let H be a maximal algebraic subgroup contained in V. Then, there exists a lattice $\Lambda \subset \mathbb{Z}^n$, generated by the vectors of the set

$$D(I) = \{\eta - \mu \in \mathbb{Z}^n \mid \eta, \mu \in I\},$$

such that $H = H_\Lambda$.

Proof. Let us denote by $X(H)$ the character group of H. For any $\lambda \in \mathbb{Z}^n$, we define $\varphi_\lambda : G \longrightarrow \mathbf{G}_m^1$ the morphism defined by $\mathbf{x} \mapsto \mathbf{x}^\lambda$. Since φ_λ is a character of G, then its restriction $\chi_\lambda := \varphi_{\lambda|H}$ to H is a character of H. So we can consider the following partition of I

$$I_\chi = \{\lambda \in I \mid \chi_\lambda = \chi\}$$

where χ runs over $X(H)$. Now, we set

$$\Lambda =< \eta - \mu \in \mathbb{Z}^n \mid \eta, \mu \in I_\chi, \chi \in X(H) > .$$

Thus, we have $H \subseteq H_\Lambda$. In fact, let $\alpha \in H$, then for any $\chi \in X(H)$ and for any $\eta, \mu \in I_\chi$ we have

$$\alpha^{\eta - \mu} = \varphi_{\eta - \mu|H}(\alpha) = \chi_{\eta - \mu}(\alpha) = 1,$$

since $\chi_\lambda = \chi = \chi_\mu$. By the maximality of H, it is enough to prove that $H_\Lambda \subset V$.

Before proving it, we remark that for any $i = 1, \cdots, k$

$$f_{i|H} = \sum_{\chi \in X(H)} (\sum_{\lambda \in I_\chi} a_{i,\lambda}) \chi = 0$$

since $H \subset V$. (We can write f_i as this form since $X(H)$ is a basis of $K[H]$). By Artin's theorem on the linear independence of characters, we deduce that for any $\chi \in X(H)$, $\sum_{\lambda \in I_\chi} a_{i,\lambda} = 0$.

Now let $\alpha \in H_\Lambda$. We have

$$f_i(\alpha) = \sum_{\chi \in X(H)} \sum_{\lambda \in I_\chi} a_{i,\lambda} \alpha^\lambda = \sum_{\chi \in X(H)} (\sum_{\lambda \in I_\chi} a_{i,\lambda}) \alpha^\lambda = 0.$$

The second equality is due to the fact that the map $I \longrightarrow \mathbf{G}_m$ defined by $\lambda \mapsto \alpha^\lambda$ is constant on I_χ. By the previous remark, we get the last equality. This proves that $H_\Lambda \subset V$. □

1.1.9 Corollary. Every algebraic subgroup H of G is of type H_Λ for some subgroup Λ of $\mathbb{Z}^n$.

Proof. If H is an algebraic subroupp of G, then H is a Zariski closed subgroup by Proposition 1.1.4. So it suffices to apply Proposition 1.1.8 with $X = H$. □

1.1.10 Corollary. For every morphism $\varphi : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^r$, there exist $\mathbf{a}_1, \cdots, \mathbf{a}_r \in \mathbb{Z}^n$ such that

$$\varphi(\mathbf{x}) = (\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_r}).$$

Proof. It suffices to prove the result for $r = 1$, that is for a character $\varphi : G \longrightarrow \mathbf{G}_m$. We consider then its graph

$$\Gamma = \{(x, \varphi(x)) \in \mathbf{G}_m^n \times \mathbf{G}_m^1 \mid x \in G\}.$$

Then Γ is a linear torus of dimension n and so it has a codimension 1 in $\mathbf{G}_m^{n+1}$. By Corollary 1.1.7, $\Gamma = H_\Lambda$ where Λ is the subgroup generated by some $\mathbf{b} \in \mathbb{Z}^{n+1}$. So by setting $y = \varphi(x)$, we have

$$x_1^{b_1} \cdots x_n^{b_n} y^{b_{n+1}} = 1.$$

This implies that $y = x_1^{a_1} \cdots x_n^{a_n}$ with $\mathbf{a} \in \mathbb{Q}^n$. Since φ is a morphism, then $\mathbf{a} \in \mathbb{Z}^n$ and we get the desired result. □

1.1.11 Corollary. Let $r, s \in \mathbb{N}$ and $\varphi : \mathbf{G}_m^r \longrightarrow \mathbf{G}_m^s$ be a homomorphism of tori. If H is an algebraic subgroup of $\mathbf{G}_m^r$, then $\varphi(H)$ is an algebraic subgroup of $\mathbf{G}_m^s$.

Proof. We need to prove that $\varphi(H)$ is a closed Zariski subgroup. By Proposition 1.1.6, we may assume that H is a torus and so we suppose that $H = \mathbf{G}_m^r$. The closure of $\varphi(H)$ is again a torus and so we can suppose that it is isomorphic to $\mathbf{G}_m^s$. Therefore, we need to prove that a dominant morphism $\varphi : \mathbf{G}_m^r \longrightarrow \mathbf{G}_m^s$ is surjective. Let $y \in Y$. As both $\mathbf{G}_m^r$ and $\mathbf{G}_m^s$ are irreducible, then $y\varphi(\mathbf{G}_m^r)$ and $\varphi(\mathbf{G}_m^r)$ have nonempty intersection. Hence, there exist $x, x' \in \mathbf{G}_m^r$ such that $y\varphi(x) = \varphi(x')$. So that $y = \varphi(x'x^{-1})$ and this gives the surjectivity. □

1.1.12 Proposition. The following statements hold

$(\mathbf{1})$ The map $\Lambda \mapsto H_\Lambda$ is a bijection between subgroups of $\mathbb{Z}^n$ and algebraic subgroups of G.

$(\mathbf{2})$ Let Λ, Λ' be two subgroups of $\mathbb{Z}^n$. Then $H_\Lambda H_{\Lambda'} = H_{\Lambda \cap \Lambda'}$ and $H_\Lambda \cap H_{\Lambda'} = H_{\Lambda + \Lambda'}$.

Proof. $(\mathbf{1})$ The surjectivity is given by Corollary 1.1.9. For the injectivity, let Λ, Λ' be two subgroups of $\mathbb{Z}^n$ such that $H_\Lambda = H_{\Lambda'}$. For any $x \in H_\Lambda$, we have

$$\forall \lambda \in \Lambda,\ \forall \lambda' \in \Lambda',\ x^\lambda = x^{\lambda'} = 1.$$

Hence $H_\Lambda \subset H_{\Lambda+\Lambda'}$. Since $\Lambda \subset \Lambda+\Lambda'$, it is clear that $H_{\Lambda+\Lambda'} \subset H_\Lambda$. So $H_\Lambda = H_{\Lambda+\Lambda'}$. By Proposition 1.1.6, we have rank(Λ) = rank$(\Lambda+\Lambda')$ and $[\overline{\Lambda} : \Lambda] = [\overline{\Lambda+\Lambda'} : \Lambda+\Lambda']$. Since $\Lambda \subset \Lambda+\Lambda'$, we get $\Lambda = \Lambda+\Lambda'$ and hence $\Lambda' \subset \Lambda$. The equality follows by symmetry.

(**2**) We can make $H_\Lambda H_{\Lambda'}$ as an algebraic subgroup of G by using multiplication as a homomorphism in Corollary 1.1.11. Therefore, $H_\Lambda H_{\Lambda'}$ is the smallest subgroup containing both H_Λ and $H_{\Lambda'}$. Also, $\Lambda \cap \Lambda'$ is the largest group contained in both Λ and Λ'. Hence $H_\Lambda H_{\Lambda'} = H_{\Lambda\cap\Lambda'}$ since $\Lambda \mapsto H_\Lambda$ reverses the inclusion and is a bijection.

Since the intersection of two algebraic subgroups of G is again an algebraic subgroup of G. Then $H_\Lambda \cap H_{\Lambda'}$ is the largest algebraic subgroup contained in both Λ and Λ'. In the same way as the previous point, we get $H_\Lambda \cap H_{\Lambda'} = H_{\Lambda+\Lambda'}$. □

1.1.13 Corollary. The set of torsion points of an algebraic subgroup H of G are Zariski dense in H.

Proof. By Proposition 1.1.12 (1), there is a subgroup Λ of $\mathbb{Z}^n$ with $H = H_\Lambda$. Let $H_i = \{x \in H \mid x_1^{i!} = 1, \cdots, x_n^{i!} = 1\}$. Then $\cup H_i$ is the set of torsion points of H. We also remark that for two integers $j > i \geq 0$, $H_i \subset H_j$. By Proposition 1.1.12 (2), we have $H_i = H_{\Lambda_i}$ with $\Lambda_i = i! \cdot \mathbb{Z}^n + \Lambda$, and $\Lambda' = \cap \Lambda_i$ is Λ (using the theorem of elementary divisors). Let denote by H' the Zariski closure of $\cup H_i$. Then H' is an algebraic subgroup of G and it is the smallest algebraic subgroup containing all subgroups H_i . By Proposition 1.1.12, we conclude that $H' = H_{\Lambda'}$, because Λ' is the largest subgroup contained in every Λ_i . Since each $H_i \subset H$, the Zariski closure of $\cup H_i$ is also contained in H and we conclude that the Zarsiki closure $\cup_i H_i$ is $H_{\Lambda'} = H_\Lambda = H$. □

1.1.14 Remark. We denote by $(\mathbf{G}_m^n)_{\text{tors}}$ the set of torsion points in $\mathbf{G}_m^n$. With this Corollary, if a curve C is the translate of an 1-dimensional algebraic subgroup by a torsion point, then $(\mathbf{G}_m^n)_{\text{tors}} \cap C$ is Zariski-dense in C. The converse implication is also if $n = 2$ by Lang's Theorem [26, Proposition 4.2, p. 136].

1.2 Heights on $\mathbf{G}_m^n$

This section forms the technical heart of this essay. In this section we develop the basic theory of the height on $G = \mathbf{G}_m^n$. To define it, let us recall the height on the projective space $\mathbb{P}^n$.

1.2.1 Definition. The Height function on $\mathbb{P}^n(\overline{\mathbb{Q}})$ is the function $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_+$ defined by:

$$h(\mathbf{x}) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \log \max(|x_0|_v, \cdots, |x_n|_v),$$

where

- K is a number field containing the homogeneous coordinates x_j's,
- M_K consists of all absolute values on K whose restriction to $\mathbb{Q}$ is one of the standard absolute values on $\mathbb{Q}$ that is the usual Euclidean absolute value $|\cdot|_\infty$ on $\mathbb{C}$ or the p-adic absolute value $|\cdot|_p$ for some prime p.
- K_v (resp $\mathbb{Q}_v$) is the completion of K (resp. $\mathbb{Q}$) with respect to $v \in M_K$ (resp. the restriction of v to $\mathbb{Q}$),

- $n_v = [K_v : \mathbb{Q}_v]$,

1.2.2 Remark. The height function h satisfies the following properties:

(a) h is independent of the choice of K,

(b) h is independent of the choice of the homogeneous coordinates (using the product formula),

(c) h is invariant under the Galois action,

(d) h is positive since any homogeneous coordinates of $\mathbf{x} \in \mathbb{P}^n$ admit one coordinate equal to 1 .

We refer to [18, Part B, p. 174-182] for the proof.

1.2.3 Example. If $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ has homogeneous coordinates with x_i's are coprime, then $h(\mathbf{x}) = \log \max(|x_0|, \cdots, |x_n|)$, where $|\cdot|$ is the usual absolute absolute value on $\mathbb{Q}$. Indeed, we have $x \in \mathbb{Z}^{n+1}$ and if p is a prime number, then $|x_i|_p \leq 1$ for all $i = 0, \cdots, n$. Since x_i's are coprime, then $|x_i|_p = 1$ for at least one i. So if v extends a p-adic absolute value, then $\log \max(|x_0|_v, \cdots, |x_n|_v) = 0$. Therefore, we have

$$h(\mathbf{x}) = \log \max(|x_0|, \cdots, |x_n|).$$

1.2.4 Definition. The field of definitions of a point $\mathbf{x} = (x_0 : \cdots : x_n) \in \mathbb{P}^n(\overline{\mathbb{Q}})$ is the number field

$$\mathbb{Q}(\mathbf{x}) = \mathbb{Q}(\frac{x_0}{x_j}, \cdots, \frac{x_n}{x_j}) \text{ for any } j \text{ such that } x_j \neq 0.$$

1.2.5 Theorem. *(Northcott) For any real numbers $d \geq 1$ and $b \geq 0$, the set*

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq d \textit{ and } h(\mathbf{x}) \leq b\}$$

is finite. In particular, for any number field K and $b \geq 0$, the set

$$\{\mathbf{x} \in \mathbb{P}^n(K) \mid h(\mathbf{x}) \leq b\}$$

is finite.

Proof. We choose homogeneous coordinates $(x_0 : \cdots : x_n)$ of x such that some coordinates equal to 1. Let K be a number field containing $x_0, \cdots, x_n$. If $j \in \{0, \cdots, n\}$ and $v \in M_K$, then

$$\max(|x_0|_v, \cdots, |x_n|_v) \geq \max(|x_j|_v, 1).$$

We sum over v and divide by $[K : \mathbb{Q}]$, so we obtain

$$h(\mathbf{x}) \geq h(x_j), \text{ where } h(x_j) \text{ is the height on } \mathbb{P}^1.$$

Moreover, $\mathbb{Q}(\mathbf{x}) \supseteq \mathbb{Q}(x_j)$. Hence, it is enough to prove that

$$\{x_j \in \overline{\mathbb{Q}} \mid [\mathbb{Q}(x_j) : \mathbb{Q}] \leq d \text{ and } h(x_j) \leq b\}$$

is finite. This amounts to saying that there are only finitely many algebraic numbers of bounded degree and bounded height [4, Theorem 1.6.8, p. 25]. This implies the finiteness in the statement. □

An immediate corollary of the finiteness property in Northcott's Theorem is the following important result due to Kronecker.

1.2.6 Theorem. *(Kronecker's Theorem) Let K be a number field, and let $\mathbf{x} = (x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$. Fix any i with $x_i \neq 0$. Then $h(\mathbf{x}) = 0$ if and only if the ratio x_j/x_i is a root of unity or zero for every $0 \leq j \leq n$.*

Proof. Without loss of generality, we may divide the coordinates of $\mathbf{x}$ by x_i and then reorder them, so we may assume that $\mathbf{x} = (1 : x_1, x_2 : \cdots : x_n)$. First suppose that every x_j is a root of unity. Then $|x_j|_v = 1$ for every absolute value on K and hence $h(\mathbf{x}) = 0$. Next suppose that $h(\mathbf{x}) = 1$. For each $r = 1, 2, \cdots$, let $\mathbf{x}^r = (x_0^r, \cdots, x_n^r)$. It is clear from the definition of the height that $(\mathbf{x}^r) = rh(\mathbf{x})$, so $h(\mathbf{x}^r) = 1$ for every $r > 1$. But $\mathbf{x} \in \mathbb{P}^n(K)$, so Northcott's Theorem tells us that the sequence $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \cdots$ contains only finitely many distinct points. Choose integers $s > r > 1$ such that $\mathbf{x}^s = \mathbf{x}^r$. This implies that $x_j^s = x_j^r$ for each $1 \leq j \leq n$ (since we have dehomogenized with $x_0 = 1$). Therefore, each x_j is a root of unity or is zero. □

Now, let us see how we can define the height function on $G = \mathbf{G}_m^n$.

1.2.7 Definition. To define the height function on $G = \mathbf{G}_m^n$, we consider the canonical embedding of G in $\mathbb{P}^n$ given by

$$\begin{array}{rcl} \iota : \mathbf{G}_m^n & \hookrightarrow & \mathbb{P}^n \\ (x_1, \cdots, x_n) & \mapsto & (1 : x_1 : \cdots : x_n) \end{array}$$

Hence, we define the height of $\mathbf{x} \in G$ as the height on $\mathbb{P}^n$ of the image of $\mathbf{x}$ by ι, that is $h(\iota(\mathbf{x}))$. By abuse of notation, we still denote it by $h(\mathbf{x})$. In this essay, we will use this height when we talk about height on $\mathbf{G}_m^n$.

Apart of the properties of height mentioned above, this height has the advantage that it plays along nicely with the group law on $\mathbf{G}_m^n(\overline{\mathbb{Q}})$. Hence we have the following properties for our heights that we call "elementary height properties" in the rest of this thesis.

1.2.8 Lemma. For all $\mathbf{x}, \mathbf{y} \in \mathbf{G}_m^n$ and $k \in \mathbb{Z}$, we have:

(a) $h(\mathbf{xy}) \leq h(\mathbf{x}) + h(\mathbf{y})$,

(b) $h(\mathbf{x}^k) = |k|h(\mathbf{x})$ if $k \geq 0$ or $n = 1$,

(c) $\max\{h(x_1), \cdots, h(x_n)\} \leq h(\mathbf{x}) \leq h(x_1) + \cdots + h(x_n)$ where $h(x_i)$ is the height on $\mathbb{P}^1$,

(d) $h(\mathbf{x}^k) \leq n|k|h(\mathbf{x})$,

(e) If $\zeta \in \mathbf{G}_m^n$ is torsion point, $h(\zeta\mathbf{x}) = h(\mathbf{x})$.

