Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Corso di Laurea Triennale in Matematica

The Brunn-Minkowski inequality and the Heisenberg group

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Anno Accademico 2022/2023

22.09.2023

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Introduction

The core of this thesis is the Brunn-Minkowski inequality.

In Chapter 1 we prove the Brunn-Minkowski inequality in the Euclidean space,

$$m(A+B)^{\frac{1}{n}} \geq m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}.$$

As a corollary, we prove the Isoperimetric inequality in the Euclidean space. At the end of this chapter we introduce the "geodesic" version of the Brunn-Minkowski inequality in the Euclidean space

$$m(Z_t(A,B))^{\frac{1}{n}} \geq (1-t)m(A)^{\frac{1}{n}} + tm(B)^{\frac{1}{n}},$$

and the so called *Measure Contraction Property.*

Subsequently, we investigate the validity of the geodesic generalization of the Brunn-Minkowski inequality in the Heisenberg group, which is the simplest example of sub-Riemannian manifond. Indeed, this space is invariant under left translations and homogeneous of degree one with respect to a family of dilations as the Euclidean space. Then, we prove that the Euclidean geodesic generalization of the Brunn-Minkowski inequality is false. Nevertheless there exists a modified version of the Brunn-Minkowski inequality which holds in the Heisenberg group,

$$m(Z_t(A,B))^{\frac{1}{3}} \geq (1-t)^{\frac{5}{3}}m(A)^{\frac{1}{3}} + t^{\frac{5}{3}}m(B)^{\frac{1}{3}},$$

but it does not imply the Isoperimetric inequality.

Furthermore, we provide geometric evidence of the exponent involved in the Measure Contraction Property in the Heisenberg group, and we give a sketch of its proof. So as to pursue our purpose, in Chapter 2, we give a brief introduction of the Heisenberg group, specifically we focus on the metric structure given by the Carnot-Carathéodory distance. In particular, we discuss the notion of horizontal curves and length minimizers and provide a sketch of the calculation thereof.

Chapter 1

The Brunn-Minkowski inequality in the Euclidean space

In this chapter we prove the Brunn-Minkowski inequality in $\mathbb{R}^n$ and the "piecewise $\mathcal{C}^2$ case" of the isoperimetric inequality. To this end, we start presenting some tools concerning the Lebesgue outer measure, namely the measurable hull and the set of points of density 1.

1.1 Notation and statement of the theorem

We begin recalling the definition of Lebesgue outer measure. Let $\mathcal{F}$ be the family of all rectangular boxes $(a_1, b_1) \times \cdots \times (a_n, b_n)$, where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i \quad \forall i \in \{1, ..., n\}$, for every $U \in \mathcal{F}$ we set $m_0(U) = \prod_{i=1}^n (b_i - a_i)$. Then the Lebesgue outer measure of $A \subset \mathbb{R}^n$ is:

$$m(A) = \inf \left\{ \sum_{j=1}^{\infty} m_0(U_j) : A \subset \bigcup_{j=1}^{\infty} U_j, \quad U_j \in \mathcal{F} \quad \forall j \in \mathbb{N} \right\}, \tag{1.1}$$

and, being the σ-algebra of the Lebesgue measurable sets denoted by $\mathcal{L}$, then $m|_{\mathcal{L}}$ is the Lebesgue measure by virtue of Carathéodory theorem.

Definition 1.1. (Minkowski sum) Let A and B be two nonempty subsets of $\mathbb{R}^n$, we define the Minkowski sum:

$$A + B := \{a + b : a \in A\,, b \in B\}.$$

Remark 1.1. Given A and B two nonempy Borel sets of $\mathbb{R}^n$, we observe that $A + B$ is a Borel set for the map $A \times B \ni (a, b) \longmapsto a + b \in A + B$ is continuous thus measurable; it is also open. Note that if two sets are only measurable, the Minkowski sum of those, in general, is not measurable as shown in [11].

With this in mind, we are ready to state the general version of the Brunn-Minkowski inequality in $\mathbb{R}^n$, in other words we make no assumption on the sets involved in the inequality, nonemptiness aside.

Theorem 1.1. (Brunn-Minkowski inequality) If A and B are two nonempty subsets of $\mathbb{R}^n$, then

$$m(A + B)^{\frac{1}{n}} \geq m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}.$$

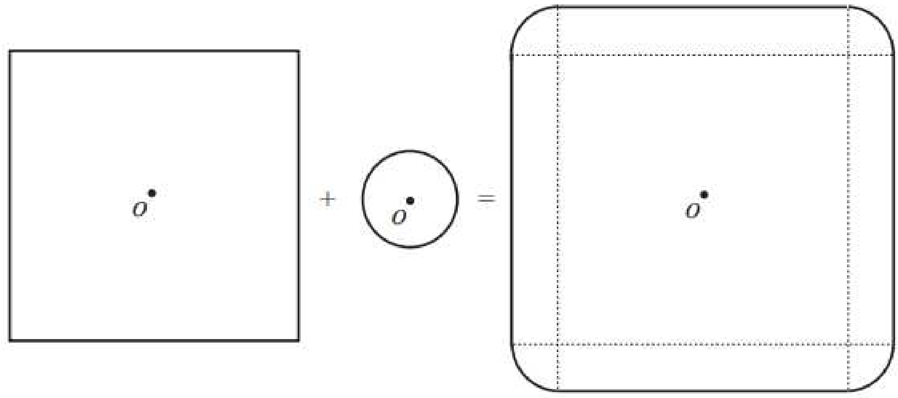


Figure 1.1: The Minkowski sum of a square and a disk. Immage rendered by [6]

1.2 Proof of the theorem

Before proving the Brunn-Minkowski inequality we need some preliminary results about the Lebesgue outer measure. The next proposition assures the existence of a measurable hull for every subsets of $\mathbb{R}^n$. We must observe that this existence is strictly related to the $\sigma-$finiteness of the Lebesgue measure and the regularity thereof.

Proposition 1.1. (Measurable hull) For every $A \subset \mathbb{R}^n$, there exists a measurable hull E of A, i.e. E is measurable, $A \subset E$, and for every F measurable such that $A \subset F$ we have that $m(E \setminus F) = 0$.

Proof. Case 1: Suppose $m(A) < \infty$, by regularity of the Lebesgue measure there for any $k \in \mathbb{N}$ there exists an open set E_k such that $A \subset E_k$ and

$$m(E_k) \leq m(A) + \frac{1}{k}.$$

Define $E = \cap_k E_k$, then E is a measurable since countable intersection of measurable and $A \subset E$. Let F measurable, $A \subset F$, then $m(E) = m(E \cap F) + m(E \setminus F) \geq m(A) + m(E \setminus F)$, hence $m(E \setminus F) = 0$.

Case 2: Due to the σ-finiteness of $\mathbb{R}^n$, there exists a family $\{R_j\}_{j=0}^{\infty}$ of Lebesgue measurable sets, which can be assumed disjoint without loss of generality, such that $\mathbb{R}^n = \bigcup_{j=1}^{\infty} R_j$ and $m(R_j) < \infty$ for all j. Given $A \subset \mathbb{R}^n$, we can write $A = \bigcup_j (A \cap R_j)$ and consider E_j the measurable hull of $A \cap R_j$ which exists by virtue of *Case 1* as $m(R_j) < \infty$. Note that if E_j is a measurable hull of $A \cap R_j$ also $E_j \cap R_j$ is a measurable hull of $A \cap R_j$. We set $E = \bigcup_j (E_j \cap R_j)$ which is measurable and $A \subset E$. Moreover if F is measurable and $A \subset F$, we have

$$m(E \setminus F) = m\left(\bigcup_j (E_j \cap R_j) \setminus \bigcup_j (A \cap R_j)\right) = m\left(\bigcup_j ((E_j \cap R_j) \setminus (A \cap R_j))\right) = 0$$

$\square$

Remark 1.2. One can readily see that if E is a measurable hull of A, we have that $m(A \cap T) = m(E \cap T)$ for every measurable set T.

Consequently we define the set of all elements of density 1 needed for the proof of the theorem.

Proposition 1.2. Let A be a subset of $\mathbb{R}^n$. We denote with $\omega_n = m(B(0,1))$, where $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$, and define

$$A^* = \left\{ y \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{m(A \cap B(y,r))}{\omega_n r^n} = 1 \right\}, \tag{1.2}$$

then A^* is measurable, $m(A \setminus A^*) = 0$ and $A \cup A^*$ is a hull of A.

