UNIVERSITÀ DEGLI STUDI DI PADOVA

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Tesi di Laurea

Radiative corrections to the vacua of broken extended supergravity

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Contents

Chapter 1

Introduction

Nowadays, it is a renowned fact that the Universe shows an accelerating expansion. This feature can be explained theoretically by introducing in the spacetime equations a small positive cosmological constant Λ. However, within the framework of High Energy Theoretical Physics and, in particular, of String Theory, which is the at the moment the only self-consistent theory of quantum gravity, it is a real dilemma to find out a mechanism which can introduce such a positive cosmological constant. To solve this problem, it should be useful to study the vacua of low-energy effective theories of String Theory, namely Supergravity theories. While it is fairly simple to generate vacua with a positive cosmological constant in minimal supergravity models (which are generically difficult to lift to full stringy backgrounds), but there are almost no examples of locally stable classical de Sitter vacua in extended models and also unstable ones are quite rare. One possible way out is to consider quantum corrections. For instance a positive cosmological constant could be generated by such corrections to a classical Minkowski vacuum. Understanding when de Sitter critical points are obtainable in supergravity models with a clear relation to string theory is especially interesting in connection with the recent conjecture [1] that theories with positive energy critical points belong to the swampland, i.e. to the set of models that do not admit a consistent ultraviolet completion in a quantum gravity theory. Finding even a single counter-example would be enough to disprove the conjecture, while better understanding the conditions underlying the lift of the value of the cosmological constant in effective theories of string theory would help making the conjecture more robust.

There is another important aspect that makes quantum corrections of extended supergravities interesting. General relativity is not renormalizable (it exhibits divergences at two-loop [2]) and its coupling to matter generically makes the situation worse, by introducing divergencies already at one-loop in the perturbative expansion. However, when we consider supergravity theories with extended supersymmetry, a very interesting feature comes out: these models are finite up to higher loop orders. To be more precise, it is well known that $N = 4$ Supergravity is finite up to three loops and it is believed that $N = 8$ Supergravity is also finite at least to seven loops [3], although it might reveal to be finite at all orders. In particular, explicit calculations in recent years have revealed the existence of hidden properties: in fact, infrared and ultraviolet cancellations exhibit remarkable features, which are not only related to supersymmetry.

Thus, even though $N = 8$ and $N = 4$ models can not be used to obtain realistic phenomenology, understanding the conditions for their perturbative behaviour might help us understand if there are still options available to build (perturbatively) finite quantum field theories of gravity, or if the only option is to resort in string theory. One point we would like to stress is that, so far, all these higher loop orders calculations have been made in the context of ungauged supergravity theories, where the background can be naturally taken to be flat and one can use standard perturbation theory about Minkowski spacetime. Also supersymmetry is fully preserved in the vacuum. However, supersymmetry can be broken spontaneously if we deform the theory by gauging a subgroup of its global symmetries, also breaking some of the global symmetries, by introducing new terms in the supersymmetry transformations and in the Lagrangian. The resulting gauged supergravities still admit Minkowski vacua, which break partially or completely supersymmetry. It is therefore interesting to study if the cancellations present in ungauged supergravity models occur also in these models, in order to better understand if some hidden symmetries play a role in these cancellations.

A recent paper following these lines [4] has shown that in $N = 8$ supergravities with arbitrary gauging there are no ultraviolet divergences in the one-loop effective potential and, in addition, there is no example of one-loop stable supersymmetry breaking vacuum with positive or vanishing vacuum energy. However, the $N = 8$ theories are very constrained by supersymmetry, thus it is even more interesting to study one-loop corrections in gauged $N = 4$ supergravities, where, in addition to the choice of the gauging, the number of matter multiplets is arbitrary, in order to better understand these hidden features.

Radiative corrections to the cosmological constant coincides with the value of the scalar potential evaluated at the vacuum. It can be shown [5] that one-loop corrections to the scalar potential can be written in terms of even powers of the *supertrace* of the *mass matrices*:

$$V(\phi_c) = V_0 + \frac{\Lambda^4}{32\pi^2}\mathrm{STr}\mathcal{M}^0 + \frac{\Lambda^2}{32\pi^2}\mathrm{STr}\mathcal{M}^2 + \frac{1}{64\pi^2}\mathrm{STr}\,\mathcal{M}^4\left(\log\frac{\mathcal{M}^2}{\Lambda^2} - \frac{1}{2}\right) , \tag{1.1}$$

where V_0 is the *classical potential*, Λ is an *ultraviolet cut-off* and we defined the *supertrace* of mass matrices as

$$\begin{aligned} \mathrm{STr}\mathcal{M}^{2n} &\equiv \sum_i (-1)^{2J_i}(2J_i+1)m_i^{2n} \\ &= \sum_k (-1)^{2J_k}(2J_k+1)\,\mathrm{Tr}\,\mathcal{M}_k^{2n} , \end{aligned} \tag{1.2}$$

where i covers the whole spectrum of the theory under consideration, J_i and m_i are the spin and the mass of the corresponding particle and $k = 0, \frac{1}{2}, 1, \frac{3}{2}$ is the spin of particles, whose mass matrix is $\mathcal{M}_k$. In particular, the second line applies to an arbitrary theory with gravitinos, gauge bosons, spin-$\frac{1}{2}$ fermions and scalars. From (1.1) we can read how $\mathcal{M}^0$, $\mathcal{M}^2$ and $\mathcal{M}^4$ control quartic, quadratic and logarithmic divergences of the one-loop effective potential, respectively. Finite corrections to the potential are managed by higher powers of the mass matrix. It is a well-known result that in any theory with exact supersymmetry all supertraces vanish on any Minkowski vacuum [6]. Therefore, we will require that supersymmetry is completely broken on the vacuum.

$\mathrm{STr}\mathcal{M}^0$ is always field independent and it counts the number of bosonic degrees of freedom minus the number of fermionic degrees. Then, it vanishes in any supersymmetric theory, where supersymmetry is realized linearly and even if it is spontaneously broken at the vacuum. But, in principle, there are no other reasons for which the supertraces of the quartic and quadratic mass matrix vanish. In the paper [4], it is shown using the general formalism of $N = 8$ *gauged* supergravities that the supertraces of the quadratic and the quartic mass matrices vanish at all classical four-dimensional Minkowski vacua with completely broken $N = 8$ supergravity:

$$\mathrm{STr}\mathcal{M}^2 = \mathrm{STr}\mathcal{M}^4 = 0 \ . \tag{1.3}$$

This makes quantum corrections to the potential finite and make possible to generate a small cosmological constant. The paper also takes into account the higher powers of the mass matrix. Under few restrictions on the gauging, which include all the known gaugings that lead to classical vacua with fully broken supersymmetry on a flat background, it is shown that

$$\mathrm{STr}\mathcal{M}^6 = 0 \ , \qquad\qquad \mathrm{STr}\mathcal{M}^8 > 0 \tag{1.4}$$

and, as a consequence, that the effective one-loop potential is negative definite, which correspond to Anti-de Sitter vacua.

Following this work, we are going to study radiative corrections to classical vacua with vanishing cosmological constant which completely break supersymmetry in the case of four-dimensional *half-maximal* supergravity with arbitrary gaugings and different couplings of matter.

Since in $N = 4$ we have a free parameter n, which is the number of matter multiplets, in principle one may expect that $N = 4$ calculations are more difficult than $N = 8$, which is totally fixed by supersymmetry. Thus, we start this work with the hope that some unexpected feature might simplify the whole calculations and higher power supertraces can be studied in the general case. The results from the $N = 8$ tell us that quantum corrections to the potential in $N = 4$ supergravities are finite in those cases which can be obtained as truncations of maximal supergravity to half-maximal one. We will give more details at the end of this work, when all the machinery will have been built and presented. Indeed, until now there does not exist in the literature the complete Lagrangian for the $N = 4$ supergravities with general gauging. Thus, in order to solve a part of the problem presented above, we need to provide a series of interaction terms which have never been written. The final result of this work is that the *quadratic divergence* of one-loop effective potential *vanishes* for *any* gauging and *any* number of matter multiplets.

We are going to present how this thesis is organized.

In the second chapter, we will show how effective potential is linked to the traces of the even powers of mass matrices, displaying the well known computations by Coleman and Weinberg [5] in the most general case. The mechanism introduced in this famous paper is presented in the case of a general non-renormalizable Yang-Mills theory. The final result will be generalized to theories containing higher-spin particle with a very simple argument.

In the third chapter, the $N = 4$ supergravity is presented: we show which is the field content of the theory, how these fields can be organized, supersymmetric field variations and some relevant terms of the Lagrangian. In extended supergravity theories ($N > 2$) the scalars parametrize some coset manifold, therefore in this chapter we will also introduce the formalism which is useful to describe this type of sigma models.

The fourth chapter contains a general review about how to gauge extended supergravity in various dimensions, by introducing the so-called *embedding tensor*. Internal consistency of the gauging and the preserving of supersymmetry require the embedding tensor to satisfy certain constraints which are both linear and quadratic. Then we present the pure-bosonic Lagrangian, which has already been found out in [7]. At this point we will introduce the T*-tensor*, whose irreducible components properties will be widely studied in appendices C and D. Fermion mass matrices will be composed out of these tensors. Finally, a series a consistency relations between scalar potential (which enters the theory when we gauge it) and T-tensor components will be pointed out. A large use of group theory will help us to do all these calculations, which may take weeks otherwise.

In the last chapter, the mass matrices of all the particles entering the theory will be written. We will also evaluate the supertrace of quadratic mass matrix and we will find out that it turns to be zero for arbitrary gaugings and number of matter multiplets n. This is the original result of this work: we would have expected that it might vanish in those cases which are truncations of $N = 8$ supergravities, but in principle there are no reasons this trace to be zero.

Most of the calculations which appear in the thesis imply the presence of multi-indices tensors with their own symmetry properties. Thus, although they are mechanical, to face them by hand will take a very long time and oversights are always lurking. Fortunately, the program *Cadabra* [8] came to the aid and we made a large use of it.

Chapter 2

Radiative corrections to effective potential

One-loop correntions to the vacuum energy in a general theory with scalars, spin-$\frac{1}{2}$ fermions and gauge bosons were computed first by Coleman and E. Weinberg in [5]. In the present chapter we will present the general ideas of this computation, which will lead to a general formula in terms of the mass matrices of the particles which are present in the theory. The main definitions and the general formalism of generating functions, which we rely on, is reviewed in appendix A. We will give the general idea of the calculations, then present a sample computation in *scalar QED* and finally reviewed the general method.

In this work we are interested in quantum corrections to the *pseudo-moduli space* of a supersymmetric theory, *i.e.* the space of non supersymmetric vacua. Associated to this we find massless scalar modes X, whose masses are not protected by any symmetry at the quantum level. Therefore, we expect these fields to acquire a mass when we take into account quantum corrections. At tree-level, the pseudo-moduli vacuum expectation value $\langle X \rangle$ labels physically inequivalent degenerate states and determines tree-level masses of the particles. The general result of Coleman-Weinberg computations can be explained schematically [9]: when we quantize the bosonic harmonic oscillator, the energy of the ground state gets a contribution $\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{k^2+m^2}$, while it gets the same contribution with an opposite sign for each fermionic mode. We sum over all particles and their spin states and introduce a cut-off Λ for momenta. Thus we have terms which diverge quadratically and logarithmically, which are proportional respectively to the quadratic and quartic supertrace of mass matrices, *i.e.* the sum of traces of mass matrices times the number of spin states number of each particle (with a sign minus for fermion masses). Finite corrections are proportional to higher powers of these supertraces: these tell us whether quantum corrections are positive or negative, that is whether they generate *de Sitter* or *anti-de Sitter* vacua when coupled to gravity.

2.1 Effective potential and loop expansion

Let us consider a general theory whose Lagrangian we denote with $\mathcal{L}(\phi_r, \partial\phi_r)$, where the r-index stands for a generic set of Lorentz and internal indices of the fields. Then the *classical* *field configuration* is set by extremizing the classical action

$$I[\phi_r] = \int \mathrm{d}^4x\, \mathcal{L}(\phi_r, \partial\phi_r) \ . \tag{2.1.1}$$

The Lagrangian generically contains kinetic terms of the fields, some interaction term and a scalar potential, which is a particular case of self-interacting terms that concerns only scalar fields. The relevance of the scalar potential becomes already clear at semi-classical level, since it determines the *vacuum* of the theory: by definition the vacuum is the state which minimizes the energy and has to be maximally symmetric. In particular, we are interested in vacua with vanishing cosmological constant

$$\Lambda = 0 \ , \tag{2.1.2}$$

which are Poincaré invariant. Lorentz invariance, in particular, requires that expectation values of fermionic and bosonic fields are zero (the same holds in maximally symmetric vacua):

$$\langle\psi\rangle = \langle\bar{\psi}\rangle = \langle A_\mu\rangle = 0 \ . \tag{2.1.3}$$

Translation invariance requires that fields are constant on the vacuum, or equivalently have zero momentum in Fourier space. Thus the vacuum of the theory is totally fixed by the value of $\langle\phi\rangle$, for which the scalar potential is on its minimum (we are not considering boson and fermion condensates $\langle A_\mu A^\mu\rangle \neq 0$ or $\langle\bar{\psi}\psi\rangle \neq 0$).

This non-vanishing expectation value may produce spontaneous symmetry breaking: *i.e.* we define a new scalar fields which vanish expectation value on the vacuum

$$\phi' = \phi - \langle\phi\rangle \ , \tag{2.1.4}$$

and rewrite the action $I[\phi_r]$ in terms of ϕ'. If $I[\phi_r]$ was manifestly invariant under the linear action of some symmetry, it is possible that the new action is not, that is the symmetry is no longer realized linearly. However $\langle\phi\rangle$, which minimizes scalar potential, determines only the *classical vacuum* of the theory: the *true vacuum* receives can be found if we take into account radiative corrections, because at quantum level the potential achieves additional terms, as we are going to show. In some theories, *classical* scalar potential is not present, or it does not produce a non-vanishing scalar expectation value. The idea of Coleman and Weinberg [5] was that quantum corrections may produce *SSB* (spontaneous symmetry breaking) in theories for which the semi-classical (tree) approximation does not indicate such breakdown. The simplest realistic model in which this phenomenon occurs is the electrodynamics of scalar massless mesons: SSB occurs at quantum level and the theory becomes that of a massive vector interacting with a massive scalar meson. We will deal with this theory in the following section. Although it is simple, the example mentioned shows all the main features of calculations in a generic non-abelian theory (either renormalizable or not), which will follow in a natural way by generalization to more fields and with fermions.

In appendix A we show how to construct the quantum action in terms of the so-called *classical field*

$$\phi_c(x) = \frac{\langle\Omega^+|\,\phi(x)\,|\Omega^-\rangle_J}{\langle\Omega^+|\Omega^-\rangle_J} \ , \tag{2.1.5}$$

which has the form

$$\Gamma[\phi_c] = \int \mathrm{d}^4x \left[-V_{\text{eff}}(\phi_c) + \frac{Z(\phi_c)}{2}\partial_\mu\phi_c\partial^\mu\phi_c + (\text{higher order derivative}) + (\text{other fields}) \right] , \tag{2.1.6}$$

where $V_{\text{eff}}(\phi_c)$ is the new *effective potential*

$$V_{\text{eff}}(\phi_c) = V(\phi_c) + \text{quantum corrections} . \tag{2.1.7}$$

In particular, the effective potential can be written as the sum of *proper vertices functions*

$$V_{\text{eff}}(\phi_c) = i\sum_n \frac{1}{n!} i\tilde{\Gamma}^{(n)}(0,\ldots,0) , \tag{2.1.8}$$

where $\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n)$ are 1-particle irreducible connected Green's functions in Fourier space, with n external scalar lines.

The scalar vacuum expectation value $\langle\phi\rangle$ is determined by

$$\left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\phi}\right|_{\phi_c=\langle\phi\rangle} = 0 , \tag{2.1.9}$$

with the further requirement that it is a minimum, in order to have *stability* of the vacuum state.

The idea of the original paper [5] was to calculate the effective potential in *loop expansion*, which can be demonstrated to be equivalent to $\hbar$ expansion:

$$V(\phi_c) = \text{tree graphs } + \text{ one closed loop graphs } + \mathcal{O}\big(\hbar^2\big) . \tag{2.1.10}$$

Let us introduce $\hbar$ in a new Lagrangian $\mathcal{L}'(\phi,\partial\phi,\hbar)$, as it appears in the definition of the generating functional (A.3),

$$\mathcal{L}'(\phi,\partial\phi,\hbar) = \frac{\mathcal{L}(\phi,\partial\phi)}{\hbar} , \tag{2.1.11}$$

and expand it in powers of $\hbar$. The power of $\hbar$, H, associated with any graph is

$$H = I - V , \tag{2.1.12}$$

where I is the number of internal lines and V the number of vertices in the graph. The propagator is, indeed, the inverse of the differential operator occurring in the quadratic term of $\mathcal{L}'$, while the vertices carry a factor $\hbar^{-1}$ (there are no propagators due to external lines attached to proper vertices). The number of loops L is the number of unconstrained momenta, *i.e.*

$$L = I - V + 1 , \tag{2.1.13}$$

because each vertex carries a δ^4 in the momentum space (at each vertex four-momentum has to be conserved) and we have to exclude the conservation of over-all momentum. Therefore

$$H = L - 1 . \tag{2.1.14}$$

This expansion is unaffected by the shift of the fields because the parameter $\hbar^{-1}$ multiplies the total Lagrangian density.

2.2 Sample computations - Massless scalar electrodynamics

We will show how computations work in the following simple case, which captures the main features of this procedure appearing also in more complicated models.

Let us consider the Lagrangian density of massless scalar electrodynamics

$$\mathcal{L}_{\text{SQED}} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 + \left(D_\mu\varphi\right)^\dagger\left(D^\mu\varphi\right) - \frac{\lambda}{3!}\left(\varphi^\dagger\varphi\right)^2 \ , \tag{2.2.1}$$

where φ is a complex scalar, D_μ is the *gauge covariant derivative*

$$D_\mu := \partial_\mu - ieA_\mu \ , \tag{2.2.2}$$

A_μ is the gauge boson field and $F_{\mu\nu}$ is its *field strength*

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.2.3}$$

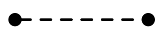
Let us write the complex field φ in terms of two real fields ϕ_1 and ϕ_2

$$\varphi := \frac{\phi_1 + i\phi_2}{\sqrt{2}} \ , \tag{2.2.4}$$

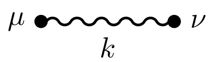
and rewrite (2.2.1) in terms of these new fields:

$$\mathcal{L}_{\text{SQED}} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_1 - eA_\mu\phi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_2 - eA_\mu\phi_1\right)^2 - \frac{\lambda}{4!}\left({\phi_1}^2 + {\phi_2}^2\right)^2 \ . \tag{2.2.5}$$

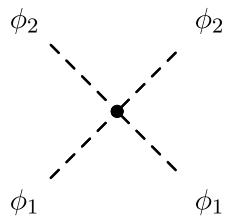
If we quantize the photon field in *Landau gauge*, the Feynman rules for this model are



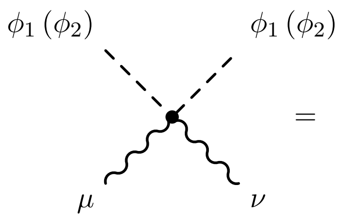
$$= \frac{i}{k^2 + i\epsilon} \ , \tag{2.2.6}$$



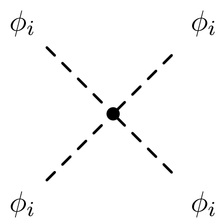
$$= -\frac{i}{k^2 + i\epsilon}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \ , \tag{2.2.7}$$



$$= -i\,\frac{\lambda}{3} \ , \tag{2.2.8}$$

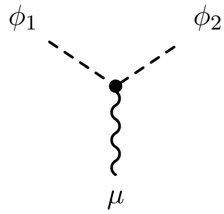


$$= 2ie^2\eta^{\mu\nu} \ , \tag{2.2.9}$$



$$= -i\,\lambda \ . \tag{2.2.10}$$

We will not consider vertices between ϕ_1, ϕ_2 and A_μ,



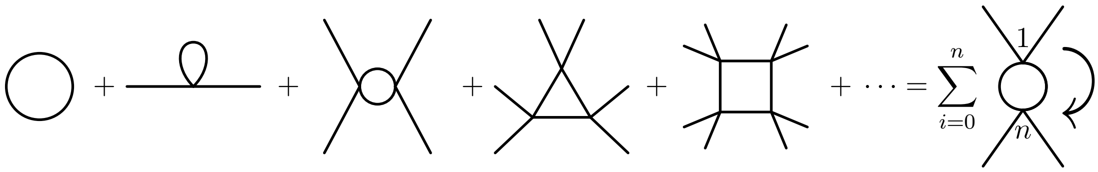
(2.2.11)

because they depend on the momentum of one of the two scalars. Indeed, we are working with external scalars with zero momenta, therefore the momenta of the photon and of the remaining scalar running around the loop are opposite and the contraction of internal momentum (coming from the vertex) with photon propagator in Landau gauge is zero.
The effective potential can only depend on $\phi_c^2 := \phi_1^2 + \phi_2^2$. Then we can calculate graphs with ϕ_1 external lines only and replace ${\phi_1}^2$ with ϕ_c^2 in the final result.

By symmetry considerations, we know that only graphs with an even number of external lines are not zero. To the lowest order the only 1PI connected graphs contributing are the vertices, thus

$$V(\phi_c) = \frac{\lambda}{4!}\left({\phi_1}^2 + {\phi_2}^2\right)^2 + \mathcal{O}(\hbar) \ . \tag{2.2.12}$$

The effective potential at one-loop contains all the interactions of the scalars (quadratic, quartic, septic, and so on, with one loop), as it can be read from (2.1.8), and we have an infinite series of polygonal graphs:



,

where internal lines can be either vectors or scalars, and external lines are scalars only.

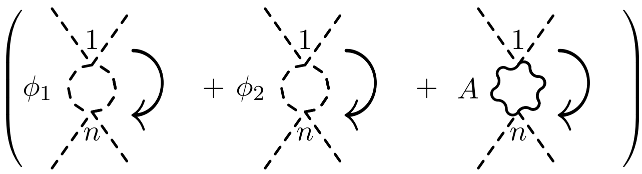
The first *vacuum diagram* gives always the contribution

$$i\Gamma^{(0)} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} = \frac{1}{8\pi^2}\int_0^\Lambda \mathrm{d}k k^3 = \frac{\Lambda^4}{32\pi^2} \ , \tag{2.2.13}$$

which is a field independent result. We have to multiply it for the number of bosonic degrees of freedom. Fermions contribute with a minus sign, due to fermionic loop. However this is a constant correction, thus in ordinary Yang-Mills field theories with a Minkowski background can be eliminated by redefining the scalar potential. Nevertheless, in the following chapters we will deal with supergravity theories and this arguments is no longer valid. Supersymmetry comes to the aid: indeed, as we said this correction only depends on the spectrum of the theory and in supersymmetric theories bosonic and fermionic degrees of freedom are always equal, no matter whether the vacuum breaks SUSY or not. Therefore, in what follows we will always ignore this term.

There are three types of graphs to compute: those with either ϕ_1, ϕ_2 or A_μ running around the polygonal loops. Thus the proper vertices are

$$i\Gamma^{(2n)} = \frac{1}{2n}\frac{(2n)!}{2^n}$$



(2.2.14)

$$= \frac{1}{2n}\frac{(2n)!}{2^n}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+i\epsilon)^n}\left[\lambda^n + \left(\frac{\lambda}{3}\right)^n + 3\left(2e^2\right)^n\right] ,$$

where the $1/2n$ is the *symmetry factor* due to the internal lines of the diagrams (the number of symmetries of a polygon with n non-oriented sides is $2n$) and $(2n)!/2^n$ are the permutations of external lines, except for the exchange of scalars at the same vertex. Coefficients $1/3$ and 2 of the second and third types of graphs respectively come from the vertices, while a coefficient 3 of the third type comes from the trace of Lorentz indices on n photon propagator numerators:

$$\mathrm{Tr}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)^n = 3 , \tag{2.2.15}$$

which is the number of degrees of freedom of a massive vector.

Now we write the potential in powers of $\hbar$

$$V(\phi_c) = \frac{\lambda}{4!}\phi_c^4 + \hbar V_1(\phi_c) + \mathcal{O}\left(\hbar^2\right) . \tag{2.2.16}$$

Remembering relation (2.1.8), at one loop the three types of graphs give

$$V_1(\phi_c) = i\sum_{n=1}^{\infty}\phi_c^{2n}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2n}\frac{1}{(k^2+i\epsilon)^n}\left[\left(\frac{2\lambda}{3}\right)^n + 3\left(e^2\right)^n\right] , \tag{2.2.17}$$

Let us solve explicitly the integral for the first term in the right hand side, since the second follows trivially. The first steps are a *Wick rotation* and the exchange of momentum integral and summation:

$$V_1(\phi_c) = \frac{1}{2}\int\frac{\mathrm{d}^4k_E}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{2\lambda\phi_c^2}{3k_E^2}\right)^n + \mathcal{O}\left(e^2\right) , \tag{2.2.18}$$

where we recognize the Taylor expansion of the logarithm:

$$V_1(\phi_c) = \frac{1}{2}\int\frac{\mathrm{d}^4k_E}{(2\pi)^2}\log\left(1+\frac{2\lambda\phi_c^2}{3k_E^2}\right) + \mathcal{O}\left(e^2\right) . \tag{2.2.19}$$

Now we introduce an *ultraviolet cut-off* Λ:

$$\begin{aligned} V_1(\phi_c) &= \frac{1}{16\pi^2}\int_0^\Lambda \mathrm{d}k\, k^3\log\left(1+\frac{2\lambda\phi_c^2}{3k^2}\right) + \mathcal{O}\left(e^2\right) \\ &= \Lambda^2\frac{\lambda\phi_c^2}{48\pi^2} + \frac{\lambda^2\phi_c^4}{144}\left(\log\left(\frac{2\lambda\phi_c^2}{3\Lambda^2}\right) - \frac{1}{2}\right) + \mathcal{O}\left(e^2\right) , \end{aligned} \tag{2.2.20}$$

where we have thrown away terms that vanish as Λ goes to infinity.

Including $\mathcal{O}\left(e^2\right)$ terms, the final expression is

$$V_1(\phi_c) = \frac{\Lambda^2}{32\pi^2}\left(\frac{2\lambda}{3}+3e^2\right)\phi_c^2 + \frac{\lambda^2}{144\pi^2}\phi_c^4\left(\log\frac{2\lambda\phi_c^2}{3\Lambda^2} - \frac{1}{2}\right) + \frac{3e^4}{64\pi^2}\phi_c^4\left(\log\frac{e^2\phi_c^2}{\Lambda^2} - \frac{1}{2}\right) . \tag{2.2.21}$$

2.3 Effective potential in general theories

We are going to compute one-loop corrections to the effective potential of the most general gauge field theory. We always keep in mind the renormalizable interaction, but what we are going to say is also valid in general. The final expression can be obtained in terms of traces of certain matrices constructed from the coupling constants of the theory. In particular, these are tree-level mass matrices of the fields which are functions of vacuum expectation values of the scalars. The Lagrangian density involves a set of real spinless boson fields ϕ_i, a set of Dirac fields ψ_i and a set of gauge bosons A^a_μ, where i and a are the indices of the representations of the gauge group.

The interactions of the scalars are quartic self-interactions, Yukawa couplings with fermions and minimal gauge-invariant couplings with vector fields. If we choose Landau gauge, the only diagrams contributing to radiative corrections are polygon graphs with either spinless bosons, fermions or gauge bosons running around the loops. Ghost fields do not directly couple at one loop with spinless fields. The effective potential will be the sum of the classical potential and quantum corrections from boson and fermion loops

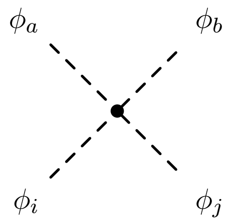
$$V = V_0 + \hbar \left(V_s + V_f + V_g \right) \ , \tag{2.3.1}$$

with an obvious notation for subscripts.