Proof. (a) We have for any $v \in M_K$:

$$\max\{1, |x_1y_1|_v, \cdots, |x_ny_n|_v\} \leq \max\{1, |x_1|, \cdots, |x_n|_v\} \max\{1, |y|_1, \cdots, |y_n|_v\}.$$

(b) If $k \geq 0$, then clearly, we have $h(\mathbf{x}) = kh(\mathbf{x})$ by definition.

If $n = 1$ and $k < 0$, then $h(x^k) = |k|h(x^{-1}) = |k|h(x)$ since $h(x^{-1}) = h(x)$ on $\mathbb{P}^1$.

(c) This is due to fact that for all $v \in M_K$ and $i \in \{1, \cdots, n\}$

$$\max\{1, |x_i|_v\} \leq \max\{1, |x_1|_v, \cdots, |x_n|_v\} \leq \max\{1, |x_1|_v\} \cdots \max\{1, |x_n|_v\}$$

(d) If $k \geq 0$, it is obvious. Assume that $k < 0$. We have

$$\begin{aligned} h(\mathbf{x}) &= |k|h(\mathbf{x}^{-1}) \leq |k|\big(h(x_1^{-1}) + \cdots + h(x_n^{-1})\big) \text{ by the second inequality of (c),} \\ &\leq |k|\big(h(x_1) + \cdots + h(x_n)\big) \text{ by the second case of (b),} \\ &\leq |k|n \max\{h(x_1), \cdots, h(x_n)\} \leq n|k|h(\mathbf{x}) \text{ by the first inequality of (c)} \end{aligned}$$

(e) Since torsion points have absolute value 1, the last assertion follows. □

1.2.9 Remark. Of course, the analogue version of Northcott's Theorem and Kronecker's Theorem on $\mathbf{G}_m^n$ hold.

1.3 Notations and Auxiliary Results

For a given positive integer $n \geq 1$, we denote $G = \mathbf{G}_m^n(\overline{\mathbb{Q}})$ the algebraic torus $(\overline{\mathbb{Q}}^\times)^n$. We will mostly work over the algebraic torus, and so for simplicity we will define and study subvarieties in that setting. A subvariety of G defined over a field K is the zero set of an ideal of

$$K[y_1^{\pm 1}, \cdots, y_n^{\pm 1}].$$

Following this convention, a subvariety is not necessarily irreducible. We also note that some auxiliary results will be needed such as Fiber Dimension Theorem, Bézout Theorem, Siegel's Lemma for the proofs of our theorems. We recall some of them in the appendices A and B.

2. Unlikely Intersections in Algebraic Tori

Our first theorem concerns the structure of "anomalous variety" from the Unlikely Intersections. Let us start with some definitions. We fix in the following a positive integer $n \geq 1$ and we denote $G = \mathbf{G}_m^n(\overline{\mathbb{Q}})$.

2.1 Structure Theorem and Anomalous Openness Conjecture

Let X be a (closed) irreducible variety in G defined over $\mathbb{C}$ for the moment. We recall that a coset is a translate of an algebraic sugbroup of G. A torus is an irreducible algebraic subbgroup of G. A torsion coset is a translate of an algebraic subgroup by a torsion point.

2.1.1 Definition. An irreducible subvariety Y of X is called *anomalous or X-anomalous* if there exists a coset K in G such that $Y \subset K$ and

$$\dim Y > \max\{0, \dim X + \dim K - n\}.$$

We define the *open anomalous* X^{oa} as the complementary in X of the union of all anomalous subvarieties of X.

2.1.2 Remark. We recall from intersection theory that if $X \cap K$ is nonempty then one would expect its dimension to be $\dim X + \dim K - n$. So the anomalous varieties are those whose dimension is strictly larger than the expected (we called such situation *unlikely intersection*). The inequality of the definition 2.1.1 can be stated as

$$\dim Y \geq \max\{1, \dim X + \dim K - n + 1\}.$$

We also remark that the definition remains unchanged when we require an equality in this relation. Indeed, if the inequality is strict then we can replace K with a larger coset to obtain an equality. In some situations, considering X^{oa} is not satisfying, hence we may also define the following subvarieties.

2.1.3 Definition. An irreducible subvariety Y of X is a *torsion-anomalous* if there exists a torsion coset K in G such that $Y \subset K$ and

$$\dim Y > \max\{0, \dim X + \dim K - n\}.$$

We define the *torsion anomalous* X^{ta} as the complementary in X of the union of all torsion-anomalous subvarieties.

2.1.4 Example. When $X = C$ is a curve, C^{oa} and C^{ta} can easily be determined. We have $C^{\mathrm{oa}} = C$ if C is not contained in a proper coset (i.e of dimension at most $n-1$), and $C^{\mathrm{oa}} = \emptyset$ otherwise. Similarly, $C^{\mathrm{ta}} = C$ if C is not contained in a proper torsion coset, and $C^{\mathrm{ta}} = \emptyset$ otherwise.

2.1.5 Example. When $X = S$ is a surface, then S^{oa} (resp. S^{ta}) is the set that remains of S after removing:

- S itself if it lies in any coset (resp. torsion coset) of dimension at most $n - 1$,
- all irreducible curves in S that lie in any coset (resp. torsion coset) of dimension at most $n - 2$.

The curves defined in the second point are called *anomalous* curves of S (resp. torsion anomalous curves). For instance, let us consider the surface S in $\mathbf{G}_m^3$ defined by $x_1 + x_2 = x_3$. In particular, S is a plane. The irreducible curves in S defined by

$$x_1 = \alpha_1 x_3, \ x_2 = \alpha_2 x_3, \tag{2.1.1}$$

with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ satisfying $\alpha_1 + \alpha_2 = 1$, are anomalous curves of S. Here S^{oa} is empty because any arbitrary point $(\xi_1, \xi_2, \xi_3) \in S$ belongs to these curves as such point verifies (2.1.1) with $\alpha_1 = \xi_1/\xi_3$ and $\alpha_2 = \xi_2/\xi_3$.

2.1.6 Example. If X is a hypersurface, that is if $\dim X = n-1$, it can be shown that $X^{\text{oa}} = X^{\circ}$ where X° is defined to be the complement in X of the union of all positive dimensional cosets contained in X [17, p. 2].

2.1.7 Remark. The condition that X is not contained in a proper algebraic subgroup (resp. in proper coset) of G means that the coordinates of X are multiplicatively independent (resp. modulo nonzero constant). We recall that $x_1, \cdots, x_n \in \overline{\mathbb{Q}}^{\times}$ are independent multiplicatively (resp. modulo constant) means that if $x_1^{a_1} \cdots x_n^{a_n} = 1$ (resp $x_1^{a_1} \cdots x_n^{a_n} = \alpha$ for some $\alpha \in \overline{\mathbb{Q}}^{\times}$) implies that $a_1 = \cdots = a_n = 0$.

2.1.8 Remark. Clearly, we have $X^{\text{oa}} \subset X^{\text{ta}}$. The natural question to ask is whether X^{oa} or X^{ta} is open. We shall see below that X^{oa} is open. X^{ta} is also open (under some assumptions) but we will not see its proof here (See [6, Theorem 4, p. 7]). The following theorem gives the structure of the anomalous subvarieties.

2.1.9 Theorem. *(Structure Theorem For an Algebraic Torus) Let G be the torus $\mathbf{G}_m^n$. Let X be an irreducible variety in G defined over $\mathbb{C}$. Then there exists a finite family Φ of tori in G such that any anomalous subvariety Y of X is a component of the intersection $X \cap gH$ for some $g \in G$ and some $H \in \Phi$ satisfying*

$$1 \leq n - \dim H \leq \dim X \ \textit{and} \ \dim Y = \dim X + \dim H - n + 1.$$

2.1.10 Remark. This theorem says that the set of anomalous subvarieties of X is contained in a finite number of translates of tori of codimension $\leq \dim X$.

2.1.11 Remark. An anomalous subvariety is *maximal* if it is not contained in a strictly larger anomalous subvariety of X. Hence it is sufficient to remove all these maximal anomalous subvarieties to obtain X^{oa}. Furthermore, in the proof of our Structure theorem, we only work with maximal anomalous subvarieties. We also remark that this theorem, although proved only over $\mathbb{C}$, remains valid over $\overline{\mathbb{Q}}$. Another important result in this section is a consequence of this theorem on the openness of X^{oa} which was called "Anomalous Openness Conjecture".

2.1.12 Theorem. *(Anomalous Openness Theorem) X^{oa} is open in X for the Zariski topology.*

The principal ingredient of the proof of Structure Theorem 2.1.9 is based on Chabauty's Result (See Appendix B). The openness of X^{oa} is deduced from the structure theorem. Indeed, we shall see that $X \setminus X^{\text{oa}}$ can be expressed as $\cup_{H \in \Phi} \mathcal{L}_H$, where Φ is the finite family defined in The Structure Theorem 2.1.9 and $\mathcal{L}_H$ are some Zariski closed in X.

2.1.13 Remark. As seen in the example above, X^{oa} can be empty. For a curve X, this is the case if X is not contained in a proper coset. For higher dimensional variety, this condition is not anymore sufficient.

2.2 Bounded Height Conjecture

Bombieri, Masser, and Zannier conjectured in [6] the following statement : for a given irreducible closed subvariety X of $\mathbf{G}_m^n$, the set of points of the open anomalous X^{oa} contained in the union of all algebraic subgroups of codimension at least $\dim X$ has bounded absolute Weil height. It was called the "Bounded Height Conjecture".

This conjecture corresponds to our problem on the boundedness of height of the intersections of X with the algebraic subgroups of G restricted by dimension. Hence we define for a given $d \in \mathbb{N}$, $G^{[d]}$ to be the union of all subgroup of G of codimension $\geq d$ (or of dimension $\leq n-d$).

In this essay, we give the height bound not only for points on X^{oa} contained in the union of such algebraic subgroups, but also for points of some generalization of X^{oa} near such subgroups with respect to the height. Namely, let be $S \subset G(\overline{\mathbb{Q}})$ and $\epsilon \geq 0$. We define *the truncated cone around* S as

$$\mathcal{C}(S,\epsilon) = \{\mathbf{x} \in G(\overline{\mathbb{Q}}) \mid \mathbf{x} = ab \text{ with } a \in S \text{ and } b \in G(\overline{\mathbb{Q}}) \text{ with } h(b) \leq \epsilon(1+h(a))\}.$$

Since a torsion point of $G(\overline{\mathbb{Q}})$ has height 0, therefore $C(S,\epsilon)$ contains S for any $\epsilon \geq 0$.

Also, we want to refine of the definition of X^{oa}. We assume for the moment that our variety is defined over $\mathbb{C}$. For a variety Y containing a point x, we recall that $\dim_x Y$ denotes the largest dimension of an irreducible component of Y passing through x. For an integer d, we define $X^{\mathrm{oa},[d]}$ as $X(\mathbb{C})$ deprived of all the set of points $x \in X(\mathbb{C})$ such that there exits an algebraic subgroup $H \subset G$ with

$$\dim_x(X \cap xH) \geq \max(1, d + \dim H - n + 1).$$

In particular, if $d = \dim X$, we have $X^{\mathrm{oa},[d]} = X^{\mathrm{oa}}$. This definition is only interesting when $d \geq \dim X$; otherwise, we clearly see that $X^{\mathrm{oa},[d]} = \emptyset$ if X has a positive dimension (we just consider $H = G$). The following result is then our main theorem in this section.

2.2.1 Theorem. *[16] Let $G = \mathbf{G}_m^n(\overline{\mathbb{Q}})$ and let X be an irreducible closed subvariety of G defined over $\overline{\mathbb{Q}}$. For any integer d, there exists $\epsilon > 0$ such that $X^{\mathrm{oa},[d]} \cap \mathcal{C}(G^{[d]},\epsilon)$ has a bounded height.*

In particular if we take $d = \dim X$, the theorem above implies the affirmation of the above the bounded height conjecture.

2.2.2 Theorem. *(Bounded Height Theorem) Let $G = \mathbf{G}_m^n$ and let X be an irreducible variety of G defined over $\overline{\mathbb{Q}}$. Then $X^{\mathrm{oa}}(\overline{\mathbb{Q}}) \cap G^{[\dim X]}$ has a bounded height.*

2.2.3 Example. As a simple example, we take X to be the line L defined by $x+y=1$ in $\mathbf{G}_m^2$. It cannot be contained in a proper coset, hence $X^{\mathrm{oa}} = L$. Algebraic subgroups of $\mathbf{G}_m^2$ can be described by at most two monomial dependence relations $x_1^{a_1}x_2^{a_2} = 1$ with integer exponents a_1 and a_2. For subgroups of dimension 1, one non-trivial relation suffices. If (x,y) is contained in such a subgroup then x and y are multiplicatively dependent. Hence the intersection of our curve with the union of all proper algebraic subgroups of $\mathbf{G}_m^2$ can be described by the solutions of

$$\tau^{a_1}(1-\tau)^{a_2} = 1.$$

The Bounded Height Conjecture now can be restated as claiming that the algebraic numbers τ such that τ and $(1-\tau)$ are multiplicatively dependent have bounded Weil height. Indeed, we have $h(\tau) \leq \log 2$ for any such algebraic numbers τ. Let us see here how we can get such bound. In this example, we consider the height function on $\mathbb{P}^1$. In what follows, ξ_i and η_i, for any $i \in \mathbb{N}$, denote roots of unity. Hence we write

$$\tau = \xi_1 t^a \text{ and } 1 - \tau = \eta_1 t^b,$$

for some nonzero algebraic numbers t, not a root of unity, and certain integers a, b. Hence we have

$$t^a = \xi_2 t^b + \eta_2. \tag{2.2.1}$$

It follows that

$$t^{-a} = \xi_3 t^{(a-b)} + \eta_3. \tag{2.2.2}$$

From (2.2.1), we may assume that $a \geq b \geq 0$. If $b = 0$, then $\tau = 1 - \eta_1$. Hence, by basic properties of heights, namely $h(x+y) \leq h(x) + h(y) + \log 2$, we have

$$h(\tau) \leq \log 2.$$

If $a = b$ and using (2.2.2), we apply the same argument to t^{-1} and use the fact that $h(\tau) = h(\tau^{-1})$ to get the same result. Now we assume that $a > b > 0$. Thus, in particular, t and t^{-1} are algebraic integers, so that t is a unit. Therefore, τ is also a unit. Let d be the degree of τ and $\sigma_1(\tau), \cdots, \sigma_d(\tau)$ the conjugates of τ. Using $h(\tau) = h(\tau^{-1})$, we have

$$h(\tau) = \frac{1}{d}\sum_{i=1}^{d} \log\max\left(1, |\sigma_i(\tau)|\right) = \frac{1}{d}\sum_{i=1}^{d} \log\max\left(1, |\sigma_i(\tau)|^{-1}\right),$$

where $|\cdot|$ denotes the usual archimedean absolute value on $\mathbb{C}$. By (2.2.1), we have $|t|^a \leq |t|^b + 1$ so that $|t|$ does not exceed the positive real root x_0 of

$$x^a = x^b + 1 \text{ (using Descartes' rules of signs, such root } x_0 \text{ exists and unique).}$$

Clearly, we have $x_0 > 1$ since $a > b$. Let $\lambda > 1$ be the real number such that $a = \lambda b$. We set $\zeta = x_0^b$. Then we have

$$\zeta^\lambda = \zeta + 1.$$

Also, we have $|t|^b \leq \zeta$, so that

$$\log|t| \leq \frac{1}{b}\log\zeta.$$

Therefore, we have

$$\log|\tau| \leq \lambda\log\zeta = \log(\zeta + 1). \tag{2.2.3}$$

We can apply the similar argument to τ^{-1}. Namely, by (2.2.2), we have

$$|t^{-1}|^a \leq |t^{-1}|^{a-b} + 1.$$

We set $\bar{\zeta}$ to be the real number > 1 satisfying

$$\bar{\zeta}^{\bar{\lambda}} = \bar{\zeta} + 1 \text{ where } \bar{\lambda} = \frac{\lambda}{\lambda - 1}.$$

Hence we have an analogue of (2.2.3), namely

$$\log|\tau^{-1}| \leq \bar{\lambda}\log\bar{\zeta} = \log(\bar{\zeta}+1). \tag{2.2.4}$$

By a similar discussion, both (2.2.3) and (2.2.4) remain true when we replace τ by any of its conjugates. Since τ is not a root of unity but invertible, we set $0<\mu<1$ be the rational number such that μd conjugates of τ that exceed 1 in absolute value. Then the number of conjugates of τ do not exceed 1 in absolute value is $(1-\mu)d$. It follows from (2.2.3) applied to the conjugates of τ exceeding 1, that

$$h(\tau) \leq \mu\log(\zeta+1).$$

From (2.2.4) applied to the conjugates of τ not exceeding 1, we have

$$h(\tau) \leq (1-\mu)\log(\bar{\zeta}+1). \tag{2.2.5}$$

For any $A, B \geq 0$, we have

$$\min(A\mu,(1-\mu)B) \leq \frac{AB}{A+B}.$$

Hence we have

$$h(\tau) \leq S(\lambda),$$

where

$$S(\lambda) = \frac{\log(\zeta+1)\log(\bar{\zeta}+1)}{\log(\zeta+1)+\log(\bar{\zeta}+1)}.$$

Since $S(\lambda)\to\log 2$ for both $\lambda\to 1^+$ and $\lambda\to+\infty$. This gives the results of the example.

