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Gravity/CMT Holography and Non-linear Electrodynamics

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Abstract

The aim of this thesis is the study of a holographic description of condensed matter systems by means of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence which involves models of non-linear electrodynamics. After exposing the main concepts of AdS/CFT and generic non-linear electrodynamics, we study effects of the recently found (non-linear) ModMax electrodynamics on the minimal holographic superconductor in the background of a Schwarzschild-AdS black hole spacetime, and compare the results with the case of Maxwell electrodynamics. This is done by introducing in the background spacetime a charged scalar field minimally coupled to the ModMax electromagnetic field, and by using the correspondence to compute the behaviour of the dual scalar operator in the field theory, whose expectation value is identified with the order parameter, distinguish between the normal and superconducting state. We find that the scalar operator acquires a non-zero vacuum expectation value below a critical temperature and the effect of ModMax is to enhance its value. The critical temperature at which this happens and the optical conductivity of the field theory are unaffected by the introduction of ModMax. We also analyze the effect of ModMax on the Reissner-Nordström AdS planar black hole since it represents the main ingredient for dualizing finite density boundary theories.

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Chapter 1

Introduction

The physical description of natural phenomena is based on two main theories: quantum field theory and general relativity. The first describes three fundamental interactions of matter - weak, strong and electromagnetic - as well as condensed matter systems, where one deals with many body physics and the quantized excitations are not necessarily elementary particles but so called *quasiparticles*. Quantum field theory makes strong and accurate predictions when the degrees of freedom are weakly coupled, since one can rely on perturbation theory. However, when a set of weakly coupled degrees of freedom cannot be identified, the theory loses most of its predictivity.

On the other hand, general relativity describes the remaining fundamental interaction, the gravitational one, and more generally the spacetime geometry, in presence of matter. General relativity is a classical theory. A quantum description of gravity capable of describing the gravitational interaction at very small scales as being mediated by corpuscular quanta called *gravitons* is a long-sought objective which is still missing. There are however theories that are candidates to unify quantum field theory and general relativity. The best known examples are *string theory* and *loop quantum gravity theory*.

String theory, in particular, is a quantum theory of gravity and other fundamental interactions in which elementary objects are one-dimensional strings rather than point-like particles. This allows one to avoid quantum divergences which appear when one tries to quantize general relativity in the framework of the conventional field theory. It is in the context of string theory where in the 1990s a less intuitive, although impressive connection was found, relating quantum field theories and gravity theory. This connection between the two seemingly unrelated areas of physics is known as *anti-de Sitter/conformal field theory correspondence*, or briefly AdS/CFT [1]. The term AdS refers to the anti-de Sitter spacetime, a solution to Einstein's field equations in the presence of a negative cosmological constant, while CFT refers to a special class of quantum theories which are invariant under the conformal group, a generalization of the Lorentz group and scale transformations. Crucially, spacetime dimensions are different in the two theories: the CFT lives in a spacetime with at least one dimension less than AdS.

Relating gravity to quantum field theory in fewer dimensions was an idea prior to AdS/CFT. In 1970s S. Hawking and J. Bekenstein discovered that black holes are endowed with thermodynamic properties as temperature and entropy, this is black holes are intrinsically thermodynamic systems [2, 3]. In particular, the entropy of a black hole is proportional to its surface area, which led 't Hooft [4] to propose a radical interpretation of the entropy law, according to which it must be possible to describe all phenomena within the volume by a set of degrees of freedom which reside on the surface. Encoding the information of a gravitational system in a reduced number of dimensions is the content of the *holographic principle* [4, 5]. For this reason, the AdS/CFT correspondence is interpreted as a concrete realization of the holographic principle according to which the gravity theory on AdS, also called the *bulk theory*, is controlled in some way by field theories on effective spacetime boundaries, and for this referred as *boundary theories*. More generally, AdS/CFT is also referred as *gauge/gravity* duality or simply as holography.

A very first glimpse of why gravity in AdS may be a good candidate to describe conformal field theories and vice-versa is that the conformal algebra is isomorphic to the algebra of the AdS isometry group. The correspondence between AdS and CFTs is understood as a *duality*, in the sense that certain types of quantum field theories admit a (dual) description in terms of stringy quantum gravity in an AdS spacetime, and vice-versa. The actual, quantitative way to relate one description to the other was found by Gubser, Klebanov, Polyakov and independently by Witten [6, 7]. They provided a "dictionary" translating the quantities in one description to corresponding quantities in the dual description. The fact that on the "left hand side" of the correspondence there is a stringy quantum gravity description is what makes AdS/CFT a somewhat mysterious tool, as that of quantum gravity is not yet a fully understood field. This would also limit its applicability. However, in a particular limit - to be discussed in a moment - the quantum gravity description reduces to a classical theory of gravity in AdS spacetime: the fact that a classical theory of gravity can provide a dual description of a quantum field theory is by itself an astonishing result! The above-mentioned limit refers to the number of colours N of the quantum field theory tending to infinity and to the 't Hooft coupling λ of this theory being large too. Since quantum field theories are generically predictive only when they are weakly coupled, the possibility of investigating the strongly coupled regime by means of the correspondence, and in particular of a classical theory of gravity, is extremely appealing. On the other hand, the $N \to \infty$ limit obliges to consider theories which are very different from real world quantum systems, for example from quantum chromodynamics, which is based on $SU(3)$ symmetry group and hence has $N = 3$. So, how can one apply the correspondence to "real world" systems? Said otherwise, how can one use classical gravity to study quantum field theories away from the large N limit? This generically depends on the questions one wants to answer using the correspondence. Remarkably, there are cases where the correspondence seems to work even for theories which are not large-N gauge theories. This is related to the so called "UV-independence" which basically is a statement about the independence of macroscopic phenomena from the details of the microscopic physics. The idea of UV-independence led theoretical physicists to apply the correspondence, or some generalization of it, to study strongly coupled quantum systems in e.g. quantum chromodynamics or condensed matter, although they are not large-N gauge theories.

A typical example of the above discussion is given by the gravity/fluid duality, or holographic hydrodynamics. With this term one refers to the capacity of holography to reproduce hydrodynamics by means of classical gravity. The fundamental equations of hydrodynamics are the Navier-Stokes equations, which govern the macroscopic behaviour of any system respecting translational and rotational symmetry. The structure of these equations is universal, in the sense that they are independent on the specifics of the underlying UV theory, which is important only in determining the coefficients in the equations. As proved by Bhattacharyya, Hubeny, Minwalla and Rangamani in [8], the Navier-Stokes equations for the finite-temperature boundary theory can be obtained, using AdS/CFT, from the dynamics of Einstein gravity. Clearly these will be the Navier-Stokes equations for a large N field theory. As already mentioned, this detail will only determine the specific values of the parameters in the equations, without changing their formal structure. The ratio between the shear viscosity of this strongly interacting fluid and its entropy density has been computed in [9] to be

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} ,$$

which is very small quantity.

Quantum chromodynamics (QCD) is the theory which describes the strong interaction and it is notably a strongly interacting theory at low energies. In this regime perturbation theory is not an option, and one has to rely on more sophisticated field theoretical approaches, as lattice calculations, or numerical methods (e.g. Monte Carlo method) which do not always work well enough. For these reasons QCD has become an important arena where to test AdS/CFT, resulting in an enormous amount of literature (for a review, see [10]). The application of the correspondence to study the strong interaction is known as AdS/QCD. Particularly interesting is the AdS/QCD description of the strongly coupled quark-gluon plasma produced in heavy-ion collisions. Quark-gluon plasma is a state of matter where quarks and gluons are not confined into hadrons, and heavy-ion collisions experiments found that its behaviour is close to a perfect fluid, having a very small viscosity. The value η/s computed using AdS/CFT is compatible with the small experimental value, whereas perturbative computations gave a significantly larger value. This was the first confirmation of the validity of AdS/CFT by an experiment. The fact that the dual description gave a result very close to the experimental value was also suggesting that the quark-gluon plasma is indeed a strongly coupled system, despite being in the deconfined phase [9, 11, 12].

In this work we will focus on applications of the AdS/CFT correspondence to condensed matter physics. This amounts to the study of large number of particles at temperatures low enough so that the quantum effects overcome the thermal fluctuations and quantum mechanics plays a significant role in determining the phase structure. In the zero temperature case, the system may undergo a phase transition by varying an external tuning parameter g, for example an applied magnetic field, pressure, etc. In this case we have a *quantum phase transition*, because it is driven by quantum (i.e. not thermal) fluctuations only. This is usually a continuous phase transition, with the coherence length diverging as g approaches a critical value g_c. At that point the system is scale invariant [13]. This situation is referred to as quantum criticality and it happens at the quantum critical point, which technically speaking is only realized at the absolute zero. However the critical behaviour can dominate in a wider region, called quantum critical region, for instance in presence of a finite temperature T, but with g close to g_c [13, 14]. Of particular interest is the understanding of the behaviour of *strange metals*, a phase of correlated electrons in solids. The name is due to important differences with usual metals described as Fermi-liquid states; one of the main differences lies in that a strange metal resistivity is proportional to the temperature [15], while that of ordinary metals is proportional to the temperature squared. Many condensed matter systems display a strange metal behaviour, in particular heavy-fermion compounds [16] and the cuprate high temperature superconductor compounds.

AdS/CMT, the application of AdS/CFT to condensed matter, is then the attempt to describe condensed matter systems in the vicinity of critical points and which are amenable to a quantum description, hoping that this framework may be valid also more generally. The first important aspect is that AdS/CFT can capture the physics of thermal matter using classical gravity. The dictionary says that this is done by considering a black hole in the AdS spacetime. As already mentioned, black holes are intrinsically thermodynamic systems and their thermodynamical quantities, as temperature and entropy, are directly related to quantities of the boundary theory. For example, the temperature of the boundary theory is the same of the black hole Hawking temperature in the deep interior of the bulk. The introduction of a black hole in the AdS spacetime clearly changes the overall geometry: while the asymptotic, near boundary geometry must remain the one of AdS as to apply the AdS/CFT dictionary, moving away from the boundary reveals a different geometry. This difference with respect to the pure AdS case is related to the renormalization group of the field theory. The important notion here is that in AdS/CMT one often considers *asymptotically* AdS spacetimes due to the presence of a black hole, in order to account for thermodynamic and finite-density properties of the boundary quantum field theory. The latter is then identified with the UV fixed point of the boundary renormalization group flow.

Holography in condensed matter is particularly powerful when dealing with strongly interacting systems at finite-density, either at zero or finite-temperature. With finite-density one means that there is a net charge density, usually associated with a $U(1)$ conserved charge, and a chemical potential. The gravitational dual description is provided by a charged black hole in AdS spacetime - usually with a planar horizon - being a generalization of the well known Reissner-Nordström black hole. Within this model one can holographically compute transport coefficients/Green's functions of primary importance as the optical and Hall conductivity in the critical phase (e.g. [17, 18]), the shear viscosity and many others and account for dissipative effects. An important aspect of the AdS/CFT correspondence is that it relates global symmetries in the CFT boundary theory to corresponding local symmetries (gauge fields) in the bulk. The immediate example of a $U(1)$ gauge field capable of reproducing finite-density in the boundary is the Maxwell field, described by the action

$$S_{\text{Maxwell}} = -\frac{1}{4}\int d^D x F_{\mu\nu}F^{\mu\nu} \ ,$$

where D is the number of bulk dimensions.

Using this particular black hole solution, and via the holographic principle, it was possible to study quantum theories exhibiting properties associated with strange metals [19, 20, 21, 22, 15, 23], Fermi surfaces [24, 25, 26, 27, 28], and superfluids/superconductors [29, 30, 31, 32, 33, 34, 35] without invoking a quasiparticle description. All of these holographic descriptions are obtained by filling the bulk with appropriate fields. For example, Fermi surfaces are reproduced by fermions in the bulk. Remarkably, holography can be used to compute fermions spectral functions of the quantum theory at finite density. Angular resolved photoemission (ARPES) and scanning tunnelling spectroscopy (STS) are two main ways of probing the spectral function of strongly interacting systems in solids; its calculation is of primary importance in order to compare with experiments. Superconductors, on the other hand, are holographically described by introducing a scalar field in the bulk which condenses below a critical temperature.

The power of AdS/CFT to investigate strongly interacting systems in condensed matter motivated people to apply the correspondence starting from various bulk theories, following a bottom-up approach. A particularly interesting modification of the bulk theory is the replacement of the Maxwell action with a non-linear electrodynamics model [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]. Examples of non-linear $U(1)$ gauge fields are Born-Infeld [51], Euler-Heisenberg [52], power-Maxwell, logarithmic and exponential.

Recently a new non-linear model of electrodynamics, named Modified Maxwell or ModMax theory [53], has been found. ModMax is the unique one-parameter non-linear extension of Maxwell theory, preserving all of its symmetries. Contrarily to above mentioned non-linear models, effects of ModMax in the context of AdS/CMT have not been investigated yet. In this thesis, we explore the main concepts of the AdS/CFT and how non-linear effects can affect the physics of the boundary theory. The focus in on ModMax electrodynamics and the consequences of its employment in AdS/CMT, in particular in the context of holographic superconductivity.

The thesis is organized as follows: in chapter 2 we review the main properties of AdS spacetime, with particular focus on black hole solutions and on the relation between AdS isometries and the conformal group. In chapter 3 we explore general notions about non-linear electrodynamics and in particular of ModMax electrodynamics. The solution for a charged black hole with planar horizon in the presence of ModMax electrodynamics is obtained. As already said, this is a master bulk system to describe holographically field theories at finite density. Chapter 4 is devoted to the study of the holographic dictionary, this is the set of rules needed to quantitatively connect bulk quantities with boundary quantities. A rigorous presentation of the dictionary may require the employment of string theory. We will sometimes rely on qualitative and intuitive argument to explain the dictionary content, in order not to depart too much from the main topic of this work. In chapter 5 we apply the correspondence to condensed matter systems. In particular we work out a holographic description of a superconductor, where the dual theory in the bulk is general relativity with minimally-coupled ModMax electrodynamics and a charged scalar field. Effects of ModMax on the formation of the condensate, critical temperature and conductivity are studied. We show that, within the approximations of our model, the ModMax parameter does affect the boundary theory for what concerns the value of the scalar condensate in the superconducting phase but it does not affect the critical temperature and the optical conductivity. Finally, in Chapter 6 we present our conclusions and provide possible future directions.

Notation and conventions

In this work we will always use units where the speed of light c, the reduced Plank constant $\hbar$ and the Boltzmann constant are equal to one:

$$c = \hbar = k_B = 1 \, .$$

With these units every physical quantity has dimensions of a length to the appropriate power. In particular the gravitational constant $G^{(D)}$ in D spacetime dimensions has length dimension

$$[G^{(D)}] = D - 2 \, .$$

The number of spacetime dimensions of the boundary CFT is denoted with $d+1$, where d is the number of spatial dimension of the field theory and the "+1" is for time. The bulk theory lives in a spacetime with an extra dimension with respect to that of the CFT; thus the bulk theory is defined on a $d+2$ dimensional manifold.

For generic tensors in the CFT theory we use Greek indices which assume the values 0,1,2,...,d, where 0 is for time. For generic tensors in the bulk theory we still use Greek indices, but this time they assume values 0,1,2,...,$d+1$. We introduce different indices for tensors in the two theories when confusion between bulk and boundary indices may arise. In both cases the metric is Lorentzian with signature $(-,+,+,\dots)$.

Physical dimensions

Another important constant in the context of AdS/CFT is the AdS curvature radius, denoted with L. This is not a fundamental constant but it appears when relating quantities of the dual descriptions using the dictionary. This constant and G are often combined with other constants. We collect many of the constants and quantities appearing in this thesis together with their length dimension in the following table, where the dimension of the bulk is $d+2$ and that of the boundary is $d+1$:

bulk quantity	symbol	length dimension
AdS curvature radius	L	1
gravitational constant	G	d
black hole mass term	M	$d-1$
black hole charge term	Q	$d-1$
black hole temperature	T	-1
scalar field mass	m	-1
scalar field charge	q	0
metric tensor	ds^2	2
gauge field (one-form)	A	$(2-d)/d$
boundary quantity		
$U(1)$ conserved current	J^μ	$-(d+2)/2$
source of the current	A^{QFT}_μ	$-d/2$
temperature	T	-1

Chapter 2

Aspects of AdS spacetime

The aim of this chapter is to provide a schematic collection of classical results regarding the structure of the anti-de Sitter (AdS) spacetime, serving as a starting point for the following chapters where the AdS/CFT paradigm and its applications to condensed matter will require many of the contents exposed here. As shown in the next section, charts on AdS usually include a time-like coordinate, a radial coordinate and a set of d "equivalent" space-like coordinates. For this reason the dimension of the manifold will be written as $d+2$ and the manifold itself will be denoted with AdS_{d+2} or with similar notations. Even if the contents of this chapter will be applied in the context of AdS/CFT (or AdS/CMT), references to the correspondence are minimized.

We start with the definition of the AdS_{d+2} spacetime together with different coordinate systems, each of which will reveal some of the manifold properties. The isometries of AdS_{d+2} and the connection to the conformal group are studied, providing a first hint for the AdS/CFT correspondence. Next, we analyse more physical aspects of this spacetime like its definition as a solution to Einstein equations and the black hole generalizations. In particular, we make contact with known results, showing that black hole solutions in AdS approach the known Schwarzschild and Reissner-Nordström in asymptotically flat space. Eventually we explore the propagation of a classical scalar field in AdS_{d+2} and its stability. Understanding the near boundary behaviour of the fields in AdS is crucial for exposing the AdS/CFT dictionary. Also, AdS-black hole solutions together with a scalar field are fundamental ingredients for the holographic description of a superconductor.

2.1 Review of AdS spacetime

In the Euclidean $\mathbb{R}^d$ space, one can define the hyperboloid as the set of points whose coordinates $(X_1, \dots, X_d)$ satisfy

$$-X_1^2 + X_2^2 + \cdots + X_d^2 = -L^2 \, . \tag{2.1}$$

The anti-de Sitter (AdS_{d+2}) space is the Lorentzian generalization of the previous hyperboloid. In order to construct it, first consider the flat $\mathbb{R}^{2,d+1}$; this is the set of points $(U, V, X_1, \dots, X_{d+1})$ equipped with the metric

$$ds^2 = -dU^2 - dV^2 + dX_1^2 + \dots dX_{d+1}^2 \, . \tag{2.2}$$

Given a positive constant L with dimensions of a length, the AdS_{d+2} spacetime is then defined as the hypersurface in $\mathbb{R}^{2,d+1}$ satisfying

$$-U^2 - V^2 + X_1^2 + X_2^2 + \cdots + X_{d+1}^2 = -L^2 \, , \tag{2.3}$$

i.e. the points at a given (pseudo)distance from the origin. So we can either see AdS_{d+2} as a kind of hyperboloid in the Euclidean space, or as a pseudosphere in flat $\mathbb{R}^{2,d+1}$. In both cases we are using a definition by embedding, in the sense that AdS_{d+2} is seen as a part of a larger space. The last point of view is the appropriate one when considering the metric, as it is induced by that of $\mathbb{R}^{2,d+1}$.

2.1.1 AdS coordinates

The next step is to introduce new coordinates $u^\mu \equiv (t, \rho, \theta_1, \ldots, \theta_d)$ on the manifold, in such a way that the above equation is satisfied. An immediate choice is

$$\begin{aligned} U &= L\cosh(\rho)\sin(t)\ , \\ V &= L\cosh(\rho)\cos(t)\ , \\ X_i &= L\sinh(\rho)\hat{X}_i\ , \qquad\quad i = 1, \ldots, d+1\ , \end{aligned} \tag{2.4}$$

with $\sum_{i=1}^{d+1}\hat{X}_i^2 = 1$. For instance

$$\begin{aligned} \hat{X}_1 &= \cos(\theta_1)\ , \\ \hat{X}_2 &= \sin(\theta_1)\cos(\theta_2)\ , \\ \hat{X}_3 &= \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\ , \\ &\vdots \\ \hat{X}_d &= \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{d-1})\cos(\theta_d)\ , \\ \hat{X}_{d+1} &= \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{d-1})\sin(\theta_d)\ . \end{aligned} \tag{2.5}$$

The ranges of the coordinates u^μ are $\rho \in [0, \infty)$, $t \in [0, 2\pi]$, $\theta_d \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$ for $i < d$. In these new $d+2$ coordinates, known as *global coordinates*, the metric induced from (2.2) on the hypersurface AdS_{d+2} is

$$ds^2 = L^2(-\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\Omega_d^2)\ , \tag{2.6}$$

with $d\Omega_d^2$ being the metric of the d-sphere. Allowing now the variable t to run from $-\infty$ to ∞ one gets the universal covering of AdS_{d+2}[1], regardless of the embedding used to get here. For simplicity this will be referred as AdS_{d+2} also in the following, with no reference to the universal covering. A similar expression is obtained by rescaling ρ and t by defining $r = \rho L \in [0, \infty)$ and $\tau = tL \in (-\infty, \infty)$, obtaining

$$ds^2 = -\cosh^2\left(\frac{r}{L}\right)d\tau^2 + dr^2 + L^2\sinh^2\left(\frac{r}{L}\right)d\Omega_d^2\ . \tag{2.7}$$

In this form, it is straightforward to see that in the $L \to \infty$ the metric reduces to Minkowski (in spherical coordinates):

$$ds^2 \to ds^2_{\text{Mink}} = -d\tau^2 + dr^2 + r^2 d\Omega_d^2\ . \tag{2.8}$$

Another useful form of the metric, especially when dealing with the boundary of the manifold, is obtained by the change the of coordinate $\sinh(\rho) = \tan(\psi)$ (or, analogously, $\cosh(\rho) = [\cos(\psi)]^{-1}$), leading to

$$ds^2 = \frac{L^2}{\cos^2(\psi)}\left(-dt^2 + d\psi^2 + \sin^2(\psi)d\Omega_d^2\right)\ , \tag{2.9}$$

[1]Some authors take this as the definition of AdS_{d+2} spacetime.

with $\psi \in [0, \pi/2)$.

However, the most useful coordinate systems for the AdS/CFT correspondence, especially for our purposes, are the following:

1. $w^\mu \equiv (z, t, x_1, \dots, x_d)$ defined by

$$\begin{aligned} U &= \frac{1}{2z}\left(L^2 + z^2 - t^2 + \sum_{i=1}^{d} x_i^2\right) , \\ V &= \frac{L}{z} t , \\ X_1 &= \frac{L}{z} x_1 , \\ &\vdots \\ X_d &= \frac{L}{z} x_d , \\ X_{d+1} &= \frac{1}{2z}\left(L^2 - z^2 + t^2 - \sum_{i=1}^{d} x_i^2\right) . \end{aligned} \tag{2.10}$$

   with $t, x_1, \dots, x_d \in (-\infty, \infty)$ and $z \in (0, \infty)$. In these coordinates (*Poincaré coordinates*) the metric is

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2\right) \tag{2.11}$$

   and it is clearly conformal to the flat metric, indeed

$$ds^2 = \Omega^2(z)\left(dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2\right) \quad ; \quad \Omega(z) = \frac{L}{z} . \tag{2.12}$$

   The interior corresponds to large values of z, while small values correspond to the the boundary (to be defined soon). Notice that this patch doesn't cover the full AdS_{d+2} but only one half of it, which is conformal to half of Minkowski.

2. $v^\mu = (r, t, x_1, \dots, x_d)$ related to the previous by $r = L^2/z$; the metric is

$$ds^2 = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2}\left(-dt^2 + \sum_{i=1}^{d} dx_i^2\right) . \tag{2.13}$$

   These are referred as Poincaré coordinates too. Now the interior corresponds to small values of r while approaching the boundary corresponds to taking $r \to \infty$. This coordinate system still covers half of AdS_{d+2}.

2.1.2 AdS boundary and compactification

AdS/CFT correspondence qualitatively asserts that the physics on the boundary of an (asymptotically) AdS_{d+2} spacetime is strictly related to the one of a conformal field theory (CFT) in $d+1$ dimensions. In this sense the CFT can be thought as living on the AdS_{d+2} boundary. Actually, AdS_{d+2} does not have a boundary in the standard sense and with the word “boundary” we refer to its *conformal boundary*. This is obtained by compactification of the original AdS_{d+2} spacetime, resulting in a new manifold $\widehat{\mathrm{AdS}}_{d+2}$. In order to get it, a finite range coordinate system for AdS_{d+2} is adopted, as the one in (2.9). The metric can then be written in the form

$$ds^2 = \frac{1}{\cos^2(\psi)} d\hat{s}^2 \tag{2.14}$$

with $\psi = \pi/2$, corresponding to an infinitely far away point, not included. However the metric $d\hat{s}^2$ is regular at $\psi = \pi/2$. The compactified spacetime $\widehat{\mathrm{AdS}}_{d+2}$ is then defined by the metric $d\hat{s}^2$ with the point $\psi = \pi/2$ included. Thus AdS_{d+2} is conformal to $\widehat{\mathrm{AdS}}_{d+2}$: the original AdS_{d+2} is mapped in the interior of $\widehat{\mathrm{AdS}}_{d+2}$ and the conformal boundary of AdS_{d+2} is defined as the boundary of $\widehat{\mathrm{AdS}}_{d+2}$, i.e. its submanifold having $\psi = \pi/2$.

