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Connecting state-based and team-based inquisitive logic

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Abstract

In this thesis, we develop entailment-preserving translations between the standard system of inquisitive first-order logic InqBQ, which uses a state-based semantic framework, and its team-semantics counterpart, InqBT. We utilize the translations to confirm intuitions regarding a connection between these systems. We first demonstrate the usefulness of the translations by establishing the equivalence of corresponding major open questions about the two systems. Then, we transpose known results about wide fragments of the language from InqBQ to the corresponding fragments of InqBT.

Contents

Introduction

One of the objectives of formal logic is the formalization and regimentation of sentences of natural language. A vast majority of the systems that have been developed and studied in this field realize this objective by dealing exclusively with the logical relations that occur between assertions. Natural language, however, includes a much wider range of expressions, such as questions, exclamations, imperatives and more. Inquisitive logic is a conservative extension of classical logic that aims to provide a logical environment where both questions and statements can be analyzed uniformly and at the same time. This generalization is achieved through both syntactical and semantical modifications of the standard systems of classical logic. The enterprise of extending logical analysis to questions is justified. The extension is motivated from various directions. One line of motivation is the important role that questions play, for instance, in natural language and database theory. However, there is also intrinsic interest in studying and formalizing the logical relations holding between questions and in the application of the tools of logic to them. The endeavour is also further supported by how natural and well-behaved the resulting extensions of classical logic turn out to be.

The key element of the inquisitive extension of classical logic is the notion of *support*. In classical logic, formulas are evaluated with respect to a relational structure (called model) and a formula can be or not be *true* at a model. Intuitively, a model can be seen as a representation of a state of affairs. In the propositional case, a model is a function assigning a truth value to the propositional atoms and this describes what propositions are true in the corresponding state of affairs. In the predicative case, models are composed of a domain and an interpretation function, which assigns an extension to predicates and gives an interpretation to functions. This represents which individuals satisfy which properties in the corresponding state of affairs.

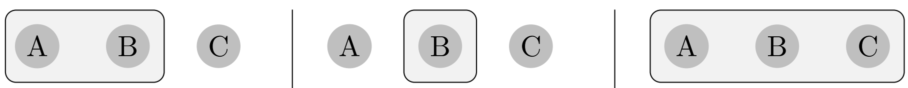
Inquisitive semantics ([CGR18]) takes a different approach: formulas are evaluated at *information states* and they can either be supported or not supported by one of these states.
Formally, every inquisitive model includes a set of possible worlds, each individually associated with some relational structure, and an information

state is any subset of this logical space. The idea behind this setup is that a state of information intuitively represents what information is available. Now, a single possible world can be assigned to a specific relational structure, which represents (just like in classical logic) a state of affairs. An information state is then intuitively composed of all possible worlds associated with a state of affairs that is compatible with the information that the state models.

To make this more concrete, let's consider some examples of information states. Consider a situation where we have three friends playing a game: Alice, Bob and Charlie. We want to capture the following possible situations: the one where Alice wins the game, the one where Bob wins the game and the one where Charlie wins the game. Then, our logical space will look like this:

Figure 1: Example of a logical space

where, intuitively, world A is the state of affairs where Alice wins, world B is the one where Bob wins and world C the one where Charlie wins.
Now, let's consider some examples of information that we could have about this situation and how they can be modeled with information states:



(a) s_1 (b) s_2 (c) s_3

Figure 2: Examples of information states

In Figure 2(a), the information represented by the state s_1 implies that Alice or Bob could have won the game, while certainly Charlie has lost. In Figure 2(b), the information that the state s_2 represents is only compatible with Bob winning and, therefore, it determines a specific state of affairs. In Figure 2(c), the information modeled by s_3 is insufficient to determine anything about the outcome of the game.

This state-based semantics can be employed to construct conservative extensions for various classical logical systems. For instance, the inquisitive approach enables the introduction of questions in propositional logic, predicate logic and a wide range of modal logics ([CR14],[Cia15],[CO21],[Cia22a]).

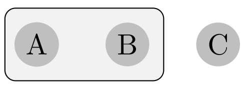
In this thesis, we will focus our attention on one of this systems, called InqBQ. InqBQ is the standard system for first order inquisitive logic.
Syntactically, the language of InqBQ extends the language of first order clas-

sical logic with two additional symbols that allow the expression of questions: the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and the inquisitive existential $\exists\!\!\exists$. Given a signature Σ, and assuming that p indicates generic atoms of Σ (as standardly defined in classical predicate logic), the resulting language $\mathcal{L}(\Sigma)$ is then defined by

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \forall x\varphi \mid \exists\!\!\exists x\varphi$$

It is customary to also include the negation symbol $\neg$, defined by $\neg\varphi := \varphi \to \bot$, the inquisitive symbol ?, defined by $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$, the classical disjunction $\vee$, given by $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, and the classical existential quantifier $\exists$, defined by $\exists x\varphi := \neg\forall x\neg\varphi$.

Semantically, recall that in classical logic formulas are given truth conditions at a model with respect to an assignment. In InqBQ, the evaluation of a formula is carried out by defining its support conditions with respect to information states under an assignment. Models are therefore constructed as sets of relational structures (given by a universe of possible worlds) with a constant domain. This allows the representation of uncertainty about the state of affairs. An inquisitive disjunction $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ is supported by a state if one of the disjuncts is supported by the whole state (in contrast, classical disjunctions $\varphi \vee \psi$ are supported if every world of the state individually supports one of the disjuncts). To see how InqBQ can separate these symbols, let us recall the example provided in Figure 1. Suppose we have three atoms in our language: a, which stands for "*Alice wins the game*", b, which stands for "*Bob wins the game*", and c, which stands for "*Charlie wins the game*". Intuitively, a is true only at world A and false at worlds B and C, b is only true at B and c is only true at C. The following figure shows an information state that differentiates between $a \vee b$ and $a \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} b$.



(a) s_1

Figure 3: A state that distinguishes $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\vee$

The state s_1, containing the worlds A and B, clearly supports $a \vee b$, since A supports a and B supports b. However, neither a nor b are supported by $s_1 = \{A, B\}$, since A doesn't support b and B doesn't support a. Therefore, s_1 doesn't support $a \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} b$.

Inquisitive and classical existentials also differ. Inquisitive existential quantifications $\exists\!\!\exists x\varphi$ are supported by a state if there is some individual d of the domain such that φ is supported by the state when x is evaluated into d (again, differing from the support clause of classical existentials $\exists x\varphi$, which only require the existence, at each world, of some individual with this property).

The resulting system is conservative over classical logic, but the added symbols, combined with the modelling capabilities of the semantical framework, enable the expression of questions.

Indeed, InqBQ proves to be rather expressively rich. The conservativity over classical logic means that the usual classical regimentation of statements can be carried over to InqBQ. For instance, formulas like $\forall x P(x)$ or $\forall x(P(x) \wedge Q(x))$, still formalize the assertions "*all objects are* P" and "*all objects are both* P *and* Q" about the state of affairs. At the same time, the inquisitive approach allows for the expression of a considerable range of questions, while also preserving intuitions about their behaviour. One example are polar questions of the form "*What is the truth value of* p*?*", expressed by the formula $?p$ or more complex versions like "*Are all objects* P*?*", expressed by $?\forall x P(x)$. InqBQ can also capture existence questions, like "*What's one instance of* P*?*", formalized by $\exists x P(x)$. Other questions involve the extension of predicates, for instance, "*What's the extension of* P*?*", expressed by $\forall x ?P(x)$.

A different, but related, approach to the conservative extension of the expressive power of classical logic is that of team semantics ([Hod97a],[Hod97b]). The name *team semantics* refers to a semantical framework where formulas of a logical system are evaluated with respect to sets of assignments, called *teams*. The main advantage that team semantics offers over classical logic, and the strongest motivation for its development, is the possibility of expressing relations between variables. These include, for instance, dependence ([Vä07],[YV16]), independence ([GV13]) and inclusion ([Gal12]).

A prominent system making use of team semantics is dependence logic ([Vä07]), a logical framework that extends the syntax and the semantics of classical logic with the goal of giving a uniform account of both classically expressible linguistic structures and variable dependencies. Syntactically, dependency atoms of the form $=(x_1, ..., x_n; y)$ are added to the language of predicate classical logic. These formulas can be read as "*the value of the variable* y *is completely determined by the values of the variables* $x_1, ..., x_n$". It is the use of team semantics (i.e. making the point of evaluation of formulas a team instead of an assignment), that allows the system to express said interpretation of the dependency atoms. Indeed, in dependence logic, an atom $=(x_1, ..., x_n; y)$ is said to be satisfied in the context of a team T when, for any two assignments $g, g' \in T$, if $g(x_1) = g'(x_1), ..., g(x_n) = g'(x_n)$ then $g(y) = g'(y)$.

Let's consider an example of teams that satisfy and do not satisfy a dependency formula to informally illustrate this idea. Suppose we have two teams with two assignments each: $T = \{g_1, g_2\}$ and $T' = \{g'_1, g'_2\}$ both evaluating into the domain of natural numbers. We will restrict our attention to their evaluation of two variables x and y, represented in Figure 4.

Team T

T	x	y
g_1	1	3
g_2	5	6

Team T'

T'	x	y
g'_1	2	4
g'_2	2	5

Figure 4: Two teams that differentiate $=(x;y)$

The team T supports $=(x;y)$: under T, knowing the value of x implies knowing the assignment and, therefore, the value of y. The team T', on the other hand, doesn't support $=(x;y)$: in the context of T', if one knows that $x=2$, the value of y could either be 4 or 5, so it is not completely determined.

As for the expressive power of dependence logic, it is known that, on the level of sentences, the system is as expressive as the existential fragment of second order classical logic (consisting of formulas of the form $\exists T_1 ... \exists T_n \varphi$ where $T_1, ..., T_n$ are second order variables and φ contains no second order quantifications). In particular, this implies that dependence logic doesn't admit a recursive axiomatization.

It is relevant at this point to observe that one can draw a parallel between team semantics and inquisitive logic ([Cia16]). In both cases, the extension of classical logic is achieved semantically by considering **sets** of classical points of evaluation of formulas. In the case of inquisitive predicate logic, one considers a set of relational structures and a single assignment, while in team semantics the evaluation is carried out with respect to a single relational structure but relative to a set of assignments. This comparison suggests a natural idea: interpreting the language of first order inquisitive logic in the context of team semantics, by adapting the semantic clauses in a straightforward way. This observation leads to the system of team-based inquisitive logic InqBT.

InqBT, first defined in [Yan14] as WID (weak intuitionistic dependence logic), has been recently revisited from this perspective in [Cia22b].
Syntactically, the language of InqBT is the language $\mathcal{L}(\Sigma)$, given above for InqBQ. Semantically, models are defined, just like in first order classical logic, as standard relational structures. The use of team semantics makes it possible to define support for formulas with respect to both a model and a team, while mirroring the semantic clauses of first order inquisitive logic. The only difference is that here the clauses refer to sets of assignments, whereas InqBQ uses sets of relational structures (the information states).
An inquisitive disjunction $\varphi \vee\!\!\!\vee \psi$ is supported under a team only if one of the disjuncts is supported by the whole team. Again, an inquisitive existential quantification $\exists x \varphi$ is supported by a team if there is some individual d of

the domain such that φ is supported by the team after we set d as the value of x for all assignments in the team. Classical universal quantifications $\forall x\varphi$ follow a similar idea, being supported by a team only if, for any choice of an individual d in the domain, φ is supported by the team after we set d as the evaluation of x for all assignments in the team.
Note that this way of dealing with quantifiers deviates from the approach taken by most systems based on team semantics. Normally, non-uniform evaluations of the variable into the domain are taken into account by the semantic clauses, weakening existential quantifications and strengthening universal quantifications. This is also the case for dependence logic.

Thanks to the addition of the inquisitive symbols, in InqBT one can formulate questions about the value of variables similar to those that InqBQ expresses about the state of affairs.
For instance, $?P(x)$ formalizes the question "*is x a P?*". Other examples include $\forall y?R(x,y)$, which captures "*what objects is x R-related to?*" and $?\forall yR(y,x)$, formalizing "*are all objects R-related to x?*".
Most importantly, in InqBT one can formulate identification questions such as "*what is the value of x?*", which is captured by $\lambda x \coloneqq \exists y(x = y)$. Thanks to this fact, in InqBT it is possible to retrieve the dependence atoms from dependence logic in the form of implications between identification questions. This follows a general pattern for dependences between questions discussed in [Cia16]. For instance, the formula $=(x_1, ..., x_n, y)$ can be expressed as $(\lambda x_1 \wedge ... \wedge \lambda x_n) \to \lambda y$.

At the moment, research around InqBQ and InqBT is at an intermediate stage. Several basic properties have been proved for both systems and some significant, more advanced results have been achieved for InqBQ.
At the same time, the fundamental metatheoretical questions for the two systems remain unresolved. The most prominent among these open problems, for both InqBQ and InqBT, is the existence of a sound and complete proof system. Moreover, definitive answers for important open questions in this direction are yet to be found for either system. These include some fundamental metatheoretical properties, such as entailment compactness (i.e. the existence, given an entailment, of a finite subset of its premises that entails the same conclusion) and the semi-decidability of validity (i.e. the existence of a computable procedure that halts when a formula is supported by all models under all assignments/teams).

While the two systems have many open questions in common, the state of research around InqBQ is considerably more advanced than that of its team semantics counterpart. In particular, the study of wide syntactical fragments of its language has proved extremely productive. In recent publications ([CG22],[Gri21]), sound and complete proof systems have been identified for the restricted existential fragment Rex, where inquisitive existential quantifiers can only appear in the antecedent of an implication, and for the classical antecedent fragment Clant, where no inquisitive symbols are allowed in the antecedents of implications. Another interesting portion of the language of InqBQ is the finitely coherent fragment. Formulas of inquisitive logics are said to be finitely coherent if they are supported by a state only when they are supported by all its finite subsets of size smaller than a certain finite cardinality. A finite model property (i.e. the existence for any non-validity of a model with a finite universe disproving it) and entailment compactness have been proved for finitely coherent formulas. Similar conclusions have not been reached for the corresponding fragments of InqBT and their validity remains an open question for the system.

The structural similarities between the two systems suggest the existence of a relation between them. A formal definition of this supposed connection can be given in terms of a two-way entailment preserving translation between InqBQ and InqBT. The problem is stated explicitly in [Cia22b] as follows (the symbols $\vDash_{\mathsf{InqBQ}}$ and $\vDash_{\mathsf{InqBT}}$ denote the entailment relations in InqBQ and InqBT respectively):

- *Open problem: existence of a translation of InqBT into InqBQ:*

  Given a signature Σ, is there a signature Σ', a decidable set $\Theta \subseteq \mathcal{L}(\Sigma')$ and a computable map $(.)^* : \mathcal{L}(\Sigma) \to \mathcal{L}(\Sigma')$ s.t. for all sets $\Phi \cup \{\psi\} \subseteq \mathcal{L}(\Sigma)$ we have
  $\Phi \vDash_{\mathsf{InqBT}} \psi \iff \Phi^*, \Theta \vDash_{\mathsf{InqBQ}} \psi^*$?

- *Open problem: existence of a translation of InqBQ into InqBT:*

  Given a signature Σ, is there a signature Σ', a decidable set $\Theta \subseteq \mathcal{L}(\Sigma')$ and a computable map $(.)^* : \mathcal{L}(\Sigma) \to \mathcal{L}(\Sigma')$ s.t. for all sets $\Phi \cup \{\psi\} \subseteq \mathcal{L}(\Sigma)$ we have
  $\Phi \vDash_{\mathsf{InqBQ}} \psi \iff \Phi^*, \Theta \vDash_{\mathsf{InqBT}} \psi^*$?

In this thesis, we define two translations satisfying these requests and further explore the consequences of their existence.

We first prove the possibility of constructing such translations by showing the definability of a support-preserving correspondence between the semantical structures in one system and those in the other. This directly connects state-assignment pairs with model-team pairs, demonstrating the strength of the connection between InqBQ and InqBT. Then, by identifying sets of formulas that characterize the translated structures, we prove the conservation of entailment under the two translations. Using these results, it is possible to connect metatheoretical properties and open questions between the two systems. In particular, we show that resolving whether entailment compactness and the semi-decidability of validity hold in either system is equivalent to settling the matter in the other. Moreover, we employ the translations to transfer from InqBQ to InqBT relevant known results about the Clant, Rex and finitely coherent fragments. We prove that entailment compactness holds in InqBT for entailments with finitely coherent conclusions, a known result in InqBQ. For the Rex fragment, we demonstrate that the finite coherence of Rex formulas, entailment compactness and the semi-decidability of validity, all established results for the fragment in InqBQ, can be transferred to InqBT. In a similar way, results of entailment compactness and semi-decidability of validity can be proved for the Clant fragment of InqBT, employing the translations and the validity of the same results in InqBQ.

These conclusions are relevant to research around both systems in multiple ways. First, they imply the equivalence of attempting to solve the now connected fundamental open questions of InqBQ and InqBT in either context. This means that the resolution of a certain problem can be tackled in the most suitable system, based on known results and intrinsic convenience for the chosen approach. Moreover, the results we have obtained better define the status of InqBT in relation with comparable logical frameworks. For dependence logic, whose expressive power InqBT partially retrieves, entailment compactness and semi-decidability of validity have been proved not to be attainable, implying the impossibility of an axiomatization. Now, for InqBT, the resolution of such problems has been equated with that of their counterparts in InqBQ, whose exact position between first order and second order classical logic has yet to be established. Finally, the existence of two-way entailment-preserving translations makes the transfer of additional results, including those beyond the scope of this thesis, a much simpler task. This more general fact is also specifically evidenced in this work by the ease with which we transferred the relevant InqBQ properties to InqBT.

The contents of the thesis will be organized as follows:

- Chapter 1 - Background: A detailed definition and an overview of both InqBQ and InqBT, accompanied by simple illustrations;
- Chapter 2 - Translations: Definition of the translations and proof of the preservation of entailment in both a simpler special case and in the general case;
- Chapter 3 - Repercussions: Transfer of open problems about metatheoretic properties of the two systems and transfer of known results from fragments of InqBQ to InqBT.

Chapter 1

Background

1.1 The system InqBQ

In this section, we introduce in detail the system InqBQ. This is the first of the two systems between which we will then define the translations. As such, it is necessary to outline a complete definition of the system as well as some short illustrations of its expressive power. We will also state the key properties of this system and some results that will be useful in the following chapters. For a more complete overview of this system, we refer to [Cia22b].

1.1.1 Syntax

The language of InqBQ is an extension of the language of first order classical logic, where we add operators to express questions. The meaning of these *inquisitive* operators will be made clear by their semantic clauses and by some simple illustrations. There are also minor notational differences with respect to the definition of a signature. We will call this language $\mathcal{L}(\Sigma)$ (where Σ will be a signature for the language) or simply $\mathcal{L}$, when no ambiguity arises.

Signature A signature for first order inquisitive logic is a set of symbols Σ which includes a set of relation symbols $\mathcal{R}_\Sigma$ and a set of function symbols $\mathcal{F}_\Sigma$. In the set of function symbols we specify a set of rigid function symbols $\mathcal{F}^R_\Sigma$. All symbols come with an arity n. The arity of a symbol will be shown as a subscript only when necessary. We will therefore write relation, function and rigid function symbols as R_n, f_n and f_n respectively.
As usual, we will refer to function symbols of arity 0 as *constants* and to relation symbols of arity 0 as *propositional atoms.*

Variables As customary, we have a countably infinite set of variables $Var := \{x_0, x_1, x_2, ...\}$. We will sometimes use meta-variables of the form $x, y, z, ..., y_1, y_2, ...$.

Terms Given a signature Σ, we define its set of terms $Ter(\Sigma)$ by induction as follows:

$$t ::= x \mid f(t, ..., t)$$

where $x \in Var$ and $f \in \mathcal{F}_\Sigma$.
We also define a set of *rigid* terms inductively:

$$\mathsf{t} ::= x \mid \mathsf{f}(\mathsf{t}, ..., \mathsf{t})$$

where $x \in Var$ and $\mathsf{f} \in \mathcal{F}_\Sigma^R$

Formulas Given a signature Σ, the formulas of InqBQ are defined inductively as follows:

$$\varphi ::= \bot \mid t_1 = t_2 \mid R(t_1, ..., t_n) \mid \varphi \land \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \forall x \varphi \mid \exists x \varphi$$

where $R \in \mathcal{R}_\Sigma$, n is the arity of R and $t_1, ..., t_n \in Ter(\Sigma)$.
The symbols $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ are called *inquisitive disjunction* and *inquisitive existential quantifier*. They will be interpreted differently than their classical counterparts, which we define below along with some additional defined operators:

- $\neg\varphi := \varphi \to \bot$
- $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$
- $\varphi \lor \psi := \neg(\neg\varphi \land \neg\psi)$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \land (\varphi \to \psi)$
- $\exists x \varphi := \neg\forall x \neg\varphi$

We also define a subset $\mathcal{L}^{cl} \subseteq \mathcal{L}$, which corresponds to the subset of classical formulas:

1.1 DEFINITION (Classical formulas)**.** We call $\mathcal{L}^{cl}(\Sigma)$ the set of formulas given by restricting the inductive definition of $\mathcal{L}(\Sigma)$ as follows:

$$\varphi ::= \bot \mid R(t_1, ..., t_n) \mid \varphi \land \varphi \mid \varphi \to \varphi \mid \forall x \varphi$$

where $R \in \mathcal{R}_\Sigma$, n is the arity of R and $t_1, ..., t_n \in Ter(\Sigma)$

1.2 DEFINITION**.** Free and bound occurrences of variables are defined as usual, as well as the set of free variables of a formula φ, which we write as $FV(\varphi)$. We also define closed formulas in the usual way as those formulas φ which satisfy $FV(\varphi) = \emptyset$.

1.1.2 Semantics

Let us define two types of models for InqBQ. We will start from models that do not consider identity:

1.3 DEFINITION (Relational information model)**.** A relational information model is a triple $M = \langle W, D, I \rangle$ where

- W is a set, whose individuals we call *possible worlds* or simply *worlds*
- D is a non-empty set, called the *domain*, whose elements we will call *individuals*
- I is a function, called the *interpretation function* from the set of possible worlds. I assigns to each possible world w a function I_w from Σ satisfying the following constraints:
  - $I_w(R_n) \subseteq D^n$, for any $R_n \in \mathcal{R}_\Sigma$
  - $I_w(f_n) : D^n \to D$, for any $f_n \in \mathcal{F}_\Sigma$
  - if $\mathsf{f} \in \mathcal{F}^R_\Sigma$ is rigid, then $I_w(\mathsf{f}) = I_{w'}(\mathsf{f})$ for all $w, w' \in W$

Often, given a relation symbol R or a function symbol f, we will use the following notation:

- R_w, to refer to $I_w(R)$
- f_w, to refer to $I_w(f)$

Now, let's extend these models to include identity. We will use both these types of structure in different contexts.

1.4 DEFINITION (Relational information model with identity)**.** A relational information model with identity is a relational information model equipped with an identity extension function. We will write these structures as quadruples $M = \langle W, D, I, \sim \rangle$. An identity extension function assigns each world w to $\sim_w$, the extension of the identity relation at w. For $\sim$ to be an identity relation, $\sim_w$ must satisfy the following constraints for any $w \in W$:

- Equivalence conditions:
  - Reflexivity: for all $d \in D$, $d \sim_w d$
  - Symmetry: for all $d_1, d_2 \in D$, $d_1 \sim_w d_2 \iff d_2 \sim_w d_1$
  - Transitivity: for all $d_1, d_2, d_3 \in D$, if $d_1 \sim_w d_2$ and $d_2 \sim_w d_3 \implies d_1 \sim_w d_3$
- Congruence conditions:
  - Relations: if $d_1 \sim_w d_1'$,...,$d_n \sim_w d_n'$, then
    $\langle d_1, ..., d_n \rangle \in I_w(R_n) \iff \langle d_1', ..., d_n' \rangle \in I_w(R_n)$

– Functions: if $d_1 \sim_w d'_1$,...,$d_n \sim_w d'_n$, then
$I_w(f_n)(d_1, ..., d_n) \sim_w I_w(f_n)(d'_1, ..., d'_n)$

In the future, we will simply refer to relational information models with identity as relational information models or models when no ambiguity arises.

Before continuing, it is important to note that this treatment of identity allows the expression of uncertainty about the extension of the identity predicate (since the extension of $\sim_w$ depends on w). This means that, while the domain D of individuals is the same across all worlds of a model, we can still represent uncertainty about, for example, the number of individuals.

Observation. As we mentioned when we introduced inquisitive logic, models of an inquisitive system can be seen as sets of classical relational structures. While in the case without identity this classical counterparts can be obtained very intuitively (we can assign each world w to a model M_w with domain D and interpretation function I_w), the general case requires more attention. We can recover a set of relational structure uniquely determined by the worlds of a relational information model with identity as follows. We assign each world $w \in W$ to the model $\tilde{M}_w = \langle \tilde{D_w}, \tilde{I}_w \rangle$, where:

- $\tilde{D}_w = D/\sim_w$
- $\tilde{I}_w(f)([d_1]_{\sim_w}, ..., [d_n]_{\sim_w}) = [I_w(f)(d_1, ..., d_n)]_{\sim_w}$
- $\langle [d_1]_{\sim_w}, ..., [d_n]_{\sim_w} \rangle \in \tilde{I}_w(R) \iff \langle d_1, ..., d_n \rangle \in I_w(R)$

$\tilde{M}_w$ is well defined thanks to the congruence conditions we imposed on $\sim$.

1.5 DEFINITION (Assignment)**.** An assignment for a model $M = \langle W, D, I, \sim \rangle$ is a function $g : Var \to D$.
We also introduce the notation $g[x \mapsto d]$ to indicate the assignment coinciding with g on all variables except for x, which it maps to $d \in D$.

The evaluation of terms and formulas takes into consideration both possible worlds and assignments.

