Università degli Studi di Padova

Università degli studi di Padova

Dipartimento di Fisica e Astronomia

Corso di Laurea Magistrale in Fisica

Tesi di Laurea Magistrale

Frame invariance of cosmological observables under disformal transformations in Scalar-Tensor theories

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Anno Accademico 2016/2017

[Though leaves are many, the root is one
W.B. Yeats]

A mia nonna che mi ha portato fin qua
e a Sara che mi ha supportato/sopportato in questi 5 anni.

Contents

Chapter 1

Introduction

Since the formulation of General Relativity (GR) a large number of possible alternatives to it has been proposed (Einstein himself modified his theory adding a cosmological constant term). This line of research has been partly motivated by *High Energy Physics* theories, such as *Superstring* or *Supersymmetric* theories, where extensions to GR arise naturally, and partly by the discovery of the “dark” sector of the Universe as well as the new insight on the early Universe gained with the new technologies. Besides a wide array of modified theories of gravity, such a development has led to an outstanding comprehension of the gravity itself and the growth of new tools to investigate it. Among them, and source of a long debate, there is the use of field transformations, i.e. transformations of the metric tensor $g_{\mu\nu}$ making up gravity as well as of the other possible fields entering the gravity action.

As early as the ’60s, a particular class of metric transformations, dubbed *conformal transformations*, has been exploited in the realm of Scalar-Tensor (S-T) theories to recast the Brans-Dicke theory in a GR-like form [20]. Afterwards, their use has become more frequent in modified gravity, also outside the Scalar-Tensor world and with the same motivation: they generally allow to make a bridge between GR and the new modified theory, or in a broader viewpoint, among different representations of modified theories of gravity.

However, the use of conformal transformation has brought with it a very heated debate about the physical interpretation of metric transformations. First of all, does the underlying physics change under metric transformations? If so, how does it change? From a Field Theory point of view, when non-singular, they are nothing else than a field redefinition and as such they shouldn’t affect the physics. Anyway, because of the privileged role of the metric tensor, the interpretation of metric transformations turns out to be very subtle and not obvious at all. In short, this is the content of the conformal transformation issue.

In general there can be different levels of invariance with respect to field transformations. A "complete" invariance at the action level, i.e. $S[g_{\mu\nu}] = S'[g'_{\mu\nu}]$, which automatically would imply physical invariance. However, apart from extraordinary cases, the transformation of the action is not trivial and this invariance is not realized. We can instead address form-invariance, i.e. invariance up to a redefinitions of the free parameters of the theory; this is what happens in the realm of canonical Scalar-Tensor theories with conformal transformations. In this sense it reasonable to ask whether the observable physics changes performing such a transformation.

In the first part of this work we review the conformal transformation issue as it arose historically and the different points of view regarding the conundrum problems of the physical invariance outlined above. We show that the different *frames*, i.e. the different representations related by a conformal transformation, are physically equivalent at the price of a re-interpretation of the physical conventions. In the second part we treat a further generalization of conformal transformations, namely the disformal transformations (DT) introduced by Bekenstein in 1992 [4]. Here the frame-invariance problem is indeed more subtle because of the field-derivative dependence within the transformation law, although the disformal invariance is generally assumed in the wake of the conformal one.

One important result in this respect is that the Horndeski theory, which represents the most general S-T theory with second order equations of motion, is invariant in form under a particular class of disformal transformations. Moreover, it has been shown that using them we can obtain healthy theories beyond Horndeski which are closed under disformal transformations; in this context they can be assumed to be an extra symmetry of the most generic S-T theories. However, no much work has been done to prove the disformal invariance of observational physics, which is usually assumed *a priori*. The aim of this thesis is therefore to analyse how the cosmological observables transform under a disformal transformation to prove that they are disformally invariant. To achieve this some simplifications are made on the disformal transformation, but this will help us to show the addressed invariance. Moreover, a new formalism is introduced through which the frame-invariance is made explicit. All the work is focused on Scalar-Tensor theories, where the disformal transformations find their proper environment.

In more detail this work is thus divided:

- In chapter 2 we are going to briefly introduce modern cosmology and the ΛCDM model;

- In chapter 3 we review the main features of General Relativity and the possible way to extend it consistently.

- In chapter 4 we review Scalar-Tensor theories starting from the Brans-Dicke prototype model up to the Horndeski theories. Some insight into other alternatives to GR are given as well as into the PPN formalism.

- In chapter 5 we introduce the conformal transformations and the conformal transformation issue in Scalar-Tensor theories.

- In chapter 6 we introduce the disformal transformations and the concept of disformally related frames; the chapter is closed with a brief discussion about the Ostrogradski's instability.

- Chapter 7 studies the conditions of invertibility of the DT. It is also shown that when the transformation is not invertible then a new physics emerges, described by the so called *Mimetic* theories of gravity.

- In chapter 8 the disformal invariance of cosmological observables is analysed. In particular we show that the Boltzmann equation is frame-invariant under a particular DT, as well as the cosmological redshift and the inflationary power spectra. A new formalism is given to explicitly show the frame-invariance at the action level.

- In chapter 9 conclusions are given.

- In appendix 1 we write down some useful calculations related to the frame-invariant construction of chapter 5.

- In appendix 2 we evaluate, at the action level, how the Klein-Gordon and Maxwell fields transform under a conformal transformation.

- In appendix 3 we explicit the transformation rule of the line-element under a full disformal transformation. We also evaluate how the (linear) cosmological perturbations change passing from one frame to another one.

- In appendix 4, following [74] the disformal invariance of continuous media is analysed.

Notation

In this work we use natural units

$$\hbar = c = 1,$$

unless specified differently.
We have chosen to work with the metric signature $(-,+,+,+)$. Greek indices such as μ, ν take the values $(0,1,2,3)$, while Latin indices such as i, j take the values $(1,2,3)$.
Conventions about derivatives and covariant derivatives are specified throughout the work.

Chapter 2

Modern Cosmology

The mystery of the Cosmos has played a key role in scientific and philosophical thinking since ancient times. The Universe, immensely far away in the past, today has become considerably closer to us thanks not only to the powerful zoom of the telescopes but also to the understanding that the underlying physics of it is the same that governs everyday reality. However, although we are aware of its many facets, the mystery surrounding the universe is still a source of intense scientific activity and if we want philosophical debate.

Cosmology has turned out to be a highly multidisciplinary science capable of embracing advanced mathematics, particle physics, statistics up to quantum mechanics. Just think that the large-scale structure of the Universe are the outcome of quantum fluctuations of a "scalar particle" in a four-dimensional curved spacetime. Cosmology has also turned out to be a very dynamic science, which has repeatedly changed the paradigm and source of many debates around its predictions, its mysteries.

Just to name a few examples, one of the key moments in the history of cosmology was the sensational discovery that the universe is expanding (*Hubble*, 1929), against the generally accepted scenario of a static and eternal universe of the early '900. Perhaps less sensational, but deeper is the rather recent discovery (*Riess et al.* 1998) that this expansion is accelerated, leading the scientific community to think to alternative forms of energy or perhaps alternative formulations of the laws of gravity ruling our universe.

A lot of Universe models have been proposed, most rejected, others under current investigation. The one that is most in agreement with experimental data is what is called the *concordance model*, namely the ΛCDM model. It is, up to now, the best description of our observable universe and has passed most of the experimental tests. For this reason it is also called the standard model of cosmology.

2.1 The ΛCDM model: an introduction

The pillar on which the ΛCDM model is based is the Einstein's theory of General Relativity (1916), which describes in an elegant way the dynamical evolution of the universe. This has to be supplemented with other main ingredients: matter, which accounts for the 30% of the energy content of the universe, shared between ordinary matter (4%) (such as baryons, leptons, radiation) and the Cold Dark Matter (26%); Dark Energy, which constitutes the remaining 70% of the energy and accounts for the current accelerated expansion of the universe. Finally, as we shall see, the standard model has to implement a viable inflationary model, i.e. a phase of cosmic acceleration in the early universe.

General Relativity(GR) gives the geometrical framework from which the evolution history of the universe can be reconstructed, dubbed the *Hot Big Bang* paradigm. The elegance of General Relativity lies in the fact that through one formula it is able to encapsulate all the variety of the universe that surrounds us simply relating the geometry of spacetime to the matter-energy content of the universe. This is achieved with the famous Einstein's equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{2.1}$$

where the *Einstein tensor* $G_{\mu\nu}$ is a function of the spacetime metric $g_{\mu\nu}$ and its derivative, G is the gravitational constant and $T_{\mu\nu}$ is the *energy-momentum tensor* (EMT) and encodes the matter-energy content of the spacetime. The last term in the RHS is the *cosmological constant* term which is the simplest mathematical description of the Dark Energy.

The 85% of the matter content of the universe appears to be in the form of an unknown kind of particles which nowadays seems to interact only via the gravitational force with the other particle species, and that is capable of clustering. For this elusive behaviour, that does not allow us to detect it directly as it happens with ordinary matter (which emits radiation), it was called Dark Matter (DM). It is common to distinguish between *Hot Dark Matter*(HDM) and *Cold Dark Matter*(CDM), meaning respectively relativistic and non-relativistic Dark Matter[1].

The first evidence of DM dates to 1933, when the Swiss astrophysicist Fritz Zwicky, using the virial theorem to infer the mass of the *Coma* galaxy cluster, found that the mass thus obtained exceeded by 400 times the expected value from its luminosity [6]. He called this undetected matter "*Dunkle Materie*", from which comes the current terminology. More important for the scientific community were the studies made by Vera Rubin in the early '70. It is known that the rotational

[1] Precisely Hot(Cold) Dark Matter is composed by those particle species that were (non-)relativistic at the the time of decoupling with the rest of the universe. Examples of HDM are neutrinos, possible candidates for CDM are instead axions or WIMPS (*Weak Interactive Massive Particles*).

velocity of a spiral galaxy composed by baryonic matter decreases with the inverse of the radius of the galaxy. What she showed was instead that the velocity profile remains constant in the outer parts of the galaxy; this can be explained if we assume that it is surrounded by an halo of (unseen) matter [7].

Today it is believed that Cold Dark Matter plays a central role in the formation and growth of large scale structures in the universe (see [8]) and this is what is assumed in the ΛCDM model. However, up to now no convincing candidates for CDM have been found, both theoretically and experimentally and it is even believed that DM could arise from modification of gravity laws.

The Dark Energy sector is as much "dark" as the Dark Matter one, or even more. We observe that our universe is experimenting an accelerated expansion phase. The source of such an acceleration is generally called Dark Energy. Whether this is due to some unknown form of energy (something that is not able to cluster as matter) or to a modification of the gravitational theory is still a great source of debate and research. Anyway it acts like a cosmological constant (from this comes the "Λ" in the name of the concordance model) and the Λ-term in the Einstein's equation offers a good description of its effect.

The state-of-the-art of cosmology is that the 96% of the total energy budget of the universe has an unknown origin and still needs an explanation. This can lead to think that we are going to enter a new exciting phase for our knowledge of the universe and probably to change our paradigm again.

2.2 The Hot Big Bang

Although we are aware that the universe is mostly "dark", and although we do not possess a full inventory of all the particles and forces that come into play in the universe, we are able to delineate the evolutionary history of the universe quite decisively and successfully. This is through the *Hot Big-Bang* paradigm.

The Big Bang cosmology, with the help of high-energy particle physics is able to reconstruct practically the entire history of the universe from the Planck time ($10^{-43}\ s$) onward. Something can also be speculated about the cosmology of pre-Planckian time, but in order to understand it we need a complete quantum gravity theory that so far it is not available.

The Big Bang paradigm is based on the fact that, looking at the large scale galaxy distribution and the CMB maps, the universe seems to be homogeneous and isotropic; in other words, on scales larger than 100 Mpc the universe looks the same in all directions, independently of the position from which it is observed. This is the content of the *Cosmological principle*, which in a more precise formulation states that for a *comoving* observer the universe looks the same no matter the position from which the observation is taken and the direction. A comoving observer is an observer at rest with respect to the cosmic microwave background.

Clearly this is not true if we limit the observations to our galaxy or nearby galaxies, where the matter distribution is highly inhomogeneous and anisotropic. Indeed this is not true if we carefully study the CMB maps, where anisotropies of the order of 10^{-5} are present. In other words we are not comoving observers; but nevertheless, at first approximation (and how cosmologists have assumed historically) the cosmological principle holds.

Homogeneity and isotropy are related to the invariance under translation and rotation of the space and therefore the assumptions of the Cosmological principle is physically important and almost of philosophical nature.

Practically, the high-level of symmetry implied by the Cosmological principle allows to solve the Einstein's equation straightforwardly. In fact, the metric felt by a comoving observer, who sees the universe expanding isotropically and with homogeneity is simply given by

$$ds^2 = -dt^2 + a(t)^2\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big], \tag{2.2}$$

where $a(t)$ is called the *scale factor* and describes the size of the universe as a function of the *cosmic time* t, k is the *curvature parameter* which can assume three different values: $k = \{0, +1, -1\}$. $k = 0$ is related to a flat universe, whereas $k = +1(-1)$ to a closed(open) universe. We refer to this metric as the *FRW*-metric by the names of the scientists who independently discovered it (Friedmann(1924), Robertson(1935), Walker(1936)).

Assuming the Cosmological principle we can easily treat the Einstein's equations (2.1), which become functions of the scale factor and the energy-matter content of the universe. The latter can be parametrized with a perfect fluid, whose energy-momentum tensor reads

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu}P, \tag{2.3}$$

where $\rho = \rho(t)$ is the energy density of the fluid, $P = P(t)$ is its (isotropic) pressure and u_μ is the four-velocity of the cosmic fluid. We note that this form is compatible with the Cosmological Principle, which requires no spatial dependence on physical (cosmological) quantities.

Inserting the FRW metric in the Einstein's equation (we neglect the cosmological constant term that is relevant only in late times) and using (2.3) we find the well-known Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{2.4}$$

and

$$\ddot{a} = -\frac{4\pi G}{3}a(\rho + 3P). \tag{2.5}$$

H is called the *Hubble parameter*, and its current value, dubbed the *Hubble constant* is parametrized as

$$H_0 = 100\, h\, kms^{-1}\, Mpc^{-1},$$

with $h \sim 0.7$. The dimension of the Hubble constant is the inverse of time, and as can be seen, its inverse is nothing else than the age of the universe.

Using the conservation-equation $T^{\mu\nu}_{;\nu} = 0$ we find the continuity equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P). \tag{2.6}$$

Thus we have three unknown variables $\{a(t), \rho(t), P(t)\}$ and three equations; in reality only two of them are independent. An equation of state $P = P(\rho)$ provides us with the extra equation necessary to close the system. This can be assumed to be $P = w\rho$ with w constant and such to describe the kind of matter we have. The most relevant values for w in cosmology are

- $w = 0$ that represents dust particles (pressureless matter) like baryons or CDM;
- $w = 1/3$ that describes radiation like photons or ultra-relativistic particles;
- $w = -1$ that is the case of the cosmological constant (note that in this case the pressure is negative).

In the ΛCDM model the curvature is assumed to be flat, therefore we can solve the Friedmann equations putting $k = 0$[2]. We find the following solutions

$$a(t) \propto t^{\frac{2}{3(1+w)}}; \tag{2.7}$$
$$\rho(t) \propto a^{-3(1+w)} \propto t^{-2}, \tag{2.8}$$

describing the so called Friedmann universes. Hence, in the case of a dust-dominated universe ρ scales as a^{-3}, while in a radiation-dominated universe $\rho \propto a^{-4}$.

In the Hot Big Bang scenario the universe is filled with a perfect fluid with several components. At early time the universe evolves from a state of very high temperature and density and therefore the dominant contribution to the energy density comes from the ultra-relativistic component. Hence $w = 1/3$ and $a(t) \propto t^{\frac{1}{2}}$. This is called the *radiation-dominated era* (RDE).

Gradually the temperature falls down with the expansion of the universe, the matter content becomes dominant with ρ that scales as a^{-3} and $a \propto t^{\frac{2}{3}}$. This is the *matter-dominated era* (MDE). Inside this epoch, when the universe was $\sim$ 380000 year-old, baryon and photon could not maintain thermal equilibrium because of the low temperatures; photons decoupled from matter and the universe becomes transparent. This is the recombination epoch, whose relic is the cosmic microwave background we observe in the sky. Afterwards the growth of the large

[2]The flat solution is the most relevant in cosmology because observations tell us that the total energy density of the universe is almost equal to the critical density. Furthermore there are convincing theoretical motivations to put $k = 0$ as we will see later.

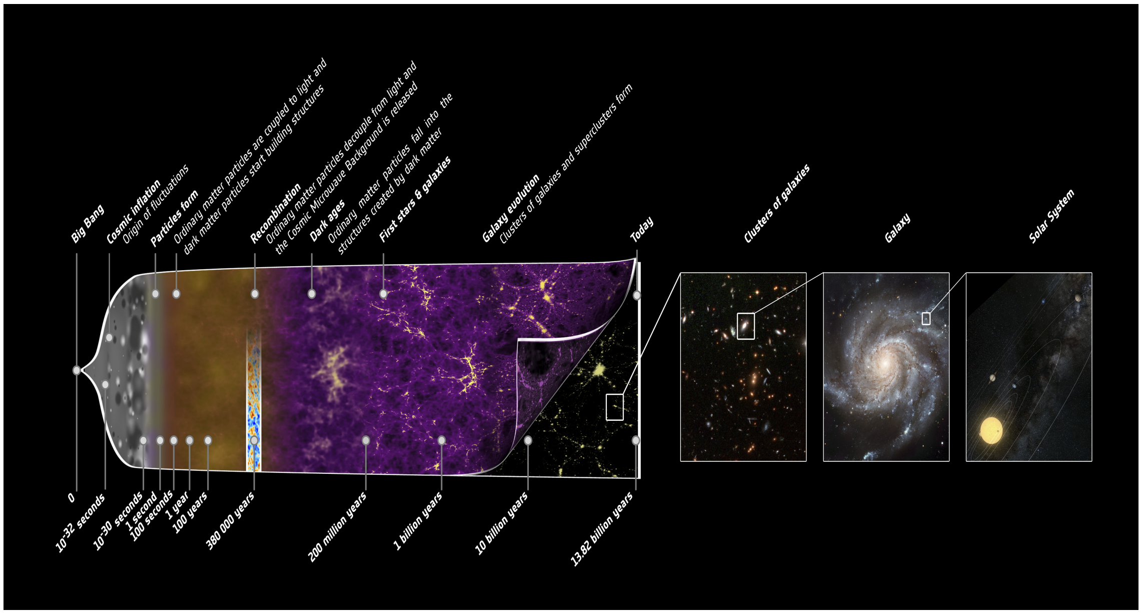

Figure 2.1: A qualitative picture describing the evolution of the universe accordingly with the standard Hot Bing-Bang evolution, supplemented with a phase of inflation at early times.

scale structures of the universe begins, with the formation of the first stars and galaxies. This model also predicts the right abundances of light-elements produced during the cosmological nucleosynthesis.

This is the basic picture depicted by the Hot Big-Bang model. In reality, it is able to predict and describe all the relevant phenomena and processes the universe undergoes during its evolution and whose traces are imprinted for example in the CMB. The CMB itself is one of the most important prediction of this model, observed for the first time in 1961 and now well-measured by several experiments.

A crucial point of the universe evolution is the transition phase between the Planck era and the radiation dominated era. The Hot Big-Bang model predicts a smooth evolution from the singularity at $t = 0$ (or $a = 0$) to the RDE, but this is unsatisfactory for many reasons we are going to explain in the following section.

2.3 Inflation

2.3.1 The shortcomings of the Hot Big Bang model

Although this seemingly simple picture offers a good description of the universe evolution starting from the early times, it suffers from some problems. The most known shortcomings of the Hot Big Bang model are: the *horizon problem*, the *flatness problem* and the *unwanted relics problem*, the latter also better known as the *monopole problem.*

The horizon problem

Let's define the *particle horizon* d_H as the greatest distance for which we can have causal connection between two points. What we expect is that regions separated by distances larger than d_H should be uncorrelated. In the meanwhile it is well-known that the CMB, that is nothing else than a snapshot of the universe when it was 380000 year-old, has a homogeneous and isotropic temperature up to corrections of order $O(10^{-5})$. But this implies that regions never been in causal contact[3], display the same thermal properties. Is this just a (extremely improbable) coincidence, or is there some underlying mechanism leading to a thermal homogenisation? This puzzling question constitutes what is known as the *horizon problem.*

Another way to think of it is the following. Let's consider the comoving Hubble radius $r_c \equiv \frac{1}{\dot{a}}$. It can be shown that it is of the same order of the comoving particle horizon, therefore we can assume they have the same causal properties. In the standard evolution of the Friedmann universes in which $\ddot{a} < 0$ we have

$$\dot{r}_c = -\frac{\dot{a}}{\ddot{a}} > 0,$$

i.e. r_c increases with time.

Let's consider a comoving scale λ. The bigger λ is, later it crosses the horizon. So we expect that the largest scales, i.e. those have crossed the horizon only recently, were uncorrelated with the lower scales. But this is not the case. The universe appears homogeneous and isotropic over all the scales (as long as they are sufficiently large).

The flatness problem

Let's consider Eq.(2.4). Dividing it by H^2 it reads

$$1 = \frac{8\pi G}{3}\frac{\rho}{H^2} - \frac{k}{a^2H^2}.$$

Defining

$$\rho_c = \frac{3H^2}{8\pi G} \qquad \text{and} \qquad \Omega = \frac{\rho}{\rho_c}, \tag{2.9}$$

the previous equation becomes

$$\Omega - 1 = \frac{k}{a^2H^2} = \frac{k}{\dot{a}^2} = kr_c^2. \tag{2.10}$$

[3]For example regions that at $t \sim 380000\ yr$ were separated by distances larger than the particle horizon. In terms of CMB this means an angular separation larger than 1 degree more or less.

ρ_c is called the *critical density* and it is the value the energy density would have if $k = 0$.

Observations tell us that the present value of Ω, namely Ω_0, is such that $|\Omega_0 - 1| < 0.005$ (C.L. 95 %). But this implies that, because in the standard picture r_H increases with time, going back to the past Ω has to approach closer and closer the unity (or, in another way, ρ is closer and closer to its critical value defined at $k = 0$). Going back until the Planck time this means that $|\Omega_{t_p} - 1| < 10^{-65}$. A very strong fine tuning is required on the initial condition in order to have the observed current value of Ω!

This is the *flatness problem*, and for the reason explained just above it is also understood as a *fine tuning problem*.

The monopole problem

It is thought that when the universe was extremely hot with temperatures of the order of $10^2 - 10^{19}$ GeV. it went through a series of high temperature phase transitions (with associated *spontaneous symmetry breakings*). In *Grand Unified Theory* (GUT), whenever these transitions happen, there is production of very massive and stable particles, capable to survive until present times. But if this were the case their contribution to Ω would be very large, and such that $\Omega \gg 1$. Such stable particles are called *topological defects* and among them we count: *magnetic monopoles*, *domain walls*, *cosmic strings* and other more.

Furthermore other kinds of objects are predicted by high energy physics theories, such as *gravitinos* in Supergravity or *moduli* in String Theory.

In this context all these objects are also known as *unwanted relics* for the fact that, so far, we have not detected them because probably their contribution to the total energy of the universe is negligible. But their production at early times seems to be unavoidable. This shortcoming of Big Bang cosmology is historically known as the *monopole problem*.

2.3.2 The basic picture of the inflation

All the previous problems, together with other shortcomings not reported here, find a solution in the context of inflation. Basically, inflation is a phase in which the universe experiences an accelerated expansion at early times, i.e. such that $\ddot{a} > 0$. Looking at (2.5) this can be achieved if and only if $w < -1/3$[4]. In most inflationary models the expansion is exponential, or almost so, that means $a \propto e^{Ht}$ with $H \simeq$ *constant*, corresponding to the effect of a cosmological constant ($w = -1$).

Such an accelerated expansion phase ensures that during inflation $\dot{r}_c < 0$. Looking at fig:(2.2) we see that, if inflation lasts enough, scales that now are

[4]This means that inflation occurs before primordial nucleosynthesis that happens at $T \sim 1\,MeV$ in the RDE when $w = 1/3$.

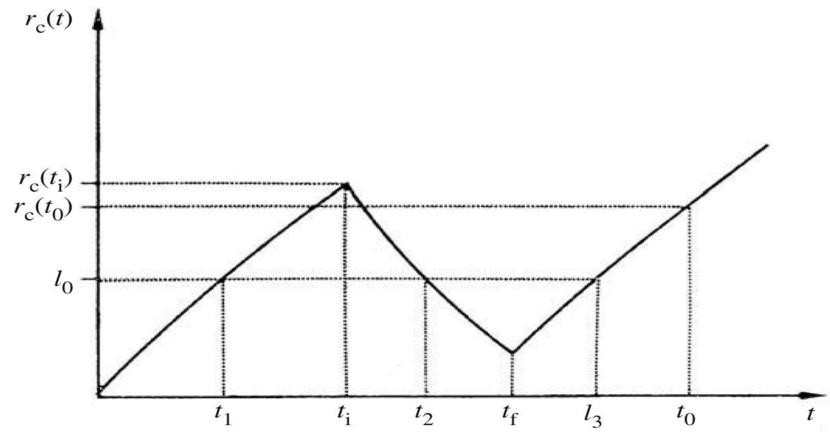

Figure 2.2: In this picture we see the time evolution of the comoving Hubble radius in an inflationary universe. Before and after inflation it increases since its time-derivative is positive; during inflation its time derivative is negative and therefore it decreases. The minimal request to solve the horizon problem is that $r_c(t_i) \simeq r_c(t_0)$; in such a way those scales entering now the horizon had already been in causal contact at $t \simeq t_i$. This picture is taken from [5].

entering the horizon could have already been in casual contact during or before inflation. Inflation may solve the horizon problem with the minimal request that scales which have just entered the horizon now, had been in causal contact at the beginning of inflation.

In a similar vein it also solves the flatness problem. In fact, considering Eq.(2.10), a decreasing r_H means that during inflation Ω is squeezed towards one. So in principle inflation could start with $\Omega - 1 \in O(1)$ (strictly speaking a non fine-tuned value); during the inflationary phase $\Omega - 1$ is suppressed and if inflation lasts enough, at the end of it $\Omega - 1$ is sufficiently small to have the current measured value $\Omega - 1 < 0.005$.

For what concerns the monopole problem, recalling that $n_X \propto a^{-3}$, where X is a particular particle specie with a number density n_X, we see that during inflation n_X decreases exponentially. If a relic is produced before inflation, its number density is strongly suppressed and at the end of inflation it is completely negligible.

Inflation must last enough to solve the previous problems: in particular, defining the number of e-folds as

$$N \equiv \ln \frac{a(t_f)}{a(t_i)} = \int_{t_i}^{t_f} H dt, \tag{2.11}$$

where t_i and t_f are respectively the initial and final time of inflation, in order to solve the flatness problem, the horizon problem and so on we must require that $N \gtrsim 60 \div 70$.

A viable inflationary model should provide a dynamical way to obtain an accelerated expansion at early times that ends with the beginning of the RDE as required by observations. We can achieve this considering the evolution of a scalar field $\phi(\vec{x}, t)$. It can be shown that if the early universe is filled with such a scalar field whose potential is flat enough, then at a certain point the universe starts to expand exponentially, or almost exponentially. A common picture is the one of a scalar field rolling down slowly towards the minimum of its potential. Inflation ends when the scalar field reaches such a minimum. In a more quantitative way a *slow-roll* inflation is achieved if

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1, \qquad \eta \equiv \frac{1}{3}\frac{V''}{H^2} \ll 1, \tag{2.12}$$

where ϵ and η are called *slow-roll parameters*, and as it can be seen directly, they are related to the form of the potential.

Furthermore inflation provides a way for generating density perturbations, the seeds of large-scale structures of the universe. Essentially density perturbations are generated from quantum fluctuations of the scalar field during inflation. It can be seen that when a particular scale crosses the horizon, the associated quantum fluctuation freezes in, becoming constant and therefore "classic". After that the scale re-enters the horizon in the RDE or in the MDE as a density perturbation.

Inflationary theories are very predictive and can be tested up to technical difficulties; however, it is not clear which model is the right one, if there is one. Up to now we have only some constraints that allow to discard some of the many proposals.

2.4 The Dark Energy Era

In 1998 Riess *et al.* [9] studying the relationship between distance and luminosity in type Ia Supernovae arrived to the outstanding discovery that our universe is subjected to an accelerated expansion. Further, the same result was achieved independently by another group one year later [10]. Afterwards new evidences coming from other kinds of experiments confirmed this result, such as through the measurements of the Hubble parameter [11] or from the Planck collaboration analysis (a list of the effects DE induces to the CMB can be found in [16]).

As we observed in the last section, in the standard FRW picture, in order to have an accelerated expansion phase in the universe we must require that $w < -1/3$. In other words, accordingly with the EoS $P = w\rho$, there must be something producing a negative pressure. This is what is called Dark Energy.

In the ΛCDM model Dark Energy is parametrised with a tiny and positive cosmological constant term added to the Einstein equations in a way that it provides the 70% of the total energy density of the universe and the EoS $P = -\rho$ $(w = -1)$.

It is commonly associated to the vacuum energy density of the universe and as such, in absence of another reference energy scale, we expected that $\Lambda \simeq M_{pl}^4$, where $M_{pl} \simeq 10^{19}\, GeV$ is the *Planck Mass*. But the measured value of Λ is 120 order of magnitude less than this expected value! Such a fine-tuning problem is commonly known as the *cosmological constant problem*, and it is certainly one of the most compelling shortcomings of the ΛCDM model.

It is worth noting that there are some mechanisms, coming from *High Energy Physics*, that allow to lower the value of Λ almost of 60 orders of magnitude, but unfortunately they are not sufficient to eliminate the fine-tuning problem consistently. For a review of the cosmological constant problem see [18] and references therein.

2.5 Summary

The standard model of cosmology is currently the best description we have of our universe. It essentially depends only on six parameters that we can measure with experiments of different nature. Among them the energy amount of baryon Ω_{0b} and Cold Dark matter Ω_{0c}, the Hubble parameter H_0, the scalar spectral index n_s, the reionization optical depth τ and the fluctuation amplitude at $8h^{-1}\, Mpc$.

Here we list them in a table with the latest measurements released by Planck, in which different sources are taken into account included external data [17][5]

Parameters	**PlanckTT,TE,EE+ +lowP+lensing+ext**
H_0	67.74 ± 0.46
Ω_{0b}	0.04860 ± 0.00051
Ω_{0c}	0.2589 ± 0.0057
n_s	0.9667 ± 0.0040
σ_8	0.8159 ± 0.0086
τ	0.066 ± 0.012

Because a flat universe is assumed, the several energy contributions Ω_{0i} have to add up to one; this imposes that $\Omega_{0\Lambda} = 0.6911 \pm 0.0062$. Together with the radiation component, although it is orders of magnitude lower with respect to the matter energy density, these are the main contributions to the total energy density of the universe.

