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Dynamics of a Tagged Particle
in the Exclusion Process

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Introduction

In probability theory, the simple exclusion process is one of the classical and most simple interacting particle systems, introduced in 1970 by Frank Spitzer [17]. Since then, it has been widely studied and has become a stochastic model for transport phenomena, such as cars in a traffic jam, but also for analyzing the evolution of some physical systems consisting of a large number of components, for example, a fluid or a gas. Due to the huge complexity of the dynamics between molecules, it becomes hard to analyze the microscopic evolution of the system and some simplifications need to be introduced. With that purpose, usually one assumes that the underlying microscopic dynamics is stochastic, so we can consider particles which exhibit interacting random walk behaviour.

Therefore, the simple exclusion process models the behaviour of infinitely many identical particles on a countable set, so that the dynamics of each particle constitute a continuous time Markov chain. However, the main feature of this motion is the exclusion interaction: transitions to occupied sites are suppressed. The process takes place on an underlying space S, which is usually the set of the vertices of a graph $G(V, E)$ or the lattice $\mathbb{Z}^d$.

In this work we study the simple exclusion process "as seen from a tagged particle". In the case of a graph as an underlying space, we initially place a particle on a distinguished site called the "root". We then follow the evolution of this tagged particle, as done in [4], where the considered graph is a rooted d-regular tree. Here, we choose $S = \mathbb{Z}^d$ and to describe the dynamics we fix a finite-range probability measure $p(\cdot)$ on $\mathbb{Z}^d$ which does not charge the origin, that is $p(0) = 0$, and distribute particles on the lattice in such a way that each site is occupied by at most one particle. A particle at x waits an exponentially distributed time with mean 1, then jumps to a site y with probability $p(x, y) = p(y - x)$. If y is vacant the jump is performed, otherwise the particle stays at x. All the holding times and choices according to p are independent. Moreover, since the holding times have a continuous distribution, only one particle moves at a time: that is the reason why we can tag an individual particle and study its motion, the so-called "tagged particle process" [7].

This informal description corresponds to a Feller process on the space $\mathbb{X} = \{0, 1\}^{\mathbb{Z}^d}$ endowed with the product topology. The states of $\mathbb{X}$ are denoted by $\{\eta_t : t \geq 0\}$, with $\eta(x)$ equal to 0 if the site $x \in \mathbb{Z}^d$ is empty, otherwise $\eta(x) = 1$. We will consider a configuration η with a tagged particle at the origin, denoting $\{Z_t : t \geq 0\}$ its position at time t and we will see that the pair $\{(Z_t, \eta_t) : t \geq 0\}$ is a Markov process.

Our goal is to study the asymptotic behaviour of the tagged particle's position Z_t. In particular we aim to verify that the Law of Large Numbers, the Central Limit Theorem and the Invariance Principle can be applied. If there were no other particles, Z_t would evolve as a continuous time random walk and all these results would hold. But in fact, the presence of the environment, consisting of all the remaining particles, affects the dynamics of Z_t and its study becomes more complicated. The conventional strategy, which we will adopt, is to consider the evolution of the environment as seen from the position of the tagged particle. For this purpose, we will place the chosen particle at the origin and after its jump to a site x, we will translate the entire configuration by $-x$, keeping the origin fixed. Due to this fact, we need the probability measure $p(\cdot)$ to be translation-invariant. Moreover, since we want to analyze the behaviour of the environment excluding the origin, which is always occupied by the tagged particle, we can assume that our process takes place on the space $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$.

The main results of this thesis are a Law of Large Numbers, a Central Limit Theorem and an Invariance Principle for the position of the tagged particle. The first two, under some assumptions, are relatively simple to derive and to prove. The arguments we will use here are from Komorowski, Landim and Olla [13] and, basically, they consist of decomposing the position Z_t into a martingale plus an additive functional, and then studying separately these two parts, that are simpler processes. In particular, we will prove the Central Limit Theorem in any dimension for the case of zero-mean ($m = 0$) process. However the result holds true in dimension $d \geq 3$ even if $m \neq 0$ [13], while, in smaller dimensions this behaviour has only been conjectured and its proof is still an open problem.

To approach the third main result, the Invariance Principle, that states the convergence of the evolution of the tagged particle's position to a Brownian motion, we are going to work in the context of Rough Path theory. The main idea of this theory is to "enhance" a path X with some additional data $\mathbb{X}$, namely the Itô integral of X against itself, in order to restore the continuity of the Itô map. To work in this context we have to consider the space of hölder continuous Rough Paths and the suitable metric, see [8] and [10]. Another main ingredient are semimartingales, which fit into the theory, as shown in [5]. Indeed, the canonical lift of a semimartingale is almost surely a Rough Path of finite p-variation, for $p > 2$.

The theory of Rough Paths was developed, in the mid-nineties, by Terry Lyons [16], in order to treat controlled differential equations of the form

$$dy_t = \sum_i f^i(y_t)dx_t^i$$

where the f^i are vector fields and the driving signal x_t is a rough path, such as a vector valued Brownian motion, a semi-martingale or any similar stochastic process. A new metric is needed to guarantee that the solution map will be a continuous function of the driving rough path. In our case, we will have to deal with càdlàg (from the French: right continuous with left limits) functions; the set of these functions on a given domain is called a Skorokhod space and when endowed with the corresponding metric, it generates

the so-called Skorokhod topology. Introducing this new metric on the space of càdlàg Rough Paths, we will be able to apply a version of the invariance principle to the position of the tagged particle, particularly using tightness in p-variation. However, we will see that the convergence to the Itô integral $\int_0^t B_s \otimes dB_s$ is not exact: in addition there will be a correction term of the form $1/2\langle B, B\rangle_t + \Gamma t$.

In detail, the thesis will be organized as follows. In the first chapter we will introduce the setting where we are going to work and the necessary main assumptions, defining the $\mathscr{H}_1$ Hilbert space and its dual $\mathscr{H}_{-1}$. Then, we will provide a formal description of the simple exclusion process and of the simple exclusion process with the tagged particle. In both processes there is a family of stationary states, the Bernoulli product measure with density $0 \le \alpha \le 1$. This means that the site where the tagged particle chooses to jump is empty with probability $1 - \alpha$. We thus expect a mean displacement in the stationary state to be equal to $(1 - \alpha)m$, with $m = \sum_{x \in \mathbb{Z}^d} xp(x)$ defined as the mean displacement of the tagged particle. This is exaclty what we will show with the Law of Large Numbers.

Before proving the main result of the second chapter, it will be necessary to rewrite the position Z_t of the tagged particle, through the definition of the jump processes, as a martingale and an additive functional. This one will be then expressed in terms of elementary ortoghonal martingales, associated to the jumps of the Markov process. The key property of these martingales is that we can write any other martingale in terms of these elementary orthogonal martingales, by the Dynkin's formula, so that by this decomposition it is possible to compute the limit, under our stationary measure. This will be enough to conclude the proof of the Law of Large Numbers.

In the third chapter, we will present the Central Limit Theorem for Z_t, the proof of which relies on the more general Central Limit Theorem for additive functionals. The aim of Section 3.1 is to use the solution of the resolvent equation, in order to rewrite the functional as a sum of a martingale and a negligible term, which we will prove to vanish, as $t \uparrow \infty$. So, the only thing left to show is the convergence of the martingale terms to a mean zero Gaussian random vector with a finite covariance matrix. The arguments used for this purpose will be treated separately in the symmetric and asymmetric case, under the assumption $m = 0$.

Finally, in the chapter concerning the invariance principle, we introduce the necessary elements and definitions about Rough Path theory, the Skorokhod topology and, in particular, the Skorokhod metric in p-variation. Moreover, we will introduce the notions of Burkhölder-Davis-Gundy (BDG) inequality and Uniformed Controlled Variation (UCV), from [5]. Having done all of this, our process X will be lifted to a rough path $\mathbb{X} = \int_s^t X_r \otimes dX_r$. We will finally prove that the pair $(X, \mathbb{X})$ converges, in the p-variation Skorokhod topology, to $\left(B; \left(\int_0^t B_s \otimes dB_s + \frac{1}{2}\langle B, B\rangle_t + \Gamma_t\right)_{t \ge 0}\right)$ and we will compute the covariance of the Brownian motion B and the correction term Γ.

Chapter 1

Exclusion Processes

1.1 Notation and Main Assumptions

In this first section we define the setting where we are going to work, taking some results from [13]. Let E be a complete and separable metric space endowed with its Borel σ-algebra $\mathscr{E}$. Denote by $B(E)$ the set of bounded measurable functions on E and by $C_0(E)$ the space of continuous functions on E which vanish at infinity, regarded as a Banach space with the norm

$$\| f \| = \sup_x | f(x) | .$$

Let $D([0, \infty), E)$ be the set of functions $X : [0, \infty) \to E$ which are right continuous and with left limits (r.c.l.l.). Denote by $\Pi_s : D([0, \infty), E) \to E$, when $s \geq 0$, the canonical projection defined by

$$\Pi_s(X) = X_s.$$

Let $\mathscr{F}^o$ be the smallest σ-algebra on $D([0, \infty), E)$ which turns the projections Π_s, $s \geq 0$, measurable and let $\mathscr{F}_t^0$ be the natural filtration, i.e. the smallest σ-algebra relative to which all the mappings Π_s, $0 \leq s \leq t$, are measurable.

Let $\{P_t : t \geq 0\}$ be a strictly Markovian, Feller semigroup of linear operators on $C_0(E)$, according to the following definition from [15, Section I.1.].

Definition 1.1. A family $\{P_t : t \geq 0\}$ of linear operators on $C_0(E)$ is called a stricly Markovian Feller semigroup if it satisfies the following conditions:

1. $P(0) = I$, the identity operator,

2. the map $t \mapsto P(t)f$, from $[0, \infty)$ to $C_0(E)$, is right continuous for all $f \in C_0(E)$,

3. $P(t + s)f = P(t)P(s)f$ for all $f \in C_0(E)$ and all $t, s \geq 0$,

4. $P(t)\mathbf{1} = \mathbf{1}$ for all $t \geq 0$, where $\mathbf{1}$ is the constant function equal to 1,

5. $P(t)f \geq 0$ for all non-negative $f \in C_0(E)$.

Note that the function $\mathbf{1}$ does not belong to $C_0(E)$ if E is not compact, but the semigroup $\{P_t : t \geq 0\}$ can clearly be extended to $B(E)$.

Consider a normal Markov process on E associated to the semigroup $\{P_t : t \geq 0\}$, this gives a family of probability measures $\{\mathbb{P}_x : x \in E\}$ defined on $(D([0,\infty), E), \mathscr{F}^o)$ such that

- $\mathbb{P}[X_0 = x] = 1$ for all $x \in E$ (normality);

- for every $A \in \mathscr{F}^o$, the map $x \to \mathbb{P}_x[A]$ is measurable;

- for all $x \in E$, $f \in C_0(E)$,

$$\mathbb{E}_x[f(X_{t+s}) \mid \mathscr{F}_s^o] = (P_t f)(X_s), \qquad \mathbb{P}_x\text{-a.s.,}$$

   where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

In chapter 1 of [1] we can find a proof that a normal Markov process associated to a Feller semigroup always exists and it is unique.

For a probability measure μ in $(E, \mathscr{E})$, denote by $\mathbb{P}_\mu$ the measure on $(D([0,\infty), E), \mathscr{F}^o)$ given by $\int \mathbb{P}_x \mu(dx)$. Expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$. Assume that a probability measure π on E is stationary for the semigroup:

$$\langle P_t f \rangle_\pi = \langle f \rangle_\pi \qquad \text{for all } f \in C_0(E),$$

where $\langle \cdot \rangle$ stands for the expectation with respect to π.

Denote by $(D([0,\infty), E), \mathscr{F}, \mathbb{P}_\pi, \{\mathscr{F}_t : t \geq 0\})$ the usual augmentation of the filtered space $(D([0,\infty), E), \mathscr{F}^o, \mathbb{P}_\pi, \{\mathscr{F}_t^o : t \geq 0\})$ which satisfies the usual conditions. We know (see Theorem 8.11 and Proposition 8.12 of [1]) that the right continuous Feller process $\{X_t : t \geq 0\}$ is strong Markov with respect to the augmented filtration.

Let $L^2(\pi)$ be the Hilbert space of π-square integrable functions and denote by $\langle \cdot, \cdot \rangle_\pi$ its scalar product, with associated norm $\| \cdot \|_\pi$. Denote by $L^p(\pi)$, with $p \geq 1$, the space of measurable functions $f : E \to \mathbb{R}$ such that $\langle \mid f \mid^p \rangle_\pi < \infty$. The semigroup $\{P_t : t \geq 0\}$ extends to a semigroup of positive contractions on any $L^p(\pi), p \geq 1$. We assume that this extension is strongly continuous for any $p \in [1, +\infty)$ and that the measure π is ergodic: any $f \in L^1(\pi)$ such that $P_t f = f$ for all $t \geq 0$ is constant π-almost everywhere.

We introduce now some basic definitions from [15].

Definition 1.2. A linear operator L on $C_0(E)$, with domain $\mathscr{D}(L)$, is said to be a Markov pregenerator if it satisfies the following properties:

1. $\mathbf{1} \in (D)(L)$ and $L\mathbf{1} = 0$,

2. $\mathscr{D}(L)$ is dense in $C_0(E)$,

3. if $f \in \mathscr{D}(L), \lambda \geq 0$ and $f - \lambda L f = g$, then

$$\min_{x \in E} f(x) \geq \min_{x \in E} g(x).$$

A linear operator L on $C_0(E)$ is said to be closed if its graph is a closed subset of $C_0(E)$. Let $\mathscr{R}(\Omega)$ denote the range of a linear operator Ω.

Definition 1.3. A Markov generator is a Markov pregenerator L, which satisfies

$$\mathscr{R}(I - \lambda L) = C_0(E),$$

for all sufficiently small positive λ.

Definition 1.4. Suppose L is a Markov generator on $C_0(E)$. A linear subspace $\mathscr{C}$ of $\mathscr{D}(L)$ is said to be a core for L if L is the closure of its restriction to $\mathscr{C}$. Of course, L is uniquely determined by its value on a core.

In our case, let L be the generator of the semigroup $\{P_t : t \geq 0\}$ in $L^2(\pi)$ with $\mathscr{D}(L)$ denoting its domain. Let $\mathscr{C} \subset \mathscr{D}(L)$ be a core for the operator L, denote by L^* the adjoint of L in $L^2(\pi)$ and assume that $\mathscr{C} \subset \mathscr{D}(L^*)$. Since π is stationary, L^* is itself the generator of a Markov process. On $\mathscr{C}$ we can define $S = (1/2)(L + L^*)$ and $A = (1/2)(L - L^*)$, respectively, the symmetric and antisymmetric parts of the generator and we suppose that S is essentially self-adjoint.

We will also need the following property. If $\mathscr{C}$ is a core for an operator G, for any function f in the domain of G, there exists a sequence $\{f_k : k \geq 1\}$ in $\mathscr{C}$ such that f_k, Gf_k converge to f, Gf, respectively.

Denoting by ω a trajectory of $D(\mathbb{R}_+, E)$, let $\{\theta_t : t \geq 0\}$ be the semigroup of shift operators $\theta_t : D(\mathbb{R}_+, E) \to D(\mathbb{R}_+, E)$, defined by

$$(\theta_t \omega)(s) = \omega(t + s).$$

Since π is stationary ergodic measure, $\mathbb{P}_\pi$ is invariant and ergodic under the flow of transformations $\{\theta_t : t \geq 0\}$. This property will play a fundamental role in the following.

We state that the space of cylinder functions $\mathscr{C}$ is a common core for the operators L and its adjoint L^*. Consider, now, the seminorm $\| \cdot \|_1$ defined on $\mathscr{C}$ by

$$\| f \|_1^2 = \left\langle f, (-L)f \right\rangle_\pi.$$

Denote by $\mathscr{G}_1$ the normed space $(\mathscr{C} |_{\sim_1}, \| \cdot \|_1)$, where $\sim_1$ is the equivalence relation in $\mathscr{C}$ defined by

$$f \sim_1 g \quad \text{if} \quad \| f - g \|_1 = 0.$$

We can called $\mathscr{H}_1$ the completion of $\mathscr{G}_1$ with respect to the norm $\| \cdot \|_1$ and, since this norm satisfies the parallelogram identity, $\mathscr{H}_1$ is a Hilbert space, where the inner product is given by polarization

$$\langle f, g \rangle_1 = 1/4 \left\{ \| f + g \|_1^2 - \| f - g \|_1^2 \right\}.$$

and here only the symmetric part of the generator, $S = 1/2(L + L^*)$, plays a role, indeed

$$\| f \|_1^2 = \langle f, (-L)f \rangle_\pi = \langle f, (-S)f \rangle_\pi.$$

So, it is possible to check that, for any $f, g \in \mathscr{C}$,

$$\langle f, g \rangle_1 = \langle f, (-S)g \rangle_\pi. \tag{1.1}$$

In general, neither $\mathscr{H}_1$ nor $L^2(\pi)$ are subspaces of each other, but when L is bounded, we have $L^2(\pi) \subset \mathscr{H}_1$.

We can note that, by definition, $\mathscr{H}_1$ consists of sequences $\{f_n : n \geq 1\}$ of functions in $\mathscr{C}$ which are Cauchy in $\mathscr{H}_1$. If such a sequence converges to some function f in $L^2(\pi)$, we will identify the sequence $\{f_n : n \geq 1\}$ with f and we will say that f belongs to $\mathscr{H}_1$.

The domain $\mathscr{D}(S)$ is contained in $\mathscr{H}_1$. Indeed, fixing a function $f \in \mathscr{D}(S)$, since $\mathscr{C}$ is a core for $\mathscr{D}(S)$, there exists a sequence $\{f_n : n \geq 1\}$ of functions in $\mathscr{C}$, such that f_n, Sf_n converge in $L^2(\pi)$, as $n \uparrow \infty$, to f, Sf, respectively. By (1.1), it follows that $\{f_n : n \geq 1\}$ is a Cauchy sequence in $\mathscr{H}_1$. Thus, we have $\mathscr{D}(S) \subset \mathscr{H}_1$. The same argument apply to $\mathscr{D}(L), \mathscr{D}(L^*) \subset \mathscr{H}_1$.

