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Explicit formula of Mumford forms on moduli space of Riemann surfaces and Polyakov measure of bosonic string

Espressione esplicita della forma di Mumford sullo spazio dei moduli delle superfici di Riemann e la misura di Polyakov della stringa bosonica

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1 Abstract

In string theory one consider one-dimensional objects instead of point-like particles. In 1981 Polyakov extended the path integral formulation from the known Feynman formulation to a sum over the surfaces traced out by the string [1]. This thesis begins with a brief overview of Feynman path integral as introduction. We then will start formulating the closed bosonic string, so the world surface traced out by the it (the "path") will be a Riemann surface [2]. Being developed for more than a century with no connection with strings, the Riemann surface theory covers a huge number of problems and results, but only compact Riemann surfaces are of physical interest, thus definition and fundamental facts about this objects will be given. Particular focus is dedicated to the uniformization theorem and the description of the moduli (parameters defining inequivalent surfaces) space, this last one is in fact the space of integration of the partition function that will be performed. The Polyakov action indeed is invariant under transformations bringing a surface to an equivalent (conformal) one, this huge gauge symmetry which would cause the integral to diverge is (factored out) eliminated by Faddeev-Popov method and by setting the dimension of the space time its critical dimension $d = 26$ and letting only integration over moduli, which parameterize variations of the metric leading to inequivalent surfaces. In this dimension it was showed by Beilinson, Manin [3] and Belavin, Knizhnik [4] that the Polyakov measure is related to the Mumford form [5] which can be expressed in term of theta functions and related quantities. These objects, also born as utilities for pure mathematics (their origin is attributed to Euler), highlight properties of the measure to be discussed at the end of the thesis. Finally more recent results will be reported.

In teoria delle stringhe si considerano oggetti unidimensionali invece di particelle puntiformi. Nel 1981 Polyakov estese la formulazione dell'integrale sui cammini, dalla nota formulazione di Feynman, a una somma sulle superfici tracciate dalla stringa [1]. Questa tesi inizia con una breve panoramica dell'integrale sui cammini di Feynman come introduzione. Successivamente, inizieremo a formulare la teoria della stringa bosonica chiusa, quindi la superficie tracciata da essa (il "cammino") sarà una superficie di Riemann [2]. Essendo stata sviluppata per più di un secolo senza alcuna connessione con le stringhe, la teoria delle superfici di Riemann contiene un enorme numero di problemi e risultati, ma solo le superfici di Riemann compatte sono di interesse fisico, quindi verranno date definizioni e fatti fondamentali su questi oggetti. Particolare attenzione è dedicata al teorema di uniformizzazione e alla descrizione dello spazio dei moduli (parametri che definiscono superfici non equivalenti), l'integrazione della funzione di partizione verrà infatti eseguita su quest'ultimo spazio. L'azione di Polyakov è infatti invariante sotto trasformazioni che trasformano una superficie in un'altra equivalente (conforme), questa grande simmetria di gauge che causerebbe la divergenza dell'integrale è eliminata mediante il metodo di Faddeev-Popov e impostando la dimensione dello spazio-tempo alla sua dimensione critica $d = 26$, lasciando solo l'integrazione sui moduli, che parametrizzano le variazioni della metrica che portano a superfici non equivalenti. In questa dimensione è stato dimostrato da Beilinson, Manin [3] e Belavin, Knizhnik [4] che la misura di Polyakov si può scrivere in funzione della forma di Mumford [5], che può essere espressa in termini di funzioni theta e quantità correlate. Questi oggetti, nati anche come strumento per la matematica pura (la loro origine è attribuita a Eulero), evidenziano proprietà della misura che saranno discusse alla fine della tesi. Infine, verranno riportati risultati più recenti.

2 Path integral

The path integral method arise from giving quantum theory a Lagrangian formulation. As stated by Dirac [6] there is no a direct way to take over the classical Lagrangian equations because the partial derivatives of the Lagrangian with respect to spatial coordinates and velocities have no meaning in quantum mechanics, however using the ideas of the classical Lagrangian theory (considering all the paths connecting two points) it will be shown that:

$$\langle q', t \,|\, q, 0 \rangle = \lim_{n\to\infty} \left(\frac{nm}{2\pi i\hbar t}\right)^{n/2} \int dq_1 \dots dq_{n-1} \exp\left\{ \frac{it}{n\hbar} \sum_{j=0}^{n-1} \left(\frac{mn^2}{2} \left(\frac{q_{j+1}-q_j}{t}\right)^2 - V(q_j) \right) \right\} \tag{1}$$

Where $q_0 \equiv q$ and $q_n \equiv q'$. It is known as the Feynman-Kac formula. The exponent is the definition of the Riemann integral $\int_0^t L(q, \dot{q}, t)dt$ where $L(q, \dot{q}, t)$ is the classical Lagrangian, thus: $\langle q', t \,|\, q, 0 \rangle = k \int_{all\ curves\ q(t)} e^{i\frac{S(q(t))}{\hbar}}$, where k is the constant that guarantees normalization. In the classical limit the mass is big enough with respect to $\hbar$ that a small changes in the trajectory produce fast oscillations in the exponent so only classical paths contribute to the integral and $\langle q', t \,|\, q, 0 \rangle = 1$ if q' lies on a classical paths passing trough q and $\langle q', t \,|\, q, 0 \rangle = 0$ if it does not.

In order to prove the Feynman-Kac formula it is recalled that writing $H \equiv H_0 + V$, where all the operators are self-adjoint, the Trotter formula [7] guarantees that:

$$e^{-i(H_0+V)t} = \lim_{n\to\infty} \left(e^{-iH_0\frac{t}{n}} e^{-iV\frac{t}{n}} \right)^n$$

Thus

$$\begin{aligned}
\langle q', t \,|\, q, 0 \rangle &= \langle q' \,|\, e^{-iHt} \,|\, q \rangle \\
&= \lim_{n\to\infty} \left\langle q' \,\middle|\, \left(e^{-iH_0\frac{t}{n}} e^{-iV\frac{t}{n}} \right)^n \,\middle|\, q \right\rangle \\
&= \lim_{n\to\infty} \left\langle q' \,\middle|\, e^{-iH_0\frac{t}{n}} e^{-iV\frac{t}{n}} I e^{-iH_0\frac{t}{n}} e^{-iV\frac{t}{n}} \dots e^{-iH_0\frac{t}{n}} e^{-iV\frac{t}{n}} \,\middle|\, q \right\rangle \\
&= \lim_{n\to\infty} \int dq_1 \dots dq_{n-1} \prod_{j=0}^{n-1} \left\langle q_{j+1} \,\middle|\, e^{-iH_0\frac{t}{n}} \,\middle|\, q_j \right\rangle e^{-iV(q_j)\frac{t}{n}}
\end{aligned}$$

where in the last equality the $n-1$ identities inserted as been written as the completeness relation $I = \int dq_j \,|q_j\rangle \langle q_j| \,, \quad j = 1, \dots, n-1$ and, as before, $q_0 \equiv q$ and $q_n \equiv q'$. Now it remains to evaluate $\left\langle q_{j+1} \,\middle|\, e^{-iH_0\frac{t}{n}} \,\middle|\, q_j \right\rangle$.

$$\begin{aligned}
\left\langle q_{j+1} \,\middle|\, e^{-iH_0\frac{t}{n}} \,\middle|\, q_j \right\rangle &= \int dp \,\langle q_{j+1} \,|\, p \rangle \langle p \,|\, q_j \rangle \, e^{-i\frac{p^2}{2m\hbar}\frac{t}{n}} \\
&= \int dp \frac{1}{2\pi\hbar} e^{i\frac{p}{\hbar}(q_{j+1}-q_j)} e^{-i\frac{p^2}{2m\hbar}\frac{t}{n}} \\
&= e^{i\frac{mn(q_{j+1}-q_j)^2}{2\hbar t}} \int dp \frac{1}{2\pi\hbar} e^{-\frac{i}{2m}\frac{t}{n}\left(p - \frac{(q_{j+1}-q_j)mn}{t}\right)^2} \\
&= \left(\frac{mn}{2\pi i\hbar t}\right)^{1/2} e^{i\frac{mn(q_{j+1}-q_j)^2}{2\hbar t}}
\end{aligned}$$

Finally the substitution in the precedent expression leads to the Feynman-Kac formula.

In quantum field theory the quantity of physical interest are the vacuum expectation values of time-ordered products of operators (called *Green functions*)

$$G(x_1, \dots, x_n) \equiv \langle 0 \,|\, T\phi(x_1) \dots \phi(x_n) \,|\, 0 \rangle \tag{2}$$

An expression for this quantity can be obtained by the quantum mechanics analog for one degree of freedom $Q(t)$ and then by taking the limit of an infinite number of degrees of freedom [8], the procedure will be sketched here. A Wick rotation is performed in the time variable, $t \to -i\tau$, so that

$$\hat{Q}(\tau) = e^{H\tau}\hat{Q}(0)e^{-H\tau} \tag{3}$$

The *correlator* is defined as

$$\left\langle q', \tau' \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, q, \tau \right\rangle, \quad \tau' > \tau_1 > \cdots > \tau_n > \tau \tag{4}$$

The insertions of $I = \sum_l |E_l\rangle \langle E_l|$, where $|E_l\rangle$ is a complete set of energy eigenstates, to the left and right of each operator leads to

$$\left\langle q', \tau' \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, q, \tau \right\rangle = \sum_{l,l'} e^{-E'_l\tau'} e^{E_l\tau} \psi_{l'}(q')\psi^*_l(q) \left\langle E_{l'} \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, E_l \right\rangle$$

where

$$\langle q, \tau \,|\, E_l\rangle = \left\langle q \,\middle|\, e^{-H\tau} \,\middle|\, E_l \right\rangle \equiv \psi_l(q)e^{-E_l\tau}$$

Now taking the limit $\tau' \to \infty$, $\tau \to -\infty$ implies that only the terms of minimum energy contribute to the sum, thus

$$\left\langle q', \tau' \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, q, \tau \right\rangle \to e^{E_0(\tau-\tau')}\psi_0(q')\psi^*_0(q) \left\langle E_{l'} \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, E_l \right\rangle \tag{5}$$

and the Green function is obtained by taking the same limit:

$$\frac{\left\langle q', \tau' \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, q, \tau \right\rangle}{\langle q', \tau' \,|\, q, \tau\rangle} \to \left\langle E_0 \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, E_0 \right\rangle \tag{6}$$

From the Feynman-Kac formula it follows that, for $\tau' \to \infty$, $\tau \to -\infty$,

$$\left\langle E_0 \,\middle|\, \hat{Q}(\tau_1) \ldots \hat{Q}(\tau_n) \,\middle|\, E_0 \right\rangle = \frac{\int DQ\, Q(\tau_1) \ldots Q(\tau_n) e^{-S[Q]}}{\int DQ\, e^{-S[Q]}} \tag{7}$$

Generalization to an infinite number of degree of freedom, and applying the inverse Wick rotation, leads to the following expression for the Green function in quantum field theory

$$\langle 0 \,|\, \phi(x_1) \ldots \phi(x_n) \,|\, 0\rangle = \frac{\int D\phi\, \phi(x_1) \ldots \phi(x_n) e^{iS[\phi]}}{\int D\phi\, e^{iS[\phi]}} \tag{8}$$

In string theory particles are defined as one-dimensional objects, thus integration must be performed over surfaces. The path integral formulation due to Polyakov [1] will be considered and the next sections deals with the mathematical background necessary for this formulation.