1.6 DEFINITION (Referent of a term)**.** The referent of a term t in a world w under an assignment g is defined inductively as an individual $[t]^w_g \in D$ by the following clauses:

•$\underline{Var}$: $[x]^w_g = g(x)$

•$\underline{f_n}$: given $t_1, ..., t_n \in Ter(\Sigma)$, $[f_n(t_1), ..., f_n(t_n)]^w_g = I_w(f_n)([t_1]^w_g, ..., [t_n]^w_g)$

Let us now finally define the support clauses for the formulas of $\mathcal{L}$

1.7 DEFINITION (Support for InqBQ)**.** Let $M = \langle W, D, I, \sim \rangle$ be a relational information model with identity, let $s \subseteq W$ be a state in M and let g be an assignment for M. Then, we define support for a formula of $\mathcal{L}$ inductively as follows:

•$\underline{\bot}$: $M, s \vDash_g \bot \iff s = \emptyset$

•$\underline{R}$: $M, s \vDash_g R(t_1, ..., t_n) \iff$ f.a. $w \in s$, $\langle [t_1]_g^w, ..., [t_n]_g^w \rangle \in I_w(R)$

•$\underline{=}$: $M, s \vDash_g t_1 = t_2 \iff$ f.a. $w \in s$, $[t_1]_g^w \sim_w [t_2]_g^w$

•$\underline{\wedge}$: $M, s \vDash_g \varphi \wedge \psi \iff M, s \vDash_g \varphi$ and $M, s \vDash_g \psi$

•$\underline{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$: $M, s \vDash_g \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff M, s \vDash_g \varphi$ or $M, s \vDash_g \psi$

•$\underline{\to}$: $M, s \vDash_g \varphi \to \psi \iff$ f.a. $t \subseteq s$, $M, t \vDash_g \varphi$ implies $M, t \vDash_g \psi$

•$\underline{\forall}$: $M, s \vDash_g \forall x \varphi \iff$ f.a. $d \in D$, $M, s \vDash_{g[x \mapsto d]} \varphi$

•$\underline{\exists}$: $M, s \vDash_g \exists x \varphi \iff$ f.s. $d \in D$, $M, s \vDash_{g[x \mapsto d]} \varphi$

Note that we will drop the reference to the model M when no confusion arises. In such cases, we will simply write $s \vDash_g \varphi$.

Observation. Given a state s in a model M, an assignment g over M, a formula $\varphi \in \mathcal{L}$ and a set of formulas $\Phi \subseteq \mathcal{L}$, we introduce the following notation:

- $M, s \vDash \varphi$ (or simply $s \vDash \varphi$) meaning that, f.a. assignments g over M, $M, s \vDash_g \varphi$;
- $M \vDash_g \varphi$, meaning $M, W \vDash_g \varphi$;
- $M, s \vDash_g \Phi$, meaning that, for all $\psi \in \Phi$, $M, s \vDash_g \psi$.

The special case of a singleton state occupies an important position in the semantics of InqBQ. We will address this fundamental role later on. In the meanwhile, let us introduce a useful definition and a first result involving the semantics at individual worlds.

1.8 DEFINITION (Truth at a world)**.** We say that a formula φ is true at a world w if it is supported by the singleton state $\{w\}$. To indicate this, we will write $M, w \vDash \varphi$ (or simply $w \vDash \varphi$) as an abbreviation for $M, \{w\} \vDash \varphi$.

Truth at a world is an important notion for working with classical formulas, as evidenced by the following proposition:

1.9 PROPOSITION (Truth conditionality of classical formulas)**.** *Let $\alpha \in \mathcal{L}^{cl}$, let s be a state in a model M and let g be any assignment over M. Then,*

$$M, s \vDash_g \alpha \iff M, w \vDash_g \alpha \textit{ for all } w \in s$$

Having introduced the notion of support for InqBQ, let us also define the notion of entailment:

1.10 DEFINITION (InqBQ entailment)**.** Let Φ and ψ be formulas of $\mathcal{L}$. We say that Φ logically entails ψ ($\Phi \vDash_{\mathsf{InqBQ}} \psi$) if, for any model M, for any state s and for any assignment g,

$$\text{if } M, s \vDash_g \Phi, \text{ then } M, s \vDash_g \psi$$

Note that we will simply write $\Phi \vDash \psi$ when no ambiguity with other entailment relations arises.

1.11 DEFINITION (Validity)**.** We say that φ is a validity of InqBQ (which we will write as $\vDash_{\mathsf{InqBQ}} \varphi$) if, for all models M,

$$M \vDash \varphi$$

In the following section, we explain this setup and show its expressive power with both direct observations and basic examples.

1.1.3 Illustration

1. Basic examples It is useful to begin our illustration from the more familiar case of classical formulas. As stated in Proposition 1.9, these formulas are truth conditional, which means that their support conditions at a state boil down to truth conditions at each world of the state. Let's see this in practice with a few examples.

1.12 Example (Classical formulas)**.** Consider the formula $p \to q$, where p and q are propositional atoms. This is a closed formula, so we can omit the reference to an assignment. Let's then unpack its support conditions:

$$\begin{aligned} M, s \vDash p \to q &\iff \text{f.a. } t \subseteq s,\ M, t \vDash p \text{ implies } M, t \vDash q \\ &\iff \text{f.a. } t \subseteq s,\ (\text{f.a. } w \in t,\ w \vDash p) \text{ implies } (\text{f.a. } w \in t,\ w \vDash q) \\ &\iff \text{f.a. } w \in s,\ w \vDash p \text{ implies } w \vDash q \\ &\iff \text{f.a. } w \in s,\ w \nvDash p \text{ or } w \vDash q \end{aligned}$$

In the last step, the $\implies$ direction is obvious, since the singletons of s are also subsets of s, while the converse holds thanks to the truth conditionality of propositional atoms.

The result confirms our expectations and, indeed, the support conditions have reduced to the classical truth conditions of $p \to q$.

To see how the system faithfully interprets classical quantification, let's consider the example of $\forall x P(x)$ for a unary predicate P. As we mentioned in the introduction, this formalizes the statement "*all objects are P.*". Let's verify this explicitly. Let $M = \langle W, D, I, \sim \rangle$ be a model and g an assignment, then

$$\begin{aligned} M, s \vDash_g \forall x P(x) &\iff \text{f.a. } d \in D,\ M, s \vDash_{g[x \mapsto d]} P(x) \\ &\iff \text{f.a. } d \in D, \text{f.a. } w \in s,\ [x]^w_{g[x \mapsto d]} \in I_w(P) \\ &\iff \text{f.a. } d \in D, \text{f.a. } w \in s,\ d \in I_w(P) \\ &\iff \text{f.a. } w \in s, \text{f.a. } d \in D,\ d \in I_w(P) \end{aligned}$$

Indeed, this is verified only when, at each world, all objects are Ps.

1.13 Example (Classical and inquisitive disjunctions)**.** To illustrate the difference in the behaviour of the inquisitive disjunction $\mathbin{\vee\mkern-10mu\vee}$ and of the classical disjunction $\vee$, let us recall the simple situation we considered earlier in Figure 3.

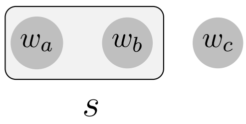


Figure 1.1: A state that distinguishes $\mathbin{\vee\mkern-10mu\vee}$ and $\vee$

Here, intuitively, $w_a \vDash a$ and $w_a \nvDash b, c$, $w_b \vDash b$ and $w_b \nvDash a, c$ and $w_c \vDash c$ and $w_c \nvDash a, b$.
This simple example is sufficient to show that the interpretations of the two symbols can differ at the level of states. The state $s = \{w_a, w_b\}$ supports $a \vee b$. Indeed, $w_a \vDash a$, so $w_a \vDash a \vee b$ and $w_b \vDash b$, so $w_b \vDash a \vee b$. However, clearly, s doesn't support $a \mathbin{\vee\mkern-10mu\vee} b$, since $w_a \nvDash b$, which implies that $s \nvDash b$, and $w_b \nvDash a$, which implies that $s \nvDash a$.

We have seen a couple of examples showing the behaviour of the system with respect to classical formulas. Now, we move on to more complex formulas involving the inquisitive symbols to illustrate the expressive capabilities of InqBQ.

2. Polar questions Before considering the case of quantifiers, let us see an example of the expressive power granted by the inquisitive disjunction $\mathbin{\vee\mkern-10mu\vee}$. In this example, we consider the derived symbol ? in conjunction with classical formulas.

1.14 Example (Polar questions)**.** Let $\alpha \in \mathcal{L}^{cl}$ be a classical formula. Then $?\alpha$ expresses the polar question "*whether* α". This question is intuitively settled by a state when the state supports either α or its negation. Since α is a classical formula, our intuition suggests that the question is then settled at a state if either α is true at every world of the state or α is false at every world of the state. Indeed, if we let s be a state in a model M, the semantics confirm these expectations:

$$\begin{aligned} M, s \vDash ?\alpha &\iff M, s \vDash \alpha \vee\!\!\!\vee \neg\alpha && \text{(By definition)} \\ &\iff M, s \vDash \alpha \text{ or } M, s \vDash \neg\alpha \\ &\iff M, w \vDash \alpha \text{ f.a. } w \in s \text{ or } M, w \vDash \neg\alpha \text{ f.a. } w \in s && \text{(By Prop. 1.9)} \\ &\iff M, w \vDash \alpha \text{ f.a. } w \in s \text{ or } M, w \nvDash \alpha \text{ f.a. } w \in s \end{aligned}$$

A particularly interesting instance of this type of questions are formulas of the form $?Px$, asking whether P holds for x or not; equivalently, this can be seen as asking whether x is in the extension of P or, in a less rigorous interpretation, whether x is a P. This specific formulation will appear in various contexts.
Notice that the formula $?\alpha$ also serves as an additional example of the difference between the semantics of $\vee\!\!\!\vee$ and the semantics of $\vee$. The support conditions of $\alpha \vee\!\!\!\vee \neg\alpha$ can clearly be falsified by a state with two worlds disagreeing on the truth of α. $\alpha \vee \neg\alpha$, on the other hand, is supported by any state, since it is a classical formula and is true at every world.

3. Mention-some questions This example shows the expressive power of the inquisitive existential quantifier $\exists\!\!\!\exists$. Even in its most basic application, this quantifier allows for the expression of a notable type of questions.

1.15 Example (Mention-some questions)**.** Given a unary predicate P, consider the formula $\exists\!\!\!\exists x Px$. Let's take a generic state s to unpack its support conditions and understand its intuitive interpretation:

$$\begin{aligned} M, s \vDash_g \exists\!\!\!\exists x Px &\iff \text{f.s. } d \in D,\ M, s \vDash_{g[x \mapsto d]} Px \\ &\iff \text{f.s. } d \in D, \text{ f.a. } w \in s,\ M, w \vDash_{g[x \mapsto d]} Px && \text{(by Prop. 1.9)} \\ &\iff \text{f.s. } d \in D, \text{ f.a. } w \in s,\ d \in I_w(P) \\ &\iff \text{there is some } d \in D \text{ such that } d \in I_w(P) \text{ for all } w \in s, \end{aligned}$$

Intuitively, this means that to answer this question one must provide an individual that satisfies the property P in every world of the state. This question then amounts to asking for an *instance of* P and is usually called a *mention-some* question.

4. Mention-all questions The interaction of classical quantifiers with inquisitive symbols also allows for the expression of interesting questions, even in very simple cases.

1.16 Example (Mention-all questions)**.** Consider, given a unary predicate P, the sentence $\forall x?Px$. Let's unravel its semantics:

$$\begin{aligned}
M,s \vDash_g \forall x?Px &\iff \text{f.a. } d \in D,\ M,s \vDash_{g[x\mapsto d]} ?Px \\
&\iff \text{f.a. } d \in D,\ M,s \vDash_{g[x\mapsto d]} Px \text{ or } M,s \vDash_{g[x\mapsto d]} \neg Px \\
&\iff \text{f.a. } d \in D, (\text{f.a. } w \in s,\ M,w \vDash_{g[x\mapsto d]} Px) \text{ or} \\
&\qquad\qquad (\text{f.a. } w \in s,\ M,w \vDash_{g[x\mapsto d]} \neg Px) \\
&\iff \text{f.a. } d \in D,\ (\text{f.a. } w \in s, d \in P_w) \text{ or } (\text{f.a. } w \in s, d \notin P_w) \\
&\iff \text{f.a. } d \in D,\ \text{f.a. } w, w' \in s, (d \in P_w \iff d \in P_{w'}) \\
&\iff \text{f.a. } w, w' \in s, \text{f.a. } d \in D, (d \in P_w \iff d \in P_{w'}) \\
&\iff \text{f.a. } w, w' \in s, P_w = P_{w'}
\end{aligned}$$

The support conditions of $\forall x?Px$ are then satisfied only when the extension of P is the same at all worlds in the state. Intuitively, to answer this question, one must then have enough information to know exactly which individuals have the property P and which individuals don't. The formula can therefore be seen as a formalization of the questions "*which individuals have the property P*" or "*what is the extension of P*". Such questions are usually referred to as *mention-all questions.*
Note that the same conclusions can also be reached for relation symbols with arity greater than 1.

5. Questions involving identity The addition of identity atoms also extends the expressive capabilities of the system. The interaction of identity formulas with inquisitive symbols makes this extension more significant than the one achieved by the same symbol in classical logic.
One example of the kinds of questions expressible in InqBQ thanks to the introduction of identity is the following:

1.17 Example (Identification questions)**.** Let t be a term not containing x. Consider the sentence $\exists x(x = t)$. Semantically, its support conditions are:

$$\begin{aligned}
M,s \vDash_g \exists x(x = t) &\iff \text{f.s. } d \in D,\ M,s \vDash_{g[x\mapsto d]} x = t \\
&\iff \text{there is a } d \in D \text{ such that for all } w \in s,\ d \sim_w [t]_g^w
\end{aligned}$$

Therefore, the formula is supported by a state only when the term t denotes the same individual d of the domain at every world in the state. Since resolving this question then amounts to providing an individual who is the t at every possible world in the state, we call these expressions *identification questions.* To make this more concrete, consider for example the question "*Who is Alice's father?*", which can be regimented as $\exists x(x = f(a))$.

These types of question will also be considered in the following sections, so let us introduce the abbreviation

$$\lambda t := \exists x(x = t)$$

for any term t not containing x.

The introduction of identity also allows for the expressions of other types of questions, which we will simply mention, such as *unique instance* questions and *mention-n* questions. These are interrogative counterparts of the *there is a unique P* and *there are at least n Ps* assertions, which can be expressed by first order classical logic.

6. Entailments As the last section of our illustration of InqBQ, let us take into consideration a few examples of entailment holding between questions.

1.18 Example. Let p and q be two propositional atoms. Consider the following entailment:

$$p \leftrightarrow \neg q,\ \ ?q \vDash ?q$$

Making the support conditions explicit easily shows the validity of the entailment. From a more intuitive point of view, however, we are also satisfied by this result. If in a state we know that the truth value of p is the opposite of that of q (a consequence of $p \leftrightarrow \neg q$) and the truth value of q is settled by the state ($?q$), it is reasonable to expect that the truth value of p is also settled at that state ($?p$).
Now, take P and Q to be unary predicates. We can then consider the similar, more complex valid entailment involving mention-all questions:

$$\forall x(P(x) \leftrightarrow \neg Q(x)), \forall x?Q(x) \vDash \forall x?P(x)$$

Intuitively, if we have settled that P and Q are complements and we have settled the extension of Q, we have also settled the extension of P.

The following example show the case of an entailment involving the inquisitive existential quantifier and identification questions.

1.19 Example. Let f be a rigid binary function symbol, let a, b, c be non-rigid constants. Then,

$$a = \mathsf{f}(b, c), \lambda b, \lambda c \vDash \lambda a$$

This is reasonable. If in a state we have settled the value of b and the value of c, it means that these values don't change across worlds. Then, if we also know that the value of a is the same as the one indicated by the interpretation of f (constant across worlds) when applied to the values of b and c (also constant across worlds), we can see that the value of a must also be constant across worlds and, therefore, λa must be settled.
The rigidity of f is crucial. Without it, the value of $\mathsf{f}(b, c)$ could change between worlds even if the value of b and c doesn't.

We have seen various examples of the expressive power of InqBQ, as well as some examples of how it confirms our intuitions about entailments between questions. Let's now move on to consider some fundamental properties of this system and some results which we will use in the following chapters.

1.1.4 Basic properties and useful results

1. Fundamental properties
To begin with, let us start from the two key properties that support must verify:

1.20 PROPOSITION.

1. ***Empty state property:*** *for all* $\varphi \in \mathcal{L}$, $\emptyset \vDash \varphi$
2. ***Persistency:*** *for all models* M, *states* s, *assignments* g *and formulas* $\varphi \in \mathcal{L}$, *if* $s \vDash_g \varphi$ *and* $t \subseteq s$, *then* $t \vDash_g \varphi$

These are considered to be the most basic, necessary features of support. Indeed, their intuitive understanding justifies this request. As mentioned in the introduction, one can see states as representing available information. In this interpretation, worlds in a state are viewed as states of affairs compatible with the given information. The support of a formula at a state then models the possibility of either settling or verifying the questions or facts that the formula expresses about the state of affairs with the information at hand. The empty state can be viewed as the representation of inconsistent information. Similarly to the *ex falso quodlibet* principle of classical logic, it is natural to assume that one can establish anything from inconsistent information. As for persistency, the restriction of a state can be seen as an expansion of the available information (which corresponds to a restriction of the set of compatible states of affairs). It is then natural that, if it is possible to establish certain properties about the state of affairs when given certain information, it will still be possible to do so when new, additional information is introduced.

Another desideratum of any inquisitive system is for entailment between classical formulas to hold if and only if it holds in standard first order classical logic. To show this, we recall the notion of truth at a world (Definition 1.8). The following proposition, together with the truth conditionality of classical formulas (stated in Proposition 1.9), ensures this property.

1.21 PROPOSITION. *For any classical formula* $\alpha \in \mathcal{L}^{cl}$, *for any model* M, *world* w, *and assignment* g,

$$M, w \vDash_g \alpha \iff \tilde{M}_w \vDash_{\tilde{g}_w} \alpha \text{ in first order classical logic}$$

where $\tilde{g}_w$ *is the assignment given by* $\tilde{g}_w(x) = [g(x)]_{\sim_w}$

Another important property of InqBQ is locality. In words, this property translates to the fact that support at a state exclusively depends on words in that state.

1.22 DEFINITION (Restriction of a model)**.** Given a model $M = \langle W, D, I, \sim\rangle$ of InqBQ and a state s in M, we define the restriction of M to s as the model

$$M|_s = \langle s, D, I|_s, \sim|_s\rangle$$

1.23 PROPOSITION (Locality)**.** *Given a model $M = \langle W, D, I, \sim\rangle$ of InqBQ, a state s in M, an assignment g and a formula $\varphi \in \mathcal{L}$,*

$$M, s \vDash_g \varphi \iff M|_s, s \vDash_g \varphi$$

Let us also state a common property of many logical systems: assignments are irrelevant when we are dealing with closed formulas.

1.24 PROPOSITION**.** *If φ is a closed formula, then for any model M, state s and assignment g,*

$$M, s \vDash_g \varphi \iff M, s \vDash \varphi$$

2. Useful results: Id-models

Although we are interested in being able to express uncertainty about the extension of the identity relation, it is also often useful to restrict our attention to models where identity has its usual extension. We call these models *id-models.*

1.25 DEFINITION (Id-models)**.** A model $M = \langle W, D, I, \sim\rangle$ is called an id-model if

$$\sim_w = id_D \text{ for all } w \in W$$

where $id_D = \{\langle d, d\rangle \mid d \in D\}$.

In cases where our interests allow us to restrict to id-models, we can also work with a stronger entailment relation.

1.26 DEFINITION (Id-entailment)**.** Given Φ and ψ formulas of $\mathcal{L}$, we say that Φ id-entails ψ (which we write as $\Phi \vDash_{id} \psi$) when for all id-models M, all states s and all assignments g, if $M, s \vDash_g \Phi$ then $M, s \vDash_g$.
If a formula φ satisfies $\vDash_{id} \varphi$, we say that φ is an id-validity.

1.27 Example. An example of an id-validity is the formula $\forall x \forall y ?(x = y)$. Indeed, let's spell out its support conditions. Let M be an id model and s a state (we can suppress the reference to an assignment since the formula is closed), then

$$
\begin{aligned}
M, s \models \forall x \forall y ?(x = y) &\iff \text{f.a. } d, d' \in D,\ M, s \models_{[x \mapsto d, y \mapsto d']} ?(x = y) \\
&\iff \text{f.a. } d, d' \in D,\ M, s \models_{[x \mapsto d, y \mapsto d']} x = y \text{ or} \\
&\qquad\qquad\qquad\qquad M, s \models_{[x \mapsto d, y \mapsto d']} \neg(x = y) \\
&\iff \text{f.a. } d, d' \in D,\ (\text{f.a. } w \in s,\ d \sim_w d'\ ) \text{ or} \\
&\qquad\qquad\qquad\qquad (\text{f.a. } w \in s,\ d \nsim_w d'\ ) \\
&\iff \text{f.a. } w, w' \in s,\ \sim_w \equiv \sim_{w'}
\end{aligned}
$$

That is, $\forall x \forall y ?(x = y)$ is supported by a state if and only if the extension of the identity relation is the same in all worlds of the state. Since this is trivially true in id-models, the formula is an id-validity.

The previous example, as one could have imagined from the short formulation of its support conditions, is actually a characterizing formula of a rather interesting class of models. Let us describe these models and their main properties.

1.28 DEFINITION (Models with decidable identity). We say that a model $M = \langle W, D, I, \sim\rangle$ of InqBQ has decidable identity if, for all $w, w' \in W$, $\sim_w \equiv \sim_{w'}$.

The role of the previous example is made explicit by the following statement:

1.29 PROPOSITION (Characterization of models with decidable identity). *A model M has decidable identity* $\iff$ $M \models \forall x \forall y ?(x = y)$

The most interesting feature of models with decidable identity is their relation with id-models. Clearly, id-models have decidable identity; however, the converse is not true in general. The following definition and proposition give a clear picture of how we can go from models with decidable identity to id-models in a natural way.

1.30 DEFINITION (Id-contract). Let $M = \langle W, D, I, \sim\rangle$ a model with decidable identity and let us write $\sim$ for any $\sim_w$ and $[d]$ for any $[d]_{\sim_w}$. We define the id-contract of M as the model $M_{id} = \langle W, D/\sim, \tilde{I}, \approx\rangle$ where

- $\tilde{I}_w(f)([d_1], ..., [d_n]) = [I_w(f)([d_1], ..., [d_n])]$
- $\langle [d_1], ..., [d_n]\rangle \in \tilde{I}_w(R) \iff \langle d_1, ..., d_n\rangle \in I_w(R)$
- $\approx_w$ is the standard identity relation on $D/\sim$ for all $w \in W$

Let us also define, given an assignment g, the assignment $g_{id} : Var \to D/\sim$ defined by $g_{id}(x) = [g(x)]$.

Id-contraction preserves the satisfaction of formulas, as formalized by the following proposition:

1.31 PROPOSITION. *Let M be a model with decidable identity, s a state and g and assignment. Then, for any formula φ,*

$$M, s \vDash \varphi \iff M_{id}, s \vDash_{g_{id}} \varphi$$

Models with decidable identity also allow us to simulate id-entailment:

1.32 PROPOSITION. *For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}$,*

$$\Phi \vDash_{id} \psi \iff \Phi, \forall x \forall y ?(x = y) \vDash \psi$$

3.Useful results: Essential equivalence

For the last set of useful results that we will consider, we will turn our attention to the fact that, based on the provided semantics of InqBQ, it is possible for two worlds to be associated the same relational structure. Intuitively, the distinction between these two worlds should then be irrelevant towards the support of formulas. Indeed, this intuition is confirmed by the theory. For a more complete treatment of this topic, we refer to [Gri20]. The results we describe here are weaker than those proved in Grilletti's presentation of essential equivalence, where he also considers a notion of essential equivalence for individuals. Nonetheless, they are sufficient for the needs of our thesis, they follow immediately from their more general counterparts as a special case and they allow for a simpler, shorter exposition.

Let us start by giving a formal definition of this equivalence between worlds:

1.33 DEFINITION (Essential equivalence between worlds). Given a model $M = \langle W, D, I, \sim \rangle$, we say that two worlds $w, w' \in W$ are *essentially equivalent* (which we write as $w \approx_e w'$) if $\tilde{M}_w = \tilde{M}_{w'}$,
where, for all $w \in W$, $\tilde{M}_w$ is the relational structure associated to w as described in the observation to Definition 1.4.
We also define W^e as the quotient $W/\approx_e$ and, for any state s, s^e as the set of equivalence classes of worlds in s, i.e. $\{[w]_{\approx_e} \mid w \in s\}$.
When no confusion arises, we will use $[w]$ to refer to $[w]_{\approx_e}$.

Using these definitions, we can construct the essential quotient of a model:

1.34 DEFINITION (Essential quotient of a model). Let $M = \langle W, D, I, \sim \rangle$ be a model, we define its essential quotient as $M^e = \langle W^e, D, I^e, \sim^e \rangle$, where, for all $w \in W$,

- $I^e_{[w]}(f)(d_1, ..., d_n) = I_w(f)(d_1, ..., d_n)$
- $\langle d_1, ..., d_n \rangle \in I^e_{[w]}(R) \iff \langle d_1, ..., d_n \rangle \in I_w(R)$

- $\sim^e_{[w]} \equiv \sim_w$

M^e is well defined thanks to the definition of essential equivalence of worlds (and, therefore, of the classes $[w]$).

The essential quotient of a model behaves exactly as expected: its states support the same propositions as their equivalent counterparts found in the original model. We make this idea more precise with the following lemma.

1.35 LEMMA. *Let M be a model, s a state and g an assignment. Then, for any formula φ,*

$$M, s \vDash_g \varphi \iff M^e, s^e \vDash_g \varphi$$

With this result, we conclude our overview of InqBQ. It is now time for the formal introduction of the second logical system that we will consider in this thesis, InqBT.

1.2 The system InqBT

This section serves as an introduction to the system InqBT. We will cover its definition and a very basic illustration. As we did in the case of InqBQ, we will also state the key properties of the system and some results which will turn out to be useful in the following chapters. For a more detailed treatment of InqBT, we refer to [Cia22b].
Before moving on to a formal description of the system, it is useful to provide an informal description of the idea behind its construction.
InqBT aims to provide a system with structure, behaviour and expressivity comparable to those of InqBQ, but equipped with team-based semantics. Intuitively, it's a good starting point to keep in mind the idea of substituting states (i.e. sets of possible worlds) with teams (i.e. sets of assignments) and, in particular, exchanging the role of a single possible world with that of an assignment. Clearly, the two entities differ in their behaviour but, as we will see, this approach produces a sufficiently similar environment.

1.2.1 Syntax

A signature for InqBT is defined as a set of symbols Σ in which we identify a subset $\mathcal{R}_\Sigma$ of relation symbols and a subset $\mathcal{F}_\Sigma$ of function symbols. There is no need to distinguish between rigid and non-rigid function symbols. This will become clear with the definition of the semantics for the system.
Given a signature Σ, the language of InqBT is simply defined as $\mathcal{L}(\Sigma)$. The definition of classical formulas ($\mathcal{L}^{cl}$) and the definition of free and bound variables also carry over straightforwardly to InqBT.

1.2.2 Semantics

The semantics of InqBT also mirrors the one provided for InqBQ. There are, however, significant differences in the overall definition.