Together to the Hilbert space $\mathscr{H}_1$, one can consider its dual space $\mathscr{H}_{-1}$ defined as follows. For $f \in L^2(\pi)$ define the norm by the variational formula

$$\| f \|_{-1}^2 = \sup_{g \in \mathscr{C}} \left\{ 2\langle f, g \rangle_\pi - \| g \|_1^2 \right\} \tag{1.2}$$

and denote with $\mathscr{G}_{-1}^*$ the subspace of $L^2(\pi)$ of all functions that have the above norm $\| \cdot \|_{-1}$ finite. Once again we introduce the normed space $\mathscr{G}_{-1} = (\mathscr{G}_{-1}^* \mid_{\sim_{-1}}, \| \cdot \|_{-1})$, where the equivalence relation $\sim_{-1}$ has the same meaning as before:

$$f \sim_{-1} g \quad \text{if} \quad \| f - g \|_{-1} = 0.$$

The completion of this space with respect to the norm $\| \cdot \|_{-1}$ is the Hilbert space $\mathscr{H}_{-1}$ with inner product defined through polarization.

It will be useful, for the next chapters, to consider some properties concerning the spaces $\mathscr{H}_1, \mathscr{H}_{-1}$.

Claim A (S can be extended as a bounded operator from $\mathscr{H}_1$ to $\mathscr{H}_{-1}$) For any $f \in \mathscr{C}$, Sf belongs to $\mathscr{H}_{-1}$. Indeed, for any $g \in \mathscr{C}$, since $-S$ is a non-negative operator, by Schwarz inequality,

$$\langle Sf, g \rangle_\pi^2 \leq \langle (-S)f, f \rangle_\pi \langle (-S)g, g \rangle_\pi = \| f \|_1^2 \| g \|_1^2.$$

In particular, by (1.2), we get $\| Sf \|_{-1} \leq \| f \|_1$.

Claim B (Extension of the scalar product $\langle \cdot, \cdot \rangle_\pi$ to $\mathscr{H}_1 \times \mathscr{H}_{-1}$) From the variational formula (1.2), it is easy to check that for every function $f \in \mathscr{C}$ and every function $g \in L^2(\pi) \cap \mathscr{H}_{-1}$

$$| \langle f, g \rangle_\pi | \leq \| f \|_1 \| g \|_{-1} \tag{1.3}$$

Indeed, fixing such functions f, g and fixing $a \in \mathbb{R}$, such that $af \in \mathscr{C}$, by the formula (1.2) we have

$$2a\langle f, g \rangle_\pi \leq a^2 \| f \|_1^2 + \| g \|_{-1}^2.$$

Dividing by a and minimizing over a we obtain (1.3). Now, fixing $g \in \mathscr{H}_{-1}$ and $f \in \mathscr{H}_1$, consider a sequence $\{g_n : n \geq 1\}$ in $L^2(\pi)$ and one $\{f_n : n \geq 1\}$ in $\mathscr{C}$, converging to $g \in \mathscr{H}_{-1}$ and to $f \in \mathscr{H}_1$, respectively. So, we can extend the inner product $\langle \cdot, \cdot \rangle_\pi$ to $\mathscr{H}_1 \times \mathscr{H}_{-1}$, setting

$$\langle f, g \rangle_\pi := \lim_{n \to \infty} \langle f_n, g_n \rangle_\pi.$$

And in the view of (1.3), this definition does not depend on the chosen sequence. Moreover, Schwarz inequality holds in this more general context, where $f \in \mathscr{H}_1$ and $g \in \mathscr{H}_{-1}$:

$$| \langle f, g \rangle_\pi | = \lim_{n \to \infty} | \langle f_n, g_n \rangle_\pi | \leq \lim_{n \to \infty} \| f_n \|_1 \| g_n \|_{-1} = \| f \|_1 \| g \|_{-1} . \tag{1.4}$$

The extension of the scalar product permits to generalize the relation (1.1) and for f, g in $\mathscr{H}_1$, we get

$$\langle f, (-S)g \rangle_\pi = \langle f, g \rangle_1. \tag{1.5}$$

Since Sg belongs to $\mathscr{H}_{-1}$ by Claim A, the left-hand side is meant to be the extended scalar product to $\mathscr{H}_1 \times \mathscr{H}_{-1}$, while the right-hand side is the usual one in $\mathscr{H}_1$.

Claim C (Sufficient conditions to belonging to $\mathscr{H}_{-1}$) Lastly, the saw variational formula permits to prove that a function f in $L^2(\pi)$ belongs to $\mathscr{H}_{-1}$ if and only if, for every $g \in \mathscr{C}$ there exists a finite constant C such that

$$\langle f, g \rangle_\pi \leq C \| g \|_1 \tag{1.6}$$

and in this case

$$\| f \|_{-1} \leq C.$$

Claim D (Sf belongs to $\mathscr{H}_{-1}$) The domains $\mathscr{D}(L)$ and $\mathscr{D}(S)$ are contained in $\mathscr{H}_1$. For any $f \in \mathscr{D}(L)$ and $g \in \mathscr{D}(S)$,

$$\| f \|_1^2 = \langle f, (-L)f \rangle_\pi, \quad \| g \|_1^2 = \langle g, (-S)g \rangle_\pi, \quad \langle f, g \rangle_1 = \langle f, (-S)g \rangle_\pi.$$

Indeed, consider sequences $\{f_n : n \geq 1\}$, $\{g_n : n \geq 1\}$ in $\mathscr{C}$ such that f_n, g_n, Lf_n, Sg_n converge in $L^2(\pi)$ to f, g, Lg, Sg, respectively. Since, by convention, f represents in $\mathscr{H}_1$ the Cauchy sequence $\{f_n : n \geq 1\}$, by (1.1),

$$\| f \|_1^2 = \lim_{n \to \infty} \| f_n \|_1^2 = \lim_{n \to \infty} \langle f_n, (-S)f_n \rangle_\pi = \lim_{n \to \infty} \langle f_n, (-L)f_n \rangle_\pi = \langle f, (-L)f \rangle_\pi.$$

The third identity follows from the fact that f_n belongs to $\mathscr{C}$ which is a core for both S and L. By similar reasons,

$$\| g \|_1^2 = \langle g, (-S)g \rangle_\pi, \qquad \langle f, g \rangle_1 = \langle f, (-S)g \rangle_\pi.$$

In conclusion, considering also Claim A, we derive that Sf belongs to $\mathscr{H}_{-1}$, for any $f \in \mathscr{D}(L)$, and $\| Sf \|_{-1} \leq \| f \|_1$.

Claim E   ($\mathscr{H}_{-1}$ is the closure of $\{Sf : f \in \mathscr{H}_{-1}\}$) We have just proved that Sf belongs to $\mathscr{H}_{-1}$, since the scalar product there is defined through polarization, we have:

$$\langle g, (-S)f \rangle_{-1} = (1/4)\{\parallel g - Sf \parallel_{-1}^2 - \parallel g + Sf \parallel_{-1}^2\}.$$

By (1.2),

$$\parallel g - Sf \parallel_{-1}^2 = \sup_{h \in \mathscr{C}} \{2\langle g - Sf.h \rangle_\pi - \parallel h \parallel_1^2\}.$$

Since f and h are in $\mathscr{C}$, $\langle (-S)f, h \rangle_\pi = \langle f, h \rangle_1$, thus

$$2\langle (-S)f, h \rangle_\pi - \parallel h \parallel_1^2 = \parallel f \parallel_1^2 - \parallel h - f \parallel_1^2$$

and so

$$\parallel g - Sf \parallel_{-1}^2 = \parallel f \parallel_1^2 + \sup_{h \in \mathscr{C}} \{2\langle g, h \rangle_\pi - \parallel h - f \parallel_1^2\}.$$

As f belongs to $\mathscr{C}$, replacing h by $h' = f - h$, one obtains that the variational term is equal to $2\langle g, f \rangle_\pi + \parallel g \parallel_{-1}^2$, and moreover, replacing f by $-f$, one gets

$$\parallel g + Sf \parallel_{-1}^2 = \parallel f \parallel_1^2 - 2\langle g, f \rangle_\pi + \parallel g \parallel_{-1}^2,$$

from which we derive:

$$\langle g, (-S)f \rangle_{-1} = \langle g, f \rangle_\pi.$$

From this last equality, it follows that $\mathscr{H}_{-1} \cap L^2(\pi)$ is contained in the $\mathscr{H}_{-1}$-closure of $\{Sf : f \in \mathscr{C}\}$. Indeed, fixing $g \in \mathscr{H}_{-1} \cap L^2(\pi)$ and assuming that $\langle g, Sf \rangle_{-1} = 0$ for all $f \in \mathscr{C}$, we get $\langle g, f \rangle_\pi = 0$. This implies that $g = 0$ in $L^2(\pi)$, because the core $\mathscr{C}$ is dense in $L^2(\pi)$, and, by (1.2), $g = 0$ in $\mathscr{H}_{-1}$. Our Claim follows from this observation, since $\mathscr{H}_{-1}$ is the $\mathscr{H}_{-1}$-closure of $\mathscr{H}_{-1} \cap L^2(\pi)$.

1.2   The Simple Exclusion Process

The simple exclusion process is an interacting particle system, which consists of continuous-time random walks on the lattice $\mathbb{Z}^d$, with particles distributed on it, such that, at every instant, each site is occupied by at most one particle. So the dynamic of this process is characterized by a hard-core interaction between particles. In order to describe it, we introduce the probability measure $p(\cdot)$ on $\mathbb{Z}^d$. Then, a particle in the site $x \in \mathbb{Z}^d$ jumps to the site y with the translation-invariant transition probability $p(x, y) = p(y - x)$ and if a particle tries to jump on a site already occupied, the jump is supressed, in order to respect the so-called exclusion rule. Moreover, the particles are allowed to jump only to adjacent sites, in other words, we have the following assumption: $p(x, y) = 0$ if $|x - y| \neq 1$.

Formally, we will consider the space $\mathbb{X} = \{0, 1\}^{\mathbb{Z}^d}$, called the configuration space, endowed with the product topology, such that it becomes a metrizable, compact space. Denoting each configuration by η, we will put $\eta(x) = 0$ if the site $x \in \mathbb{Z}^d$ is free, otherwise $\eta(x) = 1$. We can also interpret η as the subset of occupied sites of $\mathbb{Z}^d$.

The existence of the simple exclusion process was proved by Liggett [14], under the assumption $\sup_y \sum_x p(x,y) < \infty$, which is always satisfied for translation-invariant $p(x,y)$. Let $\mathcal{C}(\mathbb{X})$ be the collection of continuous functions on $\mathbb{X}$, with the norm $\| f \| = \sup_{\eta \in \mathbb{X}} | f(\eta) |$ and let $\mathscr{C}$ be its subset of cylinder functions on $\mathbb{X}$. Define the configuration $\sigma^{x,y}\eta$ obtained from η interchanging the occupation variables $\eta(x), \eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases} \tag{1.7}$$

Now we can consider the operator L defined on $\mathscr{C}$ as follows

$$(Lf)(\eta) = \sum_{x,z \in \mathbb{Z}^d} \eta(x)[1 - \eta(x+z)]p(z)[f(\sigma^{x,x+z}\eta) - f(\eta)]. \tag{1.8}$$

In [15], it is shown that this is a Markov pre-generator and its closure, still denoted by L, is a Markov generator and $\mathscr{C}$ is a core for L. Moreover, through the Hille-Yosida theorem ([15, Thm I.2.9]) we associate to the generator L the strictly Markovian Feller semigroup $\{S(t) : t \geq 0\}$ on $C(\mathbb{X})$.

We define, as in the previous section, the smallest σ-algebra $\mathscr{F}^o$ on the space $D([0,\infty), \mathbb{X})$ of r.c.l.l. trajectories $\eta : [0,\infty) \to \mathbb{X}$ and the canonical projections $\{\Pi_t : t \geq 0\}$, given by $\Pi_t(\eta) = \eta_t$.

Let $\{\mathbb{P}_\eta : \eta \in \mathbb{X}\}$ be the normal Markov process associated to the semigroup $\{S(t) : t \geq 0\}$, then it is a family of probabilities measures on $(D([0,\infty), \mathbb{X}), \mathscr{F}^o)$ characterized by the same properties shown before:

- $\mathbb{P}_\eta[\eta_0 = \eta] = 1$ for all $\eta \in \mathbb{X}$;

- for every $A \in \mathscr{F}^o$, the map $\eta \to \mathbb{P}_\eta[A]$ is measurable;

- for all $\eta \in \mathbb{X}$, $f \in C(\mathbb{X})$, $s, t \geq 0$,

$$\mathbb{E}_\eta[f(\eta_{t+s}) \mid \mathscr{F}^o_s] = (S(t)f)(\eta_s), \qquad \mathbb{P}_\eta\text{-a.s.,}$$

.

Now we present the main assumptions that the probability measure $p(\cdot)$ has to satisfy, in order to state all the results in the following chapters.

- p is irreducible: for any pair $x, y \in \mathbb{Z}^d$, there exists $M \geq 1$ and a sequence $x = x_0, \cdots, x_M = y$ such that, from definition of $p(\cdot)$, for $0 \leq i \leq M - 1$,

$$p(x_i, x_{i+1}) + p(x_{i+1}, x_i) := p(x_{i+1} - x_i) + p(x_i - x_{i+1}) > 0.$$

So that the set $\{x : p(x) + p(-x) > 0\}$ generates $\mathbb{Z}^d$.

- p has finite range: there exists $R \in \mathbb{N}$ such that $p(z) = 0$ for all z such that $| z | \geq R$.

- p does not charge the origin: $p(0) = 0$.

The conservation of the total number of particles has as a consequence the existence of a one-parameter family of invariant measures, given through the Bernoulli product measure v_α of parameter $0 \leq \alpha \leq 1$. This means that under v_α the variables $\{\eta(x), x \in \mathbb{Z}^d\}$ are independent with marginals

$$v_\alpha(\eta(x) = 1) = \alpha = 1 - v_\alpha(\eta(x) = 0).$$

Proposition 1.5. *The Bernoulli measures $\{v_\alpha, 0 \leq \alpha \leq 1\}$ are invariant for simple exclusion processes.*

Proof. By simple change of variables, for any cylinder functions f, g and any $\{x, y\}$,

$$\int f(\sigma^{x,y}\eta)g(\eta)\eta(x)[1 - \eta(y)]v_\alpha(d\eta) = \int f(\eta)g(\sigma^{x,y}\eta)\eta(y)[1 - \eta(x)]v_\alpha(d\eta) \qquad (1.9)$$

With this identity, the fact that $1 = \sum_{z \in \mathbb{Z}^d} p(z) = \sum_{z \in \mathbb{Z}^d} p(-z)$ and a change in the order of summation, we get that, for all cylinder functions f,

$$\int Lf dv_\alpha = 0.$$

By Proposition 2.13 in [15], we conclude. $\square$

For $0 \leq \alpha \leq 1$ we have the filtered space $(D([0, \infty), \mathbb{X}), \mathscr{F}^o, \mathbb{P}_{v_\alpha}, \{\mathscr{F}_t^o : t \geq 0\})$, with the usual augmentation $(D([0, \infty), \mathbb{X}), \mathscr{F}, \mathbb{P}_{v_\alpha}, \{\mathscr{F}_t : t \geq 0\})$, $\mathbb{P}_{v_\alpha}$ is the measure on $(D([0, \infty), \mathbb{X}), \mathscr{F}^o)$ given by $\int \mathbb{P}_\eta v_\alpha(d\eta)$, expectation with respect to $\mathbb{P}_{v_\alpha}$ is denoted by $\mathbb{E}_{v_\alpha}$. By theorems of Blumenthal and Getoor [1], $\{\eta_t : t \geq 0\}$ is a strong Markov process with respect to this augmented filtration. By Schwarz inequality, for any cylinder function f, $[S(t)f(\eta)]^2 \leq S(t)f^2(\eta)$ and, since v_α is a stationary measure for any $\alpha \in [0, 1]$,

$$\int [S(t)f(\eta)]^2 v_\alpha(d\eta) \leq \int f(\eta)^2 v_\alpha(d\eta).$$

Therefore the semigroup $S(t)$ extends to a Markov semigroup on $L^2(v_\alpha)$ and approximating a function in $L^2(v_\alpha)$, one can show that the semigroup $\{S(t) : t \geq 0\}$ is strongly continuous in $L^2(v_\alpha)$. Its generator is the closure of L in $L^2(v_\alpha)$, still denoted as L, and the domain of L is denoted as $\mathscr{D}(L)$.

Denote by L_{p^*} the infinitesimal generator whose action on cylinder function is defined by (1.8) with the probability measure $p^*(x) = p(-x)$ in place of p. Denote by L^* the adjoint of L, we can state the following.

Lemma 1.6. *The adjoint operator L^* is a generator and $L^* = L_{p^*}$. In particular $\mathscr{C}$ is a common core for L and L^*.*

Proof. We first note that the space $\mathscr{U} = \{(f, L^*f) : f \in (D)(L^*)\}$ is the orthogonal of the space $\mathscr{B} = \{(Lg, -g) : g \in \mathscr{D}(L)\}$, so that $\mathscr{U} = \mathscr{B}^\perp$. A simple computation shows that $(f, L^*f), f \in \mathscr{D}(L^*)$, is orthogonal to $\mathscr{B}$. On the other hand, if a pair (f, h) is orthogonal to $\mathscr{B}$, $\langle f, Lg \rangle_{v_\alpha} = \langle h, g \rangle_{v_\alpha}$ for all $g \in \mathscr{D}(L)$. This identity proves asserts that f belongs to $\mathscr{D}(L^*)$ and that $h = L^*f$, which proves the orthogonality of the spaces $\mathscr{U}$ and $\mathscr{B}$.