3 Riemann surfaces

3.1 Definition and topology

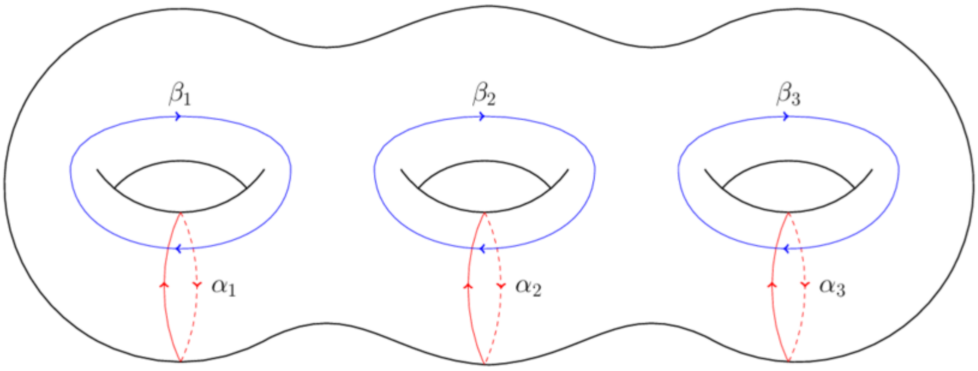

Figure 1: Riemann surface of genus 3[1]

Definition 3.1 (Riemann surface). *A Riemann surface is a one-complex-dimensional connected complex analytic manifold $\mathcal{M}$ with a set of charts $\{U_\alpha, z_\alpha\}_{\alpha\in A}$, that is, the $\{U_\alpha\}_{\alpha\in A}$ are an open cover of $\mathcal{M}$ and each $z_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto an open subset of the complex plane $\mathbb{C}$ such that each transition function*

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C} \to z_\alpha(U_\alpha \cap U_\beta)\mathbb{C}$$

is holomorphic if $U_\alpha \cap U_\beta \neq \varnothing$.

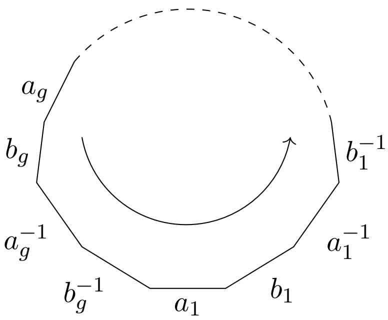

Figure 2: Riemann surface of genus g

The Euler characteristic $\chi(\mathcal{M})$ can be easily computed by triangulation: recalling $\chi(S^2) = 2$ and $\chi(T^2) = 0$ and 'pasting' a sphere with g torus is obtained $\chi(\mathcal{M}) = 2 - 2g$ (for each couple of triangles pasted the vertices eliminated cancel out with the sides in the computation of the Euler characteristic, thus the contribution comes from the two faces are eliminated). It is useful to express the Euler characteristic also for a surface with n punctures:

$$\chi(\mathcal{M}) = 2 - 2g - n \tag{9}$$

An important theorem relating the curvature of manifold with its Euler characteristic is recalled:

Theorem 3.1 (Gauss-Bonnet theorem).

$$\int R\, dv = 2\pi\chi(\mathcal{M}) \tag{10}$$

This implies that there is a topological constraint to the sign curvature, in particular in order to have negative curvature, from eq. (9) the surface must have at least three

[1] Figure taken from `https://www.researchgate.net/publication/329100697_Efficient_integration_on_Riemann_surfaces_applications`.

punctures or one handle and one puncture.
It is useful to see the surface as a $4g$ sided polygon as in figure 2, where $a_1, \ldots, a_g, b_1, \ldots, b_g$ are generators for the fundamental group $\pi_1(\mathcal{M})$ (i.e. $2g$ closed loops which are not homotopic). The generators of $\pi_1(\mathcal{M})$ satisfy the relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = I \tag{11}$$

A quadratic form can be introduced on elements of the first homology group $H_1(\mathcal{M}, \mathbb{Z})$ as:

$$(\cdot, \cdot): \quad H_1(\mathcal{M}, \mathbb{Z}) \times H_1(\mathcal{M}, \mathbb{Z}) \to \mathbb{Z}$$

that counts the number of intersection of any 1-cycles with orientation. Given a first homology group basis $\{\chi_1, \ldots, \chi_{2g}\}$, its intersection matrix N is defined as $\mathrm{N}_{ij} = (\chi_i, \chi_j)$ and if it is equal to $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ the basis is called *canonical.*

3.2 Differential forms and integration

Given the definition of Riemann surface, it is evident that a complex structure can be built on it in analogy with differential geometry, with some simplification due to holomorphicity of transition functions, and convenience of using complex notation. In the following $dz = dx + idy$, $d\bar{z} = dx - idy$.
It is convenient to define two new operators ∂, $\bar{\partial}$

$$\partial f \equiv \frac{\partial f}{\partial z} dz, \quad \frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \quad \bar{\partial} f \equiv \frac{\partial f}{\partial \bar{z}} d\bar{z}, \quad \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{12}$$

Note that this operators commute and for the exterior differential operator $d = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy$ it is true that $d = \partial + \bar{\partial}$. A $1-$form $\omega = f(x,y)dx + g(x,y)dy$, with f, g continuous ($\omega = udz + vd\bar{z}$, $u = \frac{f - ig}{2}$, $v = \frac{f + ig}{2}$ in complex notation) is called *holomorphic* if $\omega = df$ with f holomorphic, thus holomorphicity for a $1-$form implies $v = 0$ and u a holomorphic function[2]. If $d\omega = 0$, ω is called *closed*; from the previous argument holomorphic $1-$forms are closed. The space of holomorphic $1-$forms on the Riemann surface $\mathcal{M}$ is written as $H^0(K_{\mathcal{M}})$. Also it can be checked that the hodge star operator ($* : \bigwedge^k \to \bigwedge^{2-k}$; $(*\omega)_{i_{k+1} \ldots i_n} = \frac{1}{k!}\sqrt{g}\epsilon_{i_1 i_2}\omega^{i_1 \ldots i_k}$, $k = 0, 1, 2$) acts on $1-$ forms, in euclidean metric, in the following way:

$$*(fdx + gdy) = -gdx + fdy \quad * (udz + vd\bar{z}) = -iudz + ivd\bar{z} \tag{13}$$

in particular for holomorphic differentials it is true that $*\omega = -i\omega$.
Given $D \subset \mathcal{M}$ compact a scalar product between $1-$forms is defined as:

$$(\omega_1, \omega_2) \equiv \iint_D \omega_1 \wedge *\bar{\omega}_2 \tag{14}$$

[2] $\omega = df = (\partial + \bar{\partial})f = \frac{\partial f}{\partial z} dz \equiv udz$ and u is holomorphic because it is the derivative of an holomorphic function.

3.3 Complex tensor calculus

Given a patch on the Riemann surface the metric can always be written as $ds^2 = 2g_{z\bar{z}}dzd\bar{z} \equiv \rho dzd\bar{z}$. Meromorphic $(p,q)-$differentials are defined in the same way tensors are defined in differential geometry:

Definition 3.2 (Meromorphic differential)**.** *A meromorphic $(p,q)-$differential is the assignment of a meromorphic function to each local coordinates z on $\mathcal{M}$ so that*

$$f(z)\, dz^p d\bar{z}^q \tag{15}$$

is invariantly defined.
The space of meromorphic $(p,q)-$differentials is indicated $\mathcal{T}^{p,q}$ and the couple (p,q) is called weight of the differential.

It follows that given two elements U_α, U_β of the open cover of $\mathcal{M}$, on $U_\alpha \cap U_\beta$ f transforms as $f_\beta = \left(\frac{dz_\alpha}{dz_\beta}\right)^p \left(\frac{d\bar{z}_\alpha}{d\bar{z}_\beta}\right)^q f_\alpha$.
The inner product between elements T_1, T_2 of $\mathcal{T}^{p,0}$ is defined as

$$\langle T_1 \,|\, T_2 \rangle = \int d^2z \sqrt{g}\, (g^{z\bar{z}})^p T_1^* T_2 \tag{16}$$

Requiring that such inner product is a scalar (invariant under coordinate change) shows that the weights can be raised and lowered with the metric in the usual way, with the simplification that the only non-zero components of the metric are $g_{z\bar{z}}$ and its inverse.
The covariant derivative acting on $(n,0)-$differentials $\nabla^n_z : \mathcal{T}^{n,0} \to \mathcal{T}^{n+1,0}$ is:

$$\nabla^n_z (T(dz)^n) \equiv (g_{z\bar{z}})^n \partial_z ((g^{z\bar{z}})^n T)(dz)^{n+1} = (\partial_z - n\partial \log\rho) T (dz)^{n+1} \tag{17}$$

of course the definition is well-posed because the metric raises all the $z-$indices of the tensor, thus the quantity transforms as a $(n+1,0)-$differential.
Under the product (16) the adjoint operator of ∇^{n-1}_z is $\nabla^z_n : \mathcal{T}^{n,0} \to \mathcal{T}^{n-1,0}$

$$\nabla^z_n (T(dz)^n) \equiv g_{z\bar{z}} \partial_{\bar{z}} T (dz)^{n-1} \tag{18}$$

as can be checked by integration by parts.
The laplacian $\Delta_{1-n} : \mathcal{T}^{1-n,0} \to \mathcal{T}^{1-n,0}$ is defined as

$$\Delta_{1-n} = \nabla^{-n}_z \nabla^z_{1-n} \tag{19}$$

The scalar curvature is defined by

$$[\nabla^z, \nabla_z]\, T(dz)^n \equiv \frac{n}{2} R\, T(dz)^n \tag{20}$$

where $[\nabla^z, \nabla_z] = \nabla^z \nabla_z - \nabla_z \nabla^z$, using eq. (17) and (18) the first term can be rewritten:

$$\begin{aligned}
[\nabla^z, \nabla_z]\, T(dz)^n &= \nabla^z (\partial_z - n\partial_z \log\rho) T (dz)^{n+1} - \nabla_z \left(\frac{2}{\rho} \partial_{\bar{z}} T (dz)^{n-1} \right) \\
&= \frac{2}{\rho} \partial_{\bar{z}} (\partial_z - n\partial_z \log\rho) T(dz)^{n+1} - (\partial_z - (n-1)\partial_z \log\rho) \left(\frac{2}{\rho} \partial_{\bar{z}} T (dz)^{n-1} \right) \\
&= -n \frac{2}{\rho} \partial_z \partial_{\bar{z}} \log\rho
\end{aligned}$$

From the definition of scalar curvature (20)

$$R = 2K = -\frac{4}{\rho} \partial_z \partial_{\bar{z}} \log\rho \tag{21}$$

where also the Gaussian curvature $K = \frac{R}{2}$ has been introduced.

3.4 Period matrix

Firstly the following facts are recalled:

Lemma 3.1. *If θ and $\widetilde{\theta}$ are two closed differentials on $\mathcal{M}$ compact and of genus g, then*

$$\iint_{\mathcal{M}} \theta \wedge \widetilde{\theta} = \sum_{j=1}^{g} \left(\int_{a_j} \theta \int_{b_j} \widetilde{\theta} - \int_{b_j} \theta \int_{a_j} \widetilde{\theta} \right) \tag{22}$$

Lemma 3.2. *On a compact Riemann surface of genus g, the vector space of holomorphic differentials has complex dimension g.*

Given a canonical homology basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$, a dual basis of holomorphic differentials (which has dimension g) $\{\xi_1, \ldots, \xi_g\}$ can be chosen (i.e. $\int_{a_k} \xi_j = \delta_{jk}$), this determines a unique holomorphic differential ξ, so also the $b-$cycles $\int_{b_k} \xi_j$; in fact given $\theta \in H^0(K_{\mathcal{M}})$ with zero $a-$periods ($\int_{a_j} \theta = 0,\ j = 1, \ldots, g$), using $*\theta = -i\theta$ from the section (3.2)

$$\|\theta\|^2 = \iint_{\mathcal{M}} \theta \wedge *\bar{\theta} = i \iint_{\mathcal{M}} \theta \wedge \bar{\theta} = i \sum_{j=1}^{g} \left(\int_{a_j} \theta \int_{b_j} \bar{\theta} - \int_{b_j} \theta \int_{a_j} \bar{\theta} \right) = 0 \tag{23}$$

thus $\theta = 0$ and the map that assigns to each $\xi \in H^0(K_{\mathcal{M}})$ its $a-$cycles is a isomorphism between $H^1(K_{\mathcal{M}})$ and $\mathbb{C}^g$ (the two spaces has the same dimension and the map has a trivial kernel).

