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Complex Metrics in the Gravitational Path Integral

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Abstract

In general relativity, the force of gravity results from the curvature of spacetime, which is described by a tensor called the metric. One approach to quantizing gravity involves considering all possible metrics between initial and final states, called the gravitational path integral. In several important applications, complex metrics need to be considered, which arise as saddle points of the path integral. Because some complex metrics are unphysical, it is important to distinguish accceptable from unacceptable complex metrics. This thesis applies a recent admissibility condition to charged, rotating supersymmetric black holes in anti-de Sitter space, which have attracted considerable attention recently in the context of holography.

The complex metrics studied in this thesis are supersymmetric but not extremal. Supersymmetric extremal black holes, called BPS black holes, have infinite inverse temperature, making the action difficult to compute. By imposing supersymmetry without extremality, one of the parameters of the black hole becomes complex, which complexifies the metric. Unlike a metric complexified by a Wick rotation, this metric has no Lorentzian counterpart. The action for these black holes can be computed, which approaches the action for BPS black holes as the temperature goes to zero. While these black holes are important for quantum gravity, their admissibility has not yet been studied.

The admissibility condition comprises a set of different conditions, each of which must be satisfied at every spacetime point. This thesis systematically explores these conditions using a combination of analytical and numerical methods. It also evaluates an equivalent formulation of the admissibility condition in terms of the complex eigenvalues of the metric. It is shown that the metric is allowable if the radius of the event horizon is greater than a critical value, which corresponds to the BPS limit of the black hole. Because the inner and outer horizons of the black hole coincide at the BPS limit, this condition requires two distinct horizons. This demonstrates that the admissibility condition is a regularity condition on the form of the supersymmetric black hole solution.

In memoria di Bella

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Chapter 1

Introduction

Introduced by Einstein more than a century ago, general relativity is a theory of gravity that accurately describes many astrophysical phenomena, including black holes, neutron stars, and gravitational waves. In this theory, space and time are unified into a single entity that evolves depending on the matter and energy present. The resulting curvature of spacetime causes objects to move, which is experienced as the force of gravity. The properties of spacetime are encapsulated by a tensor called the metric that is used to compute lengths and durations between spacetime points. In most settings, the components of the metric are real values, but sometimes it is necessary to consider complex metrics. Usually, complex metrics are introduced to regularize an unphysical situation, but not every complex metric is physical. To apply complex metrics successfully, a condition has recently been proposed to distinguish between acceptable and unacceptable metrics [1, 2]. The main purpose of this thesis is to investigate this condition and apply it as a tool to understand better the quantum nature of gravity.

As a classical theory, general relativity describes spacetime using a single metric field. In quantum theory, fields do not exist in single configurations, but instead in a superposition of different configurations. In one approach to quantum gravity, the quantum state is computed using the gravitational path integral, which sums over all possible metrics with given boundary conditions. In general, the gravitational path integral fails to converge, but it can be regularized using various techniques. While the regularized path integral is still difficult to compute exactly, it tends to be dominated by contributions from metrics that solve the classical equations of motion. In certain situations, like rotating black holes, regularizing the path integral also makes the metric complex. When this happens, the path integral needs to be taken over the space of complex metrics. To define this integral properly, the admissibility condition can be used to decide the space over which to integrate. This demonstrates the utility of the admissibility condition for investigating quantum gravity.

This thesis focuses on complex metrics for supersymmetric black holes. Historically, supersymmetric black holes have played an important role in quantum gravity. By representing their microscopic constituents using string theory, the entropy of supersymmetric black holes in asymptotically flat space was reproduced in terms of the area of the event horizon [3]. This development led eventually to the AdS/CFT correspondence, which relates a gravitational system in asymptotically anti-de Sitter space to a conformal field theory on its boundary [4, 5, 6]. More formally, the correspondence connects the gravitational path integral to the partition function of a dual quantum field theory on the boundary.

Because of their importance to the AdS/CFT correspondence, this thesis explores specifically charged, rotating supersymmetric black holes in AdS space [7, 8, 9]. As reviewed in [10], the entropy of black holes in AdS space has been computed using the dual quantum field theory [11]. Because the path integral for rotating black holes is taken over complex metrics, understanding which complex metrics should be included is crucial to applying the AdS/CFT correspondence. In this thesis, a first step toward this problem is taken by analyzing the set of acceptable metrics for charged, rotating supersymmetric black holes in four-dimensional asymptotically AdS space.

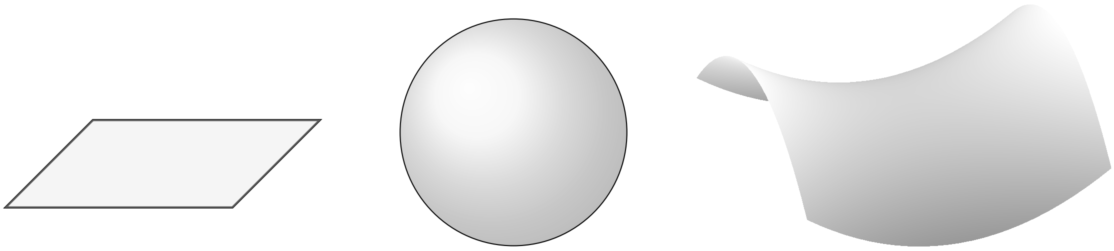
In the remainder of this introduction, the background needed to understand the role of complex metrics in the gravitational path integral is developed. First, the basics of general relativity are reviewed, including black hole solutions. Then the gravitational path integral is discussed, connecting it to supersymmetric black holes and the AdS/CFT correspondence. Finally, the admissibility condition is introduced as a way to distinguish between acceptable and unacceptable metrics in the gravitational path integral. The introduction concludes with an overview of the contents of the thesis.

1.1 The Einstein Equation

In general relativity, spacetime evolves depending on the matter and energy present. The dynamics of spacetime are captured by a rank-two tensor field $g_{\mu\nu}$ called the metric, which characterizes the geometry throughout space and time. As the spacetime geometry curves, it alters the motion of matter and energy, which is experienced as the force of gravity. The central equation relating the curvature of spacetime with the energy content is the Einstein equation, given by [12, 13]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1}$$

where G is Newton's constant that captures the strength of gravity. The first two terms on the lefthand side represent the curvature of spacetime. The tensor $R_{\mu\nu}$



(a) Flat (b) Positively curved (c) Negatively curved

Figure 1.1: When the energy-momentum tensor is zero, there are three general solutions to the Einstein equation with constant curvature. Schematically, the solutions are either flat, round, or saddle-shaped, which correspond to a zero, positive, and negative cosmological constant, respectively.

is called the Ricci tensor, while R is the Ricci scalar. They are functions of the metric and its first and second derivatives. On the righthand side, $T_{\mu\nu}$ is the energy-momentum tensor representing the strength of matter and energy. Finally, Λ represents the vacuum energy, called the cosmological constant. The metric is determined by solving the Einstein equation given $T_{\mu\nu}$ and Λ. It is often expressed in terms of the line element $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$.

One general way to classify a metric is based on its signature, which corresponds to the number of positive and negative eigenvalues it has. When all the eigenvalues are positive, the metric is called Euclidean, represented by $(++\cdots+)$. When there is a single minus sign, the metric is Lorentzian. An example of a Lorentzian signature is $(-+\cdots+)$. If the metric has any other type of signature, it is called indefinite.

A metric may possess certain symmetries, called isometries. Roughly, an isometry is a transformation that leaves the metric invariant. For instance, if the metric is independent of a coordinate x, then translations along x are an isometry. One consequence is that the momentum of a particle in that direction remains constant. In general, isometries are generated by vectors K^μ called Killing vectors [13]. An infinitesimal displacement in the direction of K^μ leaves the metric invariant. Identifying Killing vectors is important for studying the properties of a metric.

When the energy-momentum tensor is zero, there are three general solutions to the Einstein equation, called vacuum solutions. Depicted schematically in Figure 1.1, these vacuum solutions are important for future discussions. They correspond to maximally symmetric solutions with different signs of the cosmological constant, which is proportional to the curvature. When the curvature is zero, the space is flat. If the metric is Lorentzian in four dimensions, this space is called Minkowski space. When the curvature is positive, the space has the shape of a sphere. For instance, the d-sphere S^d is the Euclidean solution to the vacuum Einstein equation with positive

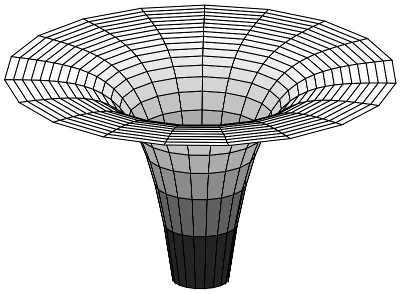


Figure 1.2: A black hole is a solution to the Einstein equation corresponding to a spherically symmetric mass that generates such a large gravitational field that, within a certain range, not even light can escape. In this example, the space outside the black hole is asymptotically flat.

cosmological constant. The Lorentzian analogue is called de Sitter space. In one set of coordinates in four dimensions, it can be interpreted as a spatial three-sphere that shrinks before expanding. When the curvature is negative, the space has a hyperbolic shape. Anti-de Sitter space is an example of a negatively curved space, which has important applications in quantum gravity.

Another important solution to the Einstein equation is a black hole, which represents a special type of spherically symmetric mass. This is represented schematically in Figure 1.2. A black hole is characterized by its mass M, charge Q, and angular momentum J. There are also different black holes depending on the geometry surrounding them. For example, a Schwarzschild black hole is one that asymptotically approaches flat space. In spherical coordinates, the metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{1.2}$$

where $\mathrm{d}\Omega^2$ is the metric on the unit two-sphere. As r approaches infinity, the flat Minkowski metric is recovered. Other examples of black holes include ones that asymptotically approach de Sitter or anti-de Sitter space.

At the center of a black hole called the singularity, the curvature becomes infinite. Surrounding the singularity is the event horizon. If a particle crosses the event horizon, it cannot escape back to the outside. The event horizon of a Schwarzschild black hole occurs at a range of $2GM$. At this range, the metric of Equation (1.2) degenerates, but this degeneracy can be removed by a suitable change of coordinates. If a black hole does not possess an event horizon, the singularity is said to be naked. According to one version of the cosmic censorship conjecture, black holes with naked singularities cannot exist in the universe [14]. Black holes can also have an inner horizon closer to the singularity, called the Cauchy horizon. For example, charged and rotating black holes possess an inner horizon in addition to the event horizon.

The properties of charged or rotating black holes are dictated by how much charge or angular momentum they have compared to their mass. For example, if a black hole has more mass than charge, it possesses an inner and outer horizon. This type of black hole is called subextremal. When the mass and charge are balanced, the black hole is called extremal, for which the inner and outer horizons coincide. When the black hole has more charge than mass, there is a naked singularity, which does not correspond to a black hole solution. Rotating black holes can be classified the same way, replacing charge with angular momentum. These classifications will become important when analyzing complex versions of black holes in subsequent chapters.

An interesting fact about black holes is that they are thermodynamic objects [15]. For example, black holes obey a version of the first law of thermodynamics given by

$$dM = T\,dS + \Omega\,dJ + \Phi\,dQ \tag{1.3}$$

where T is the temperature, Ω the angular velocity, and Φ the electrostatic potential. The temperature is proportional to the strength of gravity at the event horizon. Different types of black holes have different temperatures. For instance, extremal black holes have zero temperature. If a black hole has a temperature, it radiates energy, causing it to evaporate [16]. The quantity S is the entropy, which is related to the area of the event horizon A [17, 18]. In natural units where $c = \hbar = k_B = 1$, it is given by

$$S = \frac{A}{4G} \tag{1.4}$$

The fact that black holes carry entropy points to the existence of some microscopic constituents that make them up. While the relationship between entropy and area was derived using general relativity, the theory itself does not offer any explanation of what the underlying constituents are.

A striking feature of Equation (1.4) is that the entropy scales as a function of the area, not volume as with other extensive variables. This fact suggests black holes possess fewer degrees of freedom than corresponding systems in the same space, like a quantum field theory. This feature of black holes has been elevated to a general property of gravity called the holographic principle, by analogy with holograms which encode three-dimensional objects on two-dimensional surfaces [19, 20]. One requirement of any quantum gravity theory is that it reproduces the holographic entropy of black holes by counting the microscopic degrees of freedom. Using string theory, the entropy was derived for a class of five-dimensional extremal black holes [3]. Building on this development, the AdS/CFT correspondence was introduced, which is a concrete example of holography relating gravity in anti-de Sitter space to a quantum field theory in one fewer dimension [4, 5]. In the next section, the AdS/CFT correspondence will be discussed as a way of interpreting the gravitational path integral.

1.2 The Gravitational Path Integral

The concept of the path integral has a rich history in physics. In classical mechanics, the action principle states that a particle travels along the trajectory that extremizes the action, which is the integral of the Lagrangian over time. In quantum mechanics, a particle does not follow a single trajectory, but only the probability of transitioning from one position to another can be predicted. The path integral was introduced as a way of determining this probability by summing over all trajectories between the two points [21]. This path integral is depicted schematically in Figure 1.3. To extend this idea to quantum field theory, the path integral sums all configurations of the field between initial and final configurations. While this path integral fails to converge in general, it successfully captures many properties of a quantum field theory, including correlations between spacetime points.

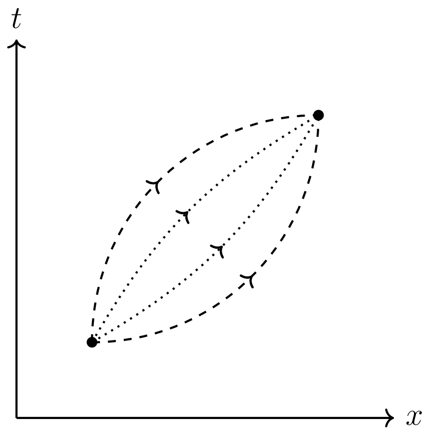


Figure 1.3: In quantum mechanics, the path integral can be used to compute the probability of a particle traveling between two points in a given amount of time. Here, four example paths are shown, where the line styles indicate their relative contribution to the path integral. To extend this concept to a field theory, the path integral needs to be computed over all possible trajectories of the field.

Because the metric in general relativity is a field, it is natural to attempt to quantize gravity by defining a path integral over all metrics, called the gravitational path integral [22]. While this path integral is difficult to compute in general, it is straightforward to define it in analogy with quantum field theory. According to this approach, the quantum amplitude of transitioning from an initial metric g_i to a final metric g_f is given by the path integral over all metric configurations connecting them:

$$\langle g_f | g_i \rangle = \int_{g_i}^{g_f} \mathcal{D}g \, e^{iS[g]} \tag{1.5}$$

The action that appears in the exponential is the gravitational action in Lorentzian signature. The simplest gravitational action is the integral of the curvature over the spacetime manifold $\mathcal{M}$, called the Einstein-Hilbert action. In four dimensions, the action is given by

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g} \, R \tag{1.6}$$

where g is the determinant of the metric. The appearance of the minus sign accounts for the fact that a Lorentzian metric has a negative determinant. Other terms can be included in the action to represent a non-zero cosmological constant or gauge fields which produce electric or magnetic charge. Other terms must also be included when the spacetime has a boundary, which will be discussed in the next chapter.

Like the path integral in quantum field theory, the gravitational path integral is an oscillatory integral that fails to converge. In one approach to quantum gravity called Euclidean quantum gravity, this problem is addressed by performing a Wick rotation [23, 24]. There are several ways to perform a Wick rotation, but one way is through the transformation $t = i\tau$. Because time is traditionally treated as a real variable, this change of coordinates makes time imaginary. Following a Wick rotation, the path integral becomes

$$\int \mathcal{D}g \, e^{-S_{\mathrm{e}}[g]} \tag{1.7}$$

where S_{e} is the Euclidean action given by

$$S_{\mathrm{e}} = \int_{\mathcal{M}} d^4x \sqrt{g} \, R \tag{1.8}$$

If the Euclidean action is positive definite, the integral may converge. Unfortunately, the gravitational action is generally unbounded from below, which prevents convergence. This will be investigated in more depth in the next chapter.

While the Euclidean path integral is difficult to compute exactly, the appearance of the minus sign in the exponential suggests that it will tend to be dominated by metrics that minimize the action. This approximation holds in the semiclassical limit where $\hbar \to 0$. Because metrics that solve the Einstein equation extremize the action, the path integral can be approximated by a sum over these metrics only. This approximation is portrayed in Figure 1.4.

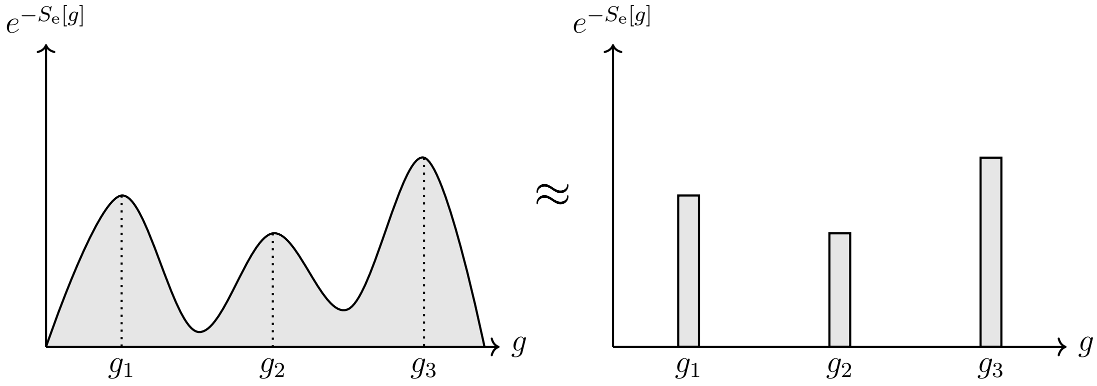


Figure 1.4: The Euclidean gravitational path integral is the integral over all metrics of the exponential of the negative Euclidean action. When the Euclidean action is a local minimum, the integrand has a local maximum. This corresponds to the solutions of the classical equations of motion. To approximate the path integral, the sum only over those metrics can be taken.

The difficulties surrounding the gravitational path integral underscore the fact that it is unclear exactly what it represents. Furnishing a clear interpretation of the gravitational path integral is one theme behind recent efforts in quantum gravity. The AdS/CFT correspondence is one step forward in providing this interpretation, which may help to compute the path integral in certain situations. As already mentioned, the AdS/CFT correspondence is a concrete example of holography that relates gravity in anti-de Sitter space to a quantum field theory in one fewer dimension. This correspondence is sketched in Figure 1.5. In the bulk, gravity is described by string theory, which is equated to quantum fields propagating on the boundary. More formally, the correspondence equates the path integral of string theory in the bulk to the path integral of a conformal field theory on the boundary:

$$Z_S(h) = Z_{\mathrm{CFT}}(h) \tag{1.9}$$

where h is the boundary metric [5]. By relating the gravitational path integral to a path integral of a quantum field theory, the correspondence provides a potential way to compute and interpret the path integral for gravity.

The original example of the AdS/CFT correspondence equates Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ to supersymmetric Yang-Mills theory in four dimensions [4]. The theories on both sides of this correspondence are supersymmetric. In a supersymmetric theory, fermions that represent massive particles can be transformed into corresponding bosons that represent massless particles. The strength of the string interaction is proportional to the square coupling constant g^2_{YM} of the Yang-Mills

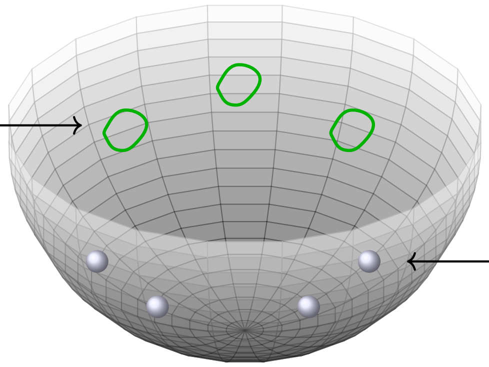


Figure 1.5: The AdS/CFT correspondence relates string theory in asymptotically anti-de Sitter space to a conformal field theory on the boundary. By relating gravity to a quantum field theory without gravity, it provides a potential way to compute and interpret the gravitational path integral.

theory, which has an SU(N) symmetry group. When N is made large while Ng_{YM}^2 is held fixed, the string theory becomes weakly coupled. In this large N limit, string theory can be approximated by supergravity, a theory combining general relativity and supersymmetry [25, 26].

Broadly, supergravity solutions can be either ungauged or gauged. Gauged solutions provide important backgrounds for the AdS/CFT correspondence. Supergravity solutions are also classified based on the number of supercharges $\mathcal{N}$, which are the generators of the supersymmetry transformations. The charged, rotating supersymmetric black holes analyzed in this thesis are solutions to gauged supergravity in four dimensions, which makes the analysis relevant to the AdS/CFT correspondence.

One effect of the Wick rotation performed to regularize the gravitational path integral is that it may make the metric complex. Not every metric becomes complex following a Wick rotation. For example, the Schwarzschild metric of Equation (1.2) becomes Euclidean outside the event horizon, but it does not become complex. If the black hole acquires angular momentum, the metric will develop cross-terms between time and angular components, leading to complexification. This is the case for the Kerr metric, which represents a rotating black hole in asymtotically flat space. In spherical coordinates, it is given by

$$\begin{aligned} \mathrm{d}s^2 = -\left(1 - \frac{2GMr}{\chi^2}\right)\mathrm{d}t^2 - \frac{2GMar\sin^2\theta}{\chi^2}\left(\mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}\phi\,\mathrm{d}t\right) + \\ \frac{\chi^2}{\Delta}\mathrm{d}r^2 + \chi^2\mathrm{d}\theta^2 + \frac{\Xi\sin^2\theta}{\chi^2}\mathrm{d}\phi^2 \end{aligned} \tag{1.10}$$

where $a \equiv J/M$ and χ, Δ, and Ξ are functions of r and θ which are unimportant for this discussion. On Wick rotation, $\mathrm{d}t = i\,\mathrm{d}\tau$, and the metric has complex components.[1] When the metric is complex, the gravitational path integral has to be taken over the space of complex metrics. As with all complex integrals, performing this integral requires defining the contour of integration in the complex plane. While this is an important issue, it will not be investigated in detail in this thesis.

Beyond the complexification introduced by Wick rotation, charged, rotating black holes that are also supersymmetric possess a natural degree of complexification. As will be discussed in Chapter 4, the supersymmetry condition makes one of the parameters characterizing the black hole complex, which complexifies the metric. Unlike a metric made complex through a Wick rotation, this metric has no Lorentzian counterpart. While these types of metrics are not physical, they may play an important role in the Euclidean gravitational path integral. The admissibility condition is one way of deciding the acceptable complex metrics to include in this integral.

Another reason why the supersymmetric metrics studied in this thesis are important is because they can be used to evaluate the action of supersymmetric extremal black holes, which can be compared to the dual quantum field theory. Supersymmetric extremal black holes, called BPS black holes, have infinite inverse temperature, making the action difficult to compute. By imposing supersymmetry without extremality, the action can be computed, which approaches the BPS limit when the temperature approaches zero. While these black holes are important for quantum gravity, their admissibility has not yet been studied. In the next section, the condition that helps distinguish between acceptable and unacceptable complex metrics will be discussed.

1.3 The Admissibility Condition

There are various ways that complex metrics can emerge, including Wick rotation and supersymmetry. Generally, complex metrics are introduced to regularize an unphysical situation, but not every complex metric is physical. Recently, a condition has been proposed that can help distinguish between acceptable and unacceptable complex metrics [2]. This condition was originally introduced as a possible way to formulate a quantum field theory without relying on traditional axioms. Although removed from its original context, this condition has also been proposed as a way to define a theory of quantum gravity [1].

How the admissibility condition is derived and how it addresses unphysical situations will be discussed in later chapters, but the condition centers around the concept of the path integral. It requires the path integral of an arbitrary gauge field coupled

[1] By sending a to ia_{e}, the metric can be made real and positive definite, but this approach will not be taken in this thesis.

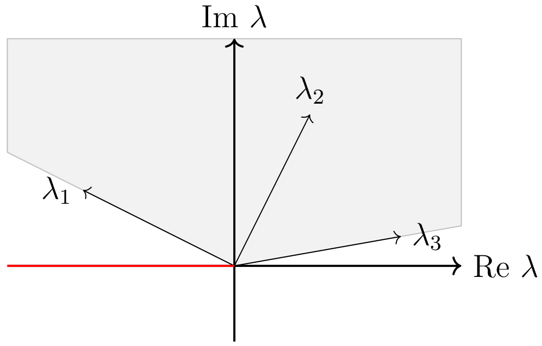


Figure 1.6: One way of expressing the admissibility condition involves the complex eigenvalues of the metric, requiring the sum of their phases in the complex plane to be less than π. It follows that the eigenvalues cannot lie on the negative real axis, shown in red. The convex cone swept out by the eigenvalues also cannot subtend more than π radians.

to the metric to converge, which ensures that the energy of the quantum field theory is positive definite. The gauge field is represented by a non-zero, real p-form F, which is an antisymmetric rank-p tensor with components $F_{i_1 \cdots i_p}$. The admissibility condition is satisfied if the action has positive real part for every p:

$$\text{Re}\left(\sqrt{g}\, g^{i_1 j_1} \cdots g^{i_p j_p} F_{i_1 \cdots i_p} F_{j_1 \cdots j_p}\right) > 0, \quad 0 \le p \le d \tag{1.11}$$

This condition must be satisfied at every spacetime point. In total, there are $d+1$ individual conditions, so in general multiple conditions need to be evaluated when assessing whether a given metric is allowable.

Another way of expressing the admissibility condition that will be developed in the next chapter involves the complex eigenvalues of the metric. If λ_i represents the complex eigenvalues of $g_{\mu\nu}$, the admissibility condition requires the sum of their angles swept out in the complex plane to be less than π radians:

$$\sum_{i=1}^{d} \left|\text{Arg}\, \lambda_i\right| < \pi \tag{1.12}$$

One consequence is that no eigenvalues can lie on the negative real axis, which corresponds to an angle of π. Another consequence of this condition is that the convex cone swept out between the minimum and maximum angles must subtend less than π radians, as shown in Figure 1.6.

In this thesis, the theory and applications of the admissibility condition will be investigated. First, the motivation behind the condition will be presented, leading to the derivation of Equation (1.12). Then various examples of spacetimes where complex metrics naturally emerge will be analyzed, demonstrating how the admissibility condition can discriminate between acceptable and unacceptable metrics. Next, the original contribution of this thesis will be presented, which consists of the application of the admissibility condition to charged, rotating supersymmetric black holes in four-dimensional anti-de Sitter space. These types of black holes are relevant to the AdS/CFT correspondence and possess an intrinsic degree of complexification, making them particularly interesting to study. It will be shown that the admissibility condition requires the radius of the event horizon to be greater than a critical value that is approached as the black hole becomes extremal. The interpretation of this result will also be discussed.

1.4 Thesis Outline

Below is an outline of the remainder of the thesis.