It is now easy to deduce that L^* is closed. Consider a sequence $\{f_n : n \geq 1\}$ of functions in $\mathscr{D}(L^*)$, such that f_n, L^*f_n converge in $L^2(v_\alpha)$ to f, h, respectively. Therefore, (f_n, L^*f_n) belongs to $\mathscr{U} = \mathscr{B}^\perp$. Since $\mathscr{B}^\perp$ is closed, $(f, h) \in \mathscr{B}^\perp = \mathscr{U}$, so that $f \in \mathscr{D}(L^*)$ and $L^*f = h$.

By (1.9), for every cylindric function f, g,

$$\langle Lf, g \rangle_{v_\alpha} = \langle f, L_{p^*}g \rangle_{v_\alpha}. \tag{1.10}$$

Since $\mathscr{C}$ is a core for L, this identity can be extended to $f \in \mathscr{D}(L)$. Hence, $\mathscr{C} \subset \mathscr{D}(L^*)$ and $L^*g = L_{p^*}g$ for $g \in \mathscr{C}$. This proves that the domain $\mathscr{D}(L^*)$ is dense in $L^2(v_\alpha)$ and that $1 \in \mathscr{D}(L^*)$ and $L^*1 = L_{p*}1 = 0$. Denote by $\{G_\lambda : \lambda > 0\}$, the resolvent associated to the generator L:

$$G_\lambda = (\lambda - L)^{-1};$$

and keep in mind that G_λ is a bounded operator: $\parallel \lambda G_\lambda \parallel \leq 1$. Let G_λ^*, with $\lambda > 0$, be the adjoint of G_λ. It is easy to show that $\mathscr{D}(G_\lambda^*) = L^2(v_\alpha)$, $\parallel G_\lambda^* \parallel = \parallel G_\lambda \parallel$, $G_\lambda^*(L^2(v_\alpha)) \subset \mathscr{D}(L^*)$ and $G_\lambda^* = (\lambda - L^*)^{-1}$. In particular, the range of $\lambda - L^*$ is $L^2(v_\alpha)$ and L^* is dissipative, since for every $f \in \mathscr{D}(L^*)$,

$$\langle f, (-L^*)f \rangle_{v_\alpha} = \lim_{\lambda \to 0} \langle f, (\lambda - L^*)f \rangle_{v_\alpha} = \lim_{\lambda \to 0} \langle G_\lambda^*(\lambda - L^*)f, (\lambda - L^*)f \rangle_{v_\alpha}$$
$$= \lim_{\lambda \to 0} \langle (\lambda - L^*)f, G_\lambda(\lambda - L^*)f \rangle_{v_\alpha}$$
$$=_{\lambda \to 0} \langle (\lambda - L)G_\lambda(\lambda - L^*)f, G_\lambda(\lambda - L^*)f \rangle_{v_\alpha} \geq 0$$

because L is dissipative. This proves that L^* is a generator.

Now it remains to show that $L^* = L_{p^*}$. Denote by $\{G_{p^*, \lambda} : \lambda > 0\}$ the resolvent associated to the generator L_{p^*}. It is enough to show that $G_{p^*, \lambda} = G_\lambda^*$, since the latter operator is the resolvent of the generator L^*. We claim that for every $f \in \mathscr{D}(L_{p^*})$, $G_\lambda^*(\lambda - L_{p^*})f = f$. Indeed, assume first that $f \in \mathscr{C}$, in this case, by definition of G_λ^*

$$\langle G_\lambda^*(\lambda - L_{p^*})f, g \rangle_{v_\alpha} = \langle (\lambda - L_{p^*})f, G_\lambda g \rangle_{v_\alpha},$$

for every $g \in L^2(v_\alpha)$. Since $G_\lambda g \in \mathscr{D}(L)$ and since $\mathscr{C}$ is a core for L, approximating $G_\lambda g$ by a sequence $\{h_n : n \geq 1\}$ in $\mathscr{C}$ such that $h_n \to G_\lambda g$, $Lh_n \to LG_\lambda g$, by (1.10) we obtain that the previous expression is equal to

$$\langle f, (\lambda - L)G_\lambda g \rangle_{v_\alpha} = \langle f, g \rangle_{v_\alpha},$$

which proves the claim for functions f in the core $\mathscr{C}$. Since this is also a core for the generator L_{p^*}, and since G_λ^* is a bounded operator, an approximation argument permits

to extend the identity $G_\lambda^*(\lambda - L_{p^*})f = f$ to functions f in the domain $\mathscr{D}(L_{p^*})$. This proves the claim.

To conclude, fix a function $h \in L^2(v_\alpha)$. Let $h_\lambda \in \mathscr{D}(L_{p^*})$ be the solution of $(\lambda - L_{p^*})h_\lambda = h$, which exists because L_{p^*} is a generator. By the previous identity and by definition of $G_{p^*,\lambda}$,

$$G_\lambda^* h = G_\lambda^*(\lambda - L_{p^*})h_\lambda = h_\lambda = G_{p^*,\lambda}(\lambda - L_{p^*})h_\lambda = G_{p^*,\lambda}h,$$

which proves that $G_{p^*,\lambda} = G_\lambda^*$ and so that $L^* = L_{p^*}$. $\qquad\qquad\square$

The simple exclusion process is said symmetric if the probability measure p is symmetric, so if $p(z) = p(-z)$, with mean zero if p has zero average, $\sum_z zp(z) = 0$, and asymmetric otherwise. In the symmetric case we have that the generator L is a symmetric operator in $L^2(v_\alpha)$, or equivalently, that v_α is a reversible measure for the symmetric simple exclusion process. From theprevious result, it follows that in fact L is self adjoint.

Denote by $s(\cdot)$ and $a(\cdot)$, respectively, the symmetric and asymmetric part of the probability measure p:

$$s(x) = 1/2\{p(x) + p(-x)\} \qquad\qquad\qquad (1.11)$$
$$a(x) = 1/2\{p(x) - p(-x)\}. \qquad\qquad\qquad (1.12)$$

Decomposing the operator L into its symmetric and asymmetric part, $L = S + A$, where for any cylinder function f,

$$(Sf)(\eta) = \sum_{x,z\in\mathbb{Z}^d} \eta(x)(1 - \eta(x+z))s(z)[f(\sigma^{x,x+z}\eta) - f(\eta)],$$
$$(Af)(\eta) = \sum_{x,z\in\mathbb{Z}^d} \eta(x)(1 - \eta(x+z))a(z)[f(\sigma^{x,x+z}\eta) - f(\eta)].$$

By the (1.9), with an elementary computation, for every f, g cylindric functions

$$\langle Sf, g\rangle_{v_\alpha} = \langle f, Sg\rangle_{v_\alpha}, \qquad \langle Af, g\rangle_{v_\alpha} = -\langle f, Ag\rangle_{v_\alpha}.$$

Moreover, we obtain

$$S = 1/2(L + L^*), \qquad A = 1/2(L - L^*),$$

and with the change of variables $x' = x + z$, $z' = -z$, for the symmetry of $s(\cdot)$, we have

$$(Sf)(\eta) = \sum_{x,z\in\mathbb{Z}^d} \eta(x+z)(1 - \eta(x))s(z)[f(\sigma^{x,x+z}\eta) - f(\eta)].$$

Adding the two previous formulas for Sf and since $\sigma^{x,x+z}\eta = \eta$, unless $\eta(x)(1-\eta(x+z)) + \eta(x+z)(1 - \eta(x)) = 1$, we deduce the simpler form

$$(Sf)(\eta) = 1/2 \sum_{x,z\in\mathbb{Z}^d} s(z)[f(\sigma^{x,x+z}\eta) - f(\eta)]$$

for the operator S, which results to be the generator of the simple exclusion process with probability measure $s(\cdot)$. Since $s(\cdot)$ is symmetric, by the previous lemma, it follows that S is a self-adjoint operator on $L^2(v_\alpha)$.

Let $D(f)$ be the Dirichlet form of a cylinder function f:

$$
\begin{aligned}
D(f) &= \langle f, (-Lf) \rangle_{v_\alpha} = \langle f, (-Sf) \rangle_{v_\alpha} \\
&= 1/4 \sum_{x,z \in \mathbb{Z}^d} s(z) \int [f(\sigma^{x,x+z}\eta) - f(\eta)]^2 v_\alpha(d\eta).
\end{aligned}
\tag{1.13}
$$

This formula holds also for functions f in the domain $\mathscr{D}(L)$ of the generator, and the series defined on the right-hand side converge absolutely.

Theorem 1.7. *For any $\alpha \in [0,1]$, v_α is ergodic for L.*

Proof. Let $f \in L^2(v_\alpha)$, such that $s(t)f = f$ for any $t \geq 0$. Then $f \in \mathscr{D}(L)$ and $Lf = 0$. Multiplying this one by f and integrating, by (1.13) for functions in $\mathscr{L}$, we get

$$
\sum_{x,z} s(z) \int [f(\sigma^{x,x+z}\eta) - f(\eta)]^2 v_\alpha(d\eta) = 0.
$$

By assumption the support of $s(\cdot)$ generates $\mathbb{Z}^d$. We deduce, then, that for any $x, y \in \mathbb{Z}^d$

$$
f(\sigma^{x,y}\eta) = f(\eta) \qquad v_\alpha\text{-a.e.}
$$

By De Finetti's theorem we conclude that f is constant v_α-a.e. $\qquad\square$

1.3 The Tagged Particle Process

In the previous section we gave a description of the simple exclusion process, but from now on we are interested in studying the dynamics of a distinguished tagged particle in this process and so the exclusion process "as seen from a tagged particle". We consider the same setup as before: the probability measure p and the process $\{\eta_t : t \geq 0\}$ on $\mathbb{X} = \{0,1\}^{\mathbb{Z}^d}$, which satisfies the assumptions of the previous section, and we denote the position, at time $t \geq 0$, of the tagged particle by Z_t. We will consider a configuration η, with this particle set at the origin.

Without the presence of other particles, Z_t would evolve as a continuous-time random walk, for which the classical results, as the Law of Large Numbers, the Central Limit Theorem and the Invariance Principle, can be easily obtained. Now our aim is to study the asymptotic behaviour of Z_t, taking into account the eventual collisions with other particles, called the environment. Clearly, these interactions alter the behaviour of the tagged particle and make the study of its dynamic more complicate. Indeed $\{Z_t : t \geq 0\}$ by itself is not a Markov process. While the pair $\{(Z_t, \eta_t) : t \geq 0\}$ it is.

In this Markov process the origin is always occupied by the tagged particle and the evolution of the environment, consisting of the remaining particles, is observed from

there. In order to keep the tagged particle at the origin, everytime this one jumps to the site x, we will translate the entire configuration by $-x$.

Denoting the group of translations on $\mathbb{X}$ by $\{\tau_x : x \in \mathbb{Z}^d\}$, for all $x, y \in \mathbb{Z}^d$ and for $\eta \in \mathbb{X}$,

$$(\tau_x \eta)(y) = \eta(x + y)$$

and being $\{\xi_t : t \geq 0\}$ the state of the process as seen from the tagged particle, where $\xi_t = \tau_{Z_t} \eta_t$, we can note that the origin is always occupied:

$$\xi_t(0) = (\tau_{Z_t} \eta_t)(0) = \eta_t(Z_t + 0) = \eta_t(Z_t) = 1.$$

Due to this fact, we can consider the set $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$ and consequently ξ_t is a configuration of $\mathbb{X}^* = \{0, 1\}^{\mathbb{Z}^d_*}$. Moreover, as already stated, to keep the tagged particle at the origine, define on $\mathbb{X}^*$ the shift operators $\{\theta_x : x \in \mathbb{Z}^d_*\}$, such that

$$(\theta_x \xi)(y) = \begin{cases} \xi(x) & \text{if } y = -x \\ \xi(x + y) & \text{otherwise.} \end{cases}$$

This means that $\theta_x(\xi)$ is the configuration where the tagged particle jumps to site x and then all the configuration is translated by $-x$.

As in the previous section, we assume $\mathbb{X}^*$ endowed with the product topology, which tunrs it into a metrizable, compact space and we will work on $C(\mathbb{X}^*)$, the Banach space of continuous functions on $\mathbb{X}^*$, with the norm $\| f \| = \sup_{\xi \in \mathbb{X}^*} | f(\xi) |$. Denoting $\mathscr{C} \subset C(\mathbb{X}^*)$ the space of cylinder functions, which depend on configuration η only through a finite number of coordinates, we define on it the sum operator $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_\theta$, with $\mathscr{L}_0$ and $\mathscr{L}_\theta$ defined as follows:

$$
\begin{aligned}
(\mathscr{L}_0 f)(\xi) &= \sum_{x, y \in \mathbb{Z}^d_*} p(y - x) \xi(x)[1 - \xi(y)][f(\sigma^{x,y}\xi) - f(\xi)], \\
(\mathscr{L}_\theta f)(\xi) &= \sum_{z \in \mathbb{Z}^d_*} p(z)[1 - \xi(z)][f(\theta_z \xi) - f(\xi)],
\end{aligned}
\tag{1.14}
$$

such that the first one describes the jumps of the environment, while the second one the jumps of the tagged particle. Here, again, we have that the operator $\mathscr{L}$ is a Markov pregenerator and its closure, still denoted by $\mathscr{L}$, is a Markov generator, with $\mathscr{C}$ a core for $\mathscr{L}$. Once again, through the Hille-Yosida theorem, we associate to the generator $\mathscr{L}$ the strictly Markovian Feller semigroup $\{S(t) : t \geq 0\}$ on $C(\mathbb{X}^*)$. Let $D([0, \infty), \mathbb{X}^*)$ be the space of right continuous left limited trajectories $\xi : [0, \infty) \to \mathbb{X}^*$ and let $\{\Pi_t : t \geq 0\}$ be the canonical projections $\Pi_t(\xi) = \xi_t$. We represent $\mathscr{F}^o_t$ the smallest σ-algebra relative to which all the projections Π_s, $0 \leq s \leq t$, are measurable.

Here, we can denote the family of stationary states, the Bernoulli product measure on $\mathbb{X}^*$, by v^*_α, with density $0 \leq \alpha \leq 1$ and with marginals given by

$$v^*_\alpha(\xi : \xi(x) = 1) = \alpha = 1 - v^*_\alpha(\xi : \xi(x) = 0) \qquad \text{for } x \in \mathbb{Z}^d_*.$$

The notation $\langle \cdot, \cdot \rangle_{v^*_\alpha}$ stands for the scalar product in $L^2(v^*_\alpha)$. We have the analogous of proposition 1.5 for the simple exclusion process, that states the first important assumption to develop the next results.

Proposition 1.8. *The Bernoulli measures $\{v_\alpha^* : 0 \leq \alpha \leq 1\}$ are invariant for the Markov process $\{\xi_t : t \geq 0\}$.*

Proof. For any cylinder functions f, g and any $z \in \mathbb{Z}_*^d$,

$$\int f(\theta_z \xi) g(\xi)[1 - \xi(z)] v_\alpha^*(d\xi) = \int f(\xi) g(\theta_{-z} \xi)[1 - \xi(-z)] v_\alpha^*(d\xi),$$

thanks to a simple change of variables. Moreover from this and from (1.9), we have for all cylinder functions f, g,

$$< g, \mathscr{L} f >_{v_\alpha^*} = < \mathscr{L}_{p^*} g, f >_{v_\alpha^*} \tag{1.15}$$

and $p^*(z) = p(-z)$. In particular, $\int \mathscr{L} f dv_\alpha^* = 0$ for any cylinder function f, and this proves the result, using Proposition 2.13 in [15]. □

Observation. The probability measures v_α^* are invariant for the operators $\mathscr{L}_0$ and $\mathscr{L}_\theta$ taken individually if and only if the probability measure p is symmetric.

For $0 \leq \alpha \leq 1$ we have the filtered space $(D([0, \infty), \mathbb{X}^*), \mathscr{F}^o, \mathbb{P}_{v_\alpha}, \{\mathscr{F}_t^o : t \geq 0\})$, with the usual augmentation $(D([0, \infty), \mathbb{X}^*), \mathscr{F}, \mathbb{P}_{v_\alpha}, \{\mathscr{F}_t : t \geq 0\})$, with respect to the which $\{\xi_t : t \geq 0\}$ is a strong Markov process.

The semigroup $\{S_t : t \geq 0\}$ extends to a Markov semigroup on $L^2(v_\alpha^*)$ whose generator $\mathscr{L}_{v_\alpha^*}$ is the closure of $\mathscr{L}$ in $L^2(v_\alpha^*)$, where the density α remains fixed. We will denote by $\mathscr{L}^*$ the adjoint of $\mathscr{L}$ in $L^2(v_\alpha^*)$ and by $\mathscr{L}_{p^*}$ the generator acting on $C(\mathbb{X}^*)$ and on $L^2(v_\alpha^*)$. Using the simple notation $\mathscr{L}$ to denote $\mathscr{L}_{v_\alpha^*}$ and being $\mathscr{D}(\mathscr{L})$ the domian of $\mathscr{L}$ in $L^2(v_\alpha^*)$, we have

Lemma 1.9. *The adjoint operator $\mathscr{L}^*$ is a generator and $\mathscr{L}^* = \mathscr{L}_{p^*}$. In particular $\mathscr{C}$ is a common core for $\mathscr{L}$ and $\mathscr{L}^*$ and $\mathscr{L}$ is self-adjoint with respect to each v_α^*, when p is symmetric.*

Proof. This result follows from Lemma 1.6 and from equation (1.15) in the previous proof. □

Moreover, for any cylinder function f,

$$\langle f, (-\mathscr{L}) f \rangle_{v_\alpha^*} =: \mathscr{D}(f) = \mathscr{D}_0(f) + \mathscr{D}_\theta(f) \tag{1.16}$$

with

$$\mathscr{D}_0(f) = (1/2) \sum_{x,y \in \mathbb{Z}_d^*} s(y - x) \int \xi(x)[1 - \xi(y)](T^{x,y} f)(\xi)^2 v_\alpha^*(d\xi)$$

$$\mathscr{D}_\theta(f) = (1/2) \sum_{z \in \mathbb{Z}_d^*} s(z) \int [1 - \xi(z)](T^z f)(\xi)^2 v_\alpha^*(d\xi),$$

where $s(\cdot)$ stands for the symmetric part of the probability $p(\cdot)$ and where, for a function $f \in L^2(v_\alpha^*)$,

$$(T^{x,y}f)(\xi) = f(\sigma^{x,y}\xi) - f(\xi)$$
$$(T^z f)(\xi) = f(\theta_z \xi) - f(\xi).$$

With this definition $(T^{x,y}f)(\xi)$ vanishes if $\xi(x) = \xi(y)$ and, since $T^{x,y}f = T^{y,x}f$, it is possible to rewrite

$$\mathscr{D}_0(f) = (1/4) \sum_{x,y \in \mathbb{Z}_d^*} s(y-x) \int (T^{x,y}f)(\xi)^2 v_\alpha^*(d\xi),$$

obtaining the Dirichlet form for cylinder functions f, in the domain $\mathscr{D}(\mathscr{L})$.