Definition 3.3 (Period matrix)**.** *Given a canonical homology basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ and a basis $\{\xi_1, \ldots, \xi_g\}$ of holomorphic differentials on $\mathcal{M}$, the $g \times g$ matrix τ with entries $\tau_{jk} = \int_{b_k} \xi_j$ is called the period matrix of the Riemann surface $\mathcal{M}$.*

Two important properties of the period matrix can be easily found:

Proposition 3.1. *Let τ be a period matrix, then:*

i) τ *is symmetric.*

ii) $\operatorname{Im} \tau > 0$.

Proof. i) Given arbitrary holomorphic differentials θ and ω, it follows that $\theta \wedge \omega = 0$, furthermore they are closed thus $\iint_{\mathcal{M}} \xi_j \wedge \xi_k = \sum_{l=1}^{g} \left(\int_{a_l} \xi_j \int_{b_l} \xi_k - \int_{b_l} \xi_j \int_{a_l} \xi_k \right)$. The left-hand side of the equation is zero, the right one (recalling $\int_{a_k} \xi_j = \delta_{jk}$) is $\int_{b_j} \xi_k - \int_{b_k} \xi_j$ and $\int_{b_j} \xi_k = \int_{b_k} \xi_j$ ($\tau_{kj} = \tau_{jk}$ is obtained.

ii) $0 < (\xi_j, \xi_k) = \iint_{\mathcal{M}} \xi_j \wedge *\overline{\xi_k} = i \iint_{\mathcal{M}} \xi_j \wedge \overline{\xi_k} = i \sum_{l=1}^{g} \left(\int_{a_l} \xi_j \int_{b_l} \xi_k - \int_{b_l} \xi_j \int_{a_l} \xi_k \right) = i \left(\overline{\int_{b_j} \xi_k} - \int_{b_k} \xi_j \right) = 2 \operatorname{Im} \tau_{jk} \quad \forall j, k.$

□

Torelli's theorem also shows that inequivalent Riemann surfaces have different period matrices, so one is tempted to represent each surface of genus g with a element of the space of all $g \times g$ symmetric matrices with positive imaginary part Z: $\mathcal{H}_g = \{Z | Z_{ij} = Z_{ji},\ \operatorname{Im} Z > 0\}$ which has dimension $\frac{1}{2} g(g+1)$ and is called Siegel's upper half plane; however it is easy to show that, given a surface, a change of the homology basis can

produce a different period matrix:
suppose that the transformation can be written as $\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ where the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ is an element of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, which conserves the intersection matrix $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$, these transformations will be called *modular transformations.*
Thus $a'_j = D_{il} a_l + C_{il} b_l$, and evaluating the new $a-$cycles gives

$$\int_{a'_i} \xi_j = D_{il} \int_{a_l} \xi_j + C_{il} \int_{b_l} \xi_j = D_{il}\delta_{lj} + C_{il}\tau_{lj} = D_{ij} + (C\tau)_{ij}$$

then requiring $\int_{a'_i} \xi'_j = \delta ij$ it is obtained

$$\xi' = \left((D + C\tau)^{-1}\right)^t \xi \tag{24}$$

Now the matrix of periods can be evaluated:

$$\int_{b'_i} \xi'_j = (D + C\tau)^{-1}_{kj} \left(B_{il} \int_{a_l} \xi_k + A_{il} \int_{b_l} \xi_k \right) = \left((B + A\tau)(D + C\tau)^{-1}\right)^{ij}$$

thus

$$\tau' = (B + A\tau)(D + C\tau)^{-1} \tag{25}$$

In fact it will be shown that the moduli space M_g that parametrizes inequivalent surfaces has dimension 1 for $g = 1$ and $3g - 3$ for $g \geq 2$, while there are no moduli for $g = 0$; so the dimensions of the Siegel upper half plane and of M_g coincides up to $g = 3$, this is known as *Schottky problem.*

3.5 Divisors and Riemann-Roch theorem

Definition 3.4 (Divisor)**.** *A divisor on $\mathcal{M}$ is a formal symbol*

$$D = \sum_{P \in \mathcal{M}} \alpha(P) P$$

where $\alpha(P) \in \mathbb{Z}$ and $\alpha(P) \neq 0$ for only finitely many $P \in \mathcal{M}$.
The sum of two divisors $D_1 = \sum_{P\in\mathcal{M}} \alpha_1(P)P$ and $D_2 = \sum_{P\in\mathcal{M}} \alpha_2(P)P$ is defined as:

$$D_1 + D_2 = \sum_{P \in \mathcal{M}} \left(\alpha_1(P) + \alpha_2(P)\right) P$$

The divisors together with this operation form a group written as $\mathrm{Div}(\mathcal{M})$.

$$\mathrm{deg} D \equiv \sum_{P \in \mathcal{M}} \alpha(P)$$

is called degree of the divisor D.

It is noted that the map deg establishes a homomorphism between $\mathrm{Div}(\mathcal{M})$ and $\mathbb{Z}$. Each meromorphic function f (differential ω) which is not identically zero determines a divisor (f) $((\omega))$ given by

$$(f) = \sum_{P \in \mathcal{M}} \mathrm{ord}_P f \, P, \quad (\omega) = \sum_{P \in \mathcal{M}} \mathrm{ord}_P \omega \, P$$

An equivalence relation $\sim$ is introduced on $\text{Div}(\mathcal{M})$ such that $D_1 \sim D_2$ if exists a meromorphic function f such that $D_1 - D_2 = (f)$; equivalent divisors are said to belong to the same divisor class, the divisor class of meromorphic differentials is called *canonical class* and is written as Z.
For $D \in \text{Div}(\mathcal{M})$ two vector spaces are introduced ($\mathcal{K}$ is the space of meromorphic functions):

$$L(D) = \{f \in \mathcal{K}(\mathcal{M}) : (f) \geq D\}$$

$$\Omega(D) = \{\omega \, \text{meromorphic differential} : (\omega) \geq D\}$$

Their dimensions are indicated respectively as $r(D)$ and $i(D)$.

Lemma 3.3.

$$i(D) = r(D - Z)$$

Proof. Given a meromorphic differential ω the map

$$\Omega(D) \ni \psi \to \frac{\psi}{\omega} \in L(D - (\omega))$$

is a isomorphism between the two spaces, which thus have the same dimension. □

Theorem 3.2 (Riemann-Roch [9])**.** *Let $\mathcal{M}$ be a compact Riemann surface and $D \in \text{div}(\mathcal{M})$. Then*

$$r(-D) = \deg D - g + 1 + i(D) \tag{26}$$

Corollary 3.2.1.

$$\deg Z = 2g - 2$$

Proof. From the Riemann-Roch theorem and the lemma (3.3)

$$r(-Z) - i(Z) = \deg Z - g + 1$$

$$I(0) - r(0) = \deg Z - g + 1$$

and the result follow from the fact that the dimension of the space of holomorphic differentials (i.e. $i(0)$) is g, and the one of holomorphic functions (i.e. $r(0)$) is 1 because, for Liouville theorem, only constants functions live in this space. □

From the fact that the line bundle of $\lambda-$differentials is the λth tensor power of K it follows that $\deg(\lambda Z) = 2\lambda(g - 1)$.
As end to this section the Riemann-Roch theorem is used to compute the dimension of the space of holomorphic $\lambda-$differentials.

Proposition 3.2 ([9])**.** *Let $\lambda \in \mathbb{Z}$. The dimension of the space of holomorphic $\lambda-$differentials on $\mathcal{M}$ is given by the following table:*

Genus	Weight	Dimension
$g = 0$	$\lambda \leq 0$	$1 - 2\lambda$
	$\lambda > 0$	0
$g = 1$	$\forall\lambda$	1
$g \geq 2$	$\lambda < 0$	0
	$\lambda = 0$	1
	$\lambda = 1$	g
	$\lambda > 1$	$(2\lambda - 1)(g - 1)$

Proof. The Riemann-Roch theorem can be written as:

$$r(-D) = \deg D - g + 1 + r(D - Z) \tag{27}$$

furthermore, as it was done to prove the lemma 3.3, given a differential ω such that $(\omega) = Z$, it is built the isomorphism between $L(-\lambda Z)$ and $H^{\lambda}(\mathcal{M})$, the vector space of holomorphic $\lambda-$differentials.

$$L(-\lambda Z) \ni f \to f\omega^{\lambda} \in H^{\lambda}(\mathcal{M}) \tag{28}$$

Thus $r(-\lambda Z)$ is the quantity to be evaluated.

- $g = 0$:
  1. $\lambda > 0$: $\deg(-\lambda Z) = -2\lambda(-1) > 0 \;\Rightarrow\; r(-\lambda Z) = 0$.
  2. $\lambda < 0$: from eq. (27) $r(-\lambda Z) - r((\lambda - 1)Z) = r(-\lambda Z) = 1 - 2\lambda$.
- $g = 1$:
  1. A holomorphic differential cannot have zeros because $\deg Z = 2(g-1) = 0$, thus $H^{q}(\mathcal{M}) \ni \xi \to \omega\xi \in H^{q+1}(\mathcal{M})$ is a isomorphism and, from $r(0) = 1$, induction proves that $r(-\lambda Z) = 1\,\forall\lambda$.
- $g \geq 2$:
  1. $\deg(-\lambda Z) = -2\lambda(g-1) > 0 \;\Rightarrow\; r(-\lambda Z) = 0$.
  2. $\lambda = 0$: $r(0) = 1$ as already stated.
  3. $\lambda = 1$: $r(Z) = i(0) = g$ as already stated.
  4. $\lambda > 1$: from eq. (27) $r(-\lambda Z) - r((\lambda-1)Z) = r(-\lambda Z) = \lambda(2g-2) - g + 1 = (2\lambda - 1)(g-1)$.

□

3.6 Jacobi map

Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a canonical homology basis, $\{\xi_1, \ldots, \xi_g\}$ its dual basis for holomorphic differentials and τ their period matrix as above. In addition denote $L(\mathcal{M})$ the lattice, over $\mathbb{Z}$, generated by the columns of the $g \times 2g$ matrix (I, τ) (i.e. the set of points given by $Im + \tau n$, with $m, n \in \mathbb{Z}^g$).

Definition 3.5 (Jacobian variety)**.**

$$J(\mathcal{M}) \equiv \mathbb{C}^g / L(\mathcal{M})$$

is called the Jacobian variety of $\mathcal{M}$*.*

Definition 3.6 (Jacobian map)**.** *Chosen a point* $P_0 \in \mathcal{M}$*, the map*

$$\phi : \mathcal{M} \to J(\mathcal{M})$$

such that

$$\phi(P) = \left(\int_{P_0}^{P} \xi_1, \ldots, \int_{P_0}^{P} \xi_g \right)$$

is called Jacobian map.

Such a map is well defined, in fact if the map is calculated along two different curves c_1 and c_2 connecting the same two points their difference is the integral of the holomorphic differentials $\xi_1, \dots, \xi_g$ along the closed curve $c_1 c_2^{-1}$, thus an element of $L(\mathcal{M})$. It is noted that the map such defined depend on P_0.
The Jacobian map is naturally extended to a map $\phi : \mathrm{Div}(\mathcal{M}) \to J(\mathcal{M})$ given by (where $D = \sum_{P\in\mathcal{M}} \alpha(P)P$

$$\phi(D) = \sum_{P\in\mathcal{M}} \alpha(P)\phi(P) \tag{29}$$

Thus the maps from the divisors of zero degree are independent from the point P_0 by linearity of the integral.
An important theorem is stated

Theorem 3.3 (Abel theorem [9]). *Let $D \in \mathrm{Div}(\mathcal{M})$. A necessary and sufficient condition for D to be the divisor of a meromorphic function is that*

$$\phi(D) = 0 \bmod L(\mathcal{M}) \quad \textit{and} \quad \deg \mathrm{D} = 0$$

3.7 Uniformization theory

The scope of this section is to give a first description of the moduli space of Riemann surfaces with the help of uniformization theorem, in particular its dimension will be obtained. The approach of Alvarez, Nelson [2] is followed, in order to do this basics facts about uniformization theory are recalled, more complete reference are Matone [10] and Farkas, Kra [9].
Given a Riemann surface $\mathcal{M}$ it is always possible to construct a new Riemann surface $\widetilde{\mathcal{M}}$, which is known as universal covering of $\mathcal{M}$, with the following properties:

- There is a surjective local homeomorphism $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$.
- $\widetilde{\mathcal{M}}$ is simply connected.
- Every closed curve on $\mathcal{M}$ is mapped by π^{-1} into an open curve on $\widetilde{\mathcal{M}}$.