Proof. If C is a hull of A, then

$$A^* = \left\{ y \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{m(C \cap B(y,r))}{\omega_n r^n} = 1 \right\},$$

thanks to Remark 1.2, observe that

$$\lim_{r \to 0^+} \frac{m(C \cap B(y,r))}{\omega_n r^n} = \lim_{r \to 0^+} \frac{1}{\omega_n r^n} \int_{B(y,r)} \chi_C dx,$$

where χ_C is the characteristic function of the set C and dx is the Lebesgue measure. Thus by virtue of the Lebesgue differentiation theorem $A^* = C$ almost everywhere, hence measurable. As a consequence, $m(C \triangle A^*) = 0$ and the assertions about A^* follow from the inclusions:

$$A \setminus A^* \subset C \setminus A^*, \quad A \subset A \cup A^* \subset C \cup (A^* \setminus C).$$

$\square$

Note that $m(A^*) \leq m(A \cup A^*) \leq m(A^*) + m(A \setminus A^*) = m(A^*)$ thence $m(A \cup A^*) = m(A^*)$ and $m(A) = m(A^*)$ due to Remark 1.2.

Now we are ready to give a proof of the Brunn-Minkowski inequality.

Proof of Theorem 1.1. Let $\mathcal{F}$ be the family of all rectangular boxes $P_1 \times \cdots \times P_n$ where $P_1, ..., P_n$ are nonempty, bounded, open subintervals of $\mathbb{R}$.

If $A = P_1 \times \cdots \times P_n \in \mathcal{F}$ and $B = Q_1 \times \cdots \times Q_n \in \mathcal{F}$, then $A + B = (P_1 + Q_1) \times \cdots (P_n + Q_n)$, and $m(P_i) + m(Q_i) = m(P_i + Q_i)$ for $i = 1, \ldots, n$; now

$$\begin{aligned}\frac{m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}}{m(A+B)^{\frac{1}{n}}} &= \prod_{i=1}^{n} \left(\frac{m(P_i)}{m(P_i + Q_i)} \right)^{\frac{1}{n}} + \prod_{i=1}^{n} \left(\frac{m(Q_i)}{m(P_i + Q_i)} \right)^{\frac{1}{n}} \\ &\leq \frac{1}{n} \sum_{i=1}^{n} \frac{m(P_i)}{m(P_i + Q_i)} + \frac{1}{n} \sum_{i=1}^{n} \frac{m(Q_i)}{m(P_i + Q_i)} = 1\end{aligned}$$

thanks to arithmetic-geometric mean inequality.

Next we treat the case when $A = \bigcup_{j=1}^{p} G_j$ and $B = \bigcup_{j=1}^{q} H_j$ for some finite disjoint subfamilies $\mathcal{G} = \{G_j\}_{j=1}^{p}$ and $\mathcal{H} = \{H_j\}_{j=1}^{q}$ of $\mathcal{F}$, by applying induction with respect to $p+q$.

If $p > 1$, we choose $i \in \{1, \dots, n\}$ and $a \in \mathbb{R}$ so that each of the two sets

$$A_1 = A \cap \{x \in \mathbb{R}^n : x_i < a\}, \quad A_2 = A \cap \{x \in \mathbb{R}^n : x_i > a\}$$

contains some elements of $\mathcal{G}$, and also choose $b \in \mathbb{R}$ such that the sets

$$B_1 = B \cap \{x \in \mathbb{R}^n : x_i < b\}, \quad B_2 = B \cap \{x \in \mathbb{R}^n : x_i > b\}$$

satisfy the equations

$$\frac{m(A_k)}{m(A)} = \frac{m(B_k)}{m(B)} \quad \text{for } k = 1, 2.$$

We see that

$$A_k = \bigcup_{j=1}^{p} (G_j \cap A_k) \quad \text{and} \quad B_k = \bigcup_{j=1}^{q} (H_j \cap B_k),$$

thanks to the above construction there exists at least one $j \in \{1, ..., p\}$, which can assumed to be $j = p$ minus reordering, such that $G_p \cap A_k = \emptyset$; for the same reason there could exist some $j \in \{1, ..., q\}$ such that $H_j \cap A_k = \emptyset$, hence we have that

$$A_k = \bigcup_{j=1}^{p'} (G_j \cap A_k), \quad \text{with } p' < p, \qquad B_k = \bigcup_{j=1}^{q'} (H_j \cap B_k), \quad \text{with } q' \leq q.$$

Since $A_1 + B_1$ and $A_2 + B_2$ are separated by $\{x \in \mathbb{R}^n : x_i = a + b\}$, induction yields

$$\begin{aligned} m(A+B) &\geq m(A_1 + B_1) + m(A_2 + B_2) \\ &\geq \left(m(A_1)^{\frac{1}{n}} + m(B_1)^{\frac{1}{n}}\right)^n + \left(m(A_2)^{\frac{1}{n}} + m(B_2)^{\frac{1}{n}}\right)^n \\ &= \left(m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}\right)^n. \end{aligned}$$

If A, B are two nonempty compact subsets of $\mathbb{R}^n$, then $A+B$ is also compact, Remark 1.1. From (1.1) we have that for every $\varepsilon > 0$ there exists a sequence $\{U_j\}_{j=1}^{\infty} \subset \mathcal{F}$, which can be assumed to have finite Measure as the measure of a compact set is finite, such that

$$\sum_{j=1}^{\infty} m(U_j) \leq m(A) + \varepsilon,$$

thus we have that $\sum_{j=1}^{\infty} m(U_j) = \sum_{j=1}^{N} m(U_j) + r_N$ with $r_N \to 0$ as $N \to \infty$, hence

$$\sum_{j=1}^{N} m(U_j) \leq m(A) + \varepsilon - r_N,$$

if we choose N big enough, any compact set can be approximated by a finite disjoint union of boxes, and as a result the Brunn-Minkowski inequality holds in this case.

If A, B are two nonempty measurable subsets of $\mathbb{R}^n$, there exist two sequence of compact sets $\{K_j\}_{j=1}^{\infty} \subset A$ and $\{H_j\}_{j=1}^{\infty} \subset B$ such that

$$m(K_j) \geq m(A) - \frac{1}{k} \quad \text{and} \quad m(H_j) \geq m(B) - \frac{1}{k},$$

then for every $j \in \mathbb{N}$ we have

$$\begin{aligned} m(A+B)^{\frac{1}{n}} &\geq m(K_j + H_j)^{\frac{1}{n}} \geq m(K_j)^{\frac{1}{n}} + m(H_j)^{\frac{1}{n}} \geq \\ &\geq \left(m(A) - \frac{1}{k}\right)^{\frac{1}{n}} + \left(m(B) - \frac{1}{k}\right)^{\frac{1}{n}} \xrightarrow[k\to\infty]{} m(A)^{\frac{1}{n}} + m(B)^{\frac{1}{n}}. \end{aligned}$$

Finally, if A and B are not measurable, it suffices to consider A^* and B^* as in (1.2) and observe that $A^* + B^* \subset (A+B)^*$. □

Remark 1.3. Henceforth we shall always assume A and B to be measurable so as to freely apply the properties of Lebesgue measure and integration, nevertheless this assumption is not restrictive in that we now know how to treat the excluded case.

For the sake of completeness, we specify the equality condition for Theorem 1.1, a detailed dissertation is available in [9].

Theorem 1.2. If A and B are two nonempty compact subsets of $\mathbb{R}^n$, then the equality in Theorem 1.1 holds if and only if A and B are homotetic and $m(conv(A) \setminus A) = m(conv(B) \setminus B) = 0$, where $conv(A)$ is the convex hull of A, i.e. the smallest convex set in which A is contained.

1.3 Applications and possible generalizations

As a corollary of the Brunn-Minkowski inequality, we are able to prove the isoperimetric inequality.

Theorem 1.3. Let A be measurable and bounded set of $\mathbb{R}^n$, with Lipschitz boundary, then

$$\mathscr{H}^{n-1}(\partial A) \geq n\omega_n^{\frac{1}{n}} m(A)^{\frac{n-1}{n}},$$

denoting with $\mathscr{H}^{n-1}$ the $n-1$-dimensional Hausdorff measure and with $\omega_n = m(B(0,1))$. Moreover the equality holds if and only if A is, up to translations and sets of measure 0, a ball.

Proof. We only prove the case in which ∂A is $\mathcal{C}^2$, and in fact piecewise $\mathcal{C}^2$. For a general proof one may consult [5].