Unlike the previous case, the probability measures $\{v_\alpha^* : 0 \leq \alpha \leq 1\}$ are ergodic in all but one degenerate case.

Theorem 1.10. *Assume $d = 1$ and $p(\cdot)$ such that $\sum_{x \neq \pm 1} p(x) > 0$ or assume $d \geq 2$. Then, for any $0 \leq \alpha \leq 1$, v_α^* is ergodic for $\mathscr{L}$.*

Proof. Fix a function $f \in L^2(v_\alpha^*)$ invariant for the semigroup generated by $\mathscr{L}$, so such that $S(t)f = f$ for any $t \geq 0$. Then f is in the domain of $\mathscr{L}$ and $\mathscr{L}f = 0$. By multiplying by f both sides of this equation andd integrting, we obtain

$$1/4 \sum_{x,y \in \mathbb{Z}_d^*} s(y-x) \int [f(\sigma^{x,y}\xi) - f(\xi)]^2 v_\alpha^*(d\xi) + 1/2 \sum_{x \in \mathbb{Z}_d^*} s(x) \int [1-\xi(x)][f(\theta_x\xi) - f(\xi)]^2 v_\alpha^*(d\xi) = 0.$$

Under the made assumptions, the support of $s(\cdot)$ generates $\mathbb{Z}_d^*$. Hence, for any $x, y \in \mathbb{Z}_d^*$,

$$f(\sigma^{x,y}\xi) = f(\xi) \qquad v_\alpha^*\text{-a.e..}$$

By De Finetti's theorem we conclude that f is constant v_α^*-a.e.. $\qquad \square$

So, from now on, we exclude the degenerate case with $d = 1$ and $\sum_{x \neq \pm 1} p(x) = 0$, which means that for all $x \neq \pm 1$, $p(x) = 0$, this is the case of the simple transition process where only nearest neighbor jumps can occur. These assumptions will be fundamental to state and prove the results in the following.

Chapter 2

Law of Large Numbers

Now we want to show the first important result for the exclusion process described in the previous chapter: the Law of Large Numbers for the position of the tagged particle Z_t.

Theorem 2.1. *Let $\alpha \in [0,1]$ and set $m = \sum_{x \in \mathbb{Z}_*^d} x p(x)$. Then, $\mathbb{P}_{\nu_\alpha^*}$-almost surely*

$$\lim_{t \to \infty} \frac{Z_t}{t} = [1 - \alpha]m.$$

In order to derive the desired result, it will be useful to represent the position of the tagged particle in terms of elementary orthogonal martingales associated to the jumps of the process.

2.1 Representation of the Tagged Particle's Position

We introduce here the definition of the jump processes $\{N_t^z : t \geq 0\}$:

Definition 2.2. For $z \in \mathbb{Z}_*^d$ such that $p(z) > 0$ and for $0 \leq s < t$, let $N_{[s,t]}^z$ be the total number of jumps of the tagged particle from the origin to the site z, in the time interval $[s,t]$. Analogously, for $x, y \in \mathbb{Z}_*^d$ such that $p(y - x) > 0$, let $N_{[s,t]}^{x,y}$ be the total number of jumps of a particle from x to y in the interval $[s,t]$.

We will set $N_{[0,t]}^z = N_t^z$ and $N_{[0,t]}^{x,y} = N_t^{x,y}$. Through the jump processes, we define in the following lemma the elementary orthogonal martingales and we will compute their quadratic variation.

Lemma 2.3. *For $x, y, z \in \mathbb{Z}_*^d$ such that $p(z) > 0$, $p(y - x) > 0$, denote*

$$
\begin{aligned}
M_t^z &= N_t^z - \int_0^t p(z)[1 - \xi_s(z)]ds \\
M_t^{x,y} &= N_t^{x,y} - \int_0^t p(y-x)\xi_s(x)[1 - \xi_s(y)]ds.
\end{aligned}
$$

$$(2.1)$$

Then, $\{M_t^z : p(z) > 0\}$ *and* $\{M_t^{x,y} : p(y-x) > 0\}$ *are orthogonal martingales with quadratic variation* $\langle M^z \rangle_t$ *and* $\langle M^{x,y} \rangle_t$, *given by*

$$\langle M^z \rangle_t = \int_0^t p(z)[1 - \xi_s(z)]ds$$

$$\langle M^{x,y} \rangle_t = \int_0^t p(y-x)\xi_s(x)[1 - \xi_s(y)]ds. \tag{2.2}$$

Proof. Fix $x, y, z \in \mathbb{Z}_*^d$ such that $p(z) > 0$, $p(y-x) > 0$, $(\xi_t, N_t^z, N_t^{x,y})$ is a Markov process on $\mathbb{X}^* \times \mathbb{Z} \times \mathbb{Z}$ with generator $\mathscr{L}_{x,y,z}$ given by

$$
\begin{aligned}
(\mathscr{L}_{x,y,z}f)(\xi, k, j) &= p(y-x)\xi(x)[1 - \xi(y)]\{f(\sigma^{x,y}\xi, k, j+1) - f(\xi, k, j)\} \\
&\quad + p(z)[1 - \xi(z)]\{f(\theta_z\xi, k+1, j) - f(\xi, k, j)\} \\
&\quad + \sum_{x',y' \in \mathbb{Z}_*^d} p(y'-x')\xi(x')[1 - \xi(y')]\{f(\sigma^{x',y'}\xi, k, j) - f(\xi, k, j)\} \\
&\quad + \sum_{z' \neq z} p(z')[1 - \xi(z')]\{f(\theta_{z'}\xi, k, j) - f(\xi, k, j)\}
\end{aligned}
$$

In this formula the first sum is over all pairs $(x', y') \in \mathbb{Z}_*^d \times \mathbb{Z}_*^d$ different from (x, y). Applying Dynkin's formula to the functions $F_1(\xi, k, j) = k$, $F_2(\xi, k, j) = j$, we show that M_t^z, $M_t^{x,y}$ are martingales with quadratic variation as stated.

To show that these martingales are orthogonal is sufficient to prove that $M_t^z M_t^{x,y}$ is a martingale too. Rewriting the processes N_t^z, $N_t^{x,y}$ as $N_t^z = M_t^z + \int_0^t p(z)[1 - \xi_s(z)]ds$ and $N_t^{x,y} = M_t^{x,y} + \int_0^t p(y-x)\xi_s(x)[1 - \xi_s(y)]ds$, we can plug these in

$$N_t^z N_t^{x,y} - \int_0^t \{N_s^z p(y-x)\xi_s(x)[1 - \xi_s(y)] + N_s^{x,y}p(z)[1 - \xi_s(z)]\}ds,$$

which is a martingale by the Dynkin's formula applied to $F(\xi, k, j) = kj$, and integrating by parts we have that

$$
\begin{aligned}
M_t^z M_t^{x,y} &+ \int_0^t \{M_s^z p(y-x)\xi_s(x)[1 - \xi_s(y)] + M_s^{x,y}p(z)[1 - \xi_s(z)]\}d \\
&+ \int_0^t p(y-x)\xi_s(x)[1 - \xi_s(y)]ds \int_0^t p(z)[1 - \xi_s(z)]ds \\
&- \int_0^t \{N_s^z p(y-x)\xi_s(x)[1 - \xi_s(y)] + N_s^{x,y}p(z)[1 - \xi_s(z)]\}ds
\end{aligned}
$$

is a martingale. Expressing the martingales M^z, $M^{x,y}$ in terms of the jump processes N^z, $N^{x,y}$ we obatin that

$$
\begin{aligned}
M_t^z M_t^{x,y} &+ \int_0^t \int_0^s p(z)[1 - \xi_r(z)]dr p(y-x)\xi_s(x)[1 - \xi_s(y)]ds \\
&+ \int_0^t \int_0^s p(y-x)\xi_r(x)[1 - \xi_r(y)]dr p(z)[1 - \xi_s(z)]ds \\
&+ \int_0^t p(y-x)\xi_s(x)[1 - \xi_s(y)]ds \int_0^t p(z)\xi_s(z)ds
\end{aligned}
$$

is a martingale. With an integration by parts the integrals cancel, so that $M_t^z M_t^{x,y}$ is a martingale, as claimed.

We can use the same argument to prove that any pair of distinct martingales in the set $\{M_t^z : p(z) > 0\} \cup \{M_t^{x,y} : x, y \in \mathbb{Z}_*^d, p(y-x) > 0\}$ is a martingale and this concludes the proof. □

These martingales associated to the jumps of the Markov process are called elementary martingales because any martingale given by the Dynkin's formula can be rewritten in terms of them. Indeed, for a cylinder function $f : \mathbb{X}^* \to \mathbb{R}$, by the Dynkin's formula, we obtain the martingale

$$M_t^f = f(\xi_t) - f(\xi_0) - \int_0^t (\mathscr{L}f)(\xi_s)ds. \tag{2.3}$$

And rewriting $f(\xi_t) - f(\xi_0)$ as the sum of all differences arising from a jump to or from a site contained in the support of f in the interval $[0,t]$, we have

$$f(\xi_t) - f(\xi_0) = \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t (T^{x,y}f)(\xi_{s-})dN_s^{x,y} + \sum_{z\in\mathbb{Z}_*^d} \int_0^t (T^z f)(\xi_{s-})dN_s^z.$$

Using this difference, we obtain

$$M_t^f = \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t (T^{x,y}f)(\xi_{s-})dM_s^{x,y} + \sum_{z\in\mathbb{Z}_*^d} \int_0^t (T^z f)(\xi_{s-})dM_s^z, \tag{2.4}$$

where $p(z) = 0$ and $p(y-x) = 0$, respectively, imply $M_t^z = 0$ and $M_t^{x,y} = 0$. So we have derived the representation of M_t^f in terms of the elementary martingles and, since these ones are orthogonal, one can easily compute the quadratic variation of the martingale M^f:

$$\begin{aligned}
\langle M^f \rangle_t &= \left\langle \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t (T^{x,y}f)(\xi_{s-})dM_s^{x,y} + \sum_{z\in\mathbb{Z}_*^d} \int_0^t (T^z f)(\xi_{s-})dM_s^z \right\rangle_t \\
&= \left\langle \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t (T^{x,y}f)(\xi_{s-})dM_s^{x,y}, \sum_{x,y\in\mathbb{Z}_*^d} \int_0^t (T^{x,y}f)(\xi_{s-})dM_s^{x,y} \right\rangle_t \\
&\quad + \left\langle \sum_{z\in\mathbb{Z}_*^d} \int_0^t (T^z f)(\xi_{s-})dM_s^z, \sum_{z\in\mathbb{Z}_*^d} \int_0^t (T^z f)(\xi_{s-})dM_s^z \right\rangle_t \\
&= \sum_{x,y\in\mathbb{Z}_*^d} p(y-x) \int_0^t \xi_s(x)[1 - \xi_s(y)](T^{x,y}f)(\xi_s)^2 ds \\
&\quad + \sum_{z\in\mathbb{Z}_*^d} p(z) \int_0^t [1 - \xi_s(z)](T^z f)(\xi_s)^2 ds,
\end{aligned} \tag{2.5}$$

where the second equality follows from orthogonality and the third one from the quadratic variation of M_t^z and $M_t^{x,y}$, seen in Lemma 2.3. By the quadratic variation of M_t^f, we get

$$
\begin{aligned}
\mathbb{E}_{v_\alpha^*}[(M_t^f)^2] = t \sum_{x,y \in \mathbb{Z}_*^d} p(y-x) \int \xi(x)[1-\xi(y)](T^{x,y}f)(\xi)^2 v_\alpha^*(d\xi) \\
+ t \sum_{z \in \mathbb{Z}_*^d} p(z) \int [1-\xi(z)](T^z f)(\xi)^2 v_\alpha^*(d\xi).
\end{aligned}
\tag{2.6}
$$

Moreover, via the change of variables $\xi' = \theta_z \xi$ and $\xi'' = \sigma^{x,y}\xi$, we can use the following identities to replace p by s:

$$
\begin{aligned}
(T^{x,y}f)(\xi) &= (T^{y,x}f)(\xi), \\
(T^z f)(\theta_{-z}\xi) &= -(T^{-z}f)(\xi), \\
(T^{x,y}f)(\sigma^{x,y}\xi) &= -(T^{x,y}f)(\xi),
\end{aligned}
$$

and also

$$
(T^{x,y}f)(\xi)^2\{\xi(x)[1-\xi(y)]\} + \xi(y)[1-\xi(x)]\} = (T^{x,y}f)(\xi)^2,
$$

at the end, from (2.6) we have

$$
\begin{aligned}
\mathbb{E}_{v_\alpha^*}[(M_t^f)^2] &= t \sum_{x,y \in \mathbb{Z}_*^d} s(y-x) \int \xi(x)[1-\xi(y)](T^{x,y}f)(\xi)^2 v_\alpha^*(d\xi) \\
&+ t \sum_{z \in \mathbb{Z}_*^d} s(z) \int [1-\xi(z)](T^z f)(\xi)^2 v_\alpha^*(d\xi) \\
&= 2t\mathscr{D}(\mathscr{L}),
\end{aligned}
\tag{2.7}
$$

where the last step holds from the definition of $\mathscr{D}(\mathscr{L})$ in (1.16).

We also want to prove that the representation (2.4) is applicable also to functions in the domain $\mathscr{D}(\mathscr{L})$.

Lemma 2.4. *Let $u \in \mathscr{D}(\mathscr{L})$. Then the martingale*

$$
M_t^u = u(\xi_t) - u(\xi_0) - \int_0^t (\mathscr{L}f)(\xi_s)ds
$$

can be represented as in (2.4) with u in place of f:

$$
M_t^u = \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t (T^{x,y}u)(\xi_{s-})dM_s^{x,y} + \sum_{z \in \mathbb{Z}_*^d} \int_0^t (T^z u)(\xi_{s-})dM_s^z
\tag{2.8}
$$

Proof. Since u belongs to the domain of the generator and the space of cylinder functions forms a core for the generator in $L^2(v_\alpha^*)$, there exists a sequence of cylinder functions $\{f_n : n \geq 1\}$ such that f_n and $\mathscr{L}f_n$ converges in $L^2(v_\alpha^*)$ to u and $\mathscr{L}u$, respectively.

Hence, for every $t > 0$ the martingale $M_t^{f_n}$, defined by (2.3), with f_n in place of f, as $n \uparrow \infty$, converges in $L^2(v_\alpha^*)$ to the martingale M_t^u, defined again by (2.3) with u in place of f. This one gives a martingale in $L^2(v_\alpha^*)$ because the partial sums form a Cauchy sequence in $L^2(v_\alpha^*)$, let call it m_t^u.

So we need to show that $M_t^u = m_t^u$: it is enough to show that $M_t^{f_n}$ converges in $L^2(v_\alpha^*)$ to m_t^u. Since f_n is a cylinder function, the martingale $M_t^{f_n}$ can be represented through the elementary martingales M^z, $M^{x,y}$ by (2.4). Since these martingales are orthogonal, by the computation performed to obtain (2.7), we have

$$\frac{1}{t} \mathbb{E}_{v_\alpha^*}[(M_t^{f_n} - m_t^u)^2] = 2\mathscr{D}(f_n - u)$$

and this expression vanishes as $n \uparrow \infty$ by the chioce of the sequence $\{f_n : n \geq 1\}$. This concludes the proof. $\qquad \square$

2.2 Proof of the Law of Large Numbers

In order to prove Theorem 2.1, which states the $\mathbb{P}_{v_\alpha^*}$-convergence of the position of the tagged particle Z_t over the time t, we need the convergence of the elementary martingales, introduced in the previous section. So, we want to examine the $L^2(\mathbb{P}_{v_\alpha^*})$-limits of the martingales M_t^f, where f is a cylinder function.

As in Section 1.1, denote by $\mathscr{H}_1$ the Hilbert space generated by the space $\mathscr{C}$ of cylinder functions endowed with the scalar product $\langle f, (-\mathscr{L})g \rangle_{v_\alpha^*}$, with associated norm $\| \cdot \|_1$, such that

$$\| f \|_1^2 = \mathscr{D}_0(f) + \mathscr{D}_\theta(f),$$

as we have seen in (1.16), for functions $f \in \mathscr{C}$. This identity extends to the domain $\mathscr{D}(\mathscr{L})$ since $\mathscr{C}$ forms a core for $\mathscr{L}$. Using the probability measure v_α^* in place of π, it is possible to define $\mathscr{H}_{-1}$ the dual space of $\mathscr{H}_1$, with norm given by the variational formula

$$\| f \|_{-1}^2 = \sup_{g \in \mathscr{C}} \{2\langle f, g \rangle_{v_\alpha^*} - \| g \|_1^2\}, \qquad f \in L^2(v_\alpha^*).$$

As we did before, from this formula it is easy to check that, for every function $f \in \mathscr{H}_1$ and every function $g \in L^2(v_\alpha^*) \cap \mathscr{H}_{-1}$ we get

$$| \langle f, g \rangle_{v_\alpha^*} | \leq \| f \|_1 \| g \|_{-1}$$

and moreover f in $L^2(v_\alpha^*)$ belongs to $\mathscr{H}_{-1}$ if and only if there exists a finite constant C such that

$$\langle f, g \rangle_{v_\alpha^*} \leq C \| g \|_1 \tag{2.9}$$

for every $g \in \mathscr{C}$. In this case $\| f \|_{-1} \leq C$. This also has been already shown in the general context in Claim C of Section 1.1.