2.2.4 Remark. Some remarks on the Bounded Height Theorem.

1. The fact that we consider X^{oa} is necessary.

For instance, let us see the case of curves. Suppose that $X=C$ is a curve such that $C\subset gH$ with H a proper subgroup of $\mathbf{G}_m^n$ (thus $X\setminus X^{\mathsf{oa}}=C$). If g is a torsion point, then C is contained in a proper subgroup H' of $\mathbf{G}_m^n$. Therefore, the set $C\cap H'(\mathbb{Q})=C(\mathbb{Q})$ has unbounded height. If instead g is not a torsion point, we proceed as follows. After a monoidal change of coordinates, we may assume that H is the subtorus $x_n=1$, gH is given by $x_n=g_n$, and x_1 is not a constant function on C. Then the torus H_m given by $x_1=x_n^m$ has a nonempty intersection with C for infinitely many integers m. Every point $\mathbf{x}$ of $C\cap H_m$ has $x_1=g_n^m$; hence $h(\mathbf{x})\geq(|m|)h(g_n)$, which is unbounded as $m\mapsto\infty$.

2. The set $X^{\mathsf{oa}}(\overline{\mathbb{Q}})\cap G^{[\dim X]}$ can be infinite in X. Indeed, in the previous example, taking $\tau\neq 1$ a root of unity gives infinitely many solutions. However, for example for an irreducible curve X, the Bounded Height Conjecture implies (through Northcott's theorem) the finiteness of the set of points in $X\cap G^{[1]}$ which are defined over a given number field, or even which have bounded degree over $\mathbb{Q}$. For instance, for the above example where $X\,:\,x+y=1$, the rational points in $X\cap G^{[1]}$ are $(-1,2),(2,-1),(1/2,1/2)$.

3. If X is curve, then X is not contained in a proper coset is necessary and sufficient for the boundness of the height on $X^{\mathsf{oa}}\cap G^{[\dim X]}$. For higher dimension variety X, this is not generally sufficient to guarantee the boundedness conclusion. Let us see that for the case of $\dim X=2$. As a simple example, let us consider the surface X defined in $\mathbf{G}_m^4$ defined by $x+y=1$ and $z+w=c$, where c is a nonzero constant. Then $X\cap G^{[2]}$ has unbounded height. In fact, for $a,b\in\mathbb{Z}$, let us consider the algebraic subgroup of codimension 2 defined by $z=x^a$ and $w=y^b$. By Zhang's theorem [26, Theorem

4.9, p. 150], for any algebraic numbers τ, not a root unity, we have $h(\tau) + h(1-\tau) \geq h_0 > 0$, where $h_0 = 1/5 \ln(5/3)$. Then we have $h(x) + h(y) \geq h_0$, thus $h(z) + h(w) \geq \min(|a|, |b|)h_0$.

4. The codimension of the involved subgroup is optimal in the sense that it is minimal as a function of $\dim X$ as soon as $X^{oa} \neq \emptyset$. The dimension cannot be reduced even if we replace X^{oa} by any non empty Zariski open subset. In fact, $U \cap G^{[\dim X - 1]}$ may have unbounded height. For instance, for a curve X, $U \cap G^{[\dim X-1]} = U$ has unbounded height for any open dense $U \subset X$.

2.2.5 Remark. These remarks also ensure that for the boundedness of height or finiteness on the intersection of a Zariski open non-empty $U \subset X$ with $G^{[d]}$, one must assume that $d \geq \dim X$ and $d \geq \dim X + 1$ respectively. The proof of Theorem 2.2.1 uses compactness argument. The latter makes the proof ineffective from a formal point of view.

2.3 Finiteness Theorem

One problem in Diophantine Geometry is to know whether a given set is finite or not. We have claimed that the set $X^{oa}(\overline{\mathbb{Q}}) \cap G^{[\dim X]}$ has a bounded height. So we may ask if such set is finite. We remark that if X is defined over a number field K, then by Northcott's Theorem, we deduce the finiteness of this set. However, we can recover the finiteness even dropping this restriction. Indeed, we shall prove the following result.

2.3.1 Theorem. *(Finiteness Theorem) Let X be a closed and irreducible variety in G defined over $\overline{\mathbb{Q}}$. Then the intersection $X^{oa}(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$ is a finite set.*

In particular, for the case of curves, we have the finiteness result under slightly weaker condition on X.

2.3.2 Theorem. *Let X be a irreducible curve in G and not be contained in any proper algebraic subgroup of G. Then the intersection $X(\overline{\mathbb{Q}}) \cap G^{[2]}$ is finite.*

2.3.3 Remark. Theorem 2.3.1 does not apply directly to get Theorem 2.3.2 since we only assume X to be a curve which is not contained in a proper algebraic subgroup (so it may lie in translate of a proper algebraic subgroup). The former theorem gives only the finiteness on the open set X^{oa}. So we need further arguments. These results give answers of our problem on the finiteness of the intersections of X with algebraic subgroups of G restricted by dimension.

2.3.4 Example. Let us consider the line X defined by $x + y = 1$ in $\mathbf{G}_m^2$. Then the points of $X \cap G^{[2]}$ correspond to the torsion points on the line X in $\mathbf{G}_m^2$. Such line has at most two torsion points: satisfying $|x| = |y| = 1$ and corresponding to the intersection of the circles in $\mathbb{C}$ with radii $|a|$,$|b|$ and centers $0, 1$. It contains the torsion points $(e^{\pm\pi i/3}, e^{\mp\pi i/3})$ and no other points.

2.3.5 Example. For each algebraic number $\tau \neq 0, \pm 1$, the points $(\tau, \tau-1, \tau+1)$ parametrizes a curve C in $\mathbf{G}_m^3$. Hence the finiteness theorem claims that the set of algebraic points τ such that $\tau, \tau+1, \tau-1$ satisfy two independent multiplicative relations is finite. Such τ satisfies $h(\tau) \leq \log 2$ by the result of Example 2.2.3. Once we show that the degree of τ is uniformly bounded, then there is only a finite list of possibilities for τ.

By the multiplicative dependence assumption, there is a non-zero algebraic number t, not a root of unity, ξ_1, ξ_2, ξ_3 roots of unity and integers a, b, c such that

$$\tau = \xi_1 t^a, \;\; \tau + 1 = \xi_2 t^b \text{ and } \tau - 1 = \xi_3 t^c.$$

We can suppose that b, c are not both 0, and that a, b, c are not all equal 1. Moreover, we can suppose that a, b, c are coprime. Assume that $a \neq 0$. Let τ' be any conjugate of τ over $\mathbb{Q}$ and put

$$\tau' = re^{i\theta},\ r > 0,\ \theta \in [-\pi, \pi).$$

Let us set $\alpha = 2b/a$ and $\beta = 2c/a$. Then we have

$$\begin{aligned} |\tau' + 1|^2 &= |\tau'|^\alpha, \\ |\tau' - 1|^2 &= |\tau'|^\beta. \end{aligned}$$

This means that

$$\begin{aligned} r^2 + 1 + 2r\cos\theta &= r^\alpha, \\ r^2 + 1 - 2r\cos\theta &= r^\beta. \end{aligned}$$

Hence, we have

$$2(r^2 + 1) = r^\alpha + r^\beta.$$

A direct calculus argument (or the Descartes' Rule of Signs) shows that there are at most two positive roots r of this equation for given α and β. Also, for r given, it is clear that $\cos\theta$ is uniquely determined so that $re^{i\theta}$ has at most two possibilities. There are therefore at most 4 possibilities for τ'. This proves that τ is of degree at most 4. Moreover, τ is necessarily a unit and so the product of the absolute values of its conjugates over $\mathbb{Q}$ equals 1. This further restricts the possibilities. When $a = 0$, so that τ is a root of unity and we may argue in a similar way with $\tau + 1$ replacing τ if $b \neq 0$ and with $\tau - 1$ replacing τ if $c \neq 0$. Again, this shows that τ must have degree at most 4.

2.3.6 Remark. As mentioned above, the condition that X is not contained in a proper subgroup is definitively necessary for the validity of this Theorem. However, the condition that X does not lie in a translate of some proper subgroup is not necessary. Let us see this situation in the case of curves. Assume that C is a curve which it is not contained in a proper subgroup but does lie in a translate of some subgroup H. Suppose that $\dim(H) = 1$. So after a monoidal transformation to change coordinates, we may assume that C is given by $x_2 = g_2, \cdots, x_n = g_n$ with $g_i \in \overline{\mathbb{Q}}^\times$. These must be multiplicatively independent since C does not lie in a strict subgroup. It follows that $C \cap H'$ is empty whenever codim $H' \geq 2$. So Finiteness Theorem does hold in this case.

2.3.7 Remark. To prove Theorem 2.3.1, the main issue is to prove that the set of points of

$$\Sigma = X^{\mathsf{oa}}(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$$

is finite. In order to apply Northcott's Theorem, we shall see that Σ has bounded height and the degree of the extension field of the points of Σ is uniformly bounded. The bounded height condition already holds by Bounded Height Theorem 2.2.2 since $G^{[d+1]} \subset G^{[d]}$ for any integer $d \geq 1$. We mention here that the proof of such result relies on a theorem of David and Amoroso on the problem of finding the best lower bounded height (called Lehmer's Problem).

2.4 Abelian Varieties and Other Algebraic Groups

The whole present context has a natural analogue for (semi)abelian varieties, that is we may replace the algebraic group $\mathbf{G}_m^n$ by a (semi)abelian variety. Recall that an abelian variety is an algebraic group whose underlying space is "geometrically integral" and projective. A *semiabelian* variety G is an extension of an abelian variety A by a torus. The multiplicative group $\mathbf{G}_m$, its powers (algebraic tori), and elliptic curves which are one-dimensional abelian varieties are special cases of semiabelian varieties.

3. Proofs of Main Results and its Generalizations

3.1 Preliminaries For The Structure Theorem

Here we give a proof of Structure Theorem 2.1.9 by following the idea from [6]. We need the notion of degree theory and Chow form of a variety. We refer to Appendix B for a brief recall. We will use as well some known independent results.

Let X be an irreducible variety in $G = \mathbf{G}_m^n$ defined over $\mathbb{C}$. Then we consider the Chow ideal $\mathcal{I}(X) = (P_1, \cdots, P_N)$ defined from the Chow form of X (Appendix B) where $P_1, \cdots, P_N$ are polynomials over $\mathbb{C}$ in $x_1, \cdots, x_n$. In particular, these polynomials define the ideal of X. Let us fix some notations. We denote by $J(X)$ the Jacobian matrix with N rows and n columns

$$J(X) = \left(\frac{\partial P_i}{\partial x_j}\right).$$

For $\mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we can define the following row

$$r(\mathbf{z}) = (\frac{z_1}{x_1}, \cdots, \frac{z_n}{x_n}).$$

Then for any integer $h \geq 1$ and $\mathbf{z}_1, \cdots, \mathbf{z}_h$ in $\mathbb{C}^n$, we define $J(\mathbf{z}_1, \cdots, \mathbf{z}_h; X)$ as the matrix with $N + h$ rows and n columns obtained by extending the rows of $J(X)$ with the h rows $r(\mathbf{z}_1), \cdots, r(\mathbf{z}_h)$.

As the entries lie in the field $\mathbb{C}(x_1, \cdots, x_n)$, then we can consider the rank of these matrices. If Y is any irreducible subvariety of X in G, we write rank_Y for the rank of $J(\mathbf{z}_1, \cdots, \mathbf{z}_n; X)$ considered as a matrix in the function field $\mathbb{C}(Y)$. In other words, we replace the x_i (considered as variables) with the x_i (considered as functions on Y).

If $\mathbf{a} = (a_1, \cdots, a_n)$ lies in $\mathbb{Z}^n$, we recall $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Then for every $i = 1, \cdots, n$

$$\frac{\partial \mathbf{x}^{\mathbf{a}}}{\partial x_i} = \mathbf{x}^{\mathbf{a}} \frac{a_i}{x_i}.$$

Hence, if $\mathbf{a}_1, \cdots, \mathbf{a}_h$ are in $\mathbb{Z}^n$, then $J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X)$ corresponds to the Jacobian matrix of the intersection of X with a translate of the subgroup defined by the $\mathbf{x}^{\mathbf{a}_1} = \cdots = \mathbf{x}^{\mathbf{a}_h} = 1$.

For any integer $h \geq 1$ and $\mathbf{a}_1, \cdots, \mathbf{a}_h$ in $\mathbb{Z}^n$, we define the map $\varphi = \varphi(\mathbf{a}_1, \cdots, \mathbf{a}_h)$ by

$$\begin{array}{rcl} \varphi : \mathbf{G}_m^n & \longrightarrow & \mathbf{G}_m^h \\ \mathbf{x} & \mapsto & (\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}) \end{array}$$

The proof our Structure Theorem is based on the following six lemmas. We denote by r the dimension of X. For any irreducible variety Y in G, the term $\mathbb{C}(Y)$ denotes the function field of Y.

3.1.1 Lemma. [6, Lemma 1] Assume that $\mathbf{a}_1, \cdots, \mathbf{a}_h \in \mathbb{Z}^n$ and $\text{rank}_X J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X) \leq n - r + h - 1$. Then $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}$ are algebraically dependent on X.

Proof. The rank condition means that there are at least $(N + h) - (n - r + h - 1) = N - n + r + 1$ linearly independent relations over $\mathbb{C}(X)$ among the rows of $J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X)$. Every of such relation is of the form

$$\gamma_1 \frac{\partial P_1}{\partial x_j} + \cdots + \gamma_N \frac{\partial P_N}{\partial x_j} + \gamma_{N+1} \frac{a_j^{(1)}}{x_j} + \cdots + \gamma_{N+h} \frac{a_j^{(h)}}{x_j} = 0 \text{ for } j = 1, \cdots, n, \tag{3.1.1}$$

where $\mathbf{a}_i = (a_1^{(i)}, \cdots, a_n^{(i)}) \in \mathbb{C}^n$ for $i = 1, \cdots, h$ and $\gamma_1, \cdots, \gamma_{N+n} \in \mathbb{C}(X)$.

Suppose by contradiction that $\varphi_1 = \mathbf{x}^{\mathbf{a}_1}, \cdots, \varphi_h = \mathbf{x}^{\mathbf{a}_h}$ are algebraically independent on X. Then $\mathbb{C}(\varphi_1, \cdots, \varphi_h)$ is a purely transcendental subfield of $\mathbb{C}(X)$. It implies that the derivations $\frac{\partial}{\partial \varphi_1}, \cdots, \frac{\partial}{\partial \varphi_h}$ extend to $\mathbb{C}(X)$. We use this fact to prove the following claim.

Claim: For every $j = 1, \cdots, n$, we have

$$\gamma_1 \frac{\partial P_1}{\partial x_j} + \cdots + \gamma_N \frac{\partial P_N}{\partial x_j} + = 0.$$

Since $P_1, \cdots, P_N$ belong to the ideal of X, then they are identically zero on X. Hence for $i = 1, \cdots, m$, we have the equation

$$0 = \frac{\partial P_i(x)}{\partial \varphi_l} = \sum_{j=1}^{N} \frac{\partial P_i}{\partial x_j} \frac{\partial x_j}{\partial \varphi_l}, \text{ for } l = 1, \cdots, h.$$

Multiplying the equation 3.1.1 by $\frac{\partial x_j}{\partial \varphi_l}$ and summing over j, we obtain

$$\gamma_{m+1} \sum_{j=1}^{n} \frac{a_j^{(1)}}{x_j} + \cdots + \gamma_{m+h} \sum_{j=1}^{n} \frac{a_j^{(h)}}{x_j} = 0, \text{ for } l = 1, \cdots, h.$$

On the other hand, we have

$$\delta_{i,l} = \frac{\partial \varphi_i}{\partial \varphi_l} = \mathbf{x}^{\mathbf{a}_i} \sum_{j=1}^{n} \frac{a_j^{(i)}}{x_j} \frac{\partial x_j}{\partial \varphi_l} = \varphi_i \sum_{j=1}^{n} \frac{a_j^{(i)}}{x_j} \frac{\partial x_j}{\partial \varphi_l}.$$

Replacing the $\sum_{j=1}^{n} \frac{a_j^{(i)}}{x_j} \frac{\partial x_j}{\partial \varphi_l}$ with $\delta_{i,l} \varphi_i^{-1}$ in the previous equation, we obtain $\gamma_{N+l} \varphi_l^{-1} = 0$ for every $l = 1, \cdots, h$. Hence we proved the claim.