Let us start with the scalar interactions. The Lagrangian density in this case is

$$\mathcal{L}_s = -V_0(\phi) \ , \tag{2.3.2}$$

then the Feynman rule becomes



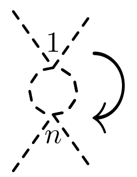
$$= \quad -i \, \frac{\delta^4 V_0}{\delta\phi_i \delta\phi_j \delta\phi_a \delta\phi_b} \ , \tag{2.3.3}$$

where, for convention, the scalars i, j are externals and a, b are those running around the loops. For each vertex there are two fields attached ϕ_i and ϕ_j: if $i = j$ the factor 2 coming from the derivatives is compensate by the factor $1/2$ due to the indistinctness of the two bosons at the same vertex. Thus, in the calculations that follow, we have to consider the matrix $\mathbf{V}$, defined by

$$V_{ab} := \frac{\delta^2 V_0}{\delta\phi_a \delta\phi_b}, \tag{2.3.4}$$

which is real and symmetric and it is a function of the scalar fields ϕ_i. Proper vertices are

$$i\Gamma^{(2n)} =$$



$$= \frac{1}{2n}(2n)! \frac{\mathrm{Tr}\, \mathbf{V}^n}{(k^2 + i\epsilon)^n} \ . \tag{2.3.5}$$

Furthermore there exists an orthogonal matrix $\mathbf{O}$, such that $\mathbf{O}^T\mathbf{O} = \mathbb{I}$ and $\mathbf{O}^T\mathbf{V}\mathbf{O} = \mathbf{D}$ is diagonal. The one-loop correction becomes

$$V_s(\phi_c) = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\frac{1}{(k^2+i\epsilon)^n}\,\mathrm{Tr}\,\mathbf{V}^n \ . \tag{2.3.6}$$

Using the properties of the trace we can do the calculation using the diagonal matrix $\mathbf{D}$, instead of $\mathbf{V}$. In this way it is easy to sum and integrate and then come back to $\mathbf{V}$:

$$V_s(\phi_c) = \frac{\Lambda^2}{32\pi^2}\,\mathrm{Tr}\,\mathbf{V}(\phi_c) + \frac{1}{64\pi^2}\,\mathrm{Tr}\left(\mathbf{V}(\phi_c)^2\log\frac{\mathbf{V}(\phi_c)}{\Lambda^2}\right) - \frac{1}{128\pi^2}\,\mathrm{Tr}\,\mathbf{V}(\phi_c)^2 \ . \tag{2.3.7}$$

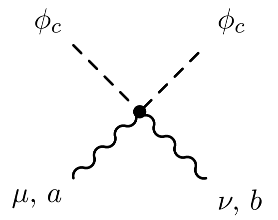
The term of the Lagrangian density which contributes to the correction due to gluon fields is

$$\mathcal{L}_g = \frac{1}{2}\sum_{ab} {M^2}_{ab}(\phi)A^a_\mu A^{\mu b} \ , \tag{2.3.8}$$

where ${M^2}_{ab}(\phi)$ is a symmetric and real matrix (the position of the indices a and b is meaningless). Its general form is

$${M^2}_{ab}(\phi) = g_a g_b T^a_{ij}T^b_{jk}\phi_i\phi_k \ , \tag{2.3.9}$$

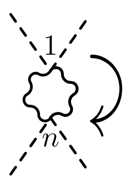
where T^a_{ij} and T^b_{ij} are the generators of infinitesimal transformations of the gauge group, g_a and g_b are the coupling constants (in the general case, if the gauge group is not simple, they can be different). The Feynman rule for the vertex is



$$= i{M^2}_{ab}\,(\phi_c) \tag{2.3.10}$$

and computations are straightforward:

$$i\Gamma^{(2n)} =$$



$$= \frac{3}{2n}(2n)!\frac{\mathrm{Tr}\left(M^2\right)^n}{\left(k^2+i\epsilon\right)^n} \ , \tag{2.3.11}$$

$$V_g(\phi_c) = \frac{3\Lambda^2}{32\pi^2}\,\mathrm{Tr}\,\mathbf{M^2}(\phi_c) + \frac{3}{64\pi^2}\,\mathrm{Tr}\left(\mathbf{M^2}(\phi_c)^2\log\frac{\mathbf{M^2}(\phi_c)}{\Lambda^2}\right) - \frac{3}{128\pi^2}\,\mathrm{Tr}\,\mathbf{M^2}(\phi_c)^2 \ . \tag{2.3.12}$$

It is identical to the previous case, except for the coefficient 3 overall coming from the Lorentz traces of propagators. Let us just notice that the degrees of freedom of a massive bosons are three. This aspect will soon assume a fundamental role.

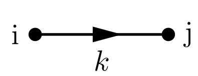
The last case we are considering is the most general Yukawa coupling:

$$\mathcal{L}_Y = -\sum_{ij} m_{ij}(\phi)\bar{\psi}_i\psi_j \ , \tag{2.3.13}$$

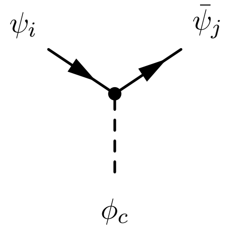
where $\mathbf{m}(\phi)$ is a matrix in Dirac space and internal space. In particular, we can decompose it as

$$m_{ij}(\phi) = A_{ij}(\phi)\mathbb{I} + iB_{ij}(\phi)\gamma^5 \ , \tag{2.3.14}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the hermitian Dirac fifth gamma matrix, $\mathbf{A}$ and $\mathbf{B}$ are hermitian matrices in the internal space. Thus, we can diagonalize them by a unitary transformation and calculations are the same as in the previous case. The Feynman rules are



$$= \frac{i}{\not k}\delta_{ij} \ , \tag{2.3.15}$$



(2.3.16)

$$= \quad -im_{ji}\left(\phi_c\right) \ . \tag{2.3.17}$$

(2.3.18)

Since the fermion propagators are massless, only graphs with an even number of internal fermions contribute to the sum. Indeed, using (2.3.14), hermiticity of $\mathbf{A}$ and $\mathbf{B}$ and the commutation rules $\{\gamma^\mu, \gamma^5\} = 0$ we have

$$\mathrm{Tr}\left[\ldots \mathbf{m}\frac{1}{\not k}\mathbf{m}\frac{1}{\not k}\ldots\right] = \mathrm{Tr}\left[\ldots \mathbf{m}\mathbf{m}^\dagger\frac{1}{k^2}\ldots\right] \ , \tag{2.3.19}$$

and

$$\mathrm{Tr}\left[\frac{1}{\not k}\gamma^5\right] = \mathrm{Tr}\left[\frac{1}{\not k}\right] = 0 \ , \tag{2.3.20}$$

where the trace runs over both internal and Dirac indices. The polygonal loops of fermions have oriented sides, thus we do not have reflection symmetry any more. A factor 1/2 comes from the sum over the even terms only, thus

$$i\Gamma^{(2n)} =$$

$$= \frac{1}{2n}(2n)!\,\mathrm{Tr}\left(\mathbf{m}\frac{1}{\not k}\right)^n \tag{2.3.21}$$

and the final contribution to the potential is

$$V_f(\phi_c) = -\frac{\Lambda^2}{32\pi^2}\,\mathrm{Tr}\,\mathbf{m}\mathbf{m}^\dagger - \frac{1}{64\pi^2}\,\mathrm{Tr}\left[\left(\mathbf{m}\mathbf{m}^\dagger\right)^2\log\frac{\mathbf{m}\mathbf{m}^\dagger}{\Lambda^2}\right] + \frac{1}{128\pi^2}\ \mathrm{Tr}\,\mathbf{m}\mathbf{m}^\dagger \ , \tag{2.3.22}$$

where an overall minus sign comes from fermion loops. This sign is the main difference with respect to the bosonic case: fermion loops can compensate the bosonic radiative corrections. If we are considering Majorana fermions, instead of Dirac, we recover reflection symmetry of polygonal loops and an overall 1/2 factor has to be added.
We can also consider Weyl fermions

$$\psi_{\mathrm{L,R}} = \mathrm{P}_{\mathrm{L,R}}\psi \ , \tag{2.3.23}$$

where ψ is a Dirac fermion and $\mathrm{P_R}$ and $\mathrm{P_L}$ are the chiral projectors:

$$\mathrm{P_{L,R}} = \frac{\mathbb{I} \mp \gamma^5}{2} \ . \tag{2.3.24}$$

It is possible to decompose $\mathbf{m}(\phi)$ (2.3.14) in an alternative way:

$$\mathbf{m}(\phi) = \mathbf{m}_\mathrm{L}(\phi)\mathrm{P_L} + \mathbf{m}_\mathrm{R}(\phi)\mathrm{P_R} \ . \tag{2.3.25}$$

If we are considering one loop contribution of Weyl spinors, only one of these two terms is not null, because of the projectors. Then the trace over Dirac indices gives an overall factor

$$\mathrm{Tr}\,\mathrm{P_L} = \mathrm{Tr}\,\mathrm{P_R} = 2 \ , \tag{2.3.26}$$

which are the degrees of freedom of a Weyl spinor.

It can be noted that all calculations we have done are also valid in a non-renormalizable theory, since we have not used this assumption in the calculations. Furthermore, when we sum (2.3.7), (2.3.12) and (2.3.22) (for Weyl fermions) the final result can be written as

$$V(\phi_c) = V_0 + \frac{\Lambda^4}{32\pi^2}\mathrm{STr}\mathcal{M}^0 + \frac{\Lambda^2}{32\pi^2}\mathrm{STr}\mathcal{M}^2 + \frac{1}{64\pi^2}\mathrm{STr}\left(\mathcal{M}^4 \log\frac{\mathcal{M}^2}{\Lambda^2}\right) - \frac{1}{128\pi^2}\mathrm{STr}\mathcal{M}^4 \ , \tag{2.3.27}$$

where $\mathrm{STr}\mathcal{M}^{2n}(\phi_c)$ is the *supertrace* over internal indices of mass matrices:

$$\mathrm{STr}\mathcal{M}^{2n} = \mathrm{Tr}\,\mathbf{V}^n - 2\,\mathrm{Tr}\,\mathbf{m}_\mathrm{H}^{2n} + 3\,\mathrm{Tr}\,\mathbf{M}^{2n} \ , \tag{2.3.28}$$

where the notation adopted is an obvious simplification of the previous one and H denotes the helicity of the Weyl fermion. The result is very general, indeed we can include particles with higher-spin, like *gravitinos*. In the general case the supertrace is defined as

$$\mathrm{STr}\mathcal{M}^{2n} = \sum_i (-1)^{2J_i}\,(2J_i+1)\,\mathrm{Tr}\,\mathcal{M}_{J_i}^{2n} \ , \tag{2.3.29}$$

where J_i are the spins of the particles ($2J_i + 1$ its degrees of freedom) and $\mathcal{M}^2_{J_i}$ their mass matrices. In (2.3.27), we also included the contribution of vacuum diagrams $\mathrm{STr}\mathcal{M}^0$ (2.2.13), which counts the degree of freedom of the theory:

$$\mathrm{STr}\mathcal{M}^0 = n_s - 2n_f + 3n_v \ , \tag{2.3.30}$$

where n_s, n_f, n_v are the number of scalars, fermions and vectors entering the theory.

We can justify this general formula as mentioned in the introduction of this chapter, that is summing over momenta and spin configurations of bosonic and fermionic harmonic oscillators ground states.

$$V_1(X) = \sum (-1)^\mathrm{F} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2 + m_i^2(X)} \ , \tag{2.3.31}$$

where $(-1)^\mathrm{F}$ is $+1$ for bosons and -1 for fermions and we are summing over all particles and their spin states. Although we present this formula from qualitative physical consideration, it can be proved [10] in a rigorous way. We introduce a cut-off Π for the three-momentum k.

$$\begin{aligned} V_1(X) &= \sum \frac{(-1)^\mathrm{F}}{(2\pi)^2}\int_0^\Pi \mathrm{d}k\, k^2 \sqrt{k^2+m_i^2} \\ &= \sum \frac{(-1)^\mathrm{F}}{32\pi^2}\left[k\sqrt{k^2+m_i^2}(2k^2+m_i^2) - m_i^4 \log\left(\sqrt{k^2+m_i^2}+k\right)\right]\Bigg|_0^\Pi \\ &= \sum \frac{(-1)^\mathrm{F}}{32\pi^2}\left[2\Pi^4 + 2m_i^2\Pi^2 + \frac{1}{2}m_i^4\left(\log\frac{m_i^2}{4\Pi^2}+\frac{1}{2}\right)\right] \ , \end{aligned} \tag{2.3.32}$$

where in the last step we have take the limit $\Pi \gg m_i$.

Thus, we can notice that the structure of this schematic quantum correction is exactly the same as (2.3.27), because the sum over spin states gives the coefficient $2J_i + 1$. Nevertheless, numerical coefficients are different. However this is an expected feature since in the formula we have just written appears a three-momentum cut-off, while in (2.3.27) Λ is a four-momentum cut-off in euclidean space. Coefficients of polynomial terms in Π (or Λ) and m_i depend on the regularization scheme that we have chosen and the relation between two different schemes is usually non-trivial. The only coefficient, which does not depend on the regularization scheme, is that of $m_i^4 \log m_i^2$ and, indeed, it coincides in the two formulae. The coefficients of the other terms are meaningless: we are considering theories coupled with gravity which are not renormalizable, then all the divergences have to disappear. That is the supertraces of quadratic and quartic mass matrix must be zero, in order the theories to be self-consistent.

Chapter 3

D=4 N=4 Supergravity

$N = 4$ pure supegravities in four dimensons were constructed in the seventies [11–14], and within the following dacade the coupling of vector multiples to these theories and some of their gaugings were worked out [15–19]. In the following chapter, we will present the field content of the theory and the *ungauged* version of $N = 4$ supergravity in $D = 4$, with an arbitrary number n of vector multiples. In particular, we will focus on few terms, that are crucial for the study of mass terms in gauged theories.

3.1 D=4 N=4 Supergravity fields content

The fields content of $N = 4$ pure supergravity in four dimensions can be organized in the so called *gravity multiplet*, which contains the metric $g_{\mu\nu}$ (or equivalently the vierbein $e^{\mathfrak{a}}_{\mu}$, with $\mathfrak{a}$ flat space-time index), four gravitinos ψ^i_μ, six massless vectors A^m_μ, four spin-1/2 fermions χ^i and one complex scalar τ, where $i = 1 \dots 4$ is a complex $SU(4)$-index and $m = 1 \dots 6$ is a real $SO(6)$-index (as it will be clarified in the following sections). The only $N = 4$ matter multiplet available is the vector multiplet [20, 21], which contains a massless gauge field, four massless spin-1/2 fermions and six real scalars. We are going to consider n vector multiplets, thus the fields will be organized as follows: the gauge fields A^a_μ ($a = 1 \dots n$ is a real $SO(n)$-index), the fermions λ^i_a and the scalars $\phi^{a[ij]}$.

	Spin-0	Spin-$\frac{1}{2}$	Spin-1	Spin-$\frac{3}{2}$	Spin-2
Gravity multiplet	2	4	6	4	1
Matter multiplet	6	4	1	0	0

The Lagrangian of the ungauged theory is totally fixed by supersymmetry and, in particular, the bosonic Lagrangian is highly constrained by the structure of an underlying *global symmetry group* $G = SL(2, \mathbb{R}) \times SO(6, n)$. In the rest of this section, we will present how the global symmetry group G determines the ungauged Lagrangian.

3.2 Scalar sectors

The scalars τ and $\phi^{a[ij]}$ are described by *non-linear sigma models*, *i.e.* they are coordinates of non-compact Riemannian differiential manifolds, *target spaces*, with dimensions 2 and $6n$, respectively:

$$\mathcal{L}_{\text{s.kin}} = -\frac{e}{2} g_{st} \partial_\mu \phi^s \partial^\mu \phi^t - \frac{e}{2} g_{xy} \partial_\mu \phi^x \partial^\mu \phi^y \ , \tag{3.2.1}$$

where $s = 1, 2$ and $x = 1 \dots 6n$ are curved indices of the two *scalar manifolds*, g_{st} and g_{xy} are their metrics and ϕ^x, ϕ^s some coordinates. The consistent coupling of the scalar fields to vectors and fermions, for a given amount of supersymmetry, requires restrictions on the scalar manifolds and additional structures defined on it. In particular, combined scalar-fermion and scalar-vector couplings with extended supergravity promote the isometry group of scalar manifolds to a global symmetry of the field equations of motion and Bianchi identities. Furthermore, the target spaces have to be *homogeneous* and *symmetric* in the case of theories with large enough supersymmetry, $N > 2$ [22]: namely, they are *coset spaces* of the form G/H with G *semisimple* and H its *maximal compact* subgroup. In $D = 4$, $N = 4$ supergravity scalar fields are described by

$$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(6, n)}{SO(6) \times SO(n)} \ . \tag{3.2.2}$$

It is convenient to formulate these sigma-models G/H in terms of a G-valued matrix representative $L(\phi)$ (ϕ's are coordinates of scalar manifold) and a *left-invariant* current

$$J = L^{-1} \mathrm{d}L \in \mathfrak{g} \ , \tag{3.2.3}$$

where $\mathfrak{g} \equiv \mathrm{Lie}(G)$. Under a left multiplication by $g \in G$, $L(\phi)$ is in general carried into another coset with representative element $L(\phi') = L(g * \phi)$ (where $g*$ denotes a non-linear action of g on ϕ):

$$gL(\phi) = L(\phi')\, h \ , \tag{3.2.4}$$

where $h \in H$ and ϕ' are functions of g and ϕ.

J can be decomposed according to

$$J = Q + P \ , \qquad Q \in \mathfrak{h} \ , \qquad P \in \mathfrak{k} \ , \tag{3.2.5}$$

where $\mathfrak{h} \equiv \mathrm{Lie}(H)$ and $\mathfrak{k}$ denotes its complement, the set of the coset generators, *i.e.* $\mathfrak{g} = \mathfrak{h} \perp \mathfrak{k}$. The generators algebra satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \ , \qquad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k} \ , \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h} \ , \tag{3.2.6}$$

where the first condition comes from the fact that H is a group and the third is the definition of symmetric coset spaces. The second is a consequence of an appropriate choice of the basis of generators.

Let us show how P and Q transform under the action of a global G transformation on the left and a *local* H compensation on the right:

$$J(\phi') = L^{-1}(\phi')\, \mathrm{d}L(\phi') = hJ(\phi)\, h^{-1} + h\mathrm{d}h^{-1} \ , \tag{3.2.7}$$

thus

$$P\left(\phi'\right) = hP\left(\phi\right)h^{-1}, \qquad Q\left(\phi'\right) = hQ\left(\phi\right)h^{-1} + h\mathrm{d}h^{-1} \ , \tag{3.2.8}$$

showing that Q is a composite connection which acts as a gauge field under H. The current P on the other hand transforms in the adjoint representation of H and it is a vielbein on G/H. Since Q is the H-*connection*, it has to be carefully inserted in the covariant derivative of fermionic fields, which transform linearly under the local H symmetry, and scalar field matrices L:

$$DL \equiv dL - LQ = LP \ . \tag{3.2.9}$$

The vielbein P can be used to construct H-invariant interaction terms of the action between scalars and fermions and builds the G-invariant kinetic term

$$\mathcal{L}_{\text{s.kin}} = -\frac{e}{2}\,\mathrm{Tr}\,P_\mu P^\mu \ , \tag{3.2.10}$$

where

$$P_\mu = J_\mu - Q_\mu = L^{-1}\partial_\mu L - Q_\mu \ . \tag{3.2.11}$$

The local H-symmetry is not a gauge symmetry associating with propagating gauge fields (Q is a composite connection), but simply takes care of the redundancy in parametrizing the coset space G/H:

$$\#\,\text{of scalars} = \dim\frac{G}{H} = \dim G - \dim H \ . \tag{3.2.12}$$

In order to construct the full supersymmetric action of the theory, it is most convenient to formulate the theory in terms of manifestly H-invariant objects:

$$M = L\Delta L^T \ , \tag{3.2.13}$$

where Δ is a constant H-invariant positive definite matrix (for both our coset spaces Δ is simply the identity matrix). Thus the Lagrangian (3.2.10) can be written as

$$\mathcal{L}_{\text{s.kin}} = \frac{1}{k}\,\mathrm{Tr}\,\partial_\mu M \partial^\mu M^{-1} \ , \tag{3.2.14}$$

with a proper normalization constant $1/k$.

3.2.1 SL(2)/SO(2) scalar sector

Let us now apply the general procedure to the scalars entering our theory. We start from the $SL(2)/SO(2)$, which is the simplest non-trivial coset space.

We can work in the fundamental representation of $SL(2)$ and choose the following basis of generators:

$$\mathfrak{sl}(2) = \{\sigma^1, i\sigma^2, \sigma^3\} = \{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\} \ . \tag{3.2.15}$$

The isotropy algebra is

$$\mathfrak{so}(2) = \{J\} = \{-i\sigma^2\} \ , \tag{3.2.16}$$

while its orthogonal complement is spanned by symmetric traceless matrices:

$$\mathfrak{k} = \{K_1, K_2\} = \{\sigma^1, \sigma^3\} \ . \tag{3.2.17}$$

Let us define the *solvable parametrization* by writing the decomposition $\mathfrak{sl}(2) = \mathfrak{so}(2) \oplus \mathcal{S}$, with $\mathcal{S}$ a solvable subalgebra which consists of the following upper-triangular generators:

$$\mathcal{S} = \{\sigma^3, \sigma^+\}, \qquad \sigma^+ = \sigma^1 + i\sigma^2 \ . \tag{3.2.18}$$

This defines the solvable parametrization $\phi^s = (\varphi, \chi)$, in which the coset representatives have the following form:

$$\mathcal{V} = e^{\chi\sigma^+} e^{\varphi\sigma^3} = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix} . \tag{3.2.19}$$

The vielbein and connection one-forms are

$$\mathcal{V}^{-1}\mathrm{d}\mathcal{V} = P^s K_s + QJ \ , \tag{3.2.20}$$

where

$$P^1 = \mathrm{d}\varphi \ , \qquad P^2 = \frac{e^{-2\varphi}}{2}\mathrm{d}\chi \ , \qquad Q = \frac{e^{-2\varphi}}{2}\mathrm{d}\chi \ , \tag{3.2.21}$$

and the metric reads

$$\mathrm{d}s^2 = 2\mathrm{d}\varphi^2 + \frac{1}{2}e^{-4\varphi}\mathrm{d}\chi^2 \ . \tag{3.2.22}$$

The matrix M for this model can be computed from (3.2.13) with $\Delta = \mathbb{I}_2$, and it is mostly compact expressed in terms of the complex scalar field

$$\tau = \chi + ie^{2\varphi} \ , \tag{3.2.23}$$

with $\operatorname{Im}\tau > 0$, giving rise to

$$M_{\alpha\beta} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} |\tau|^2 & \operatorname{Re}\tau \\ \operatorname{Re}\tau & 1 \end{pmatrix} , \tag{3.2.24}$$

where $\alpha, \beta = +, -$ are $SL(2)$-indices. The first term of kinetic Lagrangian (3.2.1) take the form

$$e^{-1}\mathcal{L}_{\text{s1.kin}} = \frac{1}{8}\partial_\mu M_{\alpha\beta}\partial^\mu M^{\alpha\beta} = -\frac{1}{4\operatorname{Im}\tau^2}\partial_\mu\tau^*\partial^\mu\tau = -\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}e^{-4\varphi}\partial_\mu\chi\partial^\mu\chi \ , \tag{3.2.25}$$

where $M^{\alpha\beta}$ is the inverse matrix of $M_{\alpha\beta}$:

$$M^{\alpha\beta} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} 1 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & |\tau|^2 \end{pmatrix} . \tag{3.2.26}$$

The $SL(2)$ symmetry action on $M_{\alpha\beta}$

$$M \to gMg^T \ , \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2) \tag{3.2.27}$$

acts on τ as a Möbius transformation $\tau \to \frac{a\tau+b}{c\tau+d}$.

When we will couple $SL(2)/SO(2)$ scalars with fermions, it will be crucial to introduce a $SL(2)$-covariant notation. Namely, it is necessary to introduce a complex $SL(2)$-vector $\mathcal{V}_\alpha$, which carries under local $U(1)$ a charge $+1$:

$$\mathcal{V}_\alpha = \begin{pmatrix} \psi \\ \phi \end{pmatrix} . \tag{3.2.28}$$

The redundant degree of freedom of $\mathcal{V}_\alpha$ is fixed by requiring the following equation to be valid:

$$M_{\alpha\beta} = \operatorname{Re}\left(\mathcal{V}_\alpha\mathcal{V}^*_\beta\right) \ . \tag{3.2.29}$$

Thus it easy to show the following relations are valid:

$$\mathrm{Im}\,\tau = \frac{1}{\phi\phi^*}\ , \qquad |\tau| = \frac{|\psi|}{|\phi|}\ , \qquad (\mathrm{Im}\,\psi^*\phi)^2 = |\psi|^2|\phi|^2 - (\mathrm{Re}\,\psi^*\phi)^2 = 1\ , \tag{3.2.30}$$

which is, by convention[1], $\mathrm{Im}\,\psi^*\phi = 1$, which is a $SL(2)$-invariant request. Furthermore, it is easy to prove that

$$\mathrm{Re}\,\tau = \frac{\psi^*}{\phi^*} - \frac{i}{\phi\phi^*} = \frac{\psi}{\phi} + \frac{i}{\phi\phi^*}\ . \tag{3.2.31}$$

The symmetric real positive defined matrix $M_{\alpha\beta}$ can be written as[2]

$$M_{\alpha\beta} = \mathcal{V}_\alpha \mathcal{V}^*_\beta + i\epsilon_{\alpha\beta}\ , \tag{3.2.32}$$

and its inverse is

$$M^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}M_{\gamma\delta}\ . \tag{3.2.33}$$

Consistently, we define an upper $SL(2)$-index as follows:

$$\mathcal{V}^\alpha = \epsilon^{\alpha\beta}\mathcal{V}_\beta\ . \tag{3.2.34}$$

It can be noted that $M_{\alpha\beta}$ acts on $\mathcal{V}_\alpha$ similarly to a metric:

$$M_{\alpha\beta}\mathcal{V}^\alpha = i\mathcal{V}_\beta\ . \tag{3.2.35}$$

The $U(1)$-connection and the vielbein are

$$Q_s = -\frac{i}{2}\epsilon^{\alpha\,\beta}\mathcal{V}_\alpha\partial_s\mathcal{V}^*_\beta\ , \qquad P_s = \frac{i}{2}\epsilon^{\alpha\,\beta}\mathcal{V}_\alpha\partial_s\mathcal{V}_\beta\ , \tag{3.2.36}$$

which are imaginary and complex quantities, respectively. Straightforwardly, covariant derivatives are

$$D_s\mathcal{V}_\alpha = (\partial_s - Q_s)\,\mathcal{V}_\alpha = P_s\mathcal{V}^*_\alpha\ , \qquad D_s\mathcal{V}^*_\alpha = (\partial_s + Q_s)\,\mathcal{V}^*_\alpha = P^*_s\mathcal{V}_\alpha\ , \tag{3.2.37}$$

and

$$D_sM_{\alpha\beta} = P_s\mathcal{V}^*_\alpha\mathcal{V}^*_\beta + P^*_s\mathcal{V}_\alpha\mathcal{V}_\beta\ . \tag{3.2.38}$$

The metric of the scalar manifold is

$$g_{st} = 2P^*_{(s}P_{t)}\ , \tag{3.2.39}$$

while the *Maurer-Cartan equations* become

$$D_{[s}P_{t]} = 0, \qquad P^*_{[s}P_{t]} = -\partial_{[s}Q_{t]}\ . \tag{3.2.40}$$

3.2.2 SO(6,n)/SO(6)×SO(n) scalar sector

We choose the metric of $SO(6,n)$ to be

$$\eta_{MN} = \eta_{\underline{M}\underline{N}} = \mathrm{diag}\,(-1,-1,-1,-1,-1,-1,1,\ldots,1)\ , \tag{3.2.41}$$

[1] This convention accords to the one adopted in [16] and [7], provided that in the former we substitute $\phi \to i\phi$.

[2] For the epsilon tensor $\epsilon_{\alpha\beta}$ we use $\epsilon_{+-} = \epsilon^{+-} = 1$ which yields $\epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta^\beta_\alpha$.

where $M, N = 1 \ldots 6+n$ and $\underline{M}, \underline{N} = 1 \ldots 6+n$ are $SO(6,n)$ and $SO(6) \times SO(n)$ indices, respectively, in the fundamental representations. We recall that $SU(4)$ is the covering group of $SO(6)$, thus at the level of the present work can be considered as the same, $SO(6) \approx SU(4)$.