Denote now by $\mathbb{L}^2(v_\alpha^*)$ the space of the sequences

$$\Psi = \{\Psi_z : \mathbb{X}^* \to \mathbb{R}; p(z) > 0\} \times \{\Psi_{x,y} : \mathbb{X}^* \to \mathbb{R}; x, y \in \mathbb{Z}_*^d; p(y - x) > 0\}$$

of functions in $L^2(v_\alpha^*)$ such that

$$\sum_{z \in \mathbb{Z}_*^d} p(z) \int [[1 - \xi(z)]\Psi_z(\xi)^2]v_\alpha^*(d\xi) + \sum_{x,y \in \mathbb{Z}_*^d} P(y-x) \int [\xi(x)[1-\xi(y)]\Psi_{x,y}(\xi)^2]v_\alpha^*(d\xi)$$

is finite. $\mathbb{L}^2(v_\alpha^*)$ is endowed with the scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \Psi, \Phi \rangle = \sum_{z \in \mathbb{Z}_*^d} p(z) \int [[1 - \xi(z)]\Psi_z(\xi)\Phi_z(\xi)]v_\alpha^*(d\xi)$$

$$+ \sum_{x,y \in \mathbb{Z}_*^d} p(y-x) \int [\xi(x)[1-\xi(y)]\Psi_{x,y}(\xi)\Phi_{x,y}(\xi)]v_\alpha^*(d\xi).$$

A function u in the domain $\mathscr{D}(\mathscr{L})$ induces a sequence in $\mathbb{L}^2(v_\alpha^*)$. Denoting this one Ψ^u, we have

$$\Psi_z^u = T^z u, \quad \Psi_{x,y}^u = T_{x,y}u.$$

Moreover, $\Psi \in \mathbb{L}^2(v_\alpha^*)$ defines the square integrable martingale M^Ψ, like in (2.4), with Ψ_z, $\Psi_{x,y}$ in place of $T^z f$, $T^{x,y} f$, such that

$$\mathbb{E}_{v_\alpha^*}[(M_t^\Psi)^2] = t\langle \Psi, \Psi \rangle.$$

Denote by $\mathbb{L}_0^2(v_\alpha^*)$ the closed subspace of $\mathbb{L}^2(v_\alpha^*)$ composed by all sequences Ψ for which the following relations are v_α^*-a.s.

$$\Psi_{x,y}(\xi) = \Psi_{y,x}(\xi)$$
$$\Psi_{x,y}(\sigma^{x,y}\xi) = -\Psi_{x,y}(\xi)$$
$$\Psi_{x,y}(\xi)^2 \{\xi(x)[1 - \xi(y)] + \xi(y)[1 - \xi(x)]\} = \Psi_{x,y}(\xi)^2$$
$$\Psi_z(\theta_{-z}\xi) = -\Psi_{-z}(\xi)$$

With these formulas, repeating the same computation in the previous section, we obtain

$$\frac{1}{t}\mathbb{E}_{v_\alpha^*}[(M_t^\Psi)^2] = \langle \Psi, \Psi \rangle$$

$$= (1/2) \sum_{x,y \in \mathbb{Z}_*^d} s(y-x) \int \Psi_{x,y}(\xi)^2 v_\alpha^*(d\xi) \qquad (2.10)$$

$$+ \sum_{z \in \mathbb{Z}_*^d} s(z) \int [1 - \xi(z)]\Psi_z(\xi)^2 v_\alpha^*(d\xi)$$

and we can see the analogy with (2.7). It follows:

Lemma 2.5. *Consider a sequence of cylinder functions $\{f_n : n \geq 1\}$ which forms a Cauchy sequence in $\mathscr{H}_1$. Then, there exists Ψ in $\mathbb{L}_0^2(v_\alpha^*)$ such that $M_t^{f_n}$ converges to M_t^Ψ in $L^2(\mathbb{P}_{v_\alpha^*})$ for all $t \geq 0$.*

Proof. A sequence $\{f_n : n \geq 1\}$ of cylinder functions forms a Cauchy sequence in $\mathcal{H}_1$ if and only if $\{\Psi^{f_n} : n \geq 1\}$ forms a Cauchy sequence in $\mathbb{L}^2(\upsilon_\alpha^*)$. In particular, Ψ^{f_n} converges in $\mathbb{L}^2(\upsilon_\alpha^*)$ to some Ψ which belongs to $\mathbb{L}_0^2(\upsilon_\alpha^*)$ because this space is closed. By (2.10), $M_t^{f_n}$ converges to M_t^{Ψ} in $\mathbb{L}^2(\upsilon_\alpha^*)$. $\qquad\square$

To conclude, we will prove the Theorem 2.1, through the obtained results. Recall that we can represent the position at time t of the tagged particle as the sum of the number of jumps multiplied by their size:

$$Z_t = \sum_{z \in \mathbb{Z}_*^d} z N_t^z = \sum_{z \in \mathbb{Z}_*^d} z M_t^z + \int_0^t V(\xi_s) ds, \qquad (2.11)$$

where the second equality derives from the definition of the jump processes $\{N_t^z : t \geq 0\}$ and V is the cylinder function given by

$$V(\xi) = \sum_{z \in \mathbb{Z}_*^d} z p(z)[1 - \xi(z)].$$

We have seen that the first term $M_t = \sum_{z \in \mathbb{Z}_*^d} z M_t^z$ is a martingale with quadratic variation bounded by Ct, under the stationary measure $\mathbb{P}_{\upsilon_\alpha^*}$. In particular, we have, almost surely,

$$\lim_{t \to \infty} \frac{M_t}{t} = 0.$$

For what concern the other factor, since $\mathbb{P}_{\upsilon_\alpha^*}$ is ergodic, almost surely,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t V(\xi_s) ds = \mathbb{E}_{\upsilon_\alpha^*}[V] = (1 - \alpha)m.$$

This proves that $\frac{Z_t}{t}$ converges $\mathbb{P}_{\upsilon_\alpha^*}$-almost surely to $(1-\alpha)m$. This factor is exactly what we expected. Indeed, since υ_α^* is stationary, the site to which the tagged particle chooses to jump is empty with probability $1-\alpha$. So, a mean displacement in the stationary state should be equal to $(1-\alpha)m$.

Chapter 3

Central Limit Theorem

Now we want to prove the Central Limit Theorem for the position of the tagged particle in an exclusion process, such that, as we assumed, the mean m is equal to zero or the dimension d is greater than 2.

Theorem 3.1. *Assume that $m = 0$ or that $d \geq 3$. Then, under $\mathbb{P}_{\upsilon_\alpha^*}$,*

$$\frac{Z_t - (1 - \alpha)mt}{t^{1/2}}$$

converges in distribution, as $t \uparrow \infty$, to a mean zero Gaussian random vector with covariance matrix $D(\alpha)$, a quadratic form strictly positive and finite. So that we can write, for all $a \in \mathbb{R}^d$,

$$C_0^{-1}\alpha(1 - \alpha) \mid a \mid^2 \leq a \cdot D(\alpha)a \leq C_0(1 - \alpha) \mid a \mid^2,$$

where C_0 is a strictly positive and finite constant, depending only on p.

The proof is obtained from the definition of the tagged particle's position Z_t seen in (2.11), from which we can derive:

$$Z_t - (1 - \alpha)tm = \sum_{z \in \mathbb{Z}_*^d} z M_t^z + \int_0^t \widetilde{V}(\xi_s)ds, \qquad (3.1)$$

where $\widetilde{V}(\xi_s)$ is the mean zero cylinder function:

$$\widetilde{V}(\xi) = V(\xi) - (1 - \alpha)m = \sum_{x \in \mathbb{Z}_*^d} x p(x)(\alpha - \xi(x))$$

and $\sum_{z \in \mathbb{Z}_*^d} z M_t^z =: M_t$ is a martingale. After rewriting the position Z_t in this way, it is sufficient to use a proof of the central limit theorem for additive functionals of Markov processes, which relies on bounds on $\mathscr{H}_{-1}$. In order to apply this argument, we need to introduce the concept of resolvent equation.

3.1   The Resolvent Equation

Let us study the general case, where we have a fixed function V in $L^2(\pi) \cap \mathscr{H}_{-1}$ and a fixed coefficient $\lambda > 0$. We can consider the resolvent equation, from [13].

$$\lambda f_\lambda - L f_\lambda = V, \tag{3.2}$$

where L is the generator of our Markov process, while $f_\lambda = (\lambda - L)^{-1} V$ belongs to the domain $\mathscr{D}(L)$ of the generator. Taking the scalar product with respect to f_λ on both sides, we get

$$\lambda \langle f_\lambda, f_\lambda \rangle_\pi + \parallel f_\lambda \parallel_1^2 = \langle V, f_\lambda \rangle_\pi. \tag{3.3}$$

Hence, by Schwarz inequality (1.4),

$$\lambda \langle f_\lambda, f_\lambda \rangle_\pi + \parallel f_\lambda \parallel_1^2 \leq \parallel f_\lambda \parallel_1 \parallel V \parallel_{-1},$$

so that

$$\parallel f_\lambda \parallel_1 \leq \parallel V \parallel_{-1}.$$

Combining these two bounds, one obtains the stronger estimate

$$\lambda \langle f_\lambda, f_\lambda \rangle_\pi + \parallel f_\lambda \parallel_1^2 \leq \parallel V \parallel_{-1}^2. \tag{3.4}$$

From where we conclude that λf_λ vanishes in $L^2(\pi)$ as $\lambda \downarrow 0$ and that $\{f_\lambda : 0 < \lambda \leq 1\}$ forms a bounded sequence in $\mathscr{H}_1$. Moreover, since $f_\lambda = (\lambda - L)^{-1} V$, from (3.4) $(\lambda - L)^{-1}$ extends from $L^2(\pi)$ to a bounded mapping from $\mathscr{H}_1$ to $\mathscr{H}_{-1}$. In conclusion, for any $V \in \mathscr{H}_{-1}$, we have

$$\parallel (\lambda - L)^{-1} V \parallel_1 \leq \parallel V \parallel_{-1}.$$

Now we can state that a central limit theorem for the additive functional $t^{-1/2} \int_0^t V(X_s) ds$ holds provided the following two conditions are satisfied

$$\lim_{\lambda \to 0} \lambda \parallel f_\lambda \parallel_\pi^2 = 0 \quad \text{and} \quad \lim_{\lambda \to 0} \parallel f_\lambda - f \parallel_1 = 0 \tag{3.5}$$

for some $f \in \mathscr{H}_1$. This will be proved in the next section for our specific case. But first, we want to show that these two conditions are implied by the following bound

$$\sup_{0 < \lambda \leq 1} \parallel L f_\lambda \parallel_{-1} < \infty.$$

Lemma 3.2. *Fix a function $V \in \mathscr{H}_{-1} \cap L^2(\pi)$ and denote by $\{f_\lambda : \lambda > 0\}$ the solution of the resolvent equation. Assume that $\sup_{\lambda > 0} \parallel L f_\lambda \parallel_{-1} \leq C_0$ for some finite constant C_0. Then, there exists $f \in \mathscr{H}_1$ such that (3.5) holds.*

Proof. We have already proved with (3.4) that

$$\sup_{0 < \lambda \leq 1} \parallel f_\lambda \parallel_1 \leq \parallel V \parallel_{-1} \quad \text{and} \quad \sup_{0 < \lambda \leq 1} \lambda \parallel f_\lambda \parallel_\pi^2 \leq \parallel V \parallel_{-1}^2.$$

In particular, λf_λ vanishes in $L^2(\pi)$, as $\lambda \downarrow 0$. The proof consists in several claims.

- Lf_λ converges weakly in $\mathscr{H}_{-1}$ to $-V$, as $\lambda \downarrow 0$.

  It is sufficient to prove that $\lim_{\lambda \to 0} \lambda f_\lambda = 0$ weakly in $\mathscr{H}_{-1}$. Since $\sup_{\lambda > 0} \parallel Lf_\lambda \parallel_{-1}$ is bounded, so is $\sup_{\lambda > 0} \parallel \lambda f_\lambda \parallel_{-1}$. As a result any sequence $\{\lambda_n f_{\lambda_n} : n \geq 1\}$ is weakly precompact in $\mathscr{H}_{-1}$, as $\lambda_n \downarrow 0$. We show that 0 is the only weak limiting point. Suppose, therefore, that $\lambda_n f_{\lambda_n}$ converges weakly in $\mathscr{H}_{-1}$ to g. Since $\langle g, (-S)f \rangle_{-1} = \langle f, g \rangle_\pi$ and λf_λ converges to 0 strongly in $L^2(\pi)$, for any $h \in \mathscr{C}$
  $$\langle g, (-S)h \rangle_{-1} = \lim_{n \to \infty} \langle \lambda_n f_{\lambda_n}, (-S)h \rangle_{-1} = \lim_{n \to \infty} \langle \lambda_n f_{\lambda_n}, h \rangle_\pi.$$

  Moreover, we can prove that $\mathscr{H}_{-1}$ is the closure of $\{Sf : f \in \mathscr{C}\}$. This implies $\{Sf : f \in \mathscr{C}\}$ is dense in $\mathscr{H}_{-1}$ so that $g = 0$.

- In the same way, since $\sup_{\lambda > 0} \parallel f_\lambda \parallel_1$ is bounded, each sequence $\lambda_n \downarrow 0$ has a subsequence, still denoted by λ_n, for which f_{λ_n} converges weakly in $\mathscr{H}_1$ to some function W. The function W satisfies the relation

  $$\parallel W \parallel_1^2 = \langle W, V \rangle_\pi.$$

  To check this identity apply Mazur's theorem to the sequences f_{λ_n} and Lf_{λ_n} to obtain sequences g_n and Lg_n which converge strongly in $\mathscr{H}_1$ to W an, respectively, in $\mathscr{H}_{-1}$ to $-V$. Keeping in mind that each g_n is obtained by finite convex combination of functions f_{λ_k} and therefore belongs to the domain $\mathscr{D}(L)$ and since g_n (resp. Lg_n) converges strongly in $\mathscr{H}_1$ (resp. $\mathscr{H}_{-1}$) to W (resp. $-V$), we have that $\langle g_n, Lg_n \rangle_\pi$ converges to $-\langle W, V \rangle_\pi$. On the other hand, since $\langle g_n, Lg_n \rangle_\pi = \parallel g_n \parallel_1^2$, it converges to $\parallel W \parallel_1^2$. It follows,

  $$\parallel W \parallel_1^2 = \langle W, V \rangle_\pi$$

  .

- $\lim_{\lambda \to 0} \lambda \parallel f_\lambda \parallel_\pi^2 = 0$.

  Suppose by contradiction that $\lambda \parallel f_\lambda \parallel_\pi^2$ does not converge to 0, as $\lambda \to 0$. In this case there exists $\epsilon > 0$ and a subsequence $\lambda_n \downarrow 0$ such that $\lambda_n \parallel f_{\lambda_n} \parallel_\pi^2 \geq \epsilon$ for all n. We have just shown the existence of a sub-subsequence $\lambda_{n'}$ for which $f_{\lambda_{n'}}$ converges weakly in $\mathscr{H}_1$ to some W satisfying the relation $\langle W, V \rangle_\pi = \parallel W \parallel_1^2$. Since $f - \lambda$ is the solution of the resolvent equation,

  $$\limsup_{n' \to \infty} \parallel f_{\lambda_{n'}} \parallel_1^2 \leq \limsup_{n' \to \infty} (\lambda_{n'} \parallel f_{\lambda_{n'}} \parallel_\pi^2 + \parallel f_{\lambda_{n'}} \parallel_1^2)$$
  $$= \limsup_{n' \to \infty} \langle f_{\lambda_{n'}}, V \rangle_\pi = \langle W, V \rangle_p i = \parallel W \parallel_1^2$$
  $$\leq \limsup_{n' \to \infty} \parallel f_{\lambda_{n'}} \parallel_1^2.$$

  These inequalities contradict the fact that $\lambda_n \parallel f_{\lambda_n} \parallel_\pi^2 \geq \epsilon$ for all n, so that $\lim_{\lambda > 0} \lambda \parallel f_\lambda \parallel_\pi^2 = 0$.

- f_λ converges strongly in $\mathscr{H}_1$ as $\lambda \downarrow 0$.

From the previous argument it follows that $f_{\lambda_{n'}}$ converges to W strongly in $\mathscr{H}_1$. In particular, all sequences λ_n have subsequences $\lambda_{n'}$ for which $f_{\lambda_{n'}}$ converges strongly in $\mathscr{H}_1$. To show that f_λ converges strongly, it remains to check uniqueness of the limit. Consider two decreasing sequences $\lambda_n \, \mu_n$, vanishing as $n \uparrow \infty$. Denote by $W_1 \, W_2$ the strong limit in $\mathscr{H}_1$ of $f_{\lambda_n} \, f_{\mu_n}$, respectively. Since f_λ is the solution of the resolvent equation,

$$\langle \lambda_n f_{\lambda_n} - \mu_n f_{\mu_n}, f_{\lambda_n} - f_{\mu_n} \rangle_\pi + \parallel f_{\lambda_n} - f_{\mu_n} \parallel_1^2 = 0$$

for all n. Since $f_{\lambda_n} \, f_{\mu_n}$, converges strongly to $W_1 \, W_2$ in $\mathscr{H}_1$,

$$\lim_{n \to \infty} \parallel f_{\lambda_n} - f_{\mu_n} \parallel_1^2 = \parallel W_1 - W_2 \parallel_1^2 .$$

On the other hand, since $\lambda \parallel f_\lambda \parallel_\pi^2$ vanishes as $\lambda \downarrow 0$,

$$\lim_{n \to \infty} \langle \lambda_n f_{\lambda_n} - \mu_n f_{\mu_n}, f_{\lambda_n} - f_{\mu_n} \rangle_\pi = - \lim_{n \to \infty} (\langle \lambda_n f_{\lambda_n}, f_{\mu_n} \rangle_\pi + \langle \mu_n f_{\mu_n}, f_{\lambda_n} \rangle_\pi).$$

Each of these terms vanish as $n \uparrow \infty$. Indeed,

$$\lambda_n \langle f_{\lambda_n}, f_{\mu_n} \rangle_\pi = \lambda_n \langle f_{\lambda_n}, f_{\mu_n} - W_2 \rangle_\pi + \lambda \langle f_{\lambda_n}, W_2 \rangle_\pi.$$

By Schwarz inequality (1.3), the first term on the right hand side is bounded above by $\parallel \lambda_n f_{\lambda_n} \parallel_{-1} \parallel f_{\mu_n} - W_2 \parallel_1$, which vanishes because λf_λ is bounded in $\mathscr{H}_{-1}$ and f_{μ_n} converges to W_2 in $\mathscr{H}_1$. To show that the second term of the previous formula also vanishes, fix $\epsilon > 0$. Since $\mathscr{H}_1$ is obtained as the completion of $\mathscr{C}$, there exists $g \in \mathscr{C}$ such that $\parallel W_2 - g \parallel_1 \leq \epsilon$. By the same Schwarz inequality, the second term is then absolutely bounded by $\sup_{0 < \lambda \leq 1} \parallel \lambda f_\lambda \parallel_{-1} \epsilon + \mid \langle \lambda_n f_{\lambda_n}, g \rangle_\pi \mid$. Since λf_λ vanishes in $L^2(\pi)$ as $\lambda \downarrow 0$, the second term on the right hand side of the previous displayed formula converges to 0 as $n \uparrow \infty$. This concludes the proof of the lemma.