Now, the claim implies that the $N - n + r + 1$ relations remain independent which shows that $\text{rank}_X J(X) \leq N - (N - n + r + 1) = n - r - 1$. Let us denote by $\mathcal{P}(X)$ the prime ideal of X. If $\mathcal{P}(X) = \mathcal{I}(X)$, then this inequality would contradict the Jacobian Criterion. In general, $\mathcal{P}(X)$ is the only isolated component of $\mathcal{I}(X)$, thus there exists $P \notin \mathcal{P}(X)$ such that $\mathcal{P}(X) \subset P^{-1}\mathcal{I}(X)$. Then the same argument applies to get the contradiction. □

In this situation, we see that the image $\varphi(X)$ has dimension at most $h-1$ in $\mathbf{G}_m^h$. For $\mathbf{a}_1, \cdots, \mathbf{a}_h \in \mathbb{Z}^n$, we define the algebraic subgroup $H(\mathbf{a}_1, \cdots, \mathbf{a}_h)$ of G defined by the equation $\mathbf{x}^{\mathbf{a}_1} = \cdots = \mathbf{x}^{\mathbf{a}_h} = 1$. If $\mathbf{a}_1, \cdots, \mathbf{a}_h$ are $\mathbb{Q}$-linearly independent, then such subgroup has dimension $n - h$.

3.1.2 Lemma. [6, Lemma 2] Assume that Y is X-anomalous subvariety of dimension s and lies in a translate of $H(\mathbf{a}_1, \cdots, \mathbf{a}_h)$ with $h = r - s + 1$. Then $\text{rank}_Y J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X) \leq n - r + h - 1$.

Proof. There are $\omega_1, \cdots, \omega_h \in \mathbb{C}$ such that the functions $\mathbf{x}^{\mathbf{a}_1} - \omega_1, \cdots, \mathbf{x}^{\mathbf{a}_h} - \omega_h$ vanish on the translate of H in question. Hence $P_1, \cdots, P_N, P_{N+1} := \mathbf{x}^{\mathbf{a}_1} - \omega_1, \cdots, P_{N+h} := \mathbf{x}^{\mathbf{a}_h} - \omega_h$ vanish on Y since Y belongs to the intersection of X and with the translate on question. By Jacobian criterion, we have

$$\text{rank}_Y \left(\frac{\partial P_i}{\partial x_j} \right)_{\substack{i=1,\cdots,N+h \\ j=1,\cdots,n}} \leq n - s = n - r + h - 1.$$

For any $\mathbf{a} \in \mathbb{Z}^n$ and $\omega \in \mathbb{C}$, we have

$$\frac{\partial(\mathbf{x}^{\mathbf{a}-\omega})}{\partial x_i} = \mathbf{a}^{\mathbf{a}} \frac{a_i}{x_i} \ (i = 1, \cdots, n).$$

Using these identities, we have

$$\text{rank}_Y \left(\frac{\partial P_i}{\partial x_j} \right)_{\substack{i=1,\cdots,N+h \\ j=1,\cdots,n}} = \text{rank}_Y J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X)$$

the lemma follows. We notice here that the rows we added for the two matrices differ from $\mathbf{x}^{\mathbf{a}} \frac{a_i}{x_i}$ by $\mathbf{x}^{\mathbf{a}}$ which are invertible functions on all of G. □

3.1.3 Lemma. [6, Lemma 3] Assume that Y is a maximal X-anomalous and lies in a translate K of $H(\mathbf{a}_1, \cdots, \mathbf{a}_h)$ with $h = r - s + 1$. Then $\dim \varphi(X) \leq h - 1$.

Proof. Under the map φ, the image of K is a point w. Assume by contradiction that $\dim \varphi(X) < h-1$. By the Fundamental Dimension Theorem (a), every component of $\varphi_{|X}^{-1}(w) = X \cap \varphi^{-1}(w)$ has dimension stricly larger than $r - (h - 1) = s$. Such component remains X-anomalous and Y lies in one of such components. This contradicts the maximality of Y. Let $y_1, \cdots, y_r$ be generic linear polynomials in $x_1, \cdots, x_n$ and let $\left(\frac{\partial}{\partial y_l}\right)_{1 \leq l \leq r}$ be r corresponding independent derivations on $\mathbb{C}(X)$. □

3.1.4 Lemma. [6, Lemma 4] Assume that there exists $\mathbf{z} \neq 0$ in $\mathbb{C}^n$ such that $\text{rank}_X J(\mathbf{z}, X) \leq n - r$. Then X is X-anomalous.

Proof. We have $\text{rank}_X J(\mathbf{z}; X) = r - n$ since $\text{rank}_X J(\mathbf{z}; X) \geq \text{rank}_X J(X) = n - r$. Hence we deduce a relation on X

$$\frac{z_j}{x_j} = \gamma_1 \frac{\partial P_1}{\partial x_j} + \cdots + \gamma_N \frac{\partial P_N}{\partial x_j}, \tag{3.1.2}$$

where $\gamma_1, \cdots, \gamma_N \in \mathbb{C}(X)$ and for every $j = 1 \cdots, n$. Since $P_i = 0$ for $i = 1, \cdots, N$, then

$$0 = \frac{\partial P_i(x)}{\partial y_l} = \sum_{j=1}^{n} \frac{\partial P_i}{\partial x_j} \frac{\partial x_j}{\partial y_l}, \text{ for } l = 1, \cdots, h.$$

Therefore, multiplying by $\frac{\partial x_j}{\partial y_l}$ the previous equation 3.1.2 and summing over j, we obtain

$$\sum_{j=1}^{n} z_j \frac{1}{x_j} \frac{\partial x_j}{\partial y_l} = 0 \ (l = 1, \cdots, r).$$

In particular, we can deduce a relation $z_1 dx_1/x_1 + \cdots + z_n dx_n/x_n = 0$ on the differential of X by setting $dx_j = \frac{\partial x_j}{\partial y_l}$ for some $l \in \{1, \cdots, r\}$. By integration, it yields a relation $z_1 \log(x_1/\zeta) + \cdots + z_n \log(x_n/\zeta_n) = 0$ which holds in any neighborhood of any nonsingular point $(\zeta, \cdots, \zeta)$ of X. By

Chabauty result mentioned above, this defines a μ-variety M of dimension $n-1$ containing $W = X$. By applying Chabauty's Theorem (mentioned above) for $I = X$ there exists an algebraic μ-variety (in Chabauty's terminology) A, which is just a coset, containing X with

$$\dim A \leq \dim M + \dim W - \dim I = n - 1.$$

This coset A gives a non-zero $\mathbf{a} \in \mathbb{Z}$ with $\mathbf{x}^{\mathbf{a}}$ being constant on X. This shows that X is X-anomalous. $\square$

3.1.5 Lemma. [6, Lemma 5] Assume that Y is X-anomalous and lies in a translate K of $H(\mathbf{a}_1, \cdots, \mathbf{a}_h)$ with $h = r - s + 1$. Moreover, assume that

- $Y \nsubseteq X_{\mathsf{Sing}}$ (singular locus of X),
- $\mathbf{a}_1, \cdots, \mathbf{a}_h$ are $\mathbb{Q}$-linearly independent,
- $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}$ have transcendence degree $h-1$,
- $\varphi(K)$ is a nonsingular point in $\varphi(X)$.

Then there exists $\mathbf{z} \neq 0$ in $\mathbb{C}^n$ such that $\mathrm{rank}_Y J(\mathbf{z}; X) \leq n - r$.

Proof. By the assumption on the transcendence degree, we deduce that $\dim \varphi(X) = h-1$. Hence $\varphi(X)$ is defined by a single polynomial equation $F = 0$ in $\mathbf{G}_m^h$. By a (classical) result from [20, Proposition 3, p. 188], we can find independent derivations $\delta_1, \cdots, \delta_r$ on $\mathbb{C}(X)$ such that $\delta_l(x_j)$ are regular on $X \setminus X_{\mathsf{Sing}}$ for $l = 1, \cdots, r$ and for $j = 1, \cdots, n$. For simplicity, we set $\varphi_1 = \mathbf{x}^{\mathbf{a}_1}, \cdots, \varphi_h = \mathbf{x}^{\mathbf{a}_h}$. Since $F(\varphi_1, \cdots, \varphi_h) = 0$ on X, we have for any $l = 1, \cdots, r$

$$0 = \delta_l(F(\varphi_1, \cdots, \varphi_h)) = \sum_{i=1}^{h} F_i(\varphi_1, \cdots, \varphi_h)\delta_l(\varphi_i) \text{ on } X,$$

where $F_i = \dfrac{\partial F}{\partial x_i}$. For any $l = 1, \cdots, r$ and $i = 1, \cdots, h$, we have

$$\delta_l(\varphi_i) = \varphi_i \sum_{j=1}^{n} \frac{a_j^{(i)}}{x_j} \delta_l(x_j),$$

where $\mathbf{a}_i = (a_1^{(i)}, \cdots, a_n^{(i)})$. Hence we have for any $l = 1, \cdots, r$

$$\sum_{j=1}^{n} \sum_{i=1}^{h} F_i(\varphi_1, \cdots, \varphi_h)\varphi_i \frac{a_j^{(i)}}{x_j} \delta_l(x_j) = 0 \text{ on } X,$$

Since Y is not contained in the singular locus of X, we can specialize to Y in K. Writing $\omega = \varphi(K) = (\omega_1, \cdots, \omega_h)$, we set $z_j = \sum_{i=1}^{h} F_i(\omega)\omega_i a_j^{(i)}$ for any $j = 1, \cdots, n$ i.e

$$\mathbf{z} = \sum_{i=1}^{h} F_i(\omega)\omega_i \mathbf{a}_i,$$

which holds on Y. We have $\mathbf{z} \neq 0$. In fact, $\mathbf{a}_1, \cdots, \mathbf{a}_h$ are linearly independent over $\mathbb{Q}$ hence also over $\mathbb{C}$. Moreover, the F_i's are not all zero since ω is a nonsingular point of $\varphi(X)$ and $\omega_i \neq 0$ since $\omega_i \in \mathbf{G}_m^1$ for all $i = 1, \cdots, h$. Let set $v_l (l = 1, \cdots, r)$ to be the columns with entries $\delta_l(x_j)$ $(j = 1, \cdots, n)$. With the same computation as in the previous lemma, we deduce that $J(\mathbf{z}; X)v_l = 0$ $(l = 1, \cdots, r)$ on Y. Finally, the linear independence of $\delta_1, \cdots, \delta_r$ implies the linear independence of $v_1, \cdots, v_r$. This implies that $\text{rank}_Y J(\mathbf{z}; X) \leq n - r$. This completes the proof of the present lemma. □

3.1.6 Lemma. There is a $c(n)$ which depends only on n, with the following property. Suppose that X is X-anomalous of positive dimension r and degree Δ. Then there exists $\mathbf{a} \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{a}}$ is constant on X and

$$0 < |\mathbf{a}| \leq c(n)\Delta^{n-1}.$$

Proof. Let $y_1, \cdots, y_r$ be a transcendence basis given by sufficiently generic linear polynomials in $x_1, \cdots, x_n$. So $\mathbb{C}(y_1, \cdots, y_r)$ is a purely transcendental extension of $\mathbb{C}$. Such field has a proper set of absolute values satisfying a product formula: the absolute values corresponding to irreducible polynomials and the absolute corresponding to the total degree (See Appendix B).

Now $\mathbb{C}(X)$ is an algebraic extension of $L = \mathbb{C}(y_1, \cdots, y_r)$ with degree Δ. We choose an embedding of $\mathbb{C}(X)$ in an algebraic closure $\bar{L}$ of L. There is a Weil logarithm function h on $\overline{L}$ extending the one on L. By restriction, we get a height on $\mathbb{C}(X)$.

Now, we claim that for all $z \in \mathbb{C}(X) \setminus \mathbb{C}$

$$h(z) \geq \frac{1}{\Delta}.$$

This will also show that the zero height group Z of $\mathbb{C}(X)$ remains Z_L. So let $z \in \mathbb{C}(X) \setminus \mathbb{C}$, then it has $m \leq \Delta$ conjugates $z_1, \cdots, z_m$ in $\overline{L}$. Since $z \notin \mathbb{C}$, then we can find an elementary symmetric function w in $z_1, \cdots, z_m$ such that $w \in L$ but not in $\mathbb{C}$. For any ultrametric absolute valuation $|\cdot|_v$, we have

$$\max\{1, |w|_v\} \leq \max\{1, |z_1|_v\} \cdots \max\{1, |z_m|_v\}.$$

Therefore, we have

$$h(w) \leq h(z_1) + \cdots + h(z_m).$$

Using $h(w) \geq 1$ and $h(z) = h(z_1) = \cdots = h(z_m)$, the claim follows.

The next claim is the following: for any $i = 1, \cdots, x_n$, we have

$$h(x_i) \leq 1.$$

Let y be any generic linear polynomial in $x_1, \cdots, x_n$. The Chow form has a degree Δ, hence $y^\Delta + w_1 y^\Delta + \cdots + w_\Delta = 0$, where $w_j \in \mathbb{C}[y_1, \cdots, y_r]$ has total degree at most $j (j = 1, \cdots, \Delta)$. Thus for any ultrametric absolute value $|\cdot|_v$, there exists j such that

$$|y|_v^\Delta \leq \max\{|w_1 y^{\Delta-1}|_v, \cdots, |w_\Delta|\} = |w_j y^{\Delta-j}| \leq E^j |y|^{\Delta-j},$$

where $E = e$ if the valuation extends the total degree valuation and $E = 1$ otherwise. It follows that $\max\{1, |y|_v\} \leq E$. As each x_i is a linear combination of such y with coefficients in $\mathbb{C}$, then we get the same upper bound for $\max\{1, |x_i|_v\}$. Hence the claims follows.

Now, we apply the Lemma 2.2 of [8, p. 457] to $x_1, \cdots, x_n$. As X is anomalous, there exists a subgroup of rank at most $n-1$ over Z. Combining with the second claim, for any positive integer T, there exists a nonzero $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ with $|a_i| \leq T (i = 1, \cdots)$ and $h(z) \leq cT^{-\frac{1}{n-1}}$ with $z = \mathbf{x}^{\mathbf{a}}$, where c depends only on n. Choosing T to be minimal to contradict the first claim, we deduce that $z = \mathbf{x}^{\mathbf{a}}$ is constant on X. This completes the proof of the lemma. □

3.2 Proof of Structure Theorem and its Abelian case

We give a generalization of the Structure Theorem. Before that, let us see the sketch of the proof of this latter theorem. We still denote by r the dimension of X and by Δ its degree. With these six lemmas, we can prove the following proposition. For $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$, we use the norm $|\mathbf{a}| = \max\{|a_1|, \cdots, |a_n|\}$.

3.2.1 Proposition. For any $1 \le r \le n$, there exist two constants $c(n,r)$ and $\mu(n,r)$ which depend only on n and r, with the following property. For any X-anomalous subvariety Y, there exists a nonzero $\mathbf{a} \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{a}}$ is constant on Y and

$$0 < |\mathbf{a}| \le c(n,r)\Delta^{\mu(n,r)}.$$

Proof. We prove it by induction on $\dim X = r$. The case $r = 1$ of curves holds with $\mu(n,r) = n-1$ by applying the Lemma 3.1.6.

Assume now that X has dimension $r \ge 2$ and suppose that the result holds for anomalous subvarieties with lower dimension. So let Y be an X-anomalous subvariety of X. We may assume that Y is maximal. Again by Lemma 3.1.6, we may suppose that $Y \ne X$ and so $s \le r-1$.

By defintion, there exists some algebraic subgroup $H = H(\mathbf{a}_1, \cdots, \mathbf{a}_h)$, with $\mathbf{a}_1, \cdots, \mathbf{a}_h$ are $\mathbb{Q}$-linearly independent in $\mathbb{Z}^n$, such that Y lies in some translate of H. Hence we have $s \ge r + (n-h) - n + 1$, and so $h \ge r - s + 1 \ge 2$. We can assume that $h = r - s + 1$ (by enlarging the subgroup if necessary). By Lemma 3.1.2, we have $\mathrm{rank}_Y J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X) \le n - r + h - 1$.

• Assume first that $\mathrm{rank}_X J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X) > n - r + h - 1$. This implies that there exists a minor F of $J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X)$ of size (number of rows) at least $n - r + h$ which does not vanish identically on X but which is identically 0 on Y. Let $\tilde{X}$ be an irreducible component of $X \cap \{F = 0\}$ containing Y. Then $\dim \tilde{X} \le r - 1$. Moreover, Y is a $\tilde{X}$-anomalous by considering the same translate of H. By the induction hypothesis, there exists a nonzero $\mathbf{a} \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{a}}$ is constant on Y. By Bézout Theorem, we have

$$\deg \tilde{X} \le \deg X \deg F \le c\Delta^2, \text{ with } c \text{ depends only on } n.$$

Hence we have the required bound for $|\mathbf{a}|$ with $\mu(n,r) = 2\mu(n, r-1)$.

• Now, we consider the case $\mathrm{rank}_X J(\mathbf{a}_1, \cdots, \mathbf{a}_h; X) \le n - r + h - 1$. By Lemma 3.1.1, $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}$ are algebraically dependent on X. Hence $\varphi(X)$ has dimension at most $h-1$ in $\mathbf{G}_m^h$. Since Y is maximal, then by Lemma 3.1.3, this dimension is exactly $h-1$. Let set W to be the closure of $\varphi(X)$ in $\mathbf{G}_m^h$. Then W is a hypersurface in $\mathbf{G}_m^h$. By The Fundamental Dimension Theorem (b) (See Appendix A), there exists an open dense subset U of W such that for every $w \in U$ and every component Z of $X \cap \varphi^{-1}(w)$, we have $\dim Z = r - (h-1) = s$. Each of such Z is X-anomalous since it also lies in a translate of H. We can assume that every point of U is nonsingular (we remove the singular point in U if necessary).