As we have already stressed the ΛCDM model has a very good agreement with observations, no matter the scale of these observations is. However, doubtless it is not the final model as it has been pointed out during this chapter: it suffers from several problems besides the fact that so far we do not possess any direct

[5] In this table are shown the best-fit values for the ΛCDM parameters and 68% confidence levels, computed from the Planck CMB power spectra, in combination with the CMB lensing likelihood ("lensing") and a compilation of external data sets ("ext").

evidence of inflation (and neither a really convincing theoretical model) and the nature of Dark Matter and Dark Energy is still a mystery.

Therefore in order to explain these two dark sectors, and to find a solution for the unavoidable shortcomings, it is necessary to go beyond this model. This is possible for example in the context of modified theories of gravity as we shall see throughout this work.

Chapter 3

General Relativity and beyond

3.1 General relativity

General Relativity (GR) is the successful theory of gravitation proposed by Einstein in 1916. He firstly established, on the basis of the *equivalence principles*, and in particular the observed equivalence between inertial and gravitational mass, that gravity is the geometrical manifestation of the spacetime curvature, this one being influenced by the energy-matter content of the universe and vice-versa.

3.1.1 Riemannian geometry

In terms of differential geometry, which is the mathematical environment of GR, Einstein theory of gravity assumes that the spacetime is a 4-D differentiable manifold provided with a Lorentzian metric $\mathbf{g}$. We indicate the spacetime with the pair $(\mathcal{M}, \mathbf{g})$. The relation between spacetime and matter is given by the so called Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3.1}$$

In order to explain the "ingredients" and the main features of this set of equations, we are going to give an insight into the basic differential geometry tools related to General Relativity.

In special relativity (SR) the spacetime is given by the pair $(\mathbb{R}^4, \eta)$, where η is the Minkowski metric. This flat spacetime has no role in gravity phenomena, and in general it is just a background space in which particle physics acts. In GR the Minkowski metric is promoted to a dynamical field which determines the geometrical structure of the spacetime and propagates the gravitational interaction accordingly with the energy-matter content of the spacetime.

As it happens in SR, all the physical quantities are described by means of *tensors*, which in some coordinates representation are generically written as $T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$.

Let's consider now the coordinate transformation $x^\mu \to x'^\mu(x)$. The transformation rule of a generic tensor under such a coordinate change is given by

$$T'^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_p}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_p}}{\partial x'^{\nu_p}}T^{\rho_1\rho_2\ldots\rho_n}_{\sigma_1\sigma_2\ldots\sigma_p}(x). \tag{3.2}$$

Such an object is called a rank-(n,p) tensor and the transformation rule (3.2) is a generalisation of the Lorentz transformation of SR.

The most used tensors we will encounter throughout this work are the following:

- scalars, i.e. $(0,0)$-tensors that transform as

$$\phi'(x') = \phi(x); \tag{3.3}$$

- controvariant vectors, i.e. $(1,0)$-tensors that accordingly with (3.2) transform as

$$A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu; \tag{3.4}$$

- covariant vectors, i.e. $(0,1)$-tensors transforming as

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}A_\nu; \tag{3.5}$$

- tensor densities of weight p, i.e. tensors whose transformation rule under a a coordinate transformation is given by

$$T'^{\mu\nu} = \left|\frac{\partial x'}{\partial x}\right|^p \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}T^{\rho\sigma}, \tag{3.6}$$

  where $|\quad|$ is the determinant of the *Jacobian* of the transformation.

The metric itself is a rank-$(0,2)$ symmetric tensor with components $g_{\mu\nu}$ such that $g_{\mu\nu} = g_{\nu\mu}$. Another important quantity we will frequently meet is the determinant of the metric tensor $g \equiv \det g_{\mu\nu}$; this is an example of a scalar density of weight "-2", whose transformation rule is

$$g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g. \tag{3.7}$$

Furthermore the infinitesimal four-volume element d^4x is a scalar density of weight "+2". From this it follows that the quantity $d^4x\sqrt{-g}$ is invariant under a generic coordinate transformation[1]. This turns out to be very useful in the Lagrangian

[1] The minus sign inside the square root is a consequence of the fact that the in GR the metric is not positive definite.

formulation of General Relativity, where d^4x is replaced by the invariant volume element $d^4x\sqrt{-g}$.

The main difference between the tensorial calculus in Minkowski space and in general curved manifolds is the concept of differentiation. While in SR the partial derivative of a tensor is still a tensor, in GR this is not the case. If we want to differentiate a tensor still obtaining an object that transforms like a tensor we must replace partial derivatives with *covariant derivatives*, with the so-called *comma-goes-to-semicolon* rule

$$\frac{\partial}{\partial x^\mu}A^\nu \equiv A^\nu_{\ ,\mu} \longrightarrow A^\nu_{\ ;\mu} \equiv \nabla_\mu A^\nu = \frac{\partial A^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\rho}A^\rho; \tag{3.8}$$

$$\frac{\partial}{\partial x^\mu}A_\nu \equiv A_{\nu\,,\mu} \longrightarrow A_{\nu\,;\mu} \equiv \nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma^\rho_{\mu\nu}A_\rho. \tag{3.9}$$

We have listed the covariant derivative for a controvariant and a covariant vector. This can easily generalised to the (n,p)-rank tensor. For a scalar tensor ϕ we have instead $\phi_; \equiv \phi_,$. The Γ coefficients are called *connections*.

So far we have said that in General Relativity the spacetime is a four-D curved manifold; this can be implemented with a connection that allows to generalise the concept of derivation in curved space. Further we can endow it with a metric $\mathbf{g}$. This permits us to calculate distances in the manifold $\mathcal{M}$ through

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{3.10}$$

where we have chosen a particular set of coordinates x^μ and associated components for $\mathbf{g}$ given by $g_{\mu\nu}(x)$.

Afterwards, without entering into the details of differential geometry, it is possible to show that under the assumption of *torsion-free* covariant derivative, that practically entails $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$, and requiring the *metric compatibility* condition: $g_{\mu\nu;\alpha} = 0$ for each value of the indexes $(\mu, \nu\,\alpha)$, we can relate the components of the connections to the components of the metric $g_{\mu\nu}$ with the crucial expression

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}). \tag{3.11}$$

Under this conditions the connections are also called *Christoffel symbols*.

Torsion-free and metric compatibility conditions are assumed in GR for the pair $(\mathcal{M}, \mathbf{g})$ and in the rest of this work unless specified differently.

The intrinsic curvature of the spacetime $\mathcal{M}$ is encoded in a special tensor called the *Riemann tensor* whose expression in terms of the connections is

$$R^\mu_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma}\Gamma^\tau_{\nu\rho}. \tag{3.12}$$

We note that the above expression does not depend on the metric. Indeed covariant derivatives, connections and Riemann tensor can be introduced without the needing of a metric over $\mathcal{M}$. However, once we have introduced a metric on the manifold, as we have already seen for the Christoffel symbols and under the same conditions, we can express the Riemann tensor as follows

$$R_{\mu\nu\rho\sigma} = g_{\mu\tau}R^{\tau}_{\nu\rho\sigma} = \frac{1}{2}(+g_{\nu\rho,\mu\sigma} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma}) + g_{\alpha\beta}\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\nu\rho} - g_{\alpha\beta}\Gamma^{\alpha}_{\mu\rho}\Gamma^{\beta}_{\nu\sigma}. \tag{3.13}$$

We can contract the first and the third index of the Riemann tensor to obtain the so called *Ricci tensor* whose expression is

$$R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\mu} - \Gamma^{\mu}_{\nu\mu,\sigma} + \Gamma^{\mu}_{\nu\sigma}\Gamma^{\rho}_{\mu\rho} - \Gamma^{\mu}_{\nu\mu}\Gamma^{\rho}_{\sigma\rho}. \tag{3.14}$$

A further contraction gives us the *Ricci scalar* $R = g^{\nu\sigma}R_{\nu\sigma}$.

The Ricci tensor and the Ricci scalar are the objects that enter the Einstein's equation, together with the energy-momentum tensor $T_{\mu\nu}$, a (0,2)-rank tensor, which describes matter.

In Minkowski space each component of the Riemann tensor vanishes. Whenever this happens we say that the space is flat[2].

An important property coming from these definitions is that, if we define the *Einstein tensor* $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, it is possible to prove that it is covariantly conserved, i.e.

$$G^{\alpha\beta}{}_{;\beta} = 0. \tag{3.15}$$

This is called the *Bianchi identities* and ensure that the energy-momentum tensor in (3.1) is also covariantly conserved, i.e. $T^{\mu\nu}{}_{;\nu} = 0$[3].

3.1.2 The Equivalence Principle

One of the fundamental pillars of Einstein's theory is the relationship between gravitation and inertia, namely the observed equivalence between gravitational and inertial mass. As pointed out by Einstein in his famous *gedanken* experiments, a free-falling system in a gravitational field is physically equivalent to a system subjected to an appropriate acceleration, of course if the system is small enough to neglect tidal (gravitational) forces. This consideration was assumed as a principle by Einstein and it put the basis for the construction of General Relativity theory. Afterwards the Einstein's idea has been developed and encapsulated in what are called the *equivalence principles.*

Because of their importance in the conception of GR and also in testing alternative theories beyond it, following [29] we list them in this way:

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[2]Of course, if we find a frame in which $R_{\mu\nu\rho\sigma} = 0$ for each index, then it remains true in every frame because of the covariant nature of the expression. Hence, the flatness of a space is an invariant property of it.

[3]The conservation of $T^{\mu\nu}$ is required by the *Einstein Equivalence Principle* as we shall see later.

- *Weak Equivalence Principle (WEP)*: All uncharged, freely falling test particles follow the same trajectories, once an initial position and velocity is fixed.

  Freely falling particles are particles not subjected to any external force and therefore follow the geodesics of the spacetime. Another way to put the WEP is to say that inertial and gravitational mass are equivalent.

- *The Einstein Equivalence Principle (EEP)*: The WEP holds, and furthermore in all freely falling frames one recovers (locally, up to tidal gravitational forces) the same laws of special relativistic physics, independent of position and velocity.

- *The Strong Equivalence Principle (SEP)*: The WEP holds for test particles as well as for massive gravitating body and in all freely falling frames one recovers (locally, up to tidal gravitational forces) the same laws of special relativistic physics, independent of position and velocity.

The importance of the EEP relies on the fact that any non gravitational law that is true in SR is still true in a local inertial frame (LIF) in GR. For example, since in SR the energy-momentum tensor $T^{\mu\nu}$ is conserved, i.e. $T^{\mu\nu}{}_{,\nu} = 0$, then it is also conserved in a LIF in GR. But this implies that this expression must be true in any other frame, where the partial derivative is in general replaced by the covariant derivative:

$$T^{\mu\nu}_{,\nu} = 0 \quad \text{in a LIF} \quad \rightarrow \quad T^{\mu\nu}_{;\nu} = 0 \quad \text{in a general frame.}$$

Due to their importance for the understanding and foundation of General Relativity a lot of tests have been performed to reveal their validity.
The most common tests of the WEP are the Eötvös-like experiments where an eventual difference between gravitational and inertial mass is measured. In this tests two different laboratory-sized bodies are considered and chosen to have different composition. The inertial mass-energy of the two bodies is constituted by several contributions: the rest-mass, the electromagnetic energy, the weak-interaction energy and so on. If one of these kinds of mass-energy contributes differently to the gravitational mass than it does to the inertial mass, in the presence of an external gravitational field the two bodies, since they differ in composition, may experience a different acceleration, causing a violation of the WEP. We can write

$$m_p = m_I + \sum_i \eta^i \frac{E^i}{c^2},$$

where E^i is the internal energy of the body generated by the interaction "i" and η^i is the dimensionless parameter that encapsulates the deviation from the WEP. The acceleration experienced by the two bodies is given by

$$a_1 = \Big(1 + \sum_i \eta^i \frac{E^i}{c^2 m_{I1}}\Big) g, \quad a_2 = \Big(1 + \sum_i \eta^i \frac{E^i}{c^2 m_{I2}}\Big) g..$$

What is usually measured in this kind of experiments is the "Eötvös-ratio"

$$\eta = \frac{2|a_1 - a_2|}{a_1 + a_2}.$$

If the WEP holds, $\eta = 0$. The most stringent null result we have has been obtained using beryllium and titanium and it is given by $\eta = (0.3 \pm 1.8) \times 10^{-13}$ [12]. This can be improved further in space based experiments since most of the noise comes from Earth ground perturbations.

Testing the EEP and the SEP is more involved, because not only we have to show that different (gravitating) test bodies follow the same trajectories, but also that locally the entire set of Special Relativistic laws holds. For a more comprehensive discussion about this issue see [29] and reference therein.

3.1.3 The Einstein's equation

We have now all the ingredients to understand the Einstein's equations (3.1). Essentially they can be inferred if one requires an expression that gives second order equations of motion as the Newtonian equation for gravity is, and such that in the weak field limit they reduce to the Poisson equation

$$\nabla^2 \Phi(\vec{x}) = 4\pi G \rho(\vec{x}), \tag{3.16}$$

where $\Phi(\vec{x})$ is the gravitational potential, while $\rho(\vec{x})$ is the mass density.

Such an expression should be written in terms of the metric tensor and should be linear in the energy-momentum tensor since ρ appears linearly in (3.16). The simplest expression that it is possible to find and that is in agreement with the EEP[4] is exactly Eq.(3.1). This is not only the simplest expression, but also the unique choice to have second order differential equations in the metric and that ensures the conservation of $T^{\mu\nu}$.

In reality an extra term is allowed, as pointed out by Einstein itself some years later the first publication, that is the cosmological constant term. The complete Einstein's equation then becomes

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3.17}$$

Λ is a constant term, and because of the metric compatibility condition the extra-term $\Lambda g_{\mu\nu}$ is also conserved, and the conservation equation is not spoiled.

[4]The EEP implies ${T^{\mu\nu}}_{;\nu} = 0$ and this is consistent with the conservation of $G^{\mu\nu}$ via the Bianchi identities. This reveals the self-consistency of GR.

3.1.4 The Lagrangian formulation of General Relativity

The vacuum Einstein's equation, given by (3.17) with $T_{\mu\nu} = 0$, can be equivalently obtained from a variational principle applied to the so called *Hilbert-Einstein (H-E) action*, that is

$$S_{HE} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g}(R - 2\Lambda), \tag{3.18}$$

where $g = \det g_{\mu\nu}$ and R is the Ricci scalar. We assume that torsion-free and metric compatibility conditions hold and therefore R is function only of $g_{\mu\nu}$. Note that the action is invariant under a generic coordinate transformation.

We now explicitly show that varying the action with respect to the metric, and imposing the least action principle then we obtain the vacuum Einstein's equation. Recalling that $R = g^{\mu\nu} R_{\mu\nu}$ the variation of the action (3.18) reads

$$\begin{aligned} \delta \int d^4x \sqrt{-g}(g^{\mu\nu} R_{\mu\nu} - 2\Lambda) &= \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int d^4x (R - 2\Lambda) \delta\sqrt{-g} + \\ &+ \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}. \end{aligned}$$

Let's start evaluating $\delta\sqrt{-g}$:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta g = -\frac{1}{2\sqrt{-g}} \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}, \tag{3.19}$$

where in the last equality we have used the relation $\det A = e^{Tr \ln A}$, with A a generic matrix, that in our case is $g_{\mu\nu}$. Moreover, since we have

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \quad \to \delta g_{\mu\nu} = -g_{\mu\sigma} g_{\nu\rho} \delta g^{\sigma\rho}$$

we can rewrite (3.1.4) as follows

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{3.20}$$

Furthermore, it is possible to show that

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla^\sigma v_\sigma, \tag{3.21}$$

i.e. it contributes with a surface term that can be put to zero, see [1].

Collecting the various pieces we find

$$\int d^4x \sqrt{-g} \big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda)\big) \delta g^{\mu\nu}.$$

The least action principle $\frac{\delta S_{HE}}{\delta g^{\mu\nu}} = 0$ gives us the vacuum Einstein's equation.

We can take into account the presence of matter by adding an action S_m of the kind

$$S_m = \frac{1}{c}\int d^4x\sqrt{-g}\mathcal{L}_m(g,\psi), \tag{3.22}$$

where $\mathcal{L}_m$ is the Lagrangian density for the generic matter field ψ[5].

Defining the energy-momentum tensor for ψ in the following way

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{3.23}$$

and taking the variation of the total action $S = S_{HE} + S_m$ with respect to $g^{\mu\nu}$, we obtain

$$\int \sqrt{-g}\big[\frac{c^3}{16\pi G}\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda)\big) - \frac{1}{2c}T_{\mu\nu}\big]\delta g^{\mu\nu}. \tag{3.24}$$

This gives us the Einstein's equation (3.17).

In this approach, the conservation of $T_{\mu\nu}$ follows from the conservation of $G_{\mu\nu} + \Lambda g_{\mu\nu}$ and therefore the Einstein Equivalence Principle is automatically satisfied. Moreover from its definition it follows that it is symmetric in (μ,ν).

It's important to underline that within this approach we can also obtain the equation of motion for the matter field by extremising S_m with respect to ψ

$$\frac{\delta S_m}{\delta\psi} = 0. \tag{3.25}$$

Whenever this condition is fulfilled we say that the matter field is *on shell*.

The second point we want to stress is that the form of the action (3.22) is the one that defines the *minimal coupling* between matter and gravity. Essentially we have minimal coupling simply taking the matter action of Special Relativity and doing the substitutions $\eta_{\mu\nu} \to g_{\mu\nu}$ and $d^4x \to d^4x\sqrt{-g}$. If no other fields enter both the gravitational part of the action and the matter part, it is possible to show that such substitutions determine the conservation of $T_{\mu\nu}$ (and consequently matter follows the geodesics of the spacetime).

3.2 Beyond General Relativity

General Relativity is one of the most successful theories of modern physics. Indeed, it has been the same since its formulation in 1915 and a very large amount of experiments has been made to test it and amazingly they have widely confirmed its validity.

[5]In GR ψ is a tensorial field on the manifold $\mathcal{M}$ and it depends on the the spacetime position, i.e. $\psi = \psi(x)$. For example it could be a scalar field or a fermionic field satisfying the Dirac equation.

We have already mentioned the Eötvös test about the foundational principle of GR in the last section. More experiments have been made on different length scales, from the microscopic world (micron) to the cosmological scales. Without entering into the details we can list some of the most known:

- *The gravitational redshift of light*: the frequency of light gets redshifted in passing through a gravitational field. One consequence is that a clock runs in different ways in a gravitational field accordingly with its position. A typical test of this is the Pound-Rebka experiment [13].
- *The bending of light due to the solar gravitational field*: in General Relativity photon trajectories are bended by the gravitational field. For a ray passing near the sun the deflection angle θ is predicted to be $\theta \simeq 1.75''$. The tightest constraint from observation says $\theta = (0.99992 \pm 0.00023) \times 1.75''$ from [31].
- *The perihelion precession of Mercury*: the observed "anomalous" perihelion precession of Mercury cannot be explained by Newtonian gravity, but it finds an explanation in GR. Current measurements of it are in agreement with the predicted value of GR, see [14].
- *The detection of Gravitational Waves*: like the first two effects listed above, gravitational waves are a peculiar feature of GR. Their detection took one entire century since their first prediction from the Einstein's equation.

Despite the observational data are in very good agreement with the predictions (and foundations) of general relativity, in the last decades the Einstein's theory has also showed its limits. First of all we know that this theory is not applicable at the quantum level. In particular it is well-known that GR is not renormalizable, and therefore it is believed it is only an effective field theory whose validity holds for low energy scales. Besides this, as we have pointed out in the last chapter, it is believed that the universe went through a phase of cosmic acceleration due to the inflaton field in the early time, and we are currently experiencing another accelerated expansion phase due to the DE. Since, up to now, no convincing arguments from theoretical physics could explain such a puzzle, it seems reasonable to think that a solution could be found within alternative theories of gravity. Moreover, we have already seen that in the universe we have another dark component, the Dark Matter, whose effects appear only gravitationally.

In principle we can explain all these dark sectors hypothesising that a modification of Einstein's equations can accommodate the observational data. For example we can hypothesise that a scalar field enters the H-E action coupled with gravity and this could be the scalar field responsible for the inflation, or otherwise for the late-time acceleration, or both of them. This is just an example; in the rest of this work we shall see some concrete models of modified gravity.

The important point to fix is that besides the numerous experiments that have tested the validity of Einstein's theory, it's widely accepted that GR is not the final theory of gravitation. Indeed some experiments have showed deviations from GR, such as the detection of a small variability of the fine-structure constant. Together with the unknown sectors of cosmology these issues are sufficient to give sense to an investigation beyond General Relativity. Before doing it, we are going to see how we can modify GR consistently.

3.3 Modifying General Relativity

Attempts to modify GR should be consistent with the well tested predictions of Einstein's theory. Because of this, deviations from it must be negligible in the solar system scales and however must check all the observational data. What we are looking for are gravity equations that differ from the Einstein's equations, but in some sense not too much in the weak-field regime of the solar-system tests.

A way to understand how to modify gravity is provided by the *Lovelock's theorem*. This states that Einstein's equations (3.17) are the unique expressions we can obtain from a variational principle applied to a scalar density $\mathcal{L}(g)$ depending only on the metric and in a four-dimensional spacetime.

Therefore, if we want to obtain gravitational equations different from the Einstein's ones we must loosen at least one of the assumptions of this theorem.

We list now some possible paths we can follow emerging from the Lovelock's theorem in modifying gravity:

1 We can deal with a $D = n > 4$ spacetime.

  This is a notable research topic and is quite natural in Grand Unified Theories. An example of a modified theory of gravity involving more than four dimensions are Kaluza-Klein models [23].

2 We can release the request to have second order differential equations, as the Einstein's equations are.

  Examples of theories that consider higher order derivatives are: $f(R)$ gravity, Gauss-Bonnet gravity and Horâva-Lifshitz gravity.

3 We can construct a gravity action depending on some extra fields other than the metric.

  This is doubtless the most followed path among the attempts to modify gravity. In general the extra field(s) is a scalar that enters the gravity equations together with the metric. An example is the Brans-Dicke theory, or more in general the so-called Scalar-Tensor theories (ST), such as the Horndeski's theories. Of course instead of a scalar it is possible to add a vector or even a tensor. For example a theory called Teves takes into account tensor, vector and scalar degrees of freedom.

For a review of the many possible modified gravity theories see [29].

Of course, these are not the unique ways to modify GR. For example we can consider a non-local theory, or give up Lorentz-invariance. Certainly it would not be so easy to accommodate such theories with observations, since locality and Lorentz-invariance are well-accepted facts.

In this work we will focus on the third path, i.e. we will consider Scalar-Tensor theories and we will investigate how these theories are modified by peculiar transformations of the metric.

Chapter 4

Scalar-Tensor theories

In our work we will focus our attention on Scalar-Tensor theories (ST) of modified gravity. In this wide class of gravitational theories a scalar field is considered other than the metric tensor, the former being generally "non-minimally" coupled to gravity and/or to the matter fields, and provided with its own dynamics.

The first remarkable attempt to construct a ST theory was performed in the '40s by P. Jordan [22] in the context of five-dimensional Kaluza-Klein's theory [23]. He first introduced a non-minimal coupling between the scalar field and the gravity sector replacing the gravitational constant of Einstein's theory with the scalar field; he was inspired by the Dirac's "large number hypothesis", i.e. the idea that the fundamental constants should vary with time. Later on in 1961 Brans and Dicke proposed what we can now call the *prototype Brans-Dicke (BD) model* [24], closely related to the Jordan's one, and that can be thought as the mother of all the subsequent ST theories.

The reason why Scalar-Tensor theories are widely studied today is related to the fact that scalar fields are ubiquitous both in cosmological and high energy physics. We need only think that almost all the inflationary models are based on one (or more) scalar field(s) as well as Dark Energy models. In high energy physics the situation is similar: Higgs bosons are scalar fields and constitute the main ingredient in unification theories of particle physics. Furthermore, the presence of a scalar field related to the metric tensor seems to be unavoidable in theories that attempt to unify gravity with the other fundamental forces, such as *Supergravity* and *Superstring* theories.

In this chapter we present the prototype Brans-Dicke model and see how it can be constrained with solar system experiments of gravity. Unfortunately they limit the possible deviations from GR to be very small and therefore it is necessary to go beyond this prototype version of ST theories. Thus, we generalize the BD model to a more generic theory involving two free parameters rather than the one of the prototype model. Further we will briefly introduce $f(R)$ theories as an example of modified theories of gravity that can be recast in a Scalar-Tensor like form and then *Quintessence* theories as models for Dark Energy which contain a scalar field. Finally we generalize the ST action to the most general action containing a scalar and a tensor field giving rise to second order equations of motion in four-dimensional spacetime, dubbed the Horndeski theory.

4.1 The prototype BD model

The model proposed by Brans and Dicke in 1961 is described by the following action

$$S_{BD} = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\Big(\phi R - \frac{\omega_0}{\phi}\nabla^\mu\phi\nabla_\mu\phi\Big) + S_m[g,\psi_m]. \tag{4.1}$$

Here ϕ is the scalar field that, as we see, plays the role of the inverse of the gravitational constant, ω_0 is a dimensionless constant called the *Brans-Dicke parameter*, and indeed is the only free parameter of the theory; ψ_m indicates generic matter fields. Let us note that the scalar field does not enter the matter action; this ensures that matter follows geodesics of the metric as it is well established from experiments and that the matter energy-momentum tensor is covariantly conserved, i.e. $T^{\mu\nu}_{m\,;\nu} = 0$. The first term in (4.1) defines what is meant with non-minimal coupling of the scalar field with gravity.

The full set of equations of motion can be obtained taking the variation of the action with respect to $g_{\mu\nu}$, ϕ and ψ_m. Varying with respect to the metric we obtain

$$G_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} + \frac{\omega_0}{\phi^2}\big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi\big) + \frac{1}{\phi}\big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\big). \tag{4.2}$$

Variation with respect to ϕ and ψ_m leads to

$$2\frac{\omega_0}{\phi}\Box\phi - \frac{\omega_0}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + R = 0; \tag{4.3}$$

$$\frac{\delta S_m}{\delta\psi_m} = 0, \tag{4.4}$$

where $\Box$ is the covariant D'Alembertian that for a scalar field reads

$$\Box\phi = \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\phi_{,\mu})_{,\nu}.$$

Remembering that $G^\mu_\mu = -R$, we can recast (4.3) in a simpler form taking the trace of (4.2) and substituting in (4.3) the expression for R thus obtained. This gives us

$$\Box\phi = \frac{8\pi}{c^4(2\omega_0+3)}T. \tag{4.5}$$

We see that the dynamics of the scalar field is sourced by the matter field, so at the level of the equations of motion matter is coupled with the scalar field.

However we note that in the limit $\omega_0 \to \infty$ the right-hand side of (4.5) vanishes and the scalar field dynamics is decoupled from matter. This shows that in the $\omega_0 \to \infty$ the BD model tends to General Relativity, plus of course the presence of a decoupled scalar field.

4.1.1 Solutions for the BD model

A huge variety of solutions of Eq.(4.2)-(4.5) has been found during the last fifty years. A first analysis in the weak field approximation is already present in the seminal paper of Brans and Dicke [24]. The latter also found a static spherically symmetric solution about a point mass given by

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}(dr^2 + r^2 d\Omega^2), \tag{4.6}$$

and

$$\phi = \phi_0 \left[\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right]^{\frac{-C}{\lambda}}, \tag{4.7}$$

with

$$\begin{aligned} e^{2\alpha} &= e^{2\alpha_0}\left[\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right]^{\frac{2}{\lambda}}; & (4.8)\\ e^{2\beta} &= e^{2\beta_0}\left(1+\frac{B}{r}\right)^4\left[\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right]^{\frac{2(\lambda-C-1)}{\lambda}}, & (4.9) \end{aligned}$$

where α_0, β_0, B and C are arbitrary constant, $\lambda = [(C+1)^2 - C(1-\frac{\omega_0}{2}C)]^{\frac{1}{2}}$.

More recently a non-static spherically symmetric vacuum solution of (4.2)-(4.5) has been found [25], pointing to the fact that the Birkhoff theorem doesn't hold in the Brans-Dicke model.

The BD cosmology has also been well-studied, starting with [24] where calculations were carried out at the background level. Extended vacuum solutions with a spatial-dependent scalar field are given in [27] (see also [26, 28]). Solutions for anisotropic universes (Bianchi-like) exist; for a list of references about these topics see [29].

4.2 The parametrised post-Newtonian formalism

One way to test a theory of gravity, that could be General Relativity or an alternative to it, is to compare its theoretical outcomes with the results obtained in solar system experiments which are currently fairly accurate.

We know that at the solar system level gravitation is well-described by the Newtonian theory. Here, a test-body feels an acceleration given by $\vec{a} = \vec{\nabla} U$ where U is the gravitational potential satisfying $\nabla^2 U = -4\pi\rho$; ρ is the rest-mass energy density of the gravitational source. In an appropriate coordinate system the Newtonian metric reads: $g_{00} = -(1-2U)$, $g_{ij} = \delta_{ij}$.

The Newtonian theory is capable to explain gravitational effects with a precision of one part in 10^5, but this is not enough to clarify phenomena like the Mercury's perihelion shift that requires a precision of the order of one part in 10^7. A more accurate description of gravitational phenomena in our solar system must go beyond the Newtonian theory and account for *post-Newtonian* effects.

A successful approach in this sense is the so-called *parametrised post-Newtonian* formalism, or *PPN* formalism, that provides a direct way to compare experimental results with a proposed gravity theory. The main ingredient of this approach is an expansion in "small" quantities of the metric, where the order of smallness is dictated by the following assumptions

$$U \sim v^2 \sim \frac{P}{\rho} \sim \Pi \sim O(2), \tag{4.10}$$

where v is the planetary velocity[1], p/ρ is the ratio of the pressure to density of the matter making up the solar system objects, Π is the ratio of energy density to rest-mass energy. U is generally of the order of 10^{-5}. Note that we are assuming a perfect-fluid form for the matter, with pressure P and density ρ. Furthermore, since $\frac{\partial}{\partial t} \sim \vec{v} \cdot \vec{\nabla}$ and $v \sim O(1)$, then

$$\frac{|\partial/\partial t|}{|\partial/\partial x|} \sim O(1). \tag{4.11}$$

For example the product $U \cdot v$ is $O(3)$, while U^2 is $O(4)$.