1.36 DEFINITION (Model)**.** Models for InqBT are nothing more than the standard relational structures used in first order classical logic. Formally, a model for InqBT is then a couple $M = \langle D, I\rangle$, where

- D is a non-empty set, whose elements we refer to as individuals
- I is a function with domain Σ, satisfying the following constraints:
  - $I(R_n) \subseteq D^n$, for any $R_n \in \mathcal{R}_\Sigma$
  - $I(f_n) : D^n \to D$, for any $f \in \mathcal{F}_\Sigma$

Assignments are defined in the usual way. However, it is useful to define the crucial notion of a team.

1.37 DEFINITION (Team)**.** Given a model M for InqBT, a team T for M is a set of assignments for M.
Given $d \in D$, we will use the notation $T[x \mapsto d] := \{g[x \mapsto d] \mid g \in T\}$

1.38 Observation. Note that, while in InqBQ different possible worlds could be associated to an identical relational structure by the interpretation and identity extension functions (rendering them indistinguishable in practice), all assignments in a team must be distinct from each other. This is not a requirement we impose on a team, but simply a direct consequence of the definition.

The evaluation of terms and formulas is performed with respect to both models and assignments.

1.39 DEFINITION**.** The referent of a term t in a model M under an assignment g is defined inductively as the individual $[t]_g^M \in D$ by the following clauses:

•$\underline{Var}$: $[x]_g^M = g(x)$

•$\underline{f_n}$: given $t_1, ..., t_n \in Ter(\Sigma)$, $[f_n(t_1), ..., f_n(t_n)]_g^M = I(f_n)([t_1]_g^M, ..., [t_n]_g^M)$

The semantic clauses for the formulas of $\mathcal{L}$ in InqBT are identical to those given in the case of InqBQ. The only difference is that now the role of the states is taken on by the teams.

1.40 DEFINITION (Support for InqBT)**.** Let $M = \langle D, I \rangle$ be a model of InqBT and let T be a team for M. Then, we define support for a formula of $\mathcal{L}$ inductively as follows:

•$\underline{\bot}$: $M \vDash_T \bot \iff s = \emptyset$

•$\underline{R}$: $M \vDash_T R(t_1, ..., t_n) \iff$ f.a. $g \in T, \ \langle [t_1]_g^M, ..., [t_n]_g^M \rangle \in I(R)$

•$\underline{=}$: $M \vDash_T t_1 = t_2 \iff$ f.a. $g \in T, \ [t_1]_g^M = [t_2]_g^M$

•$\underline{\wedge}$: $M \vDash_T \varphi \wedge \psi \iff M \vDash_T \varphi$ and $M \vDash_T \psi$

•$\underline{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$: $M \vDash_T \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff M \vDash_T \varphi$ or $M \vDash_T \psi$

•$\underline{\rightarrow}$: $M \vDash_T \varphi \rightarrow \psi \iff$ f.a. $T' \subseteq T, \ M \vDash_{T'} \varphi$ implies $M \vDash_{T'} \psi$

•$\underline{\forall}$: $M \vDash_T \forall x \varphi \iff$ f.a. $d \in D, \ M, s \vDash_{T[x \mapsto d]} \varphi$

•$\underline{\exists}$: $M \vDash_T \exists x \varphi \iff$ f.s. $d \in D, \ M, s \vDash_{T[x \mapsto d]} \varphi$

Observation. Let $\varphi \in \mathcal{L}$ and let M be a model. We introduce the notation $M \vDash \varphi$, meaning that, for any team T, $M \vDash_T \varphi$

1.41 DEFINITION (InqBT entailment)**.** Let Φ and ψ be formulas of $\mathcal{L}$. We say that Φ logically entails ψ ($\Phi \vDash_{\mathsf{InqBT}} \psi$) if, for any model M and for any team T,

$$\text{if } M \vDash_T \Phi, \text{ then } M \vDash_T \psi$$

Note that we will simply write $\Phi \vDash \psi$ when no ambiguity with other entailment relations arises.

1.42 DEFINITION (Validity)**.** We say that φ is a validity of InqBT (which we will write as $\vDash_{\mathsf{InqBT}} \varphi$) if, for all models M,

$$M \vDash \varphi$$

.

1.2.3 Properties

Basic properties Unlike in the case of InqBT, we will first describe the properties of InqBT and only later illustrate its expressive capabilities.
Let us start from the fact that support in InqBT satisfies the two fundamental properties:

1.43 PROPOSITION**.**

1. ***Empty state property:*** *for all models M and formulas $\varphi \in \mathcal{L}$, $M \vDash_{\emptyset} \varphi$*
2. ***Persistency:*** *for all models M, teams T and T' and formulas $\varphi \in \mathcal{L}$, if $M \vDash_T \varphi$ and $T' \subseteq T$, then $M \vDash_{T'} \varphi$*

Free variables On the other hand, there are properties of InqBT with no counterpart in InqBQ. This is due to the fact that InqBT only models uncertainty about the value of variables. One of these interesting features is that support conditions for a formula at a team only depend on the evaluation of its free variables. The following proposition formalizes this idea.

1.44 PROPOSITION**.** *Let $\varphi \in \mathcal{L}$, let M be a model and let T and T' be teams. If $T|_{FV(\varphi)} = T'|_{FV(\varphi)}$,*

$$M \vDash_T \varphi \iff M \vDash_{T'} \varphi$$

In particular, a notable consequence of this is the fact that closed formulas aren't affected by team evaluation (excluding the case of the empty team).

1.45 PROPOSITION (Sentences in InqBT)**.** *Let $\varphi \in \mathcal{L}$ be a closed formula and let M be a model. Then, for any T, T' non-empty teams over M,*

$$M \vDash_T \varphi \iff M \vDash_{T'} \varphi$$

Classical formulas Singleton teams, like singleton states in InqBQ, have a special position in the semantics of InqBT.

1.46 DEFINITION (Truth at a world)**.** We say that a formula φ is true at a model M under an assignment g if it is supported by M under the singleton team $\{g\}$. To indicate this, we will write $M \vDash_g \varphi$ as an abbreviation for $M \vDash_{\{g\}} \varphi$.

Again, we have that truth under an assignment is a crucial notion for understanding the behaviour of classical formulas:

1.47 PROPOSITION (Truth conditionality of classical formulas)**.** *Let $\alpha \in \mathcal{L}^{cl}$ be a classical formula and let T be a team over a model M. Then,*

$$M \vDash_T \alpha \iff M \vDash_g \alpha \textit{ for all } g \in T$$

Conservativity over classical logic follows from this proposition in conjunction with the observation that truth conditions at a model under an assignment coincide with those of standard first order classical logic. Differently from InqBQ, this observation doesn't require much attention as the association of a singleton team with an assignment is trivial.
A notable difference from InqBQ is that a much greater portion of formulas containing inquisitive symbols are truth conditional. This is due to the structure of formulas in general: while InqBQ is sensitive to the presence of any non-logical symbols, i.e. symbols from the signature and equality, InqBT is only sensitive to appearances of free variables. While non-logical symbols always appear in formulas, this is definitely not the case for free variables. This situation applies in particular to closed formulas:

1.48 DEFINITION (Classical variant)**.** Given a formula $\varphi \in \mathcal{L}$, we define its classical variant, indicated by φ^{cl} as the formula obtained by replacing each appearance of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ with, respectively, $\vee$ and $\exists$.

1.49 PROPOSITION**.** *Let φ be a closed formula. Then,*

$$M \vDash \varphi \iff M \vDash \varphi^{cl}$$

Notable examples of this property holding will be included in the following section, dedicated to an illustration of the system.

1.2.4 Illustration

1. Classical formulas Again, it is constructive to begin our illustration from simple examples of classical formulas, to see the system in action in a familiar environment.

1.50 Example (Classical formulas)**.** First, let's consider an instance of a closed classical formula, $\forall x P(x)$ for a unary predicate P, to demonstrate how both truth conditionality and the independence from teams emerge when calculating its support conditions. Let T be a team over a model $M = \langle D, I \rangle$, then

$$\begin{aligned} M \vDash_T \forall x P(x) &\iff \text{f.a. } d \in D,\ M \vDash_{T[x \mapsto d]} P(x) \\ &\iff \text{f.a. } d \in D,\ \text{f.a. } g \in T[x \mapsto d],\ [x]_g^M \in I(P) \\ &\iff \text{f.a. } d \in D, d \in I(P) \end{aligned}$$

The classical informal interpretation of the formula is still respected. $\forall x P(x)$ can be seen as a formalization of "*All objects are P.*" and, therefore, its support should only depend on the model. Accordingly, the derived conditions are not affected by changing the team.
A similar example with a free variable shows how, even when teams play a role, the evaluation of classical formulas is still preserved. Consider the formula $P(x)$. Intuitively, this formula expresses the sentence "*x is a P.*". Let's unpack its support conditions:

$$\begin{aligned} M \vDash_T P(x) &\iff \text{f.a. } g \in T, [x]_g^M \in I(P) \\ &\iff \text{f.a. } g \in T, g(x) \in I(P) \end{aligned}$$

Which means that the formula is supported when, under any assignment of the team, the evaluation of x is a P, confirming our expectations.

2. Questions As a first example of expressible questions in InqBT, let's consider the case of polar questions.

1.51 Example (Polar questions)**.** Take $\alpha \in \mathcal{L}^{cl}$. Let's expand the support conditions of $?\alpha$.

$$\begin{aligned} M \vDash_T ?\alpha &\iff M \vDash_T \alpha \text{ or } M \vDash_T \neg\alpha \\ &\iff (\text{f.a. } g \in T, M \vDash_g \alpha) \text{ or } (\text{f.a. } g \in T, M \vDash_g \neg\alpha) && \text{(By truth cond.)} \end{aligned}$$

Indeed this means that all assignments in the team must agree on the truth value of α. Therefore, for $?\alpha$ to be supported by a team, the team must settle the polar question "*whether* α".

As we mentioned in the introduction, the types of questions that InqBT can express about variables are similar to those that InqBQ expresses about the state of affairs. As our next example, we therefore consider the familiar cases of mention-some and mention-all questions. Of course, since we used closed formulas to illustrate them in InqBQ, the examples will have to be different.

1.52 Example (Mention-some and mention-all questions)**.** As an example of a mention-some question, consider the formula $\exists y R(x,y)$, where R is a binary relation symbol. Its support conditions can be obtained as follows:

$$\begin{aligned} M \vDash_T \exists y R(x,y) &\iff \text{f.s. } d \in D,\ M \vDash_{T[y \mapsto d]} R(x,y) \\ &\iff \text{f.s. } d \in D,\ \text{f.a. } g \in T,\ \langle g(x), d\rangle \in I(R) \end{aligned}$$

That is, $\exists y R(x,y)$ is supported exactly when some specific object $d \in D$ is R-related to $g(x)$ under all assignments g in T. This means that the team T settles the question "*what is an object that is R-related to x?*".
We can make this more concrete by considering an explicit example. We will look at the evaluation into the domain of natural numbers of the variable x under three teams T_1, T_2 and T_3, all composed of four assignments.

T_1	x
g_1	2
g_2	3
g_3	6
g_4	7

T_2	x
g'_1	2
g'_2	6
g'_3	22
g'_4	4

T_3	x
g''_1	2
g''_2	4
g''_3	8
g''_4	16

Figure 1.2: The values of x under the teams T_1, T_2 and T_3.

Suppose that $R(x,y)$ is the relation that holds only if y is a prime factor of x. Then, $\exists y R(x,y)$ reads as "*What is a prime factor of x?*" which is settled by a team if and only if there is a number which is a prime factor for the value of x under all assignments in the team. In Figure 1.2, T_2 and T_3 verify this condition, having 2 as the common prime factor, while T_1 clearly doesn't.

Using binary relation symbols, we can also construct mention-all question. An example is the formula $\forall y ?R(x,y)$, formalizing the question "*what objects are R-related to x?*". Let's study its semantics to verify this fact:

$$\begin{aligned} M \vDash_T \forall y ?R(x,y) &\iff \text{f.a. } d \in D,\ M \vDash_{T[y \mapsto d]} ?R(x,y) \\ &\iff \text{f.a. } d \in D,\ (M \vDash_{T[y \mapsto d]} R(x,y)) \text{ or} \\ &\qquad\qquad (M \vDash_{T[y \mapsto d]} \neg R(x,y)) \\ &\iff \text{f.a. } d \in D,\ (\text{f.a. } g \in T[y \mapsto d],\ \langle g(x), d\rangle \in I(R)) \text{ or} \\ &\qquad\qquad (\text{f.a. } g \in T[y \mapsto d],\ \langle g(x), d\rangle \notin I(R)) \end{aligned}$$

The last condition is satisfied if all assignments agree, for any object d in the domain, on whether $\langle g(x), d\rangle$ belongs to $I(R)$ or not. That is, if they agree on what objects are and what objects aren't R-related to x. Again,

taking $R(x, y)$ to mean that y is a prime factor of x, $\forall y?R(x, y)$ translates to "*What are the prime factors of x?*". In the situation represented by Figure 1.2, this is settled only by team T_3, where the only prime factor of the value of x is 2.
It is interesting to note that, in both examples, the questions that are expressed by these formulas explicitly mention a variable. This is in accordance with the fact that InqBT can only express questions about the value of variables and doesn't model uncertainty about the state of affairs.

3. Dependencies The case of identification questions, which we previously illustrated for InqBQ, takes on a more prominent role in InqBT. This is due to both the motivations behind the development of the system and its relations with dependence logic.

1.53 Example (Identification questions)**.** Identification questions of the form "*who is the t?*, where t is a generic term, can be captured by the same formulas that formalized them in InqBQ. Given a term t, we define

$$\lambda t := \exists x(x = t)$$

The semantics of these formulas shows their correspondence with the informal interpretation:

$$\begin{aligned} M \vDash_T \exists x(x = t) &\iff \text{f.s. } d \in D,\ M \vDash_{T[x \mapsto d]} x = t \\ &\iff \text{f.s. } d \in D,\ \text{f.a. } g \in T[x \mapsto d],\ [t]^M_g = d \end{aligned}$$

That is, λt is supported by a team only if there is some object in the domain that coincides with the evaluation of t under all assignments of the team. Informally, this means that one can identify t.
Notice that, when the term t is just a variable y, $[y]^M_g$ simply reduces to $g(y)$. Therefore, λy is supported when the value of y is determined by the team.

The conclusion we reached for λy makes it clear that this is exactly the type of formula that one needs to retrieve the dependency atoms of dependence logic in an inquisitive setting. As we mentioned in the introduction, one can then express dependencies such as "*the value of x determines the value of y* with the formula $\lambda x \to \lambda y$ and more general dependencies of the form "*the values of* $x_1, ..., x_n$ *determine the value of* y with $(\lambda x_1 \wedge ... \wedge \lambda x_n) \to \lambda y$.

It is worth noting, however, that the expressive power of InqBT goes further, allowing the system to capture more complex relations between variables and, especially, between questions involving them.

1.54 Example (Questions with dependencies)**.** Having seen how we can express questions about the values of variable and dependencies between them, the idea of combining the two is the most natural development. Let's see some interesting examples of relations that InqBT can formalize. This can be best achieved by referring to a specific concrete situation. Let's consider the following evaluation of x and y under four teams T_1, T_2, T_3 and T_4 into the domain of natural numbers. To avoid cluttering the picture we will not give a specific name to each assignment.

T_1	x	y
	1	2
	1	4
	2	6
	3	9
	3	15
	4	1

T_2	x	y
	4	3
	4	3
	4	5
	4	5
	6	2
	6	9

T_3	x	y
	1	3
	1	6
	1	9
	2	4
	2	8
	2	10

T_4	x	y
	0	3
	1	3
	2	6
	3	9
	4	8
	5	13

Figure 1.3: The values of x and y under the teams T_1, T_2, T_3 and T_4

Now, assume we have a unary predicate *Even*, which is satisfied by even natural numbers, a binary relation $<$, indicating the usual order on natural numbers and a binary relation R, such that $R(x, y)$ holds only when y is a prime factor of x. Let's formalize some relations involving dependencies and questions and see which of the teams in Figure 1.3 support them:

- "*The value of x determines the parity of y.*"
  - Formulation: $\lambda x \to ?Even(y)$
  - Supported by: $T_1,\ T_4$
- "*The parity of x determines the value of x.*"
  - Formulation: $Even(x) \to \lambda x$
  - Supported by: T_2
- "*The value of x and whether x is less than y determine the value of y.*"
  - Formulation: $(\lambda x \wedge ?(x < y)) \to \lambda y$
  - Supported by: $T_2,\ T_4$
- "*The value of x determines a prime factor of y.*"
  - Formulation: $\lambda x \to \exists z R(y, z)$
  - Supported by: $T_1,\ T_3,\ T_4$

This sample represents just a small fraction of the behaviours and connections of variables that InqBT is able to express. Nonetheless, it is indicative of the impressive range of situations that the system can formalize.

So far, we have observed and utilized only simple dependencies between the values of variables. InqBT, however, can capture another interesting class of dependencies.

1.55 Example (Higher order dependencies)**.** Consider the following example of a team over the domain of natural numbers, containing all assignments of the following form for all n, m natural numbers:

T	x	y	w	z
	n	m	$3n$	$5m$

Figure 1.4: The team T

Under this team (and, therefore, in any of its subteams), if there is a functional dependency between x and y, there must also be a functional dependency between w and z. Now, we have seen that a functional dependency between x and y can be expressed by $\lambda x \to \lambda y$, and the same goes for w and z. This means that InqBT can naturally capture this kind of higher order dependency with formulas such as $(\lambda x \to \lambda y) \to (\lambda w \to \lambda z)$. Clearly, the approach can also be generalized for higher order dependencies involving more than functional dependencies, for instance by integrating the formalizations of variable relations that we presented in the previous example.

The previous subsections have given us an idea of the behaviour and expressivity of InqBT. Now, it is useful to compare the characteristics of this system with those of its inquisitive counterpart.

1.2.5 Relations with InqBQ

As we mentioned, InqBT and InqBQ are constructed in very similar ways. Indeed, InqBT can be seen simply as a version of first order inquisitive logic where worlds and states are replaced by assignments and teams. Thanks to these similarities, many properties of InqBQ carry over straightforwardly. Also, as we will see in the following chapters, this structural correspondence has deep implication on how the systems are connected. It is important, however, to also mention what differences the two systems present. Essentially, these boil down to the fact that the semantic approach of the two systems is similar but applies to different objects. In particular, InqBQ models uncertainty about the state of affairs, i.e. about properties, relations and individuals while InqBT can only represent uncertainty about the value of variables. Thanks to the examples and properties we described, the consequences of this setup difference are now more clear. The way in which formulas of a formal language are constructed means that non-logical vocabulary and free variables are utilized and appear in formulas in different ways. This has important repercussions, especially on InqBT, where, for instance, sentences always coincide with their classical variant. Most other differences that the two systems have also stem from this fundamental distinction in the objects that they model. For a more complete overview of these topics, we again refer to [Cia22b].

Chapter 2

Translations

2.1 Translation without function symbols and without equality

In this section, we define translations between InqBQ and InqBT in the special case where the given signature contains only relation symbols and where models do not include an identity extension function.
We do so in order to make our presentation of the translations more clear. The specific situation we picked is sufficiently general as to allow us to demonstrate the key ideas behind the translations. At the same time, it doesn't include additional details, which, as we will see in the next section, are mostly technical in their nature. This structure should make it easier to understand the main intuitions behind the translations.
Let's outline the contents of this section more precisely. We start with an analysis of the InqBQ to InqBT direction of the problem. In particular, we first define a translation function from model-assignment pairs to model-team pairs and between formulas. We then prove that such functions are well-behaved with respect to the support relation. We continue with a proof of the entailment preservation property for this translation.
Finally, we present the InqBT to InqBQ direction with the same general structure.

2.1.1 Translating $\mathsf{InqBQ} \to \mathsf{InqBT}$

2.1.1.1 Translation of a generic model of InqBQ into an equivalent model of InqBT

2.1 DEFINITION.

Signature Let Σ be a signature consisting only of relation symbols R_n. Let's define

- $\Sigma^\sharp := \{R'_{n+1} \mid R_n \in \Sigma\}\dot\cup\{O_1\}$, where O is a unary relation symbol which will intuitively be satisfied by exactly all objects of the domain.

Model and team Let $M := \langle D, W, I\rangle$ be a model of InqBQ in the signature Σ, where $W = \{w_j \mid j \in \mathcal{I}\}$, and let $g : Var \to D$ be an assignment into M.
We define a model $M^\sharp$ of InqBT in the signature $\Sigma^\sharp$ and a team T for $M^\sharp$:

- $M^\sharp := \langle D^\sharp, I^\sharp\rangle$ where:
  - $D^\sharp := D\dot\cup W$
  - $I^\sharp(R'_{n+1}) := \{\langle d_1, d_2, ..., d_{n+1}\rangle \in (D^\sharp)^{n+1} \mid d_1 \in W, \langle d_2, ..., d_{n+1}\rangle \in I_{d_1}(R_n)\}$ f.a. $R'_{n+1} \in \Sigma^\sharp$
  - $I^\sharp(O) := D$
- $g_j : Var \to D^\sharp$ t.c. $g_j(x_0) = w_j$ e $g_j(x_{n+1}) = g(x_n)$ f.a. $n \in \mathbb{N}$, $j \in \mathcal{I}$
- $T := \{g_j \mid j \in \mathcal{I}\}$

Formulas Given any $\varphi \in \mathcal{L}$, we define $\varphi^\sharp$ inductively on the structure of φ:

•$\underline{\bot}$: $(\bot)^\sharp := \bot$

•$\underline{R_n}$: $(R_n(x_{i_1}, ..., x_{i_n}))^\sharp, := R'_{n+1}(x_0, x_{i_1+1}, ..., x_{i_n+1})$

•$\underline{\circ}$: $(\psi \circ \xi)^\sharp := (\psi^\sharp \circ \xi^\sharp)$, where $\circ \in \{\wedge, \mathbin{\mathbb{v}}, \to\}$

•$\underline{\forall}$: $(\forall x_n\psi)^\sharp := \forall x_{n+1}(O(x_{n+1}) \to \psi^\sharp)$

•$\underline{\exists}$: $(\exists x_n\psi)^\sharp := \exists x_{n+1}(O(x_{n+1}) \wedge \psi^\sharp)$

2.2 Observation. In all the translations that we will present, the worlds $w_j \in W$ and the assignments $g_j \in T$ are indexed on the same set $\mathcal{I}$. Therefore, the subsets of T and the subsets of W are in a bijection, thanks to their respective bijections to the subsets of $\mathcal{I}$.
For this reason, to denote the subsets of T and W, we will use the following notation: given any subset $\mathcal{J} \subseteq \mathcal{I}$,

- $T_\mathcal{J} := \{g_j \in T \mid j \in \mathcal{J}\}$
- $s_\mathcal{J} := \{w_j \in W \mid j \in \mathcal{J}\}$

In particular, under this notation, considering all $T_\mathcal{J}$ and all $s_\mathcal{J}$ for all $\mathcal{J} \subseteq \mathcal{I}$ is equivalent to considering, respectively, all subsets of T and all subsets of W.

2.3 PROPOSITION. *Under the hypotheses of Definition 2.1, for any* $\varphi \in \mathcal{L}$ *and for any* $\mathcal{J} \subseteq \mathcal{I}$,

$$M, s_{\mathcal{J}} \vDash_g \varphi \iff M^\sharp \vDash_{T_{\mathcal{J}}} \varphi^\sharp$$

Proof. We will prove the following, more general claim: f.a. $\varphi \in \mathcal{L}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M, s_J \vDash_{g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]} \varphi \iff M^\sharp \vDash_{T_{\mathcal{J}}[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]} \varphi^\sharp$$

We can make this statement more clear by introducing the following abbreviations:

- $g^* \coloneqq g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $g_j^* \coloneqq g_j[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $(T_{\mathcal{J}})^* \coloneqq T_{\mathcal{J}}[x_{i_1+1} \mapsto d_1, ..., x_{i_n+1} \mapsto d_n]$

It is useful to observe that, for any $\mathcal{J} \subseteq \mathcal{I}$,

$$T_{\mathcal{J}}^* = \{g_j^* \mid j \in \mathcal{J}\} \tag{2.1.1}$$

That is, $(T_{\mathcal{J}})^* = (T^*)_{\mathcal{J}}$. This allows us to omit the parentheses when referring to these sets.
Using the notation we just defined, the claim reads as follows:
f.a. $\varphi \in \mathcal{L}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M, s_J \vDash_{g^*} \varphi \iff M^\sharp \vDash_{T_{\mathcal{J}}^*} \varphi^\sharp$$

Let us now turn to proving this result by induction on the structure of φ.
First, we observe that f.a. $g_j^* \in T^*$, f.a. $i \in \mathbb{N}$,

$$g^*(x_i) = g_j^*(x_{i+1}) \tag{2.1.2}$$

Indeed,

$$g^*(x_i) = \begin{cases} d_i = g_j^*(x_{i+1}) \text{ if } x \in \bar{x} \\ g(x_i) = g_j^*(x_{i+1}) \text{ if } x \notin \bar{x} \end{cases}$$

We will use equality (2.1.2) in the proof of the base case of the induction.