Furthermore, letting $\widetilde{\mathcal{U}}$ be a generic covering of $\mathcal{U}$, it can be proved that:

- $\pi_1(\widetilde{\mathcal{U}}) \cong N$, where N is a subgroup of $\pi_1(\mathcal{U})$.
- If N is normal[3] there is a discontinuous group $G \cong \pi_1(\mathcal{U})/N$ of fixed point free automorphisms of $\widetilde{\mathcal{U}}$ such that $\widetilde{\mathcal{U}}/G \cong \mathcal{U}$.

Going back to universal covering of Riemann surfaces one thus obtain (being the fundamental group of the simply connected covering the identity, which is obviously a normal subgroup of $\pi_1(\mathcal{M})$):

$$\mathcal{M} \cong \widetilde{\mathcal{M}}/\Gamma \tag{30}$$

where Γ are fixed point free analytic (holomorphic) discontinuous automorphisms of $\widetilde{\mathcal{M}}$.
The reason for studying Riemann surfaces through their universal covering lies on the following powerful theorem:

Theorem 3.4 (Uniformization theorem). *Every simply connected Riemann surface is conformally equivalent to:*

[3]A subgroup N of G is said to be normal in G if $gng^{-1} \in N$ $\forall g \in G$ and $\forall n \in N$

i) The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} + \{\infty\}$.

ii) The complex plane $\mathbb{C}$.

iii) The upper half plane $\mathbb{H}$.

The group of analytic (this exclude the complex conjugation) automorphisms of the Riemann sphere is the Moebius group acting as

$$z \to \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \tag{31}$$

It has 1 or 2 fixed points (the solutions to $cz^2 + (d-a)z - b = 0$ or ∞ and $z = \frac{az+b}{d}$) or every points in $\hat{C}$ if the transformation is the identity. The fact that there are not fixed point free automorphisms on the Riemann sphere proves that there is only one Riemann surface of genus zero.
For the plane $\mathbb{C}$ only the translations act without fixed points, thus compositions of generic automorphisms with non-zero translations must be considered.
The group of translations is spanned by two independent vectors: $\omega_1, \omega_2 \in \mathbb{C}$, $\omega_1 \neq \lambda\omega_2$, $\lambda \in \mathbb{R}$, furthermore both ω_1 and ω_2 are chosen to be different from zero because we are interested in compact surfaces.
By means of composition with analytic automorphisms only (a rotation and a rescaling) ω_1 is set equal to 1, ω_2 equal to τ, $\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$. The lattice generated by $1, \tau$ $\mathbb{Z} + \tau\mathbb{Z}$ determines a family of tori on $\mathbb{C}$ (figure (3)), and each family is determined by the complex parameter $\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$, i.e. the space of this families is the upper half plane $\mathbb{H}$. However the map

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \tag{32}$$

is a diffeomorphism between families of tori, which then are conformally equivalent. $\mathrm{SL}(2, \mathbb{Z})$ is spanned by the identity, the translation $z \to z + 1$ and the counter-inversion $z \to -\frac{1}{z}$ (which sends points outside from the ball of radius 1 into it and vice-versa), it follows that the space that of all the inequivalent tori (the moduli space M_1) is the shaded area with identified boundaries in figure 4 and has complex dimension equal to 1.

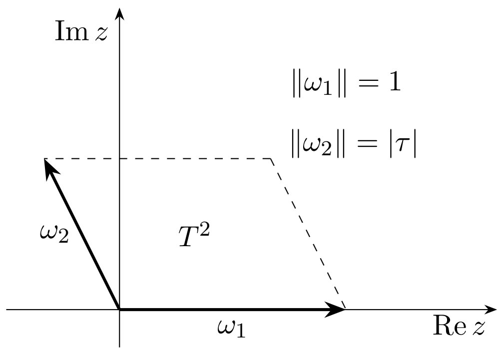

Figure 3: Generators of the automorphisms

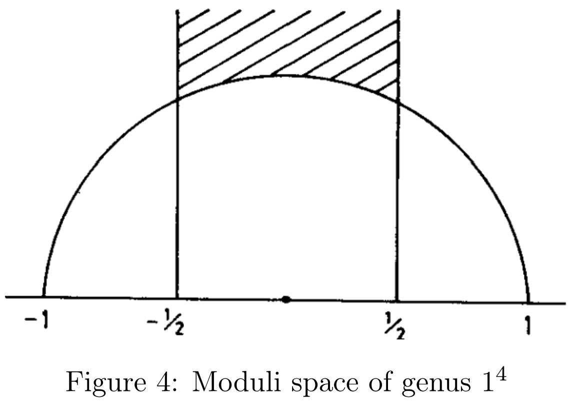

Figure 4: Moduli space of genus 1[4]

The compactified moduli space of genus 1 $\overline{M}_1$ is obtained by adding the point $\tau = \infty$ which corresponds to the family of pinched tori.
In the case $g > 1$ the covering space is $\mathbb{H}$, furthermore it is useful to recall that that Gauss-Bonnet theorem 10 implies that the curvature is negative, thus setting $R = -1$ in eq. (21) the Poincaré metric is obtained, $ds^2 = \frac{dzd\bar{z}}{Im(z)^2}$. In this settings automorphisms are isometries of $\mathbb{H}$, thus elements of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{I, -I\}$, whose action is given by

$$z \to \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \gamma \in \mathrm{PSL}(2,\mathbb{Z}) \tag{33}$$

Thus the fixed points are solution to the equation

$$cz^2 + (d-a)z - b = 0; \quad z_\pm = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c} \tag{34}$$

where it has been used that $ad - bc = 1$. This leads to the following classification of the transformations γ and their fixed points:

- $|\mathrm{Tr}\,\gamma < 2|$: z_+ is the complex conjugate of z_- and lies on $\mathbb{H}$, this fixed points are called *elliptic.*
- $|\mathrm{Tr}\,\gamma = 2|$: $z_+ = z_- \in \mathbb{R}$. γ is similar to a translation along the real axis so the order of its stabilizer is infinite and topologically they correspond to punctures.
- $|\mathrm{Tr}\,\gamma > 2|$: z_+ and z_- are distinct and lie on the real axis, thus they are not in $\mathbb{H}$. Topologically they represent handles.

The requirement that γ is fixed point free rules out the transformations with $|\mathrm{Tr}\,\gamma < 2|$, while $\gamma \cong \pi_1(\mathcal{M})$ implies that each family of inequivalent Riemann surfaces of genus g is represented by a set of $2g$ transformations $\{\gamma_1, \ldots, \gamma_{2g}\}$ determined up to a global isometry of $\mathbb{H}$ and subject to the condition (11):

$$\prod_{i=1}^{g} \gamma_{2i-1}\gamma_{2i}\gamma_{2i-1}^{-1}\gamma_{2i}^{-1} = I \tag{35}$$

Each transformation depends on 3 real parameters (a, b, c, d with the constraint of $\det\gamma = 1$) and the conditions eliminate 6 of them, thus for $g > 1$ the complex dimension of the moduli space is

$$\dim M_g = \frac{1}{2}(6g - 6) = 3g - 3 \tag{36}$$

[4]Figure taken from `https://lib-extopc.kek.jp/preprints/PDF/1987/8701/8701339.pdf`.

4 Theta functions

Definition 4.1 (Theta function). *Given a period matrix τ, a theta function with half-integer characteristic* $\begin{bmatrix} a \\ b \end{bmatrix}$ *is defined as:*

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) = \sum_{m \in \mathbb{Z}^g} exp \left\{ 2\pi i \left(\frac{1}{2}(m+a) \cdot \tau(m+a) + (m+a) \cdot (z+b) \right) \right\} \tag{37}$$

where $a, b \in \frac{1}{2}\mathbb{Z}^g$, $z \in \mathbb{C}^g$.

In the following it will be written $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z|\tau) \equiv \theta(z|\tau)$.
It can be shown by a simple calculation from the definition that theta functions are multivalued under shifts of the lattice $L_\tau = \mathbb{Z}^g + \tau\mathbb{Z}^g$:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z+n+\tau m|\tau) = exp\{2\pi i(a \cdot n - b \cdot m - \frac{1}{2} m \cdot \tau m - m \cdot z)\} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) \tag{38}$$

Note that the set of zeros is periodic on the lattice. In an analogous way the following two properties are obtained:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (-z|\tau) = exp \left\{ 2\pi i \left(2a \cdot b \right) \right\} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau)$$

$$\theta \begin{bmatrix} a+k \\ b+h \end{bmatrix} (z|\tau) = exp \left\{ 2\pi i \left(a \cdot k \right) \right\} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau)$$

where $h, k \in \mathbb{Z}^g$. Thus there are 2^{2g} theta functions with half integer characteristic which correspond to spin structures on the surface $\mathcal{M}$. Spin structures are called even or odd depending on the parity of the corresponding theta function. There are $2^{g-1}(2^g+1)$ even structures and $2^{g-1}(2^g-1)$ odd ones, as can be shown by induction.

The behaviour of the theta function under shifts of the lattice $L_\tau = \mathbb{Z}^g + \tau\mathbb{Z}^g$ (38) suggest the association between theta functions and Riemann surfaces with the substitution $z \to \phi$, $\tau \to \tau$.
Let $f \equiv \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \circ \phi$, from the definition of theta function it is clear that f has no poles, thus

$$\# \, of \, zeros \, of \, f = \frac{1}{2\pi i} \int_{\delta\mathcal{M}} \frac{df}{f} \tag{39}$$

Using the representation of a Riemann surface of figure 2 and choosing the generators of $\pi_1(\mathcal{M})$ such that no zeros lie on them

$$\begin{aligned} \frac{1}{2\pi i} \int_{\delta\mathcal{M}} \frac{df}{f} &= \frac{1}{2\pi i} \sum_{k=1}^{g} \int_{a_k+b_k+a_k^{-1}+b_k^{-1}} \frac{df}{f} \\ &= \frac{1}{2\pi i} \sum_{k=1}^{g} \left(\int_{a_k} \left(\frac{df}{f} - \frac{df^-}{f^-} \right) + \int_{b_k} \left(\frac{df}{f} - \frac{df^-}{f^-} \right) \right) \end{aligned} \tag{40}$$

where f^- is the value of the function on the curves a_k, b_k $k = 1, \ldots g$. From the multivaluedness (38) follows that ($b-$cycles are passed through after an $a-$cycle, and $a^{-1}-$cycles after a $b-$cycle)

$$f(P) = e^{2\pi i \alpha_k} f^-(P) \; if \; P \in b_k \; \Rightarrow \; \frac{df}{f} - \frac{df^-}{f^-} = 0 \; if \; P \in b_k$$

$$f^-(P) = e^{2\pi i(-\beta_k - \phi_k(P) - \tau_{kk}/2)} f(P) \; if \; P \in a_k \; \Rightarrow \frac{df}{f} - \frac{df^-}{f^-} = 2\pi i d(\phi_k(P)) \; if \; P \in a_k$$

thus

$$\# \, of \, zeros \, of \, f = \frac{1}{2\pi i} \sum_{k=1}^{g} \int_{a_k} 2\pi i d(\phi_k(P)) = \sum_{k=1}^{g} \int_{a_k} \omega_k = g \tag{41}$$