- Chapter 2 introduces the notion of complex spacetime metrics and develops the admissibility condition. In many ways, it is a more technical presentation of the material discussed in this introduction.
- Chapter 3 explores various examples where complex metrics arise naturally. The admissibility condition will be applied to distinguish between allowable and unallowable metrics.
- Chapter 4 presents the original contribution of the thesis involving charged, rotating supersymmetric black holes in four-dimensional anti-de Sitter space. The various forms of the admissibility condition will be analyzed.
- Chapter 5 concludes the thesis with some final remarks and possibilities for future research.

Chapter 2

Complex Spacetime Metrics

In the last chapter, the notion of the metric in general relativity was introduced. A dynamic field that characterizes the spacetime, the metric usually has real components, but sometimes complex metrics need to be considered. While complex metrics may play an important role in the gravitational path integral, not every complex metric is physical. Following recent proposals, an admissibility condition was introduced to discriminate between allowable and unallowable complex metrics. In this chapter, this condition is developed in detail, motivating it from a physical perspective and deriving the final result.

This chapter begins by introducing the notion of complex metrics from a more general perspective. Possible ways to construct complex metrics are discussed starting from real solutions to the Einstein equation. Next, it builds on the discussion of the last chapter by elaborating on how complex metrics emerge when evaluating the path integral in quantum field theory and quantum gravity. It then motivates the admissibility condition based on the requirement that the path integral in quantum field theory converges. As previewed in the last chapter, the form of the admissibility condition involving the complex eigenvalues of the metric is derived. Finally, the chapter concludes with some thoughts about possible integration cycles over complex metrics of the path integral.

2.1 Complex Solutions to the Einstein Equation

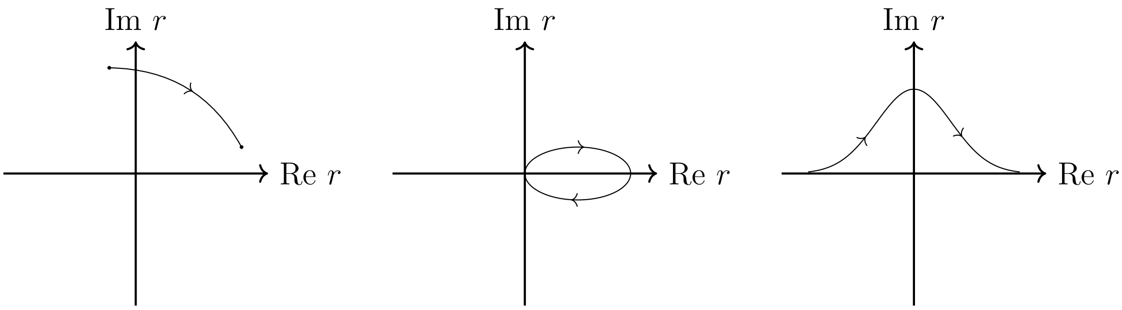
In general relativity, the central equation relating the curvature of spacetime to the energy and matter present is the Einstein equation. The solution to this equation is a rank-two tensor $g_{\mu\nu}$ called the metric, which can be used to compute lengths and durations between spacetime points. Various examples of metrics were presented in the last chapter, including different types of vacuums and black holes. In general, the metric is a function of the coordinates and other parameters that characterize the solution like charge or angular momentum. These coordinates and parameters are usually treated as real variables, which makes the metric a real-valued tensor. Complex solutions to the Einstein equation can be generated by making the coordinates or parameters complex, which can be done in several ways [1]. Consider the flat Euclidean metric in d dimensions

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{2.1}$$

where $\mathrm{d}\Omega^2$ is the round metric on the $(d-1)$-sphere, S^{d-1}. As the last chapter reviewed, this is the Euclidean solution to the vacuum Einstein equation with zero cosmological constant. Usually, r is treated as a non-negative real variable, but it can also be taken to be complex. For example, r can be a curve in the complex plane parameterized by a real variable u, as shown in Figure 2.1a. The metric becomes

$$\mathrm{d}s^2 = r'(u)^2\mathrm{d}u^2 + r(u)^2\mathrm{d}\Omega^2 \tag{2.2}$$

There are several ways to view this complexification that will be discussed later, but one way is that the manifold itself has become complex. Other complex manifolds can be generated by complexifying other variables. For example, in addition to r, the remaining variables x_i can be complexified subject to the constraint $\sum_i x_i^2 = 1$ that characterizes S^{d-1}.



(a) Example curve of r in the complex plane

(b) Complex r curve for manifold with compact boundary

(c) Complex r curve representing a wormhole

Figure 2.1: One way of complexifying a metric is by making the radial coordinate a curve in the complex plane. Depending on the type of curve that is used, the complex manifold has different properties.

Depending on how the complexification is carried out, complex manifolds with different properties can be produced. For example, to generate a compact manifold without a boundary, the parameter u can be made to run over a compact interval

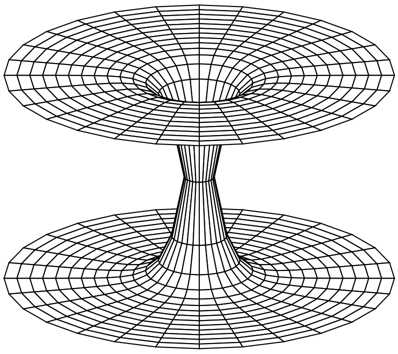


Figure 2.2: One way of creating a wormhole that connects two copies of flat Euclidean space is by complexifying the radial coordinate of the flat Euclidean metric.

while requiring r to vanish at the endpoints. This is depicted in Figure 2.1b. Another choice is to allow u to range over the interval $[0, \infty)$. In this case, the manifold is topologically equivalent to $\mathbb{R}^d$ if $r(0) = 0$. This manifold has an asymptotically Euclidean metric if $r(u) \sim u$ as $u \to \infty$. Wormhole solutions can also be constructed by allowing u to run over the whole real line, as shown in Figure 2.1c. In this example, $r(u) \to \pm\infty$ as $u \to \pm\infty$, and $r(u) \neq 0$ for all u. This produces a complex spacetime where two ends that are asymptotically $\mathbb{R}^d$ are connected through a wormhole. This is depicted in Figure 2.2.

While making the radial coordinate complex is one way of complexifying the metric, there are many other ways to complexify it. Some of these ways will be discussed in the next chapter, which focuses on potentially problematic complex metrics and how the admissibility condition addresses them. In the next section, complex metrics that arise when quantizing a field theory with the path integral will be considered.

2.2 Complex Metrics in the Path Integral

The path integral is a general technique to quantize a theory that involves summing over all trajectories between two points. Based on concepts in classical mechanics, this approach has been applied in quantum mechanics and quantum field theory. In the last chapter, the use of the path integral to quantize gravity was discussed, highlighting the various problems involved in computing this integral. Because the admissibility condition is based on requiring the path integral in quantum field theory to converge, this section begins by reviewing this integral, following the treatment in [2]. It then reexamines the gravitational path integral in Euclidean quantum gravity.

Compared to the last chapter, it offers more detail about why the gravitational action is not positive definite, first identified in [27]. It will be shown how this issue can be addressed by considering complex metrics.

2.2.1 Quantum Field Theory

In quantum field theory, the path integral is a powerful tool that can be used to compute quantum amplitudes by considering all possible trajectories between the initial and final states. For example, the quantum amplitude of a field ϕ transitioning from an initial configuration ϕ_i to a final configuration ϕ_f in time Δt can be computed by summing over all possible configurations between them. In natural units, this is given by

$$\langle \phi_f | e^{-iH\Delta t} | \phi_i \rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \, e^{iS[\phi]} \tag{2.3}$$

where H is the Hamiltonian, which evolves the state. The transition probability is given by the square modulus of the lefthand side, while the righthand is the path integral over all field configurations consistent with the initial and final states. The functional $S[\phi]$ is the action functional of the theory, defined as the volume integral over the spacetime manifold $\mathcal{M}$ of the Lagrangian density $\mathcal{L}$. For a Lorentzian manifold in d dimensions, it is given by

$$S = \int_{\mathcal{M}} d^d x \, \sqrt{-g} \, \mathcal{L} \tag{2.4}$$

where g is the determinant of the metric. The negative sign before g accounts for the fact that g is negative for a Lorentzian manifold. As an example, consider a free scalar field ϕ in Minkowski space, with signature $(-+++)$. The action is

$$S = \frac{1}{2} \int_{\mathcal{M}} d^4 x \, \sqrt{-g} \, \partial_\mu \phi \, \partial^\mu \phi = \frac{1}{2} \int d^4 x \left(-\dot{\phi}^2 + (\nabla \phi)^2 \right) \tag{2.5}$$

The path integral can also be used to compute the vacuum expectation value of observables at different spacetime points. In the path integral formulation, these observables are represented by generalized functions of the fields at different points. If O_k is the observable at the point x_k, the expectation value of this observable across different spacetime points can be computed by

$$\langle O_1 \cdots O_k \rangle = \int O_1(\phi) \cdots O_k(\phi) \, e^{iS[\phi]} \, \mathcal{D}\phi \tag{2.6}$$

While the path integral is a powerful approach that captures information about a theory beyond the perburtative level, it is difficult to evaluate because it is an oscillatory integral that does not converge in general. A popular method for addressing this issue is the Wick rotation. In a Wick rotation, time is transformed into an imaginary number. This time complexification can be viewed in two different ways, either in terms of generating a complex time manifold or a complex-valued metric [2]. These two viewpoints are summarized below.

1. ***Complex time manifold***. In this approach, time is treated as complex. This creates a complex time manifold with physical time at its boundary. Vacuum expectation values like that of Equation (2.6) are boundary values of holomorphic functions on the domain of this manifold. In principle, the same idea can be applied to other coordinates.

2. ***Complex-valued metric***. In this case, time is not viewed as complex, but as the length of an oriented time interval. The manifold is not complexified, but the metric becomes complex-valued. Not every metric becomes complex. For example, the Minkowski metric becomes a flat Euclidean metric after Wick rotation, but not complex. In general, whether a metric is complexified depends on whether it has mixed time and space components.

In the first approach, the action functional needs to be computed over the complex time manifold. This integral makes sense if the original manifold is Minkowski space, but it is not well-defined in curved spacetime. The second approach works better with the path integral formulation. Although the metric determinant g appearing in the action functional may become complex, the integral is still computed over a real manifold and is well-defined.

The second approach can be used to make the path integral of Equation (2.3) converge. Consider the Wick rotation given by $t = i\tau$. The action functional picks up an extra factor of i, transforming the path integral into

$$\int_{\phi_i}^{\phi_f} \mathcal{D}\phi \, e^{-S_\mathrm{e}[\phi]} \tag{2.7}$$

where S_e is the Euclidean version of the action, given by $S_\mathrm{e} = -iS$. For example, the Euclidean version of the action of Equation (2.5) is given by

$$S_\mathrm{e} = \frac{1}{2}\int d^4x \left(\dot{\phi}^2 + (\nabla\phi)^2\right) \tag{2.8}$$

where the dot now represents differentiation with respect to τ. Because this action is always positive, the path integral converges.

Even if the Euclidean action is positive definite, the path integral may still be difficult to compute. One approximate way to compute the integral is called the saddle point approximation [28]. Because the integrand is maximized when the action is minimized, this approach considers only field configurations that minimize the action. First, the action is expanded around a given minimum ϕ^*. Up to second order, this is given by

$$S_\mathrm{e}[\phi] \approx S_\mathrm{e}[\phi^*] + \frac{1}{2}\,(\phi - \phi^*)^2\,\frac{\delta^2 S_\mathrm{e}[\phi]}{\delta\phi^2}\bigg|_{\phi^*} \tag{2.9}$$

The term containing the first derivative of the action does not appear because it vanishes at the minimum. Using this approximation, the path integral becomes a Gaussian integral, which evaluates to

$$\int \mathcal{D}\phi\, e^{-S_\mathrm{e}[\phi]} = \sqrt{2\pi}\, e^{-S_\mathrm{e}[\phi^*]} \left(\frac{\delta^2 S_\mathrm{e}[\phi]}{\delta\phi^2}\bigg|_{\phi^*}\right)^{-1/2} \tag{2.10}$$

This approach can be extended to incorporate multiple local minima by expanding the action around the given minima. When the action is positive definite, this approach leads to better results as the expansion order is increased.

2.2.2 Euclidean Quantum Gravity

The path integral approach to quantizing a field can also be applied to quantize gravity. This time, the path integral is also taken over metrics in addition to other fields. While this integral is difficult to solve exactly, understanding gravity in terms of an ensemble of metrics provides important insights into quantum gravity [23]. Incorporating matter fields, the gravitational path integral can be written

$$\int \mathcal{D}g\, \mathcal{D}\phi\, e^{iS[g,\,\phi]} \tag{2.11}$$

Through the AdS/CFT correspondence, this type of integral can be related to a path integral of a quantum field theory without gravity, which may provide one way of evaluating and interpreting the gravitational path integral [4].

In Euclidean quantum gravity, the gravitational path integral is regularized, again using a Wick rotation. The metric can become complex, requiring the path integral to be taken over a proper contour of integration of complex metrics. Complex metrics may emerge as saddles of the gravitational path integral, which correspond to complex solutions to the Einstein equation. For example, the metric for rotating black holes becomes complex due to the presence of mixed time and angular components. Assuming this complex metric dominates the path integral, the thermodynamic relation between entropy and the area of the black hole horizon can be recovered [23]. This suggests that this complex metric plays an important role in the underlying physics, but it is unclear if other complex metrics are also acceptable.

In the previous section, it was shown that the Euclidean action of a scalar field is positive definite, which was needed to ensure that the integral converges. The Euclidean action is also positive definite for more complicated examples like Yang-Mills fields, but it is not positive definite in general relativity [27]. In natural units, the Lorentzian gravitational action is given by

$$S = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,R + 2\int_{\partial\mathcal{M}} d^3x\,\sqrt{-h}\,K \tag{2.12}$$

The second term is the Gibbons-Hawking-York term, needed to make the variational problem well-defined when the spacetime has a boundary $\partial\mathcal{M}$ [29]. The metric h is the induced metric on the boundary, and K is the trace of the extrinsic curvature [13]. A Wick rotation can be performed to obtain the Euclidean action. Under a conformal transformation, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, the Euclidean action becomes

$$S_{\mathrm{e}} = -\int_{\mathcal{M}} d^4x\,\left(\Omega^2 R + 6\sqrt{g}\,\partial_\mu\Omega\,\partial^\mu\Omega\right) - 2\int_{\partial\mathcal{M}} d^3x\,\Omega^2\sqrt{h}\,K \tag{2.13}$$

It appears that S_{e} can become arbitrarily negative provided $(\partial_\mu\Omega)^2$ is arbitrary large. To make the integral converge in this case, the contour of integration for the conformal factor has to be parallel to the imaginary axis, as shown in Figure 2.3. This suggests that for the Euclidean gravitational path integral to converge, the contour of integration needs to contain only a suitable family of complex metrics. In the next section, a possible way of determining this family is discussed, inspired by the notion of making the path integral converge.

2.3 Admissible Complex Metrics

In the last section, the path integral was introduced as a powerful approach to quantize a field theory, including general relativity where the dynamic field is the metric characterizing spacetime. To make the path integral converge, the Wick rotation was introduced, which complexifies the metric. Unlike for scalar and Yang-Mills fields, this Euclidean path integral for gravity fails to converge. The integral is a contour integral over complex metrics. If the contour of integration is defined appropriately, the integral may converge. A condition has recently been put forward to help define this contour, which distinguishes between acceptable and unacceptable metrics [1]. It requires the path integral of a gauge field coupled to the metric to converge. In this section, the motivation behind this condition will be reviewed, and the exact condition that a metric must satisfy to be allowable will be derived.

Recall that following a Wick rotation, the path integral is given by

$$\int \mathcal{D}\phi\, e^{-S_{\mathrm{e}}[\phi]} \tag{2.14}$$

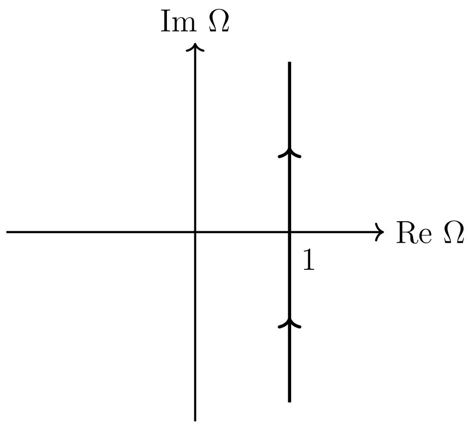


Figure 2.3: In one example of the gravitational path integral, the Euclidean action can become arbitrarily negative following a conformal transformation, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. To make the path integral converge, the contour of integration for the conformal factor Ω can be taken to be parallel to the imaginary axis.

In several applications, the Euclidean action is real, but in general it can be complex. In this case, the integral may still converge if the real part of the action is positive definite. Because S_e involves the metric, this requirement places conditions on the metric. To formalize this condition, the action functional needs to be defined precisely for a complex metric g. A complex-valued metric is a bilinear map $g : T_x \times T_x \mapsto \mathbb{C}$, where T_x is the tangent space defined at each point x of the manifold. The Euclidean action is given by

$$S_\mathrm{e} = \int_\mathcal{M} d^d x \sqrt{g}\,\mathcal{L} = \int \mathcal{L} * 1 \tag{2.15}$$

where $*1 \equiv \mathrm{vol}_g = \sqrt{g} \wedge_{i=1}^{n} \mathrm{d}x^i$ represents the volume element of the manifold. Since the Hodge star operator makes sense for a complex metric, this volume element is well-defined. In general, the square root of the determinant of a complex metric is complex, which makes the volume element complex.

Consider again the Euclidean action for a scalar field in four dimensions:

$$S_\mathrm{e} = \frac{1}{2} \int_\mathcal{M} d^4 x \sqrt{g}\, \partial_\mu \phi\, g^{\mu\nu}\, \partial_\nu \phi \tag{2.16}$$

The action is a quadratic form similar to $x^T A x$, where $\sqrt{g}\, g^{\mu\nu}$ plays the role of A. For the quadratic form S_e to be positive definite, two conditions need to be satisfied. First, the real part of the volume element vol_g needs to be a positive volume form on $\mathcal{M}$ [2]. This requires the real part of $\sqrt{g}$ to be positive. Every complex number has two square roots, and the square root with positive real part can always be chosen

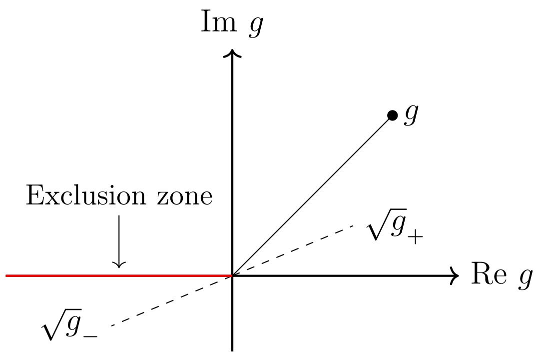


Figure 2.4: Just like the square root for reals, there are two square roots of the complex metric determinant g, which change based on the spacetime point. As long as g is never a negative real, the square root can always be chosen so that the real part is positive.

except when g is a negative real number. This condition is depicted in Figure 2.4. Because g is a field, this condition must be satisfied at every point. When moving through the state space, which square root to take to ensure $\mathrm{Re}\sqrt{g}$ is positive might change. The second condition that needs to be satisfied is that the real part of the matrix $\sqrt{g}\, g^{\mu\nu}$ is positive definite. This imposes conditions on the matrix elements, which will be discussed in later chapters.

The conditions introduced above create a space of allowable metrics among the set of all possible complex metrics. The Minkowski metric does not satisfy these conditions, but lies on the boundary of this space. The square root of the metric determinant is i, which has zero real part. The matrix $\sqrt{g}\, g^{\mu\nu}$ also has zero real part. One possible requirement in constructing the space of allowable metrics is that only Lorentzian metrics reside on the boundary.[1] Given just the two conditions above, indefinite metrics can also lie on the boundary. Consider a metric with signature $(3, d-3)$ in $d > 4$ dimensions. Although it is not Lorentzian, this metric also resides on the boundary. To restrict the boundary to only physically relevant Lorentzian metrics, other conditions need to be considered.

To impose further conditions, an electromagnetic field two-form $F_{\mu\nu}$ can be introduced, which is taken to be real. The action functional is

$$S_{\mathrm{e}} = \frac{1}{2} \int F \wedge *F \tag{2.17}$$

[1]The boundary of a complex domain relevant for this application is called the Shilov boundary [2, 30]. Let U be an open subset of a finite-dimensional complex manifold. The closure of U is a compact manifold X with a piecewise smooth boundary. The Shilov boundary of U is the smallest compact subset K of X such that, for every holomorphic function f defined in a neighbourhood of X, $\sup_U |f| = \sup_K |f|$. As discussed in the text, one important example of a Shilov boundary is the generalized half-plane of complex-valued quadratic forms on $\mathbb{R}^n$ with positive definite real part.

To compute $*F$, the two-form $F_{\mu\nu}$ can be contracted with the volume element. In d dimensions, this contraction produces a tensor of rank $d-2$:

$$(*F)_{\mu_1\cdots\mu_{d-2}} = (*1)^{\nu_1\nu_2}{}_{\mu_1\cdots\mu_{d-2}} F_{\nu_1\nu_2} \tag{2.18}$$

Notice that this operation requires two contractions with the metric, one more than that required by the scalar action of Equation (2.16). The map $F \mapsto F \wedge *F$ takes a real two-form and returns a complex d-form:

$$\wedge^2(T_x^*) \mapsto \wedge^d(T_x^*)\big|_{\mathbb{C}} \tag{2.19}$$

where the $\mathbb{C}$ on the right denotes complexification. It is natural to continue this line of reasoning further to include any real p-form F, where $0 \le p \le d$. In this case, the quadratic form is a map $\wedge^p(T_x^*) \mapsto \wedge^d(T_x^*)|_{\mathbb{C}}$. As before, the metric is allowable if the real part of $\wedge^d(T_x^*)|_{\mathbb{C}}$ is positive definite.

The allowability condition can be rewritten in a form that highlights the contractions with the complex metric. The wedge product between the p-form F and $*F$ can be written

$$F \wedge *F = \frac{1}{d!}\, g^{\nu_1\rho_1} \cdots g^{\nu_p\rho_p} F_{\nu_1\cdots\nu_p} F_{\rho_1\cdots\rho_p} \epsilon_{\mu_1\cdots\mu_d}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} \tag{2.20}$$

Because the Levi-Civita tensor ϵ is the volume element, the action can be written

$$S_\mathrm{e} = \int_{\mathcal{M}} d^d x \sqrt{g}\, g^{i_1j_1} \cdots g^{i_pj_p} F_{i_1\cdots i_p} F_{j_1\cdots j_p} \tag{2.21}$$

The metric is allowable if the action has positive real part for every non-zero, real F and $0 \le p \le d$:

$$\mathrm{Re}\left(\sqrt{g}\, g^{i_1j_1} \cdots g^{i_pj_p} F_{i_1\cdots i_p} F_{j_1\cdots j_p}\right) > 0, \quad 0 \le p \le d \tag{2.22}$$

This is a set of $d+1$ individual conditions, each of which must be satisfied at every spacetime point for a metric to be considered allowable.

The positivity condition restricts the space of metrics to only those that may make the path integral of a p-form gauge field coupled to the metric converge. The expectation is that a general quantum field theory is well-defined when the metric satisfies this condition. In this way, the positivity requirement may be a substitute for some of the axioms of quantum field theory. While the introduction of a p-form gauge field may seem arbitrary, it is actually physically motivated, as discussed in [1]. For instance, while free field theories of massless bosonic fields other than p-forms exist, they do not have gauge-invariant stress tensors, preventing them from being defined in curved spacetime [31]. There are also no known interacting ultraviolet-complete theories of massless fields which do not involve p-forms even in flat spacetime. This suggests that theories involving a p-form gauge field correspond to a broad class of physically relevant theories.

While the positivity condition is physically motivated, it has some limitations. When defining a quantum field theory, the set of states usually needs to be defined in a local Hilbert space. The positivity condition constructs the theory using a path integral in curved spacetime through analytical continuation of the metric, but this method does not guarantee a local Hilbert space. Another issue is that Wick rotations of matter fields may introduce phases in the path integral that make it diverge. This issue has to be treated carefully when considering more sophisticated theories.

The allowability condition can be cast into a more useful form involving the eigenvalues of the metric. For $p = 1$, the condition requires the real part of $W \equiv \sqrt{g}\, g^{\mu\nu}$ to be positive definite. To assess this condition, it is helpful to diagonalize the matrix. One way of diagonalizing a complex matrix is with a real basis and complex eigenvalues, which exposes the complex structure of the matrix. In the real basis, the matrix $W^{-1} = g_{\mu\nu}/\sqrt{g}$ is also diagonal, which means $g_{\mu\nu}$ is diagonal. Let the eigenvalues of $g_{\mu\nu}$ be λ_i, $i = 1, \ldots, d$. Then the allowability condition requires $\operatorname{Re}\alpha > 0$, where

$$\alpha \equiv \sqrt{g} \prod_{i \in \mathcal{S}} \lambda_i^{-1} \tag{2.23}$$

for any subset $\mathcal{S} \subseteq \{1, \ldots, d\}$. This is the same as requiring the argument of α to lie in the right half-plane:

$$-\pi/2 < \operatorname{Arg}\alpha < \pi/2 \tag{2.24}$$

Expanding the determinant in terms of the eigenvalues and using the properties of the argument function, the argument of α reads

$$\begin{aligned}
\operatorname{Arg}\alpha &= \operatorname{Arg}\left(\prod_{i=1}^{d}\sqrt{\lambda_i}\prod_{j\in\mathcal{S}}\lambda_j^{-1}\right) \\
&= \operatorname{Arg}\left(\prod_{i\in\mathcal{S}}\frac{1}{\sqrt{\lambda_i}}\prod_{j\notin\mathcal{S}}\sqrt{\lambda_j}\right) \\
&= \operatorname{Arg}\left(\prod_{i\in\mathcal{S}}\frac{1}{\sqrt{\lambda_i}}\right) + \operatorname{Arg}\left(\prod_{i\notin\mathcal{S}}\sqrt{\lambda_i}\right) \\
&= \sum_{i\in\mathcal{S}}\operatorname{Arg}\frac{1}{\sqrt{\lambda_i}} + \sum_{i\notin\mathcal{S}}\operatorname{Arg}\sqrt{\lambda_i} \\
&= -\frac{1}{2}\sum_{i\in\mathcal{S}}\operatorname{Arg}\lambda_i + \frac{1}{2}\sum_{i\notin\mathcal{S}}\operatorname{Arg}\lambda_i
\end{aligned} \tag{2.25}$$

Substituting the result into Equation (2.24) leads to the inequality

$$-\pi < \sum_{i \notin \mathcal{S}} \operatorname{Arg} \lambda_i - \sum_{i \in \mathcal{S}} \operatorname{Arg} \lambda_i < \pi \tag{2.26}$$

Both sides of this inequality contain 2^d separate inequalities, which correspond to different subsets $\mathcal{S}$ of $\{1, \ldots, d\}$. The inequality on the left for a given set $\mathcal{S}$ is the same as the inequality on the right for the complement of $\mathcal{S}$. Choosing one side of the inequality, Equation (2.26) becomes

$$\sum_{i \in \mathcal{S}} \operatorname{Arg} \lambda_i - \sum_{i \notin \mathcal{S}} \operatorname{Arg} \lambda_i < \pi \tag{2.27}$$

Consider what happens to this inequality when a given number j is present in $\mathcal{S}$. The term $\operatorname{Arg} \lambda_j$ will enter with a plus sign. When it is not in $\mathcal{S}$, it will appear with a minus sign. Taking this into account, the inequality becomes

$$\sum_{i=1}^{d} a_i \operatorname{Arg} \lambda_i < \pi \tag{2.28}$$

where

$$a_i = \begin{cases} 1, & i \in \mathcal{S} \\ -1, & i \notin \mathcal{S} \end{cases} \tag{2.29}$$

Because this must hold for every $\mathcal{S}$, all combinations of coefficients need to be considered. This amounts to the following inequality:

$$\sum_{i=1}^{d} \left| \operatorname{Arg} \lambda_i \right| < \pi \tag{2.30}$$

This inequality is also a set of 2^d inequalities depending on the imaginary parts of the eigenvalues. If a given eigenvalue λ_j has positive imaginary part, then $|\operatorname{Arg} \lambda_j| = \operatorname{Arg} \lambda_j$ and the term will enter with a plus sign. When λ_j has negative imaginary part, $|\operatorname{Arg} \lambda_j| = -\operatorname{Arg} \lambda_j$, causing the term to enter with a minus sign. This is the same as Equation (2.28).