$\square$

3.2   Proof of the Central Limit Theorem

Now the aim is to show that the previous argument is valid in our case, taking into account the (3.1). Fix a vector $a \in \mathbb{R}^d$ and let $V_a = a \cdot \widetilde{V}$. Denoting by u_λ the solution of the resolvent equation and with $\mathscr{L}$ our generator, we get

$$\lambda u_\lambda - \mathscr{L} u_\lambda = V_a. \tag{3.6}$$

In this section, we will prove Theorem 3.1, assuming that for every vector $a \in \mathbb{R}^d$

$$V_a \in \mathscr{H}_{-1} \quad \text{and} \quad \sup_{0 < \lambda \leq 1} \parallel \mathscr{L} u_\lambda \parallel_{-1} < \infty. \tag{3.7}$$

By Lemma 3.2, from these conditions it follows that

$$\lim_{\lambda \to 0} \lambda \langle u_\lambda, u_\lambda \rangle_{v_\alpha^*} = 0 \quad \text{and} \quad u_\lambda \text{ converges in } \mathscr{H}_1 \text{ as } \lambda \downarrow 0. \tag{3.8}$$

The strategy is to represent the additive functional $\int_0^t V_a(\xi_s)ds$ as the sum of a martingale m_t and a negligible term and then to use the central limit theorem for the martingale $M_t + m_t$, where $M_t = \sum_z z \cdot a M_t^z$.

For $\lambda > 0$, let m_t^λ be the martingale

$$m_t^\lambda = u_\lambda(\xi_t) - u_\lambda(\xi_0) - \int_0^t (\mathscr{L} u_\lambda)(\xi_s)ds.$$

By the result 2.4, the previous martingale can be represented in term of elementary martingales, as

$$m_t^\lambda = \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t \Psi_{x,y}^\lambda(\xi_s)dM_s^{x,y} + \sum_{z \in \mathbb{Z}_*^d} \int_0^t \Psi_z^\lambda(\xi_s)dM_s^z, \tag{3.9}$$

where $\Psi_{y,x}^\lambda = T^{x,y}u_\lambda$, $\Psi_z^\lambda = T^z u_\lambda$. The resolvent equation permits to write the position of the tagged particle Z_t as

$$Z_t \cdot a - (1-\alpha)t(m \cdot a) = M_t + m_t^\lambda + R_t^\lambda, \tag{3.10}$$

where M_t is the martingale $\sum_z (z \cdot a)M_t^z$ and where the remainder R_t^λ is given by

$$R_t^\lambda = u_\lambda(\xi_0) - u_\lambda(\xi_t) + \lambda \int_0^t u_\lambda(\xi_s)ds.$$

Lemma 3.3. *For every $t > 0$, the martingale m_t^λ converges in $L^2(\mathbb{P}_{v_\alpha^*})$ to some martingale m_t as $\lambda \downarrow 0$.*

Proof. The sequence u_λ converges in $\mathscr{H}_1$ as $\lambda \downarrow 0$ and $\mathscr{C}$ forms a core for the generator $\mathscr{L}$, so, by Lemma 2.5, the martingale m_t^λ converges in $L^2(\mathbb{P}_{v_\alpha^*})$ to a martingale $m_t = M_t^\Psi$ associated to a sequence Ψ in $\mathbb{L}_0^2(v_\alpha^*)$. $\square$

Also the remainder R_t^λ in 3.10 converges in $L^2(\mathbb{P}_{v_\alpha^*})$, as $\lambda \downarrow 0$, so that

$$Z_t \cdot a - (1-\alpha)t(m \cdot a) = M_t + m_t + R_t. \tag{3.11}$$

In order to work only with martingales, we can prove that the remainder term vanishes.

Lemma 3.4. *In $L^2(\mathbb{P}_{v_\alpha^*})$*

$$\lim_{t \to \infty} \frac{R_t}{t^{1/2}} = 0.$$

Proof. Putting together (3.1) and (3.11), we get

$$\int_0^t \widetilde{V}(\xi_s)ds = m_t + R_t.$$

Moreover, using (3.10) and from the expression of the remainder R_t^λ, we have

$$\int_0^t \widetilde{V}(\xi_s)ds = m_t^\lambda + u_\lambda(\xi_0) - u_\lambda(\xi_t) + \lambda \int_0^t u_\lambda(\xi_s)ds.$$

From these two formulas we derive

$$\frac{R_t}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left\{ m_t^\lambda - m_t + u_\lambda(\xi_0) - u_\lambda(\xi_t) + \lambda \int_0^t u_\lambda(\xi_s)ds \right\}.$$

We consider separately each term on the right hand side of the above expression. Since m_t^λ converges to m_t in $L^2(\mathbb{P}_{\upsilon_\alpha^*})$,

$$\frac{1}{t}\mathbb{E}_{\upsilon_\alpha^*}[(m_t^\lambda - m_t)^2] = \frac{1}{t}\lim_{\lambda' \to 0} \mathbb{E}_{\upsilon_\alpha^*}[(m_t^\lambda - m_t^{\lambda'})^2].$$

The computation of the quadratic variation of the martingale $M_t^\lambda - M_t^{\lambda'}$ shows that the previous expression is equal to

$$2\lim_{\lambda' \to 0} \parallel u_\lambda - u_{\lambda'} \parallel_1^2 = 2 \parallel u_\lambda - u \parallel_1^2.$$

In the last step we used the assumption that u_λ converges in $\mathscr{H}_1$ to some u, as stated in (3.8). We now estimate the term R_t^λ, considering that

$$\mathbb{E}_{\upsilon_\alpha^*}[(\sqrt{t}R_t^\lambda)^2] \le \frac{3}{t}\mathbb{E}_{\upsilon_\alpha^*}[u_\lambda(\xi_t)^2] + \frac{3}{t}\mathbb{E}_{\upsilon_\alpha^*}[u_\lambda(\xi_0)^2] + \frac{3}{t}\lambda^2\mathbb{E}_{\upsilon_\alpha^*}\left[\left\{\int_0^t u_\lambda(\xi_s)ds\right\}^2\right].$$

Since $\{\xi_t, t \ge 0\}$ is stationary with the initial distribution υ_α^* we obtain that the first two expressions on the right hand side taken together yield $6t^{-1} \parallel u_\lambda \parallel_{\upsilon_\alpha^*}^2$. On the other hand, by Schwarz inequality, the third term is bounded by $3t\lambda^2 \parallel u_\lambda \parallel_{\upsilon_\alpha^*}^2$. Putting together all the previous estimates, we obtain that, for all $\lambda > 0$,

$$\frac{1}{t}\mathbb{E}_{\upsilon_\alpha^*}[R_t^2] \le 2 \parallel u_\lambda - u \parallel_1^2 + 2(6t^{-1} + 3t\lambda^2) \parallel u_\lambda \parallel_{\upsilon_\alpha^*}^2.$$

Setting $\lambda = t^{-1}$ we conclude the proof in view of the (3.5), with u in place of f. $\square$

Since both martingales M_t and m_t are written in terms of the elementary martingales,

the quadratic variation of the sum is easy to compute:

$$
\begin{aligned}
\langle M + m \rangle_t &= \left\langle \sum_{z \in \mathbb{Z}_*^d} z \cdot a M_t^z + \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t \Psi_{x,y}(\xi_s) dM_s^{x,y} + \sum_{z \in \mathbb{Z}_*^d} \int_0^t \Psi_z(\xi_s) dM_s^z \right\rangle_t \\
&= \left\langle \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t \Psi_{x,y}(\xi_s) dM_s^{x,y}, \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t \Psi_{x,y}(\xi_s) dM_s^{x,y} \right\rangle_t \\
&\quad + \left\langle \sum_{z \in \mathbb{Z}_*^d} z \cdot a M_t^z + \int_0^t \Psi_z(\xi_s) dM_s^z, \sum_{z \in \mathbb{Z}_*^d} z \cdot a M_t^z + \int_0^t \Psi_z(\xi_s) dM_s^z \right\rangle_t \quad (3.12) \\
&= \sum_{x,y \in \mathbb{Z}_*^d} p(y - x) \int_0^t \xi_s(x)[1 - \xi_s(y)] \Psi_{x,y}(\xi_s)^2 ds \\
&\quad + \sum_{z \in \mathbb{Z}_*^d} p(z) \int_0^t [1 - \xi_s(z)] \{a \cdot z + \Psi_z(\xi_s)\}^2 ds,
\end{aligned}
$$

where the last inequality holds from the quadratic variation of the elementary martingales, defined in Lemma 2.3.

By the ergodic theorem under $\mathbb{P}_{\upsilon_\alpha^*}$, $t^{-1} \langle M + m \rangle_t$ converges a.s. and in $L^1(\mathbb{P}_{\upsilon_\alpha^*})$. So $t^{-1/2} \{M_t + m_t\}$, and therefore $t^{-1/2}[Z_t \cdot a - (1 - \alpha) t(m \cdot a)]$ converges in distribution to a mean zero Gaussian variable with variance $D(\alpha)$ satisfying

$$
\begin{aligned}
a \cdot D(\alpha) a &= \sum_{x,y \in \mathbb{Z}_*^d} p(y - x) \int \xi(x)[1 - \xi(y)] \Psi_{x,y}(\xi)^2 \upsilon_\alpha^*(d\xi) \\
&\quad + \sum_{z \in \mathbb{Z}_*^d} p(z) \int [1 - \xi(z)] \{a \cdot z + \Psi_z(\xi)\}^2 \upsilon_\alpha^*(d\xi).
\end{aligned}
\tag{3.13}
$$

Since Ψ belongs to $\mathbb{L}_0^2(\upsilon_\alpha^*)$, we can derive that

$$
\begin{aligned}
a \cdot D(\alpha) a &= \frac{1}{2} \sum_{x,y \in \mathbb{Z}_*^d} s(y - x) \int \Psi_{x,y}(\xi)^2 \upsilon_\alpha^*(d\xi) \\
&\quad + \sum_{z \in \mathbb{Z}_*^d} s(z) \int [1 - \xi(z)] \{a \cdot z + \Psi_z(\xi)\}^2 \upsilon_\alpha^*(d\xi).
\end{aligned}
\tag{3.14}
$$

In principle, we would need a strictly positive lower bound and a finite upper bound, in order to prevent the asymptotic variance $D(\alpha)$ from vanishing or being $+\infty$. Such bounds can be derived, even if we will not show it here ([13, Section 6.8]) and the proof of the central limit theorem is, therefore, complete.

However, we have to remember that this result has been achieved under the assumptions on the solution of the resolvent equation:

$$
V_a \in \mathscr{H}_{-1} \quad \text{and} \quad \sup_{0 < \lambda \le 1} \| \mathscr{L} u_\lambda \|_{-1} < \infty.
\tag{3.15}
$$

The purpose, now, is to show that these conditions hold in the mean zero case or if $d \ge 3$.

The Symmetric Case   In this paragraph it will be shown that the function V_a belongs to $\mathscr{H}_{-1}$ if $m = 0$ and that the solution of the resolvent equation satisfies the second of (3.15) provided $p(-x) = p(x)$, proving the central limit theorem for the tagged particle in the symmetric case.

Proposition 3.5. *Assume that $m = 0$. Then the cylinder function V_a belongs to $\mathscr{H}_{-1}$ and $\parallel V_a \parallel^2_{-1} \leq C\chi(\alpha) \mid a \mid^2$.*

Proof. To show that $V_a \in \mathscr{H}_{-1}$ we need to prove that there exists a finite constant C such that, for every cylinder function f,

$$\langle f, V_a \rangle_{v^*_\alpha} \leq C\sqrt{\chi(\alpha)} \parallel f \parallel_1 \mid a \mid .$$

Fix a cylinder function f. In the mean zero caso, V_a can be rewritten as

$$\sum_{x \in \mathbb{Z}^d} (a \cdot x)p(x)\{\xi(e) - \xi(x)\}$$

where e is any fixed site of $\mathbb{Z}^d_*$. Since $s(\cdot)$ generates $\mathbb{Z}^d_*$, for each x such that $p(x) > 0$, there exists a path $x = y_0, \ldots, y_n = e$ going from x to e avoinding the origin and such that $s(y_{i+1} - y_i) > 0$. Since

$$\langle \xi(e) - \xi(x), f \rangle_{v^*_\alpha} = \sum_{i=0}^{n-1} \langle \xi(y_{i+1}) - \xi(y_i), f \rangle_{v^*_\alpha}.$$

With the change of variables $\xi' = \sigma^{y_i, y_{i+1}}\xi$, we may rewrite the previous sum as

$$-\frac{1}{2}\sum_{i=0}^{n-1} \langle \xi(y_{i+1}) - \xi(y_i), T^{y_i, y_{i+1}}f \rangle_{v^*_\alpha},$$

for which we clearly have

$$-\frac{1}{2}\sum_{i=0}^{n-1} \langle \xi(y_{i+1}) - \xi(y_i), T^{y_i, y_{i+1}}f \rangle_{v^*_\alpha} \leq \frac{1}{2}nA\alpha(1 - \alpha) + \frac{1}{4A}\sum_{i=0}^{n-1} \mathbb{E}_{v^*_\alpha}[(T^{y_i, y_{i+1}}f)^2]$$

for every $A > 0$. Summing over all x such that $p(x) > 0$, we get

$$\langle V_a, f \rangle_{v^*_\alpha} \leq CA\alpha(1 - \alpha) \mid a \mid^2 + \frac{C}{A}\mathscr{D}_0(f)$$

for all $A > 0$ and some finite constant C_0. Minimizing the previous expression with respect to A, we come to the conclusion. $\square$

This proposition proves that the first assumption in (3.15) holds in the mean zero case. While, the second one is due to the reversibility of the process in the symmetric case, as we show with the following argument.

Assume that the generator $\mathscr{L}$ is self adjoint in $\mathbb{L}^2(v_\alpha^*)$. In this case, as we have seen in Claim D in Section 1.1, $\mathscr{L}u$ belongs to $\mathscr{H}_{-1}$ for any function $u \in \mathscr{D}(\mathscr{L})$ and

$$\| \mathscr{L}u \|_{-1} \leq \| u \|_1 .$$

In fact the equality holds. In particular, in the reversible case, our condition follows from the estimate

$$\sup_{0 < \lambda \leq 1} \| u_\lambda \|_1 < \infty,$$

which follows from (3.4).

The Asymmetric Case We will treat here the mean zero case $\sum_x p(x)x = 0$ in the more general asymmetric case. The first assumption, proved in Proposition 3.5, holds also for the asymmetric case, since it was a direct consequence of $m = 0$.

Assume that the generator $\mathscr{L}$ satisfies the sector condition

$$\langle f, (-\mathscr{L})g \rangle_{v_\alpha^*}^2 \leq C \langle f, (-\mathscr{L})f \rangle_{v_\alpha^*} \langle g, (-\mathscr{L})g \rangle_{v_\alpha^*} \tag{3.16}$$

for some finite constant C and for every cylinder function f, g in the domain of the generator $\mathscr{D}(\mathscr{L})$. In view of (1.6), for any function g in $\mathscr{D}(\mathscr{L})$,

$$\| \mathscr{L}g \|_{-1}^2 \leq C \| g \|_1^2$$

and the desired condition in (3.15) follows again from the estimate

$$\sup_{0 < \lambda \leq 1} \| u_\lambda \|_1 < \infty.$$

So now it is enough to show that the generator satisfies the sector condition. In order to obtain this result we will use the decomposition of a mean zero probability in cycle probability measures. The following notions are from [13, Chapter 5].

A cycle C of length n is a sequence of n sites of $\mathbb{Z}^d$ starting and ending at the same point: $(y_0, y_1, \ldots, y_{n-1}, y_n = y_0)$, $y_i \neq y_{i+1}$, $0 \leq i \leq n-1$.

Definition 3.6. A mean zero probability measure p_C on $\mathbb{Z}^d$, associated to a cycle C, which does not charge the origin is defined by

$$p_C(x) = \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}\{x = y_{j+1} - y_j\} \tag{3.17}$$

and is called a cyclic probability measure.