The scalar coset is described by representatives L that transform under global $SO(6,n)$ from the left and local composite $SO(6) \times SO(n) \approx SU(4) \times SO(n)$ from the right:

$$L \to gLh, \qquad L_M{}^{\underline{M}} = (L_M{}^m, L_M{}^a) = \left(L_M{}^{[ij]}, L_M{}^a\right) , \tag{3.2.42}$$

where $g \in SO(6,n)$, $h = h(\phi^x) \in SO(6) \times SO(n)$ and indices are those introduced in section 3.1. Thus, $\underline{M}$ can be decomposed into the fundamentals $\mathbf{6}$ of $SO(6)$ and $\mathbf{n}$ of $SO(n)$, or equivalently into the $\mathbf{6}$ (the antisymmetric part of $\mathbf{4} \times \mathbf{4}$) of $SU(4)$ and $\mathbf{n}$ of $SO(n)$.

$L_M{}^{[ij]}$ is subject to the pseudo-reality constraint

$$L_{Mij} = \left(L_M{}^{ij}\right)^* = \frac{1}{2}\epsilon_{ijkl} L_M{}^{kl} , \tag{3.2.43}$$

and the normalization of the map $L_M{}^m \to L_M^{ij}$ is fixed such that

$$L_M{}^m L_{Nm} = -L_M{}^m L_N{}^m = L_M{}^{ij} L_{Nij} . \tag{3.2.44}$$

Using the symmetry properties of $L_M{}^{ij}$ matrix, it is straightforward to show that

$$L_{(M}{}^{ik} L_{N)jk} = \frac{1}{4}\delta^i_j L_{(M}{}^{kl} L_{N)kl} , \qquad \frac{1}{2}\epsilon_{mnkl} L^{ik[N} L^{jlP]} = -\delta^{(i}_{[m} L^{[N}_{n]l} L^{j)lP]} . \tag{3.2.45}$$

$L_M{}^{\underline{M}}$ is a $SO(6,n)$-matrix, thus its inverse can be defined as follows:

$$L^{-1} = \eta L^T \eta , \qquad \left(L^{-1}\right)_{\underline{M}}{}^M = \eta^{MN} \eta_{\underline{M}\underline{N}} L^{\underline{N}}{}_N , \tag{3.2.46}$$

that is

$$L_{\underline{N}}{}^M L_M{}^{\underline{M}} = \delta^{\underline{M}}_{\underline{N}} . \tag{3.2.47}$$

Following the normalization prescription (3.2.44), we find

$$L_n{}^M L_M{}^m = \delta^m_n \quad \Rightarrow \quad L_{ij}{}^M L_M{}^{kl} = -\delta^k_{[i}\delta^l_{j]} , \qquad L_a{}^M L_M{}^b = \delta^a_b . \tag{3.2.48}$$

In terms of $\left(L_M{}^{ij}, L_M{}^a\right)$, the orthogonality condition for SO(6,n) matrix $L_M{}^{\underline{M}}$ becomes

$$- L_M{}^{ij} L_{Nij} + L_M{}^a L_N{}^a = \eta_{MN} , \tag{3.2.49}$$

where η_{MN} is the constant tensor introduced in (3.2.41).

This coset space can be also parametrized by a symmetric positive defined matrix M_{MN}. For $SO(6,n)/SO(6) \times SO(n)$, the invariant matrix in (3.2.13) is $\Delta = \delta_{\underline{M}\underline{N}}$ and $M = LL^T$:

$$M_{MN} = +L_M{}^{ij} L_{Nij} + L_M{}^a L_M{}^a . \tag{3.2.50}$$

M_{MN} acts on $L_M{}^{\underline{M}}$ similarly to the metric η_{MN}:

$$M_{MN} L^{ijM} = -L_N{}^{ij} , \qquad M_{MN} L^{aM} = L_N{}^a . \tag{3.2.51}$$

We denote its inverse $M^{-1} = \eta M \eta$ by

$$M^{MN} = \eta^{MP}\eta^{NQ}M_{PQ} \ . \tag{3.2.52}$$

The vielbein on the coset space and the composite $SU(4) \times SO(n)$-connection are determined from

$$L^{-1}\partial_x L = P_x{}^{a[ij]}T_{a[ij]} + Q_x{}^{ab}T_{ab} + Q_x{}^i{}_j T_i{}^j \ , \tag{3.2.53}$$

where $\left(T_{ab}, T_i{}^j\right)$ are the generators of the Lie algebra $\mathfrak{su}(4) \times \mathfrak{so}(n)$ and $T_{a[ij]}$ denotes generators of the coset part of the Lie algebra $\mathfrak{so}(6,n)$. More precisely,

$$P_{x\underline{N}}{}^{\underline{M}} = L_{\underline{N}}{}^M D_x L_M{}^{\underline{M}} \ , \qquad P_{x\underline{N}}{}^{\underline{M}} = -P_x{}^{\underline{M}}{}_{\underline{N}} \ , \tag{3.2.54}$$

are the vielbein on $SO(6,n)/SO(6) \times SO(n)$ scalar manifold. The vielbeins are covariantly constant with respect to the full covariant derivative:

$$D_x P_y{}^{aij} = \partial_x P_y{}^{aij} - \Gamma^z_{xy} P_z{}^{aij} - Q_{xb}{}^a P_y{}^{bij} - Q_{xk}{}^i P_y{}^{akj} - Q_{xk}{}^j P_y{}^{aik} = 0 \ . \tag{3.2.55}$$

The metric of the scalar manifold is given by

$$g_{xy} = \frac{1}{2} P_{x\underline{N}}{}^{\underline{M}} P_{y\underline{M}}{}^{\underline{N}} = P_x{}^{aij} P_y{}^a{}_{ij} \ , \tag{3.2.56}$$

and the differential equations satisfied by the coset representatives are

$$D_x L = L P_x \ , \qquad D_x L_M{}^{\underline{M}} = L_M{}^{\underline{N}} P_{x\underline{N}}{}^{\underline{M}} \ , \tag{3.2.57}$$

$$D_x L^{-1} = -P_x L^{-1} \ , \qquad D_x L_{\underline{M}}{}^M = -L_{\underline{N}}{}^M P_{x\underline{M}}{}^{\underline{N}} \ , \tag{3.2.58}$$

which in components read

$$D_x L_M{}^a = P_x{}^{aij} L_{Mij} \ , \tag{3.2.59}$$

$$D_x L_M{}^{ij} = P_x{}^{aij} L_M{}^a \ , \tag{3.2.60}$$

$$D_x L_a{}^M = P_x{}^{aij} L_{ij}{}^M \ , \tag{3.2.61}$$

$$D_x L^{ijM} = P_x{}^{aij} L_a{}^M \ . \tag{3.2.62}$$

What follows from these equations is the relation

$$D_x M_{MN} = 2\left(L_M{}^a L_{Nij} + L_N{}^a L_{Mij}\right) P_x{}^{aij} \ . \tag{3.2.63}$$

The inverse of the vielbein is defined via

$$P_{x\underline{N}}{}^{\underline{M}} P^x{}_{\underline{Q}}{}^{\underline{P}} = \delta^{\underline{M}}_{\underline{Q}} \delta^{\underline{P}}_{\underline{N}} - \eta^{\underline{MP}} \eta_{\underline{NQ}} \ , \qquad P_x{}^{aij} P^{xb}{}_{kl} = \delta^{ab} \delta^{[i}_k \delta^{j]}_l \ . \tag{3.2.64}$$

The kinetic term for these scalars is

$$\mathcal{L}_{\text{s2.kin}} = \frac{1}{16} \partial_\mu M_{MN} \partial^\mu M^{MN} \ . \tag{3.2.65}$$

We also define a scalar dependent completely antisymmetric tensor

$$\begin{aligned} M_{MNPQRS} &= \epsilon_{mnopqr} L_M{}^m L_N{}^n L_P{}^o L_Q{}^p L_R{}^q L_S{}^r \\ &= -2i \epsilon_{ijps} \epsilon_{klqt} \epsilon_{mnru} L_{[M}{}^{ij} L_N{}^{kl} L_P{}^{mn} L_Q{}^{pq} L_R{}^{rs} L_{S]}{}^{tu} \ , \end{aligned} \tag{3.2.66}$$

which will be useful to express in a compact way the scalar potential, when gauging the theory (ambiguous indices are $SO(6)$ in first line and $SU(4)$ in the second).

3.3 Vector sector and 2-forms

In four dimensions there is a duality between 1-forms and 1-forms, such that these appear in pairs of which only the first half enters in the Lagrangian and carries propagating degrees of freedom, while the second half is defined as their on-shell duals [23].

In $N = 4$ $D = 4$ supergravity multiplets carry $6 + n$ vector fields which show up in the Lagrangian, but it is only together with their $6 + n$ duals that they are in the fundamental representation $(\mathbf{2}, \mathbf{6} + \mathbf{n})$ of $SL(2) \times SO(6, n) \subset Sp(12 + n, 12 + n)$. Let us define the *symplectic form* Ω preserved by $Sp(12 + n, 12 + n)$ introducing a composite index $\mathcal{M} = (\alpha, M)$

$$\Omega_{\mathcal{M}\mathcal{N}} = \epsilon_{\alpha\beta}\eta_{MN} \ , \qquad \Omega^{\mathcal{M}\mathcal{N}} = \epsilon^{\alpha\beta}\eta^{MN} \ . \tag{3.3.1}$$

The existence of this symplectic form is a general feature of four-dimensional gauge theories. Every decomposition $A_\mu^M = \left(A_\mu^\Lambda, A_{\mu\Lambda}\right)$ such that

$$\Omega_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \Omega^{\Lambda\Sigma} & \Omega^{\Lambda}{}_{\Sigma} \\ \Omega_{\Lambda}{}^{\Sigma} & \Omega_{\Lambda\Sigma} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \tag{3.3.2}$$

provides a consistent split into an equal number of electric A_μ^Λ and magnetic $A_{\mu\Lambda}$ vector fields. That is the ungauged theory can be formulated such that the electric fields A_μ^Λ appear in the Lagrangian while their dual magnetic fields $A_{\mu\Lambda}$ are only introduced on-shell. Such a decomposition is called a symplectic frame. Every two symplectic frames are related by a symplectic rotation.

For the vector fields of the theory one can choose a symplectic frame such that the subgroup $SO(1, 1) \times SO(6, n)$ is realized off-shell. The electric vector fields $A_\mu^{M+} \approx \left(A_\mu^m, A_\mu^a\right)$ form a vector under $SO(6, n)$ and carry charge $+1$ under $SO(1, 1)$. Their magnetic duals A_μ^{M-} carry a $SO(1, 1)$ charge -1. Together they constitute an $SL(2)$ vector $A_\mu^{M\alpha} = \left(A_\mu^{M+}, A_\mu^{M-}\right)$. The kinetic term of the vectors is [3]

$$e^{-1}\mathcal{L}_{\text{v.kin}} = -\frac{1}{4}\operatorname{Im}\tau M_{MN} F_{\mu\nu}^{M+} F^{\mu\nu N+} + \frac{1}{8}\operatorname{Re}\tau \eta_{MN}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{M+} F_{\rho\sigma}^{N+} \ , \tag{3.3.3}$$

where $F_{\mu\nu}^{M+} = 2\partial_{[\mu} A_{\nu]}^{M+}$ is the abelian field-strength of A_μ^{M+}.

As said above, only $F_{\mu\nu}^{M+}$ enters the Lagrangian, but $F_{\mu\nu}^{M-}$ appears in the equations of motion. To express the latter it is also useful to define the following combination of the electric field strength:

$$\begin{aligned} G_{\mu\nu}^{M+} &\equiv F_{\mu\nu}^{M+} \ , \\ G_{\mu\nu}^{M-} &\equiv e^{-1}\eta^{MN}\varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}_{\text{kin}}}{\partial F_{\rho\sigma}^{N+}} \\ &= -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\operatorname{Im}\tau M^{MN}\eta_{NP}F_{\mu\nu}^{P+} - \operatorname{Re}\tau F_{\mu\nu}^{M+} \ . \end{aligned} \tag{3.3.4}$$

The equations of motion for the electric vectors in the lowest order in the fields take the form $\partial_{[\mu} G_{\nu\rho]}^{M-} = 0$ (the full field equations receive higher-order terms in the fermions). Introducing

[3] Space-time metric has signature $(-, +, +, +)$ and Levi-Civita is a proper space-time tensor, *i.e.* $\varepsilon^{0123} = e^{-1}$, $\varepsilon_{0123} = -e$.

consistently by hand $F_{\mu\nu}^{M-} = G_{\mu\nu}^{M-}$ (these will be equations of motion for 2-forms in the gauged theory), $G_{\mu\nu}^{M\alpha}$ and $F_{\mu\nu}^{M\alpha}$ are on-shell identical and $SL(2) \times SO(6,n)$ is a global symmetry group of equation of motions.

In the ungauged theory, 2-forms are only introduced on-shell, according to our description, and they are dual to the scalars. We introduce them because they will be fundamental when we will gauge the theory. 2-forms transform in the adjoint representation of $SL(2) \times SO(6,n)$ and since the group has two factors there are also two kinds of 2-form fields, namely $B_{\mu\nu}^{[MN]}$ and $B_{\mu\nu}^{(\alpha\beta)}$. Indeed, due to the non-linear couplings of the scalar fields, discussed in 3.2, it turns out that field strengths of 2-forms $F_{\mu\nu\rho}^{[MN]}$ and $F_{\mu\nu\rho}^{(\alpha\beta)}$ are dual to Noether-currents associated with the symmetry generated by generators of the group, which can be written more compactly as

$$(t_{\alpha\beta})_\gamma{}^\delta = \delta^\delta_{(\alpha}\epsilon_{\beta)\gamma} \quad \text{and} \quad (t_{MN})_P{}^Q = \delta^Q_{[M}\eta_{N]P} \ , \tag{3.3.5}$$

respectively for the two factors:

$$F_{\mu\nu\rho}^{[MN]} = e\varepsilon_{\mu\nu\rho\sigma}j^{\sigma[MN]}, \qquad F_{\mu\nu\rho}^{(\alpha\beta)} = e\varepsilon_{\mu\nu\rho\sigma}j^{\sigma(\alpha\beta)} \ . \tag{3.3.6}$$

The apparent mismatch between the number ($\dim G - \dim H$) of physical scalars and the number of 2-forms is explained by the fact that not all the Noether-currents $j^{[MN]}$ and $j^{(\alpha\beta)}$ are independent: it follows from the structure of the coset space sigma-model that $L^{-1}\,(jt)\,L \in \mathfrak{k}$ for the currents associated with (3.2.10). This implies $\dim K$ linear constraints between the field strengths of 2-forms.

3.4 Fermionic sector

All the fermions carry a representation of $H = U(1) \times SU(4) \times SO(n)$, as we said above. The $U(1)$ acts on the fermions as a multiplication factor with a complex phase $\exp(iq\lambda(x))$, where the charges are

$$q_\psi = -\frac{1}{2} \ , \qquad q_\chi = +\frac{3}{2} \ , \qquad q_\lambda = +\frac{1}{2} \ . \tag{3.4.1}$$

Furthermore, all fermions are chiral and conventions about chirality are linked to our previous choice about the sign of $\mathrm{Im}\,\psi^*\phi$ (3.2.30):

$$\gamma_5\psi^i_\mu = +\psi^i_\mu \ , \qquad \gamma_5\chi^i = -\chi^i \ , \qquad \gamma_5\lambda^{ai} = +\lambda^{ai} \ . \tag{3.4.2}$$

The most commonly adopted convention about the normalization of kinetic terms and the one we choose is

$$e^{-1}\mathcal{L}_{\text{f.kin}} = -\bar\psi_{\mu i}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho - \frac{1}{2}\bar\chi_i\gamma^\mu D_\mu\chi^i - \bar\lambda_{ai}\gamma^\mu D_\mu\lambda^{ai} \ , \tag{3.4.3}$$

where

$$D_\mu\lambda^i_a = \nabla_\mu\lambda^i_a + Q_{\mu a}{}^b\lambda^i_b + Q_{\mu j}{}^i\lambda^j_a - \frac{1}{2}Q_\mu\lambda^i_a \ , \tag{3.4.4}$$

$$D_\mu\psi^i_\nu = \nabla_\mu\psi^i_\nu + Q_{\mu j}{}^i\psi^j_\nu + \frac{1}{2}Q_\mu\psi^i_\nu \ , \tag{3.4.5}$$

$$D_\mu\chi^i = \nabla_\mu\chi^i + Q_{\mu j}{}^i\chi^j - \frac{3}{2}Q_\mu\chi^i \ , \tag{3.4.6}$$

with ∇_μ being the Lorentz and space-time covariant derivative and $Q_{\mu a}{}^b = Q_{xa}{}^b\partial_\mu\phi^x$, $Q_{\mu i}{}^j = Q_{xi}{}^j\partial_\mu\phi^x$ and $Q_\mu = Q_s\partial_\mu\phi^s$.

3.5 The ungauged Lagrangian and supersymmetry transformation laws

The complete ungauged Lagrangian, up to four-fermions terms, can be found in the paper [16], imposing $f_{MNP} = 0$. Most of the conventions adopted in this work match perfectly with those used by Bergshoeff, Koh and Sezgin.

We are going to concentrate only on those terms which are relevant for our analysis.

$$\mathcal{L}^{D=4}_{N=4,\text{ungauged}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{ferm.scal}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{4-\text{ferm}} \ , \tag{3.5.1}$$

in particular,

$$e^{-1}\mathcal{L}_{\text{kin}} = \frac{1}{2}R + \mathcal{L}_{\text{s1.kin}} + \mathcal{L}_{\text{s2.kin}} + \mathcal{L}_{\text{f.kin}} + \mathcal{L}_{\text{v.kin}} \ , \tag{3.5.2}$$

where R is the *Ricci scalar* for the metric $g_{\mu\nu}$.

$\mathcal{L}_{\text{Pauli}}$ contains terms of the type $F\bar{f}_1\gamma f_2 L$, where F is the abelian field strength, L a scalar manifold representative, f_{i} are generic fermions and γ is a generalized gamma matrix. $\mathcal{L}_{4-\text{ferm}}$ contains interaction terms of four fermions. Both these sectors of the Lagrangian are not relevant for our analysis.

$$e^{-1}\mathcal{L}_{\text{ferm.scal}} = s_1\bar{\chi}_i\Gamma^\mu\Gamma^\nu{\psi_\mu}^i P_\nu + s_2\bar{\lambda}^a_i\Gamma^\mu\Gamma^\nu\psi_{\mu j}{P_\nu}^{aij} + \text{h.c.} \ , \tag{3.5.3}$$

and we have to set numerical coefficients s_1 and s_2.

Once fixed the normalization of kinetic terms of all the fields, coefficients of supersymmetry variations follow trivially, by requiring that the variation of the Lagrangian vanishes up to bound terms. Supersymmetric variations are

$$\delta e^{\mathfrak{a}}_\mu = \bar{\epsilon}^i\gamma^{\mathfrak{a}}\psi_{\mu i} + \text{h.c.} \ , \tag{3.5.4}$$

$$\begin{aligned}\delta\psi^i_\mu &= 2D_\mu\epsilon^i + \frac{1}{4}i\mathcal{V}^*_\alpha {L_M}^{ij} G^{M\alpha}_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\epsilon_j + \text{fermion bilinears} \\ &= 2D_\mu\epsilon^i - \frac{1}{2\phi}{L_M}^{ij}F^{M+}_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\epsilon_j + \text{fermion bilinears} \ ,\end{aligned} \tag{3.5.5}$$

$$\delta A^{\mu M\alpha} = 2\epsilon^{\alpha\beta}\mathcal{V}_\beta {L_{ij}}^M{\bar{\psi}_\mu}{}^i\epsilon^j - \epsilon^{\alpha\beta}\mathcal{V}_\beta L^{ijM}\bar{\chi}_i\Gamma^\mu\epsilon_j + i\epsilon^{\alpha\beta}\mathcal{V}_\beta {L_a}^M\bar{\lambda}^a_i\Gamma^\mu\epsilon^i + \text{h.c.} \ , \tag{3.5.6}$$

$$\begin{aligned}\delta\chi^i &= 2P_\mu\gamma^\mu\epsilon^i + \frac{1}{2}i\mathcal{V}_\alpha {L_M}^{ij}G^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon_j + \text{fermion bilinears} \ , \\ &= 2P_\mu\gamma^\mu\epsilon^i + \frac{1}{\phi^*}{L_M}^{ij}F^{M+}_{\mu\nu}\gamma^{\mu\nu}\epsilon_j + \text{fermion bilinears} \ ,\end{aligned} \tag{3.5.7}$$

$$\begin{aligned}\delta\lambda^i_a &= 2i{P_{\mu a}}^{ij}\gamma^\mu\epsilon_j - \frac{1}{4}\mathcal{V}_\alpha L_{Ma}G^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon^i + \text{fermion bilinears} \\ &= 2i{P_{\mu a}}^{ij}\gamma^\mu\epsilon_j + \frac{i}{2\phi^*}L_{Ma}F^{M+}_{\mu\nu}\gamma^{\mu\nu}\epsilon^i + \text{fermion bilinears} \ ,\end{aligned} \tag{3.5.8}$$

$$\delta\mathcal{V}_\alpha = \mathcal{V}^*_\alpha\bar{\epsilon}_i\chi^i \ , \tag{3.5.9}$$

$$\delta {L_M}^a = -2i{L_M}^{ij}\bar{\epsilon}_i\lambda^a_j + \text{h.c.} \ , \tag{3.5.10}$$

$$\delta {L_M}^{ij} = 2i\left(\bar{\epsilon}^{[i}\lambda^{j]}_a - \frac{1}{2}\epsilon^{ijkl}\bar{\epsilon}_k\lambda_{al}\right){L_M}^a \ . \tag{3.5.11}$$

In fermion variations, we used the definitions (3.3.4), properties of scalar representatives shown in sections 3.2.1 and 3.2.2, and the gamma matrix identity

$$\frac{1}{2} i \varepsilon_{\mu\nu\rho\lambda} \gamma^{\mu\nu} = -\gamma_{\rho\lambda} \gamma^5 \ . \tag{3.5.12}$$

It will be useful to consider the supersymmetry variation of the scalars $L_M{}^{\underline{M}}$ in terms of coordinate-fields ϕ^x:

$$\delta\phi^x = \frac{1}{2} g^{xy} P_{y\underline{M}}{}^{\underline{N}} L_{\underline{N}}{}^M \delta L_M{}^{\underline{M}} = -2i P^{xaij} \bar{\epsilon}_i \lambda^a_j \ . \tag{3.5.13}$$

The coefficients s_1 and s_2 of scalar-fermion couplings were fixed by imposing that terms of the forms $\bar{\chi} \not{P} \not{D} \epsilon$ and $\bar{\lambda} \not{P} \not{D} \epsilon$ vanish in the supersymmetric variation of $\mathcal{L}^{D=4}_{N=4,\mathrm{ungauged}}$: these terms are generated by varying ψ^i_μ field in $\mathcal{L}_{\mathrm{ferm.scal}}$ and χ^i, λ^i_a fields in $\mathcal{L}_{\mathrm{f.kin}}$. The result is[4]

$$s_1 = +\frac{1}{2} \ , \qquad s_2 = +i \ . \tag{3.5.14}$$

[4] As we said, ungauged Lagrangian can be found in [16], but we have shown how to fix the coefficients s_1 and s_2 because they do not coincide with those of the paper.

Chapter 4

Gauged N=4 supergravities

In this chapter gauged versions of $N = 4$ supergravity in four dimensions are presented. Firstly, we give an overview on the standard procedure to gauge supegravity theories, which is valid in general for arbitrary dimensions and number of supercharges [24], always keeping in mind the particular case $D = 4$ and $N = 4$. Then, in section two, we give the explicit form of *linear* and *quadratic constraints* and the bosonic sector of $N = 4$ $D = 4$ gauged Lagrangian is presented [7]. In the third section, mass matrices of fermions are constructed requiring the closure of supersymmetry algebra for various terms. The last sections introduces some consistency relations quadratic in fermion mass matrices with the scalar potential and its first derivatives.

4.1 Gauging supergravities: general overview

In order to avoid unusual indices, we switch convenctions for the present section. We denote the global symmetry group of the supergravity theory as G and we will promote $G_0 \subset G$ to a *local symmetry.* We will always stay G-covariant and denote with $M, N, \ldots$ and $\alpha, \beta, \ldots$ the indices of the fundamental and adjoint representations of G, respectively. It is quite surprising to note that in even dimensions G is realized only on-shell and this construction can accommodate gaugings of a subgroup $G_0 \subset G$ that are not among the off-shell symetries of the ungauged Lagrangian.

Under the action of the non-abelian global symmetry group G, the bosonic fields of ungauged supergravity transform as

$$\delta L_M{}^{\underline{M}} = \Lambda^\alpha \left(t_\alpha\right)_M{}^N L_N{}^{\underline{M}} \ , \tag{4.1.1}$$

$$\delta A_\mu^M = -\Lambda^\alpha \left(t_\alpha\right)_N{}^M A_\mu^N \ , \tag{4.1.2}$$

with Λ^α, $\alpha = 1, \ldots \dim G$, constant parameters. In addition, the ungauged theory shows a standard abelian gauge symmetry $U(1)^{n_{\mathbf{v}}}$, where $n_{\mathbf{v}}$ is the dimension of the representation $\mathcal{R}_{\mathbf{v}}$ of G, according to which vectors transform like:

$$\delta A_\mu^M = \partial_\mu \Lambda^M (x) \ . \tag{4.1.3}$$

We want to promote $G_0 \subset G$ to a local symmetry, therefore, we have to select a subset of the generators algebra $\mathfrak{g} = \mathrm{Lie}G$. We denote these generators by X_M and introduce the standard *covariant derivative*

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu - g A_\mu^M X_M \ , \tag{4.1.4}$$

where g is the gauge coupling constant. A general set of $n_{\mathbf{v}}$ generators in $\mathfrak{g}$ can be described as

$$X_M \equiv \Theta_M{}^\alpha t_\alpha \in \mathfrak{g} \ , \tag{4.1.5}$$

by means of a constant tensor $\Theta_M{}^\alpha$, the so-called *embedding tensor*, which completely encodes the embedding of the gauge group G_0 into the global G and may be totally characterized group-theoretically. The dimension of the gauge group is given by the rank of the matrix $\Theta_M{}^\alpha$, which for the moment is simply a constant ($n_{\mathbf{v}} \times$ dimG) matrix. Deformed equations of motion remain manifestly G-covariant, provided that $\Theta_M{}^\alpha$ transforms under G according to the structure of its indices. The embedding tensor will always appear together with the coupling constant g, we have introduced above.

At this point, the theory should be invariant under the standard combined transformations

$$\delta L_{\underline{M}}{}^{\underline{M}} = g \Lambda^M(x)\, X_{MN}{}^P L_P{}^{\underline{M}} \ , \tag{4.1.6}$$

$$\delta A_\mu^M = \partial \Lambda^M(x) + g A_\mu^N X_{NP}{}^M \Lambda^P(x) = D_\mu \Lambda^M(x) \ , \tag{4.1.7}$$

where $X_{MN}{}^P \equiv \Theta_M{}^\alpha (t_\alpha)_N{}^P$. But if we keep $\Theta_M{}^\alpha$ arbitrary, this does not happen. Indeed, consistency requires that the generators (4.1.5) close into a subalgebra of $\mathfrak{g}$. This translates into a set of non-trivial *constraints* on the embedding tensor, which are *quadratic* in $\Theta_M{}^\alpha$. Furthermore, we will see how generators have to satisfy also some *linear constraints*. Every solution of this set of constraints will give rise to a consistent supersymmetric Lagrangian with a local gauge symmetry (4.1.6).

The set of quadratic constraints is very generic, since they do not depend on the dimensions of space-time and number of supercharges of the theory. They state that the tensor $\Theta_M{}^\alpha$ is invariant under the action of generators (4.1.5) of the local gauge symmetry. Due to the index structure, which are in two different representations, $\Theta_M{}^\alpha$ is almost never G-invariant. But consistency of the gauged theory requires that $\Theta_M{}^\alpha$ has to be invariant under the action of the subgroup G_0. Thus, we need to require that

$$\delta_P \Theta_M{}^\alpha = \Theta_P{}^\beta (t_\beta)_M{}^N \Theta_N{}^\alpha + \Theta_P{}^\beta f_{\beta\gamma}{}^\alpha \Theta_M{}^\gamma = 0 \ , \tag{4.1.8}$$

where $f_{\alpha\beta}{}^\gamma$ are structure constants of G. Contracting this result with a generator t_α of G, we obtain the equivalent form

$$[X_M, X_N] = -X_{MN}{}^P X_P \ , \tag{4.1.9}$$

hence gauge invariance of the embedding tensor implies the closure of the generators (4.1.5) into an algebra. However, constraint (4.1.8) is stronger than the algebra closure: indeed, equations (4.1.9) implies a relation upon symmetrization in indices M and N (the left hand side trivially vanishes, while the right hand side does not), which goes beyond the simple closure.