2.1.3 AdS isometries and relation to the conformal group

AdS_{d+2} spacetime has isometry group $SO(2, d+1)$. This is because AdS_{d+2} is a pseudo-sphere in flat $\mathbb{R}^{2,d+1}$ and, analogously to the 2-sphere embedded in $\mathbb{R}^3$ having $SO(3)$ as isometry group, it inherits the symmetry from the ambient space. Of course considering the universal covering of AdS_{d+2} does not affect this property. The group $SO(2, d+1)$ has $D(D-1)/2$ generators, with $D = (d+1)+2$ being the dimension of the ambient space. It follows that the $SO(2, d+1)$ group accounts for all the possible isometries of AdS_{d+2}. It is useful to have these symmetries acting in a clear way on the local coordinates. Unfortunately, the realization of the $SO(2, d+1)$ isometry on local coordinates is not easy to see. Nevertheless, one can recognize how do particular elements of $SO(2, d+1)$ act on, e.g. Poincaré coordinates. From (2.11), the $ISO(1, d)$ is manifest: this is the isometry group of the Minkowski spacetime $\mathbb{M}^{d+1}$, called Poincaré group, which acts as translations and usual Lorentz transformations on the coordinates $t, x_1, \ldots, x_d$ in (2.11).

Also, the subgroup $SO(1, 1)$ of the original $SO(2, d+1)$ can be shown to act on the Poincaré coordinates w^μ defined in (2.10) as a dilation

$$z \to \lambda z, \; t \to \lambda t, \; x_i \to \lambda x_i \tag{2.15}$$

(see below for an argument). In the coordinates v^μ (see (2.13)), the $SO(1, 1)$ obviously acts as

$$r \to r/\lambda, \; t \to \lambda t, \; x_i \to \lambda x_i \,. \tag{2.16}$$

Argument for the previous statement

Consider the subgroup $SO(1, 1)$ of $SO(2, d+1)$, acting on global coordinates (U, X_{d+1}) as

$$SO(1,1) : \begin{cases} U' &= \gamma(U - \beta X_{d+1}) \\ X'_{d+1} &= \gamma(X_{d+1} - \beta U) \end{cases} , \tag{2.17}$$

which, introducing null coordinates $P_\pm = (U \pm X_{d+1})/\sqrt{2}$, reads

$$SO(1,1) : P'_\pm = \gamma(1 \mp \beta) P_\pm \equiv a^{\pm 1} P_\pm \,. \tag{2.18}$$

On the other hand, using the Poincaré coordinates (2.10), it is possible to infer what is the transformation on the latter. Indeed, also using (2.18),

$$\begin{aligned} P_+ &= \frac{L^2}{z} \xrightarrow{SO(1,1)} a\frac{L^2}{z} \\ P_- &= \frac{1}{z}\left(z^2 - t^2 + \sum_{i=1}^{d} x_i^2\right) \xrightarrow{SO(1,1)} a^{-1}\frac{1}{z}\left(z^2 - t^2 + \sum_{i=1}^{d} x_i^2\right) \; ; \end{aligned} \tag{2.19}$$

from the first one, it necessarily follows that $z \to a^{-1}z$ under $SO(1,1)$, while from the second one the correct transformation is obtained if one takes $t \to a^{-1}t$ and $x_i \to a^{-1}x_i$. Upon renaming a^{-1} with λ, this is exactly (2.15).

□

It is worth mentioning that the isometry group of AdS_{d+2}, i.e. $SO(2, d+1)$, has the same dimension of the conformal group on $(d+1)$-dimensional Minkowski space, $Conf(\mathbb{M}^{d+1})$. Moreover, the algebras of the two groups are isomorphic.

Now we investigate the relation between AdS isometries and the conformal group. The conformal group of $\mathbb{M}^{d+1} \simeq \mathbb{R}^{1,d}$ contains the usual isometries $ISO(1, d)$ with generators P_μ (translations) and $M_{\mu\nu}$ (Lorentz transformations), but also dilations generated by D and special conformal transformations generated by K_μ, with $\mu, \nu = 0, 1, \ldots, d$. In order to work out the conformal algebra, consider its scalar representation. We start with a transformation of the type

$$\begin{aligned} \phi'(x') &= F(\phi(x)) \\ x' &= x + \delta x \end{aligned} \tag{2.20}$$

which we will take to be infinitesimal. This means that if ϵ is the small parameter of the transformation, then $x' - x$ and $\phi'(x) - \phi(x)$ are of order ϵ.

In general, the generator G of a transformation[2] is defined in terms of the "synchronous" variation (comparing fields at the same point) as

$$-i\epsilon G\phi(x) = \phi'(x) - \phi(x) \; . \tag{2.21}$$

Since the left hand side is of order ϵ, we can substitute x with $x' = x + \delta x$, where δx is itself of order ϵ. Thus

$$-i\epsilon G\phi(x) = -i\epsilon G\phi(x') = \phi'(x') - \phi(x') \tag{2.22}$$

at first order. For conformal transformations, the field and its transformed are related by

$$\phi'(x') = \Omega^\Delta(x)\mathcal{D}(R(x))\phi(x) \tag{2.23}$$

where Δ is the scaling dimension of the field ϕ and $\Omega(x)$ is the conformal factor implicitly defined by

$$\frac{\partial x^\mu}{\partial x'^\nu} = \Omega(x) R^\mu{}_\nu(x) \; , \quad R^\mu{}_\nu(x) \in SO(1,d) \; , \tag{2.24}$$

together with $R^\mu{}_\nu(x)$. The requirement that $R^\mu{}_\nu(x)$ is an $SO(1, d)$ matrix makes the previous definition unambiguous. The quantity $\mathcal{D}(R(x))$ is the spin representation of the

[2]Clearly, the transformation belongs to a Lie group.

group element $R(x)$, and so it brings spin indices. For a scalar field, $\mathcal{D}$ maps to the identity, so that equation (2.23) reads

$$\phi'(x') = \Omega^{\Delta}(x)\phi(x) . \tag{2.25}$$

We can now compute the generators for the different conformal transformations:

- Translations ($\delta x^\mu = a^\mu$)

  The infinitesimal parameters here are a^μ and the generators are P_μ. For translations, $\Omega(x) = 1$ and by using (2.25) we find $\phi'(x') = \phi(x)$. The generators in (2.22) are thus given by

  $$-ia^\mu P_\mu \phi(x) = \phi(x) - \phi(x') . \tag{2.26}$$

  It is now straightforward to find the generators. By Taylor expansion we get

  $$\phi(x') = \phi(x) + a^\mu \partial_\mu \phi(x) \implies -a^\mu \partial_\mu \phi(x) = \phi(x) - \phi(x') . \tag{2.27}$$

  and by comparison with (2.26),

  $$P_\mu = -i\partial_\mu . \tag{2.28}$$

- Lorentz ($\delta x^\mu = \omega^\mu{}_\nu x^\nu$, $\omega^{\mu\nu} = -\omega^{\nu\mu}$)

  The infinitesimal parameters are $\omega^{\mu\nu}$, which contract with the antisymmetric[3] generators $M_{\mu\nu}$. The conformal factor $\Omega(x)$ is again 1, and so $\phi'(x') = \phi(x)$. The generators are found similarly to what have been done for translations. Indeed, the definition of generator (2.22) and the Taylor expansion give

  $$\begin{cases} -\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} &= \phi(x) - \phi(x') \\ \phi(x') &= \phi(x) + \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \end{cases} \implies M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) , \tag{2.29}$$

  where we conventionally added a $\frac{1}{2}$ in the first line and the generators $M_{\mu\nu}$ have been already antisymmetrized.

- Dilations ($\delta x^\mu = s x^\mu$)

  The infinitesimal parameter is s and the generator is D. This time the conformal factor is not equal to one, indeed equation (2.24) reads

  $$\Omega(x) R^\mu{}_\nu(x) = \frac{\partial x^\mu}{\partial x'^\nu} = \frac{\partial(x'^\mu - s x^\mu)}{\partial x'^\nu} = \frac{\partial(x'^\mu - s x'^\mu)}{\partial x'^\nu} = (1-s)\delta^\mu_\nu \tag{2.30}$$

  and so, using (2.25), $\phi'(x') = (1-s)^\Delta \phi(x) = (1+s)^{-\Delta}\phi(x)$. The equation for the generator (2.22) is then

  $$-isD\phi(x) = (1+s)^{-\Delta}\phi(x) - \phi(x') = -s\Delta\phi(x) - s x^\mu \partial_\mu \phi(x) , \tag{2.31}$$

  where we expanded at first order in s. It follows that the generator is

  $$D = -ix^\mu \partial_\mu - i\Delta . \tag{2.32}$$

[3] A possible symmetric part will give no contribution in the contraction anyway.

- Special conformal transformation ($\delta x^\mu = 2(x \cdot b)x^\mu - x^2 b^\mu$)

  The infinitesimal parameters are the b^μ and the generators are the K_μ. As for dilation, we first compute $\Omega(x)$ at first order in b^μ as follows:

$$\begin{aligned} \Omega(x) R^\mu{}_\nu(x) = \frac{\partial x^\mu}{\partial x'^\nu} &= \frac{\partial}{\partial x'^\nu}(x'^\mu - \delta x^\mu) \\ &= \frac{\partial}{\partial x'^\nu}(x'^\mu - 2(x' \cdot b)x'^\mu + x'^2 b^\mu) \\ &= \delta^\mu_\nu - 2(x' \cdot b)\delta^\mu_\nu - 2b_\nu x'^\mu + 2x'_\nu b^\mu \\ &= \delta^\mu_\nu - 2(x \cdot b)\delta^\mu_\nu - 2b_\nu x^\mu + 2x_\nu b^\mu \, . \end{aligned} \tag{2.33}$$

  If we collect the coefficients of δ^μ_ν, we get

$$\begin{aligned} \Omega(x) R^\mu{}_\nu(x) &= (1 - 2(x \cdot b))\delta^\mu_\nu - 2(b_\nu x^\mu - x_\nu b^\mu) \\ &= (1 - 2(x \cdot b)) \left(\delta^\mu_\nu + 2\frac{x_\nu b^\mu - b_\nu x^\mu}{1 - 2(x \cdot b)} \right) \, . \end{aligned} \tag{2.34}$$

  The indexed part is clearly an $SO(1, d)$ matrix because it has the form $\delta^\mu_\nu + \omega^\mu{}_\nu$, with $\omega_{\mu\nu}$ antisymmetric. We conclude that

$$\Omega(x) = 1 - 2(x \cdot b) \, , \tag{2.35}$$

  that substituted in (2.25) returns

$$\phi'(x') = (1 + 2(x \cdot b))^{-\Delta}\phi(x) = \phi(x) - 2(x \cdot b)\Delta\phi(x) \, . \tag{2.36}$$

  This in turn can be used in the definition of the generators (2.22), to find

$$\begin{aligned} -ib^\mu K_\mu \phi(x) &= \phi(x) - 2(x \cdot b)\Delta\phi(x) - \phi(x') \\ &= -2b^\mu x_\mu \Delta\phi(x) - 2b^\mu x_\mu x^\nu \partial_\nu \phi(x) + x^2 b^\mu \partial_\mu \phi(x) \end{aligned} \tag{2.37}$$

  where in going from the first to the second line we Taylor expanded $\phi(x')$ around x. Finally, the generators K_μ are

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) - 2ix_\mu \Delta \, . \tag{2.38}$$

Using these explicit expressions for the generators P_μ, $M_{\mu\nu}$, D and K_μ, it is possible to work out the conformal algebra $conf(\mathbb{M}^{d+1})$

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\rho\sigma}] &= i\,(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \\ [D, P_\mu] &= iP_\mu \\ [K_\mu, P_\nu] &= 2i\,(\eta_{\mu\nu} D - M_{\mu\nu}) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i\,(\eta_{\mu\sigma} M_{\nu\rho} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + (\mu \leftrightarrow \nu; \rho \leftrightarrow \sigma)) \\ [D, M_{\mu\nu}] &= 0 \\ [K_\rho, M_{\mu\nu}] &= i\,(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\ [D, D] &= 0 \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, K_\nu] &= 0 \end{aligned} \tag{2.39}$$

where $\eta_{\mu\nu}$ is of course $diag(-1,1,\ldots,1)$. As usual when trying to map one algebra into another, the above generators can be mixed to form new elements of the algebra. Precisely, one defines new antisymmetric generators as

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \\ A_\nu \equiv J_{-1,\nu} &= \frac{1}{2}\left(P_\nu - K_\nu\right) \\ B_\nu \equiv J_{d+1,\nu} &= \frac{1}{2}\left(P_\nu + K_\nu\right) \\ J_{-1,d+1} &= D \end{aligned} \tag{2.40}$$

or, in matrix notation,

$$\begin{pmatrix} 0 & \boxed{\ldots \; J_{-1,\nu} \; \ldots} & D \\ \boxed{\begin{matrix} \vdots \\ J_{\mu,-1} \\ \vdots \end{matrix}} & \boxed{M_{\mu\nu}} & \boxed{\begin{matrix} \vdots \\ J_{\mu,d+1} \\ \vdots \end{matrix}} \\ -D & \boxed{\ldots \; J_{d+1,\nu} \; \ldots} & 0 \end{pmatrix} \tag{2.41}$$

which form the $SO(2,d+1)$ algebra

$$[J_{MN}, J_{RS}] = i\left(\eta_{MS}J_{NR} - (M \leftrightarrow N) - (R \leftrightarrow S) + (M \leftrightarrow N; R \leftrightarrow S)\right) \tag{2.42}$$

with $M,S,R,N = -1,0,1,\ldots,d,d+1$ and $\eta_{MS} = diag(-1,-1,1\ldots,1,1)$, showing that the algebra of the conformal group of the Minkowski spacetime is isomorphic to the isometry group of AdS_{d+2}, this is $conf(\mathbb{M}^{d+1}) \simeq so(2,d+1)$.

2.1.4 AdS as a solution of Einstein's field equations with negative cosmological constant

AdS spacetime is commonly known to be a maximally symmetric solution to the Einstein's field equations with a negative cosmological constant and vanishing energy momentum tensor:

$$R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda)g_{\mu\nu} = 0\,, \tag{2.43}$$

with

$$\Lambda = -\frac{d(d+1)}{2L^2} < 0\,, \tag{2.44}$$

obtained from the action

$$S = \frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\,(R - 2\Lambda)\ . \tag{2.45}$$

It is worth recalling that the gravitational constant G has dimension $[\text{length}]^d$, so that the action is dimensionless.

2.2 Black holes in AdS spacetime

In this section we present some black hole solutions which are asymptotically AdS. In order to make contact with well known results, before considering AdS we review the standard Schwarzschild black hole in asymptotically flat spacetime.

The 4-dimensional standard Schwarzschild black hole is a spherically symmetric solution of Einstein's field equations with vanishing energy momentum tensor and no cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0\,, \tag{2.46}$$

obtained from the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R\,. \tag{2.47}$$

The metric can be obtained by inserting in the Minkowski metric (in spherical coordinates) the *emblackening factor* $f_{\text{Sch}}(r)$ according to

$$ds^2_{\text{Sch}} = -f_{\text{Sch}}(r) dt^2 + \frac{dr^2}{f_{\text{Sch}}(r)} + r^2 d\Omega_2^2 \tag{2.48}$$

with $f_{\text{Sch}}(r) = 1 - 2Gm/r$. Here m is the mass of the black hole, which is found by matching with Newton's gravity in the weak field limit. Clearly, when $r \to \infty$ Minkowski metric is recovered, while for $r = 2Gm$ the emblackening factor vanishes and there is a coordinate singularity. This particular value of r is the horizon radius r_h.

For AdS_{d+2} black holes, we will always take a metric of the form

$$ds^2_{\text{AdS-BH}} = \frac{r^2}{L^2}\left(-f(r)dt^2 + d\Sigma^2\right) + \frac{L^2}{r^2 f(r)} dr^2\,, \tag{2.49}$$

with $d\Sigma^2 \equiv \sum_{i=1}^d h_{ii} dx^i dx^i$. Notice that $g_{ii} = r^2 h_{ii}/L^2$, so that, in matrix notation, the metric tensor $g_{\mu\nu}$ in the coordinates $(t, r, x_1, \ldots, x_d)$ reads

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r^2 f(r)}{L^2} & 0 & 0 & \dots & 0 \\ 0 & \frac{L^2}{r^2 f(r)} & 0 & \dots & 0 \\ 0 & 0 & \frac{r^2}{L^2} h_{11} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{r^2}{L^2} h_{dd} \end{pmatrix}\,. \tag{2.50}$$

We will sometimes improperly refer to $d\Sigma^2$ and h_{ii} as the *horizon metric*. Also, it will be sometimes useful to express equations using the function

$$\tilde{f}(r) = \frac{r^2}{L^2} f(r)\,, \tag{2.51}$$

referred as emblackening factor too.

2.2.1 AdS-Schwarzschild black hole and Hawking temperature

The AdS-Schwarzschild metric is a solution to the Einstein's field equations with negative cosmological constant in $d+2$ dimensions with no energy momentum tensor:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{d(d+1)}{2L^2} g_{\mu\nu} = 0 \tag{2.52}$$

and it can be obtained from the metric ansatz (2.49) mimicking the one for the asymptotically flat case. Notice that the above equation is nothing but (2.43)-(2.44). Here we present the solution for three possible topologies of the horizon[4], labelled by the curvature k. They are spherical horizon, flat horizon and hyperbolic horizon, corresponding to $k = 1, 0, -1$ respectively [54]. The metric is expressed in the form (2.49) as

$$ds^2_{\text{AdS-Sch}} = \frac{r^2}{L^2}\left(-f_{\text{AdS-Sch}}(r)dt^2 + d\Sigma_k^2\right) + \frac{L^2}{r^2}\frac{dr^2}{f_{\text{AdS-Sch}}(r)}\,, \tag{2.53}$$

with $d\Sigma_k^2 = \sum_{i=1}^d h_{ii}dx^i dx^i$ given by

$$d\Sigma_k^2 = \begin{cases} L^2 d\Omega_d^2 & k=1 \quad \text{spherical horizon} \\ d\vec{x}^2 \equiv \Sigma_{i=1}^d dx_i^2 & k=0 \quad \text{flat horizon} \\ L^2 dH_d^2 & k=-1 \quad \text{hyperbolic horizon} \end{cases}\,, \tag{2.54}$$

where $d\Omega_d^2$ and dH_d^2 are the metric on the $d-$sphere S^d and the metric on the $d-$hyperbolic space $\mathbb{H}^d$, respectively. The emblackening factor $f_{\text{AdS-Sch}}(r)$ turns out to be [54, 55]

$$f_{\text{AdS-Sch}}(r) = 1 - \frac{ML^2}{r^{d+1}} + \frac{kL^2}{r^2}\,, \tag{2.55}$$

where M is a constant with dimension $[\text{length}]^{d-1}$ and it is related to a proper mass term m (i.e. having dimension $[\text{length}]^{-1}$) and to the gravitational constant G (with dimension $[\text{length}]^d$) as

$$M = \frac{16\pi G m}{d\text{Vol}_d}\,, \tag{2.56}$$

with Vol_d being the volume of the $d-$sphere/$d-$hyperboloid or equal to $L^{-d}\int d^d x$ (infinite for an infinitely extended plane horizon), for $k = \pm 1, 0$ respectively.

The distinction of different topologies is even more evident when there is no black hole at all ($M = 0$): the $k = 0$ case is again the pure AdS_{d+2} of equation (2.13) but for $k = \pm 1$ the resulting metric is different although it still satisfies Einstein equations (2.52).

The outer horizon, or simply "horizon", corresponds to the largest zero of $f_{\text{AdS-Sch}}$, namely

$$f_{\text{AdS-Sch}}(r_h) = 0 \implies ML^2 = r_h^{d+1} + kL^2 r_h^{d-1}\,. \tag{2.57}$$

For later convenience, we also write the derivative of the emblackening factor at the horizon

$$f'_{\text{AdS-Sch}}(r_h) = (d+1)\frac{ML^2}{r_h^{d+2}} - \frac{2kL^2}{r_h^3} = \frac{(d+1)}{r_h} + \frac{(d-1)kL^2}{r_h^3}\,. \tag{2.58}$$

which is related to the temperature of the black hole, as we now discuss.

As discovered by Hawking in the 1970s [3], black holes are thermal objects and they are associated with a temperature. Indeed, they emit a black body radiation, whose temperature is known as Hawking temperature and it can also be derived from the Euclidean formulation of gravity. In this context, the Hawking temperature is identified with the inverse of the length of the imaginary time circle. In order to compute it, consider a generic AdS black hole metric of the form (2.49), perform a Wick rotation ($t \to i\tau$) and then a near horizon expansion obtained by replacing $r \to r_h$ and

[4] In asymptotically flat spacetimes, the Schwarzschild solution has a spherical horizon but in AdS we extend the name also to other topologies.

$f(r) \to f(r_h) + (r - r_h)f'(r_h) = (r - r_h)f'(r_h)$. This gives the near-horizon Euclidean metric

$$ds^2_{\text{AdS-BH},E} \approx \frac{r_h^2}{L^2} f'(r_h)(r - r_h)d\tau^2 + \frac{r_h^2}{L^2} d\Sigma^2 + \frac{L^2}{r_h^2} \frac{dr^2}{(r - r_h)f'(r_h)} \tag{2.59}$$

which, upon the change of variables

$$\begin{aligned} \tilde{\rho}^2 &= \frac{4L^2(r - r_h)}{(r_h^2 f'(r_h))} \,, \\ \tilde{\tau} &= \frac{r_h^2 f'(r_h)\tau}{(2L^2)} \,, \end{aligned} \tag{2.60}$$

reads

$$ds^2_{\text{AdS-BH},E} \approx d\tilde{\rho}^2 + \tilde{\rho}^2 d\tilde{\tau}^2 + \frac{r_h^2}{L^2} d\Sigma^2 \,, \tag{2.61}$$

resembling in part the $\mathbb{R}^2$ metric in polar coordinates, with $\tilde{\rho}$ being the radius and $\tilde{\tau}$ the angular variable. The Hawking temperature is found by requiring $\tilde{\tau}$ to have periodicity $\Delta\tilde{\tau} = 2\pi$, which, using the second of (2.60), corresponds to a periodicity of the imaginary time $\beta \equiv \Delta\tau = \frac{4\pi L^2}{r_h^2 f'(r_h)}$. The temperature $T_{\text{AdS-BH}} = \beta^{-1}$ of an AdS black holes is then

$$T_{\text{AdS-BH}} = \frac{r_h^2 f'(r_h)}{4\pi L^2} = \frac{\tilde{f}'(r_h)}{4\pi} \,, \tag{2.62}$$

where we used the definition of the function $\tilde{f}(r)$ given in (2.51).

For the AdS-Schwarzschild case, the emblackening factor is $f_{\text{AdS-Sch}}$ (equation (2.55)) and the horizon metric $d\Sigma^2$ is given by (2.54). The temperature is easily computed just by substituting (2.58) in (2.62):

$$T_{\text{AdS-Sch}} = \frac{1}{4\pi L^2} \left((d+1)r_h + \frac{kL^2(d-1)}{r_h} \right) . \tag{2.63}$$

Up to here we presented the solution of the AdS-Schwarzschild black hole for three values of k to emphasize that the difference with the standard Schwarzschild lies both in the horizon topology and in the presence of the cosmological constant.

For $k = 1$ and $d = 2$, the Schwarzschild solution (2.48) is obtain as the $L \to \infty$ limit of the AdS-Schwarzschild metric. Indeed, in this limit the Einstein equations (2.52) reduces to (2.46), admitting the 4-dimensional Schwarzschild metric as a solution. To explicitly check this, we start from the AdS_4 black hole (2.53)-(2.55) with, of course, $k = 1$ and $d = 2$:

$$ds^2_{\text{AdS-Sch}} = \left(\frac{r^2}{L^2} - \frac{M}{r} + 1 \right) dt^2 + \left(\frac{r^2}{L^2} - \frac{M}{r} + 1 \right)^{-1} dr^2 + r^2 d\Omega_2^2 \,. \tag{2.64}$$

From (2.56) we find $\text{Vol}_2 = 4\pi$ and

$$M = 2Gm \,. \tag{2.65}$$

Substituting this value of M in (2.64), it is straightforward to see that the $L \to \infty$ limit correctly gives the standard Schwarzschild solution (2.48). Also in this limit, from (2.57) we find that $M = r_h$, consistently with the last equation and with the temperature of a standard Schwarzschild black hole, which is found taking the limit of (2.63):

$$T_{\text{Sch}} = \frac{1}{8\pi Gm} \,. \tag{2.66}$$

This last result is particularly important; for a standard Schwarzschild spacetime the temperature decreases as the mass (and the radius) increases, while for an AdS-Schwarzschild with flat horizon, the temperature is an increasing function of the radius, as can be seen from (2.63).

In the remainder of this work we will mostly focus on flat horizons ($k = 0$); this is a common choice in the context of gauge/gravity duality, especially for condensed matter applications. Black holes with a flat horizon are known as *black branes*.