The cyclic probability measure has mean is zero, indeed

$$\sum_{x \in \mathbb{Z}^d} x p_C(x) = \frac{1}{n} \sum_{x \in \mathbb{Z}^d} \sum_{j=0}^{n-1} x \mathbf{1}\{x = y_{j+1} - y_j\} = \frac{1}{n} \sum_{j=0}^{n-1} (y_{j+1} - y_j) = \frac{y_n - y_0}{n} = 0.$$

Moreover, for a positive integer m, finite cycles $\mathbf{C} = \{C_1, \ldots, C_m\}$ and a probability measure $\mathbf{w} = \{w_1, \ldots, w_m\}$, let $p_{\mathbf{C},\mathbf{w}}(\cdot)$ be the probability measure on $\mathbb{Z}^d$ defined by

$$p_{\mathbf{C},\mathbf{w}}(\cdot) = \sum_{k=1}^{m} w_k p_{C_k}(\cdot).$$

We prove in the following Lemma that all finite- range, zero-mean probability measure which do not charge the origin may be written as a convex combination of cyclic probability measures.

Lemma 3.7. *Fix a finite-range mean zero probability measure p on $\mathbb{Z}^d$ which does not charge the origin. There exists $m \geq 1$ finite cycles $\mathbf{C} = \{C_1, \ldots, C_m\}$ and a probability measure $\mathbf{w} = \{w_1, \ldots, w_m\}$ such that $p = p_{\mathbf{C},\mathbf{w}}(\cdot)$, the probability measure associated to a finite collection of cycles $\mathbf{C}$ and to a probability measure $\mathbf{w}$.*

This result states that the generator L of an exclusion process associated to a mean zero probability measure p can be written as $L = \sum_{1 \leq j \leq m} w_j L_{C_j}$, for a probability $\{w_1, \ldots, w_m\}$ and cycles $\{C_1, \ldots, C_m\}$ with associated cycle generator L_{C_j}. Let L be a generator which can be decomposed as a convex combination of two generators: $L = \theta L_1 + (1 - \theta) L_2$, for some $0 < \theta < 1$. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\nu)$, where ν is a stationary state. The next result asserts that L satisfies a sector condition if both L_1 and L_2 do.

Lemma 3.8. *If there exist finite constants B_1, B_2 such that*

$$|\langle -L_j f, g \rangle| \leq B_1 \langle -L_j f, f \rangle^{1/2} \langle -L_j g, g \rangle^{1/2}$$

for $j = 1, 2$ and all functions f, g in a common core $\mathscr{C}$ for L, L_1, L_2, then

$$|\langle -L f, g \rangle| \leq B \langle -L f, f \rangle^{1/2} \langle -L g, g \rangle^{1/2}$$

for every $f, g \in \mathscr{C}$ and for $B = B_1 + B_2$.

So, in general, a sector condition holds for a finite convex combination of generators if it holds individually and with the next it holds for an exclusion process associated to a cyclic probability measure.

Lemma 3.9. *For every cylinder function f, g*

$$|\langle L_C f, g \rangle_{v_\alpha}| \leq 4n^2 \langle -L_C f, f \rangle_{v_\alpha}^{1/2} \langle -L_C g, g \rangle_{v_\alpha}^{1/2}.$$

Applying these lemmas the next proposition follows from a sector condition for generators associated to cycles.

Proposition 3.10. *There exists a finite constant C, depending only on the probability measure p, such that*

$$\langle f, (-\mathscr{L}) g \rangle_{v_\alpha^*}^2 \leq C \langle f, (-\mathscr{L}) f \rangle_{v_\alpha^*} \langle g, (-\mathscr{L}) g \rangle_{v_\alpha^*}$$

for all cylinder functions f, g.

Proof. By Lemma 3.7 and Lemma 3.8, we may assume that p is a cycle probability measure:

$$p(x) = \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}\{x = y_j - y_{j-1}\},$$

for some $n \geq 1$, the length of the cycle, and some set $C = \{y_0, \cdots, y_{n-1}, y_n = y_0\}$. We also assume that the cycle is irreducible in the sense that $y_i \neq y_j$ if $i \neq j$, for $0 \leq i, j \leq n - 1$. Let $C + x$ be the cycle $\{y_0 + x, \cdots, y_{n-1} + x, y_n + x\}$. Since the cycle C is irreducible, there are exactly n cycles of the form $C + x$ which intersect the origin: $C - y_0, \cdots, C - y_{n-1}$. For a cycle $C + x$ which does not intersect the origin, let $\mathscr{L}^0_{C+x}$ be the generator defined by

$$(\mathscr{L}^0_{C+x} f)(\xi) = \frac{1}{n} \sum_{k=0}^{n-1} \xi(y_k + x)[1 - \xi(y_{k+1} + x)]\{f(\sigma^{y_k+x,y_{k+1}+x}\xi) - f(\xi)\}.$$

On the other hand, for the cycle $C - y_j$, $0 \leq j \leq n - 1$, we have the corresponding generator

$$(\mathscr{L}^0_{C-y_j} f)(\xi) = \frac{1}{n} \sum_{\substack{0 \leq k \leq n-1 \\ k \neq j-1, j}} \xi(y_k - y_j)[1 - \xi(y_{k+1} - y_j)]\{f(\sigma^{y_k-y_j,y_{k+1}-y_j}\xi) - f(\xi)\},$$

where we ignore the jump from $y_{j-1} - y_j$ to the origin, because the origin stays always occupied, and from the origin to $y_{j+1} - y_j$, because this jump does not appear in the generator associated to the environment. Moreover, note that in both formulas, we can remove the factors $\xi(y_k + x)$ and $\xi(y_k - y_j)$, without any change.

For the probability measure p obtained from the cycle C, the generator associated to the jumps of the tagged particle is written as

$$(\mathscr{L}_\theta f)(\xi) = \frac{1}{n} \sum_{k=0}^{n-1} [1 - \xi(y_{k+1} - y_k)]\{f(\theta_{y_{k+1}-y_k}\xi) - f(\xi)\}.$$

Let $\mathscr{L}_1 = \sum_{x \notin C} \mathscr{L}^0_{C-x}$ be the piece of the generator $\mathscr{L}$ whose cycles do not intersect the origin and let $\mathscr{L}_2 = \mathscr{L}_\theta + \sum_{0 \leq j \leq n-1} \mathscr{L}^0_{C-y_j}$. By Lemma 3.18, we derive

$$\langle f, (-\mathscr{L}_1 g)\rangle^2_{\nu^*_\alpha} \leq 16n^4 \langle f, (-\mathscr{L}_1 f)\rangle_{\nu^*_\alpha} \langle g, (-\mathscr{L}_1 g)\rangle_{\nu^*_\alpha}, \tag{3.18}$$

where, on the right hand side, we can replace $\mathscr{L}_1$ by $\mathscr{L}$, since we are adding only non negative terms.

It remains to prove a similar bound for the generator $\mathscr{L}_2$, which involves $n(n-2)$ measure-preserving transformations coming from $\sum_{0 \leq j \leq n-1} \mathscr{L}^0_{C-y_j}$ and n additional measure-preserving transformations coming from the term $\mathscr{L}_\theta$. Denote by $T_1, \ldots, T_{n-1}$ these transformations. We claim that there is a permutation s of $\{1, \ldots, n(n-1)\}$ such that $T_{s(n(n-1))} \ldots T_{s(1)} \xi = \xi$, provided ξ has a hole in a specific site. Even if a rigorous proof of this property is too lenghty.

Eventually, considering the generators $\mathscr{L}_3 = \sum_{0 \leq j \leq n-1} \mathscr{L}^0_{C-y_j}$ and $\mathscr{L}_\theta$, it is possible the derive the searched bound, see Lemma 5.8 in [13], obtaining

$$\langle f, (-\mathscr{L}_3 - \mathscr{L}_\theta g) \rangle^2_{v^*_\alpha} \leq 16n^2 \langle f, (-\mathscr{L}_3 - \mathscr{L}_\theta f) \rangle_{v^*_\alpha} \langle g, (-\mathscr{L}_3 - \mathscr{L}_\theta g) \rangle_{v^*_\alpha}. \tag{3.19}$$

Here again, we may replace the generator $\mathscr{L}_3 + \mathscr{L}_\theta$ by $\mathscr{L}$ on the right hand side. This bound together with the previous one gives us the sector condition for the mean zero exclusion process as seen from the tagged particle. $\qquad\square$

In conclusion, if the mean of the process $m = \sum_{x \in \mathbb{Z}^d_*} xp(x)$ is equal to zero, we have proved a central limit theorem in both symmetric and asymmetric case. So far, this result has been established only in the mean zero case and in the asymmetric case in dimension $d \geq 3$. It is conjectured to hold also in dimensions 1 and 2 in the case $m \neq 0$, but this is still an open problem, given that even the finiteness of the asymptotic variance has not yet been proved.

Chapter 4

Invariance Principle

The fundamental result that we would like to show in this work is the Invariance Principle for the position of the tagged particle. Our aim is to state the convergence to a Brownian motion in the rough path topology, for this purpose we need to define the main notions of the rough path theory, which will be necessary.

4.1 Elements of Rough Path Theory

A rough path generalizes the notion of smooth path, in order to give meaning to integrals of the form $\int f(X)dX$ or to provide solutions to controlled differential equations of the form $dY = f(Y)dX$, for rough paths X. Often the path X is the sample path of a Wiener process and in that case we talk of stochastic differential equations.

Here, following the main arguments of Friz and Hairer in [8], we define the space of Hölder continuous rough paths, but then we will replace the "α-Hölder continuity" by the "finite p-variation" for $p = 1/\alpha$. The reasons of this choice lie in the fact that the value of such an integral or solution does not depend on the parametrisation of X, just as the p-variation of a function is independent of its parametrisation.

Definition 4.1. A rough path on an interval $[0, T]$ with values in a Banach space V consists of a continuous function $X : [0, T] \to V$ and a continuous "second order process" $\mathbb{X} : [0, T]^2 \to V \otimes V$, subject to certain algebraic and analytical conditions, such as the Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}, \tag{4.1}$$

which must hold for every triple of times (s, u, t). Since $X_{t,t} = 0$, taking $s = u = t$ it follows that we also have $\mathbb{X}_{t,t} = 0$ for every t. So we can put

$$\int_s^t X_{s.r} \otimes dX_r := \mathbb{X}_{s,t}, \tag{4.2}$$

where the right hand side is taken as a definition for the left hand side.

Note that the algebraic relations (4.1) are by themselves not sufficient to determine $\mathbb{X}$ as a function of X. Indeed, for any $V \otimes V$-valued function F, the substitution $\mathbb{X}_{s,t} \mapsto \mathbb{X}_{s,t} + F_t - F_s$ leaves the left hand side of (4.1) invariant. So it remains to discuss what are the analytical conditions one should impose.

We will assume that the path X itself is α-hölder continuous, so that $\mid X_{s,t} \mid \lesssim \mid t-s \mid^{\alpha}$ (the symbol "$\lesssim$" stands for "less equal than up to a constant"). The archetype of an α-hölder continuous function is one which is self-similar with index α, so that $X_{\lambda s, \lambda t} \sim \lambda^{\alpha} X_{s,t}$. Given (4.2), we expect $\mathbb{X}$ to be self-similar as well, but with $\mathbb{X}_{\lambda s, \lambda t} \sim \lambda^{2\alpha} \mathbb{X}_{s,t}$. This argument justifies the following definition for the space of (hölder continuous) rough paths.

Definition 4.2. For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, define the space of α-hölder rough paths, over V, in symbols $\mathscr{C}^{\alpha}([0,T], V)$, as the pairs $(X, \mathbb{X})$ such that

$$\parallel X \parallel_{\alpha} := \sup_{s \neq t \in [0,T]} \frac{\mid X_{s,t} \mid}{\mid t-s \mid^{\alpha}} < \infty, \qquad \parallel \mathbb{X} \parallel_{2\alpha} := \sup_{s \neq t \in [0,T]} \frac{\mid \mathbb{X}_{s,t} \mid}{\mid t-s \mid^{2\alpha}} < \infty, \qquad (4.3)$$

and such that the algebraic constraint (4.1) is satisfied.

Observation. Obviously, this construction is only possible if $\alpha \leq \frac{1}{2}$, indeed by (4.3) $\mathbb{X}$ has to be 2α-hölder continuous. Furthermore, we have chosen $\alpha \in (\frac{1}{3}, \frac{1}{2})$ since the Lyons-Victoir extension theorem asserts that given an arbitrary path $X \in \mathscr{C}^{\alpha}$, with values in some Banach space V, it can be always lifted to a rough path $(X, \mathbb{X}) \in \mathscr{C}^{\alpha}$ provided $\alpha \in (\frac{1}{3}, \frac{1}{2})$. In general, instead, this is not always possible.

We now want to introduce the notion of Skorokhod topology in some metric space, in view of the work of I. Chevyrev and P. Friz [5]. Denote by $C([s,t], E)$ and $D([s,t], E)$ the spaces of continuous and càdlàg functions (or paths) respectively, from an interval $[s,t]$ into a metric space (E, d). Equip $C([s,t], E)$ with the usual uniform metric

$$d_{\infty, [st]}(x, y) = \sup_{r \in [s,t]} d(x(r), y(r)).$$

While, on $D([s,t], E)$ we consider a topological structure introduced by A.V. Skorokhod as an alternative to the uniform topology, in order to study the convergence in distribution of stochastic processes with jumps.

Definition 4.3. Denote $\Lambda_{[s,t]}$ the set of all striclty increasing bijections of $[s,t]$ to itself and, for $\lambda \in \Lambda$, denote $\mid \lambda \mid := \sup_{r \in [s,t]} \mid \lambda(r) - r \mid$. For any $x, y \in D([s,t], E)$, the Skorokhod metric is given by

$$\sigma_{\infty, [s,t]}(x, y) = \inf_{\lambda \in \Lambda_{[s,t]}} \{ \mid \lambda \mid \vee d_{\infty, [s,t]}(x \circ \lambda, y) \},$$

Let $\mathcal{D} = (t_0 = s < t_1 < \cdots < t_{k-1} < t_k = t)$ denote a partition of $[s,t]$ and let $\sum_{t_i \in \mathcal{D}}$ denote summation over all points in $\mathcal{D}$, with $\mid \mathcal{D} \mid = \max_{t_i \in \mathcal{D}} \mid t_{i+1} - t_i \mid$ the mesh-size of a partition.

Definition 4.4. For $p > 0$, we define the p-variation of a path $\mathbf{X} := (X, \mathbb{X})$ in $D([s, t], E)$ by

$$\| \mathbf{X} \|_{p,[s,t]} := \sup_{\mathcal{D} \subset [s,t]} \left(\sum_{t_i \in \mathcal{D}} d(\mathbf{X}_{t_i}, \mathbf{X}_{t_i+1})^p \right)^{1/p}.$$

Denote $D^{p-var}([s, t], E)$ subspaces of path of finite p-variation.

A càdlàg rough path $\mathbf{X}^n$ is said with uniformly bounded p-variation if

$$\sup_n \| \mathbf{X}^n \|_{p,[0,T]} =: L < \infty.$$

On D^{p-var} we define a p-variation variant to the Skorokhod metric, fixing $p = \frac{1}{\alpha}$, then $p \in (2, 3)$.

Definition 4.5. For $p \in (2, 3)$ and for any càdlàg rough paths $\mathbf{X} := (X, \mathbb{X})$, $\mathbf{Z} = (Z, \mathbb{Z})$, with finite p-variation, we set

$$\sigma_{p,[0,T]}(\mathbf{X}, \mathbf{Z}) := \inf_{\lambda \in \Lambda} \{ | \lambda | \vee \| \mathbf{X} \circ \lambda; \mathbf{Z} \|_{p,[0,T]} \}$$

where we define

$$\| \mathbf{Y}; \mathbf{Z} \|_{\infty,[0,T]} := \| Y - Z \|_{\infty,[0,T]} + \| \mathbb{Y} - \mathbb{Z} \|_{\infty,[0,T]} .$$

Observation. Note that by Chen's relation (4.1)

$$\mathbb{X}_{s,t} = \mathbb{X}_{0,t} - \mathbb{X}_{0,s} - X_{0,s} \otimes X_{s,t} \quad \text{whenever} \quad 0 \leq s \leq t \leq T$$

and therefore

$$\begin{aligned}
\| \mathbb{X} - \mathbb{Z} \|_{\infty,[0,T]} &= \sup_{0 \leq s < t \leq T} | \mathbb{X}_{s,t} - \mathbb{Z}_{s,t} | \\
&\lesssim \| \mathbb{X}_{0,\cdot} - \mathbb{Z}_{0,\cdot} \|_{\infty,[0,T]} \\
&\quad + (\| X_{0,\cdot} \|_{\infty,[0,T]} \vee \| Z_{0,\cdot} \|_{\infty,[0,T]}) \| X_{0,\cdot} - Z_{0,\cdot} \|_{\infty,[0,T]} .
\end{aligned}$$

This means that it is sufficient to show the convergence in distribution in the uniform, respectively Skorokhod, topology of the processes $(X_{0,\cdot}^n, \mathbb{X}_{0,\cdot}^n)$ in order to obtain the convergence of $(X^n, \mathbb{X}^n)$ in distribution in the uniform, respectively Skorokhod, topology.

Moreover, by Lemma 8.16 in [9] which states the usual interpolation estimate for the homogeneous metric d_{p-var}, it follows

Lemma 4.6. *Fix $p \in (2, 3)$. Let $\mathbf{X}^n = (X^n, \mathbb{X}^n)$ and $\mathbf{X} = (X, \mathbb{X})$ be rough paths with uniformly bounded p-variation, then, for any $p' > p$, there exists $C = C(p, p')$ such that*

$$\| \mathbf{X}^n; \mathbf{X} \|_{p',[0,T]} \leq C L^{\frac{p}{p'}} \| \mathbf{X}^n; \mathbf{X} \|_{\infty,[0,T]}^{1-\frac{p}{p'}} .$$

As a consequence, assuming that $\mathbf{X}^n = (X^n, \mathbb{X}^n)$ converges in distribution to $\mathbf{X} = (X, \mathbb{X})$ in the uniform, respectively Skorokhod topology and that the family $(\| (X^n, \mathbb{X}^n) \|_{p,[0,T]})_n$ is tight, then $(X^n, \mathbb{X}^n)$ converges in distribution to $(X, \mathbb{X})$ in the p'-variation uniform, respectively Skorokhod topology for all $p' \in (p, 3)$.