Fix $r > 0$, consider $A_r = \{x \in \mathbb{R}^n : d(x, A) < r\}$ with $d(x, A) = \inf_{a\in A} |x - a|$. Observe that this definition is equivalent to $A_r = A + rB(0,1)$. Note that since $m(A) < \infty$ eventually $m(A_r) < \infty$, thus the following inequality is well defined

$$m(A_r \setminus A) = m(A_r) - m(A) \geq \left(m(A)^{\frac{1}{n}} + r\omega_n^{\frac{1}{n}}\right)^n - m(A) = rn\omega_n^{\frac{1}{n}} m(A)^{\frac{n-1}{n}} + o(r), \quad (1.3)$$

due to the Brunn-Minkowski inequality and Newton's Binomial Theorem. In order to prove the theorem, we have to show that

$$\lim_{r\to 0^+} \frac{m(A_r \setminus A)}{r} = \mathscr{H}^{n-1}(\partial A), \tag{1.4}$$

in that, if (1.4) were true, we would have

$$\mathscr{H}^{n-1}(\partial A) \geq \lim_{r\to 0^+} \frac{rn\omega_n^{\frac{1}{n}} m(A)^{\frac{n-1}{n}} + o(r)}{r} = n\omega_n^{\frac{1}{n}} m(A)^{\frac{n-1}{n}}$$

thanks to (1.3), as claimed.

Let f be the function $x \longmapsto d(x, A)$, one can show that if ∂A is $\mathcal{C}^2$, $f \in \mathcal{C}^2(\overline{A_r \setminus A})$ and $|\nabla f| = 1$ on $A_r \setminus A$. Firstly observe that

$$\lim_{r\to 0^+} \int_{A_r\setminus A} \Delta f(x)dx \leq \lim_{r\to 0^+} \|\Delta f\|_\infty m(A_r \setminus A) = 0, \tag{1.5}$$

as $f \in \mathcal{C}^2(\overline{A_r \setminus A})$. Recall that $\Delta f = \operatorname{div}(\nabla f)$, being div the divergence operator. Thanks to (1.5) we have

$$\begin{aligned} 0 &= \lim_{r\to 0^+} \int_{A_r\setminus A} \Delta f(x)dx = \lim_{r\to 0^+} \left(\int_{\{f=r\}} \langle \nabla f, \nabla f\rangle d\mathscr{H}^{n-1} - \int_{\partial A} \langle \nabla f, \nabla f\rangle d\mathscr{H}^{n-1} \right) = \\ &= \lim_{r\to 0^+} \mathscr{H}^{n-1}(\{f = r\}) - \mathscr{H}^{n-1}(\partial A), \end{aligned}$$

by virtue of the Divergence Theorem. Hence the function $r \longmapsto \mathscr{H}^{n-1}(\{f = r\})$ is continuous at $r = 0$. We conclude due to the Fundamental Theorem of Calculus and Coarea Formula:

$$\frac{m(A_r \setminus A)}{r} = \frac{1}{r} \int_{A_r\setminus A} |\nabla f(x)|dx = \frac{1}{r} \int_0^r \mathscr{H}^{n-1}(\{f = t\})dt \underset{r\to 0^+}{\longrightarrow} \mathscr{H}^{n-1}(\partial A).$$

From Theorem (1.2) we know that the Brunn-Minkowski equality condition holds if and only if A is, up to translations and sets of measure 0, a ball, then (1.3) is an equality and we conclude. $\square$

Now we give a more "dynamical" version of Theorem 1.1. In so doing, we need the following

Definition 1.2. (t-intermediate set) Let $A, B \subset \mathbb{R}^n$ be measurable nonempty sets and $t \in [0, 1]$. The set $Z_t(A, B)$ of *t-intermediate points* is the set of all points $\gamma(t)$, where $\gamma \colon [0, 1] \to \mathbb{R}^n$ is a minimizing geodesic such as $\gamma(0) \in A$ and $\gamma(1) \in B$.

Recall that the minimizing geodesic in $\mathbb{R}^n$ joining two points x and y is

$$\begin{aligned} \gamma \colon [0, 1] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto x + t(y - x). \end{aligned}$$

As a consequence, in the Euclidean case the explicit formula for the set $Z_t(A, B)$ of Definition 1.2 is

$$Z_t(A,B) = A + t(B - A) = \{a + t(b - a) \colon a \in A, b \in B\}.$$

Now we can state the "geodesic" version of the Brunn-Minkowski inequality, and in particular we show that it is equivalent to Theorem 1.1.

Proposition 1.3. Let $A, B \subset \mathbb{R}^n$ be nonempty and measurable, then:
Theorem 1 is equivalent to

$$m(Z_t(A,B))^{\frac{1}{n}} \geq (1-t)m(A)^{\frac{1}{n}} + tm(B)^{\frac{1}{n}} \quad \forall t \in [0,1]. \tag{1.6}$$

Proof. For the necessity: adopt the following substitution: $A \to (1-t)A$ and $B \to tB$ then the assertion is true by virtue of Lebesgue measure property under linear transformation. For the sufficiency: consider the time $t = 1/2$, then substitute $A \to 2A$ and similarly for B. □

Remark 1.4. If in the (1.6) A is a singleton $\{x\}$ with $x \in \mathbb{R}^n$ we have the so called *Measure Contraction Property*

$$m(Z_t(x,B)) \geq t^n m(B) \quad \forall t \in [0,1]. \tag{1.7}$$

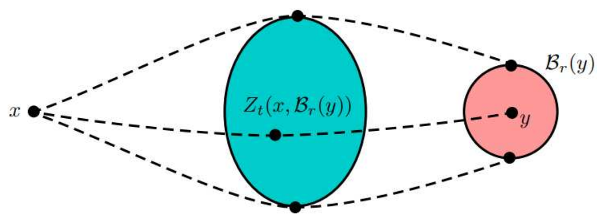


Figure 1.2: A graphic representation of the Measure Contraction Property, general case. This picture is courtesy of [4].

Furthermore, we note that (1.7) in the Euclidean case is an equality which readily descends from the invariance under translation of the Lebesgue measure.

In Chapter 3, we study the "geodesic" Brunn-Minkowski inequality in the Heisenberg group.

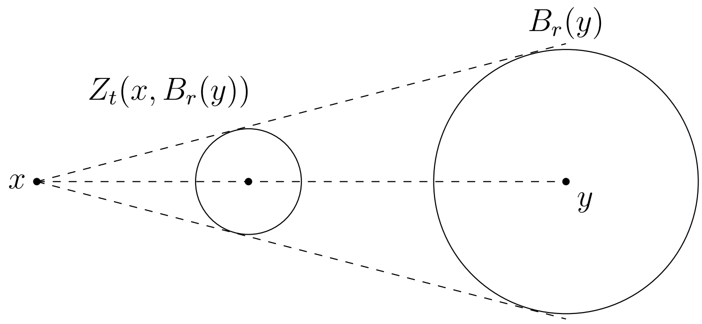


Figure 1.3: A visual interpretation of the Measure Contraction Property in $\mathbb{R}^2$.

Chapter 2

The Heisenberg Group

In this chapter we introduce the Heisenberg group viewed as a sub-Riemannian manifold. Once that the basic notions are given, we shall focus on the concept of length-minimizers.

2.1 Definition of the Heisenberg group

Definition 2.1. The Heisenberg group is the manifold $\mathbb{H} = \mathbb{R}^3$ endowed with the group product

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right). \tag{2.1}$$

The Heisenberg group is a non commutative Lie group, the identity element is $0 = (0,0,0)$ and the inverse of x is $-x = (-x_1, -x_2, -x_3)$.

We define the *left translation* by $p \in \mathbb{H}$ to be the map $\tau_p : \mathbb{H} \longrightarrow \mathbb{H}$ where $\tau_p(x) = p \cdot x$. Observe that the left-translation map can be written as

$$\tau_p(x) = p + d\tau_p x \quad \text{where} \quad d\tau_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{p_2}{2} & \frac{p_1}{2} & 1 \end{pmatrix}$$

In addition, for every $\lambda > 0$ we define the map $\delta_\lambda : \mathbb{H} \longrightarrow \mathbb{H}$ where $\delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$ is called *dilation.* Note that $\delta_\lambda(x) = \Delta x$ where $\Delta = diag(\lambda, \lambda, \lambda^2)$.

Let $E \subset \mathbb{H}$ be measurable, then by applying the change of variables theorem and the invariance under tlanslations of the Lebesgue measure we have that

$$m(\tau_p(E)) = m(E) \quad \text{and} \quad m(\delta_\lambda(E)) = \lambda^4 m(E).$$

We introduce the Lie algebra of left-invariant vector fields of $\mathbb{H}$. X is a left-invariant vector field if $X(p) = d\tau_p X(0)$. Those vectors together with the Lie bracket form a Lie algebra $\mathfrak{h}$ called Heisenberg Lie algebra. The algebra $\mathfrak{h}$ is spanned by the vector fields

$$X_1 = \partial_1 - \frac{1}{2} x_2 \partial_3, \quad X_2 = \partial_2 + \frac{1}{2} x_1 \partial_3, \quad T = \partial_3. \tag{2.2}$$

The vector fields (2.2) satisfy

$$X_i(p) = d\tau_p X_i(0) \quad \text{for } i = 1, 2 \quad \text{and } T(p) = d\tau_p T(0).$$

The distribution $D_p = \text{span}\{X_1(p), X_2(p)\}$ is called *horizontal distribution* and it is bracket-generating for $[X_1, X_2] = T$ and $\dim_{\mathbb{R}}\text{span}\{X_1, X_2, [X_1, X_2]\} = 3$.