Using the new version of the admissibility condition formulated in terms of the eigenvalues of the metric, it is possible to reason about which metrics reside on the boundary of allowable metrics. Consider the set of real metrics, which possess only real eigenvalues. If the eigenvalue is positive, its argument is zero. When it is negative, its argument is π. Equation (2.30) implies that on the boundary of allowable metrics, the sum of the arguments must be π. It follows that for a real metric to reside on

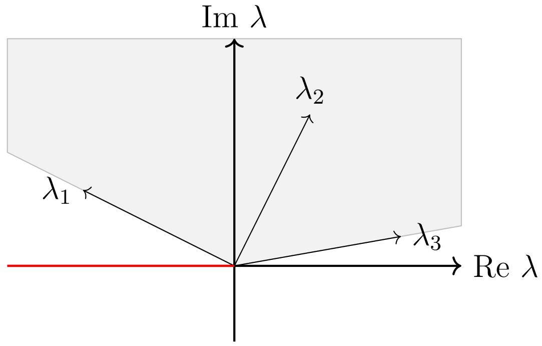


Figure 2.5: The angles of the complex eigenvalues of the metric sweep out a convex cone in the complex plane. To satisfy the admissibility condition, the cone must sweep out less than a half-plane.

the boundary, only one eigenvalue can be negative. The only kind of real metric that satisfies this condition is a Lorentzian metric. Other real metrics cannot lie on the boundary, including the metric discussed before with signature $(3, d-3)$ in $d > 4$. This was one feature that the space of allowable metrics should possess, which motivated the introduction of higher p-forms in the action. Although it was derived initially considering the case of $p = 1$, the admissibility condition in terms of the eigenvalues encapsulates the various conditions across different values of p.

Another consequence of the admissibility condition is that the metric eigenvalues sweep out less than a half-plane:

$$\max_i(\operatorname{Arg}\lambda_i) - \min_i(\operatorname{Arg}\lambda_i) < \pi \tag{2.31}$$

This corresponds to a closed convex cone in $\mathbb{C}$. Because this condition is violated if any of the eigenvalues is negative, this cone cannot intersect the negative real axis. Reproduced from Chapter 1, Figure 2.5 shows an example of the region of the complex plane swept out by the eigenvalues.

Another advantage of the admissibility condition involving the complex eigenvalues is that it allows admissibility to be verified for any subspace of the original manifold. Consider a subspace in one fewer dimension. Let $\{\theta_k\}_{k=1}^{d}$ represent the angles of the metric eigenvalues in descending order in the space V, and $\{\theta'_k\}_{k=1}^{d-1}$ the angles corresponding to the subspace W. It can be shown that the angles for W interleave those for V:

$$\theta_1 \geq \theta'_1 \geq \theta_2 \geq \theta'_2 \geq \cdots \geq \theta_{d-1} \geq \theta'_{d-1} \geq \theta_d \tag{2.32}$$

This implies $\sum|\theta'_k| < \sum|\theta_k|$. It follows that if the metric in V is allowable, the metric in W is also allowable. This holds for every subspace W [2].

To make the restriction on the eigenvalues more concrete, an expression for the complex eigenvalues can be obtained given a general complex metric of the form $g_{\mu\nu} = A + iB$. This is done in detail in Appendix B. To summarize the argument, the real part of the metric can be diagonalized:

$$A = QMQ^T \tag{2.33}$$

Assuming A is positive definite, its eigenvalues are positive reals, so M can be written as the square of another real matrix, $M = D^2$. The eigenvalues of the metric are then given by $\lambda_i = 1 + in_i$, where n_i are the eigenvalues of the real matrix $(QD^{-1})^T B(QD^{-1})$. In later chapters, these eigenvalues will be computed explicitly for different metrics, which enables the allowability condition to be assessed.

2.4 Defining a Complex Integration Cycle

In the last section, a condition was derived that could distinguish between acceptable and unacceptable metrics when evaluating the Euclidean path integral in a quantum field theory. Computing the path integral for gravity requires more than knowing what constitutes a good class of complex metrics. The integration cycle over which to perform the path integral also has to be known. This section presents some thoughts about defining the integration cycle taken from [1].

To define the integration cycle in a general way, a set of integration variables Φ_i, $i = 1, \ldots, N$ can be introduced. The action is a functional of these integration variables. The path integral is an integral over a region U of the integration variables:

$$\int_U d\Phi_1 \cdots d\Phi_N \, e^{-S[\Phi]} \tag{2.34}$$

This integral may fail to converge, but the goal is to generalize it to become a convergent complex contour integral. To start, Φ_i can be analytically continued to complex variables $\tilde{\Phi}_i$ and the action to the holomorphic function $\mathcal{S}(\tilde{\Phi})$. The integration region becomes a complexification $\mathcal{U}$ of U. The task is then to find an integration cycle $\Gamma \subset \mathcal{U}$ such that the complex integral over Γ converges.

In general, it is difficult to define an integration cycle so that the path integral always converges, but one way to do it involves gradient flow [32, 33]. In this approach, a positive definite metric G on $\mathcal{U}$ is chosen. Then $\tilde{\Phi}_i$ can be parameterized by a variable s that satisfies the following gradient flow equation

$$\frac{d\tilde{\Phi}_i}{ds} = G^{ij} \frac{\partial(\mathrm{Re}\,\mathcal{S})}{\partial \tilde{\Phi}^{j*}} \tag{2.35}$$

In a d-dimensional field theory on a manifold $\mathcal{M}$, this equation is a differential equation on a $(d+1)$-manifold $\mathcal{M} \times \mathbb{R}$, where $\mathbb{R}$ is parameterized by s. The solutions to this equation in which $\tilde{\Phi}$ is independent of s are critical points of the action, which correspond to classical solutions of the equations of motion. Let p be such a critical point of the action. The parameter s can be defined on $(-\infty, 0]$ such that $\tilde{\Phi}$ starts at p when $s \to -\infty$. Under mild assumptions, $\mathrm{Re}\,\mathcal{S}$ grows at infinity and the complex integral converges.

Unfortunately, with a simple choice of the metric G, there is no guarantee that the integration cycle remains in the space of allowable metrics. If the metric is restricted to be a certain type of metric called a Kahler metric, it will remain in the space of allowable metrics, but the real part of the action might not go to infinity as required above. An alternative is to define an integration cycle not just for gravity but for the combined system of gravity and matter fields. In this case, all critical points are potentially allowed, but another mechanism would have to be found to exclude undesirable metrics. Another issue with this approach is that Wick rotating the matter fields may multiply the path integral by a potentially ill-defined phase, as mentioned before.

This brief discussion highlights some of the difficulties involved in defining the integration cycle that allows the Euclidean gravitational path integral to converge. This is a topic of ongoing research, but in the next chapter, the admissibility condition developed here will be applied to various important applications in which complex metrics naturally emerge. It will be shown how the admissibility condition can be used to select between physical and unphysical metrics in these settings. This is a first step toward constructing a consistent theory that can be applied to a wide range of spacetimes.

Chapter 3

Applications of the Admissibility Condition

In the last chapter, the notion of complex spacetime metrics was reviewed. Different ways of complexifying the metric were discussed, including performing a Wick rotation to regularize the path integral. While allowing the metric to become complex may help the path integral converge, not every complex metric is physically relevant, motivating the need for a way to decide between physical and unphysical metrics. By requiring the path integral to converge when the metric is coupled to a generic gauge field, conditions were derived that the metric must satisfy to be considered acceptable. In this chapter, this admissibility condition will be applied to various examples in which complex metrics naturally arise. These examples are taken from [2] and accompanying references.

The first example that will be considered is the Hartle-Hawking wavefunction of the universe, which describes large semiclassical spacetimes. To arrive at sensible results, complex solutions of the Einstein equation need to be considered. The next example involves spacetimes where the topology changes, including the splitting of a closed universe in two. In these settings, the metric degenerates at the point where the topology changes, but this degeneracy can be removed by making the metric complex. Next, time folds and double cones will be discussed. In these geometries, the path integral propagates forward and backward in time, which can be used to compute observables in quantum field theory or the spectral form factor in a holographic theory. Finally, rotating black holes will be considered, which become complex after a Wick rotation due to mixed time and angular components in the metric. This last example will pave the way for the discussion of supersymmetric black holes in asymptotically anti-de Sitter space in the next chapter.

3.1 Hartle-Hawking Wavefunction of the Universe

To apply quantum mechanics on cosmological scales, the wavefunction describing the whole universe needs to be computed [34]. In quantum field theory, the wavefunction of the ground state can be computed using a Euclidean path integral, which sums all possible trajectories of the field on a given manifold. To extend this idea to the universe as a whole, the sum needs to be performed over all possible manifolds the universe can assume over its history which are consistent with its initial and final shape [35, 36]. In this approach, the wavefunction is given by the path integral

$$\Psi = \sum_{\mathcal{M}} \int \mathcal{D}g\, \mathcal{D}\phi\, e^{-S[g,\,\phi,\,\mathcal{M}]} \tag{3.1}$$

Very similar to the gravitational path integral introduced before, this integral differs only in the sum over different manifolds, which is restricted to those that share the same boundary $\partial\mathcal{M}$.

A problem occurs when trying to evaluate this path integral. The minus sign in the exponential means the integral will be dominated by metrics that minimize the action. The metrics satisfying the classical equations of motion extremize the action. This allows the integral to be approximated by a sum only over those metrics that solve the Einstein equation. The problem is that there are no known Euclidean solutions to the Einstein equation with a given boundary. For example, consider the Einstein equation with a positive cosmological constant Λ, where the boundary is a $(d-1)$-sphere. In natural units, the action is given by

$$S = -\int_{\mathcal{M}} d^d x\, \sqrt{g}\,(R - 2\Lambda) \tag{3.2}$$

Provided the radius of the sphere is larger than the length scale set by the cosmological constant, $1/\sqrt{G\Lambda}$, there is no real Euclidean solution.

While there are no real solutions for the scenario presented above, there are complex solutions that lead to sensible results for the wavefunction of the universe. Complex solutions can be created starting with the metric of the d-sphere of radius ρ:

$$\mathrm{d}s^2 = \rho^2(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\Omega^2) \tag{3.3}$$

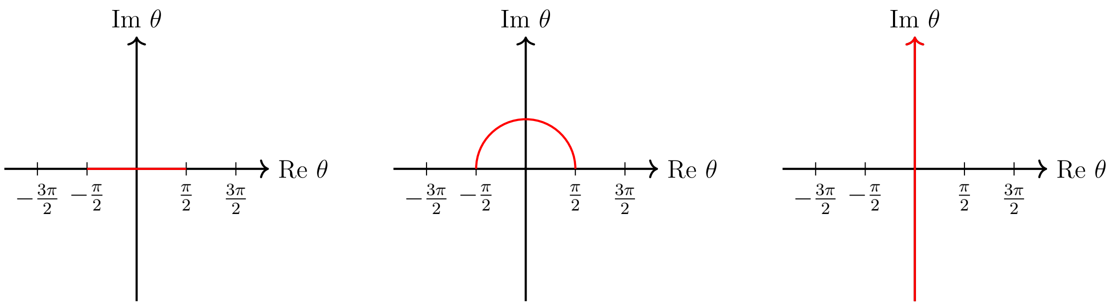
where $\mathrm{d}\Omega^2$ is the metric on the $(d-1)$-sphere. This metric is a Euclidean solution to the vacuum Einstein equation with a positive cosmological constant. To make this metric complex, θ can be chosen to be a curve in the complex plane parameterized by a real variable u. The metric becomes

$$\mathrm{d}s^2 = \rho^2\left(\theta'(u)^2\mathrm{d}u^2 + \cos^2\theta(u)\,\mathrm{d}\Omega^2\right) \tag{3.4}$$

Different curves generate different types of manifolds. When θ is a straight line on the real axis between $-\pi/2$ and $\pi/2$, the manifold is a d-sphere equipped with a real metric, as shown in Figure 3.1a. The endpoints of this line are zeros of the function $\operatorname{Re}\cos\theta$, which occur at $\operatorname{Re}\theta = (n+1/2)\pi$, $n \in \mathbb{Z}$. Any other curve connecting two zeros, while avoiding all other zeros, will also generate a manifold that is topologically S^d. This is shown in Figure 3.1b. Another type of manifold is produced when $\theta = iu$. In this case, the metric becomes

$$\mathrm{d}s^2 = \rho^2\left(-\mathrm{d}u^2 + \cosh^2 u\, \mathrm{d}\Omega^2\right) \tag{3.5}$$

which corresponds to de Sitter space, depicted in Figure 3.1c. This metric is also a solution to the Einstein equation with a positive cosmological constant, this time in Lorentzian signature.



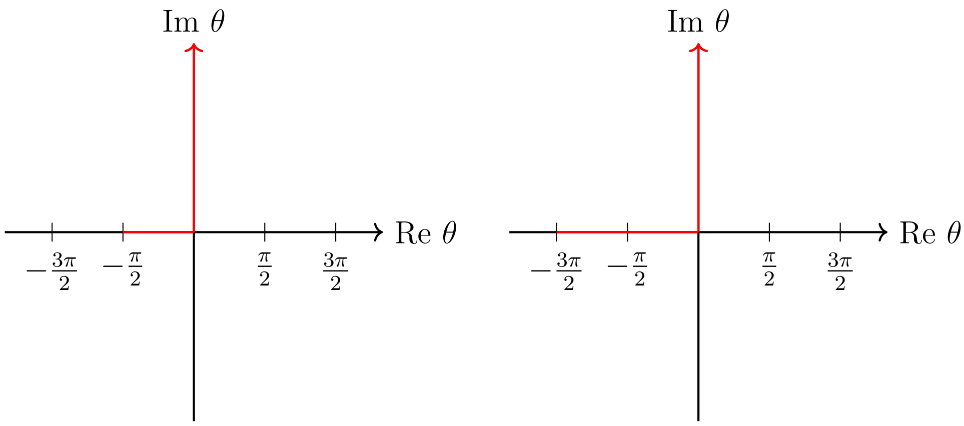
(a) S^d with real metric (b) Topologically S^d (c) de Sitter space

Figure 3.1: The round metric can be complexified by making θ a curve in the complex plane, shown in red. Different versions of the d-sphere can be generated by connecting zeros of $\operatorname{Re}\cos\theta$, while de Sitter space can be realized by taking θ to be purely imaginary.

By making the metric complex, a solution to the problem above involving a positive cosmological constant and $(d-1)$-sphere boundary can be produced. An example is shown in Figure 3.2a. The curve consists of two segments, one corresponding to a hemisphere in Euclidean signature, the other to half of de Sitter spacetime. Together, they describe creation of a closed universe that then expands exponentially fast. To determine the contribution of this solution to the wavefunction of the universe, the action of the two segments needs to be computed. The action of the Euclidean hemisphere is real, while the action of the de Sitter spacetime is complex. In general, the action takes the form, $S/2 + iI(u)$. The contribution to the wavefunction is then

$$e^{-S/2-iI(u)} \tag{3.6}$$

The factor $e^{-S/2}$ captures the norm of the state, while the oscillatory factor $e^{-iI(u)}$ describes the real time evolution of de Sitter space.



(a) Euclidean hemisphere, followed by de Sitter expansion

(b) Euclidean solution with even more negative action

Figure 3.2: A Euclidean solution to the Einstein equation with a positive cosmological constant and $(d-1)$-sphere boundary can be produced using a suitably chosen θ curve in the complex plane, shown in red.

To determine the magnitude of the state, the action of the Euclidean solution S needs to be computed. Because the Ricci scalar is constant, the action is given by

$$S = (2\Lambda - R)V \tag{3.7}$$

where V is the spacetime volume. The Ricci scalar can be expressed in terms of the cosmological constant using the Einstein equation. For the d-sphere, the Ricci tensor is $R_{\mu\nu} = kg_{\mu\nu}$. The constant of proportionality k is related to the Ricci scalar, $R = kd$. Substituting these values into the vacuum Einstein equation, the cosmological constant is $\Lambda = (d-2)k/2$, which is positive if $d > 2$. The Ricci scalar is then $R = 2d\Lambda/(d-2)$. Substituting this into Equation (3.7), the action becomes

$$S = -\frac{4\Lambda V}{d-2} \tag{3.8}$$

Crucially, the action is negative, which means in the expression for the wavefunction, the exponential is raised to a positive power.

To arrive at the final expression for the action, the volume needs to be computed. For the metric of Equation (3.3), the determinant is $g = \rho^{2d}\cos^{2(d-1)}\theta$. When θ is a line segment between the nth and mth zeros of $\mathrm{Re}\cos\theta$, the volume is given by

$$V = \int_{\mathcal{M}} d^d x\,\sqrt{g} = \rho^d\, V_{d-1} \int_{(n+1/2)\pi}^{(m+1/2)\pi} \cos^{d-1}\theta\, d\theta \tag{3.9}$$

where V_{d-1} is the volume of S^{d-1} with unit radius. When $n = -1$ and $m = 0$, the volume of the d-sphere V_d is recovered. For the Euclidean hemisphere of Figure 3.2a, $n = -1$ and $m = -1/2$. The volume depends on the values of n and m, but also on d. When d is even, $\cos\theta$ in the integrand is raised to an odd power. This means if $m - n$ is even, the integral will vanish. If $m - n$ is odd, the volume is $V_d \operatorname{sign}(m - n)$. In this case, the magnitude of the volume remains unchanged as n and m vary. If d is odd instead, the integrand is an even power of $\cos\theta$, and the volume is $(m - n)V_d$, which changes in magnitude as the endpoints vary.

While the action of the Euclidean hemisphere is finite, it can be made arbitrarily negative when d is odd by shifting one of the endpoints. Consider what happens when the left endpoint is shifted farther to the left, which corresponds to making n more negative. This scenario is depicted in Figure 3.2b. By shifting the endpoint more, the action can be made as negative as desired, causing the wavefunction to blow up. The question is whether the admissibility condition can be used to eliminate unacceptable metrics like this from contributing to the path integral.

One of the admissibility conditions introduced in the last chapter required the real part of $\sqrt{g}$ to be positive, which prevents the determinant from being a negative real number. Because the radius is real, this requires $\operatorname{Re}\cos^{d-1}\theta > 0$. Consider for simplicity the case where $d = 3$, although the argument is the same in other dimensions. Defining $\theta = a + ib$, the admissibility condition is

$$\text{Re}\cos^2\theta = \cos^2 a\,\cosh^2 b - \sin^2 a\,\sinh^2 b > 0 \tag{3.10}$$

What distinguishes the unphysical solution in Figure 3.2b is that it passes through one of the zeros of $\operatorname{Re}\cos\theta$. When this happens, $\text{Re}\cos^2\theta = -\sin^2 a\,\sinh^2 b$, which violates the admissibility condition. To avoid this, θ must be confined to a strip $(n - 1/2)\pi \leq \operatorname{Re}\theta \leq (n + 1/2)\pi$. This is true of the curve in Figure 3.2a, but not others that start farther away on the real θ axis. In summary, the admissibility condition provides one way of distinguishing between physical and unphysical metrics on cosmological scales.

3.2 Topology-changing Processes

The creation of the universe from nothing discussed in the last section is actually part of a class of spacetimes in which the topology changes. In a topology-changing process, the metric is smooth, but it is not invertible at the point at which the topology changes [37, 38]. Because of this degeneracy, coupling the metric to quantum fields is not well-defined. By making the metric complex, the degeneracy can be removed, but at the expense of potentially introducing unphysical metrics. This section will again discuss how the admissibility condition can be used to address this problem.

Although the formalism can accommodate other scenarios, this section will focus on the topology-changing process in which a closed universe splits in two.

To analyze topology-changing processes, it is helpful to introduce a Morse function f and positive definite metric h_{ab}. Often called the time function, f defines the causal structure of the manifold.[1] From these two ingredients, it is possible to construct a metric that is Lorentzian almost everywhere. This metric is

$$g_{ab} = h_{ab}h^{cd}(\partial_c f)(\partial_d f) - \zeta\,(\partial_a f)(\partial_b f) \tag{3.11}$$

where ζ is a parameter which controls the metric signature, as will be discussed later.

To describe a wide range of topology-changing processes, the metric h_{ab} can be chosen to be flat and f can be a quadratic polynomial in the coordinates. Consider a process in two dimensions with coordinates x and y. Let $h_{ab} = \delta_{ab}$ and $f = \frac{1}{2}(x^2+\epsilon y^2)$, where $\epsilon = \pm 1$ is the Morse index of f. The metric becomes

$$\mathrm{d}s^2 = \left(x^2 + y^2\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - \zeta\left(x\,\mathrm{d}x + \epsilon y\,\mathrm{d}y\right)^2 \tag{3.12}$$

This metric is invertible everywhere except $x = y = 0$, where it vanishes. Depending on the value of ϵ, different processes can be described. When $\epsilon = 1$, the metric describes the creation of a closed universe from nothing. When $\epsilon = -1$, the spacetime corresponds to the splitting of a closed universe in two, depicted in Figure 3.3. In the remainder of this section, ϵ will be taken to be -1.

As mentioned before, the parameter ζ controls the signature of the metric. This can be better seen by making the coordinate transformation

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned} \tag{3.13}$$

In these coordinates, the metric becomes

$$\mathrm{d}s^2 = \frac{1}{4}(1-\zeta)\mathrm{d}u^2 + \frac{1}{4}\mathrm{d}v^2 \tag{3.14}$$

When $\zeta > 1$, the metric is Lorentzian, otherwise it is Euclidean. In either case, the metric degenerates at $x = y = 0$. The coordinate u plays the role of time, being proportional to the Morse function f. The metric describes a hyperbola that grows with time, producing the geometry shown schematically in Figure 3.3. The geometry degenerates at the critical point of u.

[1] More generally, a Morse function is a smooth real function on a manifold $f : \mathcal{M} \to [0, 1]$ that has no degenerate critical points [39]. At a critical point, the gradient of f vanishes. It is degenerate if the matrix of second derivatives, called the Hessian, is singular. The number of negative eigenvalues of the Hessian is called the Morse index of the critical point. The Morse function and index can be used to study the topology of a manifold.

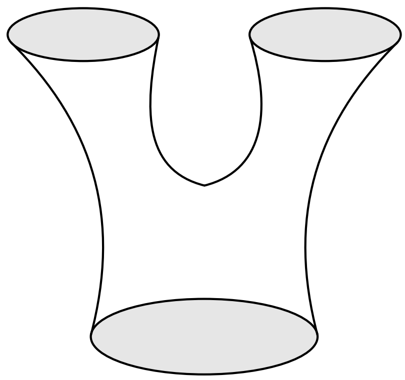


Figure 3.3: One example of a topology-changing process is when a closed universe splits in two. At the point at which it splits, the metric becomes degenerate, which prevents it from being coupled to quantum fields.

Because of the degeneracy, the metric cannot be coupled to quantum fields. For instance, the kinetic term of a two-form gauge field $F_{\mu\nu}F^{\mu\nu}$ requires contracting the field with the inverse metric. When the metric degenerates, the inverse blows up, making this term ill-defined. To address this problem, the metric can be complexified. One possible complex metric is [1]

$$\mathrm{d}s^2 = (x^2 + y^2 + \gamma)(\mathrm{d}x^2 + \mathrm{d}y^2) - (\zeta \pm i\delta)(x\,\mathrm{d}x - y\,\mathrm{d}y)^2 \tag{3.15}$$

The parameters γ and δ were introduced, which serve different purposes. The role of γ is to make the metric non-degenerate everywhere. A function of x and y, it should be positive at $x = y = 0$, but can vanish everywhere else. The parameter δ complexifies the metric. When δ goes to zero, the metric becomes Lorentzian. This satisfies a requirement introduced in the last chapter, which says Lorentzian metrics should lie on the boundary of the space of complex metrics. For the metric to be complex, δ should be nonzero.

In summary, to cure the degeneracy, two requirements need to be met, $\gamma > 0$ at $x = y = 0$ and $\delta \neq 0$. Again, the question is whether the admissibility condition is equivalent to these requirements. To simplify the analysis, the metric of Equation (3.15) can be recast into a slightly different form. Let V be the one-form $x\,\mathrm{d}x - y\,\mathrm{d}y$, and W be a one-form orthogonal to it. Then the metric has the form

$$\mathrm{d}s^2 = -(A \pm i\delta)V \otimes V + B(W \otimes W) \tag{3.16}$$

where $A > 0$. For the admissibility condition to be satisfied, the angles of the metric eigenvalues should sum to less than π. This is possible if $B > 0$ and $\delta \neq 0$. In this case, one of the eigenvalues is positive, while the other has an argument less than

π. For $B > 0$ to be satisfied, γ needs to be positive when $x = y = 0$. These two conditions are exactly the ones needed to remove the degeneracy.

This section focused on one example of a topology-changing process, but there are many similar processes that can be explored. Even within the current framework, different values of ϵ and ζ can be chosen, leading to different singularities. The example of a closed universe splitting in two hints at the possibility of using the admissibility condition to address non-invertible metrics in similar processes where the topology changes.