It can be shown (see [30]) that in order to go beyond the Newtonian approximation and approach the post-Newtonian limit we need to expand the metric components as follows

$$g_{00} \quad \text{to O(4)}; \qquad g_{0i} \quad \text{to O(3)}; \qquad g_{ij} \quad \text{to O(2)}, \tag{4.12}$$

if we want to describe massive particle motion; if instead we want to analyse light-ray propagation it suffices the following expansion

$$g_{00} \quad \text{to O(2)}; \qquad g_{ij} \quad \text{to O(2)}. \tag{4.13}$$

Also the perfect-fluid energy momentum tensor components must be expanded accordingly, i.e.

$$T^{00} \quad \text{to } \rho\, O(2); \qquad T^{0i} \quad \text{to } \rho\, O(3); \qquad T^{ij} \quad \text{to } \rho\, O(4). \tag{4.14}$$

How does this approach work? We have a post-Newtonian system (our solar system) with a metric $g_{\mu\nu}$. We require that far from such a system the metric

[1]The virial theorem ensures that U and v^2 are of the same order.

tends asymptotically to the Minkowskian one. Therefore, since we are in a weak field regime we can expand the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small. Then we write the metric correction $h_{\mu\nu}$ in terms of metric functionals (or potentials) generated by U, rest-mass energy and velocity taking into account all the possible viable combinations, with arbitrary parameters in front of them. Thus, choosing an appropriate coordinate system and an appropriate gauge[2] the PPN test metric reads

$$\begin{aligned}
g_{00} &= -1 + 2GU - 2\beta G^2 U^2 - 2\xi G^2 \Phi_W + (2\gamma + 2 + \alpha_3 + \beta_1 - 2\xi) G\Phi_1 \\
&\quad 2(1 + 3\gamma - 2\beta + \beta_2 + \xi) G^2 \Phi_2 + 2(1 + \beta_3) G\Phi_3 - (\beta_1 - 2\xi) GA \\
&\quad 2(2\gamma + 3\beta_4 - 2\xi) G\Phi_4; \\
g_{0i} &= -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \beta_1 - 2\xi) GV_i - \frac{1}{2}(1 + \alpha_2 - \beta_1 + 2\xi) GW_i; \\
g_{ij} &= (1 + 2\gamma GU)\delta_{ij}.
\end{aligned}$$

Here $\beta, \gamma, \xi, \beta_1, \beta_2, \beta_3, \beta_4, \alpha_1, \alpha_2$ and α_3 are the post-Newtonian parameters, while $\Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4, A, V_i$ and W_i are the post-Newtonian gravitational potential constructed using ρ, P, U, Π. Their definitions are given in [30].

Of course this metric does not depend on a particular theory and indeed its parameters are constrained by experiments only. For instance, the γ parameter can be constrained through measurements of the bending of light by sun. Using the PPN test metric the predicted bending of light is

$$\theta = \frac{1+\gamma}{2}\theta_{GR}, \tag{4.15}$$

where θ_{GR} is the GR prediction for θ ($\theta_{GR} = 1$). Using the tightest constraint on θ given by the experiment due to Shapiro, David, Lebach and Gregory [31], we have

$$\gamma - 1 = (-1.7 \pm 4.5) \times 10^{-4}. \tag{4.16}$$

A tighter constraint on the γ parameter can be obtained measuring the Shapiro delay of time. It gives [32]

$$\gamma - 1 = (-2.1 \pm 2.3) \times 10^{-5}. \tag{4.17}$$

The next step is to understand how we can relate the PPN formalism to a given gravity theory. The procedure is the following: we consider the dynamical variables of the theory expanding them at the required order around their background value dictated by the asymptotic conditions. For example in a generic

[2] Generally it is considered a quasi-Cartesian coordinate system at rest with the universe (the universe rest frame is the frame in which the universe appears isotropic). The gauge choice is called *standard gauge*, see [30].

Scalar-Tensor theory, where the dynamical variables are the metric tensor and the scalar field, we should expand

$$\begin{aligned} h_{00} &\sim O(2) + O(4); \qquad h_{0i} \sim O(3); \qquad h_{ij} \sim O(2); && (4.18) \\ \phi &\equiv \phi_0 + \varphi \quad \rightarrow \quad \varphi \sim O(2) + O(4). && (4.19) \end{aligned}$$

Then we substitute such expansions in the field equations, retaining only those terms consistent with the post-Newtonian order considered and solve the equations for h_{00}, h_{0i} and h_{ij}. To compare the metric tensor thus obtained with the PPN test metric it is necessary to transform the coordinates and the gauge to the standard form adopted by the PPN metric. At this point it is possible to compare the two metrics and constrain the theory parameters with the PPN ones.

4.3 Constraints on the prototype BD model

We can follow the procedure outlined in the previous section for the prototype Brans-Dicke model. In this theory we have only one free parameter (ω_0) that can be constrained comparing the post-Newtonian BD metric with the PPN test metric. It can be shown that the post-Newtonian limit of the BD model gives the following metric components [30]

$$\begin{aligned} g_{00} &= -1 + 2GU - 2G^2U^2 + 4\Big(\frac{3+2\omega_0}{4+2\omega_0}\Big)G\Phi_1 + \\ &\qquad +4\Big(\frac{1+2\omega_0}{4+2\omega_0}\Big)G^2\Phi_2 + 2G\Phi_3 + 6\Big(\frac{1+\omega_0}{2+\omega_0}\Big)G\Phi_4; \\ g_{0i} &= -\frac{1}{2}\Big(\frac{10+7\omega_0}{2+\omega_0}\Big)GV_i - \frac{1}{2}GW_i; \\ g_{ij} &= \Big(1 + 2\Big(\frac{1+\omega_0}{2+\omega_0}\Big)GU\Big)\delta_{ij}. \qquad (4.20) \end{aligned}$$

From this we can read off the value of the PPN parameters in terms of ω_0:

$$\gamma = \frac{1+\omega_0}{2+\omega_0}; \qquad \beta = 1; \qquad \xi = \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.$$

The Shapiro delay constraint on γ is translated into $\omega_0 \gtrsim 500$. However, there is a tighter constraint coming from observations carried out by the Cassini satellite of the time delay of radio signals; this gives $\omega_0 \gtrsim 40000$.

We have said that the more ω_0 is, the more the BD model approaches General Relativity. Indeed if $\omega_0 \to \infty$ then $\gamma \to 1$, that is the value of γ in GR.

The observational constraints on ω_0 put strong limits on the possible deviations from GR provided by this model. $\omega_0 \gtrsim 40000$ means that deviations from GR are very small on all the scales, and therefore the two theories look phenomenological very similar. This problem can be circumvented allowing for a variable ω and introducing a potential term for the scalar field. This is the topic of the next section.

4.4 Extension of the Brans-Dicke model

A natural extension of the prototype Brans-Dicke theory is to allow for a varying ω_0, which can be promoted to a function of the scalar field $\omega = \omega(\phi)$. Further, we can introduce a cosmological constant term and also account for a ϕ-dependence of it. This leads to the following action

$$S = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\Big(\phi R - \frac{\omega(\phi)}{\phi}\nabla^\mu\phi\nabla_\mu\phi - 2\Lambda(\phi)\Big) + S_m[g, \psi_m]. \tag{4.21}$$

Note that the generalised cosmological constant term can also be interpreted as a potential for the scalar field, that therefore acquires a truly complete dynamics.

Allowing for a scalar field redefinition: $\phi \to f(\phi)$, this action can be recast in the following form

$$S = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\Big(f(\phi)R - g(\phi)\nabla^\mu\phi\nabla_\mu\phi - 2\Lambda(\phi)\Big) + S_m[g, \psi_m], \tag{4.22}$$

with a redefinition of $\Lambda(\phi)$ and the new function $g(\phi)$.

The latter is a quite general form for a generic Scalar-Tensor theory whose action is quadratic in the first derivatives of the scalar field, up to boundary terms. In particular no second-derivatives or higher than second derivatives appear in the action[3].

The equations of motion derived from (4.21) are similar to the ones obtained for the prototype version, with the addition of extra terms due to the ϕ-dependence on ω and the extra Λ-term; therefore we omit them. Anyway, within this theory we can still perform a post-Newtonian expansion and solve for the metric correction $h_{\mu\nu}$. This leads to the following PPN-parameters

$$\gamma = \frac{1+\omega}{2+\omega}, \qquad \beta = 1 + \frac{d\omega/d\phi}{(4+2\omega)(3+2\omega)^2}. \tag{4.23}$$

The γ-parameter is the same of the prototype version, except that now ω is not constant and therefore the previous constraints based on solar-system experiments only bind ω at the present time in our solar system. The β-parameter is instead different from its prototype counterpart. Of course we note that in the limit $\omega \to$ constant it reduces to one as in the BD model. Measurements of Mercury's perihelion precession can constraint β to be $\beta - 1 \simeq O(3)$ or $O(4)$. The strong constraint $\omega \gtrsim 40000$ coming from γ measurements still holds, although as we have already noted, it concerns only the local value of ω.

A key feature of this class of theories is that there can be mechanisms for which the presence of the scalar field is hidden/screened in weak-field regimes

[3]This is a sufficient condition to have second-order equations of motion, but not necessary. In fact, as we shall see in the next section, this action can be generalised further to include higher order derivatives of the scalar field.

(leading to a GR-like theory) but at cosmological scales it is still alive. Such mechanisms generally permit to satisfy all the solar system tests of gravity, still leaving the pssibility of GR deviations at cosmological levels. This is exactly what we want, since up to now no deviations from GR have been measured in the weak field regime of the solar system. Among these mechanisms there are the "spontaneous scalarization" [33], the "chameleon mechanism"[34] and the "symmetron mechanism"[35].

A note is in order. We can further generalize the action (4.21) or (4.22) allowing for a coupling of the scalar field to the matter action. A natural way to perform this is to introduce the following coupling

$$S_m[g_{\mu\nu}, \psi_m] \quad \longrightarrow \quad S_m[e^{2\alpha(\phi)} g_{\mu\nu}, \psi_m], \tag{4.24}$$

where $\alpha(\phi)$ is an arbitrary function of ϕ. This is a conformal-type coupling arising performing a conformal transformation of the metric $g_{\mu\nu}$. We shall analyse conformal transformations and their effect on Scalar-Tensor theories in the next chapter.

4.5 f(R) and *quintessence* theories

Historically, besides the Scalar-Tensor theories *a là* Brans-Dicke, many other alternatives to general Relativity have been considered. Fortunately, it has been showed that some of them, that apparently seem to be unrelated to a Scalar theory, indeed can be reformulated eating the extra degree of freedom with a scalar field. This is the case for example of $f(R)$ theories that we are going to introduce briefly.

Furthermore we will introduce *quintessence* theory and its extension, as theories for Dark Energy. These theories contemplate the presence of a scalar field, but it is not coupled to gravity; in other words the scalar field sources the RHS of the Einstein's equation as an ordinary matter field.

4.5.1 f(R) theories

$f(R)$ gravity is a natural extension of General Relativity in which the extra degrees of freedom come from higher order terms in metric derivatives in the EoM. The basic formulation replaces the Ricci scalar of GR with a generic scalar function of it, which is commonly indicated with $f(R)$. That is, the linear dependence on R of the Einstein-Hilbert action is replaced with a generic function of R that leads to higher order terms in metric derivatives. Higher-order curvature terms in the action can appear when quantum corrections are considered.

A notable example of $f(R)$ gravity is the Starobinsky model [40], where $f(R) = R + \alpha R^2$, $(\alpha > 0)$. This model provides an early phase of accelerated expansion and in fact in its first formulation in 1980 it was used as a model of inflation. A review of $f(R)$ gravity can be found in [41].

The action for a generic $f(R)$ theory is

$$S = \int d^4x\sqrt{-g}f(R) + S_m[g, \psi_m]. \tag{4.25}$$

Varying the action with respect to $g_{\mu\nu}$ we obtain

$$\delta S = \int d^4x\sqrt{-g}\big[\frac{1}{2}f(R)g^{\mu\nu}\delta g_{\mu\nu} + f_R\delta R + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}\big], \tag{4.26}$$

where $f_R \equiv \partial f/\partial R$, with (up to boundary terms)

$$f_R\delta R = -\big[f_R R^{\mu\nu} + g^{\mu\nu}\Box f_R - f_{R;\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\big]\delta g_{\mu\nu}. \tag{4.27}$$

Therefore the equations of motion read

$$f_R R_{\mu\nu} + g_{\mu\nu}\Box f_R - \frac{1}{2}f(R)g_{\mu\nu} - f_{R;\mu\nu} = \frac{1}{2}T_{\mu\nu}. \tag{4.28}$$

We note that if $f(R) = R$ then (4.28) reduces to the Einstein equations.

A key feature of this theory is that by an opportune field redefinition it is possible to recast it to a Scalar-Tensor theory. In fact, introducing a scalar field ψ we can write the action (4.25) in the following dynamically equivalent form

$$S = \int d^4x\sqrt{-g}\big[f(\psi) + f'(\psi)(R - \psi)\big] + S_m, \tag{4.29}$$

where the prime denotes derivatives with respect to ψ.

Varying this action with respect to ψ we find $f''(\psi)(R-\psi) = 0$, that means, if $f'' \neq 0$ then $R = \psi$ and with this constraint we re-obtain the $f(R)$ action (4.25).

Now let us define a new scalar field

$$\phi \equiv f'(\psi), \tag{4.30}$$

with the hypothesis that $\phi(\psi)$ is an invertible function. If we define

$$V(\phi) \equiv \psi(\phi)\phi - f(\psi(\phi)),$$

then (4.29) becomes

$$S = \int d^4x\sqrt{-g}\big[\phi R - V(\phi)\big] + S_m, \tag{4.31}$$

that has the standard form (4.21) of a ST theory.

This result shows that an higher-order theory propagates extra degrees of freedom, that in the case of a $f(R)$ theory is in the form of a scalar field.

4.5.2 *Quintessence* theories

Quintessence models were historically introduced to provide an alternative explanation with respect to the current ΛCDM model for the late phase of cosmic acceleration.

In the FRW cosmology, a cosmological constant term provides a late-time accelerated expansion with a constant equation of state $w_\Lambda = -1$. A very similar behaviour can be achieved in the context of quintessence and *k-essence* theories where a scalar field minimally coupled to gravity is responsible for a late time cosmic inflation. The key feature of these models is that the equation of state describing Dark Energy evolves dynamically with time and it is not plagued by the too-small value problem of the cosmological constant term.

The action for the quintessence model is given by

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{2k} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big] + S_m, \tag{4.32}$$

with $k = 8\pi G/c^3$. We note that the scalar field is minimally coupled to gravity and the kinetic term is of the standard form.

It can be shown that, as in the case of slow-roll inflation, the equation of state for such a scalar field reads

$$w_\phi = \frac{\dot{\phi}^2 - V(\phi)}{\dot{\phi}^2 + V(\phi)}, \tag{4.33}$$

If we require that at late times the potential is sufficiently flat, i.e. $\dot{\phi}^2 \ll V(\phi)$, then $w \simeq -1$: the universe undergoes a phase of accelerated expansion.

Besides the technical aspects of this kind of models, we immediately note the similarity between the action (4.32) and that of a Scalar-Tensor theory. Indeed the borderline between the two theories is quite faint and it can be shown that models of quintessence can be recast in a ST-like theory *a là* Brans-Dicke after a Weyl rescaling of the metric.

A further extension of quintessence models is the k-essence model [42], where now the kinetic term of the scalar field is non-standard, i.e.

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \equiv X \to P(X,\phi) = K(\phi)X + L(\phi)X^2, \tag{4.34}$$

leading to the action

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{2k} + P(X,\phi)\Big] + S_m. \tag{4.35}$$

K-essence models are motivated by low-energy effective string theory where a P-like term of the form (4.34) appears [43]. Anyway, in these models the effects of the quintessence potential are reproduced by the non-standard kinetic term, which as it can be seen, leads to the equation of state

$$w_\phi = \frac{1-X}{1-3X},$$

where in order to have an accelerated expansion phase ($\omega_\phi < -1/3$) we must require $X < 2/3$.

4.6 The Horndeski action

A further generalization of the action (4.22) was found by Horndeski in the 1970s. In his work [39] he determined the most general Scalar-Tensor action yielding second order equations of motion both for the metric and the scalar field in four-dimension spacetime. The importance of this work has remained unnoticed for many years and rediscovered only recently in the context of Galileon Theory [36, 37, 38]. In the language of the latter theory the Horndeski action reads

$$S_H = \int d^4x\sqrt{-g}\sum_{i=2}^{5} L_i, \tag{4.36}$$

where

$$\begin{aligned} L_2 &= G_2(\phi, X); && (4.37)\\ L_3 &= G_3(\phi, X)\Box\phi; && (4.38)\\ L_4 &= G_4(\phi, X)R - 2G_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right]; && (4.39)\\ L_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}\left[(\Box\phi)^3 - 3\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}_{\sigma}\right]. && (4.40) \end{aligned}$$

where $\phi_\mu = \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$, $X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$, $Gi, X = \partial G_i/\partial X$ and R and $G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor respectively. We note that not only we have non-minimal coupled Ricci scalar, but also a non-minimal coupling to the Einstein tensor. This is a new feature for Scalar-tensor theories with respect to *a là* Brans-Dicke models.

We easily see that putting $G_2 = G_3 = G_5 = 0$ and $G_4 = 1$ it reduces to the GR Lagrangian (up to multiplicative constant factors), whereas for $G_2 = 2\Lambda(\phi)$, $G_3 = -g(\phi)$, $G_4 = f(\phi)$, $G_5 = 0$ it reduces to (4.22). In general the Horndeski action contains a huge quantities of Scalar-Tensor models, like *k-essence* theories, $f(R)$ gravity other than some Galileon theories.

Although the Horndeski Lagrangian contains order-two derivatives, it gives rise to second order field equations. This is due to a fine cancellation between higher derivatives coming from the R and $G_{\mu\nu}$ terms and those generated by the derivative counterterms. The price to pay is a highly non-linear structure of the action and more involved equations of motion. Another important key feature of Horndeski Lagrangian is that every single sub-Lagrangian L_i yields second order field equations, therefore we can set one (or more) of them to zero without damaging the second order nature of the EoMs.

Chapter 5

Conformal transformations

Soon after the formulation of the Brans-Dicke theory, Dicke himself in 1961 showed that under the metric transformation $g_{\mu\nu} \to e^{\alpha(\phi)} g_{\mu\nu}$ and a proper scalar field redefinition, it is possible to convert the BD action to a General Relativity like theory, for what concerns the gravitational sector.

The metric transformations he exploited are called *conformal transformations* or *Weyl transformations*. They are widely used in General Relativity, e.g. in the theory of asymptotic flatness [1] or in Black-Hole physics. In the context of modified theories of gravity they are mainly employed to recast a starting theory to a GR like form (as in the case of the Brans-Dicke model), for which we know solutions to the equations of motion.

However, such an use of conformal transformations has brought with it a very heated debate about the physical interpretation of these metric transformations. However, this problem seems to have been solved today even though recently the issue has come back to life because of the introduction of a new class of metric transformations which extend the conformal ones: the *disformal transformations.*

In this chapter we will introduce the conformal transformations and see how they can be implemented to simplify a given theory. This will lead us to define the concept of representation, or frame, for a Scalar-Tensor theory. The next step will be to analyse the physical interpretation of the different frames, which goes under the name of conformal frame's issue. We will show that a clever solution could be to define a conformal invariant theory, as in [19, 54, 55]. Finally we will do some considerations about the relationships between conformal transformations and quantization in Scalar-Tensor theories of gravity.

5.1 Conformal transformations: an introduction

Consider a n-dimensional manifold $\mathcal{M}$ endowed with a Riemannian metric $g_{\mu\nu}$. A *conformal* transformation (CT) of the metric $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{5.1}$$

where $\Omega(x)$ is a smooth, strictly positive function of the spacetime points.

A conformal transformation is therefore the map $(\mathcal{M}, g_{\mu\nu}) \to (\mathcal{M}, \tilde{g}_{\mu\nu})$, with $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ *conformally* related via (5.1). We also require that this represents a map between two Riemannian space.

By definition a conformal transformation does not act on the coordinate-separations, i.e. after a CT $dx^\mu \to dx^\mu$ and therefore

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = \Omega^2(x)ds^2. \tag{5.2}$$

We argue that after a conformal transformation times and lengths are stretched by a factor Ω. Nevertheless we note that the "stretching" is isotropic: the rate of stretching in the $\hat{x}$ direction is equal to the rate in the $\hat{y}$ and $\hat{z}$ direction. However, the causal behaviour is not modified: a space-like (time-like) vector remains space-like (time-like) as well as null vectors remains null vector; furthermore photons still follow null trajectories.

Given two four-vectors A^μ and B^μ, the angle between them is preserved: indeed, defining the angle to be

$$\theta \equiv \frac{A \cdot B}{|A||B|} = \frac{g_{\mu\nu}A^\mu B^\nu}{\sqrt{g_{\alpha\beta}A^\alpha A^\beta}\sqrt{g_{\alpha\beta}B^\alpha B^\beta}}, \tag{5.3}$$

it is straightforward to see that under (5.1) θ does not change.

Moreover we notice that in general a coordinate-invariant, i.e. a scalar, gets transformed as we have seen for the spacetime interval.

Now, we are going to write down the transformation property of some of the basic geometrical quantities of metric theories of gravity. First of all, denoting with $g^{\mu\nu}$ the inverse metric, then $\tilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}$; in this way $\tilde{g}^{\mu\nu}\tilde{g}_{\nu\sigma} = \delta^\mu_\sigma$.

The Christoffel symbols of the metric transform as

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \Omega^{-1}\big(\delta^\alpha_\beta\nabla_\gamma\Omega + \delta^\alpha_\gamma\nabla_\beta\Omega - g_{\beta\gamma}\nabla^\alpha\Omega\big), \tag{5.4}$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are the Christoffel symbols associated with the transformed metric $\tilde{g}_{\mu\nu}$.

The Ricci scalar transforms as

$$\tilde{R} \equiv \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} = \Omega^{-2}\Big[R - 2(n-1)\frac{\Box\Omega}{\Omega} - (n-1)(n-4)\frac{g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2}\Big], \tag{5.5}$$

which in the case of $n = 4$ reads

$$\tilde{R} = \Omega^{-2}\Big[R - \frac{6\Box\Omega}{\Omega}\Big] = \Omega^{-2}\Big[R - \frac{12\Box\sqrt{\Omega}}{\sqrt{\Omega}} - \frac{3g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2}\Big]. \tag{5.6}$$

It is also possible to show that for $n = 4$

$$R = \Omega^2\Big[\tilde{R} + 6\frac{\tilde{\Box}\Omega}{\Omega} - 12\tilde{g}^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\Big]. \tag{5.7}$$

The transformation rule of the determinant of the metric is

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g}. \tag{5.8}$$

Let's consider a generic field Ψ. We say that an equation for Ψ is conformally invariant if a number $s \in \mathbb{R}$ exists such that Ψ is a solution with $g_{\mu\nu}$ if and only if $\tilde{\Psi} \equiv \Omega^s \Psi$ is a solution with the metric $\tilde{g}_{\mu\nu}$. From this follows that the Klein-Gordon (K-G) equation for a massless scalar field: $\Box\phi = 0$ is not conformally invariant if $n \neq 2$. However if we modified it as follows[1]

$$\Box\phi - \frac{n-2}{4(n-1)} R\phi = 0, \tag{5.9}$$

then this new K-G equation is conformally invariant with $s = 1 - n/2$.

For $n = 4$ (and $s = 0$) the Maxwell equations $\nabla^\mu F_{\mu\nu} = 0$ are conformally invariant, i.e. $\tilde{\nabla}^\mu \tilde{F}_{\mu\nu} = 0$.

The energy-momentum conservation equation $\nabla_\mu T^{\mu\nu} = 0$ is conformally invariant only if $T^{\mu\nu}$ is symmetric and trace-free (and $s = -n - 2$). In fact it can be shown that, choosing $s = -n - 2$, then

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = -\Omega^{-7} T \nabla^\nu \Omega = -\frac{\tilde{\nabla}^\nu \Omega}{\Omega} \tilde{T}, \tag{5.10}$$

that vanishes if $T \equiv g_{\mu\nu} T^{\mu\nu} = 0$.

5.2 Conformal transformation in Scalar-Tensor theories

Conformal transformations are widely used in Scalar-Tensor theories of modified gravity. Here, the conformal factor Ω is assumed to have a functional dependence on the scalar field that appears in the theory, i.e. $\Omega = \Omega(\phi(x))$. Usually such CTs are used to recast a generic Scalar-Tensor action in a form which resembles the Hilbert-Einstein action. In this way, instead of solving the more involved field equations of the S-T theory, we can solve the Einstein-like equations and then come back to the original solutions with the inverse transformation; however, as we shall see, this comes with a price.

This seemingly powerful technique was firstly exploited by Dicke in the prototype Brans-Dicke model in 1961 [20], who also explained clearly the physical meaning of such a conformal transformation. Let's now look at the B-D model and see how the CT can be employed to recast it in a more suitable form.

The prototype B-D action is given by Eq.(5.11)

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big(\phi R - \frac{\omega_0}{\phi} \nabla^\mu \phi \nabla_\mu \phi \Big) + S_m[g, \psi_m], \tag{5.11}$$

where we have set $c = 1$.

Let's consider the conformal transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = (G\phi) g_{\mu\nu}, \tag{5.12}$$

where G is the gravitational constant. Note that we have chosen $\Omega = \sqrt{G\phi}$.

Defining $f = \ln \Omega = \ln \sqrt{G\phi}$ we can rewrite Eq.(5.7) as follows

$$R = G\phi \Big[\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{\mu\nu} f_{,\mu} f_{,\nu}\Big], \tag{5.13}$$

and using this, the B-D action reads

$$\begin{aligned} S_{BD} &= \frac{1}{16\pi}\int d^4x \frac{\sqrt{-\tilde{g}}}{G^2\phi^2}\Big[G\phi^2\big(\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{\mu\nu} f_{,\mu} f_{,\nu}\big) - \frac{\omega_0}{\phi}\nabla^\mu\phi\nabla_\mu\phi\Big] + S_m[\tilde{g},\phi,\psi_m] \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\Big[\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{\mu\nu} f_{,\mu} f_{,\nu} - \frac{\omega_0}{G\phi^3}\nabla^\mu\phi\nabla_\mu\phi\Big] + S_m[\tilde{g},\phi,\psi_m] \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\Big[\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{\mu\nu} f_{,\mu} f_{,\nu} - 4\omega_0\tilde{g}^{\mu\nu} f_{,\mu} f_{,\nu}\Big] + S_m[\tilde{g},\phi,\psi_m], \end{aligned}$$

where in the last step we have used that $\nabla^\mu\phi\nabla_\mu\phi = \phi G\tilde{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = 4\phi^3 G\tilde{g}^{\mu\nu} f_{,\mu} f_{,\nu}$.

If we redefine the scalar field as follows

$$\phi = e^{\tilde{\phi}\sqrt{\frac{8\pi}{\omega_0+3/2}}}, \tag{5.14}$$

for $\omega_0 > -3/2$, then $f = \frac{1}{2}\ln G + \frac{1}{2}\tilde{\phi}\sqrt{\frac{8\pi}{\omega_0+3/2}}$. Moreover, after integration by part we can discard $\tilde{\Box} f$; therefore the B-D action becomes

$$S_{BD} = \int d^4x\sqrt{-\tilde{g}}\Big[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\mu\tilde{\phi}\Big] + S_m[\tilde{g},\phi,\psi_m]. \tag{5.15}$$

We see that the gravitational part of this new action is exactly the same as the Hilbert-Einstein one, with the addition of a scalar field which possesses a standard kinetic energy, and which can be treated as a new form of matter minimally coupled to the gravitational metric. Furthermore, we notice that in the matter action a new dependence on the scalar field ϕ appears due to the conformal transformation (5.12). We explore the effect of this non-minimal coupling between matter and the scalar field in the next section.

The gravitational field equations of the new "representation" are given by

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 8\pi G\big(T^\phi_{\mu\nu} + \tilde{T}^m_{\mu\nu}\big), \tag{5.16}$$

$$T^\phi_{\mu\nu} \equiv \tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\phi}\tilde{\nabla}_\beta\tilde{\phi}, \tag{5.17}$$

$$\tilde{T}^m_{\mu\nu} \equiv -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_m}{\delta\tilde{g}^{\mu\nu}}. \tag{5.18}$$

We immediately note that Eq.(5.16) looks like the Einstein equations with an extra matter source given by $T^{\phi}_{\mu\nu}$. The scalar field does not allow to have the vacuum solutions $\tilde{R}_{\mu\nu} = 0$; it fills the spacetime also when matter is not present. In other words, a vacuum solution is mapped into a non-vacuum solution by means of the conformal transformation (5.1).

The new matter energy-momentum tensor can be related to the untilded one by means of the chain rule:

$$\tilde{T}^{m}_{\mu\nu} = \frac{\sqrt{-g}}{\sqrt{-\tilde{g}}}\frac{\delta g^{\rho\sigma}}{\delta \tilde{g}^{\mu\nu}}\frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\rho\sigma}} = \sqrt{\frac{g}{\tilde{g}}}\frac{\delta g^{\rho\sigma}}{\delta \tilde{g}^{\mu\nu}} T^{m}_{\rho\sigma} = \Omega^{-2} T^{m}_{\mu\nu}, \tag{5.19}$$

with $\Omega^2 = G\phi$.

Furthermore, the coupling of the scalar field to the matter action spoils the conservation of the matter energy-momentum tensor. Indeed, from eq.(5.16) and the Bianchi identities follows that

$$\tilde{\nabla}^{\mu} T^{\phi}_{\mu\nu} = -\tilde{\nabla}^{\mu}\tilde{T}^{m}_{\mu\nu} \neq 0, \tag{5.20}$$

and consequently matter does not follow the geodesics of the metric $\tilde{g}_{\mu\nu}$.

For this reason the new representation cannot be interpreted simply as General Relativity plus a scalar field, unless our theory contemplates only conformally-invariant fields such as the electromagnetic field, or unless the matter dynamics can be neglected as it is the case when the scalar field drives the inflation.

5.2.1 The Jordan and the Einstein frames

What we have done is to take the B-D action (5.11) written in terms of $g_{\mu\nu}$ and rewrite it in terms of $\tilde{g}_{\mu\nu}$, obtaining (5.15). In the literature it is common to interpret these two different representations as different conformally-related "*frames*". This is in analogy with the coordinate transformations which allow to pass from one coordinate reference frame to another one.

It is obvious that there are infinite conformally-related frames accordingly with the choice of the conformal factor Ω. In this section we have instead picked out two precise conformal frames:

- the starting frame, in which the action is given by (5.11), characterised by the fact that the matter field is minimally coupled to the scalar field (as it is in GR), but in which there is a non-minimal coupling between the scalar field and the curvature term. We call it the **Jordan frame**.

- Choosing $\Omega = \sqrt{G\phi}$ and with the field redefinition (5.14) we have obtained an action which resembles the Hilbert-Einstein action, and therefore characterised by a minimal coupling between the curvature term and the scalar field, but where the matter field now is non-minimally coupled to the scalar. We call this representation the **Einstein frame**.

We also note that in the Jordan frame the kinetic term has a non-standard form; this can lead to a wrong sign in front of the kinetic term accordingly with the sign of ω_0, causing problems of instability to the theory. At the same time an anomalous coupling to matter can spoil the Einstein equivalence principles, besides the fact that the matter energy-momentum tensor is generally not conserved in the Einstein frame.