Assume that $Z \not\subseteq X_{\mathrm{Sing}}$:

By Lemma 3.1.5, there exists $\mathbf{z}_Z \ne 0$ in $\mathbb{C}^n$ such that $\mathrm{rank}_Z J(\mathbf{z}_Z; X) \le n - r$. On the other hand, we have $\mathrm{rank}_X J(\mathbf{z}_Z; X) > n - r$. Indeed, if $\mathrm{rank}_X J(\mathbf{z}_Z; X) \le n - r$, then Lemma 3.1.4 would imply that X was X-anomalous. As Y is maximal, it would give $Y = X$, but we already exclude this case above. These rank conditions mean that there exists a minor G which does not vanish on X but is identically zero on Z.

Assume that $Z \subseteq X_{\text{Sing}}$:

In this case, as the locus is defined in X by the vanishing of polynomials of total degree at most $c\Delta$ (using arguments on Chow forms) for some c depending only on n, we can use one of these equations in place of G as above.

Hence, in any case, we have an equation G which does not vanish on X but is identically zero on Z. So as above, we can find a nonzero $\mathbf{a}_Z \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{a}_Z}$ is constant on Z with $|\mathbf{a}_Z| \leq c\Delta^{2\mu(r,n-1)}$ and where c depends only on n. Now, the problem is to find a nonzero $\mathbf{a} \in \mathbb{Z}^n$ which works for "almost all" the Z's. We remark the union of all Z above is $X \cap \varphi^{-1}(U)$, which is dense in X since $\varphi_{|X}$ is an open map. Now $\mathbf{a}_Z$ depends on Z, and there is no way to change this fact. However, $|\mathbf{a}_Z|$ is bounded uniformly in $\mathbb{Z}$. By Pigeonhole principle applied to the finitely many possibilities $\mathbf{a}_Z$, there exist a dense subset T of $X \cap \varphi^{-1}(U)$ and a nonzero $\mathbf{a} \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{a}}$ is constant on T and $|\mathbf{a}| \leq c\Delta^{2\mu(r,n-1)}$.

Next, we claim that $\mathbf{x}^{\mathbf{a}}$ is algebraically dependent on $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}$ on X i.e $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}$ with $\mathbf{x}^{\mathbf{a}}$ have the same transcendence degree $h-1$. We can suppose that $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_{h-1}}$ are algebraically independent on X. Assume by contradiction that $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_{h-1}}, \mathbf{x}^{\mathbf{a}}$ are algebraically independent. By Lemma 3.1.1, it implies that $\text{rank}_X J(\mathbf{a}_1, \cdots, \mathbf{a}_{h-1}, \mathbf{a}) > n-r+h-1$. Hence this gives a minor G of size at least $n-r+h$ that does not vanish identically on X. As T is dense in X, the there exists $t \in T$ such that $G(t) \neq 0$. However, t lies in one the Z's above, which are all X-anomalous of dimension s. It also lies in the translate of $H(\mathbf{a}_1, \cdots, \mathbf{a}_{h-1}, \mathbf{a})$ since $\mathbf{x}^{\mathbf{a}}$ is constant on T. Therefore, by Lemma 3.1.2 we have $\text{rank}_Z J(\mathbf{a}_1, \cdots, \mathbf{a}_{h-1}, \mathbf{a}) \leq n-r+h-1$. As the above minor G vanishes on Z and so at t. This gives a contradiction and established the claim.

Lastly, we proceed to prove that $\mathbf{x}^{\mathbf{a}}$ is constant on Y. This will complete the induction step and so the proof of the Proposition. We remark that $\varphi = \varphi(\mathbf{a}_1, \cdots, \mathbf{a}_h)$ is constant on each translate of $H = H(\mathbf{a}_1, \cdots, \mathbf{a}_h)$, hence on Y. We set $w = \varphi(Y)$, so Y lies in $X \cap \varphi^{-1}(w)$. Hence Y must be a component of $X \cap \varphi^{-1}(w)$. Otherwise, Y would lie in a component of dimension strictly larger than s. Such component would still be X-anomalous, contradicting maximality. Now, we have $X \cap \varphi^{-1}(w) = Y \cup X_0$ for some subvariety X_0 of X not containing Y. Let us choose $y \in Y \setminus X_0$. We set $\lambda = y^{\mathbf{a}}$. We have the morphism $\psi : \varphi(\mathbf{a}_1, \cdots, \mathbf{a}_h, \mathbf{a}) : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^{h+1}$. By definition of λ, $y \in X \cap \psi^{-1}(\tilde{\omega})$ where $\tilde{\omega} = (\omega, \lambda)$. Let $\tilde{Y}$ be a component of $X \cap \psi^{-1}(\tilde{\omega})$ containing y. Again by definition of y, we have $\tilde{Y} \subset Y$ since $\tilde{Y} \subset X\varphi^{-1}(w) = Y \cup X_0$. On the other hand, the claim above and the algebraoc dependence of $\mathbf{x}^{\mathbf{a}_1}, \cdots, \mathbf{x}^{\mathbf{a}_h}$ on X imply that $\dim \psi(X) \leq h-1$. By The Fundamental Degree Theorem (a), we have $\dim \tilde{Y} \geq r-(h-1) = s$. By dimension argument, we have $\tilde{Y} = Y$. As $\mathbf{x}^{\mathbf{a}} = \lambda$ on $\tilde{Y}$, $\mathbf{x}^{\mathbf{a}}$ is also constant on Y. This completes the proof. □

Proof of The Structure Theorem 2.1.9. We prove the existence of the finite collection Φ by induction on $\dim X = r$ with the following extra condition on degree:

$$\deg H \leq c(n,r)(\deg X)^{\kappa(n,r)} \tag{3.2.1}$$

for every $H \in \Phi$. The extra condition ensures the finiteness of the family Φ since every $H \in \Phi$ is defined by equations of the forms $\mathbf{x}^{\mathbf{a}} = 1$ with $|\mathbf{a}| \leq c(n)(\deg H)^{\lambda(n)}$ by Proposition 1.1.8.

If $r = 1$ then X is a curve. If X is not anomalous, there is nothing to prove. Assume X is anomalous. Then X is lies in a proper coset gH. By Lemma 3.1.6, there exists $\mathbf{a} \in \mathbb{Z}^n$ such that the coset gH is defined by some equation $\mathbf{x}^{\mathbf{a}} = \lambda$ with $|\mathbf{a}| \leq c(n)(\deg X)^{n-1}$ for some $\lambda \in G$. We can moreover suppose that X belongs to a finite family of cosets gH such that the degree of H satisfies (3.2.1).

Next, assume that $r \geq 2$ and suppose that the result holds for maximal anomalous subvarieties with lower dimension. let Y be a maximal anomalous subvariety of X of dimension s. By Proposition

3.2.1, there exists a non-trivial relation $\mathbf{x}^{\mathbf{a}} = \lambda$ that holds on Y with $|a| \leq c(n,r)(\deg X)^{\mu(n,r)}$. We can assume that $\mathbf{a}$ is primitive and then find $\mathbf{a}_1, \cdots, \mathbf{a}_{n-1}$ satisfying the same bound and such that $\alpha = \varphi(\mathbf{a}_1, \cdots, \mathbf{a}_{n-1}, \mathbf{a})$ is an automorphism. Therefore, as we $\deg \alpha(X) \leq a(\deg X)^b$ for a, b depending only on n, and by the equation (3.2.1), we may assume that α is identity and $\mathbf{a} = (0, \cdots, 0, 1)$.

If $x_n = \lambda$ on X, then X is X-anomalous and by maximality, we have $Y = X$. We then finish as in the case of curves above. Otherwise, we have $X' := X \cap \{x_n = \lambda\}$, with $\dim X' = r - 1$. Since $x_n = \lambda$, then the projection $\tilde{Y}$ of Y to $\mathbf{G}_m^{n-1}$ is still of dimension s. Moreover, $\tilde{Y}$ lies in some component of the projection $\tilde{X}$ of X' to $\mathbf{G}_m^{n-1}$. Also, we still have $\dim \tilde{X} = r - 1$. Now, we claim that $\tilde{Y}$ is $\tilde{X}$-anomalous. Indeed, by definition Y lies in a coset in $\mathbf{G}_m^n$ of dimension at most $n - r + s - 1$. Projecting this coset to $\mathbf{G}_m^{n-1}$, we obtain a coset of dimension at most $n - r + s - 1 = (n-1) - (r-1) + \dim \tilde{Y} - 1$. This latter coset contains $\tilde{Y}$ and this proves that $\tilde{Y}$ is anomalous in $\tilde{X}$. Next, we also claim that $\tilde{Y}$ is a maximal anomalous in $\tilde{X}$. In fact, let Y' be anomalous in $\tilde{X}$ such that

$$\tilde{Y} \subset Y' \subset \tilde{X} \subset \mathbf{G}_m^{n-1}.$$

This implies that

$$Y = \tilde{Y} \times \{\lambda\} \subset Y' \times \{\lambda\} \subset \tilde{X} \times \{\lambda\}.$$

We note that $\tilde{X} \times \{\lambda\} \subset X' \subset X$. Then $Y' \times \{\lambda\}$ is anomalous in X since it is contained in a coset of dimension $(n-1) - (r-1) + \dim Y' - 1 = n - r + \dim(Y' \times \{\lambda\}) - 1$. By the maximality of Y, we have $\tilde{Y} \times \{\lambda\} = Y' \times \{\lambda\}$. Thus $\tilde{Y} = Y'$ and $\tilde{Y}$ is a maximal anomalous in $\tilde{X}$.

Now, by induction hypothesis, $\tilde{Y}$ lies in a translate of a torus $\tilde{H}$ of $\mathbf{G}_m^{n-1}$ with $\dim \tilde{H} = (n-1) - (r-1) + s - 1 = n - r + s - 1$ and $\deg \tilde{H} \leq c(n-1, r-1)(\deg X)^{\kappa(n-1,r-1)}$. Hence we set $H = \tilde{H} \times \{1\}$, then $Y = \tilde{Y} \times \{\lambda\}$ lies in a translate gH with the similar bound for $\deg H$. Finally Y is a component of $X \cap gH$, otherwise there would be a component containing Y, thus it would contradict the maximality of Y. $\square$

We mention here that the analogue version of this Structure Theorem in the abelian varieties.

3.2.2 Definition. Let G be a semiabelian variety defined over $\mathbb{C}$. Let X be a closed and irreducible subvariety in G. Let K be a closed subvariety of G. A component Y of $X \cap K$ is called *atypical component* if its dimension satisfies

$$\dim Y > \dim X + \dim K - \dim G.$$

When G is an algebraic torus, an atypical component of strictly positive dimension is anomalous. For the abelian case, we have the following result due to J. Kirby [19].

3.2.3 Theorem. *(Structure Theorem for semiabelian varieties) Let G be a semi-abelian variety and X a (closed) irreducible subvarieties defined over $\mathbb{C}$. Then there exists a finite family Φ of proper semi-abelian subvarieties of G such that for any translate of a semi-abelian subvariety K and any atypical component of the intersection $X \cap gK$, there exists $H \in \Phi$ and $h \in G$ such that $Y \subset hH$ and*

$$\dim(H) + \dim(Y) = \dim(K) + \dim(X \cap hH).$$

3.2.4 Remark. The last dimension condition means that if $K \subset H$, then Y is a typical component of the intersection $(X \cap hH) \cap gK$ in the translate gH. Indeed, we have

$$\dim(Y) = \dim(X \cap hH) + \dim(K) - \dim(H) = \dim(X \cap hH) + \dim(gK) - \dim(gH).$$

3.3 The Anomalous Openness Theorem and its Generalization

Now we want to see that X^{oa} is Zariski open in X. The idea of following proof is based on the paper of [6].

Proof of The Anomalous Openness Theorem 2.1.12. We recall that X is an irreducible variety in $\mathbf{G}_m^n$. For a torus H in G, we define $\mathcal{L}_H$ to be the union of all anomalous subvarieties Y in X satisfying

$$\dim Y = \dim X + \dim H - n + 1. \tag{3.3.1}$$

We prove the theorem in two steps. The first step is to prove that for any such torus H, $\mathcal{L}_H$ is a Zariski closed in X. The second step is to show that $X \setminus X^{\mathrm{oa}} = \cup_{H\in\Phi}\mathcal{L}_H$, where Φ is the finite family in Structure Theorem 2.1.9. This will imply that X^{oa} is an open subset of X.

- $\mathcal{L}_H$ is a Zariski closed in X:

Let H be a torus in G. We set $h := n - \dim H$ and $s := \dim X - h + 1$. As any torus is of dimension $n-h$ is ismorphic to $\mathbf{G}^{n-h}$, then we identify H with $\{1\}^h \times \mathbf{G}_m^{n-h}$. Let us we consider the projection map $\pi_h : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^h$. Then we define

$$\mathcal{L}_h = \{\mathbf{x} \in X \mid \dim_{\mathbf{x}}(X \cap \pi_h^{-1}(\pi_h(\mathbf{x}))) \geq s\}.$$

By Chevalley's Semi-continuity Theorem (Fundamental Dimension Theorem (c) in Appendix A), $\mathcal{L}_h$ is a closed subset in X. Now let us check that $\mathcal{L}_h = \mathcal{L}_H$.

If $\mathbf{x} \in \mathcal{L}_h$, then there is a component $Y_{\mathbf{x}}$ of $X \cap \pi_h^{-1}(\pi_h(\mathbf{x}))$ through x whose dimension is at least s. If $\dim Y_{\mathbf{x}} = s$, then $Y_{\mathbf{x}}$ satisfies the equation (3.3.1). It is clear that any subvarieties satisfying such equality is anomalous and thus $\mathbf{x} \in \mathcal{L}_H$ in this case. Now, assume that $\dim Y_{\mathbf{x}} > s$. Then we can find an irreducible subvariety $Y'_{\mathbf{x}}$ through $\mathbf{x}$ of dimension s. As above, such subvariety $Y'_{\mathbf{x}}$ is anomalous and we conclude that $\mathbf{x} \in \mathcal{L}_H$.

Conversely, if $\mathbf{x} \in \mathcal{L}_H$, then $\mathbf{x}$ lies in a anomalous subvariety Y of X. Such subvariety Y has dimension $\dim X - h + 1 = s$ and lies in a coset of H. But $\pi_h^{-1}(\pi_h(\mathbf{x})) = \mathbf{x}H$, so it is the only coset in G containing $\mathbf{x}$. So the dimension of $\pi_h^{-1}(\pi_h(\mathbf{x}))$ through $\mathbf{x}$ is at least $\dim Y = s$. This shows that $\mathbf{x} \in \mathcal{L}_h$. This gives the result of the first step.

- $X \setminus X^{\mathrm{oa}} = \cup_{H\in\Phi}\mathcal{L}_H$:

Clearly, we have $X \setminus X^{\mathrm{oa}} \supseteq \cup_{H\in\Phi}\mathcal{L}_H$. It remains to show the converse. Let Y be an anomalous subvariety of X. By The Structure Theorem 2.1.9, Y is a component of the intersection $X \cap H$, for some $g \in G$ and some $H \in \Phi$ such that

$$\dim Y = \dim X + \dim H - n + 1, \text{ which is exactly the equation (3.3.1).}$$

This gives the equality. As mentioned above, this completes the proof the Openness Theorem. $\square$

The generalization to the case of semiabelian variety has been proved due to the result in Theorem 3.2.3. Indeed, for be a complex semi-abelian variety G and a (closed and irreducible) subvariety X in G, we have the following result.

3.3.1 Theorem. *[11, p. 7] Let X^{oa} be the complementary in X of the union of atypical component of dimension strictly positive, of the intersection $X \cap K$, where K runs over the translate nonzero semi-abelian subvarieties of G of codimension $\leq \dim(X)$. Then X^{oa} is open in X for the Zariski topology.*

Proof. The previous proof can be adapted for the semiabelian version. □

3.4 The Bounded Height Theorem and its Abelian Case

Let us see now the proof of Bounded Height Theorem 2.2.2. As we mentioned in Chapter 2, Section 2.2, it is sufficient to prove Theorem 2.2.1. For that, we follow an approach due to P. Habegger in [16]. So let X be an irreducible closed subvariety of G defined over $\overline{\mathbb{Q}}$. The following lemma is the main key of our proof.

3.4.1 Lemma. Let $Y \subset X$ be an irreducible subvariety of positve dimension and let d be an integer such that $\dim Y \leq d \leq n$. If $Y \cap X^{\mathrm{oa},[d]} \neq \emptyset$ then there exists $\epsilon > 0$ and a Zariski open dense $U \subset Y$ such that

$$U(\overline{\mathbb{Q}}) \cap \mathcal{C}(G^{[d]}, \epsilon) \text{ has a bounded height.}$$

3.4.2 Remark. We will see the proof later, but let us see why this result leads to our theorem. Let d be an integer. From this lemma, if $X^{\mathrm{oa},[d]} \neq \emptyset$, then there exists an open dense subset U of X such that the height of points of $U(\overline{\mathbb{Q}}) \cap \mathcal{C}(G^{[d]}, \epsilon)$ is bounded above. The following proposition deals with the complementary case. We denote $r := \dim X$.