As we said, $\Theta_M{}^\alpha$ have to satisfy another linear constraint, which is implied by the closure of supersymmetry. Contrary to quadratics, linear constraints depend strongly on the number of space-time dimensions and supercharges considered. In half-maximal theories in four-dimensions, vector fields transform in the bi-fundamental $(\mathbf{2},\Box)$ of $SL(2) \times SO(6,n)$ as shown in chapter 3. The embedding tensor a priori transforms in the tensor product of fundamental and adjoint representation, which decomposes according to

$$(\mathbf{2},\Box) \otimes \left((\mathbf{3},\bullet) \oplus \left(\mathbf{1},\substack{\Box\\\Box}\right)\right) = 2 \cdot (\mathbf{2},\Box) \oplus \left(\mathbf{2},\substack{\Box\\\Box\\\Box}\right) \oplus \left(\mathbf{2},\substack{\Box\Box\\\Box}\right) \oplus (\mathbf{4},\Box) \ . \tag{4.1.10}$$

Supersymmetry restricts the embedding tensor to $(\mathbf{2},\Box) \oplus \left(\mathbf{2},\substack{\Box\\\Box\\\Box}\right)$, *i.e.* forbits the last two contributions in (4.1.10) and poses a linear constraint among the two terms in the $(\mathbf{2},\Box)$ representation [25].

In general, the linear representation constraints schematically take the form

$$\mathbb{P}\Theta = 0 \ . \tag{4.1.11}$$

4.1.1 Covariant field strengths and linear constraints in D=4

Apart from the minimal coupling induced by (4.1.4), the field strengths of the vectors need to be modified in order to capture the non-abelian nature of G_0 and write down covariant couplings to the fermion fields.

The standard ansatz for the non-abelian field strengths is

$$\mathcal{F}^M_{\mu\nu} = 2\partial_{[\mu} A^M_{\nu]} + g X_{[NP]}{}^M A^N_\mu A^P_\nu \ , \tag{4.1.12}$$

but we will show that it is not sufficient. The *structure constants* X_M (4.1.5) can be written as

$$X_{MN}{}^P = X_{[MN]}{}^P + Z^P{}_{MN} \ , \tag{4.1.13}$$

with $Z^P{}_{MN} = X_{(MN)}{}^P$, which is in general non-vanishing. Quadratic constraints (4.1.9) reduce to

$$Z^P{}_{MN} X_P = 0 \ , \tag{4.1.14}$$

and proper structure constants $X_{[MN]}{}^P$ fails to satisfy the *Jacobi identities*:

$$X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R = -Z^R{}_{P[Q} X_{MN]}{}^P \ , \tag{4.1.15}$$

which can be satisfied upon contraction with a generator X_R, as a consequence of (4.1.14). Equation (4.1.15) implies that standard $\mathcal{F}^M_{\mu\nu}$ (4.1.12) turns out to be not fully covariant:

$$\delta\mathcal{F}^M_{\mu\nu} = -g\Lambda^P X_{PN}{}^M \mathcal{F}^N_{\mu\nu} + 2gZ^M{}_{PQ}\left(\Lambda^P \mathcal{F}^Q_{\mu\nu} - A^P_{[\mu}\delta A^Q_{\nu]}\right) \ , \tag{4.1.16}$$

of which only the first term would correspond to a standard homogeneous covariant transformation. We can notice that unwanted terms are proportional to $Z^M{}_{NP}$, thus the combination $\mathcal{F}^M_{\mu\nu} X_M$ is fully covariant. But conceivable covariant kinetic terms constructed from this object such as $\mathrm{Tr}\left[\mathcal{F}^M_{\mu\nu} X_M \mathcal{F}^{\mu\nu N} X_N\right]$ can not be smooth deformations of the standard kinetic term in ungauged theories, *e.g.* in $D > 4$ space-time dimensions:

$$\mathcal{L}^{\text{std}}_{\text{v.kin}} = -\frac{1}{4} e M_{MN} F^M_{\mu\nu} F^{\mu\nu N} \ , \tag{4.1.17}$$

where $F^M_{\mu\nu}$ is the abelian field strength and M_{MN} is the scalar matrix (3.2.13).

This problem is linked to the choice of staying G-covariant, with a redundant description in terms of $n_{\mathbf{v}}$ vectors while G_0 has dimensions smaller than $n_{\mathbf{v}}$. Indeed, A_μ^M can be splitted into A_μ^m, transforming in the adjont of G_0, and A_μ^i, transforming in some representation of G_0: with respect to this splitting

$$Z^m{}_{PQ} = 0 \quad \text{and} \quad Z^i{}_{PQ} \neq 0 \ , \tag{4.1.18}$$

creating the problem manifest in (4.1.16).

The covariant ansatz intertwines 1-forms and 2-forms $B_{\mu\nu}^{MN} = B_{\mu\nu}^{(MN)}$ and defines the full covariant field strengths as [26] [27]

$$\mathcal{H}_{\mu\nu}^M = \mathcal{F}_{\mu\nu}^M + gZ^M{}_{PQ} B_{\mu\nu}^{PQ} \ . \tag{4.1.19}$$

The non-covariant term in (4.1.16) can then be absorbed by postulating the following transformation laws

$$\delta A_\mu^M = D_\mu \Lambda^M - gZ^M{}_{PQ} \Xi_\mu^{PQ} \ , \tag{4.1.20}$$

$$\delta B_{\mu\nu}^{MN} = 2D_{[\mu} \Xi_{\nu]}^{MN} - 2\Lambda^{(M} \mathcal{H}_{\mu\nu}^{N)} + 2A_{[\mu}^{(M} \delta A_{\nu]}^{N)} \ , \tag{4.1.21}$$

where Ξ_μ^{MN} labels the tensor gauge transformations associated with the 2-forms. These gauge 2-forms $B_{\mu\nu}^{MN}$ can not be added random, since the number of bosonic and fermionic degrees of freedom is balanced by supersymetry. Rather, these have to be (a subset of) 2-forms that are already present in the unguaged supergravity (which are dual to the scalars). The index structure of $B_{\mu\nu}^{(MN)}$ shows that they generically appear in some representation of G which is contained in the product $(\mathcal{R}_{\mathbf{v}} \otimes \mathcal{R}_{\mathbf{v}})_{\text{sym}}$. In turns, this constrains the tensor $Z^P{}_{MN}$, which in its indices (MN) should project only onto those representations filled by the 2-forms in the ungauged theory. As $Z^P{}_{MN}$ is a function of the embedding tensor $\Theta_M{}^\alpha$,this leads to a linear constraint of the type (4.1.11) on $\Theta_M{}^\alpha$.

Let us restrict to the case of $D = 4$ dimensions and remember that 2-forms are dual to scalars and transform in the adjoint representation, as shown above in section 3.3:

$$(\mathcal{R}_{\mathbf{v}} \otimes \mathcal{R}_{\mathbf{v}})_{\text{sym}} = \mathcal{R}_{\mathbf{adj}} \oplus \dots \ , \tag{4.1.22}$$

and the remaining representations are excluded. Actually, in four dimensions $G \subset Sp(m,m)$,, *i.e.*

$$(t_\alpha)_{[M}{}^P \, \Omega_{PN]} \equiv (t_\alpha)_{[MN]} = 0 \ . \tag{4.1.23}$$

where Ω_{MN} is the symplectic matrix of $Sp(m,m)$ which raise and lower $M, N, \dots$ indices, as $\Theta^{M\alpha} = \Omega^{MN} \Theta_N{}^\alpha$, ecc. Multiplying the previous equation by the embedding tensor $\Theta_M{}^\alpha$, we obtain the first linear constraint on the generators $X_{MN}{}^P$:

$$X_{M[N}{}^Q \Omega_{P]Q} = 0 \ , \tag{4.1.24}$$

which tells that symplectic form Ω_{MN} is invariant under gauge transformation and a gauge transformation can not change the previously chosen symplectic frame (the choice of electric and magnetic 1-forms). Moreover

$$\begin{aligned} Z^P{}_{MN} = X_{(MN)}{}^P &= \frac{1}{2} \Theta_M{}^\alpha \, (t_\alpha)_N{}^P + \frac{1}{2} \Theta_N{}^\alpha \, (t_\alpha)_M{}^P \\ &= -\frac{1}{2} \Theta^{P\alpha} \, (t_\alpha)_{MN} + \frac{3}{2} X_{(MNL)} \Omega^{PL} \ . \end{aligned} \tag{4.1.25}$$

If the embedding tensor satisfies the linear constraint

$$X_{(MNL)} = 0 \ , \tag{4.1.26}$$

the 2-forms appear always under the projection

$$Z^P{}_{MN} B^{MN}_{\mu\nu} = -\frac{1}{2}\Theta^{P\alpha} B_{\mu\nu\alpha} \quad \text{with} \quad B_{\mu\nu\alpha} = (t_\alpha)_{MN}\, B^{MN}_{\mu\nu} \ . \tag{4.1.27}$$

Thus, condition (4.1.26) is a sufficient (and also necessary) condition in onder to preserve supersymmetric degrees of freedom. This result was firstly found in [28].

It will be important to mention for future purpose that the new full covariant field strength $\mathcal{H}^M_{\mu\nu}$ (4.1.19) does not satisfy the standard Bianchi identities, but rather its deformed version

$$D_{[\mu}\mathcal{H}^M_{\nu\rho]} = \frac{1}{3} g Z^M{}_{NP}\mathcal{H}^{NP}_{\mu\nu\rho} \ , \tag{4.1.28}$$

where $\mathcal{H}^{MN}_{\mu\nu\rho}$ denotes the properly covariantized field strength of the 2-forms.

4.1.2 Lagrangians of gauged theories

Now we describe how to obtain a Lagrangian that is compatible with the new local symmetry (4.1.20) as a deformation of the ungauged Lagrangian. The first trivial step is to covariantize all derivatives and substitute abelian field strengths with the new non-abelian full covariant ones.

Since the construction of the deformation is manifestly G-covariant, the gauged theories in even dimensions generically carry the full G-representation of forms, rather then only electric half. Let us always keep in mind the case of 1-forms in four dimensions. Magnetic fields do not possess kinetic terms, but only appear in covariant derivatives, in new field strengths and in new *topological terms* which are required by gauge invariance: *e.g.*

$$D_\mu \equiv \partial_\mu - g A^M_\mu \Theta_M{}^\alpha t_\alpha = \partial_\mu - g A^\Lambda_\mu \Theta_\Lambda{}^\alpha - g A_{\mu\Lambda}\Theta^{\Lambda\alpha} \ , \tag{4.1.29}$$

where we have split the $2m$-dimensional irreducible linear representation $A^M_\mu = \left(A^\Lambda_\mu, A_{\mu\Lambda}\right)$ of the symmetry group G, with A^Λ_μ the electric half which appears off-shell in the ungauged theory and $A_{\mu\Lambda}$ the magnetic-half which appears only on-shell. In principle, this might lead to inconsistent additional field equations. However, it turns out that various contributions from kinetic and topological terms precisely combine into *first-order* field equations for the additional fields. Then, these fields do not constitute additional degrees of freedom but are on-shell duals of the fields of the ungauged theory. This highlights the importance of the covariantization of topological terms. In $D = 4$, it follows from the variation of $B_{\mu\nu\alpha}$ that

$$g\Theta^{\Lambda\alpha}\left(\mathcal{H}_{\mu\nu\Lambda} + e\varepsilon_{\mu\nu\rho\sigma}\frac{\partial L_{\text{kin}}}{\partial\mathcal{H}^\Lambda_{\rho\sigma}}\right) \ , \tag{4.1.30}$$

which reproduce the covariant version of duality equations for 1-forms. Like-wise, variation with respect to magnetic vector fields induces the duality equations between scalars and 2-forms (3.3.6). Therefore, in gauged theories, duality equations arise as true field equations of motion projected with the matrix $\Theta^{\Lambda\alpha}$. In particular, in the limit $g \to 0$, all dual fields disappear from the action and duality equations decouple.

At this point the deformed Lagrangian

$$\mathcal{L}' = \mathcal{L}_{\text{ung}} \left[\partial \to D, F \to \mathcal{H}\right] + \mathcal{L}_{\text{top}} \tag{4.1.31}$$

is no longer supersymmetric, due to extra terms which arise from the variation of vector fields in the covariant derivatives and from modified Bianchi identities (4.1.28) in the new Lagrangian. Anyway, it can be shown that, in linear order of the gauge constant g, these extra terms can be compensated by the introduction of fermionic masses of the type:

$$e^{-1}\mathcal{L}_{\text{f.mass}} = gA_{ij}\bar{\psi}^i_\mu \gamma^{\mu\nu}\psi^j_\nu + gB_{Ai}\bar{\chi}^A\gamma^\mu\psi^i_\mu + gC_{AB}\bar{\chi}^A\chi^B + \text{h.c.} \ , \tag{4.1.32}$$

where ψ^i_μ and χ^A denote spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$, respectively, with $i, j, \ldots$ and $A, B, \ldots$ labelling respective representations of H, maximal compact subgroup of G. A_{ij}, B_{Ai} and C_{AB} have to be linear in the embedding tensor and depend on scalar fields, whose coset space representatives are the only objects relating representations of G and H, since transform according to (3.2.4). More precisely they are composed out of irreducible components of the T-*tensor*, which is the embedding tensor dressed with the scalar group matrix L, evaluated in the fundamental and in the adjoint representations of G:

$$T_{\underline{N}}{}^{\underline{\beta}} = \Theta_M{}^\alpha L_{\underline{N}}{}^M L_\alpha{}^{\underline{\beta}} \ . \tag{4.1.33}$$

The T-tensor inherits all the information about Θ, but it is scalar dependent and transforms under H. Every G-irreducible component of Θ branches into one or more H-irreducible component of T, which can be used to build up the fermionic mass tensors A_{ij}, B_{Ai} and C_{AB}.

The T-tensor has to satisfy the linear constraint

$$\mathbb{P}T = 0 \ , \tag{4.1.34}$$

which now holds for any value of the scalar fields on which T depends. Turning this argument around, this shows the origin of linear representation constraints from supersymmetry. The supersymmetry violating term at order g can be cancelled by the variation of additional fermionic mass terms (4.1.32) if and only if the tensor $T_{\underline{M}}{}^{\underline{\beta}}$ can be built from the representations of proper fermionic mass tersors. It can be proved that, in $D = 4$, supersymmetry imposes precisely the same linear constraints found from purely bosonic considerations, such that no further restriction descends from compatibility with supersymmetry.

In order to cancel completely supersymmetry-violating terms in linear order of g, such as $D_\mu\epsilon_i$ contributions descending from (4.1.32) by varying gravitinos, we have to introduce *fermion-shifts* in fermionic supersymmetry transformations (schematically):

$$\delta\psi^i_\mu = \delta_0\psi^i_\mu + gA^{ij}\gamma_\mu\epsilon_j \ , \tag{4.1.35}$$

$$\delta\chi^A = \delta_0\chi^A - gB^{Ai}\epsilon_i \ , \tag{4.1.36}$$

where δ_0 denotes the properly covariantized supersymmetry transformations of the ungauged theory.

Finally, supersymmetry in second order g^2 of the deformation requires the appearance of a new scalar potential

$$\mathcal{L}_{\text{pot}} = -g^2 eV \ , \tag{4.1.37}$$

in order to cancel g^2 contributions descending from supersymmetric variations (4.1.35) and (4.1.36) of the Lagrangian sector (4.1.32). Consistent cancellation of all supersymmetry variations at order g^2 requires a number of non-trivial algebraic identities to be satisfied by fermionic mass tensors A_{ij}, B_{Ai} and C_{AB}. In particular, one needs the traceless condition (schematically)

$$B^{Ai}B_{Aj} - A^{ik}A_{jk} \approx \frac{1}{N}\delta^i_j V \ , \tag{4.1.38}$$

with N the number of supersymmetries and the scalar potential V, which is the so-called *supersymmetric Ward identity*. This condition is quadratic in the embedding tensor, hence the only way it can be satisfied without imposing additional constraints on the gauging is as a consequence of the quadratic constraints (4.1.9). Indeed, in general, these consistency relations can be obtained by (4.1.9) upon dressing the latter with the scalar matrices L and breaking it into its H-irreducible parts.

4.2 Bosonic Lagrangian and field equations

Let us switch back to our previous $D = 4$, $N = 4$ indices conventions. The pure bosonic Lagrangian were constructed in [7]. Now we give a general review of the results of this article. As shown in section 4.1.1 the gauge group generators in the vector field representation, with the composite index $\mathcal{M} = (M, \alpha)$ introduced in (3.3.1), have to satisfy equations (4.1.24) and (4.1.26), which read

$$X_{\mathcal{M}[\mathcal{N}}{}^{\mathcal{Q}}\Omega_{\mathcal{P}]\mathcal{Q}} = 0 \ , \qquad X_{(\mathcal{M}\mathcal{N}}{}^{\mathcal{Q}}\Omega_{\mathcal{P})\mathcal{Q}} = 0 \ . \tag{4.2.1}$$

In order to be solution of these constraints, generators $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = X_{M\alpha N\beta}{}^{P\gamma}$ can be written as

$$X_{M\alpha N\beta}{}^{P\gamma} = -\delta^\gamma_\beta f_{\alpha MN}{}^P + \frac{1}{2}\left(\delta^P_M\delta^\gamma_\beta\xi_{\alpha N} - \delta^P_N\delta^\gamma_\alpha\xi_{\beta M} - \delta^\gamma_\beta\eta_{MN}\xi_\alpha{}^P + \epsilon_{\alpha\beta}\delta^P_N\xi_{\delta N}\epsilon^{\delta\gamma}\right) \ , \tag{4.2.2}$$

where $\xi_{\alpha M}$ and $f_{\alpha MNP}$ are the irreducible components of the embedding tensor which survive the linear constraints, as said above after decomposing the $\Theta_M{}^\alpha$ in its irreducible parts (4.1.10). $\xi_{\alpha M}$ and $f_{\alpha MNP} = f_{\alpha[MNP]}$ have to be treated as objects which transform in the bi-fundamental of $SL(2) \times SO(6, n)$ and in the fundamental of $SL(2)$ and three-fold antisymmetric vector representation of $SO(6, n)$, respectively.

$SL(2)$ and $SO(6, n)$ generators can be read in (3.3.5) and the *covariant derivative* takes the form

$$D_\mu \equiv \nabla_\mu - gA^{M\alpha}_\mu\Theta_{M\alpha}{}^{NP}t_{NP} - gA^{M\alpha}_\mu\Theta_{M\alpha}{}^{\beta\gamma}t_{\beta\gamma} \ , \tag{4.2.3}$$

where ∇_μ is the Lorentz and space-time covariant derivative. The two components $\Theta_{M\alpha}{}^{NP}$ and $\Theta_{M\alpha}{}^{\beta\gamma}$ further decompose into the irreducible representations of G_0 just founded.

According to the decomposition

$$X_{M\alpha N\beta}{}^{P\gamma} = \Theta_{M\alpha}{}^{QR}\,(t_{QR})_N{}^P\,\delta^\gamma_\beta + \Theta_{M\alpha}{}^{\delta\epsilon}\,(t_{\delta\epsilon})_\beta{}^\gamma\,\delta^P_N \ , \tag{4.2.4}$$

$f_{\alpha MNP}$ and $\xi_{\alpha M}$ tensors constitute the two components of the embedding tensor as follows

$$\Theta_{M\alpha}{}^{NP} = f_{\alpha M}{}^{NP} + \frac{1}{2}\delta^{[N}_M\xi_\alpha{}^{P]} \ , \tag{4.2.5}$$

$$\Theta_{M\alpha}{}^{\beta\gamma} = \frac{1}{2}\xi_{\delta M}\epsilon^{\delta(\beta}\delta^{\gamma)}_\alpha \ . \tag{4.2.6}$$

Working out the quadratic constraint (4.1.9) on $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}} \tag{4.2.7}$$

in terms of $\xi_{\alpha M}$ and $f_{\alpha MNP}$ yields the following set of constraints:

$$\xi_\alpha{}^M \xi_{\beta M} = 0 \ , \tag{4.2.8}$$

$$\xi_{(\alpha}{}^P f_{\beta)PMN} = 0 \ , \tag{4.2.9}$$

$$3 f_{\alpha R[MN} f_{\beta PQ]}{}^R + 2\xi_{(\alpha[M} f_{\beta)NPQ]} = 0 \ , \tag{4.2.10}$$

$$\epsilon^{\alpha\beta}\left(\xi_\alpha{}^P f_{\beta PMN} + \xi_{\alpha M}\xi_{\beta N}\right) = 0 \ , \tag{4.2.11}$$

$$\epsilon^{\alpha\beta}\left(f_{\alpha MNR} f_{\beta PQ}{}^R - \xi_\alpha{}^R f_{\beta R[M[P}\eta_{Q]N]} - \xi_{\alpha[M} f_{\beta N][PQ]} + \xi_{\alpha[P} f_{\beta Q][MN]}\right) = 0 \ . \tag{4.2.12}$$

The deformation of the theory is consistent if and only if these conditions, which guarantee the closure of the gauge group, are satisfied, as explained above. They are covariant under the global symmetry group $SL(2) \times SO(6,n)$: given one particular solution one can create another acting with a G transformation. Trivial solutions are the *purely electric gaugings*, according to which only electric vector fields A_μ^{M+} enter the Lagrangian:

$$\xi_{\alpha M} = 0 \ , \tag{4.2.13}$$

$$f_{-MNP} = 0 \ , \tag{4.2.14}$$

and the constraint (4.2.10) simplifies to the Jacobi identity

$$f_{+R[MN} f_{+PQ]}{}^R = 0 \ , \tag{4.2.15}$$

where indices are contracted by η^{MN} metric of $SO(6,n)$, instead of the ordinary Cartan-Killing form, and $f_{+MNP} = f_{+MN}{}^Q \eta_{QP}$ are the structure constants of the gauge group. This $N = 4$, $D = 4$ gauged supergravities were firstly found out by Bergshoeff, Koh and Sezgin [16], and the general deformation have to reproduce the Lagrangian of this article, provided that we impose (4.2.13), (4.2.14) and

$$f_{+MNP} \to -f_{MNP} \ , \tag{4.2.16}$$

where f_{MNP} are the structure constants used in the article.

The following combination of the generalized structure constants occurs regularly:

$$\hat{f}_{\alpha MNP} = f_{\alpha MNP} - \xi_{\alpha[M}\eta_{P]N} - \frac{3}{2}\xi_{\alpha N}\eta_{MP} \ . \tag{4.2.17}$$

We give the full covariant field strengths of 1-form and of 2-form up to terms of order g:

$$\mathcal{H}_{\mu\nu}^{M\alpha} = 2\partial_{[\mu} A_{\nu]}^{M\alpha} - g\hat{f}_{\beta NP}{}^M A_{[\mu}^{N\beta} A_{\nu]}^{P\alpha} + \frac{g}{2}\epsilon^{\alpha\beta}\Theta_\beta{}^M{}_{NP} B_{\mu\nu}^{NP} + \frac{g}{2}\xi_\beta{}^M B_{\mu\nu}^{\alpha\beta} \ , \tag{4.2.18}$$

$$\mathcal{H}_{\mu\nu\rho}^{MN} = 3\partial_{[\mu} B_{\nu\rho]}^{MN} + 6\epsilon_{\alpha\beta} A_{[\mu}^{\alpha[M}\partial_\nu A_{\rho]}^{\beta N]} + \mathcal{O}(g) \ , \tag{4.2.19}$$

$$\mathcal{H}_{\mu\nu\rho}^{\alpha\beta} = 3\partial_{[\mu} B_{\nu\rho]}^{\alpha\beta} + 6\eta_{MN} A_{[\mu}^{(\alpha M}\partial_\nu A_{\rho]}^{\beta)N} + \mathcal{O}(g) \ . \tag{4.2.20}$$

Only the electric field strength $\mathcal{H}_{\mu\nu}^{M+}$ enters the Lagrangian, but the magnetic and the 2-form field strengths appear in the equations of motion.

It is useful to covariantize the combination of the electric field strength defined in (3.3.4):

$$\begin{aligned}\mathcal{G}^{M+}_{\mu\nu} &\equiv \mathcal{H}^{M+}_{\mu\nu} \ , \\ \mathcal{G}^{M-}_{\mu\nu} &\equiv e^{-1}\eta^{MN}\varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}_{\text{kin}}}{\partial\mathcal{H}^{N+}_{\rho\sigma}} \\ &= -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Im}\,\tau M^{MN}\eta_{NP}\mathcal{H}^{P+}_{\mu\nu} - \mathrm{Re}\,\tau\mathcal{H}^{M+}_{\mu\nu} \ .\end{aligned} \tag{4.2.21}$$

We have to insert $\mathcal{G}^{M\alpha}_{\mu\nu}$, instead of $\mathcal{H}^{M\alpha}_{\mu\nu}$, in the covariant variations in order to find a formulation entirely in terms of the electric vector fields in the limit $g \to 0$. Then, the gauge transformations of the vector and *covariant variations* of the 2-forms gauge fields

$$\Delta B^{MN}_{\mu\nu} = \delta B^{MN}_{\mu\nu} - 2\epsilon_{\alpha\beta}A^{[M\alpha}_{[\mu}\delta A^{N]\beta}_{\nu]} \quad \text{and} \quad \Delta B^{\alpha\beta}_{\mu\nu} = \delta B^{\alpha\beta}_{\mu\nu} + 2\eta_{MN}A^{M(\alpha}_{[\mu}\delta A^{N\beta)}_{\nu]} \tag{4.2.22}$$

are

$$\delta A^{M\alpha}_{\mu} = D_\mu\Lambda^{M\alpha} - \frac{g}{2}\epsilon^{\alpha\beta}\Theta_{\beta}{}^{M}{}_{NP}\Xi^{NP}_{\mu} + \frac{g}{2}\xi_{\beta}{}^{M}\Xi^{\alpha\beta}_{\mu} \ , \tag{4.2.23}$$

$$\Delta B^{MN}_{\mu\nu} = 2D_{[\mu}\Xi^{MN}_{\nu]} - 2\epsilon_{\alpha\beta}\Lambda^{[M\alpha}\mathcal{G}^{N]\beta}_{\mu\nu} \ , \tag{4.2.24}$$

$$\Delta B^{\alpha\beta}_{\mu\nu} = 2D_{[\mu}\Xi^{\alpha\beta}_{\nu]} + 2\eta_{MN}\Lambda^{M(\alpha}\mathcal{G}^{N\beta)}_{\mu\nu} \ , \tag{4.2.25}$$

where the $\Lambda^{M\alpha}$, $\Xi^{MN}_{\mu} = \Xi^{[MN]}_{\mu}$ and $\Xi^{\alpha\beta}_{\mu} = \Xi^{(\alpha\beta)}_{\mu}$ are the gauge parameters of the group.