2.2.2 AdS-Reissner-Nordström black hole

Consider the Einstein-Maxwell theory in presence of negative cosmological constant, whose action is[5]

$$S = \int d^{d+2}x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R + \frac{d(d+1)}{L^2}\right) - \frac{1}{4g_F^2}F_{\mu\nu}F^{\mu\nu}\right] . \tag{2.67}$$

The classical equations of motion admit an asymptotically AdS charged black hole/brane solution for different k's, being a generalisation of the Reissner-Nordström (RN) black hole in asymptotically flat spacetime. The "structure" of the metric is the same as in the AdS-Schwarzschild case

$$ds^2_{\text{AdS-RN}} = \frac{r^2}{L^2}(-f_{\text{AdS-RN}}(r)dt^2 + d\Sigma_k^2) + \frac{L^2}{r^2}\frac{dr^2}{f_{\text{AdS-RN}}(r)} \tag{2.68}$$

with $d\Sigma_k^2$ defined as in (2.54) and the emblackening factor is now given by [56]

$$f_{\text{AdS-RN}}(r) = 1 + \frac{Q^2L^2}{r^{2d}} - \frac{ML^2}{r^{d+1}} + \frac{kL^2}{r^2} . \tag{2.69}$$

The constants Q and M represent the charge and the mass and they both have the dimensions of $[\text{length}]^{d-1}$. To make contact with RN in asymptotically flat spacetime, it is useful to keep in mind that, similarly to (2.56), $Q^2 \propto Gq^2$,[6] with q an "honest" charge with dimensions $[\text{length}]^{(d-2)/2}$. Regarding the gauge field, a solution is provided by [56, 24]

$$A_t = \mu\left(1 - \frac{{r_h}^{d-1}}{r^{d-1}}\right) , \qquad \text{with } \mu \equiv \frac{g_F Q}{2c_d\sqrt{\pi G}r_h^{d-1}} , \ c_d \equiv \sqrt{\frac{2(d-1)}{d}} . \tag{2.70}$$

As usual, the metric is asymptotically (locally isometric to) AdS and the horizon is identified with the events at fixed $r = r_h$, with r_h largest positive root of $f_{\text{AdS-RN}}$:

$$f_{\text{AdS-RN}}(r_h) = 0 \implies ML^2 = r_h^{d+1} + \frac{Q^2L^2}{r_h^{d-1}} + kL^2r_h^{d-1} . \tag{2.71}$$

We now focus on the black brane solution, this is $k = 0$. In this case the above equation is accompanied by the inequality

$$ML^2 \geq \frac{2d}{d-1}\left(\frac{d-1}{d+1}\right)^{(d+1)/2d}(QL)^{(d+1)/d} , \tag{2.72}$$

[5] The normalization of the fields is the one used in [55].

[6] The proportionality constant is set by the actual definition of the charge.

needed in order not to have any naked singularity. When the inequality is saturated, we have the extremal black brane solution such that equation (2.71) is now

$$\begin{aligned} \frac{2d}{d-1}\left(\frac{d-1}{d+1}\right)^{(d+1)/2d}(QL)^{(d+1)/d} &= r_h^{d+1}+\frac{Q^2L^2}{r_h^{d-1}} \\ \implies r_h\equiv r_* &= \left(\frac{d-1}{d+1}\right)^{1/2d}(QL)^{1/d} \end{aligned} \tag{2.73}$$

This particular black brane has zero temperature but finite mass and charge. Indeed, in the general case,

$$f'_{\text{AdS-RN}}(r_h)=\frac{d+1}{r_h}-\frac{Q^2L^2}{r_h^{2d+1}}(d-1) \tag{2.74}$$

which substituted in (2.62) gives the Hawking temperature for the AdS-RN black brane

$$T_{\text{AdS-RN}}=\frac{r_h^2 f'_{\text{AdS-RN}}(r_h)}{4\pi L^2}=\frac{(d+1)r_h}{4\pi L^2}\left(1-\frac{(d-1)Q^2L^2}{(d+1)r_h^{2d}}\right). \tag{2.75}$$

From the rightmost term it is evident that the temperature vanishes in the extremal case $r_h=r_*$. The fact that for an extremal black brane the first derivative of the emblackening factor vanishes at the horizon strongly affects the near horizon expansion, as now the dominant term is the one containing $f''_{\text{AdS-RN}^*}(r_*)$.[7] Explicitly one has

$$\begin{aligned} f''_{\text{AdS-RN}^*}(r_*) &= \frac{2d(d+1)}{r_*^2}, \\ ds^2_{\text{AdS-RN}^*} &\approx -\frac{d(d+1)(r-r_*)^2}{L^2}dt^2+\frac{L^2}{d(d+1)(r-r_*)^2}dr^2+\frac{r_*^2}{L^2}d\vec{x}^2 . \end{aligned} \tag{2.76}$$

The change of coordinates

$$r-r_*=\lambda L_2^2/\zeta\,,\qquad t=\lambda^{-1}\theta\,,\qquad \lambda\to 0\,, \tag{2.77}$$

with $L_2=L/\sqrt{d(d+1)}$, reveals that the near horizon metric of an extremal black brane is that of $\text{AdS}_2\times\mathbb{R}^d$

$$ds^2_{\text{AdS-RN}^*}=\frac{L_2^2}{\zeta^2}\left(-d\theta^2+d\zeta^2\right)+\frac{r_*^2}{L^2}d\vec{x}^2 . \tag{2.78}$$

with the AdS_2 sector given in Poincaré coordinates (2.11).

Finally, to make contact with known results, we briefly discuss the asymptotically flat RN limit. The limit is $L\to\infty$ in $d=2$, $k=1$ and $g_F=1$ for simplicity. The metric is obtained from the limit of (2.68) and (2.69)

$$ds^2_{\text{RN}}=\left(1-\frac{M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega_2^2 \tag{2.79}$$

while the A_t is the same of (2.70), but we remove the constant term and change sign (this does not affect the energy momentum tensor and so Einstein equations are still satisfied):

$$A_t=\frac{Q}{2\sqrt{\pi G}r}. \tag{2.80}$$

[7]The subscript AdS-RN* indicates the extremal solution.

We can explicitly write down the relation between Q and the dimensionless q. Consider a hypersurface Σ enclosing the charge, basically the filled 2-sphere centered in $r = 0$ with infinite radius. The timelike unit normal vector is $\sigma^\mu = (\frac{L}{r}, 0, 0, 0)$ (indeed $\sigma^t\sigma^t g_{tt} = -1$ at radial infinity). Then consider the boundary of Σ, the 2-sphere $\partial\Sigma$, with outward normal unit vector given by $n^\nu = (0, \frac{r}{L}, 0, 0)$ (indeed $n^r n^r g_{rr} = 1$ at radial infinity). If $\gamma^{(\partial\Sigma)}$ denotes the metric on $\partial\Sigma$, then from Gauss law [57]

$$\begin{aligned} q &\equiv \int_{\partial\Sigma} d^2x \sqrt{|\gamma^{(\partial\Sigma)}|} \sigma^\mu n^\nu \left(-2\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\right) \\ &= \int_{\partial\Sigma} d\theta d\varphi \, r^2 \sin\theta \, \sigma^t n^r F_{tr} \,, \end{aligned} \tag{2.81}$$

where the term in brackets is the generalization of $F_{\mu\nu}$ valid also for non-linear electrodynamics.

$$q = \int_{\partial\Sigma} d\theta d\varphi \, r^2 \sin\theta \left(\frac{Q}{2\sqrt{\pi G} r^2}\right) = \frac{2\sqrt{\pi}Q}{\sqrt{G}} \implies Q = q\sqrt{\frac{G}{4\pi}} \,. \tag{2.82}$$

Substituting in (2.79) and (2.80) and using (2.65) for the mass term, we get the RN solution

$$ds^2_{\text{RN}} = \left(1 - \frac{2Gm}{r} + \frac{\frac{Gq^2}{4\pi}}{r^2}\right) dt^2 + \left(1 - \frac{2Gm}{r} + \frac{\frac{Gq^2}{4\pi}}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \,, \tag{2.83}$$

$$A_t = \frac{q}{4\pi r} \,. \tag{2.84}$$

2.3 Exact solution for a scalar field in AdS

In this section we consider a pure AdS_{d+2} with metric given by (2.13) and study the behaviour of a massive scalar field in this fixed background, i.e. the backreaction on the spacetime is ignored. We will have more to say about the mass of the scalar field at end of this section. The coordinates x_i are collectively denoted $\mathbf{x}$. The scalar field action is

$$\begin{aligned} S_\phi &= \frac{1}{2}\int_{\text{AdS}} dr\, dt\, d^dx \sqrt{-g}\left[-(\nabla\phi)^2 - m^2\phi^2\right] + S_{\text{ct}} \\ &= \int_{\text{AdS}} dr\, dt\, d^dx \sqrt{-g}\, \frac{1}{2}\phi(\nabla_\mu\nabla^\mu - m^2)\phi + S_{\text{bdy}} + S_{\text{ct}}, \end{aligned} \tag{2.85}$$

and the explicit expression for the metric components is

$$g_{\mu\nu} = diag\left(-\frac{r^2}{L^2}, \frac{L^2}{r^2}, \frac{r^2}{L^2}, \dots\right) . \tag{2.86}$$

Here we are only concerned with solving the equation of motion and so the boundary term will not be considered, as its presence does not change the equations. The term S_{ct} is the counterterm action, and it is inserted to have a finite action, but does not contribute to the equations. The dynamics of the field $\phi(r, t, \mathbf{x})$ is dictated by the Klein-Gordon like equation

$$\nabla_\mu\nabla^\mu\phi - m^2\phi = 0 \,. \tag{2.87}$$

We anticipate that the mass parameter m^2 can also be negative, but not too much. Using the relation $\nabla_\mu \nabla^\mu \phi = (\sqrt{-g})^{-1}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)$, straightforward calculations lead to the equation

$$\left(r^2\partial_r^2 + (2+d)r\partial_r - \frac{L^4}{r^2}(\partial_t^2 - \partial_{\mathbf{x}}^2) - m^2L^2\right)\phi = 0\,. \tag{2.88}$$

At this point it is convenient to Fourier transform the t and $\mathbf{x}$ variables, so that $\phi(r,t,\mathbf{x}) \to \phi(r,\omega,\mathbf{k})$ and $(\partial_t^2 - \partial_{\mathbf{x}}^2 \to -\omega^2 + \mathbf{k}^2 = k^2)$, resulting in the following equation

$$\left[\partial_r^2 + \frac{d+2}{r}\partial_r - \left(\frac{k^2L^4}{r^4} + \frac{m^2L^2}{r^2}\right)\right]\phi = 0\,. \tag{2.89}$$

The change of variable $z = kL^2/r$ will show that this is nothing but a "cousin" of the Bessel equation:

$$\left[z^2\partial_z^2 - dz\partial_z - \left(z^2 + m^2L^2\right)\right]\phi = 0 \tag{2.90}$$

the difference lying in that the middle term in the Bessel equation is $+z\partial_z$. To solve this we look for a solution of the form $\phi = z^\alpha B_\nu(z)$ with $B_\nu(z)$ the solution of the Bessel equation

$$\left[z^2\partial_z^2 + z\partial_z - (z^2 + \nu^2)\right]B_\nu = 0\,. \tag{2.91}$$

The parameter ν^2 has not to be chosen equal to m^2L^2 as equation (2.90) at first sight would suggest. Instead, set $\nu^2 = m^2L^2 + \beta^2$. By substitution in (2.90) one gets

$$\left[z^2\partial_z^2 - dz\partial_z - (z^2 + \nu^2 - \beta^2)\right]z^\alpha B_\nu = 0 \tag{2.92}$$

and on using (2.91)

$$z^{\alpha+1}\partial_z B_\nu\left(2\alpha - (d+1)\right) + z^\alpha B_\nu\left(\beta^2 + \alpha^2 - \alpha(d+1)\right) = 0\,. \tag{2.93}$$

A solution can be easily found just by requiring that the two terms separately vanish: from the first term, α is found to be $(d+1)/2$, which, once set in the second term, determines $\beta = (d+1)^2/4$. The solution is then

$$\phi = z^{(d+1)/2}B_\nu(z) \qquad \text{with } \nu = \sqrt{\frac{(d+1)^2}{4} + m^2L^2}\,. \tag{2.94}$$

Recall that z depends on k; for $k^2 > 0$ equation (2.91) is properly called *modified* Bessel equation, whose solution is

$$B_\nu(z) = a_K K_\nu(z) + a_I I_\nu(z)\,, \tag{2.95}$$

with $K_\nu(z), I_\nu(z)$ being modified Bessel functions and a_K, a_I being constants. On the other hand, if $k^2 < 0$, then z is purely imaginary and the solution is given by

$$B_\nu(z) = a_+ H_\nu^{(1)}(\mathrm{Im}z) + a_- H_\nu^{(2)}(\mathrm{Im}z) \tag{2.96}$$

where $H_\nu^{(1)}, H_\nu^{(2)}$ are the Hankel functions and a_+, a_- are constants. In this case, $\mathrm{Im}z = \sqrt{\omega^2 - \mathbf{k}^2}L^2/r$. At this point it is worth to note that $I_\nu(z)$ diverges approaching the interior of AdS$_{d+2}$, corresponding to $z \to \infty$ or, equivalently, $r \to 0$. For this reason one usually sets $a_I = 0$. We then have three independent solutions, one for $k^2 > 0$ and two for $k^2 < 0$.

We now restore the original variable $r = kL^2/z$ and study the asymptotic ($r \to \infty$) behaviour of the solutions. Starting from the $k^2 > 0$ case (2.95), with $a_I = 0$, the solution (2.94) is

$$\phi = a_K \left(\frac{kL^2}{r}\right)^{(d+1)/2} K_\nu \left(\frac{kL^2}{r}\right) \qquad (k^2 > 0) \,, \tag{2.97}$$

with near boundary behavior obtained by the known expansion of the modified Bessel function

$$K_\nu \left(\frac{kL^2}{r}\right) = \left(\frac{kL^2}{r}\right)^{-\nu} \frac{\Gamma(\nu)}{2} + \left(\frac{kL^2}{r}\right)^{\nu} \frac{\Gamma(-\nu)}{2} + \dots \qquad (r \to \infty) \,, \tag{2.98}$$

resulting in

$$\phi = A(k) \left(\frac{r}{L}\right)^{-\frac{d+1}{2}+\nu} + B(k) \left(\frac{r}{L}\right)^{-\frac{d+1}{2}-\nu} \qquad (k^2 > 0, r \to \infty) \,. \tag{2.99}$$

For the case $k^2 < 0$ we consider the two independent solutions separately. The first one is proportional to $H^{(1)}_\nu \left(\sqrt{\omega^2 - \mathbf{k}^2} L^2/r\right)$, whose asymptotic expansion is given by

$$H^{(1)}_\nu \left(\frac{qL^2}{r}\right) = \left(\frac{qL^2}{2r}\right)^{-\nu} \left(-\frac{i\Gamma(\nu)}{\pi}\right) + \left(\frac{qL^2}{2r}\right)^{\nu} \left(-\frac{i\Gamma(-\nu)}{\pi} e^{-i\pi\nu}\right) + \dots \qquad (r \to \infty) \,. \tag{2.100}$$

For the solution proportional to $H^{(2)}_\nu \left(\sqrt{\omega^2 - \mathbf{k}^2} L^2/r\right)$ the expansion is similar:

$$H^{(2)}_\nu \left(\frac{qL^2}{r}\right) = \left(\frac{qL^2}{2r}\right)^{-\nu} \left(\frac{i\Gamma(\nu)}{\pi}\right) + \left(\frac{qL^2}{2r}\right)^{\nu} \left(\frac{i\Gamma(-\nu)}{\pi} e^{i\pi\nu}\right) + \dots \qquad (r \to \infty) \,. \tag{2.101}$$

In both cases the solution ϕ has a near boundary behavior of the form (2.99), but this time with $k^2 < 0$. In the context of AdS/CFT, the exponents in (2.99) are usually written in terms of

$$\Delta_\pm = \frac{d+1}{2} \pm \sqrt{\frac{(d+1)^2}{4} + m^2 L^2} \,, \tag{2.102}$$

so that the asymptotic behavior can be written as

$$\phi = \frac{\phi_+}{r^{\Delta_+}} + \frac{\phi_-}{r^{\Delta_-}} \qquad (r \to \infty) \,. \tag{2.103}$$

Here we did not specify the argument as both $\phi(r, t, \mathbf{x})$ and the Fourier transformed $\phi(r, \omega, \mathbf{k})$ have this asymptotic behaviour. That this works also for $\phi(r, t, \mathbf{x})$ can be checked by substitution in (2.88) and neglecting the r^{-2} term. Note that $\Delta_- \leq \Delta_+$ and so the ϕ_- piece is the *leading* term at the boundary (also called *non-normalizable*[8] or *slow falloff*), while the ϕ_+ term is *subleading* (also called *normalizable* or *fast falloff*). For future convenience, notice that $\Delta_\pm = d + 1 - \Delta_\mp$.

About the Breitenlohner-Freedman bound on m^2

In [58] Breteinlohner and Freedman showed that a negative mass squared scalar field propagating in AdS does not necessarily give rise to an instability (tachyon propagation).

[8]It is often the case that this term is actually non-normalizable. Nonetheless, there are cases in which the leading behaviour is normalizable, and for which this terminology is not very appropriate.

Indeed, while in Minkowski spacetime a negative m^2 makes the potential not bounded from below, AdS geometry contributes with a "binding" term to the effective potential, with the result that m^2 can be negative without causing an instability, provided that the square root in (2.102) is real. Explicitly, in order to avoid instability, the following inequality must hold

$$m^2 L^2 \geq -\frac{(d+1)^2}{4}\,, \tag{2.104}$$

known as Breitenlohner-Freedman (BF) bound.

Chapter 3

Non-linear electrodynamics and ModMax

Models of non-linear electrodynamics have been extensively studied and applied in different areas, from cosmology [59, 60] to condensed matter (e.g. [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]). Properties of strongly correlated condensed matter system at finite density can be holographic captured by introducing a $U(1)$ gauge field in the bulk. This does not necessarily need to be linear, so that one can try to holographically simulate different behaviours of the strongly interacting system by means of a non-linear $U(1)$ field. Examples of quantities which are susceptible to non-linear electromagnetic effects are the conductivity, the Hall angle, phase transitions and Meissner effect in superconductors.

In this chapter we analyze general properties of non-linear models of electrodynamics. In order to make contact with well known results, we start from Maxwell case and see how one can extend the theory to construct non-linear models, including the new ModMax theory. We will mostly consider a fixed four dimensional Minkowski metric and, as in the previous chapter, references to the AdS/CFT correspondence are minimized. Then in section (3.3) we will derive asymptotically AdS black hole solutions in presence of ModMax electrodynamics and make some comments on what the correspondence will imply in that specific case.

3.1 Maxwell electrodynamics and its non-linear extensions

We start from Maxwell electrodynamics in absence of sources. The main ingredient is the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, , \tag{3.1}$$

using which one constructs the action in a manifestly Lorentz-invariant and gauge-invariant form[1]:

$$S = -\frac{1}{4} \int d^4x \; F_{\mu\nu} F^{\mu\nu} \, . \tag{3.2}$$

[1]Sometimes the field strength is rescaled by the coupling g_F; this leads to an extra g_F^2 in the denominator below. This was the convention used in chapter 2 when discussing the AdS-RN black hole.

Equation (3.1) is actually the solution of the Bianchi identity. Putting together the equations that are derived from (3.2) and the Bianchi identity, we get, respectively

$$\begin{aligned} &\partial_\mu F^{\mu\nu} = 0 \,, \\ &\partial_\mu {}^*F^{\mu\nu} = 0 \,, \qquad\qquad {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,. \end{aligned} \tag{3.3}$$

Free Maxwell theory posses four important symmetries: Lorentz invariance, gauge invariance, conformal invariance and electromagnetic duality invariance. With the latter we mean the $SO(2)$ transformation

$$\begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} F'_{\mu\nu} \\ {}^*F'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \tag{3.4}$$

valid for the Maxwell case. This definition will be generalized to the case of non-linear models. The Lorentz invariance is easily seen from all the indices in the action (3.2) being contracted. The gauge invariance is a consequence of the fact that the action (3.2) is built with $F_{\mu\nu}$. The latter is given by (3.1), which is invariant under gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$. Invariance of the equations of motion under conformal transformation have been shown in [61] while invariance of the equations of motion under electromagnetic duality is seen by applying (3.4) on (3.3). In this way the Maxwell equations (3.3) are transformed into

$$\begin{aligned} \partial_\mu F^{\mu\nu}\cos\theta + \partial_\mu {}^*F^{\mu\nu}\sin\theta &= 0 \,, \\ -\partial_\mu F^{\mu\nu}\sin\theta + \partial_\mu {}^*F^{\mu\nu}\cos\theta &= 0 \,, \end{aligned} \tag{3.5}$$

which is equivalent to (3.3).

We now move to non-linear extensions of Maxwell theory. The first properties that such extensions must have are Lorentz and gauge invariance. These requirements are satisfied if the action is a functional of the two independent Lorentz invariants, a scalar and a pseudoscalar, defined as

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \,, \qquad \mathcal{P} = \frac{1}{2} F_{\mu\nu} {}^*F^{\mu\nu} \,. \tag{3.6}$$

where ${}^*F^{\mu\nu}$ is still defined by the third equation in (3.3). Hence, the generic action for a non-linear model of electrodynamics takes the form

$$S_{\text{NED}} = \int d^4x \, \mathcal{L}_{\text{NED}}(\mathcal{S}, \mathcal{P}) \,, \tag{3.7}$$

where NED stays for non-linear electrodynamics. We will omit this subscript when it does not cause any confusion. The gauge invariance is again a consequence of the definition of the field strength which is the building block of the NED action. Non-linear electrodynamics models should satisfy certain conditions. For instance, causality and unitarity of the theory constrain a possible form of its Lagrangian. Also, the NED is usually expect to reduce to the Maxwell electrodynamics in some limit, e.g. for weak fields. The generalization of the equation of motion for a non-linear model is obtained by replacing $F^{\mu\nu}$ in the first of (3.3) with (see e.g. [62])

$$E^{\mu\nu} = -2\frac{\partial \mathcal{L}_{\text{NED}}}{\partial F_{\mu\nu}} = -2\left(\mathcal{L}_{\mathcal{S}} F^{\mu\nu} + \mathcal{L}_{\mathcal{P}} {}^*F^{\mu\nu}\right) \tag{3.8}$$

where $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}_\mathcal{P}$ are the partial derivatives of $\mathcal{L}_{\rm NED}$ with respect to the invariants $\mathcal{S}$ and $\mathcal{P}$ respectively. It is straightforward to check that $E^{\mu\nu} = F^{\mu\nu}$ in the Maxwell case. The equations of motion are then

$$\begin{aligned} \partial_\mu E^{\mu\nu} &= 0\,, \\ \partial_\mu {}^*F^{\mu\nu} &= 0\,. \end{aligned} \tag{3.9}$$

The first one is, of course, the Euler-Lagrange equation derived from the action (3.7).

Keeping the discussion general, it is interesting to characterize non-linear models exhibiting electromagnetic duality invariance. We say that a there is duality invariance if the equations of motion are invariant under

$$\begin{pmatrix} E_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} E'_{\mu\nu} \\ {}^*F'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix}, \tag{3.10}$$

and

$$E'_{\mu\nu} = -2\frac{\partial \mathcal{L}_{\rm NED}(F')}{\partial F'^{\mu\nu}}\,. \tag{3.11}$$

Here F' is in general a function of F and $\mathcal{L}_{\rm NED}(F')$ has the same form as $\mathcal{L}_{\rm NED}(F)$. Notice that equations (3.10) and (3.11) are in general not equivalent. Indeed, E' defined as the upper component of the $SO(2)$ transformed doublet will not in general satisfy the constitutive relation (3.11) similar to (3.8). The duality invariance requires that the constitutive relation must be satisfied. This restricts the form of the Lagrangian $\mathcal{L}_{\rm NED}$. As proven in [62], the theory has duality invariance if it satisfies

$$4(\mathcal{L}_\mathcal{S}^2 - \mathcal{L}_\mathcal{P}^2)\mathcal{P} - 8\mathcal{L}_\mathcal{S}\mathcal{L}_\mathcal{P}\mathcal{S} = \mathcal{P}. \tag{3.12}$$

The Maxwell Lagrangian $\mathcal{L}_{\rm Maxwell} = -\mathcal{S}/2$ satisfies this equation.

3.1.1 Born-Infeld electrodynamics

Born-Infeld electrodynamics [51] was the first proposed model of non-linear electrodynamics. It was constructed in order to have a finite electron self energy, but it ended up being used in many different context, in particular in AdS/CMT. The Lagrangian density of Born-Infeld is

$$\begin{aligned} \mathcal{L}_{\rm BI} &= T - \sqrt{T^2 + \frac{T}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}\left(F_{\mu\nu}{}^*F^{\mu\nu}\right)^2} \\ &= T - \sqrt{T^2 + T\mathcal{S} - \frac{\mathcal{P}^2}{4}}\,. \end{aligned} \tag{3.13}$$

The parameter T is the characteristic parameter of the theory. It has dimension of an energy density. The above Lagrangian satisfies (3.12) and so the theory is duality invariant. However, it is not scale invariant (and so neither conformal invariant). If we rescale the fields, the whole Lagrangian does not get rescaled uniformly and therefore cannot compensate the rescaling of the measure in the action integral. In the weak field limit (equivalent to taking $T \to \infty$) $\mathcal{L}_{\rm BI}$ can be expanded as

$$\mathcal{L}_{\rm BI\,|\,weak\ fields} \approx T - T\left(1 + \frac{\mathcal{S}}{2T}\right) = -\frac{\mathcal{S}}{2}\,, \tag{3.14}$$

which is the Maxwell Lagrangian.