•$\underline{\bot}$: it suffices to observe $s_{\mathcal{J}} = \emptyset \iff T_{\mathcal{J}}^* = \emptyset$

•$\underline{R_n}$:
$$\begin{aligned} M, s_{\mathcal{J}} \vDash_{g^*} R_n(x_{i_1}, ..., x_{i_n}) &\iff \text{f.a. } w_j \in s_{\mathcal{J}},\ \langle g^*(x_{i_1}), ..., g^*(x_{i_n})\rangle \in I_{w_j}(R_n) \\ &\iff \text{f.a. } j \in \mathcal{J},\ \langle g^*(x_{i_1}), ..., g^*(x_{i_n})\rangle \in I_{w_j}(R_n) \\ &\overset{(2.1.2)}{\iff} \text{f.a. } j \in \mathcal{J},\ \langle g_j^*(x_{i_1+1}), ..., g_j^*(x_{i_n+1})\rangle \in I_{w_j}(R_n) \\ &\iff \text{f.a. } j \in \mathcal{J},\ \langle g_j^*(x_0), g_j^*(x_{i_1+1}), ..., g_j^*(x_{i_n+1})\rangle \in I^\sharp(R'_{n+1}) \\ &\overset{(2.1.1)}{\iff} \text{f.a. } g_j^* \in T_{\mathcal{J}}^*,\ \langle g_j^*(x_0), g_j^*(x_{i_1+1}), ..., g_j^*(x_{i_n+1})\rangle \in I^\sharp(R'_{n+1}) \\ &\iff M^\sharp \vDash_{T_{\mathcal{J}}^*} R'_{n+1}(x_0, x_{i_1+1}, ..., x_{i_n+1}) \end{aligned}$$

$$\bullet\underline{\wedge}:\quad M, s_{\mathcal{J}} \vDash_{g^*} (\psi \wedge \xi) \iff M, s_{\mathcal{J}} \vDash_{g^*} \psi \ e\ M, s_{\mathcal{J}} \vDash_{g^*} \xi$$
$$\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} \psi^\sharp \ e\ M^\sharp \vDash_{T^*_{\mathcal{J}}} \xi^\sharp$$
$$\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} (\psi \wedge \xi)^\sharp$$

$$\bullet\underline{\mathbin{\mathbb{V}}}:\quad M, s_{\mathcal{J}} \vDash_{g^*} (\psi \mathbin{\mathbb{V}} \xi) \iff M, s_{\mathcal{J}} \vDash_{g^*} \psi \ o\ M, s_{\mathcal{J}} \vDash_{g^*} \xi$$
$$\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} \psi^\sharp \ o\ M^\sharp \vDash_{T^*_{\mathcal{J}}} \xi^\sharp$$
$$\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} (\psi \mathbin{\mathbb{V}} \xi)^\sharp$$

$$\bullet\underline{\to}:\quad M, s_{\mathcal{J}} \vDash_{g^*} (\psi \to \xi) \overset{\text{(Obs 2.2)}}{\iff} \text{f.a. } \mathcal{J}' \subseteq \mathcal{J},\ M, s_{\mathcal{J}'} \vDash_{g^*} \psi \text{ implies } M, s_{\mathcal{J}'} \vDash_{g^*} \xi$$
$$\iff \text{f.a. } \mathcal{J}' \subseteq \mathcal{J},\ M^\sharp \vDash_{T^*_{\mathcal{J}'}} \psi^\sharp \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}'}} \xi^\sharp \qquad \text{(induction)}$$
$$\overset{\text{(Obs 2.2)}}{\iff} \text{f.a. } T^*_{\mathcal{J}'} \subseteq T^*_{\mathcal{J}},\ M^\sharp \vDash_{T^*_{\mathcal{J}'}} \psi^\sharp \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}'}} \xi^\sharp$$
$$\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} (\psi \to \xi)^\sharp$$

$$\bullet\underline{\forall}:\quad M, s_{\mathcal{J}} \vDash_{g^*} \forall x_n \psi \iff \text{f.a. } d \in D,\ M, s_{\mathcal{J}} \vDash_{g^*[x_n \to d]} \psi$$
$$\iff \text{f.a. } d \in D,\ M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1} \mapsto d]} \psi^\sharp \qquad \text{(induction)}^1$$
$$\iff \text{f.a. } d^\sharp \in D^\sharp,\ d^\sharp \in D \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1} \mapsto d^\sharp]} \psi^\sharp$$
$$\iff \text{f.a. } d^\sharp \in D^\sharp,\ d^\sharp \in O_{M^\sharp} \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1} \mapsto d^\sharp]} \psi^\sharp$$
$$\iff \text{f.a. } d^\sharp \in D^\sharp,$$
$$M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1} \mapsto d^\sharp]} O(x_{n+1}) \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1} \mapsto d^\sharp]} \psi^\sharp$$
$$\overset{(!)}{\iff} \text{f.a. } d^\sharp \in D^\sharp, \text{ f.a. } \mathcal{J}' \subseteq \mathcal{J},$$
$$M^\sharp \vDash_{T^*_{\mathcal{J}'}[x_{n+1} \mapsto d^\sharp]} O(x_{n+1}) \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}'}[x_{n+1} \mapsto d^\sharp]} \psi^\sharp$$
$$\iff \text{f.a. } d^\sharp \in D^\sharp,$$
$$M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1} \mapsto d^\sharp]} (O(x_{n+1}) \to \psi^\sharp)$$
$$\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} \forall x_{n+1}(O(x_{n+1}) \to \psi^\sharp)$$

(!): This step isn't as straightforward as the rest. While the direction $\Longleftarrow$ is obvious, the inverse direction really isn't. Indeed, it is not true in general that, if an implication holds for a specific team, then it must hold for its subsets. However, in the specific case in question this is true, due to the structure of the antecedent. Let's show this explicitly:

We proceed by contraposition. Let $d^* \in D^\sharp$ and $\mathcal{J}' \subseteq \mathcal{J}$ s.t.

$$M^\sharp \vDash_{T^*_{\mathcal{J}'}[x_{n+1} \mapsto d^*]} O(x_{n+1}) \quad \text{and} \quad M^\sharp \nvDash_{T^*_{\mathcal{J}'}[x_{n+1} \mapsto d^*]} \psi^\sharp$$

Then the following statements are true:

1. $M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^*]} O(x_{n+1})$, since the only free variable of $O(x_{n+1})$ is x_{n+1}, which is assigned to the same value by all assignments in $T_{\mathcal{J}}[x_{n+1} \mapsto d^\sharp]$.
2. $M^\sharp \nvDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^*]} \psi^\sharp$, by persistency

From 1. and 2. we obtain, using the existence of such a $d^* \in D^\sharp$,

$$\text{f.s. } d^\sharp \in D^\sharp,\ M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} O(x_{n+1}) \text{ and } M^\sharp \nvDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} \psi^\sharp$$

which is the negation of

$$\text{f.a. } d^\sharp \in D^\sharp,\ M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} O(x_{n+1}) \text{ implies } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} \psi^\sharp$$

□

•∃: $$\begin{aligned} M, s_{\mathcal{J}} \vDash_{g^*} \exists x_n \psi &\iff \text{f.s. } d \in D,\ M, s_{\mathcal{J}} \vDash_{g^*[x_n \to d]} \psi \\ &\iff \text{f.s. } d \in D,\ M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d]} \psi^\sharp && \text{(induction)} \\ &\iff \text{f.s. } d^\sharp \in D^\sharp,\ d^\sharp \in D \text{ e } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} \psi^\sharp \\ &\iff \text{f.s. } d^\sharp \in D^\sharp,\ d^\sharp \in O_{M^\sharp} \text{ e } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} \psi^\sharp \\ &\iff \text{f.s. } d^\sharp \in D^\sharp,\ M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} O(x_{n+1}) \text{ e } M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} \psi^\sharp \\ &\iff \text{f.s. } d^\sharp \in D^\sharp,\ M^\sharp \vDash_{T^*_{\mathcal{J}}[x_{n+1}\mapsto d^\sharp]} (O(x_{n+1}) \wedge \psi^\sharp) \\ &\iff M^\sharp \vDash_{T^*_{\mathcal{J}}} \exists x_{n+1}(O(x_{n+1}) \wedge \psi^\sharp) \end{aligned}$$

□

2.1.1.2 Entailment preservation

2.4 THEOREM. *Let Σ be a signature containing only relation symbols, let Φ be a set of formulas of $\mathcal{L}$ and let ψ be a formula of $\mathcal{L}$ in Σ.*
Then,

$$\Phi \vDash_{\mathsf{InqBQ}} \psi \iff \Gamma_{\Sigma^\sharp}, \Phi^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp$$

where $\Gamma_{\Sigma^\sharp}$ is the set of formulas given by the following formulas to express the following notions:

to express the fact that x_0 is always evaluated as a world,

$$\neg O(x_0) \tag{2.1.3}$$

to describe how all other variables are evaluated as elements of the domain,

$$O(x_{n+1}) \tag{2.1.4}$$

[1] Observe that $T^*_{\mathcal{J}}[x_i \mapsto d] = T_{\mathcal{J}}[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n][x_i \mapsto d] = T_{\mathcal{J}}[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n, x_i \mapsto d]$. This tells us that $T^*_{\mathcal{J}}[x_i \mapsto d]$ is indeed a team of the type that can be used to apply the inductive hypothesis.

to ensure that all tuples in an $(n+1)$*-ary relation have the structure* $\langle$*world,* n *elements of the domain*$\rangle$,

$$\forall x \forall y_1 ... \forall y_n (R'_{n+1}(x, y_1, ..., y_n) \to \neg O(x) \wedge \bigwedge_{i=1,...,n} O(y_i)) \tag{2.1.5}$$

and to guarantee that the evaluation of all variables other than x_0 *is constant,*

$$\exists x (x_{n+1} = x) \tag{2.1.6}$$

In (2.1.5)*,* R'_{n+1} *varies over* $\Sigma^\sharp$ *while in* (2.1.4) *and* (2.1.6)*,* n *varies over* $\mathbb{N}$*.*
Note that, whenever it isn't explicitly relevant, we will omit the reference to $\Sigma^\sharp$ *in future occurrences of* $\Gamma_{\Sigma^\sharp}$*.*

2.5 Observation (Meaning of Γ)**.** At this point, it is useful to explain in detail what it means for a model $M = \langle D, I \rangle$ to satisfy Γ with respect to a team T.

- $M \vDash_T$ (2.1.3) $\iff$ f.a. $g_j \in T,\ g_j(x_0) \notin O_M$
  Intuitively, x_0 is evaluated into a world by each assignment in the team T.

- $M \vDash_T$ (2.1.4) $\iff$ f.a. $g_j \in T$, f.a. $n \in \mathbb{N},\ g_j(x_{n+1}) \in O_M$
  Intuitively, every x_i is evaluated, f.a. $i > 0$, into an object of the domain by each assignment in the team T.

- $M \vDash_T$ (2.1.5) $\iff$ $I(R'_{n+1}) \subseteq (D \setminus O_M) \times (O_M)^n$
  Intuitively, relations can only subsist at a world (set as the first argument) and between n elements of the domain.

- $M \vDash_T$ (2.1.6) $\iff$ f.s. $d \in D,\ M \vDash_{T[x \mapsto d]} x_{n+1} = x$
  $\iff$ f.s. $d \in D$, f.a. $g_j \in T[x \mapsto d],\ g_j(x_{n+1}) = d$
  $\iff$ $g_j(x_n) = g_{j'}(x_n)$ f.a. $g_j, g_{j'} \in T$ e f.a. $n > 0$
  That is, over the team T all variables are assigned constantly, except for x_0.

The idea is for these formulas and their respective properties to characterize the model-team pairs of InqBT that are obtained by applying the translation $\sharp$.

To prove the preservation of the entailment we will need the following result:

2.1.1.3 Encoding an InqBT model in InqBQ

2.6 LEMMA. *Let $M = \langle D, I\rangle$ be a model of InqBT and let T be a team over M such that $M \vDash_T \Gamma$ relative to the signature $\Sigma^\sharp$.*
Then, there exist an InqBQ model $M^- := \langle D^-, W^-, I^-\rangle$, where $W = \{w_j \mid j \in \mathcal{I}\}$, a state $s_T \subseteq W^-$ and an assignment g^- in the signature Σ such that

$$M^{-\sharp} = M \text{ and } T_{s_T} = T$$

where T_{s_T} is defined by the condition (f.a. $j \in \mathcal{I}$, $g_j \in T_{s_T} \iff w_j \in s_T$). Intuitively, T_{s_T} is the team corresponding to s_T after applying the translation $\sharp$ to the model M^- and to the assignment g^-.

Proof. First, let's define $M^- := \langle D^-, W^-, I^-\rangle$:

•$\underline{D^-}$: $D^- := I(O)$

•$\underline{W^-}$: $W^- := D \setminus I(O)$

•$\underline{I^-}$: $I^-_w(R_n) := \{\langle d^-_1 ..., d^-_n\rangle \in (D^-)^n \mid \langle w, d^-_1, ..., d^-_n\rangle \in I(R'_{n+1})\}$

The condition that D^- and W^- must be non-empty sets is satisfied because $M \vDash_T \Gamma$ and therefore, in particular, $M \vDash_T$ (2.1.3), (2.1.4).
Now, let's define g^- and s_T:

•$\underline{g^-}$: $g^- : x_n \mapsto g(x_{n+1})$ f.s. $g \in T$

•$\underline{s_T}$: $s_T := T[x_0]$

In the definition of g^-, as we will see, g can be chosen arbitrarily in T because $M \vDash$ (2.1.6) which implies that all assignment g of T coincide on every variable x_i for $i > 0$. The well-definedness of g^-, i.e. the fact that $g^-(x_i) \in D^-$ f.a. $i \in \mathbb{N}$, is guaranteed by the meaning of (2.1.4).
The fact that s_T is a state, i.e. that it is a subset of W^-, follows from $M \vDash$ (2.1.3), which is equivalent to $g_j(x_0) \in (D \setminus O_M) = W^-$ f.a. $g_j \in T$.

At this point it suffices to prove that the starting objects are equal to the translations of the ones we just defined.

•$\underline{D}$: $D^{-\sharp} = D^- \dot\cup W^- = I(O) \dot\cup (D \setminus I(O)) = D$

•$\underline{I}$: let $d_1, ..., d_{n+1} \in (D^{-\sharp})^n = D^n$, then

$$\begin{aligned}\langle d_1, ..., d_{n+1}\rangle \in I^{-\sharp}(R'_{n+1}) &\iff d_1 \in W^-,\ \langle d_2, ..., d_{n+1}\rangle \in I^-_{d_1}(R_n)\\ &\iff d_1 \notin O_M,\ \langle d_2, ..., d_{n+1}\rangle \in (O_M)^n \text{ and}\\ &\qquad\qquad \langle d_1, d_2, ..., d_{n+1}\rangle \in I(R'_{n+1})\\ &\iff \langle d_1, ..., d_{n+1}\rangle \in I(R'_{n+1})\end{aligned}$$

Therefore, $I^{-\sharp}(R'_{n+1}) = I(R'_{n+1})$.
The last equivalence is guaranteed by the definition of I^{-} and by the fact that $M \vDash (2.1.5)$.
Indeed, as we saw in Observation 2.5, if an $(n+1)$-tuple belongs to the interpretation of a relation symbol for a model M that satisfies all formulas of Γ, then the first element of the tuple is not in O_M while all the other elements belong to O_M.
This allows us to suppress the first two conditions in the last equivalence, since they are implied by the third condition $\langle d_1, ..., d_{n+1}\rangle \in I(R'_{n+1})$.

•$\underline{T}$: Let's prove the two directions of $T_{s_T} = T$ separately.
Before continuing with the proof, let us observe that s_T can be seen in the following way:
$s_T \coloneqq \{w_j \in W \mid g(x_0) = w_j \text{ f.s. } g \in T\} = \{w_j \mid j \in \mathcal{I} \text{ and } g(x_0) = w_j \text{ f.s. } g \in T\}$.

($\subseteq$): let $g_j \in T_{s_T}$, then

$$g_j \in T_{s_T} \implies \begin{cases} g_j(x_0) = w_j \text{ f.s. } w_j \in s_T \\ g_j(x_{i+1}) = g^{-}(x_{i+1}) \text{ f.a. } i \in \mathbb{N} \end{cases} \qquad \text{(by def of } g_j\text{)}$$

$$\implies \begin{cases} \text{f.s. } g \in T,\ g_j(x_0) = w_j = g(x_0) \\ \text{f.a. } g \in T,\ g_j(x_{i+1}) = g^{-}(x_{i+1}) = g(x_{i+1}) \text{ f.a. } i \in \mathbb{N} \end{cases} \qquad \text{(by def of } s_T\text{)}$$

$$\implies g_j = g \text{ f.s. } g \in T$$

□

($\supseteq$): let $g \in T$, then

$$
\begin{aligned}
g \in T \implies & \begin{cases} g(x_0) = w_j \text{ f.s. } w_j \in s_T & \text{(by def of } s_T\text{)} \\ g(x_{i+1}) = g^-(x_{i+1}) \text{ f.a. } i \in \mathbb{N} & \text{(by def of } g^-\text{)} \end{cases} \\
\implies & \begin{cases} g(x_0) = w_j \text{ f.s. } w_j \in s_T \\ g(x_{i+1}) = g^-(x_{i+1}) \text{ f.a. } i \in \mathbb{N} \end{cases} \\
\implies & \begin{cases} g(x_0) = w_j = g_j(x_0) \text{ f.s. } w_j \in s_T \\ g(x_{i+1}) = g^-(x_{i+1}) = g_j(x_{i+1}) \text{ f.a. } i \in \mathbb{N}, \text{ f.a. } j \in \mathcal{I} \end{cases} \text{(by def of } g_j\text{)} \\
\implies & \text{ f.s. } j \in \mathcal{I},\ g = g_j \text{ and } w_j \in s_T \\
\implies & g = g_j \text{ f.s. } g_j \in T_{s_T}
\end{aligned}
$$

□

Having verified the equality of the two components of $M^{-\sharp}$ and M and the equality of the two teams T_{s_T} and T we can conclude the proof. □

2.1.1.4 Proof of Theorem 2.4

We start by proving the $\Longleftarrow$ direction of Theorem 2.4, which only requires the good behaviour of the translation $\sharp$.

Proof. By contraposition: assume $\Phi \nvDash_{\mathsf{InqBQ}} \psi$. Then, there exist a model M, a state $s_{\mathcal{J}}$ and an assignment g such that $M, s_{\mathcal{J}} \vDash_g \Phi$ and $M, s_{\mathcal{J}} \nvDash_g \psi$. Now, by Proposition 2.3, we get

$$M, s_{\mathcal{J}} \nvDash \psi \iff M^\sharp \nvDash_{T_{\mathcal{J}}} \psi^\sharp$$

$$M, s_{\mathcal{J}} \vDash \Phi \iff M^\sharp \vDash_{T_{\mathcal{J}}} \Phi^\sharp$$

Clearly $M^\sharp \vDash_{T_{\mathcal{J}}} \Gamma$ (since $M^\sharp$ and T satisfy by construction the semantic conditions corresponding to Γ that we described in Observation 2.5), so we have a counterexample to show that $\Gamma, \Phi^\sharp \nvDash_{\mathsf{InqBT}} \psi^\sharp$. □

To prove the $\Longrightarrow$ direction of the Theorem we will also need Lemma 2.6:

Proof. By contraposition: assume $\Gamma, \Phi^\sharp \nvDash_{\mathsf{InqBT}} \psi^\sharp$. Then, there exist a model of InqBT M and a team T such that $M \vDash_T \Gamma$, $M \vDash_T \Phi^\sharp$ e $M \nvDash_T \psi^\sharp$. Then we have that if we define M^-, s_T and g^- as in Lemma 2.6,

$$M \nvDash_T \psi^\sharp \iff M^-, s_T \nvDash_{g^-} \psi$$

$$M \vDash_T \Phi^\sharp \iff M^-, s_T \vDash_{g^-} \Phi$$

This proves that $\Phi \nvDash_{\mathsf{InqBQ}} \psi$. □

2.1.2 Translating $\mathsf{InqBT} \to \mathsf{InqBQ}$

2.1.2.1 Translation of a generic model of InqBT into an equivalent model of InqBQ

2.7 DEFINITION.

Signature Let Σ be a signature containing only relation symbols R_n. We define:

- $\Sigma^\flat := \Sigma \cup \{a_i \mid i \in \mathbb{N}\}$, where the a_i are non-rigid constants.

Model Let $M := \langle D, I \rangle$ be a model of InqBT in the signature Σ and let $T = \{g_j \mid j \in \mathcal{I}\}$ be a team for M.
Let's define the model $M_T^\flat$ of InqBQ in the signature $\Sigma^\flat$:

- $M_T^\flat := \langle D, W^\flat, I^\flat \rangle$ where:
  - $W^\flat := \{w_j \mid j \in \mathcal{I}\}$
  - $I_{w_j}^\flat(R) := I(R)$ *f.a.* $j \in \mathcal{I}$, *f.a.* $R \in \Sigma^\flat$
  - $I_{w_j}^\flat(a_i) := g_j(x_i)$ *f.a.* $j \in \mathcal{I}$, *f.a.* $i \in \mathbb{N}$

Formulas For any $\varphi \in \mathcal{L}$, we define $\varphi^\flat$ inductively on the structure of φ:

•$\underline{\bot}$: $(\bot)^\flat := \bot$

•$\underline{R}$: $(R(x_{i_1}, ..., x_{i_n}))^\flat := R(a_{i_1}, ..., a_{i_n})$

•$\underline{\circ}$: $(\psi \circ \xi)^\flat := (\psi^\flat \circ \xi^\flat)$, where $\circ \in \{\wedge, \vee\!\!\!\vee, \to\}$

•$\underline{\forall}$: $(\forall x_n \psi)^\flat := \forall x_n \psi^\flat[x_n/a_n]$

•$\underline{\exists}$: $(\exists x_n \psi)^\flat := \exists x_n \psi^\flat[x_n/a_n]$

2.8 PROPOSITION. *Under the same hypotheses of Definition 2.7, f.a. $\varphi \in \mathcal{L}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$,*

$$M \vDash_{T_\mathcal{J}} \varphi \iff M_T^\flat, s_\mathcal{J} \vDash \varphi^\flat$$

where $T_\mathcal{J}$ and $s_\mathcal{J}$ are defined as the subsets of T and $W^\flat$ indexed by $\mathcal{J}$ as per Observation 2.2.

Proof. First, we choose any assignment g over the model $M_T^\flat$.
To prove the proposition, we prove the validity of the following, more general claim:

f.a. $n \in \mathbb{N}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M \vDash_{T_\mathcal{J}[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]} \varphi \iff M_T^\flat, s_\mathcal{J} \vDash_{g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]} \varphi^\flat[x_{i_1}/a_{i_1}, ..., x_{i_n}/a_{i_n}]$$

Similarly to what we did for the proof of proposition 2.3, we will make this statement more readable by introducing the following notation:

- $g^* \coloneqq g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $g_j^* \coloneqq g_j[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $(T_{\mathcal{J}})^* \coloneqq T_{\mathcal{J}}[x_{i_1+1} \mapsto d_1, ..., x_{i_n+1} \mapsto d_n]$
- $\bar{x} \coloneqq \{x_{i_1}, ..., x_{i_n}\}$, $\bar{a} \coloneqq \{a_{i_1}, ..., a_{i_n}\}$
- $\varphi[\bar{x}/\bar{a}] \coloneqq \varphi[x_{i_1}/a_{i_1}, ..., x_{i_n}/a_{i_n}]$

In the new notation, the claim reads as follows:
f.a. $n \in \mathbb{N}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M \vDash_{T^*_{\mathcal{J}}} \varphi \iff M_T^\flat, s_{\mathcal{J}} \vDash_{g^*} \varphi^\flat[\bar{x}/\bar{a}]$$

Let's proceed by induction on the structure of φ.
First, note that, f.a. $\mathcal{J} \subseteq \mathcal{I}$,

$$T^*_{\mathcal{J}} = \{g_j^* \mid g_j \in T_{\mathcal{J}}\} = \{g_j^* \mid j \in \mathcal{J}\} \tag{2.1.7}$$

Then, let's observe that, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $j \in \mathcal{J}$ and f.a. $i \in \mathbb{N}$,

$$g_j^*(x_i) = [a_i[\bar{x}/\bar{a}]]^{w_j}_{g^*} \tag{2.1.8}$$

Indeed,

$$g_j^*(x_i) = \begin{cases} d_i = g^*(x_i) = [x_i]^{w_j}_{g^*} = [a_i[\bar{x}/\bar{a}]]^{w_j}_{g^*} & \text{if } x_i \in \bar{x} \\ \\ I^\flat_{w_j}(a_i) = [a_i[\bar{x}/\bar{a}]]^{w_j}_{g^*} & \text{if } x_i \notin \bar{x} \end{cases}$$

This equality will be useful in proving the base case of the induction.

•$\underline{R}$: $\quad$ $$\begin{aligned} M \vDash_{T^*_{\mathcal{J}}} R(x_{i_1}, ..., x_{i_n}) &\iff \text{f.a. } \tilde{g}_j \in T^*_{\mathcal{J}},\ \langle \tilde{g}_j(x_{i_1}), ..., \tilde{g}_j(x_{i_n})\rangle \in I(R) \\ &\overset{(2.1.7)}{\iff} \text{f.a. } j \in \mathcal{J},\ \langle g_j^*(x_{i_1}), ..., g_j^*(x_{i_n})\rangle \in I(R) \\ &\overset{(2.1.8)}{\iff} \text{f.a. } j \in \mathcal{J},\ \langle [(a_{i_1})[\bar{x}/\bar{a}]]^{w_j}_{g^*}, ..., [(a_{i_n})[\bar{x}/\bar{a}]]^{w_j}_{g^*}\rangle \in I(R) \\ &\iff \text{f.a. } w_j \in s_{\mathcal{J}},\ \langle [(a_{i_1})[\bar{x}/\bar{a}]]^{w_j}_{g^*}, ..., [(a_{i_n})[\bar{x}/\bar{a}]]^{w_j}_{g^*}\rangle \in I^\flat_{w_j}(R) \\ &\iff M_T^\flat, s_{\mathcal{J}} \vDash_{g^*} R(a_{i_1}, ..., a_{i_n})[\bar{x}/\bar{a}] \\ &\iff M_T^\flat, s_{\mathcal{J}} \vDash_{g^*} (R(x_{i_1}, ..., x_{i_n}))^\flat[\bar{x}/\bar{a}] \end{aligned}$$

•$\underline{\wedge, \vee\!\!\!\!\vee, \bot}$: Analogous to Proposition 2.3

•$\underline{\forall}$: There is a small difference between the following two cases:

− if $x_i \notin \bar{x}$, then

$$\begin{aligned}
M \vDash_{T^*_{\mathcal{J}}} \forall x_i \psi &\iff \text{f.a. } d_{n+1} \in D,\ (M \vDash_{T^*_{\mathcal{J}}[x_i \mapsto d_{n+1}]} \psi) \\
&\iff \text{f.a. } d_{n+1} \in D,\ (M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \psi^\flat[\bar{x}/\bar{a}][x_i/a_i]) \quad \text{(induction}^2\text{)} \\
&\iff \text{f.a. } d_{n+1} \in D,\ (M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \psi^\flat[\bar{x}/\bar{a}, x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \forall x_i(\psi^\flat[\bar{x}/\bar{a}, x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \forall x_i(\psi^\flat[\bar{x}/\bar{a}][x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*} (\forall x_i \psi)^\flat[\bar{x}/\bar{a}]
\end{aligned}$$

− if $x_i \in \bar{x}$, then

$$\begin{aligned}
M \vDash_{T^*_{\mathcal{J}}} \forall x_i \psi &\iff \text{f.a. } d_{n+1} \in D,\ (M \vDash_{T^*_{\mathcal{J}}[x_i \mapsto d_{n+1}]} \psi) \\
&\iff \text{f.a. } d_{n+1} \in D,\ (M \vDash_{T_{\mathcal{J}}[x_{i_1} \mapsto d_1, \ldots, x_i \mapsto d_{i_k}, \ldots, x_{i_n} \mapsto dn][x_i \mapsto d_{n+1}]} \psi) \\
&\iff \text{f.a. } d_{n+1} \in D,\ (M \vDash_{T_{\mathcal{J}}[x_{i_1} \mapsto d_1, \ldots, x_i \mapsto d_{n+1}, \ldots, x_{i_n} \mapsto d_{n+1}]} \psi) \\
&\iff \text{f.a. } d_{n+1} \in D, \\
&\qquad M^\flat_T, s_{\mathcal{J}} \vDash_{g^\flat[x_{i_1} \mapsto d_1, \ldots, x_i \mapsto d_{n+1}, \ldots, x_{i_n} \mapsto d_{n+1}]} \\
&\qquad\qquad \psi^\flat[x_{i_1}/a_{i_1}, \ldots, x_i/a_i, \ldots, x_{i_n}/a_n] \quad \text{(induction)} \\
&\iff \text{f.a. } d_{n+1} \in D, \\
&\qquad M^\flat_T, s_{\mathcal{J}} \vDash_{g^\flat[x_{i_1} \mapsto d_1, \ldots, x_i \mapsto d_{i_k}, \ldots, x_{i_n} \mapsto d_{n+1}][x_i \mapsto d_i]} \\
&\qquad\qquad \psi^\flat[x_{i_1}/a_{i_1}, \ldots, x_i/a_i, \ldots, x_{i_n}/a_n][x_i/a_i] \\
&\iff \text{f.a. } d_{n+1} \in D,\ (M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \psi^\flat[\bar{x}/\bar{a}][x_i/a_i]) \\
&\iff \text{f.a. } d_{n+1} \in D,\ (M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \psi^\flat[\bar{x}/\bar{a}, x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \forall x_i(\psi^\flat[\bar{x}/\bar{a}, x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i \mapsto d_{n+1}]} \forall x_i(\psi^\flat[\bar{x}/\bar{a}][x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*} (\forall x_i \psi)^\flat[\bar{x}/\bar{a}]
\end{aligned}$$

[2] As noted in the proof of Proposition 2.3, the inductive hypothesis can be applied to $T^*_{\mathcal{J}}[x_i \mapsto d_{n+1}]$ and $g^*[x_i \mapsto d_{n+1}]$.

$$\begin{aligned}
\bullet\exists:\quad M \vDash_{T^*_{\mathcal{J}}} \exists x_i\psi &\iff \text{f.s. } d_{n+1} \in D,\ (M \vDash_{T^*_{\mathcal{J}}[x_i \mapsto d_{n+1}]} \psi) \\
&\iff \text{f.s. } d_{n+1} \in D,\ (M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i\mapsto d_{n+1}]} \psi^\flat[\bar{x}/\bar{a}][x_i/a_i]) \ \text{(induction)} \\
&\iff \text{f.s. } d_{n+1} \in D,\ (M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i\mapsto d_{n+1}]} \psi^\flat[\bar{x}/\bar{a}, x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i\mapsto d_{n+1}]} \exists x_i(\psi^\flat[\bar{x}/\bar{a}, x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*[x_i\mapsto d_{n+1}]} \exists x_i(\psi^\flat[\bar{x}/\bar{a}][x_i/a_i]) \\
&\iff M^\flat_T, s_{\mathcal{J}} \vDash_{g^*} (\exists x_i\psi)^\flat[\bar{x}/\bar{a}]
\end{aligned}$$

□

2.1.2.2 Entailment preservation

2.9 THEOREM. *Let Σ be a signature containing only relation symbols, let Φ be a set of formulas of $\mathcal{L}$ and let ψ be a formula of $\mathcal{L}$ in Σ.*
Then

$$\Phi \vDash_{\mathsf{InqBT}} \psi \iff \Delta_{\Sigma^\flat}, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat$$

where $\Delta_{\Sigma^\flat}$ is the set of formulas given by:

$$\forall x_1 ... \forall x_n ?R(x_1, ..., x_n) \tag{2.1.9}$$

where in (2.1.9) *we include a formula for each $R_n \in \Sigma$.*
Note that, whenever it isn't explicitly relevant, we will omit the reference to $\Sigma^\flat$ in future occurrences of $\Delta_{\Sigma^\flat}$.