Before introducing a metric structure on $\mathbb{H}$, we need some definitions and remarks.

Definition 2.2. Let $I \subset \mathbb{R}$ be a closed interval. A Lipschitz curve $\gamma : I \longrightarrow \mathbb{H}$ is *horizontal* if $\dot{\gamma}(t) \in D_{\gamma(t)}$ for a.e. $t \in I$. Equivalently γ is horizontal if there exist functions $u_1, u_2 \in L^\infty(I)$ such that

$$\dot{\gamma} = u_1 X_1(\gamma) + u_2 X_2(\gamma) \quad \text{a.e. on } I. \tag{2.3}$$

Observe that Definition 2.2 is equivalent to ask the curve to have their velocities in the kernel of $\omega = -dx_3 + \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$.

We denote the set of horizontal curve connecting x and y with $H_{x,y}$. Note that in the latter set we are considering only the support of the curves, not the parametrizations.

We set the sub-Riemannian norm in the Heisenberg group by the relation

$$\|a_1 X_1 + a_2 X_2\|_{\mathbb{H}}^2 = |a_1^2 + a_2^2|^2,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$. As a consequence, we define the sub-Riemannian length of an horizontal curve to be

$$\ell(\gamma) = \int_I \|\dot{\gamma}(t)\|_{\mathbb{H}} dt = \int_I \sqrt{u_1^2(t) + u_2^2(t)} dt.$$

Moreover, observe that the sub-Riemannian length of an horizontal curve is exactly the Euclidean length of its projection onto the plane $\{x_3 = 0\}$:

$$\ell(\gamma) = \int_0^1 \sqrt{u_1^2 + u_2^2} dt = \int_0^1 \sqrt{\dot{x}_1^2 + \dot{x}_2^2} dt = \int_0^1 |(\pi(\gamma))'| dt = \ell_e(\pi(\gamma))$$

With the following proposition, we characterise the behaviour of the length of an horizontal curve under some geometric transformations.

Proposition 2.1. Let γ be an horizontal curve, then

1. $\ell(\tau_p(\gamma)) = \ell(\gamma)$ for every $p \in \mathbb{H}$,
2. $\ell(\delta_\lambda(\gamma)) = \lambda \ell(\gamma)$ for every $\lambda > 0$,
3. $\ell(\rho_\theta(\gamma)) = \ell(\gamma)$ for every $\theta \in \mathbb{R}$,
4. $\ell(sym(\gamma)) = \ell(\gamma)$

where ρ_θ is the rotation around the x_3-axis of on angle θ and $sym(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$.

Proof. 1. True by definition as X_1 and X_2 are left-invariant vector fields.

2. From the identities: $D\delta_\lambda(p)X_i = \lambda X_i(\delta_\lambda(p))$ for $i = 1, 2$.
3. From the identities $D\rho_\theta(p)X_1 = \cos(\theta)X_1(\rho_\theta(p)) + \sin(\theta)X_2(\rho_\theta(p))$ and $D\rho_\theta(p)X_2 = -\sin(\theta)X_1(\rho_\theta(p)) + \cos(\theta)X_2(\rho_\theta(p))$.
4. From the identities $Dsym(p)X_1 = X_1(sym(p))$ and $Dsym(p)X_2 = -X_2(sym(p))$

$\square$

Now we try to understand the geometry of horizontal curves starting from the origin. Given $\gamma : [0, 1] \to \mathbb{H}$ such that $\gamma(0) = 0$, and $\gamma(1) = p$, by Definition 2.2 we have

$$\begin{cases} \dot\gamma_i = u_i \quad i = 1, 2 \\ \dot\gamma_3 = -\frac{\gamma_2}{2}u_1 + \frac{\gamma_1}{2}u_2. \end{cases}$$

In particular, given u_1 and u_2 we find:

$$\begin{aligned} \gamma_i(t) &= \int_0^t u_i(s)ds \quad i = 1, 2 \\ \gamma_3(t) &= \int_0^t \left(-\frac{\gamma_2(s)}{2}u_1(s) + \frac{\gamma_1(s)}{2}u_2(s)\right) ds = \frac{1}{2}\int_0^t (\gamma_1\dot\gamma_2 - \dot\gamma_1\gamma_2)ds = \\ &= \int_\gamma \frac{1}{2}(x_1dx_2 - x_2dx_1) = \int_{\gamma\eta} \frac{1}{2}(x_1dx_2 - x_2dx_1) = \int_R dx_1 \wedge dx_2 = \mathscr{H}^2(R), \end{aligned}$$

where $[0, 1] \ni t \longmapsto \eta(t) = (1 - t)p \in \mathbb{H}$ whose contribution to the integral is null as $\eta_1\dot\eta_2 - \dot\eta_1\eta_2 = 0$, $\gamma\eta$ is the composition of paths, and R is the region enclosed by $\gamma\eta$.

With the following remark, we show that the Heisenberg group is connected via horizontal curves.

Remark 2.1. (Connectivity of $\mathbb{H}$) Observe that due to the left-invariance of the vector fields X_1 and X_2, so as to find an horizontal curve joining p and q, it suffices to calculate the curve from 0 to $p^{-1} \cdot q$ and then apply τ_p to that curve. Now we exhibit an horizontal curve of finite length joining 0 and p.

To start with, consider $p = (x, 0, z)$ with $z \geq 0$. Define $\kappa : [0, 1] \longrightarrow \mathbb{H}$

$$\kappa_{x,z}(t) = \begin{cases} 5t(x, 0, 0) & t \in \left[0, \frac{1}{5}\right] \\ (x, 0, 0) + (5t - 1)(0, -\sqrt{z}, -\sqrt{z}\frac{x}{2}) & t \in \left[\frac{1}{5}, \frac{2}{5}\right] \\ (x, -\sqrt{z}, -\sqrt{z}\frac{x}{2}) + (5t - 2)(\sqrt{z}, 0, \frac{z}{2}) & t \in \left[\frac{2}{5}, \frac{3}{5}\right] \\ (x + \sqrt{z}, -\sqrt{z}, \frac{z - x\sqrt{z}}{2}) + (5t - 3)(0, \sqrt{z}, \frac{z + x\sqrt{z}}{2}) & t \in \left[\frac{3}{5}, \frac{4}{5}\right] \\ (x + \sqrt{z}, 0, z) + (5t - 4)(-\sqrt{z}, 0, 0) & t \in \left[\frac{4}{5}, 1\right], \end{cases} \tag{2.4}$$

is horizontal since

$$\dot\kappa_{x,z}(t) = \begin{cases} 5xX_1(\kappa) & t \in \left[0, \frac{1}{5}\right] \\ -5\sqrt{z}X_2(\kappa) & t \in \left[\frac{1}{5}, \frac{2}{5}\right] \\ 5\sqrt{z}X_1(\kappa) & t \in \left[\frac{2}{5}, \frac{3}{5}\right] \\ 5\sqrt{z}X_2(\kappa) & t \in \left[\frac{3}{5}, \frac{4}{5}\right] \\ -5\sqrt{z}X_1(\kappa) & t \in \left[\frac{4}{5}, 1\right]. \end{cases} \tag{2.5}$$

In addition, we see that $\ell(\kappa) = |x| + 4\sqrt{z}$. If $z < 0$ we can consider the curve $sym(\kappa_{x,-z})$ which satisfies our purpose. For a generic point $p = (x, y, z)$, supposing $z \geq 0$, consider the curve $\rho_\theta(\kappa_{r,z})$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

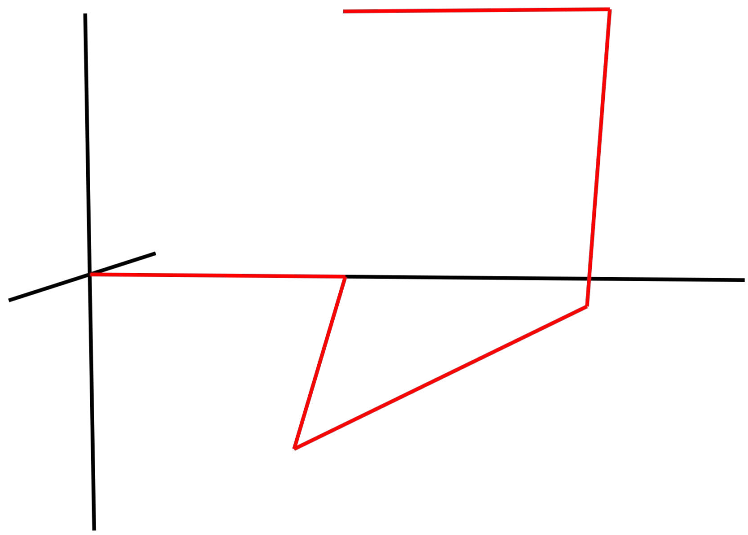


Figure 2.1: Graphic representation of $\kappa_{1,1}$.

