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Credit Crises and Large Portfolio Losses

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Introduction

Credit risk refers to the possibility of a default on a debt resulting from a borrower's failure to repay a loan or meet contractual obligations.
It is extremely difficult and complex to pinpoint exactly how likely a firm is to default on their loan. At the same time, properly assessing credit risk can reduce the likelihood of losses from default and delayed repayment.
Credit risk modeling is the best way for lenders to understand how likely a particular loan is to get repaid. In other words, it's a tool to understand the credit risk of a borrower.
In this thesis we consider a network of firms linked by business relationships and a financial institution, we can think of a bank, holding a large portfolio with positions issued by the firms. We are in a situation where a bank buy debts (that is equivalent to say that lends money) to a large number of firms and then these firms should give the money back to the lender.
As the firms are linked among each others, a financial distress in one of these firms can spread to the others causing the so called credit contagion effect. Credit contagion indeed refers to the propagation of economic distress from one firm or sovereign government to another. To understand how the financial distress spreads in our network of firms, we study a credit contagion model proposed by Dai Pra, Runggaldier, Sartori and Tolotti [3].
This model, despite being only a qualitative model, describes the phenomenon of credit risk and credit contagion well enough: we deal with a model where the variables are binary and the interaction is supposed to be of the mean field type.
Using this type of framework we explain the idea of credit crisis, how this phenomenon depends on our model parameters and the consequences that the clustering of defaults have on the portfolio hold by a bank with positions issued by the firms.
With credit crises we mean that there is evidence, looking at real data, of periods in which many firms end up in financial distress in a very short time. More specifically, looking at the dynamics of particular variables and looking at their variances, we are able to observe this phenomenon: how financial

contagion can cause a sudden period characterized by a financial crisis.
We can observe for example how, during certain period of time, the variance of different processes may increase sharply removing the possibility of predicting what will happen in the future.
After a good understanding about the meaning and the nature of a credit crisis through mathematical approaches and numerical simulations, we shall provide formulas to compute quantiles of the probability of excess losses in the context of our contagion model. In other words we try to estimate the fluctuation of aggregate credit losses for a bank holding large portfolios of financial positions.
To study properly the portfolio losses topic we propose different types of simulations varying the parameters and the initial conditions. Our work consists in giving first theoretical results that later have to be supplemented with numerical simulations. In this way we are able to show the main topics of this thesis completely.
To have some possible intuitions about the consistency of our model with real market data, we show what happens historically to credit and financial markets during period of credit crises.

The outline of the paper is as follows. In Chapter 1 we introduce the already existing contagion models literature and we present our model. Chapter 2 is devoted to stating the main limit theorems on the stochastic dynamics, in particular a Law of Large Numbers and a Central Limit Theorem. In Chapter 3 we introduce the phenomenon of credit crises and we start showing numerical simulations related to these. The financial application and the idea of portfolio losses is explained in Chapter 4 through theoretical results and numerical simulations. We end the paper with some concluding remarks in particular related to the consistency of our model with real market data.
Appendix A and Appendix B contain respectively the main theoretical tools used during the thesis and the Matlab codes to generate all the numerical simulations.

Chapter 1

A model for contagion

1.1 Existing literature

Credit risk models play a fundamental role in properly assessing a firm's credit risk, reducing in this way the likelihood of losses due to defaults or delayed repayments. A credit risk model is used by a bank to estimate a credit portfolio's PDF (the probability density function). In this regard, credit risk models can be divided into two main classes: structural and reduced form models. Structural models are used to calculate the probability of default for a firm based on the value of its assets and liabilities. The basic idea is that a company (with limited liability) defaults if the value of its assets is less than the debt of the company. Reduced form models typically assume an exogenous cause of default. They model default as a random event without any focus on the firm's balance sheet. This random event of default is described as a Poisson event. As Poisson models look at the arrival rate, or intensity, of a specific event, this approach to credit risk modelling is also referred to as default intensity modelling.
In other words, comparing structural and reduced form credit risk models from an information based perspective, we can say that the difference between these two approaches can be characterized in terms of the information assumed known by the modeler. Structural models assume that the modeler has the same information set as the firm's manager—complete knowledge of all the firm's assets and liabilities. In contrast, reduced form models assume that the modeler has the same information set as the market—incomplete knowledge of the firm's condition. Structural models were originated with Black and Scholes (1973), Merton (1974) whereas reduced form models were due to Jarrow and Turnbull (1992), and subsequently studied by Jarrow and Turnbull (1995), Duffie and Singleton (1999) among others.

The model we are going to study in this thesis is to be considered within the class of reduced form models and is based on interacting intensities. The probability of having a default somewhere in the network depends also on the state of the other obligors.
Our model is based on different rating classes: the rating classes are published by credit rating agencies and used by investment professionals to assess the likelihood the debt of a certain firm will be repaid. The "Big Three" credit rating agencies are Fitch Ratings, Moody's and Standard & Poor's (S&P) — controlling approximately 95% of the ratings business.
Through the existing literature, different credit risk models can be taken into consideration. Some of these are, for example, the "Bernoulli mixture models" that, in the context of contagion-based models, was first introduced by Giesecke and Weber [8], a cascade contagion process described by Horst [9] or the large-deviations approach by Dembo, Deuschel and Duffie [5].

"Bernoulli mixture models" is one of the credit risk models we are going to use in the following chapters. In this approach, fluctuations of aggregate losses are due to the fluctuation in some exogenous macro-economic variables. This kind of modeling has been used in the context of cyclical correlations, where cyclical correlation are due to the dependence of the firms on common economic exogenous factors. This type of model is really useful to emphasize both contagion and cyclical effects on the rating probabilities.
Horst's model is an interactive model of credit ratings where external shocks spread, by a contagious chain reaction, to the entire economy. Counterparty relationships along with discrete adjustments of credit ratings, generate a transition mechanism that allows the financial distress of one firm to spill over to its business partners. Here it is emphasized how even small shocks, initially affecting only a small number of firms, spread by a contagious chain reaction to the rest of the economy, creating the so called cascade process.
In a large economy, this cascade can be described by a branching process. Under the assumption that the interaction between different firms is weak enough, the distribution of the total number of defaults can be given in closed form.
In 2004, Dembo, Deuschel and Duffie provided in their paper a large-deviations approximation of the tail distribution of total financial losses on large portfolios of heterogenous credit securities. They depart from the classical ruin-theory approach, as it ignores unrealistically the re-capitalization at low levels of capital.

On the other hand, the paper we have taken into consideration for the framework we are going to adopt, is *Large Portfolio Losses: A dynamic contagion*

model written by Dai Pra, Runggaldier, Sartori, Tolotti in 2009 (see [3]). This paper describes the problem of contagion in a different way: it uses methodologies that belong to statistical mechanics, using particle systems as a way to describe the idea of a network where agents interact with each other. The next sections focus on introducing the main assumptions and the methodology used in the contagion model that we take into account.
Our model is a mean field interacting model of the Curie-Weiss type, where each particle interacts with all the others in the same way.

1.2 The mean field hypothesis

The mean field approach studies the behavior of high-dimensional random systems by studying a simpler system that approximates the original one by averaging over degrees of freedom.
What characterizes a mean field model, within a large class of particle systems, is the absence of a "geometry" in the configuration space, meaning that each particle interacts with all the others in the same way. This approximation amounts to assume that each unit interacts with the rest of the network in a homogeneous "average" way: instead of following the evolution of each single unit, we describe the system through the evolution of a probability measure which takes into consideration the fraction of the population sharing a certain state.
Although it is a strong assumption, it can be seen quite reasonable in this setting; the market, due to the technology that characterizes our present, is deeply interconnected.
If we are considering a large group of firms belonging to the same sector, then the ability of generating cash flows or the capacity of raising capital from financial institutions may be considered as "homogeneous" characteristics within the group. This approach allows us to describe the dynamics of a large system with few aggregate statistics. The final aim of this work is to study aggregate quantities for a large economy such as the expected global health of the system and large portfolio losses as well as related quantities.
In literature other assumptions, different from the mean field one, have been largely used: for example a local-interaction model, where the particles are supposed to interact with a fixed number d of neighbors (see the Voter model described by Giesecke and Weber [8]), or some random graph approach type, meaning that the connections are randomly generated with some distribution functions.

1.3 Coming back to the Curie-Weiss Model

The Curie-Weiss model is an exactly solvable model of Ferromagnetism that allows to study in detail the thermodynamic functions.
We start showing the main aspects of this model as, the one that we will use in this paper, is a generalization of the Curie-Weiss one. The model assumptions are the following:

- We have N firms (we will think of N particles), with N large.
- State variables $\sigma_1, \dots, \sigma_N \in$ {-1,1} called "spins".
- Time interval: $t \geq 0$ (continuous time)
  $\underline{\sigma}(t) = (\sigma_1(t), \dots, \sigma_N(t))$ represents the configuration at time t of the system.
- The mean field assumption: this approach leads to let the interaction depends on the "magnetization" (e.g. the empirical mean)

  $$m_N^\sigma := \frac{1}{N}\sum_{i=1}^{N}\sigma_i \quad \in \{-1, -1+\frac{2}{N}, \dots, 1-\frac{2}{N}, 1\} \subseteq [-1,1].$$

  This variable plays a fundamental role on the study of the model; it is the only aggregate variable that allows to describe the dynamics of the whole system.
  The transition from $\sigma_i \mapsto -\sigma_i$ is given by the following intensity

  $$\sigma_i \mapsto -\sigma_i \quad \text{with intensity } e^{-\beta\sigma_i m_N^\sigma}, \quad \beta > 0$$

  where β represents the inverse of the temperature.
- The state variables form a continuous-time Markov Chain on the configuration space $\{-1,1\}^2$. Its evolution is described by the infinitesimal generator $\mathcal{L}$ acting on functions $f : \{-1,1\}^2 \to \mathbb{R}$ as

  $\mathcal{L}f(\underline{\sigma}) = \sum_{i=1}^{N} e^{-\beta\sigma_i m_N}[f(\underline{\sigma}^i) - f(\underline{\sigma})]$

  where $\underline{\sigma}^i = (\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots, \sigma_N)$ represents the configuration after the jump of the i^{th} component.

1.4 A more general model

Let consider a network of N firms. The state of each firm is identified by two variables, (σ_i, ω_i), that denotes the state of the ith firm, $i = 1, \ldots, N$. The variable σ may be interpreted as the rating class indicator: a low value reflects a bad rating class, that is, a higher probability of not being able to pay back obligations; this variable represents the perceived/observed state.
The variable ω represents a more intrinsic indicator of the financial health of the firm and typically is not directly observable on the market. We can think at ω as the real state (may be the liquidity or a well-being factor).
Inspired by the Curie-Weiss model, we assume the two indicators σ_i, ω_i are binary variables, i.e. they can take only two values, that we label by 1 ("good" financial state) and -1 (financial distress). Recalling that the final aim of this work is to describe propagation of financial distress in a network of firms linked by business relationships, we are naturally led to an interacting intensity model, where we have to specify the intensities or rates (inverse of the average waiting times) at which the transitions $\sigma_i \mapsto -\sigma_i$ and $\omega_i \mapsto -\omega_i$ take place. Again, as in Curie-Weiss model, if we neglect direct interactions between the ω_i's and we make the mean field assumption that the interaction between different firms only depends on the value of the global financial health indicator

$$m_N^\sigma := \frac{1}{N} \sum_{i=1}^{N} \sigma_i$$

we are let to consider the following intensities

$$\begin{aligned} \sigma_i \mapsto -\sigma_i & \quad \text{with intensity } e^{-\beta \sigma_i \omega_i}, \quad \beta > 0 \\ \omega_i \mapsto -\omega_i & \quad \text{with intensity } e^{-\gamma \omega_i m_N^\sigma} \quad \gamma > 0 \end{aligned}$$

where β represents how much the market gets aware of the change of ω_i, whereas γ indicates the strength of the contagion.
The variable m_N^σ is a global financial health indicator that is simply an empirical mean of the different perceived states of the N firms. It gives an idea on how bad or good the firms are doing in their perceived states.

In order to have a full understanding of the parameters, we show how the contagion works; the vehicle of contagion is given by:

$$\omega_i \xrightarrow{1} \sigma_i \longrightarrow m_N^\sigma \xrightarrow{2} \omega_j$$

First the real state ω influences the perceived state σ (1), then σ, by definition of m_N^σ, changes m_N^σ itself. As last, if m_N^σ gets better/worse, all firms are affected in their real state (2).
From a theoretical perspective we are dealing with a $\{-1,1\}^2$-valued continuous Markov Chain $(\sigma_i[0,T], \omega_i[0,T])_{i=1}^N$ with the following infinitesimal generator:

$$\mathcal{L}f(\underline{\sigma},\underline{\omega}) = \sum_{i=1}^N e^{-\beta\sigma_i\omega_i}\nabla_i^\sigma f(\underline{\sigma},\underline{\omega}) + \sum_{i=1}^N e^{-\gamma\omega_i m_N^\sigma}\nabla_i^\omega f(\underline{\sigma},\underline{\omega}) \tag{1.1}$$

where $\nabla_i^\sigma f(\underline{\sigma},\underline{\omega}) = f(\underline{\sigma}^i,\underline{\omega}) - f(\underline{\sigma},\underline{\omega})$ (analogously for ∇_i^ω) , and where the j^{th} component of $\underline{\sigma}^i$ is

$$\sigma_j^i = \begin{cases} \sigma_j, & \text{for } j \neq i \\ -\sigma_i, & \text{for } j = i \end{cases}$$

1.5 Methodology

Our interacting particle system is described by the two variables (σ,ω). What we are going to show now is that, however, our system is nonreversible: the lack of reversibility will lead us to have an unusual approach to study the behavior of our system, different from the classical approach in Statistical Mechanics. First of all let's recall some useful notions about stationary distributions.
All the following Definitions and Theorems about stochastic processes that we are going to recall are at discrete time; the Markov process above instead is in continuous time. However all the following statements can simply be translated in similar definitions and theorems at continuous time too.

Definition 1.1. *A probability distribution $\pi = (\pi(1),\ldots,\pi(k))$ is a stationary distribution (or invariant distribution) for $M = (m_{ij})_{ij\in E}$ if*

$$\pi\ M = \pi$$

or equivalently

$$\sum_{i\in E}\pi(i)\ m_{ij} = \pi(j) \quad \forall j$$

where E is the configuration space and M the transition matrix.

In other words a stationary distribution is a probability distribution that remains constant as the time of the Markov Chain evolves.
To show that our model is nonreversible, let's consider equation (1.1): the

operator $\mathcal{L}$ given in (1.1) defines an irreducible, finite-state Markov chain. It follows, from the following three theorems,

- Theorem Irreducible, finite-state Markov chain $\Rightarrow$ positively recurrent
- Theorem Irreducible, homogeneous Markov chain is:
  positively recurrent $\Longleftrightarrow$ has a stationary distribution
- Theorem The invariant measure of a irreducible, recurrent Markov chain is unique except for a multiplicative factor.

that the process admits a unique stationary distribution μ_N.
From Kolmogorov's equations, we know that if π is invariant then $\pi\ A = 0$ where A is the infinitesimal generator. By this implication we can state that μ_N is a distribution such that, for each function f on the configuration space of (σ, ω),

$$\sum_{\underline{\sigma},\underline{\omega}} \mu_N(\underline{\sigma}, \underline{\omega}) \mathcal{L} f(\underline{\sigma}, \underline{\omega}) = 0. \tag{1.2}$$

This distribution reflects the long-time behavior of the system, in the sense that, for each f and any initial distribution,

$$\lim_{t \to +\infty} E[f(\underline{\sigma}(t), \underline{\omega}(t))] = \sum_{\underline{\sigma},\underline{\omega}} \mu_N(\underline{\sigma}, \underline{\omega}) f(\underline{\sigma}, \underline{\omega}).$$

The previous stationary condition (1.2) is equivalent to

$$\sum_{i=1}^{N} \left[\mu_N(\underline{\sigma}^i, \underline{\omega}) e^{\beta \sigma_i \omega_i} - \mu_N(\underline{\sigma}, \underline{\omega}) e^{-\beta \sigma_i \omega_i} \right]$$
$$+ \sum_{i=1}^{N} \left[\mu_N(\sigma, \underline{\omega}^i) e^{\gamma \omega_i m_N^\sigma} - \mu_N(\underline{\sigma}, \underline{\omega}) e^{-\gamma \omega_i m_N^\sigma} \right] = 0$$

for every $\underline{\sigma}, \underline{\omega} \in \{-1, 1\}^N$.

A simpler sufficient condition for stationarity is the detailed balance condition.

Definition 1.2. *Given a Markov Chain with transition matrix $M = (m_{ij})_{ij \in E}$, we say that π satisfy DBE if*

$$\pi(i)\ m_{ij} = \pi(j)\ m_{ji}.$$

In other words we say that a probability v on $\{-1,1\}^{2N}$ satisfies the detailed balance condition for the generator $\mathcal{L}$ if

$$\begin{aligned} v(\underline{\sigma}^i,\underline{\omega})e^{\beta\sigma_i\omega_i} &= v(\underline{\sigma},\underline{\omega})e^{-\beta\sigma_i\omega_i} \quad \text{and} \\ v(\underline{\sigma},\underline{\omega}^i)e^{\gamma\omega_i m_N^\sigma} &= v(\underline{\sigma},\underline{\omega})e^{-\gamma\omega_i m_N^\sigma} \end{aligned} \tag{1.3}$$

for every $\underline{\sigma},\underline{\omega}$.
To understand in a better way, we can say that, for example, $v(\underline{\sigma}^i,\underline{\omega})e^{\beta\sigma_i\omega_i}$ is the probability to start in $(\underline{\sigma}^i,\underline{\omega})$ $(v(\underline{\sigma}^i,\underline{\omega}))$ and go to $(\underline{\sigma},\underline{\omega})$ (in fact the intensity of $\underline{\sigma}^i \mapsto \underline{\sigma}$ is $e^{\beta\sigma_i\omega_i}$)
The utility of the DBE is given by the following Theorem:

<u>Theorem</u> If π satisfy a DBE for $M = (m_{ij})_{ij\in E} \Rightarrow \pi$ is invariant for the matrix M.
About terminology, when the detailed balance conditions (1.3) hold, we say the system is reversible: in the case (1.3) admits a solution, they usually allow to derive the stationary distribution explicitly. The following proposition show us that our system is not reversible, showing that the detailed balance equation (1.3) doesn't admit a solution.

Proposition 1.1. *The detailed balance equations* (1.3) *admit no solution, except at most for one specific value of N.*

Proof. By way of contradiction, assume a solution v of (1.3) exists. Then one easily obtains

$$\begin{aligned} \nabla_i^\sigma \log v(\underline{\sigma},\underline{\omega}) &= -2\beta\sigma_i\omega_i \\ \nabla_i^\omega \log v(\underline{\sigma},\underline{\omega}) &= -2\gamma\omega_i m_N^\sigma \end{aligned}$$

Indeed, for example,

$$v(\underline{\sigma}^i,\underline{\omega})e^{\beta\sigma_i\omega_i} = v(\underline{\sigma},\underline{\omega})e^{-\beta\sigma_i\omega_i} \quad \Rightarrow \quad \frac{v(\underline{\sigma}^i,\underline{\omega})}{v(\underline{\sigma},\underline{\omega})} = \frac{e^{-\beta\sigma_i\omega_i}}{e^{\beta\sigma_i\omega_i}} \quad \Rightarrow$$

$$\log\left(\frac{v(\underline{\sigma}^i,\underline{\omega})}{v(\underline{\sigma},\underline{\omega})}\right) = -2\beta\sigma_i\omega_i \quad \Rightarrow \quad \nabla_i^\sigma \log v(\underline{\sigma},\underline{\omega}) = -2\beta\sigma_i\omega_i$$

and the same computation for the second equation.
The previous two equations imply

$$\begin{aligned} \nabla_i^\omega\nabla_i^\sigma \log v(\underline{\sigma},\underline{\omega}) &= 4\beta\sigma_i\omega_i \\ \nabla_i^\sigma\nabla_i^\omega \log v(\underline{\sigma},\underline{\omega}) &= 4N^{-1}\gamma\omega_i\sigma_i \end{aligned}$$

This is not possible since $\nabla_i^\omega\nabla_i^\sigma \log v(\underline{\sigma},\underline{\omega}) \equiv \nabla_i^\sigma\nabla_i^\omega \log v(\underline{\sigma},\underline{\omega})$ $\square$

The nonreversibility of our model implies that an explicit formula for the stationary distribution ($t \to +\infty$) and its $N \to +\infty$ asymptotics is not available. Then, to understand the long-time behavior of our dynamic system, we have to follow a more specific approach: instead of using a so-called "static" approach we use a more dynamic one which we prefer, since we want to describe the dynamics of our system. We are indeed interested in describing dynamically the credit quality of a large number of firms. First we study the $N \to +\infty$ limiting distributions on the configuration space. To do this we use a of Law of Large Numbers. Then we consider the equilibria of the limiting dynamics ($t \to +\infty$); this leads to the study of phase transitions whose main goal is to capture the transitions from one equilibrium to another and to study the nature of such transitions. Finally we study the finite volume approximation of the limiting dynamics using a specific version of the Central Limit Theorem that allows to analyze the fluctuations around the limit.
Summarizing, our approach proceeds according to the following three steps:

1. Study of the limit dynamics of the system ($N \to +\infty$) obtaining evolution equations for the asymptotic system (with an infinite number of firms).