2.10 Observation (Meaning of Δ)**.** The formulas (2.1.9) encode the idea that in a model of InqBQ the interpretation of relation symbols does not vary between worlds[3].
Intuitively, the aim of these formulas is to characterize all models that are obtained by applying the translation $\flat$. These models, as we will see, turn out to be completely generic aside from this property.

To prove the conservation of the entailment we will need the following result:

2.1.2.3 Encoding an InqBQ model in InqBT

2.11 LEMMA. *Let $M = \langle D, W, I\rangle$ be a model of InqBQ, where $W = \{w_j \mid j \in \mathcal{I}\}$, such that $M, W \vDash \Delta$ in the signature $\Sigma^\flat$.*
Then, there exist a model $M^+ := \langle D^+, I^+\rangle$ and a team T^+ in the signature Σ such that for its translation $(M^+)^\flat_{T^+} = \langle D^{+\flat}, W^\flat, I^{+\flat}\rangle$ the following properties are true:

- $D^{+\flat} = D$

[3] Recall example 1.16, where we covered mention-all questions.

- $W^\flat \subseteq W$
- $W^\flat \approx_e W$

2.12 COROLLARY. *In the hypotheses of the previous lemma, if we take a model M^+ such as described in the statement, we have that for any $\varphi \in \mathcal{L}$,*

$$M \vDash \varphi \iff (M^+)^\flat_{T^+} \vDash \varphi$$

Proof of the lemma. First we define $M^+ := \langle D^+, I^+ \rangle$:

•$\underline{D^+}$: $D^+ := D$

•$\underline{I^+}$: $I^+(R) := I_{w_j}(R)$ f.s. $j \in \mathcal{I}$

Note that in the definition of the interpretation function I^+ the choice of j in $\mathcal{I}$ is irrelevant, thanks to what we pointed out in Observation 2.10 and to the fact that $M, W \vDash \Delta$.
Let's now define the team T^+. This definition requires more attention than the other ones. First, we consider the set W^e, i.e. the quotient modulo essential equivalence of W. We can then define a set W^+ of representatives for the equivalence classes in W^e. From these representatives we define $\mathcal{I}^+ := \{j \in \mathcal{I} \mid w_j \in W^+\}$.
We now have what we need to define T^+:

•$\underline{g_j}$: $g_j(x_i) = I_{w_j}(a_i)$ f.a. $i \in \mathbb{N}$, f.a. $j \in \mathcal{I}$

•$\underline{T^+}$: $T^+ := \{g_j \mid j \in \mathcal{I}^+\}$

We can now turn to proving that the stated properties are verified by these objects.

•$\underline{D}$: $D^{+\flat} = D^+ = D$

•$\underline{W}$: $W^\flat = \{w_j \mid j \in \mathcal{I}^+\} \subseteq \{w_j \mid j \in \mathcal{I}\} = W$. Observe that $W^\flat = W^+$.

•$\underline{I}$: Additionally, we prove that, for all $w_j \in W^\flat$, $I^{+\flat}_{w_j} = I_{w_j}$:

$I^{+\flat}_{w_j}(R) = I^+(R) = I_{w_j}(R)$

$I^{+\flat}_{w_j}(a_i) = g_j(x_i) = I_{w_j}(a_i)$

We obtain the first equality from Definition 2.7 and the second equality from the definitions given above.

The three equalities involving D, W and I imply that $(M^+)^\flat_{T^+} = M|_{W^\flat}$. Therefore, we can consider essential equivalence between worlds in $W^\flat$ and worlds in W.

•$\underline{\approx_e}$: We need to show that for all $w_{j'} \in W^\flat$ there is some $w_j \in W$ such that $w_{j'} \approx_e w_j$ and for all $w_j \in W$ there is some $w_{j'} \in W^\flat$ such that $w_{j'} \approx_e w_j$.
The first claim is evident, since $W^\flat \subseteq W$.
The second claim is also quite straightforward: for any world $w_j \in W$, we find in $W^\flat = W^+$ some representative $w_{j'}$ of its equivalence class with respect to essential equivalence. Obviously this means that $w_{j'} \approx_e w_j$.

□

2.1.2.4 Proof of Theorem 2.9

Let us start by proving the $\Longleftarrow$ direction of 2.9, which only relies on the good behaviour of $\flat$.

Proof. By contraposition: assume that $\Phi \nvDash_{\mathsf{InqBT}} \psi$. Then, there exist a model M and a team T such that $M \vDash_T \Phi$ e $M \nvDash_T \psi$. Now, thanks to proposition 2.8, we have

$$M \nvDash_T \psi \iff M_T^\flat \nvDash \psi^\flat$$

$$M \vDash_T \Phi \iff M_T^\flat \vDash \Phi^\flat$$

Clearly $M_T^\flat \vDash \Delta$, since $M_T^\flat$ satisfies by construction the semantic conditions that correspond to the formulas of Δ as pointed out in Observation 2.10. This provides us with a counterexample that proves $\Delta, \Phi^\flat \nvDash_{\mathsf{InqBQ}} \psi^\flat$. □

To prove the $\Longrightarrow$ direction of the theorem we employ Lemma 2.11:

Proof. By contraposition: assume that $\Delta, \Phi^\flat \nvDash_{\mathsf{InqBQ}} \psi^\flat$. Then[4] there exists a model of InqBQ M such that $M \vDash \Delta$, $M \vDash \Phi^\flat$ and $M \nvDash \psi^\flat$. Then, by defining M^+ and T^+ as in Lemma 2.11,

$$M \nvDash \psi^\flat \iff (M^+)^\flat_{T^+} \nvDash \psi^\flat \iff M^+ \nvDash_{T^+} \psi$$

$$M \vDash \Phi^\flat \iff (M^+)^\flat_{T^+} \vDash \Phi^\flat \iff M^+ \vDash_{T^+} \Phi$$

The first biconditional of each line holds thanks to Corollary 2.12, while the second equivalence is an application of Proposition 2.8.
The two statements provide a counterexample, which proves $\Phi \nvDash_{\mathsf{InqBT}} \psi$. □

[4]Since the relevant formulas are all closed, there is no need to refer to a specific assignment g in the three expressions that follow.

2.2 Translation for the general case

In this section, we cover the definition of the two translations and the proof of their required properties for the general case. In particular, we now allow our signatures to have function symbols and we consider models equipped with an identity extension functions.
The structure of the contents of the section is identical to that of the previous case. Nonetheless, there are some additions that turn out to be necessary for this generalization.
Obviously, we must now include in our definitions clauses for function symbols, generic terms and identity formulas. In these definitions, we will denote with ★ the items that only appear in the general case. Another difference can be found in the definition of the translation $\sharp$ from InqBQ to InqBT. While in the special case it was both possible and reasonable to give a function associating each model-assignment pair to a single model-team pair, it would be needlessly restrictive to do so in the general case. While the corresponding team is still uniquely determined, we choose to define a more natural relation, holding between a certain model-assignment pair and various model-team pairs with the same team. The proofs of the good behaviour of the translations remain very similar to the special case, with the main difference being the addition of a proof of the good behaviour with respect to terms.
As for the proof of entailment preservation, it is necessary to substantially expand the sets of formulas which characterize the translated structures. Moreover, the proofs of the existence of a translation-preserved encoding of generic structures of each system turns out to be slightly more complicated, due to the specific properties of the translated structures.

2.2.1 Translating InqBQ → InqBT

2.2.1.1 Translation of a generic model of InqBQ into an equivalent model of InqBT

2.13 DEFINITION. Most of this definition is basically a restatement of Definition 2.1. For ease of consultation, we still present the whole definition in detail.

Signature Let Σ be a signature consisting of relation symbols R_n, function symbols f_n and symbols for rigid functions h_n. We define

★ $\Sigma^\sharp \coloneqq \{R'_{n+1} \mid R_n \in \Sigma\} \dot\cup \{f'_{n+1} \mid f_n \in \mathcal{F}_\Sigma\} \dot\cup \{h'_n \mid \mathbf{h}_n \in \mathcal{F}^R_\Sigma\} \dot\cup \{O_1\} \dot\cup \{R^=_3\}$

Model and team Let $M \coloneqq \langle D, W, I, \sim\rangle$ be a model of InqBQ in the signature Σ, where $W = \{w_j \mid j \in \mathcal{I}\}$, and let $g : Var \to D$ be an assignment

for M.
Let's define the team T_M:

- $g_j : Var \to (D\dot{\cup}W),\ g_j(x_0) \coloneqq w_j,\ g_j(x_{n+1}) \coloneqq g(x_n)$ f.a. $n \in \mathbb{N}$
- $T_M \coloneqq \{g_j \mid j \in \mathcal{I}\}$

Now, we define a relation $\sim_\sharp$ between model-assignment pairs of InqBQ over a signature Σ and model-team pairs of InqBT over the signature $\Sigma^\sharp$. This differs from our approach for the special case of Definition 2.1. The difference, as we will see, is motivated by a greater degree of freedom in the construction of a model that corresponds to M. In particular, this concerns the definition of the interpretation of function symbols. The idea for functions is analogous to the one we had for relations in Definition 2.1: we emulate the dependence on worlds of the interpretation of functions in InqBQ by introducing an additional argument to functions. Since we are extending the domain from D to $D\dot{\cup}W$, we get all the $n+1$-tuples we need to realize this emulation, i.e. those composed of one world (in first position) and n elements of D. For these tuples the interpretation of the functions in InqBT must behave just as it did in InqBQ. For all other tuples, there is no intuitive mapping. Indeed, we will see that defining the interpretation of a function symbol on these values has no effect on the satisfaction of translated formulas. Note that we could easily make this relation functional by picking specific mappings. This, however, would be less natural and lead to a more restrictive definition with no added benefits.
Let us then proceed with the definition of $\sim_\sharp$.
A model $M^\sharp = \langle D^\sharp, I^\sharp\rangle$ of InqBT in the signature $\Sigma^\sharp$ satisfies

$$\langle M, g\rangle \sim \langle M^\sharp, T_M\rangle$$

if the following conditions hold:

- $D^\sharp = D\dot{\cup}W$
- $I^\sharp(R'_{n+1}) \;=\; \{\langle d_1, d_2, ..., d_{n+1}\rangle \in (D^\sharp)^{n+1} \mid$
  $d_1 \in W, \langle d_2, ..., d_{n+1}\rangle \in I_{d_1}(R_n)\}$
  f.a. $R_{n+1} \in \Sigma^\sharp$

★ $I^\sharp(f'_{n+1})(w_j, d_1, ..., d_n) = I_{w_j}(f_n)(d_1, ..., d_n)$ f.a. $w_j \in W$
f.a. $d_1, ..., d_n \in D$

★ $I^\sharp(h_n)(d_1, ..., d_n) = I(\mathbf{h'_n})(d_1, ..., d_n)$ f.a. $d_1, ..., d_n \in D$

★ $I^\sharp(R^=) = \{\langle d_1, d_2, d_3\rangle \mid d_1 \in W,\ d_2 \sim_{d_1} d_3\}$

- $I^\sharp(O) = D$

Terms For any term $t \in Ter(\Sigma)$, we define $t^\sharp \in Ter(\Sigma^\sharp)$ inductively on the structure of t:

★$\underline{\text{var}}$: $(x_i)^\sharp = x_{i+1}$ f.a. $i \in \mathbb{N}$

★$\underline{f_n}$: $(f_n(t_1, ..., t_n))^\sharp := f'_{n+1}(x_0, (t_1)^\sharp, ..., (t_n)^\sharp)$

★$\underline{\mathbf{h}_n}$: $(\mathbf{h}_n(t_1, ..., t_n))^\sharp := h'_n((t_1)^\sharp, ..., (t_n)^\sharp)$

Formulas For any $\varphi \in \mathcal{L}$, we define $\varphi^\sharp$ inductively on the structure of φ:

•$\underline{\bot}$ $(\bot)^\sharp := \bot^\sharp$

★$\underline{=}$ $(t = t')^\sharp := R^=(x_0, (t)^\sharp, (t')^\sharp)$

•$\underline{R_n}$ $(R_n(x_{i_1}, ..., x_{i_n}))^\sharp, := R'_{n+1}(x_0, x_{i_1+1}, ..., x_{i_n+1})$

•$\underline{\circ}$ $(\psi \circ \xi)^\sharp := (\psi^\sharp \circ \xi^\sharp)$, where $\circ \in \{\wedge, \mathbin{\vee\!\!\!\vee}, \rightarrow\}$

•$\underline{\forall}$ $(\forall x_n \psi)^\sharp := \forall x_{n+1}(O(x_{n+1}) \rightarrow \psi^\sharp)$

•$\underline{\exists}$ $(\exists x_n \psi)^\sharp := \exists x_{n+1}(O(x_{n+1} \wedge \psi^\sharp))$

2.14 PROPOSITION. *Under the hypotheses of Definition 2.13, if* $\langle M, g\rangle \sim_\sharp \langle M^\sharp, T_M\rangle$, *then f.a.* $\varphi \in \mathcal{L}$, *f.a.* $\mathcal{J} \subseteq \mathcal{I}$,

$$M, s_\mathcal{J} \vDash_g \varphi \iff M^\sharp \vDash_{T_\mathcal{J}} \varphi^\sharp$$

Note that to simplify the appearance of the expressions we suppress the reference to M in the notation $T_\mathcal{J}$ in the previous statement and in the following proof.

Proof. Let $M^\sharp$ be some model that satisfies the hypotheses.
We prove the following, more general claim:
f.a. $n \in \mathbb{N}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M, s_\mathcal{J} \vDash_{g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]} \varphi \iff M^\sharp \vDash_{T_\mathcal{J}[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]} \varphi^\sharp \qquad (2.2.1)$$

Just as we did in the proof of Proposition 2.3, let's introduce the usual notation

- $g^* := g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $g^*_j := g_j[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $(T_\mathcal{J})^* := T_\mathcal{J}[x_{i_1+1} \mapsto d_1, ..., x_{i_n+1} \mapsto d_n]$

With this notation, the statement can be expressed more concisely as follows: f.a. $n \in \mathbb{N}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M, s_{\mathcal{J}} \vDash_{g^*} \varphi \iff M^\sharp \vDash_{T^*_{\mathcal{J}}} \varphi^\sharp$$

The main difference from the proof of the special case is in the interpretation of the terms. Let's verify that the translation behaves well in this regard. The good behaviour can be formalized as follows:
f.a. $t \in Ter[\Sigma]$, f.a. $j \in \mathcal{J}$,

$$[t]^{w_j}_{g^*} = [t^\sharp]^{M^\sharp}_{g^*_j}$$

We prove this claim by induction on the structure of t:

•<u>var</u>: $$[x_i]^{w_j}_{g^*} = g^*(x_i) = \begin{cases} d_i = g^*_j(x_{i+1}) = [x_{i+1}]^{M^\sharp}_{g^*_j} = [(x_i)^\sharp]^{M^\sharp}_{g^*_j} & \text{if } x_i \in \bar{x} \\ g(x_i) = g_j(x_{i+1}) = g^*_j(x_{i+1}) = [x_{i+1}]^{M^\sharp}_{g^*_j} = [(x_i)^\sharp]^{M^\sharp}_{g^*_j} & \text{if } x_i \notin \bar{x} \end{cases}$$

•<u>f_n</u>: $$\begin{aligned} [(f_n(t_1, ..., t_n))^\sharp]^{M^\sharp}_{g^*_j} &= I^\sharp(f'_{n+1})([x_0]^{M^\sharp}_{g^*_j}, [(t_1)^\sharp]^{M^\sharp}_{g^*_j}, ..., [(t_n)^\sharp]^{M^\sharp}_{g^*}) \\ &= I^\sharp(f'_{n+1})(w_j, [t_1]^{w_j}_{g^*}, ..., [t_n]^{w_j}_{g^*}) && \text{(induction)} \\ &= [f_n(t_1, ..., t_n)]^{w_j}_{g^*} \end{aligned}$$

•<u>$\mathbf{h}_n$</u>: $$\begin{aligned} [(\mathbf{h}_n(t_1, ..., t_n))^\sharp]^{M^\sharp}_{g^*_j} &= I^\sharp(h'_n)([(t_1)^\sharp]^{M^\sharp}_{g^*_j}, ..., [(t_n)^\sharp]^{M^\sharp}_{g^*}) \\ &= I^\sharp(h'_n)([t_1]^{w_j}_{g^*}, ..., [t_n]^{w_j}_{g^*}) && \text{(induction)} \\ &= [\mathbf{h}_n(t_1, ..., t_n)]^{w_j}_{g^*} \end{aligned}$$

Proving this claim is the key step for the proof of the overall statement. Now that we have obtained this equality, the rest of the proof is either straightforward or completely analogous to the special case.
We can once again proceed by induction over the structure of φ and the only differences are in the base cases. For the case of relation atoms, the structure of the proof remains unchanged, as we only need to swap the variables with general terms. The case of identity atoms is analogous to the one for relations, especially since identity is interpreted as a ternary relation in $\varphi^\sharp$. □

2.2.1.2 Entailment preservation

2.15 THEOREM. *Let Σ be a signature, let Φ be a set of formulas of $\mathcal{L}$ and let ψ be a formula of $\mathcal{L}$ in Σ.*
Then,

$$\Phi \vDash_{\mathsf{InqBQ}} \psi \iff \Gamma_{\Sigma^\sharp}, \Phi^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp$$

where $\Gamma_{\Sigma^\sharp}$ is the set composed of the following formulas[5]*:*

$$\neg O(x_0) \tag{2.2.2}$$

[5] We do not restate previous explanations of the formulas.

$$O(x_{n+1}) \tag{2.2.3}$$

$$\forall x \forall y_1 ... \forall y_n \left(R'_{n+1}(x, y_1, ..., y_n) \rightarrow \neg O(x) \wedge \bigwedge_{i=1,...,n} O(y_i) \right) \tag{2.2.4}$$

a variation of the previous formulas for the case of the relation symbol $R^{=}$,

$$\forall x \forall y_1 \forall y_2 (R^{=}(x, y_1, y_2) \rightarrow (\neg O(x) \wedge O(y_1) \wedge O(y_2))) \tag{2.2.5}$$

three formulas to describe the reflexive, symmetric and transitive property of $R^{=}$,

$$\forall x \forall y ((\neg O(x) \wedge O(y)) \rightarrow R^{=}(x, y, y)) \tag{2.2.6}$$

$$\forall x \forall y_1 \forall y_2 (R^{=}(x, y_1, y_2) \rightarrow R^{=}(x, y_2, y_1)) \tag{2.2.7}$$

$$\forall x \forall y_1 \forall y_2 \forall y_3 ((R^{=}(x, y_1, y_2) \wedge R^{=}(x, y_2, y_3)) \rightarrow R^{=}(x, y_1, y_3)) \tag{2.2.8}$$

three formulas to describe the congruence conditions for $R^{=}$,

$$\forall x \forall y_1 \forall y_n ... \forall z_1 \forall z_n \\ (R^{=}(x, y_1, z_1) \wedge ... \wedge R^{=}(x, y_n, z_n) \rightarrow f'_{n+1}(x, y_1, ..., y_n) = f'_{n+1}(x, z_1, ..., z_n)) \tag{2.2.9}$$

$$\forall x \forall y_1 \forall y_n ... \forall z_1 \forall z_n \\ (R^{=}(x, y_1, z_1) \wedge ... \wedge R^{=}(x, y_n, z_n) \rightarrow h'_n(y_1, ..., y_n) = h'_n(z_1, ..., z_n)) \tag{2.2.10}$$

$$\forall x \forall y_1 ... \forall y_n \forall z_1 \forall z_n \\ ((R^{=}(x, y_1, z_1) \wedge ... \wedge R^{=}(x, y_n, z_n) \wedge R'_{n+1}(x, y_1, ..., y_n))) \rightarrow R'_{n+1}(x, z_1, ..., z_n)) \tag{2.2.11}$$

$$\exists x (x_{n+1} = x) \tag{2.2.12}$$

formulas to express how functions map tuples of the form ⟨world, n elements of the domain⟩ to elements of the domain,

$$\forall x \forall y_1 ... \forall y_n \forall z \left(\left(f'_{n+1}(x, y_1, ..., y_n) = z \wedge \neg O(x) \wedge \bigwedge_{i=1,...,n} O(y_i) \right) \rightarrow O(z) \right) \tag{2.2.13}$$

and analogous formulas for the case of rigid functions.

$$\forall y_1 ... \forall y_n \forall z \left(\left(h'_n(y_1, ..., y_n) = z \wedge \bigwedge_{i=1,...,n} O(y_i) \right) \rightarrow O(z) \right) \tag{2.2.14}$$

In the above list, we take a formula for all $R_n \in \Sigma$, for all $f_n \in \mathcal{F}_\Sigma$, $\mathbf{h}_n \in \mathcal{F}^R_\Sigma$ and for all $n \in \mathbb{N}$.

Note that, whenever it isn't explicitly relevant, we will omit the reference to $\Sigma^\sharp$ in future occurrences of $\Gamma_{\Sigma^\sharp}$.

2.16 Observation (Meaning of Γ)**.** Once again, it is useful to describe explicitly what it means for a model $M = \langle D, I \rangle$ to satisfy Γ with respect to a team T.
The semantic conditions corresponding to (2.2.2), (2.2.3), (2.2.4) and (2.2.12), and their intuitive meaning, remain unchanged from Observation 2.5. Therefore, we only present the three cases of formulas concerning equality and the two sorts of function symbols.

- $M \vDash_T$ (2.2.5) $\iff \langle d_1, d_2, d_3 \rangle \in I(R^=)$ implies
$$d_1 \notin O_M \text{ e } d_2, d_3 \in O_M$$
That is, $I(R^=) \subseteq (D \setminus O_M) \times (D^2)$.
The intuitive meaning behind this expression is that the relation that represents equality can only be verified at a world (set as the first argument) and by two elements of the domain.

- $M \vDash_T$ (2.2.6), (2.2.7), (2.2.8), (2.2.9), (2.2.10), (2.2.11)
$\iff$
for any world $d \in (D \setminus O_M)$, $\langle d, ., . \rangle \in I(R^=)$ is an equivalence relation on O_M and a congruence with respect to the interpretation of function and relation symbols.
These are precisely the restrictions we imposed on a binary relation of the domain to be an identity relation in Definition 1.4.

- $M \vDash_T$ (2.2.13) $\iff$ the image of $I(f_{n+1})|_{(D \setminus O_M) \times (O_M)^n}$ is contained in D
Intuitively, this means that non-rigid functions map n elements of the domain to an element of the domain, in a way that depends on a world (set as the first argument). This, correctly, doesn't imply anything about the mapping of other $n+1$-tuples.

- $M \vDash_T$ (2.2.14) $\iff$ the image of $I(h_n)|_{(O_M)^n}$ is contained in D
Intuitively, rigid functions associate n elements of the domain to an element of the domain. Again, this doesn't impose any conditions on how other n-tuples could be mapped.