Finally we are ready to define a metric structure on $\mathbb{H}$.

Proposition 2.2. (Carnot-Carathéodory distance) The function $d : \mathbb{H} \times \mathbb{H} \longrightarrow [0, \infty)$

$$d(x, y) = \inf\{\ell(\gamma) : \gamma \in H_{x,y}\} \tag{2.6}$$

is a distance on $\mathbb{H}$.

Proof. For any $x, y \in \mathbb{H}$ we have that $d(x, y) < \infty$, thus the function d is well defined by virtue of Remark 2.1. $d(x, y) \geq 0$ for any $x, y \in \mathbb{H}$. Moreover if $d(x, y) = 0$, we have that for any $\gamma \in \mathbb{H}_{x,y}$, $\dot{\gamma} = 0$ a.e. on I thus γ is constant as it is continuous. Symmetry follows from the definition of $H_{x,y}$. For the triangular inequality consider $x, y, z \in \mathbb{H}$. For any $\gamma \in H_{x,y}$, $\gamma' \in H_{x,z}$, $\gamma'' \in H_{y,z}$, we have that the composition of path $\gamma'\gamma'' \in H_{x,y}$, and $\ell(\gamma'\gamma'') = \ell(\gamma') + \ell(\gamma'')$, hence the conclusion.

□

Thanks to Proposition 2.1 the distance d is left invariant and homogeneous of degree 1 with respect to dilations, i.e. for every $x, y, z \in \mathbb{H}$ we have:

$$d(x \cdot y, x \cdot z) = d(y, z), \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y).$$

2.2 Length-minimizers in the Heisenberg group

In this section we consider the problem of finding length-minimizers, namely the shortest curve joining two points in $\mathbb{H}$ with respect to the Carnot-Carathéodory metric introduced in Proposition 2.2.

Remark 2.2. As the length of an horizontal curve is invariant up to reparametrization, we can freely assume that it is parametrized on $[0, 1]$.

Definition 2.3. An horizontal curve $\gamma : [0, 1] \longrightarrow \mathbb{H}$ is a length minimizer if $\ell(\gamma) = d(\gamma(0), \gamma(1))$.

Remark 2.3.
- Every horizontal curve γ is the reparametrization of an horizontal curve with constant speed and $\ell(\gamma)$ is invariant up to reparametrization, thus henceforth we shall deal with curves constant speed parametrized.

- Constant speed curves realize the following equality:

$$\left(\int_0^1 \|\dot{\gamma}\|_{\mathbb{H}} dt\right)^2 = \int_0^1 \|\dot{\gamma}\|^2_{\mathbb{H}} dt, \tag{2.7}$$

note that in general is only true with $\leq$. Defining the *sub-Riemannian energy* $J(\gamma) = \frac{1}{2}\int_0^1 \|\dot{\gamma}\|^2_{\mathbb{H}} dt$, (2.7) becomes $\ell(\gamma)^2 = 2J(\gamma)$. Hence, in order to find length-minimzers from x to y we have to compute

$$\inf\{J(\gamma) : \gamma \in H_{x,y}\}$$

where $H_{x,y}$ is the set of all horizontal curves joining x and y. Observe that this is an optimal control problem:

$$\begin{cases} \inf \int_0^1 (u_1^2 + u_2^2) dt \\ x(0) = 0, \quad x(1) = y \\ \dot{x}(t) = u_1(t) X_1(x(t)) + u_2(t) X_2(x(t)). \end{cases}$$

Now, we would like to compute the length-minimizers, we give a brief scketch of the computations, a complete and exhaustive formulation of this optimal control problem can be found in [1].

We write the sub-Riemannian Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(h_1^2 + h_2^2) \quad \text{where} \\ h_1 &= p \cdot X_1(x) = p_1 - \frac{x_2}{2} p_3 \\ h_2 &= p \cdot X_2(x) = p_2 + \frac{x_1}{2} p_3. \end{aligned}$$

It is a good idea to write down the equation with these coordinates $(x_1, x_2, x_3, h_1, h_2, h_0)$ where

$$h_0 = p \cdot [X_1, X_2](x) = p_3.$$

The Hamiltonian equations are:

$$\begin{cases} \dot{x}_1 = h_1 \\ \dot{x}_2 = h_2 \\ \dot{x}_3 = \frac{1}{2}(x_1 h_2 - x_2 h_1) \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_0 h_2 \\ \dot{h}_2 = h_0 h_1 \\ \dot{h}_0 = 0 \end{cases}$$

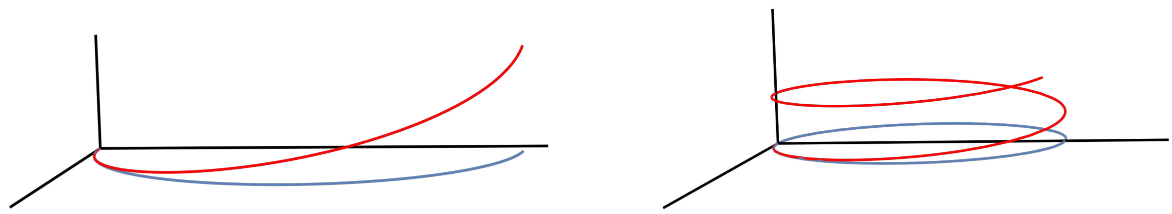
Solving the vertical part with the initial condition $(\cos\theta_0, \sin\theta_0, h_0)$ we obtain:

$$\begin{cases} h_1(t) = \cos(\theta_0 + h_0 t) \\ h_2(t) = \sin(\theta_0 + h_0 t) \\ h_0(t) = h_0 \end{cases}$$

If $h_0 = 0$ we observe that the horizontal equations describe a straight line starting from the origin, otherwise we have:

$$\begin{cases} x_1(t) = \frac{1}{h_0}(\sin(\theta_0 + h_0 t) - \sin(\theta_0)) \\ x_2(t) = -\frac{1}{h_0}(\cos(\theta_0 + h_0 t) - \cos(\theta_0)) \\ x_3(t) = \frac{1}{2h_0^2}(h_0 t - \sin(h_0 t)). \end{cases} \tag{2.8}$$

Remark 2.4. We observe that the projection of (2.8) onto the x_1x_2-plane describes an arc of circle whose center is $C = \frac{1}{h_0}(-\sin\theta_0, \cos\theta_0)$ and radius $\rho = \frac{1}{|h_0|}$. Moreover one can show that the curve in (2.8) is a length minimizer if and only if $h \in [-2\pi, 2\pi]$ and it is unique if and only if the extremes of the interval are excluded; a proof thereof can be found in [2].



(a) Solution to (2.8) with $h_0 = 3$, and $\theta_0 = 0$, length-minimizer

(b) Solution to (2.8) with $h_0 = 9$, and $\theta_0 = 0$, not a length-minimizer

Figure 2.2: Examples of solution of (2.8)

We now characterise a family of horizontal curves needed in Chapter 3.

Let $v = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$ and $h \in (-2\pi, 2\pi)$ we can define the following family of curves:

$$\gamma_{v,h}(t) = \begin{cases} (A\frac{v}{h}, \frac{|v|^2}{2}\frac{ht - \sin(ht)}{h^2}) & h \neq 0 \\ (tv, 0) & h = 0 \end{cases} \quad \text{where } A = \begin{pmatrix} \cos(ht) - 1 & \sin(ht) \\ \sin(ht) & 1 - \cos(ht) \end{pmatrix}. \tag{2.9}$$

With no difficulties, one computes

$$\dot{\gamma}(t) = (-\sin(ht)v_1 + \cos(ht)v_2)X_1(\gamma) + (\cos(ht)v_1 + \sin(ht)v_2)X_2(\gamma),$$

thus every curve in this family is horizontal. Furthermore the length of $\gamma_{v,h}$ between a and b is $|v|(b-a)$. Note that if in 2.9 we use $v = (\cos(\theta), \sin(\theta))$, they are exactly those in (2.8).

We define the map $\Gamma_t(v,h) := \gamma_{v,h}(t)$ which is analytic on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. In particular we shall make use of it for $t = 1$. Note that $\Gamma_t(v,h) = \Gamma_1(tv, th)$ for $0 < |t| \leq 1$.