5.2.2 More conformal frames

In section 4.4 we wrote the action for a generic Scalar-Tensor theory, Eq.(4.22), in terms of the unspecified parameters $f(\phi)$, $g(\phi)$ and $\Lambda(\phi)$. We also noticed that we can extend it taking into account a conformal coupling between the matter and the scalar field. In this way, using the notation of [44], the action for a generic S-T theory reads

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G}A(\phi)R - \frac{1}{2}B(\phi)\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\Big] + S_m\big[e^{2\alpha(\phi)}g_{\mu\nu},\psi_m\big]. \tag{5.21}$$

The action is written in terms of four parameters that must be specified in order to specify the "physical" theory itself. We will come back to this issue later when we shall try to interpret the physical meaning of such "specifications". Now let's see what the effect of a conformal transformation on this action is.

In [44] the author points out that by means of a conformal transformation plus a scalar field redefinition the structure of the action (5.21) remains unaltered, i.e. applying the following redefinitions

$$\begin{aligned} g_{\mu\nu} &= e^{2\gamma(\phi)}\tilde{g}_{\mu\nu}, \\ \phi &= f(\tilde{\phi}), \end{aligned} \tag{5.22}$$

the action gets transformed into

$$S = \int d^4x\sqrt{-\tilde{g}}\Big[\frac{1}{16\pi G}\tilde{A}(\tilde{\phi})\tilde{R} - \frac{1}{2}\tilde{B}(\tilde{\phi})\tilde{\nabla}_\mu\phi\tilde{\nabla}^\mu\tilde{\phi} - \tilde{V}(\tilde{\phi})\Big] + S_m\big[e^{2\tilde{\alpha}(\tilde{\phi})}\tilde{g}_{\mu\nu},\psi_m\big], \tag{5.23}$$

where

$$\begin{aligned} \tilde{A}(\tilde{\phi}) &= e^{2\gamma(\tilde{\phi})}A[f(\tilde{\phi})], \\ \tilde{\alpha}(\tilde{\phi}) &= \alpha[f(\tilde{\phi})] + \gamma(\tilde{\phi}), \\ \tilde{V}(\tilde{\phi}) &= e^{4\gamma(\tilde{\phi})}V[f(\tilde{\phi})], \\ \tilde{B}(\tilde{\phi}) &= e^{2\gamma(\tilde{\phi})}\big[f'(\tilde{\phi})^2B[f(\tilde{\phi})] - \frac{6}{8\pi G}f'(\tilde{\phi})^2\gamma'(\tilde{\phi})A[f(\tilde{\phi})] - \frac{6}{8\pi G}\gamma'(\tilde{\phi})^2A[f(\tilde{\phi})]\big]. \end{aligned}$$

We say that the action (5.21) is closed under the field transformations (5.22), whose effect is to redefine the four free parameters of the theory. In this sense the two representations, before and after the fields redefinition, are *mathematically* equivalent. If this also implies *physical* equivalence will be discussed later; indeed physical equivalence turns out to be more subtle compared to mathematical equivalence. Anyway since these redefinitions contain two degrees of freedom, then actions differing only by the fixing of two out of the four free parameters in (5.21) describe only different representations of the same theory. Whether these representations are physically equivalent will be analysed in the next sections.

The important thing for our current discussion is that appropriately choosing the functions in (5.22) we can pick up different frames out of this theory. Since by means of (5.22) we can fix two free parameters, we can select the following frames

- Fixing $\alpha = 0$ and $B = 1$ we obtain the Jordan frame. Here, as it happens with the prototype B-D model, matter follows the geodesics of the (gravitational) metric, but a non-minimal coupling between the scalar field and the curvature remains. Note that we can also discard the request $B = 1$ still having a Jordan frame in the sense of the B-D model.

- Fixing $A = 1$ and $B = 1$ we select the Einstein Frame, where now matter does not follow the geodesics of the (gravitational) metric.

- Fixing $A = 1$ and $\alpha = 0$ we have an Einstein frame theory with a non-standard kinetic term for the scalar field.

- Fixing $A = 1$ and $B = 0$ we encounter the case of an EF-like theory with no kinetic term for the scalar field, that means no dynamical degree of freedom of the scalar field.

In principle any other choice of the parameters is possible. However usually the choice is made for the sake of convenience.

5.3 The conformal transformations issue

5.3.1 The issue

In the previous sections we have seen that given a S-T theory, we are able to rewrite it in a different representation or frame by means of a conformal transformation. This turned out to be a very powerful tool because it allowed us to put a generic S-T theory in a GR-like form moving to the Einstein frame. A question then arises, i.e. whether different conformally related frames are physically equivalent. We have seen that in the context of the S-T theories given by the action (5.21), conformally related frames are mathematically equivalent. Are they also physically equivalent?

In order to investigate this issue we must give a precise definition that clarifies what we mean with physical equivalence. We say that two conformally related frames are physically equivalent if the value of the physical observables does not change going from one frame to the other one. This means that in order to check the equivalence we should compare the entire set of observables among different frames. This has turned out to be a very subtle task because not always the interpretation of results obtained in one frame is straightforward. For this reason a very long debate has been raised in the last fifty years and despite this we think the issue is still open.

There were (and still there are) different lines of thought about this problem. Those who think that conformally related frames are not physically equivalent and therefore among them only one has to be considered as physical. Among these people, some of them believed (believes) that only the Jordan frame version is the correct one, while the Einstein frame version is unphysical (viewpoint one). Others assumed the physicality of the Einstein frame version and discarded the Jordan's one (viewpoint two). On the other hand there were and still there are people that believed all the frames are physically equivalent, and therefore the entire set of observables does not depend on the chosen representation (viewpoint three). This line of reasoning is the most adopted now, though other interpretations have been raised recently.

In order to understand the criticality of this issue let's now briefly list some of the motivations that led physicists to accept one point of view rather than the other ones listed above.

Viewpoint one: This includes people who assume the Jordan representation to be the physical one; they are usually motivated by the fact that in this frame matter follows the geodesics of the gravitational metric (for example in [45]) and both the WEP and the EEP are satisfied. On the other hand, in the EF the scalar field is non-minimally coupled to matter and consequently the latter does not follow the geodesics of the gravitational metric and the Einstein's equivalence principle can be violated. Moreover in the EF the matter energy-momentum tensor is not conserved and this could favour the JF with respect to the EF.

Viewpoint two: This viewpoint is supported by the fact that in the Jordan frame the scalar field possesses a non-standard kinetic energy with an indefinite sign, which led physicists to think the EF as the physical one (e.g. in [48]). In fact a wrong sign in the kinetic energy causes an unstable ground state, with the system that decays in a lower state *ad infinitum*. This, with the fact that in the JF there is violation of the *weak energy condition* [46] led to discard the Jordan frame in favour of the Einstein frame. Other arguments are given along this line, for example the fact that the standard procedure to quantize the fluctuations of the scalar field in the linear regime does not work in the JF.

For a more comprehensive review about the different arguments supporting the above points of view see [46, 47].

Viewpoint three: This point of view has become widely accepted by the community in the last 15 years. Accordingly with it, those differences that arise among conformally related frames are only apparent and due to a misleading interpretation of the underlying physics. Actually in the seminal paper of Dicke [20], where firstly the CT is introduced to pass to the Einstein frame description of the Brans-Dicke model [24], he clearly elucidates the physical meaning of such a transformation underlining the physical equivalence between the Jordan and the Einstein descriptions. His reasoning is based on the fact that a CT can be interpreted as a local change of units, and therefore if we agree that physics must be invariant under change of units, then the physical equivalence follows directly from this. This reasoning has been straightforwardly generalized in subsequent works to most general Scalar-Tensor theories ([44, 19, 49]). They also show the physical equivalence of observables in conformally related frames. More recent papers about this are [67, 53].

Other interpretations of the issue appeared, for example in [51], which is based on the geometrical aspect of the transformation. Needless to say, the conformal transformation issue is a part of a more general discussion arisen only recently, i.e. the disformal transformation issue. This will be treated in the next chapters, through which we also hope to elucidate the conformal "aspect".

5.4 Conformal transformation as a change of units

In recent years the confusion around the conformal transformation issue has started to fade away, with the community quite entirely in agreement with the third viewpoint explained above.

In reality it is interesting to note that a clear solution to this issue was given formerly in the '60 by the same author who introduced the conformal transformations in the Scalar-Tensor theories context. In fact, in [20] Dicke (1961) put the CT in a close relationship with local changes of units, indeed showing that the effect of a conformal transformation is the same of a local rescaling of the units. We are going to analyse his reasoning and how it can be extended in a more general framework also in the cosmological context.

5.4.1 Local rescaling of units

Let's suppose to perform a local rescaling of the units of measure such that the units of time and length are rescaled by a common factor $\Omega^{-1}(x)$, with $\Omega(x)$ a smooth, non-vanishing function. For instance we want to measure a spatial distance $\bar{AB}$ in units of length that we indicate with $\mathbf{u_l}$. The outcome of the measure will be the dimensionless number l such that $\bar{AB} = l \cdot \mathbf{u_l}$, where l is nothing else than the number of units required to cover the interval $\bar{AB}$. If we rescale the units by a factor of $\Omega^{-1}(x)$ then we see that l transforms as $l \to \tilde{l} = \Omega(x)l$ since $\bar{AB}$ has to remain unaltered.

After such a transformation c does not change. Therefore it is convenient to choose units such that $c = 1$. Another convenient choice is to fix $\hbar = 1$. In this way lengths, time and reciprocal masses[1] have the same dimension that we can fix to be the dimension of time. Further, we require that under such a change of units the coordinate system is held fixed and therefore $dx^\mu \to dx^\mu$.

Now, since

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$

has the dimension of time squared, it scales with a factor $\Omega^2(x)$. But since the coordinates dx^μ do not change, it follows that the metric tensor transforms as

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}.$$

I.e. we see that under a local rescaling of the units the metric tensor is subjected to a point-dependent conformal rescaling. In other words, a conformal transformation of the metric tensor is equivalent to a local rescaling of the units of measure.

Consequently, let's suppose to have a certain Scalar-Tensor theory and to perform a conformal transformation of the metric. If we agree with the fact that the physics must be invariant after a change of units, no matter whether such a change is local, then we must agree with the fact that conformally related frames are physically equivalent, provided that we remember to rescale the units accordingly in passing from one frame to the other one. So, if we choose constant units in the Jordan frame description of some ST theory, say for instance the Brans-Dicke model, after a CT we expect to have new units that now change in space and time. In this viewpoint the two representations are physically equivalent, but in order to compare the observables we must remember to rescale the units in the transformed frame.

Let's see how this reasoning work. After a CT lengths and times take a factor $\Omega(x)$, while the particle masses scale as

$$m \to \tilde{m} = \frac{m}{\Omega}. \tag{5.24}$$

This means that constant masses can acquire a spacetime dependence after a CT. This is just a reminiscence of the fact that starting with the Jordan version of a theory, where the masses are truly constant, and moving to the Einstein frame (or to a generic frame) then the matter Lagrangian is non-minimally coupled with the scalar field (or in other words with $\Omega(\phi)$). Of course at the same time all the

[1] Since $c = 1$ and $[\hbar] = ML^2T^{-1} = 1$ then from a dimensional analysis follows that $[m] = L^{-1}$, i.e. it has the dimension of a reciprocal length.

dimensionful quantities scale accordingly. In particular, performing a straightforward dimensional analysis the following transformation rules can be proved

$$
\begin{aligned}
\tilde{c} &= c, \\
\tilde{\hbar} &= \hbar, \\
\tilde{m} &= \Omega^{-1} m, \\
\tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu}, \\
d\tilde{s} &= \Omega ds, \\
d\tilde{\tau} &= \Omega d\tau, \\
\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{\tau}} &= \Omega^{-1} u^\mu, \\
\tilde{e} &= e,
\end{aligned}
$$

where $d\tau$ is the proper time, e is the electric charge and of course u^μ is the particle four-velocity. This is only an incomplete list of rules we have to remember when we perform a conformal transformation in the Dicke's viewpoint.

A natural question then arises, i.e. what is the link between this line of reasoning and the problems we have faced in the previous section?

5.4.2 The equivalence between EF and JF

In the Dicke's paper he implements the rescaling of units in the prototype Brans-Dicke model, that we remember is conceived in the Jordan frame, to find the Einstein frame version of it. However his reasoning is general and can be applied to any S-T theory; this is what has been made in subsequent works. Hence, for the moment we do not assume any specific theory; we only assume to have a generic ST theory given in the Jordan frame, that for our purposes means with constant units and consider a conformal transformation to the Einstein frame by means of a local rescaling of the units (we assume that such a transformation does exist; in general, this is not the case of Horndeski theory).

For what we have said above the two representations must be physically equivalent, and therefore the observables, i.e. the outcome of the experiments, must be the same.

First of all we note that in the Dicke's viewpoint masses acquire a spacetime dependence. So it seems that if we measured the mass of a particle in the JF and then in the EF we would obtain different values. This is not true, because we must take into account the rescaling of the mass unit.

Let's indicate with m_u the unit of mass. Suppose we want to measure the electron mass that in the JF has the constant value $m_e = 9.11 \times 10^{-31} kg$. In the Einstein frame it becomes $\tilde{m}_e = \Omega^{-1}(\phi) m_e$, i.e. it acquires a dependence on ϕ. However, when we measure this mass, what we really measure is the ratio between the electron mass and the arbitrary chosen mass unit, that is m_e/m_u in the JF and $\tilde{m}_e/\tilde{m}_u$ in the EF. But this dimensionless ratio is frame independent because

$$\frac{\tilde{m}_e}{\tilde{m}_u} = \frac{\Omega^{-1}m_e}{\Omega^{-1}m_u} = \frac{m_e}{m_u}. \tag{5.25}$$

The observable value of a particle mass is therefore frame-independent as it should be if conformal frames are physically equivalent. However, one should check also the other observables in order to verify such an equivalence, and often this is not a straightforward task. Anyway from this calculation we learn that the outcome of measurements are dimensionless quantities, that are naively invariant under a change of units and therefore automatically frame-independent. It is also clear that if we did not rescale the units then we would face with an inequivalent theory, because for example the above ratio would depend on the frame.

At this point we can proceed in two ways. We can continue to evaluate explicitly the frame-invariance of observables comparing them in the two frames. Alternatively we can rewrite our theory (namely the action) in an explicit frame-invariant way exploiting the fact that dimensionless quantities are explicitly frame-invariant. This is done for example in [19, 55], or in a slightly different way in [54].

We conclude this section evaluating an important observable in cosmology, namely the redshift, showing that its measurement does not depend on the conformal frame. This calculation will be useful to introduce a little bit of the formalism of the next section, in which we will write down a frame-invariant formalism for ST theories.

We have said that units of time, length and reciprocal mass have the same dimension. Therefore we can introduce a unique reference unit, l_r, which as we know, transforms as $\tilde{l}_r = \Omega l_r$. It should be clear that l_r has a practical meaning: if we are dealing with wavelength measurements as happens in astrophysics, then it's convenient to choose l_r to be the wavelength of some typical atomic transition easily accessible in the lab. In cosmology it will be useful to choose l_r to be the inverse of the Planck mass; this is what is done in [55]. Anyway we can specify it according to our needs.

Let's consider the FRW line element

$$ds^2 = -a^2(\tau)\big(d\tau^2 - \delta_{ij}dx^i dx^j\big),$$

where τ is the conformal time. After a conformal transformation the FRW line element transforms as

$$d\tilde{s} = -\Omega^2(\tau)a^2(\tau)\big(d\tau^2 - \delta_{ij}dx^i dx^j\big) = -\tilde{a}(\tau)\big(d\tau^2 - \delta_{ij}dx^i dx^j\big),$$

where we have introduced $\tilde{a} \equiv \Omega a$. Note that we have dropped the spatial-dependence on the scale factor. The reason is that we are at the zero-order and the universe is assumed to be isotropic and homogeneous.

Using the standard FRW metric the gravitational redshift of photons travelling towards us is given by

$$z \equiv 1 + \frac{\lambda_0}{\lambda_e} = 1 + \frac{a_0}{a_e},$$

where λ is the wavelength of the photon, the subscript "0" indicates quantities evaluated at $\tau = \tau_0$ (observation time), whereas "e" indicates quantities at the time of the emission.

If instead we use the transformed metric, clearly the result changes

$$\tilde{z} = 1 + \frac{\tilde{a}_0}{\tilde{a}_e} = 1 + \frac{a_0}{a_e}\frac{\Omega_0}{\Omega_e}. \tag{5.26}$$

Actually this is only apparent, since we have to take into account that when we perform a measurement we only measure dimensionless ratios of the observed quantities and reference units, as we saw in the case of the electron mass. In this case the measured quantities are the dimensionless (and frame-invariant) ratios

$$\frac{\lambda(\tau_0)}{\bar{l}_r(\tau_0)} = \frac{a(\tau_0)}{\bar{l}_r(\tau_0)}; \quad \frac{\lambda(\tau_e)}{\bar{l}_r(\tau_e)} = \frac{a(\tau_e)}{\bar{l}_r(\tau_e)}. \tag{5.27}$$

Note that we put a bar over the reference unit l_r; this to stress that we are evaluating it at the background level.

In this case it is convenient to choose l_r to be the wavelength of an atomic transition we can reproduce in a lab. In any case, the truly measured redshift is

$$1 + z = \frac{a(\tau_0)}{\bar{l}_r(\tau_0)}\frac{\bar{l}_r(\tau_e)}{a(\tau_e)}, \tag{5.28}$$

that is explicitly frame-invariant.

This calculation shows us the concept we have mentioned before: not always the interpretation of the results is straightforward. We must remember that following the Dicke's point of view each unit has to be rescaled when passing from one frame to another one, otherwise we would obtain frame-dependent results, and conformal frames would not be physically equivalent.

5.5 A frame-invariant approach

5.5.1 A frame-invariant action

A way to remove the ambiguities coming from the local rescaling of units could be to rewrite a ST theory in terms of dimensionless quantities. Actually this approach would give us a frame-invariant theory or in other words a conformally invariant theory.

The main point of this strategy is to define a frame-invariant metric with which we can rewrite all the geometrical quantities of the ST theory. Such a metric can be defined as follows

$$\bar{g}_{\mu\nu} \equiv \frac{g_{\mu\nu}}{l_r^2}. \tag{5.29}$$

Under a change of units $\bar{g}_{\mu\nu}$ is manifestly invariant. Moreover it allows us to define the frame-invariant line-element

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = \frac{ds^2}{l_r^2}.$$

Once we have determined a frame-invariant metric we need to specify our gravity theory, that we want to be explicitly frame-invariant. We can start from a known action, e.g. the BD model, and find a manner to rewrite it in terms of the frame invariant metric $\bar{g}_{\mu\nu}$. This must be done in such a way that different choices of the unit l_r return us different conformal frames[2]. In particular we expect at least to reproduce the Jordan and the Einstein frame of the starting theory.

The starting point is to note that in the Einstein frame of any Scalar-Tensor theory the Planck mass is constant. Choosing the unit of length to be $l_r = m_{Pl}^{-1} = const$ we expect that a generic ST theory reduces to its EF version with the curvature R minimally coupled to the scalar field, the latter possessing a standard kinetic-term and a potential, the matter conformally related to the scalar field.

Therefore a good strategy is to consider a generic EF version of a Scalar-Tensor theory, rewrite it in the terms of frame-invariant quantities in such a way that choosing $l_r = m_{Pl}^{-1}$ it remains invariant up to constant multiplicative terms. This heuristic reasoning leads us to the following action

$$S = \int d^4x\sqrt{-\bar{g}}\, k^2\big[R(\bar{g}) - 2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 4\bar{V}(\phi)\big] + S_m\big[\bar{g}_{\mu\nu}e^{-2b(\phi)}, \bar{\psi}_m\big]. \tag{5.30}$$

First we explain the meaning of each term, then we show its consistency with the above heuristic reasoning. The first term we encounter is k^2. This is a constant that replaces the gravitational constant term appearing in the common gravity actions. Inside the first square brackets there is the gravity action, where $R(\bar{g})$ is the Ricci scalar as a function of the metric $\bar{g}$, then there is the usual kinetic term for the scalar field with the covariant derivative $\bar{\nabla}$ associated to $\bar{g}_{\mu\nu}$; $\bar{V} = l_r^2 V$ is a dimensionless potential. The matter action is coupled to the scalar field via $e^{-2b(\phi)}$, where $b(\phi)$ is some function of the scalar field ϕ, $\bar{\psi}_m$ indicates the dimensionless matter field, i.e. normalized with the appropriate power of l_r accordingly with its dimension. In the appendix we give an explicit example which shows how to obtain a dimensionless matter field in our language.

Now, if we choose $l_r = l_{Pl} = constant$, since

$$\bar{g}^{\mu\nu} = l_r^2 g^{\mu\nu}, \quad \sqrt{-\bar{g}} = l_r^4\sqrt{-g}, \quad R(\bar{g}) = l_r^2 R(g),$$

[2]Since each conformal frame corresponds to a particular choice of units.

the action (5.30) acquires the usual form of an Einstein-frame-like action, with $k^2 l_{Pl}^{-2} = (16\pi G)^{-2}$.

The Jordan frame can be extracted form the above action choosing units such that the matter field is minimally coupled to the JF metric. This is straightforwardly achieved choosing $l_r = l_{Pl} e^{-b(\phi)}$.

Let's calculate how the gravity action with this choice of units becomes: first of all the metric reads

$$\bar{g}_{\mu\nu} = g_{\mu\nu} e^{2b(\phi)}/l_{Pl}^2 \equiv \Omega^2 g_{\mu\nu}.$$

The barred metric and the unbarred one are related by a conformal-like term. We can therefore use eq. (5.6) to infer how $R(\bar{g})$ can be written in terms of $g_{\mu\nu}$, that is

$$R(\bar{g}) = l_{Pl}^2 e^{-2b(\phi)} \big[R(g) - 6 g^{\mu\nu} \big(\frac{db}{d\phi}\big)^2 \, \nabla_\mu \phi \nabla_\nu \phi \big],$$

up to boundary terms. Consequently the gravity action in this units becomes

$$S = k^2 l_{Pl}^{-2} \int d^4x \sqrt{-g} e^{2b} \big[R(g) - 2(1 - 3\alpha^2) g^{\mu\nu} \, \nabla_\mu \phi \nabla_\nu \phi - 4V(\phi) \big],$$

that clearly resembles the action (4.22) with the appropriate redefinition of the free parameters. Note we have defined $\alpha = db/d\phi$.

We stress that the action (5.30) is invariant under a units transformation since it is written in terms of explicitly frame-invariant quantities. However, the choice of particular units picks up one precise representation of the theory. The observables have to be extracted from (5.30), no matter what is the choice of units we do.

The first thing we can do is to find out the cosmological solutions to the equations of motion of this theory. After that we can explore the frame-invariant physics described by (5.30), focusing on the particles equations of motion and the Boltzmann equations.

5.5.2 Frame-invariant equations of motion

A good feature of this model is that we can evaluate and solve its equations of motion without worrying about the conformal issue.

Now we will focus on the cosmological solutions of the EoMs. First of all we shall define the frame-invariant perturbations of the metric and after that we shall write down the full EoMs related to the action (5.30). Finally we shall evaluate them at the background level in the FRW spacetime. Here we shall report the results, the calculations are carried out explicitly in Appendix 1.

We choose to work with the conformal time τ and in the Newtonian gauge for simplicity. Thus, the line-element reads

$$ds^2 = a(\tau)^2 \big[-(1 + 2\Psi) d\tau^2 + (1 - 2\Phi) \delta_{ij} dx^i dx^j \big].$$

In our frame-invariant language it becomes

$$d\bar{s}^2 = \frac{ds^2}{l_r^2} = \frac{a^2}{\bar{l}_r^2}\Big[-\Big(1+2\Psi-2\frac{\delta l_r}{l_r}\Big)d\tau^2 + \Big(1-2\Phi-2\frac{\delta l_r}{l_r}\Big)\delta_{ij}dx^i dx^j\Big].$$

As we did in the previous section, we have put a bar over l_r to indicate it is evaluated at the background level. Now it is convenient to define the following frame invariant quantities

$$\bar{a} \equiv \frac{a}{\bar{l}_r}, \qquad \bar{\Psi} \equiv \Psi - \frac{\delta l_r}{l_r}, \qquad \bar{\Phi} \equiv \Phi + \frac{\delta l_r}{l_r}. \tag{5.31}$$

In terms of these, the frame-invariant line-element simply reads

$$d\bar{s}^2 = \bar{a}(\tau)^2\big[-(1+2\bar{\Psi})d\tau^2 + (1-2\bar{\Phi})\delta_{ij}dx^i dx^j\big].$$

Varying the action (5.30) with respect to $\bar{g}_{\mu\nu}$ we obtain

$$\begin{aligned} R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) &= 2\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{g}_{\mu\nu}\bar{V} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\bar{\nabla}_\sigma\phi\bar{\nabla}_\rho\phi + \frac{1}{2k^2}\bar{T}_{\mu\nu}, \quad (5.32)\\ \bar{T}_{\mu\nu} &\equiv -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\bar{g}^{\mu\nu}}. \end{aligned}$$

A straightforward calculation gives also

$$\bar{T}^\mu_{\nu;\mu} = -\alpha\bar{\nabla}_\nu\phi\bar{T}^\mu_\mu. \tag{5.33}$$

Let's suppose that the energy-momentum tensor is that of a perfect fluid with energy density $\bar{\rho}$ and pressure $\bar{P} = w\bar{\rho}$. Since from (5.19)

$$\tilde{T}^\mu_\nu = \Omega^{-4}T^\mu_\nu,$$

it follows that the frame-invariant energy-momentum tensor with mixed indices can be defined as $\bar{T}^\mu_\nu \equiv l_r^4 T^\mu_\nu$.

Explicitly we have

$$\begin{aligned} \bar{T}^0_0 &= -\bar{\rho} = -(\bar{\rho}_0 + \delta\bar{\rho}), & (5.34)\\ \bar{T}^i_j &= \bar{P}\delta^i_j = (\bar{P}_0 + \delta\bar{P})\delta^i_j, & (5.35) \end{aligned}$$

where the subscript 0 indicates that we are evaluating them at the zero-order, the dot means derivatives with respect to the conformal time and the overbars mean that we are considering frame-invariant quantities

$$\bar{\rho} = l_r^4\rho, \quad \bar{P} = l_r^4 P.$$

From the (00)-component of Eq.(5.32) at the background level we have the frame-invariant Friedmann equation

$$\Big(\frac{\dot{\bar{a}}}{\bar{a}}\Big)^2 - \frac{2}{3}\Big(\frac{1}{2}\dot{\phi}^2 + \bar{a}^2\bar{V}\Big) = \frac{1}{6k^2}\bar{\rho}_0\bar{a}^2 \tag{5.36}$$

The trace of Eq.(5.32) and the 0-component of the conservation equation (5.33) give also

$$\frac{\ddot{\bar{a}}}{\bar{a}^2} + \frac{1}{3}(\dot{\phi}^2 - 4\bar{a}^2\bar{V}) = -\frac{1}{12k^2}\bar{\rho}_0\bar{a}^2(1-3w), \tag{5.37}$$

$$\dot{\bar{\rho}}_0 + 3\bar{\rho}_0(1+w)\frac{\dot{\bar{a}}}{\bar{a}} = -\alpha\dot{\phi}\bar{\rho}_0(1-3w). \tag{5.38}$$

We will not concern here with the solutions of such equations. The important thing is that once we have these equations we can solve them in the most convenient frame; indeed we are sure that the physical results do not depend on the choice of units.

For completeness we are going to give also the zero-order equation of motion for the scalar field, obtained minimizing the action (5.30) with respect to ϕ and evaluating it at the background level. It reads

$$\ddot{\phi} + 2\frac{\dot{\bar{a}}}{\bar{a}}\dot{\phi} + \bar{a}^2\frac{\partial\bar{V}}{\partial\phi} = \frac{\alpha}{4k^2}\bar{a}^2\bar{\rho}_0(1-3w). \tag{5.39}$$

We could proceed calculating the first order equations of motion using the frame-invariant quantities defined in (5.31). We will not do it here and we refer the reader to [19]. Now we will try to build a frame-invariant Boltzmann equation in the context of our approach.

5.5.3 Frame-invariant Boltzmann equation

In the early universe many of the relevant processes happened out of the equilibrium. Such processes can be studied through the Boltzmann equation formalism.

Schematically the Boltzmann equation can be written as

$$\hat{L}[f] = \hat{C}[f],$$

where f is the distribution function of the particle specie we want to study, $\hat{L}$ is the Liouville's operator and $\hat{C}$ is the collisional term which takes into account all the interactions of the considered specie with other particles; as such, $\hat{C}$ has a functional dependence on all the f_i of the particle species involved.

Essentially the Boltzmann equation is the evolution equation of the distribution function of the considered particle specie. It represents the building block of the study of perturbations in cosmology and therefore it would be interesting to investigate how under a conformal transformation it changes. This can be done quite simply using the formalism developed in the previous section.

So, let's focus on the evolution of some particle specie with mass m. We consider the six-dimensional phase space given by the three coordinates x^i and their conjugate momenta P_i. A generic distribution function f will be a function of the spacetime coordinates $(\tau, \vec{x})$ and the momentum P_i, i.e. $f = f(\tau, \vec{x}, P_i)$.

The conjugate momentum has the property to be the spatial part of the particle covariant four-momentum, namely $P_i = m dx_i/d\tau$, where τ is the proper time with which we parametrize the particle's path.

It is worth noticing that considering the particle four-momentum with low indices

$$P_\mu = m g_{\mu\nu}\frac{dx^\mu}{ds},$$

with $ds = \sqrt{-ds^2} = d\tau$, and performing the usual change of units, then it gets transformed into

$$\tilde{P}_\mu = \tilde{m}\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\tilde{s}} = \frac{m}{\Omega}\Omega^2 g_{\mu\nu}\frac{dx^\mu}{\Omega ds} = m g_{\mu\nu}\frac{dx^\mu}{ds} = P_\mu,$$

i.e. it is manifestly frame invariant. This is due to the fact that in our picture, under a local change of units all the dimensionful quantities get transformed, included the particle masses. This is not the case for the controvariant four-momentum that instead transforms as

$$P^\mu \to \tilde{P}^\mu = \Omega^{-2}P^\mu, .$$

However, the spatial part of the four-momentum with low indices is frame-invariant. We use this to define the frame-invariant distribution function as follows

$$dN = f(\vec{x}, P_i, \tau)dx^1dx^2dx^3dP_1dP_2dP_3,$$

where dN is the number of particles in the phase space volume $dx^1dx^2dx^3dP_1dP_2dP_3$.

Once we have introduced the distribution function we can define the comoving number density

$$n_c(\vec{x}, \tau) = g_s\int\frac{d^3P}{(2\pi)^3}f(\vec{x}, P_i, \tau), \tag{5.40}$$

where g_s is the number of spin degrees of freedom of the particle specie and $d^3P = dP_1dP_2dP_3$. Since f is frame-invariant, the same holds for n_c.