To prove the preservation of entailments under $\sharp$ we will make use of the following result:

2.2.1.3 Encoding an InqBT model in InqBQ

2.17 LEMMA. *Let $M = \langle D, I\rangle$ be a model of InqBT and let T be a team over M such that $M \vDash_T \Gamma$ in the signature $\Sigma^\sharp$.*
Then, there exist a model $M^- := \langle D^-, W^-, I^-, \sim^-\rangle$, where $W = \{w_j \mid j \in \mathcal{I}\}$, a state $s_T \subseteq W^-$ and an assignment g^- in the signature Σ such that

$$T = T_{s_T}$$

where T_{s_T} is defined by the condition (f.a. $j \in \mathcal{I},\ g_j \in T_{s_T} \iff w_j \in s_T$), and such that

$$\langle M^-, g^-\rangle \sim_\sharp \langle M, T_{M^-}\rangle$$

Proof. We start with the definition of $M^- := \langle D^-, W^-, I^-, \sim^-\rangle$:

•$\underline{D^-}$: $D^- := O_M$

•$\underline{W^-}$:$W^- := D \setminus O_M$

•$\underline{I^-}$: (R_n): $I^-_{w_j}(R_n) := \{\langle d_1^- ..., d_n^-\rangle \in (D^-)^n \mid \langle w_j, d_1^-, ..., d_n^-)\rangle \in I(R_{n+1})\}$

(f_n): f.a. $w_j \in W^-$, f.a. $d_1, ..., d_n \in D^-$,

$$I^-_{w_j}(f_n)(d_1, ..., d_n) := I(f'_{n+1})(w_j, d_1, ..., d_n)$$

$(\mathbf{h}_n)$: f.a. $d_1, ..., d_n \in D^-$,

$$I^-_{w_j}(\mathbf{h}_n)(d_1, ..., d_n) := I(\mathbf{h}_n)(d_1, ..., d_n)$$

$(\sim)$: $d_1(\sim^-)_{w_j} d_2 \iff \langle w_j, d_1, d_2\rangle \in I(R^=)$

The condition that D^- and W^- are non-empty sets is fulfilled because $M \vDash_T (2.2.2), (2.2.3)$.
The interpretations of (f_n) and $(\mathbf{h}_n)$ are well defined because of what we said in Observation 2.16 about the formulas in (2.2.13) and (2.2.14). Indeed, for the specific values we consider in their definition, the images of the interpretation of function symbols are all elements of $O_M = D^-$.
Also note that $(\sim^-)_{w_j}$ is a valid identity relation for all $w_j \in W^-$ thanks to the meaning of formulas (2.2.6),(2.2.7),(2.2.8),(2.2.9),(2.2.11). As pointed out in observation 2.16, these formulas are sufficient constraints on $R^=$ to ensure that it induces an equivalence relation which is a congruence with respect to the interpretation function.
Moreover, since $M \vDash (2.2.12)$, all assignments of T coincide on every variable x_i for all $i > 0$ and, since $M \vDash (2.2.3)$, the images of all variables x_{n+1} under

any assignment $g \in T$ are elements of $O_M = D^-$. These properties are useful to define g^-.
Let's then define g^- and s_T:

•$\underline{g^-}$: $g^- : x_n \mapsto g(x_{n+1})$ f.s. $g \in T$

•$\underline{s_T}$: $s_T \coloneqq T[x_0]$

At this point, we only need to prove that equality holds between T and T_{s_T} and that the conditions for the validity of $\langle M^-, g^- \rangle \sim_\sharp \langle M, T \rangle$ are fulfilled. For the parts of the proof that involve D, the symbols R_{n+1}, $I(O)$ and T we refer to the analogous proofs in Lemma 2.6. Thus, we only show the proof of the claims about equality and function symbols.

•$\underline{R^=}$: We need to show that

$$I(R^=) = \{\langle d_1, d_2, d_3 \rangle \mid d_1 \in W^-, d_2(\sim^-)_{d_1} d_3\} :$$

$$(\subseteq): \quad \langle d_1, d_2, d_3 \rangle \in I(R^=) \implies \begin{cases} d_1 \in W^- & \text{(by (2.2.5))} \\ d_2(\sim^-)_{d_1} d_3 & \text{(by def. of } \sim^-\text{)} \end{cases}$$

$\square$

$$\begin{aligned}(\supseteq): \quad & \langle d_1, d_2, d_3 \rangle \in \{\langle d_1, d_2, d_3 \rangle \mid d_1 \in W^-, d_2(\sim^-)_{d_1} d_3\} \\ & \implies d_2(\sim^-)_{d_1} d_3 \\ & \implies \langle d_1, d_2, d_3 \rangle \in I(R^=)\end{aligned}$$

$\square$

•$\underline{f_n}$: let $w_j \in W^-$, $d_1, ..., d_n \in D^-$, then

$$I^-_{w_j}(f_n)(d_1, ..., d_n) = I(f'_{n+1})(w_j, d_1, ..., d_n)$$

•$\underline{\mathbf{h}_n}$: let $d_1, ..., d_n \in D^-$, then

$$I^-_{w_j}(\mathbf{h}_n)(d_1, ..., d_n) = I(h'_n)(d_1, ..., d_n)$$

2.2.1.4 Proof of Theorem 2.15

The $\Longleftarrow$ direction of Theorem 2.15 only requires the good behaviour of the translation $\sharp$:

Proof. By contraposition: assume that
$\Phi \not\models_{\mathsf{InqBQ}} \psi$, then, there exist a model M, a state $s_{\mathcal{J}}$ and an assignment g such that $M, s \models_g \Phi$ and $M, s \not\models_g \psi$.

Now let $M^\sharp$ be any model[6] that satisfies $\langle M, g\rangle \sim_\sharp \langle M^\sharp, T_M\rangle$. From Proposition 2.14 we obtain

$$M, s_{\mathcal{J}} \nvDash_g \psi \iff M^\sharp_{(M,g)} \nvDash_{T_{\mathcal{J}}} \psi^\sharp$$

$$M, s_{\mathcal{J}} \vDash_g \Phi \iff M^\sharp \vDash_{T_{\mathcal{J}}} \Phi^\sharp$$

Clearly $M^\sharp \vDash_{T_{\mathcal{J}}} \Gamma$, since it satisfies by construction the semantic properties corresponding to Γ. Therefore, we have shown a counterexample that validates $\Gamma, \Phi^\sharp \nvDash_{\mathsf{InqBT}} \psi^\sharp$. $\square$

The $\implies$ direction makes use of Lemma 2.17:

Proof. By contraposition: assume that $\Gamma, \Phi^\sharp \nvDash_{\mathsf{InqBT}} \psi^\sharp$.Then, there exist a model of InqBT M and a team T such that $M \vDash_T \Gamma$, $M \vDash_T \Phi^\sharp$ and $M \nvDash_T \psi^\sharp$. By defining M^-, W^-, s_T and g^- as in Lemma 2.17, we get

$$\langle M^-, g^-\rangle \sim_\sharp \langle M, T_{M^-}\rangle$$

Therefore, since $T = T_{s_T} \subseteq T_{M^-}$, by application of Proposition 2.14 we get

$$M \nvDash_T \psi^\sharp \iff M^-, s_T \nvDash_{g^-} \psi$$

$$M \vDash_T \Phi^\sharp \iff M^-, s_T \vDash_{g^-} \Phi$$

This implies $\Phi \nvDash_{\mathsf{InqBQ}} \psi$. $\square$

2.2.2 Translating InqBT → InqBQ

2.2.2.1 Translation of a generic model of InqBT into an equivalent model of InqBQ

2.18 DEFINITION. Most of the definition has already been presented in Definition 2.7. We still detail all the components of the construction, while marking new items with ★.

Signature Let Σ be a signature comprising relation symbols R_n and function symbols f_n. Let's define a signature

★ $\Sigma^\flat := \{R \mid R \in \Sigma\} \cup \{f \mid f \in \Sigma\} \cup \{a_i \mid i \in \mathbb{N}\}$ where the f are rigid function symbols and the a_i are non-rigid constants.

[6] We haven't provided an explicit proof of the existence of such a model, but it's easy to show that any choice of how $I^\sharp(f'_{n+1})$ and $I^\sharp(h')$ map those values that are not included in the conditions given by the relation $\sim^\sharp$ generates a satisfying model.

Model Let $M = \langle D, I\rangle$ be a model of InqBT in the signature Σ and let $T = \{g_j \mid j \in \mathcal{I}\}$ be a team over M.
We define the model $M_T^\flat$ of InqBQ in the signature $\Sigma^\flat$:

- $M_T^\flat \coloneqq \langle D, W_T, I^\flat, \sim^\flat\rangle$ where:
  - $W_T \coloneqq \{w_j \mid j \in \mathcal{I}\}$
  - $I_{w_j}^\flat(R_n) \coloneqq I(R_n)$ f.a. $j \in \mathcal{I},$ f.a. $R_n \in \Sigma^\flat$
  - ★ $I_{w_j}^\flat(f_n) \coloneqq I(f_n)$ f.a. $j \in \mathcal{I},$ f.a. $f_n' \in \Sigma^\flat$
  - $I_{w_j}^\flat(a_i) \coloneqq g_j(x_i)$ f.a. $j \in \mathcal{I},$ f.a. $i \in \mathbb{N}$
  - ★ $d(\sim^\flat)_{w_j} d' \iff d = d'$ f.a. $j \in \mathcal{I}$

Terms For any $t \in Ter(\Sigma)$, we define $t^\flat \in Ter(\Sigma^\flat)$ inductively on the structure of t:

•<u>Var</u>: $(x_i)^\flat \coloneqq a_i$ f.a. $i \in \mathbb{N}$

★<u>f_n</u>: $(f_n(t_1, ..., t_n))^\flat \coloneqq f_n((t_1)^\flat, ..., (t_n)^\flat)$

Formulas For any $\varphi \in \mathcal{L}$, we define $\varphi^\flat$ inductively on the structure of φ:

•<u>$\bot$</u>: $(\bot)^\flat \coloneqq \bot$

★<u>$=$</u>: $(t = t')^\flat \coloneqq (t)^\flat = (t')^\flat$

•<u>R_n</u>: $(R_n(t_1, ..., t_n))^\flat \coloneqq R_n(t_1^\flat, ..., t_n^\flat)$

•<u>$\circ$</u>: $(\psi \circ \xi)^\flat \coloneqq (\psi^\flat \circ \xi^\flat)$, where $\circ \in \{\wedge, \vee\!\!\!\vee, \to\}$

•<u>$\forall$</u>: $(\forall x_n \psi)^\flat \coloneqq \forall x_n(\psi^\flat[x_n/a_n])$

•<u>$\exists$</u>: $(\exists x_n \psi)^\flat \coloneqq \exists x_n(\psi^\flat[x_n/a_n])$

2.19 Observation. It's important to notice that all models generated by the translation $\flat$ are id-models. As we will see, this is not the only property of these models. It is however very significant, as the results on id-models presented in Section 1.1.4 will prove very useful when working with the translated structures.

2.20 PROPOSITION. *Under the hypotheses of Definition 2.18, f.a. $\varphi \in \mathcal{L}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$,*

$$M \vDash_{T_{\mathcal{J}}} \varphi \iff M_T^\flat, s_{\mathcal{J}} \vDash \varphi^\flat$$

Proof. First, let g be any assignment over the model $M_T^\flat$.
As we did in the other cases, we prove the following, more general claim: f.a. $n \in \mathbb{N}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M \vDash_{T_\mathcal{J}[x_{i_1}\mapsto d_1,...,x_{i_n}\mapsto d_n]} \varphi \iff M_T^\flat, s_\mathcal{J} \vDash_{g[x_{i_1}\mapsto d_1,...,x_{i_n}\mapsto d_n]} \varphi^\flat[x_{i_1}/a_{i_1}, ..., x_{i_n}/a_{i_n}]$$

As usual, let's make this statement more readable by introducing some abbreviations:

- $g^* \coloneqq g[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $g_j^* \coloneqq g_j[x_{i_1} \mapsto d_1, ..., x_{i_n} \mapsto d_n]$
- $(T_\mathcal{J})^* \coloneqq T_\mathcal{J}[x_{i_1+1} \mapsto d_1, ..., x_{i_n+1} \mapsto d_n]$
- $\bar{x} \coloneqq \{x_{i_1}, ..., x_{i_n}\}$, $\bar{a} \coloneqq \{a_{i_1}, ..., a_{i_n}\}$
- $\varphi[\bar{x}/\bar{a}] \coloneqq \varphi[x_{i_1}/a_{i_1}, ..., x_{i_n}/a_{i_n}]$

In the new notation, the claim reads as follows:
f.a. $n \in \mathbb{N}$, f.a. $\mathcal{J} \subseteq \mathcal{I}$, f.a. $i_1, ..., i_n \in \mathbb{N}$, f.a. $d_1, ..., d_n \in D$,

$$M \vDash_{T^*_\mathcal{J}} \varphi \iff M, s_\mathcal{J} \vDash_{g^*} \varphi^\flat[\bar{x}/\bar{a}] \tag{2.2.15}$$

The key step of the proof is again the interpretation of terms. We formalize the fact that the translation $\flat$ is well-behaved on the terms as follows:
f.a. $t \in Ter(\Sigma)$ and f.a. $j \in \mathcal{J}$,

$$[t]^M_{g_j^*} = [t^\flat[\bar{x}/\bar{a}]]^{w_j}_{g^*}$$

We prove this claim by induction on the structure of t:

•<u>Var</u>: $[x_i]^M_{g_j^*} = \begin{cases} d_i = g^*(x_i) = [a_i[\bar{x}/\bar{a}]]^{w_j}_{g^*} = [(x_i)^\flat[\bar{x}/\bar{a}]]^{w_j}_{g^*} & \text{if } x_i \in \bar{x} \\ g_j^*(x_i) = g_j(x_i) = I^\flat_{w_j}(a_i) = [a_i[\bar{x}/\bar{a}]]^{w_j}_{g^*} = [(x_i)^\flat[\bar{x}/\bar{a}]]^{w_j}_{g^*} & \text{if } x_i \notin \bar{x} \end{cases}$

•<u>f_n</u>: $$\begin{aligned}[(f_n(t_1, ..., t_n))^\flat[\bar{x}/\bar{a}]]^{w_j}_{g^*} &= I^\flat_{w_j}(f_n)([(t_1)^\flat[\bar{x}/\bar{a}]]^{w_j}_{g^*}, ..., [(t_n)^\flat[\bar{x}/\bar{a}]]^{w_j}_{g^*}) \\ &= I^\flat_{w_j}(f_n)([t_1]^M_{g_j^*}, ..., [t_n]^M_{g_j^*}) \\ &= I(f_n)([t_1]^M_{g_j^*}, ..., [t_n]^M_{g_j^*}) \\ &= [f_n(t_1, ..., t_n)]^M_{g_j^*}\end{aligned}$$

Just like in the case of the translation $\sharp$, we omit the rest of the proof, which again would be carried out by induction on the structure of φ. Indeed, in the base case of relation atoms the difference is only in the substitution of variables with terms. The case of identity atoms is analogous to the case of relations. $\square$

2.2.2.2 Entailment preservation

2.21 THEOREM. *Let Σ be a signature, let Φ be a set of formulas of $\mathcal{L}$ and let ψ be a formula of $\mathcal{L}$ in Σ. Then*

$$\Phi \vDash_{\mathsf{InqBT}} \psi \iff \Delta_{\Sigma^\flat}, \Phi \vDash_{\mathsf{InqBQ}} \psi^\flat$$

where $\Delta_{\Sigma^\flat}$ is the set of formulas where we include the following:

to express the fact that relations are interpreted rigidly,

$$\forall x_1 ... \forall x_n \; ?R_n(x_1, ..., x_n) \tag{2.2.16}$$

to express the fact that functions are interpreted rigidly,

$$\forall x_1 ... \forall x_n \forall x \; ?(f_n(x_1, ..., x_n) = x) \tag{2.2.17}$$

and to characterizes models with decidable identity,

$$\forall x \forall y \; ?(x = y) \tag{2.2.18}$$

In $\Delta_{\Sigma^\flat}$, we consider a formula f.a. $R_n \in \Sigma^\flat$ and f.a. $f_n \in \Sigma^\flat$.
Note that, whenever it isn't explicitly relevant, we will omit the reference to $\Sigma^\flat$ in future occurrences of $\Delta_{\Sigma^\flat}$.

2.22 Observation (Meaning of Δ)**.** As noted in Observation 2.10, the formulas (2.2.16) characterize models of InqBQ where the interpretation of relation symbols does not vary between worlds.
The formulas (2.2.17) have the same meaning for the interpretation of function symbols.
The formula (2.2.18) is a characteristic formula for models with decidable identity. Actually, all models generated by the translation $\flat$ are part of the smaller class of id-models. For our purposes, this distinction becomes irrelevant, since entailment for id-models is equivalent to entailment for models with decidable identity.[7]

Now let's state the encoding Lemma that we will use to prove the preservation of entailment.

2.2.2.3 Encoding an InqBQ model in InqBT

2.23 LEMMA. *Let $M = \langle D, W, I, \sim\rangle$, where $W = \{w_j \mid j \in \mathcal{I}\}$, be an id-model of InqBQ such that $M, W \vDash \Delta$ in the signature $\Sigma^\flat$.*
Then, there exist a model $M^+ := \langle D^+, I^+\rangle$ and a team T^+ in the signature Σ such that their translation $(M^+)^\flat_{T^+} = \langle D^{+\flat}, W^\flat, I^{+\flat}, \sim^\flat\rangle$ is an id-model and for its components the following properties are true:

[7]Recall Proposition 1.32.

- $D^{+\flat} = D$
- $W^\flat \subseteq W$
- $W^\flat \approx_e W$

2.24 COROLLARY**.** *In the hypotheses of the previous lemma, if we take a model M^+ such as the one described in the statement, we have that for any $\varphi \in \mathcal{L}$,*

$$M \vDash \varphi \iff (M^+)^\flat_{T^+} \vDash \varphi$$

Proof of the lemma. First, let's define $M^+ := \langle D^+, I^+ \rangle$:

•$\underline{D^+}$: $D^+ := D$

•$\underline{I^+}$: (R_n): $I^+(R_n) := I_{w_j}(R_n)$ f.s. $j \in \mathcal{I}$
(f_n): $I^+(f_n) := I_{w_j}(f_n)$ f.s. $j \in \mathcal{I}$

Note that in the definition of I^+ we can choose j arbitrarily in $\mathcal{I}$ thanks to what we pointed out in Observation 2.22.
Now, let's define the team T^+. As we did in the proof of Lemma 2.11, we choose a set of representatives W^+ for the equivalence classes of $\approx_e$. We then define the set of indexes $\mathcal{I}^+ := \{j \in \mathcal{I} \mid w_j \in W^+\}$. Now we can define T^+.

•$\underline{g_j}$: $g_j : Var \to D^+$ t.c. $g_j(x_i) = I_{w_j}(a_i)$ f.a. $i \in \mathbb{N}$

•$\underline{T^+}$: $T^+ := \{g_j \mid j \in \mathcal{I}^+\}$

The only thing left to do is verifying that the elements we just defined have the desired properties. We will refer to the proof of Lemma 2.11 for the omitted parts of the proof, which remain unchanged in this general case.
Once again we start from the properties concerning D, W and I:

•$\underline{D}$: omitted

•$\underline{W}$: omitted

•$\underline{I}$: again, we prove additionally that, for all $w_j \in W^\flat$, $I^{+\flat}_{w_j} = I_{w_j}$

$$\begin{aligned} I^{+\flat}_{w_j}(R_n) &= I^+(R_n) = I_{w_j}(R_n) \\ I^{+\flat}_{w_j}(f_n) &= I^+(f_n) = I_{w_j}(f_n) \\ I^{+\flat}_{w_j}(a_i) &= g_j(x_i) = I_{w_j}(a_i) \end{aligned}$$

These equalities allow us to state the claim of essential equivalence between $W^\flat$ and W. We omit the proof of this fact, as it is completely analogous to the one we provided for the special case.
We can now conclude the proof by verifying that $(M^+)^\flat_{T^+}$ is an id-model:

$\bullet$<u>id</u>: $d_1 \sim_{w_j} d_2 \overset{(id-mod)}{\iff} d_1 = d_2 \overset{(\text{Def } 2.18)}{\iff} d_1(\sim^\flat)_{w_j} d_2$

$\square$

2.2.2.4 Proof of Theorem 2.21

For the $\Longleftarrow$ direction of Theorem 2.21 we only need the good behaviour of the translation $\flat$:

Proof. By contraposition: assume $\Phi \nvDash_{\mathsf{InqBT}} \psi$, then, there exist a model M and a team T such that $M \vDash_T \Phi$ and $M \nvDash_T \psi$. Now, based on Proposition 2.20, we have

$$M \nvDash_T \psi \iff M_T^\flat \nvDash \psi^\flat$$

$$M \vDash_T \Phi \iff M_T^\flat \vDash \Phi^\flat$$

From the definition of $M_T^\flat$ and from Observation 2.22, $M_T^\flat \vDash \Delta$. Thus, we have shown a counterexample that proves $\Delta, \Phi^\flat \nvDash_{\mathsf{InqBQ}} \psi^\flat$. $\square$

The $\Longrightarrow$ direction relies on the results of Lemma 2.23 and on the properties of id-models:

Proof. By contraposition: assume that $\Delta, \Phi^\flat \nvDash_{\mathsf{InqBQ}} \psi^\flat$. Then, there exists a model of InqBQ M such that $M \vDash \Delta$, $M \vDash \Phi^\flat$ and $M \nvDash \psi^\flat$. In particular, $M \vDash \forall x \forall y ?(x = y)$, so it has decidable identity. Then, by Proposition 1.31, we have

$$M \nvDash \psi^\flat \iff M_{id} \nvDash \psi^\flat$$

$$M \vDash \Phi^\flat \iff M_{id} \vDash \Phi^\flat$$

Note that, since $\psi^\flat$ and $\Phi^\flat$ are composed of closed formulas only, we can suppress the reference to the assignments g and g_{id} of Proposition 1.31.
Now, by defining $(M_{id})^+$ and T^+ as in Lemma 2.23, we get

$$M_{id} \nvDash \psi^\flat \iff ((M_{id})^+)^\flat_{T^+} \nvDash \psi^\flat \iff (M_{id})^+ \nvDash_{T^+} \psi$$

$$M_{id} \vDash \Phi^\flat \iff ((M_{id})^+)^\flat_{T^+} \vDash \Phi^\flat \iff (M_{id})^+ \nvDash_{T^+} \Phi$$

The first biconditional of each line holds thanks to Corollary 2.24, while the second equivalence is an application of Proposition 2.20.
Together, the two statements provide a counterexample, which proves $\Phi \nvDash_{\mathsf{InqBT}} \psi$.
$\square$

Chapter 3

Repercussions

3.1 Transferring meta-theoretic properties

In this section, we finally cover the first application of the translations that we presented in the previous chapter. The system InqBQ, although extensively studied, still maintains various open questions of fundamental importance. InqBT, on the other hand, has received significantly less attention and the study of its properties is therefore much more limited. In particular, there are two major open questions about the metatheoretic behaviour of the two systems: entailment compactness and the recursive enumerability of their validities. These are extremely important characteristics of a logical system. Additionally, a proof of such properties (or the opposite result) would represent a significant step forward towards proving whether the natural proof system for InqBQ (as defined in [Cia22b]) is complete.
With the results we outline in this chapter, we establish the equivalence of the resolution of these open questions in the two systems. Proving this equivalence provides future research work on these topics with the possibility of addressing the open problems in either system. This means that it is possible to pick what system to work in based both on its suitability to the specific approach in question and on known results about the system.

3.1.1 Entailment compactness

The first such property we set out to consider is entailment compactness. We recall that a logical system S is compact with respect to its entailment relation, say $\vDash_{\mathsf{S}}$, if for any set $\Phi \cup \{\psi\}$ of formulas of S, whenever $\Phi \vDash_{\mathsf{S}} \psi$ holds then there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vDash_{\mathsf{S}} \psi$. Entailment compactness can be a fundamental tool in the construction of a completeness proof for a logical system. As such, it is considered one of the main meta-theoretical properties of any formal system and is therefore of great interest for research in this direction. It is for this reason that we investigate

the relation between its validity in InqBQ and in InqBT.
Whether this property holds in general for the two systems is still an open question. However, let us now show that, thanks to the translations provided in the previous chapter, finding an answer to this question in InqBQ is equivalent to doing so in InqBT.

3.1 PROPOSITION. *Entailment compactness holds for InqBQ if and only if it holds for InqBT.*

Proof. For the left to right direction, suppose that entailment compactness holds for InqBQ. We have to prove that, for any $\Phi \cup \{\psi\} \subseteq \mathcal{L}$,

$$\Phi \vDash_{\mathsf{InqBT}} \psi \text{ implies that f.s. } \Phi_0 \subseteq \Phi,\ \Phi_0 \vDash_{\mathsf{InqBT}} \psi$$

$$\begin{aligned}
\Phi \vDash_{\mathsf{InqBT}} \psi &\iff \Delta, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Thm. 2.21)}\\
&\implies \text{f.s. finite subsets } \Delta_0 \subseteq \Delta \text{ and } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta_0, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by compactness)}\\
&\implies \text{f.s. finite subset } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Phi_0 \vDash_{\mathsf{InqBT}} \psi && \text{(by Thm. 2.21)}
\end{aligned}$$

For the right-to-left direction, the proof is analogous, but we swap the roles of InqBQ and InqBT and use the opposite result of entailment preservation. Suppose that entailment compactness holds in InqBT, then

$$\begin{aligned}
\Phi \vDash_{\mathsf{InqBQ}} \psi &\iff \Delta, \Phi^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp && \text{(by Thm. 2.15)}\\
&\implies \text{f.s. finite subsets } \Delta_0 \subseteq \Delta \text{ and } \Phi_0^\sharp \subseteq \Phi^\flat,\ \Delta_0, \Phi_0^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp && \text{(by compactness)}\\
&\implies \text{f.s. finite subset } \Phi_0^\sharp \subseteq \Phi^\sharp,\ \Delta, \Phi_0^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Delta, \Phi_0^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Phi_0 \vDash_{\mathsf{InqBQ}} \psi && \text{(by Thm. 2.15)}
\end{aligned}$$

□

3.1.2 Semi-decidability of validity

Let us now turn our attention to another metatheoretic property: the semi-decidability of validity. We recall that validity is semi-decidable in a logical system if there is an effective, computable procedure that halts on a formula only if the formula is a validity of the system. We also recall the fact that the semi-decidability of validity is equivalent to the recursive enumerability of validities, which holds for a system when there is an effective, computable procedure that lists all its validities. We will use the two notions interchangeably in the following sections.

This property holds for first order classical logic, but fails for second order classical logic. It is still an open problem for InqBQ and for InqBT. As we did for entailment compactness, we will show how finding an answer to this question in one of these systems is equivalent to resolving it in the other one.

Before proceeding with the proof of the equivalence, let us state some technical definitions and two consequences of Theorems 2.15 and 2.21. These lemmas will be useful in more than one context.