We can now present the bosonic Lagrangian of the gauged theory

$$\mathcal{L}_{\text{bos}} = \mathcal{L}_{\text{b.kin}} + \mathcal{L}_{\text{top}} + \mathcal{L}_{\text{pot}} \ . \tag{4.2.26}$$

It consists of a bosonic kinetic term

$$\begin{aligned}e^{-1}\mathcal{L}_{\text{b.kin}} =& \frac{1}{2}R + \frac{1}{16}D_\mu M_{MN}D^\mu M^{MN} + \frac{1}{8}D_\mu M_{\alpha\beta}D^\mu M^{\alpha\beta} \\ &- \frac{1}{4}\,\mathrm{Im}\,\tau M_{MN}\mathcal{H}^{M+}_{\mu\nu}\mathcal{H}^{N+}_{\mu\nu} + \frac{1}{8}\eta_{MN}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}^{M+}_{\mu\nu}\mathcal{H}^{N+}_{\rho\sigma} \ ,\end{aligned} \tag{4.2.27}$$

a topological term for the vector and tensor gauge fields

$$\begin{aligned}e^{-1}\mathcal{L}_{\text{top}} =& -\frac{g}{2}\epsilon^{\mu\nu\rho\sigma} \\ &\Big[\xi_{+M}\eta_{NP}A^{M-}_{\mu}A^{N+}_{\nu}\partial_\rho A^{P+}_{\sigma} - \left(\hat{f}_{-MNP} + 2\xi_{-N}\eta_{MP}\right)A^{M-}_{\mu}A^{N+}_{\nu}\partial_\rho A^{P-}_{\sigma} \\ &- \frac{g}{4}\hat{f}_{\alpha MNR}\hat{f}_{\beta PQ}{}^{R}A^{M\alpha}_{\mu}A^{N+}_{\nu}A^{P\beta}_{\rho}A^{Q-}_{\sigma} + \frac{g}{16}\Theta_{+MNP}\Theta_{-}{}^{M}{}_{QR}B^{NP}_{\mu\nu}B^{QR}_{\rho\sigma} \\ &- \frac{1}{4}\left(\Theta_{-MNP}B^{NP}_{\mu\nu} + \xi_{\alpha M}B^{+\alpha}_{\mu\nu}\right)\left(2\partial_\rho A^{M-}_{\sigma} - g\hat{f}_{\alpha QR}{}^{M}A^{Q\alpha}_{\rho}A^{R-}_{\sigma}\right)\Big] \ ,\end{aligned} \tag{4.2.28}$$

and a scalar potential

$$\begin{aligned}e^{-1}\mathcal{L}_{\text{pot}} =& -g^2 V \\ =& -\frac{g^2}{16}f_{\alpha MNP}f_{\beta QRS}M^{\alpha\beta}\left[\frac{1}{3}M^{MQ}M^{NR}M^{PS} + \left(\frac{2}{3}\eta^{MQ} - M^{MQ}\right)\eta^{NR}\eta^{PS}\right] \\ &+ \frac{g^2}{36}f_{\alpha MNP}f_{\beta QRS}\epsilon^{\alpha\beta}M^{MNPQRS} - \frac{3}{16}g^2\xi_{\alpha M}\xi_{\beta N}M^{\alpha\beta}M^{MN} \ .\end{aligned} \tag{4.2.29}$$

It has to be noted that in the case of purely electric gaugings, topological terms are identically zero.

The explicit action of the covariant derivative (4.2.3) on the scalar matrices is

$$D_\mu M_{\alpha\beta} = \partial_\mu M_{\alpha\beta} + gA_\mu^{M\gamma}\xi_{(\alpha M}M_{\beta)\gamma} - gA_\mu^{M\delta}\xi_{\epsilon M}\epsilon_{\delta(\alpha}\epsilon^{\epsilon\gamma}M_{\beta)\gamma} \ , \tag{4.2.30}$$

$$D_\mu M_{MN} = \partial_\mu M_{MN} + 2gA_\mu^{P\alpha}\Theta_{\alpha P(M}{}^{Q}M_{N)Q} \ . \tag{4.2.31}$$

Under general variations of vectors and covariant variations of 2-form gauge fields, the bosonic Lagrangian varies as [28]

$$\begin{aligned}
e^{-1}\delta\mathcal{L}_{\text{bos}} = & \frac{1}{8}g\left(\Theta_{-MNP}\Delta B_{\mu\nu}^{NP} + \xi_{\alpha M}\Delta B_{\mu\nu}^{+\alpha}\right)\varepsilon^{\mu\nu\rho\sigma}\left(\mathcal{H}_{\rho\sigma}^{M-} - \mathcal{G}_{\rho\sigma}^{M-}\right) \\
& + \frac{1}{2}\delta A_\mu^{M+}\left(g\xi_{\beta M}M_{+\gamma}D^\mu M^{\beta\gamma} + \frac{g}{2}\Theta_{+MP}{}^{N}M_{NQ}D^\mu M^{QP} - \varepsilon^{\mu\nu\rho\sigma}\eta_{MN}D_\nu G_{\rho\sigma}^{N-}\right) \\
& + \frac{1}{2}\delta A_\mu^{M-}\left(g\xi_{\beta M}M_{-\gamma}D^\mu M^{\beta\gamma} + \frac{g}{2}\Theta_{-MP}{}^{N}M_{NQ}D^\mu M^{QP} + \varepsilon^{\mu\nu\rho\sigma}\eta_{MN}D_\nu G_{\rho\sigma}^{N+}\right) \\
& + \text{total derivatives} \ ,
\end{aligned} \tag{4.2.32}$$

which encodes the gauge field equations of motion of the theory at linear order in the fields (other sectors of the gauged Lagrangian give higher order contributions, which are not relevant for the present and future analysis). As anticipated in sections 3.3 and 4.1.2, variation of the 2-forms yields a projected version of the duality equation $\mathcal{H}_{\mu\nu}^{M-} = \mathcal{G}_{\mu\nu}^{M-}$, between electric and magnetic vector fields: $\mathcal{H}_{\mu\nu}^{M\alpha}$ and $\mathcal{G}_{\mu\nu}^{M\alpha}$ are on-shell identical. By varying the electric vectors, one obtains a field equation for the electric vectors themselves which contains scalar currents as a source term. Finally, the variation of the magnetic vectors gives a duality equation between scalars and 2-forms: this is straightforward once introduced the modified Bianchi identity for $\mathcal{G}_{\mu\nu}^{M+} = \mathcal{H}_{\mu\nu}^{M+}$, which reads

$$D_{[\mu}\mathcal{H}_{\nu\rho]}^{M+} = \frac{g}{6}\left(\Theta_{-}{}^{M}{}_{PQ}\mathcal{H}_{\mu\nu\rho}^{PQ} + \xi_\alpha{}^{M}\mathcal{H}_{\mu\nu\rho}^{+\alpha}\right) \ . \tag{4.2.33}$$

Thus we found that the tensors $f_{\alpha MNP}$ and $\xi_{\alpha M}$ do not only specify the gauge group but also organize the couplings of the various fields. They determine which vector gauge fields appear in the covariant derivatives, how the field strengths have to be modified , which magnetic vectors and 2-forms enter the Lagrangian and how they become dual to electric vectors and scalars via equation of motion.

4.3 Fermion mass matrices and fermionic shifts

4.3.1 T-tensor irreducible components

Fermion mass matrices are composed out of irreducible components of the *T-tensor*, *i.e.* of the embedding tensor dressed with scalar group representative. Then, our first step has the identification of the $U(1) \times SO(n) \times SU(4)$-irreducible part of these new tensors. Let us concentrate first on the simplest component of the embedding tensor, that is $\xi_{\alpha M}$. $SO(6,n)$-index M can be contracted either with $L_a{}^M$ or $L_{ij}{}^M$. While $SL(2)$-index can be contracted either with $\mathcal{V}^\alpha$ or its complex conjugate, which fix the $U(1)$-phase to be $+1$ or -1, respectively. It will be useful to introduce both T-tensor components and *partially dressed embedding tensor*, where only $SO(6,n)$ are contracted. We introduce the compact notations $\left(\mathcal{R}_{SL(2)}, \mathcal{R}_{SO(n)}, \mathcal{R}_{SU(4)}\right)$ for the latter and $\left(\mathcal{R}_{SO(n)}, \mathcal{R}_{SU(4)}\right)_{q_{U(1)}}$ for the former in order to identify the representations for the two compact components of $SO(6,n)$ and three-components of the H group, when the $U(1)$-charge is also fixed. We will label the $SO(n)$ representations with *Young tableaux*.

Partially-dressed embedding tensors and corresponding representations are

$$(\mathbf{2}, \square, \mathbf{1}) \qquad E_{\alpha a} \equiv \xi_{\alpha M} L_a{}^M \ , \tag{4.3.1}$$

$$(\mathbf{2}, \bullet, \mathbf{6}) \qquad E_{\alpha ij} \equiv \xi_{\alpha M} L_{ij}{}^M \ , \tag{4.3.2}$$

and the various components of the T-tensor

$$(\square, \mathbf{1})_{+1} \qquad E_a \equiv \mathcal{V}^\alpha \xi_{\alpha M} L_a{}^M \ , \tag{4.3.3}$$

$$(\square, \mathbf{1})_{-1} \qquad E_a^\dagger \equiv (\mathcal{V}^\alpha)^* \xi_{\alpha M} L_a{}^M \ , \tag{4.3.4}$$

$$(\square, \mathbf{1})_{+1} \qquad E_{ij} \equiv \mathcal{V}^\alpha \xi_{\alpha M} L_{ij}{}^M \ , \tag{4.3.5}$$

$$(\square, \mathbf{1})_{-1} \qquad E_{ij}^\dagger \equiv (\mathcal{V}^\alpha)^* \xi_{\alpha M} L_{ij}{}^M \ . \tag{4.3.6}$$

Now we consider the $f_{\alpha MNP} = f_{\alpha[MNP]}$ component of $\Theta_\alpha{}^M$, which is slightly more complex because of the structure of the three $SO(6,n)$-vector indices which are completely antisymmetrized. When we dress this tensor with three $L_{ij}{}^M$, it transforms in the H-representation

$$\Big(\mathbf{2}, \bullet, (\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6})_{\text{asym}}\Big) \ , \tag{4.3.7}$$

where $\mathbf{10}$ is the symmetric part of $\mathbf{4} \otimes \mathbf{4}$ representation product.

If we contract $f_{\alpha[MNP]}$ with two $L_{ij}{}^M$ tensors and a $L_a{}^M$ matrix, the $SU(4)$-indices of the partially dressed embedding tensor transform in $(\mathbf{6} \otimes \mathbf{6})_{\text{asym}}$ representation:

$$(\mathbf{6} \otimes \mathbf{6})_{\text{asym}} = \mathbf{15} \ , \tag{4.3.8}$$

where $\mathbf{15}$ is the traceless part of $\mathbf{4} \otimes \overline{\mathbf{4}}$ representation product. Hence we can classify the irreducible parts of partially dressed $f_{\alpha MNP}$ as follows:

$$(\mathbf{2}, \bullet, \mathbf{10}) \qquad F_\alpha{}^{ij} \equiv f_{\alpha M}{}^{NP} L_{kl}{}^M L_N{}^{ik} L_P{}^{jl} \ , \tag{4.3.9}$$

$$\big(\mathbf{2}, \bullet, \overline{\mathbf{10}}\big) \qquad F_{\alpha ij} = \left(F_\alpha{}^{ij}\right)^* = f_{\alpha MN}{}^P L_{ik}{}^M L_{jl}{}^N L_P{}^{kl} \ , \tag{4.3.10}$$

$$(\mathbf{2}, \square, \mathbf{15}) \qquad F_{\alpha ai}{}^j = f_{\alpha MN}{}^P L_a{}^M L_{ik}{}^N L_P{}^{jk} \ , \tag{4.3.11}$$

$$\left(\mathbf{2}, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \mathbf{6}\right) \qquad F_{\alpha abij} = f_{\alpha MNP} L_a{}^M L_b{}^N L_{ij}{}^P \ , \tag{4.3.12}$$

$$\left(\mathbf{2}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \mathbf{1}\right) \qquad F_{\alpha abc} = f_{\alpha MNP} L_a{}^M L_b{}^N L_c{}^P \ . \tag{4.3.13}$$

Irreducible components of T-tensor are obtained contracting the just defined tensors with $\mathcal{V}^\alpha$ or $(\mathcal{V}^\alpha)^*$, and we will denote them with a † when they transform with a -1 charge under $U(1)$ and without it when have charge $+1$: *e.g.* $F^{ij} \equiv \mathcal{V}^\alpha F_\alpha{}^{ij}$, $F^{\dagger ij} \equiv (\mathcal{V}^\alpha)^* F_\alpha{}^{ij}$, $F_{ij} \equiv \mathcal{V}^\alpha F_{\alpha ij}$, ecc.

For completeness, let us note that $\mathbf{6} \simeq \overline{\mathbf{6}}$ and $\mathbf{15} \simeq \overline{\mathbf{15}}$ due to the pseudo-reality constraint (3.2.43): for example

$$E_\alpha{}^{ij} = \left(E_{\alpha ij}\right)^* = \frac{1}{2}\epsilon^{ijkl}E_{\alpha kl}\ , \tag{4.3.14}$$

$$\begin{aligned} F_\alpha{}^i{}_j = \left(F_{\alpha i}{}^j\right)^* =& \frac{1}{4}\epsilon^{ilmk}\epsilon_{jnok}f_{\alpha MN}{}^P L_a{}^M L_{lm}{}^N L_P{}^{no} = \\ =& f_{\alpha MN}{}^P L_a{}^M L_{jk}{}^N L_P{}^{jk} = -F_{\alpha aj}{}^i = -\delta^i_m\delta^n_j F_{\alpha an}{}^m\ . \end{aligned} \tag{4.3.15}$$

In appendix D we present some useful relations between partially dressed components of the embedding tensor, or equivalently between irreducible components of T-tensor. In appendix C we express the quadratic constraints (4.2.8)-(4.2.12) in terms of T-tensor components.

4.3.2 Fermion shift matrices and consistency relations at linear order

Let us introduce the sector of the Lagrangian which contains Yukawa interactions, that is fermionic mass matrices:

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{f.mass}} =& gM_1^{ij}\bar\psi_{\mu i}\Gamma^{\mu\nu}\psi_{\nu j} + gM_2^{ij}\bar\psi_{\mu i}\Gamma^\mu\chi_j + gM_{2ai}{}^j\bar\psi_\mu{}^i\Gamma^\mu\lambda^a_j + \\ &+ gM_3^{ij}\bar\chi_i\chi_j + gM_{3ai}{}^j\bar\chi^i\lambda^a_j + gM_{3ab}{}^{ij}\bar\lambda^a_i\lambda^b_j + \text{h.c.}\ , \end{aligned} \tag{4.3.16}$$

where the tensors M have to be linear combinations of irreducible components of T-tensor:

$$M \approx \sum E + \sum F, \qquad \text{or} \qquad M \approx \sum E^\dagger + \sum F^\dagger\ , \tag{4.3.17}$$

and from symmetry properties of fermion bilinears we have $M_1^{ij} = M_1^{(ij)}$, $M_3^{ij} = M_3^{(ij)}$ and $M_{3ab}{}^{ij} = M_{3(ab)}{}^{(ij)} + M_{3[ab]}{}^{[ij]}$. As explained in section 4.1.2, fermion shifts have to be introduced in supersymmetry variations of the spinors in order to cancel terms which contain $D_\mu\epsilon^i$ at order g^1, coming from the gravitino variations in the Lagrangian sector of fermion masses (4.3.16):

$$\delta\psi^i_\mu = \cdots + \alpha g M_1^{ij}\gamma_\mu\epsilon_j\ , \tag{4.3.18}$$

$$\delta\chi^i = \cdots + \beta g M_2^{ji}\epsilon_j\ , \tag{4.3.19}$$

$$\delta\lambda^i_a = \cdots + \gamma g M_{2aj}{}^i\epsilon^j\ . \tag{4.3.20}$$

The coefficients α, β and γ have to be fixed. In order to find their values, we have to study supersymmetric variations at order g for terms with the structure $gM\bar f\gamma D\epsilon$, with f one out of the three types of fermion and which only originate from kinetic terms and fermion masses:

$$\begin{aligned} e^{-1}\delta\mathcal{L}_{\text{f.mass}} &= 2gM_1^{ij}\bar\psi_{\mu i}\gamma^{\mu\nu}\delta\psi_{\nu j} - gM_2^{ji}\bar\chi_i\gamma^\mu\delta\psi_{\mu j} - gM_{2aj}{}^i\bar\lambda^a_i\gamma^\mu\delta\psi^j_\mu + \text{h.c.} + \dots \\ &= 4gM_1^{ij}\bar\psi_{\mu i}\gamma^{\mu\nu}D_\nu\epsilon_j - 2gM_2^{ji}\bar\chi_i\gamma^\mu D_\mu\epsilon_j - 2gM_{2aj}{}^i\bar\lambda^a_i\gamma^\mu D_\mu\epsilon^j + \text{h.c.} + \dots\ , \end{aligned} \tag{4.3.21}$$

and

$$\begin{aligned} e^{-1}\delta\mathcal{L}_{\text{f.kin}} &= -\bar\psi_{\mu i}\gamma^{\mu\nu\rho}D_\nu\delta\psi^i_\rho - \frac{1}{2}\bar\chi_i\gamma^\mu D_\mu\delta\chi^i - \bar\lambda_{ai}\gamma^\mu D_\mu\delta\lambda^{ai} + \text{h.c.} \\ &= -2\alpha g\bar\psi_{\mu i}\gamma^{\mu\nu}M_1^{ij}D_\nu\epsilon_j - \frac{\beta}{2}g\bar\chi_i\gamma^\mu M_2^{ji}D_\mu\epsilon_j - \gamma g\bar\lambda_{ai}\gamma^\mu M_{2aj}{}^iD_\mu\epsilon^j + \text{h.c.} + \dots\ , \end{aligned} \tag{4.3.22}$$

where we used the gamma matrix property

$$\gamma^{\mu\nu\rho}\gamma_\rho = 2\gamma^{\mu\nu} \ . \tag{4.3.23}$$

Therefore the numerical coefficients are

$$\alpha = 2, \qquad \beta = -2, \qquad \gamma = -4 \ . \tag{4.3.24}$$

Now we are going to fix the gravitinos mass matrix M_1^{ij}. Group theoretically, the fermion bilinear $\bar{\psi}_{i\mu}\gamma^{\mu\nu}\psi_{\nu j}$ transforms in the

$$\left(\bullet, \overline{\mathbf{4}}\right)_{+\frac{1}{2}} \otimes_{\text{sym}} \left(\bullet, \overline{\mathbf{4}}\right)_{+\frac{1}{2}} = \left(\bullet, \overline{\mathbf{10}}\right)_{+1} \ , \tag{4.3.25}$$

thus M_1^{ij} transforms in the $(\bullet, \mathbf{10})_{-1}$, *i.e.* the only T-tensor component that transforms according to this representation is $F^{\dagger ij}$:

$$M_1^{ij} = aF^{\dagger ij} \ , \tag{4.3.26}$$

where a is a constant to be fixed. In order to set the value of a, we recall the paper by Bergshoeff, Koh and Sezgin [16]: we restrict M_1^{ij} to the case of purely electric gaugings (4.2.13), (4.2.14), (4.2.15) remembering the change of conventions (4.2.16) and

$$\phi \to i\phi \ . \tag{4.3.27}$$

$$M_1^{ij}\bigg|_{\text{Elec}} = a\epsilon^{+-}\mathcal{V}_-{}^* f_{+M}{}^{NP} L_{kl}{}^M L_N{}^{ik} L_P{}^{jl} \to ai\phi^* f_M{}^{NP} L_{kl}{}^M L_N{}^{ik} L_P{}^{jl} \ , \tag{4.3.28}$$

hence[1]

$$a = \frac{1}{3} \ . \tag{4.3.29}$$

Before working out consistency constraints that completely fix the mass matrices once M_1^{ij} is known, let us have a group theoretically look at the structure of the other mass matrices as we have just done for M_1^{ij}. We decompose the representations of bilinear fermions in their irreducible parts and predict the form of the matrices except for numerical coefficients. Let us start with mixing terms between spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fermions:

$$\bar{\psi}_{\mu i}\gamma^\mu\chi_j \ : \qquad \left(\bullet, \overline{\mathbf{4}}\right)_{+\frac{1}{2}} \otimes \left(\bullet, \overline{\mathbf{4}}\right)_{-\frac{3}{2}} = (\bullet, \mathbf{6})_{-1} \oplus \left(\bullet, \overline{\mathbf{10}}\right)_{-1} \ , \tag{4.3.30}$$

$$\bar{\psi}^i_\mu\gamma^\mu\lambda^a_j \ : \qquad (\bullet, \mathbf{4})_{-\frac{1}{2}} \otimes \left(\Box, \overline{\mathbf{4}}\right)_{-\frac{1}{2}} = (\Box, \mathbf{1})_{-1} \oplus (\Box, \mathbf{15})_{-1} \ , \tag{4.3.31}$$

thus it follows that

$$M_2^{ij} = b_1 F^{ij} + b_2 E^{ij} \ , \tag{4.3.32}$$

$$M_{2ai}{}^j = b_3 F_{ai}{}^j + b_4\delta_i^j E_a \ . \tag{4.3.33}$$

The most interesting case is the matrix M_3^{ij}.

$$\bar{\chi}_i\chi_j \ : \qquad \left(\bullet, \overline{\mathbf{4}}\right)_{-\frac{3}{2}} \otimes_{\text{sym}} \left(\bullet, \overline{\mathbf{4}}\right)_{-\frac{3}{2}} = \left(\bullet, \overline{\mathbf{10}}\right)_{-3} \ , \tag{4.3.34}$$

[1] There is a sign minus with respect to the gravitinos mass matrix of Schön and Weidner [7], which would lead to inconsistencies in the relations that we will find at linear order.

which implies

$$M_3^{ij} = 0 \ , \tag{4.3.35}$$

because there is not a component of T-tensor which transforms with a charge $+3$ under $U(1)$. We will see how consistency relations give the same result for M_3^{ij}.

Finally, the remaining two matrices:

$$\bar{\chi}^i\lambda^a_j \ : \qquad (\bullet,\mathbf{4})_{+\frac{3}{2}} \otimes \left(\square,\overline{\mathbf{4}}\right)_{-\frac{1}{2}} = (\square,\mathbf{1})_{+1} \oplus (\square,\mathbf{15})_{+1} \ , \tag{4.3.36}$$

$$\bar{\lambda}^a_i\lambda^b_j \ : \qquad \left(\square,\overline{\mathbf{4}}\right)_{-\frac{1}{2}} \otimes_{\rm sym} \left(\square,\overline{\mathbf{4}}\right)_{-\frac{1}{2}} = \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix},\mathbf{6}\right)_{-1} \oplus \left(\square\!\square,\overline{\mathbf{10}}\right)_{-1} \oplus \left(\bullet,\overline{\mathbf{10}}\right)_{-1} \ . \tag{4.3.37}$$

There are no components of T-tensor transforming in the $\square\!\square$ for $SO(n)$ and the only two-indices tensor transforming trivially under $SO(n)$ is the Cartan-Killing metric of $SO(n)$:

$$M_{3ai}{}^{j} = c_1 {F^\dagger}_{ai}{}^{j} + c_2\delta_i^j {E^\dagger}_a \ , \tag{4.3.38}$$

$$M_{3ab}{}^{ij} = c_3 F_{ab}{}^{ij} + c_4\delta_{ab}F^{ij} \ . \tag{4.3.39}$$

In order to express linear and quadratic consistency relations and match results with paper [7], it is convenient to introduce the following rescaled mass matrices:

$$M_1^{ij} = \frac{1}{3}A_1^{ij} \ , \tag{4.3.40}$$

$$M_2^{ij} = +\frac{1}{3}A_2^{ij} \ , \tag{4.3.41}$$

$$M_{2ai}{}^{j} = -iA_{2ai}{}^{j} \ , \tag{4.3.42}$$

$$M_3^{ij} = A_3^{ij} \ , \tag{4.3.43}$$

$$M_{3ai}{}^{j} = iA_{3ai}{}^{j} \ , \tag{4.3.44}$$

$$M_{3ab}{}^{ij} = -A_{3ab}{}^{ij} \ , \tag{4.3.45}$$

where $A_1^{ij} = F^{\dagger ij}$ and $\bar{A}_{1ij} \equiv \left(A_1^{ij}\right)^* = F_{ij}$, as we just found out.

We rewrite (4.3.16) and the fermion supersymmetry variations in terms of this new matrices:

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm f.mass} &= \frac{1}{3}gA_1^{ij}\bar{\psi}_{\mu i}\Gamma^{\mu\nu}\psi_{\nu j} + \frac{1}{3}gA_2^{ij}\bar{\psi}_{\mu i}\Gamma^{\mu}\chi_j - igA_{2ai}{}^{j}\bar{\psi}_\mu{}^{i}\Gamma^\mu\lambda^a_j + \\
&\quad + gA_3^{ij}\bar{\chi}_i\chi_j + igA_{3ai}{}^{j}\bar{\chi}^i\lambda^a_j - gA_{3ab}{}^{ij}\bar{\lambda}^a_i\lambda^b_j + \text{h.c.} \ , \\
\delta\psi^i_\mu &= 2D_\mu\epsilon^i + \frac{i}{4}\mathcal{V}^*_\alpha L_M{}^{ij}\mathcal{G}^{M\alpha}_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\epsilon_j + \frac{2}{3}gA_1^{ij}\gamma_\mu\epsilon_j + \text{bilinear fermions} \ , \\
\delta\chi^i &= 2P_\mu\gamma^\mu\epsilon^i + \frac{i}{2}\mathcal{V}_\alpha L_M{}^{ij}\mathcal{G}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon_j - \frac{4}{3}gA_2^{ji}\epsilon_j + \text{bilinear fermions} \ , \\
\delta\lambda^i_a &= 2iP_{\mu a}{}^{ij}\gamma^\mu\epsilon_j - \frac{1}{4}\mathcal{V}_\alpha L_{Ma}\mathcal{G}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon^i + 2igA_{2aj}{}^{i}\epsilon^j + \text{bilinear fermions} \ .
\end{aligned} \tag{4.3.46}$$

Consistency relations for the fermion mass matrices are needed in order to have *supersymmetry closure* for terms proportional to $\bar{\psi}_\mu\gamma^{\mu\nu}\epsilon$, $\bar{\psi}^\mu\epsilon$, $\bar{\lambda}\gamma^\mu\epsilon$ and $\bar{\chi}\gamma^\mu\epsilon$ at linear order in g. Therefore we will have four relations: the closure for $\bar{\psi}_\mu\gamma^{\mu\nu}\epsilon$ term fixes b_1 and b_3, $\bar{\psi}^\mu\epsilon$ fixes b_2 and b_4. c_1 and

c_2 are set by both $\bar\lambda\gamma^\mu\epsilon$ and $\bar\chi\gamma^\mu\epsilon$ terms, while the former also establishes the value of c_3 and c_4. M_3^{ij} is set at zero by $\bar\chi\gamma^\mu\epsilon$.

We are going to present individually the sectors of the Lagrangian whose supersymmetric variations at order g give rise to the terms mentioned above. First consider the fermion kinetic terms:

$$\begin{aligned} e^{-1}\delta\mathcal{L}_{\text{f.kin}} &= -\bar\psi_{\mu i}\gamma^{\mu\nu\rho}D_\nu\delta\psi^i_\rho - \frac{1}{2}\bar\chi_i\gamma^\mu D_\mu\delta\chi^i - \bar\lambda_{ai}\gamma^\mu D_\mu\delta\lambda^{ai} + \dots \\ &= g\bar\psi_{\mu i}\gamma^{\mu\nu}\epsilon_j\left(-\frac{4}{3}D_\nu A_1^{ij}\right) + g\bar\chi_i\gamma^\mu\epsilon_j\left(\frac{2}{3}D_\mu A_2^{ji}\right) + ig\bar\lambda^a_i\gamma^\mu\epsilon^j\left(-2D_\mu A_{2aj}{}^i\right) + \dots \end{aligned} \tag{4.3.47}$$

Then we look at fermion masses:

$$\begin{aligned} e^{-1}\delta\mathcal{L}_{\text{f.mass}} =& \frac{1}{3}gA_2^{ij}\bar\psi_{\mu i}\gamma^\mu\delta\chi_j + ig\bar A_{2a}{}^i{}_j\bar\psi_{\mu i}\gamma^\mu\delta\lambda^{aj} + 2gA_3^{ij}\bar\chi_i\delta\chi_j + \\ &+ igA_{3ai}{}^j\bar\lambda^a_j\delta\chi^i - ig\bar A_{3a}{}^i{}_j\bar\chi_i\delta\lambda^{aj} - 2gA_{3ab}{}^{ij}\bar\lambda^a_i\delta\lambda^b_j + \dots = \\ =& g\bar\psi_{\mu i}\gamma^\mu\gamma^\nu\epsilon_j\left(\frac{2}{3}A_2^{ij}P^*_\nu - 2\bar A_{2a}{}^i{}_kP_\nu{}^{akj}\right) + \\ &+ g\bar\chi_i\gamma^\mu\epsilon_j\left(4A_3^{ij}P^*_\mu + 2\bar A_{3a}{}^i{}_kP_\mu{}^{akj}\right) + \\ &+ ig\bar\lambda^a_i\gamma^\mu\epsilon^j\left(4A_{3ab}{}^{ik}P_\mu{}^b{}_{jk} + 2A_{3aj}{}^iP_\mu\right) + \dots \end{aligned} \tag{4.3.48}$$

The gravitinos variation at order g gives a contribution proportional to $D\epsilon$, which has just been compensated by introducing the fermion-shifts.