Further, we can give an expression for the energy-momentum tensor in terms of the distribution function as follows

$$T_{\mu\nu} = g_s\int\frac{d^3P}{(2\pi)^3}(-g)^{-\frac{1}{2}}\frac{P_\mu P_\nu}{P^0}f(\vec{x}, P_i, \tau), \tag{5.41}$$

which is not frame-invariant as expected. Note that it is consistent with the fact that under a conformal transformation $T_{\mu\nu} \to \tilde{T}_{\mu\nu} = \Omega^{-2}T_{\mu\nu}$ (see eq.(5.19)) .

Let's consider a particle specie with distribution function f_ψ. Our frame-invariant definition of the phase space distribution allows us to write down a frame-invariant version of the Boltzmann equation. In fact, the Liouville's operator applied to f_ψ

$$\hat{L}[f_\psi] = \frac{df_\psi}{d\tau} = \frac{\partial f_\psi}{\partial\tau} + \frac{dx^i}{d\tau}\frac{\partial f_\psi}{\partial x^i} + \frac{dP_i}{d\tau}\frac{\partial f_\psi}{\partial P_i}, \tag{5.42}$$

is explicitly frame-invariant.

For what concerns the collisional term, in order to write it explicitly we have to specify which kind of processes we are dealing with. We can consider the generic process $\psi + a + b + ... \rightarrow i + j + ...$ where ψ label our particle with distribution function f_ψ. For such a process the RHS of the Boltzmann equation reads

$$\begin{aligned}\hat{C}[f_\psi] &= \frac{1}{2P_0}\int d\Pi^a d\Pi^b \cdots d\Pi^i d\Pi^j \cdots (2\pi)^4 \delta^4(P^\psi + P^a + P^b + \cdots - P^i - P^j \cdots) \\ &\times \left[|M|^2_{\psi+a+b+\ldots\rightarrow i+j+\ldots} f_\psi f_a f_b \cdots (1 \pm f_i)(1 \pm f_j)\cdots\right. \\ &- \left.|M|^2_{i+j+\ldots\rightarrow \psi+a+b+\ldots} f_i f_j \cdots (1 \pm f_\psi)(1 \pm f_a)(1 \pm f_b)\cdots\right], \end{aligned} \tag{5.43}$$

where

$$d\Pi \equiv l_r^2 \frac{d^4P}{(2\pi)^3}(-g)^{-\frac{1}{2}}\delta(P^2 - m^2)\Theta(P^0) = l_r^2 \frac{d^3P}{(2\pi)^3}(-g)^{-\frac{1}{2}}\frac{1}{2P^0}$$

is frame invariant, and the sign "$+$" applies to bosons, whereas the sign "-" applies to fermions. Moreover the delta-function in (5.43) depends on the covariant four-momenta (i.e. with low indices). $|M|$ is the transition amplitude of the considered process which depends on the fundamental physics.

We have therefore succeeded in writing a frame-invariant Boltzmann equation with which analyse the out-of-equilibrium phenomena of the early universe, in a way that is manifestly frame-invariant. Our formalism can be applied to processes like decay or scattering events.

For instance, let's consider the decay process $\psi \rightarrow a + b$. With the formalism developed above it is possible to show that the evolution of the ψ-particles number density is given by

$$\frac{dn_c^\psi}{d\log x} = -\frac{\Gamma a}{H(1 + m'_\psi/m_\psi)}(n_c^\psi - n_c^{\psi 0}).$$

The variable x is defined as $x = m_\psi/T$, Γ is the decay rate, the prime denotes the derivative with respect $\log a$ and finally $n_c^{\psi 0}$ is the equilibrium comoving number density.

The remarkable thing is that in the JF, where $m'_\psi = 0$, we obtain the usual expression for decay processes in the Boltzmann formalism. Moreover we note that the freeze-out condition $\Gamma \lesssim H/a$ gets transformed into

$$\Gamma a \lesssim H(1 + m'_\psi/m_\psi).$$

This is a frame-invariant expression: Γ has the dimension of inverse of time and therefore scales opposite to a, whereas, noting that

$$\frac{m'_\psi}{m_\psi} = \frac{1}{m_\psi}\frac{dm_\psi}{d\log a} = \frac{1}{H}\frac{\dot{m}_\psi}{m_\psi},$$

it is easy to show that $H + \dot{m}_\psi / m_\psi$ is frame-invariant.

It is worth doing a final consideration. The results we have obtained in this subsection are not really dependent on the reference unit l_r except in the definition of $d\Pi$ above. In other words the frame-invariance of the Boltzmann equation comes automatically without the needing of a normalization factor in the physical quantities. This is a countercheck of our understanding that the physics should not depend on the conformal frame we choose.

5.6 Quantum mechanical considerations

The central result we have obtained is that conformally related frames seem to be physically equivalent. We could expect this from the fact that conformal transformations are nothing else than field redefinitions. But so far our treatment has been fully classical. Therefore we could ask ourselves if this result still holds when we try to quantize the field within our theory.

First of all, should we expect frame-invariance at the quantum level? It is known that classical theories that are equivalent could be inequivalent at the quantum level. For this reason we do not expect to have frame-invariance when quantization is taken into account.

Indeed, yet at the semi-classical level, in which both the matter and the scalar fields are quantized but not the metric degrees of freedom, it seems that the change of frame and the quantization process do not commute. In fact, as it is shown in [56], once we have (semi)quantized the theory, the field redefinitions related to the change of frame induce a coupling of the scalar field with the kinetic term of any gauge field which is coupled to fermions.

Indeed we know that, for example, the electromagnetic field is conformally-invariant in classic field theory (see Appendix 2) and therefore if this field is absent in one frame it would be absent in all the other frame. For what we have just said this is not true in quantum theory where a conformal transformation can generate a coupling between the scalar field and the Maxwell kinetic term.

The problem does not change going to a full quantum gravity description, where it has been showed that conformal related theories are inequivalent.

A significant result is instead that in the framework of effective field theories, where we only consider low-energy processes but all the degrees of freedom are quantized, we recover conformal equivalence. This is due to the *equivalence theorem* of Lagrangian field theory, which says that the S-matrix is invariant under non-linear local field redefinitions (and a conformal transformation is a non-linear metric tensor redefinition). However, this is true only for small perturbations of the fields, and in the gravity case, only for small perturbations around the flat Minkowski metric. For a more general treatment of this issue see [44] and reference therein.

To conclude this section we may wonder in which way the quantization procedure could affect our dimensionless treatment given in section (5.5). Accordingly with [55] such an explicit frame-invariant approach based on dimensionless quantities should be free of the frame-dependence problems arising in the quantization process. However, we think that this issue needs further investigations and this is beyond the scope of this work.

Chapter 6

Disformal transformations

We have seen how a special kind of metric transformations, namely the conformal transformations, can help to shed light on a vast class of Scalar-Tensor theories defined by the action (5.21). We could try to apply the same reasoning to a more general class of S-T theories, such as those included in the Horndeski action. However, here the conformal transformations cannot be as efficient as they are for standard Scalar-Tensor theories specified by (5.21) because of the kinetic dependence in the free parameters of the Horndeski theory.

In this chapter we will introduce a broader class of metric transformations, dubbed the *disformal* transformations. They were introduced by Bekenstein in 1992 [4] and applied two years later by himself and Sanders to explain the measured bending of light by gravitating objects in the context of Scalar-Tensor theories [58]; nowadays they are widely used in cosmology, e.g. in effective field theories for inflation [59], in varying speed of light theories [60] as well as in models for Dark Energy and Dark Matter [61, 62]. Here we will see how they can be used to gain further knowledge about Horndeski theories and their healthy extensions. In particular we will see that the Horndeski action is invariant under a sub-class of such disformal transformations (as shown in [57]), which therefore have the same role that the conformal transformations have towards standard Scalar-Tensor theories. We will investigate the issue of disformally related frames as we did in the conformal case, with a special attention to physical interpretations.

6.1 Beyond conformal transformations

In the previous chapter we considered the simplest kind of suitable metric transformations, consisting in a point-dependent rescaling of the metric tensor. There, for dynamical reasons we assumed that the conformal factor has a functional dependence on the scalar field appearing in the theory. A conservative way to extend this transformation, still remaining in the realm of conformal transformations, is to allow the conformal factor to have a functional dependence on the derivatives of the scalar field. Of course, if we introduce higher-order derivatives at the action level we should face the problem to have higher-order (than two) EoMs[1]. Therefore a simple choice is to allow the conformal factor to depend only on the first derivatives of the scalar field. In reality, as we shall see, also this choice introduces higher than two time derivatives of the scalar field in the equations of motion, though they can be eliminated exploiting "hidden constraints" coming from the equations of motion themselves.

The transformation we are going to consider is

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(\phi, X) g_{\mu\nu}, \qquad X \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi. \tag{6.1}$$

Here X is the simplest coordinate-invariant we can think of using only the metric tensor and the scalar field. However, this transformation is general enough to enclose much of the issues we are going to treat in this and in the next chapter. Note also that the X-term introduces a metric dependence in the conformal factor.

In sec. 5.2.2 we saw that the Scalar-Tensor theory specified by the action (5.21)(from now on we refer to it as the "standard" Scalar-Tensor theory) is closed under the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$. Is this also true for our "extended conformal transformation"? The answer is negative. Indeed, the X-dependence introduces new terms which cannot be brought back to the action form (5.21).

In order to see this, let's consider the simple case of an Hilbert-Einstein like action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_m[g_{\mu\nu}, \psi_m].$$

Let's write this action in terms of the tilded metric, using the following relations

$$g_{\mu\nu} = \Omega^{-2}(\phi, \tilde{X})\tilde{g}_{\mu\nu}, \quad \tilde{X} \equiv \Omega^{-2}X, \qquad g^{\mu\nu} = \Omega^2(\phi, \tilde{X})\tilde{g}^{\mu\nu}, \tag{6.2}$$

and the well-known $\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$.

Using the relation (5.7) and integrating by part, the action above takes the form

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[\Omega^{-2}\tilde{R} + 6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega\Big] + S_m[\tilde{g}_{\mu\nu}\Omega^{-2}, \psi_m]. \tag{6.3}$$

Let's focus on the second term. The quantity $\tilde{\nabla}_\mu\Omega$ explicitly reads

$$\tilde{\nabla}_\mu\Omega = \Omega_{,\phi}\tilde{\nabla}_\mu\phi + \Omega_{,X}\tilde{\nabla}_\mu X,$$

where the subscripts in Ω indicate the derivatives with respect to ϕ and X. We also note that the term $\tilde{\nabla}_\mu X$ is manifestly of second order in the scalar field.

[1]However, we should expect to have all higher order derivatives of the scalar field in a UV complete theory, unless some symmetry forbids them.

In particular it can be shown that the second term in (6.3) produces at the action level the quantity

$$6\Omega_{,X}\tilde{\nabla}_{\mu}\phi\tilde{\nabla}^{\mu}\tilde{\nabla}^{\nu}\phi\tilde{\nabla}_{\nu}\phi,$$

which can not be put in the form of the terms present in the action (5.21). This means that the action (5.21) is not closed under the extended conformal transformation (6.1). Moreover, as it is shown in [57, 63], the terms generated by $(\Omega_{,X})^2\tilde{\nabla}_{\mu}X\tilde{\nabla}^{\mu}X$ can not be even recast in a Horndeski-like form, i.e. neither the Horndeski's action is closed under (6.1).

This knowledge will help us to understand under which metric transformations the Horndeski's action is form-invariant. Moreover it will be exploited to investigate theories beyond Horndeski[2].

Moreover the fact that $\tilde{\nabla}_{\mu}X$ is second-order in the derivative of the scalar field ensures that in the equations of motion appear terms up to fourth order in field derivatives. We should expect this result since the Horndeski Lagrangian is the most general theory which produces second order EoMs. In reality, as explicitly calculated in [63], a constraint coming directly from the EoM kills higher than two time-derivatives of the scalar field, leaving second-order equations of motion. Roughly speaking, the Horndeski theorem fails to determine the most general action giving rise to second order equations of motion; this because in the Horndeski's analysis lacks the consideration of "hidden constraints" that could eat higher order terms in the EoMs.

To conclude this section we mention that inside the domain of this extended conformal transformations lie the theories of *Mimetic Gravity*. Roughly speaking, such theories arise from the fact that not always the CT (6.1) is invertible. If this is the case then to the non-invertibility condition corresponds an extra degree of freedom in the theory, and in fact a new physically different theory with respect to the untransformed one. An example of conformal map giving rise to a Mimetic degree of freedom is the following

$$g_{\mu\nu} = \big(-\tilde{g}^{\mu\nu}\tilde{\nabla}_{\mu}\phi\tilde{\nabla}_{\nu}\phi\big)\tilde{g}_{\mu\nu} \equiv P\tilde{g}_{\mu\nu},$$

that in our language corresponds to $\Omega^2 = X^{-1}$.

Actually this was the first example of Mimetic gravity, which appeared in [65]. Further extensions will be explored afterwards, when we will give a more complete description of the invertibility issue just outlined above.

6.2 Disformal transformations

In the previous section we showed a simple and natural extension of the field-dependent conformal transformation. A further extension has been introduced in 1992 by Bekestein [?] in the context of *Finsler* geometry.

[2]Actually beyond-Horndeski theories can be generated by means of these "extended" metric transformations, as we shall see in the rest of the work.

He considers gravitational theories supplied by two geometries, one for the gravity sector and one for the matter sector[3]. Assuming a *Finslerian* geometry for the matter sector, he finds that in order to preserve both the weak equivalence principle and the causal structure it has to reduce to a Riemannian geometry, whose metric $\tilde{g}_{\mu\nu}$ is related to the gravitational one $g_{\mu\nu}$ by

$$\tilde{g}_{\mu\nu} = \alpha^2(\phi, X) g_{\mu\nu} + \beta(\phi, X)\partial_\mu\phi\,\partial_\nu\phi. \tag{6.4}$$

This defines the so-called *disformal* transformation[4].

Here α and β are scalar parameters, called the *conformal* and *disformal* factor respectively. If $\beta = 0$ it reduces to a conformal transformation and the square in α guarantees it behaves well; for this reason it represents a generalization of the conformal metric transformation.

Since we are going to use it extensively we now give some useful formulas and features regarding the disformal transformations. Qualitatively we note that it corresponds not only to a uniform stretching of the metric as in the case of CTs, but also to a translation along the directions in which the scalar field is changing. One could ask if such a translation term spoils the causal structure of the original metric. Actually, suppose we have a four-vector v^μ that in the starting metric is a null vector, i.e. $g_{\mu\nu}v^\mu v^\nu = 0$. Once we perform a disformal transformation we have

$$\tilde{g}_{\mu\nu}v^\mu v^\nu = \alpha^2 g_{\mu\nu}v^\mu v^\nu + \beta\partial_\mu\phi\,\partial_\nu\phi\,v^\mu v^\nu = \beta\partial_\mu\phi\,\partial_\nu\phi\,v^\mu v^\nu.$$

Now the causal property of the vector v will depend on the sign of β, and *a priori* after the disformal transformation it can be timelike or even spacelike.

This means that the light-cones get modified by such a transformation. Indeed

$$ds^2 \to d\tilde{s}^2 = \alpha^2 g_{\mu\nu}dx^\mu dx^\nu + \beta(\partial_\mu\phi\,dx^\mu)^2 = \alpha^2 ds^2 + \beta(\partial_\mu\phi\,dx^\mu)^2. \tag{6.5}$$

The light-cone becomes wider or tighter accordingly with the sign of β. In order to ensure causal behaviour for particles we require that $d\tilde{s}^2 < 0$, which corresponds to the condition $\beta < 0$ everywhere.

Another feature we want to maintain from the original metric is the Lorentzian signature. In particular we want that for each value of the scalar field $\tilde{g}_{00} < 0$, i.e.

$$\tilde{g}_{00} = \alpha^2 g_{00} + \beta\dot{\phi}^2 < 0. \tag{6.6}$$

Let's consider a reference frame in which $\partial_\mu\phi = (\dot{\phi}, 0, 0, 0)$. Multiplying Eq.(6.6) by g^{00} (which is negative), we have

$$\alpha^2 + \beta g^{00}\dot{\phi}^2 = \alpha^2 + \beta X > 0, \tag{6.7}$$

[3]This happens for example in Brans-Dicke like theories, where the matter metric is related to the gravity metric by a conformal transformation. In this case the geometries are both Riemannian. In Bekenstein's work he starts considering a *Finslerian* geometry for the matter sector.

[4]The term "disformal" has been coined by Bekestein himself in [4], where he says "the term is meant as a contrasting one to *conformal* transformation".

that is the condition α and β must fulfil in order to preserve the Lorentzian signature (note that it is a reference frame invariant condition).

Another aspect we are aware to be necessary to have a good metric is the existence of the inverse metric $\tilde{g}^{\mu\nu}$ which must be non singular as well. Requiring that $\tilde{g}^{\mu\nu}\tilde{g}_{\nu\sigma} = \delta^{\mu}_{\sigma}$ as well as $g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}_{\sigma}$, we find that

$$\tilde{g}^{\mu\nu} = \frac{1}{\alpha^2(\phi, X)}g^{\mu\nu} - \frac{\beta(\phi, X)}{\alpha^2(\phi, X)}\frac{\nabla^{\mu}\phi\,\nabla^{\nu}\phi}{\big(\alpha^2(\phi, X) + \beta(\phi, X)X\big)}. \tag{6.8}$$

It can be shown by inspection that with this form for the inverse metric we have $\tilde{g}^{\mu\nu}\tilde{g}_{\nu\sigma} = \delta^{\mu}_{\sigma}$. In order for $\tilde{g}^{\mu\nu}$ to be non singular we have to require

$$\alpha^2 \neq 0, \qquad \alpha^2 + \beta X > 0,$$

which are automatically satisfied by means of (6.7).

Finally we require the volume-element to be non-singular. It is given by

$$\sqrt{-\tilde{g}} = \alpha^4\Big(1 + \frac{\beta X}{\alpha^2}\Big)^{\frac{1}{2}}\sqrt{-g}, \tag{6.9}$$

that is non-singular in virtue of the previous constraint.

Summarizing, in order to have a well-defined metric disformally related to a well-defined Riemannian metric, we have required:

- It must be causal.
- It must preserve Lorentz signature.
- The inverse must exist and be non-singular.
- The volume element must be non-singular.

We have seen that to fulfil such conditions it is sufficient to require $\alpha^2 + \beta X > 0$ with the inverse metric given by Eq.(6.8) and of course $\alpha \neq 0$.

6.3 Disformally related frames

Are disformally related frames physically equivalent? In order to answer this question we need to take a step back. In sec (5.2.2) we showed that the standard Scalar-Tensor theories are form-invariant under a conformal transformation plus a scalar field redefinition. As pointed out in [44] structure-invariance under conformal transformations implies physical invariance provided that we properly interpret physical results moving from one frame to the other. E.g. we noticed that in the EF units of measure are now spacetime-dependent as a consequence of the conformal coupling to the matter fields.

At the beginning of this chapter we have understood that if we exit the realm of simple conformal transformations, then standard Scalar-Tensor theories are not generic enough to accomodate the new terms coming from the metric transformation. A more promising environment in this sense could be the Horndeski theory (and of course its healthy extensions). Indeed in [57] it is shown that the Horndeski theory, given by eq.(4.36) is invariant in structure only under the following sub-class of disformal transformations

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \alpha^2(\phi) g_{\mu\nu} + \beta(\phi)\partial_\mu\phi\, \partial_\nu\phi, \tag{6.10}$$

where the X-dependence in the coefficients has been dropped. We now review the argument given in [57]; this will shed light on the property of the Horndeski action and the disformal frames related to it by (6.10).

The Horndeski action is known to be the most generic S-T theory giving rise to second order equations of motion. This property is due to a precise cancellation between higher derivatives coming from non-minimal coupling terms (such as $G_4(\phi, X)R$ and $G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi$) and those coming from the derivative "counterterms" (the second terms in 4.39 and 4.40). This is possible because of the antisymmetric structure of $\mathcal{L}_4$ and $\mathcal{L}_5$. For example $\mathcal{L}_4$ can be rewritten in the following way

$$\mathcal{L}_4 = \left(g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\beta\alpha}\right)\left[G_4(\phi, X)R_{\mu\nu\alpha\beta} - G_{4,X}(\phi, X)\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi\right],$$

that is now manifestly antisymmetric.

This antisymmetric property is spoiled whenever we have a disformal transformation involving X-dependent coefficients. Even in the simple case $\tilde{g}_{\mu\nu} = \alpha^2(X)g_{\mu\nu}$ it can be shown that the antisymmetric structure of $\mathcal{L}_4$ is broken, e.g. by the term

$$4G_{4,X}\left(\frac{2\alpha_{,X}}{\alpha}\right)^2 \phi^\mu\phi^\nu\phi^\alpha\phi^\beta\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi,$$

clearly symmetric and which can not be cancelled by other terms coming from the transformation.

However, in [57] it is showed that the antisymmetric structure is preserved under eq. (6.10), and therefore this transformation only accounts for a redefinition of the coefficients and the addition of surface terms (irrelevant for the dynamics). In this sense the disformal transformations represent a symmetry of the Horndeski action as the conformal transformations were with respect to the standard Scalar-Tensor theories given by the action (5.21).

Consider now the generic class of theories defined by

$$S_T = S_H[g, \phi] + S_m[g', \psi_m], \tag{6.11}$$

where

$$g'_{\mu\nu} = C^2(\phi)g_{\mu\nu} + D(\phi)\nabla_\mu\phi\nabla_\nu\phi. \tag{6.12}$$

Notice that we have allowed a disformal dependence on the matter sector. Since after a disformal transformation of the gravitational metric $g_{\mu\nu}$, the matter metric $g'_{\mu\nu}$ is mapped into

$$\begin{aligned}\tilde{g}'_{\mu\nu} &= C^2(\alpha^2 g_{\mu\nu} + \beta\nabla_\mu\phi\nabla_\nu\phi) + D\nabla_\mu\phi\nabla_\nu\phi \\ &= C^2\alpha^2 g_{\mu\nu} + (\beta C^2 + D)\nabla_\mu\phi\nabla_\nu\phi \equiv \gamma^2 g_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi,\end{aligned}$$

then S_T is invariant in form under the disformal transformation (6.10), since both S_H and the matter metric are.

In standard Scalar-Tensor theories we saw there are four free parameters in the action, all depending on the scalar field ϕ. The conformal transformation plus the scalar field redefinition allowed us to fix two of them. Therefore we concluded that actions defined up to the fixing of two parameters are just equivalent representations of the same physical theory.

In the case of Horndeski theories, S_T depends on six parameters $\big(K(\phi, X)$, $G_3(\phi, X)$, $G_4(\phi, X)$, $G_5(\phi, X)$, $C(\phi)$ and $D(\phi)\big)$. The ϕ-dependent disformal transformation has two degrees of freedom given by the coefficients $\alpha(\phi)$ and $\beta(\phi)$, and we have also the freedom the rescale the scalar field as follows

$$\phi \to \phi s(\phi), \tag{6.13}$$

leaving the action invariant in form.

Naively one could think that, as it happens in the case of CTs, theories differing for the fixing of three out of the six free parameters of S_T are physically equivalent. In reality, this reasoning is complicated by the fact that the free parameters of the Horndeski action have the kinetic dependence on X. The fixing is not efficient if it is done through the disformal transformation (6.10).

Anyway we still have the freedom to eliminate the disformal coupling in the matter metric since C and D only depend on the scalar field. In this way from the generic action (6.11) with a non-minimal coupling of the scalar field to matter, we can define a "Jordan frame" version of the theory

$$\tilde{S}_T = \tilde{S}_H[\tilde{g}, \phi] + S_m[\tilde{g}, \psi_m],$$

with the matter sector minimally coupled to the new metric $\tilde{g}$.

Summarizing, disformally related frames are physically equivalent. This can be exploited to eliminate the disformal coupling to matter (if any) to obtain a Jordan frame version of the theory. Could they be exploited to obtain an Einstein frame version too, i.e. with the gravitational sector simply given by the Ricci scalar plus quintessence/cosmological constant terms?

To achieve an Einstein frame starting from the generic Horndeski action we must move to a frame where $G_4 = const$ and $G_5 = 0$. For the reasons explained above this is not possible using only the ϕ-dependent disformal transformation and the scalar field redefinition. However, if we start from the onset with a simpler

Horndeski action, with the paramters G_4 and G_5 defined *ad hoc*, then it can be shown (see [57]) that for the action

$$S = \int d^4x\sqrt{-g}\Big[G_E(\phi, X)R - G_{E,X}\left[(\Box\phi)^2 - (\nabla^\mu\nabla_\mu\phi)^2\right] + \mathcal{L}_2 + \mathcal{L}_3\Big],$$

where

$$G_E = \big(1 - 2B(\phi)X\big)^{\frac{1}{2}},$$

it is possible to define an Einstein Frame performing a disformal transformation.

6.4 Ostrogradski ghosts and physical degrees of freedom

One may wonder if disformal transformations more general than Eq.(6.10) can be exploited to obtain healthy theories beyond Horndeski's, i.e. if performing a general DT to the Horndeski's action (or to a subset of it) we obtain a new viable class of Scalar-Tensor theories. In reality extensions to Horndeski's theories, and in general to any alternative to GR, are strictly limited by *Ostrogradski's theorem.* This states that the Hamiltonian of any non-degenerate Lagrangian depending on more than first-order time-derivatives is not bounded from below, leading to instability problems which render the theory unphysical.

We now review the basics of the Ostrogradski's theorem and instability in order to understand how it can be avoided in the context of Scalar-Tensor theories of gravity.

Suppose to have a classical system described by the Lagrangian $\mathcal{L}(q, \dot{q}, \ddot{q})$ with $q = (q_1, q_2, ..., q_N)$, which is *non-degenerate*, i.e. such that

$$\frac{\partial^2\mathcal{L}}{\partial\ddot{q}_i\partial\ddot{q}_j} \neq 0. \tag{6.14}$$

The Euler-Lagrange (EL) equations are

$$\frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{q}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} + \frac{\partial\mathcal{L}}{\partial q} = 0. \tag{6.15}$$

With the hypothesis of non-degeneracy the EL equations become fourth order differential equations: $q^{(4)} = \mathcal{F}(q, \dot{q}, \ddot{q}, q^{(3)})$ which therefore need $4N$ initial conditions. In order to switch to the Hamiltonian formulation we have to define $4N$ canonical variables

$$Q_1 = q, \quad Q_2 = \dot{q}, \quad P_1 = \frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\ddot{q}}, \quad P_2 = \frac{\partial\mathcal{L}}{\partial\ddot{q}}. \tag{6.16}$$

The condition of non-degeneracy allows us to invert these transformations and explicit $\ddot{q}$ in terms of P_2, Q_1 and Q_2

$$\ddot{q} = \mathcal{Q}(Q_1, Q_2, P_2). \tag{6.17}$$

The Hamiltonian is given by the *Legendre transform* of the Lagrangian and reads

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &= \sum_{i=1}^{2} P_i q^{(i)} - L\big(Q_1, Q_2, \mathcal{Q}(Q_1, Q_2, P_2)\big) \\ &= P_1 Q_2 + P_2 \mathcal{Q}(Q_1, Q_2, P_2) - L\big(Q_1, Q_2, \mathcal{Q}(Q_1, Q_2, P_2)\big). \end{aligned}$$

The Hamilton-Jacobi equations $\dot{Q}_i = \frac{\partial H}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial H}{\partial Q_i}$ are equivalent to the EL equations and generate time evolution.

The thing we want to stress here is that the Hamiltonian above depends linearly on P_1 since it appears only in the first term of the RHS. This means that the Hamiltonian is not bounded from below and can take arbitrary large negative values. In principle this is not a problem, but when the theory is interacting then an empty state decays *ad infinitum* into couples of positive and negative energy particles and this is clearly non-physical. This is the so-called *Ostrogradski's instability*. Generally speaking, it depends on the fact that higher order EOMs need extra initial conditions than the usual dynamic system, and this translates to the presence of extra degrees of freedom in the Hamiltonian formulation. It can be shown that attempts to quantize such d.o.f. lead either to negative norm states (and therefore to a non-unitary theory) or to negative energy eigenstates and this is the onset of instability. Such states are also called ghosts in quantum mechanics.

Our analysis was restricted to second order time-derivatives in the Lagrangian, but the reasoning can be easily generalized to higher orders theories. Indeed the situation is even worse when we consider higher orders dependence on $\mathcal{L}$. In fact, in this case more than one canonical momentum appears linearly in the Hamiltonian. In such a case we have that $\mathcal{L} = \mathcal{L}(q, \dot{q}, ..., q^{(N)})$ and the condition of non-degeneracy entails that the EL equations

$$\sum_{i=1}^{N} \Big(-\frac{d}{dt}\Big)^i \frac{\partial \mathcal{L}}{\partial q^{(i)}} = 0 \tag{6.18}$$

contains $q^{(2N)}$. As in the simpler case, the phase space depends on $2N^2$ variables and we need to define $2N^2$ canonical variables

$$Q_i = q^{(i-1)}, \qquad P_i = \sum_{j=1}^{N} \Big(-\frac{d}{dt}\Big)^{j-i} \frac{\partial \mathcal{L}}{\partial q^{(j)}}. \tag{6.19}$$

Due to the non-degeneracy condition we can write $q^{(N)} = \mathcal{Q}(Q_1, ..., Q_N, P_N)$ and the Hamiltonian reads

$$\begin{aligned} H &= \sum_{i=1}^{N} P_i q^{(i)} - L(Q_1, ..., Q_N, \mathcal{Q}) \\ &= P_1 Q_2 + P_2 Q_3 + ... + P_{N-1} Q_N + P_N \mathcal{Q} - L(Q_1, ..., Q_N, \mathcal{Q}) \end{aligned} \tag{6.20}$$

We see that now half of canonical conjugate momenta appears linearly and therefore the instability comes from several channels.

The crucial point is that for non-degenerate system with higher than first order time-derivatives the theory suffers from Ostrogradski's instability. This result is really general and it also explains why nature is described by second order differential equations like Newton forces. A theory that produces second order differential equations is degenerate and therefore avoids the Ostrogradski instability. This is the case of General Relativity even though the Hilbert-Einstein action contains second order derivatives of the metric tensor. The same holds for the Horndeski theory which in addition contains second order time-derivatives of the scalar field. However, due to the structure of the Horndeski Lagrangians, they generate second-order equations of motion and therefore escape the Ostrogradski's theorem.

However, we understand why the space of possible alternative theories of gravity (and in our case Scalar-Tensor theories) is limited by this no-go theorem: apart from particular cases (such as $f(R)$-gravity or Horndeski theories themselves), whenever we have higher than one time-derivatives of the fields in the Lagrangian the theory is unstable and therefore unphysical.

Hence we should expect that beyond Horndeski theories, for instance those obtained via a disformal transformation to the Horndeski lagrangians are plagued by the Ostrogradski ghosts and present more than the $2+1$ degrees of freedom of healthy Scalar-Tensor theories. Actually, they produce higher order equations of motion, but as it is shown in [63], they can be made second order in time-derivatives exploiting hidden constraints coming from the EoMs themselves. This gives a loophole in the Horndeski's theorem, since it fails to provide the most general Scalar-Tensor theory with second order equations of motion free of Ostrogradski instability, essentially because it does not take into account the presence of such hidden constraints.