3.3 Time Folds and Double Cones

The previous sections demonstrated how the admissibility condition can be applied to spacetimes where the topology evolves in time. In the creation of a closed universe from nothing, spacetime starts as a hemisphere, then expands exponentially. In the other topology-changing process, the universe splits in two at a certain point in time, the two branches evolving independently into the future. In either case, time moves in one direction. In other applications, it is important to consider both forward and backward propagation in time. For instance, this has to be done to compute the time-dependent average of an observable in quantum field theory:

$$\langle \mathcal{O} \rangle = \langle \Psi | \mathcal{O}(t) | \Psi \rangle = \langle \Psi | e^{iHt} \mathcal{O} e^{-iHt} | \Psi \rangle \tag{3.17}$$

This type of observable can be computed using path integrals that move forward and backward in time. Using the completeness relation twice, the expectation value can be written

$$\langle \mathcal{O} \rangle = \iint d\Psi_1 \, d\Psi_2 \, \langle \Psi | e^{iHt} | \Psi_2 \rangle \langle \Psi_2 | \mathcal{O} | \Psi_1 \rangle \langle \Psi_1 | e^{-iHt} | \Psi \rangle \tag{3.18}$$

The integrand consists of three components. Starting from the right, the first expectation value is a path integral that propagates the state forward by a time t to the intermediate state Ψ_1. Next, the average of $\mathcal{O}$ between Ψ_1 and a second intermediate state Ψ_2 is taken. Finally, Ψ_2 is propagated backward to the initial state. This is an example of a time fold, depicted schematically in Figure 3.4.

Another example of a computation that involves forward and backward time propagation is the time-dependent density matrix, which has many applications including calculation of the gravitational entropy [40, 41, 42]. The density matrix is a description of a quantum state that allows for the possibility for the state to exist as a mixture of different wavefunctions. For a given wavefunction $|\Psi(t)\rangle$, the density matrix is given by

$$\rho = |\Psi(t)\rangle \langle \Psi(t)| = e^{-iHt} |\Psi_0\rangle \langle \Psi_0| e^{iHt} \tag{3.19}$$

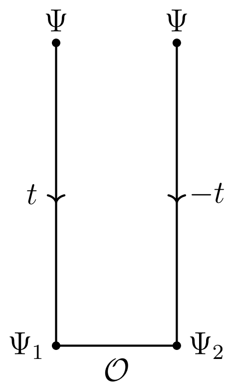


Figure 3.4: Time folds are processes that involve forward and backward propagation in time. This example computes the average of an observable $\mathcal{O}$ in a state Ψ. To do this, the state is propagated forward and backward in time by an amount t, which can be achieved with two path integrals. The operator $\mathcal{O}$ connects the two intermediate states.

where $|\Psi_0\rangle$ is the initial state. To compute each element of this matrix ρ_{ij}, two path integrals can be used:

$$\rho_{ij} = \langle\Psi_i| e^{-iHt} |\Psi_0\rangle \langle\Psi_0| e^{iHt} |\Psi_j\rangle \tag{3.20}$$

Similar to before, the first path integral propagates the initial state forward to state Ψ_i, and the second propagates the state Ψ_j backward to the initial state. This is another example of a time fold.

The notion of forward and backward propagation in time is intimately connected to complex spacetime metrics. In particular, forward and backward time propagation is a consequence of regularizing a Lorentzian metric two different ways [43, 44, 45]. This can be seen by revisiting the concept of the Feynman propagator in quantum field theory [46]. The propagator captures the probability amplitude for a particle to travel from one position to another in a given amount of time. This process is shown in Figure 3.5. For a massive scalar field, the Feynman propagator in momentum space is given by

$$\Delta_F(p) = \frac{1}{p^2 - m^2} = \frac{1}{-E^2 + \vec{p}^2 - m^2} \tag{3.21}$$

where p is the norm of the particle four-momentum and m is its mass. To expand the square momentum in terms of the energy E and three-momentum $\vec{p}$, the Minkowski metric with signature $(-+++)$ was used.

To compute the amplitude in position space, the Feynman propagator needs to be integrated over all values of the four-momentum. This integral fails to converge because the propagator is ill-defined when $E = \pm\sqrt{\vec{p}^2 - m^2} \equiv \pm\omega_p$. The usual

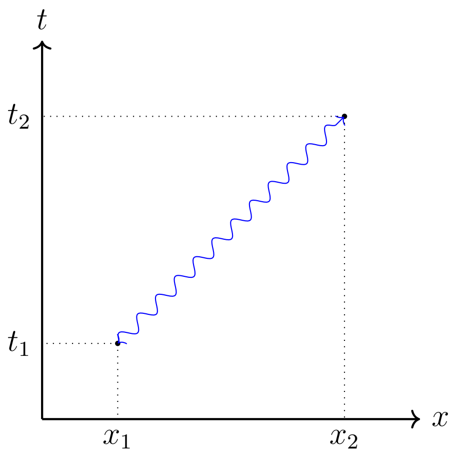


Figure 3.5: The Feynman propagator carries information about the likelihood for a particle to travel between two spacetime points. Depending on the sign of the ϵ prescription that is used, forward or backward time propagation is encoded, which corresponds to two versions of complexified Minkowski space.

method to remove the degeneracy is to add a small imaginary quantity $i\epsilon$ to the denominator. The propagator then develops poles at $E = \pm(\omega_p + i\epsilon)$. The integral over energy is transformed into a complex integral, which avoids the poles when taken only over real values. A similar result can be achieved by adding $-i\epsilon$ to the propagator instead, which suggests the general form

$$\Delta_F(p) = \frac{1}{p^2 - m^2 \pm i\epsilon} \tag{3.22}$$

The ϵ prescription is usually viewed as a method to regularize the propagator, but it also carries information about the time direction of the propagation. When $+i\epsilon$ is used, the propagator moves a state forward in time. When $-i\epsilon$ is used, the state is propagated backward in time.

The two versions of the ϵ prescription correspond to different regularizations of the metric. This can be seen by rewriting the propagator of Equation (3.22) as

$$\Delta_F(p) = \frac{1}{E^2(-1 \pm i\epsilon) + \vec{p}^2 - m^2} = \frac{1}{p^\mu p^\nu \eta_{\mu\nu} - m^2} \tag{3.23}$$

where the metric $\eta_{\mu\nu}$ is a small ϵ away from the usual Minkowski metric. In d dimensions, it is given by

$$\mathrm{d}s^2 = (-1 \pm i\epsilon)\mathrm{d}x_1^2 + \sum_{i=2}^{d} \mathrm{d}x_i^2 \tag{3.24}$$

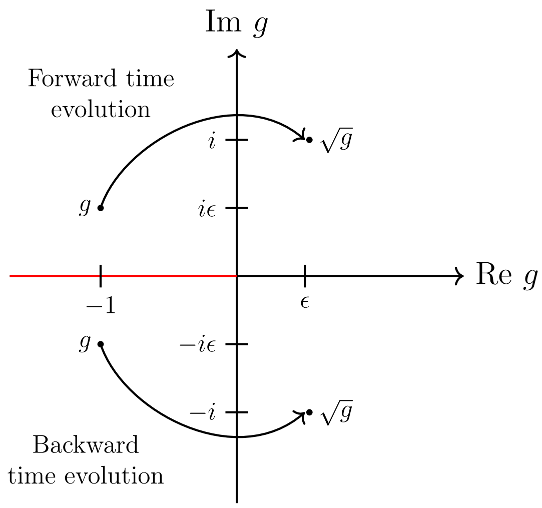


Figure 3.6: The admissibility condition requires the square root of the metric determinant to have positive real part. When applied to two different versions of a metric that is ϵ away from being Lorentzian, forward and backward time evolution is implemented.

While the Minkowski metric is not allowable, the ϵ prescription produces a complex metric that is allowable. Reinterpreting the ϵ prescription in terms of complex metrics allows it to be generalized to curved spacetimes [44].

The fact that the ϵ prescription made the metric allowable suggests a connection between time propagation and the allowability condition. The admissibility condition requires the square root of the metric determinant to be chosen so that it has positive real part. The determinant of Equation (3.24) is $g = -1 \pm i\epsilon$. For a given choice of $\pm i\epsilon$, there are two possible square roots. The square roots with positive real part are given by $\sqrt{g} = \epsilon \pm i$, which are close to the positive and negative imaginary axes. These square roots that satisfy the admissibility condition are related to forward and backward time propagation through the time evolution operator U, which involves the metric determinant. Over a small time interval t, U is given by

$$\begin{aligned} U &= \exp(-\sqrt{g}Ht) \\ &= \exp\left(-Ht(\epsilon \pm i)\right) \\ &= \exp(\mp iHt - \epsilon H) \end{aligned} \tag{3.25}$$

where ϵ has absorbed the time t in the last line. The conclusion is that $\sqrt{g}$ close to the positive imaginary axis generates forward time evolution, while $\sqrt{g}$ close to the negative imaginary axis generates backward evolution. This is depicted in Figure 3.6.

Depending on the sign of $i\epsilon$ used in the metric, forward or backward time evolution is implemented. To model a situation in which a state evolves forward then backward

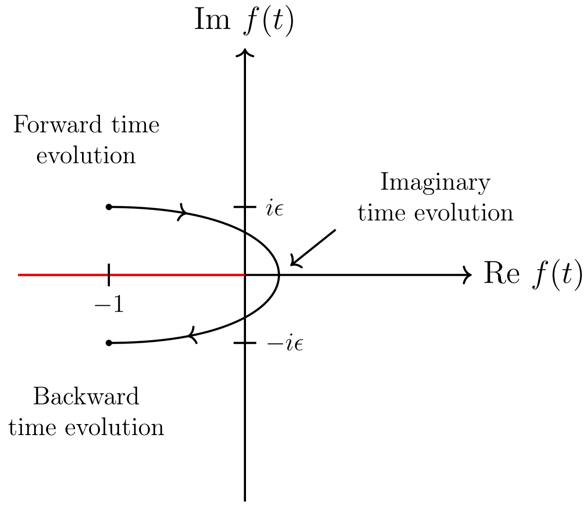


Figure 3.7: To evolve a state forward and backward in time, a metric can be used that smoothly connects two versions of a regularized Lorentzian metric. The function $f(t)$ is the time component of the metric, whose imaginary part goes from positive to negative. When the real part of f becomes positive, the path integral is regularized.

in time, the two different metrics can be stitched together in the following way:

$$\mathrm{d}s^2 = f(t)\,\mathrm{d}t^2 + \sum_{i=2}^{d} \mathrm{d}x_i^2 \tag{3.26}$$

where $f(t)$ is the discontinuous function

$$f(t) = \begin{cases} -1 + i\epsilon, & t < 0 \\ -1 - i\epsilon, & t > 0 \end{cases} \tag{3.27}$$

Because coupling quantum fields to a discontinuous metric is problematic, the function $f(t)$ can be varied continuously, as shown in Figure 3.7. It must take the long way around because if it crosses the negative real axis, the metric is no longer allowable. When the curve crosses over into the positive real axis, the real part of the metric is positive definite. This provides some imaginary time propagation that regularizes the path integral.

Similar to a time fold, a double cone can be used to compute observables in quantum field theory using path integrals. While a time fold involves propagating a state forward and backward in time, a double cone computes observables by propagating a state around a time circle. This type of path integral is useful for computing the spectral form factor of a quantum system, which captures correlations in its energy spectrum [47, 48]. The spectral form factor is given by $\left\langle \mathrm{Tr}\, e^{iHT}\, \mathrm{Tr}\, e^{-iHT} \right\rangle$. The traces in this expression can be computed using a path integral:

$$\begin{aligned} \mathrm{Tr}\, e^{-iHT} &= \int d\Psi \, \langle \Psi |\, e^{-iHt} \,| \Psi \rangle \\ &= \int d\Psi \int_{\phi_i=\Psi}^{\phi_f=\Psi} \mathcal{D}\phi \, e^{iS[\phi]} \\ &= \int \mathcal{D}\phi \, e^{iS[\phi]}, \quad \phi(t+T) = \phi(t) \end{aligned} \tag{3.28}$$

The path integral in the last line is taken only over fields that are periodic over time T. To compute the spectral form factor, two path integrals over periodic fields need to be computed, one traveling forward over the time circle, the other backward. These two circles lie on the boundary of a double cone geometry. According to the AdS/CFT correspondence, the spectral form factor of the quantum fields on the boundary is related to the partition function over geometries in the interior. One of the saddle points of this partition function corresponds to the double cone.

The double cone geometry can be constructed starting from the metric

$$\mathrm{d}s^2 = -\sinh^2 r \, \mathrm{d}t^2 + \mathrm{d}r^2 \tag{3.29}$$

where t is a real variable with identification $t \cong t + T$. When r is non-negative, this metric describes a single cone. By allowing r to become negative, it describes a double cone joined at $r = 0$, shown in Figure 3.8. The double cone metric degenerates at $r = 0$. To regulate it, r can be complexified so that it does not pass through zero:

$$r = u - i\epsilon \tag{3.30}$$

where u is real and ϵ is a small nonzero real number. Without loss of generality, ϵ can be taken to be positive. The metric becomes

$$\mathrm{d}s^2 = -\sinh^2(u - i\epsilon)\, \mathrm{d}t^2 + \mathrm{d}r^2 \tag{3.31}$$

As was done with time folds, the admissibility condition can be used to guide the choice of $\sqrt{g}$ that corresponds to forward and backward time propagation. Using $\sinh(u - i\epsilon) = \cos\epsilon \sinh u - i \sin\epsilon \cosh u$, the metric determinant is

$$g = (2\sin^2\epsilon - 1)\sinh^2 u + \sin^2\epsilon + \frac{1}{2}\sin(2\epsilon)\sinh(2u) i \tag{3.32}$$

When u is positive, g lies just above the negative real axis, and $\sqrt{g}$ is close to the positive imaginary axis. When u is negative, $\sqrt{g}$ is close to the negative imaginary axis. This is similar to the situation in Figure 3.6, where u plays the role of $i\epsilon$ controlling the time evolution. The positive u cone is related to the holographic description of the forward time evolution, $\mathrm{Tr}\, e^{-iHT}$, and the negative u cone to the backward time evolution, $\mathrm{Tr}\, e^{iHT}$.

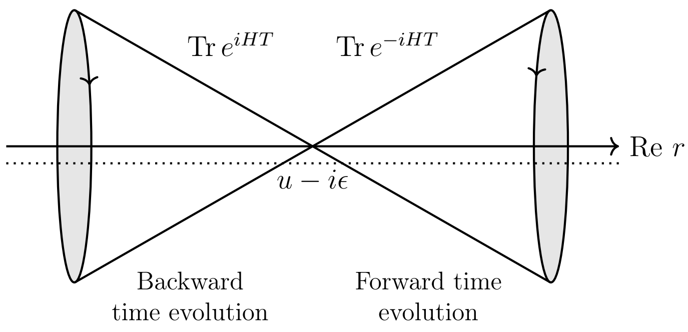


Figure 3.8: The regularized double cone geometry can be used to compute the spectral form factor of the dual conformal field theory on the boundary. The right cone corresponds to forward time evolution, the left cone to backward evolution.

3.4 Rotating Black Holes

The last application of the admissibility condition presented in this chapter involves rotating black holes. As the last chapter touched on, a Wick rotation makes the metric for a rotating black hole complex, which does not happen for a black hole without angular momentum. By assuming this metric dominates the path integral, the thermodynamics of rotating black holes can be derived [23]. While this result suggests that this metric is somehow important, introducing complex metrics can lead to unphysical situations.

To see how complex metrics can lead to unphysical results, consider a rotating black hole in asymptotically flat space. This black hole has two conserved quantities, the energy H and angular momentum J. These quantities can be combined into a single conserved charge $Q \equiv H - \Omega J$, where Ω is the fixed angular velocity. Following a Wick rotation, the Euclidean path integral is related to the partition function

$$Z = \mathrm{Tr}\, e^{-\beta Q} \tag{3.33}$$

where β is the inverse temperature. The conserved charge Q has an associated Killing vector field, which in asymptotically flat space is

$$K^\mu = \partial_t - \Omega \left(x\, \partial_y - y\, \partial_x \right) \tag{3.34}$$

When this vector field is spacelike, a particle localized in the field can have an arbitrarily negative value of Q. This occurs when $K^\mu K_\mu > 0$, which corresponds to $\Omega^2 \left(x^2 + y^2 \right) > 1$. When this happens, the partition function blows up, making the thermodynamic ensemble unstable. This prevents quantum corrections to the ensemble from being applied, which is often done in quantum gravity. One way of avoiding this situation is to require the Killing vector to be everywhere timelike outside the horizon. In this section, it will be shown that this is the same as requiring the Euclidean form of the metric to be allowable.

There are many different types of rotating black holes that can be considered, including ones that asymptotically approach flat or anti-de Sitter space [49]. In general, the metric of a rotating black hole takes the form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \rho^2(N^\phi\mathrm{d}t + \mathrm{d}\phi)^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 \tag{3.35}$$

where N, N^ϕ, ρ, g_{rr}, and $g_{\theta\theta}$ are functions of r and θ only. The outer horizon is the outermost surface for which $N^2 = 0$. On the horizon, the function N^ϕ is equal to the angular velocity Ω. If the space is asymptotically flat or anti-de Sitter, N^ϕ approaches zero as r approaches infinity. One example of a metric that has this form is the Kerr metric, which describes a rotating black hole in asymptotically Minkowski space. Introduced in Chapter 1, the Kerr metric is reproduced below:

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1 - \frac{2GMr}{\chi^2}\right)\mathrm{d}t^2 - \frac{2GMar\sin^2\theta}{\chi^2}\left(\mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}\phi\,\mathrm{d}t\right) + \\ &\frac{\chi^2}{\Delta}\mathrm{d}r^2 + \chi^2\mathrm{d}\theta^2 + \frac{\Xi\sin^2\theta}{\chi^2}\mathrm{d}\phi^2\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}\chi^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2GMr + a^2 \\ \Xi &= (r^2 + a^2)^2 - a^2\Delta\sin^2\theta\end{aligned} \tag{3.37}$$

The radius of the outer horizon r_+ is the larger root of $\Delta(r)$. At the outer horizon, the function N^2 given by

$$N^2 = 1 - \frac{2GMr(r^2 + a^2)}{\Xi} \tag{3.38}$$

vanishes. The function N^ϕ is

$$N^\phi = -\frac{2GMar}{\Xi} \tag{3.39}$$

which is equal to $\Omega = -a/(r_+^2 + a^2)$ at the outer horizon. The other functions ρ, g_{rr}, and $g_{\theta\theta}$ can be obtained similarly by matching the metric coefficients between Equation (3.35) and Equation (3.36).

As mentioned before, it needs to be shown under what conditions the Killing vector field is timelike outside the horizon. To compute the Killing vector, it is convenient to introduce a new angular coordinate, $\tilde{\phi} \equiv \phi + \Omega t$. The metric is then

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \rho^2\left((N^\phi - \Omega)\mathrm{d}t + \mathrm{d}\tilde{\phi}\right)^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 \tag{3.40}$$

In these new coordinates, the Killing vector is simply $K^\mu = \partial_t$, which is timelike if $g_{tt} = -N^2 + \rho^2(N^\phi - \Omega)^2 < 0$. This means the partition function is well-defined if

$$N^2 - \rho^2(N^\phi - \Omega)^2 > 0 \tag{3.41}$$

This is a condition involving the coordinates and parameters that must be satisfied at every spacetime point outside the horizon. If the angular velocity is too large, this condition is violated. This suggests that for the Killing vector field to be timelike outside the horizon, the black hole cannot rotate too quickly. This agrees with the requirement that the charge $Q \equiv H - \Omega J$ should not be arbitrarily negative, which spoils the thermodynamic ensemble.

To establish the connection between a timelike Killing vector field and an allowable metric, a Wick rotation first needs to be performed, $t = i\tau$. In most physical situations, including in asymptotically flat and anti-de Sitter space, g_{rr} and $g_{\theta\theta}$ are positive outside the horizon. In this case, the r and θ components of the metric are already positive definite and can be ignored when assessing admissibility. The remaining components of the metric are given by

$$\mathrm{d}s^2 = N^2 \mathrm{d}\tau^2 + \rho^2 \left(i(N^\phi - \Omega)\mathrm{d}\tau + \mathrm{d}\tilde{\phi} \right)^2 \tag{3.42}$$

On the horizon, $N^2 = 0$ and $N^\phi = \Omega$, and the metric becomes $\mathrm{d}s^2 = \rho^2 \mathrm{d}\tilde{\phi}^2$. Because the determinant of this metric in the larger two-dimensional space is zero, the metric lies on the boundary of the space of positive definite metrics. This agrees with the fact that the Killing vector is null-like on the horizon, lying on the boundary of the condition in Equation (3.41).

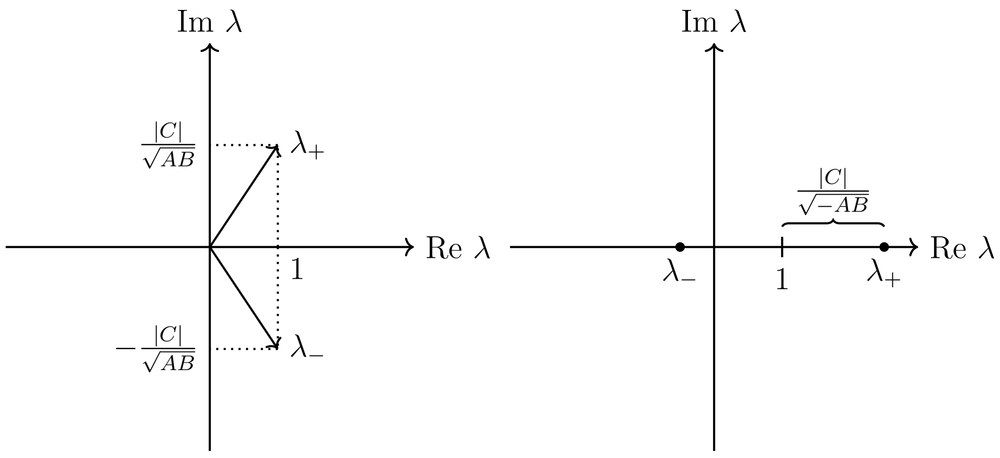
The allowability of the metric away from the horizon needs to be analyzed. Outside the horizon, the metric takes the form

$$\mathrm{d}s^2 = A\,\mathrm{d}\tau^2 + B\,\mathrm{d}\tilde{\phi}^2 + 2iC\,\mathrm{d}\tau\mathrm{d}\tilde{\phi} \tag{3.43}$$

where

$$\begin{aligned} A &= N^2 - \rho^2(N^\phi - \Omega)^2 \\ B &= \rho^2 \\ C &= \rho^2(N^\phi - \Omega) \end{aligned} \tag{3.44}$$

The admissibility condition for $p = 1$ requires $\mathrm{Re}(\sqrt{g}\, g^{\mu\nu})$ to be positive definite. As the last chapter discussed, this is the same as requiring $\mathrm{Re}(g_{\mu\nu}/\sqrt{g})$ to be positive definite. The metric determinant is $g = AB + C^2 = (N\rho)^2 > 0$. In this case, the condition requires $\mathrm{Re}\ g_{\mu\nu}$ to be positive definite. This happens when the trace and determinant of this matrix are positive. These conditions require $A + B > 0$ and



(a) Complex eigenvalues (b) Real eigenvalues

Figure 3.9: Depending on the coefficients of the rotating black hole metric, its eigenvalues can be complex or real. When $AB > 0$, the eigenvalues lie on either side of the positive real axis. When $AB < 0$, the eigenvalues lie on either side of 1. At least one of them becomes negative, violating the admissibility condition.

$AB > 0$, respectively. Combined, they require $A, B > 0$. The condition $B = \rho^2 > 0$ is satisfied outside the horizon. Otherwise, the metric would not be Lorentzian. The condition $A > 0$ is the same as Equation (3.41).

Imposing the allowability condition for $p = 1$ leads to the same results as requiring a timelike Killing vector field outside the horizon, but the same conclusion can be reached by computing the complex eigenvalues of the metric. Following the procedure outlined in Appendix B, the complex eigenvalues of Equation (3.43) are

$$\lambda_\pm = 1 \pm \frac{C}{\sqrt{AB}} i \tag{3.45}$$

When $AB > 0$, the eigenvalues lie on either side of the positive real axis, depicted in Figure 3.9a. The angle subtended by the two eigenvalues cannot exceed π, which satisfies the admissibility condition. Because $B > 0$, this requires $A > 0$. Another scenario is that $AB = -\gamma^2 < 0$, where $\gamma > 0$. In this case, the eigenvalues are real, as shown in Figure 3.9b. The admissibility condition requires $|C| < \gamma$. For this to be satisfied, the determinant of $g_{\mu\nu}$ needs to be negative, which is not the case. This proves that the admissibility condition involving the complex eigenvalues of the metric requires $A, B > 0$, just like the $p = 1$ condition.

Not every rotating black hole satisfies the admissibility condition derived in this chapter. For instance, in asymptotically flat space, the admissibility condition is not satisfied far from the black hole. This can be seen by considering the norm of the

Killing vector, which is $\Omega^2(x^2+y^2)-1$ as discussed above. Because the norm is positive at infinity, the Killing vector is spacelike regardless of the angular velocity, violating the admissibility condition. Viewed another way, the black hole cannot be in thermal equilibrium because the thermal radiation would need to corotate faster than light far from the black hole. In contrast, in asymptotically anti-de Sitter space, a black hole can be in equilibrium if the angular velocity is small enough [1, 49]. This is another reason why the admissibility condition for rotating black holes in AdS space is interesting to study.

One example of an admissible black hole in asymptotically anti-de Sitter space is a BTZ black hole in three dimensions [50, 51]. A BTZ black hole is characterized by its mass M and angular momentum J in a world with a negative cosmological constant. The radius of curvature l is related to the cosmological constant by $\Lambda = -1/l^2$. When $|J| < Ml$, black hole solutions exist, with the following functions

$$\begin{aligned} N^2 &= \left(\frac{r}{l\rho}\right)^2 (r^2 - r_+^2) \\ N^\phi &= -\frac{4GJ}{\rho^2} \\ \rho^2 &= r^2 + 4GMl^2 - \frac{1}{2}r_+^2 \end{aligned} \tag{3.46}$$

The outer horizon occurs at $r_+^2 = 8Gl\sqrt{M^2l^2 - J^2}$, and the angular velocity Ω is $-8GJ/(r_+^2 + 8GMl^2)$. The allowability condition requires

$$N^2 - \rho^2(N^\phi - \Omega)^2 = \frac{r^2 - r_+^2}{\rho^2}\left(\frac{r^2}{l^2} - \frac{(4GJ)^2(r^2 - r_+^2)}{(\frac{1}{2}r_+^2 + 4GMl^2)^2}\right) > 0 \tag{3.47}$$

Because $|J| < Ml$, the second term is less than r^2/l^2. This means that outside the horizon, $r > r_+$, the condition is satisfied.

This chapter presented examples of complex metrics that emerge for various reasons, including to regularize a degenerate metric or make the path integral converge. In general, the admissibility condition was used to identify the set of metrics that provide consistent physical results. In the last example of rotating black holes, acceptable metrics corresponded to those that kept the thermodynamic partition function bounded. In the next chapter, rotating black holes will again be studied, focusing specifically on charged, rotating supersymmetric black holes in four-dimensional anti-de Sitter space. Supersymmetric AdS black holes like these are especially important in quantum gravity because of the AdS/CFT correspondence, which relates them to quantum field theories without gravity. The admissibility condition will be analyzed in depth, providing preliminary results on the space of complex metrics over which to define a quantum theory of black holes.