2. Study of the equilibria of the limiting dynamics.

3. Study of the finite volume approximation: the system is not truly with infinite dimension, meaning that the number of firms is large, but finite, so we analyze the rate of convergence/the error that we commit using the limiting system in place of the finite one.

Chapter 2

Going deeper in the model

2.1 Limiting Dynamics

In order to analyze the long-time behavior of our system, first of all, we study the limit dynamics: we send N, the number of firms, to infinity.
Let the $\{-1,1\}^2$-valued continuous Markov Chain $\left(\sigma_i^{(N)}[0,T], \omega_i^{(N)}[0,T]\right)_{i=1}^{N}$ with infinitesimal generator:

$$\mathcal{L}f(\underline{\sigma}, \underline{\omega}) = \sum_{i=1}^{N} e^{-\beta\sigma_i\omega_i} \nabla_i^{\sigma} f(\underline{\sigma}, \underline{\omega}) + \sum_{i=1}^{N} e^{-\gamma\omega_i m_N^{\sigma}} \nabla_i^{\omega} f(\underline{\sigma}, \underline{\omega}). \tag{2.1}$$

We now rewrite the previous equation (2.1) in terms of the following statistics:

- $m_N^{\sigma}(t) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i(t)$.
- $m_N^{\omega}(t) = \frac{1}{N}\sum_{i=1}^{N} w_i(t)$.
- $m_N^{\sigma\omega}(t) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i(t) w_i(t)$.

This allows to describe the dynamics of 2N variables $((\sigma_i, \omega_i)_{i=1}^{N})$ using only 3 order processes $(m_N^{\sigma}$,m_N^{ω},$m_N^{\sigma\omega})$.
Now we apply the infinitesimal generator to a function φ depending on these 3 processes and rewrite the infinitesimal generator as follows:

$$\mathcal{L}_N \varphi\left(m_N^{\sigma}, m_N^{\omega}, m_N^{\sigma\omega}\right) = \left(\mathcal{K}_N \varphi\right)\left(m_N^{\sigma}, m_N^{\omega}, m_N^{\sigma\omega}\right),$$

where φ depends on σ and ω through the 3 order processes. In particular this dependence is given by the fact that we are interested in the collective behavior of the system.

$$\mathcal{L}_N\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right) = \sum_{i=1}^{N}\left\{e^{-\beta\sigma_i\omega_i}\left[\varphi\left(m_N^\sigma - \frac{2}{N}\sigma_i, m_N^\omega, m_N^{\sigma\omega} - \frac{2}{N}\sigma_i\omega_i\right)\right.\right.$$
$$\left. -\ \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]$$
$$\left.+e^{-\gamma\omega_i m_N^\sigma}\left[\varphi\left(m_N^\sigma, m_N^\omega - \frac{2}{N}\omega_i, m_N^{\sigma\omega} - \frac{2}{N}\sigma_i\omega_i\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right\} = (*).$$

To rewrite the infinitesimal generator using only $m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}$, let's split the previous equation into these four different cases for the couple (σ,ω):
(1,1); (1,-1); (-1,1); (-1,-1)

Then:

$$\begin{aligned}
(*) = &\sum_{i=1}^{N}\frac{1+\sigma_i+\omega_i+\sigma_i\omega_i}{4}\left\{e^{-\beta}\left[\varphi\left(m_N^\sigma - \frac{2}{N}, m_N^\omega, m_N^{\sigma\omega} - \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right.\\
&\left.+e^{-\gamma m_N^\sigma}\left[\varphi\left(m_N^\sigma, m_N^\omega - \frac{2}{N}, m_N^{\sigma\omega} - \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right\}\\
&+\sum_{i=1}^{N}\frac{1+\sigma_i-\omega_i-\sigma_i\omega_i}{4}\left\{e^{\beta}\left[\varphi\left(m_N^\sigma - \frac{2}{N}, m_N^\omega, m_N^{\sigma\omega} + \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right.\\
&\left.+e^{\gamma m_N^\sigma}\left[\varphi\left(m_N^\sigma, m_N^\omega + \frac{2}{N}, m_N^{\sigma\omega} + \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right\}\\
&+\sum_{i=1}^{N}\frac{1-\sigma_i+\omega_i-\sigma_i\omega_i}{4}\left\{e^{\beta}\left[\varphi\left(m_N^\sigma + \frac{2}{N}, m_N^\omega, m_N^{\sigma\omega} + \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right.\\
&\left.+e^{-\gamma m_N^\sigma}\left[\varphi\left(m_N^\sigma, m_N^\omega - \frac{2}{N}, m_N^{\sigma\omega} + \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right\}\\
&+\sum_{i=1}^{N}\frac{1-\sigma_i-\omega_i+\sigma_i\omega_i}{4}\left\{e^{-\beta}\left[\varphi\left(m_N^\sigma + \frac{2}{N}, m_N^\omega, m_N^{\sigma\omega} - \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right.\\
&\left.+e^{\gamma m_N^\sigma}\left[\varphi\left(m_N^\sigma, m_N^\omega + \frac{2}{N}, m_N^{\sigma\omega} - \frac{2}{N}\right) - \varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right\}.
\end{aligned} \tag{2.2}$$

Now we can take the $\{.....\}$ outside the sums $\sum_{i=1}^{N}$ and notice that:

1. $\sum_{i=1}^{N} \frac{1+\sigma_i+\omega_i+\sigma_i\omega_i}{4} = \frac{N}{4}\left(1+m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}\right)$, if $(\sigma_i,\omega_i)=(1,1)$
2. $\sum_{i=1}^{N} \frac{1+\sigma_i-\omega_i-\sigma_i\omega_i}{4} = \frac{N}{4}\left(1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}\right)$, if $(\sigma_i,\omega_i)=(1,-1)$
3. $\sum_{i=1}^{N} \frac{1-\sigma_i+\omega_i-\sigma_i\omega_i}{4} = \frac{N}{4}\left(1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)$, if $(\sigma_i,\omega_i)=(-1,1)$
4. $\sum_{i=1}^{N} \frac{1-\sigma_i-\omega_i+\sigma_i\omega_i}{4} = \frac{N}{4}\left(1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)$, if $(\sigma_i,\omega_i)=(-1,-1)$

Equation (2.2) becomes:

$$\sum_{(j,k)\in\{-1,1\}} \frac{N}{4}\left(1+j\ m_N^\sigma+k\ m_N^\omega+jk\ m_N^{\sigma\omega}\right)\left\{e^{-\beta jk}\left[\varphi\left(m_N^\sigma-\frac{2}{N}j, m_N^\omega, m_N^{\sigma\omega}-\frac{2}{N}jk\right)\right.\right.$$
$$\left.\left.-\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]+e^{-\gamma k m_N^\sigma}\left[\varphi\left(m_N^\sigma, m_N^\omega-\frac{2}{N}k, m_N^{\sigma\omega}-\frac{2}{N}jk\right)-\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)\right]\right\}$$

We apply, now, the first order Taylor expansion centered at $(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega})$ to the terms $\varphi\left(m_N^\sigma-\frac{2}{N}j, m_N^\omega, m_N^{\sigma\omega}-\frac{2}{N}jk\right)$ and $\varphi\left(m_N^\sigma, m_N^\omega-\frac{2}{N}k, m_N^{\sigma\omega}-\frac{2}{N}jk\right)$. For example, considering the first term above, we get:

$$\varphi\left(m_N^\sigma-\frac{2}{N}j, m_N^\omega, m_N^{\sigma\omega}-\frac{2}{N}jk\right)=\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)+\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^\sigma}\left(-\frac{2}{N}j\right)$$

$$+\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^\omega}(0)+\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^{\sigma\omega}}\left(-\frac{2}{N}jk\right)+o\left(\sqrt{\left(-\frac{2}{N}j\right)^2+\left(-\frac{2}{N}jk\right)^2}\right)$$

where $o\left(\sqrt{\left(-\frac{2}{N}j\right)^2+\left(-\frac{2}{N}jk\right)^2}\right) = o\left(\sqrt{\frac{8}{N^2}}\right) = o\left(\frac{1}{N}\right)$. After having replaced the Tayolr expansion in the last equation we get:

$$\sum_{(j,k)\in\{-1,1\}} \frac{N}{4}\left(1+j\ m_N^\sigma+k\ m_N^\omega+jk\ m_N^{\sigma\omega}\right)\left\{e^{-\beta jk}\left[\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^\sigma}\left(-\frac{2}{N}j\right)\right.\right.$$
$$\left.+\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^{\sigma\omega}}\left(-\frac{2}{N}jk\right)+o\left(\frac{1}{N}\right)\right]$$
$$\left.+e^{-\gamma k m_N^\sigma}\left[\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^\omega}\left(-\frac{2}{N}k\right)+\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^{\sigma\omega}}\left(-\frac{2}{N}jk\right)+o\left(\frac{1}{N}\right)\right]\right\}$$

If we consider again the 4 different cases (1,1), (1,-1), (-1,1), (-1,-1) and we write, for simplicity, $\frac{\partial\varphi}{\partial m_N^\sigma}$ in place of $\frac{\partial\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)}{\partial m_N^\sigma}$ (the same thing for the other magnetizations) we get

$$\frac{N}{4}\left(1+m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}\right)\left\{e^{-\beta}\left[\frac{\partial\varphi}{\partial m_N^\sigma}\left(-\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\sigma\omega}}\left(-\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right.$$
$$\left.+e^{-\gamma m_N^\sigma}\left[\frac{\partial\varphi}{\partial m_N^\omega}\left(-\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\omega\sigma}}\left(-\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right\}$$
$$+\frac{N}{4}\left(1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}\right)\left\{e^{\beta}\left[\frac{\partial\varphi}{\partial m_N^\sigma}\left(-\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\sigma\omega}}\left(\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right.$$
$$\left.+e^{\gamma m_N^\sigma}\left[\frac{\partial\varphi}{\partial m_N^\omega}\left(\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\omega\sigma}}\left(\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right\}$$
$$+\frac{N}{4}\left(1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\left\{e^{\beta}\left[\frac{\partial\varphi}{\partial m_N^\sigma}\left(\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\sigma\omega}}\left(\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right.$$
$$\left.+e^{-\gamma m_N^\sigma}\left[\frac{\partial\varphi}{\partial m_N^\omega}\left(-\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\omega\sigma}}\left(\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right\}$$
$$+\frac{N}{4}\left(1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\left\{e^{-\beta}\left[\frac{\partial\varphi}{\partial m_N^\sigma}\left(\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\sigma\omega}}\left(-\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right.$$
$$\left.+e^{\gamma m_N^\sigma}\left[\frac{\partial\varphi}{\partial m_N^\omega}\left(\frac{2}{N}\right)+\frac{\partial\varphi}{\partial m_N^{\omega\sigma}}\left(-\frac{2}{N}\right)+o\left(\frac{1}{N}\right)\right]\right\}.$$

Now we group the terms together, finding the coefficients for $\frac{\partial\varphi}{\partial m_N^\sigma}, \frac{\partial\varphi}{\partial m_N^\omega}, \frac{\partial\varphi}{\partial m_N^{\sigma\omega}}$.

$$\text{Coefficient of } \frac{\partial\varphi}{\partial m_N^\sigma}: \qquad -\frac{1}{2}e^{-\beta}\frac{\partial\varphi}{\partial m_N^\sigma}\left(1+m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}\right)$$
$$-\frac{1}{2}e^{\beta}\frac{\partial\varphi}{\partial m_N^\sigma}\left(1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}\right)+\frac{1}{2}e^{\beta}\frac{\partial\varphi}{\partial m_N^\sigma}\left(1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)$$
$$+\frac{1}{2}e^{-\beta}\frac{\partial\varphi}{\partial m_N^\sigma}\left(1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)$$
$$=\frac{1}{2}\frac{\partial\varphi}{\partial m_N^\sigma}\left[e^{-\beta}\left(-1-m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}+1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\right.$$
$$\left.+e^{\beta}\left(-1-m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}+1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\right]$$
$$=\frac{\partial\varphi}{\partial m_N^\sigma}\left[e^{\beta}\left(-m_N^\sigma+m_N^\omega\right)+e^{-\beta}\left(-m_N^\sigma-m_N^\omega\right)\right].$$

$$\begin{aligned}
&\text{Coefficient of } \frac{\partial \varphi}{\partial m_N^\omega}: \quad -\frac{1}{2}e^{-\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^\omega}\left(1+m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}\right)\\
&+\frac{1}{2}e^{\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^\omega}\left(1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}\right)-\frac{1}{2}e^{-\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^\omega}\left(1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\\
&+\frac{1}{2}e^{\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^\omega}\left(1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\\
&=\frac{1}{2}\frac{\partial \varphi}{\partial m_N^\omega}\left[e^{-\gamma m_N^\sigma}\left(-1-m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}-1+m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\right.\\
&\qquad\qquad\left.+e^{\gamma m_N^\sigma}\left(+1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}+1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\right]\\
&=\frac{\partial \varphi}{\partial m_N^\omega}\left[e^{-\gamma m_N^\sigma}\left(-1-m_N^\omega\right)+e^{\gamma m_N^\sigma}\left(1-m_N^\omega\right)\right].
\end{aligned}$$

$$\begin{aligned}
&\text{Coefficient of } \frac{\partial \varphi}{\partial m_N^{\sigma\omega}}: \quad -\frac{1}{2}e^{-\beta}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1+m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}\right)\\
&-\frac{1}{2}e^{-\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1+m_N^\sigma+m_N^\omega+m_N^{\sigma\omega}\right)+\frac{1}{2}e^{\beta}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}\right)\\
&+\frac{1}{2}e^{\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}\right)+\frac{1}{2}e^{\beta}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\\
&+\frac{1}{2}e^{-\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)-\frac{1}{2}e^{-\beta}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\\
&-\frac{1}{2}e^{\gamma m_N^\sigma}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left(1-m_N^\sigma-m_N^\omega+m_N^{\sigma\omega}\right)\\
&=\frac{1}{2}\frac{\partial \varphi}{\partial m_N^{\sigma\omega}}\left[e^{-\beta}\left(-1-m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}-1+m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\right.\\
&\qquad\qquad+e^{\beta}\left(+1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}+1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\\
&\qquad\qquad+e^{-\gamma m_N^\sigma}\left(-1-m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}+1-m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\\
&\qquad\qquad\left.+e^{\gamma m_N^\sigma}\left(+1+m_N^\sigma-m_N^\omega-m_N^{\sigma\omega}-1+m_N^\sigma+m_N^\omega-m_N^{\sigma\omega}\right)\right]\\
&=\frac{\partial \varphi}{\partial m_N^\sigma}\left[e^{-\beta}\left(-1-m_N^{\sigma\omega}\right)+e^{\beta}\left(1-m_N^{\sigma\omega}\right)\right.\\
&\qquad\left.+e^{-\gamma m_N^\sigma}\left(-m_N^\sigma-m_N^{\sigma\omega}\right)+e^{\gamma m_N^\sigma}\left(m_N^\sigma-m_N^{\sigma\omega}\right)\right].
\end{aligned}$$

Using the hyperbolic functions we rewrite the relations above and we get that the infinitesimal generator $\mathcal{L}_N\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)$ is given by:

$$\begin{aligned}
&\mathcal{L}_N\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right) = \\
&= \frac{\partial\varphi}{\partial m_N^\sigma}\left[2\sinh(\beta)m_N^\omega - 2\cosh(\beta)m_N^\sigma\right] + \frac{\partial\varphi}{\partial m_N^\omega}\left[2\sinh\left(\gamma m_N^\sigma\right) - 2\cosh\left(\gamma m_N^\sigma\right)m_N^\omega\right] \\
&+ \frac{\partial\varphi}{\partial m_N^{\sigma\omega}}\left[2\sinh(\beta) + 2\sinh\left(\gamma m_N^\sigma\right)m_N^\sigma - 2\left(\cosh(\beta) + \cosh\left(\gamma m_N^\sigma\right)\right)m_N^{\sigma\omega}\right] + o(1)
\end{aligned} \tag{2.3}$$

If now we consider the $\lim_{N\to\infty}\mathcal{L}_N\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)$, as we are dealing with continuous objects, we obtain:

$$\begin{aligned}
&\mathcal{L}\varphi\left(m^\sigma, m^\omega, m^{\sigma\omega}\right) = \frac{\partial\varphi\left(m^\sigma, m^\omega, m^{\sigma\omega}\right)}{\partial m^\sigma}\left[2\sinh(\beta)m^\omega - 2\cosh(\beta)m^\sigma\right] \\
&+ \frac{\partial\varphi\left(m^\sigma, m^\omega, m^{\sigma\omega}\right)}{\partial m^\omega}\left[2\sinh\left(\gamma m^\sigma\right) - 2\cosh\left(\gamma m^\sigma\right)m^\omega\right] \\
&+ \frac{\partial\varphi\left(m^\sigma, m^\omega, m^{\sigma\omega}\right)}{\partial m^{\sigma\omega}}\left[2\sinh(\beta) + 2\sinh\left(\gamma m^\sigma\right)m_N^\sigma\right. \\
&\left.-2\left(\cosh(\beta) + \cosh\left(\gamma m^\sigma\right)\right)m^{\sigma\omega}\right],
\end{aligned} \tag{2.4}$$

where $(m^\sigma, m^\omega, m^{\sigma\omega}) \in [-1,1]^3$
$\left((m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}) \in \left\{-1, -1+\frac{2}{N}, \cdots, 1-\frac{2}{N}, 1\right\}^3 \subset [-1,1]^3\right)$

Now $\lim_{N\to\infty}\sup_{x\in E^3}|\mathcal{L}_N\varphi(x) - \mathcal{L}\varphi(x)| = 0$
then by Theorem A.1 (see Appendix A) that allows us to deduce, from the convergence of the infinitesimal generators, the convergence in distribution of the processes we are considering, we obtain:

$$(m_N^\sigma(t), m_N^\omega(t), m_N^{\sigma\omega}(t)) \xrightarrow[N\to\infty]{d} (m^\sigma(t), m^\omega(t), m^{\sigma\omega}(t))$$

In particular as holds that

$$\mathcal{L}f(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j}\left(\sigma(x)\sigma(x)^t\right)\frac{\partial^2 f}{\partial x_i\partial x_j}(x)$$

for x a stochastic process such that $dx(t) = b(x(t))\,dt + \sigma(x(t))\,dB(t)$, where $B(t)$ is the Brownian motion, we get:

$$\begin{aligned}
\dot{m}_t^\sigma &= 2\sinh(\beta)m_t^\omega - 2\cosh(\beta)m_t^\sigma \\
\dot{m}_t^\omega &= 2\sinh\left(\gamma m_t^\sigma\right) - 2\cosh\left(\gamma m_t^\sigma\right)m_t^\omega \\
\dot{m}_t^{\sigma\omega} &= 2\sinh(\beta) + 2\sinh\left(\gamma m_t^\sigma\right)m_t^\sigma - 2\left(\cosh(\beta) + \cosh\left(\gamma m_t^\sigma\right)\right)m_t^{\sigma\omega}
\end{aligned} \tag{2.5}$$

2.2 Equilibria of the limiting dynamics

After having obtained the differential equations that characterize the three order processes for $N \to \infty$, we provide a Theorem to study in detail what happens to the equilibria of the limiting dynamic depending on the parameters we consider.
We analyze three different possible choices for the parameters:

- $\gamma \leq \frac{1}{\tanh(\beta)}$: the case where the level of contagion is less or equal than the critical value;
- $\gamma = \frac{1}{\tanh(\beta)}$ where they coincide;
- $\gamma \geq \frac{1}{\tanh(\beta)}$: where the contagion γ is greater than the critical value.

Theorem 2.1. *(i) Suppose $\gamma \leq \frac{1}{\tanh(\beta)}$. Then $(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega)$ with $V(x, y) = (2\sinh(\beta)y - 2\cosh(\beta)x, 2\sinh(\gamma x) - 2y\cosh(\gamma x))$ has (0,0) as a unique equilibrium solution, which is globally asymptotically stable, that is, for every initial condition (m_0^σ, m_0^ω), we have*

$$\lim_{t\to+\infty}(m_t^\sigma, m_t^\omega) = (0,0)$$

(ii) For $\gamma < \frac{1}{\tanh(\beta)}$ the equilibrium (0,0) is linearly stable. For $\gamma = \frac{1}{\tanh(\beta)}$ the linearized system has a neutral direction, that is, $DV(0,0)$ has one zero eigenvalue (where $DV(0,0)$ is the Jacobian matrix evaluated in $(0,0)$).
(iii) For $\gamma > \frac{1}{\tanh(\beta)}$ the point (0,0) is still an equilibrium for $(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega)$ but it is a saddle point for the linearized system, that is, the matrix $DV(0,0)$ has two nonzero real eigenvalues of opposite sign. Moreover the previous two-dimensional system has two linearly stable solutions $(m_*^\sigma, m_*^\omega), (-m_*^\sigma, -m_*^\omega)$, where m_*^σ is the unique strictly positive solution of the equation

$$x = \tanh(\beta)\tanh(\gamma x) \tag{2.6}$$

and

$$m_*^\omega = \frac{1}{\tanh(\beta)} m_*^\sigma \tag{2.7}$$

(iv) For $\gamma > \frac{1}{\tanh(\beta)}$, the phase space $[-1,1]^2$ is bipartitioned by a smooth curve Γ containing (0,0) such that $[-1,1]^2 \backslash \Gamma$ is the union of two disjoint sets Γ^+, Γ^- that are open in the induced topology of $[-1,1]^2$. Moreover

$$\lim_{t\to+\infty}(m_t^\sigma, m_t^\omega) = \begin{cases} (m_*^\sigma, m_*^\omega), & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^+ \\ (-m_*^\sigma, -m_*^\omega), & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^- \\ (0,0), & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma \end{cases}$$

Proof. We first observe that the square $[-1,1]^2$ is stable for the flow of $(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega)$, since the vector field
$V(x,y) = (2\sinh(\beta)y - 2\cosh(\beta)x, 2\sinh(\gamma x) - 2y\cosh(\gamma x))$ points inward at the boundary of $[-1,1]^2$.
Let's see why $V(x,y)$ points inward at the boundary of $[-1,1]^2$. Let consider the case where $y = 1$ is fixed and the other cases are similar.
$V(x,1) = (2\sinh(\beta) - 2\cosh(\beta)x, 2\sinh(\gamma x) - 2\cosh(\gamma x))$ where
$\dot{x} = 2\sinh(\beta) - 2\cosh(\beta)x$. We consider the two following cases:

- if x is close to 1, we get that
  $2\sinh(\beta) - 2\cosh(\beta)x < 0 \iff x > \tanh(\beta)$
  but, as x is close to 1, the previous inequalities hold and we get $\dot{x} < 0$. This means that x decreases and so tends to go inward;

- if x is close to -1, we get that
  $2\sinh(\beta) - 2\cosh(\beta)x > 0 \iff x < \tanh(\beta)$
  but, as x is close to -1, the previous inequalities hold and we get $\dot{x} > 0$. This means that x increases and so tends to go inward.