A class of theories free of the Ostrogradski's instability is the so-called $G^3/GLPV$ theories as showed in [71] which constitute an healthy extension of Horndeski theory. The key of the avoidance of ghosts lies in the degeneracy of the Lagrangian and this issue has been widely studied, for instance in [64], which includes an analysis of cubic degenerate higher-order scalar tensor (DHOST) theories beyond Horndeski.

A very general result in this sense is that given in [73]. Here it is proved that general invertible transformations does not alter the number of physical degrees of freedom, even if the transformation involves derivatives of fields, such as the case of disformal transformations. This means that if we start with an healthy theory, such as the Horndeski theory, and we perform a generic invertible transformation then we end up with a new class of healthy theories.

One should have expected this result since an invertible transformation is a one-to-one correspondence between the starting and final variables and as such it does not introduce new information to the theory. The disformal transformations are generally invertible as we are going to see. Clearly, when this does not happen we end up with a new physically different theory with an extra number of d.o.f. This will be the subject of the next chapter.

Chapter 7

Disformal invariance and Mimetic Gravity

In the last chapter we found that the Horndeski action is invariant under a special class of metric transformations. At the same time we have showed that performing a generic disformal transformation to the Horndeski action a richer class of theories can be obtained.

Here we will explore the issue of the invertibility of the disformal transformations, when they are seen as a parametrisation of the physical metric in terms of an auxiliary metric and a scalar field. We will see that the request of the invertibility translates to a condition on the coefficients appearing in the definition of the transformation. The crucial fact is that when the transformation is invertible then the equations of motion derivable from a generic Scalar-Tensor theory are invariant under such a parametrisation [67, 68].

On the other hand, when the transformation is not invertible, a new degree of freedom appears in the theory when the transformation is performed. In other words, we end up with a new theory, physically inequivalent respect to the original one (before the transformation) and characterised by different equations of motion. This defines a new class of theories called *Mimetic Gravity* theories [65, 66], which has received a lot of attention since their first appearance in 2012. We shall review the key features of them, with a special attention on their relationship with the disformal transformation.

7.1 Non-invertibility condition

In the previous chapter we saw some properties of the disformal transformation

$$\tilde{g}_{\mu\nu} = \alpha^2(\phi, X)g_{\mu\nu} + \beta(\phi, X)\partial_\mu\phi\,\partial_\nu\phi.$$

In particular we found that in order to be a "good" metric, the disformal metric must satisfy the constraint $\alpha^2 + \beta X > 0$, $\alpha \neq 0$. However, no discussion was made about the invertibility condition of such a transformation, i.e. the possibility of inverting the relation above to $g_{\mu\nu} = g_{\mu\nu}(\tilde{g})$.

We are now going to investigate under which conditions the inversion can be done, regardless of the underlying Scalar-Tensor theory; we are going to see that the invertibility condition translates to a condition on the coefficients α and β.

In order to do it we use a slightly different notation in line with [68]; in particular we consider two metrics: the "physical" metric $g_{\mu\nu}$ and an "auxiliary" metric $l_{\mu\nu}$, which are related by the disformal transformation

$$g_{\mu\nu} = A(\phi, X)l_{\mu\nu} + B(\phi, X)\partial_\mu\phi\,\partial_\nu\phi, \tag{7.1}$$

where $X = l^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$, $A > 0$ and $A + BX > 0$.

What we would like to do is to write $l_{\mu\nu}$ as a function of $g_{\mu\nu}$ and the scalar field ϕ; however, this is complicated by the presence of $l_{\mu\nu}$ in X. If we could express X as a function of $g_{\mu\nu}$ (and the scalar field) only, then we would be able to invert the relation (7.1) to $l_{\mu\nu} = l_{\mu\nu}(g_{\mu\nu}, \phi)$.

In order to do this we start writing the inverse metric $g^{\mu\nu}$ as follows

$$g^{\mu\nu} = \frac{1}{A(\phi, X)}l^{\mu\nu} + \frac{B(\phi, X)}{B(\phi, X)g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - 1}g^{\alpha\mu}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi, \tag{7.2}$$

where we require $B(\phi, X)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - 1 \neq 0$. Note that it is such that $l^{\mu\nu}l_{\nu\sigma} = \delta^\mu_\sigma$ (as can be proved by inspection). Using that $g^{\mu\nu}g_{\mu\nu} = 4 = l^{\mu\nu}l_{\mu\nu}$ we have

$$\begin{aligned}
g^{\mu\nu}g_{\mu\nu} &= \Big(\frac{1}{A}l^{\mu\nu} + \frac{B}{Bg^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - 1}g^{\alpha\mu}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi\Big)g_{\mu\nu} \\
4 &= \frac{1}{A}l^{\mu\nu}\big(Al_{\mu\nu} + B\partial_\mu\phi\,\partial_\nu\phi\big) + \frac{B}{Bg^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - 1}g_{\mu\nu}g^{\alpha\mu}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi \\
4 &= 4 + \frac{B}{A}X + \frac{B}{Bg^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - 1}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi \\
&\Rightarrow \quad X = \frac{A(\phi, X)g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi}{B(\phi, X)g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - 1}.
\end{aligned} \tag{7.3}$$

Eq.(7.3) can be rewritten in the following way

$$G(\phi, X) = g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi, \tag{7.4}$$

where

$$G(\phi, X) \equiv \frac{X\big(B(\phi, X)g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - 1\big)}{A(\phi, X)}. \tag{7.5}$$

If we were able to invert Eq.(7.4) for a given ϕ, then we could express X as a function of $g_{\mu\nu}$: $X = G^{-1}(g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi)$. We are not able to do it explicitly, but fortunately we can use the inverse function theorem, which states that if $\frac{dG(\phi,X)}{dX}\big|_{X=X_*} \neq 0$ then G^{-1} exists in a neighbourhood of X_* for any fixed ϕ.

Therefore, whenever $\frac{dG(\phi,X)}{dX}\big|_{X=X_*} \neq 0$, we can write X as a function of $g_{\mu\nu}$ and consequently we can write $l_{\mu\nu}$ as a function of the physical metric.

What happens if $\frac{dG(\phi,X)}{dX}\big|_{X=X_*} = 0$? This equation can be easily solved noticing that if $\frac{dG(\phi,X)}{dX}\big|_{X=X_*} = 0$ then G is a function of ϕ only and we can write it as

$$G(\phi, X) = \frac{1}{b(\phi)}. \tag{7.6}$$

Using Eq.(7.4) we immediately find $1/b(\phi) = g^{\alpha\beta}\partial_\alpha\phi\, \partial_\beta\phi$, that combined with Eq.(7.3) gives

$$B(\phi, X) = -\frac{A(\phi, X)}{X} + b(\phi). \tag{7.7}$$

This is the non-invertibility condition translated to the disformal coefficients $A(\phi, X)$ and $B(\phi, X)$. We notice that in order to have a non-invertible disformal transformation, at least one of the two coefficients must depend on X. This is not the case of the class of disformal transformations under which the Horndeski theory is invariant, neither of the conformal transformations considered in chapter 5.

7.2 Disformal invariance of the equations of motion

Let's consider a generic Scalar-Tensor theory defined by the action

$$S = \int d^4x\sqrt{-g}\mathcal{L}\left[g_{\mu\nu}, \partial_{\lambda_1}g_{\mu\nu}, \dots \partial_{\lambda_p}g_{\mu\nu}, \phi, \partial_{\lambda_1}\phi, \dots \partial_{\lambda_q}\phi\right] + S_m[g_{\mu\nu}, \psi_m], \tag{7.8}$$

with $p, q \geq 2$. We do not need to specify the form of the lagrangian $\mathcal{L}$, but we notice that it includes for example the standard S-T theories analysed before and the Horndeski theory (where $q = 2$).

Given that $g_{\mu\nu}$ is related to the auxiliary metric $l_{\mu\nu}$ via Eq.(7.1), we will show that the EoMs obtained varying the previous action with respect to $g_{\mu\nu}$ or with respect to $l_{\mu\nu}$ and ϕ are the same, provided that the disformal relation is invertible.

We can formally write

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}\left(E^{\mu\nu} + T^{\mu\nu}\right)\delta g_{\mu\nu} + \int d^4x\Omega_\phi\delta\phi + \int d^4x\Omega_{\psi_m}\delta\psi_m, \tag{7.9}$$

where

$$\begin{aligned}
E^{\mu\nu} &\equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\Big(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}} + \sum_{h=1}^{p}(-1)^h\frac{d}{dx^{\lambda_1}}\cdots\frac{d}{dx^{\lambda_h}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_{\lambda_1}\ldots\partial_{\lambda_h}g_{\mu\nu})}\Big),\\
T^{\mu\nu} &\equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}, \qquad \Omega_{\psi_m} \equiv \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi},\\
\Omega_\phi &\equiv \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\phi} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\phi} + \sum_{h=1}^{q}(-1)^h\frac{d}{dx^{\lambda_1}}\cdots\frac{d}{dx^{\lambda_h}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_{\lambda_1}\ldots\partial_{\lambda_h}\phi)}.
\end{aligned}$$

Taking into account Eq.(7.1) we can write the variation of $g_{\mu\nu}$ as

$$\begin{aligned}
\delta g_{\mu\nu} &= (\delta A)l_{\mu\nu} + A\delta l_{\mu\nu} + (\delta B)\partial_\mu\phi\partial_\nu\phi + B\delta(\partial_\mu\phi\partial_\nu\phi)\\
&= A\delta l_{\mu\nu} - \Big(l_{\mu\nu}\frac{\partial A}{\partial X} + \partial_\mu\phi\partial_\nu\phi\frac{\partial B}{\partial X}\Big)\big[(l^{\alpha\rho}\partial_\alpha\phi)(l^{\beta\sigma}\partial_\beta\phi)\delta l_{\rho\sigma} - 2l^{\rho\sigma}(\partial_\rho\phi)(\partial_\sigma\delta\phi)\big]\\
&+ \Big(l_{\mu\nu}\frac{\partial A}{\partial\phi} + \partial_\mu\phi\partial_\nu\phi\frac{\partial B}{\partial\phi}\Big)\delta\phi + B\big[(\partial_\mu\delta\phi)(\partial_\nu\phi) + (\partial_\nu\delta\phi)(\partial_\mu\phi)\big]. \qquad (7.10)
\end{aligned}$$

Using this expression in Eq.(7.9) and minimizing respect to $l_{\mu\nu}$ we obtain

$$A(E^{\mu\nu} + T^{\mu\nu}) = \Big(\alpha_1\frac{\partial A}{\partial X} + \alpha_2\frac{\partial B}{\partial X}\Big)(l^{\mu\rho}\partial_\rho\phi)(l^{\nu\sigma}\partial_\sigma\phi), \qquad (7.11)$$

with

$$\alpha_1 \equiv (E^{\mu\nu} + T^{\mu\nu})l_{\mu\nu}, \quad \alpha_2 \equiv (E^{\mu\nu} + T^{\mu\nu})\partial_\mu\phi\,\partial_\nu\phi. \qquad (7.12)$$

The equation of motion (7.11) can be contracting with $l_{\mu\nu}$ and $\partial_\mu\phi\,\partial_\nu\phi$ to give respectively the following scalar equations

$$\alpha_1\Big(A - X\frac{\partial A}{\partial X}\Big) - \alpha_2 X\frac{\partial B}{\partial X} = 0, \quad \alpha_1 X^2\frac{\partial A}{\partial X} - \alpha_2\Big(A - X^2\frac{\partial B}{\partial X}\Big). \qquad (7.13)$$

This system of two linear equations with unknown α_1 and α_2 can be written in matrix form as

$$M\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = 0, \quad \text{with} \quad M \equiv \begin{pmatrix} A - X\frac{\partial A}{\partial X} & -X\frac{\partial B}{\partial X}\\ X^2\frac{\partial A}{\partial X} & -A + X^2\frac{\partial B}{\partial X}\end{pmatrix}. \qquad (7.14)$$

Let us now consider the determinant of M, which is

$$det(M) = X^2 A\frac{\partial}{\partial X}\Big(B + \frac{A}{X}\Big). \qquad (7.15)$$

If $det(M) \neq 0$ we have only the trivial solutions $\alpha_1 = 0 = \alpha_2$, which imply

$$E^{\mu\nu} + T^{\mu\nu} = 0, \qquad (7.16)$$

i.e. we obtain the same EoMs we would obtain varying directly with respect to $g_{\mu\nu}$.

Looking at Eq.(7.15) it is straightforward to recognize the condition $det(M) = 0$ as the condition of invertibility of the disformal transformation found in the previous section, i.e. $B + A/X = b(\phi)$.

In other words if the transformation is invertible ($det(M) \neq 0$) then it does not alter the equations of motion (written in terms of the physical metric). On the other hand if the non-invertibility condition is satisfied ($det(M) = 0$), varying with respect to the auxiliary metric and the scalar field gives us a new set of equations of motion, that after some algebraic manipulations can be expressed as [68]

$$E^{\mu\nu} + T^{\mu\nu} = \frac{\alpha_1}{X}(l^{\mu\rho}\partial_\rho\phi)(l^{\nu\sigma}\partial_\sigma\phi). \tag{7.17}$$

In practice, a non-invertible disformal transformation introduces an extra degree of freedom which alters the original equations of motion, producing an inequivalent physical theory. This is the so-called mimetic degree of freedom, which we shall analyse in the next section in the simple (and first) example with $B(\phi, X) = 0$ and $b(\phi) = -1$.

7.3 Mimetic gravity

We have already encountered an example of mimetic gravity theory when we analysed the extended conformal transformations. There, using the current language, we had

$$g_{\mu\nu} = -Xl_{\mu\nu} = (-l^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi)l_{\mu\nu}, \tag{7.18}$$

which satisfies the relation (7.7), with $B(\phi, X) = 0$ and $b(\phi) = -1$.

This is exactly the first example of an entire class of theories currently named "mimetic gravity theories"; it was introduced by Mukhanov and Chamseddine in 2013, who coined the term "mimetic dark matter" in relation to the new degree of freedom embodied by the parametrisation of the physical metric Eq.(7.18), which, as we are going to see, mimics the effect of dark matter.

We will now review their argument based on Eq.(7.18) applied to the Hilbert-Einstein action and then we will briefly discuss mimetic gravity in Horndeski theories.

First of all we notice that the parametrisation $g_{\mu\nu} = -Xl_{\mu\nu}$ implies that $g^{\mu\nu} = (-1/X)l^{\mu\nu}$ and therefore

$$g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = -1, \tag{7.19}$$

as can be seen contracting $g^{\mu\nu} = (-1/X)l^{\mu\nu}$ with $\partial_\mu\phi\,\partial_\nu\phi$.

Let's now consider a Hilbert-Einstein Lagrangian plus a generic matter sector

$$S = \frac{1}{2}\int d^4x\sqrt{-g(l_{\mu\nu},\phi)}\big[R(g_{\mu\nu}(l_{\mu\nu},\phi) + \mathcal{L}_m\big], \tag{7.20}$$

where the physical metric is parametrised via eq.(7.18) and we have set $8\pi G = 1$.

Varying the action with respect to the metric $g_{\mu\nu}$ gives

$$\delta S = \frac{1}{2}\int d^4x\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\int d^4x\sqrt{-g}\big[G^{\mu\nu} - T^{\mu\nu}\big]\delta g_{\mu\nu}. \tag{7.21}$$

We now consider the fact that the variation of the physical metric can be expressed through the variations of the auxiliary metric and the scalar field as in Eq.(7.10)

$$\begin{aligned}
\delta g_{\mu\nu} &= -X\delta l_{\mu\nu} - \delta X l_{\mu\nu} = \\
&= -X\delta l_{\mu\nu} - l_{\mu\nu}\big(- l^{\alpha\sigma}l^{\beta\rho}\delta l_{\sigma\rho}\partial_\alpha\phi\,\partial_\beta\phi + 2l^{\alpha\beta}\partial_\alpha\delta\phi\,\partial_\beta\phi\big) \\
&= -X\delta l_{\sigma\rho}\big(\delta^\sigma_\mu\delta^\rho_\nu + g_{\mu\nu}g^{\alpha\sigma}g^{\beta\rho}\partial_\alpha\phi\,\partial_\beta\phi\big) + g_{\mu\nu}2g^{\alpha\beta}\partial_\alpha\delta\phi\,\partial_\beta\phi,
\end{aligned}$$

where in the last equality we have used that $g^{\mu\nu} = \frac{1}{-X}l^{\mu\nu}$ and therefore $l_{\alpha\beta}l^{\mu\nu} = g_{\alpha\beta}g^{\mu\nu}$.

Taking the variation with respect to $\delta l_{\mu\nu}$ and $\delta\phi$ in (7.21) we find

$$G_{\mu\nu} - T_{\mu\nu} = -\big(G - T)\partial_\mu\phi\,\partial_\nu\phi, \tag{7.22}$$

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}(G-T)g^{\mu\nu}\partial_\nu\phi)_{,\mu} = 0 \tag{7.23}$$

While the auxiliary metric does not appear explicitly in the equations of motion (although the physical metric depends on it), the scalar field contributes explicitly as we can see in the equations above. Its contribution can be expressed through the following energy-momentum tensor

$$T^\phi_{\mu\nu} = -\big(G - T\big)\partial_\mu\phi\,\partial_\nu\phi, \tag{7.24}$$

which assumes the form of a new matter component. Notice that it is also covariantly conserved, i.e. $\nabla^\mu T^\phi_{\mu\nu} = 0$.

The trace of eq.(7.22) gives

$$\big(G - T\big)\big(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + 1\big) = 0 \tag{7.25}$$

In virtue of (7.19) this relation is satisfied even if $G - T \neq 0$. Therefore Eq.(7.22) admits non-trivial solutions even in the absence of matter ($T = 0$) where we have $G_{\mu\nu} = -G\partial_\mu\phi\,\partial_\nu\phi$. In other words the parametrisation (7.18) has introduced a new degree of freedom which propagates dynamically.

The next step is to understand what the physical interpretation of this new matter component is. Actually (7.24) can be rewritten as the energy-momentum tensor of a perfect fluid

$$T^\phi_{\mu\nu} = (\rho + p)u_\mu u_\nu + g_{\mu\nu}p, \quad u^\mu u_\mu = -1, \tag{7.26}$$

with $\rho = -(G-T)$, $p = 0$ and $u_\mu = \partial_\mu \phi$.

This is a pressureless perfect fluid with four-velocity equals to the gradient of the scalar field. We expect that in a cosmological environment it describes dust with equation of state $p = 0$. In order to verify this let us solve the equation of motion (7.23) in the FRW metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$. The presence of the scalar field suggests us to choose the hypersurfaces of constant time to correspond to the hypersurfaces of constant ϕ; i.e. up to a constant (which we put to zero) we can identify ϕ with the time t. We note that this choice satisfies the constraint (7.19).

Now, using that $\sqrt{-g} = a^3$, Eq.(7.23) becomes

$$\begin{aligned} \partial_0\big(-a^3(G-T)\big) &= 0 \\ \Rightarrow G-T &= \frac{C}{a^3}, \end{aligned} \tag{7.27}$$

where $C = C(x^i)$ is an integration constant with only spatial dependence.

Recalling that in a FRW universe the energy density of a fluid with equation of state $p = w\rho$ scales as $\rho \propto a^{-3(1+w)}$, we see that the mimetic degree of freedom behaves like dust. The fact that it is subjected only by the gravitational interaction suggests that it mimics the behaviour of dark matter. From this comes the name "mimetic dark matter". The amount of it is encoded in the integration constant $C(x^i)$.

Chapter 8

Cosmological observables and disformal invariance

In this chapter we are going to investigate the physical invariance under disformal transformations directly looking at the transformations laws for the relevant observables in cosmology. In particular we are going to show that with the proper assumption on the scalar field ϕ, which corresponds to the choice of the comoving gauge to the scalar field, we are able to interpret the disformal transformation as a local change of units in the same fashion as it has been done in the conformal case. In this framework we have calculated how the particle four momentum changes under a disformal transformation and have exploited it to show that the Boltzmann formalism is unaltered by such a transformation. Being relevant for cosmological applications, we have inferred the frame-invariance of the cosmological redshift as well as the scalar field and Maxwell actions.

Subsequently we have found convenient to introduce a frame-invariant formalism in the context of beyond-Horndeski theories. Within this approach we have written a frame-invariant action encompassing the GLPV theories and parametrised the frame choice with two extra parameters corresponding to the physical units of time and length. This is nothing else than an extension of the frame invariant approach of [19] reviewed in chapter 5.

To close the chapter we have reported an important result obtained in [59] and [72], i.e. the disformal invariance of power spectra from inflation and other relevant inflationary observables like the tensor-to-scalar ratio and the spectral indexes.

All the calculations are carried in the comoving or unitary gauge where the disformal transformation is considerably simpler to analyse. Moreover, in what follows it is assumed that the disformal transformation is invertible in the sense of Sec.(7.1); if not, the frame-invariance could not be addressed.

8.1 Disformal transformation as a change of units

In the previous chapters we have showed that conformally related frames are physically equivalent, though this equivalence turns out to be not obvious when we have to interpret the results from one frame to another. We have also stated that the same should hold in the disformal case. Now we want to make it crystal-clear trying to interpret the disformal transformation as a change of units, exactly in the same fashion of the conformal case.

We start considering the generic disformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \alpha^2(\phi, X) g_{\mu\nu} + \beta(\phi, X) \partial_\mu \phi \, \partial_\nu \phi. \tag{8.1}$$

We have already seen the effect of this *disformation* on the line-element, that we recall

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \alpha^2 ds^2 + \beta \partial_\mu \phi \, \partial_\nu \phi \, dx^\mu dx^\nu.$$

Most of the difficulties in treating disformal transformations and the physics related to them come from the new scalar field derivatives term proportional to $\beta(\phi, X)$. A reasonable procedure to understand the effects of a disformal transformation would be to manage this term in order to put it in a simpler form. For instance, we can assume that β is constant, and/or that ϕ is only time-dependent, as it is at the background level in cosmology. If we want to investigate disformally related frames the latter assumption, despite being very limiting, is indeed really powerful because the effects of the disformal transformations would be very clear.

An alternative approach strictly related to it is provided remembering that in a cosmological environment, where $\partial_\mu \phi$ is timelike, we have the gauge-freedom to put to zero the perturbations of the scalar field, i.e. to choose the so-called *comoving* or *unitary* gauge [3] in which $\delta\phi = 0$ and therefore $\phi = \phi(t)$. This choice is convenient whenever the scalar field gives the dominant contribution to the energy density of the universe as it is the case during inflation. Another relevant case in which the comoving gauge is usually exploited is in dark energy models where the scalar field is responsible for the late-time accelerated expansion[1].

In what follows we will consider as a starting point the following action

$$S_T = \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{\tilde{g}}[\tilde{g}, \phi] + \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_m[\tilde{g}, \psi_m],$$

where ψ_m is a generic matter field, $\tilde{g}_{\mu\nu}$ is the matter metric disformally coupled to $g_{\mu\nu}$ via Eq.(8.1). with $X \equiv g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi$.

For the moment we do not assume any gravity theory, but we can think of a Horndeski-like theory or one of its healthy extensions, like GLPV theories.

[1] We can consider the comoving gauge also when the scalar field is sub-dominant, e.g. in the matter-dominated era, but this choice would not be so practical at all. However, the use of disformal transformations turns out to be convenient especially in situations like inflation, or in dark energy models.

We refer to $g_{\mu\nu}$ as the gravity metric. In the gravity frame the theory is written in the gravity metric and therefore matter does not follow geodesics of this metric because of the disformal coupling (8.1). In the matter frame the theory is written using the tilde metric as above and in this frame matter follows the geodesics of the gravity-sector metric. The latter is the analogous of the Jordan frame, in which matter follows the geodesics of the metric, but generally the gravity sector is different from the Einstein-Hilbert one.

Throughout this chapter we shall assume that the disformal transformation is invertible in the sense of Sec.(7.1), i.e. we can express $g_{\mu\nu}$ as a function of $\tilde{g}_{\mu\nu}$ and the scalar field derivatives. In particular we require that $X = X(\tilde{X})$ with $\tilde{X} = \tilde{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$.

In any case the gravity and the matter frame are related by a disformal transformation and we should expect invariance of the underlying physics between the two frames.

Let us move to the comoving gauge in which $\partial_\mu\phi = \big(\dot{\phi}(t), 0, 0, 0\big)$. The action of the disformal transformation on the line-element now is simply

$$d\tilde{s}^2 = \alpha^2(\phi, X)ds^2 + \beta(\phi, X)\dot{\phi}^2 dt^2, \tag{8.2}$$

where $\phi = \phi(t)$, but X can have a spatial dependence through the metric tensor. Considering the flat FLRW metric for the sake of clarity, we have explicitly

$$\begin{aligned} d\tilde{s}^2 &= -\alpha^2 dt^2 + \beta\dot{\phi}^2 dt^2 + \alpha^2 a^2(t)\delta_{ij}dx^i dx^j \\ d\tilde{s}^2 &= -\alpha^2\Big(1 - \frac{\beta\dot{\phi}^2}{\alpha^2}\Big)dt^2 + \alpha^2 a^2(t)\delta_{ij}dx^i dx^j \\ d\tilde{s}^2 &\equiv -\alpha^2\gamma^2 dt^2 + \alpha^2 a^2(t)\delta_{ij}dx^i dx^j, \end{aligned} \tag{8.3}$$

where we have defined $\gamma^2 \equiv 1 - \beta\dot{\phi}^2/\alpha^2$. In order to preserve the Lorentzian signature we must require $\gamma^2 > 0$ as we pointed out in Sec.(6.2).

We see that the disformal transformation acts differently on the time and spatial part of the metric. In particular we can split up the effect of the DT into two parts: a pure disformal part induced by the second term in (8.2) giving rise to the γ^2-factor in front of dt^2, and a conformal part induced by the α^2-factor. We analyse the former putting the latter equals to unity. In this case

$$d\tilde{s}^2 \equiv -\gamma^2 dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{8.4}$$

and we immediately note that the effect of the pure disformal term is to rescale times by a factor γ.

We know from the previous chapters that the conformal transformation can be interpreted as a local change of units. In the same fashion the disformal transformation can be interpreted as a local change of units, where though units of time and length are rescaled differently. From this point of view disformally related frames are linked by a local change of units and therefore they are physically equivalent.

8.2 Disformal invariance of particle physics

In order to investigate how the particle physics changes under the disformal transformation considered in this chapter let us summarize what we have said. In the unitary gauge the effect of a DT on the line-element is a conformal transformation of the metric plus a rescaling of times. In particular we have seen that in the ADM formalism (in an opportune set of spatial coordinates in which $N^i = 0$)

$$d\tilde{s}^2 = \alpha^2\big(- N^2\gamma^2 dt^2 + h_{ij}dx^i dx^j\big). \tag{8.5}$$

We shall assume that the particle masses are constant in the matter frame, while they can acquire a spacetime dependence in the gravity frame as it happens in the conformal case.

The first aim is to understand how the particle four-momentum changes when we perform the disformal transformation. In order to achieve this we keep the conformal and the disformal effects separated. Considering only the latter we see that, if in the untilde frame t is the cosmic proper time, then in the matter frame it is $d\tilde{t} = \gamma dt$ as follows from Eq.(8.5) putting $\alpha = 1$ (and with $\gamma^2 = 1 - \beta\dot{\phi}^2/N^2$) . Therefore the disformal transformation amounts for a rescaling of the cosmic proper time, and this has to be taken into account when we interpret the cosmological observables.

From this it can be argued that such a time rescaling affects only the time component of vectors (and therefore modifies all the dispersion relations) and it can be eaten with a suitable coordinate-redefinition; as such it should leave (coordinate) scalars invariant. The effect of the conformal part, as we have already understood in the previous chapters, can be interpreted as a rescaling of all the masses by a factor α^2 such that $\tilde{m} = m/\alpha$. Furthermore we assume that the spatial component are affected only by the conformal rescaling $\tilde{P}^i = P^i/\alpha^2$.

This leads to

$$\begin{aligned}
\tilde{g}_{\mu\nu}\tilde{P}^\mu\tilde{P}^\nu &= -\tilde{m}^2 \\
-N^2\alpha^2\gamma^2\big(\tilde{P}^0\big)^2 + \alpha^2 h_{ij}\tilde{P}^i\tilde{P}^j &= -\tilde{m}^2 \\
-N^2\alpha^2\gamma^2\big(\tilde{P}^0\big)^2 + \frac{1}{\alpha^2}h_{ij}P^iP^j &= -\frac{m^2}{\alpha^2}.
\end{aligned}$$

If $\tilde{P}^0 = \frac{1}{\alpha^2\gamma}P^0$ then the last line becomes

$$\begin{aligned}
-N^2(P^0)^2 + h_{ij}P^iP^j &= -m^2 \\
g_{\mu\nu}P^\mu P^\nu &= -m^2.
\end{aligned}$$

We can therefore assume that the transformation rules of the four-momentum are

$$\tilde{P}^\mu = \Big(\frac{P^0}{\alpha^2\gamma}, \frac{P^i}{\alpha^2}\Big), \tag{8.6}$$

$$\tilde{P}_\mu = \Big(\gamma P_0, P_i\Big), \tag{8.7}$$

and the mass transformation $\tilde{m} = m/\alpha$ with $\tilde{m}$ constant.

We note that the covariant three-momentum is frame-invariant. Exactly as we did in the conformal case we can define the frame-invariant distribution function for a particle specie in the matter frame as follows

$$d\tilde{N} = dx^1 dx^2 dx^3 d\tilde{P}_1 d\tilde{P}_2 d\tilde{P}_3 f(\tilde{t}, x^i, \tilde{P}_i), \tag{8.8}$$

where $d\tilde{N}$ is the (frame-independent) number of particle in the frame-invariant phase-space volume $dx^1 dx^2 dx^3 d\tilde{P}_1 d\tilde{P}_2 d\tilde{P}_3$ and $\tilde{t}$ is the cosmic proper time in the matter frame.

The frame-invariant distribution function allows us to define a frame-invariant comoving number density

$$n_c(\tilde{t}, \vec{x}) = g_s \int \frac{d^3\tilde{P}}{(2\pi)^3} f(\vec{x}, \tilde{P}_i, \tilde{t}), \tag{8.9}$$

where $d^3\tilde{P} = d\tilde{P}_1 d\tilde{P}_2 d\tilde{P}_3$ is frame-invariant.