3.4.3 Proposition. Let d be an integer such that $r \leq d \leq n$. Assume that there exist $\epsilon > 0$ and a subset $S \subsetneq X^{\mathrm{oa},[d]}$ such that $S \cap \mathcal{C}(G^{[r]}, \epsilon)$ has a bounded height. Then there exist $\epsilon' > 0$ and $S' \subset X^{\mathrm{oa},[\mathrm{d}]}$ containing S such that

$$\overline{X^{\mathrm{oa},[\mathrm{d}]} \setminus S'} \subsetneq \overline{X^{\mathrm{oa},[\mathrm{d}]} \setminus S} \text{ and } S' \cap \mathcal{C}(G^{[d]}, \epsilon') \text{ has a bounded height.}$$

Proof. We set $W = X^{\mathrm{oa},[d]}$. By assumption, $W \setminus S \neq \emptyset$. Hence its Zariski closure has an irreducible component Y with $Y \cap W \neq \emptyset$. We can write thus $\overline{W \setminus S} = Y \cup Z$ for some closed irreducible subset Z with $Y \nsubseteq Z$. As $\dim Y \geq 0$, by Lemma 3.4.1, there exist $\epsilon' > 0$ and $U \subset Y$ such that

$$U \cap \mathcal{C}(G^{[d]}, \epsilon') \text{ has a bounded height.}$$

We can assume that $\epsilon' \leq \epsilon$. Let us consider the set $S' = S \cup (U \cap W)$. Then $S' \cap \mathcal{C}(G^{[d]}, \epsilon')$ has bounded height. Indeed, $(U \cap W) \cap \mathcal{C}(G^{[d]}, \epsilon') \subset U \cap \mathcal{C}(G^{[d]}, \epsilon')$ and the latter one has bounded height by previous paragraph. Since $\epsilon' \leq \epsilon$ and $G^{[d]} \subset G^{[r]}$, we have $S \cap \mathcal{C}(G^{[d]}, \epsilon') \subset S \cap \mathcal{C}(G^{[r]}, \epsilon)$ and the latter has a bounded height by assumption. Next, we claim that $W \setminus S' \subset (Y \setminus U) \cup Z$. In, fact, let $x \in W$ but $x \notin S'$. This latter means that $x \notin S$ and $x \notin U \cap Z$. Since $x \in W$, then $x \notin U$. The fact that $x \in \overline{W \setminus S'}$ implies that $x \in Y \cup Z$. From this claim, we get $\overline{W \setminus S'} \subset (Y \setminus U) \cup Z$. Since $U \neq \emptyset$, then $\overline{W \setminus S'} \neq \overline{W \setminus S}$. □

Now, let us present the proof of the main result.

Proof of Theorem 2.2.1. We may assume that $r = \dim X \geq 1$ and $r \leq d \leq n$. We set $S_0 = \emptyset$ and $\epsilon_0 = 1$. Using Proposition 3.4.3 by induction, we can find S_{k-1} and $\epsilon_{k-1} > 0$ such that $S_{k-1} \cap \mathcal{C}(G^{[d]}, \epsilon_{k-1})$ has a bounded height. If $S_{k-1} = X^{\mathrm{oa},[d]}$, then the theorem follows. Assume $S_{k-1} \subsetneq X^{oa,[d]}$. As $G^{[d]} \subset G^{[r]}$, using again Proposition above, there exist $\epsilon_k > 0$ and $S_k \subset X^{oa,[d]}$ such that $S_k \cap \mathcal{C}(G^{[d]}, \epsilon_k)$ has a bounded height. Therefore, we obtain a chain of Zariski closed:

$$X \supset \overline{X \setminus S_0} \supsetneq \overline{X \setminus S_1} \supsetneq \cdots \supsetneq \overline{X \setminus S_k} \supsetneq \cdots .$$

As X is a Noetherian topological space, then it satisfies the descending chain condition for Zariski closed. In this situation, it means that $S_k = X^{\mathrm{oa},[d]}$ for some integer k and the theorem follows with $\epsilon = \epsilon_k$. □

Now, let us present some preliminaries for the proof of Lemma 3.4.1. Let $d \geq 1$ be an integer such that $1 \leq d \leq n$. We recall that a morphism $\varphi : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^d$ is given by d monomials in n variables. Therefore, we identify φ to a $d \times n$ matrix with integer coefficients. We denote $\mathbf{M}_{dn}(\mathbb{Z})$ the group of such matrices. We denote by $||\varphi||$ the Euclidean matrix norm. The following three lemmas give a height upper bound of the image of $\mathbf{G}_m^n(\overline{\mathbb{Q}})$ by φ.

Height Upper Bound:

3.4.4 Lemma. Let d be an integer such that $1 \leq d \leq n$. For any morphism $\varphi : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^d$ and $p \in \mathbf{G}_m^n(\overline{\mathbb{Q}})$, we have

$$h(\varphi(p)) \leq \sqrt{dn}||\varphi||h(p).$$

Proof. The morphism φ has row $\mathbf{u}_i = (u_1^{(i)}, \cdots, u_n^{(i)}) \in \mathbb{Z}^n$ for $i = 1, \cdots, d$. We write $p = (p_1, \cdots, p_n)$. Then by the elementary height properties above, we have

$$h(\varphi(p)) = h(p^{\mathbf{u}_1}, \cdots, p^{\mathbf{u}_d}) \leq h(p^{\mathbf{u}_1}) + \cdots + h(p^{\mathbf{u}_d}) \leq \sum_{i,j} |u_j^{(i)}| h(p_j).$$

By the Cauchy Schwartz Inequality, we have

$$(\sum_{i,j} |u_j^{(i)}|)^2 \leq dn \sum_{i,j} |u_j^{(i)}|^2.$$

Combining these equalities and using again the elementary height properties, we deduce that

$$h(\varphi(p)) \leq \sqrt{dn}||\varphi||h(p).$$

This proves the present lemma. □

We denote by $\mathbf{K}_{dn} \subset \mathbf{M}_{dn}(\mathbb{R})$ the compact set of all matrices whose rows are orthonormal. Such matrices are of rank d.

3.4.5 Lemma. Let $\Omega \subset \mathbf{M}_{dn}(\mathbb{R})$ be an open neighborhood of $\mathbf{K}_{dn}$. There exists a real number Q_0 with the following properties. Let $Q > Q_0$ be a real number and let $\varphi_0 \in \mathbf{M}_{dn}(\mathbb{R})$ with rank d, there exist $q \in \mathbb{Z}$, $\varphi \in \mathbf{M}_{dn}(\mathbb{Z})$, and $\theta \in \mathbf{GL}_d(\mathbb{Q})$ such that

$$1 \leq q \leq Q, \ \varphi \in q\Omega, \ ||q\theta\varphi_0 - \varphi|| \leq \sqrt{dn}Q^{-1/(dn)} \text{ and } ||\varphi|| \leq (d+1)q.$$

Proof. Since $\mathbf{K}_{dn}$ is compact, then there exists $\epsilon > 0$ with the following property: if $f \in \mathbf{K}_{dn}$ and $g \in \mathbf{M}_{dn}(\mathbb{R})$ with $||f - g|| < \epsilon$ then $g \in \Omega$. We may clearly assume $\epsilon \leq 1$. We choose Q_0 such that

$$\sqrt{dn}Q_0^{-1/(dn)} = \epsilon/2, \text{ so } Q_0 \geq 1.$$

Let be $Q > Q_0$. As φ_0 has rank d, the $\mathbb{R}$-vector space generated by the rows of φ_0 has dimension d and admits an orthonormal basis. In other words, there exists an invertible $\theta_0 \in \mathbf{GL}_d(\mathbb{R})$ such that $\theta_0\varphi_0 \in \mathbf{K}_{dn}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $\theta \in \mathbf{GL}_d(\mathbb{Q})$ such that

$$||\theta\varphi_0 - \theta_0\varphi_0|| \leq \frac{\epsilon}{2}. \tag{3.4.1}$$

By Dirichlet's Approximation Theorem, we can approximate $\theta\varphi_0$ to get an integer q with $1 \leq q \leq Q$ and $\varphi \in \mathbf{M}_{dn}(\mathbb{Z})$ such that

$$||q\theta\varphi_0 - \varphi|| \leq \sqrt{dn}Q^{-1/(dn)}.$$

Furthermore, we have

$$||\theta\varphi_0 - \frac{\varphi}{q}|| \leq \frac{\sqrt{dn}}{Q^{1/(dn)}q} \leq \frac{\sqrt{dn}}{Q_0^{1/(dn)}q} \leq \frac{\epsilon}{2}. \tag{3.4.2}$$

These inequalities with the triangle inequality imply that

$$||\theta_0\varphi_0 - \varphi/q|| < \epsilon.$$

So $\varphi/q \in \Omega$ since $\theta_0\varphi_0 \in \mathbf{K}_{dn}$. Hence we get all the assertions in the Lemma except for the last inequality. We apply the triangle inequality to the inequality (3.4.1) to get $||\theta\varphi_0|| < ||\theta_0\varphi_0|| + \epsilon/2$. As the rows of $\theta_0\varphi_0$ have norm 1, so $||\theta_0\varphi_0|| = \sqrt{d}$. The triangle inequality applied to (3.4.2) gives the desired assertion. □

3.4.6 Lemma. Let $\Omega \subset \mathbf{M}_{dn}(\mathbb{R})$ be an open neighborhood of $\mathbf{K}_{dn}$. There exists a real number $Q_0 > 0$ such that for all real number $Q > Q_0$, the following statement holds. If $p \in G^{[d]}$, there exist $q \in \mathbb{Z}$ and $\varphi \in \mathbf{M}_{dn}(\mathbb{Z})$ such that

$$1 \leq q \leq Q, \ \varphi \in q\Omega, \ ||\varphi|| \leq (d+1)q \text{ and } h(\varphi(p)) \leq \frac{dn}{Q^{1/dn}}h(p).$$

Proof. The previous lemma gives $Q_0 > 0$. Let $Q > Q_0$ and $p \in G^{[d]}$. By definition, p belongs to an algebraic subgroup of codimension at least d. Hence there exists a homomorphism of algebraic groups $\varphi_0 : \mathbf{G}_m^n \longrightarrow \mathbf{G}_d^n$ with rank d such that $\varphi_0(p) = 1$. By the previous lemma, we can find $q \in \mathbb{Z}$, $\varphi \in \mathbf{M}_{dn}(\mathbb{Z})$ and $\theta \in \mathbf{GL}_d(\mathbb{Q})$ satisfying the assertions. It remains to show the bound on the height. For brevity, we write $\delta = \varphi - q\theta\varphi_0$. Hence its norm is

$$||\delta|| \leq \sqrt{dn}Q^{-1/(dn)}. \tag{3.4.3}$$

Since θ has rational coefficients, there is a positive integer N with $N\theta \in \mathbf{M}_{dn}(\mathbb{Z})$ and so $N\theta \in \mathbf{M}_{dn}(\mathbb{Z})$. Hence we have

$$(N\delta)(p) = \varphi(p^N)(q(N\theta)\varphi_0)(p)^{-1} = \varphi(p^N) = \varphi(p)^N,$$

since $\varphi_0(p) = 1$. Taking heights and using homogeneity, we get

$$Nh(\varphi(p)) = h(\varphi(p)^N) = h(N\delta(p)).$$

By Lemma 3.4.4, we have

$$h(\varphi(p)) = N^{-1}h(N\delta(p)) \leq \sqrt{dn}N^{-1}||N\delta||h(p) = \sqrt{dn}||\delta||h(p).$$

Combining with (3.4.3), we get the desired result. □

Similarly, let us see an analogue version for the lower bound.

Height Lower Bound:

Let X be an irreducible closed variety of G defined over $\overline{\mathbb{Q}}$ and of dimension $r \geq 1$. For a morphism $\varphi : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^r$, we denote $\Delta_X(\varphi)$ the degree of $\varphi_{|X}$.

3.4.7 Proposition. There exists a constant $c_1 > 0$ depending only on X with the following properties. For any homomorphism $\varphi : \mathbf{G}_m^n \longrightarrow \mathbf{G}_m^r$ with $\varphi \neq 0$, there exists Zariski open dense subset $U_\varphi \subset X$ and a constant $c_2 = c_2(\varphi)$ which depends on φ such that for all $p \in U_\varphi(\overline{\mathbb{Q}})$,

$$h(\varphi(p)) \geq \frac{r}{2c_1} \frac{\Delta_X(\varphi)}{||\varphi||^{r-1}} h(p) - c_2.$$

Sketch of the proof. The existence of c_1 and such bound is an application of Philippon's theorem [22, Prop 3.3] (The main tools used there are the so called First Chern Class of a line bundle). The existence of the U_φ is due to the fact that a certain line bundle on X has a nonzero global section. Siu's Theorem [21, p. 143] guarantees the existence of a such nonzero global section on such line bundle under a numerical criterion. □

The following proposition is a uniform lower bound. Assume for the moment that X is defined over $\mathbb{C}$. Furthermore, let d be an integer with $r \leq d \leq n$. For any $s < d$, we define Π_{sd} to be the set of morphisms $\mathbf{G}_m^d \longrightarrow \mathbf{G}_m^s$ defined by projecting to s distinct coordinates of $\mathbf{G}_m^d$.

3.4.8 Proposition. Let $K \subset \mathbf{M}_{dn}(\mathbb{R})$ be compact and such that all its elements have rank d. Let Y be an irreducible closed subvariety of X such that $Y \cap X^{\mathrm{oa},[d]} \neq \emptyset$. Then there exists a constant $c_3 > 0$ and an open neighborhood $\Omega \subset \mathbf{M}_{dn}(\mathbb{R})$ of K such that for all $\varphi \in \Omega \cap \mathbf{M}_{dn}(\mathbb{Q})$, there exists $\pi \in \Pi_{sd}$ with $\Delta_Y(\pi\varphi) \geq c_3$ and where $s = \dim Y$.

Proof. We refer to [16, Section 6-7] for the proof. It is mainly an application of Ax's Theorem in [3, Theorem 1] □

Now let us give the proof of our lemma. All varieties in this section are assumed to be defined over $\overline{\mathbb{Q}}$. So let X be an irreducible closed variety of $G = \mathbf{G}_m^n(\overline{\mathbb{Q}})$ defined over $\overline{\mathbb{Q}}$.

Proof of Lemma 3.4.1. Let $Y \subset X$ be an irreducible subvariety of dimension s and let d an integer such that $s \leq d \leq n$. We apply Proposition 3.4.8 to $K = \mathbf{K}_{dn}$ (defined as before). Hence there exist a constant $c_3 > 0$ and an open neighborhood $\Omega \subset \mathbf{M}_{dn}(\mathbb{R})$ of K such that for all $\varphi \in \Omega \cap \mathbf{M}_{dn}(\mathbb{Q})$, there exists $\pi \in \Pi_{sd}$ with $\Delta_Y(\pi\varphi) \geq c_3$. By Lemma 3.4.6, there exists $Q_0 > 0$. Let be $Q > Q_0$ that we will fix later. It depends only on X and Y.

Let Φ be the set of all matrices $\varphi \in \mathbf{M}_{dn}(\mathbb{Z})$ such that there exists $q \in \mathbb{Z}$ with $1 \leq q \leq Q$, $\varphi \in q\Omega$ and $||\varphi|| \leq (d+1)q$. It is clearly a finite set. Therefore, for any $\varphi \in \Phi$, there exists $\pi \in \Pi_{sd}$ such that $\Delta_Y(\varphi'/q) \geq c_3$ where $\varphi' = \pi\varphi$. Since $c_3 > 0$, then $\varphi' \neq 0$. By homogeneity of $\Delta_Y(\varphi')$ [16, Lemma 3.1 (iii), p. 866-867], we have

$$\Delta_Y(\varphi') = q^s \Delta_Y(\varphi'/q) \geq q^s c_3.$$

Since $\varphi \neq 0$, then $||\varphi|| \geq 1$. So we obtain the following lower bound for the coefficient in front $h(p)$ of Proposition 3.4.7

$$\frac{s}{2c_1} \frac{\Delta_Y(\varphi')}{||\varphi'||^{s-1}} = \frac{s}{2c_1} \frac{||\varphi'||\Delta_Y(\varphi')}{||\varphi'||^s} \geq \frac{sc_3}{2c_1} \frac{q^s}{||\varphi'||^s}.$$

where c_1 depends only on Y. Since $||\varphi'|| \leq ||\varphi|| \leq (d+1)q$, then the latter inequality becomes

$$\frac{s}{2c_1}\frac{\Delta_Y(\varphi')}{||\varphi'||^{s-1}} \geq \frac{sc_3}{2c_1(d+1)^s}.$$

We denote by c_4 this last bound. It is positive and independent of Q and $\varphi \in \Phi$. Now, we choose Q and ϵ as follows:

$$Q = \max\{Q_0+1, \left(8dnc_4^{-1}\right)^{dn}\} > Q_0 \text{ and } \epsilon = \min\{\frac{1}{2n}, \frac{\sqrt{dn}}{d+1}\frac{1}{Q^{1+1/dn}}\} \in \left(0, \frac{1}{2n}\right]. \tag{3.4.4}$$

Let us denote by U_φ the Zariski open dense of Y given by Proposition 3.4.7 applied to φ'. Then we have the intersection

$$U = \cap_{\varphi\in\Phi} U_\varphi.$$

U is open and dense in Y since φ is finite. So for all $p \in U(\overline{\mathbb{Q}})$ and all $\varphi \in \Phi$, we deduce

$$h(\varphi'(p)) \geq c_4 h(p) - C(Q), \tag{3.4.5}$$

where $C(Q)$ depends neither on p nor on φ.