Moreover the two scalar-fermion interaction terms generate contributions for each of the term considered.

$$\begin{aligned} e^{-1}\delta\mathcal{L}_{\text{ferm.scal}} =& \frac{1}{2}\bar\psi_{\mu i}\gamma^\nu\gamma^\mu\delta\chi^iP^*_\nu + \frac{1}{2}\bar\chi_i\gamma^\mu\gamma^\nu\delta\psi^i_\mu P_\nu + \\ &+ i\bar\psi_{\mu j}\gamma^\nu\gamma^\mu\delta\lambda^a_iP_\nu{}^{aij} + i\bar\lambda^a_i\gamma^\mu\gamma^\nu\delta\psi_{\mu j}P_\nu{}^{aij} + \dots \\ =& g\bar\psi_{\mu i}\gamma^\nu\gamma^\mu\epsilon_j\left(-\frac{2}{3}A_2^{ji}P^*_\nu\right) + g\bar\chi_i\gamma^\mu\epsilon_j\left(-\frac{2}{3}A_1^{ij}P_\mu\right) + \\ &+ g\bar\psi_{\mu i}\gamma^\nu\gamma^\mu\epsilon_j\left(2\bar A_{2a}{}^j{}_kP_\nu{}^{aki}\right) + ig\bar\lambda^a_i\gamma^\mu\epsilon^j\left(-\frac{4}{3}\bar A_{1jk}P_\mu{}^{aik}\right) + \dots\ , \end{aligned} \tag{4.3.49}$$

where we used the gamma matrices identity

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu\ . \tag{4.3.50}$$

Finally, the last contributions comes from the variation of vectors in the covariantized scalar kinetic terms:

$$\begin{aligned}
\delta\left(\frac{1}{16}D_\mu M_{MN}D^\mu M^{MN}\right) =& -\frac{1}{4}gD_\mu M_{MN}\delta A^{\mu P\alpha}\theta_{\alpha PQ}{}^{(M}M^{N)Q}+\dots\\
=& -\frac{1}{2}g\delta A^{\mu P\alpha}P_\mu{}^{aij}\theta_{\alpha PQM}\left(-L_a{}^M L_{ij}{}^Q + L_a{}^Q L_{ij}{}^M\right)+\dots\\
=& g\bar\psi_i^\mu\epsilon_j\left(4F^\dagger{}_{ak}{}^{[i}P_\mu{}^{aj]k} - E^\dagger{}_a P_\mu{}^{aij}\right)+ \\
&+g\bar\chi_i\gamma^\mu\epsilon_j\left(-2F_{ak}{}^{[i}P_\mu{}^{aj]k}+\frac{1}{2}E_a P_\mu{}^{aij}\right)+\\
&+ig\bar\lambda_i^a\gamma^\mu\epsilon^j\left(-\delta^i_j F_{abkl}P_\mu{}^{bkl}-\frac{1}{2}\delta^i_j E_{kl}P_\mu{}^{akl}\right)+\dots
\end{aligned} \tag{4.3.51}$$

$$\begin{aligned}
\delta\left(\frac{1}{8}D_\mu M_{\alpha\beta}D^\mu M^{\alpha\beta}\right) =& \frac{1}{4}g\left(\delta A_\mu{}^{M\gamma}\xi_{(\alpha M}M_{\beta)\gamma} - \delta A_\mu{}^{M\delta}\xi_{\epsilon M}\epsilon_{\delta(\alpha}\epsilon^{\epsilon\gamma}M_{\beta)\gamma}\right)\left(P^\mu\mathcal{V}^{*\alpha}\mathcal{V}^{*\beta}+\text{h.c.}\right)+\dots\\
=& -\frac{1}{2}ig\delta A_\mu^{M\alpha}\mathcal{V}^*_\alpha\xi_{\beta M}\mathcal{V}^{*\beta}P^\mu + \text{h.c.}+\dots\\
=& g\bar\psi_i^\mu\epsilon_j\left(2E^{ij}P^*_\mu\right)+g\bar\chi_i\gamma^\mu\epsilon_j\left(-E^{\dagger ij}P_\mu\right)+ig\bar\lambda_i^a\gamma^\mu\epsilon^j\left(\delta^i_j E^\dagger_a P_\mu\right)+\dots
\end{aligned} \tag{4.3.52}$$

Now consistency relations can be extrapolated:

$$D_\mu A_1^{ij} = A_2^{(ij)}P^*_\mu - 3\bar A_{2a}{}^{(i}{}_k P_\mu{}^{akj)}\ , \tag{4.3.53}$$

$$\frac{4}{3}A_2^{[ij]}P^*_\mu - 4\bar A_{2a}{}^{[i}{}_k P_\mu{}^{akj]} = +E^\dagger{}_a P_\mu{}^{aij} - 4F^\dagger{}_a{}^{[i}{}_k P_\mu{}^{akj]} - 2E^{ij}P^*_\mu\ , \tag{4.3.54}$$

$$4A_3^{ij}P^*_\mu + 2\bar A_{3a}{}^i{}_k P_\mu{}^{akj} = \frac{2}{3}A_1^{ij}P_\mu - \frac{2}{3}D_\mu A_2^{ji} + 2F_{ak}{}^{[i}P_\mu{}^{aj]k} - \frac{1}{2}E_a P_\mu{}^{aij} + E^{\dagger ij}P_\mu\ , \tag{4.3.55}$$

$$4A_{3ab}{}^{ik}P_\mu{}^b{}_{jk} + 2A_{3aj}{}^i P_\mu = 2D_\mu A_{2aj}{}^i + \frac{4}{3}\bar A_{1jk}P_\mu{}^{aik} + \delta^i_j F_{abkl}P_\mu{}^{bkl} + \frac{1}{2}\delta^i_j E_{kl}P_\mu{}^{akl} - \delta^i_j E^\dagger_a P_\mu\ . \tag{4.3.56}$$

Actually, these are eight consistency constraints if we consider separately the two types of scalar:

$$D_\mu = D_\mu\phi^x D_x + D_\mu\phi^s D_s\ . \tag{4.3.57}$$

From (4.3.53) we read the symmetric part of matrix A_2^{ij}:

$$A_2^{(ij)} = F^{ij}\ , \tag{4.3.58}$$

while (4.3.54) fixes the antisymmetric part[2]

$$A_2^{[ij]} = -\frac{3}{2}E^{ij}\ . \tag{4.3.59}$$

A combination of (4.3.53) and (4.3.54) projected on the $SO(6,n)/SO(n)\times SO(6)$ scalar manifold also completely defines the second mixing mass matrix:

$$A_{2ai}{}^j = F_{ai}{}^j - \frac{1}{4}\delta^i_j E_a\ . \tag{4.3.60}$$

[2]It has to be noted that the relative sign between symmetric and antisymmetric parts is different from [7].

Spin-1/2 fermions mass matrices can be find out using (4.3.55) and (4.3.56). As we anticipated by group theoretical considerations, equation (4.3.55) tells us that

$$A_3^{ij} = 0 \ . \tag{4.3.61}$$

${A_{3ai}}^j$ mass matrix is equivalently determined by either (4.3.55) or (4.3.56):

$$ {A_{3ai}}^j = {F^\dagger_{ai}}^j - \frac{3}{4}\delta_i^j E^\dagger_a \ . \tag{4.3.62}$$

Finally, considering (4.3.56) on $SO(6,n)/SO(n)\times SO(6)$ scalar manifold and after projecting with an inverse of the vielbein ${P^{xb}}_{jk}$, it can be shown that

$$ {A_{3ab}}^{ij} = {F_{ab}}^{ij} + \frac{1}{3}\delta_{ab}F^{ij} \ . \tag{4.3.63}$$

4.4 Quadratic closure relations

We have to require that supersymmetry variations close also at order g^2. In particular, if we consider terms proportional to $\bar\psi_\mu\gamma^\mu\epsilon$, $\bar\chi\epsilon$ and $\bar\lambda\epsilon$ at second order, we find consistency relations which are quadratic in the mass matrices (as sketched in section 4.1.2). In particular, these terms come from the supersymmetric variations of the fermion masses (4.3.46) and of the scalar potential sector:

$$\begin{aligned} e^{-1}\delta\left(\mathcal{L}_{\text{f.mass}}\right) =& g^2\bar\psi_{\mu i}\gamma^\mu\epsilon^j\left(\frac{4}{3}{A_1}^{ik}\bar A_{1jk} - \frac{4}{9}{A_2}^{ik}\bar A_{2jk} - 2{A_{2aj}}^k{\bar A_2}{}^i{}_k\right) + \text{h.c.}\\ &+g^2\bar\chi_i\epsilon^j\left(-\frac{8}{9}{A_2}^{ki}\bar A_{1kj} + 2{A_{2aj}}^k{\bar A_{3a}}{}^i{}_k\right) + \text{h.c.}\\ &+ig^2\bar\lambda_i^a\epsilon_j\left(\frac{8}{3}{A_{2ak}}^i{A_1}^{jk} + 4{A_{3ab}}^{ik}{\bar A_{2b}}{}^j{}_k - \frac{4}{3}{A_2}^{jk}{A_{3ak}}^i\right) + \text{h.c.}\ , \end{aligned} \tag{4.4.1}$$

and

$$\begin{aligned} \delta\left(\mathcal{L}_{\text{pot}}\right) &= -g^2\delta eV - g^2e\delta V\\ &= -g^2\left(-\bar\psi_{\mu i}\gamma^\mu\epsilon^i + \text{h.c.}\right)eV - g^2e\frac{\partial V}{\partial\phi^x}\delta\phi^x - g^2e\left(\frac{\partial V}{\partial\mathcal{V}_\alpha}\delta\mathcal{V}_\alpha + \text{h.c.}\right)\ , \end{aligned} \tag{4.4.2}$$

where we used the vielbein variation (3.5.4). Recalling the scalar variations (3.5.9) and (3.5.13), the quadratic consistency constraints read

$$\frac{1}{9}{A_2}^{ik}\bar A_{2jk} + \frac{1}{2}{A_{2aj}}^k{\bar A_{2a}}{}^i{}_k - \frac{1}{3}{A_1}^{ik}\bar A_{1kj} = \frac{1}{4}\delta^i_jV\ , \tag{4.4.3}$$

$$2{A_{2aj}}^k{\bar A_{3a}}{}^i{}_k - \frac{8}{9}{A_2}^{ki}\bar A_{1kj} = \delta^i_j\frac{\partial V}{\partial\mathcal{V}^*_\alpha}\mathcal{V}_\alpha\ , \tag{4.4.4}$$

$$\frac{8}{3}{A_{2ak}}^i{A_1}^{jk} + 4{A_{3ab}}^{ik}{\bar A_{2b}}{}^j{}_k - \frac{4}{3}{A_2}^{jk}{A_{3ak}}^i = 2i{P^x_a}^{ij}\frac{\partial V}{\partial\phi^x}\ . \tag{4.4.5}$$

For consistency, these relations have to be a linear composition of quadratic constraints in terms of T-tensor components. Indeed, it is easy to verify that on the left hand side of (4.4.3) only the trace component survives: we can express the A matrices in terms of partially dressed embedding tensor and use the quadratic constraints (D.1), (D.9), (D.13) and (D.23). The scalar potential in terms of partially dressed embedding tensor and scalar matrices reads

$$V = M^{\alpha\beta}\left(\frac{3}{8}E_{\alpha a}E_{\beta a} - \frac{2}{9}F_{\alpha ij}F_{\beta}{}^{ij} - \frac{1}{2}F_{\alpha ai}{}^{j}F_{\beta aj}{}^{i}\right) + \frac{4}{9}i\epsilon^{\alpha\beta}F_{\alpha ij}F_{\beta}{}^{ij} \ , \tag{4.4.6}$$

and it can be easily verified that this result coincides exactly with (4.2.29), by using properties of scalar representatives. The equation (4.4.3) is the so-called *supersymmetric Ward identity.*

Similarly, (4.4.4) and (4.4.5) are also consequences of the quadratic constraint. However, they show precise relations between fermion mass matrices and the first derivative of the potential. Let us stress important feature: quadratic constraints impose that traceless components of the left hand side of (4.4.4) is zero. Furthermore, they require that the symmetric part in i, j of the left hand side of (4.4.5) is identically zero, because the right hand side is vanishing. Thus, we can split (4.4.5) into:

$$\frac{8}{3}A_{2ak}{}^{(i}A_{1}{}^{j)k} + 4A_{3ab}{}^{(ik}\bar{A}_{2b}{}^{j)}{}_{k} - \frac{4}{3}A_{2}{}^{(jk}A_{3ak}{}^{i)} = 0 \tag{4.4.7}$$

and

$$\frac{8}{3}A_{2ak}{}^{[i}A_{1}{}^{j]k} + 4A_{3ab}{}^{[ik}\bar{A}_{2b}{}^{j]}{}_{k} - \frac{4}{3}A_{2}{}^{[jk}A_{3ak}{}^{i]} = 2iP^{x}{}_{a}{}^{ij}\frac{\partial V}{\partial\phi^{x}} \ . \tag{4.4.8}$$

We are interested in vacuum configurations. This means that the vacuum expectation values of the scalars are such that

$$\left.\frac{\partial V}{\partial\phi^{\underline{x}}}\right|_{\langle\phi^{\underline{x}}\rangle} = 0 \ , \tag{4.4.9}$$

for any $\underline{x} = x,\ s$, where $\langle\phi^{\underline{x}}\rangle$ is the vacuum expectation value of the scalar fields. In particular, when considering *Minkowski vacua* (that is with zero *cosmological constant* $\Lambda = \langle V\rangle = 0$), equations (4.4.3), (4.4.4) and (4.4.5) become

$$\left.\frac{1}{9}A_{2}{}^{ij}\bar{A}_{2ij} + \frac{1}{2}A_{2ai}{}^{j}\bar{A}_{2a}{}^{i}{}_{j} - \frac{1}{3}A_{1}{}^{ij}\bar{A}_{1ij}\right|_{\langle\phi^{\underline{x}}\rangle} = 0 \ , \tag{4.4.10}$$

$$\left.2A_{2ai}{}^{j}\bar{A}_{3a}{}^{i}{}_{j} - \frac{8}{9}A_{2}{}^{ij}\bar{A}_{1ij}\right|_{\langle\phi^{\underline{x}}\rangle} = 0 \ , \tag{4.4.11}$$

$$\left.\frac{8}{3}A_{2ak}{}^{i}A_{1}{}^{jk} + 4A_{3ab}{}^{ik}\bar{A}_{2b}{}^{j}{}_{k} - \frac{4}{3}A_{2}{}^{jk}A_{3ak}{}^{i}\right|_{\langle\phi^{\underline{x}}\rangle} = 0 \ . \tag{4.4.12}$$

Let us characterize group theoretically these conditions. The condition (4.4.10), which gives Minkowski vacua, has zero charge under $U(1)$ transformations and it is a singlet of $SO(n)\times SO(6)$. The condition (4.4.11) is also a singlet of $SO(n)\times SO(6)$ but has a non-trivial charge under $U(1)$: under the action of H it transforms according to the representation

$$(\bullet, \mathbf{1})_{+2} \ , \tag{4.4.13}$$

which means that it is a complex condition on scalars. Equation (4.4.12) is a constraint on the antisymmetric part of the indices i, j, because (4.4.7) holds thanks to the quadratic constraint. The representation of $U(1)\times SO(n)\times SO(6)$, according to which (4.4.8) transforms, is

$$(\square, \mathbf{6})_{0} \ , \tag{4.4.14}$$

and, since $\mathbf{6}$ and $\square$ are both real, we have $6n$ real conditions on the scalar.

Chapter 5

Mass matrices and supertraces

In chapter 2 we explained how quantum corrections to the potential are linked to the traces of mass matrices of the different fields in the theory we are considering. In particular, divergent corrections depend on the traces of quadratic and quartic mass matrices and finite corrections are related to the traces of higher even powers of these matrices. In this chapter, we are going to define a general method to identify mass matrices for various fields, with arbitrary spin. Then, we will give the explicit forms of the bosons and fermions entering the Lagrangian of $N = 4$, $D = 4$ supergravity theories. Finally, we will check that all the machinery we have shown until now gives

$$\mathrm{STr}\mathcal{M}^2 = 0 \ , \tag{5.0.1}$$

not only for those cases which are truncations of $N = 8$ supergravities, but for *any* number of vector multiplets and *any* gauging.

5.1 Mass matrices - General overview

The computation of supertrace mass formulae requires the knowledge of the spectrum of quadratic fluctuations at the maximally symmetric critical points described by the equations (4.4.10)-(4.4.12). In general, the matrices determining such spectrum can be computed by *linearizing* the field equations of motion. But this needs further speculations in order to determine the mass matrix of the spin-$\frac{1}{2}$ fermions, for which we will give a detailed analysis in the following and in the next section.

As we showed in the previous chapters, bosons fields can be expressed in different, but equivalent ways. In order to avoid multi-indices structures and, as a consequence, the possible overcounting of field when taking traces, we believe it is more convenient to express mass matrices in the following representations of the fields: we consider the $6n + 2$ coordinate fields $\phi^{\underline{x}} = \phi^x$, ϕ^s of target manifolds for the scalars and $A_\mu^{M\alpha}$ for electric and magnetic vectors, instead of the alternative

$$\left(A_\mu^{a\alpha},\, A_\mu^{ija}\right) \equiv \left({L_M}^a,\, {L_M}^{ij}\right) A_\mu^{M\alpha} \ , \tag{5.1.1}$$

used in [29] for *maximal supergravity* in four dimensions.

Trivially, $\left(\frac{1}{\sqrt{2}}\chi_i,\, \lambda_a^i\right)$ is the spin-$\frac{1}{2}$ fermion multiplet (the structure of the indices is such that the multiplet has defined chirality). Gravitinos are redefined:

$$\tilde{\psi}_\mu^i = \psi_\mu^i + \alpha\gamma_\mu\zeta^i \ , \tag{5.1.2}$$

where α is a numerical coefficient and ζ^i are linear combinations of spin-$\frac{1}{2}$ fermions such that mass mixing terms between spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ vanish and which are massless on vacua completely breaking supersymmetry and are the so-called *goldstinos*.

Let us recall that kinetic terms of bosons entering the $N = 4$ supergravity Lagrangian are not canonical, as it can be immediately seen from (3.2.1) and (3.3.3). Then a closer look tells us that mass matrices do not coincides with those of equations (2.3.4) and (2.3.8), but we need to give a correction. The right way of identifying bosonic squared mass matrices is to write linearized equations of motion, that is for a generic field φ^I:

$$\Box\varphi^I = M_b^{2\,I}{}_J\varphi^J + \text{higher order terms} \ , \tag{5.1.3}$$

where M_b^2 is the bosonic quadratic mass matrix.

Similarly, for fermionic degrees of freedom, proper mass matrices M_f are defined by

$$\partial\!\!\!/\varphi^I = M_f{}^{IJ}\varphi_J \ , \tag{5.1.4}$$

and their squares are

$$M_f^{2\,I}{}_J = M_f{}^{IK} M_{fJK} \ , \tag{5.1.5}$$

where $M_{fIJ} = \left(M_f{}^{IJ}\right)^*$.

5.2 Mass matrices in half-maximal supergravities

5.2.1 Scalar fields

We need only to consider the following sector of the Lagrangian

$$e^{-1}\mathcal{L}_s = -\frac{1}{2} g_{\underline{x}\,\underline{y}} D_\mu\phi^{\underline{x}} D^\mu\phi^{\underline{y}} - g^2 V\left(\phi^{\underline{x}}\right) \ , \tag{5.2.1}$$

because other terms give higher order contributions in the fields which are not relevant for the present analysis, as we explained above. We have to stress some considerations: in the ungauged theories there is not a scalar potential, thus one cannot have supersymmetry breaking Minkowski vacua. When we gauged the theory, the scalar potential necessarily appears to preserve the invariance under properly covariantized supersymmetric variations, thus scalar expectation values on the vacuum can lead to spontaneous supersymmetry breaking.

Let us denote by $\langle\phi^{\underline{x}}\rangle$ the scalar vacuum expectation values. As previously discussed, we are interested in Minkowski vacua that completely break supersymmetry. We expand the scalar potential in powers of $\phi^{\underline{x}}$ around the expectation value $\langle\phi^{\underline{x}}\rangle$, which has to be a solution of equations (4.4.10)-(4.4.12):

$$
\begin{aligned}
V\left(\phi^{\underline{x}}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\, D_{(\underline{x}_1} \cdots D_{\underline{x}_n)} V \bigg|_{\langle\phi^{\underline{x}}\rangle} \tilde{\phi}^{\underline{x}_1} \cdots \tilde{\phi}^{\underline{x}_n} \\
&= \langle V \rangle + \frac{\partial V}{\partial \phi^{\underline{x}}}\bigg|_{\langle\phi^{\underline{x}}\rangle} \tilde{\phi}^{\underline{x}} + \frac{1}{2}\, D_{(\underline{x}_1} D_{\underline{x}_2)} V \bigg|_{\langle\phi^{\underline{x}}\rangle} \tilde{\phi}^{\underline{x}_1} \tilde{\phi}^{\underline{x}_2} + \ldots \\
&= \frac{1}{2}\, D_{(\underline{x}_1} D_{\underline{x}_2)} V \bigg|_{\langle\phi^{\underline{x}}\rangle} \tilde{\phi}^{\underline{x}_1} \tilde{\phi}^{\underline{x}_2} + \ldots\ ,
\end{aligned}
\tag{5.2.2}
$$

where we have defined $\tilde{\phi}^{\underline{x}} = \phi^{\underline{x}} - \langle \phi^{\underline{x}} \rangle$ and in the third line we have imposed the Minkowski vacua and critical points conditions. We can write

$$
e^{-1}\mathcal{L}_s = -\frac{1}{2} g_{\underline{x}\,\underline{y}} D_\mu \tilde{\phi}^{\underline{x}} D^\mu \tilde{\phi}^{\underline{y}} - \frac{1}{2}\, D_{(\underline{x}_1} D_{\underline{x}_2)} V \bigg|_{\langle\phi^{\underline{x}}\rangle} \tilde{\phi}^{\underline{x}_1} \tilde{\phi}^{\underline{x}_2} + \text{higher order terms}\ , \tag{5.2.3}
$$

and the Euler–Lagrange equations are

$$
D_\mu D^\mu \tilde{\phi}^{\underline{x}} = g^{\underline{x}\,\underline{y}} D_{(\underline{y}} D_{\underline{z})} V\, \tilde{\phi}^{\underline{z}}\ . \tag{5.2.4}
$$

Therefore, the scalar mass matrix is

$$
{\mathcal{M}_0^2}^{\underline{x}}{}_{\underline{y}} = g^{\underline{x}\,\underline{z}} D_{(\underline{z}} D_{\underline{y})} V \bigg|_{\langle\phi^{\underline{x}}\rangle} \equiv \begin{pmatrix} {M^x}_y & {M^x}_t \\ {M^s}_y & {M^s}_t \end{pmatrix}\ , \tag{5.2.5}
$$

where

$$
{M^x}_y = g^{xz} D_{(z} D_{y)} V \bigg|_{\langle\phi^{\underline{x}}\rangle}\ , \tag{5.2.6}
$$

$$
{M^x}_s = g^{xy} D_y D_s V \bigg|_{\langle\phi^{\underline{x}}\rangle}\ , \tag{5.2.7}
$$

$$
{M^s}_x = g^{st} D_t D_x V \bigg|_{\langle\phi^{\underline{x}}\rangle}\ , \tag{5.2.8}
$$

$$
{M^s}_t = g^{su} D_{(u} D_{t)} V \bigg|_{\langle\phi^{\underline{x}}\rangle}\ . \tag{5.2.9}
$$

To give the explicit form of this matrices is a mere computational task, once we recall the explicit form of the scalar potential in terms of the partially-dressed embedding tensors (4.4.6) and use derivative relations of these tensors, shown in appendix C. We will not give here the explicit form, which is very long and not illuminating.

5.2.2 Vector fields

We have already written the equations of motion for vector fields in (4.2.32), but now we have to pay attention to some details. Let us consider first the kinetic part only, which can be rewritten in a more useful and compact way:

$$
\begin{aligned}
e^{-1}\, \delta\mathcal{L}_{\text{bos}}\big|_{\text{kinetic}} &= \delta A_\mu^{M+} \left(-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} \eta_{MN} D_\nu \mathcal{G}_{\rho\sigma}^{N-}\right) + \delta A_\mu^{M-} \left(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} \eta_{MN} D_\nu \mathcal{G}_{\rho\sigma}^{N+}\right) \\
&= \delta A_\mu^{M\alpha} M_{\alpha\beta} M_{MN} D_\nu \mathcal{G}^{\nu\mu N\beta} + \ldots \\
&= \delta A_\mu^{M\alpha} M_{\alpha\beta} M_{MN} D_\nu \mathcal{H}^{\nu\mu N\beta} + \ldots\ ,
\end{aligned}
\tag{5.2.10}
$$

where in the last step we have used the on-shell equality between the 2-forms $\mathcal{G}^{\mu\nu M\alpha}$ and $\mathcal{H}^{\mu\nu M\alpha}$.

As it can be seen from (2.3.8) or equivalently from the equations of motion (4.2.32), the mass term for gauge bosons comes from the covariant kinetic terms of scalar fields and, to be more precise, from terms of order g^2:

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{s.kin}} =& \frac{1}{16}D_\mu M_{MN}D^\mu M^{MN} + \frac{1}{8}D_\mu M_{\alpha\beta}D^\mu M^{\alpha\beta} \\ \overset{(g^2)}{=}& \frac{1}{8}A^{\mu M\alpha}A_\mu^{N\beta}\left(\eta^{PR}\eta^{QS}\Theta_{\alpha MPQ}\Theta_{\beta NRS} - M^{PR}M^{QS}\Theta_{\alpha MPQ}\Theta_{\beta NRS}\right) + \\ &+\frac{1}{4}A^{\mu M\alpha}A_\mu^{N\beta}\left(2\xi_{\alpha[M}\xi_{\beta N]} - M_{\alpha\beta}M^{\gamma\delta}\xi_{\gamma M}\xi_{\delta N}\right) \ . \end{aligned} \tag{5.2.11}$$

Therefore, the squared mass matrix $\mathcal{M}_1^{2\,\mathcal{M}}{}_{\mathcal{N}}$ of the electric and magnetic electric fields is

$$\begin{aligned} \mathcal{M}_1^{2\,M\alpha}{}_{N\beta} = M^{MP}M^{\alpha\gamma}\bigg(&\frac{1}{4}M^{QS}M^{RT}\Theta_{\gamma PQR}\Theta_{\beta NST} - \frac{1}{4}\eta^{QS}\eta^{RT}\Theta_{\gamma PQR}\Theta_{\beta NST} + \\ &+\frac{1}{2}M_{\gamma\beta}M^{\delta\epsilon}\xi_{\delta P}\xi_{\epsilon N} - \xi_{\gamma[P}\xi_{\beta N]}\bigg) \ . \end{aligned} \tag{5.2.12}$$

This is a $12+2n \times 12+2n$ matrix. Since $6+n$ vectors are not physical, when diagonalizing such matrix we should find $6+n$ null eigenvalues at least.

5.2.3 Fermionic fields

Kinetic terms of fermionic fields are standard, but we have to pay attention to numerical factors, which are different for the three types of fermions entering the Lagrangian.

The complex feature of this calculation is the presence of mass mixing terms between spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fermions. Indeed, as we said before, it is necessary to introduce new gravitino fields in order to diagonalize the fermions mass matrix. This is the so-called *Super-Higgs Mechanism*: for ordinary global symmetries, the *Goldstone theorem* states that for every broken generator there is a massless scalar particle in the spectrum. When global symmetry is gauged, the *Higgs mechanism* describes how these massless modes are *eaten* by gauge bosons, which become massive. In supersymmetric theories, fermionic generators are broken and in non-supersymmetric vacua a massless fermions enter the spectrum, the so-called *goldstinos*. If supersymmetry is local, gravitinos play the role of gauge fields and on these vacua they *eat* goldstinos, such that they acquire $\pm\frac{1}{2}$ polarizations and become massive.