Now it is immediately seen that the equation $V(x,y) = 0$ holds if and only if

$$\begin{cases} 2\sinh(\beta)y - 2\cosh(\beta)x = 0 \\ 2\sinh(\gamma x) - 2y\cosh(\gamma x)) = 0 \end{cases}$$

and so if and only if $x = \tanh(\beta)\tanh(\gamma x)$ and $y = \frac{1}{\tanh(\beta)}x$. Moreover a simple convexity argument shows that $x = \tanh(\beta)\tanh(\gamma x)$ has $x = 0$ as unique solution for $\gamma \leq \frac{1}{\tanh(\beta)}$, while for $\gamma > \frac{1}{\tanh(\beta)}$ a strictly positive solution, and its opposite, bifurcate from the null solution.
Let consider the fixed point equation $x = \tanh(\beta)\tanh(\gamma x)$; we want to find how many solution this equation has. Let $g(x) = x$ and $f(x) = \tanh(\beta)\tanh(\gamma x)$; we have that $f'(x) = \frac{\tanh(\beta)}{\cos^2 h(\gamma x)}\gamma$, $g'(x) = 1$. Comparing now $f'(0) = \tanh(\beta)\gamma, \quad g'(0) = 1$ we get the following statements:

$$\begin{aligned} \tanh(\beta)\gamma \leq 1 \Rightarrow \quad & \exists! \quad x = 0 \\ \tanh(\beta)\gamma > 1 \Rightarrow \quad & \exists \quad x = 0 \text{ and } x^*, -x^* \end{aligned}$$

We have therefore found all equilibria of $(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega)$. We now remark that The Poincaré-Bendixson Theorem states that every trajectory of a 2-dimensional autonomous system, converges either to an equilibrium point or to a periodic solution (see Theorem A.4 Appendix A). Here to prove that $(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega)$ has no cycles (periodic solutions), we use the

Divergence Theorem. Indeed, suppose (x_t, y_t) is a cycle of period T. Then by a 2-dimensional version of the Divergence Theorem (that is equivalent to Green's Theorem) we get

$$\int_0^T \left[V_1(x_t, y_t)\,\dot{x}_t + V_2(x_t, y_t)\,\dot{y}_t\right] dt = \int_C \operatorname{div} V(x, y) dx dy, \tag{2.8}$$

where V_1, V_2 are the components of V and C is the open set enclosed by the cycle. But a simple direct computation shows that

$$\operatorname{div} V(x, y) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = -2\cosh(\beta) - 2\cosh(\gamma x) < 0 \text{in all of} [-1, 1]^2.$$

On the contrary

$$\int_0^T \left[V_1(x_t, y_t)\,\dot{x}_t + V_2(x_t, y_t)\,\dot{y}_t\right] dt = \int_0^T \left[V_1^2(x_t, y_t) + V_2^2(x_t, y_t)\right] dt \geq 0$$

so that (2.8) cannot hold.

It follows by the Poincaré-Bendixson theorem that every solution must converge to an equilibrium as $t \to +\infty$. This completes the proof of (i). If we consider the matrix of the linearized system $(\dot{x}, \dot{y}) = \mathrm{DV}(\bar{x}, \bar{y})(x - \bar{x}, y - \bar{y})$, where

$$DV(x, y) = \begin{pmatrix} \frac{\partial V_1}{\partial x} & \frac{\partial V_1}{\partial y} \\ \frac{\partial V_2}{\partial x} & \frac{\partial V_2}{\partial y} \end{pmatrix} \text{ where } \begin{array}{l} V_1 = 2\sinh(\beta)y - 2\cosh(\beta)x \\ V_2 = 2\sinh(\gamma x) - 2y\cosh(\gamma x) \end{array}$$

and we evaluate it at (0,0), we get

$$DV(0, 0) = \begin{pmatrix} -2\cosh(\beta) & 2\sinh(\beta) \\ 2\gamma & -2 \end{pmatrix}$$

from which we obtain the following two eigenvalues:

$$\lambda_{1,2} = \frac{-2(\cosh(\beta) + 1) \pm \sqrt{4\left(1 + \cosh^2(\beta) + 2\cosh(\beta)\right) - 16\cosh(\beta) + 16\gamma\sinh(\beta)}}{2}$$

$$\lambda_{1,2} = -(\cosh(\beta) + 1) \pm \sqrt{1 + \cosh^2(\beta) - 2\cosh(\beta) + 4\gamma\sinh(\beta)}$$

We have 3 different cases:

- if $\gamma = \frac{1}{\tanh(\beta)}$ $\lambda_{1,2} = -(\cosh(\beta)+1) \pm (\cosh(\beta)+1)$, then we have that DV(0,0) has one zero eigenvalues;

- if $\gamma < \frac{1}{\tanh(\beta)}$ we have already understood what happens previously in the proof;

- if $\gamma \geq \frac{1}{\tanh(\beta)}$, then DV(0,0) has two nonzero real eigenvalues of opposite sign ($\lambda_1 > 0, \;\; \lambda_2 < 0$).

In this way we have shown also (ii) and (iii). It remains to show (iv).
For $\gamma > \frac{1}{\tanh(\beta)}$, we let v_s be an eigenvector of the negative eigenvalue of $DV(0,0)$. By the Stable Manifold Theorem (see Theorem A.5 in Appendix A), the set of initial conditions that are asymptotically driven to (0,0) forms a 1-dimensional manifold Γ that is tangent to v_s at $(0,0)$. Since any solution converges to an equilibrium point, and solutions starting at Γ^+ cannot cross Γ (otherwise uniqueness would be violated), the remaining part of statement (iv) follows. □

2.3 Central Limit Theorem

The main purpose of this section is to understand at which rate $m_N^\sigma(t)$ converges to m_t^σ and the same for the other two order parameters $m_N^\omega(t)$ and $m_N^{\sigma\omega}(t)$. In the previous sections we have studied the case $N \to \infty$, but we have to consider that N can be a very large number, but finite since we are dealing with a finite dimensional system.
Then we describe the finite volume approximations of the limiting dynamics via a suitable version of the Central Limit Theorem (CLT). So, let, first, define the 3 following variables:

$$\begin{aligned} x_N(t) &:= \sqrt{N}\,(m_N^\sigma(t) - m_t^\sigma) \\ y_N(t) &:= \sqrt{N}\,(m_N^\omega(t) - m_t^\omega) \\ z_N(t) &:= \sqrt{N}\,(m_N^{\sigma\omega}(t) - m_t^{\sigma\omega}) \end{aligned}$$

In other words we amplify the error that we do considering the infinite dimensional case instead of the large but finite one; this error depends obviously on N. We add the right noise $\sqrt{N}$ to see the fluctuation of the differences $(m_N(t) - m_t)$.
We state and prove the following theorem:

Theorem 2.2. *Consider the following 3-dimensional fluctuation process:*

$$\begin{aligned} x_N(t) &:= \sqrt{N}\left(m_N^{\sigma}(t) - m_t^{\sigma}\right) \\ y_N(t) &:= \sqrt{N}\left(m_N^{\omega}(t) - m_t^{\omega}\right) \\ z_N(t) &:= \sqrt{N}\left(m_N^{\sigma\omega}(t) - m_t^{\sigma\omega}\right) \end{aligned}$$

Then $(x_N(t), y_N(t), z_N(t))$ converges as $N \to \infty$, in the sense of weak convergence of stochastic processes, to a limiting 3-dimensional Gaussian process $(x(t), y(t), z(t))$ which is the unique solution of the following linear stochastic differential equation:

$$\begin{pmatrix} dx(t) \\ dy(t) \\ dz(t) \end{pmatrix} = A(t) \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} dt + D(t) \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{pmatrix}, \tag{2.9}$$

where B_1, B_2, B_3 are independent, standard Brownian motions,

$$A(t) = 2 \begin{pmatrix} -\cosh(\beta) & \sinh(\beta) & 0 \\ -\gamma m_t^{\omega} \sinh\left(\gamma m_t^{\sigma}\right) + \gamma \cosh\left(\gamma m_t^{\sigma}\right) & -\cosh\left(\gamma m_t^{\sigma}\right) & 0 \\ \sinh\left(\gamma m_t^{\sigma}\right) + \gamma m_t^{\sigma} \cosh\left(\gamma m_t^{\sigma}\right) - \gamma m_t^{\sigma\omega} \sinh\left(\gamma m_t^{\sigma}\right) & 0 & -\left(\cosh(\beta) + \cosh\left(\gamma m_t^{\sigma}\right)\right) \end{pmatrix},$$

$$\frac{D(t)D'(t)}{2}$$

$$= \begin{pmatrix} -m_t^{\sigma\omega}\sinh(\beta) + \cosh(\beta) & 0 & -m_t^{\sigma}\sinh(\beta) + m_t^{\omega}\cosh(\beta) \\ 0 & -m_t^{\omega}\sinh\left(\gamma m_t^{\sigma}\right) + \cosh\left(\gamma m_t^{\sigma}\right) & m_t^{\sigma}\cosh\left(\gamma m_t^{\sigma}\right) - m_t^{\sigma\omega}\sinh\left(\gamma m_t^{\sigma}\right) \\ -m_t^{\sigma}\sinh(\beta) + m_t^{\omega}\cosh(\beta) & m_t^{\sigma}\cosh\left(\gamma m_t^{\sigma}\right) - m_t^{\sigma\omega}\sinh\left(\gamma m_t^{\sigma}\right) & -m_t^{\sigma\omega}\sinh(\beta) + \cosh(\beta) - m_t^{\omega}\sinh\left(\gamma m_t^{\sigma}\right) + \cosh\left(\gamma m_t^{\sigma}\right) \end{pmatrix},$$

and $(x(0), y(0), z(0))$ have a centered Gaussian distribution with covariance matrix

$$\begin{pmatrix} 1-(m_\lambda^{\sigma})^2 & m_\lambda^{\sigma\omega} - m_\lambda^{\sigma} m_\lambda^{\omega} & m_\lambda^{\omega} - m_\lambda^{\sigma} m_\lambda^{\sigma\omega} \\ m_\lambda^{\sigma\omega} - m_\lambda^{\sigma} m_\lambda^{\omega} & 1-(m_\lambda^{\omega})^2 & m_\lambda^{\sigma} - m_\lambda^{\sigma\omega} m_\lambda^{\omega} \\ m_\lambda^{\omega} - m_\lambda^{\sigma} m_\lambda^{\sigma\omega} & m_\lambda^{\sigma} - m_\lambda^{\sigma\omega} m_\lambda^{\omega} & 1-(m_\lambda^{\sigma\omega})^2 \end{pmatrix} \tag{2.10}$$

Proof. The first part of this proof is quite similar to the one that we did studying the limiting dynamics topic. We want to check that, if we apply the generator $\mathcal{L}$ in (2.1) to a function of the form $\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)$, we obtain again a function of $(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega})$. From the computation that we did previously we know that we have

$$\mathcal{L}_N\varphi\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right) = \left(\mathcal{K}_N\varphi\right)\left(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega}\right)$$

where

$$\begin{aligned}
&\left(\mathcal{K}_N\varphi\right)(\xi,\eta,\theta)\\
&\quad= \sum_{(j,k)\in\{-1,1\}} \frac{N}{4}\left(1 + j\,\xi + k\,\eta + jk\,\theta\right)\left\{e^{-\beta jk}\left[\varphi\left(\xi - \frac{2}{N}j, \eta, \theta - \frac{2}{N}jk\right)\right.\right.\\
&\quad\left.\left. -\varphi\left(\xi,\eta,\theta\right)\right] + e^{-\gamma k\xi}\left[\varphi\left(\xi, \eta - \frac{2}{N}k, \theta - \frac{2}{N}jk\right) - \varphi\left(\xi,\eta,\theta\right)\right]\right\}.
\end{aligned} \tag{2.11}$$

This implies that $\mathcal{K}_N$ is the infinitesimal generator of the 3-dimensional Markov process $(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega})$. Notice, now, that $(x_N(t), y_N(t), z_N(t))$ is obtained from $(m_N^\sigma, m_N^\omega, m_N^{\sigma\omega})$ through a time-dependent, linear invertible transformation. We call T_t this transformation, that is,

$$T_t(\xi,\eta,\theta) = \left(\sqrt{N}\left(\xi - m_t^\sigma\right), \sqrt{N}\left(\eta - m_t^\omega\right), \sqrt{N}\left(\theta - m_t^{\sigma\omega}\right)\right) = (x,y,z).$$

The 3-dimensional process $(x_N(t), y_N(t)\;\; z_N(t))$ is itself a (time inhomogeneous) Markov process, whose infinitesimal generator $\mathcal{H}_{N,t}$ can be obtained from (2.11) as follows:

$$\mathcal{H}_{N,t}f(x,y,z) = \mathcal{K}_N\left[f\circ T_t\right]\left(T_t^{-1}(x,y,z)\right) + \frac{\partial}{\partial t}\left[f\circ T_t\right]\left(T_t^{-1}(x,y,z)\right)$$

First of all we show some useful computations for $\mathcal{K}_N\left[f\circ T_t\right]\left(T_t^{-1}(x,y,z)\right)$ noticing that:

$$(f\circ T_t)(\xi,\eta,\theta) = f(x,y,z)$$

$$\begin{aligned}
&\mathrm{T}_t\left(\xi - \tfrac{2}{N}j, \eta, \theta - \tfrac{2}{N}jk\right)\\
&\quad= \left(\sqrt{N}\left(\xi - \frac{2}{N}j - m_t^\sigma\right), \sqrt{N}\left(\eta - m_t^\omega\right), \sqrt{N}\left(\theta - \frac{2}{N}jk - m_t^{\sigma\omega}\right)\right)\\
&\quad= \left(x - \frac{2}{\sqrt{N}}j, y, z - \frac{2}{\sqrt{N}}jk\right)
\end{aligned}$$

and so

$(f \circ T_t)\left(\xi - \frac{2}{N}j, \eta, \theta - \frac{2}{N}jk\right) = f\left(x - \frac{2}{\sqrt{N}}j, y, z - \frac{2}{\sqrt{N}}jk\right).$
On the other hand

$$\frac{\partial}{\partial t}\left[f \circ T_t\right]\left(T_t^{-1}(x,y,z)\right) = \frac{\partial}{\partial t}\left(f \circ T_t\right)(\xi,\eta,\theta) = \frac{\partial}{\partial t} f(x,y,z)$$

$$= f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} + f_z \frac{\partial z}{\partial t},$$

where $x = \sqrt{N}\left(\xi - m_t^\sigma\right) \Rightarrow \frac{\partial x}{\partial t} = -\sqrt{N}\dot{m}_t^\sigma$, and exactly in the same way for the others.
With the previous calculations we get that:

$$\begin{aligned}
&\mathcal{H}_{N,t} f(x,y,z) \\
&\quad = \frac{N}{4} \sum_{(j,k)\in\{-1,1\}^2} \left[j\frac{x}{\sqrt{N}} + k\frac{y}{\sqrt{N}} + jk\frac{z}{\sqrt{N}} + jm_t^\sigma + km_t^\omega + jkm_t^{\sigma\omega} + 1 \right] \\
&\quad \times \left\{ e^{-\beta jk} \left[f\left(x - \frac{2}{\sqrt{N}}j, y, z - \frac{2}{\sqrt{N}}jk\right) - f(x,y,z) \right] \right. \\
&\quad \left. + e^{-\gamma\left(x/\sqrt{N} + m_t^\sigma\right)k} \left[f\left(x, y - \frac{2}{\sqrt{N}}k, z - \frac{2}{\sqrt{N}}jk\right) - f(x,y,z) \right] \right\} \\
&\quad - \sqrt{N}\dot{m}_t^\sigma f_x(x,y,z) - \sqrt{N}\dot{m}_t^\omega f_y(x,y,z) - \sqrt{N}\dot{m}_t^{\sigma\omega} f_z(x,y,z),
\end{aligned} \tag{2.12}$$

where f_x stands for $\frac{\partial f}{\partial x}$, and similarly for the other derivatives. At this point we compute the asymptotics of $\mathcal{H}_{N,t} f(x,y,z)$ as $N \to +\infty$, assuming $f : \mathbb{R}^3 \to \mathbb{R}$ a $\mathcal{C}^3$ function with compact support. First of all we apply a Taylor expansion to the following terms:

$$\begin{aligned}
&f\left(x - \frac{2}{\sqrt{N}}\, j, y, z - \frac{2}{\sqrt{N}} jk\right) - f(x,y,z) \\
&\qquad = -\frac{2}{\sqrt{N}}\, j\ f_x(x,y,z) - \frac{2}{\sqrt{N}}\, jk\ f_z(x,y,z) \\
&\qquad + \frac{2}{N} f_{xx}(x,y,z) + \frac{2}{N} f_{zz}(x,y,z) + \frac{4}{N} f_{xz}(x,y,z) + o\left(\frac{1}{N}\right)
\end{aligned}$$

and

$$e^{-\gamma(x/\sqrt{N})} = 1 - \gamma\left(\frac{x}{\sqrt{N}}\right) + o\left(\frac{1}{\sqrt{N}}\right)$$

Now we can plug the two Taylor's expansions above into (2.12) obtaining the following equation:

$$
\begin{aligned}
&\mathcal{H}_{N,t} f(x,y,z) \\
&\quad = \frac{N}{4} \sum_{(j,k)\in\{-1,1\}^2} \left[j\frac{x}{\sqrt{N}} + k\frac{y}{\sqrt{N}} + jk\frac{z}{\sqrt{N}} + jm_t^\sigma + km_t^\omega + jkm_t^{\sigma\omega} + 1 \right] \\
&\quad \times \Bigg\{ e^{-\beta jk} \left[-\frac{2}{\sqrt{N}}\, j\, f_x - \frac{2}{\sqrt{N}}\, jk\, f_z + \frac{2}{N} f_{xx} + \frac{2}{N} f_{zz} + \frac{4}{N} f_{xz} + o\left(\frac{1}{N}\right) \right] \\
&\quad + e^{-\gamma m_t^\sigma k} \left(1 - \gamma\, k\frac{x}{\sqrt{N}} + o\left(\frac{1}{N}\right) \right) \left[-\frac{2}{\sqrt{N}}\, j\, f_x - \frac{2}{\sqrt{N}}\, jk\, f_z + \frac{2}{N} f_{yy} \right. \\
&\quad \left. + \frac{2}{N} f_{zz} + \frac{4}{N} f_{yz} + o\left(\frac{1}{N}\right) \right] \Bigg\} - \sqrt{N}\dot{m}_t^\sigma f_x - \sqrt{N}\dot{m}_t^\omega f_y - \sqrt{N}\dot{m}_t^{\sigma\omega} f_z.
\end{aligned} \tag{2.13}
$$

As we did in the proof of the Limiting Dynamic case, we separate the 4 different cases (1,1), (1,-1), (-1,1), (-1,-1).
In the following lines we give an idea on how we have to collect all the different terms: in first place we show how the 3 terms $-\sqrt{N}\dot{m}_t^\sigma f_x$, $-\sqrt{N}\dot{m}_t^\omega f_y$, $-\sqrt{N}\dot{m}_t^{\sigma\omega} f_z$ are cancled with other terms of order $\sqrt{N}$ coming from the sum over $(j,k) \in \{-1,1\}^2$.