Actually the position of the zeros of the theta function can also be described thanks to the following theorem

Theorem 4.1 (Riemann vanishing theorem [9]). *The function*

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\phi(z) + \xi|\tau) \,, \quad z, P_k \in \mathcal{M} \tag{42}$$

either vanishes identically or else it has h zeros $z = P_1, \ldots, P_g$ *which satisfy*

$$\phi\left(\sum_{i=1}^{g} P_i\right) = -\xi - Ib - \tau a + \Delta \tag{43}$$

Where Δ *is called vector of Riemann constants and is given by*

$$\Delta_j = \frac{1}{2} - \frac{1}{2}\tau_{jj} + \sum_{i \neq j} \int_{a_i} \left(\omega_i(z) \int_{z_0}^{z} \omega_j \right). \tag{44}$$

In order to construct single-valued differentials it will be useful to know the expression of a differential vanishing at just one point in terms of theta functions, such differential is called *prime form*. To obtain it the Verlinde E., Verlinde H. construction [11] is followed: consider the theta function with odd characteristic

$$f([a,b], z, w) \equiv \theta \begin{bmatrix} a \\ b \end{bmatrix} (\phi(z) - \phi(w)|\tau) = \theta \begin{bmatrix} a \\ b \end{bmatrix} \left(\int_w^z \omega|\tau \right) \tag{45}$$

Being an odd function it has a zero for $z = w$, while the other $g-1$ zeros (with respect to both z and w) due to Riemann vanishing are called r_i, $i = 1, \ldots, g-1$. It follows that close to each of the r_i the behaviour of f is $f \sim const(z-w)(z-r_i)(w-r_i)$, thus the $1-$form

$$g([a,b], z) \equiv -\partial_w f([a,b], z, w)|_{w=z} = \sum_{k=1}^{g} \omega_k(z) \partial_{u_k} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] (u|\tau)|_{u_k=0}$$

behaves as $g \sim const(z-r_i)^2$, and doesn't have other zeros because has degree equal to $2g-2$. The prime form is thus defined as

$$E(z,w) = \frac{f([a,b], z, w)}{\sqrt{g(z)}\sqrt{g(w)}} \tag{46}$$

and as required it has a zero only for $z = w$, furthermore it is a differential of weight $(-1/2, 0)$ for both z and w.
Also the $g/2$-differential with empty divisor $\sigma(z)$ will be used

$$\sigma(z) = exp\left\{ -\sum_{k=1}^{g} \int_{a_k} \omega_k(w) \log E(z,w) \right\}. \tag{47}$$

Their multivaluednesses are obtained by theta function's one:

$$\begin{aligned} E(z+n\cdot a+m\cdot b,w) &= e^{\pi i m\cdot\tau m-2\pi i m\cdot(\phi(z)-\phi(w))}E(z,w) \\ \sigma(z+n\cdot a+m\cdot b) &= e^{\pi i(h-1)m\cdot\tau m-2\pi i m\cdot(\Delta-(h-1)\phi(z))}\sigma(z) \end{aligned} \tag{48}$$

Their values also depends on the choice of the homology basis and their behaviours under modular transformations are

$$\hat{E}(z,w) = exp\left\{\pi i\int_z^w \omega\cdot(C\tau+D)^{-1}C\int_z^w\right\}E(z,w) \tag{49}$$

$$\hat{\sigma}(z) = exp\left\{\frac{i\pi}{g-1}\Delta\cdot(C\tau+D)^{-1}C\Delta\right\}\sigma(z) \tag{50}$$

5 Gaussian integrals

5.1 Commuting variables

From the well-known formula, using Lebesgue measure,

$$\int dx\, e^{-\frac{1}{2}x^2} = \sqrt{\pi} \tag{51}$$

it is seen that a normalization of the measure can be chosen in an indirect way by setting (for later convenience)

$$\int Dx\, e^{-\frac{1}{2}x^2} \equiv 1 \tag{52}$$

Given an operator A acting on a finite-dimensional space, it is useful to compute the integral $\int Dxe^{-\frac{1}{2}x\cdot Ax}$. It is immediately seen that only the symmetric part of A contributes to the integral, thus, after diagonalizing A, one get from the previous integral

$$\int Dx\, e^{-\frac{1}{2}x\cdot Ax} = (\det(Sym\, A))^{-1/2} \tag{53}$$

5.2 Grassmann variables

Definition 5.1 (Grassmann algebra)**.** *The set of elements $\{\theta_1, \ldots, \theta_n\}$ is said to be a set of generators of a Grassmann algebra if they anticommute, i.e.*

$$\theta_i, \theta_j \equiv \theta_i\theta_j + \theta_i\theta_j = 0 \tag{54}$$

An element of the Grassmann algebra is defined to be a (necessary finite) power series of the generators:

$$f(\theta) = f_0 + \sum_i f_i\theta_i + \sum_{i\neq j} f_{ij}\theta_i\theta_j + \cdots + f_{1\ldots n}\theta_1\ldots\theta_n \tag{55}$$

Integration over a Grassmann variable is determined by the following rules [12]:

i) $\int d\theta_i = 0$.

ii) $\int d\theta_i\theta_i = 1$.

Using this rules one can evaluate a Gaussian integral of the form

$$I = \int \prod_{l=1}^{n} d\bar{\theta}_l d\theta_l\, e^{-\sum_{ij}\bar{\theta}_i A_{ij}\theta_j} \tag{56}$$

First of all it is noted that for Grassmann variables, because of $\theta_i^2 = 0$ the exponential is just its expansion to first order

$$e^{-\sum_{ij}\bar{\theta}_i A_i j\theta_j} = \prod_i e^{-\sum_j \bar{\theta}_i A_i j\theta_j} = \prod_i (1 - \sum_j \bar{\theta}_i A_i j\theta_j) \tag{57}$$

then the first integration rule states that only the terms with all the variables give a non-zero contribution, thus

$$
\begin{aligned}
I &= \int \prod_{l=1}^{n} d\bar{\theta}_l d\theta_l \, \prod_i (-\sum_j \bar{\theta}_i A_{ij} \theta_j) \\
&= \int \prod_{l=1}^{n} d\bar{\theta}_l d\theta_l \, \sum_{j_1 \cdots j_n} \theta_{j_1}\bar{\theta}_1 \ldots \theta_{j_n}\bar{\theta}_n A_{1j_1} \ldots A_{nj_n} \\
&= \int \prod_{l=1}^{n} d\bar{\theta}_l d\theta_l \, \theta_1\bar{\theta}_1 \ldots \theta_n\bar{\theta}_n \sum_{j_1 \cdots j_n} \epsilon_{j_1 \ldots j_n} A_{1j_1} \ldots A_{nj_n} \\
&= \det A
\end{aligned} \tag{58}
$$

The evaluation of the same integral, but with commuting variables, gives (as can be seen performing an orthonormal transformation to diagonalize the matrix A)

$$
I_c = \int \prod_{l=1}^{n} d\bar{z}_l dz_l \, e^{-\sum_{ij} \bar{z}_i A_{ij} z_j} = (\det A)^{-1} \tag{59}
$$

From comparison between I and I_c it will be used the fact that by replacing commuting variables with anticommuting ones in integrals in the form of (59) leads to the evaluation of the inverse of the value at the LHS.

6 Polyakov string

In the following the bosonic Polyakov action is considered.

Definition 6.1 (Bosonic Polyakov action)**.**

$$
S_P = \frac{T}{8\pi} \int d^2\xi \, \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \tag{60}
$$

where T is called string tension and will be set equal to 1, x^μ, $\mu = 1, \ldots, d$ and $G_{\mu\nu}(x)$ are respectively coordinates and metric of the space-time, ξ^m, $m = 1, 2$ and $g_{mn}(x)$ coordinates and metric of the Riemann surface traced out by the string.

The following transformations are symmetries of the Polyakov action:

i) The group Diff($\mathcal{M}$) of diffeomorphisms on $\mathcal{M}$.

ii) The group Weyl($\mathcal{M}$) of local rescalings of the metric tensor given by $g' = e^{2\sigma} g$.

iii) The Poincaré group is a symmetry for the space time.

The physical quantities of interest are the partition functions, one could be tempted to write (Wick rotation has been performed):

$$
Z = \sum_{h=0}^{\infty} \int D g_{mn} D x^\mu e^{-S(x,g)} \tag{61}
$$

however there is a huge overcounting due to the fact that surfaces parameterised by x^μ, g_{mn} related by Weyl transformations and reparametrization describe the same physical state, the Polyakov action being invariant under these transformations suggest that this problem can be solved by factoring out the volume of these symmetry groups, thus the partition function is formally written

$$Z = \sum_{h=0}^{\infty} \int \frac{Dg_{mn} Dx^\mu}{V_{Diff} V_{Weyl}} e^{-S(x,g)} \tag{62}$$

6.1 Integration over spce-time coordinates

The integration over x^μ reduces to a Gaussian integral: first by integration by parts the Polyakov action is rewritten as

$$S_p = \frac{1}{8\pi} \langle x \,|\, \Delta_g x \rangle \tag{63}$$

where

$$\Delta_g = -\frac{1}{\sqrt{g}} \partial_m \sqrt{g} g^{mn} \partial_n \tag{64}$$

Thanks to Sturm-Liouville theorem the x variable can always be split into the constant zero mode x_0, and the other modes orthogonal to it x'^μ, so that $x^\mu = x_0^\mu + x'^\mu$. As in section (5.1) the measure is indirectly chosen by setting $1 = \int Dx^\mu \, e^{-\|x\|^2/8\pi}$.

$$\begin{aligned} \int Dx^\mu \, e^{-S_p} &= \int Dx_0^\mu \int Dx'^\mu \, e^{-\langle (x_0 + x') \,|\, \Delta_g (x_0 + x') \rangle / 8\pi} \\ &= \int Dx_0^\mu \int Dx'^\mu \, e^{-\langle x' \,|\, \Delta_g x' \rangle / 8\pi} \\ &= (\det{}' \Delta_g)^{-d/2} \int Dx_0^\mu \int Dx'^\mu \, e^{-\|x'\|^2/8\pi} \end{aligned} \tag{65}$$

$\int Dx_0^\mu \equiv \Omega$ is the volume of space time, while $\int Dx'^\mu \, e^{-\|x'\|^2/8\pi}$ is evaluated by

$$\begin{aligned} 1 &= \int Dx^\mu \, e^{-\|x\|^2/8\pi} = \int Dx_0^\mu \int Dx'^\mu \, e^{-\|x_0\|^2/8\pi - \|x'\|^2/8\pi} \\ &= \left(\frac{8\pi^2}{\int_{\mathcal{M}} d^2\xi \, \sqrt{g}} \right)^{d/2} \int Dx'^\mu \, e^{-\|x'\|^2/8\pi} \end{aligned} \tag{66}$$

substituting in (65) it is finally obtained

$$\int Dx^\mu \, e^{-S_p} = \Omega \left(\frac{8\pi^2}{\int_{\mathcal{M}} d^2\xi \, \sqrt{g}} \det{}' \Delta_g \right)^{-d/2} \tag{67}$$

6.2 Integration over metrics

As mentioned before the strategy to handle the integration over the metrics avoiding the divergence of the partition function is to factor out the volume of the gauge group. Following Friedan [13] and Polchinski [14] the idea is to fixing the gauge in order to perform the integration over just a gauge slice and obtaining the correct measure on this space by the Faddeev-Popov method [15].