3.2 DEFINITION. Let Σ be a signature for InqBQ, let ψ be a formula of $\mathcal{L}$ in the signature Σ. We define the following signature:

$$\Sigma^\sharp_{\psi^\sharp} := \{\sigma \in \Sigma^\sharp \mid \sigma \text{ appears in } \psi^\sharp\}$$

3.3 DEFINITION. Let Σ be a signature for InqBQ, let ψ be a formula of $\mathcal{L}$ in the signature Σ. We define the following set of formulas:

$$\Gamma_{\psi^\sharp} := \Gamma_{\Sigma^\sharp_{\psi^\sharp}}|_{FV(\psi^\sharp)}$$

where $\Gamma_{\Sigma^\sharp_{\psi^\sharp}}|_{FV(\psi^\sharp)} = \{\varphi \in \Gamma_{\Sigma^\sharp_{\psi^\sharp}} \mid FV(\varphi) \subseteq FV(\psi^\sharp)\}$

3.4 LEMMA. *In the same hypotheses of Theorem 2.15, as a consequence of the theorem, we have the following result:*

$$\vDash_{\mathsf{InqBQ}} \psi \iff \Gamma_{\psi^\sharp} \vDash_{\mathsf{InqBT}} \psi^\sharp \iff \vDash_{\mathsf{InqBT}} \bigwedge \Gamma_{\psi^\sharp} \to \psi^\sharp$$

Proof. For the first biconditional, we can separate the proof in two parts: the first part is a direct consequence of Theorem 2.15, the second part, although very intuitive, requires additional results.
We can apply the theorem to state that

$$\vDash_{\mathsf{InqBQ}} \psi \iff \Gamma_{\Sigma^\sharp_{\psi^\sharp}} \vDash_{\mathsf{InqBT}} \psi^\sharp$$

Now, we just need to show the following:

$$\Gamma_{\Sigma^\sharp_{\psi^\sharp}} \vDash_{\mathsf{InqBT}} \psi^\sharp \iff \Gamma_{\psi^\sharp} \vDash_{\mathsf{InqBT}} \psi^\sharp$$

The right-to-left direction is obvious, as one of the sets of premises is a subset of the other. Let's prove the left-to-right direction of the claim:
by contraposition, suppose $\Gamma_{\psi^\sharp} \nvDash_{\mathsf{InqBT}} \psi^\sharp$. Then, there exist a model M and a team T such that $M \vDash_T \Gamma_{\psi^\sharp}$ and $M \nvDash_T \psi^\sharp$. Then, we can consider an arbitrary team T' such that

1. $T'|_{FV(\psi^\sharp)} = T|_{FV(\psi^\sharp)}$
2. f.a. $n \in \mathbb{N}$, x_{n+1} is evaluated constantly by T' into an element of O_M

3. f.a. $g \in T'$, $g(x_0) \in D \setminus O_M$

Such a team exists because the three conditions are compatible. The only potential problem is the compatibility of conditions 1. and 2. but, since $M \vDash_T \Gamma_{\psi^\sharp}$, all variables in $FV(\psi^\sharp)$ are evaluated constantly[1].
Now, combining conditions 2. and 3. with the fact that $\Gamma_{\Sigma^\sharp_{\psi^\sharp}}$ and $\Gamma_{\psi^\sharp}$ differ on formulas that only include variables and the symbol O, we get that $M \vDash_{T'} \Gamma_{\Sigma^\sharp_{\psi^\sharp}}$.
Also, $M \nvDash_{T'} \psi^\sharp$ since, by Proposition 1.44, for any formula φ, $T'|_{FV(\varphi)} = T|_{FV(\varphi)}$ implies $M \vDash_T \varphi \iff M \vDash_{T'} \varphi$.
Therefore, we have a counterexample that proves $\Gamma_{\Sigma^\sharp_{\psi^\sharp}} \nvDash_{\mathsf{InqBT}} \psi^\sharp$.

For the second biconditional, it suffices to note that $\Gamma_{\psi^\sharp}$ is a finite set of formulas. Notice that the starting set $\Gamma_{\Sigma^\sharp}$ is always infinite, as it contains multiple formulas for each one of the countably many variables. We can check the finiteness of $\Gamma_{\psi^\sharp}$ explicitly. First, observe that $\Sigma^\sharp_{\psi^\sharp}$ and $FV(\psi^\sharp)$ are finite. Then let's verify that the number of formulas in $\Gamma_{\psi^\sharp}$ that contain a certain symbol or variable is finite:

- $\underline{R}$: for each relation symbol of the form $R'_{n+1} \in \Sigma^\sharp_{\psi^\sharp}$, we have two formulas: (2.2.4) and (2.2.11)
- $\underline{f}$: for each function symbol of the form $f'_{n+1} \in \Sigma^\sharp_{\psi^\sharp}$, we have two formulas: (2.2.13) and (2.2.9)
- $\underline{h}$: for each rigid function symbol of the form $h'_n \in \Sigma^\sharp_{\psi^\sharp}$, we have two formulas: (2.2.14) and (2.2.10)
- $\underline{R^=}$: for the symbol $R^=$, we have the four formulas (2.2.5), (2.2.6), (2.2.7), (2.2.8), one formula (2.2.11) for each relation symbol of $\Sigma^\sharp_{\psi^\sharp}$, one formula (2.2.9) for each function symbol and one formula (2.2.10) for each rigid function symbol
- $\underline{O}$: for the symbol O, we have the four formulas (2.2.2), (2.2.3), (2.2.5), (2.2.6), one formula (2.2.4) for each relation symbol of $\Sigma^\sharp_{\psi^\sharp}$, one formula (2.2.13) for each function symbol and one formula (2.2.14) for each rigid function symbol
- $\underline{x_0}$: for the variable x_0, we have the formula (2.2.2)
- $\underline{x_{n+1}}$: for each variable of the form x_{n+1}, we have the formulas (2.2.3) and (2.2.12)

[1] Recall Observation 2.16.

Since $\Gamma_{\psi^\sharp}$ is finite, the biconditional $\Gamma_{\psi^\sharp} \vDash_{\mathsf{InqBT}} \psi^\sharp \iff \vDash_{\mathsf{InqBT}} \bigwedge \Gamma_{\psi^\sharp} \to \psi^\sharp$ holds. $\square$

The motivation for this formulation of the preservation of validity by the translation $\sharp$ lies in the finiteness of $\Gamma_{\Sigma^\sharp_{\psi^\sharp}}$. Indeed, entailments with finitely many premises have special properties when compared to general entailments. This will be useful later.

3.5 DEFINITION. Let Σ be a signature for InqBQ, let ψ be a formula of $\mathcal{L}$ in the signature Σ. We define the following signature:

$$\Sigma^\flat_{\psi^\flat} \coloneqq \{\sigma \in \Sigma^\flat \mid \sigma \text{ appears in } \psi^\flat\}$$

3.6 LEMMA. *In the same hypotheses of Theorem 2.21, as a consequence of the theorem, we have the following result:*

$$\vDash_{\mathsf{InqBT}} \psi \iff \Delta_{\Sigma^\flat_{\psi^\flat}} \vDash_{\mathsf{InqBQ}} \psi^\flat \iff \vDash_{\mathsf{InqBQ}} \bigwedge \Delta_{\Sigma^\flat_{\psi^\flat}} \to \psi^\flat$$

Proof. The first biconditional is a direct consequence of Theorem 2.21.
For the second biconditional, we proceed as above. First, we note that $\Sigma^\flat_{\psi^\flat}$ is finite. Then, we can observe that $\Delta_{\Sigma^\flat_{\psi^\flat}}$ contains the formula (2.2.18), exactly one formula for each relation symbol of $\Sigma^\flat_{\psi^\flat}$, namely (2.2.16), and exactly one formula for each function symbol of $\Sigma^\flat_{\psi^\flat}$, namely (2.2.17). Since $\Delta_{\Sigma^\flat_{\psi^\flat}}$ is finite, the biconditional $\Delta_{\Sigma^\flat_{\psi^\flat}} \vDash_{\mathsf{InqBQ}} \psi^\flat \iff \vDash_{\mathsf{InqBQ}} \bigwedge \Delta_{\Sigma_{\psi^\flat}} \to \psi^\flat$ holds. $\square$

3.7 PROPOSITION. *The validities of InqBQ are recursively enumerable if and only if the validities of InqBT are recursively enumerable*

Proof. We prove this claim by showing that, given a formula in one of the systems, there is a computable way to determine a formula of the other system which is a validity if and only if the original formula is. The procedure relies on Lemmas 3.4 and 3.6, which in turn make use of the translations. Showing the existence of such a procedure is enough to prove semi-decidability of whether a formula is a validity of one system, assuming we have this property in the other system. The procedure to semi-decide this problem would go as follows: given a formula in one of the two systems, we compute its corresponding formula in the other system; at this point, we know that whether this resulting formula is a validity or not (and therefore whether this is true of the starting formula) is a semi-decidable matter.
For the right to left direction, by Lemma 3.4, for any $\psi \in \mathcal{L}$,

$$\vDash_{\mathsf{InqBQ}} \psi \iff \vDash_{\mathsf{InqBT}} \bigwedge \Gamma_{\psi^\sharp} \to \psi^\sharp$$

So ψ is a validity of InqBQ if and only if $\bigwedge \Gamma_{\psi^\sharp} \to \psi^\sharp$ is a validity of InqBT. The resulting formula is computable from the syntax of ψ. Indeed, $\Gamma_{\psi^\sharp}$ is both finite and computable from the syntax of $\psi^\sharp$, which in turn is computable from the syntax of ψ.

For the left to right direction, by Lemma 3.6, for any $\xi \in \mathcal{L}$,

$$\vDash_{\mathsf{InqBT}} \xi \iff \vDash_{\mathsf{InqBQ}} \bigwedge \Delta_{\Sigma_{\xi^\flat}} \to \xi^\flat$$

So ξ is validity of InqBT if and only if $\bigwedge \Delta_{\Sigma_{\xi^\flat}} \to \xi^\flat$ is a validity of InqBQ. As before, this formula is computable from the syntax of ξ. $\square$

3.2 Transferring results about properties of fragments

We now turn our attention to some notable fragments of the two systems. So far, the fragments of InqBQ have been a research area of great interest and they have proved to be easier to approach in comparison with the complete system. In particular, an axiomatization has been found for two wide syntactical fragments, called Rex and Clant. It is therefore interesting to investigate the relation of certain fragments of InqBQ with their correspectives in InqBT. We will transfer known results that have already been proved for the finitely coherent, Rex and Clant fragments of InqBQ to the corresponding fragments of InqBT by employing the translations.

In particular, we will show how using the translations we can prove entailment compactness for the finitely coherent fragment. We then transfer results of finite coherence, entailment compactness and semi-decidability of validity to the Rex fragment of InqBT. Finally, we use the translations to transfer the existence of an entailment preserving translation from the Clant fragment to first order classical logic and, again, entailment compactness and semi-decidability of validity for the fragment.

The section's contents will be structured as follows: first we define a fragment, property or concept, then we outline the already known results of InqBQ that involve said definition and, finally, we transfer these results to InqBT by using the translations. Note that we will omit proofs of known properties concerning fragments of InqBQ. Unless explicitly stated otherwise, we refer to [Cia22b] for a more complete treatment of these matters.

3.2.1 The finitely coherent fragment

Finite coherence is one of the properties of inquisitive formulas that have no counterparts in classical logic. Its consequences on the behaviour of formulas are however very significant, as we will see. It is therefore interesting to study if the translations respect this property for formulas in the two systems and the consequences of this fact.

Let's start by defining k-coherence in InqBQ and in InqBT.

3.8 DEFINITION. For a cardinal k and a formula $\varphi \in \mathcal{L}$, we say that φ is k-coherent in InqBQ if, for all models M, for all states s and for all assignments g,

$$M, s \vDash_g \varphi \iff (\text{f.a. } t \subseteq s \text{ s.t. } \#t \leq k,\ M, t \vDash_g \varphi)$$

For a cardinal k and a formula $\varphi \in \mathcal{L}$, we say that φ is k-coherent in InqBT if, for all models M and for all teams T,

$$M \vDash_T \varphi \iff (\text{f.a. } T' \subseteq T \text{ s.t. } \#T' \leq k,\ M \vDash_{T'} \varphi)$$

Let's start with a basic application of the translations:

3.9 PROPOSITION. *For any formula $\varphi \in \mathcal{L}$, for any cardinal k,*

$$\varphi^\sharp \text{ is } k\text{-coherent in } \mathsf{InqBT} \implies \varphi \text{ is } k\text{-coherent in } \mathsf{InqBQ}$$

$$\varphi^\flat \text{ is } k\text{-coherent in } \mathsf{InqBQ} \implies \varphi \text{ is } k\text{-coherent in } \mathsf{InqBT}$$

Proof. We only prove the first claim, as the second claim is completely analogous.
Suppose that $\varphi^\sharp$ is k-coherent in InqBT. Let M be a model of InqBQ where W is indexed on some $\mathcal{I}$, let g be an assignment over M. Then, for all $\mathcal{J} \subseteq \mathcal{I}$, for all states $s_{\mathcal{J}} \subseteq W$ and for any model $M^\sharp$ such that $\langle M, g\rangle \sim_\sharp \langle M^\sharp, T_M\rangle$,

$$\begin{aligned}
M, s_{\mathcal{J}} \vDash_g \varphi &\iff M^\sharp \vDash_{T_{\mathcal{J}}} \varphi^\sharp && \text{(by Prop. 2.14)}\\
&\iff \text{f.a. } (T_{\mathcal{J}})' \subseteq T_{\mathcal{J}} \text{ with } \#(T_{\mathcal{J}})' \leq k,\ M^\sharp \vDash_{(T_{\mathcal{J}})'} \varphi^\sharp && (k\text{-coherence})\\
&\iff \text{f.a. } \mathcal{J}' \subseteq \mathcal{J} \text{ with } \#\mathcal{J}' \leq k,\ M^\sharp \vDash_{(T_{\mathcal{J}})'} \varphi^\sharp && \text{(by Obs. 2.2)}\\
&\iff \text{f.a. } \mathcal{J}' \subseteq \mathcal{J} \text{ with } \#\mathcal{J}' \leq k,\ M, s_{\mathcal{J}'} \vDash_g \varphi && \text{(by Prop. 2.14)}\\
&\iff \text{f.a. } (s_{\mathcal{J}})' \subseteq s_{\mathcal{J}} \text{ with } \#(s_{\mathcal{J}})' \leq k,\ M, (s_{\mathcal{J}})' \vDash_g \varphi && \text{(by Obs. 2.2)}
\end{aligned}$$

This proves k-coherence in InqBQ for φ. □

This property is not too informative; however, it can help with the transfer of some properties between the two systems.
We can obtain a similar but weaker result in the opposite direction. As we should expect from the characterization of the translated models, we have that k-coherence for a formula only implies k-coherence over, respectively, Γ-models or Δ-id-models for its translation. Let's formalize this claim.

3.10 DEFINITION. For a cardinal k and a set of formulas $\Phi \cup \{\psi\} \subseteq \mathcal{L}$, we say that ψ is k-coherent over Φ-models in InqBQ if, for all models M, for all states s and for all assignments g such that $M, s \vDash_g \Phi$,

$$M, s \vDash_g \varphi \iff (\text{f.a. } t \subseteq s \text{ s.t. } \#t \leq k,\ M, t \vDash_g \psi)$$

For a cardinal k and a set of formulas $\Phi \cup \{\psi\} \subseteq \mathcal{L}$, we say that ψ is k-coherent over Φ-models in InqBT if, for all models M and for all teams T such that $M \vDash_T \Phi$,

$$M \vDash_T \psi \iff (\text{f.a. } T' \subseteq T \text{ s.t. } \#T' \leq k,\ M \vDash_{T'} \psi)$$

3.11 PROPOSITION. *For any formula $\varphi \in \mathcal{L}$, for any cardinal k,*

$$\varphi \text{ is } k\text{-coherent in } \mathsf{InqBQ} \implies \varphi^\sharp \text{ is } k\text{-coherent over } \Gamma\text{-models in } \mathsf{InqBT}$$

$$\varphi \text{ is } k\text{-coherent in } \mathsf{InqBT} \implies \varphi^\flat \text{ is } k\text{-coherent over } \Delta\text{-id-models in } \mathsf{InqBQ}$$

where Γ and Δ are the sets of characteristic formulas for the translated models that we defined in Chapter 2.2.

Proof. Let's prove the first claim. We need to show that, if we suppose φ to be k-coherent in InqBQ, then for any model M and for any team T such that $M \vDash_T \Gamma$,

$$M \vDash_T \varphi^\sharp \iff (\text{f.a. } T' \subseteq T \text{ s.t. } \#T' \leq k,\ M \vDash_{T'} \varphi^\sharp)$$

We show both directions of the biconditional simultaneously, although only the right-to-left direction actually needs to be proven.
First, we take M^- and g^- to be such that $\langle M^-, g^- \rangle \sim_\sharp \langle M, T_{M^-} \rangle$. The existence of such two objects follows from Lemma 2.17. Then, if we define s_T as in Lemma 2.17, we have that

$$\begin{aligned}
M \vDash_T \varphi^\sharp &\iff M^-, s_T \vDash_{g^-} \varphi && \text{(by Prop. 2.14)}\\
&\iff \text{f.a. } (s_T)' \subseteq s_T \text{ s.t. } \#(s_T)' \leq k,\ M^-, (s_T)' \vDash_{g^-} \varphi && \text{(by } k\text{-coherence)}\\
&\iff \text{f.a. } T' \subseteq T \text{ s.t. } \#T' \leq k,\ M \vDash_{T'} \varphi^\sharp && \text{(by Prop. 2.14)}
\end{aligned}$$

The proof of the second claim is slightly more complicated, but follows a similar structure.
We need to show that, assuming φ to be k-coherent in InqBT, then, for any id-model M and for any state s such that $M, s \vDash \Delta$ [2],

$$M, s \vDash \varphi^\flat \iff (\text{f.a. } t \subseteq s \text{ s.t. } \#t \leq k,\ M, s \vDash \varphi^\flat)$$

Again, we show both directions of this biconditional at the same time.
Note that in the following step we will consider a model $(M|_s)^+$, a team T^+ and a state $s^\flat$. These are defined in accordance with Lemma 2.23.

$$\begin{aligned}
M, s \vDash \varphi^\flat &\iff M|_s \vDash \varphi^\flat && \text{(by Prop. 1.23)}\\
&\iff ((M|_s)^+)^\flat_{T^+} \vDash \varphi^\flat && \text{(by Cor. 2.24)}\\
&\iff (M|_s)^+ \vDash_{T^+} \varphi && \text{(by Prop. 2.20)}\\
&\iff \text{f.a. } T' \subseteq T^+ \text{ s.t. } \#T' \leq k,\ (M|_s)^+ \vDash_{T'} \varphi && \text{(by } k\text{-coherence)}\\
&\iff \text{f.a. } t \subseteq s \text{ s.t. } \#t \leq k,\ ((M|_s)^+)^\flat_{T^+}, t \vDash \varphi^\flat && \text{(by Prop. 2.20)}\\
&\overset{(!)}{\iff} \text{f.a. } t \subseteq s \text{ s.t. } \#t \leq k,\ M|_s, t \vDash \varphi^\flat\\
&\iff \text{f.a. } t \subseteq s \text{ s.t. } \#t \leq k,\ M, t \vDash \varphi^\flat
\end{aligned}$$

(!): this step is more delicate then the rest and, while intuitive, it isn't a direct consequence of any result that we have explicitly stated. Let's unpack its validity. First, note that $((M|s)^+)^\flat_{T^+} = \langle D, s^\flat, I|^\flat_s, \sim^\flat \rangle$, where $s^\flat \approx_e s$ and, therefore, $s^\flat \subseteq s$ also. Therefore, thanks to the definition of essential equivalence, we have that

$$\text{f.a. } t^\flat \subseteq s^\flat,\ ((M|_s)^+)^\flat_{T^+} \vDash \varphi^\flat \iff M|_s, t^\flat \vDash \varphi^\flat \tag{3.2.1}$$

With this observation we can prove the biconditional.

[2] Note that, since all formulas of Δ and φ are sentences, we can omit the reference to a specific assignment.

($\implies$) By contraposition, suppose there is $t \subseteq s$ s.t. $\#t \leq k$ and $M|_s, t \nvDash \varphi^\flat$.
Then, since $s^\flat \approx_e s$, there is $t^\flat \subseteq s^\flat$ s.t. $t^\flat \approx_e t$.
Therefore, there is $t^\flat \subseteq s^\flat$ s.t. $\#t^\flat \leq \#t \leq k$ and $M|s, t^\flat \nvDash \varphi^\flat$.
By (3.2.1), this implies that there is $t^\flat \subseteq s^\flat$ s.t. $\#t^\flat \leq k$ and $((M|_s)^+)^\flat_{T^+}, t^\flat \nvDash \varphi^\flat$.

($\impliedby$) By contraposition, suppose there is $t^\flat \subseteq s^\flat$ s.t. $\#t^\flat \leq k$ and $((M|_s)^+)^\flat_{T^+}, t^\flat \nvDash \varphi^\flat$.
Then, there is $t^\flat \subseteq s$ s.t. $\#t^\flat \leq k$ and, by (3.2.1), $M|s, t^\flat \nvDash \varphi^\flat$.

$\square$

We can now move on to the consequences of the good behaviour of finite coherence with respect to the translations. The first key property we can transfer is entailment compactness:

3.12 PROPOSITION (Compactness for finitely coherent conclusions, InqBQ - [CG22], Theorem 4.3)**.**
Let $\Phi \subseteq \mathcal{L}$ and let $\psi \in \mathcal{L}$ be a finitely coherent formula. Then,

If $\Phi \vDash_{\mathsf{InqBQ}} \psi$, there exists a finite subset $\Phi_0 \subseteq \Phi$ s.t. $\Phi_0 \vDash_{\mathsf{InqBQ}} \psi$

3.13 PROPOSITION (Compactness for finitely coherent conclusions, InqBT)**.**
Let $\Phi \subseteq \mathcal{L}$ and let $\psi \in \mathcal{L}$ be a finitely coherent formula. Then,

If $\Phi \vDash_{\mathsf{InqBT}} \psi$, there exists a finite subset $\Phi_0 \subseteq \Phi$ s.t. $\Phi_0 \vDash_{\mathsf{InqBT}} \psi$

Proof. To prove this proposition, we transfer the InqBQ result to InqBT by integrating the proof of Proposition 3.1 with the preservation of finite coherence by the translation $\flat$. Indeed, keeping in mind that ψ is finitely coherent and, therefore, $\psi^\flat$ is also finitely coherent by Proposition 3.9, we have

$$
\begin{aligned}
\Phi \vDash_{\mathsf{InqBT}} \psi &\iff \Delta, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Thm. 2.21)}\\
&\implies \text{f.s. finite subsets } \Delta_0 \subseteq \Delta \text{ and } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta_0, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Prop. 3.12)}\\
&\implies \text{f.s. finite subset } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Phi_0 \vDash_{\mathsf{InqBT}} \psi && \text{(by Thm. 2.21)}
\end{aligned}
$$

$\square$

3.2.2 The Restricted existential (Rex) fragment

The Rex fragment is a well-behaved and extensive syntactical fragment of $\mathcal{L}$ where inquisitive existentials can only appear in the antecedent of an implication. A sound and complete axiomatization has been established for the Rex fragment of InqBQ in [CG22]. Here, we show that our translation preserve the fragment between the two systems and how this helps us transfer some of its most important properties from InqBQ to InqBT. We also refer to [CG22] for any omitted proofs not found in this subsection.
Let's begin our presentation with a formal definition of the fragment.

3.14 DEFINITION. Let Σ be a signature. The set of rex formulas $\mathcal{L}_{Rex}(\Sigma)$ is given by the following syntax:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \psi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \forall x \varphi$$

where p are atomic sentences in the signature Σ and ψ are arbitrary formulas of $\mathcal{L}_\Sigma$.
Note that, by definition, an implication is a rex formula if and only if its consequent is a rex formula. We will use this fact throughout this section.

Clearly, the fragment remains unchanged between the two systems. One may still wonder whether the translations of rex formulas are rex formulas themselves. This can be proven quite easily:

3.15 PROPOSITION. *Let $\varphi \in \mathcal{L}$ in a signature Σ, then*

$$\varphi \in \mathcal{L}_{Rex}(\Sigma) \iff \varphi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp)$$

$$\varphi \in \mathcal{L}_{Rex}(\Sigma) \iff \varphi^\flat \in \mathcal{L}_{Rex}(\Sigma^\flat)$$

Proof. Let's start with the first claim. We prove both directions simultaneously by induction on the structure of φ:

•<u>atoms</u>: the atomic cases are obviously verified.

•<u>$\wedge$</u>:
$$\begin{aligned} \psi \wedge \xi \in \mathcal{L}_{Rex}(\Sigma) &\iff \psi \in \mathcal{L}_{Rex}(\Sigma) \text{ and } \xi \in \mathcal{L}_{Rex}(\Sigma) \\ &\iff \psi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \text{ and } \xi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \quad \text{(induction)} \\ &\iff \psi^\sharp \wedge \xi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \\ &\iff (\psi \wedge \xi)^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \end{aligned}$$

•<u>$\vee\!\!\!\vee$</u>: analogous, omitted

•<u>$\to$</u>:
$$\begin{aligned} \psi \to \xi \in \mathcal{L}_{Rex}(\Sigma) &\iff \xi \in \mathcal{L}_{Rex}(\Sigma) \\ &\iff \xi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \quad \text{(induction)} \\ &\iff \psi^\sharp \to \xi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \\ &\iff (\psi \to \xi)^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \end{aligned}$$

$$
\begin{aligned}
\bullet\underline{\forall}: \quad \forall x_i \psi \in \mathcal{L}_{Rex}(\Sigma) &\iff \psi \in \mathcal{L}_{Rex}(\Sigma) \\
&\iff \psi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) && \text{(induction)} \\
&\iff O(x_{i+1}) \to \psi^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp) \\
&\iff \forall x_{i+1}(O(x_{i+1}) \to \psi^\sharp) \in \mathcal{L}_{Rex}(\Sigma^\sharp) \\
&\iff (\forall x_i \psi)^\sharp \in \mathcal{L}_{Rex}(\Sigma^\sharp)
\end{aligned}
$$

$\bullet\underline{\exists}$: $\exists x_i \psi$ and $\exists x_{i+1}(O(x_{i+1}) \wedge \psi^\sharp)$ are never rex formulas.

We can then move on to the second claim, which we also prove by induction on the structure of φ:

$\bullet$<u>atoms</u>: the atomic cases are obviously verified.

$\bullet\underline{\wedge, \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}, \to}$: omitted

$$
\begin{aligned}
\bullet\underline{\forall}: \quad \forall x_i \psi \in \mathcal{L}_{Rex}(\Sigma) &\iff \psi \in \mathcal{L}_{Rex}(\Sigma) \\
&\iff \psi^\flat \in \mathcal{L}_{Rex}(\Sigma^\flat) && \text{(induction)} \\
&\iff \psi^\flat[x_i/a_i] \in \mathcal{L}_{Rex}(\Sigma^\flat) \\
&\iff \forall x_i(\psi^\flat[x_i/a_i]) \in \mathcal{L}_{Rex}(\Sigma^\flat) \\
&\iff (\forall x_i \psi)^\flat \in \mathcal{L}_{Rex}(\Sigma^\flat)
\end{aligned}
$$

$\bullet\underline{\exists}$: $\exists x_i \psi$ and $\exists x_i(\psi^\flat[x_i/a_i])$ are never rex formulas.