3.2 ModMax electrodynamics

In 2020 Bandos, Lechner, Sorokin and Townsend found [53] a non-linear extension of Maxwell electrodynamics being both conformal and duality invariant (and, of course, gauge invariant). It is the unique one-parameter extension of Maxwell electrodynamics preserving all of its symmetries. Using the electromagnetic invariants (3.6) the Lagrangian density of the ModMox theory is written as

$$\mathcal{L}_{\text{ModMax}} = -\frac{\cosh\gamma}{2}\mathcal{S} + \frac{\sinh\gamma}{2}\sqrt{\mathcal{S}^2+\mathcal{P}^2}\,. \tag{3.15}$$

The Lagrangian of the Maxwell theory is recovered for $\gamma = 0$, while conditions of causality and unitarity require $\gamma \geq 0$ [53, 63], at least in flat spacetime. It is worth stressing that Maxwell theory is not recovered in the weak field limit, as happens for many other non-linear models of electrodynamics, but in the limit of vanishing coupling constant. The equations of motion are (3.9)

$$\nabla_\mu\left(-2\frac{\partial\mathcal{L}_{\text{ModMax}}}{\partial F_{\mu\nu}}\right) = 0\,, \tag{3.16}$$

$$\nabla_\mu{}^*F^{\mu\nu} = 0\,, \tag{3.17}$$

where we generalized them to curved spacetimes by using covariant derivatives. They are invariant under the duality $SO(2)$ rotations [53, 64, 65] given by (3.10) and (3.11). However, this is not a symmetry of the ModMax Lagrangian. We can explicitly prove the duality invariance by checking that (3.12) is sastisfied. The Lagrangian derivatives are

$$\begin{aligned}\mathcal{L}_\mathcal{S} &\equiv \frac{\partial\mathcal{L}_{\text{ModMax}}}{\partial\mathcal{S}} = \frac{1}{2}\left(-\cosh\gamma + \sinh\gamma\frac{\mathcal{S}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\right)\,,\\ \mathcal{L}_\mathcal{P} &\equiv \frac{\partial\mathcal{L}_{\text{ModMax}}}{\partial\mathcal{P}} = \frac{\sinh\gamma}{2}\frac{\mathcal{P}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\,,\end{aligned} \tag{3.18}$$

Evaluation of the first term in (3.12) gives

$$4(\mathcal{L}_\mathcal{S}^2 - \mathcal{L}_\mathcal{P}^2)\mathcal{P} = \cosh^2\gamma\mathcal{P} + \sinh^2\gamma\left(\frac{\mathcal{S}^2-\mathcal{P}^2}{\mathcal{S}^2+\mathcal{P}^2}\right)\mathcal{P} - 2\sinh\gamma\cosh\gamma\frac{\mathcal{S}\mathcal{P}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}} \tag{3.19}$$

while the second term gives

$$8\mathcal{L}_\mathcal{S}\mathcal{L}_\mathcal{P}\mathcal{S} = -2\sinh\gamma\cosh\gamma\frac{\mathcal{S}\mathcal{P}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}} + 2\sinh^2\gamma\frac{\mathcal{S}^2\mathcal{P}}{\mathcal{S}^2+\mathcal{P}^2} \tag{3.20}$$

subtracting the above two equations exactly gives $\mathcal{P}$, hence equation (3.12) is satisfied and the theory has duality invariance.

The energy momentum tensor of ModMax theory, this time in a generic $d+2$ dimensional spacetime, is

$$T^{\text{ModMax}}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{ModMax}}}{\delta g^{\mu\nu}} = \left(\mathcal{S}g_{\mu\nu} - 2F_{\mu\sigma}F_\nu{}^\sigma\right)\mathcal{L}_\mathcal{S} \tag{3.21}$$

where $S_{\text{ModMax}} = \int d^{d+2}x\,\sqrt{-g}\mathcal{L}_{\text{ModMax}}$ and $\mathcal{L}_\mathcal{S}$ is given by (3.18)

$$\mathcal{L}_\mathcal{S} \equiv \frac{\partial\mathcal{L}}{\partial\mathcal{S}} = \frac{1}{2}\left(-\cosh\gamma + \sinh\gamma\frac{\mathcal{S}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\right)\,; \tag{3.22}$$

when $\gamma = 0$ it is equal to $-1/2$. In order to make contact with the Maxwell theory, we factorize a -2 in the rightmost term of (3.21). In this way the energy-momentum tensor reads

$$T_{\mu\nu}^{\text{ModMax}} = -2\left(F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{2}\mathcal{S}g_{\mu\nu}\right)\mathcal{L}_{\mathcal{S}} = -2\mathcal{L}_{\mathcal{S}}T_{\mu\nu}^{\text{Mawell}}\,, \tag{3.23}$$

which is a particularly useful form. It is straightforward to check that the two tensors are equal in the $\gamma = 0$ case.

In [53, 66, 67] Born-Infeld electrodynamics was combined with ModMax. By looking at the Born-Infeld Lagrangian (3.13), the middle term in the square root is $-2T\mathcal{L}_{\text{Maxwell}}$. Replacing it with $-2T\mathcal{L}_{\text{ModMax}}$ gives

$$\mathcal{L}_{\gamma\text{BI}} = T - \sqrt{T^2 - 2T\mathcal{L}_{\text{ModMax}} - \frac{1}{16}(F_{\mu\nu}{}^*F^{\mu\nu})^2}\,. \tag{3.24}$$

It can be checked that this Lagrangian density also satisfies (3.12) and hence the theory is duality-invariant. The weak field limit ($T \to \infty$) of $\mathcal{L}_{\gamma\text{BI}}$ is $\mathcal{L}_{\text{ModMax}}$.

3.3 Dyonic AdS black brane with ModMax

Black holes in the presence of ModMax electrodynamics have been studied in a number of papers. In particular, AdS black holes were explored in e.g. [68]. There, the solution for a dyonic spherical black hole with both electric and magnetic charges was given. In this section we derive the solution for a dyonic ModMax $d = 2$ black brane ($k = 0$) in AdS.

The procedure to find such a solution is the following: we first investigate how a black hole/brane solution in presence of Maxwell electrodynamics is related to the solution with ModMax. This will provide us with a "rule", or prescription, to move between the two cases. Next, we apply the so obtained "rule" to the dyonic AdS black brane in presence of Maxwell electrodynamics, found in [18] or [29], to obtain the solution in the ModMax case.

3.3.1 Relation between the Maxwell and the ModMax case

In order to find the "rule", we start by considering the action

$$S_X = \int d^4x\sqrt{-g}\left[R + \frac{d(d+1)}{L^2} + \mathcal{L}_X\right]\,, \tag{3.25}$$

where $\mathcal{L}_X$ can be $\mathcal{L}_{\text{ModMax}}$ given by (3.15), or the usual Maxwell Lagrangian with $g_F = 1$, this is $-F_{\mu\nu}F^{\mu\nu}/4$. The equations derived from the above action are[2].

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{L^2}g_{\mu\nu} = \frac{1}{2}T_{\mu\nu}^X\,. \tag{3.26}$$

and $T_{\mu\nu}^X$ is the energy momentum tensor in presence of Maxwell or ModMax electrodynamics, and we look for black holes solutions (but the results we will obtain can be generalized

[2] Notice that we normalized the fields in the action is such a way that there is no $16\pi G$. Formally this is equivalent to setting $16\pi G = 1$ or to absorb it in the energy-momentum tensor, so that the right hand side of Einstein equations will have a $\frac{1}{2}$ instead of $8\pi G$. As a consequence, the gauge field A_μ gets scaled by a factor $4\sqrt{\pi G}$, turning dimensionless

also for branes) of the form:

$$\begin{aligned} ds_X^2 &= \frac{r^2}{L^2} f_X(r) dt^2 + \frac{L^2}{r^2} \frac{dr^2}{f_X(r)} + r^2 d\Omega_2^2 \\ A^X &= A_t^X dt + A_\varphi^X d\varphi \,, \end{aligned} \tag{3.27}$$

and the equations for the gauge field are

$$-2\nabla_\mu \frac{\partial \mathcal{L}_X}{\partial F_{\mu\nu}} = 0 \,. \tag{3.28}$$

In general, in order compare two systems, we have to identify a set of parameters that make them look "similar". For the black holes under considerations we have four parameters which can be used to compare the two black holes. They are Q_e, Q_m, M and r_h and they are not independent of each other. Before proceeding, we must define the above quantities:

- the charges $Q_{e,m;X}$ are defined through Gauss law;
- the mass M_X is defined as the coefficient of $-1/r$ in $\frac{r^2}{L^2} f_X(r)$;
- $r_{h;X}$ is defined as the largest zero of $\frac{r^2}{L^2} f_X(r)$.

Here X stays for Maxwell or ModMax.

For the Maxwell case[3], the solution of (3.26) will be fully characterized by the parameters

$$M_M \,, \quad Q_{e,m;M} \,, \quad r_{h;M} \,, \tag{3.29}$$

each having the dimension of $[\text{length}]^1$, as discussed in chapter 2, and they are related by [56]

$$\begin{aligned} \frac{r^2}{L^2} f_M(r) &= \frac{r^2}{L^2} - \frac{M_M}{r} + \frac{Q_{e;M}^2 + Q_{m;M}^2}{r^2} + 1 \,, \\ r_{h;M} &\iff \frac{r^2}{L^2} f_M(r_{h;M}) = 0 \,. \end{aligned} \tag{3.30}$$

The $+1$ is due to the positive curvature of the horizon. The black hole mass is related to charges and horizon radius as

$$\frac{r^2}{L^2} f_M(r_{h;M}) = 0 \implies M_M(Q_{e,m;M}, r_{h;M}) = \frac{r_{h;M}^3}{L^2} + \frac{(Q_{e;M}^2 + Q_{m;M}^2)}{r_{h;M}} + r_{h;M} \,. \tag{3.31}$$

This is nothing but the RN-AdS black hole found in chapter 2, but with the magnetic charge.

On the other hand, the ModMax solution is fully characterized by the parameters

$$M_{MM} \,, \quad Q_{e,m;MM} \,, \quad r_{h;MM} \,, \tag{3.32}$$

related by [68]

$$\begin{aligned} \frac{r^2}{L^2} f_{MM}(r) &= \frac{r^2}{L^2} - \frac{M_{MM}}{r} + \frac{e^{-\gamma}(Q_{e;MM}^2 + Q_{m;MM}^2)}{r^2} + 1 \,, \\ r_{h;MM} &\iff \frac{r^2}{L^2} f_{MM}(r_{h;MM}) = 0 \,. \end{aligned} \tag{3.33}$$

[3]We use the subscript M for the Maxwell case and the subscript MM for the ModMax case.

In particular, the black hole mass is related to charges and horizon radius as

$$\frac{r^2}{L^2} f_{MM}(r_{h;MM}) = 0$$
$$\Longrightarrow \; M_{MM}(Q_{e,m;MM}, r_{h;MM}) = \frac{r^3_{h;MM}}{L^2} + \frac{e^{-\gamma}(Q^2_{e;MM} + Q^2_{m;MM})}{r_{h;MM}} + r_{h;MM} \, . \tag{3.34}$$

If we want to compare the two black holes, we should set a "reference" with respect to which measuring the deviations of one with respect to the other. There are three possible ways to proceed:

1. to compare two black holes with equal masses and charges;
2. to compare two black holes with equal radii and charges;
3. to compare two black holes with equal masses and radii.

We now investigate the second one, namely we set

$$Q_{e,m;M} = Q_{e,m;MM} \equiv Q_{e,m} \quad , \quad r_{h;M} = r_{h;MM} \equiv r_h \tag{3.35}$$

and the masses M_M and M_{MM} of the two black holes are not independent parameters, but rather functions of r_h and $Q_{e.m}$ via (3.31) and (3.34) respectively.

For the Maxwell case we use (3.31)

$$M_M(Q_{e,m}, r_h) = \frac{r_h^3}{L^2} + \frac{(Q_e^2 + Q_m^2)}{r_h} + r_h \tag{3.36}$$

and analogously for ModMax we use (3.34)

$$M_{MM}(Q_{e,m}, r_h) = \frac{r_h^3}{L^2} + \frac{e^{-\gamma}(Q_e^2 + Q_m^2)}{r_h} + r_h \tag{3.37}$$

and the difference of the masses is

$$\Delta M \equiv M_M(Q_{e,m}, r_h) - M_{MM}(Q_{e,m}, r_h) = (1 - e^{-\gamma})\frac{(Q_e^2 + Q_m^2)}{r_h} \, . \tag{3.38}$$

The expressions of M_M and M_{MM} can be substitutes in the first of (3.30) and (3.33) respectively. In this way we find the emblackening factors in term of the three independent parameters $Q_{e,m}$ and r_h:

$$\frac{r^2}{L^2} f_M(r) = \frac{r^2}{L^2} - \frac{\frac{r_h^3}{L^2} + \frac{(Q_e^2+Q_m^2)}{r_h} + r_h}{r} + \frac{(Q_e^2 + Q_m^2)}{r^2} + 1 \, , \tag{3.39}$$

$$\frac{r^2}{L^2} f_{MM}(r) = \frac{r^2}{L^2} - \frac{\frac{r_h^3}{L^2} + \frac{e^{-\gamma}(Q_e^2+Q_m^2)}{r_h} + r_h}{r} + \frac{e^{-\gamma}(Q_e^2 + Q_m^2)}{r^2} + 1 \, . \tag{3.40}$$

Summarizing: using specific definitions of the parameters of a black hole, two black holes in AdS, both with horizon r_h and charges $Q_{e,m}$, but in one the gauge field dynamics is governed by $\mathcal{L}_M$ while for the other by $\mathcal{L}_{MM}$, have the same metric, apart from the fact that in the ModMax case the charges in the metric get a factor $e^{-\gamma/2}$. Regarding their masses, they are different but, since the mass is not an independent parameter, this information is already encoded in the screening of the charges at the level of the metric. If we define $Q^{eff}_{e,m} = e^{-\gamma/2} Q_{e,m}$, using (3.31) and (3.34), it is straighforward to check that

$$M_{MM}(Q_{e,m}, r_h) = M_M(Q^{eff}_{e,m}, r_h) \ , \tag{3.41}$$

and so using (3.33) and (3.30) we get a relation between metric in the two cases:

$$\begin{aligned} \frac{r^2}{L^2} f_{MM}(r; Q_{e,m}, r_h) &= \frac{r^2}{L^2} - \frac{M_{MM}(Q_{e,m}, r_h)}{r} + \frac{e^{-\gamma}(Q_e^2 + Q_m^2)}{r^2} + 1 \\ &= \frac{r^2}{L^2} - \frac{M_M(Q^{eff}_{e,m}, r_h)}{r} + \frac{(Q^{eff}_e)^2 + (Q^{eff}_m)^2}{r^2} + 1 \\ &= \frac{r^2}{L^2} f_M(r; Q^{eff}_{e,m}, r_h) \ , \end{aligned} \tag{3.42}$$

meaning that the metric of an AdS-RN black hole with ModMax, having charges $Q_{e,m}$ and radius r_h is exactly the metric of an AdS-RN black hole with Maxwell, having charges $Q^{eff}_{e,m}$ and the same radius.

Regarding the gauge field, for the Maxwell case a solution is given by[4]

$$A_M(Q_e, Q_m) = \frac{2Q_e}{r} dt + 2Q_m \cos\theta d\varphi \ , \tag{3.43}$$

while for ModMax we have [68]

$$A_{MM}(Q_e, Q_m) = e^{-\gamma} \frac{2Q_e}{r} dt + 2Q_m \cos\theta d\varphi = A_M(e^{-\gamma} Q_e, Q_m) \ , \tag{3.44}$$

meaning that, the ModMax gauge field is given by the Maxwell gauge field (3.43) with $e^{-\gamma} Q_e$ in place of Q_e.

With the conventions used ($16\pi G = 1$) everything turned dimensionless. To restore the proper units, the right hand side of (3.43) is divided by $4\sqrt{\pi G}$, and similarly for (3.44). Once physical dimensions are restored, recall that the charges $Q_{e,m}$ in $d = 2$ are lengths, and they are related to a proper dimensionless charges $q_{e,m}$ by $Q^2_{e,m} \propto G q^2_{e,m}$, as discussed in chapter 2.

3.3.2 Application to the black brane

Summarizing the results obtained above, in switching from Maxwell to ModMax, keeping the radius and the charges fixed, is equivalent to apply the following substitution

$$\begin{aligned} Q^2_{e,m} &\to e^{-\gamma} Q^2_{e,m} \quad \text{in the metric} \ , \\ Q_e &\to e^{-\gamma} Q_e \quad \text{in the gauge field} \ . \end{aligned} \tag{3.45}$$

With this prescription we can obtain the black brane solution in presence of ModMax, where we still use the formal convention $16\pi G = 1$.

The ansatz for the metric is the usual one:

$$ds^2 = -\frac{r^2}{L^2} f(r) dt^2 + \frac{L^2}{r^2} \frac{dr^2}{f(r)} + r^2(du^2 + u^2 d\varphi^2) \ . \tag{3.46}$$

[4]The factors 2 in front of $Q_{e,m}$ are due to the *relative* normalization of the gravity action with respect to the electromagnetic action in (3.25).

where we switched to polar coordinates for the boundary field dimensions, this is $(dx^2 + dy^2) = L^2(du^2 + u^2 d\varphi^2)$. To solve the equations, we start from the Maxwell case, studied in [29, 18]. The emblackening factor is given by

$$\frac{r^2}{L^2} f_M(r) = \frac{r^2}{L^2} - \frac{\frac{r_h^3}{L^2} + \frac{Q_e^2 + Q_m^2}{r_h}}{r} + \frac{Q_e^2 + Q_m^2}{r^2} , \tag{3.47}$$

with Q_e and Q_m the electric and magnetic charges respectively, defined by Gauss law. Using (2.69) and the first of (2.71), we recognize that the middle term is nothing but the typical mass term in the emblackening factor. The gauge field is given by

$$A_M(r, u) = \frac{2Q_e}{r_h}\left(1 - \frac{r_h}{r}\right) dt + Q_m u^2 d\varphi . \tag{3.48}$$

If instead of using polar coordinates, we use dimensional Cartesian coordinates defined by $(dx^2 + dy^2) = L^2(dx'^2 + dy'^2)$, the last term is can be taken to be $2Q_m x' dy'$.

In order to switch to ModMax keeping the radius and the charges (defined through Gauss law) fixed, from (3.45) we see that the effect of introducing ModMax is to screen both Q_e and Q_m by a factor $e^{-\gamma/2}$ in the metric and to screen only Q_e by $e^{-\gamma}$ in the gauge field (while Q_m is untouched there). What we want to show is that the particular structure of ModMax is capable of restoring an equal screening when considered at the level of the energy-momentum tensor. In this way, both sides of Einstein equations will reduce to the Maxwell case, but with screened charges. If we use the definition of emblackening factor given in (2.51), equation (3.45) directly implies that

$$\tilde{f}_{\text{ModMax}}(r) = \tilde{f}_{\text{Maxwell}}(r) \begin{bmatrix} Q_e^2 \to e^{-\gamma} Q_e^2 \\ Q_m^2 \to e^{-\gamma} Q_m^2 \end{bmatrix} \tag{3.49}$$

and

$$A_{\text{ModMax}}(r, u) = A_{\text{Maxwell}}(r, u) \left[Q_e \to e^{-\gamma} Q_e\right] . \tag{3.50}$$

The square brackets contain the instructions for the replacements (i.e. screenings).

The ModMax energy-momentum tensor structure is given by (3.23): one factor has exactly the same form of the Maxwell energy-momentum tensor. Clearly, this is just a factorization and it does not relate the two solutions, but only the analytic expressions of the two tensors. Since we want to relate the Maxwell solution to the ModMax solution, we have to perform the screenings according to (3.45). Thus we obtain

$$T_{\mu\nu}^{\text{ModMax}} = -2\mathcal{L}_{\mathcal{S}} \bar{T}_{\mu\nu}^{\text{Maxwell}} . \tag{3.51}$$

where the bar means that $T_{\mu\nu}^{\text{Maxwell}}$, solution of the Einstein-Maxwell theory (i.e. on-shell), is acted on by the transformations (3.45)[5]. Now the only thing left is to compute $\mathcal{L}_{\mathcal{S}}$. Using (3.18) leads us to compute $\mathcal{S}$ and $\mathcal{P}$. The only non-vanishing components of the field strength are

$$\begin{aligned} F_{tr} = & -\partial_r A_t & = -2e^{-\gamma} Q_e / r^2 , \\ F_{u\varphi} = & \ \partial_u A_\varphi & = 2Q_m u , \end{aligned} \tag{3.52}$$

and $\mathcal{S}$ is found as

$$\begin{aligned} \mathcal{S} = \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} &= F_{tr} F^{tr} + F_{u\varphi} F^{u\varphi} \\ &= F_{tr}^2 g^{tt} g^{rr} + F_{u\varphi}^2 g^{uu} g^{\varphi\varphi} \\ &= 4 \frac{Q_m^2 - e^{-2\gamma} Q_e^2}{r^4} . \end{aligned} \tag{3.53}$$

[5]Indeed the energy momentum tensor is a function of the metric and of the gauge field only, thus making unambiguous how to apply the transformations (3.45).

For $\mathcal{P}$ we must first evaluate ${}^*F^{\mu\nu}$. Again, the only non-vanishing components are

$$\begin{aligned} {}^*F^{tr} &= \epsilon^{tru\varphi}F_{u\varphi} = 2Q_m/r^2 \,, \\ {}^*F^{u\varphi} &= \epsilon^{u\varphi tr}F_{tr} = -2\frac{e^{-\gamma}Q_e}{r^4 u} \,, \end{aligned} \tag{3.54}$$

where we used that $\epsilon^{\mu\nu\rho\sigma} = |g|^{-1/2}\,\tilde{\epsilon}^{\mu\nu\rho\sigma}$ (see e.g. section 2.8 of [57]), and thus

$$\begin{aligned} \mathcal{P} = \frac{1}{2}F_{\rho\sigma}{}^*F^{\rho\sigma} &= F_{tr}{}^*F^{tr} + F_{u\varphi}{}^*F^{u\varphi} \\ &= -8\frac{e^{-\gamma}Q_eQ_m}{r^4} \,. \end{aligned} \tag{3.55}$$

Now it is straightforward to compute

$$\mathcal{S}^2 + \mathcal{P}^2 = 16\left(\frac{e^{-2\gamma}Q_e^2 + Q_m^2}{r^4}\right)^2 \tag{3.56}$$

and

$$\frac{\mathcal{S}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}} = \frac{Q_m^2 - e^{-2\gamma}Q_e^2}{Q_m^2 + e^{-2\gamma}Q_e^2} \,. \tag{3.57}$$

With a bit of manipulation of the hyperbolic functions, we eventually get $\mathcal{L}_\mathcal{S}$:

$$-2\mathcal{L}_\mathcal{S} = \frac{e^{-\gamma}(Q_m^2 + Q_e^2)}{Q_m^2 + e^{-2\gamma}Q_e^2} \,. \tag{3.58}$$

Now we must compute T^{ModMax} and show that it is just T^{Maxwell} with both charges screened. The only non-vanishing components of $T_{\mu\nu}$ are the diagonal terms. Starting from the tt component, and without doing the explicit calculation of the tensor in the Maxwell case, we have (using (3.49), (3.51) and (3.58))

- tt

$$T_{tt}^{\text{Maxwell}} = \frac{2\tilde{f}_{\text{Maxwell}}}{r^4}(Q_e^2 + Q_m^2) \,, \tag{3.59}$$

$$\begin{aligned} T_{tt}^{\text{ModMax}} &= -2\mathcal{L}_\mathcal{S}\frac{2\tilde{f}_{\text{ModMax}}}{r^4}(e^{-2\gamma}Q_2^2 + Q_m^2) \\ &= \frac{2\tilde{f}_{\text{ModMax}}}{r^4}e^{-\gamma}(Q_m^2 + Q_e^2) \\ &= T_{tt}^{\text{Maxwell}}\left[\begin{array}{c} Q_e^2 \to e^{-\gamma}Q_e^2 \\ Q_m^2 \to e^{-\gamma}Q_m^2 \end{array}\right] \,; \end{aligned} \tag{3.60}$$

- rr

$$T_{rr}^{\text{Maxwell}} = -\frac{2}{\tilde{f}_{\text{Maxwell}}r^4}(Q_e^2 + Q_m^2) \,, \tag{3.61}$$

$$\begin{aligned} T_{rr}^{\text{ModMax}} &= -2\mathcal{L}_\mathcal{S}\left[-\frac{2}{\tilde{f}_{\text{ModMax}}r^4}(e^{-2\gamma}Q_e^2 + Q_m^2)\right] \\ &= -\frac{2}{\tilde{f}_{\text{ModMax}}r^4}e^{-\gamma}(Q_m^2 + Q_e^2) \\ &= T_{rr}^{\text{Maxwell}}\left[\begin{array}{c} Q_e^2 \to e^{-\gamma}Q_e^2 \\ Q_m^2 \to e^{-\gamma}Q_m^2 \end{array}\right] \,; \end{aligned} \tag{3.62}$$

- uu

$$T_{uu}^{\text{Maxwell}} = 2\frac{Q_e^2 + Q_m^2}{r^2}\,, \tag{3.63}$$

$$\begin{aligned} T_{uu}^{\text{ModMax}} &= -2\mathcal{L}_{\mathcal{S}} \cdot 2\frac{e^{-2\gamma}Q_e^2 + Q_m^2}{r^2} \\ &= 2\frac{e^{-\gamma}(Q_e^2 + Q_m^2)}{r^2} \\ &= T_{uu}^{\text{Maxwell}} \left[\begin{array}{c} Q_e^2 \to e^{-\gamma}Q_e^2 \\ Q_m^2 \to e^{-\gamma}Q_m^2 \end{array} \right]\,; \end{aligned} \tag{3.64}$$

- $\varphi\varphi$

$$T_{\varphi\varphi}^{\text{Maxwell}} = 2\frac{u^2(Q_e^2 + Q_m^2)}{r^2}\,, \tag{3.65}$$

$$\begin{aligned} T_{\varphi\varphi}^{\text{ModMax}} &= -2\mathcal{L}_{\mathcal{S}} \cdot 2\frac{u^2(e^{-2\gamma}Q_e^2 + Q_m^2)}{r^2} \\ &= 2\frac{u^2e^{-\gamma}(Q_e^2 + Q_m^2)}{r^2} \\ &= T_{\varphi\varphi}^{\text{Maxwell}} \left[\begin{array}{c} Q_e^2 \to e^{-\gamma}Q_e^2 \\ Q_m^2 \to e^{-\gamma}Q_m^2 \end{array} \right]\,. \end{aligned} \tag{3.66}$$

We stress that the result obtained is not the same of (3.51). The latter is a factorization of T^{ModMax} in terms of $\bar{T}^{\text{Maxwell}}$, where the bar means acted on by (3.45), which is a somewhat unnatural transformation. What we have shown here is that when computing T^{ModMax} starting from the Einstein-Maxwell solution T^{Maxwell}, one does not even need the factorization or to "put the bar" using the transformations (3.45), but only to replace $Q_{e,m} \to e^{-\gamma/2}Q_{e,m}$ in T^{Maxwell}. This is the same screening acting at the level of the metric, this is occurring on the left hand side of Einstein equations. We conclude that for a ModMax black brane, both sides of Einstein equations are exactly the ones of Maxwell case (studied in [29]), apart from the charges being screened, and so the equations are still satisfied. Said otherwise, if $\mathcal{L}_{\mathcal{S}}$ happened to have a different form from the one we computed in (3.58), T^{ModMax} would have transformed differently from the left-hand side, and Einstein equations wouldn't have held true anymore.