Subsequently we compute the Jacobian determinant of Γ_t which is needed in later computations.

Proposition 2.3. The value of the Jacobian determinant of Γ_t is

$$\det(D\Gamma_t)(v,h) = \begin{cases} -\frac{t|v|^2}{h^4}(th\sin(th) + 2\cos(th) - 2) & \text{for } h \neq 0 \\ -\frac{t^5|v|^2}{12} & \text{for } h = 0. \end{cases} \tag{2.10}$$

thus it does not vanish on $D = \mathbb{R}^2 \setminus \{0\} \times (-2\pi, 2\pi)$.

Proof. We split the proof in two cases.

Case 1: $t = 1$. Suppose $h \neq 0$, the case $h = 0$ will be obtained as a limit. To begin with, we prove that if $|v| = |v'|$, then $\det(D\Gamma_1)(v,h) = \det(D\Gamma_1)(v',h)$. Given $T \in SO_3(\mathbb{R})$ so that $Tv = v'$. Consider now T' defined by $T'(v,h) = (Tv,h)$. As $\det(A) \neq 0$ for $h \neq 0$ then one readily sees that $\Gamma_1 \circ T' = T' \circ \Gamma_1$, hence our assertion. We use this relation to simplify the computation by choosing $v' = (|v|, 0)$:

$$\det(D\Gamma_1) = \begin{vmatrix} (\cos(h)-1)/h & \sin(h)/h & |v|\left(-\frac{\sin(h)}{h} - \frac{\cos(h)-1}{h^2}\right) \\ \sin(h)/h & (1-\cos(h))/h & |v|\left(\frac{\cos(h)}{h} - \frac{\sin(h)}{h^2}\right) \\ |v|\frac{h-\sin(h)}{h^2} & 0 & \frac{|v|^2}{2}\left(\frac{2\sin(h)}{h^3} - \frac{1+\cos(h)}{h^2}\right) \end{vmatrix} =$$

$$= \frac{|v|^2}{h^4}\begin{vmatrix} \cos(h)-1 & \sin(h) & -\sin(h) \\ \sin(h) & 1-\cos(h) & \cos(h) \\ h-\sin(h) & 0 & \frac{1-\cos(h)}{2} \end{vmatrix} = \frac{|v|^2}{h^4}(h\sin(h) + 2\cos(h) - 2).$$

In addition we have that its continuous limit is

$$\lim_{h\to 0} \frac{|v|^2}{h^4}(h\sin(h) + 2\cos(h) - 2) = -\frac{|v|^2}{12}.$$

Now we study the sign on D_1. Observe that

$$h\sin(h) + 2\cos(h) - 2 = 2\sin\left(\frac{h}{2}\right)\left(\frac{h}{2}\cos\left(\frac{h}{2}\right) - \sin\left(\frac{h}{2}\right)\right),$$

if we prove that the odd function $f(u) := u\cos(u) - \sin(u)$ is strictly negative on $(0,\pi)$ the proof is complete. Observe that $f(0) = 0$ and $f'(u) = -u\sin(u)$ which is negative on $(0,\pi)$. On this interval f is non-increasing and does not vanish. Thus we have:

$$\det(D\Gamma_1)(v,h) = \begin{cases} \frac{|v|^2}{h^4}(h\sin(h) + 2\cos(h) - 2) & \text{for } h \neq 0 \\ -\frac{|v|^2}{12} & \text{for } h = 0, \end{cases}$$

and it does not vanish on $D_t = \mathbb{R}^2 \setminus \{0\} \times (-2\pi t, 2\pi t)$.

Case 2 $0 < |t| \leq 1$. As $\Gamma_t(v,h) = \Gamma_1(tv, th) = \Gamma \circ (t\mathbb{I}_3)(v,h)$ where $\mathbb{I}_3$ is the identity matrix of $\mathbb{R}^3$, thus

$$\det(D\Gamma_t)(v,h) = \det(D\Gamma_1)(tv,th)\det(t\mathbb{I}_3)(v,h) = t^3 \det(D\Gamma_1)(tv,th),$$

we conclude by virtue of *Case 1*.

□

2.3 Measure Contraction Property in the Heisenberg group

In this section we study the Measure Contraction Property in the Heisenberg group. So as to do so, we need to better define the "t-intermediate set" of Definition 1.2.

Henceforth we define $L = \{x \in \mathbb{H} : (x_1, x_2) = 0\}$ and $U = \{(x,y) \in \mathbb{H}^2 : x^{-1}y \notin L\}$

From the previous section we know that the minimizing geodesic in $\mathbb{H}$ joining two points x and y is:

$$\begin{aligned} \gamma\colon [0,1] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \tau_x \circ \Gamma_t \circ \Gamma_1^{-1} \circ \tau_{x^{-1}}(y). \end{aligned} \tag{2.11}$$

One must observe that (2.11) is well define if and only if $(x,y) \in U$. Nevertheless this in not an obstacle to our discussion as $m(L) = 0$.

Thus, given $A, B \in \mathbb{H}$ with positive measure, the explicit formula for the t-intemediate set is

$$Z_t(A,B) = \{\tau_a \circ \Gamma_t \circ \Gamma_1^{-1} \circ \tau_{a^{-1}}(b)\colon a \in A, b \in B\}. \tag{2.12}$$

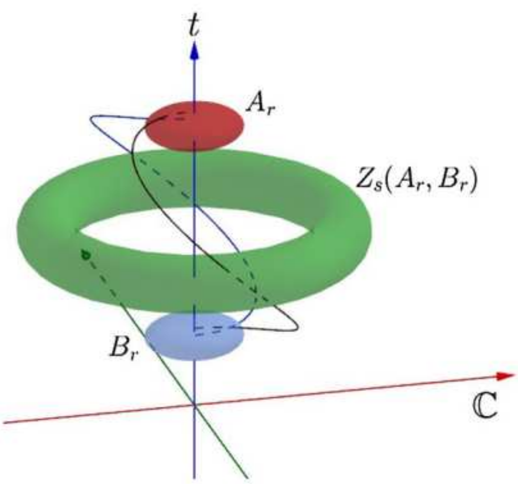


Figure 2.3: An example of s-intermediate set in $\mathbb{H}$. This picture is courtesy of [3].

In the Heisenberg group the Measure Contraction Property holds with different exponents with respect to the Euclidean case.

In the following computation we set $Z_t(p,q) = Z_t(\{p\},\{q\})$ and the map

$$\begin{aligned} \zeta_t \colon \mathbb{H} &\longrightarrow \mathbb{H} \\ p &\longmapsto Z_t(0,p). \end{aligned}$$

Theorem 2.1. (Measure Contraction Property in the Heisenberg group) Let $A \subset \mathbb{H}$ be a set with positive measure, $x \in \mathbb{H}$ and $t \in [0,1]$, then

$$m(Z_t(x,A)) \geq t^N m(A) \tag{2.13}$$

holds if and only if $N \geq 5$.

Proof. Step 1: the Measure Contraction Property does not hold if $N < 5$. Consider $N < 5$, we have that $Z_t(\tau_p q, \tau_p q') = \tau_p \circ Z_t(q,q')$ for every $p, q \in \mathbb{H}$ as the Lebesgue measure, the Carnot-Carathéodory distance and the geodesics are left-invariant. Thus is is sufficient to prove that (2.13) does not hold in 0. Let $p = (1,0,0) = \Gamma((1,0),0)$ and K_r the Euclidean ball with center p and radius $r < 1$. Fix $t \in (0,1)$, define the set $E_r = \zeta_t(K_r)$. Since K_r is contained in $\mathbb{H} \setminus L$ where $\zeta_t(p)$ is a diffeomorphism, we have:

$$m(E_r) = \int_{K_r} |\det(D\zeta_t)(q)|\, dq,$$

in addition we have

$$\det(D\zeta_t)(p) = \frac{\det(D\Gamma_t)}{\det(D\Gamma_1)} \circ \Gamma_1^{-1}(p) = t^5,$$

thanks to Proposition 2.3. As a consequence, $\det(D\zeta_t)(p) < t^N$ and by continuity it is possible to find a radius $r > 0$ small enough such that $\det(D\zeta_t)(q) < t^N$ holds for every $q \in K_r$. With this choice of r we have $t^5 m(K_r) > m(E_r)$ which contradicts the Measure Contraction Property.