The Boltzmann equation for a particle specie ψ with mass $\tilde{m}_\psi$ is given by

$$\frac{df_\psi}{d\tilde{t}} = \hat{C}[f_\psi], \tag{8.10}$$

which apparently seems to be frame-dependent since in our understanding times get rescaled and therefore

$$\frac{df_\psi}{d\tilde{t}} \to \frac{df_\psi}{\gamma dt}. \tag{8.11}$$

Anyway we are going to see that the γ-factor is compensated by another γ-factor in the collisional term and the resultant Boltzmann equations are frame-invariant. This happens if we consider, for instance, the very generic process $\psi + a + b + ... \to i + j + ...$

For such a process the RHS of the Boltzmann equation reads

$$\begin{aligned} \hat{C}[f_\psi] \;&=\; \frac{1}{2\tilde{P}_0} \int d\tilde{\Pi}^a d\tilde{\Pi}^b \cdots d\tilde{\Pi}^i d\tilde{\Pi}^j \cdots (2\pi)^4 \delta^4(\tilde{P}^\psi + \tilde{P}^a + \tilde{P}^b + \cdots - \tilde{P}^i - \tilde{P}^j \cdots) \\ &\times\; \big[- |M|^2_{\psi+a+b+\ldots \to i+j+\ldots} f_\psi f_a f_b \cdots (1 \pm f_i)(1 \pm f_j) \cdots \\ &+\; |M|^2_{i+j+\ldots \to \psi+a+b+\ldots} f_i f_j \cdots (1 \pm f_\psi)(1 \pm f_a)(1 \pm f_b) \cdots \big], \end{aligned} \tag{8.12}$$

where

$$d\tilde{\Pi} \equiv l_r^2 \frac{d^4\tilde{P}}{(2\pi)^3} (-\tilde{g})^{-\frac{1}{2}} \delta(\tilde{P}^2 - \tilde{m}^2)\Theta(\tilde{P}^0) = l_r^2 \frac{d^3\tilde{P}}{(2\pi)^3} (-\tilde{g})^{-\frac{1}{2}} \frac{1}{2\tilde{P}^0}$$

is frame invariant, and the sign "+" applies to bosons, whereas the sign "-" applies to fermions. $|M|$ is the transition amplitude of the considered process which depends on the fundamental physics. l_r is the length unit defined in 5.4.2.

The factor $1/2\tilde{P}_0$ in front of the integral in Eq.(8.12) produces a γ factor in the transformed frame which cancels out the one in the LHS. So we have that the Boltzmann equation is frame-invariant with respect to our particular disformal transformation. provided that $|M|^2$ can be written in an explicitly frame-invariant way.

8.2.1 The frame-invariant redshift

Let's suppose to have a light-emitter, like a distant galaxy, and we want to measure the redshift of photons which come to us. Following [2], an observer with four momentum u^μ will measure a photon energy given by $E = h\omega = -u^\mu k_\mu$, where ω is the frequency of photons, whereas k^μ is the photon four-momentum. The redshift of photons is defined as follows

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{\omega_{em}}{\omega_{obs}} = \frac{(u^\mu k_\mu)_{em}}{(u^\mu k_\mu)_{obs}} \equiv 1 + z. \tag{8.13}$$

In the standard FRW cosmology $\dot{E}/E = -H$ and therefore $E \propto 1/a$, giving rise to the standard relation $1 + z = a_{obs}/a_{em}$.

To evaluate the redshift in the matter frame metric (8.5) in a flat FRW background, we must move to the cosmic proper time where the metric reads

$$d\tilde{s}^2 = -\alpha^2\big(d\tilde{t}^2 + a^2\delta_{ij}dx^i dx^j\big), \tag{8.14}$$

where of course $a = a(\tilde{t}) = a(t(\tilde{t}))$ and $\alpha = \alpha(t(\tilde{t}))$. In terms of the conformal time it reads

$$d\tilde{s}^2 = -\tilde{a}^2\big(d\tilde{\eta}^2 + \delta_{ij}dx^i dx^j\big), \tag{8.15}$$

where we have defined $\tilde{a} = \alpha(\tilde{t})a(\tilde{t})$.

The only thing that changes with respect to the usual case is that a gets rescaled by α as in the conformal case and that all the quantities depends on the matter cosmic proper time $\tilde{t}$.

Since comoving observers and photons follow the same metric, we have that $\tilde{E} \propto 1/\tilde{a}$ and therefore

$$1 + z = \frac{\tilde{a}_{obs}}{\tilde{a}_{em}} = \frac{a_{obs}}{a_{em}}\frac{\alpha_{obs}}{\alpha_{em}}. \tag{8.16}$$

Again, the frame-invariant expression for the redshift is given by

$$1 + z = \frac{a_{obs}}{l_{r,obs}}\frac{l_{r,em}}{a_{em}}, \tag{8.17}$$

which takes into account the rescaling of units due to the conformal transformation (see 5.4.2 for the proper interpretation).

8.2.2 The frame-invariance of the matter action

Scalar field action

Let's now see how the action of a generic scalar field transforms under the disformal transformation considered up to now. In the matter frame it reads

$$\tilde{S}_{KG} = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\big(\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\psi}\tilde{\nabla}_\nu\tilde{\psi} + \tilde{m}^2\tilde{\psi}^2\big) \tag{8.18}$$

In terms of the ADM variables it becomes

$$\begin{aligned}\tilde{S}_{KG} &= \frac{1}{2}\int d^3x dt\tilde{N}\sqrt{-\tilde{h}}\Big(-\frac{1}{\tilde{N}^2}(\partial_t\tilde{\psi})^2 + \tilde{h}^{ij}\partial_i\tilde{\psi}\partial_j\tilde{\psi} + \tilde{m}^2\tilde{\psi}^2\Big)\\ &= \frac{1}{2}\int d^3x dt\gamma N\alpha^4\sqrt{-h}\Big(-\frac{1}{\alpha^2\gamma^2N^2}(\partial_t\tilde{\psi})^2 + \frac{h^{ij}}{\alpha^2}\partial_i\tilde{\psi}\partial_j\tilde{\psi} + \frac{m^2}{\alpha^2}\tilde{\psi}^2\Big)\end{aligned}$$

Letting $\tilde{\psi} = \psi/\alpha$, and introducing the cosmic proper time for the untilde frame we have

$$\tilde{S}_{KG} = \frac{1}{2}\int d^3x d\tilde{t}N\sqrt{-h}\Big(-\frac{1}{N^2}(\partial_{\tilde{t}}\psi)^2 + h^{ij}\partial_i\tilde{\psi}\partial_j\tilde{\psi} + m^2\psi^2\Big) \tag{8.19}$$

up to spacetime derivatives of the conformal factor which we can assume to be negligible. In fact the space and time scales of variation of α are of cosmological and/or astrophysical size and therefore spacetime variation of the conformal factor is surely much smaller with respect to the variation of particle physics quantities, such as decay rates, cross sections and so on. Hence, when we deal with particle physics quantities it is quite reasonable to neglect terms like $\partial_\mu\alpha(x)$. So for our purposes the Klein-Gordon scalar field action is invariant under disformal transformations provided that we interpret correctly the cosmic proper time passing from one frame to another.

The electromagnetic field

Let's consider the action for the electromagnetic field in the matter frame

$$S_{EL} = -\int d^4x\sqrt{-\tilde{g}}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -\int d^4x\sqrt{-\tilde{g}}\tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}\tilde{F}_{\mu\nu}\tilde{F}_{\alpha\beta},$$

where $\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$. In the untilde frame it becomes

$$S_{EL} = -\int d^3x\, dt\,\gamma N\sqrt{-h}\big[2\tilde{F}_{ti}(\frac{1}{N^2\gamma^2}\partial_t\tilde{A}_i - \frac{1}{N^2\gamma^2}\partial_i\tilde{A}_t) + \tilde{F}_{ij}\tilde{F}^{ij}\big]. \tag{8.20}$$

Note that the conformal factor does not appear; this is due to the conformal invariance of the Maxwell field. However we have an explicit dependence on γ that can be reabsorbed with the following assumptions

$$\tilde{A}_i = A_i, \qquad \tilde{A}_t = \gamma A_t, \tag{8.21}$$

that lead to

$$S_{EL} = \int d^3x \, d\tilde{t} \, N\sqrt{-h}\big[2F_{\tilde{t}i}F^{\tilde{t}i} + F_{ij}F^{ij}\big], \tag{8.22}$$

where again we have introduce the cosmic proper time $\tilde{t}$ and up to spacetime derivatives of γ.

8.3 Disformal invariance in beyond-Horndeski theories

We can now exploit the knowledge gained in the conformal case to write down a frame-invariant model for disformal transformations. The trick there was to consider from the onset only dimensionless quantities, which are not affected by a change of units and which are the actual outcome of observations. Here we will do the same but in the context of Horndeski and beyond Horndeski theories where the disformal transformations find their proper environment.

It is convenient to work in the ADM formalism, where the line-element reads

$$d\tilde{s}^2 = -\tilde{N}^2 dt^2 + \tilde{h}_{ij}\big(dx^i + \tilde{N}^i dt\big)\big(dx^j + \tilde{N}^j dt\big), \tag{8.23}$$

where $\tilde{N}$ is the *lapse function*, $\tilde{h}_{ij}$ is the three-dimensional (spatial) metric and $\tilde{N}^i$ is the *shift vector*.

In the comoving gauge and in terms of the disformal metric $g_{\mu\nu}$ it becomes

$$d\tilde{s}^2 = -N^2\alpha^2\gamma^2 dt^2 + \alpha^2 h_{ij}\big(dx^i + N^i dt\big)\big(dx^j + N^j dt\big), \tag{8.24}$$

with $\gamma^2 = 1 + \frac{\beta\dot{\phi}^2}{\alpha^2 N^2} = 1 + \frac{\beta X}{\alpha^2}$ and of course $\tilde{N} \equiv \alpha\gamma N$ and $\tilde{h}_{ij} \equiv \alpha^2 h_{ij}$. We note that $\tilde{N}^i = N^i$, i.e. the shift vector is invariant under such a disformal transformation.

Following the conformal case we can introduce two arbitrary measuring sticks t_r and l_r that under a disformal change of units transform as

$$\tilde{t}_r = \alpha\gamma t_r, \qquad \tilde{l}_r = \alpha l_r. \tag{8.25}$$

In the conformal case units of time and length scale equally and therefore we chose $t_r = l_r$, which is consistent with the fact that in the pure conformal case $\gamma = 1$. Here we need to keep distinct units of time and length since they rescale differently.

However, we can exploit them to write down an explicitly frame-invariant metric as follows

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -\bar{N}^2 dt^2 + \bar{h}_{ij}\big(dx^i + \bar{N}^i dt\big)\big(dx^j + \bar{N}^j dt\big), \tag{8.26}$$

with

$$\bar{N} \equiv \frac{N}{t_r}, \quad \bar{h}_{ij} \equiv \frac{h_{ij}}{l_r^2}, \quad \bar{N}^i = N^i. \tag{8.27}$$

This corresponds to the explicit dimensionless construction of [19, 55], but extended to the disformal change of units. In that case the arbitrary unit l_r depends on ϕ, but here in principle we could allow also for a X-dependence.

Anyway we must pay attention on which gravitational theory we are considering. Indeed it has been showed that the Horndeski action is closed under the special disformal transformation given by Eq.(6.10). On the other hand if we allow for a X-dependence in β, after the DT we end up in a new class of theories which are enclosed in the so-called G^3 or $GLPV$ theories (Gleyzes, Langlois, Piazza, and Vernizzi)[70]. The field dependence of l_r and t_r must be chosen accordingly with the theory considered. For instance, if we are in the realm of Horndeski theory, then $t_r = t_r(\phi)$ and $l_r = l_r(\phi)$, which in this gauge means they depend only on time. For the G^3 theories we let t_r to depend on X in virtue of the disformal transformations under which they are form-invariant

$$\tilde{g}_{\mu\nu} = \alpha^2(\phi)g_{\mu\nu} + \beta(\phi, X)\partial_\mu\phi\,\partial_\nu\phi. \tag{8.28}$$

What we want to do now is to rewrite the general GLPV action in a frame-invariant form using the frame-invariant metric defined above. This will ensure that all the frames related by the disformal transformation (8.28) in the unitary gauge are physically equivalent since they are related by a change of units.

8.3.1 The GLPV action

The GLPV action is given by [70, 71]

$$S_g = \int d^4x\sqrt{-g}\sum_{i=2}^{5} L_i, \tag{8.29}$$

where

$$\begin{aligned}
L_2 &= G_2(\phi, X); && (8.30)\\
L_3 &= G_3(\phi, X)\Box\phi; && (8.31)\\
L_4 &= G_4(\phi, X)^{(4)}R - 2G_{4,X}(\phi, X)\left(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right) + \\
&+ F_4(\phi, X)\epsilon^{\mu\nu\rho}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}; && (8.32)\\
L_5 &= G_5(\phi, X)^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}\left(\Box\phi^3 - 3\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + \right.\\
&+ \left. 2\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}_{\sigma}\right) + F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'}, && (8.33)
\end{aligned}$$

and $\phi_\mu = \nabla_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$. With respect to the Horndeski lagrangian here we have two extra terms, those proportional to F_4 and F_5. Therefore if we want to restrict to the Horndeski sub-class we must require $F_4 = F_5 = 0$.

We now use the ADM formalism introduced above and move to the unitary gauge $\phi = \phi(t)$. Let's define the constant-time hypersurface Σ_t in the foliation of our manifold $\mathcal{M} = \mathbb{R} \times \Sigma_t$. Further, we assume that the constant-field hypersurfaces ($\phi = constant$) coincide with the constant-time hypersurfaces Σ_t. On such hypersurfaces we introduce the spatial metric $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, where $n_\mu = -\frac{\phi_\mu}{\sqrt{-X}} = (-N, 0, 0, 0)$ is the unit vector orthogonal to Σ_t and such that $n^\mu = (1/N, -N^i/N)$.

In the usual manner we can define the intrinsic curvature of Σ_t through the spatial metric $h_{\mu\nu}$: we define a covariant derivative D_μ such that $D_\mu h_{\alpha\beta} = 0$ for each index and through it the Riemann tensor ${}^{(3)}\mathcal{R}^d_{abc}\omega_d = D_a D_b \omega_c - D_b D_a \omega_c$, for all 1-form ω_c such that $\omega_c n^c = 0$. Contracting properly the indexes we can define the 3-Ricci tensor ${}^{(3)}\mathcal{R}_{\mu\nu} \equiv R_{\mu\nu}$. Finally we define the *extrinsic curvature* of Σ_t as follows:

$$K_{\mu\nu} \equiv h^\lambda_\mu n_{\nu;\lambda},$$

which lives in Σ_t since $K_{\mu\nu} n^\mu = 0$.

After some algebraic manipulations (which exploit for example the Gauss-Codazzi relations), it can be showed that in ADM variables the action (8.29) has the more compact form [70, 71]

$$S_g = \int d^4x N\sqrt{h}\big[A_2 + A_3 K + A_4(K^2 - S) + B_4 R + A_5 K_3 + B_5(U - \frac{1}{2}KR)\big], \quad (8.34)$$

where A_2, A_3, A_4, A_5, B_4 and B_5 are coefficients depending on X and ϕ, or, since we are in the unitary gauge, depending on N and t, while R, K, S, K_3, U are geometrical quantities defined as

$$\begin{aligned} K &\equiv h^{ij}K_{ij}; \qquad S \equiv K_{ij}K^{ij}; \qquad R \equiv h^{ij}R_{ij}; \\ K_3 &\equiv K^3 - 3KS + 2K_{ij}K^{il}K^j_l; \qquad U \equiv R_{ij}K^{ij}, \end{aligned} \quad (8.35)$$

with K_{ij} explicitly given by

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - D_i N_j - D_j N_i). \quad (8.36)$$

The coefficients A_i and B_i depend on the original coefficients G_i and F_i and kinetic term X. Moreover the conditions $F_4 = F_5 = 0$ that select the Horndeski sub-class, can be given through the new coefficients

$$A_4 = -B_4 + 2XB_{4,X}, \qquad A_5 = -\frac{1}{3}XB_{5,X}. \quad (8.37)$$

8.3.2 Frame-invariant version

We have already introduced a frame-invariant metric, namely $\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}(g_{\mu\nu}, l_r, t_r)$, whereof we allowed a $\vec{x}$-dependence in l_r and t_r. But we have stated that only the disformal transformation (8.28) leaves the GLPV action invariant in form. Therefore, to keep track of this fact we restrict the spacetime-point dependence of

l_r to time-dependence only, i.e. $l_r = l_r(t)$. In this way we are sure that we remain on the GLPV domain under a change of units.

All the quantities in (8.34) are geometrical quantities constructed from the metric. So we can use our frame-invariant metric $\bar{g}_{\mu\nu}$ to construct by hand frame-invariant version of the geometrical quantities $\bar{R}_{ij}$, $\bar{K}_{ij}$ and all the barred scalars from them.

Moreover it will be useful to relate the barred quantities, normalized with the arbitrary units t_r and l_r, with the unbarred ones. For instance, exploiting that the 3D curvature is built via the spatial metric $h_{\mu\nu}$ and that $l_r = l_r(t)$, we have

$$\bar{R}_{ij} = R_{ij}; \qquad \bar{R} = \bar{h}^{ij}\bar{R}_{ij} = l_r^2\, h^{ij} R_{ij} = l_r^2 R. \tag{8.38}$$

More difficult is to express $\bar{K}_{ij}$ in terms of K_{ij} and the arbitrary scales t_r and l_r, because of the presence of the spatial covariant derivative and the time-derivative of $\bar{h}_{ij}$ (Eq.(8.36). Since $\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}$ for every spatial index, we have $\bar{D}_i\bar{N}_j = \frac{1}{l_r^2} D_i N_j$. Further, using that $\partial_t \bar{h}_{ij} = \frac{1}{l_r^2}\dot{h}_{ij} - \frac{2h_{ij}}{l_r^3}\dot{l}_r$ we finally have

$$\bar{K}_{ij} = \frac{t_r}{2N}\Big[\frac{1}{l_r^2}\dot{h}_{ij} - \frac{2h_{ij}}{l_r^3}\dot{l}_r - \frac{1}{l_r^2}(D_i N_j + D_j N_i)\Big], \tag{8.39}$$

or equivalently

$$\bar{K}_{ij} = \frac{t_r}{l_r^2} K_{ij} - \frac{t_r}{N}\frac{\dot{l}_r}{l_r^3} h_{ij}. \tag{8.40}$$

We are ready to introduce our choice for the GLPV frame-invariant action, $\bar{S}_g = \int d^4x\sqrt{-\bar{g}}\bar{L}$ that reads

$$\bar{S}_g = \int d^4x \bar{N}\sqrt{\bar{h}}\big[\bar{A}_2 + \bar{A}_3\bar{K} + \bar{A}_4(\bar{K}^2 - \bar{S}) + \bar{B}_4\bar{R} + \bar{A}_5\bar{K}_3 + \bar{B}_5(\bar{U} - \frac{1}{2}\bar{K}\bar{R})\big], \tag{8.41}$$

where the coefficients $\bar{A}_i$ and $\bar{B}_i$ depends on t and $\bar{N}$.

For completeness we could add a matter action that for consistency must be manifestly frame-invariant. It will be constructed using frame-invariant quantities, such as $\bar{g}_{\mu\nu}$ and the dimensionless matter field $\bar{\psi}_m = \bar{\psi}_m(\psi_m, t_r, l_r)$. The explicit expression for $\bar{\psi}_m$ in terms of the arbitrary scales t_r and l_r depends on the kind of matter we are considering as we have already seen in the conformal case. For the moment we keep it general, we just assume it describes a barotropic perfect fluid, with energy density $\bar{\rho}$ and pressure $\bar{P}$. Hence the frame-invariant matter action can be written as

$$\bar{S}_m = \int d^4x\sqrt{-\bar{g}} L_m(\bar{g}_{\mu\nu}, \bar{\psi}_m), \tag{8.42}$$

and the energy-momentum tensor is given by

$$\bar{T}^{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}}\frac{\delta\bar{S}_m}{\delta\bar{g}_{\mu\nu}} \tag{8.43}$$

and such that $\bar{T}^0_0 \equiv -\bar{\rho}$ and $\bar{T}^i_j \equiv \bar{P}\delta^i_j$.

8.3.3 Frame-invariant equations of motion

Once we have built our frame-invariant model for disformal transformations, it is a good exercise to evaluate the corresponding equations of motion, at least for the metric degrees of freedom. In order to do it let us consider the perturbed line-element around the flat FLRW background

$$d\bar{s}^2 = -(\bar{N}_b^2 + 2\bar{A})dt^2 + 2\bar{\psi}_{;i}dx^i dt + \bar{a}^2(t)\big[(1+2\bar{\zeta})\delta_{ij} + 2\bar{E}_{;ij} + \bar{\gamma}_{ij}\big]dx^i dx^j, \quad (8.44)$$

where $\bar{N}_b$ is the background normalized lapse function, $\bar{a}(t) = a(t)/l_r$ is the normalized scale factor, $\bar{A}$, $\bar{\psi}$, $\bar{\zeta}$ and $\bar{E}$ are the normalized scalar perturbations and finally $\bar{\gamma}$ is the normalized tensor perturbation. We have not considered vector-like perturbations because the disformal transformation acts only on the scalar part (in fact as we shall see below neither the tensor perturbation $\bar{\gamma}_{ij}$ is affected by the disformal rescaling).

A note is in order: though l_r depends only on t and therefore it can be reabsorbed into the scale factor with no consequences, t_r has also a spatial coordinates dependence. Hence, if we define $\bar{N} = \bar{N}_b + \delta\bar{N}$ we must pay attention to the definitions of $\bar{N}_b$ and $\delta\bar{N}$. In fact, defining the perturbation $\delta t_r = t_r - t_r^b$ with $t_r^b = t_r^b(t)$ the background value of the unit time-scale, we have at the first order

$$\bar{N} = \frac{N}{t_r} = \frac{N_b + \delta N}{t_r^b + \delta t_r} \simeq \frac{N_b}{t_r^b} + \frac{\delta N}{t_r^b} - \frac{N_b}{(t_r^b)^2}\delta t_r \equiv \bar{N}_b + \delta\bar{N}_0 - \bar{N}_b\delta\bar{t_r} \equiv \bar{N}_b + \delta\bar{N}, \quad (8.45)$$

where we have defined $\bar{N}_b \equiv N_b/t_r^b$, $\delta\bar{N}_0 \equiv \delta N/t_r^b$, $\delta\bar{t_r} \equiv \delta t_r/t_r^b$ and $\delta\bar{N} \equiv \delta\bar{N}_0 - \bar{N}_b\delta\bar{t_r}$.

In agreement with (8.26) we see that $2\bar{A} = \bar{N}^2 - \bar{N}_b^2 - \bar{h}_{ij}N^iN^j$, $\bar{\psi}_{|i} = \bar{h}_{ij}N^j$ and

$$\bar{h}_{ij} = \bar{a}^2(t)\big[(1+2\bar{\zeta})\delta_{ij} + 2\bar{E}_{|ij} + \bar{\gamma}_{ij}\big]. \quad (8.46)$$

Since the spatial normalization is absorbed into the scale factor, it follows that

$$\bar{\zeta} = \zeta; \qquad \bar{E} = E; \qquad \bar{\gamma}_{ij} = \gamma_{ij}, \quad (8.47)$$

in particular this tells us that the tensor perturbation is invariant under the disformal transformation considered here[2]., and the same holds for the *comoving curvature perturbation* $\bar{\mathcal{R}}_c \equiv -\bar{\zeta}$. Moreover, remembering that $l_r = l_r(t)$ it also follows that $\bar{\psi} = \psi/l_r^2$.

To fix the gauge we also require $E = 0$; hence, our gauge choice is given by the conditions

$$\delta\phi = 0, \qquad E = 0. \quad (8.48)$$

This ensures that the equations of motion we will obtain fixing the gauge at the action level will be the same we would obtain from the full action fixing the gauge at the EOM level [15].

[2]This is true also in a generic gauge, as shown in Appendix .3

The following step is to obtain the frame-invariant background equations of motion from the action (8.41). At the background level, the metric (8.44) reads

$$d\bar{s}^2 = -\bar{N}_b^2 dt^2 + \bar{a}^2\delta_{ij}dx^i dx^j. \tag{8.49}$$

Furthermore, the geometrical quantities evaluated at the background level read

$$R_{ij}^b = 0; \qquad \bar{K}_{ij}^b = \bar{H}\bar{h}_{ij}, \tag{8.50}$$

and therefore $R^b = 0 = \bar{U}^b$ and $\bar{S}^b = 3\bar{H}^2$, where we have defined

$$\bar{H} \equiv \frac{\dot{\bar{a}}}{\bar{N}^b\bar{a}}. \tag{8.51}$$

The superscript b denotes that we are considering background quantities. In terms of the unbarred quantities and the unit scales, $\bar{H}$ reads

$$\bar{H} = \frac{t_r}{N_b}\Big(H - \frac{\dot{l}_r}{l_r}\Big). \tag{8.52}$$

We can expand the action (8.41) up to the first order in scalar perturbations and then extract from it the background equations of motion. It can be shown that [72]

$$\delta\bar{S}_g = \int d^4x\big[\bar{a}^3(\bar{L}^b + \bar{N}^b\bar{L}_{,\bar{N}} - 3\bar{H}\mathcal{F})\delta\bar{N} + 3\bar{a}^2\bar{N}^b\big(\bar{L}^b - \frac{\dot{\mathcal{F}}}{\bar{N}^b} - 3\bar{H}\mathcal{F}\big)\delta\bar{a}\big], \tag{8.53}$$

where $\mathcal{F} \equiv \bar{L}_{,\bar{K}} + 2\bar{H}\bar{L}_{,\bar{S}}$ and up to total derivatives that we have neglected[3] (note that $\bar{L}$ is the frame-invariant Lagrangian defined above). Since we are at the first order, all the quantities in front of the perturbations $\delta\bar{N}$ and $\delta\bar{a}$ have to be evaluated on the background.

Expanding the matter action (8.42) up to the first order we obtain

$$\delta\bar{S}_m = \int d^4x(-\bar{a}^3\bar{\rho}\delta\bar{N} + 3\bar{a}^2\bar{N}^b\bar{P}\delta\bar{a}). \tag{8.54}$$

Considering the total action $\bar{S} = \bar{S}_g + \bar{S}_m$ and taking its variation with respect to $\delta\bar{N}$ and $\delta\bar{a}$ we obtain the background equations of motion

$$\bar{L}^b + \bar{N}^b\bar{L}_{,\bar{N}} - 3\bar{H}\mathcal{F} = \bar{\rho} \tag{8.55}$$

$$\bar{L}^b - \frac{\dot{\mathcal{F}}}{\bar{N}^b} - 3\bar{H}\mathcal{F} = -\bar{P}. \tag{8.56}$$

We can also add the continuity equation for the matter field that, since it is minimally coupled to $\bar{g}_{\mu\nu}$, simply reads

$$\frac{\dot{\bar{\rho}}}{\bar{N}^b} + 3\bar{H}(\bar{\rho} + \bar{P}) = 0. \tag{8.57}$$

[3]The derivative $\bar{L}_{,\bar{N}} \equiv \frac{\partial\bar{L}}{\partial\bar{N}}$ is understood keeping $\bar{S}$ and $\bar{K}$ constant.

8.3.4 Frame-invariant description of GR

Up to now our treatment has been fully general within the domain of GLPV theories in unitary gauge. For concreteness we can apply our results to the Hilbert-Einstein (H-E) action that is included among the GLPV theories. This will give us a frame-invariant theory that in one specific frame reduces to the usual General Relativity theory. Of course, since all the frames are related by a change of units this implies that all the frames are physically equivalent.

The H-E action is recovered putting $G_2 = G_3 = G_5 = F_4 = F_5 = 0$ and $G_4 = 1/16\pi G$ that imply $A_2 = A_3 = A_5 = B_5 = 0$ and $B_4 = -A_4 = 1/16\pi G$. So, in ADM formalism the H-E action reads

$$S_{HE} = \frac{1}{16\pi G}\int d^4xN\sqrt{h}\big(S - K^2 + R\big), \tag{8.58}$$

where we remember that we have defined $R \equiv {}^{(3)}\mathcal{R}$.

To see that this is the usual Hilbert-Einstein action (up to boundary terms) is sufficient to use the Gauss-Codazzi relation

$${}^{(4)}R = R - K^2 + K_{\mu\nu}K^{\mu\nu} + 2\nabla_\mu\big(Kn^\mu - n^\rho\nabla_\rho n^\mu\big).$$

Our frame invariant Hilbert-Einstein action is given by the dimensionless action

$$\bar{S}_{HE} = \int d^4x\bar{N}\sqrt{\bar{h}}\big(\bar{S} - \bar{K}^2 + \bar{R}\big). \tag{8.59}$$

We now evaluate the background equations found above for this particular case. Let us start calculating $\bar{L}^b$:

$$\bar{L}^b = \bar{S}_b - \bar{K}_b^2 + \bar{R}_b = 3\bar{H}^2 - 9\bar{H}^2 = -6\bar{H}^2, \tag{8.60}$$

where we have used that $\bar{K}_b^2 = \big(\bar{K}^i_{b\,i}\big)^2 = \big(\bar{H}\delta^i_i\big)^2 = 9\bar{H}^2$.

Further, using that $\bar{L}_{,\bar{K}} = -2\bar{K}$ and $\bar{L}_{,\bar{S}} = 1$ we have that

$$\mathcal{F} = -2\bar{K} + 2\bar{H} = -4\bar{H}, \qquad \dot{F} = -4\dot{\bar{H}}. \tag{8.61}$$

Then Eq.(8.55) and Eq.(8.56) read

$$6\bar{H}^2 = \bar{\rho} \tag{8.62}$$

$$6\bar{H}^2 + 4\frac{\dot{\bar{H}}}{\bar{N}_b} = -\bar{P} \tag{8.63}$$

These are our generalised Friedmann equations together with the continuity equation Eq.(8.57). It is straightforward to see that if we use constant units $t_r = l_r = const$ with the appropriate constant value and we put $N = 1$, then these equations are nothing but the usual Friedmann equations for a flat universe.

On the other hand we can choose physical units such that Eqs.(8.62)-(8.63) depart from the usual GR Friedmann equations. For instance, putting

$$l_r = t_r = e^{-b(\phi)}\sqrt{16\pi G},$$

which means our length scale is given by the Planck mass, then Eq.(8.62) becomes

$$H^2 + 2H\frac{db}{dt} + \left(\frac{db}{dt}\right)^2 = \frac{8\pi G}{3}e^{-2b}\rho,$$

where we have used $\bar{H} = l_r\big(H + db/dt\big)$, with $N_b = 1$ and $b(\phi) = b(t)$ in the unitary gauge.

This equation looks like very different from the standard expression, but they only differ for the choice of the units of measure. Therefore the underlying physics is automatically equivalent (if we remember to rescale properly all the dimensionful quantities in the frame with running units).

8.4 Disformal invariance of power spectra from inflation

Before closing this chapter we would like to mention one important result obtained in the context of beyond-Horndeski theory, that is the disformal invariance of both scalar and tensor power spectra from inflation.

Because of the importance of such a result we are going to prove it extensively without appealing to our frame-invariant formalism; i.e. we are going to see that moving from one to another the predicted power spectra remain unchanged. Actually this is an extra confirmation of the previous argument about the frame invariance of physics.