In order to diagonalize the mass matrix we define:

$$\tilde{\psi}^i_\mu = \psi^i_\mu + \frac{1}{6}\gamma_\mu\zeta^i \ , \tag{5.2.13}$$

where ζ^i are the goldstinos, which are a proper combination of spin-$\frac{1}{2}$ fermions:

$$\zeta^i = A_1^{-1ij}A_{2jk}\chi^k - 3iA_1^{-1ij}A_{2aj}{}^k\lambda^a_k \ . \tag{5.2.14}$$

The mass terms for gravitinos become

$$\begin{aligned} (ge)^{-1}\mathcal{L}_{\psi.\text{mass}} =& \frac{1}{3}A_1^{ij}\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j} + \frac{1}{3}A_2^{ij}\bar{\psi}_{\mu i}\gamma^\mu\chi_j + iA_{2a}{}^i{}_j\bar{\psi}^i_\mu\gamma^\mu\lambda^a_j + \text{h.c.} \\ =& \frac{1}{3}A_1^{ij}\bar{\tilde{\psi}}_{\mu i}\gamma^{\mu\nu}\tilde{\psi}_{\nu j} - \frac{1}{9}A_1^{ij}\bar{\zeta}_i\zeta_j + \frac{2}{9}A_2^{ij}\bar{\zeta}_i\chi_j + \frac{2}{3}iA_{2a}{}^i{}_j\bar{\zeta}_i\lambda^j_a \ . \end{aligned} \tag{5.2.15}$$

Linearized equations of motion for gravitinos then become:

$$\gamma^{\mu\nu\rho} D_\nu \tilde{\psi}^i_\rho = \frac{2}{3} A^{ij} \gamma^{\mu\nu} \tilde{\psi}_{\nu j} \ , \tag{5.2.16}$$

and the squared mass matrix of gravitinos is

$$\mathcal{M}^{2\,i}_{\frac{3}{2}\ \ j} = \frac{4}{9} A_1^{ik} A_{1jk} \ . \tag{5.2.17}$$

Let us stress a fundamental feature: A_1^{-1ij} have to exist, *i.e.* all its four eigenvalues, which coincide to the masses of gravitinos, have to be different from zero. That is the vacua, with which we are dealing, completely break supersymmetry. Indeed, the supersymmetric variations of the fermions are proportional to shift matrices A_1 and A_2 and requiring A_1 to be invertible is equivalent to say that it does not exist a fermion variation ϵ^i such that

$$A_1^{ij} \epsilon_j = 0 \ . \tag{5.2.18}$$

For ordinary fermions the form of the mass matrix is more complicated. In order to take into account the different coefficients of the two kinetic terms, we have to consider the fermion multiplet

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \chi^i \\ \lambda^a_i \end{pmatrix} , \tag{5.2.19}$$

the mass matrix of which takes the form

$$\mathcal{M}_{\frac{1}{2}} = \begin{pmatrix} 0 & \sqrt{2} i A_{3ai}{}^{j} \\ \sqrt{2} i A_{3aj}{}^{i} & -2 A_{3ab}{}^{ij} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} A_1^{-1kl} A_{2kj} A_{2li} & -\frac{2\sqrt{2}}{3} i A^{-1kl} A_{2bk}{}^{j} A_{2li} \\ -\frac{2\sqrt{2}}{3} i A^{-1kl} A_{2kj} A_{2bl}{}^{i} & -2 A^{-1kl} A_{2ak}{}^{j} A_{2bl}{}^{i} \end{pmatrix} , \tag{5.2.20}$$

where the second term comes from the interactions in (5.2.15) containing goldstinos.

The goldstinos will be null eigenvectors of this matrix, hence they will not contribute to the supertrace formulas. We note that (5.2.20) contains the inverse of gravitinos shift and this can bring some problems in the computation of the traces. But we are going to present a clever method to compute the traces of even powers of this matrix using only the original matrix

$$\mathbf{M}_{\frac{1}{2}} = \begin{pmatrix} 0 & \sqrt{2} i A_{3ai}{}^{j} \\ \sqrt{2} i A_{3aj}{}^{i} & -2 A_{3ab}{}^{ij} \end{pmatrix} , \tag{5.2.21}$$

provided that a suitable factor, which is proportional to the trace of the corresponding power of the mass matrix of gravitinos, is subtract from the final result [4]. The argument uses the fact that the term we added to the spin-$\frac{1}{2}$ mass matrix is fully projected in the goldstino directions in order to make their masses vanish. Let us give some more details. First, we diagonalize the gravitino mass matrix by means of a unitary matrix U, such that $A_1 = U^T D U$ and D is diagonal whose entries are the gravitino masses $m_A > 0$, where $A = 1, \ldots 4$. The eigenvectors of A_1 can be constructed by applying the matrix U to the orthonormal real basis of eigenvectors of D, which we write as $e^i_A = \delta^i_A$. Indeed, the vectors $V_A = U^{-1} e_A$ satisfy

$$\frac{2}{3} A_{1ij} V^j_A = m_A V_{Ai} \ , \tag{5.2.22}$$

where there is no sum on index A on the right hand side and $V_{Ai} = \left(V^i_\alpha\right)^*$. As follows from (5.2.14), the goldstino directions are given by

$$\begin{pmatrix} A_2^{ik} \\ -3i A_{2a}{}^{k}{}_{i} \end{pmatrix} V_{Ak} \ , \tag{5.2.23}$$

which are eigenvectors of the original spin-$\frac{1}{2}$ mass matrix with eigenvalue $-2m_A$, where m_A is the corresponding gravitino mass. This can be shown using quadratic constraints and equation (5.2.22):

$$\mathbf{M}_{\frac{1}{2}}\begin{pmatrix} A_2^{ik} \\ -3iA_{2a}{}^{k}{}_{i} \end{pmatrix} V_{Ak} = -2m_A \begin{pmatrix} A_{2ik} \\ -3iA_{2ak}{}^{i} \end{pmatrix} V_A^k \ . \tag{5.2.24}$$

Also, recalling that vacua has vanishing cosmological constant (4.4.10), we can show that goldstino is a null eigenvector of $\mathcal{M}_{\frac{1}{2}}$:

$$\left(\mathcal{M}_{\frac{1}{2}} - \mathbf{M}_{\frac{1}{2}}\right)\begin{pmatrix} A_2^{ik} \\ -3iA_{2a}{}^{k}{}_{i} \end{pmatrix} V_{Ak} = 2m_A \begin{pmatrix} A_{2ik} \\ -3iA_{2ak}{}^{i} \end{pmatrix} V_A^k \ . \tag{5.2.25}$$

By construction $\left(\mathcal{M}_{\frac{1}{2}} - \mathbf{M}_{\frac{1}{2}}\right)$ is fully projected on the goldstino directions and therefore orthogonal directions are null eigenvectors for such matrix. This implies that when we compute the sum of the eigenvalues of $\mathcal{M}_{\frac{1}{2}}^{2n}$, we can actually compute the trace of $\mathbf{M}_{\frac{1}{2}}^{2n}$ and subtract from the result 2^{2n} times the trace of gravitinos mass matrix to the same power. Explicitly

$$\mathrm{Tr}\,\mathcal{M}_{\frac{1}{2}}^2 = \mathrm{Tr}\,\mathbf{M}_{\frac{1}{2}}^2 - 4\,\mathrm{Tr}\,\mathcal{M}_{\frac{3}{2}}^2 \ . \tag{5.2.26}$$

5.3 Supertrace of quadratic mass matrix

We are now in position that allows us to compute the quadratic supertrace

$$\mathrm{Str}\mathcal{M}^2 = \mathrm{Tr}\,\mathcal{M}_0^2 - 2\,\mathrm{Tr}\,\mathcal{M}_{\frac{1}{2}}^2 + 3\,\mathrm{Tr}\,\mathcal{M}_1^2 - 4\,\mathrm{Tr}\,\mathcal{M}_{\frac{3}{2}}^2 = 0 \ . \tag{5.3.1}$$

In particular, we will show that it is zero, which proves that there are no quadratic divergences in fully broken half-maximal supergravity at one-loop level.

These traces have no free index, thus they are $SO(6)\times SO(n)$ singlets. Obviously the charge for $U(1)$ phase transformations is zero: adopting the conventions of chapter 4, we can say that the supertrace transforms according to $(\bullet,\mathbf{1})_0$. Moreover it will be shown that these traces can be expressed in terms of linear combination of fully contracted quadratic partially dressed embedding tensors.

In order to simplify the supertrace formula, we can use the quadratic constraint, the condition that imposes $V=0$ and the critical point conditions $DV=0$. However, since $\mathrm{Str}\mathcal{M}^2$ is quadratic in the irreducible components of the embedding tensor and it transforms in the trivial representation of H, the only useful quadratic constraint is (D.2), with $SL(2)$-indices contracted with $M^{\alpha\beta}$, such that the $U(1)$ charge is zero:

$$M^{\alpha\beta}E_{\alpha ij}E_{\beta}{}^{ij} = M^{\alpha\beta}E_{\alpha a}E_{\beta a} \ . \tag{5.3.2}$$

The critical point conditions (4.4.11) and (4.4.12) transform according to $(\bullet,\mathbf{1})_{+2}$ (and its complex conjugate) and $(\square,\mathbf{6})_0$ respectively. Therefore they will be useless for the present analysis. Nevertheless, the vanishing cosmological constant condition transforms in an appropriate way:

$$\frac{3}{8}E_{\alpha a}E_{\beta a}M^{\alpha\beta} - \frac{2}{9}F_{\alpha ij}F_{\beta}{}^{ij}M^{\alpha\beta} - \frac{1}{2}F_{\alpha ai}{}^{j}F_{\alpha aj}{}^{i}M^{\alpha\beta} + \frac{4}{9}iF_{\alpha ij}F_{\beta}{}^{ij}\epsilon^{\alpha\beta} = V = 0 \ . \tag{5.3.3}$$

Let us start from

$$\mathrm{Tr}\,\mathcal{M}_0^2 = \mathcal{M}_0^{2\,\underline{x}}{}_{\underline{x}} = M^x{}_x + M^s{}_s \ . \tag{5.3.4}$$

Using derivative properties of partially dressed embedding tensors (appendix C) and scalar matrices (3.2.37) and (3.2.38), we obtain a linear combination of terms proportional to quadratic partially dressed embedding tensors and to

$$g^{xy} P_{xaij} P_{yb}{}^{kl} = \delta_{ab} \delta_i^{[k} \delta_j^{l]} \ , \qquad g^{st} P_s P_t^* = 1 \ . \tag{5.3.5}$$

Thus we immediately obtain the trace in terms of $E_{\alpha\bullet}$ and $F_{\alpha\bullet}$ only:

$$\begin{aligned} M^x{}_x =& \frac{9}{2} E_{\alpha a} E_{\beta a} M^{\alpha\beta} + 2 F_{\alpha a i}{}^j F_{\beta a j}{}^i M^{\alpha\beta} + 5 F_{\alpha a b i j} F_{\alpha a b}{}^{ij} M^{\alpha\beta} + \frac{3}{4} n E_{\alpha a} E_{\beta a} M^{\alpha\beta} \\ & - 2n F_{\alpha a i}{}^j F_{\beta a j}{}^i M^{\alpha\beta} + \frac{8}{3} i n F_{\alpha i j} F_\beta{}^{ij} \epsilon^{\alpha\beta} \ , \end{aligned} \tag{5.3.6}$$

and

$$M^s{}_s = \frac{3}{2} E_{\alpha a} E_{\beta a} M^{\alpha\beta} - \frac{8}{9} F_{\alpha i j} F_\beta{}^{ij} M^{\alpha\beta} - 2 F_{\alpha a i}{}^j F_{\beta a j}{}^i M^{\alpha\beta} \ . \tag{5.3.7}$$

Consider now the trace of the mass matrix of the gauge bosons:

$$\begin{aligned} \mathrm{Tr}\,\mathcal{M}_1^2 = \mathcal{M}_1^{2\,\mathcal{M}}{}_{\mathcal{M}} =& \frac{1}{4} M^{\alpha\beta} M^{MN} \Big(M^{PR} M^{QS} \Theta_{\alpha MPQ} \Theta_{\beta NRS} - \eta^{PR} \eta^{QS} \Theta_{\alpha MPQ} \Theta_{\beta NRS} \Big) \\ & + M^{MN} M^{\alpha\beta} \xi_{\alpha M} \xi_{\beta N} \ . \end{aligned} \tag{5.3.8}$$

We will now use the relations (3.2.49) and (3.2.50), *i.e.* we expand scalar matrices M^{MN} and η^{MN} in terms of representatives of $SO(6,n)$. To express the trace in terms of the T-tensor irreducible components it is a matter of distributing, substituting and recalling properties in appendix C. We found

$$\mathrm{Tr}\,\mathcal{M}_1^2 = \frac{7}{2} E_{\alpha a} E_{\beta a} M^{\alpha\beta} - 2 F_{\alpha a i}{}^j F_{\beta a j}{}^i M^{\alpha\beta} + F_{\alpha a b i j} F_{\alpha a b}{}^{ij} M^{\alpha\beta} + \frac{1}{4} n E_{\alpha a} E_{\beta a} M^{\alpha\beta} \ . \tag{5.3.9}$$

The trace of gravitinos mass matrix squared is the simplest one:

$$\mathrm{Tr}\,\mathcal{M}_{\frac{3}{2}}^2 = \frac{4}{9} A_1^{ij} A_{1ij} = \frac{4}{9} F_{\alpha i j} F_\beta{}^{ij} M^{\alpha\beta} - \frac{4}{9} i F_{\alpha i j} F_\beta{}^{ij} \epsilon^{\alpha\beta} \ . \tag{5.3.10}$$

Finally, the trace of spin-$\frac{1}{2}$ fermions mass matrix is

$$\mathrm{Tr}\,\mathcal{M}_{\frac{1}{2}}^2 = \mathrm{Tr}\,\mathbf{M}_{\frac{1}{2}}^2 - 4\,\mathrm{Tr}\,\mathcal{M}_{\frac{3}{2}}^2 \ , \tag{5.3.11}$$

where, from equation (5.2.21), the explicit form of the first term is

$$\begin{aligned} \mathrm{Tr}\,\mathbf{M}_{\frac{1}{2}}^2 &= 4 A_{3ai}{}^j A_{3aj}{}^i + 4 A_{3abij} A_{3ab}{}^{ij} \\ &= -4 F_{\alpha a i}{}^j F_{\beta a j}{}^i M^{\alpha\beta} + 9 E_{\alpha a} E_{\beta a} M^{\alpha\beta} + 4 F_{\alpha a b i j} F_{\beta a b}{}^{ij} M^{\alpha\beta} \\ &\quad + \frac{4}{9} n F_{\alpha i j} F_\beta{}^{ij} M^{\alpha\beta} + \frac{4}{9} i n F_{\alpha i j} F_\beta{}^{ij} \epsilon^{\alpha\beta} \ . \end{aligned} \tag{5.3.12}$$

The final step is to sum properly all these traces:

$$\mathrm{STr}\mathcal{M}^2 = 4(n-1)V = 0 \ , \tag{5.3.13}$$

where we used the condition about vanishing cosmological constant (5.3.3). In particular, it can be highlighted how this result depends neither on which are the generators of the gauge group and nor on the number of the vector multiplets when we impose the condition $\langle V \rangle = 0$. This is a non-trivial result.

Chapter 6

Summary and conclusions

In this work, we began the study of one-loop effective potential in the half-maximal supergravity theories in four dimensions. The potential at one-loop can be expressed in terms of the supertraces of even powers of the mass matrix. Therefore, first we had to find out some proper terms of the Lagrangian, which gave us the exact form of the mass matrix of the entire spectrum. Then, we obtained a wide series of identities of the dressed embedding tensor, which are not known until now. In particular, in section 4.3.1 we found out group theoretically which are the irreducible components of the T-tensor and defined the partially dressed embedding tensors. These components allowed us to re-do many calculations of the paper [7] in a quicker way. Moreover, we showed how this machinery determines the handing down of quadratic constraints to T-tensor components and how mass matrices and their traces can be easily written down from it. Finally, we showed that there is no quadratic divergence of one-loop effective potential in half-maximal supergravities, no matter which are the generators of gauge group or the matter coupling.

Unfortunately, we were not able to go further with calculations because of the number of typos we encountered in starting this work. Until now we learned that in $N = 4$ gauged supergravities the one-loop effective potential has not quartic and quadratic divergences. Using the formalism we built up, it will be possible to extend the calculation to higher even powers of mass matrix. It will be interesting to study the behaviour of the supertrace of quartic mass matrix. We have no evidence to argue about the finiteness of one-loop potential. Thus, in principle we can not expect this supertrace to be zero in general.

However, the result for $N = 8$ gauged supergravities is well-known, as well as is known how to halve maximal supergravities in four dimensions [30]:

$$\mathrm{E}_{7(7)} \supset SL(2) \times SO(6,6) \ , \tag{6.1}$$

where $\mathrm{E}_{7(7)}$ is the global symmetry group of the $N = 8$, $D = 4$ supergravities and $SL(2) \times SO(6,6)$ is the global symmetry group of half-maximal ones, with $n = 6$ matter multiplets. It can be shown that not all $N = 4$ supergravities with $n = 6$ are related to the $N = 8$ via truncation. Indeed, in order to require this, the irreducible component of the embedding tensor $f_{\alpha MNP}$ has to satisfy additional quadratic constraints:

$$f_{\alpha MNP} f_{\beta}{}^{MNP} = 0 \ , \qquad\qquad \epsilon^{\alpha\beta} f_{\alpha[MNP} f_{\beta QRS]} = 0 \ . \tag{6.2}$$

Imposing these further constraints to be valid necessarily implies that quartic mass matrix has vanishing supertrace. However, if this supertrace will be not identically zero, we should find which conditions on the gaugings could turn this to be zero. Equations (6.2) are obviously a subset of these.

Finally we will have to study properties of supertrace of higher powers, *e.g.* positivity of $\mathrm{STr}\mathcal{M}^6$ or, when it vanishes, $\mathrm{STr}\mathcal{M}^8$, in order to study whether a small cosmological constant could be generated at quantum level or not. In particular, it will be interesting to know if the corrections to the potential are *always* negative, or positive. Probably the increasing of calculations amount will need new techniques to evaluate these traces, or we will be forced to analyse only known gaugings, which are simpler than the general formalism with embedding tensor. Then, the following step might be to analyse the effective potential of both $N = 8$ and $N = 4$ supergravities at two-loops, in order to make more clear the vanishing mechanism of higher loop divergences, if any, also in gauged supergravities.

Appendix A

Functional methods and the effective potential

We are going to introduce briefly some of the formalism that we used in the chapter 2.

A classical field theory is described by a *Lagrangian density* $\mathcal{L}(\phi_r, \partial\phi_r)$ and the classical field configuration can be found by extremizing the *classical action* functional defined by

$$I[\phi_r] \equiv \int \mathrm{d}^4x \, \mathcal{L}(\phi_r, \partial\phi_r) \ . \tag{A.1}$$

From now on, let us suppress the index r and consider only a real scalar field to simplify the notation. The general case can be trivially obtained from what follows. Let us introduce a *source term* in the Lagrangian

$$\begin{aligned} &\mathcal{L}(\phi, \partial\phi) \to \mathcal{L}(\phi, \partial\phi) + \phi(x)J(x) \ , \\ &I[\phi] \to I[\phi] + \int \mathrm{d}^4x \, \phi(x)J(x) \ . \end{aligned} \tag{A.2}$$

Now we can define the *generating functional* $Z[J]$ of *Green's functions*

$$Z[J] \equiv \int \mathcal{D}\phi \ e^{i\left(I[\phi] + \int \mathrm{d}^4x \, \phi(x)J(x)\right)/\hbar} = \left\langle \Omega^+ \middle| \Omega^- \right\rangle_J \ , \tag{A.3}$$

which is the transition amplitude from the *vacuum state* in the far past to the vacuum state in the far future in the presence of a source $J(x)$ of the field $\phi(x)$.

$$e^{iW[J]/\hbar} \equiv Z[J] \tag{A.4}$$

defines the *connected generating functional* $W[J]$. Indeed, we can expand it in Taylor series

$$W[J] = \sum_n \frac{1}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n G^{(n)}(x_1, \ldots, x_n) \, J(x_1) \ldots J(x_n) \ , \tag{A.5}$$

where $G^{(n)}(x_1, \ldots, x_n)$ are *connected Green's functions* with n external lines. We also define the *classical field*

$$\phi_c(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \frac{\langle \Omega^+ | \, \phi(x) \, | \Omega^- \rangle_J}{\langle \Omega^+ | \Omega^- \rangle_J} \ . \tag{A.6}$$

It is possible to invert this relation and to express $J(x)$ in terms of $\phi_c(x)$. By means of a functional Legendre transformation we define the quantum *effective action*

$$\Gamma[\phi_c] \equiv W[J(\phi_c)] - \int \mathrm{d}^4x J(x)\phi_c(x) \ , \tag{A.7}$$

which does not depend on $J(x)$.

It is easy to verify that

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x) \ , \tag{A.8}$$

which will be useful in what follows. We also expand the effective action in Taylor series

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n \Gamma^{(n)}(x_1, \dots, x_n)\, \phi_c(x_1) \dots \phi_c(x_n) \ , \tag{A.9}$$

and it can be shown that $\Gamma^{(n)}(x_1, \dots, x_n)$ are *1 particle irreducible connected Green's functions* with n external lines. We alternatively expand the effective action in powers of momentum of the field and we obtain

$$\Gamma[\phi_c] = \int \mathrm{d}^4x \left[-V(\phi_c) + \frac{Z(\phi_c)}{2} \partial_\mu\phi_c\partial^\mu\phi_c + (\text{higher order derivative}) \right] \ , \tag{A.10}$$

where $V(\phi_c)$ is the *effective potential* function. By comparing these two different expansions, it can be noted that the n^{th} derivative of $V(\phi_c)$ is the sum of all 1PI graphs with n vanishing external momenta (ϕ_c is constant). Indeed, the sum of Feynman graphs with n external lines correspond to $i-$times the Fourier transformation of the Green's functions $\tilde{\Gamma}^{(n)}(p_1, \dots, p_n)$, also called *proper vertices*:

$$(2\pi)^4\, \tilde{\Gamma}^{(n)}(p_1, \dots, p_n)\, \delta^4\left(\sum_{i=1}^n p_i\right) = \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n\, e^{i(x_1p_1 + \dots + x_np_n)}\, \Gamma^{(n)}(x_1, \dots, x_n) \ , \tag{A.11}$$

where δ^4 is due to translational invariance and $\tilde{\Gamma}^{(n)}(p_1, \dots, p_n)$ are evaluated with no propagators on the external lines. Thus, by equating expansions (A.9) and (A.10) with $p_i = 0\ \forall i$ ($\phi_c(x) = \phi_c$) it is easy to find

$$V(\phi_c) = i \sum_n \frac{1}{n!}\, i\tilde{\Gamma}^{(n)}(0, \dots, 0) \ . \tag{A.12}$$

This is important in the study of *spontaneous symmetry breaking*. Let us suppose that our Lagrangian density has an internal symmetry. SSB occurs if the field ϕ_c develops a *vacuum expectation value* $\langle\phi\rangle$ which does not respect the symmetry, even when the source $J(x)$ vanishes

$$\left.\frac{\delta\Gamma}{\delta\phi_c}\right|_{\phi_c = \langle\phi\rangle} = 0 \ . \tag{A.13}$$

Vacuum state has to be Poincaré invariant, *i.e.* Lorentz invariant $\langle\psi\rangle = \langle A_\mu\rangle = 0$, where ψ and A_μ are fermion and gauge fields respectively, and translationally invariant

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\phi_c}\right|_{\phi_c = \langle\phi\rangle} = 0 \ . \tag{A.14}$$

In order to require *stability*, the stationary point has to be a minimum of the effective potential. To explore the properties of the SSB theory we define a new field with vanishing expectation value $\phi' = \phi - \langle\phi\rangle$, which corresponds to the classical field $\phi'_c = \phi_c - \langle\phi\rangle$.

Appendix B

Fermion identities and conventions

For space-time flat metric mostly positive convention is chosen:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & \\ & \mathbb{I} \end{pmatrix} , \tag{B.1}$$

and the Levi-Civita is a totally antisymmetric proper space-time tensor such that

$$\epsilon^{0123} = e^{-1} , \qquad \epsilon_{0123} = -e . \tag{B.2}$$

As usual gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} , \qquad (\gamma_\mu)^\dagger = \eta^{\mu\nu}\gamma_\nu , \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 . \tag{B.3}$$

Vector indices of $SU(4)$ are raised and lowered by complex conjugation. However, for fermions we need the matrix

$$B = i\gamma_5\gamma_2 , \tag{B.4}$$

for example to define $\chi_i = B\big(\chi^i\big)^*$, in order to ensure that χ_i transforms as a Dirac spinor when χ^i does. The complex conjugate of chiral spinors has opposite chirality, *e.g.* χ_i is right-handed. For $\bar{\chi}_i = \big(\chi^i\big)^\dagger \gamma_0$, we define $\bar{\chi}^i = (\bar{\chi}_i)^* B$.

Right-handed spinors can be described by Weyl-spinors χ^α, and left-handed ones then turn to conjugate Weyl-spinors $\chi_{\dot{\alpha}}$. Here α and $\dot{\alpha}$ are (conjugate) $SL(2,\mathbb{C})$ vector indices. In the chiral representation of the gamma matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma_\mu & 0 \end{pmatrix} , \tag{B.5}$$

and

$$\gamma_5 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} , \qquad B = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} . \tag{B.6}$$

where ϵ is the two-dimensional epsilon-tensor and $\sigma^\mu = (\mathbb{I}, \overrightarrow{\sigma})$, $\sigma_\mu = \eta_{\mu\nu}\sigma^\mu = (-\mathbb{I}, \overrightarrow{\sigma})$ contains the Pauli matrices. We find right-handed spinors to have the form $\chi = (\chi^\alpha, 0)^T$, while left-handed ones look like $\chi = (0, \chi_{\dot{\alpha}})^T$. For example, if χ^i are the fermions of the gravity multiplet, we have $\chi^i = \big(0, \chi^i_{\dot{\alpha}}\big)^T$ and its complex conjugate is given by $\chi_i = (\chi_i^\alpha, 0)^T$ where the Weyl-spinors are related by $\chi_i^\alpha = \epsilon^{\alpha\beta}\Big(\chi^i_{\dot{\beta}}\Big)^*$.

The *charge conjugation* matrix satisfies

$$B = B^T = B^{-1} = B^* \ , \tag{B.7}$$

and

$$B\gamma_\mu B = {\gamma_\mu}^* \ . \tag{B.8}$$

Adopting the convention $(\eta_1\eta_2)^* = {\eta_1}^*{\eta_2}^*$, where $\eta_{\rm i}$ are generic Grassmann variables, this machinery yields

$$\bar\lambda_i\chi_j = \bar\chi_j\lambda_i = \left(\bar\lambda^i\chi^j\right)^* = \left(\bar\chi^j\lambda^i\right)^* \ , \tag{B.9}$$

$$\bar\lambda_i\gamma^\mu\chi_j = -\bar\chi_j\gamma^\mu\lambda_i = \left(\bar\lambda^i\gamma^\mu\chi^j\right)^* = -\left(\bar\chi^j\gamma^\mu\lambda^i\right)^* \ , \tag{B.10}$$

$$\bar\lambda_i\gamma^{\mu\nu}\chi_j = \bar\chi_j\gamma^{\nu\mu}\lambda_i = \left(\bar\lambda^i\gamma^{\mu\nu}\chi^j\right)^* = \left(\bar\chi^j\gamma^{\nu\mu}\lambda^i\right)^* \ , \tag{B.11}$$

$$\bar\lambda_i\gamma^{\mu\nu\rho}\chi_j = -\bar\chi_j\gamma^{\rho\nu\mu}\lambda_i = \left(\bar\lambda^i\gamma^{\mu\nu\rho}\chi^j\right)^* = -\left(\bar\chi^j\gamma^{\rho\nu\mu}\lambda^i\right)^* \ , \tag{B.12}$$

and similar relations with an upper and a lower indices.

Appendix C

Useful properties of T-tensor

In the section 4.3.1 we introduced the irreducible parts of partially dressed components of the embedding tensor $E_{\alpha\bullet}$ and $F_{\alpha\bullet}$ (with $\bullet$ a irreducible set of $SO(n) \times SU(4)$ indices) and the irreducible components of T-tensor. However, when we dress the tensor $f_{\alpha MNP}$ with $SO(6,n)$ representative matrices $L_{\bullet}{}^{M}$, we do get reducible tensors. Hence, first we have to decompose these tensors in the their irreducible part. In the second part of this appendix we will provide expressions of either $SL(2)/SO(2)$ or $SO(6,n)/SO(n) \times SO(6)$ covariant derivative of T-tensor as a linear combination of its components.

C.1 Irreducible components of $F_{\bullet}$

In the following, we will only deal with $F_{\alpha\bullet}$, but what we are going to say is easily generalizable for $F_{\bullet}$ (and $F_{\bullet}^{\dagger}$).

When we dress the tensor $f_{\alpha MNP}$ with $L_{ij}{}^{M}$ (and $L_{a}{}^{M}$) matrices, we obtain in principle tensors with two, four, or six $SU(4)$-indices. The former transforms trivially according to the **6**, hence it is irreducible: while the latter two are not.