We recall that $\dot{m}_t^\sigma = 2\sinh(\beta)m_t^\omega - 2\cosh(\beta)m_t^\sigma$, then

$$-\sqrt{N}\dot{m}_t^\sigma f_x = -\sqrt{N} f_x[(e^\beta - e^{-\beta})m_t^\omega - (e^\beta - e^{-\beta})m_t^\sigma].$$

Now considering the following terms from (2.13)

$$
\begin{aligned}
&\frac{N}{4} m_t^\sigma e^{-\beta} \left(-\frac{2}{\sqrt{N}}\right) f_x + \frac{N}{4} m_t^\omega e^{-\beta} \left(-\frac{2}{\sqrt{N}}\right) f_x \\
&+ \frac{N}{4}(m_t^\sigma - m_t^\omega)e^\beta \left(-\frac{2}{\sqrt{N}}\right) f_x + \frac{N}{4}(-m_t^\sigma + m_t^\omega)e^\beta \frac{2}{\sqrt{N}} f_x \\
&+ \frac{N}{4}(-m_t^\sigma - m_t^\omega)e^{-\beta} \frac{2}{\sqrt{N}} f_x \\
&= \sqrt{N} f_x[(e^\beta - e^{-\beta})m_t^\omega - (e^\beta - e^{-\beta})m_t^\sigma]
\end{aligned}
$$

we can see they get cancled with $-\sqrt{N}\dot{m}_t^\sigma f_x$; the same thing for the other two terms.

If now we look at the coefficients of the partial derivatives of f, it's possible to rewrite entirely equation (2.13). For example let's do the computations for the coefficients of f_x, these are (without counting those that get cancled with $-\sqrt{N}\dot{m}_t^\sigma f_x$):

$$\frac{N}{4}\left[\frac{x}{\sqrt{N}}+\frac{y}{\sqrt{N}}+\frac{z}{\sqrt{N}}\right]e^{-\beta}\left(-\frac{2}{\sqrt{N}}\right)f_x$$
$$+\frac{N}{4}\left[\frac{x}{\sqrt{N}}-\frac{y}{\sqrt{N}}-\frac{z}{\sqrt{N}}\right]e^{\beta}\left(-\frac{2}{\sqrt{N}}\right)f_x$$
$$+\frac{N}{4}\left[-\frac{x}{\sqrt{N}}+\frac{y}{\sqrt{N}}-\frac{z}{\sqrt{N}}\right]e^{\beta}\left(\frac{2}{\sqrt{N}}\right)f_x$$
$$+\frac{N}{4}\left[-\frac{x}{\sqrt{N}}-\frac{y}{\sqrt{N}}+\frac{z}{\sqrt{N}}\right]e^{-\beta}\left(\frac{2}{\sqrt{N}}\right)f_x$$
$$=-x(e^{-\beta}+e^{\beta})+y(e^{\beta}-e^{-\beta})$$
$$=2f_x[-x\cosh(\beta)+y\sinh(\beta)].$$

The other cases have long but straightforward computations; in this way we get a rewrite of (2.13). Taking now the limit for $N\to\infty$ of $\mathcal{H}_{N,t}f(x,y,z)$, as we are dealing with continuous objects, we get easily $\mathcal{H}_t f(x,y,z)$. It follows then:

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\sup_{x,y,z\in\mathbb{R}^3}|\mathcal{H}_{N,t}f(x,y,z)-\mathcal{H}_t f(x,y,z)|=0$$

where

$$\begin{aligned}\mathcal{H}_t f(x,y,z)=&2\{f_x[-x\cosh(\beta)+y\sinh(\beta)]\\&+f_y\left[-\gamma x m_t^\omega\sinh\left(\gamma m_t^\sigma\right)+\gamma x\cosh\left(\gamma m_t^\sigma\right)-y\cosh\left(\gamma m_t^\sigma\right)\right]\\&+f_z\left[x\sinh\left(\gamma m_t^\sigma\right)+\gamma x m_t^\sigma\cosh\left(\gamma m_t^\sigma\right)\right.\\&\left.-\gamma x m_t^{\sigma\omega}\sinh\left(\gamma m_t^\sigma\right)-z\cosh(\beta)-z\cosh\left(\gamma m_t^\sigma\right)\right]\\&+f_{xx}\left[-m_t^{\sigma\omega}\sinh(\beta)+\cosh(\beta)\right]\\&+f_{yy}\left[-m_t^\omega\sinh\left(\gamma m_t^\sigma\right)+\cosh\left(\gamma m_t^\sigma\right)\right]\\&+f_{zz}\left[-m_t^{\sigma\omega}\sinh(\beta)+\cosh(\beta)\right.\\&\left.-m_t^\omega\sinh\left(\gamma m_t^\sigma\right)+\cosh\left(\gamma m_t^\sigma\right)\right]\\&+2f_{xz}\left[-m_t^\sigma\sinh(\beta)+m_t^\omega\cosh(\beta)\right]\\&+2f_{yz}\left[m_t^\sigma\cosh\left(\gamma m_t^\sigma\right)-m_t^{\sigma\omega}\sinh\left(\gamma m_t^\sigma\right)\right]\}\end{aligned}\tag{2.14}$$

is the infinitesimal generator of the linear diffusion process (2.9).
To complete the proof we need to show that $(x_N(0),y_N(0),z_N(0))$ converges as $N\to\infty$, in distribution to $(x(0),y(0),z(0))$. This last statement follows by the standard Central Limit Theorem for i.i.d. random variables; indeed,

by assumption, $(\sigma_i(0), \omega_i(0))$ are independent with law λ, and (2.10) is just the covariance matrix under λ of $(\sigma(0), \omega(0), \sigma(0)\omega(0))$.
First of all we check that the matrix coincides with the one of the statement of the Theorem; for example consider $Var[\sigma(0)] = E[\sigma^2(0)] - E[\sigma(0)]^2 = E[1] - (m_\lambda^\sigma)^2 = 1 - (m_\lambda^\sigma)^2$ or $Cov[\sigma(0)\ \omega(0)] = E[\sigma(0)\ \omega(0)] - E[\sigma(0)]E[\omega(0)] = m_\lambda^{\sigma\omega} - m_\lambda^\sigma m_\lambda^\omega$ and in this way for all the entrances of the matrix (2.10).
Then we show how to apply the CLT Theorem. Let:

$$
\begin{aligned}
Y_N &= \frac{\sum_{i=1}^N \sigma_i(0) - NE[\sigma_i(0)]}{\sqrt{1-(m_\lambda^\sigma)^2}\sqrt{N}} = \frac{\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i(0) - m_\lambda^\sigma\right)}{\sqrt{1-(m_\lambda^\sigma)^2}} \\
&= \frac{\sqrt{N}\,(m_N^\sigma(0) - m_\lambda^\sigma)}{\sqrt{1-(m_\lambda^\sigma)^2}}
\end{aligned}
$$

by the Central Limit Theorem $Y_N \longrightarrow Y \sim N(0,1)$
then $x_N(0) \xrightarrow{\mathrm{d}} x(0) \sim N(0, 1-(m_\lambda^\sigma)^2)$ and the same method for $y_N(0)$ and $z_N(0)$.
It should be pointed out that here we are dealing with time-dependent generators. To fix this point is enough to introduce an additional variable $\tau(t) := t$, and consider the process $\alpha(t) := (x(t), y(t), z(t), \tau(t))$, whose generator is time-homogeneous. This argument, together with the fact that the convergence of $\mathcal{H}_{N,t}f(x,y,z)$ to $\mathcal{H}_t f(x,y,z)$ is uniform in both (x,y,z) and t, completes the proof.

□

2.4 Covariance Matrix

Proposition 2.1. *Denote by Σ_t the covariance matrix of $(x(t), y(t), z(t))$, then Σ_t solves the Lyapunov equation:*

$$
\frac{d\Sigma_t}{dt} = A(t)\Sigma_t + \Sigma_t A'(t) + D(t)D'(t) \tag{2.15}
$$

Proof. Denote by $X(t) = (x(t), y(t), z(t))'$. Then, by (2.9)

$$
\begin{aligned}
dX(t) &= A(t)X(t)dt + D(t)dB(t) \\
dX'(t) &= (A(t)X(t))'dt + (D(t)dB(t))'
\end{aligned}
$$

Thus,

$$
d\left(X(t)X'(t)\right) = X(t)dX'(t) + X'(t)dX(t) + D(t)D'(t)dt
$$

where the equality comes from the Part Formula for stochastic processes:

$d\left(X(t)Y(t)\right) = X(t)dY(t) + Y(t)dX(t) + d\langle X, Y\rangle_t$

where,

$dXt = \varphi_t \; dB(t) + \xi_t dt$
$dYt = f_t \; dB(t) + g_t dt$
$d\langle X, Y\rangle_t = \varphi_t f_t \; dt.$

In this way we get

$$\begin{aligned} d\left(X(t)X'(t)\right) &= X(t)\left[X'(t)A'(t)dt + dB'(t)D'(t)\right] \\ &\quad + X'(t)\left[A(t)X(t)dt + D(t)dB(t)\right] + D(t)D'(t)dt \\ &= X(t)X'(t)A'(t)dt + X(t)dB'(t)D'(t) + A(t)X(t)X'(t)dt \\ &\quad + D(t)dB(t)X'(t) + D(t)D'(t)dt \end{aligned}$$

Now, $\Sigma_t = E[(X(t)X'(t))] - E[X(t)]\; E[X(t)]'$
so,

$$d\Sigma_t = d\left(E[X(t)X'(t)]\right) - d\left(E[X(t)]\; E[X(t)]'\right) \tag{2.16}$$

• We have that for the first term we can bring the derivative inside the expected value and we get:

$d\left(E[X(t)X'(t)]\right) = E[X(t)X'(t)]A'(t)dt + A(t)E[X(t)X'(t)]dt + D(t)D'(t)dt$

since $E\left(X(t)dB'(t)\right) = 0$ and $E\left(dB(t)X'(t)\right) = 0$

• For the second term we have that:

$X(t) = \int A(x)X(t)dt + \int D(t)dB(t)$
$X'(t) = \int X(t)'A(t)'dt + \int dB'(t)D(t)$

then $E[X(t)] = E\left[\int A(t)X(t)dt\right] = \int A(t)E[X(t)]dt$

since $E\left[\int D(t)dB(t)dt\right] = 0$ and $d\left(E[X(t)]\right) = A(t)E[X(t)]dt$

and the same calculus for $X'(t), thus$

$E[X'(t)] = \int E[X'(t)]\ A'(t)dt$

$d\left(E[X'(t)]\right) = E[X'(t)]\ A'(t)dt.$

The second term can be rewritten as follow:

$d\left(E[X(t)]\ E[X(t)]'\right) = d\left(E[X(t)]\right)\ E[X'(t)] + E[X(t)]\ d\left(E[X'(t)]\right)$

so $d\left(E[X(t)]\ E[X(t)]'\right) = A(t)E[X(t)]E[X'(t)]dt + E[X(t)]E[X'(t)]\ A'(t)dt$

Thus, from equation (2.16),

$d\Sigma_t = \left(E\left[X(t)X'(t)\right] - E[X(t)]E\left[X'(t)\right]\right)A'(t)dt$

$+A(t)\left(E\left[X(t)X'(t)\right] - E[X(t)]E\left[X'(t)\right]\right)dt + D(t)D'(t)dt$

then,

$$\frac{d\Sigma_t}{dt} = A(t)\Sigma_t + \Sigma_t A'(t) + D(t)D'(t).$$

□

In order to solve (2.15) it is convenient to interpret Σ as a vector in $\mathbb{R}^{3\times 3} = \mathbb{R}^3 \otimes \mathbb{R}^3$. To avoid ambiguities, for a 3×3 matrix C we write vec (C) whenever we interpret it as a vector.

Proposition 2.2. *Equation* (2.15) *can be rewritten as follows:*

$$\frac{d\left(\mathrm{vec}\left(\Sigma_t\right)\right)}{dt} = \left(A(t)\otimes I + I \otimes A(t)\right)\mathrm{vec}\left(\Sigma_t\right) + \mathrm{vec}\left(D(t)D^*(t)\right) \qquad (2.17)$$

where "$\otimes$" denotes the tensor product of matrices (Kronecker product).

Proof. We can write (2.15) as

$$\frac{d\left(\mathrm{vec}\left(\Sigma_t\right)\right)}{dt} = \mathrm{vec}\left(A(t)\ \Sigma_t\ I\right) + \mathrm{vec}\left(I\ \Sigma_t\ A'(t)\right) + \mathrm{vec}\left(D(t)D^*(t)\right)$$

and, as the following property holds $\mathrm{vec}\,(A\ B\ C) = (C' \otimes A)\,\mathrm{vec}\,(B)$, we get

$$\frac{d\,(\mathrm{vec}\,(\Sigma_t))}{dt} = (A(t) \otimes I + I \otimes A(t))\,\mathrm{vec}\,(\Sigma_t) + \mathrm{vec}\,(D(t)D^*(t))$$

□

We, now, state a Corollary of Theorem 2.2 concerning the fluctuations of the global health indicator; this will be usefulin the application on Portfolio Losses.

Corollary 2.1. *As $N \to \infty$ we have that*

$$\sqrt{N}\,[m_N^\sigma(t) - m_t^\sigma]$$

converges in law to a centered Gaussian random variable Z with variance

$$V(t) = \Sigma_{11}(t), \tag{2.18}$$

where $\Sigma(t)$ solves (2.15) *and m_t^σ solves* (2.5).

The variable V(t), as in (2.18), is the variance of the process x(t) and will be largely used to show simulations concerning different behaviors of the particle system under certain parameters. Often, instead of considering the whole covariance matrix, we use V(t) as point of reference to understand how the model we are considering evolves.

Now we study the behavior of Σ_t for large t.
In particular, equation (2.17) is linear, so its solution can be given with an explicit expression and can be computed after having solved (2.5). The behavior of Σ_t for large t can be obtained explicitly as follows.

- *Case* $\gamma < \frac{1}{\tanh(\beta)}$. As we have shown in Theorem 2.1 the solution $(m_t^\sigma, m_t^\omega, m_t^{\sigma\omega})$ of (2.5) converges to $(0, 0, tanh(\beta))$ as $t \to +\infty$. In particular we can easily compute the following limits:

  $$A := \lim_{t\to+\infty} A(t), \quad DD^* := \lim_{t\to+\infty} D(t)D^*(t)$$

  It follows from (2.17) that $\lim_{t\to+\infty} \Sigma_t = \Sigma$.

- *Case* $\gamma > \frac{1}{\tanh(\beta)}$. Also in this case, by Theorem 2.1, the limit

$$\lim_{t\to+\infty} (m_t^\sigma, m_t^\omega, m_t^{\sigma\omega})$$

exists. Disregarding the exceptional case in which the initial condition of (2.5) belongs to the stable manifold Γ introduced in Theorem 2.1, the limit above equals either $(m_*^\sigma, m_*^\omega, m_*^{\sigma\omega})$ or $(-m_*^\sigma, -m_*^\omega, m_*^{\sigma\omega})$, depending on the initial conditions. In both cases one obtains, as in the previous case, the limits A and DD^*.

- *Case* $\gamma = \frac{1}{\tanh(\beta)}$. In this case, the limiting matrix A is singular; it follows that the limit $\lim_{t\to+\infty} \Sigma_t$ does not exist. The meaning of this, is the following: when we consider critical values of the parameters the size of normal fluctuations around the deterministic limit grows in time.

Chapter 3

Credit Crises

3.1 Intuitions

Before going into detail, presenting some simulation results related to the behavior of our particle system and its financial meaning, we give some intuitions about the quantities studied so far.
The main objects we are going to deal with are m_t^σ and $V(t)$.

The variable m_t^σ represents the global financial health indicator for $N \to +\infty$ and takes values in the interval $[-1, 1]$.
We can think of this variable as the quality of the debt of a certain set of firms: for example $m_t^\sigma \to -0.5$ for t that goes to $+\infty$ means the 75% of the firms are in financial distress: the probability of default for the 75% of the obligors (firms) is quite high, in other words a high probability of not being able to pay back obligations.
If we have for example that $\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = (0, 0)$, this means that almost half of the firms are in a good financial state and almost half of the them are in a bad financial state both in the perceived and real state.

The other important variable that we introduced in the previous chapter is $V(t)$, the variance of the process x(t) from Theorem 2.2. The meaning of this variable is the following: V(t) is the variance of the deviations from the average. It tells us how significant the average m_t^σ is; if V(t) is big, the average is not very reliable. We will show how the variance evolve through different simulations: we are interested in understanding its limit behavior and how the dynamics evolve before the stationarity.
To have a first intuition about $V(t)$ unit of measure, we can think about the standard deviation $\sqrt{V(t)}$: this variable represents how much we deviate

from 0 regarding the mistake we make considering m_t^σ in place of $m_N^\sigma(t)$. In other words, if $V(t) = 0.3$ for a certain time t, this means that we have to consider the dynamics of $m_t^\sigma \pm \sqrt{0.3}$ and this is how much the system differs considering the N-finite dimensional case, in place of the infinite dimensional one.

3.2 Simulations

This section is going to be really useful to fully understand the meaning of what we have found until now. We recall shortly the different steps we have followed:

1. We have described the system through the process $\left(\sigma_i^{(N)}[0,T], \omega_i^{(N)}[0,T]\right)_{i=1}^{N}$ and then have taken into consideration the three statistics (m_N^σ,m_N^ω,$m_N^{\sigma\omega}$).

2. We have found the Limiting Dynamics sending $N \to +\infty$ and obtaining (m_t^σ,m_t^ω,$m_t^{\sigma\omega}$).

3. Then we have studied the equilibria of the limiting dynamics sending $t \to +\infty$ (Theorem 2.1).

4. As last, we have used a specific version of the Central Limit Theorem to go back to a finite-dimension N, to fully understand the error that we commit considering the limit process in place of the N-finite process.

With that said, we now show some examples of how the dynamics of m_t^σ and of the variance of the process x(t) (V(t)), evolve. We show results only for these two variables, avoiding to plot the other ones, just for clarity.
Let start giving some intuitions from the financial point of view that we will explore in detail after having introduced the concept of portfolio losses.
The main aim here is to show and introduce the concept of credit crises based on the different simulations we have done.
The function that we use to plot these graphs depends on 5 different inputs: $(t_N, \beta, \gamma, m_0^\sigma, m_0^\omega)$; we will consider a short analysis on how the simulations change, based on the different parameters.
We distinguish the already mentioned 3 cases:

- *Case* $\gamma < \frac{1}{\tanh(\beta)}$.

– m_t^σ converges to 0 as $t \to +\infty$.

– The closer we are to the critical case, the slower we get to the stable equilibrium 0. In other words, if we have that $\gamma \ll \frac{1}{\tanh(\beta)}$, we get to 0 faster.

– V(t) as $t \to +\infty$ stabilizes at a certain value.

– The closer we are to the critical case, the more time V(t) needs to get to a limit value and the bigger this limit value will be.

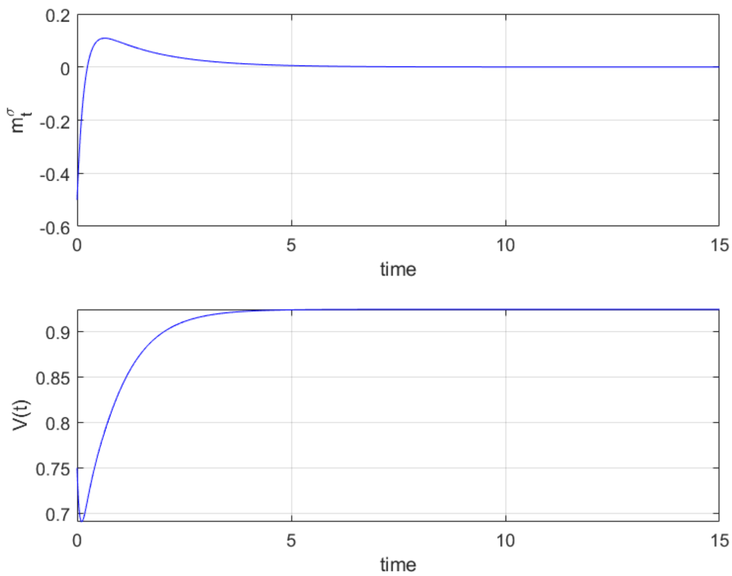

Figure 3.1: Trajectories of m_t^σ and V(t) for the following parameters $(t_N, \beta, \gamma, m_0^\sigma, m_0^\omega)$=(15,1.5,-0.5,-0.5,0.395). In the subcritical case the variance V(t) is not a bell curve.

- *Case* $\gamma = \frac{1}{\tanh(\beta)}$

– m_t^σ converges to 0 as $t \to +\infty$ but really slowly.

– V(t) grows linearly with respect to time for short time frames and has a polynomial growth with bigger time frames.

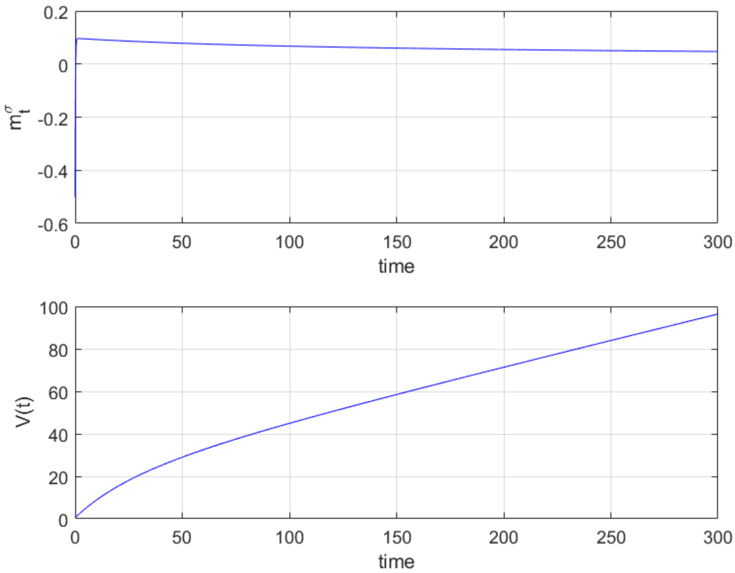

Figure 3.2: Trajectories of m_t^σ and V(t) for the following parameters $(t_N, \beta, \gamma, m_0^\sigma, m_0^\omega)$=(300,1.5,0,-0.5,0.395).