Chapter 4

Holography and AdS/CFT dictionary

This chapter is devoted to holographic methods in condensed matter physics. We first give a glimpse on the idea of holography in theoretical physics and to its best known example, the AdS/CFT correspondence. A proper presentation of AdS/CFT inevitably requires string theory, which however is far from the scope of this work. For this reason the focus will be on general features, rather than on rigour and details. After that, we discuss how the correspondence can be used to compute physical quantities.

In theoretical physics, the word *holography* generically indicates that the information of a system can be encoded in a different system (or spacetime region) having at least one dimension less. This is exactly what happens with a hologram: a three dimensional image is obtained from the interference pattern lying on a two dimensional film. The idea of holography in theoretical physics originates from black hole thermodynamics, where the fact that the black hole Bekenstein-Hawking entropy is proportional to the horizon area ($S_{\text{Schw}} = A/4G_N$ for a (3+1) Schwarzschild black hole) suggests that the information of the microstates of the black hole is stored on its surface rather than in its volume, as first proposed by 't Hooft [4]. Indeed, the entropy is a measure of the number of microscopic states corresponding to a given macroscopic state described by thermodynamic variables, such as temperature T, and for any "standard" system it grows as the volume. This is a first hint for holography, although at this point its realization remains undefined. As discussed in the following section, string theory concretely provides a realization of holography.

4.1 The AdS/CFT correspondence

In the previous section we discussed the main, qualitative idea of holography. In practice, the best known realization of holography is given by the AdS/CFT correspondence, which is found in the context of string theory. This means that by "looking" at a CFT in p dimensions one sees a gravity theory in (at least) $p+1$ dimensions.

In the original formulation of the correspondence [1], Maldacena showed that $\mathcal{N} = 4\, U(N)$ super Yang-Mills (SYM herafter) theory[1] in (3+1) dimensions is dual (in a sense to be specified soon) to type IIB supergravity on $\text{AdS}_5 \times S^5$. The $\mathcal{N} = 4$ SYM is a CFT with a peculiar matter content: a gauge field $A_\mu = A^a_\mu T^a$, six scalars and four fermions, all in the adjoint representation, and it seems very different from any field theory appearing in "real world" and condensed matter systems. Nevertheless, as anticipated before, it is not crucial to have such a theory on one side of the correspondence; we will answer the

[1]Here $\mathcal{N}$ is the number of supercharges and N is the number of colours.

question "why is it ok to use AdS/CFT in condensed matter?" in the following sections. Before proceeding, we want to fix the terminology: the gravity theory in AdS is referred as the *bulk* theory, while the dual field theory is the *boundary* theory.

Coming back to the Maldacena example, the important point is that taking the large N (number of colours) limit in the boundary theory implies going to a classical theory of gravity on the gravitational side of the correspondence. Moving away from this limit makes it necessary to consider "stringy" properties of gravity in the bulk. At the same time, the 't Hooft coupling of the field theory $\lambda = g_{\rm YM}^2 N$ should be kept large[2], so to have a derivative expansion in the bulk action (see e.g. Chapter 1 of [69]). When these two limits are realized, a strongly coupled ($\lambda \gg 1$) field theory is described by a classical gravity theory (because of the large N), and vice versa. In general it is also true the opposite, that is a strongly coupled "stringy" gravity theory is described by a weakly coupled field theory. It is then clear that AdS/CFT correspondence is precisely a strong-weak duality, which is the reason why it became a powerful tool in studying condensed matter problems.

Another crucial aspect is that the gravity theory lives in higher number of dimensions. For example in the Maldacena work the bulk spacetime is AdS_5 (the S^5 is responsible for the appearance of massive modes with masses proportional to their angular momentum along the sphere and will not be considered, see [55] Chapter 4) while the boundary theory lives in a (3+1)-dimesional spacetime.

Summarizing the aspects of interest for us, AdS/CFT relates a strongly coupled quantum field theory in p dimensions to classical gravity in (at least) $p+1$ dimensions, provided that $N \gg \lambda \gg 1$. The extra dimension, which in our case will always be the AdS radial direction r, is related to the energy scale and it geometrises the renormalization group flow. The remaining p dimensions are usually $x \equiv t, x^1, \ldots, x^{p-1}$.

4.2 AdS/CFT as a computational tool: the dictionary

Up to now we did not give any explicit formula relating the two theories in AdS/CFT. In this section we clarify what the duality is about. Most of the content in this section is written following the exposition in the textbooks [55, 69, 70].

The GKPW relation

The duality between the boundary field theory and the bulk gravity theory lies in the equality of their partition functions. The QFT partition function in the presence of sources $h_i(x)$ coupled to operators $\mathcal{O}_i(x)$ is

$$Z_{\rm QFT} = \langle e^{i\int d^{d+1}x\, h_i(x)\mathcal{O}_i(x)}\rangle_{\rm QFT} \tag{4.1}$$

where the right hand side subscript means that the path integral is performed with the (strongly interacting) QFT action, and thus generically quite difficult to compute. The gravity partition function to be equated with $Z_{\rm QFT}$ is given by

$$Z_{\rm gravity} = \int^{\phi_i \to h_i} \prod_i \mathscr{D}\phi_i \, e^{iS_{\rm bulk}[\phi]} \tag{4.2}$$

[2]Note that in the large N limit, theories with matrix valued fields in the adjoint representation have as effective coupling the 't Hooft coupling. The theory can remain (strongly) interacting even in the large N limit. For a theory with fields in the vector representation, howerver, the large N limit inevitably leads to a free/weakly interacting theory.

where S_{bulk} is the action in the bulk spacetime. This is not necessarily AdS, but generically an asymptotically AdS spacetime. The bulk fields ϕ_i, collectively denoted ϕ, are taken to coincide with the field theory sources h_i when evaluated on the spacetime boundary. This is represented by the $\phi_i \to h_i$ notation taken from [69]. In this sense the QFT is said to "live" on the boundary. In the large N limit, the theory of gravity is a classical theory, and so the partition function can be evaluated on the saddle point, this is (the underscored means on-shell)

$$Z_{\text{gravity}} = e^{i\underline{S_{\text{bulk}}}[\phi^* \to h]} \tag{4.3}$$

and ϕ^* is the solution of the classical equations of motion with the boundary conditions that $\phi_i^*(x)$ coincides with $h_i(x)$ when evaluated on the boundary (we will be more precise about this statement later in this section). Then AdS/CFT is the statement that

$$Z_{\text{QFT}} = Z_{\text{gravity}} \tag{4.4}$$

This was first formulated (in Euclidean formalism) by Witten [7] and Gubser-Klebanov-Polyakov [6], or briefly GKPW. From (4.1) and (4.2) we notice that a field ϕ_i in the bulk acts as a source h_i for an operator $\mathcal{O}_i$ of the boundary. To illustrate the exact relation between fields in the bulk and operators on the boundary, we take the example of a scalar field in AdS.

Green functions: scalar field example

Consider the scalar field in pure AdS of Section 2.3. As mentioned there, the scalar field action should contain a counterterm, as the on-shell action diverges. A geometrical regulator to treat the divergence is introduced by shifting the boundary at $r = \epsilon^{-1}$ (instead of $r = \infty$). After regularization, the action (2.85) is

$$S_{\text{bulk}} \equiv S_\phi = \int_{r \le \epsilon^{-1}} dr\, dt\, d^d x \sqrt{-g} \frac{1}{2} \phi (\nabla_\mu \nabla^\mu - m^2) \phi + S_{\text{bdy}} + S_{\text{ct}} \tag{4.5}$$

with S_{bdy} being the usual boundary term coming from the integration by parts of $-\frac{1}{2}(\nabla\phi)^2$,

$$S_{\text{bdy}} = -\frac{1}{2} \oint_{r=\epsilon^{-1}} dt\, d^d x \sqrt{-\gamma} \phi\, \partial_n \phi\,, \tag{4.6}$$

and S_{ct} is the counterterm action [71], needed to cancel the divergence when the action is evaluated on-shell:

$$S_{\text{ct}} = a \oint_{r=\epsilon^{-1}} dt\, d^d x \sqrt{-\gamma} \phi^2\,. \tag{4.7}$$

In the above formulas, a is a constant to be set in order to cancel a possible divergence, while γ is the (determinant of the) induced metric on the boundary. For pure AdS, it is given by

$$\gamma_{\mu\nu} = \frac{r^2}{L^2} \eta_{\mu\nu} \Big|_{r=\epsilon^{-1}} \implies \gamma = -\left(\frac{r^2}{L^2}\right)^{d+1}_{r=\epsilon^{-1}} \implies \sqrt{-\gamma} = \left(\frac{r}{L}\right)^{d+1}_{r=\epsilon^{-1}}\,. \tag{4.8}$$

The differential operator $\partial_n = n^\mu \partial_\mu$ is the normal derivative, and n^μ is the unit vector, normal to the boundary and pointing outward. For pure AdS, the only non-vanishing component of n^μ is

$$n^r = \frac{r}{L} \implies \partial_n = n^r \partial_r = \frac{r}{L} \partial_r \tag{4.9}$$

and it is correctly normalized since

$$n^\mu n_\mu = n^\mu n^\nu g_{\mu\nu} = n^r n^r g_{rr} = \frac{r}{L}\frac{r}{L}\frac{L^2}{r^2} = 1\,. \tag{4.10}$$

The on-shell action is obtained by substituting the exact solution found in Section 2.3 in (4.5). However, the first term on the right-hand-side vanishes on-shell, while the boundary term and the counterterm must be evaluated at $r = \epsilon^{-1}$ which is supposed to be a very large value of the coordinate r. Therefore, one just has to substitute (2.103) in (4.6), resulting in

$$\begin{aligned}
\underline{S_\phi} &= \underline{S_{\text{bdy}}} + \underline{S_{\text{ct}}}\,,\\
\underline{S_{\text{bdy}}} &= \frac{1}{2L^{d+2}}\oint_{r=\epsilon^{-1}} dt\, d^dx\, \big[\phi_+^2 \Delta_+ r^{d+1-2\Delta_+} + \phi_-^2\Delta_- r^{d+1-2\Delta_-}\\
&\qquad\qquad + (\Delta_+ + \Delta_-)\phi_+\phi_- r^{d+1-\Delta_+-\Delta_-}\big]\\
&= \frac{1}{2L^{d+2}}\oint_{r=\epsilon^{-1}} dt\, d^dx\, \big[\phi_+^2\Delta_+ r^{\Delta_- - \Delta_+} + \phi_-^2\Delta_- r^{\Delta_+-\Delta_-} + (d+1)\phi_+\phi_-\big]\,,\\
\underline{S_{\text{ct}}} &= \frac{a}{L^{d+1}}\oint_{r=\epsilon^{-1}} dt\, d^dx\, \big[\phi_+^2 r^{d+1-2\Delta_+} + \phi_-^2 r^{d+1-2\Delta_-} + 2\phi_+\phi_- r^{d+1-\Delta_- - \Delta_+}\big]\\
&= \frac{a}{L^{d+1}}\oint_{r=\epsilon^{-1}} dt\, d^dx\, \big[\phi_+^2 r^{\Delta_- - \Delta_+} + \phi_-^2 r^{\Delta_+ - \Delta_-} + 2\phi_+\phi_-\big]\,.
\end{aligned} \tag{4.11}$$

where the underscore means evaluation along the solution (i.e. on-shell) and we used $\Delta_\pm = d + 1 - \Delta_\mp$. Since $\Delta_+ \geq \Delta_-$, the ϕ_-^2 term in $\underline{S_{\text{boundary}}}$ is problematic because it leads to a divergence when $\epsilon \to 0$.[3] However, in $\underline{S_{\text{ct}}}$ there is a similar term, and by setting a one can cancel the divergence in $\underline{S_{\text{boundary}}}$. Precisely, setting $a = -\Delta_-/2L$ gives a finite on-shell action

$$\underline{S_\phi} = \frac{\Delta_+ - \Delta_-}{2L^{d+2}}\oint_{r=\epsilon^{-1}} dt\, d^dx\, \big[\phi_+^2 r^{\Delta_- - \Delta_+} + \phi_+\phi_-\big] \xrightarrow{\epsilon\to 0} \frac{\Delta_+ - \Delta_-}{2L^{d+2}}\oint dt\, d^dx\, \phi_+\phi_- \tag{4.12}$$

which is regular for $\epsilon \to 0$.

Now we can be more precise about the identification of the scalar field on the boundary with the local source $h(x)$ of the field theory. The prescription is that one should identify the leading behaviour ϕ_- (up to a normlization constant) with the local source h. With this prescription the one-point function in the field theory is obtained using (4.1), (4.3) and (4.4).

$$\langle O\rangle_h = \frac{1}{iZ_{\text{QFT}}}\frac{\delta}{\delta h} Z_{\text{QFT}}[h] = e^{-i\underline{S_\phi}[\phi]}\frac{1}{i}\frac{\delta}{\delta\phi_-}e^{i\underline{S_\phi}[\phi]} = \frac{\Delta_+ - \Delta_-}{2L^{d+2}}\phi_+\,. \tag{4.13}$$

For our purposes it is enough to keep in mind that

$$\begin{aligned} h &\propto \phi_-\\ \langle \mathcal{O}\rangle_h &\propto \phi_+\,. \end{aligned} \tag{4.14}$$

This is true even for other kind of fields, but generically the definition of $\Delta_\pm$ is different. We conclude that a scalar (fermionic, vector...) field in the bulk is dual to a scalar

[3] Also the ϕ_+^2 term could be problematic if $\Delta_- - \Delta_+ \geq 1$ but this is not possible thanks to the BF bound.

(fermionic, vector...) operator in the boundary theory, in the sense that the slow falloff gives the source of the operator and the fast falloff gives its vev, as expressed by (4.14). Notice that $\phi_\pm$ correspond to the two falloffs after having removed the $r^{\Delta_\pm}$: this is crucial (and general) as quantities in the boundary theory must be "unaware" of the radial direction!

Actually, the duality is even more tight, as both the conformal dimension and the retarded Green's function of the operator are obtained from the bulk field. From linear response theory, the momentum space retarded Green's function of the operator $\mathcal{O}$ in the field theory, defined as

$$G^R_{\mathcal{O}\mathcal{O}}(\omega,\mathbf{k}) = -i\int dt\, d\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\,\theta(t)\,\langle[\mathcal{O}(t,\mathbf{x}),\mathcal{O}(0,\mathbf{0})]\rangle \ , \tag{4.15}$$

is given by the ratio of the expectation value by the source

$$G^R_{\mathcal{O}\mathcal{O}} = \frac{\langle\mathcal{O}\rangle}{h} \propto \frac{\phi_+}{\phi_-} \, . \tag{4.16}$$

From the exact solution of the scalar field (formulas (2.97) and followings), we can see that $\phi_\pm \sim k^{\Delta_\pm}$ and so

$$G^R_{\mathcal{O}\mathcal{O}} \sim k^{\Delta_+ - \Delta_-} = (k^2)^{\Delta_+ - (d+1)/2} \tag{4.17}$$

which, upon taking the inverse Fourier transform in the boundary field theory (coming back to spacetime coordinates), gives

$$G^R_{\mathcal{O}\mathcal{O}}(x-y) \sim \frac{1}{|x-y|^{2\Delta_+}} \tag{4.18}$$

with $x-y = (x^0-y^0, \mathbf{x}-\mathbf{y})$. The right hand side of the above equation is exactly how the Green's function of a conformal dimension Δ_+ operator behaves in a CFT. So we showed that the conformal dimension of $\mathcal{O}$ is Δ_+ and it is determined by the mass of the bulk field. Summarizing, the scalar field example has been used to show that

Bulk		**Boundary**
scalar (fermion, vector...) field ϕ	$\Longleftrightarrow$	scalar (fermion, vector...) operator $\mathcal{O}$
slow falloff ϕ_-	$\Longleftrightarrow$	source h of the operator $\mathcal{O}$
fast falloff ϕ_+	$\Longleftrightarrow$	expectation value $\langle\mathcal{O}\rangle$
fast to slow ratio $\frac{\phi_+}{\phi_-}$	$\Longleftrightarrow$	mom. space ret. Green's function $G^R_{\mathcal{O}\mathcal{O}}$
field mass m^2	$\Longleftrightarrow$	operator dimension Δ_+

(4.19)

Alternative "quantization" and unitarity bound

Equation (4.18) shows that Δ_+ is the dimension of the boundary operator $\mathcal{O}$ dual to the bulk field ϕ. The BF inequality (2.104) sets the bound on Δ_+

$$\Delta_+ = \frac{d+1}{2} + \sqrt{\left(\frac{d+1}{2}\right)^2 + m^2L^2} \geq \frac{d+1}{2}\, , \tag{4.20}$$

seemingly implying that it is not possible to have boundary operators with dimension smaller than $(d+1)/2$.

On the other hand, unitarity of the boundary CFT implies that operators must have dimension greater than $(d-1)/2$ (see e.g. [72]). This rises the question of how to dualise a boundary operator with dimension Δ in the range $(d-1)/2 < \Delta < (d+1)/2$ without violating the BF bound. The answer is that there is a mass window such that both the leading and the subleading behaviours are normalizable. Indeed, the fact that the leading behaviour grows near the boundary means that they are non-fluctuating classical backgrounds of the field theory and so they cannot correspond to an operator, but only to a source [73]. But when this is not the case, one can use the subleading behaviour (fast falloff) as the source and the leading, but normalizable, behaviour (slow falloff) as the vev of the operator, which now has dimension $\Delta = \Delta_-$. This procedure is sometimes referred as *alternative quantization*, as opposed to the standard quantization discussed in previous pages. It is important to notice that the two quantizations are incompatible, and one has to make a choice. The operator in the boundary field theory depends on this choice and in general the two quantizations correspond to two different CFTs.

To determine the mass window we impose that the operator dimension in the alternative quantization belongs to the range mentioned above:

$$\frac{d-1}{2} < \Delta_- < \frac{d+1}{2}\,, \tag{4.21}$$

Recalling that $\Delta_- = (d+1)/2 - \sqrt{(d+1)^2/4 + m^2L^2}$ we solve for m^2L^2 and get

$$-\left(\frac{d+1}{2}\right)^2 < m^2L^2 < -\left(\frac{d+1}{2}\right)^2 + 1\,. \tag{4.22}$$

The left inequality in (4.22) is again the BF bound not being violated (i.e. no tachyon propagation). If also the inequality on the right is satisfied, then Δ_- lies in the range (4.21), which is not accessible to Δ_+. In this range the slow falloff is normalizable too and within the alternative quantization, the dictionary becomes

$$\begin{aligned} &\phi_+ \propto h^{\text{alt}} \\ &\phi_- \propto \langle \mathcal{O}_{\text{alt}} \rangle \\ &\frac{\phi_-}{\phi_+} \propto G^R_{\mathcal{O}_{\text{alt}}\mathcal{O}_{\text{alt}}} \end{aligned} \tag{4.23}$$

and $\mathcal{O}_{\text{alt}}$ has dimension Δ_-.

Symmetries, source fields and currents

A global symmetry in the boundary theory corresponds to a local symmetry in the bulk. In particular, the Noether currents J^a_μ in the boundary[4] correspond to the gauge field in the bulk.

As a clarifying example, useful for the following, consider the case of a global $U(1)$ in the boundary. The Noether current J^μ couples to the electromagnetic source A^{QFT}_μ. The latter is nothing but the slow falloff of the bulk field A_M, which in turn is the gauge field associated with the local $U(1)$ in the bulk. We stress that the gauge, electromagnetic field in the bulk does not dualise in a dynamical electromagnetic field (photons) in the boundary theory, clearly because there is no gauge symmetry there. The correspondence

[4]In this part we use capital Latin indices for the bulk and Greek indices for the boundary. Lowercase Latin is for color indices.

is between the gauge field in the bulk and the conserved current in the boundary. To prove this, consider a $U(1)$ gauge field in the bulk. The bulk theory is invariant under $A_M \to A_M + \nabla_M \chi$. If χ is finite at the boundary, the field theory action will have a term like

$$\int d^{d+1}x\sqrt{-\gamma}\,(A_\mu^{\rm QFT} + \nabla_\mu \chi)J^\mu = \int d^{d+1}x\sqrt{-\gamma}\,(A_\mu^{\rm QFT} J^\mu - \chi\nabla_\mu J^\mu) \tag{4.24}$$

and the invariance implies $\nabla_\mu J^\mu = 0$.

Finite temperature, finite density and background magnetic field

In chapter 2 we have seen that (AdS) black holes are thermal objects and that their temperature is taken to be the Hawking temperature, which can be computed from the length of the compact time (τ) circle. The AdS/CFT dictionary provides a straightforward way to encode the temperature of the boundary theory in the bulk system. A finite temperature in field theory can be obtained by using a black hole geometry in the bulk [74]. The boundary theory temperature is nothing but the Hawking temperature of a black hole in the deep interior of the AdS bulk, or, analogously, the length of the compact time circle of the bulk. Thus

$$T_{\rm QFT} = T_{\rm BH}\,. \tag{4.25}$$

Indeed, the time coordinate t in the bulk is also the time coordinate of the boundary theory, consequently the periodicity of the imaginary time is the same in the two theories.

Another thermodynamic quantity of interest is the (Gibbs) free energy. In the Euclidean signature, the free energy is obtained directly from the GKPW discussed above. Indeed,

$$F = -T_{\rm QFT}\ln Z_{\rm QFT} = -T_{\rm BH}\ln Z_{\rm gravity} = -T_{\rm BH}\ln e^{-\underline{S_{\rm E,bulk}}} = T_{\rm BH}\underline{S_{\rm E,bulk}}\,, \tag{4.26}$$

where $\underline{S_{\rm E,bulk}}$ is the on-shell, Euclidean action of the bulk system.

Particularly important in the context of AdS/CMT is the dualisation of QFTs at finite-density. With finite-density QFT we mean that the time component of the global $U(1)$ conserved current, which is the charge density, has non-zero expectation value: $\langle J^t\rangle \neq 0$. Starting from a zero-density QFT, one can induce finite-density by adding to the QFT Lagrangian the term $\Delta\mathcal{L}_{\rm QFT} = \mu_{\rm QFT}J^t$ [75], where $\mu_{\rm QFT} \equiv A_t^{\rm QFT}$ is the *chemical potential* of the boundary theory and it is the source of the operator J^t. As discussed in the previous section, the conserved operator J^μ associated with global $U(1)$ in the boundary is dual to $U(1)$ gauge field A_M in the bulk. In particular, J^t is dual to A_t, in the sense of the dictionary (4.19).

In chapter 2 the AdS-RN black hole/brane (with Maxwell electrodynamics) was studied. There, the time component of the $U(1)$ field was found to be

$$A_t = \mu\left(1 - \frac{r_h^{d-1}}{r^{d-1}}\right)\,. \tag{4.27}$$

Notice that the above solution is still the sum of a normalizable mode $A_+ r^{-(d-1)}$ plus a non-normalizable $A_- r^0$. The dictionary rules (4.19) still apply (with $A_\pm$ in place of $\phi_\pm$). Since the slow falloff here is μ and the source in the QFT is $\mu_{\rm QFT}$ we conclude that

$\mu = \mu_{\rm QFT}$. This justifies why we used μ in (4.27). The fast falloff is $A_+ = \mu r_h^{d-1}$ and it is related to the charge density as

$$\rho \equiv \langle J^t \rangle = \frac{(d-1)\mu r_h^{d-1}}{g_F^2 L^d} \, . \tag{4.28}$$

The is nothing but the third entry of the dictionary (4.19), where the normalization constant is the one appropriate for gauge fields and it is taken from [13, 69] written in our conventions. The chemical potential has units of $[\text{length}]^{-d/2}$ and the charge density has dimensions of $[\text{length}]^{-(d+2)/2}$, but $[\mu\rho]$=$[\text{length}]^{-(d+1)}$ as the QFT Lagrangian[5]. In particular, for a three dimensional QFT ($d = 2$) with $g_F = 1$ one gets

$$\rho = \frac{\mu r_h}{L^2}. \tag{4.29}$$

Also notice that, in $d = 2$, $[\mu] = [\text{length}]^{-1}$ and $[\rho] = [\text{length}]^{-2}$, as expected for a chemical potential and a charge density respectively. The above argument holds in the same way when the $U(1)$ field is the ModMax (and other kinds of non-linear electrodynamics, but *not* for power-Maxwell electrodynamics [50] because the gauge field falls off with different powers of r). As discussed in the previous chapter, what changes is the relation between the slow falloff μ and the bulk charge Q. The same argument holds for the fast falloff.

The background magnetic field in the boundary is obtained similarly to μ. Let us consider for simplicity $d = 2$. Given a $U(1)$ field A_M and the corresponding antisymmetric field strength F_{MN} in the bulk, the radial component of the magnetic field is given by F_{xy}. For the examples considered in the previous chapter, F_{xy} is a constant in r and so it is a non-normalizable mode, exactly as μ. For this reason the magnetic field on the boundary has to be considered as a source.