Step 2(Sketch): the Measure Contraction Property does not hold if $N \geq 5$. Let N be greater that 5. As in *Step 1*, we only need to prove (2.13) for $x = 0$. Consider E a measurable set with positive measure, and $T \in (0,1)$. The map ζ_t is a diffeomorphism on $\mathbb{H} \setminus L$ where it is equal to $\Gamma_t \circ \Gamma_1^{-1}$. If we denote $F = \zeta_t^{-1}(E)$, then we have

$$m(E) = \int_{F \setminus L} |\det(D\zeta_t)(q)| dq,$$

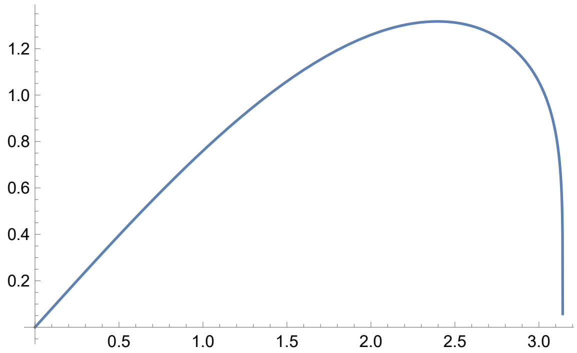
as $m(L) = 0$. Hence, to obtain (2.13) it is enough to prove that

$$|\det(D\zeta_t)(q)| = \frac{\det(D\Gamma_t)}{\det(D\Gamma_1)} \circ \Gamma_1^{-1}(q) = \frac{\det(D\Gamma_t)}{\det(D\Gamma_1)}(v,h) \geq t^N, \tag{2.14}$$

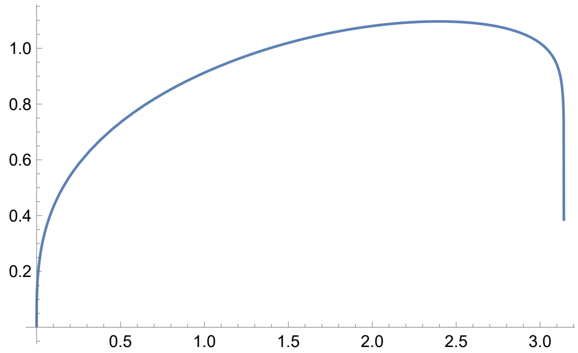
when $(v,h) \in D$ (in the case $h \neq 0$). If $h = 0$ we have

$$\frac{\det(D\Gamma_t)}{\det(D\Gamma_1)}(v,0) = t^5 \geq t^N,$$

which is obviously true. Both side of (2.14) are 0 at $t=0$ and 1 at $t=1$. Thus we want to prove that $t \longmapsto \frac{\det(D\Gamma_t)}{\det(D\Gamma_1)}^{\frac{1}{N}}(v,h)$ is concave in t for each $(v,h) \in D$. This last assertion is equivalent to the concavity of the even function $g_N(u) = (u\sin(u)(\sin(u) - u\cos(u)))^{\frac{1}{N}}$ on $(0,\pi)$. We do not prove this fact, instead we plot g_N in two cases in Figure 2.4. A formal proof can be found in [8].



(a) Plot of g_5 on $(0,\pi)$.



(b) Plot of g_{15} on $(0,\pi)$.

Figure 2.4

□

We now provide geometric evidence of the exponent involved in the Measure Contraction Property showing how it arises for the unit ball $B_1^{\mathbb{H}} = \{x \in \mathbb{H} : d(x,0) < 1\}$.

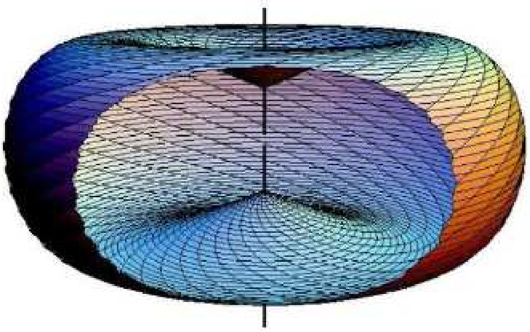


Figure 2.5: Graphic representation of $B_1^{\mathbb{H}}$. This picture is courtesy of D. Barilari.

Recall that the map $\delta_t(x)$ is defined by $\delta_t(x) = (tx_1, tx_2, t^2x_3)$ for every $x \in \mathbb{H}$. For $t \in (0,1)$, the set $\zeta_t(B_1^{\mathbb{H}})$ is certainly contained in the contraction $B_t^{\mathbb{H}} = \delta_t(B_1^{\mathbb{H}})$ whose volume is $t^4 m(B_1^{\mathbb{H}})$. Nevertheless, Theorem 2.1 asserts that $m(\zeta_t(B_1^{\mathbb{H}})) \geq t^5 m(B_1^{\mathbb{H}})$, rescaling we get

$$m(\delta_{1/t}\zeta_t(B_1^{\mathbb{H}})) \geq t m(B_1^{\mathbb{H}}). \tag{2.15}$$

We compute explicitly the value of the left hand side of (2.15) for small t.

We can visualize the Heisenberg ball to be the set of the points of the geodesic curve of length 1, namely

$$B_1^{\mathbb{H}} = \{\Gamma_1(v,h) : |v| < 1, h \in (-2\pi, 2\pi)\}.$$

Since $m(L) = 0$ we can work on $\mathbb{H} \setminus L$. Let V be the set $\{v \in \mathbb{R}^2 : |v| < 1\} \times (-2\pi, 2\pi)\}$.

$$\begin{aligned}
m(\delta_{1/t}\zeta_t(B_1^{\mathbb{H}})) &= \int_{B_1^{\mathbb{H}}} \left| \det(D\delta_{1/t})\Big|_{\zeta_t(q)} \det(D\zeta_t)\Big|_q \right| dq = \\
&= \frac{1}{t^4}\int_V \left| \left(\frac{\det(D\Gamma_t)}{\det(D\Gamma_1)} \det(D\Gamma_1) \right) \Big|_{(v,h)} \right| dvdh = \\
&= \int_V \frac{t|v|^2}{t^4h^4}(2 - th\sin(th) - 2\cos(th))dvdh = \\
&= \int_V \frac{|v|^2t^4h^4 + o(t^4)}{12t^3h^4} dvdh = \frac{t}{12}\int_0^1 \left(\int_0^{2\pi} \left(\int_{-2\pi}^{2\pi} \rho^3 + o(t)dh \right) d\theta \right) d\rho = \\
&= t\frac{\pi^2}{6} + o(t).
\end{aligned}$$

This result justifies the factor t in (2.15) and thus it is a geometric evidence of the fact that the exponent involved in the Measure Contraction Property (2.1) must be greater than 5.

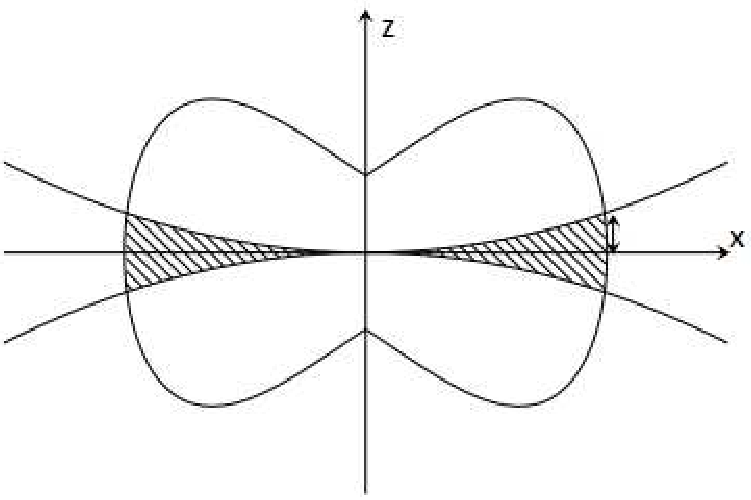


Figure 2.6: xz-section of the Heisenberg ball, the shaded area is the section of the set $\delta_{1/t}\zeta_t(B_1^{\mathbb{H}})$. This picture is courtesy of [8].

Chapter 3

On the Brunn-Minkowski inequality in the Heisenberg group

In this chapter we study one possible generalization of the Brunn-Minkowski inequality in the Heisenberg group. In particular we analyse the geodesic Brunn-Minkowski inequality.

Remark 3.1. In the Heisenberg group there exists an other way to generalise the Brunn-Minkowski inequality different from considering the geodesic version. This is possible as the Heisenberg group is endowed with an operation, i.e. the Heisenberg product of Definition 2.1.

Let A and B two nonempty sets of $\mathbb{H}$, we define $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ where "$\cdot$" is the Heisenberg product defined in (2.1). Then the following holds

$$m(A \cdot B)^{\frac{1}{3}} \geq m(A)^{\frac{1}{3}} + m(B)^{\frac{1}{3}}.$$

In order to prove the last inequality one has to follow the proof of Theorem 1.1 substituting the "+" with the Heisenberg product. An explicit proof can be found in [10].