Chapter 4

Charged Rotating Black Holes in 4D Anti-de Sitter Space

In this chapter, the original contribution of the thesis is presented, which involves the analysis of the admissibility condition for charged, rotating supersymmetric black holes in four-dimensional anti-de Sitter space. As discussed in the last chapter, rotating spacetimes exhibit complex saddles in the gravitational path integral. Because not every complex metric is physically relevant, the admissibility condition provides a way to discriminate between acceptable and unacceptable metrics. It will be shown that black holes that are supersymmetric and extremal, called BPS black holes, reside on the boundary of the space of allowable metrics. To be allowable, a black hole must have an outer horizon larger than that of these BPS black holes.

The charged, rotating supersymmetric black holes analyzed here are solutions to supergravity, a theory that combines general relativity with the principles of supersymmetry [25, 26]. Broadly, supergravity solutions can be either ungauged or gauged. Charged, rotating black hole solutions of ungauged supergravity play an important role in the microscopic counting of black hole entropy, while gauged solutions provide important backgrounds for the AdS/CFT correspondence [52]. In this thesis, solutions in gauged supergravity are considered. These solutions were first constructed within $\mathcal{N} = 4$, SO(4) supergravity [53]. They can be obtained by a reduction of 11-dimensional supergravity on S^7 [54]. To simplify the analysis, two of the charges will be equated, which reduces the solution to a Kerr-Newman-AdS black hole [55, 56]. This type of black hole can be viewed as a solution to $\mathcal{N} = 2$ supergravity.

This chapter focuses on charged, rotating black holes that are supersymmetric but not extremal. Unlike BPS black holes, the metrics for these black holes are complex without performing a Wick rotation. Compared to the black holes studied in the last chapter, they have no Lorentzian counterpart, but they may play an important role in the gravitational path integral. Another reason why these metrics are important is that they can be used to evaluate the action of BPS black holes, which can be compared to the dual quantum field theory. Because BPS black holes have infinite inverse temperature, their action is difficult to compute. By imposing supersymmetry without extremality, the action can be computed, which approaches the BPS limit as the temperature vanishes. In this chapter, the admissibility of these supersymmetric, non-extremal black holes will be studied for the first time.

This chapter begins by applying the same analysis as the last chapter to non-supersymmetric Kerr-Newman-AdS black holes, demonstrating that the admissibility condition for $p = 1$ corresponds to the Killing vector field being timelike outside the horizon [1]. Next, it is shown that for a BPS black hole, the admissibility condition is satisfied throughout the entire spacetime, which is the same as the Killing vector field being everywhere timelike. Then the admissibility condition for supersymmetric, non-extremal black holes is analyzed. The conditions for different values of p are explored analytically and numerically. It will be shown that an allowable black hole must have an outer horizon at least as big as its corresponding BPS black hole. The physical meaning of this conclusion will also be discussed.

4.1 Non-supersymmetric Black Holes

In this section, the admissibility condition for non-supersymmetric black holes is compared to the requirement that the Killing vector field is timelike outside the horizon. After introducing the form of the metric in a frame rotating at infinity, an angular change of variables is made so that the Killing vector has a single component. To assess admissibility of complex metrics, a Wick rotation is performed. Because the metric determinant is always positive, the admissibility condition reduces to the requirement that the real part of the metric is positive definite. It will be shown that this condition is the same as the previous condition on the Killing vector field.

In four-dimensional $\mathcal{N} = 4$ supergravity in asymptotically anti-de Sitter space, there are solutions representing charged, rotating black holes [7, 53]. These solutions contain two gauge fields in the $\mathrm{U}(1) \times \mathrm{U}(1)$ Abelian subgroup of the $\mathrm{SO}(4)$ gauge group. They can also be viewed as solutions in $\mathcal{N} = 8$ gauged supergravity, where the four charges associated with the $\mathrm{U}(1)^4$ subgroup of $\mathrm{SO}(8)$ are set pairwise equal [57, 58]. The Lagrangian for the $\mathcal{N} = 4$ solutions is given by [59]

$$\begin{aligned}\mathcal{L} = (R - 2\mathcal{V}) * 1 - \frac{1}{2}\mathrm{d}\xi \wedge *\mathrm{d}\xi - \frac{1}{2}e^{2\xi}\,\mathrm{d}\chi \wedge *\mathrm{d}\chi - \frac{1}{2}e^{-\xi}F_3 \wedge *F_3 - \\ \frac{1}{2}\chi\, F_3 \wedge F_3 - \frac{1}{2(1+\chi^2 e^{2\xi})}\left(e^{\xi}F_1 \wedge *F_1 - e^{2\xi}\chi\, F_1 \wedge F_1\right)\end{aligned} \tag{4.1}$$

where F_1 and F_3 are the field strengths of the Abelian gauge potentials, A_1 and A_3. The remaining gauge potentials are given by $A_2 = A_1$ and $A_4 = A_3$. The parameter g

represents the AdS length scale, which will be set to one in this analysis. The scalar fields χ and ξ are axion and dilaton fields. The potential for these fields is given by

$$\mathcal{V} = -\frac{1}{2}g^2\left(4 + 2\cosh\xi + e^{\xi}\chi^2\right) \tag{4.2}$$

The solution to the equations of motion is controlled by four parameters: a, m, δ_1, and δ_2. Roughly, these parameters correspond to the four conserved charges of the solution. The parameter a corresponds to the angular momentum J, m to the energy E, and δ_1 and δ_2 to the electric charges Q_1 and Q_3. Because the analysis with two separate electric charges is cumbersome, they will be equated, $Q_1 = Q_3 \equiv Q$, which corresponds to $\delta_1 = \delta_2 \equiv \delta$. As will be seen, this simplification does not impact admissibility because it does not depend on the charge. When the charges are equal, there is a single gauge field and the axion and dilaton fields vanish [60, 61]. This black hole is called a Kerr-Newman-AdS black hole.

For a Kerr-Newman-AdS black hole, the Lagrangian simplifies to

$$\mathcal{L} = R + 6 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.3}$$

which is the bosonic sector of minimal gauged $\mathcal{N} = 2$ supergravity. The equations of motion, found by extremizing the action, are given by

$$R_{\mu\nu} + 3g_{\mu\nu} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{4.4}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{4.5}$$

These are the Einstein-Maxwell field equations with a negative cosmological constant. In a frame rotating at infinity, the solution can be written

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta_r}{W}\left(\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\phi\right)^2 + W\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}\theta^2}{\Delta_\theta}\right) + \\ &\frac{\Delta_\theta\sin^2\theta}{W}\left(a\,\mathrm{d}t - \frac{\tilde{r}^2 + a^2}{\Xi}\mathrm{d}\phi\right)^2 \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \tilde{r} &= r + 2m\sinh^2\delta \\ \Delta_r &= r^2 + a^2 - 2mr + \tilde{r}^2(\tilde{r}^2 + a^2) \\ \Delta_\theta &= 1 - a^2\cos^2\theta, \quad W = \tilde{r}^2 + a^2\cos^2\theta, \quad \Xi = 1 - a^2 \end{aligned} \tag{4.7}$$

The gauge field is given by

$$A = \frac{\sqrt{2}m\sinh(2\delta)\tilde{r}}{W}\left(\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\phi\right) \tag{4.8}$$

The parameters are restricted to the ranges $a \geq 0$, $m \geq 0$, and $\delta \geq 0$. To keep the solution from becoming singular, the condition $a^2 < 1$ must be imposed.

The conserved charges can be expressed in terms of the solution parameters. The energy can be computed using the first law of thermodynamics, briefly reviewed in Chapter 1. The angular momentum and electric charges can be evaluated using Komar and Maxwell integrals [13]. The conserved charges evaluate to [7, 59]

$$E = \frac{m}{\Xi^2}\left(1 + 2\sinh^2\delta\right), \quad J = \frac{ma}{\Xi^2}\left(1 + 2\sinh^2\delta\right), \quad Q = \frac{m\sinh(2\delta)}{4\Xi} \tag{4.9}$$

The roots of the function Δ_r define the black hole horizon. When there are two real roots, the black hole has an inner and outer horizon. As discussed in Chapter 1, this is called a subextremal black hole. When there is a double root, the black hole is extremal. When there are no real roots, there is a naked singularity, which does not correspond to a black hole solution. Which classification a black hole falls into depends on whether m falls above or below a certain critical mass parameter [62].

The condition for the Killing vector field to be timelike outside the horizon can be determined. The Killing vector is given by

$$K^\mu = \partial_t + \Omega\,\partial_\phi \tag{4.10}$$

where Ω is the angular velocity at the outer horizon, $\tilde{r} = \tilde{r}_+$:

$$\Omega = \frac{a\Xi}{\tilde{r}_+^2 + a^2} \tag{4.11}$$

To analyze the condition, it is convenient to introduce a new angular coordinate, $\tilde{\phi} \equiv \phi - \Omega t$. The Killing vector becomes ∂_t, which is timelike if $g_{tt} < 0$. Using Equation (4.6) and making the change of variables, the condition is

$$\Delta_r\left(\Xi - a\Omega\sin^2\theta\right)^2 - \Delta_\theta\sin^2\theta\left(a\Xi - \Omega(\tilde{r}^2 + a^2)\right)^2 > 0 \tag{4.12}$$

It can be shown that the admissibility condition leads to the same condition [1]. First, a Wick rotation can be performed, $t = i\tau$. The r and θ components of the metric are positive definite outside the horizon because W, Δ_r, and Δ_θ are always positive. At fixed r and θ, the metric becomes

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta_r}{W}\left[i\left(1 - \frac{a\Omega\sin^2\theta}{\Xi}\right)\mathrm{d}\tau - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\tilde{\phi}\right]^2 + \\ &\frac{\Delta_\theta\sin^2\theta}{W}\left[i\left(a - \frac{\Omega(\tilde{r}^2 + a^2)}{\Xi}\right)\mathrm{d}\tau - \frac{\tilde{r}^2 + a^2}{\Xi}\mathrm{d}\tilde{\phi}\right]^2 \end{aligned} \tag{4.13}$$

This metric has the form

$$\mathrm{d}s^2 = A\,\mathrm{d}\tau^2 + B\,\mathrm{d}\tilde{\phi}^2 + 2iC\,\mathrm{d}\tau\mathrm{d}\tilde{\phi} \tag{4.14}$$

with determinant

$$g = \frac{\Delta_r \Delta_\theta \sin^2\theta}{\Xi^2} \tag{4.15}$$

The admissibility condition requires that $\mathrm{Re}(g_{\mu\nu}/\sqrt{g}) > 0$. Because the determinant is always positive, this reduces to the condition that the real part of the metric is positive definite, Re $g_{\mu\nu} > 0$. As demonstrated in the last chapter, this is true if $A, B > 0$. The condition $A > 0$ is the same as Equation (4.12). The coefficient B is

$$B = \frac{\Sigma \sin^2\theta}{W\Xi^2} \tag{4.16}$$

where

$$\Sigma \equiv (\tilde{r}^2 + a^2)^2\Delta_\theta - \Delta_r a^2 \sin^2\theta \tag{4.17}$$

The condition $B > 0$ is then $\Sigma > 0$. The t and ϕ parts of the original metric of Equation (4.6) can be rewritten in terms of Σ as

$$\mathrm{d}s^2 = -\frac{W\Delta_r\Delta_\theta}{\Sigma}\mathrm{d}t^2 + \frac{\Sigma\sin^2\theta}{W\Xi^2}\left[\mathrm{d}\phi - \frac{a\Xi}{\Sigma}\left(\Delta_\theta(\tilde{r}^2 + a^2) - \Delta_r\right)\mathrm{d}t\right]^2 \tag{4.18}$$

This form makes it clear that Σ must be greater than zero outside the horizon. Otherwise, the signs of the metric components will flip and the metric will no longer be Lorentzian. It follows that the admissibility condition is the same as requiring the Killing vector ∂_t to be timelike outside the horizon.

4.2 BPS Black Holes

In the last section, the admissibility condition of a non-supersymmetric charged, rotating black hole in four-dimensional anti-de Sitter space was analyzed. In this section, the same condition is analyzed for the supersymmetric, extremal version of this black hole, called a BPS black hole [63]. It will be shown that the admissibility condition is satisfied in all of spacetime, or equivalently that the Killing vector field is timelike everywhere.

The charged, rotating black hole of Equation (4.6) is supersymmetric if the parameters satisfy [7, 64][1]

$$a = \frac{2}{e^{4\delta} - 1} \tag{4.19}$$

This reduces the number of free parameters to two, for example m and δ. This choice of a has implications for the existence of the horizon. Substituting Equation (4.19) into Δ_r, the equation $\Delta_r = 0$ can be written as the sum of two squares [7]

$$\left(\tilde{r}^2 - \frac{2}{e^{4\delta} - 1}\right)^2 + \coth^2(2\delta)\left(r - \frac{2m\sinh^2\delta}{\cosh(2\delta)}\right)^2 = 0 \tag{4.20}$$

The black hole is BPS if m and δ are taken to be real. In this case, Equation (4.20) has a solution only if the two terms vanish separately. The second term vanishes at the radius of the event horizon for the BPS black hole, denoted by r_*:

$$r_* = \frac{2m\sinh^2\delta}{\cosh(2\delta)} \tag{4.21}$$

The first term vanishes when $a = \tilde{r}^2$. At $r = r_*$, this occurs only if

$$m^2 = m_*^2 = \frac{\cosh^2(2\delta)}{e^{2\delta}\sinh^5(2\delta)} \tag{4.22}$$

where m_* is the BPS value of m. Substituting this value of m into Equation (4.21), the radius of the event horizon is

$$r_* = \frac{e^{-\delta}}{2\cosh^2\delta\sqrt{\sinh(2\delta)}} \tag{4.23}$$

Because the horizon radius r_* is a double root of Δ_r, the black hole is extremal as well as supersymmetric. These BPS black holes are described by a single parameter, δ. Note that supersymmetric black holes must only satisfy (4.19), while BPS black holes must also satisfy (4.22). This particular choice of m resulted from assuming m and δ are real.

[1] As shown in [7], the supersymmetry condition can be derived by determining under what conditions the Bogomolny matrix $g^{-1}M_\alpha{}^\beta$ has a zero eigenvalue. The matrix M is the commutator of the supercharges, $M = \{\mathcal{Q}, \overline{\mathcal{Q}}\}$. The eigenvalues of the Bogomolny matrix are $\lambda = E \pm gJ \pm \sum_i Q_i$. Substituting expressions for the conserved charges, the supersymmetric condition can be recovered. In general, a supersymmetric solution also satisfies the Killing spinor equation. For the Lagrangian of Equation (4.3), the Killing spinor equation is $\left(\nabla_\mu - iA_\mu + \frac{1}{2}\Gamma_\mu + \frac{i}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon = 0$, where ϵ is the Dirac spinor, ∇_μ is the covariant derivative, and Γ_μ represents the set of gamma matrices [65].

The admissibility condition of Equation (4.12) can be analyzed for BPS black holes. It is convenient to make a change of coordinates so the frame is not rotating at infinity [66]. Making the change of variables $\phi' \equiv \phi + at$, the Killing vector is

$$K^\mu = \partial_t + \Omega' \partial_{\phi'} \tag{4.24}$$

where the angular velocity is now

$$\Omega' = \frac{a(1+\tilde{r}_+^2)}{\tilde{r}_+^2 + a^2} \tag{4.25}$$

Substituting the BPS value r_* from Equation (4.21), and exploiting the supersymmetry condition of Equation (4.19), results in $\Omega' = 1$. This implies

$$a = \tilde{r}_+^2 \tag{4.26}$$

which was already mentioned above. This condition can be seen as another parameter constraint characterizing a BPS black hole.

The new angular coordinate ϕ' can again be translated so that the Killing vector points along the time direction. The admissibility condition becomes

$$\Delta_r \left(\Xi + a(a-1)\sin^2\theta\right)^2 - \Delta_\theta \sin^2\theta \left(a\Xi + (a-1)(\tilde{r}^2 + a^2)\right)^2 > 0 \tag{4.27}$$

Substituting the expression of Δ_r in terms of $\tilde{r}$ and using Equation (4.26), the condition is a fourth-order polynomial in $\tilde{r}$ given by

$$\begin{aligned} \tilde{r}^4\cos^2\theta + (\tilde{r}_+^4\cos^4\theta + 1)\tilde{r}^2 - {} & 2\tilde{r}_+(\tilde{r}_+^2\cos^2\theta + 1)^2\tilde{r} + \\ & \tilde{r}_+^2\left[\tilde{r}_+^2\cos^2\theta + (\tilde{r}_+^2\cos^2\theta + 1)^2\right] > 0 \end{aligned} \tag{4.28}$$

Completing the square, this becomes

$$(\tilde{r}^2 - \tilde{r}_+^2)^2\cos^2\theta + (\tilde{r}_+^2\cos^2\theta + 1)^2(\tilde{r}_+ - r)^2 > 0 \tag{4.29}$$

Like Δ_r, the lefthand side has a double root at $\tilde{r}_+$. The other two roots are given by $-\tilde{r}_+ \pm i\sec\theta(\tilde{r}_+^2\cos^2\theta + 1)$. Assuming $\tilde{r}$ is real, the lefthand side is always non-negative and the admissibility condition is satisfied everywhere. Equivalently, the Killing vector field is everywhere timelike.

4.3 Supersymmetric Black Holes

In the last section, the admissibility condition for BPS black holes was analyzed. These black holes resulted from imposing supersymmetry while keeping m and δ

real. If these parameters are allowed to become complex, the admissibility of general supersymmetric solutions can be explored. As already stated, one important reason for considering supersymmetric black holes is that they can be used to evaluate the action of BPS black holes. While these supersymmetric metrics do not correspond to physical black holes, they can play an important role in the Euclidean gravitational path integral. The admissibility condition is one way of deciding the acceptable complex metrics to include in this integral.

In this section, it will first be demonstrated how m can be complexified, which complexifies the metric. The admissibility condition is analyzed for every p from zero to four. Through a combination of analytical and numerical methods, it will be shown that allowable metrics correspond to ones satisfying $\tilde{r}_+ > \sqrt{a}$. BPS black holes with positive $\tilde{r}_+$ lie on the boundary of the space of allowable metrics. It will also be shown that the form of the admissibility condition in terms of complex eigenvalues leads to the same conclusion. In the following analysis, the radial coordinate $\tilde{r}$ will be used instead of r throughout, and it is assumed to be real.

The parameter m can be made complex by turning again to the equation defining the horizon, $\Delta_r = 0$. This equation can be seen as an equation in m, which can be expressed in terms of the other parameters. The function Δ_r is given by

$$\Delta_r = \sinh^2(2\delta)\, m^2 - 2\tilde{r}\cosh(2\delta)\, m + \left(\tilde{r}^2 + 1\right)\left(\tilde{r}^2 + a^2\right) \tag{4.30}$$

Enforcing this function to vanish at the outer horizon, $\tilde{r} = \tilde{r}_+$, and using the supersymmetric condition of Equation (4.19), the solution for m is given by [59, 60]

$$m = \frac{\tilde{r}_+ \coth(2\delta) \pm i(\tilde{r}_+^2 - \coth(2\delta) + 1)}{\sinh(2\delta)} \tag{4.31}$$

Note that in this formulation, $\tilde{r}_+$ is treated as a parameter on the same footing as δ.

Making m complex introduces other complex components into the metric. Specifically, Δ_r becomes complex, which introduces new complex elements in the $\tilde{r}$, τ, and $\tilde{\phi}$ components. After complexification, the metric takes the general form

$$\mathrm{d}s^2 = (A + iB)\,\mathrm{d}\tau^2 + 2(C + iD)\,\mathrm{d}\tau\mathrm{d}\tilde{\phi} + (E + iF)\,\mathrm{d}\tilde{\phi}^2 + (G + iH)\,\mathrm{d}\tilde{r}^2 + I\,\mathrm{d}\theta^2 \tag{4.32}$$

The various metric coefficients are real functions of the coordinates. Table 4.1 provides the expressions for these coefficients in terms of the real and imaginary parts of Δ_r. Substituting Equation (4.31) into Equation (4.30), the real and imaginary parts of Δ_r are given by

$$\begin{aligned} \operatorname{Re}\Delta_r &= \tilde{r}^4 + (\tilde{r}_+ - \tilde{r})^2(a+1)^2 + 2a(\tilde{r}_+^2 - \tilde{r}^2) - \tilde{r}_+^4 \\ \operatorname{Im}\Delta_r &= 2(\tilde{r} - \tilde{r}_+)(a+1)\left(a - \tilde{r}_+^2\right) \end{aligned} \tag{4.33}$$

Table 4.1: Imposing the supersymmetric condition causes Δ_r to become complex, which introduces an additional level of complexification into the metric. In this table, the metric coefficients of Equation (4.32) are provided in terms of the real and imaginary parts of Δ_r. The functions Λ and Γ are given by $\Lambda \equiv \Xi - a\Omega \sin^2\theta$ and $\Gamma \equiv a\Xi - \Omega(\tilde{r}^2 + a^2)$.

Coefficient	**Value**	**Coefficient**	**Value**
A	$\frac{\Lambda^2 \operatorname{Re}\Delta_r - \Delta_\theta \Gamma^2 \sin^2\theta}{W\Xi^2}$	F	$-\frac{a^2 \sin^4\theta \operatorname{Im}\Delta_r}{W\Xi^2}$
B	$\frac{\Lambda^2 \operatorname{Im}\Delta_r}{W\Xi^2}$	G	$\frac{W \operatorname{Re}\Delta_r}{\lvert\Delta_r\rvert^2}$
C	$-\frac{a\Lambda \sin^2\theta \operatorname{Im}\Delta_r}{W\Xi^2}$	H	$-\frac{W \operatorname{Im}\Delta_r}{\lvert\Delta_r\rvert^2}$
D	$\frac{\left(a\Lambda \operatorname{Re}\Delta_r - \Delta_\theta \Gamma(\tilde{r}^2+a^2)\right)\sin^2\theta}{W\Xi^2}$	I	$\frac{W}{\Delta_\theta}$
E	$\frac{\left(\Delta_\theta(\tilde{r}^2+a^2)^2 - a^2\sin^2\theta \operatorname{Re}\Delta_r\right)\sin^2\theta}{W\Xi^2}$		

This additional level of complexification alters the admissibility conditions. Recall that a metric is allowable if the action has positive real part for every non-zero, real F and $0 \leq p \leq d$:[2]

$$\operatorname{Re}\left(\sqrt{g}\, g^{i_1 j_1} \cdots g^{i_p j_p} F_{i_1 \cdots i_p} F_{j_1 \cdots j_p}\right) > 0, \quad 0 \leq p \leq d \tag{4.34}$$

In $d = 4$, five conditions need to be considered. In the following sections, these five conditions are analyzed. Appendix A provides the general form of these conditions given the metric of Equation (4.32), provided the determinant is a positive real. The required conditions are summarized in Table A.1.

An alternative way of testing the same admissibility condition involves the sum of the arguments of the complex eigenvalues:

$$\sum_{i=1}^{d} \left|\operatorname{Arg}\lambda_i\right| < \pi \tag{4.35}$$

To determine the complex eigenvalues, the metric needs to be diagonalized using a real basis. This procedure is discussed in Appendix B. In addition to Equation (4.34), this condition will also be explored.

[2]It is important to stress that the field strength tensor F appearing in Equation (4.34) is different than the one appearing in the Lagrangian of Equation (4.3) that generates the charge. The first field strength tensor is a test field given by a p-form, used to evaluate the admissibility condition. It is always real. The second is the physical field strength two-form, which becomes complex because it is a function of m, as Equation (4.8) indicates.

4.3.1 $p = 0$

The admissibility condition for $p = 0$ is $\mathrm{Re}\,\sqrt{g} > 0$. This condition is satisfied because the determinant is given by

$$g = \frac{W^2 \sin^2\theta}{\Xi^2} \tag{4.36}$$

which is positive except at a few specific points like the poles, $\theta = 0, \pi$. These points are also present in the non-supersymmetric form of the metric. For this reason, pathological points like these will not be considered when examining future conditions.

4.3.2 $p = 1$

As reported in Appendix A, the admissibility conditions for $p = 1$ are

$$A, E, G, I > 0 \text{ and } AE - C^2 > 0 \tag{4.37}$$

Based on Table 4.1, the condition $I > 0$ is always satisfied. The conditions $A, E, G > 0$ relate to the τ, $\tilde{\phi}$, and $\tilde{r}$ components of the metric, while $AE - C^2 > 0$ ensures that the determinant of the real part of the metric is positive. These conditions are analyzed individually.

Radial condition. The condition $G > 0$ is equivalent to $\mathrm{Re}\,\Delta_r > 0$, which requires

$$\tilde{r}^4 + (\tilde{r}_+ - \tilde{r})^2(a+1)^2 + 2a(\tilde{r}_+^2 - \tilde{r}^2) - \tilde{r}_+^4 > 0 \tag{4.38}$$

Note that for a BPS black hole, where $a = \tilde{r}_+^2$, this equation becomes

$$(\tilde{r}^2 - \tilde{r}_+^2)^2 + (\tilde{r}_+^2 + 1)^2(\tilde{r}_+ - \tilde{r})^2 > 0 \tag{4.39}$$

which is always non-negative, as expected.

To analyze the more general case of a supersymmetric black hole that is not extremal, it is instructive to obtain the roots of $\mathrm{Re}\,\Delta_r$, which are given by

$$\begin{aligned} \tilde{r}_1^* &= \tilde{r}_+ \\ \tilde{r}_2^* &= -\frac{1}{3}\left(\tilde{r}_+ + \sqrt[3]{J} - \frac{K}{\sqrt[3]{J}}\right) \\ \tilde{r}_{3,4}^* &= -\frac{1}{3}\left(\tilde{r}_+ - \frac{\sqrt[3]{J}(1 \pm \sqrt{3}i)}{2} + \frac{(1 \mp \sqrt{3}i)K}{2\sqrt[3]{J}}\right) \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} J &= L + \sqrt{L^2 + K^3} \\ K &= 2\tilde{r}_+^2 + 3(a^2+1) \\ L &= 2\tilde{r}_+\left(5\tilde{r}_+^2 - 9(a^2 + 3a + 1)\right) \end{aligned} \tag{4.41}$$

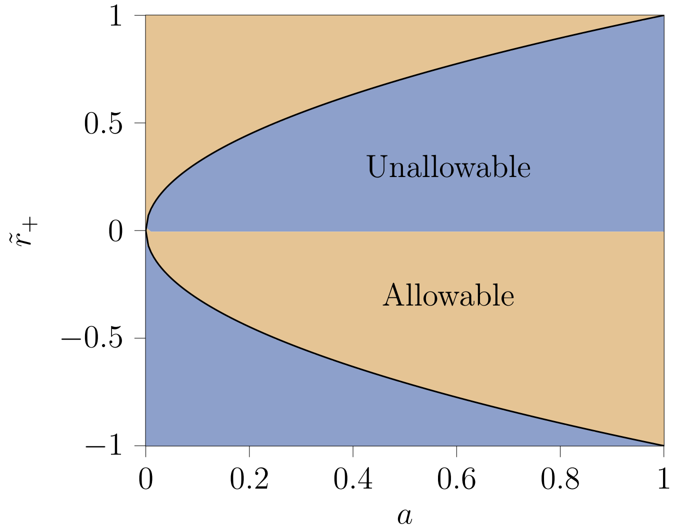


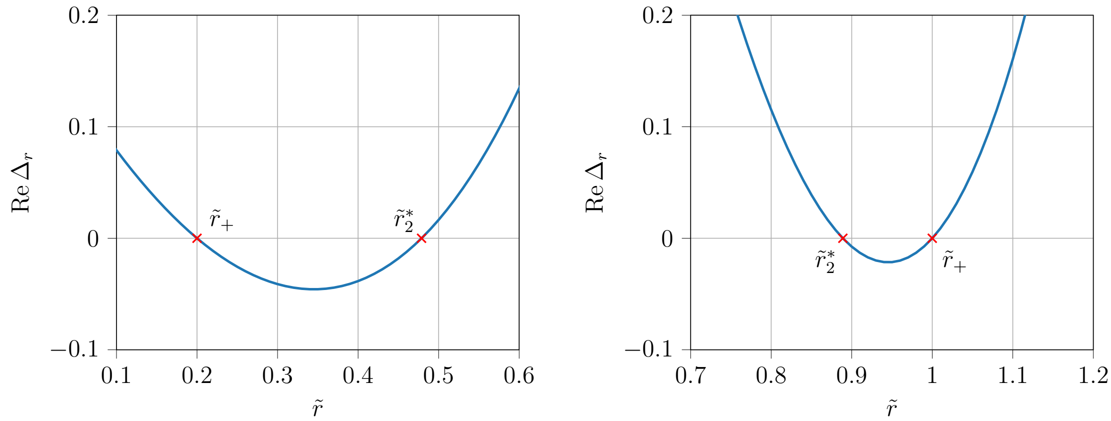
Figure 4.1: The admissibility condition on the radial component of the metric, $G > 0$, restricts the set of allowable parameter values. According to this condition, the metric is allowable if $\tilde{r}_+ > \sqrt{a}$ or $-\sqrt{a} < \tilde{r}_+ < 0$.