Now let us assume that $p \in U(\overline{\mathbb{Q}}) \cap \mathcal{C}(G^{[d]}, \epsilon)$. Then there exist $a \in G^{[d]}$ and $b \in G(\overline{\mathbb{Q}})$ such that $p = ab$ and $h(b) \leq \epsilon(1+h(a))$.

By Lemma 3.4.6, there exists $\varphi \in \Phi$ such that

$$h(\varphi(a)) \leq \frac{dn}{Q^{1/dn}} h(a).$$

Using the elementary height properties and the fact that $\epsilon \leq 1/2n$, we have

$$h(a) \leq (1+2h(p)) \text{ and } h(b) \leq 2\epsilon(1+h(p)). \tag{3.4.6}$$

With the first inequality, we deduce that

$$h(\varphi(a)) \leq \frac{2dn}{Q^{1/dn}}(1+h(p)).$$

By Lemma 3.4.4, we have the upper bound for $h(\varphi(b))$

$$h(\varphi(b)) \leq \sqrt{dn}||\varphi||h(b).$$

With the second inequality of (3.4.6) and the fact that $||\varphi|| \leq (d+1)q \leq (d+1)Q$, we deduce

$$h(\varphi(b)) \leq 2\epsilon\sqrt{dn}(d+1)Q(1+h(p)).$$

Using the elementary height properties and combining these upper bounds, we have

$$h(\varphi(p)) = h(\varphi(ab)) \leq h(\varphi(a)) + h(\varphi(b)) \leq \left(2dnQ^{-1/dn} + 2\epsilon\sqrt{dn}(d+1)Q\right)(1+h(p)).$$

The choice of ϵ in (3.4.4) implies that

$$h(\varphi(p)) \leq 4dnQ^{-1/dn}(1+h(p)).$$

The choice of Q in (3.4.4) implies that

$$h(\varphi(p)) \leq \frac{c_4}{2}(1 + h(p)).$$

As $h(\varphi'(p)) \leq h(\varphi(p))$, we have

$$h(\varphi'(p)) \leq \frac{c_4}{2}(1 + h(p)).$$

Comparing with (3.4.5), we have

$$h(p) \leq 1 + 2c_4^{-1}C(Q).$$

This ends the proof of the present lemma and so the end of the proof of Bounded Height Conjecture. □

A generalization of such theorem to the abelian case has been already established by P. Habegger in [15]. Namely, we have the following result.

3.4.9 Theorem. *Let G be an abelian variety and X a closed irreducible subvariety of G. Then $X^{oa} \cap G^{[\dim X]}$ has a bounded height.*

Proof. We refer to [15] for the proof. □

3.5 The Finiteness Theorem and Zilber-Pink Conjecture

The proof that we present here for the Finiteness Theorem is related to Lehmer's Problem. Such problem is asking the following question: can we find a constant c such that $M(\alpha) \geq c > 1$ for any $\alpha \in \overline{\mathbb{Q}}$ which is not 0 and not a root of unity?

The term $M(\alpha)$ denotes the Mahler measure of α. It can be defined by

$$M(\alpha) := M(f_\alpha) = |a_d| \prod_{i=1}^{d} \max(1, |\alpha_i|),$$

where f_α is the minimal polynomial of α over $\mathbb{Q}$ and

$$f_\alpha(x) = |a_d| \prod_{i=1}^{d} (x - \alpha_i) \text{ over } \mathbb{C}.$$

Moreover, its satisfies the functional equation $d \cdot h(\alpha) = \ln M(f_\alpha)$ [4, Proposition 1.6.6, p. 23]. Hence, in terms of height function, it is equivalent to ask for a constant $c' > 0$ such that

$$h(\alpha) \geq \frac{c'}{d} > 0.$$

This question is easy to pose, since it concerns just heights of numbers, not points on abelian varieties or other general commutative algebraic groups. However, the conjecture is still open in its full generality. One result in this direction is the following (due to Amoroso and David).

3.5.1 Theorem. *[1, Theorem 2] Let $\eta_1, \cdots, \eta_r$ be multiplicatively independent algebraic numbers generating a number field of degree $\leq d$ and let $\epsilon > 0$. Then*

$$h(\eta_1) \cdots h(\eta_r) \geq c d^{-1-\epsilon},$$

where c depends only on r and ϵ.

Proof. We refer to [1, Theorem 2] after replacing a power of $\log d$ by d^{ϵ}. $\square$

We now present the proof of Theorem 2.3.1. As we have mentioned, we need to prove that the set of the points of

$$\Sigma = X^{\mathsf{oa}}(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$$

is finite. The following proposition is the key to give this finiteness result.

3.5.2 Proposition. [7, Lemma 8.1] Let X be an irreducible variety in $\mathbf{G}_m^n$ of dimension at most $n-1$. Then for any $B \geq 0$, the set of points $p \in X^{\mathsf{ta}}(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$ such that $h(p) \leq B$ is finite.

Proof. Let us set $m = \dim X$. We define the norm $|\cdot|$ of a vector as the maximum of the modulus of its coordinates. Let $p = (\xi_1, \cdots, \xi_n) \in X^{\mathsf{ta}}(\overline{\mathbb{Q}}) \cap G^{[1+m]}$ with $h(p) \leq B$. Let r be the rank of the multiplicative subgroup Γ_p of $\overline{\mathbb{Q}}^{\times}$ generated by the coordinates of p. By assumption, we have $\operatorname{codim} \Gamma_p \geq m+1$, thus $r \leq n-m-1$. By Schlickewei's Theorem [13, Lemma 2, p. 1130] applied to Γ_p, there are elements $\eta_1, \cdots, \eta_r \in \mathbb{Q}(p)$ satisfying the following properties:

- There exist integers a_{ij} (for $1 \leq i \leq n$ and $1 \leq j \leq r$) and roots of unity $\zeta_1, \cdots, \zeta_n \in \mathbb{Q}(p)$ such that

$$\xi_i = \zeta_i \eta_1^{a_i 1} \cdots \eta_r^{a_i r} \text{ for all } 1 \leq i \leq n. \tag{3.5.1}$$

- For any vector $(e_1, \cdots, e_r) \in \mathbb{Z}^r$, we have

$$h(\eta_1^{e_1}) \cdots \eta_r^{e_r} \geq c(r) \sum_{j=1}^{r} |e_j| h(\eta_j),$$

where $c(r) = r^{-1} 4^{-r}$. In particular, we deduce that

$$h(\xi_i) \geq c(r) \sum_{j=1}^{r} |a_{ij} h(\eta_j)|, \text{ for } 1 \leq i \leq n.$$

Furthermore, we can find a constant c_1 depending only on X such that

$$|a_{ij}| h(\eta_j) \leq c_1 h(p) \leq c_1 B, \text{ for } 1 \leq i \leq n, \text{ since } h \text{ is positive.}$$

Setting $\mathbf{a}_j = (a_{1j}, \cdots, a_{nj})$, we have

$$|\mathbf{a}_j| h(\eta_j) \leq c_2 B, \quad \text{for } j = 1, \cdots, r. \tag{3.5.2}$$

where c_2 is constant depending only on X. In the following, all c_i's are constant depending only on X.

Let L be the lowest common multiple of the order of ζ_i and let $\zeta \in \mathbb{Q}(p)$ a L-th root of unity. We define an integer l_i such that $0 \leq l_i < L$ and $\zeta_i = \zeta^{l_i}$ for all i. Then we consider the following linear forms in $n+1$ variables $x_0, \cdots, x_n$:

$$\varphi_0(\mathbf{x}) = -Lx_0 + \sum_{i=1}^{n} l_i x_i \text{ and } \varphi_j(\mathbf{x}) = \sum_{i=1}^{n} a_{ij} x_i \text{ (for } 1 \leq j \leq r).$$

By definition of r, these forms are linearly independent. By Siegel's Lemma, there exist linear independent vectors $\mathbf{b}_1, \cdots, \mathbf{b}_{n-r} \in \mathbb{Z}^{n+1}$ such that $\varphi_j(\mathbf{b}_k) = 0$ for all j, k and

$$\prod_{k=1}^{n-r} |\mathbf{b}_k| \leq L c_3 \prod_{j=1}^{r} |\mathbf{a}_j|.$$

We set $A = \prod_{j=1}^{r} |\mathbf{a}_j|$ to simplify the notation. We may assume that $|\mathbf{b}_1| \leq \cdots |\mathbf{b}_{n-r}|$. We write $(b_{0k}, \cdots, b_{nk})$ the coordinates of $\mathbf{b}_k$. We deduce for the first $m \leq n-r-1$ vectors that

$$|\mathbf{b}_1| \cdots |\mathbf{b}_m| \leq (c_3 LA)^{\frac{m}{n-r}} \leq (c_3 LA)^{\frac{m}{m+1}}.$$

By the equation (3.5.1), we have

$$\xi_1^{b_{1k}} \cdots \xi_n^{b_{nk}} = 1, \text{ for } 1 \leq k \leq m.$$

These say that p lies in a certain algebraic subgroup H of $\mathbf{G}_m^n$. Since the points $(b_{k_1}, \cdots, b_{kn})$ (for $k = 1, \cdots, m$) are also independent, so $\dim H = n - m$.

Now, we use the standard degree theory in $\mathbf{G}_m^n$ inside the projective $\mathbb{P}^n$. By Bézout's Theorem, we have

$$\deg H \leq c_4 \prod_{i=1}^{m} |\mathbf{b}_k| \leq c_5 (LA)^{\frac{m}{m+1}}.$$

Let us consider the component Y passing through p of the intersection $X \cap H$. We cannot have $\dim Y \geq 1$, otherwise H would have been removed to make X^{ta}. Then $Y = p$ is an isolated component of $X \cap H$. Again by Bézout's Theorem, the number of such p does not exceed $\deg X \cdot \deg H \leq c_6 (LA)^{m/m+1}$. As all conjugates of p over the field of definition of X belong to $X \cap H$, we have

$$D := [\mathbb{Q}(p) : \mathbb{Q}] \leq c_7 (LA)^{\frac{m}{m+1}}.$$

It is at this stage that the effective lower bound concerning the Lehmer's problem plays a role. Such bounds permit to deduce an upper bound of the degree of p. The elements $\eta_1, \cdots, \eta_r$ of $\mathbb{Q}(p)$ are multiplicatively independent and ζ is a root of unity in $\mathbb{Q}(p)$. By Theorem 3.5.1, we have

$$h(\eta_1) \cdots h(\eta_r) \geq c_8(\epsilon) [\mathbb{Q}(\eta_1, \cdots, \eta_r) : \mathbb{Q}]^{-1-\epsilon}.$$

Since $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(L) \geq c_9(\epsilon) L^{1-\epsilon}$, we deduce that

$$A h(\eta_1) \cdots h(\eta_r) \geq c_{10}(\epsilon) (LA)^{(1-m\epsilon)/(m+1)}.$$

Comparing with (3.5.2), we have

$$c_{11} LA \leq 1.$$

This means that LA is bounded uniformly independently of p, so is $[\mathbb{Q}(p) : \mathbb{Q}]$. By Northcott's theorem, we get the proposition. □

This Proposition with the Bounded Height Theorem 2.2.2 imply that $\Sigma = X^{\mathsf{oa}}(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$ is finite. Hence we proved the Finiteness Theorem 2.3.1. In particular, we also observe that if X is not contained in a proper subgroup of G, then X^{oa} is a proper non empty open subset of X and $X(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$ is not dense subset in X for the Zariski topology. Indeed, if $X(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$ is dense in X, then its intersection with X^{oa} is dense in X^{oa} because it will meet any non empty open subset of X^{oa}. This intersection, being equal to $X^{\mathsf{oa}}(\overline{\mathbb{Q}}) \cap G^{[1+\dim X]}$, is finite. This would imply that X^{oa} has dimension 0 (we use the fact that X^{oa} is an algebraic variety), hence X has dimension 0.

Now, let us see the sketch of the proof of Theorem 2.3.2. We recall that we would like to show that if X is a curve defined over $\overline{\mathbb{Q}}$ in $G = \mathbf{G}_m^n$ and X is not contained in a proper subgroup of G, then $X(\overline{\mathbb{Q}}) \cap G^{[2]}$ is finite.

Proof of Theorem 2.3.1. [11, Section 4.3, p. 23-25] The proof is by induction on n. The case $n < 2$ is trivial and the case $n = 2$ results the finiteness of the number of torsion points in X (Corollary of [9]). If X is not contained in any translate of a proper algebraic subgroup, then Theorem 2.3.1 (or the finiteness of Σ defined above) gives the result. So we assume that X is contained in a translate of a proper algebraic subgroup of G. By a change of coordinates on G, there exist an integer $r \in \{1, \cdots, n-1\}$, a curve $C \subset \mathbf{G}_m^r$ and a point $P_0 \in \mathbf{G}_m^{n-r}$ such that $X = C \times \{P_0\}$. We may suppose that C is not contained in a translate of a proper algebraic subgroup of $\mathbf{G}_m^r$, otherwise we chose a smaller r. Since X is not contained in a proper algebraic subgroup of G, so is P_0 in $\mathbf{G}_m^{n-r}$. We may assume $m \geq 2$, otherwise $X \cap G^{[2]} = \emptyset$. Let us consider the two morphisms φ and ψ as follows:

$$\begin{array}{rcl} \varphi : X \times X & \longrightarrow & \mathbf{G}_m^n \\ (P, Q) & \mapsto & P \cdot Q^{-1} \end{array}$$

and

$$\begin{array}{rcl} \psi : C \times C & \longrightarrow & \mathbf{G}_m^r \\ (P, Q) & \mapsto & P \cdot Q^{-1} \end{array}$$

Then we have $\varphi((P, P_0), (Q, P_0)) = (\psi(P, Q), 1, \cdots, 1)$ for any $(P, Q) \in C \times C$. We denote by S the Zariski closure of $\varphi(C \times C)$ in $\mathbf{G}_m^r$. It is a surface and is not contained in a translate of a proper algebraic subgroup of $\mathbf{G}_m^r \times \{(1, \cdots, 1)\}$ (otherwise, C would be contained in a translate of algebraic subgroup of $\mathbf{G}_m^r$). Using some argument in birational geometry [5, Lemma 2, p. 2250], we deduce that for any $(P, Q) \in X(\overline{\mathbb{Q}}) \times X(\overline{\mathbb{Q}})$ such that $P \neq Q$,

$$h(\varphi(P, Q)) \geq C(h(P) + h(Q) - 1), \text{ for some constant } C > 0. \tag{3.5.3}$$

Furthermore, for any point $(P, Q) \in X \times X$, we have the following cases:

- $P = Q$;
- $P \neq Q$ and $\varphi(P, Q) \in S^{\mathrm{oa}}$;
- $P \neq Q$ and $\varphi(P, Q)$ belongs to a maximal anomalous subvariety of S.

It is sufficient to prove the finiteness of $X \cap G^{[2]}$ by considering a point P in this set and by chosing Q of the form $\sigma(P)$, where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Let us see the finiteness of the second case (the other cases are needed some another results as proved in [11, p. 24-25]). We need to prove that the set of points $P \in X \cap G^{[2]}$ such that there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and $\varphi(P, \sigma(P)) \in S^{\mathrm{oa}}$ is finite. By the Bounded Height Theorem, there exists a constant $B > 0$ such that $h(\varphi(P, \sigma(P))) \leq B$. Since $h(\sigma(P)) = h(P)$, the inequality (3.5.3) implies that $h(P)$ is also bounded by some B'. Using Proposition 3.5.2, we conclude the finiteness. □

Finally, we remark that there is an equivalent formulation of Theorem 2.3.2 which can be stated as follows. In this form, our Theorem 2.3.2 is a simple formulation in the case of algebraic torus $\mathbf{G}_m^n$ of so called Zilber-Pink's Conjecture [23].

3.5.3 Theorem. *Let C be an irreducible curve defined over $\mathbb{C}$, and let $f_1, \cdots, f_n$ be nonzero rational functions in $\mathbb{C}(C)$, multiplicatively independent. Then the points $x \in C$, for which $f_1(x), \cdots, f_n(x)$ verify at least two independent multiplicative dependence relations, form a finite set.*

3.5.4 Remark. The equivalence can be checked as follows. We consider the rational map $C \longrightarrow \mathbf{G}_m^n$ given by $x \mapsto (f_1(x), \cdots, f_n(x))$. Every proper algebraic subgroup of $\mathbf{G}_m^n$ is contained in a subgroup $x_1^{a_1} \cdots x_n^{a_n} = 1$. The condition that f_i's are multiplicatively independent means that there is no nontrivial $(a_1, \cdots, a_n) \in \mathbb{Z}^n$ such that product $f_1^{a_1} \cdots f_n^{a_n} = 1$. This amounts to say that the image of C is not contained in a proper algebraic subgroup of $\mathbf{G}_m^n$.