Before discussing in detail the geodesic Brunn-Minkowski inequality we need the following

Definition 3.1. We define the *geodesic-inversion map* $\mathcal{I}$ on $\mathbb{H} \setminus L$ by $\mathcal{I}(p) = \Gamma_{-1} \circ \Gamma_1^{-1}(p)$.

The name derives from the fact that for $(v, h, t) \in D_1 \times [-1, 1]$ we have that

$$\mathcal{I}(\gamma_{v,h}(t)) = \mathcal{I}(\Gamma(tv, th)) = \Gamma_{-1}(tv, th) = \gamma_{v,h}(-t).$$

We see that $\mathcal{I} \circ \mathcal{I}$ is the identity on $\mathbb{H} \setminus L$ and this is why the pair $(p, \mathcal{I}(p))$ will be called $\mathcal{I}$-conjugate points. Now we establish the connection between Z and $\mathcal{I}$.

Proposition 3.1. Let $p \in \mathbb{H} \setminus L$ then $Z_{1/2}(\mathcal{I}(p), p)$ is well defined and is 0 if and only if the h-coordinate of $\Gamma_1^{-1}(p)$ verifies $|h| < \pi$, i.e. $p \in \Gamma_{1/2}(D_1)$.

Proof. The proof can be found in [8]. $\square$

3.1 Geodesic Brunn-Minkowski inequality in the Heisenberg group

In this section we disprove the "natural" generalization of the Brunn-Minkowski inequality in the Heisenberg group, and show that the version which holds in the Heisenberg group does not imply the isoperimetric inequality.

Theorem 3.1. Given $A, B \subset \mathbb{H}$ with positive measure, then the geodesic Brunn-Minkowski inequality

$$m(Z_t(A,B))^{\frac{1}{N}} \geq (1-t)m(A)^{\frac{1}{N}} + tm(B)^{\frac{1}{N}} \tag{3.1}$$

is false for any $N \in \mathbb{N}$.

Proof. It is enough to show that there are two compact sets $K, K' \subset \mathbb{H}$ such that

$$m(K) = m(K') > m(Z_{1/2}(K,K')), \tag{3.2}$$

As a consequence, if (3.1) were true, thanks to (3.2) we would have

$$m(Z_{1/2}(K,K'))^{\frac{1}{N}} \geq \frac{1}{2}m(K)^{\frac{1}{N}} + \frac{1}{2}m(K')^{\frac{1}{N}} > m(Z_{1/2}(K,K'))^{\frac{1}{N}},$$

which is a contradiction.

Now we build the two sets K and K'. Let us consider $\gamma = \gamma_{((1,0),0)}$ on the interval $[-1,1]$. As $0 < 2\pi$ this is the unique geodesic defined on $[-1,1]$ from $p' = (-1,0,0)$ to $p = (1,0,0)$: the points p and p' are $\mathcal{I}$-conjugate, thus their midpoint is 0. Consider $K_r = \overline{B(p,r)} = \{x \in \mathbb{H} : |x-p| \leq r\}$ and $K'_r = \mathcal{I}(K_r)$. By continuity we can choose r small enough such that $K_r \subset \Gamma_{1/2}(D_1)$ and $K_r \times K'_r \subset U$. As Γ_1 and Γ_{-1} are diffeomorphisms between the same sets we have:

$$m(K'_r) = m(\Gamma_{-1}(\Gamma_1^{-1}(K_r))) = m(\Gamma_1(\Gamma_1^{-1}(K_r))) = m(K_r). \tag{3.3}$$

Note that:

$$Z_{1/2}(K_r, K'_r) = \bigcup_{a,b\in K_r} Z_{1/2}(\mathcal{I}(a), b) = \bigcup_{a,b\in K_r} Z_{1/2}(\mathcal{I}(a), a+(b-a)). \tag{3.4}$$

For any $q \in \mathbb{H} \setminus L$ let $M_q = Z_{1/2}(q, \cdot)$. We now write

$$\begin{aligned} Z_{1/2}(\mathcal{I}(a), a+(b-a)) &= Z_{1/2}(\mathcal{I}(a), a+(b-a)) + \left[DM_{\mathcal{I}(a)}(a)(b-a) - DM_{\mathcal{I}(a)}(a)(b-a)\right] + \\ &+ \left[DM_{p'}(p)(b-a) - DM_{p'}(p)(b-a)\right] = DM_{p'}(p)(b-a) + \\ &+ \left[\left(DM_{\mathcal{I}(a)}(a) - DM_{p'}(p)\right)(b-a)\right] + \left[Z_{1/2}(\mathcal{I}(a), a+(b-a)) - DM_{\mathcal{I}(a)}(a)(b-a)\right]. \end{aligned} \tag{3.5}$$

For a and b close to p, and for r close to zero we have

$$\sup_{a,b\in K_r} \left|\left(DM_{\mathcal{I}(a)}(a) - DM_{p'}(p)\right)(b-a) + Z_{1/2}(\mathcal{I}(a), a+(b-a)) - DM_{\mathcal{I}(a)}(a)(b-a)\right| = o(r).$$

Therefore, as $\{a-b \in \mathbb{R}^3 : a, b \in \overline{B(p,r)}\} = \overline{B(0,2r)}$, (3.4) and (3.5) give the following set inclusion

$$Z_{1/2}(K_r, K'_r) \subset DM_{p'}(p)\overline{B(0,2r)} + \overline{B(0,\varepsilon(r)r)}, \tag{3.6}$$

where $\varepsilon(r)$ is a non-negative function which tends to zero as r tends to zero.

Recall that $\tau_p(q) = p \cdot q$ and then $\det(D\tau_p)(q) = 1$ for any $p, q \in \mathbb{H}$. In addition $\tau_{p'^{-1}} = \tau_{p'}^{-1}$ hence we have

$$\begin{aligned} &\det(DM_{p'})(p) = \det(D(\tau_{p'} \circ \Gamma_{1/2} \circ \Gamma_1^{-1} \circ \tau_{p'^{-1}}))(p) = \det(D(\Gamma_{1/2} \circ \Gamma_1^{-1}))(p'^{-1} \cdot p) = \\ &\det(D(\Gamma_{1/2} \circ \Gamma_1^{-1}))(\Gamma_1(((2,0),0))) = -\frac{1}{2^5}. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) we have

$$m(Z_{1/2}(K_r, K'_r)) \leq m(DM_{p'}(p)\overline{B(0,2r)}) + o(r) = \frac{1}{2^5} m(K_r)(1 + o(r)),$$

we can choose r small enough and the proof is finished. □

Notwithstanding Theorem 3.1, there exist geodesic version of the Brunn-Minkowski inequality which holds in the Heisenberg group given by the following statement.

Theorem 3.2. Let $t \in [0,1]$ and A, B nonempty measurable sets of $\mathbb{H}$. Then the following inequality holds

$$m(Z_t(A,B))^{\frac{1}{3}} \geq (1-t)^{\frac{5}{3}} m(A)^{\frac{1}{3}} + t^{\frac{5}{3}} m(B)^{\frac{1}{3}}. \tag{3.8}$$

Proof. The proof uses heavily the theory of optimal transport and can be found in [3]. □

We observe that Theorem 3.2 implies directly the sharper Measure Contraction Property in $\mathbb{H}$, i.e. Theorem 2.1.

Remark 3.2. Differently from the Euclidean case, in the Heisenberg group it is not possible to derive the isoperimetric inequality from (3.8). Suppose A to be a measurable subset of $\mathbb{H}$ with $\mathcal{C}^2$ boundary. Working along the lines of Theorem 1.3, we set $A_r = \{x \in \mathbb{H} : d_{\mathbb{H}}(x, A) < r\} = A + \zeta_r(B_1^{\mathbb{H}})$, then

$$\begin{aligned} m(A_r \setminus A) = m(A + \zeta_r(B_1^{\mathbb{H}})) - m(A) &\geq (m(A)^{\frac{1}{3}} + m(\zeta_r(B_1^{\mathbb{H}})^{\frac{1}{3}})^3 - m(A) \geq \\ &\geq 3r^{\frac{5}{3}} m(A)^{\frac{2}{3}} m(B_1^{\mathbb{H}})^{\frac{1}{3}} + o(r^{\frac{5}{3}}). \end{aligned}$$

Hence we only get

$$\mathscr{H}^2(\partial A) = \lim_{r \to 0^+} \frac{m(A_r \setminus A)}{r} \geq 0$$

which is trivial.

As can be seen by the latter example, despite the fact that Theorem 3.2 is a possible generalization of the Brunn-Minkowski inequality in the Heisenberg group, it does not bring as much information as in the Euclidean case since it does not imply the isoperimetric inequality.

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