Note that $J, K > 0$. Since $\tilde{r}$ is assumed to be real, the complex roots can be ignored. The condition $\operatorname{Re}\Delta_r > 0$ is satisfied for $\tilde{r} > \tilde{r}_+$ if $\tilde{r}_+$ is the larger real root, $\tilde{r}_+ > \tilde{r}_2^*$. This condition is useful because it is a restriction on the parameter space alone, not the coordinates. Using the expression for $\tilde{r}_2^*$ and exploiting the fact that $J > 0$, the condition becomes

$$\sqrt[3]{J}\left(4\tilde{r}_+ + \sqrt[3]{J}\right) - K > 0 \tag{4.42}$$

Because the condition given by Equation (4.42) is difficult to analyze analytically, it was evaluated numerically for different values of $\tilde{r}_+$ and a to determine whether the sign is positive or negative. Figure 4.1 shows the sign of Equation (4.42) for a range of parameters. A BPS black hole corresponding to $a = \tilde{r}_+^2$ is on the boundary between allowable and unallowable metrics. This can also be seen by noting that for a BPS black hole, the two real roots coincide, $\tilde{r}_{1,2}^* = \tilde{r}_+$. In general, the metric is allowable if $\tilde{r}_+ > \sqrt{a}$ or $-\sqrt{a} < \tilde{r}_+ < 0$. One final thing to notice is the reflection symmetry in $\tilde{r}_+$. When $\tilde{r}_+ \to -\tilde{r}_+$, the metric goes from allowable to unallowable and vice versa. While this symmetry is manifested in the radial condition, in more elaborate cases examined later, the symmetry is not present.

To demonstrate how $\operatorname{Re}\Delta_r$ can be negative for some $\tilde{r} > \tilde{r}_+$, Figure 4.2a shows $\operatorname{Re}\Delta_r$ as a function of $\tilde{r}$ when $\tilde{r}_+ = 0.2$ and $a = 0.8$, which corresponds to the unallowable region. For these parameter values, $\tilde{r}_2^* > \tilde{r}_+$ and $\operatorname{Re}\Delta_r$ is negative between the two roots. Figure 4.2b shows $\operatorname{Re}\Delta_r$ in a different region of the parameter space



(a) $\tilde{r}_+ = 0.2$, $a = 0.8$

(b) $\tilde{r}_+ = 1$, $a = 0.8$

Figure 4.2: Different values of the parameters $\tilde{r}_+$ and a are allowable or not depending on whether $\tilde{r}_+$ is the larger real root of $\mathrm{Re}\,\Delta_r$. The boundary between allowable and unallowable parameters is given by $a = \tilde{r}_+^2$.

where the metric is potentially allowable outside the horizon, $\tilde{r}_+ = 1$ and $a = 0.8$. The metric crosses from unallowable to allowable and vice versa when $a = \tilde{r}_+^2$.

Time condition. A similar analysis can be performed for the time component, $A > 0$. From Table 4.1, this condition reads

$$\Lambda^2 \,\mathrm{Re}\,\Delta_r - \Delta_\theta \Gamma^2 \sin^2\theta > 0 \tag{4.43}$$

This can also be obtained by taking the real part of Equation (4.12), which was used to determine if the Killing vector of the Lorentzian metric was timelike outside the horizon. As with the radial condition, $\tilde{r}_+$ is a root of the lefthand side of Equation (4.43) because $\Delta_r(\tilde{r}_+) = 0$ and the second term also vanishes. For a BPS black hole, the condition simplifies to

$$(\tilde{r}^2 - \tilde{r}_+^2)^2 \cos^2\theta + (\tilde{r}_+^2 \cos^2\theta + 1)^2 (\tilde{r}_+ - r)^2 > 0 \tag{4.44}$$

which is satisfied for all real $\tilde{r} \neq \tilde{r}_+$. This is the same condition as Equation (4.29) encountered when demonstrating that a BPS black hole satisfies the admissibility condition everywhere.

There are two differences between this case and the radial one analyzed before. First, there is an angular dependence in this condition, which requires exploring different angular slices when assessing the condition. Second, the leading-order coefficient in $\tilde{r}$ of Equation (4.43) is not always positive. The coefficient is given by

$$D_\tau = \tilde{r}_+^4 + 2\tilde{r}_+^2 a^2 \cos^2\theta + a^4 \cos^2\theta + a^2 \cos^2\theta - a^2 \tag{4.45}$$

When checking whether the condition is satisfied, it must be checked that $\tilde{r}_+$ is greater than all real roots, and that D_τ is positive.

Figure 4.3 shows the allowable and unallowable regions in the parameter space for different values of θ. When $\theta = 0$, the boundaries are the same as Figure 4.1. As θ increases, for example when $\theta = \pi/4$, the unallowable region enlarges, reflecting the fact that D_τ becomes negative. The moving boundary between unallowable and allowable metrics is given by one root of $D_\tau(\tilde{r}_+) = 0$. When $\theta = \pi/2$, the metric is unallowable for all $\tilde{r}_+ < 0$. For larger values of θ, the allowable region expands again until at $\theta = \pi$, it is the same as for $\theta = 0$. The conclusion is that the metric is allowable across all values of θ only if $\tilde{r}_+ > \sqrt{a}$. This excludes negative values of $\tilde{r}_+$.

Angular condition. The angular condition $E > 0$ reads

$$\Delta_\theta(\tilde{r}^2 + a^2)^2 - a^2 \sin^2\theta \,\mathrm{Re}\,\Delta_r > 0 \tag{4.46}$$

Figure 4.4 shows this condition for different values of θ. Similar to the time component, the shifting boundary between allowable and unallowable metrics as θ varies is given by one of the roots of the polynomial

$$D_\phi = \frac{1}{27}\left(\frac{\tilde{J}^2}{12} + \frac{\tilde{K}a^2}{1-a^2}\right)^3 - \frac{1}{4}\left(\frac{\tilde{J}^3}{108} - \frac{\tilde{J}\tilde{K}a^2}{3(1-a^2)} + \frac{\tilde{L}^2}{2(a-1)^2}\right)^2 \tag{4.47}$$

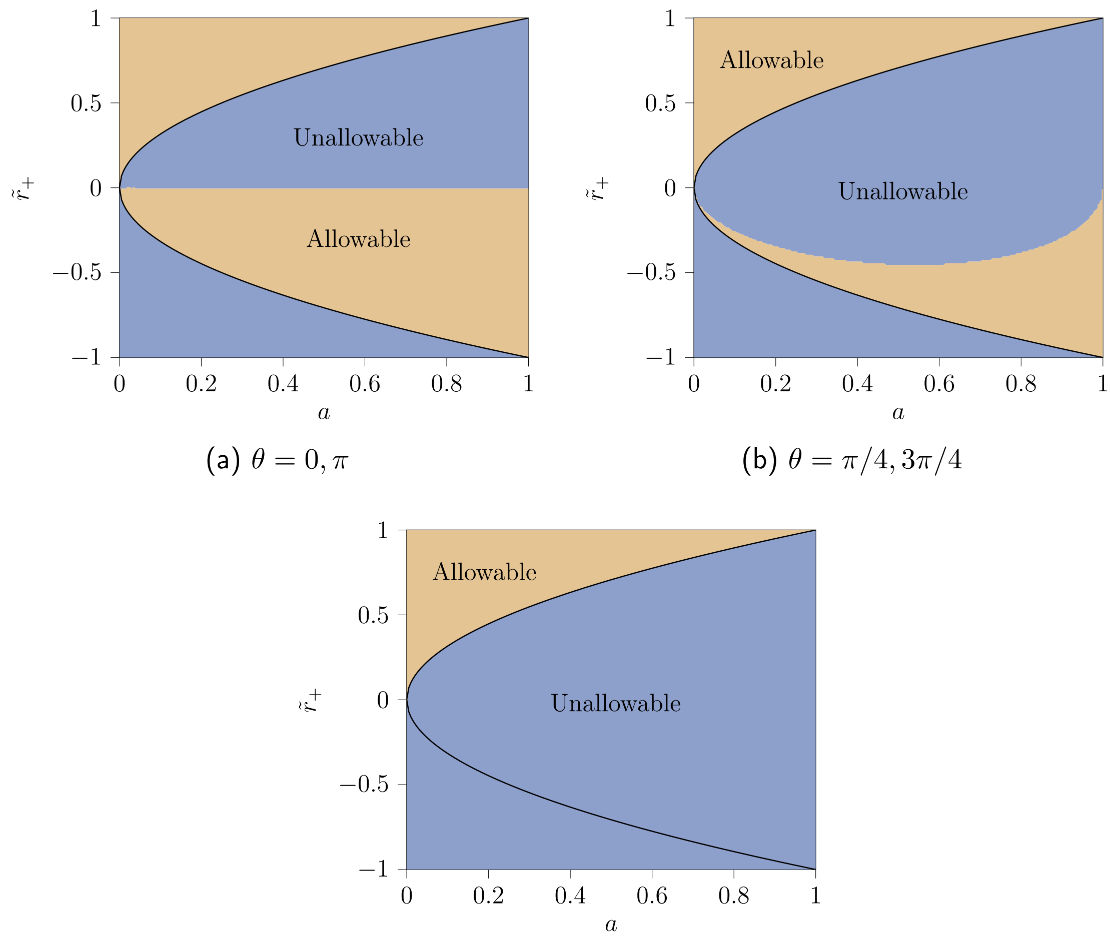
where

$$\begin{aligned} \tilde{J} &= a^2(2 - \sin^2\theta) \\ \tilde{K} &= \tilde{r}_+^4 \sin^2\theta - \tilde{r}_+^2(a^2 + 4a + 1)\sin^2\theta - a^4\cos^2\theta + a^2 \\ \tilde{L} &= \tilde{r}_+ a^2(a+1)\sin^2\theta \end{aligned} \tag{4.48}$$

Although the region of allowable metrics appears larger than that given by $\tilde{r}_+ > \sqrt{a}$, the plots of Figure 4.4 are consistent with the metric being allowable there.

Determinant condition. This condition requires $AE - C^2 > 0$. The lefthand side is an eighth-order polynomial in $\tilde{r}$ with at least one root at $\tilde{r}_+$ because $A, C = 0$ at $\tilde{r} = \tilde{r}_+$. Figure 4.5 shows the allowable regions in the parameter space for this condition. Note that the condition reduces to $\tilde{r}_+ > \sqrt{a}$ for θ as low as $\pi/4$.

Conclusion. The admissibility condition for $p = 1$ involves combining all four of the previous conditions together. When this is done, the same allowable regions as Figure 4.5 are traced out.

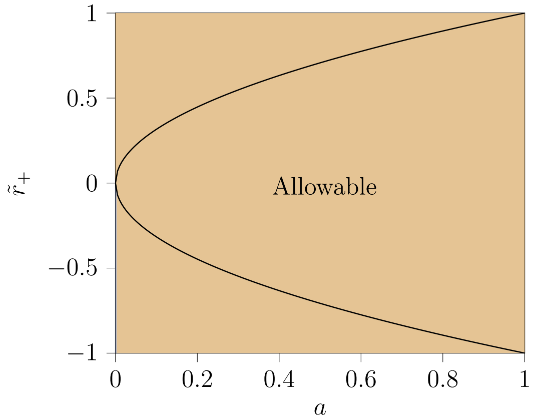


(a) $\theta = 0, \pi$

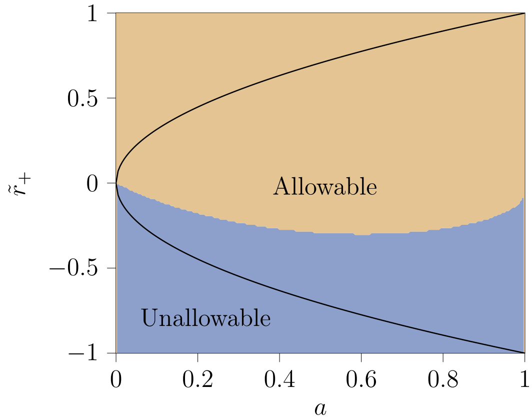
(b) $\theta = \pi/4, 3\pi/4$

(c) $\theta = \pi/2$

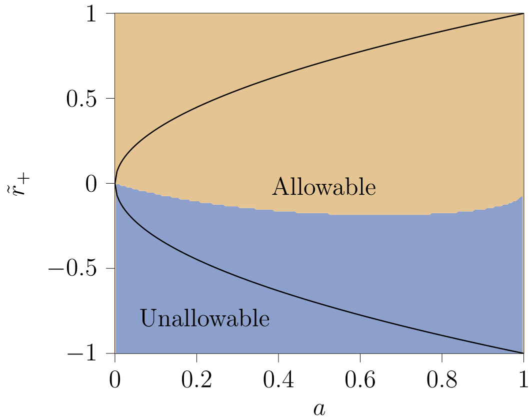
Figure 4.3: The admissibility condition on the time component leads to different allowable regions in the parameter space depending on θ. The only allowable region across all values of θ is $\tilde{r}_+ > \sqrt{a}$.



(a) $\theta = 0, \pi$

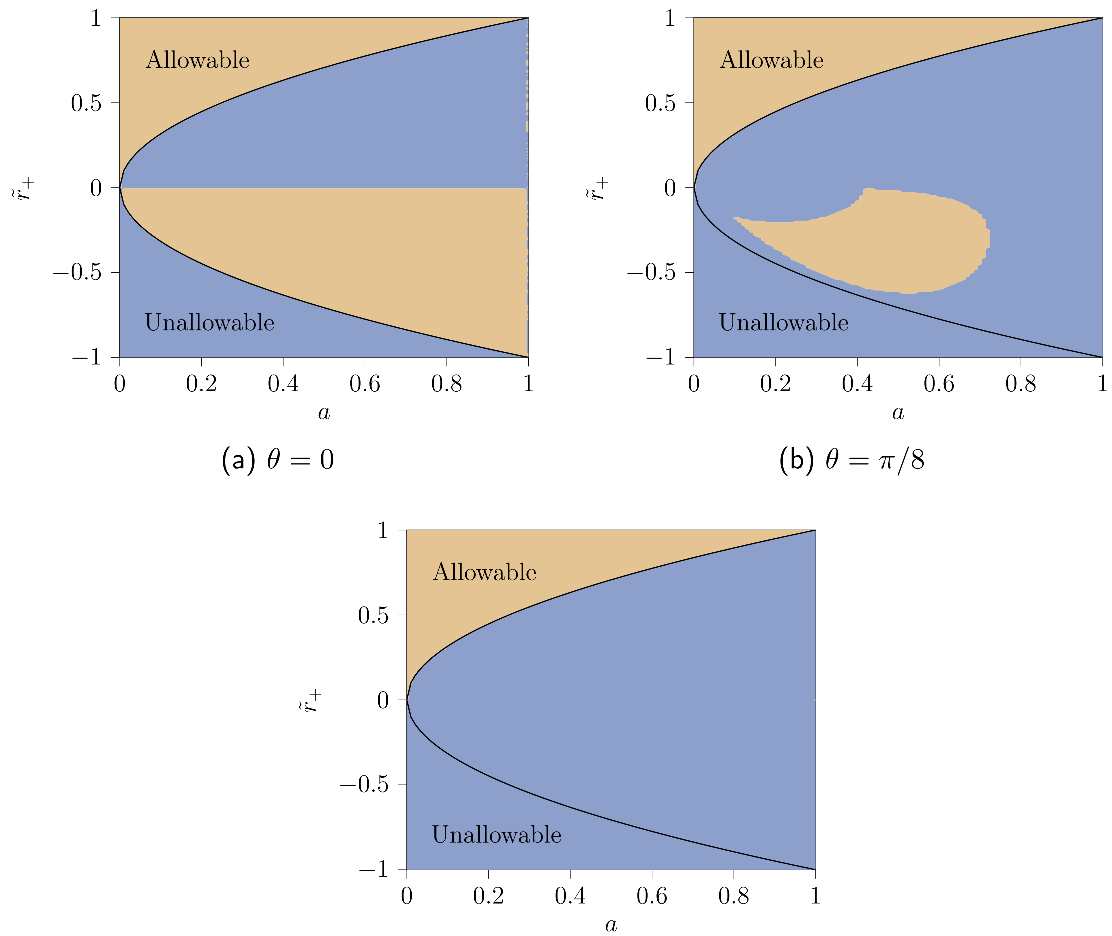


(b) $\theta = \pi/4, 3\pi/4$



(c) $\theta = \pi/2$

Figure 4.4: The admissibility condition on the angular component leads to different allowable regions as θ is swept out. Although the allowable region is larger, the region $\tilde{r}_+ > \sqrt{a}$ is allowable according to this condition.



(a) $\theta = 0$

(b) $\theta = \pi/8$

(c) $\theta = \pi/4$

Figure 4.5: The admissibility condition based on requiring the determinant of the real part of the metric to be positive reinforces the conclusion that $\tilde{r}_+ > \sqrt{a}$ is the only allowable region across all of spacetime.

4.3.3 $p = 2$

As discussed in Appendix A, the admissibility conditions for $p = 2$ involve all the conditions for $p = 1$ and the following additional four conditions:

$$\begin{aligned} &AE - C^2 > BF - D^2, \quad AG > BH, \quad EG > FH \\ &(AG - BH)(EG - FH) > (CG - DH)^2 \end{aligned} \tag{4.49}$$

Without entering into all the details, evaluating these conditions and combining them with the $p = 1$ conditions leads to the same allowable regions as Figure 4.5.

4.3.4 $p = 3$

For $p = 3$, the admissibility conditions require all the additional conditions for $p = 2$ to be satisfied and impose the additional constraint:

$$G\Big(AE - C^2 - (BF - D^2)\Big) > H(AF - 2CD + BE) \tag{4.50}$$

This constraint simplifies to $W\Delta_\theta > 0$, which is satisfied except at a few special points. The allowable regions are the same as the $p = 2$ case.

4.3.5 $p = 4$

The admissibility condition is given by $\mathrm{Re}\, g > 0$, which is satisfied.

4.3.6 Complex Eigenvalues

The alternative admissibility condition of Equation (4.35) involves computing the complex eigenvalues of the metric and ensuring that the sum of their angles sweeps out less than a half-plane. The angle corresponding to the θ component is zero because the θ component does not get complexified. As shown in Appendix B, the remaining three eigenvalues are given by

$$\begin{aligned} \lambda_\pm &= 1 + i\alpha_\pm \\ \lambda_r &= 1 + i\frac{H}{G} \end{aligned} \tag{4.51}$$

where $\alpha_\pm$ are the solutions of the quadratic equation

$$(AE - C^2)\alpha_\pm^2 + (AF - 2CD + BE)\alpha_\pm + BF - D^2 = 0 \tag{4.52}$$

With these eigenvalues, Equation (4.35) becomes

$$\sum_{i=\pm} |\texttt{arctan2}(\operatorname{Im}\lambda_i, \operatorname{Re}\lambda_i)| + |\texttt{arctan2}(H/G, 1)| < \pi \tag{4.53}$$

The function $\texttt{arctan2}(y, x)$ returns the argument of the complex number $x + iy$ between $(-\pi, \pi]$, where the angle is positive if $y > 0$ and negative if $y < 0$. This single equation captures all the information present in the various conditions for different p values analyzed above. The only additional requirement, ensuring the metric can be diagonalized with a real basis, is that the real part of the metric is positive definite, which is the same as the $p = 1$ case.

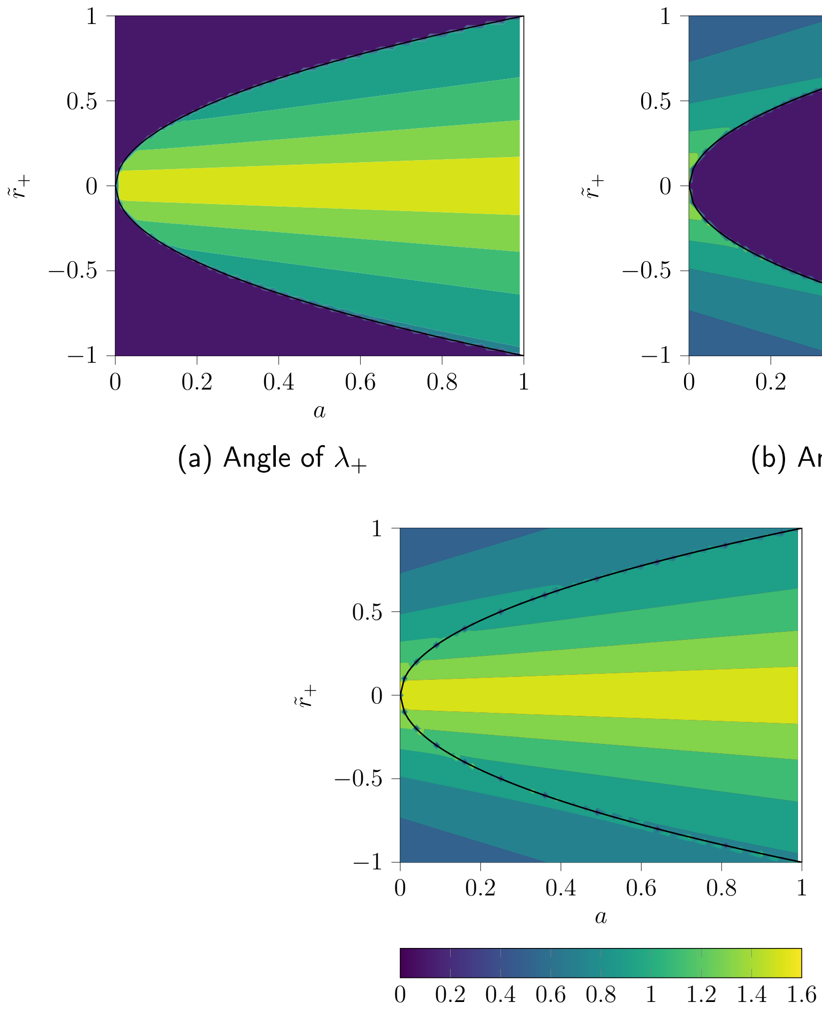
To analyze Equation (4.53), it is helpful to determine its roots, as was done for the radial condition of $p = 1$. Because it is a complicated non-linear equation, it is not easy to find the roots even numerically for given values of the parameters $\tilde{r}_+$ and a. To determine whether a given set of parameters corresponds to an allowable metric, different values of $\tilde{r} > \tilde{r}_+$ were swept over and the admissibility condition was evaluated. If the condition was violated for any sampled value of $\tilde{r}$, the metric was deemed unallowable. Since this method relies on sampling, it can sometimes be inaccurate, potentially resulting in an unallowable metric being deemed allowable.

To give a feel for the solution, Figure 4.6 displays the angles of the three eigenvalues across the parameter space for a slice of spacetime where $\theta = \pi/4$ and $\tilde{r}$ is 0.1% above the outer horizon. The angle of λ_+ is zero for $\tilde{r}_+^2 > a$, while the angle of λ_- is zero for $\tilde{r}_+^2 < a$. Similar to λ_+, the angle of λ_r increases moving toward $\tilde{r}_+ = 0$. From this single slice, the most promising candidate for a region where the angles sum to less than π is the region $\tilde{r}_+^2 > a$. Taking into account other values of $\tilde{r}$ farther from the horizon, the same allowable regions as Figure 4.5 were produced. Again, the conclusion is that for the metric to be allowable, $\tilde{r}_+$ must be greater than $\sqrt{a}$.

4.4 Discussion

In the last section, a careful treatment of the admissibility condition revealed that a charged, rotating supersymmetric black hole is allowable if the radius of the event horizon is greater than $\sqrt{a}$. In this section, the physical interpretation of this conclusion will be discussed.

The admissibility condition for supersymmetric black holes derived in this chapter is very similar to the regularity condition for non-supersymmetric black holes. Without supersymmetry, the regularity of the horizon is based on the roots of Δ_r. If there are two roots, the black hole has an inner and outer horizon. When the two roots coincide, the black hole has a single horizon. This corresponds to an extremal black hole. All other solutions do not correspond to a black hole. With supersymmetry,



(a) Angle of λ_+

(b) Angle of λ_-

(c) Angle of λ_r

Figure 4.6: The complex eigenvalues of the metric have phases that vary across spacetime. The admissibility condition says that the sum of the phases cannot sweep out more than a half-plane. In this figure, $\theta = \pi/4$ and $\tilde{r}$ is 0.1% above the outer horizon $\tilde{r}_+$.

the condition $\tilde{r}_+ > \sqrt{a}$ leads to a black hole with two horizons. When $\tilde{r}_+ = \sqrt{a}$, the black hole is extremal, or BPS. This suggests that the admissibility condition is like a regularity condition on the form of the complex supersymmetric solution.

While the admissibility condition is more complicated than an analysis of the roots of Δ_r, there are some similarities that can be drawn. The admissibility condition encompasses a set of different conditions for different values of p. One of them required $\operatorname{Re}\Delta_r > 0$. It was shown that this function has two real roots. Like the non-supersymmetric case, one of the roots is $\tilde{r}_+$, which coincides with the other root in the extremal limit. Unlike the non-supersymmetric case, $\tilde{r}_+$ is not always the larger root. Assessing this condition involved checking the relative magnitude of the roots. The admissibility condition was satisfied if $\tilde{r}_+$ is the larger real root, which agrees with the non-supersymmetric case.