Since the f_i's are nonzero on C, then the condition that $f_1(x), \cdots, f_n(x)$ verify at least two independent multiplicatively dependence relations means that the point $(f_1(x), \cdots, f_n(x)) \in \mathbf{G}_m^n$ is contained in some algebraic subgroup of codimension at least 2.

3.5.5 Example. Let us consider $n = 2$ and $C = \mathbb{P}^1(\mathbb{C})$. Then the field of rational functions $K(C)$ is $\mathbb{C}(T)$. Let us consider the two multiplicatively independent rational functions f_1 and f_2 defined by $f_1(T) = T$ and $f_2(T) = T + 1$.

For any $x \sim (x : 1) \in \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$, we have $M_x = \{(a_1, a_2) \in \mathbb{Z}^2 \mid x^{a_1}(x+1)^{a_2} = 1\}$. Hence the rank of M_x is 2 if and only if $x \neq 1$ is a 3$^{\text{rd}}$ root of unity. Indeed, if $\text{rk}(M_x) = 2$, then $\mathbb{Z}^2/M_x$ is a finite group (as they have the same rank) and there exists $k \in \mathbb{Z}$ such that $k\mathbb{Z}^2 \subseteq M_x$. So $(k, 0)$ and $(0, k)$ belong to M_x. Then we have

$$\begin{cases} x^k & = 1 \\ (x+1)^k & = 1 \end{cases}$$

Put $x = a + ib$, for some $a, b \in \mathbb{R}$, and take the absolute value in $\mathbb{C}$, then we have

$$\begin{cases} a^2 + b^2 & = 1 \\ (a+1)^2 + b^2 & = 1 \end{cases}$$

This means that $a = -\dfrac{1}{2}$ and $b = \mp\dfrac{\sqrt{3}}{2}$. Hence x is a 3$^{\text{rd}}$ root of unity and $x \neq 1$. Conversely, if $x \neq 1$ is the 3$^{\text{rd}}$ root of unity, then we have $x = \exp(\dfrac{2ki\pi}{3})$ for $k = 1, 2$ and

$$x + 1 = \begin{cases} \exp(\frac{i\pi}{3}) \text{ if } k = 1, \\ \exp(\frac{5i\pi}{3}) \text{ if } k = 2. \end{cases}$$

Then we have $(x+1)^6 = 1$. This means that $(3, 0), (0, 6)$ belong to M_x. Since they are linearly independent, then $\text{rk}(M_x) = 2$.

For the point at infinity $\infty = (1 : 0)$, we need to homogenize the two polynomials f_1 and f_2. This gives $f_1(T, U) = T$ and $f_2(T, U) = T + U$. So we get $f_1(\infty) = f_2(\infty) = 1$. Hence, for this case, we still have rk(M_∞)=2. So this gives the finiteness as in the theorem.

3.5.6 Remark. One possible extension to the abelian case of Theorem 2.3.1 could deal with the intersection of a curve C in an abelian variety A with the family of all proper algebraic subgroups in A. As we introduced before, Theorem 2.3.1 relies on finding an lower bound for the height on $\mathbf{G}_m^n$. For the abelian case, the analogue of such theorem is not yet known in full generality, due in particular to the absence of lower bounds for the *Néron-Tate height*[1] analogous to the ones of Amoroso and David for the Weil height (Theorem 3.5.1).

[1]the corresponding height for abelian varieties

4. Conclusion

The goal of this essay was to give an overview of some problems of unlikely intersections. We are specifically interested on the intersection of a subvariety X in $\mathbf{G}_m^n$ with the union of all algebraic subgroups of $\mathbf{G}_m^n$ restricted by dimension. The questions that we asked were: when such intersection exists? does this set has a bounded height? Or is it finite? We expect that such set is finite unless there is a special structure relating X with $\mathbf{G}_m^n$ which forces the contrary to happen. We have seen that removing the unlikely intersection parts, we were able to give answers of such questions.

In Chapter 1, we presented some properties of algebraic subgroups of $\mathbf{G}_m^n(\overline{\mathbb{Q}})$. Precisely, we have seen that every algebraic subgroup H of $\mathbb{G}_m^n$ can be defined by equations $\mathbf{x}^a = \mathbf{1}$ where we denote $\mathbf{x}^a := x_1^{a_1} \cdots x_n^{a_n}$ for the vector $a = (a_1, \cdots, a_n)$ runs through a lattice $\Lambda = \Lambda_H \subset \mathbb{Z}^n$. Hence, we showed that the correspondence $H \longleftrightarrow \Lambda_H$ is one-to-one. Furthermore, Λ_H has rank r if and only if H has dimension $n - r$. We also give a definition of the Weil height on $\mathbf{G}_m^n$. Namely, if $\mathbf{x} \in \mathbf{G}_m^n$, we define it as the height on $\mathbb{P}^n$ of the image of $\mathbf{x}$ by some open immersion from $\mathbf{G}_m^n$ to $\mathbb{P}^n$.

Chapter 2 described some structures of certain subvarieties, called anomalous subvariety, of an irreducible variety X defined over $\mathbb{C}$ in the algebraic torus $\mathbf{G}_m^n$. We also proved that the complement X^{oa} in X of the union of all anomalous subvariety in X is a Zariski open in X. Furthermore, we showed that intersecting X^{oa} with union of all algebraic subgroups of $\mathbf{G}_m^n$ of codimension $\geq \dim X$ has bounded Weil height. This Bounded Height Theorem implies, in particular, a Finiteness Theorem. Precisely, we have proved that the intersection of X^{oa} with the union of algebraic subgroups of dimension $\geq 1+\dim X$ is finite. In particular, we saw that if X is an irreducible curve defined over $\overline{\mathbb{Q}}$ and is not contained in a proper algebraic subgroups of $\mathbf{G}_m^n$ then the intersection of X with the union of all algebraic subgroups of $\mathbf{G}_m^n$ of codimension ≥ 2 is finite.

In Chapter 3, we gave the proofs of the main results and presented some related problems and their generalizations. In particular, we have seen the analogue abelian versions of Structure Theorem, the Bounded Height Conjecture and Finiteness Theorem. We also mentioned an equivalent reformulation of Finiteness Theorem for Curves which can be seen as a simple version of Zilber-Pink's Conjecture.

The problems of Unlikely Intersections have been studied in general for most of the commutative algebraic groups. For other algebraic groups, for instance linear noncommutative ones, no development seems to have actually been taken into account in the literature. In other cases, like the additive group $\mathbf{G}_a^n$, the situation is distinctly of different flavor. For instance, for a subvariety X of $\mathbf{G}_a^n$, the algebraic subgroups are not a discrete family but are parametrized by the *Grassmannian*. Hence, the set analogous to $X \cap G^{[d]}$ would be the intersection of X with the union of linear spaces of codimension at least d. This problem is classical in geometry and has no arithmetical ingredient unless we add the restriction to subspaces defined over a number field. In this case, when X is a curve and $d = n - 2$, the issue boils down to Mordell conjecture when $n = 3$, whereas for $n \geq 4$ partially new issues seem to emerge. For $\dim X > 1$ and $d = n - \dim X - 1$, the problems are nowadays beyond hope.

Appendix A. Some Classical Results

The Fiber Dimension Theorem

Let V and W be two irreducible varieties. Let $\varphi : V \longrightarrow W$ be a dominant morphism. Then

(a) For all $v \in V$, we have

$$\dim_v \varphi^{-1}(\varphi(v)) \geq \dim V - \dim W.$$

In particular, for all $w \in W$, every component of the fiber $\varphi^{-1}(w)$ has dimension at least $\dim V - \dim W$.

(b) There exits an open dense subset $U \subset W$ such that for all $w \in U$, we have

$$\dim \varphi^{-1}(w) = \dim V - \dim W.$$

(c) (Chevalley's Semicontinuity Theorem) For every integer k, the set

$$V_k = \{v \in V \mid \dim_v(\varphi^{-1}\varphi(v)) \geq k\}$$

is a Zariski closed in V.

For the proofs, we refer to [12, p. 228].

Siegel's Lemma

Let $n > r$ be two positive integers, and let $A = (a_{ij})$ be a nonzero $r \times n$ matrix with coefficients in $\mathbb{Z}$. Then the system of equations defined by $A\mathbf{x} = 0$ admits at least a nonzero solution $(b_1, \cdots, b_n) \in \mathbb{Z}^n$ satisfying

$$\max_{1 \leq j \leq n} |b_j| \leq \big(n \cdot \max_{1 \leq i \leq r, 1 \leq j \leq n} |a_{ij}|\big)^{\frac{r}{n-r}}.$$

This lemma says something quite simple. The system of homogeneous linear equations has more variables than equations, so we know it has nontrivial solutions. Since the coefficients are integers, there will be rational solutions and by clearing the denominators of the rational solutions, we can find integer solutions. So it is obvious that there are nonzero integer solutions. The last part of the lemma then says that we can find some solution that is not too large. For the proof, we refer to [18, Section D, p. 316].

Appendix B. Some Notions for Chapter 3

Degree of a variety

The degree of a variety X embedded in a projective space $\mathbb{P}^n$ can be seen as the second numerical invariant of X after its dimension. It reflects its position in $\mathbb{P}^n$.

If $r = \dim X$, then the *degree* $\deg X$ is the number of intersection points of X with a "general" linear subspace[1] L of dimension $n - r$. For example, if X is a hyperplane defined by a homogeneous polynomial F, then the degree of X is the same as the total degree of F. If X meets an $(n-r)$-plane L in more than $\deg X$ points then $X \cap L$ is infinite. For more details, we refer to [12, p. 250-255]. We note that the degree depends on the ambient projective space.

There is a natural embedding $\iota : \mathbb{G}_m^n \longrightarrow \mathbb{P}^n$ given by taking a point $(p_1, \cdots, p_n)$ to $(1 : p_1 : \cdots : p_n)$. Using this embedding, if $Z \subset \mathbb{G}_m^n$ is Zariski dense in an irreducible subvariety defined over $\overline{\mathbb{Q}}$, we define $\deg(Z) = \deg(\overline{\iota(Z)})$.

The following theorem called Generalized Bézout Theorem provides an upper bound for the degrees of the irreducible components of the set theoretic intersection $X_1 \cap \cdots \cap X_s$ for some varieties X_i's.

B.0.7 Theorem. *(Bézout Theorem) Let $X_1, \cdots, X_s$ be varieties in $\mathbb{P}^n$ and let $Z_1, \cdots, Z_t$ be the irreducible components of $X1 \cap \cdots \cap X_s$. Then*

$$\sum_{i=1}^{t} \deg Z_i \leq \sum_{j=1}^{s} \deg X_j.$$

Proof. We refer to [12, p. 251] □

Chow Form of a variety

We recall that *the Grassmann variety or or Grassmannian* $G(k, n)$ is the set of all k-dimensional vector subspaces in $\mathbb{C}^n$. For example if $k = 1$, this is the projective space $\mathbb{P}^{n-1}$.

Let $X \subset \mathbb{P}^n$ be an irreducible subvariety of dimension r and of degree Δ. Let $\mathcal{Z}(X)$ be the set of all $(n - r - 1)$- dimensional linear subspaces L of $\mathbb{P}^n$ such that $X \cap L$ is nonempty. Then the set $\mathcal{Z}(X)$ is an irreducible hypersurface of degree Δ in $G(n - r, n + 1)$ ([14, Chap 3, Proposition 2.2, p. 99]).

Thus $\mathcal{Z}(X)$ is defined by the vanishing of some homogeneous polynomial F_X which is unique up to a constant factor. This polynomial is called the *Chow form of* X. The Chow form of X is unique up to multiplication by a nonzero constant. Moreover, it is possible to derive equations defining the variety X from the Chow form of X ([14, Chap 3, Corollary 2.6, p. 102]). Precisely, these equations are obtained by substituting linear polynomials with coefficients taken from generic skew-symmetric matrices into the Chow form and its corresponding variety is precisely X. We denote by $P_1, \cdots, P_N \in \mathbb{C}[x_1 \cdots, x_n]$ the corresponding polynomials (after de-homogenize at $x_0 = 1$) of such equations and we call $\mathcal{I}(X) = (P_1, \cdots, P_N)$ the *Chow ideal of* X. Moreover, we can assume that P_i's form a basis of $\mathcal{I}(X)$. For a more detailed account we refer to [25, p. 37-58]

[1] linear subspace in general position that meets X transversally

Fields with Product Formula

We use here the notion of "field with a proper set of absolutes values satisfying the product formula" in the sense of [8, p. 453]. It can be explained as follows. Let L be such a field and we denote by M_L the set of the corresponding absolute values. We define the *zero height group* Z as

$$Z = Z_L := \{x \in L^\times \mid |x|_v = 1, \ \forall v \in M_L\}.$$

So Z is the subgroup of $L^\times$ made up of elements with trivial valuation everywhere. We also define a *Weil (logarithmic) height on* L by setting

$$h(x) = \sum_{v \in M_L} \log \max(1, |x|_v).$$

This height has the following properties: for any $x, y \in L$ and $\zeta \in Z, m \in \mathbb{Z}$

- $h(\zeta x) = h(x)$,
- $h(xy) \leq h(x) + h(y)$,
- $h(x^{-1}) = h(x)$ if $x \neq 0$
- $h(x^m) = |m| h(x)$.

The height also can be canonically extended to the algebraic closure $\overline{L}$ of L. Namely, if $x \in \overline{L}$ lies in an extension K of L of degree d, then every valuation v extends to a finite number of valuations $w|v$ on K, and

$$h(x) = \frac{1}{d} \sum_{v \in M_F} \sum_{w|v} e_w \log \max(1, |x|_w),$$

for appropriate multiplicities e_w.

B.0.8 Example. Let $y_1, \cdots, y_r$ be a transcendence basis given by sufficiently generic linear polynomials in $x_1, \cdots, x_n$. So $\mathbb{C}(y_1, \cdots, y_r)$ is a purely transcendental extension of $\mathbb{C}$. Such field has a proper set of absolute values satisfying a product formula. Indeed, for any irreducible polynomial P over $\mathbb{C}$ nonconstant, we consider the valuation $v_P : \mathbb{C}(y_1, \cdots, y_r) \longrightarrow \mathbb{Z}$ defined as follows: for any rational function $R = F/G \in \mathbb{C}(y_1, \cdots, y_r)$, with $F, G \in \mathbb{C}[y_1, \cdots, y_r]$ and $G \neq 0$, we can write $R = P_1^{l_1} \cdots P_k^{l_k}$ where $l_1, \cdots, l_k \in \mathbb{Z}$. Then we set

$$v_P(\frac{F}{G}) = \begin{cases} l_j \deg P_j \text{ if } P = P_j \\ 0 \text{ otherwise} \end{cases}.$$

Also, we define the total degree valuation $v_\infty : \mathbb{C}(y_1, \cdots, y_r) \longrightarrow \mathbb{Z}$ defined as follows: for any rational function $R = F/G$ written as $R = P_1^{l_1} \cdots P_k^{l_k}$, we set

$$v_\infty(R) = -\deg R.$$

For every of such valuations, we define the corresponding absolute valuation $|R|_v = e^{-v(R)}$ for any rational function by setting $v(0) = \infty$. Such absolute values are ultrametric. Then clearly, such absolute values satisfy the product formula.

Chabauty's result

Let $\mathbf{b}_i = (b_{i1}, \cdots, b_{in}) \in \mathbb{C}^n$ for $i = 1, \cdots, r$ and $\mathbf{q} = (q_1, \cdots, q_n) \in \mathbb{C}^{\times n}$. Then, following Chabauty [10, p. 144], we call the local analytic subvariety M in $\mathbb{C}^n$ at $\mathbf{q} = (q_1, \cdots, q_n)$ defined by

$$\sum_{j=1}^{n} b_{ij} \log(x_j/q_j) = 0$$

a $\mu-$*variety*. If the b_{ij} are in $\mathbb{Z}$, we shall call M an *algebraic* μ*-variety* and in this case, M is the local analytic variety at $\mathbf{q}$ defined by the algebraic variety with defining equations

$$\prod_{j=1}^{n} (x_j/q_j)^{b_{ij}} = 1, \text{ for } i = 1, \cdots, r.$$

Theorem (Chabauty) *Let M be a* μ*-variety at q and* W *be an algebraic variety containing* q*. Then for each component* I *of* $W \cap M$ *there exists an algebraic* μ*-variety* A *such that* $A \supseteq I$*, and we have*

$$\dim A \leq \dim M + \dim W - \dim I.$$

The proof can be found in [2, p. 263].

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"Ny fahatahorana an'i Jehovah no fiandoham-pahalalana", Ohabolana 1:7a.

"La crainte de l'Éternel est le commencement de la science", Proverbes 1:7a.

"The fear of the Lord is the beginning of knowledge", Proverbs 1:7a.

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