Let us first consider the tensor

$$F_{\alpha\bullet ij}{}^{kl} \equiv f_{\alpha MN}{}^{P} L_{\bullet}{}^{M} L_{ij}{}^{N} L_{P}{}^{kl} \ , \tag{C.1}$$

where $\bullet$ can be either a $SO(n)$-index in the fundamental or $SU(4)$-indices in **6** representation. Group theory tells us that this object transforms in the $(\mathbf{6} \otimes \mathbf{6})_{\text{asym}} = \mathbf{15}$, thus it can be written as a linear combination

$$F_{\alpha\bullet i}{}^{j} \equiv f_{\alpha MN}{}^{P} L_{\bullet}{}^{M} L_{ik}{}^{N} L_{P}{}^{jk} \ , \tag{C.2}$$

which is traceless. Indeed

$$\begin{aligned} F_{\alpha\bullet ij}{}^{kl} &= \frac{1}{4}\epsilon_{ijmn}\epsilon^{klop} f_{\alpha M}{}^{N}{}_{P} L_{\bullet}{}^{M} L_{N}{}^{mn} L_{op}{}^{P} \\ &= 4 f_{\alpha MN}{}^{P} L_{\bullet}{}^{M} L_{[im}{}^{N} L_{P}{}^{[km} \delta^{l]}_{j]} - f_{\alpha MN}{}^{P} L_{\bullet}{}^{M} L_{ij}{}^{N} L_{P}{}^{kl} \\ &= 4 F_{\alpha\bullet[i}{}^{[k} \delta^{l]}_{j]} - F_{\alpha\bullet ij}{}^{kl} \ , \end{aligned} \tag{C.3}$$

where in the first step we have used pseudo-reality condition (3.2.43). Rearranging left and right sides, we obtain what we expected

$$F_{\alpha\bullet ij}{}^{kl} = 2F_{\alpha\bullet[i}{}^{[k}\delta_{j]}^{l]} \ . \tag{C.4}$$

Now, we have to consider terms with six $SU(4)$-indices

$$F_{\alpha ijkl}{}^{mn} \equiv f_{\alpha MN}{}^{P} L_{ij}{}^{M} L_{kl}{}^{N} L_{P}{}^{mn} \ , \tag{C.5}$$

which can be interpreted as a particular case of the tensor shown above, with $\bullet = ij$. Then, without lost of generality we can restrict to tensors of the form

$$F_{\alpha ijk}{}^{l} \equiv f_{\alpha MN}{}^{P} L_{ij}{}^{M} L_{km}{}^{N} L_{P}{}^{lm} \ . \tag{C.6}$$

Group theoretically $(\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6})_{\text{asym}} = \mathbf{10}\oplus\overline{\mathbf{10}}$ and then $F_{\alpha ijk}{}^{l}$ is a linear combination of $F_{\alpha}{}^{ij}$ and $F_{\alpha ij}$ defined in (4.3.9) and (4.3.10) respectively:

$$\begin{aligned} F_{\alpha ijk}{}^{l} &= \frac{1}{4}\epsilon_{iji_1i_2}\epsilon^{lmi_3i_4} f_{\alpha}{}^{M}{}_{NP} L_{M}{}^{i_1i_2} L_{km}{}^{N} L_{i_3i_4}{}^{P} \\ &= -f_{\alpha MN}{}^{P} L_{ij}{}^{M} L_{km}{}^{N} L_{P}{}^{lm} + 2 + 2f_{\alpha MN}{}^{P} L_{[im}{}^{M} L_{kn}{}^{N} L_{P}{}^{mn}\delta_{j]}^{l} + 2f_{\alpha MN}{}^{P} L_{k[i}{}^{M} L_{j]m}{}^{N} L_{P}{}^{lm} \\ &= -F_{\alpha ijk}{}^{l} + 2F_{\alpha k[i}\delta_{j]}^{l} + 2F_{\alpha k[ij]}{}^{l} \ , \end{aligned} \tag{C.7}$$

where we proceed as in the previous case. Then

$$\begin{aligned} F_{\alpha k[ij]}{}^{l} &= \frac{1}{4}\epsilon_{i_1i_2k[i}\epsilon_{j]mi_3i_4} f_{\alpha}{}^{MNP} L_{M}{}^{i_1i_2} L_{N}{}^{i_3i_4} L_{P}{}^{lm} \\ &= \frac{1}{2}\epsilon_{ijko} f_{\alpha MN}{}^{P} L_{mn}{}^{M} L_{N}{}^{lm} L_{P}{}^{on} - f_{\alpha MN}{}^{P} L_{k[i} L_{j]m} L_{P}{}^{lm} - \frac{1}{2}\epsilon_{ijmo} f_{\alpha MN}{}^{P} L_{kn}{}^{M} L_{N}{}^{lm} L_{P}{}^{on} \\ &= \frac{1}{2}\epsilon_{ijkm} F_{\alpha}{}^{lm} - F_{\alpha k[ij]}{}^{l} + \frac{1}{2}F_{\alpha k[ij]}{}^{l} - \frac{1}{2}F_{\alpha k[i}\delta_{j]}^{l} \ , \end{aligned} \tag{C.8}$$

where, in the first step, we used the symmetry property

$$T_{[ijklm]} = 0 \qquad \Rightarrow \qquad \epsilon_{[ijkl}\epsilon_{m]nop} = 0 \ , \tag{C.9}$$

and in the second we recover the symmetry property (3.2.45). The final result is

$$F_{\alpha ijk}{}^{l} = -\frac{2}{3}\delta_{[i}^{l} F_{\alpha j]k} + \frac{1}{3}\epsilon_{ijkm} F_{\alpha}{}^{lm} \ . \tag{C.10}$$

C.2 Derivative properties

Let us focus on derivative relations between T-tensor components or partially dressed $f_{\alpha MNP}$, $\xi_{\alpha M}$.

$SL(2)/SO(2)$ covariant derivative D_s turns $E_\bullet$ ($F_\bullet$) tensors into $E_\bullet^\dagger$ ($F_\bullet^\dagger$), and vice versa. This is straightforward to prove recalling the properties (3.2.37):

$$D_s E_\bullet = P_s E_\bullet^\dagger \ , \qquad D_s E_\bullet^\dagger = P_s^* E_\bullet \ , \tag{C.1}$$

$$D_s F_\bullet = P_s F_\bullet^\dagger \ , \qquad D_s F_\bullet^\dagger = P_s^* F_\bullet \ . \tag{C.2}$$

Studying properties of $E_{\alpha\bullet}$ under the action of $SO(6,n)/SO(n)\times SO(6)$ covariant derivative D_x is simple as well:

$$D_x E_\alpha{}^a = P_x{}^{aij} E_{\alpha ij} \ , \qquad D_x E_\alpha{}^{ij} = P_x{}^{aij} E_{\alpha a} \ , \tag{C.3}$$

where we used derivative conditions (3.2.59)-(3.2.62) on $SO(6,n)/SO(n)\times SO(6)$ representatives. While D_x action on $F_{\alpha\bullet}$ tensors shows a more complicated feature:

$$\begin{aligned}
D_x F_\alpha{}^{ij} &= f_{\beta M}{}^{NP} L_a{}^M \left(\frac{1}{2}\epsilon_{klmn} L_N{}^{ik} L_P{}^{jl}\right) P_x{}^{amn} - 2 f_{\beta MN}{}^P L_a{}^M L_{kl}{}^N L_P{}^{(jl} P_x{}^{ai)k} \\
&= -3 F_{\alpha a}{}^{(i}{}_k P_x{}^{akj)} = -3 F_{\alpha ak}{}^{(i} P_x{}^{aj)k} \ , \\
D_x F_{\alpha ai}{}^j &= F_{\alpha mni}{}^j P_{xa}{}^{mn} + F_{\alpha ab}{}^{jk} P_x{}^b{}_{ik} - F_{\alpha abik} P_x{}^{bjk} \\
&= \frac{2}{3} F_\alpha{}^{jk} P_{xaik} - \frac{2}{3} F_{\alpha ik} P_{xa}{}^{ik} + F_{\alpha ab}{}^{jk} P_x{}^b{}_{ik} - F_{\alpha abik} P_x{}^{bjk} \ , \\
D_x F_{\alpha ab}{}^{ij} &= F_{\alpha abc} P_x{}^{cij} + f_{\alpha MN}{}^P D_x \left(L_a{}^M L_b{}^N\right) L_P{}^{ij} \\
&= F_{\alpha abc} P_x{}^{cij} + 2 F_{\alpha[akl}{}^{ij} P_{xb]}{}^{kl} \\
&= F_{\alpha abc} P_x{}^{cij} - 4 F_{\alpha[ak}{}^{[i} P_{xb]}{}^{j]k} \ .
\end{aligned} \tag{C.4}$$

Appendix D

Quadratic constraints on T-tensor

In section 4.1 we have shown that the embedding tensor has to satisfy some quadratic constraints in order to obtain a theory which is invariant under the local action of a subgroup G_0 of the global symmetry group G. In four dimensional half-maximal supergravities this constraints can be read as a set of quadratic constraints on the irreducible components $f_{\alpha MNP}$ and $\xi_{\alpha M}$.

These irreducible tensors split into one or more irreducible components of T-tensor and quadratic constraints are handed down to these new tensor. In what follows, we will give with every quadratic constraint on partially dressed embedding tensor, its corresponding representation under the action of $SL(2) \times SO(n) \times SO(6)$.

Let us consider the simplest quadratic constraint (4.2.8):

$$\eta^{MN} \xi_{\alpha M} \xi_{\beta N} = 0 \ , \tag{D.1}$$

and using the decomposition (3.2.49) for the $SO(6, n)$ metric, it straightforwardly translates into a quadratic constraint on partially dressed embedding tensors $E_{\alpha a}$ and $E_{\alpha ij}$:

$$(\mathbf{3}, \bullet, \mathbf{1}) \qquad E_{\alpha a} E_{\beta a} = E_{\alpha ij} E_{\beta}^{ij} \ . \tag{D.2}$$

Things get more complicated when we consider the remaining constraints (4.2.9)-(4.2.12), but we can make our life a little bit better by making extensive use of symmetry properties and *Clebsch-Gordon decomposition* of tensor products of representations.

The second constraint reads

$$\eta^{PQ} \xi_{\alpha P} f_{\alpha PMN} = 0 \ , \tag{D.3}$$

where free-indices M, N are antisymmetric. Thus we can dress them with either ${L_{ij}}^M {L_{kl}}^N$, ${L_{ij}}^M {L_a}^N$ or ${L_a}^M {L_b}^N$. Let us remember that the $SU(4)$ representation product

$$\mathbf{6} \otimes_{\text{asym}} \mathbf{6} = \mathbf{15} \ , \tag{D.4}$$

which is the traceless part of $\mathbf{4} \otimes \overline{\mathbf{4}}$. Then, without lost of generality M, N indices can be dressed directly with ${L_{ik}}^{[M} L^{jkN]}$, instead of ${L_{ij}}^M {L_{kl}}^N$ (these kind of tricks simplify a lot the amount of computations needed to obtain the final results). Finally, with a little effort and the combined use of properties of scalar manifold representatives (sec. 3.2.2) and partially dressed embedding tensor (app. C) we find the three independent constraints:

$$(\mathbf{3},\bullet,\mathbf{15}) \qquad E_{(\alpha a}F_{\beta)ai}{}^{j} = \frac{2}{3}E_{(\alpha ik}F_{\beta)}{}^{jk} - \frac{2}{3}E_{(\alpha}{}^{jk}F_{\beta)ik} \ , \tag{D.5}$$

$$(\mathbf{3},\square,\mathbf{6}) \qquad E_{(\alpha b}F_{\beta)abij} = -2E_{(\alpha[ik}F_{\beta)j]}{}^{k} \ , \tag{D.6}$$

$$\left(\mathbf{3},\substack{\square\\\square},\mathbf{1}\right) \qquad E_{(\alpha c}F_{\beta)abc} = E_{(\alpha}{}^{ij}F_{\beta)abij} \ . \tag{D.7}$$

This procedure is almost identical for the fourth constraint (4.2.11)

$$\epsilon^{\alpha\beta}\left(\eta^{PQ}\xi_{\alpha P}f_{\beta QMN} + \xi_{\alpha M}\xi_{\beta N}\right) = 0 \ , \tag{D.8}$$

which differs only for the symmetry of $SL(2)$-indices and has a extra terms, that is very simple to treat:

$$(\mathbf{1},\bullet,\mathbf{15}) \qquad \epsilon^{\alpha\beta}\left(E_{\alpha a}F_{\beta ai}{}^{j} + E_{\alpha ik}E_{\beta}{}^{jk}\right) = \frac{2}{3}\epsilon^{\alpha\beta}\left(E_{\alpha ik}F_{\beta}{}^{jk} - E_{\alpha}{}^{jk}F_{\beta ik}\right) \ , \tag{D.9}$$

$$(\mathbf{1},\square,\mathbf{6}) \qquad \epsilon^{\alpha\beta}\left(-2E_{\alpha[ik}F_{\beta j]}{}^{k} + E_{\alpha a}E_{\beta ij}\right) = \epsilon^{\alpha\beta}E_{\alpha b}F_{\beta abij} \ , \tag{D.10}$$

$$\left(\mathbf{1},\substack{\square\\\square},\mathbf{1}\right) \qquad \epsilon^{\alpha\beta}\left(E_{\alpha a}E_{\beta b} + E_{\alpha c}F_{\beta abc}\right) = \epsilon^{\alpha\beta}E_{\alpha}{}^{ij}F_{\beta abij} \ . \tag{D.11}$$

Now we consider constraint (4.2.10):

$$3\eta^{RS}f_{\alpha R[MN}f_{\beta PQ]S} + 2\xi_{(\alpha[M}f_{\beta)NPQ]} = 0 \ . \tag{D.12}$$

The four $SO(6,n)$-indices are totally antisymmetrized. From dimensional analysis it is a trivial result to find how the following representations product decomposes:

$$(\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6})_{\text{asym}} = \mathbf{15} \ . \tag{D.13}$$

Then we can dress the constraint with $L_{ik}{}^{M}L^{lkN}L_{lm}{}^{P}L^{jmQ}$, without loosing generality, and the resulting constraint will be traceless:

$$E_{(\alpha ik}F_{\beta)}{}^{jk} + E_{(\alpha}{}^{jk}F_{\beta)ik} + \frac{4}{3}F_{(\alpha ik}F_{\beta)}{}^{jk} + 3F_{(\alpha ai}{}^{k}F_{\beta)k}{}^{j} = \delta_i^j\left(\frac{1}{3}F_{(\alpha kl}F_{\beta)}{}^{kl} + \frac{3}{4}F_{(\alpha ak}{}^{l}F_{\beta)al}{}^{k} \ ,\right) \tag{D.14}$$

transforming in the

$$(\mathbf{3},\bullet,\mathbf{15}) \ . \tag{D.15}$$

We recall also the previous result

$$(\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6})_{\text{asym}} = \mathbf{10} + \overline{\mathbf{10}} \ , \tag{D.16}$$

thus we can contract (D.12) with $L_{ik}{}^{M}L_{jl}{}^{N}L^{klP}L_{a}{}^{Q}$ and the final formula of quadratic constraint in terms of T-tensor will be symmetric in the indices i, j: we will have 10 complex constraints from this structure, times n real ones from the free index a:

$$(\mathbf{3},\square,\mathbf{10}) \qquad E_{(\alpha a}F_{\beta)ij} = 3E_{(\alpha(ik}F_{\beta)aj)}{}^{k} + 4F_{(\alpha(ik}F_{\beta)aj)}{}^{k} + 6F_{(\alpha b(i}{}^{k}F_{\beta)abj)k} \ . \tag{D.17}$$

In working out this constraint we have has to pay attention to some subtlety: indeed when we consider expressions like $L_{ik}{}^{M}L_{jl}{}^{N}L^{klP}$ and resort to pseudo-reality condition (3.2.43), indices symmetry have to be taken into account:

$$L_{ik}{}^{M}L_{jl}{}^{N}L^{klP} = \frac{1}{2}\epsilon_{(ikmn}L^{mnM}L_{j)l}{}^{N}L^{klP} \ , \tag{D.18}$$

otherwise we will end up with an expression in the right hand side that will be no longer symmetric in i, j.

Remaining contractions with either two, three or four ${L_a}^M$ representatives will be trivial:

$$
\left(\mathbf{3},\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array},\mathbf{15}\right) \qquad \begin{aligned} & 2E_{(\alpha[a}{F_{\beta)b]i}}^{j} + E_{(\alpha ik}{F_{\beta)ab}}^{jk} - {E_{(\alpha}}^{jk}F_{\beta)abik} + \frac{4}{3}F_{(\alpha ik}{F_{\beta)ab}}^{jk} \\ & - \frac{4}{3}{F_{(\alpha}}^{jk}F_{\beta)abik} + 2F_{(\alpha abc}{F_{\beta)ci}}^{j} - 4{F_{(\alpha[ai}}^{k}{F_{\beta)b]k}}^{j} - 4F_{(\alpha[acik}{F_{\beta)b]c}}^{jk} = 0 \ , \end{aligned} \tag{D.19}
$$

$$
\begin{aligned} & \left(\mathbf{3},\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array},\mathbf{6}\right) \qquad E_{(\alpha ij}F_{\beta)abc} + 6F_{(\alpha[abd}F_{\beta)c]dij} = 3E_{(\alpha[a}F_{\beta)bc]ij} + 12{F_{(\alpha[a[i}}^{k}F_{\beta)bc]j]k} \ , \\ & \left(\mathbf{3},\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array},\mathbf{1}\right) \qquad 3F_{\alpha e[ab}F_{\beta cd]e} + 2E_{(\alpha[a}F_{\beta)bcd]} = 3F_{\alpha[abij}{F_{\beta cd]}}^{ij} \ . \end{aligned} \tag{D.20}
$$

Finally, we have to reduce the last quadratic constraint (4.2.12):

$$
\epsilon^{\alpha\beta}\left(\eta^{RS}f_{\alpha MNR}f_{\beta PQS} - \eta^{RS}\xi_{\alpha R}f_{\beta S[M[P}\eta_{Q]N]} - \xi_{\alpha[M}f_{\beta N][PQ]} + \xi_{\alpha[P}f_{\beta Q][MN]}\right) = 0 \ . \tag{D.21}
$$

This will give five (reducible) constraints on T-tensor components, because of the structure of its indices: it is antisymmetric in indices M, N and P, Q, but it is also antisymmetric in the exchange of the couples (M, N) and (P, Q). Then, dressing it with four ${L_{ij}}^M$ representatives, we have a quadratic constraint in representation product

$$
\mathbf{15} \otimes_{\text{asym}} \mathbf{15} = \mathbf{15} \oplus \mathbf{45} \oplus \overline{\mathbf{45}} \ . \tag{D.22}
$$

The relation between partially dressed embedding tensor (we are dressing the relation with ${L_{im}}^M L^{lmN} {L_{kn}}^P L^{jnQ}$) reads

$$
\begin{aligned} & \epsilon^{\alpha\beta}\Big(\frac{1}{3}E_{\alpha ik}{F_\beta}^{jl} - \frac{1}{3}{E_\alpha}^{jl}F_{\beta ik} + {F_{\alpha ai}}^{l}{F_{\beta ak}}^{j} - \frac{1}{4}E_{\alpha a}{F_{\beta ai}}^{j}\delta^l_k + \frac{1}{4}E_{\alpha a}{F_{\beta ak}}^{l}\delta^j_i + \frac{1}{3}E_{\alpha(im}F_{\beta k)n}\epsilon^{jlmn} + \\ & -\frac{1}{3}{E_\alpha}^{(lm}{F_\beta}^{j)n}\epsilon_{ikmn} + \frac{2}{9}F_{\alpha im}{F_\beta}^{jm}\delta^l_k - \frac{2}{9}F_{\alpha km}{F_\beta}^{lm}\delta^j_i + \frac{2}{9}{F_\alpha}^{jm}{F_\beta}^{ln}\epsilon_{ikmn} - \frac{2}{9}F_{\alpha im}F_{\beta kn}\epsilon^{jlmn}\Big) = 0 \ , \end{aligned} \tag{D.23}
$$

which transforms according to

$$
(\mathbf{1}, \bullet, \mathbf{15} \otimes_{\text{asym}} \mathbf{15}) \ . \tag{D.24}
$$

We can write explicitly the **15** components on the constraint by tracing with δ^k_l:

$$
(\mathbf{1}, \bullet, \mathbf{15}) \qquad \epsilon^{\alpha\beta}\left({F_{\alpha ai}}^{k}{F_{\beta ak}}^{j} - E_{\alpha a}{F_{\beta ai}}^{j} + \frac{8}{9}F_{\alpha ik}{F_\beta}^{jk}\right) = \frac{2}{9}\delta^j_i\epsilon^{\alpha\beta}F_{\alpha lk}{F_\beta}^{lk} \ . \tag{D.25}
$$

Contracting with a single L^M_a representative, the descending constraint will transform in representation

$$
(\mathbf{1}, \square, \mathbf{6} \otimes \mathbf{15}) = (\mathbf{1}, \square, \mathbf{6}) \oplus (\mathbf{1}, \square, \mathbf{10}) \oplus \left(\mathbf{1}, \square, \overline{\mathbf{10}}\right) \oplus (\mathbf{1}, \square, \mathbf{64}) \ . \tag{D.26}
$$

If we dress (D.21) with ${L_a}^M {L_{ij}}^N {L_{km}}^P L^{lmQ}$, the **6** and **10** follow straightforwardly by tracing over l and j (or i) and considering the antisymmetric and symmetric part in the remaining two indices, respectively. The general constraint is

$$
\begin{aligned}
\epsilon^{\alpha\beta}\Big(&\frac{1}{2}E_{\alpha ij}F_{\beta ak}{}^{l}+\frac{1}{2}E_{\alpha[ik}F_{\beta aj]}{}^{l}+\frac{2}{3}F_{\alpha[ik}F_{\beta aj]}{}^{l}+F_{\alpha bk}{}^{l}F_{\beta abij}-\frac{1}{3}E_{\alpha a}F_{\beta[ik}\delta^{l}_{j]}-\\
&-\frac{1}{6}E_{\alpha a}F_{\beta}{}^{lm}\epsilon_{ijkm}-\frac{1}{4}E_{\alpha b}F_{\beta ab[ik}\delta^{l}_{j]}+\frac{1}{8}E_{\alpha b}F_{\beta ab}{}^{lm}\epsilon_{ijkm}-\frac{1}{4}E_{\alpha[im}F_{\beta ak}{}^{m}\delta^{l}_{j]}-\\
&-\frac{1}{4}E_{\alpha km}F_{\beta a[i}{}^{m}\delta^{l}_{j]}-\frac{1}{4}E_{\alpha}{}^{lm}F_{\beta ak}{}^{n}\epsilon_{ijmn}+\frac{1}{8}E_{\alpha}{}^{lm}F_{\beta am}{}^{n}\epsilon_{ijkn}-\\
&-\frac{1}{8}E_{\alpha}{}^{mn}F_{\beta am}{}^{l}\epsilon_{ijkn}+\frac{2}{3}F_{\alpha km}F_{\beta a[i}{}^{m}\delta^{l}_{j]}-\frac{2}{3}F_{\alpha}{}^{lm}F_{\beta a[i}{}^{n}\epsilon_{j]kmn}\Big)=0\ ,
\end{aligned}
\tag{D.27}
$$

and its irreducible parts which are relevant for our analysis are:

$$
(\mathbf{1},\square,\mathbf{10})\qquad \epsilon^{\alpha\beta}\left(-\frac{1}{2}E_{\alpha(ik}F_{\beta aj)}{}^{k}+\frac{2}{3}F_{\alpha(ik}F_{\beta aj)}{}^{k}+F_{\alpha b(i}{}^{k}F_{\beta abj)k}-\frac{1}{2}E_{\alpha a}F_{\beta ij}\right)=0\ ,
\tag{D.28}
$$

and

$$
\begin{aligned}
(\mathbf{1},\square,\mathbf{6})\qquad \epsilon^{\alpha\beta}\Big(&\frac{1}{4}E_{\alpha[ik}F_{\beta aj]}{}^{k}+\frac{2}{3}F_{\alpha[ik}F_{\beta aj]}{}^{k}+\frac{1}{2}F_{\alpha b[i}{}^{k}F_{\beta abj]k}+\frac{5}{8}E_{\alpha b}F_{\beta abij}\\
&-\frac{1}{3}F_{\alpha}{}^{kl}F_{\beta ak}{}^{m}\epsilon_{ijlm}\Big)=0\ .
\end{aligned}
\tag{D.29}
$$

There are two inequivalent ways to dress the constraint (D.21) with two $L_a{}^M$ and two $L_{ij}{}^M$ representatives. We contract first with $L_{ij}{}^M L_a{}^N L^{klP} L_b{}^Q$ and the resulting expression transforms into the representation

$$
(\mathbf{1},\square,\mathbf{6})\otimes_{\mathrm{asym}}(\mathbf{1},\square,\mathbf{6})=\left(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{1}\right)\oplus\left(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{20'}\right)\oplus(\mathbf{1},\bullet,\mathbf{15})\oplus(\mathbf{1},\square\square,\mathbf{15})\ .
\tag{D.30}
$$

Its explicit form is

$$
\begin{aligned}
\epsilon^{\alpha\beta}\Big(&\frac{1}{2}E_{\alpha ij}F_{\beta ab}{}^{kl}+\frac{1}{2}E_{\alpha}{}^{kl}F_{\beta abij}+2F_{\alpha a[i}{}^{[k}F_{\beta bj]}{}^{l]}+F_{\alpha acij}F_{\beta bc}{}^{kl}+2E_{\alpha(a}F_{\beta b)[i}{}^{[k}\delta^{l]}_{j]}+\\
&+2F_{\alpha a[i}{}^{m}F_{\beta bm}{}^{[k}\delta^{l]}_{j]}+\frac{1}{4}E_{\alpha c}F_{\beta abc}\delta^{[k}_{i}\delta^{l]}_{j}-\frac{1}{2}\delta_{ab}E_{\alpha c}F_{\beta c[i}{}^{[k}\delta^{l]}_{j]}-\frac{1}{3}\delta_{ab}E_{\alpha[im}F_{\beta}{}^{[km}\delta^{l]}_{j]}+\\
&+\frac{1}{3}\delta_{ab}E_{\alpha}{}^{[km}F_{\beta[im}\delta^{l]}_{j]}-\frac{1}{4}E_{\alpha}{}^{mn}F_{\beta abmn}\delta^{[k}_{i}\delta^{l]}_{j}\Big)=0\ .
\end{aligned}
\tag{D.31}
$$

Then we dress it with $L_a{}^M L_b{}^N L_{ik}{}^P L^{jkQ}$ and the following constraint is

$$
\begin{aligned}
\left(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{15}\right)\qquad \epsilon^{\alpha\beta}\Big(&\frac{1}{2}E_{\alpha ik}F_{\beta ab}{}^{jk}-\frac{1}{2}E_{\alpha}{}^{jk}F_{\beta abik}-\frac{2}{3}F_{\alpha ik}F_{\beta ab}{}^{jk}+\frac{2}{3}F_{\alpha}{}^{jk}F_{\beta abik}+\\
&-E_{\alpha[a}F_{\beta b]i}{}^{j}+F_{\alpha abc}F_{\beta ci}{}^{j}\Big)=0
\end{aligned}
\tag{D.32}
$$

The remaining constraints are simpler:

$$
\begin{aligned}
\left(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{6}\right)\qquad \epsilon^{\alpha\beta}\Big(&-E_{\alpha[a}F_{\beta b]cij}+\frac{1}{2}E_{\alpha c}F_{\beta abij}-\frac{1}{2}E_{\alpha ij}F_{\beta abc}-F_{\alpha abd}F_{\beta cdij}\\
&-2F_{\alpha c[i}{}^{k}F_{\beta abj]k}-\frac{1}{2}E_{\alpha d}F_{\beta[adij}\delta_{b]c}-E_{\alpha[ik}F_{\beta[aj]}{}^{k}\delta_{b]c}\Big)=0\ ,
\end{aligned}
\tag{D.33}
$$

and

$$
\left(\mathbf{1},\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\mathbf{1}\right)\qquad \epsilon^{\alpha\beta}\left(F_{\alpha abe}F_{\beta cde}-E_{\alpha e}F_{\beta e[a[c}\delta_{d]b]}-E_{\alpha[a}F_{\beta b][cd]}+E_{\alpha[c}F_{\beta d][ab]}\right)=0\ .
\tag{D.34}
$$

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