Before to go on with this method is useful to analyze the first order variation of the metric tensor under an infinitesimal reparametrization $\delta z \equiv v^\mu(z, \bar{z})$, $\mu = z, \bar{z}$ for later necessity. These are obtained recalling that, for the metric tensor, the Lie derivative along the vector field $v^\mu(z, \bar{z})$, $\mu = z, \bar{z}$ can be written as $\pounds_{v^\mu} g_{mn} = \nabla_m v_n + \nabla_n v_m$. The

Weyl transformation $g \to e^{2\sigma} g$ induce the variation $\delta g_{mn} = 2\delta\sigma g_{mn}$, putting everything together:

$$\delta g_{mn} = (2\delta\sigma + \nabla^p v_p) g_{mn} + (P_1 v)_{mn} \tag{68}$$

where

$$(P_1 v)_{mn} \equiv \nabla_m v_n + \nabla_n v_m - g_{mn} \nabla^p v_p \tag{69}$$

is the traceless part of the deformation and is determined only by the reparametrization. The total trace part can therefore always be obtained by a Weyl transformation. In order to find inequivalent metrics for the surface (i.e. which are not obtained by reparametrization or Weyl transformations) we look for symmetric traceless quadratic differentials, let's say h_{ab}, that are not in the image of P_1:

$$\langle h \,|\, P_1 v \rangle = 0 = \left\langle P_1^\dagger h \,\middle|\, v \right\rangle$$

The space of variation of the metric is thus decomposed in the three orthogonal subspaces:

$$\{\delta g_{mn}\} = \{\delta\sigma g_{mn}\} \otimes \text{Range} P_1 \otimes \text{Ker} P_1^\dagger \tag{70}$$

The dimension of $\text{Ker}P_1^\dagger$ is easier to calculate in complex coordinates once that the metric is written, at least locally, as $ds^2 = 2g_{z\bar{z}} dz d\bar{z}$: noting that $P_1 = \nabla_z^1$ it is immediately seen that

$$h \in \text{Ker} P_1^\dagger \iff \nabla^z h = 0 \iff \partial_{\bar{z}} h = 0 \iff h \text{ holomorphic} \tag{71}$$

The dimension of holomorphic quadratic differentials was found to be 0 for $g = 0$, 1 for $g = 1$ and $3g - 3$ for $g > 1$ in proposition 3.5, confirming the counting of the dimension of the moduli space of section 3.7 .

Now we can proceed with the gauge fixing. We focus on conformal metrics, leaving the discussion of integration over moduli ($\text{Ker}P_1^\dagger$) for later. Recalling that at least locally the metric can be brought in the form $g_{ab} = \delta_{ab}$ by diffeomorphisms, Vol_{Diff} can be cancelled by choosing the slice over which integration is performed as the set of the metrics obtained by Weyl transformations from $g_{ab} = \delta_{ab}$, i.e. $g_{ab} = e^\phi \delta_{ab}$, thus the chosen gauge slice reads $[\hat{g}] = \{g_{ab} = e^\phi \delta_{ab}\}$.

The usual Faddeev-Popov machinery consists in inserting the identity

$$1 = \Delta_{FP}(g) \int d\delta\sigma dv \, \delta(g - \hat{g}) \tag{72}$$

in the partition function. The so-called Faddeev-Popov determinant Δ_{FP} is interpreted (due to the properties of the delta function under change of coordinates) as the Jacobian of the transformation made to integrate only over the gauge slice. $\Delta_{FP}(g)$ is evaluated by first computing its inverse by expanding the delta function near the identity with (68) ([14])

$$\begin{aligned}
\Delta_{FP}^{-1} &= \int d\delta\sigma dv \, \delta((2\delta\sigma + \hat{\nabla}^p v_p)\hat{g} + (\hat{P}_1 v)) \\
&= \int d\delta\sigma dv d\beta' \, exp \left\{ 2\pi i \int d^2\xi \sqrt{\hat{g}} \beta'^{mn} ((2\delta\sigma + \hat{\nabla}^p v_p)\hat{g}_{mn} + (\hat{P}_1 v)_{mn}) \right\} \\
&= \int dv d\beta \, exp \left\{ 2\pi i \int d^2\xi \sqrt{\hat{g}} \beta^{mn} (\hat{P}_1 v)_{mn} \right\}
\end{aligned} \tag{73}$$

where in the second equality the symmetric tensor field β'^{mn} has been introduced to write the integral representation of the delta functional, while integration over the generator of

Weyl transformations $\delta\sigma$ forced to consider a symmetric tensor field β^{mn} perpendicular to the full-trace part of the variation, thus to be traceless for the above arguments ((70)). An expression for Δ_{FP}^{-1} has thus been obtained as a path integral over commuting variables, from the results of (5) the evaluation of its inverse Δ_{FP} is obtained by the substitutions of the commuting fields v^m, β^{mn} with anticommuting ones:

$$v^m \to c^n, \quad \beta_{mn} \to b_{mn} \tag{74}$$

Finally in complex notation the obtained expression is

$$\Delta_{FP}(g) = \int DbD\bar{b}DcD\bar{c} \prod_{j=1}^{3g-3} b(z_j)\bar{b}(z_j)e^{-S_g(b,c)} \tag{75}$$

where

$$S_g(b,c) = \frac{1}{2\pi}\int d^2z\sqrt{g}(b_{zz}\nabla^z c^z + c.c.) \tag{76}$$

The insertion of $\prod_{i=1}^{3g-3} b(z_i)\bar{b}(z_i)$ in eq. (75) has been made to avoid the integral to vanish for the presence of $3g-3$ zero modes for the operator ∇^z (we are restricting to the case $g>1$ here).
Considering the more general case of anticommuting fields of weight n and $1-n$ the Faddeev-Popov action reads

$$S_g(b,c) = \frac{1}{2\pi}\int d^2z\sqrt{g}(b\nabla^z c + c.c.) \tag{77}$$

The problem here is that the operator ∇^z acts between spaces of differentials of different weights thus it is not possible to chose eigenvectors with scalar eigenvalues to reduce to a Gaussian integral, progresses to the formal solution to this problem was made by Quillen [16] and the value of the integral is given in an heuristic way by (for example) Verlinde E., Verlinde H. [11], Belavin, Knizhnik [17]

$$|\det\omega_j^n(z_k)|^2 \frac{\det'\Delta_{1-n}}{\det\left\langle \omega_j^n \middle| \omega_k^n\right\rangle} = \int DbD\bar{b}DcD\bar{c} \prod_{j=1}^{(2n-1)(g-1)} b(z_j)\bar{b}(z_j)e^{-\frac{1}{2\pi}\int d^2z(\sqrt{g}b\nabla^z c + c.c.)} \tag{78}$$

Writing $(\mathcal{N}_n)_{jk} \equiv \left\langle \omega_j^n \middle| \omega_k^n\right\rangle = \int \bar{\omega}_j^n \rho^{1-n}\omega_k^n$ the substitution of the Faddeev-Popov determinant in the partition function leads to the following expression

$$Z = \int_{M_g} \prod_{j=1}^{3g-3} dm_j \wedge d\overline{m}_j \frac{|\det\omega_j^2(z_k)|^2}{V_{Weyl}} \int d\delta\sigma \frac{\det'\Delta_{-1}}{\det\mathcal{N}_2}\left(\frac{8\pi^2\det'\Delta_0}{\mathcal{N}_0}\right)^{-d/2} \tag{79}$$

where m_j, $j=1,\ldots,3g-3$ are analytic coordinates on the moduli space.
The last integrand is not Weyl invariant, in fact under a Weyl transformation $g \to \hat{g} = e^{-2\sigma}g$ with heat-kernel, short-time cutoff procedure ([18]), it is found

$$\begin{aligned} \frac{\det'\Delta_{-1}}{\det\mathcal{N}_2} &= \frac{\det'\hat{\Delta}_{-1}}{\det\hat{\mathcal{N}_2}}e^{-26S_L(\sigma)} \\ \left(\frac{8\pi^2\det'\Delta_0}{\mathcal{N}_0}\right)^{1/2} &= \left(\frac{8\pi^2\det'\hat{\Delta}_0}{\hat{\mathcal{N}_0}}\right)^{1/2} e^{-S_L(\sigma)} \end{aligned} \tag{80}$$

Where $S_L(\sigma)$ is called *Liouville action*, but its explicit expression is irrelevant because we are interested in the critical dimension, where Weyl dependence is eliminated, by substitution of (80) into (79) it is found that the coefficient multiplying the Liouville action is $d - 26$, thus the critical dimension is 26. In critical dimension the integration over Weyl transformations cancel out with V_{Weyl}, and the computation of the partition function has reduced to an integration over moduli space M_g:

$$Z = \int_{M_g} \prod_{j=1}^{3g-3} dm_j \wedge d\overline{m}_j \, |\det \omega_j^2(z_k)|^2 \frac{\det' \Delta_{-1}}{\det \mathcal{N}_2} \left(\frac{8\pi^2 \det' \Delta_0}{\mathcal{N}_0} \right)^{-13} \tag{81}$$

7 Mumford forms

We will define and express in terms of theta functions the Mumford forms for a generic $n \in \mathbb{N}$, $n > 0$, while in the last paragraph it will be given the relation between the Polyakov measure and the Mumford form for $n = 2$.

7.1 Mumford forms

Let Let $\mathcal{C}_g \xrightarrow{\pi} M_g$ be the universal curve over moduli space M_g, $g \geq 2$, defined by letting π be the association between each point of M_g and the corresponding equivalent Riemann surfaces. Consider for $n \geq 1$ the determinant line bundle $\lambda_n = \det L_n$ where $L_n = R\tau_* \left(K^n_{\mathcal{C}_g/M_g} \right)$ is the vector bundle on M_g of rank $N = (2n-1)(g-1) + \delta_{1n}$ for $n > 1$ ((3.2)) with fiber $H^0\left(K^n_C\right)$ at $C \in M_g$, C compact Riemann surface. The Mumford isomorphism is

$$\lambda_n \cong \lambda_1^{c_n}, \quad c_n \equiv 6n^2 - 6n + 1 \tag{82}$$

the Mumford form $\mu_{g,n}$ for $g \geq 3$ is then the unique (up to a constant) holomorphic section of the bundle $\lambda_n \otimes \lambda_1^{-(6n^2-6n+1)}$ nowhere vanishing on M_g (it has poles at the boundary of M_g). To find an expression of the Mumford form the following lemma is needed.

Lemma 7.1. *Given an arbitrary holomorphic function $t(z)$ non-vanishing on $\mathcal{M}$ such that $t(z_0) = 1$ for a fixed $z_0 \in \mathcal{M}$, $\{\omega_i\}_{i=1,\ldots,m}$ a basis for $H^0(K^n)$, $\{\phi_i\}_{i=1,\ldots,m}$ a basis for $H^0(K^{1-n})$ and let D be a divisor of degree d, then*

$$f\left(L, z_0, \{\omega_i\}, \{\phi_i\}\right) \equiv \frac{\theta\left(\phi\left(D - \sum_1^m x_i + \sum_1^n y_i\right) - \Delta\right) \prod_{i<j}^m E\left(x_i, x_j\right) \prod_{i<j}^n E\left(y_i, y_j\right)}{\det\left(\omega_i\left(x_j\right)\right) \det\left(\phi_i\left(y_j\right)\right) \prod_{i,j} E\left(x_i, y_j\right)} \cdot \frac{\prod_1^m t\left(x_i\right) \prod_1^n \sigma\left(y_j, z_0\right)}{\prod_1^m \sigma\left(x_i, z_0\right) \prod_1^n t\left(y_j\right)} \tag{83}$$

does not depend on the points x_i, y_j, $i = 1\ldots, m, j = 1, \ldots, n$

Proof. The idea is to prove that f is a holomorphic, non-vanishing function, thus it is a constant with respect to each variables x_i, y_j. Firstly f is considered as a function of x_1 only, with x_i, $i = 2, \ldots, m$, y_j, $j = 1, \ldots, n$ fixed. The divisor of the theta function is found by application of Riemann vanishing theorem (4.1): writing $\theta\left(\phi\left(\sum_1^d a_i - \sum_2^m x_i + \sum_1^n y_i\right) - \phi(x_1) - \Delta\right)$ it is seen that there are n zeros at y_j, $j = 1, \ldots n$ and each η of the others $g - n$ (recall that the Riemann-Roch theorem imply that $m - n = d - g + 1$) satisfy

$$\phi(D) + \phi\left(\sum_1^n y_i\right) - \phi\left(\sum_2^m x_i\right) = \phi\left(\sum_1^n y_i\right) + \eta$$