- *Case* $\gamma > \frac{1}{\tanh(\beta)}$
  This is the most interesting case, the one that we will discuss more because it represents the financial situation we want to explain.
  As we showed in Theorem 2.1 there are two symmetric equilibrium configurations in the space (m^σ, m^ω), that we have defined as (m_*^σ, m_*^ω) and $(-m_*^\sigma, -m_*^\omega)$. We stated also that there exists a curve Γ that allows to characterize the domains of attraction of these two equilibria or rather the set of initial conditions that lead the trajectory to one of the equilibria. The curve Γ can be obtained by simulations: we take different initial conditions and we study which one of these leads the trajectory to (m_*^σ, m_*^ω) or to $(-m_*^\sigma, -m_*^\omega)$. The following images are the curve Γ for two specific choices of the parameters:

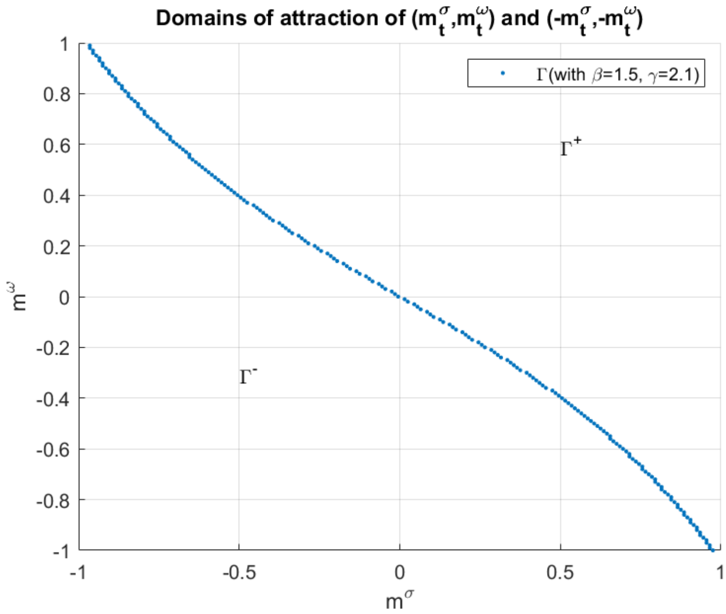

Figure 3.3: The boundary Γ curve and the two domains of attraction Γ^+ for (m_*^σ, m_*^ω) and Γ^- for $(-m_*^\sigma, -m_*^\omega)$ for $\beta = 1.5$ and $\gamma = 2.1$.

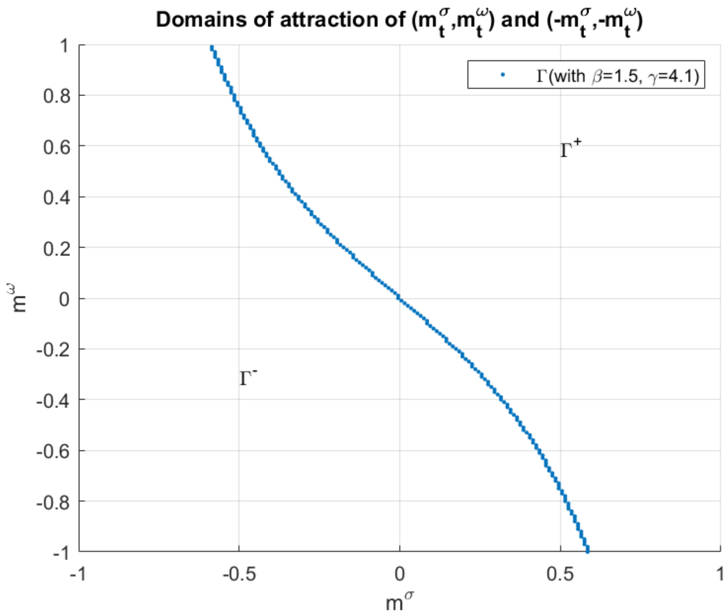

Figure 3.4: The boundary Γ curve and the two domains of attraction Γ^+ for (m_*^σ, m_*^ω) and Γ^- for $(-m_*^\sigma, -m_*^\omega)$ for $\beta = 1.5$ and $\gamma = 4.1$.

Here we emphasize some important facts regarding the supercritical case for the particle system; we will discuss the financial main aspects in the following section:

- The Γ curve depends on β and γ, but not on m_0^σ, m_0^ω.
- The dynamics of the variable m_t^σ depend on the initial choices (m_0^σ, m_0^ω), but the equilibrium points don't; depending on m_0^σ, m_0^ω and keeping fixed β and γ, m_t^σ will get to a fixed m_*^σ or to a fixed -m_*^σ possibly changing the trajectory.
- If we take t_N big enough, we fix β, m_0^σ, m_0^ω and we change γ we notice a lot of significant behaviors:
  the closer we get to the critical case the smaller the absolute value $\mid m_*^\sigma \mid$ becomes. The closer we are to the critical case the slower m_t^σ tends to its equilibrium (the equilibrium will change because m_0^σ, m_0^ω are fixed, but γ changes).
- If γ is big enough, we notice that the variance V(t) forms a bell curve; the bell happens earlier and earlier as we increase γ.
- The stationary value of $V(t)$ doesn't depend on the initial values (m_0^σ, m_0^ω), on the contrary instead the trajectories depend on the initial conditions.
- About the size of the variance $V(t)$ we will spend a lot of time trying to explain the credit crises concept. The closer we get to the critical case, the bigger the limit variances become; the closer the initial conditions are to the Γ curve, the bigger the peaks of the curve become.

Here we present some simulations just to give some intuitions about what we just said:

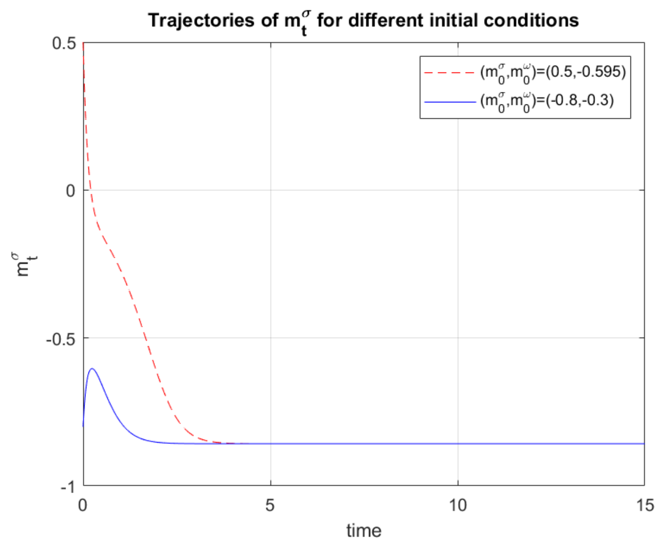

Figure 3.5: Trajectory of m_t^σ for different initial conditions. As we stated before, the trajectories change, but the limit value does not if the initial conditions belong to the same manifold Γ^+ or Γ^-.

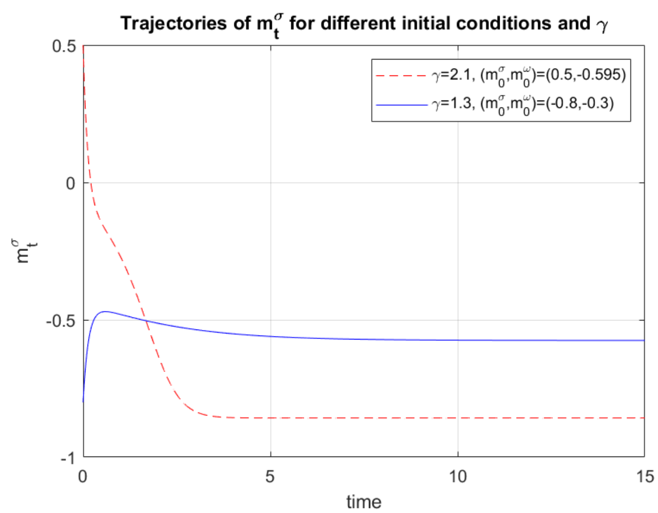

Figure 3.6: Trajectory of m_t^σ for different initial conditions and different γ. The absolute value $\mid m_*^\sigma \mid$ decreases as we get closer to the critical case $1/\tanh(\beta) = 1.1048$.

From a financial point of view, the previous two images are representing the following phenomenon: as we increase γ, we are increasing the interaction among the firms. This means that the system is more unbalanced towards particularly good situations (overall good financial state) or particularly bad situations (overall financial distress).

In the last simulation of this section we show two variances curve, using the same initial conditions but different γ.

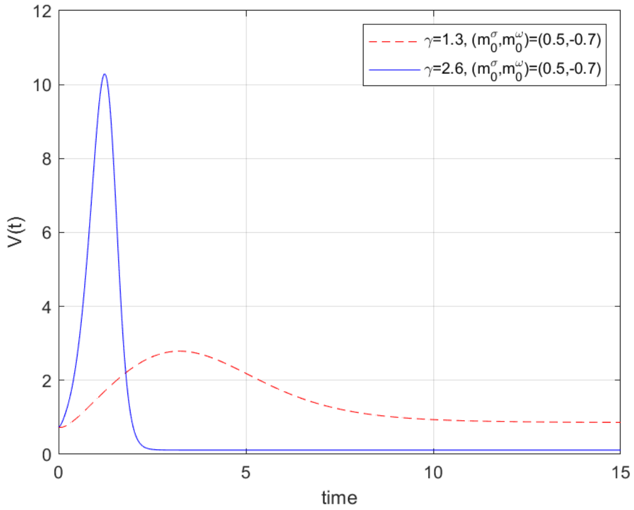

Figure 3.7: Trajectory of $V(t)$ for different values of γ.

In the simulation above we can observe the following behaviors: as we increase γ, the bell of the Γ curve happens earlier and is narrower. We have this behavior because, as already mentioned, with bigger γ, m_t^σ converges faster in time to an equilibrium. We can see also, just to have a first intuition, how, as we get closer to the critical case, the variances to the limit are bigger and bigger.

3.3 Credit Crises

In this section we analyze in detail the dynamics of the variance V(t), depending on the parameters we choose. As anticipated before, V(t) is an important variable that allows us to describe the volatility of a market like the one introduced in our model.

The model describes the phenomenon of credit crises, from a qualitative point of view, well enough: we are able to observe some situations where the variance increases a lot during a crisis and then re-stabilizes itself.
Before introducing the idea of credit crises, we define V^* as the asymptotic variance ($V(t)$ for $t \to +\infty$) and we introduce the two significant behaviors that the variable $V(t)$ has:

1. V^* grows as we get closer to the critical parameters.

2. The peaks of the variances depend on the initial conditions: the closer we are to the Γ curve with m_0^σ, m_0^ω, the higher the value in the peak is.

The first of the two behaviors is well explained by the following simulations: we plot the variance V(t) at time t_N with respect to different values of γ. The function we use has indeed $(t_N, \beta, m_0^\sigma, m_0^\omega)$ as input arguments.
Now, instead of considering γ, we consider ξ such that $\gamma = 1/\tanh(\beta) + \xi$. In this way for $\xi = 0$ we are in the critical case, if $\xi > 0$ we are in the supercritical case and for $\xi < 0$ we are dealing with the subcritical one.
Figure 3.8 is obtained using $(t_N, \beta, m_0^\sigma, m_0^\omega) = (50, 1.5, -0.5, 0.395)$. We can observe more than one particular aspect from this graph. First of all we consider $t_N = 50$ and not a bigger value because already at time 50, also for ξ really small in absolute value, we have reached the asymptotic variance limit (we already showed previously that for parameters close to the critical values the variance V(t) converges slower than for parameters far from the critical ones).
In financial terms we see how, in a "stationary" condition, the variances at the end time, become bigger as we get closer to the critical parameters; moreover the values in the subcritical case are a little bit higher than the supercritical one. For $\xi = 0$, if we increase t_N we get always higher values as we saw when we analyzed $V(t)$ behavior depending on the three different cases.
Another important point, is the fact that the variances to the limit don't depend on m_0^σ, m_0^ω: we can observe this, simply changing the initial conditions

in the following simulation (this is true for supercritical and subcritical case as at time, for example, $t_N = 50$, we already reached a stationary value, but not for the critical case that keeps getting bigger).

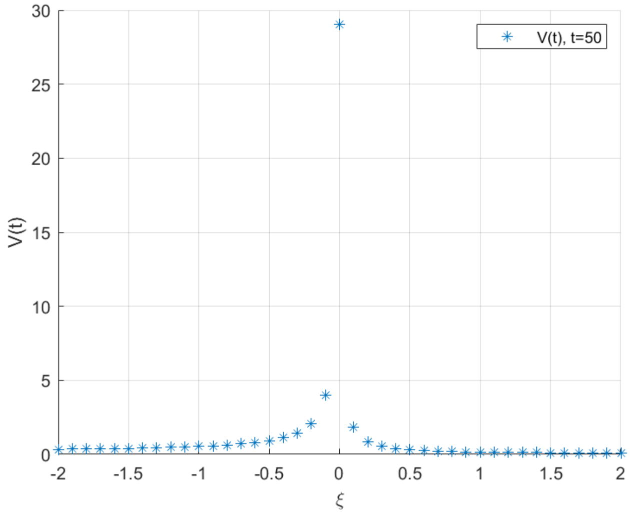

Figure 3.8: Asymptotic values of $V(t)$ for $(t_N, \beta, m_0^\sigma, m_0^\omega) = (50, 1.5, -0.5, 0.395)$.

The second of the two behaviors is probably the most important one, as it explains the idea of credit crises.
In the model, using particle system methodologies, we study the propagation of financial distress in a network of firms facing credit risk.
Credit risk is the possibility of a loss resulting from a borrower's failure to repay a loan or meet contractual obligations. Traditionally, it refers to the risk that a lender may not receive the owed principal and interest, which results in an interruption of cash flows and increased costs for collection. In our framework, the lender is a bank that, holding a large portfolio with positions issued by the firms, may suffer some losses due to the default of some of these firms.
In the previous sections we have shown how financial distress may propagate in a network of firms linked by business relationships. What we are interested in understanding now, is what is usually referred to as the clustering of defaults (or credit crises), meaning that there is evidence, looking at real data, of periods in which firms end up in financial distress in a short time. More specifically, looking at the dynamics of m_t^σ, the mean among an infinite

quantity of variables that represent firms perceived financial states, and looking at the variance $V(t)$, we are able to see this phenomenon: how financial contagion can cause a sudden period characterized by a financial crisis.
The model we propose exhibits the idea of a credit crisis with the following connotation: for certain values of the initial conditions the system is driven toward a symmetric equilibrium, in which half of the firms are in good financial health. After a certain time that depends on the initial state, the system is "captured" by an unstable direction of this symmetric equilibrium, and moves toward a stable asymmetric equilibrium; during the transition to the asymmetric equilibrium, the volatility of the system increases sharply, before decaying to a stationary value.

To capture this phenomenon, we show some numerical simulations that detect the crises when the values of the parameters are supercritical and the initial conditions are close to the boundary of the domains of attraction, that is, to Γ. The model is symmetric, this means that the behavior of the system is perfectly symmetric when starting in either Γ^+ or Γ^-. The typical connotation of a credit crisis is referred to what happens in Γ^-, that's why we consider, for the following simulations, this latter case.

In Figure 3.9 we can observe the dynamics of m_t^σ and $V(t)$ with respect to time, changing slightly the initial conditions.

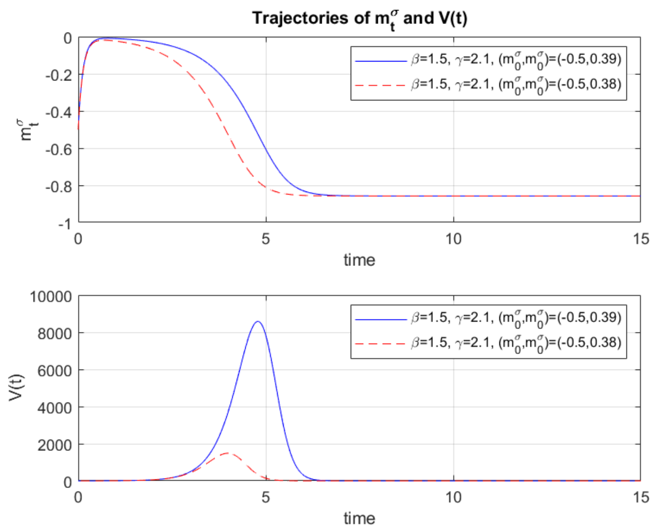

Figure 3.9: Credit Crises phenomenon. The parameters are the one reported in the figure. We can see how getting closer to the Γ curve, the credit crisis is accentuated.

In Figure 3.9 we use $\beta = 1.5$ and $\gamma = 2.1$ that are exactly the parameters used in Figure 3.3; in this way we have a better understanding on what "close to Γ" means.
In the m_t^σ graph, we have plotted two trajectories starting at $(m_0^\sigma, m_0^\omega) = (-0.5, 0.39) \in \Gamma^-$ and $(m_0^\sigma, m_0^\omega) = (-0.5, 0.38) \in \Gamma^-$ but near the boundary (the first one closer to the boundary than the second one).
Note that the variable m_t^σ (the same would happen also with m_t^ω that for clarity is not plotted) is first attracted to the unstable value zero, around which it spends a long time before moving to the stable equilibrium value -m_*^σ. This can be explained, in financial terms, as follows:

If we suppose that at the initial time the market conditions are such that (m_0^σ, m_0^ω) are in the manifold Γ^- but close to the curve Γ, then for a while the system moves toward $(0, 0)$, where half of the firms are in a good financial state and half are in a financial distress. This behavior lasts until the system gets "captured" by the unstable direction of the equilibrium point $(0, 0)$. Since the system configuration belongs to Γ^-, the new stable equilibrium that the system is attracted to is given by $(-m_*^\sigma, -m_*^\omega)$.
This situation represents, in a stylized manner, the idea of credit crises: the unstable equilibrium $(0, 0)$ can be seen as a 'credit bubble' and the decay toward the stable equilibrium mimics a credit crisis, a crash in the credit market.
With "bubble" or "credit bubble" we mean an economic cycle characterized by the rapid escalation of asset prices followed by a contraction. It is created by a surge in asset prices unwarranted by the fundamentals of the asset and driven by exuberant market behavior. When no more investors are willing to buy at the elevated price, a massive sell-off occurs, causing the bubble to deflate.
As soon as the system moves away from (0, 0), the uncertainty (volatility) increases quickly and the credit quality indicators move to the stable configuration changing completely the picture of the market (the speed of the convergence depends on γ, the level of interaction among the firms).
This situation is well illustrated by Figure 3.9. As we can observe, getting closer to the curve Γ with the initial conditions, we spend more and more time around the unstable equilibrium and this means in terms of variances that the peaks will get higher and higher. In other words, the volatility of the market keeps increasing as we get closer to the boundary of the manifolds Γ^- and Γ^+.

Another interesting simulation that gives the idea of a credit crisis is the following one: with reference to Figure 3.9, we look at the variance $V(t)$ at time $t = 5$. Obviously we have not reached yet the asymptotic values for the variances but we are in the middle of a credit crisis for certain parameters ($\beta = 1.5$,$\gamma = 2.1$). Letting change the value of ξ, we observe that for the particular initial conditions $(-0.5, 0.39)$, when $\xi = 1$ (this means $\gamma = 1/\tanh(\beta) + \xi = 1.1 + 1 = 2.1$), we get a huge value for the variance; we are exactly in the middle of a credit crisis.

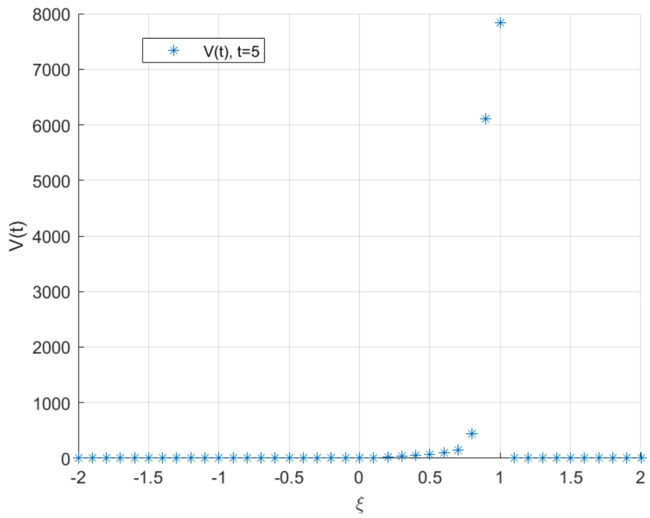

Figure 3.10: Credit Crises phenomenon. We can observe how the variance reaches high values for certain parameters. The inputs used here are the following: $(t_N, \beta, \gamma, m_0^\sigma, m_0^\omega) = (5, 1.5, 2.1, -0.5, 0.39)$.