Another similarity that can be drawn involves the condition that the Killing vector is timelike outside the horizon. Without supersymmetry, this condition on the Lorentzian metric was shown to be the same as the admissibility condition for $p = 1$. With supersymmetry, one of the admissibility conditions involving the time component of the metric corresponded to taking the real part of the Killing vector condition. This condition led to the allowable regions in Figure 4.3, which is consistent with the condition $\tilde{r}_+ > \sqrt{a}$. In the BPS limit, the condition is always satisfied. In the non-supersymmetric limit where m becomes real, the condition is the same as the timelike Killing vector condition. This demonstrates consistency between the timelike Killing vector condition and the admissibility condition.

The parameter a is related to the angular momentum of the black hole. As the angular momentum increases, the condition $\tilde{r}_+ > \sqrt{a}$ is satisfied for a smaller range of $\tilde{r}_+$. As the angular momentum becomes arbitrarily large, the black hole is not allowable. This is consistent with the discussion from Chapter 3, which concluded that for the thermodynamic ensemble to be well-defined, the black holes that make it up cannot rotate too quickly.

This discussion suggests that the admissibility condition is a generalization of the regularity condition for the existence of the horizon applied to complex metrics.

Chapter 5

Conclusion

In this thesis, a recently proposed condition was analyzed that can distinguish between acceptable and unacceptable complex metrics to include in the gravitational path integral. Different ways that complex metrics can emerge were discussed, including Wick rotation and supersymmetry. Some complex metrics may dominate the integral, but not every metric corresponds to a physically relevant situation. The need to distinguish between these two kinds of metrics is important for quantizing gravity using the gravitational path integral.

The first contribution of this thesis involved a thorough exploration of this admissibility condition, motivating it physically and deriving the major results. Following prior work, examples of spacetimes were presented where complex metrics both address and introduce physical problems. These examples included the Hartle-Hawking wavefunction of the universe, topology-changing processes, and time folds. In each of these cases, it was shown that the admissibility condition can help discriminate between physical and unphysical metrics. While this part of the thesis was not original, prior findings were revisited and brought together in a comprehensive way.

The second contribution of this thesis was the novel application of the admissibility condition to supersymmetric black holes with charge and spin in four-dimensional anti-de Sitter space. The analysis focused on black holes that are supersymmetric but not extremal. Extremal black holes have zero temperature, or infinite inverse temperature, and computing their action is difficult. By imposing supersymmetry without extremality, the metric becomes complex and has no well-defined Lorentzian counterpart. The action for these black holes can be computed and the thermodynamics can be verified. As the temperature approaches zero, results for extremal black holes can be computed. This thesis explored admissibility of these complex, supersymmetric but non-extremal metrics.

Depending on the number of spacetime dimensions, the admissibility condition involves several different conditions, which must be satisfied for every spacetime point.

This thesis systematically explored each condition. Some of the conditions were analyzed using analytical methods, while others had to be explored numerically because they were analytically intractable. The admissibility condition in terms of the complex eigenvalues of the metric was also explored, which is a single condition that encompasses the various conditions described above. Combining all the results, it was shown that the metric is allowable if the radius of the event horizon is greater than a critical value, which corresponds to the BPS limit of the black hole. Because the inner and outer horizons of the black hole coincide at the BPS limit, this condition requires two distinct horizons. In this way, it is a sort of regularity condition on the supersymmetric black hole solution.

There are several ways the research presented in this thesis can be continued in the future. Three of these ways are described below.

Black Holes in Five-dimensional Anti-de Sitter Space

This thesis analyzed AdS black holes in four dimensions, but a similar analysis can be repeated for black holes in other dimensions, for instance five dimensions [8, 67, 68]. The analysis becomes more cumbersome because the metric components are more involved, which may necessitate various approximations and simplifications. One possible simplification is to equate the two angular momenta carried by the black hole in [8]. Black holes in five-dimensional anti-de Sitter space are particularly important because they appear in the original formulation of the AdS/CFT correspondence, which relates them to a conformal field theory in four dimensions. It is expected that similar conclusions to the ones presented in this thesis will also hold in this case.

Uplift to Eleven Dimensions

To connect the results presented here for four-dimensional AdS black holes to the AdS/CFT correspondence, they need to be embedded in 11-dimensional supergravity [65]. In this embedding, the four dimensions constitute the external space, while the remaining seven make up the internal space. The metric $\mathcal{G}(Y_{11})$ in 11 dimensions can be written [69]

$$\mathcal{G}(Y_{11}) = \mathcal{G}(Y_4) + 4\left[\left(\mathrm{d}\psi + \sigma + \frac{1}{2}A\right)^2 + \mathcal{G}(\mathbb{CP}^3)\right] \tag{5.1}$$

The term $\mathcal{G}(Y_4)$ is the four-dimensional metric. The second term is the local form of the metric on a seven-dimensional Sasaki–Einstein space. Within this space, $\mathcal{G}(\mathbb{CP}^3)$ represents the metric on a complex projective space in three complex dimensions. If J is the Kahler form on the Kahler–Einstein base $\mathbb{CP}^3$, then σ is a one-form such that

$d\sigma = 2J$. The coordinate ψ parameterizes the S^1 fiber over $\mathbb{CP}^3$ which produces the Sasaki-Einstein space, while A is the potential of the gauge field.

Similar to the metric in four dimensions, admissibility of the metric in 11 dimensions can be analyzed. Because it is a fixed Kahler-Einstein metric, the metric on $\mathbb{CP}^3$ has real components and can be ignored in the analysis. The one-form σ is also not complexified. Because $\mathbb{CP}^3$ is six-dimensional, the remaining space is five-dimensional, given by $Y_4 \times S^1$. Because the gauge potential A is a function of the parameter m, it becomes complex, which produces an additional degree of complexification in the metric. Admissibility of the five-dimensional complex metric can be analyzed, which involves six separate conditions. Because Equation (5.1) introduces one new vielbein $d\psi$, which has a single component equal to one, the determinant is the same as the four-dimensional case. This means the $p = 0$ condition is satisfied, but the other five conditions need to be studied in more detail.

Admissibility of Orbifolds

When embedding the four-dimensional black holes analyzed here in 11-dimensional supergravity, the resulting theory does not asymptotically approach the same space as the direct product $\mathrm{AdS}_4 \times S^7$ [65]. The metric can be made to asymptote locally to this space, provided the following identification holds:

$$(\tau, \phi, \psi) \sim \left(\tau + \frac{\beta}{a}, \phi - \frac{2\pi b}{a}, \psi - \frac{2\pi c}{a}\right) \sim (\tau, \phi + 2\pi, \psi) \sim (\tau, \phi, \psi + 2\pi) \qquad (5.2)$$

where a, b, and c are integers. The resulting space is called an orbifold, which is a space locally based on quotients of Euclidean space by finite groups [70]. This orbifold has the same local form as the 11-dimensional solution, but a different global form. Orbifolds correspond to different saddles of the gravitational path integral, which have been matched to saddles in the superconformal index of the dual quantum field theory [65, 71].

Because of their importance, it is interesting to analyze admissibility for these orbifold geometries. Since the admissibility condition is local, the admissibility results are expected to be the same. This conclusion is reinforced by the fact that the orbifold involves identifications over τ and ϕ, which do not enter into the metric. While it appears the same admissibility conditions may hold for orbifolds, global conditions may alter this conclusion, warranting a more thorough analysis of admissibility.

Appendix A

Admissibility of 4D AdS Supersymmetric Black Holes

In this appendix, conditions are developed that a black hole in four-dimensional anti-de Sitter space must satisfy for it to be allowable. Recall that a metric is allowable if the action has positive real part for every non-zero, real p-form F:

$$\mathrm{Re}\left(\sqrt{g}\, g^{i_1 j_1} \cdots g^{i_p j_p} F_{i_1 \cdots i_p} F_{j_1 \cdots j_p}\right) > 0, \quad 0 \leq p \leq d \tag{A.1}$$

where g is the determinant of the metric. This condition must be satisfied at every spacetime point. The inverse metric for a charged, rotating supersymmetric black hole in 4D AdS space takes the general form

$$g^{\mu\nu} = \begin{pmatrix} A+iB & C+iD & 0 & 0 \\ C+iD & E+iF & 0 & 0 \\ 0 & 0 & G+iH & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{A.2}$$

where the indices μ, ν run over $\{t, \phi, r, \theta\}$. The coefficients $A, B, \ldots$ are real functions of the coordinates. For their explicit expressions, see Table 4.1 in the main text. Using this metric, the condition of Equation (A.1) will be evaluated for $0 \leq p \leq 4$, arriving at conditions on the metric coefficients.

A.1 $p = 0$

The admissibility condition for $p = 0$ is $\mathrm{Re}\,\sqrt{g} > 0$. For a black hole in 4D AdS, g is a positive real number, satisfying this condition. In the remainder of this appendix, the determinant is assumed to be a positive real.

A.2 $p = 1$

The admissibility condition for $p = 1$ is given by

$$\mathrm{Re}\left(\sqrt{g}\, g^{i_1 j_1} F_{i_1} F_{j_1}\right) > 0 \tag{A.3}$$

Because the determinant is a positive real, this condition is the same as requiring the real part of the inverse metric to be positive definite, which happens if and only if the eigenvalues are positive. Two of the eigenvalues are G and I, which must be positive. The other eigenvalues are positive if the trace and determinant of

$$\begin{pmatrix} A & C \\ C & E \end{pmatrix} \tag{A.4}$$

are positive. This is true if $A + E > 0$ and $AE - C^2 > 0$. Because the product AE and sum $A + E$ must both be positive, it follows $A, E > 0$. In summary, the conditions for $p = 1$ are

$$A, E, G, I > 0 \text{ and } AE - C^2 > 0 \tag{A.5}$$

These are conditions on the coefficients of the inverse metric, $g^{\mu\nu}$. It is interesting to determine if these conditions also apply to the metric two-form, $g_{\mu\nu}$. This is because the black hole metric is often expressed more conveniently with lower indices, and computing its inverse $g^{\mu\nu}$ from there casts it into a more complicated form. The inverse of the metric of Equation (A.2) is given by

$$\mathrm{d}s^2 = \frac{1}{\kappa}\Big[(A'+iB')\,\mathrm{d}t^2 + 2(C'+iD')\,\mathrm{d}t\mathrm{d}\phi + (E'+iF')\,\mathrm{d}\phi^2\Big] + \frac{G - iH}{\delta}\,\mathrm{d}r^2 + \frac{1}{I}\,\mathrm{d}\theta^2 \tag{A.6}$$

where

$$\begin{aligned}
A' &= A(E^2 + F^2) + E(D^2 - C^2) - 2CDF \\
B' &= F(D^2 - C^2) - B(E^2 + F^2) + 2CDE \\
C' &= B(CF - DE) - A(CE + DF) + C(C^2 + D^2) \\
D' &= A(CF - DE) + B(CE + DF) - D(C^2 + D^2) \\
E' &= E(A^2 + B^2) + A(D^2 - C^2) - 2BCD \\
F' &= B(D^2 - C^2) - F(A^2 + B^2) + 2ACD
\end{aligned} \tag{A.7}$$

The variables κ and δ are positive. To show that the admissibility conditions on $g_{\mu\nu}$ are the same as those on $g^{\mu\nu}$, it is necessary to show that imposing the conditions on the inverse metric fulfills the same conditions on the metric, and vice versa.

For the first half of this relation, it is necessary to show that Equation (A.5) leads to the same conditions for the metric:

$$A', E', G, I > 0 \text{ and } A'E' - C'^2 > 0 \tag{A.8}$$

The conditions $G, I > 0$ are already satisfied. To show $A' > 0$, first write

$$\begin{aligned} A' &= A(E^2 + F^2) + E(D^2 - C^2) - 2CDF \\ &> A(E^2 + F^2) + E(D^2 - AE) - 2CDF \\ &= AF^2 + ED^2 - 2CDF \end{aligned} \tag{A.9}$$

since $AE > C^2$ and $E > 0$. When $CDF < 0$, it follows that $A' > 0$ since AF^2 and ED^2 are also greater than zero. When $CDF > 0$, the cases $C > 0$ and $C < 0$ need to be considered. When $C > 0$, $DF > 0$ and one can write

$$\begin{aligned} A' &> AF^2 + ED^2 - 2CDF \\ &> AF^2 + ED^2 - 2\sqrt{AE}DF \\ &= (\sqrt{A}F - \sqrt{E}D)^2 \end{aligned} \tag{A.10}$$

When $C < 0$, $DF < 0$ and one gets

$$\begin{aligned} A' &> AF^2 + ED^2 - 2CDF \\ &> AF^2 + ED^2 + 2\sqrt{AE}DF \\ &= (\sqrt{A}F + \sqrt{E}D)^2 \end{aligned} \tag{A.11}$$

In either case, $A' > 0$. In the same way, it follows that $E' > 0$. Finally, the condition $A'E' - C'^2 > 0$ is satisfied since it may be written

$$A'E' - C'^2 = \left(AE - C^2\right)\left(\beta^2 + \gamma^2\right) > 0 \tag{A.12}$$

where $\beta \equiv AE - C^2 - (BF - D^2)$ and $\gamma \equiv AF - 2CD + BE$. This condition is satisfied when $AE - C^2 > 0$.

For the second half of the relation, it is necessary to show that the conditions on the metric also imply the conditions on the inverse metric. Because the metric and inverse metric are inverses of each other, the argument proceeds in exactly the same way. This demonstrates that the conditions on $g^{\mu\nu}$ and $g_{\mu\nu}$ are equivalent, which holds regardless of whether the determinant is real or complex.

A.3 $p = 2$

The admissibility condition for $p = 2$ is

$$\mathrm{Re}\left(\sqrt{g}\, g^{i_1 j_1} g^{i_2 j_2} F_{i_1 i_2} F_{j_1 j_2}\right) > 0 \tag{A.13}$$

In this case, the field strength tensor is

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} \\ -F_{01} & 0 & F_{12} & F_{13} \\ -F_{02} & -F_{12} & 0 & F_{23} \\ -F_{03} & -F_{13} & -F_{23} & 0 \end{pmatrix} \tag{A.14}$$

Again assuming the determinant g is positive, Equation (A.13) becomes

$$\begin{pmatrix} F_{01} & F_{23} \end{pmatrix} M_{01}^{23} \begin{pmatrix} F_{01} \\ F_{23} \end{pmatrix} + \begin{pmatrix} F_{02} & F_{12} \end{pmatrix} M_{02}^{12} \begin{pmatrix} F_{02} \\ F_{12} \end{pmatrix} + \begin{pmatrix} F_{03} & F_{13} \end{pmatrix} M_{03}^{13} \begin{pmatrix} F_{03} \\ F_{13} \end{pmatrix} > 0 \tag{A.15}$$

where

$$\begin{aligned} M_{01}^{23} &= \begin{pmatrix} \beta & 0 \\ 0 & GI \end{pmatrix} \\ M_{02}^{12} &= \begin{pmatrix} AG - BH & CG - DH \\ CG - DH & EG - FH \end{pmatrix} \\ M_{03}^{13} &= I \begin{pmatrix} A & C \\ C & E \end{pmatrix} \end{aligned} \tag{A.16}$$

are symmetric matrices that mix various real and imaginary components of the metric. For Equation (A.15) to be satisfied, the matrices need to be positive definite, resulting in the following conditions:

$$\begin{aligned} &AE - C^2 > BF - D^2, \quad GI > 0 \\ &AG > BH, \quad EG > FH, \quad (AG - BH)(EG - FH) > (CG - DH)^2 \\ &AI > 0, \quad EI > 0, \quad AE > C^2 \end{aligned} \tag{A.17}$$

In the case of a charged, rotating black hole in four-dimensional anti-de Sitter space, the coefficient I is positive, as Table 4.1 indicates. Then the last three conditions of Equation (A.17), as well as $GI > 0$, are the same as the conditions for $p = 1$. The other four conditions are unique to $p = 2$, and are summarized below:

$$\begin{aligned} &AE - C^2 > BF - D^2, \quad AG > BH, \quad EG > FH \\ &(AG - BH)(EG - FH) > (CG - DH)^2 \end{aligned} \tag{A.18}$$

When the metric is real, $B = D = F = H = 0$ and the conditions for $p = 2$ are the same as those for $p = 1$. When the metric is complex, this is not in general the case and the additional conditions of Equation (A.18) need to be considered.

Like the $p = 1$ case, it needs to be seen whether using $g_{\mu\nu}$ instead of $g^{\mu\nu}$ leads to the same conditions. In this case, the computations become analytically intractable and numerical methods need to be relied on. A large sample of random matrices satisfying Equation (A.17) were generated. It was found that all of these matrices had inverses which also satisfied the same conditions. This demonstrates that the conditions on the metric and the inverse metric are equivalent.

A.4 $p = 3$

The admissibility condition for $p = 3$ reads

$$\mathrm{Re}\left(\sqrt{g}\, g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} F_{i_1 i_2 i_3} F_{j_1 j_2 j_3}\right) > 0 \tag{A.19}$$

The antisymmetric tensor F has four independent components: F_{012}, F_{013}, F_{023}, and F_{123}. The admissibility condition in terms of these and the metric coefficients is

$$(G\beta - H\gamma)F_{012}^2 + I\left[\beta F_{013}^2 + \begin{pmatrix} F_{023} & F_{123} \end{pmatrix} M_{02}^{12} \begin{pmatrix} F_{023} \\ F_{123} \end{pmatrix}\right] > 0 \tag{A.20}$$

This condition requires all the extra conditions for $p = 2$ given by Equation (A.18) to be satisfied. It also imposes one additional condition given by $G\beta > H\gamma$. Expressed explicitly in terms of the metric coefficients, it is given by

$$G\Big(AE - C^2 - (BF - D^2)\Big) > H(AF - 2CD + BE) \tag{A.21}$$

When the metric is real, the conditions for $p = 1$ are recovered. The reason for this can be seen by considering the form of the action

$$S = \int F \wedge *F \tag{A.22}$$

where F is a real three-form. When the metric is real, F can be expressed as the Hodge star of a real one-form $\tilde{F}$. The action becomes

$$S = -\int *\tilde{F} \wedge \tilde{F} = \int \tilde{F} \wedge *\tilde{F} \tag{A.23}$$

where, in this case, $**\tilde{F} = -\tilde{F}$ and $*\tilde{F} \wedge \tilde{F} = -\tilde{F} \wedge *\tilde{F}$ were used. The same admissibility conditions as $p = 1$ are recovered. If instead the metric is complex, F can no longer be expressed as the Hodge dual of a real one-form. Because the Hodge star operator requires contraction with the metric, applying it to a real form makes it complex, violating the requirement that F be real.

Table A.1: Admissibility requires a set of conditions involving the coefficients of the metric to be satisfied. This table summarizes the various conditions for a rotating supersymmetric black hole in four-dimensional anti-de Sitter space.

p	**Conditions**
0	$\mathrm{Re}\sqrt{g} > 0$
1	Equation (A.5)
2	$p = 1$ conditions and Equation (A.18)
3	$p = 2$ extra conditions and Equation (A.21)
4	$\mathrm{Re}\, g > 0$

A.5 $p = 4$

The admissibility condition for $p = 4$ is

$$\mathrm{Re}\left(\sqrt{g}\, g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} g^{i_4 j_4} F_{i_1 i_2 i_3 i_4} F_{j_1 j_2 j_3 j_4}\right) > 0 \tag{A.24}$$

In four dimensions, F has one independent component, which gets pulled out of the condition since it is always squared. After simplifying, the condition becomes

$$I\,\mathrm{Re}\Big((A+iB)(E+iF)(G+iH) - (C+iD)^2(G+iH)\Big) > 0 \tag{A.25}$$

This is the same as

$$\mathrm{Re}\, g > 0 \tag{A.26}$$

which is similar to the $p = 0$ condition, $\mathrm{Re}\sqrt{g} > 0$. Because g is a positive real, both conditions are satisfied.

A.6 Summary

Table A.1 summarizes the admissibility conditions for the different values of p. Notice that the conditions for $p = 2$ are the same as for $p = 1$, plus some additional conditions. A similar thing holds for $p = 3$, which relies on the extra conditions for $p = 2$. Through analytical and numerical methods, the same conditions were shown to hold for both the metric and its inverse.

Appendix B

Admissibility Analysis using Complex Eigenvalues

In this appendix, the admissibility condition involving the complex eigenvalues of the metric is developed for the form of a black hole in four-dimensional anti-de Sitter space. First, it is shown how the metric can be diagonalized by a real basis with complex eigenvalues. This diagonalization procedure is used to generate the eigenvalues of the black hole metric. Finally, the eigenvalues are used to obtain a final expression for the admissibility condition in terms of the metric coefficients.

B.1 Diagonalization using a Real Basis

A complex metric can be diagonalized by a complex basis and real eigenvalues. Another way to diagonalize it is with a real basis and complex eigenvalues [1]. How to perform this second procedure will be demonstrated.

Consider a complex metric of the form

$$W = A + iB \tag{B.1}$$

where the real part A is positive definite. Both A and B are real symmetric matrices. The metric can be diagonalized using a real basis, producing complex eigenvalues. First, A is diagonalized:

$$A = QMQ^T \tag{B.2}$$

Because A is positive definite, the eigenvalues are positive reals, and the matrix M can be factored using another real matrix, $M \equiv D^2$. It follows that A can be transformed into the identity $\mathbb{I}$ by

$$(QD^{-1})^T A(QD^{-1}) = \mathbb{I} \tag{B.3}$$

Introducing $X \equiv QD^{-1}$, Equation (B.1) can be written

$$X^T W X = \mathbb{I} + i X^T B X \tag{B.4}$$

This amounts to changing the basis of the imaginary part of the metric. With the real part diagonalized, the imaginary part can now be diagonalized, producing

$$X^T B X = P N P^T \tag{B.5}$$

Finally, the metric becomes

$$(QD^{-1}P)^T W (QD^{-1}P) = \mathbb{I} + iN \equiv \Lambda \tag{B.6}$$

The matrix $QD^{-1}P$ represents the real basis, and the eigenvalues are $\lambda_i = 1 + iN_{ii}$, where N_{ii} are the eigenvalues of $(QD^{-1})^T B (QD^{-1})$.

B.2 Diagonalization of a Supersymmetric Black Hole Metric

The metric for a charged, rotating supersymmetric black hole in four-dimensional anti-de Sitter space takes the general form

$$\mathrm{d}s^2 = (A + iB)\,\mathrm{d}t^2 + 2(C + iD)\,\mathrm{d}t\mathrm{d}\phi + (E + iF)\,\mathrm{d}\phi^2 + (G + iH)\,\mathrm{d}r^2 + I\,\mathrm{d}\theta^2 \tag{B.7}$$

Going through the calculations above, the complex eigenvalues are given by

$$\begin{aligned} \lambda_\pm &= 1 + i\alpha_\pm \\ \lambda_r &= 1 + i\frac{H}{G} \\ \lambda_\theta &= 1 \end{aligned} \tag{B.8}$$

where

$$\alpha_\pm \equiv \frac{AF - 2CD + BE \pm \sqrt{(AF - 2CD + BE)^2 - 4(AE - C^2)(BF - D^2)}}{2(AE - C^2)} \tag{B.9}$$

The values $\alpha_\pm$ satisfy

$$(AE - C^2)\alpha_\pm^2 + (AF - 2CD + BE)\alpha_\pm + BF - D^2 = 0 \tag{B.10}$$

When $AE = C^2$ and $BF = D^2$, the equation reduces to

$$(\sqrt{AF} - \sqrt{BE})^2 \alpha_\pm = 0 \tag{B.11}$$

which is solved by a double root $\alpha_\pm = 0$. In this case, $\lambda_\pm = 1$. Three of the eigenvalues are real, and the only eigenvalue contributing to the admissibility condition comes from the radial coordinate.

The eigenvalues $\lambda_\pm$ depend on the discriminant

$$\beta \equiv (AF - 2CD + BE)^2 - 4(AE - C^2)(BF - D^2) \tag{B.12}$$

When $\beta > 0$, the real part of $\lambda_\pm$ is one, but they have different imaginary parts, as shown in Figure B.1a. When $\beta = 0$, $\alpha_\pm$ is given by

$$\alpha_\pm = \pm\sqrt{\frac{BF - D^2}{AE - C^2}} \tag{B.13}$$

In this case, $\lambda_\pm$ lie on either side of the real axis, as in Figure B.1b. When $\beta < 0$, the real part of $\lambda_\pm$ is given by

$$1 \mp \frac{\sqrt{-\beta}}{2(AE - C^2)} \tag{B.14}$$

but they share the same imaginary part, depicted in Figure B.1c.

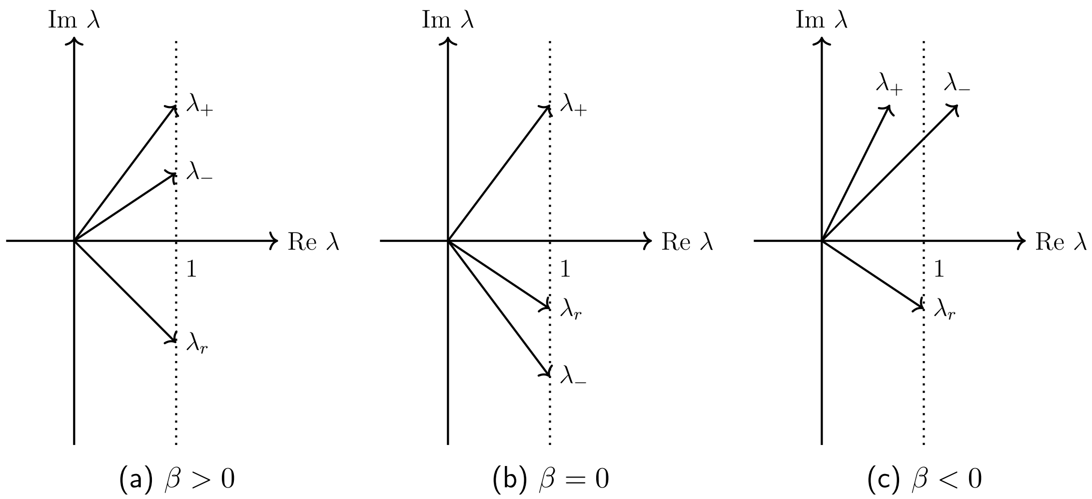


Figure B.1: Depending on the value of the discriminant β, the complex eigenvalues of the metric can lie in different places on the complex plane.

B.3 Admissibility Condition

The admissibility condition involving the complex eigenvalues requires

$$\sum_{i=1}^{d} \left|\operatorname{Arg}\lambda_i\right| < \pi \tag{B.15}$$

There are several ways to define the argument function, but one way is with the function `arctan2`. The function $\texttt{arctan2}(y, x)$ returns the argument of the complex number $x + iy$ between $(-\pi, \pi]$, where the angle is positive if $y > 0$ and negative if $y < 0$. Using this definition and the eigenvalues above, the admissibility condition can be written

$$\sum_{i=\pm} |\texttt{arctan2}(\operatorname{Im}\lambda_i, \operatorname{Re}\lambda_i)| + |\texttt{arctan2}(H/G, 1)| < \pi \tag{B.16}$$

When $AE = C^2$ and $BF = D^2$, the condition reduces to

$$-\pi < \texttt{arctan2}(H/G, 1) < \pi \tag{B.17}$$

which is automatically satisfied.