The determinant $\det\left(\omega_i\left(x_j\right)\right)$ has trivial zeros at x_i, $i = 2, \ldots m$, and the others $d - (m - 1) = g - n$ (again for Riemann-Roch) at $\sum_{i=2}^m x_i + \xi = D$; thus $\xi = \eta$, while the other zeros cancel out with the prime forms. For the above statement f is constant with respect x_1 (i.e. does not depend on it). The same argument applies to the other variables. □

In particular the expressions needed are ($t(z)$ is chosen to be $t(z) = \sigma(z, z_0)^{2n}$)

$$\begin{aligned} f(K^n, z_0, \{\omega_i^n\}, \varnothing) &= \frac{\theta\left(\phi(\sum_1^{N_n} x_i) - (2n-1)\Delta\right) \prod_1^N \sigma(x_i)^{2n-1} \prod_{i<j}^N E(x_i, x_j)}{\det \omega_i^n(x_j) \sigma(z_0)^{(1-2n)N}} \\ &\equiv \frac{1}{\kappa[\omega^n] \sigma(z_0)^{(1-2n)N}} \end{aligned} \tag{84}$$

and

$$f(K^n, z_0, \{1\}, \{v_j\}) = \frac{\theta\left(\phi(\sum_1^g y_i - x - \Delta)\right) \prod_1^g \sigma(y_i) \prod_{i<j}^g E(y_i, y_j)\sigma(z_0)^{g-1}}{\det \omega_i(y_j)\sigma(x) \prod_1^g E(x, y_i)} \equiv \frac{\sigma(z_0)^{g-1}}{\kappa[\omega]} \tag{85}$$

Also the following quantity is introduced

$$c(p) \equiv \frac{\theta\left(\sum_1^g x_i - q - \Delta\right) \prod_{i<j}^g E\left(x_i, x_j\right) \prod_1^g \sigma\left(x_j, p\right)}{\prod_1^g E\left(q, x_i\right) \det\left(\omega_i\left(x_j\right)\right) \sigma(q, p)} \tag{86}$$

Now the Mumford form in terms of theta functions is given by the following theorem

Theorem 7.1 ([19])**.** *Let $\{\omega_1^n, \ldots, \omega_N^n\}$ be any basis of $H^0\left(K_C^n\right)$ for $n \geq 2$ and $N = (2n-1)(g-1) + \delta_{1n}$. Then for any points $p, x_1, \ldots, x_N \in H$, the Mumford form is, up to a universal constant:*

$$\mu_{g,n} = \frac{\theta\left(\sum_1^N x_i - (2n-1)\Delta\right) \prod_{i<j}^N E\left(x_i, x_j\right)}{\det\left(\omega_i^n\left(x_j\right)\right) \prod_1^N c\left(x_i\right)^{(2n-1)/(g-1)}} \cdot \frac{\omega_1^n \wedge \cdots \wedge \omega_N^n}{\left(\omega_1 \wedge \cdots \wedge \omega_g\right)^{6n^2-6n+1}} \tag{87}$$

The form has a pole of order $\frac{1}{2}n(n-1)$ at the boundary $\partial M_g = \overline{M}_g - M_g$.

Proof. Using the quantities introduced above $\mu_{g,n}$ can be written as:

$$\mu_{g,n} = \mu_{g,n}\left\{\omega_i^n\right\} \frac{\omega_1^n \wedge \cdots \wedge \omega_N^n}{\left(\omega_1 \wedge \cdots \wedge \omega_g\right)^{6n^2-6n+1}},$$

$$\mu_{g,n}\left\{\omega_i^n\right\} = \frac{f\left(K^n, z_0, \left\{\omega_i^n\right\}, \varnothing\right)}{f\left(I, z_0, \{1\}, \{v_j\}\right)^{(2n-1)^2}} = \frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\omega^n]}$$

By lemma 7.1 $\mu_{g,n}$ is a holomorphic, nowhere vanishing form depending on the marking of C, thus is the Mumford form. For later use its behaviour under modular transformations is given by the behaviour of

$$\theta\left(\sum_1^N x_i - (2n-1)\Delta\right) \prod_{i<j}^N E\left(x_i, x_j\right) \prod_1^N c\left(x_i\right)^{\frac{1-2n}{g-1}}.$$

A computation ([19]) shows that this quantity picks up the factor $(\varepsilon)^{-4n(n-1)}$ $(\det(c\tau + d))^{-\left(6n^2-6n+1\right)}$, where ε is an eighth root of 1. Thus for n integer

$$\hat{\mu}_{g,n} = \det(c\tau + d)^{-\left(6n^2-6n+1\right)} \mu_{g,n}. \tag{88}$$

The boundary of the moduli space $\overline{M}_g - M_g$ is a union of divisors $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\left[\frac{g}{2}\right]}$ where each one of the points of Δ_0 corresponds to the set of equivalent pinched Riemann surfaces obtained by identifying two points a, b on a smooth genus $g-1$ surface C_*, while the points of Δ_k for $k > 0$ correspond to pinched surfaces obtained by identifying the points a, b on two Riemann surfaces C_1, C_2 of genus $g_1 = k, g_2 = g - k$ respectively. This construction is represented in figures 5 and 6 for the case $g = 2$.

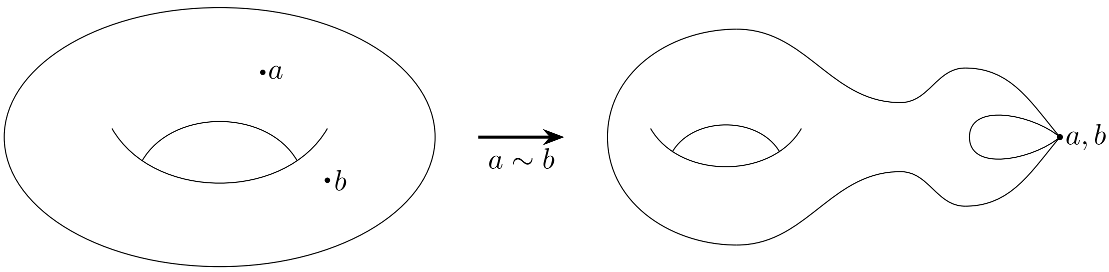

Figure 5: Construction of $\overline{M}_2 - M_2$: Δ_0

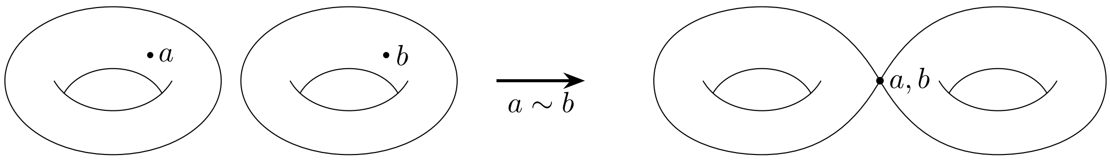

Figure 6: Construction of $\overline{M}_2 - M_2$: Δ_1

A family of surfaces C_t is built in the following way: let C_0 be one of the pinched surfaces (i.e. $C_0 \in \Delta_k$), while for $t \neq 0$ C_t is a non-singular Riemann surface of genus g. It is clear that a basis $\{\omega^n_{i,t}\}$ for $H^0(K^n_{C_t})$ has dimension $(2n-1)(g-1)+\delta_{1n}$ (proposition 3.2), while for $t = 0$ one can chose only $(2n-1)((g-1)-1)+\delta_{1n}$ independent holomorphic $n-$differentials for $k = 0$ and $(2n-1)(g_1-1+\delta_{1n}+g_2-1+\delta_{1n}) = (2n-1)(g-2+2\delta_{1n})$ for $k > 0$. Thus (assume $n > 1$, the case of $n = 1$ requires straightforward modification due to the δ_{1n} which will be clear in the following arguments, in particular note that for $K > 0$ and $n = 1$ there are still g holomorphic differentials at $t = 0$) $\{\omega^n_{i,0}\} = \{\omega^n_j\} \cup \{\eta_m\}$ where ω_j are the $(2n-1)(g-2)$ holomorphic elements and the other $2n-1$ $n-$differentials η_m has singularities at a, b given by $\frac{1}{(z-a)^m}$ for $1 \leq m \leq n-1$, $\frac{1}{(z-b)^{m-n+1}}$ for $n \leq m \leq 2n-2$ and $\left(\frac{1}{(z-b)^n} + \frac{(-1)^n}{(z-a)^n}\right)$ for $m = 2n-1$. The cases of $k = 0$ and $k > 0$ require different treatment:

- **CASE I ($k = 0$):** a basis $\omega_{1,t}, \ldots, \omega_{g,t}$ for $H^0(K_{C_t})$ at $t = 0$ become $v_1, \ldots, v_{g-1}$, $\partial_z \log \frac{E(z,b)}{E(z,a)}$, in fact close to a, b, thanks to the construction of the prime form (4), the last differential is $\partial_z \log \frac{z-b}{z-a} = \frac{1}{z-b} - \frac{1}{z-a}$ as expected. To compute the period matrix for $t \to 0$, in addition to the expression of ω_g, also the last $b-$'cycle' now goes from a to b must be taken in account. From these considerations the period matrix is:

$$\tau_{jk}(t) \sim \begin{cases} \tau_{jk} & j,k < g \\ \int_a^b \omega_j & j < g, k = g \end{cases}, \tau_{gg}(t) = \frac{1}{2\pi i}\log t + O(t)$$

Expanding up to the leading term $\mu_{g,n}\{\omega_{i,t}\}$ leads to

$$\mu_{g,n}\{\omega_{i,t}\} \sim t^{-n(n-1)/2}\frac{(-1)^n\sigma(a,b)^{(2n-1)(n-1)}\prod_{i<j}^N E(x_i,x_j)}{\det(\omega_j(x_i)\eta_k(x_i))\prod_1^N [E(x_i,a)E(x_i,b)]^n}$$
$$\cdot\frac{\theta(s)\sigma(a,b)^{2n-1}\prod_1^N E(x_i,a) - \theta(s+a-b)\prod_1^N E(x_i,b)}{(2\pi i)^{(2n-1)^2}\prod_1^N [c(x_i)\sigma(a,x_i)]^{(2n-1)/(g-1)}}$$

where $s = (2n-1)\Delta_* - \sum_1^N x_i + n(a+b) - a \in \mathbf{C}^{g-1}$. Note that since $\mu_{g,n}$ is independent of x_i, we can let $x_{N-n+1}, \dots, x_N \to b$ and then $x_{N-2n+2}, \dots, x_{N-n} \to a$ to conclude for $g \geq 3$:

$$\mu_{g,n}\{\omega_{i,t}\} \sim t^{-n(n-1)/2} \frac{E(a,b)^{n-n^2}}{(2\pi i)^{(2n-1)^2}} \mu_{g-1,n}\{\omega_i\} \tag{89}$$

- **CASE II ($k > 0$):** in this case, as already noted, a basis of holomorphic differentials only can be chosen, so

$$\{\omega_{i,0}\} = \{\omega_j, 1\} \cup \{\omega_m, 2\} \tag{90}$$

with $\{\omega_j, 1\}$ and $\{\omega_m, 2\}$ basis for $H^0(K_{C_1})$ and $H^0(K_{C_1})$ respectively. Also the $b-$cycles can be chosen to meet no modification for $t \to 0$, thus the period matrix behaves in the following way:

$$\tau \to \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \tag{91}$$

with τ_1 and τ_2 period matrices for C_1 and C_2.
Again expanding $\mu_{g,n}\{\omega_{i,t}\}$ it is obtained

$$\mu_{g,n}\{\omega_{i,t}\} \sim \varepsilon t^{-n(n-1)/2} \mu_{g_1,n}\{\omega_{j,1}\} \mu_{g_2,n}\{\omega_{m,2}\} \tag{92}$$

where ε is a fixed $(2g-2)$th root of 1 .