Chapter 4

Portfolio Losses

4.1 Credit risk modeling

As already mentioned at the beginning of this thesis, the main purpose of this work is to describe the propagation of financial distress in a network of firms linked by business relationships.
Once the model for financial contagion has been described, we quantify the impact of contagion on the losses suffered by a financial institution holding a large portfolio with positions issued by the firms.
To properly introduce the concept of portfolio losses, we first recall the idea of credit risk; credit risk modelling is the best way for lenders to understand how likely a particular loan is to get repaid. In other words, it's a tool to understand the credit risk of a borrower.
It is extremely difficult and complex to pinpoint exactly how likely a firm is to default on their loan. At the same time, properly assessing credit risk can reduce the likelihood of losses from default and delayed repayments.
There are many different factors that affect a firm's credit risk. This makes assessing a borrower's credit risk a highly complex task; this is why credit risk modeling comes into play.
Credit risk modelling refers to the process of using data models to evaluate the probability of the borrower defaulting on the loan and the impact on the financials of the lender if this default occurs.
Financial institutions rely on credit risk models to determine the credit risk of potential borrowers. They make decisions on whether or not to sanction a loan as well as on the interest rate of the loan based on the credit risk model validation.
There are two major factors affecting the credit risk of a borrower (exactly the two main aspects that credit risk modelling can analyze):

- Probability of Default: This refers to the likelihood that a borrower will default on their loans and it is obviously the most important part of a credit risk model.

- Loss Given Default: This refers to the total loss that the lender will suffer if the debt is not repaid. This is a critical component in credit risk modeling.

There are many different credit risk models; when we will introduce the concept and the definitions related to portfolio losses, we will show some possible approaches that can be used.
There are two credit risk models in which we are particularly interested: CreditMetrics of J.P. Morgan (1997) and CreditRisk+ of Credit Suisse Financial Products(1997). CreditRisk+ is a Default Model, under this approach credit risk is the risk that security's borrower defaults on their promised obligations. Therefore, only borrowers' defaults can cause losses in the portfolio. On the other hand, CreditMetrics is a Rating Model. This approach defines credit risk as the risk that the security holder does not materialise the expected value of the security due to the deterioration of the borrower's credit quality. Therefore in CreditMetrics, not only default can cause losses but also downgrading in the credit quality of borrowers.

In our model each firm is identified by two variables, σ and ω, respectively the perceived and the real financial state of the firm. The variable σ may be interpreted as the rating class indicator. This variable represents exactly the borrower's credit quality: a low value reflects a bad rating class, that is, a higher probability of not being able to pay back obligations.
We have to pay attention to one more detail. In our model companies don't fail but simply go through periods of crisis: if the firm "i" at time t is such that $\sigma_i(t) = -1$, this means that the firm struggles more to repay the debt to the bank.

4.2 Portfolio Losses

We address now one of the major topics of the thesis: computing losses in a portfolio of positions issued by N firms. We give some definitions and useful Theorems to understand from a theoretical point of view the concept of portfolio losses. Then we show simulations to explain, from a financial point of view, the correlation between portfolio losses and credit crises.
We consider the total loss that a bank may suffer due to a risky portfolio at time t as a random variable defined by

$$L^N(t) = \sum_i L_i(t).$$

It is easy to understand that $L^N(t)$ is the total loss that a bank may suffer considering all the N firms the bank lent money to, and $L_i(t)$ is the single marginal loss caused by the i^{th} firm.
Obviously different specifications for the marginal losses $L_i(t)$ can be chosen; in this work the idea is to compute the aggregate loss as a sum of marginal losses $L_i(t)$, of which the distribution is supposed to depend on the realization of the variable σ_i, that is, on the rating class (the only observable process).
We assume that the marginal losses are independent and identically distributed if conditioned on σ.
We introduce now a conditional distribution function G_x, $x \in \{-1, 1\}$,

$$G_x(u) := P(L_i(t) \leq u \mid \sigma_i(t) = x). \tag{4.1}$$

Then we define the first and the second moments as follow,

$$l_1 := E\left(L_i(t) \mid \sigma_i(t) = 1\right) < E\left(L_i(t) \mid \sigma_i(t) = -1\right) =: l_{-1}, \tag{4.2}$$

where the inequality comes from the fact that the expected value of the marginal losses conditioned on $\sigma_i(t) = 1$ is less than what we expect to lose when we are in financial distress ($\sigma_i(t) = -1$), and

$$v_1 := \mathrm{Var}\left(L_i(t) \mid \sigma_i(t) = 1\right), \quad v_{-1} := \mathrm{Var}\left(L_i(t) \mid \sigma_i(t) = -1\right). \tag{4.3}$$

The aggregate loss of a portfolio of N firms at time t is then defined as mentioned before:

$$L^N(t) = \sum_{i=1}^{N} L_i(t)$$

We introduce now a deterministic time function (there's no more dependence on N) that represents the 'asymptotic' total loss when the number of firms goes to infinity. In other words, we are in a situation where a bank lends money to N firms and then these firms should give the money back to the lender. The portfolio we are working with is a dynamic portfolio, so $L(t)$, defined below, is the loss at time t when the number of firms tends to infinity. Let

$$L(t) = \frac{(l_1 - l_{-1})}{2} m_t^\sigma + \frac{(l_1 + l_{-1})}{2}. \tag{4.4}$$

We now state and prove the main result of this section that will be useful to give a good approximation for the losses suffered by the bank portfolio.

Theorem 4.1. *Assume $L_i(t)$ has a distribution of the form (4.1). Then for $t \in [0,T]$ with $T > 0$ and for any value of the parameters $\beta > 0$ and $\gamma > 0$, we have*

$$\sqrt{N}\left(\frac{L^N(t)}{N} - L(t)\right) \xrightarrow[N\to\infty]{d} Y \sim N(0, \hat{V}(t))$$

in distribution, where $L(t)$ is defined in (4.4) and

$$\hat{V}(t) = \frac{(l_1 - l_{-1})^2 V(t)}{4} + \frac{(1 + m_t^\sigma)\, v_1}{2} + \frac{(1 - m_t^\sigma)\, v_{-1}}{2} \tag{4.5}$$

with $V(t)$ as defined in (2.18).

Proof. To prove this Theorem we need, first of all, the following Lemma:

Lemma 4.1. *For $t \in [0,T]$,*

$$\sqrt{N}\left(\frac{\sum_j l_{\sigma_j(t)}}{N} - L(t)\right) \xrightarrow[N\to\infty]{d} X \sim N\left(0, \frac{(l_1 - l_{-1})^2 V(t)}{4}\right)$$

where $L(t)$ is defined in (4.4), l_1, l_{-1} in (4.2) and $V(t)$ in (2.18).

Proof. Let's define, for $x \in \{-1, 1\}$, the quantity $A_x^N(t)$ as the number of σ_i that, at a given time t, are equal to x . Then we write $\frac{1+m_N^\sigma}{2} = \frac{A_1^N(t)}{N}$ and $\frac{1-m_N^\sigma}{2} = \frac{A_{-1}^N(t)}{N}$. Recall, moreover, that for $N \to \infty, m_{\frac{\sigma}{N}}(t) \to m_t^\sigma$ from the

Limiting Dynamic Theorem. Then we have:

$$\begin{aligned}
&\sqrt{N}\left(\frac{\sum_j l_{\sigma_j(t)}}{N} - L(t)\right) \\
&= \sqrt{N}\left(\frac{l_1 A_1^N(t) + l_{-1}A_{-1}^N(t)}{N} - L(t)\right) \\
&= \sqrt{N}\left(l_1\frac{1-m_N^\sigma}{2} + l_{-1}\frac{1-m_N^\sigma}{2} - L(t)\right) \\
&= -\sqrt{N}\left(\frac{(l_1+l_{-1})}{2} + \frac{(l_1-l_{-1})}{2}m_N^\sigma(t) - \frac{(l_1-l_{-1})}{2}m_t^\sigma - \frac{(l_1+l_{-1})}{2}\right) \\
&= \sqrt{N}\left(\frac{(l_1-l_{-1})}{2}\left(m_N^\sigma(t) - m_t^\sigma\right)\right) \to X \sim N\left(0, \frac{(l_1-l_{-1})^2 V(t)}{4}\right),
\end{aligned}$$

where the last convergence follows from Corollary 2.1. □

Now let's come back to the proof of Theorem 4.1: we want to check that

$$\sqrt{N}\left(\frac{L^N(t)}{N} - L(t)\right) \xrightarrow[N\to\infty]{d} Y \sim N(0, \hat{V}(t))$$

where $V(t)$ is defined in (4.5).
We separate the firms according to whether $\sigma_j(t)$ is +1 or -1.

$$\sqrt{N}\left(\frac{\sum_j L_j(t)}{N} - L(t)\right) = \sqrt{N}\left(\frac{\sum_{j:\sigma_j(t)=1} L_j(t) + \sum_{j:\sigma_j(t)=-1} L_j(t)}{N} - L(t)\right)$$

We then add and subtract $\sum_j l_{\sigma_j(t)}$ to obtain

$$\begin{aligned}
&\sqrt{N}\left(\frac{\sum_{j:\sigma_j(t)=1}\left(L_j(t) - l_1\right)}{N}\right. \\
&\left.+\frac{\sum_{j:\sigma_j(t)=-1}\left(L_j(t) - l_{-1}\right)}{N} + \frac{\sum_j l_{\sigma_j(t)}}{N} - L(t)\right)
\end{aligned} \tag{4.6}$$

We recall that, conditioned on the realization of σ, the marginal losses are iid.. Let's apply Levy's Theorem to show that the sequence of random variables converges in distribution to Y, proving the convergence of the corresponding characteristic functions:

$$
\begin{aligned}
E\left[\exp\left\{ir\frac{L^N(t)-NL(t)}{\sqrt{N}}\right\}\right] = \\
= E\left[E\left[\exp\left\{ir\left(\frac{\sum_{j:\sigma_j(t)=1}\left(L_j(t)-l_1\right)}{\sqrt{N}}\right.\right.\right.\right. \\
+\frac{\sum_{j:\sigma_j(t)=-1}\left(L_j(t)-l_{-1}\right)}{\sqrt{N}} \\
\left.\left.+\frac{\sum_j l_{\sigma_j(t)}-NL(t)}{\sqrt{N}}\right)\right\} \mid \underline{\sigma}(t)\right]
\end{aligned}
\tag{4.7}
$$

where the equality comes from the Tower property:
For sub-sigma-algebras $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{F}$, $E\left(E\left(X \mid \mathcal{H}_2\right) \mid \mathcal{H}_1\right) = E\left(X \mid \mathcal{H}_1\right)$. We consider now only the conditional expected value in (4.7): the last of the three terms is measurable with respect to the sigma algebra generated by $\sigma(t)$ so we can take it out from the conditional expected value (that's true because $l_{\sigma_j(t)} = E\left(L_i(t) \mid \sigma_i(t)=\sigma_j(t)\right)$). The remaining terms can now be splitted using independence hypothesis in the product of conditional expectations:

$$
\begin{aligned}
E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j(t)=1}\left(L_j(t)-l_1\right)}{\sqrt{N}}\right\} \mid \underline{\sigma}(t)\right] \\
\times E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j(t)=-1}\left(L_j(t)-l_{-1}\right)}{\sqrt{N}}\right\} \mid \underline{\sigma}(t)\right]
\end{aligned}
$$

By conditional independence and Taylor series,

$$
\begin{aligned}
E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j(t)=1}\left(L_j(t)-l_1\right)}{\sqrt{N}}\right\} \mid \underline{\sigma}(t)\right] \\
= \prod_{j=1}^{A_1^N(t)} E\left[\exp\left\{ir\frac{L_j(t)-l_1}{\sqrt{N}}\right\} \mid \underline{\sigma}(t)\right] = \left[1-\frac{v_1}{2}\frac{r^2}{N}+o\left(\frac{1}{N}\right)\right]^{A_1^N(t)},
\end{aligned}
$$

where the first equality comes from conditional independence and the second one from the following Taylor expansion:

$$
E\left[\exp\left\{ir\frac{L_j(t)-l_1}{\sqrt{N}}\right\} \mid \underline{\sigma}(t)\right] = E\left[1+\frac{L_j(t)-l_1}{\sqrt{N}}-\frac{1}{2}\frac{r^2}{N}\left(L_j(t)-l_1\right)^2 \mid \underline{\sigma}(t)\right]
$$

$$
= 1+ir\frac{E\left[L_j(t) \mid \underline{\sigma}(t)\right]-l_1}{\sqrt{N}}-\frac{1}{2}\frac{r^2}{N}\left(E\left[L_j^2(t)-2L_j(t)l_1+l_1^2 \mid \underline{\sigma}(t)\right]\right)
$$

$$= 1 - \frac{1}{2}\frac{r^2}{N}\left(E\left[L_j^2(t) \mid \underline{\sigma}(t)\right] - l_1^2\right) = 1 - \frac{v_1}{2}\frac{r^2}{N} + o\left(\frac{1}{N}\right),$$

where we remember that l_1 and v_1 are the first two conditional moments of $L_j(t)$ and $o\left(\frac{1}{N}\right)$ is the remainder obtained using Taylor series.

Recalling that $\frac{A_1^N(t)}{N} = \frac{1+m_N^\sigma(t)}{2}$ converges almost surely to $\frac{1+m_t^\sigma}{2}$ and that $\lim_{N\to\infty}\left[1+\frac{K}{N}\right]^N = e^K$ we have that:

$$\begin{aligned}\lim_{N\to\infty}\left[1 - \frac{v_1}{2}\frac{r^2}{N} + o\left(\frac{1}{N}\right)\right]^{A_1^N(t)} &= \lim_{N\to\infty}\left[1 - \frac{v_1}{2}\frac{r^2}{A_1^N(t)}\frac{A_1^N(t)}{N} + o\left(\frac{1}{N}\right)\right]^{A_1^N(t)} \\ &= \exp\left[-\frac{r^2}{2}\frac{1+m_t^\sigma}{2}v_1\right].\end{aligned}$$

In the same way for $\sigma_j(t) = -1.$, since $\frac{A_{-1}^N(t)}{N} \to \frac{1-m_t^\sigma}{2}$, we get

$$\lim_{N\to\infty}\left[1 - \frac{v_{-1}}{2}\frac{r^2}{N} + o\left(\frac{1}{N}\right)\right]^{A_{-1}^N(t)} = \exp\left[-\frac{r^2}{2}\frac{1-m_t^\sigma}{2}v_{-1}\right].$$

Recalling from Lemma 4.1 that $\frac{\sum_j l_{\sigma_j(t)} - NL(t)}{\sqrt{N}}$ converges to $X \sim N\left(0, \frac{(l_1-l_{-1})^2 V(t)}{4}\right)$, we have, from the Inverse of Levy's Theorem, that

$$\begin{aligned}&\lim_{N\to\infty} E\left[\exp\left\{ir\frac{\sum_j l_{\sigma_j(t)} - NL(t)}{\sqrt{N}}\right\}\right] \\ &\qquad = E\left[\exp\left\{irX\right\}\right] = \exp\left[-\frac{r^2}{2}\frac{\left(l_1 - l_{-1}\right)^2 V(t)}{4}\right],\end{aligned}$$

where the second term is exactly the characteristic function of the limit variable X and the last term is the computed characteristic function for a variable with Normal distribution.
We recall briefly how the characteristic function for a Gaussian variable can be computed:
if we have the variable $Z \sim N\left(\mu, \sigma^2\right)$, its characteristic function is $\phi(r) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$.

Thus, $E\left[\exp\left\{ir\frac{L^N(t)-NL(t)}{\sqrt{N}}\right\}\right]$ in (4.7) can be written as follows:

$$E\left[\exp\left\{ir\frac{L^N(t) - NL(t)}{\sqrt{N}}\right\}\right] = E\left[\exp\left\{ir\frac{\sum_j l_{\sigma_j(t)} - NL(t)}{\sqrt{N}}\right\}\right.$$

$$\times \left[1 - \frac{v_1}{2}\frac{r^2}{N} + o\left(\frac{1}{N}\right)\right]^{A_1^N(t)} \times \left[1 - \frac{v_{-1}}{2}\frac{r^2}{N} + o\left(\frac{1}{N}\right)\right]^{A_{-1}^N(t)}\Bigg].$$

By the Dominated Convergence Theorem, taking the limit $N \to \infty$, we can interchange the limit with the expectation. As the three limits are all finite, we send $N \to \infty$ for the last two terms and, as they become constant, we can bring them out the expected value. Applying again the Dominated Convergence Theorem we get:

$$\lim_{N\to\infty} E\left[\exp\left\{ir\frac{L^N(t) - NL(t)}{\sqrt{N}}\right\}\right] = \lim_{N\to\infty} E\left[\exp\left\{ir\frac{\sum_j l_{\sigma_j(t)} - NL(t)}{\sqrt{N}}\right\}\right] \times$$

$$\times \exp\left[-\frac{r^2}{2}\frac{1+m_t^\sigma}{2}v_1\right] \times \exp\left[-\frac{r^2}{2}\frac{1-m_t^\sigma}{2}v_{-1}\right]$$

$$= \exp\left[-\frac{r^2}{2}\hat{V}(t)\right]$$

The last term is exactly the characteristic function of Y.
The proof is completed using Levy's Theorem (see Theorem A.6 in "Useful Tools" Appendix A); from the pointwise convergence of characteristic functions we obtain the convergence in distribution that we are aiming to prove. □

Now, to have a more concrete tool to compute the probability that the total losses, suffered by the bank, are greater than a certain value α, we point out the following calculus:

$$P\left(L^N(t) \geq \alpha\right) = P\left(\frac{L^N(t)}{\sqrt{N}} - \frac{N\ L(t)}{\sqrt{N}} \geq \frac{\alpha}{\sqrt{N}} - \frac{N\ L(t)}{\sqrt{N}}\right)$$

$$= P\left(\frac{L^N(t) - N\ L(t)}{\sqrt{N}\ \sqrt{\hat{V}(t)}} \geq \frac{\alpha - N\ L(t)}{\sqrt{N}\ \sqrt{\hat{V}(t)}}\right) \approx P\left(Z \geq \frac{\alpha - N\ L(t)}{\sqrt{N}\ \sqrt{\hat{V}(t)}}\right) = (*)$$

where $Z \sim N(0,1)$, $Z = \frac{Y}{\sqrt{\hat{V}(t)}}$ and the approximation comes from the Gaussian approximation of Theorem 4.1.
Moreover by the symmetry of the Gaussian distribution, we get that

$$(*) = P\left(Z \leq \frac{N\ L(t) - \alpha}{\sqrt{N}\ \sqrt{\hat{V}(t)}}\right) = \mathcal{N}\left(\frac{NL(t) - \alpha}{\sqrt{N}\sqrt{\hat{V}(t)}}\right)$$

In this way we get the following approximation, useful to compute portfolio losses:

$$P\left(L^N(t) \geq \alpha\right) \approx \mathcal{N}\left(\frac{NL(t)-\alpha}{\sqrt{N}\sqrt{\hat{V}(t)}}\right). \tag{4.8}$$

To have a more concrete idea of what kind of results we obtain through the model, we show some examples for possible specifications of the marginal losses.
In all the following examples we describe large portfolio losses at a predetermined time horizon T for different specifications of the conditional loss distribution, for specific parameters β, γ and for specific initial conditions.

EXAMPLE 1.

Let consider the following marginal loss:

$$L_i(t) = \begin{cases} 1, & \text{if } \sigma_i(t) = -1 \\ 0, & \text{if } \sigma_i(t) = 1 \end{cases},$$

where we have that the expected marginal losses, conditioned on $\sigma_i(t) = -1$ or $\sigma_i(t) = 1$ are:

$$l_1 := E\left(L_i(t) \mid \sigma_i(t) = 1\right) = 0 < 1 = E\left(L_i(t) \mid \sigma_i(t) = -1\right) =: l_{-1}.$$

The total loss is given by:

$$L^N(t) = \sum_{i=1}^{N} \frac{1-\sigma_i(t)}{2}.$$

As we did for equation (4.8) we compute the following probability:

$$P\left(L^N(t) \geq \alpha\right) = P\left(\frac{N - Nm_{\frac{\sigma}{N}}(t)}{2} \geq \alpha\right) = P\left(m_N^\sigma(t) \leq \frac{N-2\alpha}{N}\right)$$

$$= P\left(\frac{\sqrt{N}\left(m_N^\sigma(t) - m_t^\sigma\right)}{\sqrt{V(t)}} \leq \frac{\sqrt{N}}{\sqrt{V(t)}}\left(\frac{N-2\alpha}{N} - m_t^\sigma\right)\right)$$

$$\approx P\left(Z \leq \frac{N\ (1-m_t^\sigma) - 2\alpha}{\sqrt{N}\sqrt{V(t)}}\right),$$

where $\frac{\sqrt{N}\left(m_N^\sigma(t)-m_t^\sigma\right)}{\sqrt{V(t)}} \xrightarrow[N\to\infty]{d} \frac{Y}{\sqrt{V(t)}}$ with $Y \sim N(0,1)$ so,

$$\approx \mathcal{N}\left(\frac{-2\alpha + (1-m_t^\sigma)\,N}{\sqrt{N}\sqrt{V(t)}}\right) = \mathcal{N}\left(\frac{-2\alpha + 2L^\infty(t)N}{\sqrt{N}\sqrt{V(t)}}\right),$$

where $L^\infty(t) := \lim_{N\to\infty} \frac{L^N(t)}{N} = \lim_{N\to\infty} \sum_i \frac{1-\sigma_i(t)}{2N} = \frac{1-m_t^\sigma}{2}$.