In the previous section we expanded the GLPV action in perturbations up to the first-order following the EFT approach of [69][4]. Going beyond the first order perturbations in the action they also showed that, for scalar modes, the second order action reads

$$S^{(2)} = \int d^4x\mathcal{L}_2 = \int d^4x\, a^3 q_s\Big[\dot{\zeta}^2 - \frac{c_s^2}{a^2}\big(\partial\zeta\big)^2\Big], \tag{8.64}$$

up to boundaries terms irrelevant for the dynamics, and where

$$\begin{aligned} q_s &= \frac{2L_{,S}\Big[4L_{,S}\big(2N_bL_{,N} + N_b^2L_{,NN}\big) + 3\big(N_bW - 4HL_{,S}\big)^2\Big]}{N_b^3W^2}, \\ c_s^2 &= \frac{2N_b}{q_s}\Big(\frac{\dot{G}}{N_b} + HG - \Sigma\Big), \end{aligned}$$

[4] Actually we have expanded our frame-invariant version, but the expansion proceeds equally to the original case for what we have explained before. Moreover, the dictionary used there, is almost the same.

with

$$
\begin{aligned}
G &\equiv \frac{4L_{,s}}{N_b W}\Big(L_{,R} + N_b L_{,RN} + \frac{3}{2}HL_{,U} + N_b H L_{,NU}\Big) \\
\Sigma &\equiv L_{,R} + \frac{\dot{L}_{,U}}{2N_b} + \frac{3}{2}HL_{,U}.
\end{aligned}
$$

Moreover we have defined $(\partial\zeta)^2 \equiv \delta^{ij}\partial_i\zeta\,\partial_j\zeta$.

From $\mathcal{L}_2$ we can easily obtain the equation of motion for the scalar perturbation ζ

$$\frac{d}{dt}\left(a^3 q_s \dot{\zeta}\right) - a q_s c_s^2\,\partial^2\zeta = 0. \tag{8.65}$$

We require that $q_s > 0$ in order to avoid ghost instabilities.

The dimensionless scalar power spectrum is given in the usual way by the vacuum expectation value after the quantization of the scalar fluctuations, and it is

$$< 0|\tilde{\zeta}(\vec{k},0)\tilde{\zeta}(\vec{k}',0)|0 >\equiv \frac{2\pi^2}{k^3}\delta^{(3)}(\vec{k}+\vec{k}')P_\zeta(k), \tag{8.66}$$

where the delta function ensures homogeneity and $\tilde{\zeta}(k,0)$ are the Fourier modes we are going to introduce.

In order to calculate $P_\zeta(k)$ we must solve Eq.(8.65) in the Fourier space. We adopt the following convention for the Fourier expansion

$$\zeta(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}}\tilde{\zeta}(\vec{k},\tau), \tag{8.67}$$

and we assume that during inflation H is nearly constant as well as the slow roll parameters we are going to define later.

As it is usually done we quantize the scalar perturbations as follows

$$\tilde{\zeta}(k,\tau) = \zeta(k,\tau)a(\vec{k}) + \zeta^*(k,\tau)a^\dagger(-\vec{k}),$$

where we have introduced the coefficients $\zeta(k,\tau)$ for each Fourier mode k, whereas $a(\vec{k})$, $a^\dagger(-\vec{k})$ are the annihilation and creation operator respectively.

In order to proceed with the quantization we have to rescale ζ to obtain a canonical scalar field: $v_s(k,\tau) = z_s\zeta(k,\tau) = a\sqrt{2q_s}\zeta(k,\tau)$. In terms of v_s the Eq.(8.65) becomes

$$v_s'' + \left(c_s^2 k^2 - \frac{z_s''}{z_s}\right)v_s = 0, \tag{8.68}$$

where the primes are derivatives with respect to the conformal time τ.

This is the equation we should solve. Once we have found the solution we should take the limit of large scales we are interested in and then we evaluate the two-point correlation function to extract the power spectrum.

We shall omit these calculations, which are the standard ones to obtain the power spectra in inflationary context. However, they can be found in [72]. The final result, i.e. the power spectrum for scalar perturbations for large scales is given by

$$P_\zeta(k) = \frac{N_b^2 H^2}{8\pi^2 q_s c_s^3}\left[1 - 2(C+1)\epsilon - C\epsilon_s - (3C+2)\eta_s\right]\Big|_{c_s k = ah}, \tag{8.69}$$

where $C \equiv \gamma - 2 + \ln 2$ (γ is the Euler-Mascheroni constant). In the expression just above we have introduced the slow-roll parameters

$$\epsilon \equiv -\frac{\dot{h}}{h^2}, \quad \epsilon_s \equiv \frac{\dot{q}_s}{q_s h}, \quad \eta_s \equiv \frac{\dot{c}_s}{c_s h}, \tag{8.70}$$

with $h = N_b H = \dot{a}/a$. They are assumed to be nearly constant and much less than 1.

It is not difficult to prove the frame-invariance of the scalar power spectrum. First of all we have to evaluate how the slow-roll parameters change under the disformal transformation. We will indicate with a tilde the transformed quantities as usual. Since $\tilde{h} = \dot{\tilde{a}}/\tilde{a}$, then we have

$$\tilde{h} = \frac{1}{\alpha a}\frac{d}{dt}(\alpha a) = h + \frac{\dot{\alpha}}{\alpha} \equiv h(1 + \epsilon_\alpha).$$

In the same fashion it can be proved that

$$\tilde{q}_s = \frac{1}{\alpha^3} q_s, \quad \tilde{c}_s^2 = \alpha^2 c_s^2. \tag{8.71}$$

Therefore we have

$$\tilde{\epsilon} = \epsilon - \epsilon \cdot \epsilon_\alpha - \frac{\dot{\epsilon}_\alpha}{h} + O(\epsilon^3), \tag{8.72}$$

$$\tilde{\epsilon}_s = \epsilon_s - 3\epsilon_\alpha \epsilon_s + 3\epsilon_\alpha^2 + O(\epsilon^3), \tag{8.73}$$

$$\tilde{\eta}_s = \eta_s + \epsilon_\alpha - \epsilon_\alpha \eta_s - \epsilon_\alpha^2 + O(\epsilon^3). \tag{8.74}$$

In the transformed frame the next-to-leading order power spectrum reads

$$\tilde{P}_\zeta = \frac{\tilde{N}_b^2 \tilde{H}^2}{8\pi^2 \tilde{q}_s \tilde{c}_s^3}\left[1 - 2(C+1)\tilde{\epsilon} - C\tilde{\epsilon}_s - (3C+2)\tilde{\eta}_s\right]\Big|_{\tilde{c}_s k = \tilde{h}\tilde{a}}. \tag{8.75}$$

Substituting Eqs.(8.71-8.74) in Eq.(8.75), with a little bit of algebra it is straightforward to see that $\tilde{P}_\zeta(k) = P_\zeta(k)$.

We note that the condition $\tilde{c}_s k = \tilde{h}\tilde{a}$ gets transformed to $c_s k = ah(1 + \epsilon_\alpha)$. However, this correction only affects the higher order contributions and therefore it can be neglected.

Analogous calculations show that also the tensor power spectrum is frame-invariant as well as the spectral scalar and tensor indexes together with the tensor-to-scalar ratio r [72]. This tells us that all the sensible inflationary observables are invariant under disformal transformations in the unitary gauge as expected.

Chapter 9

Conclusions

In this work we have dealt with the issue of cosmological disformal invariance in Scalar-Tensor theories of gravity. In this context we started showing that the Horndeski action is closed under a sub-class of disformal transformation, Eq. (6.10) [57]. We also claimed that, whenever the generic disformal transformation is invertible it does not add any new degree of freedom and therefore it can be used to healthy extend Horndeski theory as it is done in [70]. On the other hand when the transformation is not invertible a new class of theories is reached after the transformation, namely the Mimetic Gravity theories which, as we have seen, can mimic the behaviour of dark matter or in general of the dark sector of the Universe as well as inflation [75].

The results reported in chapters 6 and 7 show that from a mathematical point of view in Horndeski and beyond-Horndeski theories disformally related frames are equivalent. However, due to the special meaning of the metric tensor, invoking the physical equivalence from the mathematical one is not obvious. In fact, transforming the metric tensor means transforming the geometrical character of the spacetime. One of the consequences of this is that in disformally related frames particles follow different trajectories, or in other way particles do not follow geodesics of the spacetime. This is something we already tasted in the conformal case, but there, thanks to the simplicity of the transformation, we understood that a conformal transformation is nothing else than a local change of units and as such it does not affect physical result.

In the disformal case, the analysis of the physical effect of the transformation is considerably complicated by the non trivial character of the transformation itself, due to the presence of the first derivatives of the scalar field. For this reason, in order to gain some knowledge about the properties of observable physics under DT, our study has been simplified moving to the comoving gauge where essentially the scalar field retains only the time dependency. In this gauge it is straightforward to note that the effect of the disformal transformation is that of a rescaling of times other than the already known conformal effect. In this sense the disformal transformation can be interpreted as a change of units, where units of time and space scale differently. Adopting this point of view we obtained the transformation rule for the particle four-momentum and exploited it to infer the frame-invariance of the Boltzmann formalism. Another important result is that, under certain conditions both the scalar field and the Maxwell actions are disformally invariant. We also showed that the cosmological redshift does not depend on the frame. Furthermore, we wrote down a frame invariant formalism that makes explicit use of the change of units interpretation: we built an explicitly frame-invariant action encompassing the GLPV theories and where the choice of a frame corresponds to the choice of the units of time and space. The price to pay is the introduction of two arbitrary units (but which do not represent new propagating fields) in the action which have a precise physical meaning, i.e. they are the units of measure of time and length. Finally we exploited this formalism to write down a frame-invariant version of General Relativity and calculate its frame-invariant equations of motion.

We underline that our analysis is restricted to the comoving gauge where the effects of the disformal transformations are easier to interpret. We also underline that anyway they are also valid and exact at the background level and can be used in the background cosmology.

In conclusion it is clear that a deeper analysis is necessary in order to investigate how the observables change under the full disformal transformation, i.e. without any gauge choice. Moreover we also think that so far a further analysis of the geometrical aspects of the transformation misses. For instance, [51] interprets the conformal transformation as a mapping between Riemannian and Weyl-integrable geometries shedding light on the affine and geodesical aspects of the transformation. Something similar can be done for the disformal transformation as noted by [52].

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Appendix 1: The conformally frame-invariant EoM

Let us consider our conformal frame-invariant action (5.30)

$$S = \int d^4x\sqrt{-\bar{g}}\, k^2\big[R(\bar{g}) - 2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 4\bar{V}(\phi)\big] + S_m\big[\bar{g}_{\mu\nu}e^{-2b(\phi)}, \bar{\psi}_m\big].$$

The variation of this action with respect to the metric tensor $\bar{g}_{\mu\nu}$ is straightforward and leads to

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R \;=\; 2\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{g}_{\mu\nu}\bar{V} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\bar{\nabla}_\sigma\phi\bar{\nabla}_\rho\phi + \frac{1}{2k^2}\bar{T}_{\mu\nu}. \qquad (1)$$

Let's now focus on the background dynamics, where $\phi = \phi(t)$ and the line-element is given by

$$d\bar{s}^2 = \bar{a}^2(\tau)\big[-d\tau^2 + \delta_{ij}dx^i dx^j\big].$$

In this metric, the (00)-component of the EoM reads

$$R_{00} - \frac{1}{2}\bar{g}_{00}R = 2\dot{\phi}^2 + 2\bar{a}^2\bar{V} - \dot{\phi}^2 + \frac{1}{2k^2}\bar{T}_{00}, \qquad (2)$$

where the dot are the derivatives with respect to the conformal time τ. Using the definitions (5.34)-(5.35) for the energy-momentum tensor, and that

$$\bar{T}_{00} = \bar{g}_{0\rho}\bar{T}^\rho_0 = \bar{a}^2\bar{\rho}_0, \qquad R_{00} - \frac{1}{2}\bar{g}_{00}R = 3\Big(\frac{1}{\bar{a}}\frac{d\bar{a}}{dt}\Big)^2 = 3\Big(\frac{\dot{\bar{a}}}{\bar{a}^2}\Big)^2, \qquad (3)$$

Eq.(2) becomes

$$3\Big(\frac{\dot{\bar{a}}}{\bar{a}^2}\Big)^2 = \dot{\phi}^2 + 2\bar{a}^2\bar{V} + \frac{\bar{a}^2}{2k^2}\bar{\rho}_0,$$

that is exactly eq.(5.36).

Let's now take the trace of Eq.(1). It reads

$$\begin{aligned} -R &= 2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 8\bar{V} - 4\bar{g}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \frac{1}{2k^2}(-\bar{\rho}_0 + 3\bar{P}_0) \\ R &= 8\bar{V} + 2\bar{g}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \frac{1}{2k^2}(\bar{\rho}_0 - 3\bar{P}_0). \qquad (4) \end{aligned}$$

At the background level it becomes

$$6\frac{\ddot{\bar{a}}}{a^3} = -\frac{2}{\bar{a}^2}\dot{\phi}^2 + 8\bar{V} + \frac{1}{2k^2}(\bar{\rho}_0 - 3\bar{P}_0),$$

that is nothing else than eq.(5.37).

We now consider the conservation equation $\bar{T}^\mu_{\nu;\mu} = -\alpha\bar{\nabla}_\nu\phi\bar{T}^\mu_\mu$ (5.33), which descends from eq. (5.10) putting $\Omega = e^{-b}$. We focus on the $\nu = 0$-component at the background level. The LHS is given by

$$\bar{T}^\mu_{0;\mu} = \bar{T}^\mu_{0,\mu} - \bar{T}^\mu_\alpha\bar{\Gamma}^\alpha_{0\mu} + \bar{T}^\alpha_0\bar{\Gamma}^\mu_{\alpha\mu},$$

with $\bar{\Gamma}^{\mu}_{0\mu} = 4\frac{\dot{\bar{a}}}{\bar{a}}$ and $\bar{\Gamma}^{0}_{00} = \bar{\Gamma}^{\alpha}_{0i} = \frac{\dot{\bar{a}}}{\bar{a}}$. With a little bit of algebra Eq.(5.33) becomes

$$-\dot{\bar{\rho}}_0 - 3\frac{\dot{\bar{a}}}{\bar{a}}(\bar{\rho}_0 + 3\bar{P}_0) = -\alpha\dot{\phi}(-\bar{\rho}_0 + 3\bar{P}_0), \tag{5}$$

that is exactly Eq.(5.38).

The last expression we want to prove is Eq.(5.39). We have already sketched the procedure to obtain it: first of all we minimize the action (5.30) with respect to ϕ. This gives us

$$\bar{\Box}\phi - \frac{\partial\bar{V}}{\partial\phi} = -\frac{1}{4k^2\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\phi}. \tag{6}$$

Using the chain rule we can express the RHS in the following way

$$-\frac{1}{4k^2\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\phi} = -\frac{1}{4k^2}\frac{e^{-4b}}{\sqrt{-\bar{g}}e^{-4b}}\frac{\delta S_m}{\delta(e^{2b}\bar{g}^{\mu\nu})}\frac{\delta(e^{2b}\bar{g}^{\mu\nu})}{\delta\phi}. \tag{7}$$

Defining $h_{\mu\nu} \equiv e^{-2b}\bar{g}_{\mu\nu} \equiv \Omega^2\bar{g}_{\mu\nu}$ and using that $\bar{T}_{\mu\nu}(\bar{h}) = \Omega^{-2}\bar{T}_{\mu\nu}(\bar{g})$ we have

$$\begin{aligned}
-\frac{1}{4k^2\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\phi} &= \frac{1}{4k^2}\frac{1}{2}e^{-4b}\Big(-\frac{2}{\sqrt{-\bar{h}}}\frac{\delta S_m}{\delta\bar{h}^{\mu\nu}}\Big)\,\bar{g}^{\mu\nu}\frac{\partial e^{2b}}{\partial\phi} \\
&= \frac{1}{8k^2}\bar{g}^{\mu\nu}e^{-4b}\bar{T}_{\mu\nu}(\bar{h})2e^{2b}\alpha \\
&= \frac{1}{4k^2}\bar{T}_{\mu\nu}(\bar{g})\bar{g}^{\mu\nu}\alpha = \frac{1}{4k^2}\bar{T}\alpha = -\frac{\alpha}{4k^2}(\bar{\rho}_0 - 3\bar{P}_0).
\end{aligned}$$

Finally we find

$$\bar{\Box}\phi - \frac{\partial\bar{V}}{\partial\phi} = -\frac{\alpha}{4k^2}(\bar{\rho}_0 - 3\bar{P}_0).$$

Using the expression for the D'Alembertian in four dimension for a scalar field we recover Eq.(5.39).

Appendix 2: The frame-invariant particle physics under CTs

In chapter 5 we built an explicitly frame-invariant action expressing the geometrical quantities in terms of the dimensionless metric $\bar{g}_{\mu\nu}$. In order to have a full frame-invariant theory we stated that also the matter action must be written in terms of dimensionless quantities. We now want to express explicitly the rules out of which we define the dimensionless matter fields.

The matter action S_m defined in eq. (5.30) can be the Standard Model of particle physics action, with the Minkowski metric replaced by our frame-invariant metric plus a non-minimal coupling with the gravitational scalar field. Typical matter fields entering such an action are: scalar Klein-Gordon (KG) fields, Maxwell fields, fermions and so on. Here we will focus on KG and the Maxwell fields.

In the formalism of chapter 5 we have chosen units such that $\hbar = 1$. This means that under a local change of units the action should remain unchanged, i.e.

$$\int d^4x\sqrt{-g}\mathcal{L},$$

should be frame-invariant.

Let's now consider the action of a massive Klein-Gordon field ψ, whose action is given by

$$S_{KG} = \frac{1}{2}\int d^4x\sqrt{-g}\big(g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi + m^2\psi^2\big). \tag{8}$$

Under a local change of units it gets transformed into

$$\begin{aligned}\tilde{S}_{KG} &= \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\big(\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\psi}\tilde{\nabla}_\nu\tilde{\psi} + \tilde{m}^2\tilde{\psi}^2\big) = \\ &= \frac{1}{2}\int d^4x\sqrt{-g}\Omega^4\big(\Omega^{-2}g^{\mu\nu}\nabla_\mu\tilde{\psi}\nabla_\nu\tilde{\psi} + \Omega^{-2}m^2\tilde{\psi}^2\big) \\ &= \frac{1}{2}\int d^4x\sqrt{-g}\Omega^2\big(g^{\mu\nu}\nabla_\mu\tilde{\psi}\nabla_\nu\tilde{\psi} + m^2\tilde{\psi}^2\big).\end{aligned} \tag{9}$$

If we put $\tilde{\psi} = \Omega^{-1}\psi$ then we have that $\tilde{S}_{KG} = S_{KG}$ up to spacetime derivatives of Ω.

In this way we have inferred how a KG matter field transforms consistently under a change of units. In particular it transforms like a mass as we expected from the usual knowledge $[\psi] = M$. Hence, the frame-invariant KG field is given by $\bar{\psi} = l_r\psi$.

Let's now consider the extra terms appearing in (9) that come from the derivatives of the conformal factor $\Omega(x)$. Generally, the space and time scales of variation of Ω are model dependent and can be obtained solving the equations of motion of the model. In any case, we can assume that they are of cosmological and/or astrophysical size and therefore spacetime variation of the conformal factor is surely much smaller with respect to the variation of particle physics quantities, such as decay rates, cross sections and so on. Hence, when we deal with particle physics quantities it is quite reasonable to neglect terms like $\partial_\mu\Omega(x)$.

The same reasoning holds for the electromagnetic field whose action is given by

$$S_{EL} = -\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. After a change of units the electromagnetic action becomes

$$\tilde{S}_{EL} = -\int d^4x\sqrt{-\tilde{g}}\tilde{F}_{\mu\nu}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma}\tilde{F}_{\rho\sigma} = -\int d^4x\sqrt{-g}\tilde{F}_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}\tilde{F}_{\rho\sigma}.$$

Simply letting $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, that means $\tilde{A}_\mu = A_\mu = \bar{A}_\mu$ we have the two actions are equivalent. We should expect this result since the electromagnetic action is conformally invariant.

Appendix 3: Line-elements in the gravity and matter frames

Let's suppose that in the gravity frame the perturbed line-element (around the flat FLRW background) reads

$$ds^2 = -(1+2\psi)dt^2 + 2a\omega_{,i}dtdx^i + a^2\big[(1-2\Phi)\delta_{ij} + 2E_{,ij} + \gamma_{ij}\big]dx^i dx^j, \qquad (10)$$

where ψ, ω, Φ and E are the scalar perturbations, whereas γ_{ij} is the tensor perturbation and such that $\gamma^i_i = 0$ and $\gamma_{ij} = \gamma_{ji}$. We will not consider here the vector-like perturbations.

We now want to express the matter frame line-element in the same form of the untilted one. First of all let's write down it as follows

$$\begin{aligned} d\tilde{s}^2 &= A ds^2 + B\partial_\mu\phi\,\partial_\nu\phi dx^\mu dx^\nu = \\ &= -A(1+2\psi)dt^2 + 2Aa\omega_{,i}dtdx^i + Aa^2\big[(1-2\Phi)\delta_{ij} + 2E_{,ij} + \gamma_{ij}\big]dx^i dx^j \\ &+ B\partial_\mu\phi\,\partial_\nu\phi dx^\mu dx^\nu. \end{aligned} \qquad (11)$$

We want to keep only first order perturbations, therefore we need to perturb the coefficients A and B and the scalar field ϕ. We can write

$$\phi = \phi_0(t) + \delta\phi, \qquad A = \bar{A}(t) + \delta A, \qquad B = \bar{B}(t) + \delta B, \qquad (12)$$

where the overbar means we are considering background quantities.

In order to compute δA and δB we have to take into account that they depend on ϕ and on the metric $g^{\mu\nu}$ via X. Therefore we have

$$\delta A = \frac{\partial A}{\partial \phi}\delta\phi + \frac{\partial A}{\partial X}\delta X \equiv A_{,\phi}\delta\phi + A_{,X}\delta X, \qquad (13)$$

where the quantities in front of the perturbations are evaluated in the background, and at the first order

$$\delta X = -2\dot{\phi}_0\delta\dot{\phi} + 2\dot{\phi}_0^2\psi, \qquad (14)$$

and $\bar{X} = -\dot{\phi}_0^2$. For B the same reasoning holds. Therefore the last term of (11) becomes

$$\begin{aligned} B\partial_\mu\phi\,\partial_\nu\phi dx^\mu dx^\nu &= (\bar{B}+\delta B)\partial_\mu(\phi_0+\delta\phi)\,\partial_\nu(\phi_0+\delta\phi)dx^\mu dx^\nu = \\ &= \bar{B}\dot{\phi}_0^2\,dt^2 + 2\bar{B}\dot{\phi}_0\delta\dot{\phi}\,dt^2 + 2\bar{B}\dot{\phi}_0\partial_i\delta\phi\,dtdx^i + \delta B\,\dot{\phi}_0^2\,dt^2. \end{aligned}$$

Hence we have

$$\begin{aligned} d\tilde{s}^2 &= -\bar{A}\big(1 + 2\psi + \frac{\delta A}{\bar{A}} - \frac{\bar{B}}{\bar{A}}\dot{\phi}_0^2 - 2\frac{\bar{B}}{\bar{A}}\dot{\phi}_0\delta\dot{\phi} - \frac{\delta B}{\bar{A}}\,\dot{\phi}_0^2\big)dt^2 + 2\bar{A}a\partial_i\big(\omega + \frac{\bar{B}\dot{\phi}_0\delta\phi}{a\bar{A}}\big)dtdx^i + \\ &+ +Aa^2\big[\big(1 - 2\Phi + \frac{\delta A}{\bar{A}}\big)\delta_{ij} + 2E_{,ij} + \gamma_{ij}\big]dx^i dx^j. \end{aligned} \qquad (15)$$

It is convenient to redefine the time coordinate in the following way

$$dt^2 \longrightarrow \quad d\tilde{t}^2 = \bar{A}\Big(1 - \frac{\bar{B}}{\bar{A}}\dot{\phi}_0^2\Big)dt^2 \equiv \bar{A}\bar{\gamma}^2 dt^2, \tag{16}$$

where $\bar{\gamma}^2 = \big(1 + \bar{B}\bar{X}/\bar{A}\big) > 0$ in virtue of Eq.(6.7).

Finally, we can write

$$d\tilde{s}^2 = -\big(1 + 2\tilde{\psi}\big)d\tilde{t}^2 + 2\tilde{a}\tilde{\omega}_{,i}d\tilde{t}dx^i + \tilde{a}^2\big[(1 - 2\tilde{\Phi})\delta_{ij} + 2\tilde{E}_{,ij} + \tilde{\gamma}_{ij}\big]dx^i dx^j \tag{17}$$

where

$$\begin{aligned} \tilde{\psi} &\equiv \frac{1}{\bar{\gamma}^2}\Big(\psi + \frac{\delta A}{2\bar{A}} - 2\frac{\bar{B}}{2\bar{A}}\dot{\phi}_0\delta\dot{\phi} - \frac{\delta B}{2\bar{A}}\,\dot{\phi}_0^2\Big); \qquad \tilde{a}^2 \equiv \bar{A}a(t(\tilde{t}))^2; \\ \tilde{\omega} &\equiv \frac{1}{\bar{\gamma}}(\omega + \frac{\bar{B}\dot{\phi}_0\delta\phi}{a\bar{A}}); \qquad \tilde{\Phi} \equiv \Phi - \frac{\delta A}{2\bar{A}}; \qquad \tilde{E} = E; \qquad \tilde{\gamma}_{ij} = \gamma_{ij}. \end{aligned} \tag{18}$$

So, we are able to write the perturbed matter frame line-element in the usual form, at the price of coordinate and time redefinitions. We immediately note that neither E nor γ_{ij} are affected by the disformal transformation. We note that in principle we could have started imposing this form for the line-element in the matter frame from the onset and see how it would have transformed in the gravity frame.

Appendix 4: Disformal invariance of continuous media

We are going to show that every Lagrangian invariant under one-parameter disformal transformations is associated with an energy momentum tensor with a linear equation of state $P = w\rho$ [74], with w constant.

Let's consider the generic action specified by

$$S = \int d^4x\sqrt{-g}\mathcal{L}(g_{\mu\nu}, \phi^j, \partial\phi^j, \partial_1 \ldots \partial_n\phi^j), \tag{19}$$

where $j = 1, \ldots, N$ and ϕ^j describes a generic matter field.

In what follows we are going to consider the one-parameter family of disformal transformations given by

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{\sigma}g_{\mu\nu} + \big(e^{\sigma} - e^{3w\sigma}\big)u_\mu u_\nu, \tag{20}$$

where u_μ is taken to be the four-velocity associated with the matter's action, σ is the parameter of the transformation and w is a constant. The choice of the factor 3 reflects the fact that for $w = 1/3$ Eq.(20) reduces to the usual conformal transformation; therefore we can think (20) as a deformation of a CT.

The crucial point is that we can define a four-velocity from the matter action. In particular we can do the simple choice to assume that the energy-momentum tensor (EMT) associated with the action Eq.(19) describes a single component fluid of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\, h_{\mu\nu} + \pi_{\mu\nu}, \tag{21}$$

where we have used the $3+1$ decomposition of the metric $g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$. ρ, p and $\pi_{\mu\nu}$ represent respectively the energy density, the isotropic pressure and the anisotropic stress of the fluid, with $\pi^\mu_\mu = 0$ by construction. They are defined such that

$$\rho = T_{\mu\nu}u^\mu u^\nu, \quad p = \frac{1}{3}T_{\mu\nu}h^{\mu\nu}, \quad \pi_{\mu\nu} = h^\alpha_\mu h^\beta_\nu T_{\alpha\beta} - \frac{1}{3}h_{\mu\nu}h_{\alpha\beta}T^{\alpha\beta}, \tag{22}$$

as can be easily seen using $h_{\mu\nu}u^\mu = 0$ and $u^\mu u_\mu = -1$.

In terms of this decomposition the disformal transformation considered above reads

$$\begin{aligned}\tilde{g}_{\mu\nu} &= e^\sigma\big(h_{\mu\nu} - u_\mu u_\nu\big) + \big(e^\sigma - e^{3w\sigma}\big)u_\mu u_\nu = e^\sigma h_{\mu\nu} - e^{3\sigma w}u_\mu u_\nu \\ &\equiv Ah_{\mu\nu} - Cu_\mu u_\nu. \end{aligned}\tag{23}$$

We can consider the infinitesimal transformation connected with the identity, which corresponds to $\sigma = 0$

$$\delta\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu} = \big(g_{\mu\nu} + (1-3w)u_\mu u_\nu\big)\delta\sigma. \tag{24}$$

Requiring that the action above, associated with the EMT (21), is invariant under such a transformation we obtain

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\int d^4x\sqrt{-g}\big(T^\mu_\mu + (1-3w)\rho\big)\delta\sigma = 0, \tag{25}$$

that is

$$-\rho + 3p = (3w-1)\rho \quad \Leftrightarrow \quad p = w\rho. \tag{26}$$

Thus we have found that a single-fluid whose action is invariant under the one-parameter family of disformal transformations is described by a linear equation of state $p(\rho) = w\rho$ with a constant sound speed $c_s^2 = w$. We note also that invariance under 20 in the case $w = 1/3$ (which is nothing else than invariance under conformal transformations) implies a traceless EMT as it is well-known.

Up to now we have been quite generic, but we can apply these results to some interesting sub-cases such as perfect fluids. In this case it can be shown (see [74] and reference therein) that, using Eulerian coordinates specified by four scalar field $\Phi^A(x,\tau)$, the most general class of perfect fluids can be described by the following action (at leading order in a derivative expansion)

$$S = \int d^4x\sqrt{-g}U(b,Y), \tag{27}$$

where

$$b \equiv \sqrt{det\mathbf{B}}, \quad B^{mn} \equiv g^{\mu\nu}\partial_\mu\Phi^m\partial_\nu\Phi^n, \quad m,n = 1,2,3, \quad Y \equiv u^\mu\partial_\mu\Phi^0, \tag{28}$$

and $u^\mu\partial_\nu\Phi^m = 0$. U is a generic function and the action (27) is a sub-case of (19).

Since we are dealing with perfect fluids the EMT is $T_{\mu\nu} = \rho u_\mu u_\nu + p\, h_{\mu\nu}$, with

$$\rho = YU_Y - U, \quad p = U - bU_b. \tag{29}$$

In general the equation of state related with Eqs.(29) is not that of a barotropic fluid unless we choose particular forms for U. However, imposing invariance under the one-parameter group of disformal transformation Eq.(20) we find that the action 27 is invariant if

$$U - bU_b = w\big(YU_Y - U\big), \tag{30}$$

that with Eqs.(29) gives $p(\rho) = w\rho$, with of course w constant. Choosing w properly we can obtain all the relevant case in cosmology, like dust ($w = 0$), cosmological constant ($w = -1$) and so on.

The perfect fluid case can be straightforwardly generalized to irrotational perfect fluids as it has been done in [74], where they also consider isotropic and homogeneous solids.