7.2 Relation with Polyakov measure

Recall that it had been found that the partition function of the bosonic string in critical dimension $d = 26$ is the following integral over moduli space

$$Z = const \int_{M_g} \prod_{j=1}^{3g-3} dm_j \wedge d\overline{m}_j |\det \omega_j^2(z_k)|^2 \frac{\det' \Delta_{-1}}{\det \mathcal{N}_2} \left(\frac{\det' \Delta_0}{\mathcal{N}_0} \right)^{-13} \tag{93}$$

Also in the proof of proposition 3.1 it was proved that $\mathcal{N}_1 \propto \operatorname{Im}\tau$, thus the integrand can be written as

$$\frac{\det' \Delta_{-1}}{\det \mathcal{N}_2} \left(\frac{\det' \Delta_0}{\mathcal{N}_0 \, \det \mathcal{N}_1} \right)^{-13} |\det \omega_j^2(z_k)|^2 (\det \operatorname{Im}\tau)^{-13} \tag{94}$$

$\frac{\det' \Delta_{-1}}{\det \mathcal{N}_2} \left(\frac{\det' \Delta_0}{\mathcal{N}_0 \, \det \mathcal{N}_1} \right)^{-13} \equiv e^W$ was found by Belavin, Knizhnik [4] to be the square of an holomorphic non-zero function, also note that $\det \omega_j^2(z_k) \in \lambda_2$ and, from the definition,

$$\det \operatorname{Im}\tau = \det \frac{1}{2} \int_C \omega_i \wedge \overline{\omega}_j = \frac{1}{2^g} \det \int_C \omega_i \wedge \overline{\omega}_j$$

thus using (Matone, Volpato [20])

$$\det \int_C \omega_i \wedge \overline{\omega}_j = \frac{1}{g!} \int_{C^g} \prod_k |dz_k|^2 |\det \omega_i(z_j)|^2$$

it follows that under modular transformations $\det \operatorname{Im}\tau$ transforms as $|\det \omega_j(z_k)|^2$, so the map from the modulo square of the wedge product into a scalar

$$|\det \omega_j(z_k)|^2 \to \det \operatorname{Im}\tau \tag{95}$$

leads to the association between the modulo square of the Mumford form and the Polyakov measure multiplied by $(\det \mathcal{N}_1)^{13}$ for $n = 2$.
As first application of this fact it is noted that in critical dimension also modular invariance is guaranteed, in fact the factor $(\det(c\tau + d)^{-13})^2$ from (88) is eliminated by its inverse coming from $(\det \operatorname{Im} \tau)^{-13}$ due to (24).
Additionally the 'double pole' theorem of Belavin, Knizhnik [17] implying the divergence of the partition function at the boundary of the moduli space, interpreted as the tachyon, follows as a particular case of theorem 7.1 for $n = 2$.
Finally the factorization property of the Mumford form (92) together with the 'splitting' of the basis for $H^0(K_C)$ (90) and the period matrix (91) when the surface is pinched not in a $a-$cycle, allow the factorization of a generic partition function $Z_n(z_1, \ldots, z_I)$ [11]. The result depends on how the points z_i split over C_1 or C_2. Let $x_1, \ldots, x_{I_1+q_1}$ be the points C_1 and $y_1, \ldots, y_{I_2+q_2}$ the ones on C_2, where $I_i = (2n-1)(g_i - 1)$ and $q_1 + q_2 = (2n-1)$. Thus the factorization for $t \to 0$ reads

$$Z_n(z_1, \ldots, z_I)(t) \rightarrow t^{-q_1 q_2/2} Z_n(x_k, a; q_1) Z_n(y_j, b; q_2). \tag{96}$$

7.3 Expression of the partition function without points

The expression of the Polyakov measure in terms of theta functions still isn't satisfactory because contains points on which it has been proved to be independent (lemma 7.1), the solution to this redundancy can be found in Matone [21] where the expression for $g = 4$, conjectured by Belavin, Knizhnik [17] was proved and a general structure for the partition function derived for $g > 4$. Before to go on with this results it should be noted that removing the points in the Polyakov measure could lead to modular or Weyl anomaly, this problem is illustrated in detail and solved in Matone [22] for the more general case:

$$\left(\frac{\det' \Delta_0}{\mathcal{N}_0}\right)^{-c_n} \frac{\det' \Delta_{1-n}}{\det \mathcal{N}_n} |\det \omega_j^n(z_k)|^2 = \left|\frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\omega^n]}\right|^2 \frac{|\det \omega_j^n(z_k)|^2}{(\det \tau)^{c_n}} \tag{97}$$

The problem of eliminating the points arise from the fact that a positive definite $(1,1)$ form used to integrate over the zero mode insertions in (78) (thus, as a metric, should take the factor e^σ under Weyl transformation) can be expressed in terms of Weyl and modular invariant quantities. Anyway in the same article [22] all the Weyl invariant forms were classified, in particular it was proved that the map

$$\frac{|\wedge^{max} \omega_j^n|^2}{|\wedge^g \omega_j|^{2c_n}} \rightarrow \frac{\det \mathcal{N}_n}{(\det \tau)^{c_n}} \tag{98}$$

which associates the modulo square of the wedge products in the Mumford forms to $(0,0)$ forms, allows to express the corresponding partition function

$$Z_n = \left|\frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\omega^n]}\right|^2 \frac{\det \mathcal{N}_n}{(\det \tau)^{c_n}} \tag{99}$$

as a Weyl and modular invariant, point independent quantity when $d = 2c_n$. Of course for the bosonic string this condition is satisfied in critical dimension.
Now let, as above, $m_1, \ldots, m_{3g-3}$ some complex analytic coordinates on the moduli space of genus $g \geq 2$ compact Riemann surfaces M_g. According to the results of the previous section the genus g partition function of the Polyakov bosonic string is

$$Z_g = \int_{M_g} \frac{F \wedge \bar{F}}{(\det \operatorname{Im}\tau)^{13}} , \tag{100}$$

where

$$F \equiv F(m_1, \ldots, m_{3g-3}) dm_1 \wedge \cdots \wedge dm_{3g-3} \ ,$$

is a holomorphic $(3g-3,0)$ form nowhere vanishing on M_g with a double pole at the boundary $\overline{M}_g - M_g$.
Set $[\delta] \equiv [{a \atop b}]$ for the theta characteristic, the Thetanullwerte $\chi_k(Z)$ is

$$\chi_k(Z) \equiv \prod_{\delta \text{ even}} \theta[\delta](0,Z), \quad Z \in \mathcal{H}_g, \, k = 2^{g-2}(2^g+1) \tag{101}$$

Note that in definition (101) it makes sense to consider only the $2^{g-1}(2^g+1)$ even characteristics because the inclusion of the odd ones would make the product vanishing for each Z. For $g=2$ and $g=3$ the following relations hold

$$F(g=2) \propto \frac{\wedge^2_{i \leq j} d\tau_{ij}}{\chi_5^2(\tau)}, \quad F(g=3) \propto \frac{\wedge^3_{i \leq j} d\tau_{ij}}{\chi_{18}^{1/2}(\tau)} \tag{102}$$

The problem for $g \geq 4$ turned out to be of more difficult solution due to the Schottky problem, in fact as noted at the end of section 3.4, the dimension of the Siegel upper half space is $\frac{1}{2}g(g+1)$ and the one of the moduli space is 1 for $g=1$, $3g-3$ for $g>1$, so that their dimension coincide only up to $g=3$.
Introducing the quantity

$$F_g(Z) \equiv 2^g \sum_{\delta \text{ even}} \theta^{16}[\delta](0,Z) - \big(\sum_{\delta \text{ even}} \theta^8[\delta](0,Z) \big)^2$$

for $g=4$ it was conjectured ([17]) that

$$F = \frac{d\tau_{11} \wedge \cdots \wedge \widehat{d\tau_{ij}} \wedge \cdots \wedge d\tau_{44}}{S_{4ij}(\tau)} \tag{103}$$

where the derivative of $F_4(Z)$ is written as

$$S_{4ij}(Z) \equiv \frac{1+\delta_{ij}}{2} \frac{\partial F_4(Z)}{\partial Z_{ij}}$$

It turns out that $F_4(Z)$ has first order zeros only when Z correspond to the period matrix of a Riemann surface.
These facts allow to write the partition function as a residue formula:

$$Z_4 = \int_{\mathcal{H}_4} \frac{1}{(\det \operatorname{Im} Z)^{13}} \left| \frac{\wedge^4_{i \leq j} dZ_{ij}}{F_4(Z)} \right|^2 \tag{104}$$

The above property of $F_4(Z)$ and the fact that the residues are evaluated by taking the derivative of the vanishing denominator show that the last equation is indeed the substitution of (103) into (100).
As said above this conjecture was proved by Matone [21] by finding for the Mumford form $\mu_{4,2}$ the expression (c is a constant):

$$\mu_{4,2} = \pm \frac{1}{cS_{4ij}} \frac{\omega_1\omega_1 \wedge \cdots \wedge \widehat{\omega_i\omega_j} \wedge \cdots \wedge \omega_4\omega_4}{(\omega_1 \wedge \cdots \wedge \omega_4)^{13}} \tag{105}$$

in fact the square of (105) is associated to the Polyakov measure by two maps. One of them is (95). The other one acts between the wedge product at the numerator of (105) and the one at (103), thus can be seen as a map from $\omega_i\omega_j$ to $d\tau_{ij}$, where $d\tau_{ij}$ is the differential of a matrix in the 9th dimensional moduli space. In order to construct such a map, Beltrami differentials μ_i are introduced as the dual elements to holomorphic quadratic differentials:

$$\frac{1}{2\pi i}\int_{C_g}\mu_i\omega^j=\delta_i^j \tag{106}$$

it follows that there are $3g-3$ independent Beltrami differentials. Now consider the Rauch formula

$$d\tau_{ij}(\mu)=\int_{C_4}\mu\omega_i\omega_j \tag{107}$$

It can be proved (see [23]) that this changes the complex structure of C_4, thus the LHS is the differential of a period matrix in the moduli space, so the isomorphism (107) is the map we were looking for to complete the association between the Mumford form and the Polyakov measure.
Using this map Matone [24] found an infinite class of $b-c$ (anticommuting) and $\beta-\gamma$ (commuting) string-like theories corresponding volume forms on M_g without poles (so with no tachyon). In particular the product of Mumford forms $\mu_{g,n}$ with arbitrary n was considered:

$$\mu_g\equiv\frac{\prod_k\left(\mu_{g,k}^{n_k}\{\omega^k\}\wedge^{max}\omega_j^k\right)}{(\omega_1\wedge\cdots\wedge\omega_g)^{d/2}},\quad n_k\in\mathbb{Z} \tag{108}$$

where, according to theorem 7.1,

$$d=2\sum_k n_kc_k \tag{109}$$

Recalling the results (92) and (89) of theorem 7.1, which imply that each $\mu_{g,k}$ has a pole of order $\frac{1}{2}k(k-1)$ at ∂M_g, it is found that μ_g has no poles when

$$\sum_k n_kk(k-1)=\frac{1}{6}\sum_k n_k(c_k-1)\leq 0 \tag{110}$$

Substitution of (109) in (110) leads to

$$d\leq 2\sum_k n_k \tag{111}$$

The associations between the Mumford forms and measures on M_g allows to write the following integrand for the partition functions

$$\mathcal{Z}=\prod_k\mathcal{Z}_k^{n_k} \tag{112}$$

where

$$\mathcal{Z}_k=\prod_k\frac{|\mu_{g,k}\{\omega^k\}|^2}{(\det\tau)^{c_k}} \tag{113}$$

The argument at the end of section 5.2 shows that $n_k>0$ correspond to $b-c$ systems, while $n_k<0$ to $\beta-\gamma$ ones.

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