Now, to give a first intuition, we plot the probability to lose, in total, more than x, with respect to different values of x.
In other words we compute the quantiles $P\left(L^N(t) \geq \alpha\right)$, where α is a "large" integer. We consider a portfolio of $N = 10,000$ firms and we show how the excess loss probability changes for different values of the parameters.

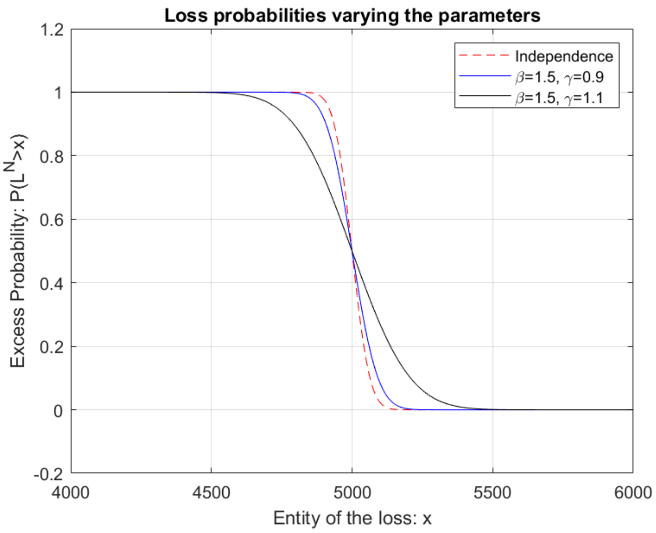

Figure 4.1: Large Portfolio Losses in the subcritical case. The inputs used here are the following: $(t_N, \beta, m_0^\sigma, m_0^\omega) = (300, 1.5, -0.5, 0.395)$, where we change γ.

In Figure 4.1 we consider the subcritical case ($\gamma < 1/\tanh(\beta) = 1.1048$); what we can observe is that, when the dependence increases, i.e., the interaction among the firms, described by γ, increases, the variance $\hat{V}(t)$ and risk measure increase as well. In the figure we consider, first of all, the independence case, where there is no interaction at all, corresponding to $\beta = \gamma = 0$. As we are dealing here with parameters belonging to the subcritical case, it makes sense to consider t_N big enough: when we studied the behaviors of $m^\sigma(t)$ and $V(t)$, we observed that the variance $V(t)$ in the subcritical case,

increases until it reaches a stationary value after a while. In this way, as $\hat{V}(t)$ depends linearly on $V(t)$, the bigger the variance $V(t)$ gets, the bigger the "asymptotic" variance when the number of firms goes to infinity ($\hat{V}(t)$), becomes.
We know also, as shown before, that the closer we are to the critical case (in the subcritical case this means get a bigger γ) the bigger the limit value of $V(t)$ is.
Note moreover, that here, unlike other future examples for the supercritical case, the mean, for the different cases based on the parameters, is the same. For a better comprehension, with mean we are talking about the variable $L(t)$ defined in (4.4): it is the "asymptotic" loss when the number of firms goes to infinity. As we are in the subcritical case, if we consider time t big enough, m_t^σ reaches the limit value 0 and $L(t)$ gets to the same value for all the three choices of the parameters.

The next example is obtained starting from a very tractable class of models, the "Bernoulli Mixture Models". Bernoulli mixture models have become a standard for the measurement and management of credit loss risk in financial institutions; we will apply our approach to this type of model.

We introduce this type of model because we aim at unifying two complementary approaches.
There are classes of models where the fluctuation of credit losses is due to the variation of economic fundamentals only (σ and ω), so that firms' interdependence is related to cyclical correlation effects only. In other words we neglect contagion effects; such an approach might underestimate the degree of loss fluctuation to be expected. On the other hand, an approach focusing exclusively on the contagion effects, as in Giesecke and Weber (2002), does not explicitly account for cyclical correlation effects.
We model aggregate credit losses on large portfolios of financial positions contracted with firms subject to both cyclical default correlation and direct default contagion processes. Cyclical correlation is due to the dependence of firms on common economic factors. Contagion is associated with the local interaction of firms with their business partners.

Let the losses depend not only on the realization of σ, but also on a random exogenous factor Ψ; more precisely, the marginal losses $L_i(t)$ are independent and identically distributed conditionally to the realizations of the $\sigma_i(t)$ and of Ψ. We consider G_x, $x \in \{-1, 1\}$, the conditional distribution function, to be

$$G_x(u) := P(L_i(t) \leq u \mid \sigma_i(t) = x, \Psi). \tag{4.9}$$

Both G_x and l_1, l_{-1}, v_1, v_{-1} are random variables.

EXAMPLE 2.

We assume that the marginal losses $L_i(t)$ are Bernoulli Mixtures, that is,

$$L_i(t) = \begin{cases} 1, & \text{with probability } P(\sigma_i(t), \Psi) \\ 0, & \text{with probability } 1 - P(\sigma_i(t), \Psi) \end{cases},$$

where $P(\sigma_i(t), \Psi)$ is the probability that the marginal loss at time t caused by the i^{th} firm is 1. The same thing for the other case.
As we can notice, the marginal loss doesn't depend only on the rating class indicator $\sigma_i(t)$ anymore, but also on an exogenous factor Ψ, where the mixing derives not only from the rating class indicator $\sigma_i(t)$ of firm i, but also from an exogenous factor $\Psi \in \mathbb{R}^p$ that represents macroeconomic variables that reflect the business cycle and thus allow for both contagion and cyclical effects on the rating probabilities.
With the above specification, we get that the variables l_1, l_{-1}, v_1, v_{-1} depend on the random factor Ψ:

$$l_1 = E(L_i(t) \mid \sigma_i(t) = 1, \Psi) = 1 \cdot P(1, \Psi) + 0 \cdot (1 - P(1, \Psi)) = P(1, \Psi)$$

$$v_1 = 1^2 \cdot P(1, \Psi) + 0 \cdot (1 - P(1, \Psi)) - P(1, \Psi)^2 = P(1, \Psi)(1 - P(1, \Psi))$$

and analogously for l_{-1}, v_{-1}.
Now, as the quantities just described depends on Ψ, also the asymptotic loss function $L(t)$ depends on the new exogenous factor. To be able to give some concrete results and simulations to better understand what we are studying, we give a possible expression for the mixing distribution for $P(\sigma, \Psi)$. Let a and $b_i, i = 1, 2$, be nonnegative weight factors. We can now consider Ψ to be a Gamma distributed random variable. Let

$$P(\sigma, \Psi) = 1 - \exp\left\{-a\Psi - b_1\left(\frac{1-\sigma}{2}\right) - b_2\right\}.$$

The previous specification follows from some ideas behind Credit Metrix risk models as we are in a framework where we deal with different rating classes.

We show now a simulation: in Figure 4.2 we plot excess loss probability.

The main financial interpretation here is quite similar to the one of the previous picture: the loss may be higher in the case of high uncertainty about the value of the macroeconomic factor $\Psi \sim \Gamma(1.25, 0.1)$ and in the case of high level of interaction (level of contagion γ).

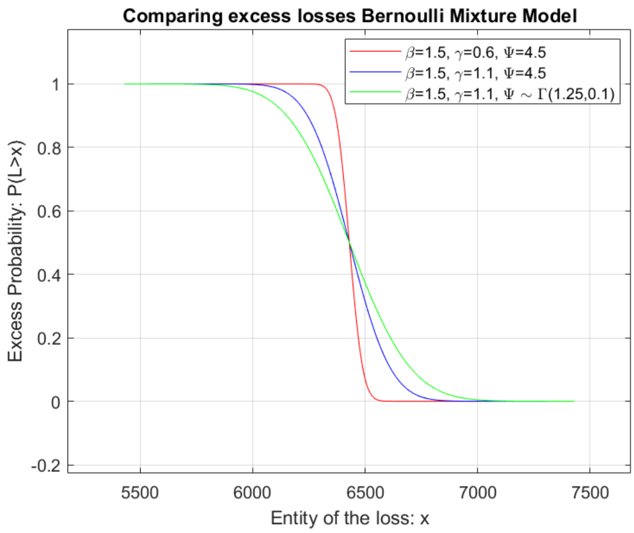

Figure 4.2: Large Portfolio Losses using Bernoulli Mixture Model. The values are rescaled to have the same average and to better compare the variances. The input used here are the following: $(t_N, \beta, m_0^\sigma, m_0^\omega, a, b1, b2) = (300, 1.5, -0.5, 0.395, 0.1, 1, 0.2)$ where we change γ and Ψ.

In the Figure below we are dealing again with parameters belonging to the subcritical case, but there is a big difference with the previous example. As already mentioned, in the subcritical case, $m^\sigma(t)$ converges to 0, but this time $L(t)$ depends on Ψ so we are not sure anymore that the means are the same. The picture is obtained avoiding to rescale.

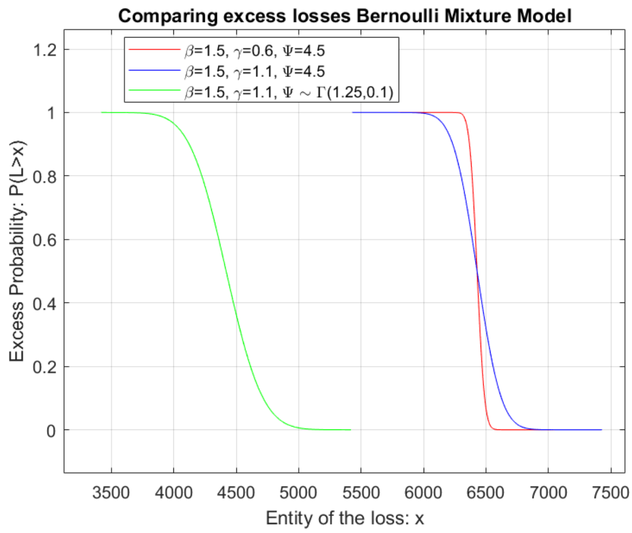

Figure 4.3: Large Portfolio Losses using Bernoulli Mixture Model. Here we can observe the true values of the averages. The inputs used here are the following: $(t_N, \beta, m_0^\sigma, m_0^\omega, a, b1, b2) = (300, 1.5, -0.5, 0.395, 0.1, 1, 0.2)$ where we change γ and Ψ.

4.3 Numerical Simulations

In this section we go into details about portfolio losses and in particular about the phenomenon of credit crises: how they influence the quantification of losses in a portfolio issued by a large number of firms. We already explained what we mean with credit crises; here we emphasize how the possibility of having a credit crisis is strictly related to the existence of particular conditions, especially the levels of interaction between the firms.
We compute portfolio losses at a specific time horizon T; we show, then, how the quantiles $P\left(L^N(t) \geq \alpha\right)$ change depending on the time horizon that we consider.
We are able to capture completely different behaviors about the loss estimate by the bank, depending on the fact that the bank does the estimations before, during, or after a credit crisis.
The simulation in Figure 4.5 is related to what we have seen in Figure 3.9 and what is reported in Figure 4.4: the input parameters are the same.

Recapping, we can say that the dynamics $m^\sigma(t)$ and $V(t)$, shape the state of a market that is undergoing a crisis. A possible idea is the following: we consider some initial conditions m_0^σ, m_0^ω such that the firms' state is not the worst possible one. We choose then some parameters such that, the initial state is not an equilibrium state, but the system goes through a credit crisis to stabilize then in an equilibrium where many other firms are in financial distress (m_t^σ as t increases, reaches a bigger value in absolute value; see Figure 4.4). The concept above is exactly what we have called credit crisis.
At this point, if the bank wants to compute an estimate of the possible loss that is going to suffer, the choice of the time at which compute this loss becomes crucial. As we can observe in Figure 4.5, the quantiles $P\left(L^N(t) \geq \alpha\right)$ have completely different behaviors depending on t_N. First of all the average loss in the three cases is different: this phenomenon is due to the dynamics of $m^\sigma(t)$. We start with $m^\sigma(0) = -0.5$, we get closer to the unstable equilibrium 0 and then we change direction to reach the stable equilibrium -0.8574 (this means that for different time horizons the average loss changes). For these specific input parameters, we see how, as we increase t_N, we increase also the expected average loss.
In Figure 4.6 we simply rescale the means, to better compare the variances $\hat{V}(t)$ for different final times.

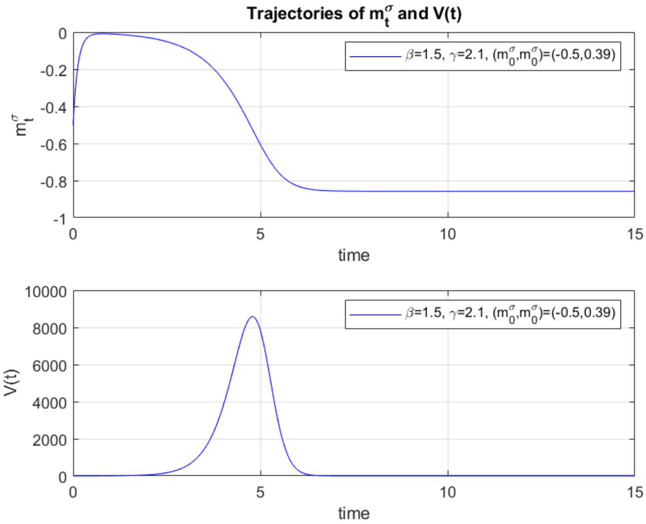

Figure 4.4: Trajectory of $m^\sigma(t)$ and $V(t)$ with initial conditions (m_0^σ, m_0^ω)=$(-0.5, 0.39)$ when $\beta = 1.5$ and $\gamma = 2.1$ (here the critic γ is $1/\tanh(\beta) = 1.105$).

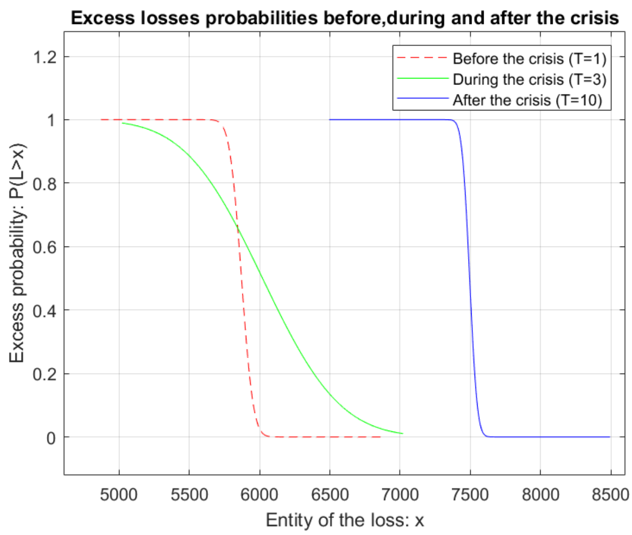

Figure 4.5: Large Portfolio Losses at different time horizons t_N. Here we can observe the true values of the average losses. The inputs used here are the following: $(\beta, \gamma, m_0^\sigma, m_0^\omega, a, b1, b2) = (1.5, 2.1, -0.5, 0.39, 0.1, 1, 0.2)$ where t_N gets values in $\{1,3,10\}$.

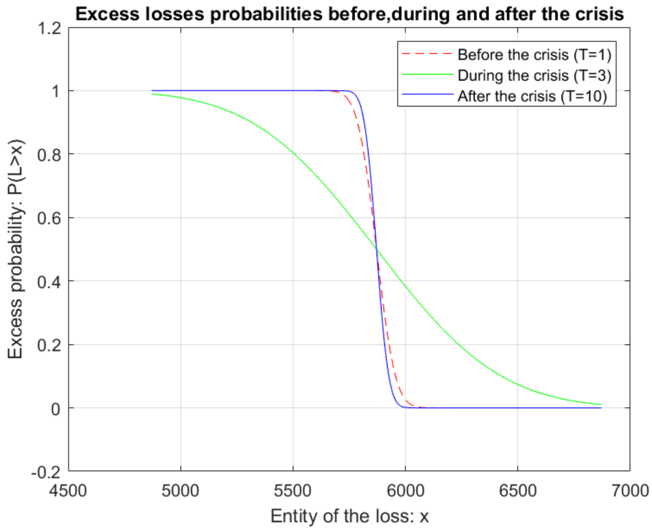

Figure 4.6: Large Portfolio Losses at different time horizons t_N. Here we force the curves to have the same average loss to better observe the variances. The inputs used here are the following: $(\beta, \gamma, m_0^\sigma, m_0^\omega, a, b1, b2) = (1.5, 2.1, -0.5, 0.39, 0.1, 1, 0.2)$ where t_N gets values in {1,3,10}.

More in financial terms, what the previous simulations want to emphasize are the following ideas.
If the bank want to make some predictions about the possible losses that may suffer and the prediction is done in the middle of a credit crisis, the simulations above show that it is more difficult to have a reliable prediction. During the crisis the volatility of the distribution of portfolio losses increases extremely: if we interpret the transition from one equilibrium to another one as a crisis, this phenomenon is captured by the model and during these periods it is more difficult to have a reliable idea of what is going to happen. On the other hand, if we wait until the market stabilizes, the number of firms in financial distress may be higher (it is exactly the case), this means the average loss gets bigger, but at the same time the variance decreases because we are outside the crisis period due to the change of equilibrium.

Now, observing closer the variances values $\hat{V}(t)$ and the standard deviations $\sqrt{\hat{V}(t)}$ at $t_N = 1, 3, 10$, we see that:

- $\hat{V}(1) = 0.44$ and $\sqrt{\hat{V}(1)} = 0.66$

- $\hat{V}(3) = 18.8$ and $\sqrt{\hat{V}(3)} = 4.34$

- $\hat{V}(10) = 0.19$ and $\sqrt{\hat{V}(10)} = 0.43$.

The standard deviation is a symmetric measure of dispersion around the average portfolio value. The greater the dispersion around the average value, the larger the standard deviation, and the greater the risk. If the portfolio values are expressed in dollars, this standard deviation calculation also results in a dollar amount.
In our case, before and after the crisis the standard deviations have the same order of magnitude, while during the crisis, at time $t_N = 3$ for example, the standard deviation grows by a multiplier almost 7 with respect to $t_N = 1$ and 10 with respect to $t_N = 10$.

Even if this model gives only a qualitative description of a real world market, we can see how the standard deviation values are in line, more or less, with what one would expect.
In the following chapter we are going to consider a particular volatility index to show how the model is somehow reliable and consistent with what happens in real world markets.

One last important simulation that we consider relevant to show is Figure 4.7. We are in the supercritical case and the two dynamics below emphasize how the possibility of having a credit crisis is strictly related to the existence of particular conditions, especially the levels of interaction between the firms.
For different levels of interaction we can distinguish between two types of crises: a smoothly varying business cycle and a real credit crisis. A business cycle, also known as an economic cycle or trade cycle, is the downward or upward movement of gross domestic product (GDP), a monetary measure of the market value of all the final goods and services produced in a specific time period.
As we can observe in Figure 4.7, considering sufficiently small parameters, $m^\sigma(t)$ reaches the stationary value slowly and smoothly and the variance $V(t)$, the level of uncertainty about the number of bad rated firms, is lower and comes later compared to the crisis case.

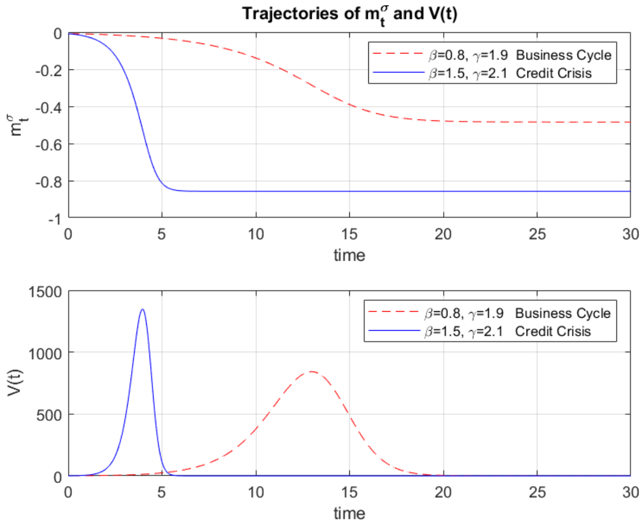

Figure 4.7: Trajectories of m_t^σ and $V(t)$ for different values of β and γ. In the case of smaller values the number of bad rated firms decreases smoothly to a new equilibrium, that is, toward a bad business cycle. In the case of higher values, we see a crisis and a corresponding peak in the uncertainty in the market. The critical values for γ are, respectively, $1/\tanh(1.5) \simeq 1.105$ and $1/\tanh(0.9) \simeq 1.396$.

Chapter 5

Consistency with real data and conclusions

5.1 Consistency with real market data

In this chapter we give a first look to the consistency between the model and a real financial market: the aim is to show that the model, and the simulations related to it, give an insight to what happens in real contexts. The approach that we have used is qualitative but, at the same time, if we compare the results with some real data, we get something sensible. What we would like to observe is mainly if the unit of measure of the variance $\hat{V}(t)$ studied in the previous chapters, has a reasonable unit of size.
We would like to observe if some real volatility indexes, during period of crises, crash as the model predicts.