□

3.2.2.1 Finite coherence and compactness

All formulas in Rex are finitely coherent with respect to InqBQ. As we observed, this property has important repercussions on the behaviour of formulas. Thanks to the translations, we can transfer both the finite coherence of Rex formulas and its consequences to InqBT.

3.16 PROPOSITION (Finite coherence, InqBQ - [CG22], Proposition 5.3)**.** *For every formula $\varphi \in \mathcal{L}_{Rex}$ there is a natural number n_φ, computable from the syntax of φ, such that φ is n_φ-coherent in InqBQ.*

Proof. We refer to [CG22] for a full proof of this result. Here, we limit the presentation to the inductive definition of the optimal estimate n_φ for the coherence degree of any given rex formula φ.

$\bullet$<u>atoms</u>: $n_\varphi = 1$ for any atomic formula;

$\bullet\underline{\wedge}$: $n_{(\psi \wedge \xi)} = max\{n_\psi, n_\xi\}$;

$\bullet\underline{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$: $n_{(\psi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \xi)} = n_\psi + n_\xi$;

•$\underline{\rightarrow}$: $n_{(\psi \rightarrow \xi)} = n_\xi$;

•$\underline{\forall}$: $n_{(\forall x_i \psi)} = n_\psi$.

□

3.17 PROPOSITION (Finite coherence, InqBT). *For every formula $\varphi \in \mathcal{L}_{Rex}$ there is a natural number n_φ, computable from the syntax of φ, such that φ is n_φ-coherent in InqBT. For all formulas, the estimate given for InqBQ in the above proposition is still a good estimate.*

Proof. We can use the translation $\sharp$ to transfer to InqBT the same estimates for the coherence degree. To this end, we will make use of Proposition 3.9. Our claim is that $n_\varphi = n_{\varphi^\sharp}$ is a good estimate and that it is computable from the syntax of φ.
Obviously, from Proposition 3.9, φ is always $n_{\varphi^\sharp}$-coherent. Therefore, we only need to prove the computability of the estimate, which we do by induction on the structure of φ:

•<u>atoms</u>: again, the atomic case is obvious.

•$\underline{\wedge}$: $n_{(\psi \wedge \xi)^\sharp} = max\{n_{\psi^\sharp}, n_{\xi^\sharp}\}$. By induction, both $n_{\psi^\sharp}$ and $n_{\xi^\sharp}$ are computable from the syntax of $\varphi = \psi \wedge \xi$.

•$\underline{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$: analogous, omitted

•$\underline{\rightarrow}$: $n_{(\psi \rightarrow \xi)^\sharp} = n_{\xi^\sharp}$. By induction, $n_{\xi^\sharp}$ is computable from the syntax of $\varphi = \psi \rightarrow \xi$

•$\underline{\forall}$: $n_{(\forall x_i \psi)^\sharp} = n_{(\forall x_{i+1}(O(x_{i+1}) \rightarrow \psi^\sharp))} = n_{(O(x_{i+1}) \rightarrow \psi^\sharp)} = n_{\psi^\sharp}$. Again, this is computable from the syntax of $\varphi = \forall x_i \psi$ by induction.

□

3.18 PROPOSITION (Compactness for rex conclusions, InqBQ). *Entailment compactness holds for all InqBQ entailments where the conclusion is a rex formula.*

3.19 PROPOSITION (Compactness for rex conclusions, InqBT). *Entailment compactness holds for all InqBT entailments where the conclusion is a rex formula.*

Proof. To prove this proposition, we transfer the InqBQ result to InqBT by integrating the proof of Proposition 3.1 with the closure of $\mathcal{L}_{Rex}$ under the translation $\flat$. Indeed, since $\psi \in \mathcal{L}_{Rex}$ and, therefore, $\psi^\flat \in \mathcal{L}_{Rex}$ by

Proposition 3.15, we have

$$
\begin{aligned}
\Phi \vDash_{\mathsf{InqBT}} \psi &\iff \Delta, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Thm. 2.21)}\\
&\implies \text{f.s. finite subsets } \Delta_0 \subseteq \Delta \text{ and } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta_0, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Prop. 3.18)}\\
&\implies \text{f.s. finite subset } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Phi_0 \vDash_{\mathsf{InqBT}} \psi && \text{(by Thm. 2.21)}
\end{aligned}
$$

□

3.2.2.2 Recursive enumerability of validities

Another fundamental metatheoretical property of the Rex fragment of InqBQ that we can easily prove for InqBT is the recursive enumerability of validities:

3.20 PROPOSITION (Recursive enumerability of Rex validities, InqBQ - [CG22], Theorem 5.4)**.** *The set $Val^Q_{Rex} \coloneqq \{\varphi \in \mathcal{L}_{Rex} \mid \varphi$ is valid in InqBQ$\}$ is recursively enumerable.*

3.21 PROPOSITION (Recursive enumerability of Rex validities, InqBT)**.** *The set $Val^T_{Rex} \coloneqq \{\varphi \in \mathcal{L}_{Rex} \mid \varphi$ is valid in InqBT$\}$ is recursively enumerable.*

Proof. We can transfer the result about Val^Q_{Rex} to Val^T_{Rex} by use of the translations. The proof is an adaptation of a part of the proof we gave for Proposition 3.7, with the added information of $\mathcal{L}_{Rex}$ being closed under the translations (Prop 3.15).
In the proof of Proposition 3.7, we showed that for all $\varphi \in \mathcal{L}$, φ is a validity of InqBT if and only if $\bigwedge \Delta_{\Sigma^\flat_{\varphi^\flat}} \to \varphi^\flat$ is a validity of InqBQ. Recall also that $\bigwedge \Delta_{\Sigma^\flat_{\varphi^\flat}} \to \varphi^\flat$ is computable from the syntax of φ.
Now, to allow the use of the first claim we need to prove that this resulting formula is still a rex formula. Recall that an implication is a rex formula if and only if its consequent is a rex formula. Therefore,

$$
\begin{aligned}
\bigwedge \Delta_{\Sigma^\flat_{\varphi^\flat}} \to \varphi^\flat \text{ is a rex formula} &\iff \varphi^\flat \text{ is a rex formula}\\
&\iff \varphi \text{ is a rex formula}
\end{aligned}
$$

This implies that

$$\varphi \in Val^T_{Rex} \iff \bigwedge \Delta_{\Sigma^\flat_{\varphi^\flat}} \to \varphi^\flat \in Val^Q_{Rex}$$

and the latter is a semi-decidable matter, thanks to the result about Val^Q_{Rex}.
□

3.2.3 The Classical antecedent (Clant) fragment

The second syntactical fragment of the language $\mathcal{L}$ that we take into consideration is the Classical antecedent fragment. As the name suggests, in Clant only classical formulas are allowed to be used as antecedents for implications. As for Rex, a complete and sound proof system has been established for Clant. Here, we show how the good behaviour of the translations with respect to its formulas allows for the transfer of various important results that are known in InqBQ for the fragment.
Let us begin the discussion of this topic from a formal definition of the fragment.

3.22 DEFINITION. Let Σ be a signature. The set of clant formulas $\mathcal{L}_{Clant}(\Sigma)$ is given by the following syntax:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \alpha \to \varphi \mid \varphi \mathbin{\mathbb{V}} \varphi \mid \forall x\varphi \mid \exists\varphi$$

where p is an atomic formula in the signature Σ and α is a classical formula in the signature.

To show that formulas are Clant if and only if their translations are, we will need the following lemma:

3.23 LEMMA. *Let $\varphi \in \mathcal{L}$, then*

$$\varphi \textit{ is a classical formula} \iff \varphi^\sharp \textit{ is a classical formula}$$

$$\varphi \textit{ is a classical formula} \iff \varphi^\flat \textit{ is a classical formula}$$

Proof. The proof of both claims is a straightforward proof by induction. It suffices to observe that the translations $\sharp$ and $\flat$ neither add nor remove any inquisitive symbols. $\square$

We can now prove the conservativeness of the translations over Clant:

3.24 PROPOSITION. *Let Σ be a signature and let $\varphi \in \Sigma$. Then*

$$\varphi \in \mathcal{L}_{Clant}(\Sigma) \iff \varphi^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp)$$

$$\varphi \in \mathcal{L}_{Clant}(\Sigma) \iff \varphi^\flat \in \mathcal{L}_{Clant}(\Sigma^\flat)$$

Proof. Let's start by proving the first claim by induction over the structure of φ.

•<u>atoms</u>: the atomic cases are obviously verified.

•<u>$\wedge$, $\mathbb{V}$</u>: omitted

$$\bullet\underline{\to}:\quad \psi \to \xi \in \mathcal{L}_{Clant}(\Sigma) \iff \psi \text{ is classical and } \xi \in \mathcal{L}_{Clant}(\Sigma)$$
$$\iff \psi^\sharp \text{ is classical and } \xi^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp) \quad \text{(induction + Lemma 3.23)}$$
$$\iff \psi^\sharp \to \xi^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp)$$
$$\iff (\psi \to \xi)^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp)$$

$$\bullet\underline{\forall}:\quad \forall x_i\psi \in \mathcal{L}_{Clant}(\Sigma) \iff \psi \in \mathcal{L}_{Clant}(\Sigma)$$
$$\iff \psi^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp) \quad \text{(induction)}$$
$$\iff O(x_{i+1}) \to \psi^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp) \quad \text{(classical antecedent)}$$
$$\iff \forall x_{i+1}(O(x_{i+1}) \to \psi^\sharp) \in \mathcal{L}_{Clant}(\Sigma^\sharp)$$
$$\iff (\forall x_i\psi)^\sharp \in \mathcal{L}_{Clant}(\Sigma^\sharp)$$

•$\underline{\exists}$: Analogous, omitted

The proof of the second claim is almost identical, except for the cases involving the quantifiers.

•<u>atoms</u>: the atomic cases are obviously verified.

•$\underline{\wedge, \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}, \to}$: omitted

$$\bullet\underline{\forall}:\quad \forall x_i\psi \in \mathcal{L}_{Clant}(\Sigma) \iff \psi \in \mathcal{L}_{Clant}(\Sigma)$$
$$\iff \psi^\flat \in \mathcal{L}_{Clant}(\Sigma^\flat) \quad \text{(induction)}$$
$$\iff \psi^\flat[x_i/a_i] \in \mathcal{L}_{Clant}(\Sigma^\flat)$$
$$\iff \forall x_i\psi^\flat[x_i/a_i] \in \mathcal{L}_{Clant}(\Sigma^\flat)$$
$$\iff (\forall x_i\psi)^\flat \in \mathcal{L}_{Clant}(\Sigma^\flat)$$

•$\underline{\exists}$: analogous, omitted

□

3.2.3.1 Entailment-preserving translation to first order logic

In InqBQ, a key result for the Clant fragment is the existence of an entailment-preserving translation to two-sorted first order logic. This makes it possible to prove many fundamental metatheoretical properties of Clant. Our translations between InqBQ and InqBT allow us to transfer this result to InqBT without defining a new translation from scratch.

3.25 DEFINITION. Let Σ be a signature for InqBQ. We define the signature Σ^* over two sorts **w** and **e** as follows:

- For every n-ary relation symbol $R \in \Sigma$, Σ^* contains an $(n{+}1)$-ary relation symbol R^*, having the first argument of sort **w** and the remaining arguments of sort **e**.

- For every non-rigid n-ary function symbol $f \in \Sigma$, Σ^* contains an $(n+1)$-ary function symbol f^*, having the first argument of sort **w**, the remaining arguments of sort **e** and the output of sort **e**.

- For every rigid n-ary function symbol $\mathsf{f} \in \Sigma$, Σ^* contains an n-ary function symbol f^*, having all of its arguments and its output of sort **e**.

Then, we define $\mathcal{L}^{FOL}_{w,e}(\Sigma^*)$ as the language of two-sorted first order logic over the signature Σ^*.

3.26 PROPOSITION ([Cia22b], Proposition 5.7.8)**.** *Let Σ be a signature for InqBQ. There is a computable translation $tr^Q : \mathcal{L}_{Clant}(\Sigma) \to \mathcal{L}^{FOL}_{w,e}(\Sigma^*)$ such that, for any*
$\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}(\Sigma)$,

$$\Phi \vDash_{\mathsf{InqBQ}} \psi \iff tr^Q(\Phi) \vDash_{FOL} tr^Q(\psi)$$

3.27 PROPOSITION**.** *Let Σ be a signature for InqBT. Then, there is a computable translation $tr^T : \mathcal{L}_{Clant}(\Sigma) \to \mathcal{L}^{FOL}_{w,e}((\Sigma^\flat)^*)$ such that for any $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}(\Sigma)$,*

$$\Phi \vDash_{\mathsf{InqBT}} \psi \iff tr^Q(\Delta), tr^T(\Phi) \vDash_{FOL} tr^T(\psi)$$

where Δ is the set of formulas defined in Chapter 2.2 and tr^Q is the translation defined in the previous proposition.

Proof. To prove the proposition we employ two previous results. These enable us to transfer the translation provided for InqBQ to a translation for InqBT, without needing to construct a proof from scratch.
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}(\Sigma)$, then

$$\begin{aligned} \Phi \vDash_{\mathsf{InqBT}} \psi &\iff \Delta, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Thm 2.21)} \\ &\iff tr^Q(\Delta), tr^Q(\Phi^\flat) \vDash_{FOL} tr^Q(\psi^\flat) && \text{(by Prop. 3.26)} \end{aligned}$$

Note that the last equivalence relies on the fact that the translation $\flat$ maps clant formulas into clant formulas (Prop. 3.24) and on the fact that all formulas of Δ are clant.
We also remark that the results of the application of tr^T are indeed in the correct signature: the translation of formulas via $\flat$ is expressed in $\Sigma^\flat$; these formulas are then mapped by tr^Q to formulas in the signature $(\Sigma^\flat)^*$.
As for the computability of the translation tr^T, it follows immediately from the computability of $\flat$ and tr^Q. □

3.2.3.2 Compactness

Once again, the first property that we transfer to InqBT is entailment compactness, a known result for Clant formulas in InqBQ.

3.28 PROPOSITION (Compactness for clant entailments, InqBQ - [Cia22b], Theorem 5.7.9)**.**
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}$. Then,

If $\Phi \vDash_{\mathsf{InqBQ}} \psi$, there exists a finite subset $\Phi_0 \subseteq \Phi$ s.t. $\Phi_0 \vDash_{\mathsf{InqBQ}} \psi$

3.29 PROPOSITION (Compactness for clant entailments, InqBT)**.**
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}$. Then,

If $\Phi \vDash_{\mathsf{InqBT}} \psi$, there exists a finite subset $\Phi_0 \subseteq \Phi$ s.t. $\Phi_0 \vDash_{\mathsf{InqBT}} \psi$

Proof. We transfer the InqBQ result to InqBT by integrating the proof of Proposition 3.1 with the closure of Clant under the translation $\flat$. In particular, in our hypotheses, this implies that both $\Phi^\flat$ and $\psi^\flat$ are clant formulas. Also, note that Δ formulas are all clant formulas since they do not contain any implications with an inquisitive antecedent. These facts guarantee the validity of the second step.

$$\begin{aligned}
\Phi \vDash_{\mathsf{InqBT}} \psi &\iff \Delta, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Thm. 2.21)}\\
&\implies \text{f.s. finite subsets } \Delta_0 \subseteq \Delta \text{ and } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta_0, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat && \text{(by Prop. 3.28)}\\
&\implies \text{f.s. finite subset } \Phi_0^\flat \subseteq \Phi^\flat,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Delta, \Phi_0^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat\\
&\iff \text{f.s. finite subset } \Phi_0 \subseteq \Phi,\ \Phi_0 \vDash_{\mathsf{InqBT}} \psi && \text{(by Thm. 2.21)}
\end{aligned}$$

□

3.2.3.3 Recursive enumerability of entailments

The last know property of Clant in InqBQ that we prove for the system in InqBT is the recursive enumerability of validities:

3.30 PROPOSITION (Recursive enumerability of Clant entailments, InqBQ - [Cia22b], Theorem 5.7.10)**.** *The set*

$$\{\langle \varphi_1, ..., \varphi_n, \psi\rangle \mid n \in \mathbb{N},\ \varphi_1, ..., \varphi_n, \psi \in \mathcal{L}_{Clant},\ \varphi_1, ..., \varphi_n \vDash_{\mathsf{InqBQ}} \psi\}$$

is recursively enumerable.

3.31 PROPOSITION (Recursive enumerability of Clant entailments, InqBT)**.** *The set*

$$\{\langle \varphi_1, ..., \varphi_n, \psi\rangle \mid n \in \mathbb{N},\ \varphi_1, ..., \varphi_n, \psi \in \mathcal{L}_{Clant},\ \varphi_1, ..., \varphi_n \vDash_{\mathsf{InqBT}} \psi\}$$

is recursively enumerable.

Proof. Similarly to what we did in previous proofs, we transfer the InqBQ result to InqBT by use of the translation. We do this by computing, given $\varphi_1, ..., \varphi_n, \psi \in \mathcal{L}_{Clant}$, a corresponding finite set of formulas $\Phi' \cup \{\psi'\} \subseteq \mathcal{L}_{Clant}$ such that $\varphi_1, ..., \varphi_n$ entail ψ in InqBT if and only if Φ' entails ψ' in InqBQ. If we can compute such a set, the claim follows immediately thanks to the recursive enumerability of clant entailments for InqBQ.

Let's see how we can compute both Φ' and ψ'. We will use the abbreviation $\Phi \coloneqq \{\varphi_1, ..., \varphi_n\}$.

To begin with, we can apply Lemma 3.6 to get the following:

$$\Phi \vDash_{\mathsf{InqBT}} \psi \iff \Delta_{\Sigma^\flat_{\psi^\flat}}, \Phi^\flat \vDash_{\mathsf{InqBQ}} \psi^\flat$$

From the proof of Proposition 3.7, we know that $\Delta_{\Sigma^\flat_{\psi^\flat}}$ is computable from the syntax of ψ. From the definition of the translation $\flat$, it is obvious that $\Phi^\flat$ and $\psi^\flat$ are computable from Φ and φ respectively.

Now, let's prove that $\Phi' \coloneqq \Delta_{\Sigma^\flat_{\psi^\flat}} \cup \Phi^\flat$ and $\psi' \coloneqq \psi^\flat$ satisfy the other two required properties.

First, all formulas involved are clant formulas. Since $\Delta_{\Sigma^\flat_{\psi^\flat}}$ is a subset of $\Delta_{\Sigma^\flat}$, whose formulas don't contain any implication symbols, both sets are composed of clant formulas. As for $\Phi^\flat$ and $\psi^\flat$, we can apply Proposition 3.24 to Φ and ψ, which we assumed to be clant formulas.

On the other hand, $\Delta_{\Sigma^\flat_{\psi^\flat}} \cup \Phi^\flat$ is clearly a finite set.

Having proved all the properties that were required of our proposed Φ' and ψ', we can conclude the proof of the initial claim. $\square$

Conclusions

We began our investigation with the question of whether it was possible to define a two-way entailment-preserving translation between InqBQ and InqBT. In Chapter 2, we gave a positive resolution to this problem, while also showing that such translations can be defined in a natural way, building on the fundamental ideas that connect the two systems. This is especially evident in the first, special case that we focused our attention on, but it is also the starting point for the setup in the general case. Indeed, the key idea of both translations is the emulation in one system of the expressivity allowed by the semantics of the other system. In InqBT, one can recover the role of possible worlds by use of a selected variable. In InqBQ, assignments can be emulated with possible worlds by introducing constants that act as variables. All other formal modifications and constructions that need to be introduced are simply a necessary byproduct of implementing these basic ideas, which emerge in a natural way from observing the relation between the two systems. Therefore, besides confirming our initial hypothesis, the specifics of the translations are also evidence of how reasonable and well-motivated our initial question was.

Then, in Chapter 3, we explored various consequences of the existence of the translations and of their particular properties. We first drew a connection between major open problems of InqBQ and InqBT, showing that entailment compactness and semi-decidability of validity hold in one system if and only if they hold in the other. Thanks to the properties of our translations, we were able to prove these facts with relative ease, as evidenced by the brevity and simplicity of the proofs provided.

Then, we shifted our focus towards another natural application of the translations: transferring known results from the fragments of InqBQ to the corresponding fragments of InqBT. For the finitely coherent fragment, we first determined to what extent the coherence degree of a formula is preserved when translating it and then we proved that, as it does in InqBQ, entailment compactness holds in InqBT for entailments with finitely coherent conclusions. We then redirected our attention to two syntactical fragments of the language, Rex and Clant. We started out by showing that Rex is closed under the translations, thanks to how they operate on formulas. Using this fact, we were able to straightforwardly adapt proofs from the section on open problems to transfer both entailment compactness and semi-decidability of validity from the InqBQ Rex fragment to the InqBT one. Our work on the Clant fragment followed a similar structure, starting from its closure under the translations and then utilizing this result to prove both entailment compactness and the semi-decidability of validity for the InqBT fragment. For the Clant fragment of InqBT we were able to transfer an additional property: the existence of an entailment-preserving translation from the fragment to a two-sorted first order language.

The results in Chapter 3 are relevant in multiple ways. First, the equivalence of certain open problems, combined with the existence of the translations, better defines the position of InqBT relative to InqBQ. Such a close relation, however intuitive, is still of great interest, as InqBT's expressive capabilities fall outside of the scope of inquisitive logic and are of relevance to other connected project, such as dependence logic. Moreover, the results on finitely coherent, Rex and Clant formulas obviously provide important advancements in the study of the fragments of InqBT, by demonstrating that they satisfy fundamental metatheoretical properties. These are encouraging partial results towards potentially identifying an axiomatization of these fragments in InqBT. They also further reinforce the plausibility, suggested by known results for the InqBQ fragments, of finding said axiomatization. Lastly, the details of the proofs that we exhibited for these connections of problems and transfers of properties represent concrete evidence of the effectiveness and usefulness of the translations.

Future work

Aside from their significance for the understanding of InqBT and of its fragments, the results of this work also represent a useful foundation for future approaches to the shared metatheoretical open questions of the two systems. Since the state-based and team-based semantical frameworks present important differences, each system can be intrinsically more suited to certain methods and approaches. Now, the resolution of one of this open problems for a system, with either a positive or a negative answer, immediately translates to its resolution in the other system. Therefore, one can tackle these questions in either environment, without reducing the significance of potential results. From this perspective, an interesting route for expanding the results of this thesis would be the connection, across the two systems, of partial results towards proving or disproving entailment compactness and the semi-decidability of validity. These would provide further stepping stones towards the resolution of these important open questions.

Without taking away from the merits of the results that we obtained, it is still necessary to note that the work is far from over. There are natural and promising directions in which the translations and the methods applied in this thesis can be further exploited. First, the most prominent potential development of these results would be the identification of a sound and complete proof system for the Rex and Clant fragments of InqBT. As we mentioned, these have already been obtained for the corresponding fragments of InqBQ in [CG22] and [Gri21] respectively. The objective seems achievable thanks to various factors: first, the structural similarities between the two systems; second, the validity of entailment compactness and semi-decidability of validity for the two fragments in InqBT; and finally, the relative ease of transferring results with the translations, evidenced by both their properties and by the ways in which we utilized them in this work. The transfer of some secondary properties of the finitely coherent fragment of InqBQ that we excluded from our analysis also seems promising, but is yet to be addressed.

The connection of open questions about the two systems also allows for expansion. For instance, we have not determined either an equivalence, or a way to easily transfer, between an axiomatization of InqBQ and one of InqBT (or vice versa). Additionally, the two-way transferability of an open question about the expressive power of the Rex fragment remains to be explored.

There is also great potential in applying the methods utilized here to obtain comparable results for other systems. Indeed, the fundamental ideas that the translations rely upon are not dependent on the specific systems in question. Essentially, we have identified a natural and general way to emulate the behaviour of possible worlds with teams and vice versa. The same approach should apply straightforwardly to other team semantics logical systems generated similarly to InqBT, which could be developed from already established inquisitive logics. This line of study remains unexplored, but it is reasonable to assume that research in this direction would prove productive.

Bibliography

[CG22] Ivano Ciardelli and Gianluca Grilletti. Coherence in inquisitive first-order logic. *Annals of Pure and Applied Logic*, 173(9):103155, 2022.

[CGR18] Ivano Ciardelli, Jeroen Groenendijk, and Floris Roelofsen. *Inquisitive Semantics*. Oxford University PressOxford, 11 2018.

[Cia15] I. Ciardelli. *Questions in Logic*. PhD thesis, Institute for Logic, Language and Computation, Amsterdam, 2015.

[Cia16] Ivano Ciardelli. *Dependency as Question Entailment*, page 129–181. Springer International Publishing, 2016.

[Cia22a] Ivano Ciardelli. Describing neighborhoods in inquisitive modal logic. In *Advances in Modal Logic, volume 14*, 2022.

[Cia22b] Ivano Ciardelli. *Inquisitive Logic: Consequence and Inference in the Realm of Questions*. Springer International Publishing, 2022.

[CO21] Ivano Ciardelli and Martin Otto. Inquisitive bisimulation. *The Journal of Symbolic Logic*, 86(1):77–109, 2021.

[CR14] Ivano A. Ciardelli and Floris Roelofsen. Inquisitive dynamic epistemic logic. *Synthese*, 192(6):1643–1687, 2014.

[Gal12] Pietro Galliani. Inclusion and exclusion dependencies in team semantics — on some logics of imperfect information. *Annals of Pure and Applied Logic*, 163(1):68–84, 2012.

[Gri20] Gianluca Grilletti. *Questions & quantification: A study of first order inquisitive logic*. Phd thesis, University of Amsterdam, 2020.

[Gri21] Gianluca Grilletti. Completeness for the classical antecedent fragment of inquisitive first-order logic. *Journal of Logic, Language and Information*, 30:1–27, 2021.

[GV13] Erich Grädel and Jouko Väänänen. Dependence and independence. *Studia Logica*, 101(2):399–410, 2013.

[Hod97a] Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL*, 5(4):539–563, 1997.

[Hod97b] Wilfrid Hodges. *Some strange quantifiers*, page 51–65. Springer Berlin Heidelberg, 1997.

[Vä07] Jouko Väänänen. *Dependence Logic: A New Approach to Independence Friendly Logic*. London Mathematical Society Student Texts. Cambridge University Press, 2007.

[Yan14] Fan Yang. *On Extensions and Variants of Dependence Logic*. PhD thesis, University of Helsinki, 2014.

[YV16] Fan Yang and Jouko Väänänen. Propositional logics of dependence. *Annals of Pure and Applied Logic*, 167(7):557–589, July 2016.