4.3 From AdS/CFT to AdS/CMT

Now we address the problem of justifying why AdS/CFT can be applied to condensed matter systems. Indeed, these systems are not in general described by large N field theories, so it is not clear why AdS/CFT, as we explored in previous sections, has a reason to be applied. The answer is that one may rely on the so called "UV independence", which often appears in physics. With this we mean that phenomenological theories are generically rather independent of the microscopic underlying physics, as general long-range physics manifests itself in hopefully a universal way. Thus we overcome the problem of the large N gauge theory because we are confident that the emergent collective phenomena are independent of the details of the microscopic theory. This aspect of field theories is also known as "strong emergence".

In general what we have to care about is the kind of condensed matter system we want to describe. In this work we will focus on superfluids/superconductors. For this reason we must ensure to work with a finite temperature and finite density boundary theory. From the discussion of section 4.2, this is obtained using a black brane to introduce temperature and a $U(1)$ field to reproduce a chemical potential. A detailed description of how it is possible to build a holographic superconductor will be given in chapter 5.

Coming back to the general case, we want to discuss what is the effect of introducing a black hole and why it is meaningful for condensed matter applications. The introduction

[5] A more natural definition is to identify the source as $\mu_{\rm QFT} = \mu L^{(d-2)/2}$, which by differentiation of the action implies an extra $L^{-(d-2)/2}$ in (4.28). In this way $[\mu]$=$[\text{length}]^{-1}$ and $[\rho]$=$[\text{length}]^{-d}$ always.

of a black hole/brane in the AdS spacetime changes the metric to that of an asymptotically AdS manifold. This modification of the geometry is particularly important for AdS/CMT. Pure AdS spacetime remains invariant under the rescaling of the coordinates $(z, t, x_i) \to \lambda(z, t, x_i)$ as it is seen from the metric (2.11). Regarding the coordinates (t, x_i), this is just the invariance under the rescaling of the boundary theory coordinates. However, the coordinate z is exclusive of the bulk theory and does not correspond to any direction in the boundary theory space. The interpretation of the radial bulk direction z (or analogously, the coordinate r used so far) is that of energy scale at which the boundary theory is probed (e.g. [76] and references therein). Following the flow along the radial direction means following the renormalization group of the boundary theory. In particular small values of z (large r) correspond to approaching the UV theory, while large values of z (small r) correspond to the IR theory. Said otherwise, processes in the bulk interior determine long distance physics, the IR, of the dual field theory while processes near the boundary control the short distance, or UV, physics.

Pure AdS spacetime is the geometrization of scale invariance for a relativistic quantum field theory, since it is invariant under the rescaling of (z, t, x_i) (the bulk interior and near boundary looks the same). The dual theory exhibits no renormalization group flow at all. An asymptotic AdS spacetime, instead, has this symmetry only for $z \to 0$, meaning that the boundary theory corresponds to the fixed UV point of its renormalization group flow [75]. An important consequence of the duality between the radial direction and the energy scale of the boundary theory is the emergence of *quantum criticality* of the field theory when probed in the IR [24]. To see this, consider the IR geometry of the RN-AdS black brane given in formula (2.78): this geometries has a scaling invariance only in time direction. This critical behaviour can be related with observed scaling symmetry of empirical strange metals (see e.g. Chapter 3 of [55] for a discussion). Also, the existence of Fermi surfaces in the IR is predicted by holography [24], while behaviour of holographic superconductors in the IR have been explored in e.g. [77]. We conclude that the introduction of a black hole in the bulk gives a way of discerning between UV and IR regimes of the boundary theory.

IR physics is the one of interest in condensed matter. However, as anticipated before, some phenomena can be observed also in the UV theory. Phenomena as the Hall effect and Nerst effect have a dual gravitational description (e.g. [18, 78]). In the specific case of interest to us, i.e. holographic superconductivity, we have in mind the condensation of a scalar operator. As will be shown in chapter 5, the dual UV theory (the one living on the AdS boundary) will exhibit such condensation and also infinite DC conductivity, even though they occur in the CFT UV boundary theory.

Chapter 5

Holographic superconductivity with ModMax

Holographic superconductors are gravitational systems dual to quantum field theories exhibiting properties commonly associated with superconductors. Real world superconductors are materials with a complex structure, usually doped and engineered to study their properties. For example, many high-T_c superconductors are cuprates, this is crystals made of layers of copper oxides, as La_2CuO_4. Still taking high-T_c superconductors as an example, they usually have a rich phase diagram: when appropriately doped and with tuned external parameters, they can transit in antiferromagnetic phase, Fermi liquid phase or pass in a strange metal phase. Finally, the microscopic mechanism determining the superconductive phase is still debated.

Reproducing such a complex phenomenology holographically is a hard task, as one has to add more and more structure to the bulk. The first description of a holographic superconductor was given in [29, 30], where the condensation of a scalar operator, infinite DC conductivity and a Meissner-like effect in the boundary have been described with holography. Starting from here, many other aspects were accounted using AdS/CFT. The introduction of a Josephson junction into the dualised picture was carried out in [79, 80]. In [34, 32, 33] the effect of magnetic fields was studied for the first time. Unconventional p-wave superconductors, i.e. superconductors where the strongly paired electrons are type p wave functions, was first studied in [35]. To add more structure in the bulk, one can use massive gravity in order to break translational symmetry [81]. Indeed, real superconductors do have a lattice structure, i.e. they are not translational invariant.

Among all the aspects regarding the description of superconductors using holography, we are interested in non-linear effects which can be mimicked by non-linear electrodynamics in the bulk. These effects have been studied over a wide range of models. The list includes Born-Infeld, power-Maxwell, exponential, logarithmic, arcsin and others non-linear models. A number of publications is furnished in the bibliography and in the introduction. Calculations are usually oriented in computing boundary quantities as e.g. optical conductivity, the formation of the condensate and magnetic effects. The choice of the electrodynamics model plays a crucial role, because it will directly influence the above quantities. Inspired by the cuprates being superconducting layers, much of the research regards $d = 2$ superconductor holographically dual to a gravity theory in an asymptotically AdS_4 bulk.

In this chapter our superconductor is associated with a quantum field theory in which a scalar operator (order parameter) acquires non vanishing expectation value when the temperature drops below a critical value. This phase transition is a second order phase transition and it leads to infinite DC conductivity and Meissner effect, which are two phenomenon associated with all superconductors. In particular we will first expose how the RN-black brane instability against perturbation of a scalar field is dual to spontaneous symmetry breaking in the boundary theory, that is to a scalar operator acquiring non-zero vev. Next we show how this idea leads to a holographic description of superconductivity, as discovered by Hartnoll, Herzog and Horowitz in 2008. Eventually we promote the Maxwell gauge field in the bulk to the ModMax gauge field and we show the effect of the ModMax non-linear parameter on the order parameter and optical conductivity.

5.1 Instability of the RN black brane

The main idea behind holographic superconductivity is that a spontaneous symmetry breaking of a global symmetry in the boundary can be encoded in the physics of a "hairy" black hole in the bulk. The word "hairy" refers to the existence of some kinds of non-trivial fields outside a black hole. For asymptotically flat spacetimes there is a class of theorems called "no-hair theorems", stating that a black hole solution is characterized only by the mass M, the charge Q, and the spin J (see e.g. [82]). The physical content of these theorems is that eventually matter falls in the black hole or is radiated to infinity, so that nothing can remain near the horizon. For example there cannot be a massive scalar field outside the black hole. However, in an asymptotically AdS spacetime these theorems are no longer valid and so there is a chance to have stationary fields outside a black hole.

If one wishes to holographically describe a superconductor and in particular the normal-to-superconductor phase transition via AdS/CFT, a hairy black hole is needed. In the phase transition a scalar operator of the field theory acquires a finite vev by spontaneous symmetry breaking. On the other hand, as shown by Gubser [83], it is possible to spontaneously break an Abelian gauge symmetry in the vicinity of a black hole horizon, in the sense that a Higgs-like scalar field condenses (classically) near the horizon. This would correspond to the breaking of the corresponding global symmetry in the boundary, thus providing a possible holographic description for the phase transition.

In practice, a bulk system in which a local $U(1)$ is spontaneously broken can be built as follows. Consider the action[1]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R+\frac{6}{L^2}\right)-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-|\nabla_\mu\psi-iqA_\mu\psi|^2-m^2|\psi|^2\right] \tag{5.1}$$

and a black brane solution of the corresponding equations of motion. This is similar to the black brane studied in chapter 2, but this time the gauge field is coupled to a charged massive scalar field ψ. We now show that the electrostatic potential acts as an effective mass squared for the scalar field, which may trigger a Higgs-like mechanism in the bulk.

Defining

$$\begin{aligned}\mathcal{L}_\psi &= -|\nabla_\mu\psi-iqA_\mu\psi|^2-m^2|\psi|^2\\ &= -(\nabla_\mu\psi^*+iqA_\mu\psi^*)(\nabla^\mu\psi-iqA^\mu\psi)-m^2\psi^*\psi\,,\end{aligned} \tag{5.2}$$

the equation for ψ is

$$\begin{aligned}0 &= -\nabla_\mu\frac{\partial\mathcal{L}_\psi}{\partial\nabla_\mu\psi^*}+\frac{\partial\mathcal{L}_\psi}{\partial\psi^*}\\ &= \nabla_\mu(\nabla^\mu\psi-iqA^\mu\psi)-iqA_\mu\nabla^\mu\psi-(q^2A_\mu A^\mu+m^2)\psi\,.\end{aligned} \tag{5.3}$$

[1] We specialize to the $d=2$ case, but the results hold also for higher dimensions.

The last term in on the right hand side of this equations contains the "effective mass"

$$m^2_{\text{eff}} = m^2 + q^2 A_\mu A_\nu g^{\mu\nu} \,. \tag{5.4}$$

To illustrate how the instability comes in, consider A_t as the only non vanishing component; the effective mass becomes

$$m^2_{\text{eff}} = m^2 + q^2 A_t^2 g^{tt} \tag{5.5}$$

with $g^{tt} < 0$ as the metric is Lorentzian. The mass receives a negative contribution, which may cause the mass to go below the BF bound, thus causing an instability (see section 2.3). Since the spacetime is asymptotically AdS_4, this means that near the boundary (and only there) the BF bound is $m^2_{\text{eff}} L^2 = -\frac{9}{4}$. However, as we approach the boundary $g^{tt} \sim -r^{-2}$ and $A_t \sim \mu$ (from the AdS-RN solution (2.68)-(2.70) with $k = 0$), so that the negative mass contribution goes like

$$q^2 A_t^2 g^{tt} \sim -\frac{\mu^2}{r^2} \qquad (r \to \infty) \tag{5.6}$$

that in general is a very small contribution, so that $m^2_{\text{eff}}(r \to \infty) = m^2$. So, if m^2 is above the boundary BF bound, then m^2_{eff} considered at the boundary will also be above that BF bound. In this case there is no instability near the boundary. Nevertheless, it is possible to argue, at least in the specific case of an extremal black brane geometry, that an instability can be triggered in proximity of the horizon. Indeed, from (2.78) we know that the near horizon topology is $\text{AdS}_2 \times \mathbb{R}^2$ with AdS curvature radius $L/\sqrt{6}$, and thus the BF bound near the horizon is the one of this specific AdS_2, this is $m^2_{\text{eff}} \frac{L^2}{6} = -\frac{1}{4}$ (by using equation (2.104) with the substitutions $d \to 0$ and $L \to L/\sqrt{6}$). Thus in order to cause a near horizon instability we need

$$m^2_{\text{eff}} L^2 < -\frac{3}{2} \,. \tag{5.7}$$

The negative contribution, which was negligible near the boundary, can be evaluated near the horizon. Using the results from chapter 2, in particular (2.69) with $k = 0$, (2.70), (2.73) and (2.72) with the equality sign, we can easily compute

$$\begin{aligned} Q^2 L^2 &= 3 r_h^4 \,, \\ M L^2 &= 4 r_h^3 \,, \\ \mu^2 &= \frac{3 g_F^2 r_h^2}{4\pi G L^2} \,. \end{aligned} \tag{5.8}$$

The shift in the mass (5.5) is thus given by

$$
\begin{aligned}
q^2 A_t^2 |g^{tt}| = q^2 \frac{A_t^2}{|g_{tt}|} &= q^2 \lim_{r\to r_h} \left[\frac{\mu^2 \left(1-\frac{r}{r_h}\right)^2}{\frac{r^2}{L^2} f(r)} \right] \\
&\overset{\mathrm{H}}{=} q^2 \mu^2 \lim_{r\to r_h} \left[\frac{-2\frac{r_h^2}{r^3} + 2\frac{r_h}{r^2}}{\left(\frac{r^2}{L^2} f(r)\right)'} \right] \\
&\overset{\mathrm{H}}{=} q^2 \mu^2 \lim_{r\to r_h} \left[\frac{6\frac{r_h^2}{r^4} - 4\frac{r_h}{r^3}}{\left(\frac{r^2}{L^2} f(r)\right)''} \right] \\
&= q^2 \mu^2 L^2 \lim_{r\to r_h} \left[\frac{3\frac{r_h^2}{r^4} - 2\frac{r_h}{r^3}}{1 + 3\frac{Q^2 L^2}{r^4} - \frac{M L^2}{r^3}} \right] \\
&\overset{(5.8)}{=} q^2 L^2 \left(\frac{3 g_F^2 r_h^2}{4\pi G L^2} \right) \lim_{r\to r_h} \left[\frac{3\frac{r_h^2}{r^4} - 2\frac{r_h}{r^3}}{1 + 9\frac{r_h^4}{r^4} - 4\frac{r_h^3}{r^3}} \right] \\
&= q^2 L^2 \left(\frac{3 g_F^2 r_h^2}{4\pi G L^2} \right) \left[\frac{1}{6 r_h^2} \right] = \frac{q^2 g_F^2}{8\pi G} \, .
\end{aligned}
\tag{5.9}
$$

In going from the first to the third line in the above equation we used two times L'Hôpital's rule. This is permitted because both $r^2 f(r)$ and its derivative (related to the temperature) vanish for the extremal black brane. Eventually we get the following near horizon effective mass:

$$
m_{\text{eff}}^2(r \to r_h) = m^2 - q^2 A_t^2 |g^{tt}| = m^2 - \frac{q^2 g_F^2}{8\pi G} \tag{5.10}
$$

We now see that it possible to produce a near horizon instability while preserving the stability at the boundary. This happens when the near horizon BF is violated, but the near boundary BF is not:

$$
\begin{aligned}
m_{\text{eff}}^2(\infty) L^2 &= m^2 L^2 > -\frac{9}{4} \\
m_{\text{eff}}^2(r_h) L^2 &= m^2 L^2 - \frac{q^2 g_F^2 L^2}{8\pi G} < -\frac{3}{2} \, .
\end{aligned}
\tag{5.11}
$$

As a consequence the Higgs-like scalar may condense near the horizon if thermodynamically favourable. To see the effect one has to rely on numerics. It turns out that the solution with a scalar profile is favourite with respect to the solution with no scalar. The falloffs of this solution are the ones expressed in (2.103) and in general both ϕ_+ and ϕ_- will be different from zero, corresponding to a having sourced scalar operator with finite vev in the boundary theory. Now, it is true that the source provides an explicit symmetry breaking but it is possible to find configurations of the scalar field which realize a genuine spontaneous symmetry breaking on the boundary, this is a finite vev with no external source.

5.2 Minimal holographic superconductor

In 2008 Horowitz, Hartnoll and Herzog [30] proposed a way to describe superconductivity using AdS/CFT correspondence. Since in a zero-density CFT there is no preferred scale, every finite temperature is equivalent and there cannot be a definite temperature at which the transition to the superconductive phase occurs. However, as discussed in chapter 4, AdS/CFT correspondence can be applied to to finite-density CFTs as well. To do so, a finite chemical potential (or charge density) is added, effectively introducing a new scale. The simplest model is the one studied in [30], featuring a non-dynamical AdS_4-Schwarzschild black brane metric (see section 2.2)

$$\begin{aligned} ds^2_{\text{AdS-Sch}} &= -\tilde{f}(r)dt^2 + \frac{dr^2}{\tilde{f}(r)} + \frac{r^2}{L^2}(dx^2+dy^2)\,, \\ \tilde{f}(r) &= \frac{r^2}{L^2}\left(1-\frac{r_h^3}{r^3}\right) = \frac{r^2}{L^2} - \frac{r_h^3}{rL^2}\,, \end{aligned} \tag{5.12}$$

with $r_h^3 = ML^2$ and temperature given by (2.63)

$$T = \frac{3r_h}{4\pi L^2}\,. \tag{5.13}$$

As seen in chapter 4, this is also the boundary field theory temperature.

The Lagrangian density in this background is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |\nabla_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2\,. \tag{5.14}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with A_μ being the $U(1)$ Maxwell gauge field, and ψ is a charged massive scalar field, just as in the previous discussion. This is known as minimal holographic superconductor, where *minimal* stays for "no self-interactions of the scalar field". Also, the fact that the spacetime geometry is not dynamical comes from the assumption that the fields do not backreact on the spacetime. This is the so called *probe limit*. Allowing the fields to backreact on the metric will produce a RN black brane with scalar hair. This is possible because the spacetime is asymptotically AdS, and so the no-hair theorem does not apply. Strictly speaking, the probe limit corresponds to the large values of the charge q. Indeed, by rescaling the fields as $A_\mu \to A_\mu/q$ and $\psi \to \psi/q$, the part of the full action (5.1) containing A_μ and ψ gets a factor $1/q^2$. In the large q limit, the action reduces to the Hilbert-Einstein, without any matter content. As explored in [29, 84], this limit retains most of the interesting physics. Restricting to the case in which A_μ and ψ depend only on r and setting $A_r = A_x = A_y = 0$ and $A_t \equiv \phi$, the equations of motion are[2]

$$\psi'' + \left(\frac{\tilde{f}'}{\tilde{f}} + \frac{2}{r}\right)\psi' + \left(\frac{q^2\phi^2}{\tilde{f}^2} - \frac{m^2}{\tilde{f}}\right)\psi = 0\,, \tag{5.15}$$

and

$$\phi'' + \frac{2}{r}\phi' - \frac{2q^2\psi^2}{\tilde{f}}\phi = 0\,, \tag{5.16}$$

For the one-form $A = \phi dt$ to be well defined at $r = r_h$, $\phi(r_h) = 0$ is required. This is naively understood recalling that at the horizon dt is infinite; a better argument can be found in [84]. Also, equation (5.15) in the near-horizon ($r \sim r_h$) gives $\psi'(r_h) = m^2\psi(r_h)/\tilde{f}'$. Indeed, in proximity of the horizon, equation (5.15) reduces to

$$\frac{\tilde{f}'}{\tilde{f}}\psi' - \frac{m^2}{\tilde{f}}\psi = 0 \implies \psi'(r_h) = \frac{m^2L^2}{3r_h}\psi(r_h)\,, \tag{5.17}$$

[2] Although we used to denote scalar fields with ϕ in the previous chapters, in the literature of holographic superconductors the scalar and the potential are always denoted with ψ and ϕ, respectively.

because all the other terms are finite (non-diverging) or zero at the horizon. Notice that the fact that ψ and ψ' are proportional at the horizon is a regularity condition. As it is well known, in general a second order differential equation has two independent boundary conditions. Finally, the asymptotic behaviours for ψ and ϕ can be taken from (2.103) and (2.70) with $d = 2$, respectively. It is a common choice in the literature to set $m^2L^2 = -2$ (above the boundary BF bound) and $L = 1$. With this convention all the quantities become dimensionless. For clarity we explicitly write down the above statements: the near boundary behaviour of the fields is[3]

$$\begin{aligned} \psi(r \to \infty) &\sim \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} , \\ \phi(r \to \infty) &\sim \phi^{(0)} - \frac{\phi^{(1)}}{r} , \end{aligned} \tag{5.18}$$

and the conditions to be imposed at the horizon are

$$\text{bcs at } r_h : \begin{cases} \phi(r_h) = 0 \\ \psi'(r_h) = -\dfrac{2}{3r_h}\psi(r_h) \end{cases} . \tag{5.19}$$

From the results of chapter 4, in particular from (4.27) and (4.28), we see that $\phi^{(0)} = \mu$ is the chemical potential of the field theory and that $\phi^{(1)} = \mu r_h = \rho$ is the charge density. With the above choice of the scalar field mass, equation (2.102) gives $\Delta_\pm = 1, 2$. As discussed in Chapter 4, there are two available correspondences for this bulk system. The first one where the falloff $\psi^{(1)}$ is dual to a scalar operator $\mathcal{O}_1$ and $\psi^{(2)}$ is the corresponding source $h_{\mathcal{O}_1}$, and the second one where $\psi^{(2)}$ is dual to a scalar operator $\mathcal{O}_2$ and $\psi^{(1)}$ is the source $h_{\mathcal{O}_2}$. The switch to the superconducting phase is signalised by having $\langle \mathcal{O} \rangle_{h=0} \neq 0$ when $T < T_c$, for a given choice of the correspondence. A final remark regards the ensemble of the boundary field theory. If one wants to work with a grand canonical ensemble, μ must be fixed. Otherwise, in the canonical ensemble, ρ must be fixed. The requirement of vanishing boundary source and the specification of the ensemble of the boundary theory, translate directly into boundary conditions at $r = \infty$ in the bulk:

$$\text{bcs at } \infty : \begin{cases} \phi^{(0)} = \mu \quad \text{or} \quad \phi^{(1)} = \rho \\ \psi^{(1)} = 0 \quad \text{or} \quad \psi^{(2)} = 0 \end{cases} . \tag{5.20}$$

We will work in the grand canonical ensemble, i.e. we set $\phi^{(0)} = \mu$ with a fixed μ, and set $\psi^{(1)} = 0$, i.e. choose the correspondence with $\psi^{(2)} \propto \langle \mathcal{O}_2 \rangle$ and $\psi^{(1)} \propto h_{\mathcal{O}_2}$.

With these choices, after imposing conditions (5.19) and (5.20), equations (5.15) and (5.16) can be solved (for instance, numerically) and afterwards one can read the value of $\psi^{(2)}$ from the falloff of the solution $\psi(r)$. Similarly, ρ can be read off the asymptotic behavior of the solution ϕ. What one finds is that $\langle \mathcal{O}_2 \rangle$ acquires finite values below a critical temperature T_c. An analogous procedure is performed for different choices in (5.20).

To conclude this section, we explain in which sense this bulk system is dual to a superconductor. First of all, as discussed in chapter 4, the boundary theory is not gauged, so it would be better to talk of a superfluid rather than a superconductor. Second, since the boundary theory has a net charge density ρ it cannot describe a neutral material. The point here is that we are ignoring the opposite charged lattice, so that we are "dualising"

[3]We label fast and slow falloffs with the inverse power of r.

only the moving charges. In order to describe the lattice, one should at least break translation invariance. However, surprisingly enough, the normal-to-superconducting phase still occurs, even if, at lest for low temperature superconductors described by BCS theory [85], the superconducting phase is microscopically due to a pair mechanism, for which the flowing charges (electrons) pair together, mediated by vibrations of the lattice (phonons). This is an example of what we referred as "strong emergence" in chapter 4, namely the independence on the microscopic details of the physical phenomena as the phase transition. Finally, the two main characteristics of a superconductor, infinite DC conductivity and Meissner effect, must occur in the boundary; this will be shown in the next chapter.

5.3 Holographic superconductor in presence of ModMax

As an application of ModMax theory in the AdS/CMT framework, we extend the results from section 5.2 to the case in which the $U(1)$ gauge field is described by the ModMax Lagrangian (3.15). We choose to study a minimal holographic superconductor, i.e. self-interactions of the scalar field are ignored. The bulk action in four dimensions is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R+\frac{6}{L^2}\right)+\mathcal{L}_{\text{ModMax}}-|\nabla_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2\right]. \tag{5.21}$$

Although one can solve the equations derived from this action with particular ansatz on the fields, it is easier to solve them in the probe limit. This limit retains much of the physics and it definitely simplifies the equations. Therefore, the metric is just the AdS_4-Schwarzschild black brane (5.12) and we again restrict to the case in which both ψ and A_t depend only on the radial coordinate r and set $A_r = A_x = A_y = 0$ and $A_t \equiv \phi$. The equations of motion for ψ and ϕ are

$$\psi'' + \left(\frac{\tilde{f}'}{\tilde{f}}+\frac{2}{r}\right)\psi' + \left(\frac{q^2\phi^2}{\tilde{f}^2}-\frac{m^2}{\tilde{f}}\right)\psi = 0 \tag{5.22}$$

and

$$\phi'' + \frac{2}{r}\phi' - e^{-\gamma}\frac{2q^2\psi^2}{\tilde{f}}\phi = 0\,, \tag{5.23}$$

respectively. The first of these two equations is independent of the electrodynamics model employed, as a consequence of the fact that the scalar field ψ is minimally coupled to the gauge field. The second equation features the factor $e^{-\gamma}$, characteristic of the ModMax theory. When $\gamma = 0$, the equation reduces to (5.16), where the gauge field is described by Maxwell theory. A complete derivation of these equations can be found in the Appendix B; as discussed there, the crucial relations to obtain the above equations (in particular the latter) are $\mathcal{S} < 0$ and $\mathcal{P} = 0$.