Appendix C

Software

To produce the results in this thesis, analytical calculations involving the metric had to be performed. Because these calculations are time-consuming and error-prone to conduct by hand, programs were written to perform them automatically. In many cases, calculations became analytically intractable, requiring numerical methods to be used. All the programs were written in Python, which provides the package `sympy` to perform symbolic computation, including tensor algebra calculations [72]. The code can be found on GitHub [73]. Producing the results found in Chapter 4 consisted of three basic steps.

1. ***Metric coefficients***. The admissibility condition involves functions of the metric coefficients. A class was created to determine these coefficients, which was utilized by various programs that needed them.

2. ***Admissibility conditions***. The admissibility condition consists of a set of different conditions for different p-forms F. For each condition, p copies of the metric had to be contracted with two copies of F. Explicit expressions were derived for each p given the general form of the metric, which required performing tensor contractions using `sympy`. The admissibility condition can also be expressed in terms of the complex eigenvalues of the metric. The final form of this condition was also derived using symbolic computations.

3. ***Admissibility evaluation***. To determine the final form of the admissibility condition for each p, the results of steps 1 and 2 were combined. Sometimes, the resulting condition could be solved analytically to determine the admissible range of the parameters. Most of the time, the condition was so complex that numerical methods had to be used. Different classes were created to conduct sweeps over the parameters to determine admissible parameter ranges.

These three steps are discussed in detail in the following sections.

C.1 Metric Coefficients

To evaluate the admissibility condition, the coefficients of the metric need to be determined. Because the metric is complex, the real and imaginary parts of each component have to be computed. A class called `Metric3D` was created to compute and return these coefficients, which allows them to be utilized by multiple programs. Listing C.1 is a snippet of the class definition. In the first few lines, the various symbolic variables are defined, including the coordinates and functions of the coordinates like Δ_r, Δ_θ, and W. Then in the constructor of the class, the metric is constructed, performing a change of coordinates and Wick rotation. The method to return the coefficients resolves the real and imaginary parts of each metric component and returns them in a dictionary.

```python
# import packages
import sympy

# define needed symbolic variables

# coordinates
r_tilde, theta = sympy.symbols("\\tilde{r} theta", real=True)

# differentials
dt, dr_tilde, dphi, dphi_tilde, dtau = sympy.symbols(
    r"dt d\tilde{r} d\phi d\tilde{\phi} d\tau"
)

# functions of coordinates
delta_r = sympy.symbols("Delta_r")
delta_theta, W = sympy.symbols("Delta_theta W", real=True)

# parameters
a, xi, omega, r_tilde_plus = sympy.symbols("a Xi Omega \\tilde{r}_+", real=True)

# real and imaginary metric coefficients
A, B, C, D, E, F, G, H = sympy.symbols("A B C D E F G H")

class Metric3D:
    """Class for representing a supersymmetric black hole in AdS4."""

    def __init__(self):
        """Construct the metric."""
        self.metric = (
            -delta_r / W * (dt - a * sympy.sin(theta) ** 2 * dphi / xi) ** 2
        )
        self.metric += (
            delta_theta * sympy.sin(theta) ** 2 / W *
            (a * dt - (r_tilde ** 2 + a ** 2) * dphi / xi) ** 2
        )
        self.metric += W * dr_tilde ** 2 / delta_r

        self._omega_shift()
        self._wick_rotate()

```

```python
    def coefficients(self):
        """Get metric coefficients involving real and imaginary parts."""
        metric_mat = self._metric_3d_to_matrix()

        A_val, B_val = self._get_real_and_imag(metric_mat[0, 0])
        C_val, D_val = self._get_real_and_imag(metric_mat[0, 1])
        E_val, F_val = self._get_real_and_imag(metric_mat[1, 1])
        G_val, H_val = self._get_real_and_imag(metric_mat[2, 2])

        return {
            A: A_val, B: B_val, C: C_val, D: D_val,
            E: E_val, F: F_val, G: G_val, H: H_val
        }
```

Listing C.1: The package sympy was used to determine the coefficients of the metric, decomposing each coefficient into its real and imaginary parts.

C.2 Admissibility Conditions

To derive expressions for the admissibility condition for every p, the metric g had to be contracted with the electromagnetic field strength tensor F. An example of this calculation for $p = 2$ is shown in Listing C.2, which computes the condition in Equation (A.13). On lines 11–16, the general form of the complex metric is defined for a supersymmetric black hole in four-dimensional anti-de Sitter space. The two-form tensor F is defined on lines 19–24. On lines 27–29, the needed tensor indices are defined. There are four i indices and four j indices. Two pairs each of i and j indices are used by g and F. The product between g and F is created on lines 32–37, then the contraction is performed on lines 40–42. Finally, the real part is taken on line 46. The final result is Equation (A.15). Similar calculations are performed for each p.

```python
1  # import packages
2  import sympy
3  from sympy.tensor.tensor import TensorIndexType, TensorHead, tensor_indices
4
5  A, B, C, D, E, F, G, H, I = sympy.symbols("A B C D E F G H I")
6  F_01, F_02, F_03, F_12, F_13, F_23 = sympy.symbols(
7      "F_{01} F_{02} F_{03} F_{12} F_{13} F_{23}"
8  )
9
10 # general form of the metric
11 g_mat = sympy.Matrix([
12     [A + sympy.I * B, C + sympy.I * D, 0, 0],
13     [C + sympy.I * D, E + sympy.I * F, 0, 0],
14     [0, 0, G + sympy.I * H, 0],
15     [0, 0, 0, I]
16 ])
17
18 # electromagnetic field strength
19 F_mat = sympy.Matrix([
```

```
    [0, F_01, F_02, F_03],
    [-F_01, 0, F_12, F_13],
    [-F_02, -F_12, 0, F_23],
    [-F_03, -F_13, -F_23, 0]
])

# create tensor indices
Lorentzian = TensorIndexType('Lorentzian', dummy_name='L')
i0, i1, i2, i3 = tensor_indices('i0:4', Lorentzian)
j0, j1, j2, j3 = tensor_indices('j0:4', Lorentzian)

# create tensor product between metric and field strength
ggFF_mat = sympy.tensorproduct(
    sympy.tensorproduct(g_mat, g_mat),
    sympy.tensorproduct(F_mat, F_mat)
)
ggFF_tensor = TensorHead('ggFF', [Lorentzian] * 8)
repl = {ggFF_tensor(i0, j0, i1, j1, -i2, -j2, -i3, -j3): ggFF_mat}

# perform contraction
quadratic_form = ggFF_tensor(
    i0, j0, i1, j1, -i0, -i1, -j0, -j1
).replace_with_arrays(repl)
quadratic_form = quadratic_form.expand().simplify().collect(sympy.I)

# take real part
real_quadratic_form = quadratic_form - quadratic_form.coeff(sympy.I) * sympy.I
```

Listing C.2: The form of the admissibility condition needs to be derived for every p. In this example, the condition for $p = 2$ is derived, which involves contracting two copies of the metric with two copies of the electromagnetic field strength tensor.

The admissibility condition involving the complex eigenvalues of the metric was also determined automatically using symbolic calculations. This is shown in Listing C.3, which computes the imaginary part of the eigenvalues N in Equation (B.6) given the form of the metric in Equation (B.7). After defining the real and imaginary parts of the metric, the real part is diagonalized on lines 12–31. This results in the matrix X, which is used to transform the imaginary part of the metric. On lines 35–54, the transformed imaginary part is diagonalized, leading to the final form of N. The diagonal elements of this matrix correspond to $\alpha_\pm$ and H/G found in Equation (B.8).

```
1 # import packages
2 import sympy
3
4 A, B, C, D, E, F, G, H = sympy.symbols("A B C D E F G H")
5
6 # real and imaginary parts of the metric
7 g_mat_real = sympy.Matrix([[A, C, 0], [C, E, 0], [0, 0, G]])
8 g_mat_imag = sympy.Matrix([[B, D, 0], [D, F, 0], [0, 0, H]])
9
10 # diagonalize real part
11
12 eigenvects_real = g_mat_real.eigenvects()
13
```

```
eig1 = eigenvects_real[2][2][0]
eig1 /= sympy.sqrt(eig1.dot(eig1))

eig2 = eigenvects_real[1][2][0]
eig2 /= sympy.sqrt(eig2.dot(eig2))

eig3 = eigenvects_real[0][2][0]
eig3 /= sympy.sqrt(eig3.dot(eig3))

Q = sympy.Matrix.hstack(eig1, eig2, eig3)

M = sympy.Matrix([
    [eigenvects_real[2][0], 0, 0],
    [0, eigenvects_real[1][0], 0],
    [0, 0, eigenvects_real[0][0]]
])

X = Q @ M.inv().applyfunc(sympy.sqrt)

# diagonalize imaginary part

g_mat_imag_trans = X.T @ g_mat_imag @ X

eigenvects_imag = g_mat_imag_trans.eigenvects()

eig_imag_1 = eigenvects_imag[2][2][0]
eig_imag_1 /= sympy.sqrt(eig_imag_1.dot(eig_imag_1))

eig_imag_2 = eigenvects_imag[1][2][0]
eig_imag_2 /= sympy.sqrt(eig_imag_2.dot(eig_imag_2))

eig_imag_3 = eigenvects_imag[0][2][0]
eig_imag_3 /= sympy.sqrt(eig_imag_3.dot(eig_imag_3))

# final eigenvalues

N = sympy.Matrix([
    [eigenvects_imag[2][0], 0, 0],
    [0, eigenvects_imag[1][0], 0],
    [0, 0, eigenvects_imag[0][0]]
])
```

Listing C.3: To compute the complex eigenvalues of the metric, the real part of the metric needs to be diagonalized, which can be used to transform the imaginary part. Then the imaginary part needs to be diagonalized, leading to the imaginary part of the eigenvalues.

C.3 Admissibility Evaluation

To derive the final expressions for the admissibility condition, the coefficients from the first step are inserted into the expressions from the second step. These final expressions are complicated functions of the coordinates and parameters, which must be greater than zero for the metric to be allowable. To assess whether the condition is satisfied, it is convenient to determine the roots of these expressions. If the expression is relatively simple, this may be possible, but in general the expression is too complicated for the roots to be determined. There are three different solution methods depending on the extent to which the roots can be determined.

- ***Analytical roots***. In this case, the condition is simple enough for the roots to be determined as functions of the parameter values. The roots are computed symbolically using `sympy.solve(expression, coordinate)`, where `expression` is the admissibility condition and `coordinate` is the coordinate over which to determine the roots. In this application, the radial coordinate is used.
- ***Pointwise analytical roots***. This method is used when the roots cannot be determined symbolically for arbitrary values of the parameters, but they can be for given parameter values. In this case, there are different sets of roots for each point in the parameter space.
- ***Range sweep***. This method applies to the most intractable admissibility conditions, where the roots cannot be determined either over the entire parameter space or for particular values. This method involves conducting a numerical sweep of different radial coordinates to check if the admissibility condition is satisfied. Because it relies on sampling, the results of this method are much more approximate.

Listing C.4 shows a simplified version of the code that implements the first solution. In this class called `AnalyticalRoots`, the main method is `admissibility`, which evaluates the admissibility condition for a grid of parameter values `pgrid`. For the black hole analyzed in the text, the parameters are $\tilde{r}_+$ and a. The method accepts the precomputed `roots` and an additional discriminant function, described in the main text. If the roots contain an angular dependence, it can also accept a certain `theta_val` over which to conduct the sweep. On lines 15–22, the various parameter values in the grid are swept over, calling the helper function defined below to check if the parameters are allowable. This helper function iterates through the roots, checking whether any is greater than $\tilde{r}_+$. If so, the admissibility expression switches signs for some $r > \tilde{r}_+$, causing the admissibility condition to be violated. If no roots exceed $\tilde{r}_+$, the function returns a value of `True`. The `admissibility` method returns the full map of results over the parameter grid.

```
1 # import packages
2 import numpy as np
3 import complex_spacetime_metrics.metric as mt
4
5 class AnalyticalRoots:
6
7     def admissibility(self, pgrid, roots, discrim, theta_val=None):
```

```
        # restrict to given theta
        if theta_val is not None:
            roots = [root.subs({mt.theta: theta_val}) for root in roots]
            discrim = discrim.subs({mt.theta: theta_val})

        admissible_map = np.zeros(pgrid.grid_size, dtype=bool)

        for i in range(pgrid.grid_size[0]):
            for j in range(pgrid.grid_size[1]):
                admissible_map[i, j] = self._check_if_admissible(
                    pgrid.a_vals_grid[i, j],
                    pgrid.r_tilde_plus_vals_grid[i, j],
                    roots,
                    discrim
                )

        return admissible_map

    def _check_if_admissible(self, a_val, r_tilde_plus_val, roots, discrim):
        # check the sign of leading term in r_tilde
        D = discrim.subs({mt.a: a_val, mt.r_tilde_plus: r_tilde_plus_val}).evalf()
        if D < 0:
            return False

        for root in roots:
            r = root.subs({mt.a: a_val, mt.r_tilde_plus: r_tilde_plus_val}).evalf()

            if r == sympy.zoo:
                return False

            if r != sympy.nan:
                if np.abs(r.coeff(sympy.I)) < 1e-10:
                    real_part = r if r.is_real else r.args[0]
                    if real_part > r_tilde_plus_val:
                        return False

        return True
```

Listing C.4: In certain instances, the admissibility condition is simple enough for the roots to be solved for in advance for arbitrary parameter values. When this happens, this class can be used to sweep over a parameter grid to determine whether the admissibility condition is satisfied, which corresponds to $\tilde{r}_+$ being greater than the largest root.

To use this class and the others defined for the other solution methods, Jupyter notebooks were created, one for each of the admissibility conditions analyzed in Chapter 4. Listing C.5 shows the code for one of the simplest conditions, the radial condition $G > 0$ that corresponds to Equation (4.38). This condition was simple enough to be analyzed by `AnayticalRoots`. On line 8, the `Metric3D` class is used to get the metric coefficients. The admissibility condition is defined on line 11, which just involves G. In more complicated examples, other metric components would enter into this expression. The function `sympy.fraction` helps extract the numerator, which must be positive for the condition to hold. The roots of G are computed symbolically on line 12. Finally, the admissibility map is generated using `AnalyticalRoots`, passing in the roots, a fine parameter grid, and the leading coefficient, which is 1 in this case. This map is plotted in Figure 4.1.

```
# import packages
import sympy
from complex_spacetime_metrics import config
from complex_spacetime_metrics import metric as metric_utils
from complex_spacetime_metrics.admissibility import AnalyticalRoots

# get metric coefficients
metric_coeffs = metric_utils.Metric3D().coefficients()

# evaluate expression and get its roots
G = sympy.fraction(metric_coeffs[metric_utils.G])[0]
roots = sympy.solve(G, metric_utils.r_tilde)

# generate admissibility map
admissible_map = AnalyticalRoots().admissibility(
    config.grids['fine'], roots, leading_coeff=1
)
```

Listing C.5: Scripts can be written to combine the results of the previous sections, including metric coefficients, admissibility conditions, and evaluation. The final result is a map of admissibility results across the parameter grid.

In more complicated examples, the other solution methods need to be used. These examples usually involve an angular dependence, which requires the parameter sweep to be conducted across different θ values. The value of the discriminant also plays a role, as discussed in the main text.

Bibliography

[1] E. Witten, *A Note on Complex Spacetime Metrics*, *Frank Wilczek: 50 Years of Theoretical Physics* (2022) [2111.06514].

[2] M. Kontsevich and G. Segal, *Wick Rotation and the Positivity of Energy in Quantum Field Theory*, *The Quarterly Journal of Mathematics* **72** (2021) 673 [2105.10161].

[3] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein-Hawking Entropy*, *Physics Letters B* **379** (1996) 99 [hep-th/9601029].

[4] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, *International Journal of Theoretical Physics* **38** (1999) 1113 [hep-th/9711200].

[5] E. Witten, *Anti de Sitter Space and Holography*, *Advances in Theoretical and Mathematical Physics* **2** (1998) 253 [hep-th/9802150].

[6] S. Gubser, I. Klebanov and A. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, *Physics Letters B* **428** (1998) 105 [hep-th/9802109].

[7] M. Cvetič, G. W. Gibbons, H. Lu and C. N. Pope, *Rotating Black Holes in Gauged Supergravities; Thermodynamics, Supersymmetric Limits, Topological Solitons and Time Machines*, hep-th/0504080.

[8] Z.-W. Chong, M. Cvetič, H. Lü and C. N. Pope, *General Non-Extremal Rotating Black Holes in Minimal Five-Dimensional Gauged Supergravity*, *Physical Review Letters* **95** (2005) 161301 [hep-th/0506029].

[9] J. B. Gutowski and H. S. Reall, *Supersymmetric AdS_5 Black Holes*, *Journal of High Energy Physics* **2004** (2004) 6 [hep-th/0401042].

[10] A. Zaffaroni, *Lectures on AdS Black Holes, Holography and Localization*, *Living Reviews in Relativity* **23** (2020) 2 [1902.07176].

[11] F. Benini, K. Hristov and A. Zaffaroni, *Black Hole Microstates in AdS_4 from Supersymmetric Localization*, *Journal of High Energy Physics* **2016** (2016) 54 [1511.04085].

[12] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation.* W. H. Freeman, San Francisco, 1973.

[13] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity.* Benjamin Cummings, 2003.

[14] R. M. Wald, *Gravitational Collapse and Cosmic Censorship*, gr-qc/9710068.

[15] J. M. Bardeen, B. Carter and S. W. Hawking, *The Four Laws of Black Hole Mechanics*, *Communications in Mathematical Physics* **31** (1973) 161.

[16] S. W. Hawking, *Particle Creation by Black Holes*, *Communications in Mathematical Physics* **43** (1975) 199.

[17] J. D. Bekenstein, *Black Holes and the Second Law*, *Lettere al Nuovo Cimento* **4** (1972) 737.

[18] J. D. Bekenstein, *Black Holes and Entropy*, *Physical Review D* **7** (1973) 2333.

[19] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, *Conf. Proc. C* **930308** (1993) 284 [gr-qc/9310026].

[20] L. Susskind, *The World as a Hologram*, *Journal of Mathematical Physics* **36** (1995) 6377 [hep-th/9409089].

[21] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, International Series in Pure and Applied Physics. McGraw-Hill, New York, NY, 1965.

[22] S. W. Hawking, *Quantum Gravity and Path Integrals*, *Physical Review D* **18** (1978) 1747.

[23] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Physical Review D* **15** (1977) 2752.

[24] S. W. Hawking, *Euclidean Quantum Gravity*, *NATO Science Series B* **44** (1979) 145.

[25] D. Freedman and A. Van Proeyen, *Supergravity.* Cambridge University Press, 2012.

[26] G. Dall'Agata and M. Zagermann, *Supergravity: From First Principles to Modern Applications*. Springer Berlin, Heidelberg, 2021.

[27] G. Gibbons, S. Hawking and M. Perry, *Path Integrals and the Indefiniteness of the Gravitational Action*, *Nuclear Physics B* **138** (1978) 141.

[28] C. Goutis and G. Casella, *Explaining the Saddlepoint Approximation*, *The American Statistician* **53** (1999) 216.

[29] S. W. Hawking and G. T. Horowitz, *The Gravitational Hamiltonian, Action, Entropy and Surface Terms*, *Classical and Quantum Gravity* **13** (1996) 1487 [gr-qc/9501014].

[30] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland Mathematical Library. North-Holland, 1973.

[31] S. Weinberg and E. Witten, *Limits on Massless Particles*, *Physics Letters B* **96** (1980) 59.

[32] E. Witten, *Analytic Continuation of Chern-Simons Theory*, *AMS/IP Studies in Advanced Mathematics* **50** (2011) 347 [1001.2933].

[33] J. Feldbrugge, J.-L. Lehners and N. Turok, *Lorentzian Quantum Cosmology*, *Physical Review D* **95** (2017) 103508 [1703.02076].

[34] J. J. Halliwell and J. B. Hartle, *Integration Contours for the No-boundary Wave Function of the Universe*, *Physical Review D* **41** (1990) 1815.

[35] J. B. Hartle and S. W. Hawking, *Wave Function of the Universe*, *Physical Review D* **28** (1983) 2960.

[36] S. Hawking, *The Quantum State of the Universe*, *Nuclear Physics B* **239** (1984) 257.

[37] J. Louko and R. D. Sorkin, *Complex Actions in Two-dimensional Topology Change*, *Classical and Quantum Gravity* **14** (1997) 179 [gr-qc/9511023].

[38] F. Dowker and S. Surya, *Topology Change and Causal Continuity*, *Physical Review D* **58** (1998) 124019 [gr-qc/9711070].

[39] R. Bott, *Lectures on Morse Theory, Old and New*, *Bulletin (New Series) of the American Mathematical Society* **7** (1982) 331.

[40] X. Dong, A. Lewkowycz and M. Rangamani, *Deriving Covariant Holographic Entanglement*, *Journal of High Energy Physics* **2016** (2016) 28 [1607.07506].

[41] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, *Replica Wormholes and the Black Hole Interior*, *Journal of High Energy Physics* **3** (2022) 205 [1911.11977].

[42] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *Replica Wormholes and the Entropy of Hawking Radiation*, *Journal of High Energy Physics* **5** (2020) 13 [1911.12333].

[43] M. Visser, *How to Wick Rotate Generic Curved Spacetime*, 1702.05572.

[44] M. Visser, *Feynman's $i\epsilon$ Prescription, Almost Real Spacetimes, and Acceptable Complex Spacetimes*, *Journal of High Energy Physics* **8** (2022) 129 [2111.14016].

[45] V. D. Ivashchuk, *Wick Rotation, Regularization of Propagators by a Complex Metric and Multidimensional Cosmology*, *Gravitation and Cosmology* **3** (1997) 8 [gr-qc/9705008].

[46] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory.* Addison-Wesley, Reading, USA, 1995.

[47] Y.-N. Zhou, T.-G. Zhou and P. Zhang, *Universal Properties of the Spectral Form Factor in Open Quantum Systems*, 2303.14352.

[48] P. Saad, S. H. Shenker and D. Stanford, *A Semiclassical Ramp in SYK and in Gravity*, 1806.06840.

[49] S. W. Hawking, C. J. Hunter and M. M. Taylor-Robinson, *Rotation and the AdS/CFT Correspondence*, *Physical Review D* **59** (1999) 064005 [hep-th/9811056].

[50] M. Bañados, C. Teitelboim and J. Zanelli, *Black Hole in Three-dimensional Spacetime*, *Physical Review Letters* **69** (1992) 1849 [hep-th/9204099].

[51] I. Basile, A. Campoleoni and J. Raeymaekers, *A Note on the Admissibility of Complex BTZ Metrics*, *Journal of High Energy Physics* **3** (2023) 187 [2301.11883].

[52] M. Cvetič and D. Youm, *Entropy of Non-Extreme Charged Rotating Black Holes in String Theory*, *Physical Review D* **54** (1996) 2612 [hep-th/9603147].

[53] Z.-W. Chong, M. Cvetič, H. Lü and C. N. Pope, *Charged Rotating Black Holes in Four-dimensional Gauged and Ungauged Supergravities*, *Nuclear Physics B* **717** (2005) 246 [hep-th/0411045].

[54] M. Cvetič, H. Lü and C. N. Pope, *Four-dimensional* $\mathcal{N}=4$*, SO(4) Gauged Supergravity from* $D=11$, *Nuclear Physics B* **574** (2000) 761 [hep-th/9910252].

[55] B. Carter, *A New Family of Einstein Spaces*, *Physics Letters A* **26** (1968) 399.

[56] B. Carter, *Hamilton-Jacobi and Schrödinger Separable Solutions of Einstein's Equations*, *Communications in Mathematical Physics* **10** (1968) 280.

[57] B. de Wit and H. Nicolai, $N=8$ *Supergravity with Local* $SO(8)\times SU(8)$ *Invariance*, *Physics Letters B* **108** (1982) 285.

[58] B. de Wit and H. Nicolai, $N=8$ *Supergravity*, *Nuclear Physics B* **208** (1982) 323.

[59] D. Cassani and L. Papini, *The BPS Limit of Rotating AdS Black Hole Thermodynamics*, *Journal of High Energy Physics* **9** (2019) 79 [1906.10148].

[60] N. Bobev and P. M. Crichigno, *Universal Spinning Black Holes and Theories of Class* $\mathcal{R}$, *Journal of High Energy Physics* **12** (2019) 54 [1909.05873].

[61] M. M. Caldarelli, G. Cognola and D. Klemm, *Thermodynamics of Kerr-Newman-AdS Black Holes and Conformal Field Theories*, *Classical and Quantum Gravity* **17** (1999) 399 [hep-th/9908022].

[62] M. M. Caldarelli and D. Klemm, *Supersymmetry of Anti-de Sitter Black Holes*, *Nuclear Physics B* **545** (1999) 434 [hep-th/9808097].

[63] V. Kostelecký and M. J. Perry, *Solitonic Black Holes in Gauged* $\mathcal{N}=2$ *Supergravity*, *Physics Letters B* **371** (1996) 191 [hep-th/9512222].

[64] D. D. Chow and G. Compère, *Dyonic AdS Black Holes in Maximal Gauged Supergravity*, *Physical Review D* **89** (2014) 065003 [1311.1204].

[65] P. B. Genolini, A. Cabo-Bizet and S. Murthy, *Supersymmetric Phases of* AdS_4/CFT_3, *Journal of High Energy Physics* **6** (2023) 125 [2301.00763].

[66] G. W. Gibbons, M. J. Perry and C. N. Pope, *The First Law of Thermodynamics for Kerr-Anti-de Sitter Black Holes*, *Classical and Quantum Gravity* **22** (2005) 1503 [hep-th/0408217].

[67] M. Cvetič, H. Lü and C. N. Pope, *Charged Kerr-de Sitter Black Holes in Five Dimensions*, *Physics Letters B* **598** (2004) 273 [hep-th/0406196].

[68] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic Origin of the Bekenstein-Hawking Entropy of Supersymmetric* AdS_5 *Black Holes*, *Journal of High Energy Physics* **10** (2019) 62 [1810.11442].

[69] J. P. Gauntlett and O. Varela, *Consistent Kaluza-Klein Reductions for General Supersymmetric AdS Solutions*, *Physical Review D* **76** (2007) 126007 [0707.2315].

[70] F. C. Caramello Jr., *Introduction to Orbifolds*, 1909.08699.

[71] O. Aharony, F. Benini, O. Mamroud and P. Milan, *A Gravity Interpretation for the Bethe Ansatz Expansion of the* $\mathcal{N} = 4$ *SYM Index*, *Physical Review D* **104** (2021) 086026 [2104.13932].

[72] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin et al., *Sympy: Symbolic Computing in Python*, *PeerJ Computer Science* **3** (2017) e103.

[73] J. Chryssathacopoulos, *Complex Spacetime Metrics*, https://github.com/jchryssanthacopoulos/complex_spacetime_metrics.