In this chapter we just want to give an idea of the consequences that a real world credit crisis may have on the fluctuation of variance/standard deviation; to do this we resort to some type of volatility indexes.
Just to simplify, we are going to consider only one type of volatility index: The CBOE Volatility Index, or VIX.
Created by the Chicago Board Options Exchange (CBOE), the Volatility Index, or VIX, is a real-time market index that represents the market's expectation of 30-day forward-looking volatility. Derived from the price inputs of the S&P 500 index options, it provides a measure of market risk and investors' sentiments. It is also known by other names like "Fear Gauge" or "Fear Index." Investors, research analysts and portfolio managers look to VIX values as a way to measure market risk, fear and stress before they take investment decisions.

Figure 5.1 is the chart of the VIX behavior from 2004 to 2020. As we can observe the volatility index increases significantly during the years 2008, due to the global crisis, and 2020, due to Covid-19. What we want to emphasize is the fact that the ratio between two VIX values, one captured during a "normal period" and the other during a crisis, has the same magnitude of the ratio that we obtain in Chapter 4, Section 4.3. To give a better explanation we recall that in the previous chapter we analyzed the multiplicative factor for the standard deviations to go from a "normal period" to a period characterized by a credit crisis. We obtain that this multiplicative factor is a value between 7 and 10 . Here, looking at the figure below, we can consider a VIX=80 in 2008 (crisis period) and a VIX=10∼15 during a "quiet" period; in this way we see how the difference between a period of time during a crisis and one not during a crisis are characterized by a 7∼10 time, bigger/smaller standard deviation.

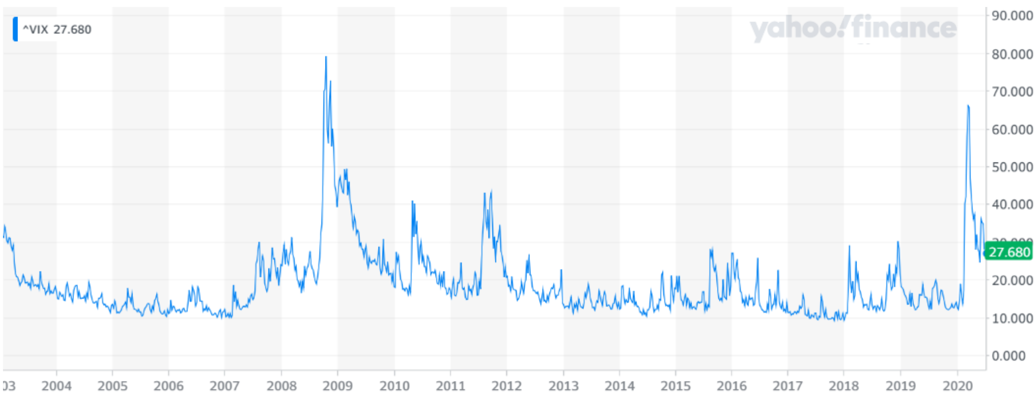

Figure 5.1: CBOE Volatility Index (VIX) 2004–2020 (daily closings). The chart is about VIX historical data from 2004 to 2020; the data are obtained from Yahoo-Finance. We can see how, during certain periods of time, the graph has some peaks in volatility due to period of crises.

We, now, want to point out that in the previous example we use data from a financial market instead of data from a credit market. The reason is why it is difficult to obtain data for credit markets and so we decided to use stock markets volatility as an approximation. In the following figure, we show a statistic that presents the mortgage delinquency rates for subprime conventional loans in the United States from 2000 to 2016. Subprime (Subprime lending), B-Paper, near-prime or second chance are terms that indicate those loans that, in the US financial context, are granted to a person who cannot access market interest rates, as he had previous problems, in his history, as a debtor. The term subprime refers to a variety of credit instruments, such as subprime mortgages, subprime auto loans, subprime credit cards. What we are aiming to show is how many defaults the banks had in their credit portfolios during the American subprime crisis started at the end of 2006, when the US housing bubble began to deflate and simultaneously, many subprime mortgage holders became insolvent due to rising interest rates. In Figure 5.2 we show some data about subprime mortgages.

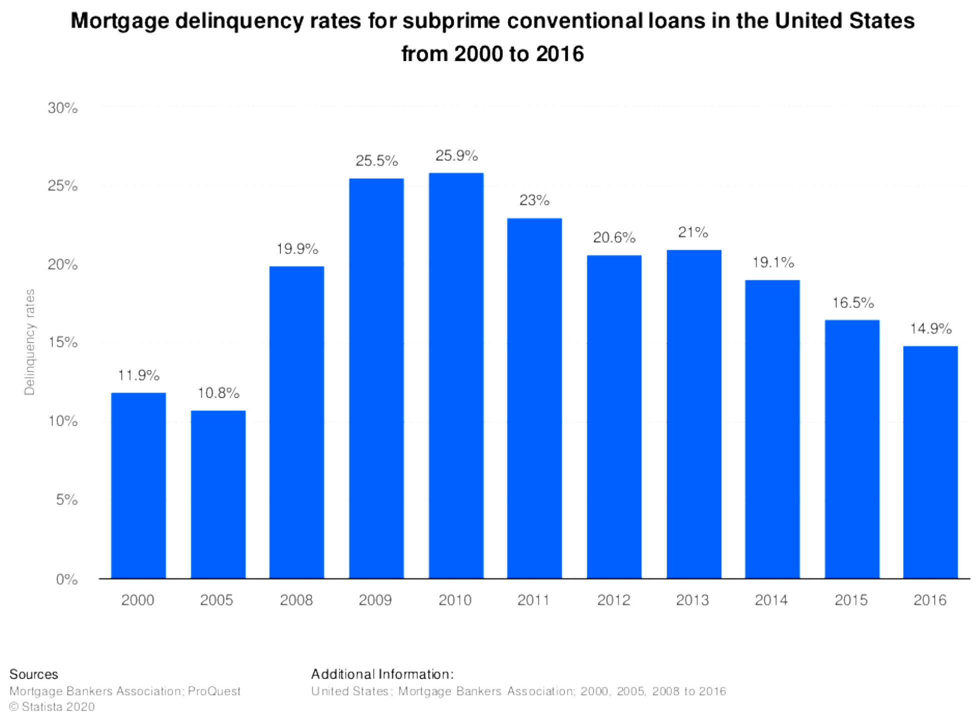

Figure 5.2: Subprime Mortages 2000–2016. The chart is about mortgage delinquency rates for subprime conventional loans in the United States from 2000 to 2016.

With Figure 5.2, we just want to emphasizes in a qualitative way, how the delinquency rates for some risky loans (subprime mortgages) change drastically during a period of crisis: the behavior of the delinquency rates doesn't follow exactly the dynamics of m_t^σ, meaning the quality of the debt of a cer-

tain set of firms. We can observe that, recalling for example Figure 3.9, the graph above shows that the period before the 2008 crisis is characterized by a housing bubble, that starts deflating causing many mortgage holders to become insolvent due to the rising rates. Before the crisis, several banks had many "junk bonds" in their portfolios, often priced as safe bonds. There were lots of defaults, but limited (10 % of the graphs that we make correspond to m_t^σ about 0). At some point the contagion starts, the variances of our model increase and the defaults explode. m_t^σ leads to a highly negative value and stays constant to that value for t that goes to infinity.
In the figure we presented before, we can observe the credit bubble and credit contagion phenomena with the main difference that, as time goes by, the delinquency rates start to touch some normal values again instead of at 25 % of defaults as stationary value. The main reasons are two: first of all m_t^σ does not represent a delinquency rate but the percentage of the N firms that have a higher probability to default. Secondly, the model for losses describes the performance of a bank portfolio formed at $t = 0$ with a certain number of loans and monitors its performance; Figure 5.2, instead, describes a reality where, as time goes by, new firms may come into play, allowing the delinquency rates to drop.

5.2 Conclusions

The main aim of this work is to introduce a direct contagion model, where a large number of firms interact with each other and quantify the losses suffered by a bank holding a large portfolio with positions issued by those firms after a credit crisis. Through theoretical results and numerical simulations we have been able to show the idea of credit crises and to provide formulas to compute quantiles of the probability of excess losses in the context of our contagion model. We have shown how, both for financial and credit markets, the results obtained with our model, even if we deal with a qualitative model, are consistent with real world market data.
The peculiarity of our model is that the changes in rating class (the variable σ) are related to the degree of health of the system (the global indicator m^σ). The firms are described also by a second characteristic that is summarized by the variable ω that describes the real state of the firms (a liquidity indicator). Our model, unlike many others, uses methodologies that belong to statistical mechanics, in fact we have used an interacting particle system to represent the different firms that interact with each other.
The model we have proposed in this thesis exhibits some stylized facts typical to financial settings. We now present some possible extensions that allow the

model to be more realist and flexible.
In this thesis we do not consider the issue of calibration to real data, but rather present some numerical simulation results related to credit crises and portfolio losses behaviors for different values of the parameters. Only in the end of our work we touch the topic of consistency with real data giving a quick look at what has happened historically, during times of crisis, to financial and credit markets.
Some possible extensions are the following:

- The mean field assumption may be weakened by considering less restrictive assumptions.

- In real applications, the pair (σ, ω) is not binary. Although the restriction to only two possible values may be unrealistic, we believe that many aspects of the qualitative behavior of the system we consider, do not really depend on this choice. However the results shown by our model can easily be extended to the case where the pair above assumes an arbitrary finite number of values.

- Instead of considering the intensities in Section 1.4 as deterministic, we can use random functions to represent the transitions $\sigma_i \mapsto -\sigma_i$ and $\omega_i \mapsto -\omega_i$.

Appendix A

Useful Tools

In this chapter we list some useful tools to approach to the different chapters of this work.
We state a useful Theorem about convergence of stochastic processes knowing the convergence of their infinitesimal generators.
In Theorem 2.1 we use the 2-dimensional Divergence Theorem, the Poincare'-Bendixson Theorem and the Stable Manifold Theorem.
In Theorem 4.1 we use Levy's Continuity Theorem and its inverse.
Here we write all the statements and some useful ideas behind them:

The following Theorem on stochastic processes gives us a result about convergence of stochastic processes from convergence of their generators:

Theorem A.1. *Let $(X_n(t))_{t\in[0,T]}$ be a sequence of Markov processes with values in E_n and denote by $\mathcal{L}_N$ the infinitesimal generator of $(X_n(t))$ (defined on $\mathcal{D}(L_N) : [0,T] \to E_n$ that denotes the space of right-continuous piecewise constant functions).*
Let $\mathcal{L}$ be the infinitesimal generator of another Markov process $X(t)$ with values in E (defined on $\mathcal{D}(L)$).
Assume that $\forall n \;\; E_n \subset E$. If the condition:

$$\lim_{N\to\infty} \sup_{x\in E_n} |\mathcal{L}_N f(x) - \mathcal{L} f(x)| = 0$$

holds $\forall f \in \mathcal{C}_b^1$ and if $X_n(0) \xrightarrow{d} X(0)$, then the sequence of processes $X_n(t)$ converges in distribution to the process $X(t)$.

The original Divergence Theorem states:

Theorem A.2. *The superficial integral of a vector field over a closed surface, which is called the flux through the surface, is equal to the volume integral of the divergence over the region inside the surface.*

$$\int_V \nabla \cdot F dV = \oint_{\partial V} F \cdot d\vec{S} \quad \text{where } d\vec{S} = \vec{n} ds$$

Now we want to consider a 2-dimensional version of the Divergence Theorem. The Divergence Theorem in 2-dimension is equivalent to Green's Theorem. We recall this Theorem below:

Theorem A.3. *Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and having continuous partial derivatives there, then*

$$\oint_C (Ldx + Mdy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dxdy$$

where the path of integration along C is anticlockwise.

Considering only two-dimensional vector fields, Green's Theorem is equivalent to the 2-dimensional version of the Divergence Theorem; here we prove one direction of this fact (Green implies Divergence). To see this, consider the unit normal $\vec{n}$ in Theorem A.2. Since in Green's Theorem dr=(dx,dy) is a vector pointing tangential along the curve and the curve C is the positively oriented (i.e. anticlockwise) curve along the boundary, an outward normal would be a vector which points 90 degrees to the right of this; one choice would be (dy,-dx). The length of this vector is $\sqrt{dx^2 + dy^2} = ds$. So (dy,-dx)=$\vec{n}$ ds.
If we consider $F = (L, M)$ we get that:

$$\oint_C F \cdot \vec{n}\, ds = \oint_C -Mdx + Ldy = \iint_D \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_D \nabla \cdot F\, dA \quad \Rightarrow \text{ we get } \quad \oint_C F \cdot \vec{n}\, ds = \iint_D \nabla \cdot F\, dA$$

that is exactly the 2-dimensional Divergence Theorem.

The Poincare' Bendixson Theorem is a classical result in the study of continuous dynamical system. This Theorem states that every trajectory of a

2-dimensional autonomous system converges either to an equilibrium or to a periodic solution.

Theorem A.4. *Suppose f a $\mathcal{C}^2$-function and consider the dynamical system*

$$\dot{y} = f(y) \quad \text{in } 2\ D$$

If y(t) is a bounded solution that does not approach to a steady state, then y(t) is either a periodic solution or it approaches a periodic solution.

About the Stable Manifold Theorem:

Theorem A.5 (Stable Manifold Theorem)**.** *Let E be an open subset of R^n containing the origin, let $f \in \mathcal{C}^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x)$. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k -dimensional differentiable manifold Γ tangent to the stable subspace E^Γ of the linear system $\dot{x} = Ax$ (where $A = Df(0)$) at 0 such that for all $t \geq 0$, $\phi_t(\Gamma) \subset \Gamma$ and for all $x_0 \in \Gamma$*

$$\lim_{t \to \infty} \phi_t(x_0) = 0$$

and there exists a $n - k$ dimensional differentiable manifold U tangent to the unstable subspace E^u of $\dot{x} = Ax$ at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$

$$\lim_{t \to -\infty} \phi_t(x_0) = 0$$

The Levy's continuity Theorem connects convergence in distribution of a sequence of random variables, with pointwise convergence of their characteristic functions:

Theorem A.6 (Levy's continuity Theorem)**.** *Let $\{X_n\}$ be a sequence of random variables with characteristic functions $\{\varphi_n(t)\}$ and X a random variable with a characteristic function $\varphi(t)$, pointwise convergence $\varphi_n(t) \to \varphi(t)$ implies convergence in distribution $X_n \xrightarrow{d} X$.*

The converse result holds too and is really easy to prove:

Lemma A.1. *Let* $\{X_n\}$ *be a sequence of random variables with characteristic functions* $\{\varphi_n(t)\}$ *and* X *a random variable with a characteristic function* $\varphi(t)$*, convergence in distribution* $X_n \xrightarrow{d} X$ *implies pointwise convergence* $\varphi_n(t) \to \varphi(t)$ *of the characteristic functions.*

Proof. It follows from the fact that if $X_n \to X$ in distribution then $E\left[f\left(X_n\right)\right] \to E[f(X)]$ for every bounded continuous f (this is often taken as the definition of convergence in distribution). If we take $f(x) = e^{itx}$ we obtain immediately that $\varphi_n \to \varphi$ pointwise. $\square$

Appendix B

Matlab Codes

```
function[y]=DensityFunction(x,m,v)
%m="mean"
%v="variance"
xx=x-m;
z=xx.*xx;
y=  (1/sqrt(2*pi*v)) *exp(-(z)/(2*v));
end
```

Note: This function is useful to write the density function for a Gaussian variable.

```
function [x,y,yy]= Norm(N,NN,r,v,t0,x0,xN)
%r= what we have called in the thesis L(t)
%x=what we have called in the thesis alpha
h=(xN-x0)/N;
x=0;
y=0;
yy=0;
x(1)=x0;
yy=(sqrt(NN)*r-NN^(-0.5)*x0)/(sqrt(v));
y(1)=integral(@(t)DensityFunction(t,0,1),t0,yy);
for i=2:N
x(i)=x0+h*i;
yy(i)=(sqrt(NN)*r-NN^(-0.5)*x(i))/(sqrt(v));
y(i)=integral(@(t)DensityFunction(t,0,1),t0,yy(i));

end
end
```

Note: This piece of code uses the previous function DensityFunction and corresponds exactly to the transposition of what we observe in Equation

(4.8).

```
function [x,y,yy] = CovarianceMatrix(tN,b,e,ms,mw,A1,A2,A3,K)

g=((exp(2*b)+1)/(exp(2*b)-1))+e;

%tN="Final time"
%b="beta"
%g="gamma"
%e="epsilon"
%ms=m_sigma_t
%mw=m_omega_t
%mp=m_sigma omega _t=E[sigma x omega]
%cs=Cov[sigma,sigma]=Var[sigma]=E[sigma^2]-E[sigma]^2=1-ms^2
%cp=Cov[sigma x omega,sigma x omega]
%csw=Cov[sigma,omega]=E[sigma omega]-E[sigma]E[omega]
%csp=Cov[sigma ,sigma x omega]
%cwp=Cov[omega ,sigma x omega]


mp=ms*mw;
cs=1-ms^2;
cw=1-mw^2;
cp=1-mp^2;
csw=mp-ms*mw;
csp=mw-ms*mp;
cwp=ms-mw*mp;

options= odeset('RelTol',1e-4,'AbsTol',1e-6);
[T,M]=ode45(@medvar,[0 ...
    tN],[ms,mw,mp,cs,csp,csw,cp,cwp,cw],options);


function dm = medvar(s,m);

dm=zeros(9,1);
dm(1)=2*[sinh(b)   *m(2)  - cosh(b)*       m(1)]; %ms(t)
dm(2)=2*[sinh(g*m(1))    - cosh(g*m(1))* m(2)]; %mw(t)
dm(3)=2*[sinh(g*m(1)) *m(1)   + sinh(b)- (cosh(g*m(1))+ ...
    cosh(b))*m(3)]; %msw(t)
dm(4)=2*[m(4)*2*(-C)+m(6)*2*sinh(b)-m(3)*S+C]; %Var(x(t))
dm(5)=2*[m(5)*(-2*C-cosh(g*m(1)))+m(4)*(sinh(g*m(1))+ ...
     +g*m(1)*cosh(g*m(1))-g*m(3)*sinh(g*m(1)))+m(8)*sinh(b)+...
     -m(1)*S+m(2)*C];
dm(6)=2*[m(6)*(-C-cosh(g*m(1)))+m(4)*(-g*m(2)*sinh(g*m(1))+...
     +g*cosh(g*m(1)))+m(9)*S];
dm(7)=2*[m(7)*2*(-C-cosh(g*m(1)))+m(5)*2*(sinh(g*m(1))+...
      g*m(1)*cosh(g*m(1))-g*m(3)*sinh(g*m(1)))-m(3)*S+C+...
```

```
      -m(2)*sinh(g*m(1))+cosh(g*m(1))];
dm(8)=2*[m(8)*(-C-2*cosh(g*m(1)))+m(6)*(sinh(g*m(1))+...
      +g*m(1)*cosh(g*m(1))-g*m(3)*sinh(g*m(1)))+...
      +m(5)*(-g*m(2)*sinh(g*m(1))+g*cosh(g*m(1)))+...
      +m(1)*cosh(g*m(1))-m(3)*sinh(g*m(1))];
dm(9)=2*[m(9)*(2*-cosh(g*m(1)))+2*m(6)*(-g*m(2)*sinh(g*m(1))+..
      g*cosh(g*m(1)))-m(2)*sinh(g*m(1))+cosh(g*m(1))];

end
%Where:
%m(1)=ms(t)
%m(2)=mw(t)
%m(3)=msw(t)
%m(4)=Var(x(t)) where x(t) is the process described in ...
    Theorem 2.2
%m(5)=Cov(x(t),z(t))=Cov(z(t),x(t))
%m(6)=Cov(x(t),y(t))=Cov(y(t),z(t))
%m(7)=Var(z(t)) where z(t) is the process described in ...
    Theorem 2.2
%m(8)=Cov(y(t),z(t))=Cov(z(t),y(t))
%m(9)=Var(y(t)) where y(t) is the process described in ...
    Theorem 2.2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                %Portfolio Losses

MM=M(length(M),1);
CC=M(length(M),4);
%MM is the value of ms(t) for t=tN
%CC is the value of Var(x(t)) for t=tN

g=((exp(2*b)+1)/(exp(2*b)-1))+e;


% We compute the Portfolio Losses for a bynomial model as ...
    in the thesis.
% We assume the probability of default to be of ...
    exponential trend
% LP = Probability of default for the case "sigma=1"
% LN = Probability of default for the case "sigma=-1"
% A1,A2,A3 positive parameters.
% K = varibale that for the moment we suppose to be constant

LP=1-exp(-(A1*K+A3));   % LP corresponds to the expected ...
    marginal loss
LN=1-exp(-(A1*K+A2*(1)+A3));
%LP=0;   % LP for the non stochastic case!!!!!
%LN=1;

VP=LP*(1-LP);   % marginal variance expected for the ...
```

```
    single loss
VN=LN*(1-LN);

D=(LP-LN)/2;
L=D*MM+(LP+LN)/2;  % loss asintotica L(t)
V=D^2*CC+(1+MM)/2*VP+(1-MM)/2*VN  % asymptotic variance V(t)

%We calculate the distribution of joint losses as in the ...
    example in the thesis on a portfolio of 10,000 entries ...
    and compare it with a loss distribution where there is ...
    no interaction (indipendence case).

[x,y,yy]= Norm(1000,10000,L,V,-10,L*10000-1000,L*10000+1000);
end
```

Note: Here, using the Covariance Matrix introduced in 2.4, first we obtain the values of $m^\sigma(t)$ and $V(t)$ at different times and then we compute portfolio losses both for stochastic and non stochastic cases, depending which example we are studying.

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