In section 5.2 we described the asymptotic behavior of ψ and ϕ; the introduction of the ModMax action in place of the standard Maxwell one's does not affect the asymptotic behavior, at least in the probe limit, which is still given by (5.18)

$$\begin{aligned}\psi(r\to\infty) &\sim \frac{\psi^{(1)}}{r}+\frac{\psi^{(2)}}{r^2}\,,\\ \phi(r\to\infty) &\sim \phi^{(0)} - \frac{\phi^{(1)}}{r}\,.\end{aligned} \tag{5.24}$$

The boundary conditions to be imposed are the same as were used in section 5.1, namely

$$\text{bcs at } r_h : \begin{cases} \phi(r_h) = 0 \\ \psi'(r_h) = \dfrac{m^2L^2}{3r_h}\psi(r_h) \end{cases} , \tag{5.25}$$

needed to have a well defined one-form $A_\mu dx^\mu$ and to satisfy (5.22) near the horizon, and

$$\text{bcs at } \infty : \begin{cases} \phi^{(0)} = \mu \\ \psi^{(1)} = 0 \end{cases} , \tag{5.26}$$

corresponding to the boundary field theory being in the grand canonical ensemble and to setting the source to zero in one of the two possible quantizations, as discussed in section 5.2.

5.3.1 Condensation in the probe limit

The first quantity of interest to study when dealing with a phase transition is the order parameter. In the case of a superconductor we look for a scalar operator that condenses when the temperature goes below a critical temperature. This is the main motivation for studying how $\langle \mathcal{O}_2 \rangle$ varies with the temperature.

Equations (5.22) and (5.23) are solved numerically with the shooting method: one imposes conditions (5.25) and $\phi(\infty) = \mu$ for some fixed μ (first of (5.26)) and then varies $\psi(r_h)$ until the corresponding numerical solution satisfies also the second of (5.26). Doing this way, all the four boundary conditions are satisfied. The values of ρ and $\psi^{(2)}$ can then be read off the asymptotics of the solutions ϕ and ψ respectively. The above procedure is carried on for different values of r_h (at fixed μ), corresponding to different temperatures thanks to (5.13). Actually, for every fixed (μ, r_h), the above procedure provides us with a countable family of solutions, corresponding to several values of $\psi^{(2)}$. Each solution can be labelled with its number of nodes, i.e. the number of times the solution crosses the zero. However, it has been argued in [83] that there is only one solution $\psi(r)$ which minimizes the free energy and it is the one with no nodes. Calculations have been performed by setting $L = 1$, $q = 0.5, 1, 2$, $m^2L^2 = -2$ and $\mu = 1, 2, 10, 31$. The numerical procedure is reported in the Appendix C.

In order to compute physical quantities in the field theory, recall that AdS/CFT dictionary relates $\psi^{(2)}$ to the expectation value of a scalar operator $\mathcal{O}_2$ of the boundary field theory. For a holographic superconductor this is nothing but the order parameter. We adopt the same convention as in [30]:

$$\langle \mathcal{O}_2 \rangle = \sqrt{2}\psi^{(2)} . \tag{5.27}$$

The numerical analysis showed that in order for the operator $\mathcal{O}_2$ to acquire a non-zero expectation value, a minimal temperature is required; above this temperature $\langle \mathcal{O}_2 \rangle$ is zero. The results are shown in Figure 5.1. For a given μ, the critical temperature can be found fitting $\langle \mathcal{O}_2 \rangle$ in the $T \sim T_c$ region, as

$$\langle \mathcal{O}_2 \rangle = cT_c^2(1 - T/T_c)^{1/2} , \qquad (T \to T_c) . \tag{5.28}$$

We found that

1. $T_c/q\mu \approx 0.058748$ regardless of the value of γ, among the one tested. We then conclude that the ModMax parameter γ does not affect the critical temperature significantly.

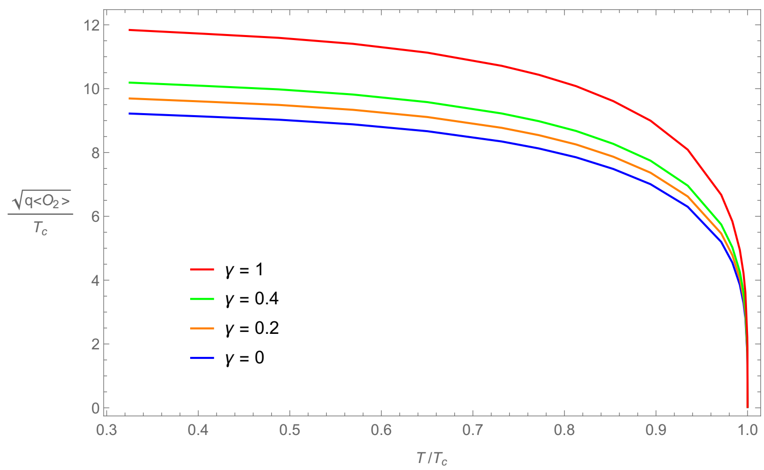

Figure 5.1: Expectation value of $\mathcal{O}_2$ as a function of the temperature in the grand canonical ensemble for different value of γ. Here $m^2L^2 = -2$, $q = 1$ and $\mu = 1$.

2. below the critical temperature, the value of the condensate $\langle \mathcal{O}_2 \rangle$ is an increasing function of γ.

An analytic expression relating the value of the condensate (at a given temperature and chemical potential) for different choices of γ is obtained from equations (5.22) and (5.23); indeed, by defining $\Psi = e^{-\gamma/2}\psi$, the equations, the boundary conditions (5.25, 5.26) and the asymptotic behaviour (5.24) stay the same, provided that one performs the substitutions $\psi \to \Psi$, $\psi^{(i)} \to \Psi^{(i)}$, $\gamma \to 0$. The bulk field Ψ then gives the value of the condensate with Maxwell theory at work ($\gamma = 0$). The definition of Ψ is natural because ψ appears always in the combination $e^{-\gamma/2}\psi$. Explicitly, the calculation is as follows. First we just write down all the equations in terms of Ψ: equations (5.22) and (5.23) read

$$\begin{aligned} &\Psi'' + \left(\frac{\tilde{f}'}{\tilde{f}} + \frac{2}{r}\right)\Psi' + \left(\frac{q^2\phi^2}{\tilde{f}^2} - \frac{m^2}{\tilde{f}}\right)\Psi = 0\,, \\ &\phi'' + \frac{2}{r}\phi' - \frac{2q^2\Psi^2}{\tilde{f}}\phi = 0\,, \end{aligned} \tag{5.29}$$

while the asymptotic behaviours (5.24) read

$$\begin{aligned} &\Psi(r\to\infty) \sim \frac{e^{-\gamma/2}\psi^{(1)}}{r} + \frac{e^{-\gamma/2}\psi^{(2)}}{r^2} \equiv \frac{\Psi^{(1)}}{r^1} + \frac{\Psi^{(2)}}{r^2}\,, \\ &\phi(r\to\infty) \sim \phi^{(0)} + \frac{\phi^{(1)}}{r}\,. \end{aligned} \tag{5.30}$$

The boundary conditions at infinity (5.26) read[4]

$$\begin{aligned} &\Psi^{(1)} = 0\,, \\ &\phi^{(0)} = \mu\,, \end{aligned} \tag{5.31}$$

[4]Notice that the first condition in (5.31) is nothing but the absence of the source $h_{\mathcal{O}_2}$ as expressed by the second equation in (5.26). However, if we allowed for a source $\psi^{(1)} = h$, the first of (5.31) would have been $\Psi^{(1)} = e^{-\gamma/2}h$, making this boundary condition model dependent.

while the boundary conditions at the horizon (5.25) read

$$\begin{aligned} \Psi'(r_h) &= -\frac{2}{3r_h}\Psi(r_h) \, , \\ \phi(r_h) &= 0 \, . \end{aligned} \tag{5.32}$$

Now, forgetting about ψ, we can solve the system (5.29)-(5.32) for Ψ and ϕ. This is the same system presented in section 5.2 and studied in [29, 30] (but in the grand canonical ensemble) and coincides with the Maxwell case ($\gamma = 0$). Thus, regardless of the value of γ, we always reduce to the minimal holographic superconductor of section 5.2, but with Ψ in place of ψ. Ψ and ϕ are then uniquely determined by the condition for Ψ of having zero nodes, and $\Psi^{(2)}$ and $\phi^{(1)}$ are read off the asymptotics of Ψ and ϕ respectively. In particular, $\sqrt{2}\Psi^{(2)}$ gives the condensate $\langle \mathcal{O}_2 \rangle_{\gamma=0}$ of the boundary CFT (at chemical potential μ and charge density $\rho = \phi^{(1)}$) in the Maxwell case, basically the blue line in figure 5.1.

Coming back to ψ, AdS/CFT tells that $\sqrt{2}\psi^{(2)}$ is the condensate $\langle \mathcal{O}_2 \rangle_{\gamma}$ at finite γ of the boundary CFT (at chemical potential μ and charge density $\rho = \phi^{(0)}$), then the definition of $\Psi = e^{-\gamma/2}\psi$ (or, analogously, the first of (5.30)) implies

$$\langle \mathcal{O}_2 \rangle_{\gamma} = e^{\gamma/2} \langle \mathcal{O}_2 \rangle_{\gamma=0} \, . \tag{5.33}$$

The manifestation of ModMax can be seen in (5.23), where $e^{-\gamma}$ multiplies ψ^2 and so we can remove it introducing Ψ^2, reducing to the $\gamma = 0$ case, as we did. The remnant of γ is in the relation between Ψ and the "original" bulk scalar field ψ.

An important consequence of the above discussion is that the charge density ρ (at a given T and μ) does not depend on the value of γ. Indeed we managed to reduce to the Maxwell ($\gamma = 0$) case just by rewriting the equations in terms of Ψ, without altering the solution ϕ. As we will discuss briefly, this is something not trivial and in general not possible with other non-linear electrodynamics. Numerical result confirm this prediction; for example at $T/T_c = 0.4$ and $\mu = 1$, $q = 1$, the charge density is $\rho \approx 0.291500$ for every value of γ considered.

This does not happen with other types of non-linear electrodynamics (see e.g. [46]) where there is a remnant of the non-linear parameter at the level of the equations. For example in Born-Infeld electrodynamics (3.13), the equation for ψ is the same as (5.22), but the equation for ϕ is [46]

$$\phi'' + \frac{2}{r}(1 - \beta^2\phi'^2)\phi' - \frac{2q^2\psi^2}{\tilde{f}}(1 - \beta^2\phi'^2)^{3/2}\phi = 0 \, , \tag{5.34}$$

where β is the non-linear parameter and it is related to the tension T used in (3.13) by $\beta^2 = T^{-1}$. It is clear that in this case one cannot do any rescaling such that the equations (and boundary conditions) are the same as in the Maxwell case. Indeed, even defining $\Psi^2 = \psi^2(1 - \beta^2\phi'^2)^{3/2}$ there will still be a β in the ϕ' term and, most importantly, the equation for the scalar will be different as we rescaled it by a non constant term. This model can be extended to the combined Born-Infeld-ModMax theory (3.24) resulting in the equation for ϕ (see Appendix B, in particular (B.35))

$$\phi'' + \frac{2}{r}(1 - e^{\gamma}\beta^2\phi'^2)\phi' - \frac{2q^2\psi^2}{\tilde{f}}e^{-\gamma}(1 - e^{\gamma}\beta^2\phi'^2)^{3/2}\phi = 0 \, . \tag{5.35}$$

In this case one can define $\Psi = e^{-\gamma/2}\psi$ (as did in the pure ModMax case) and $\tilde{\beta} = e^{\gamma/2}\beta$. The system of equations and boundary conditions for Ψ and ϕ are then the same one would get by starting in the pure Born-Infeld case with a parameter $\tilde{\beta}$.

About the difference between q and $e^{-\gamma/2}$

Here we would like to emphasise the different consequences of varying the charge q and the value of γ. These two parameters can be removed from the equations (5.22) and (5.23) by making the re-definitions $\Psi = qe^{-\gamma/2}\psi$ and $\Phi = q\phi$. The effect on the boundary conditions is only due to q which rescales the chemical potential to $\tilde{\mu} = q\mu$. Thus, one is left with studying the same system of equations (but with no γ and no q) for the unknowns Ψ and Φ, at the chemical potential $\Phi(\infty) = \tilde{\mu} = q\mu$, which is then the only scale available other than the temperature. This is the reason why we focused on the ratio $\frac{T_c}{\tilde{\mu}} = \frac{T_c}{q\mu}$. Still focusing on the rescaled fields Ψ and Φ, varying q does change the critical temperature at which Ψ (and so ψ) condenses. On the other hand, varying γ has no affect at all on Ψ and Φ. It is only when we come back to our "original" system of ψ and ϕ that we see the effect of γ: it just enhanced ψ by a factor of $e^{\gamma/2}$, everything else being unaltered with respect to the case $\gamma = 0$.

5.4 Conductivity in the probe limit

In order to compute the conductivity in the dual CFT as a function of frequency, we first add a perturbation[5] A_x in the bulk. We assume $A_x(r,t) = a_x(r)e^{-i\omega t}$ and use results from Appendix B to find

$$a_x'' + \frac{\tilde{f}'}{\tilde{f}}a_x' + \left(\frac{\omega^2}{\tilde{f}^2} - e^{-\gamma}\frac{2q^2\psi^2}{\tilde{f}}\right)a_x = 0\,, \tag{5.36}$$

which coincides with the one found in previous works (e.g. [30]), with the difference that the charge q^2 now comes with a screening factor $e^{-\gamma}$. In the above equation, ψ is the solution of (5.22). Equation (5.36) can be solved numerically. However, in analogous way to what we discussed before, defining $\Psi = e^{-\gamma/2}\psi$ results in a set of equations for Ψ, ϕ, a_x and a set of boundary conditions/asymptotic behaviours which are exactly the one for the Maxwell case. Precisely, Ψ and ϕ are determined by (5.29)-(5.32) and a_x by (5.36) which now reads

$$a_x'' + \frac{\tilde{f}'}{\tilde{f}}a_x' + \left(\frac{\omega^2}{\tilde{f}^2} - \frac{2q^2\Psi^2}{\tilde{f}}\right)a_x = 0\,. \tag{5.37}$$

Once again we reduced to Maxwell ($\gamma = 0$) case. Indeed Ψ is the scalar field in Maxwell electrodynamics, namely the solution found by Hartnoll, Herzog and Horowitz. This means that regarding a_x the dependence on γ drops out and the optical conductivity is the same of the Maxwell case: numerical results confirm this statement. This is analogous to what happened with ϕ and ρ and it is due to ψ appearing in (5.36) only in the combination $e^{-\gamma/2}\psi$.

We now review how to compute the optical conductivity. The boundary condition at the horizon to be imposed is the same of Maxwell case (infalling wave) and it is given by [86]

$$a_x(r) \propto \tilde{f}(r)^{-i\omega/3r_h} \qquad (r \to r_h)\,, \tag{5.38}$$

while the asymptotic behaviour is

$$A_x(r\to\infty, t) \sim A_x^{(0)}(t) + \frac{A_x^{(1)}(t)}{r}\,. \tag{5.39}$$

[5]By "perturbation" we mean that ψ and ϕ are already determined as in the previous section and they are used as inputs to determine A_x.

The time dependence on both sides of the above equation is of course given by $e^{-i\omega t}$. Differently from the scalar field case, this time the solution behaves at radial infinity as the sum of a normalisable mode plus a non-normalisable, and AdS/CFT dictionary states that they are related to the expectation value of the dual operator in the boundary field theory and to its source, respectively. Precisely,

$$A_x^{(0)} = A_x^{\mathrm{QFT}}\,, \quad A_x^{(1)} = \langle J_x \rangle\,. \tag{5.40}$$

with A_x^{QFT} source of the operator J_x. In order to compute the optical conductivity in the boundary theory, we use Ohm's law

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = \frac{A_x^{(1)}}{-\partial_t A_x^{\mathrm{QFT}}} = \frac{A_x^{(1)}}{-\partial_t A_x^{(0)}} = -\frac{i a_x^{(1)}}{\omega a_x^{(0)}} \tag{5.41}$$

where $a_x^{(0)}$ and $a_x^{(1)}$ are defined by

$$A_x^{(i)}(t) = a_x^{(i)} e^{-i\omega t} \quad i = 0, 1\,. \tag{5.42}$$

Given that equation (5.36) is linear, the conductivity is independent of the proportionality constant chosen in (5.38). Physically, since we are dealing with a perturbation, it seems reasonable to set the proportionality constant to be sufficiently small. Equation (5.36) with condition (5.38) are solved numerically. As anticipated before, the conductivity found in this way is exactly the one for the Maxwell case. The results are shown in figure 5.2. Following [29, 30], we notice that the imaginary part of the conductivity has a pole in $\omega = 0$ for $T < T_c$. Using one of the Kramers-Kronig relations, holding for any causal quantity, in particular for the optical conductivity, we see that the real and imaginary part of σ are related by

$$\mathrm{Im}[\sigma(\omega)] = -\frac{1}{\pi}\mathbf{PV}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Re}[\sigma(\omega')]d\omega'}{\omega' - \omega}\right] \tag{5.43}$$

where **PV** denoted the principal value of the integral. In particular, if $\mathrm{Im}[\sigma(\omega)]$ has a pole in $\omega = 0$ of the type c/ω, then $\mathrm{Re}[\sigma(\omega)]$ must have contain a $c\pi\delta(\omega)$, with c a generic constant. This delta is invisible to the numerics because of its infinitesimal width, as explained in [29]. This is characteristics of perfect conductors, and in particular of superconductors[6].

[6]The infinite DC, i.e. $\omega = 0$, conductivity is a characteristic of translationally invariant systems. In such cases the DC infinite conductivity is present at all the temperatures, since the system stays invariant under translations. However, in this case the translation invariance is implicitly broken by adopting the probe limit (see [29] for an argument). For this reason the infinite conductivity is associated directly with the superconducting phase, and it is indeed present below T_c only.

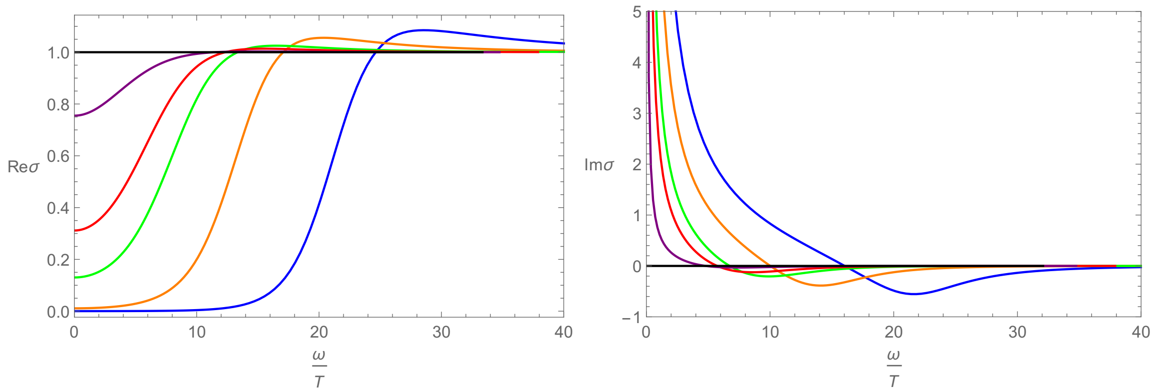

Figure 5.2: Real (left) and imaginary (right) part of the conductivity as function of ω/T, with $L = 1$, $m^2L^2 = -2$, $q = 1$ and $\mu = 1$. In both graphs, the temperature increases going from the rightmost curve (blue) to the horizontal line (black). The horizontal line represents temperatures at and above T_c. The rightmost curve is for $T/T_c = 0.4$.

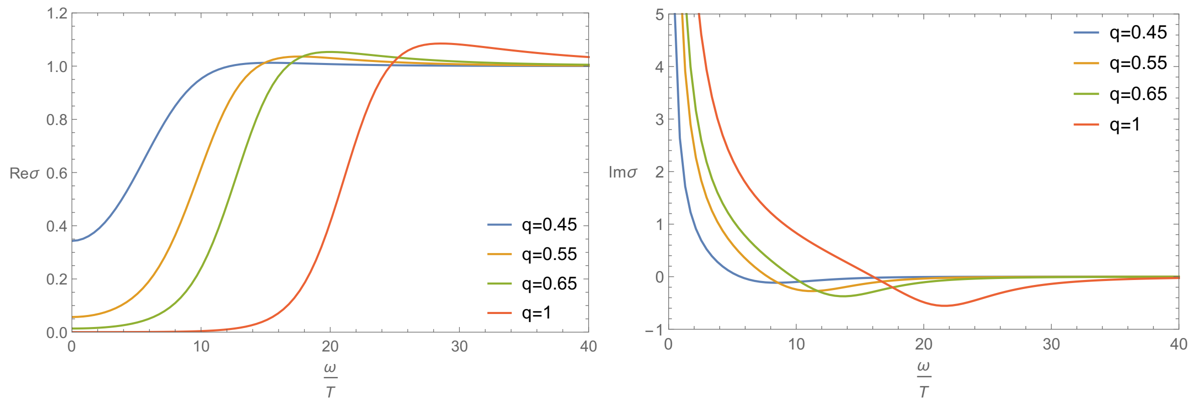

Figure 5.3: Real (left) and imaginary (right) part of the conductivity for different values of the charge q, independently of the value of γ. Here $L = 1$, $m^2L^2 = 1$, $\mu = 1$ and $T/T_c = 0.4$.

Chapter 6

Conclusions and future directions

In this thesis we have studied applications of non-linear electrodynamics to condensed matter in the framework of AdS/CMT. We first reviewed the main aspects of the AdS spacetime and black hole solutions within it. The connection of its symmetries with the conformal group was highlighted. Then we moved to the main proprieties of generic non-linear electrodynamics, in view of their employment in the bulk system. In chapter 4 we exposed how AdS/CFT provides a dual description of some strongly interacting quantum field theories in terms of classical gravity, and justified its application in studying condensed matter problems. In chapter 5, following the master works of Hartnoll, Herzog and Horowitz [29, 30], we provided a minimal holographic description of a 2+1 superconductor with the employment of the recently found ModMax electrodynamics [53]. The characteristic parameter γ of this theory was expected to modify in some way the holographic superconductor model with respect to the standard one, where Maxwell electrodynamics was adopted. Our computations were performed in the probe limit, where the matter fields in the bulk do not back-react on the spacetime geometry, which was fixed to be the AdS-Schwarzschild black brane. Also we worked in the grand canonical ensemble, meaning that the chemical potential of the dual field theory was kept fixed while varying the temperature. As expected, we found that the scalar operator dual to the scalar field in the bulk acquires a finite vev when the temperature drops below a critical T_c, via a second order phase transitions. At a given temperature, the introduction of ModMax in place of Maxwell increases the value of the condensate in the boundary theory by a factor of $e^{\gamma/2}$, but it does not affect the critical temperature T_c. This two aspects are of course related: since at a given temperature the values of the condensates in the two models are proportional, they both vanish at the critical temperature. The latter is fixed by the chemical potential μ and the charge q of the scalar via

$$\frac{T_c}{\mu q} \approx 0.058748$$

according to already known results (as reported in e.g. [70]).

We then analyzed transport properties, in particular the optical conductivity, by perturbing the probe limit bulk with a harmonic perturbation $a_x(r)e^{-i\omega t}$. The results obtained show that the optical conductivity is independent of the value of γ. The real part of the conductivity exhibits a Dirac delta for $\omega = 0$ when the temperature is below T_c. This was signalized by the imaginary part developing a pole at $\omega = 0$ for $T < T_c$. The infinite DC conductivity is characteristic of perfect conductors, and in particular of superconductors.

We found that the critical temperature and the optical conductivity do not depend on γ by observing that the scalar field can be rescaled as $\psi = e^{\gamma/2}\Psi$ in such a way that the whole system of equations and boundary conditions become unaware of γ. The information due to γ is stored only in the proportionality between ψ and Ψ. This is a consequence of conformal invariance of ModMax. It is important to recall that we worked in the probe limit, as the rescaling may not have the same effect away from this limit. We briefly investigated the $2+1$ minimal holographic superconductor in presence of the Born-Infeld-ModMax in the probe limit. In this case, the rescaling gives back the equations that can be found starting with Born-Infeld alone, but with a rescaled non linear parameter equal to $\tilde{\beta} = e^{\gamma/2}\beta$.

There are several ways in which this study can be continued. A first one is the analysis of magnetic properties of the minimal holographic superconductor, such as Meissner effect, vortex solutions and critical magnetic fields. In this regard, we have performed preliminary calculations using the charged black brane solution found in chapter 3 that point out that ModMax may predict lower critical magnetic fields with respect to Maxwell.

Another obvious extension of this work is the investigation of holographic superconductors in presence of ModMax away from the probe limit, where it may not be possible to reduce to the Maxwell case by means of the rescaling of ψ. Similarly, one could investigate the Born-Infeld-ModMax superconductor away from the probe limit. Numerical methods may be replaced by analytical techniques often used in the context of holographic superconductivity as the matching method and the Sturm-Liouville eigenvalue problem.

Finally, this work places itself in the wider context of AdS/CFT with non-linear electrodynamics. Holographic superconductors are an important subcategory but surely not the only one. Application of ModMax electrodynamics in the holographic description of other condensed matter systems (e.g. Fermi surfaces), are appealing ideas for future research. The study of effects due to non-minimal couplings of the electromagnetic field to gravity (as studied in e.g. [87, 88]) is a tantalizing future direction, in particular in the case in which the electromagnetic field is ModMax, thus extending the results obtained in this work.

Appendix

Appendix A

Christoffel symbols for black brane metric

Since it is often required to compute Christoffel symbols when performing explicit calculations, we collect them here and they will serve as a reference. For simplicity we consider a two-dimensional horizon and we use $f(r)$ for the coefficient of $-dt^2$, instead of $\tilde{f}(r)$. We distinguish two cases:

1. Cartesian coordinates on the horizon, corresponding to

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2) \,. \tag{A.1}$$

   The non-vanishing Christoffel symbols are given by

$$\begin{aligned} &\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{f'}{2f} \,; \\ &\Gamma^r_{tt} = \frac{f'f}{2} \,; \quad \Gamma^r_{rr} = -\frac{f'}{2f} \,; \quad \Gamma^r_{xx} = \Gamma^r_{yy} = -rf \,; \\ &\Gamma^x_{rx} = \Gamma^x_{xr} = \frac{1}{r} \,; \\ &\Gamma^y_{ry} = \Gamma^y_{yr} = \frac{1}{r} \,. \end{aligned} \tag{A.2}$$

2. Polar coordinates on the horizon, corresponding to

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(du^2 + u^2 d\phi^2) \,. \tag{A.3}$$

   In this case the non-vanishing Christoffel symbols are

$$\begin{aligned} &\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{f'}{2f} \,; \\ &\Gamma^r_{tt} = \frac{f'f}{2} \,; \quad \Gamma^r_{rr} = -\frac{f'}{2f} \,; \quad \Gamma^r_{uu} = -rf \,; \quad \Gamma^r_{\phi\phi} = -ru^2 f \,; \\ &\Gamma^u_{ru} = \Gamma^u_{ur} = \frac{1}{r} \,; \quad \Gamma^u_{\phi\phi} = -u \,; \\ &\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \,; \quad \Gamma^\phi_{u\phi} = \Gamma^\phi_{\phi u} = \frac{1}{u} \,. \end{aligned} \tag{A.4}$$

Appendix B

Holographic superconductors - explicit calculations

B.1 ModMax case

The probe limit equations of motion (5.22) and (5.23) are explicitly derived. The metric used is (5.12). Since in this limit the metric is not dynamical, the action reduces to

$$S_{\text{probe}} = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left[\mathcal{L}_{\text{ModMax}} - |\nabla_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2\right] , \qquad \text{(B.1)}$$

$\mathcal{L}_{\text{ModMax}}$ being given by (3.15). The equations of motion derived from this action are

$$\nabla_\mu \frac{\partial\mathcal{L}}{\partial\nabla_\mu A_\nu} - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \qquad \text{(B.2)}$$

and

$$\nabla_\mu \frac{\partial\mathcal{L}}{\partial\nabla_\mu \psi^*} - \frac{\partial\mathcal{L}}{\partial \psi^*} = 0\,, \qquad \text{(B.3)}$$

and of course the one for ψ^*.

ModMax equations

The first term in (B.2) can be computed as follows:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\nabla_{[\mu}A_{\nu]}} = \frac{\partial\mathcal{L}_{\text{ModMax}}}{\partial F_{\mu\nu}} &= -\frac{\cosh\gamma}{2}\frac{\partial\mathcal{S}}{\partial F_{\mu\nu}} + \frac{\sinh\gamma}{2\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\left(\mathcal{S}\frac{\partial\mathcal{S}}{\partial F_{\mu\nu}} + \mathcal{P}\frac{\partial\mathcal{P}}{\partial F_{\mu\nu}}\right) \\ &= -\cosh\gamma F^{\mu\nu} + \sinh\gamma\left(\frac{\mathcal{S}F^{\mu\nu} + \mathcal{P}\,{}^*F^{\mu\nu}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}}\right)\,,\end{aligned} \qquad \text{(B.4)}$$

where $F_{\mu\nu} = -F_{\nu\mu}$ and the definition of ${}^*F_{\mu\nu}$ have been used in the intermediate passages, resulting in

$$\frac{\partial\mathcal{S}}{\partial F_{\mu\nu}} = 2F^{\mu\nu} \quad ; \quad \frac{\partial\mathcal{P}}{\partial F_{\mu\nu}} = 2\,{}^*F^{\mu\nu}\,. \qquad \text{(B.5)}$$

Having the antisimmetry of $F_{\mu\nu}$ is analogous to solve $\nabla_\mu {}^*F^{\mu\nu} = 0$; this produces the extra factors of 2 in (B.5). Taking the covariant derivative of the first and last term in (B.4), gives

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_{[\mu} A_{\nu]}} = -\cosh\gamma \nabla_\mu F^{\mu\nu} + \sinh\gamma \left[\frac{\mathcal{S}\nabla_\mu F^{\mu\nu} + \mathcal{P}\nabla_\mu {}^*F^{\mu\nu}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}} + F^{\mu\nu}\partial_\mu \left(\frac{\mathcal{S}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}} \right) + {}^*F^{\mu\nu}\partial_\mu \left(\frac{\mathcal{P}}{\sqrt{\mathcal{S}^2+\mathcal{P}^2}} \right) \right] . \tag{B.6}$$

The equation is highly non-linear, but it linearizes for field configurations for which $\mathcal{P} \sim \mathcal{S}$ [89]. For instance, specializing to the case in which $\psi = \psi(r), A_t = A_t(r), A_r = A_x = A_y = 0$, the only non vanishing components of F are $F_{rt} = -F_{tr} = \partial_r A_t$. With this ansatz, the invariant $\mathcal{S}$ is negative:

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu}F^{\mu\nu} = F_{rt}F^{rt} = (F_{rt})^2 g^{tt} g^{rr} = -(\partial_r A_t)^2 < 0 . \tag{B.7}$$

The dual tensor ${}^*F$ and the invariant $\mathcal{P}$ are found to be

$$\begin{aligned} {}^*F^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \epsilon^{\mu\nu tr} F_{tr} , \\ \mathcal{P} &= \frac{1}{2} F_{\mu\nu} {}^*F^{\mu\nu} = F_{rt} {}^*F^{rt} = 0 , \end{aligned} \tag{B.8}$$

the last equation coming from the fact that ${}^*F^{rt} = 0$. Equation (B.6) reads

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_{[\mu} A_{\nu]}} = -(\cosh\gamma + \sinh\gamma)\nabla_\mu F^{\mu\nu} = -e^\gamma \nabla_\mu F^{\mu\nu} . \tag{B.9}$$

On the other hand, the second term in (B.2) is given by

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -2q^2 A^\nu |\psi|^2 + iq(\psi \nabla^\nu \psi^* - \psi^* \nabla^\nu \psi) . \tag{B.10}$$

- Setting $\nu = t$ in (B.9) and (B.10) gives the Euler-Lagrange equation for the A^t:

  $$-e^\gamma \nabla_\mu F^{\mu t} + 2q^2 A^t |\psi|^2 - iq(\psi \nabla^t \psi^* - \psi^* \nabla^t \psi) = 0 . \tag{B.11}$$

  The term $\psi \nabla^t \psi^*$ is zero and the covariant derivative is

  $$\begin{aligned} \nabla_\mu F^{\mu t} &= \partial_\mu F^{\mu t} + \Gamma^\mu_{\mu\alpha} F^{\alpha t} + \Gamma^t_{\mu\alpha} F^{\mu\alpha} \\ &= \partial_r F^{rt} + \Gamma^\mu_{\mu r} F^{rt} \\ &= \partial_r F^{rt} + \frac{2}{r} F^{rt} , \end{aligned} \tag{B.12}$$

  where in the last step we used (A.2). Using $F^{rt} = -\partial_r A_t$ and $A^t = g^{tt} A_t = -A_t/\tilde{f}$ (with $\tilde{f}$ defined as in (5.12)), from (B.11) we finally get the equation for A_t

  $$A_t'' + \frac{2}{r} A_t' - e^{-\gamma} \frac{2q^2 |\psi|^2}{\tilde{f}} A_t = 0 . \tag{B.13}$$

- Setting $\nu = r$, the Euler-Lagrange equation is

  $$-iq(\psi \nabla^r \psi^* - \psi^* \nabla^r \psi) = 0 \tag{B.14}$$

  obtained from (B.9) and (B.10) by using $\nabla_\mu F^{\mu r} = 0$, which in turn comes from $\Gamma^\mu_{\mu t} = 0$ and $\Gamma^r_{\mu\alpha} F^{\mu\alpha} = 0$ (contraction of symmetric and antisymmetric indices). The above equation implies that the phase of ψ is constant and so it can be removed with a global transformation. Indeed, setting $\psi = \rho e^{i\theta}$ gives

  $$\partial^r \rho - i\partial^r \theta - \partial^r \rho + i\partial^r \theta = 0 \implies \partial^r \theta = 0 . \tag{B.15}$$

Scalar equation

In analogy to the above derivation, the first term of (B.3) is obtained starting from

$$\frac{\partial \mathcal{L}}{\partial \nabla_\mu \psi^*} = -\nabla^\mu \psi + iq\psi A^\mu \tag{B.16}$$

and then taking the covariant derivative

$$\begin{aligned} \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \psi^*} &= -\nabla_\mu \nabla^\mu \psi + iq\nabla_\mu(\psi A^\mu) \\ &= -\partial_\mu \partial^\mu \psi - \Gamma^\mu_{\mu\alpha}\partial^\alpha \psi + iqA^\mu \partial_\mu \psi + iq\psi \partial_\mu A^\mu + iq\psi \Gamma^\mu_{\mu\alpha} A^\alpha \\ &= -\partial_r \partial^r \psi - \Gamma^\mu_{\mu r}\partial^r \psi + iq\psi \Gamma^\mu_{\mu t} A^t \\ &= -\tilde{f}\psi'' - \tilde{f}'\psi' - \frac{2\tilde{f}}{r}\psi' \,, \end{aligned} \tag{B.17}$$

where in going from the second to the third line we exploited the hypothesis of the fields discussed above, while in going from the third to the fourth we used $\Gamma^\mu_{\mu t} = 0$ and $\Gamma^\mu_{\mu r} = 2/r$. The second term in (B.3) is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi^*} &= -(m^2 + q^2 A_\mu A^\mu)\psi \\ &= -\left(m^2 - \frac{q^2 A_t^2}{\tilde{f}}\right)\psi \,, \end{aligned} \tag{B.18}$$

directly giving the equation for ψ

$$\psi'' + \left(\frac{\tilde{f}'}{\tilde{f}} + \frac{2}{r}\right)\psi' - \left(\frac{m^2}{\tilde{f}} - \frac{q^2 A_t^2}{\tilde{f}^2}\right)\psi = 0 \,. \tag{B.19}$$

Adding the perturbation A_x

In order to compute the optical conductivity we used equation (5.36). We now prove how it is obtained. The equation for the perturbation $A_x(r,t) = a_x(r)e^{-i\omega t}$ are obtained by linearizing (B.2) around the background solution found above, where ψ and A_t satisfy (B.19) and (B.13), respectively. The addition of A_x implies that we have to keep F_{rx} and F_{tx} at first order. The invariants $\mathcal{S}$ and $\mathcal{P}$ are unaffected by the perturbation as they are quadratic in the fields. Thus we start from the kinetic term with $\mathcal{S} < 0$ and $\mathcal{P} = 0$:

$$\begin{aligned} \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_{[\mu} A_{x]}} &= -e^\gamma \nabla_\mu F^{\mu x} \\ &= -e^\gamma (\partial_\mu F^{\mu x} + \Gamma^\mu_{\mu\alpha} F^{\alpha x}) \\ &= -e^\gamma \left(\partial_r F^{rx} + \partial_t F^{tx} + \frac{2}{r}F^{rx}\right) \,. \end{aligned} \tag{B.20}$$

The potential term is taken from (B.10)

$$\frac{\partial \mathcal{L}}{\partial A_x} = -2q^2 A^x |\psi|^2 \,, \tag{B.21}$$

where the derivatives of ψ do not appear as the background field ψ depends only on r. The Euler-Lagrange equations obtained from (B.20) and (B.21) are

$$\partial_r F^{rx} + \frac{2}{r} F^{rx} + \partial_t F^{tx} - 2e^{-\gamma} q^2 A^x |\psi|^2 = 0 \,. \tag{B.22}$$

By lowering the indices of F^{rx}

$$\partial_r (F_{rx} g^{rr} g^{xx}) + \frac{2}{r} (F_{tx} g^{rr} g^{xx}) + \partial_t (F_{tx} g^{tt} g^{xx}) - 2e^{-\gamma} q^2 A_x g^{xx} |\psi|^2 = 0 \,. \tag{B.23}$$

This is simplified observing that $\partial_r g^{xx} = -\frac{2}{r} g^{xx}$, leading to

$$F'_{rx} g^{rr} + F_{rx} (g^{rr})' + \dot{F}_{tx} g^{tt} - 2e^{-\gamma} q^2 A_x |\psi|^2 = 0 \,, \tag{B.24}$$

where the prime is ∂_r and the dot is ∂_t. Now using $g^{rr} = \tilde{f}$, $g^{tt} = -\tilde{f}^{-1}$ and that is $A_x = a_x(r) e^{-i\omega t}$ we get the final equation

$$a''_x + \frac{\tilde{f}'}{\tilde{f}} a'_x + \frac{\omega^2}{\tilde{f}^2} a_x - e^{-\gamma} \frac{2q^2 |\psi|^2}{\tilde{f}} a_x = 0 \,. \tag{B.25}$$

B.2 Generalized Born-Infeld case

We compute the equation for ϕ as given in (5.35). We do the calculations assuming ${}^*F = 0$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F_{rt} = \phi'$ is the only non-vanishing component of F. As before, the electromagnetic invariants are given by

$$\mathcal{S} = -F_{rt}^2 = -\phi'^2 \,, \qquad \mathcal{P} = 0 \,, \tag{B.26}$$

and the ModMax Lagrangian is

$$\mathcal{L}_{\text{ModMax}} = -\frac{\cosh\gamma}{2} \mathcal{S} + \frac{\sinh\gamma}{2} \sqrt{\mathcal{S}^2 + \mathcal{P}^2} = -e^{\gamma} \frac{\mathcal{S}}{2} \tag{B.27}$$

In order to write the Euler-Lagrande equations for $\mathcal{L}_{\gamma\text{BI}}$ of (3.24) we start computing

$$\begin{aligned}
\frac{\partial \mathcal{L}_{\gamma\text{BI}}}{\partial F_{\mu\nu}} &= \frac{\partial}{\partial F^{\mu\nu}} \left(T - \sqrt{T^2 - 2T\mathcal{L}_{\text{ModMax}}} \right) \\
&= T \frac{\partial}{\partial F^{\mu\nu}} \left(1 - \sqrt{1 - 2\mathcal{L}_{\text{ModMax}}/T} \right) \\
&= -\frac{1}{2} \frac{-2 \frac{\partial \mathcal{L}_{\text{ModMax}}}{\partial F_{\mu\nu}}}{\sqrt{1 - 2\mathcal{L}_{\text{ModMax}}/T}} \\
&= -\frac{e^{\gamma}}{2} \frac{\frac{\partial \mathcal{S}}{\partial F_{\mu\nu}}}{\sqrt{1 - 2\mathcal{L}_{\text{ModMax}}/T}} \\
&= -\frac{2e^{\gamma}}{2} \frac{F^{\mu\nu}}{\sqrt{1 - 2\mathcal{L}_{\text{ModMax}}/T}} \,.
\end{aligned} \tag{B.28}$$

In going from the third to the forth line we used (B.27) and from the fourth to the last the derivative of $\mathcal{S}$ produced an extra factor of 2 because we already accounted for the antisymmetry of F. The kinetic term of the Euler-Lagrange equation is

$$\nabla_\mu \frac{\partial \mathcal{L}_{\gamma\text{BI}}}{\partial F_{\mu\nu}} = \partial_\mu \frac{\partial \mathcal{L}_{\gamma\text{BI}}}{\partial F_{\mu\nu}} + \Gamma^{\mu}_{\mu\alpha} \frac{\partial \mathcal{L}_{\gamma\text{BI}}}{\partial F_{\alpha\nu}} \tag{B.29}$$

The only contribution from the Christoffel symbol is when $\alpha = r$, and $\Gamma^{\mu}_{\mu r} = 2/r$. There would have been another contraction with Christoffel symbols but it vanishes because of the antisymmetry of F. The $\nu = t$ kinetic term is then

$$\partial_r \frac{\partial \mathcal{L}_{\gamma\mathrm{BI}}}{\partial F_{rt}} + \frac{2}{r}\frac{\partial \mathcal{L}_{\gamma\mathrm{BI}}}{\partial F_{rt}} = -e^{\gamma}\left(\frac{\partial_r F^{rt}}{\sqrt{1-2\mathcal{L}_{\mathrm{ModMax}}/T}} + \frac{F^{rt}\partial_r \mathcal{L}_{\mathrm{ModMax}}/T}{(1-2\mathcal{L}_{\mathrm{ModMax}}/T)^{3/2}} + \frac{2}{r}\frac{F^{rt}}{\sqrt{1-2\mathcal{L}_{\mathrm{ModMax}}/T}}\right). \tag{B.30}$$

From (B.26) and (B.27),

$$\begin{aligned}\mathcal{L}_{\mathrm{ModMax}} &= \frac{e^{\gamma}}{2}\phi'^2\,,\\ \partial_r \mathcal{L}_{\mathrm{ModMax}} &= e^{\gamma}\phi'\phi''\,,\\ F^{rt} = F_{rt}g^{tt}g^{rr} &= -\phi'\,,\\ \partial_r F^{rt} &= -\phi''\,,\end{aligned} \tag{B.31}$$

giving for the kinetic term the final expression

$$\begin{aligned}\nabla_{\mu}\frac{\partial \mathcal{L}_{\gamma\mathrm{BI}}}{\partial F_{\mu t}} &= \frac{-e^{\gamma}}{\sqrt{1-e^{\gamma}\phi'^2/T}}\left(-\phi'' - \frac{e^{\gamma}\phi'^2\phi''/T}{1-e^{\gamma}\phi'^2/T} - \frac{2}{r}\phi'\right)\\ &= \frac{e^{\gamma}}{\sqrt{1-e^{\gamma}\phi'^2/T}}\left(\frac{\phi''}{1-e^{\gamma}\phi'^2/T} + \frac{2}{r}\phi'\right)\end{aligned} \tag{B.32}$$

The potential term in the same found in the previous section

$$\frac{\partial \mathcal{L}}{\partial A_t} = -2q^2 A^t|\psi|^2 = \frac{2q^2|\psi|^2}{\tilde{f}}\phi\,. \tag{B.33}$$

The equation for ϕ is then

$$\phi'' + \frac{2}{r}\left(1-\frac{e^{\gamma}\phi'^2}{T}\right)\phi' - \frac{2q^2|\psi|^2}{\tilde{f}}e^{-\gamma}\left(1-\frac{e^{\gamma}\phi'^2}{T}\right)^{3/2}\phi = 0 \tag{B.34}$$

or, introducing $\beta^2 = T^{-1}$

$$\phi'' + \frac{2}{r}\left(1-e^{\gamma}\beta^2\phi'^2\right)\phi' - \frac{2q^2|\psi|^2}{\tilde{f}}e^{-\gamma}\left(1-e^{\gamma}\beta^2\phi'^2\right)^{3/2}\phi = 0\,. \tag{B.35}$$

In the weak field limit $\beta^2 \to 0$, ModMax equation (5.23) is correctly recovered. While for $\gamma \to 0$ we recover the usual Born-Infeld equation and equation (5.34) is recovered.

Appendix C

Detailed numerical analysis

Here we describe in detail how the numerical analysis used for holographic superconductors was performed. We refer to the ModMax superconductor introduced in section 5.3. Firstly, our input parameters are

- the AdS curvature radius L;
- the horizon radius r_h;
- the charge q of the scalar;
- the mass m^2 of the scalar;
- the boundary chemical potential μ;
- the value γ of the ModMax coupling constant.

Secondly, in all the numerical calculations we used $L = 1$, which makes everything turn dimensionless, and $m^2L^2 = m^2 = -2$. Thirdly, we used the variable $z = r^{-1}$. With this choice we have a finite range for the radial variable as the horizon is at $z_h = r_h^{-1}$ and the boundary is at $z = 0$.

Before discussing the numerics, we rephrase the problem in terms of the variable z. The system we want to solve is that of section 5.3, which consists of a second order differential equation (5.22) for ψ, a second order differential equation (5.23) for ϕ. We do not write down the equations for ψ and ϕ in the variable z as it is not particularly illuminating. The asymptotic behaviours (5.24) in the variable z read

$$\begin{aligned} \psi(z \to 0) &= \psi^{(1)} z + \psi^{(2)} z^2 \, , \\ \phi(z \to 0) &= \phi^{(0)} - \phi^{(1)} z \, . \end{aligned} \tag{C.1}$$

Notice that in this variable the falloffs can be easily expressed in terms of derivatives of the fields. Indeed,

$$\begin{aligned} \psi^{(1)} &= \psi'(0) \, , \qquad & \psi^{(2)} &= \frac{1}{2}\psi''(0) \, , \\ \phi^{(0)} &= \phi(0) \, , \qquad & \phi^{(1)} &= -\phi'(0) \, , \end{aligned} \tag{C.2}$$

the prime being differentiation with respect to z. Recall that these are not conditions to be imposed, but just the analytic solution in a region close to the boundary. In the variable z, the boundary conditions to be imposed at the horizon (5.25) and at infinity (5.26) are much easier to handle. For the boundary conditions at the horizon we have

$$\begin{cases} \psi'(z_h) &= -\dfrac{m^2}{3z_h}\psi(z_h) \\ \phi(z_h) &= 0 \end{cases} , \tag{C.3}$$

while at the boundary

$$\begin{cases} \psi'(0) &= 0 \\ \phi(0) &= \mu \end{cases} . \tag{C.4}$$

We stress again that the proportionality between $\psi(z_h)$ and $\psi'(z_h)$ is a regularity requirement. Actually, there can be different values of $\psi(z_h)$ for which the system can be solved. The first boundary condition in $z = 0$ tells that the function ψ is a parabola near $z = 0$.

Now we move to the literal numerical procedure. The numerical analysis was done using `Wolfram Mathematica`. The boundary conditions at the horizon cannot literally be set there, as the emblackening factor in the equation is divergent and the numerics fail. We thus set the boundary conditions just off the horizon, at $z_{hb} = r_{hb}^{-1}$ with

$$r_{hb} = r_h(1 + 10^{-6}). \tag{C.5}$$

We considered 10^{-6} small enough for the precision needed. Similarly, the boundary conditions at the boundary are not set at 0 but at $\epsilon =$ `$MachineEpsilon` $\approx 2 \cdot 10^{-16}$. In order to solve the system we set the following boundary conditions

$$\begin{aligned} \psi(z_{hb}) &= k \, , \\ \psi'(z_{hb}) &= -\frac{m^2}{3z_h}k \, , \\ \phi(z_{hb}) &= 0 \, , \\ \phi(\epsilon) &= \mu \, . \end{aligned} \tag{C.6}$$

Notice that we gave up the boundary condition $\psi'(\epsilon) = 0$, but we now have a precise boundary condition at z_{hb} for ψ and ψ', parametrized by k. `Mathematica` can solve this parametric system using `ParametricNDSolve` (it does so by "shooting" ϕ) and gives a numerical solutions $\{\psi_k(z), \phi_k(z)\}$ in the interval $[\epsilon, z_{hb}]$. Clearly this solution will not satisfy $\psi_k'(\epsilon) = 0$, as we would like. In order to find the solution with the desired property, we construct the function of k

$$h(k) = \psi_k'(\epsilon) \tag{C.7}$$

and find the roots of $h(k)$.[1] The value $k = 0$ is always a root, and it corresponds to having no scalar field at all. Below a critical value of r_h (basically, of the temperature) there will be more roots k_i, each of which will correspond to a solution $\psi_{k_i}(z)$ with the desired property that $\psi_{k_i}'(\epsilon) = 0$. We chose the root $\bar{k}$ for which corresponding solution $\psi_{\bar{k}}(z)$ has zero nodes. This is usually the largest of the k_i's, but we always checked that the sign of $\psi_{\bar{k}}$ did not change (i.e. it stayed positive) in $[\epsilon, z_{hb}]$. Once obtained the desired solution satisfying all of the required properties ($\psi_{\bar{k}}$ has no nodes, $\psi_{\bar{k}}'(\epsilon) = 0$, and all other boundary conditions satisfied), the fast falloffs $\psi^{(2)}$ and $\phi^{(1)}$ can be computed using (C.2). The solution $\{\psi_k(z), \phi_k(z)\}$ can also be used to compute the optical conductivity. About

[1] We use the letter h because this is basically (proportional to) the source of $\mathcal{O}_2$.

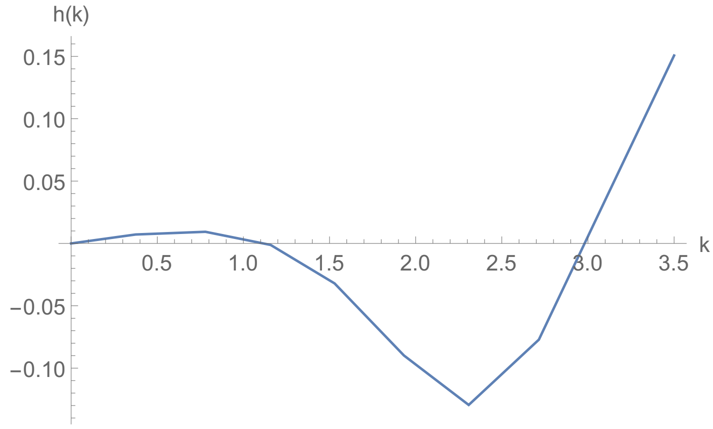

Figure C.1: Example of $h(k)$ and its roots. There are three roots. When $k \approx 3$ the solution $\psi_3(z)$ has no nodes. When $k \approx 1.1$ the solution $\psi_{1.1}(z)$ has one node. When $k = 0$ the solution $\psi_0(z)$ is trivial. The exact roots are found using `FindRoot`.

this, the numerics is very standard. We just mention that the boundary conditions for $a_x(z)$ are taken from (5.38) and in our code read

$$\begin{aligned} a_x(z_{hb}) &= \tilde{f}(z_{hb})^{-i\omega z_h/3}\,, \\ a_x'(z_{hb}) &= \left[\tilde{f}(z)^{-i\omega z/3}\right]'_{z=z_{hb}}\,. \end{aligned} \tag{C.8}$$

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