Now we can focus only on semimartingales X, that we want to lift to a rough path

$$\mathbf{X} = \left(X, \int X^- \otimes dX \right).$$

We will call $\mathbf{X}$ the Itô lift.

Theorem 4.7. *With probability one, the Itô lift $\mathbf{X}$ of a $\mathbb{R}^d$-valued r.c.l.l. semimartingale X is a r.c.l.l. p-rough path, with $p \in (2,3)$.*

Furthermore, taking into account Rough Differential Equations and Stochastic Differential Equations, one has the following equivalence driven by semimartingales, proved in [10].

Proposition 4.8. *Assume X is a semimartingale and $\mathbf{X} = \left(X, \int X^- \otimes dX \right)$ is its Itô lift. Let $F \in C_b^3$. Then the solution for the RDE*

$$dY_t = F(Y_t)^l d\mathbf{X}_t. \qquad Y_0 = y_0,$$

is equivalent with probability one to the solution of the Itô SDE

$$dY_t = F(Y_t^-)dX_t, \qquad Y_0 = y_0.$$

As a consequence of the previous proposition, we get

Theorem 4.9. *Consider the rough path equation with solution Y:*

$$dY_t = F(Y_{t-})d(X,\mathbb{X})_t, \qquad Y_0 = y_0.$$

If X is a semimartingale and $\mathbb{X}_{s,t} = \int_s^t X_{r-} \otimes dX_r + \Gamma(t-s)$ for $\Gamma \in \mathbb{R}^{d \times d}$, then

$$dY_t = F(Y_{t-})dX_t + \sum_{i,j=1}^{d} (DF(Y_{t-})F(Y_t))_{i,j}\Gamma_{i,j}dt.$$

BDG inequality and UCV condition   In this paragraph, following the results in [5], we will see an extension of Lépingle's Burkhölder-Davis-Gundy inequality to our context. The BDG inequality, applied for continuous local martingale rough paths, turns out to be a fundamental tool, together with the *uniformly controlled variation* (UCV), to derive basic convergence theorems for stochastic integrals in Skorokhod topology.

Consider a $\mathbb{R}^d$-valued semimartingale $(X_t, \mathcal{F}_t)$, $t \geq 0$, such that $X_0 = 0$ and the quadratic variation is denoted by $\langle X \rangle$. We have the following inequality in L_p spaces:

Lemma 4.10. *For $1 \leq p < \infty$, there exist c_p, C_p, positive constants depending only on p, such that*

$$c_p \mathbb{E}[\langle X \rangle^p] \leq \mathbb{E}\big[\sup_{0 \leq s \leq t} \mid X_s \mid^p \big] \leq C_p \mathbb{E}[\langle X \rangle^p].$$

This inequality has been proved in three steps: Burkhölder in [2] proved the cases $1 < p < \infty$, Burkhölder and Gundy in [3] the cases $0 \leq p < 1$ for a large class of martingales and Davis [6] the case $p = 1$ for all martingales.

From [5] we recall the next lemma, which is crucial in establishing finite p-variation of the lift of a local martingale, for the proof see [5, Lemma 4.2].

Lemma 4.11. *For every $2 < q < p < r$ there exists $C = C(p, q, r)$ such that for every $\mathbb{R}^d$-valued martingale X*

$$\mathbb{E}\big[\ \|\ \mathbb{X}\ \|_p^p\ \big] \leq C\mathbb{E}\big[\ |\ \langle X\rangle_\infty\ |^{q/2}\ \big] + C\mathbb{E}\big[\ |\ \langle X\rangle_\infty\ |^{r/2}\ \big].$$

Corollary 4.12. *For every $\mathbb{R}^d$-valued semimartingale X, $p > 2$ and $T > 0$, it holds that a.s.*

$$\|\ \mathbb{X}\ \|_{p,[0,T]} < \infty,$$

so $\mathbb{X}$ has finite p-variation.

Recall that a function $F : [0, \infty) \to [0, \infty)$ is called moderate if it is continuous, increasing $F(x) = 0$ if and only if $x = 0$, and there exists $c > 0$ such that $F(2x) \leq cF(x)$ for all $x > 0$. Now we can state the p-variation rough path BDG inequality which will be used in the next sections.

Theorem 4.13. *For every convex moderate function F and $p > 2$ there exists $c, C > 0$ such that for every $\mathbb{R}^d$-valued local martingale X*

$$c\mathbb{E}\big[F(|\ \langle X\rangle_\infty\ |^{1/2})\big] \leq \mathbb{E}\big[F(\|\ \mathbb{X}\ \|_p)\big] \leq C\mathbb{E}\big[F(\|\|\ \langle X\rangle_\infty\ |^{1/2})\big].$$

As an application of the BDG inequality, we obtain a convergence criterion for lifted local martingales in the rough path space. We first recall, from [5], the uniformly controlled variation (UCV) condition for a sequence of semimartingales.

Definition 4.14. *A sequence of semimartingales $(X^n)_{n \geq 1}$ satisfies the UCV condition if there exists $\delta > 0$ such that for all $\alpha > 0$ there exist decompositions $X^{n,\delta} = M^{n,\delta} + K^{n,\delta}$ and stopping times $\tau^{n,\alpha}$ such that for all $t \geq 0$*

$$\sup_{n \geq 1} \mathbb{P}[\tau^{n,\alpha} \leq \alpha] \leq \frac{1}{\alpha} \quad \text{and}$$

$$\sup_{n \geq 1} \mathbb{E}\big[\langle M^{n,\delta}\rangle_{t \wedge \tau^{n,\alpha}} + |\ K^{n,\delta}\ |_{1-var,[0,t \wedge \tau^{n,\alpha}]}\ \big] < \infty.$$

Now we can state the main result that allows us to pass from convergence in the Skorokhod topology to convergence in rough path topology.

Theorem 4.15. *Let $(X^n)_{n \geq 1}$ be a sequence of semimartingales such that X^n converges in law (respectively in probability) to a semimartingale X in the Skorokhod topology. Assume that $(X^n)_{n \geq 1}$ satisfies the UCV condition. Then the lifted processes $(\mathbb{X}^n)_{n \geq 1}$ converge in law (resp. in probability) to the lifted process $\mathbb{X}$ in the Skorokhod space $D([0, T], E)$. Moreover, for every $p > 2$, $(\|\ \mathbb{X}^n\ \|_p)_{n \geq 1}$ is a tight sequence of real random variables.*

4.2   Convergence in Rough Path Topology

As before, the apporach here is based on decomposing the position of the tagged particle, that we will now denote X_t, as a martingale plus an additive functional of a certain Markov process. Therefore, recalling (3.1), we have

$$X_t := Z_t - (1-\alpha)tm = \sum_{z \in \mathbb{Z}_*^d} zM_t^z + \int_0^t \widetilde{V}(\xi_s)ds =: M_t + Y_t.$$

We also define the rescaled process

$$X_t^n = \frac{1}{\sqrt{n}}X_{nt} = M_t^n + Y_t^n, \tag{4.4}$$

with the obvious definition of the rescaled processes M^n and Y^n. Moreover, from (4.2), we have

$$\mathbb{X}_{s,t} = \int_s^t X_{s,r} \otimes dX_r \quad \text{and so} \quad \mathbb{X}_{s,t}^n = \int_s^t X_{s,r}^n \otimes dX_r^n. \tag{4.5}$$

The main idea is to apply the martingale central limit theorem to M^n and the invariance principle for additive functionals to Y^n and the result that we aim to obtain is the following invariance principle.

Theorem 4.16. *The process $(X^{\lambda,n}, \mathbb{X}^{\lambda,n})$ converges in distribution in the p-variation Skorokhod topology to*

$$\left(B, \left(\int_0^t B_s \otimes dB_s + \frac{1}{2}\langle B, B \rangle_t + \Gamma t \right)_{t \geq 0} \right),$$

where B is a Browninan motion with covariance

$$
\begin{aligned}
\langle B, B \rangle_t = {}& t \lim_{\lambda \to 0} \sum_{z \in \mathbb{Z}_*^d} \int \left[p(z)[1 - \xi(z)][z + \Psi_z^\lambda(\xi)]^2 \right] \upsilon_\alpha^*(d\xi) \\
& + t \lim_{\lambda \to 0} \sum_{x,y \in \mathbb{Z}_*^d} \int \left[p(y-x)\Psi_{x,y}^\lambda(\xi)^2 \xi(x)[1 - \xi(y)] \right] \upsilon_\alpha^*(d\xi)
\end{aligned}
\tag{4.6}
$$

and where

$$\Gamma = -\frac{1}{2} \sum_{z \in \mathbb{Z}_*^d} \int \left[z^2 p(z)[1 - \xi(z)] \right] d\upsilon_\alpha^*.$$

With this we want to show the convergence in rough path topology. Recalling Lemma 4.6, the proof can be separated in two problems: identifying the limit of the sequence of processes $(X^{\lambda,n}, \mathbb{X}^{\lambda,n})$ and showing the tightness of $(X^{\lambda,n}, \mathbb{X}^{\lambda,n})$ in the p-variation Skorokhod topology.

4.2.1 Identification of the Limit

First of all we generalize Theorem 3.1 to obtain the following:

Lemma 4.17. *Under the probability measure $\mathbb{P}_{\upsilon_\alpha^*}$, the pair (M^n, Y^n) converges in distribution in the Skorokhod topology on $D(\mathbb{R}_+, \mathbb{R}^{2d})$ to a 2d-dimensional Brownian motion (B^M, B^Y) with covariance*

$$
\begin{aligned}
\langle B^M + B^Y, B^M + B^Y \rangle_t &= t \lim_{\lambda \to 0} \sum_{z \in \mathbb{Z}_*^d} \int [p(z)[1 - \xi(z)][z + \Psi_z^\lambda(\xi)]^2] \upsilon_\alpha^*(d\xi) \\
&\quad + t \lim_{\lambda \to 0} \sum_{x,y \in \mathbb{Z}_*^d} \int [p(y-x)\Psi_{x,y}^\lambda(\xi)^2 \xi(x)[1 - \xi(y)]] \upsilon_\alpha^*(d\xi).
\end{aligned}
\tag{4.7}
$$

In order to understand the joint convergence of (M^n, Y^n) we must consider the predictable quadratic covariation between M^n and the martingale that results from the decomposition of the additive functional, as seen in Section 3.2. That is the sum of a martingale m_t^λ and a neglegible term R_t^λ:

$$
\int_0^t \widetilde{V}(\xi_s) ds = m_t^\lambda + u_\lambda(\xi_0) - u_\lambda(\xi_t) + \lambda \int_0^t u_\lambda(\xi_s) ds =: m_t^\lambda + R_t^\lambda,
\tag{4.8}
$$

where

$$
m_t^\lambda = u_\lambda(\xi_t) - u_\lambda(\xi_0) - \int_0^t (\mathscr{L} u_\lambda)(\xi_s) ds
$$

is a martingale in $L^2(\mathbb{P}_{\upsilon_\alpha^*})$ with $m_0^\lambda = 0$. We write $m_t^{\lambda,n} := \frac{1}{\sqrt{n}} m_{nt}^\lambda$ and $R_t^{\lambda,n} := \frac{1}{\sqrt{n}} R_{nt}^\lambda$ Since m_t^λ can be expressed in terms of elementary martingales which are orthogonal, it is easy to compute the predictable quadratic covariation of the martingales $M_j(t) + m_j^\lambda(t)$. In order to do that, we need to recall some of the previous definitions.

While $M_{t,j} = \sum_{z \in \mathbb{Z}_*^d} z \cdot M_{t,j}^z$, the martingale m_j^λ, expressed in terms of the elementary martingales M^z and $M^{x,y}$, is given in (3.9), as follows:

$$
m_j^\lambda = \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t \Psi_{x,y}^\lambda(\xi_s) dM_{s,j}^{x,y} + \sum_{z \in \mathbb{Z}_*^d} \int_0^t \Psi_z^\lambda(\xi_s) dM_{s,j}^z,
$$

where $\Psi_{y,x}^\lambda = T^{x,y} u_\lambda$, $\Psi_z^\lambda = T^z u_\lambda$. Moreover, recalling Lemma 2.3, M_t^z and $M_t^{x,y}$ are orthogonal martingales with quadratic variation given by

$$
\langle M^z \rangle_t = \int_0^t p(z)[1 - \xi_s(z)] ds
$$

$$
\langle M^{x,y} \rangle_t = \int_0^t p(y-x)\xi_s(x)[1 - \xi_s(y)] ds
$$

Now, we can finally compute the covariation using the polarization identity,

$$
\langle M_j + m_j^\lambda, M_i + m_i^\lambda \rangle_t = \frac{1}{2} \left(\langle M_j + m_j^\lambda + M_i + m_i^\lambda \rangle_t - \langle M_j + m_j^\lambda \rangle_t - \langle M_i + m_i^\lambda \rangle_t \right)
$$

Then, with a straight forward computation, the only remaining terms are the covariations

$$\langle M_j, M_i \rangle_t + \langle M_j, m_i^\lambda \rangle_t + \langle m_j^\lambda, M_i \rangle_t + \langle m_j^\lambda, m_i^\lambda \rangle_t. \tag{4.9}$$

Taking into account the quadratic variations for the elementary martingales and the fact that they are orthogonal, for the first term in (4.9) we get

$$\langle M_j, M_i \rangle_t = \sum_{z \in \mathbb{Z}_*^d} \int_0^t (e_j \cdot e_i) z^2 p(z)[1 - \xi_s(z)] ds,$$

while the second and the third ones yield

$$\langle M_j, m_i^\lambda \rangle_t + \langle M_i, m_j^\lambda \rangle_t = 2 \sum_{z \in \mathbb{Z}_*^d} \int_0^t (e_j \cdot e_i) z \Psi_z^\lambda(\xi_s) p(z)[1 - \xi_s(z)] ds$$

and last one gives

$$\langle m_j^\lambda, m_i^\lambda \rangle = \sum_{z \in \mathbb{Z}_*^d} \int_0^t (e_j \cdot e_i) \Psi_z^\lambda(\xi_s)^2 p(z)[1 - \xi_s(z)] ds$$
$$+ \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t (e_j \cdot e_i) \Psi_{x,y}^\lambda(\xi_s)^2 p(y-x) \xi_s(x)[1 - \xi_s(y)] ds.$$

In conclusion, altogether

$$\langle M_j + m_j^\lambda, M_i + m_i^\lambda \rangle_t = \sum_{z \in \mathbb{Z}_*^d} \int_0^t (e_j \cdot e_i) p(z)[1 - \xi_s(z)][z + \Psi_z^\lambda(\xi_s)]^2 ds$$
$$+ \sum_{x,y \in \mathbb{Z}_*^d} \int_0^t (e_j \cdot e_i) p(y-x) \Psi_{x,y}^\lambda(\xi_s)^2 \xi_s(x)[1 - \xi_s(y)] ds. \tag{4.10}$$

From the predictable quadratic covariation, we derive the covariance matrix $\sigma^2 = \{\sigma_{i,j}^2 : 1 \leq i, j \geq d\}$ given by

$$\sigma_{i,j}^2 = \lim_{\lambda \to 0} \sum_{z \in \mathbb{Z}_*^d} \int [(e_j \cdot e_i) p(z)[1 - \xi(z)][z + \Psi_z^\lambda(\xi)]^2] v_\alpha^*(d\xi)$$
$$+ \lim_{\lambda \to 0} \sum_{x,y \in \mathbb{Z}_*^d} \int [(e_j \cdot e_i) p(y-x) \Psi_{x,y}^\lambda(\xi)^2 \xi(x)[1 - \xi(y)]] v_\alpha^*(d\xi).$$

Note that, when $i = j$, we get the variance (3.13) of the random Gaussian vector defined by the Central Limit Theorem 3.1, otherwise it is equal to zero. So this formula coincides with the (4.7) of Lemma 4.17.

At this point, we work with the decomposition of the position of the tagged particle as a martingale and an additive functional, seen at the beginning of the section:

$$X_t = M_t + Y_t, \tag{4.11}$$

where

$$M_t = \sum_{z \in \mathbb{Z}_*^d} z M_t^z \qquad \text{and} \qquad Y_t = \int_0^t \widetilde{V}(\xi_s) ds.$$

Firstly, our aim is to show a scaling limit for the additive functional seen as rough path, namely

$$Y_{s,t}^n := \frac{1}{\sqrt{n}} \int_{ns}^{nt} \widetilde{V}(\xi_r) dr, \qquad \mathbb{Y}_{s,t}^n := \int_s^t Y_{s,r}^n \otimes dY_r^n. \qquad (4.12)$$

And later, we will strengthen Lemma 4.17 to derive the joint convergence of $(M^n, Y^n, \mathbb{Y}^n)$.

In order to obtain a convergence result for the additive functional, we must take into account its decomposition based on the resolvent equation, as we have already mentioned in section 3.2, even if without proving all the details. From now on we will write V, in place of $\widetilde{V}$ and we will denote $\mathscr{L}_S$ the symmetric part of the generator $\mathscr{L}$.

Lemma 4.18. *Consider $V \in L^2(\mathbb{P}_{v_\alpha^*}) \cap \mathscr{H}_{-1}$. For $\lambda > 0$, let u_λ be the solution of the resolvent equation $\lambda u_\lambda - \mathscr{L} u_\lambda = V$. Then*

$$\lambda \parallel u_\lambda \parallel_{v_\alpha^*}^2 + \parallel u_\lambda \parallel_1^2 \leq \parallel V \parallel_{-1}^2$$

and there exists a martingale m^λ with $m_0^\lambda = 0$ and with $\mathbb{E}_{v_\alpha^*}\big[\langle m^\lambda \rangle_t\big] = 2t \int (u_\lambda \otimes (-\mathscr{L}_S) u_\lambda) dv_\alpha^*$, such that

$$\int_0^t V(\xi_s) ds = u_\lambda(\xi_0) - u_\lambda(\xi_t) + \int_0^t \lambda u_\lambda(\xi_s) ds + m_t^\lambda =: R_t^\lambda + m_t^\lambda. \qquad (4.13)$$

We write $m_t^{\lambda,n} := n^{-1/2} m_{nt}^\lambda$ and $R_t^{\lambda,n} := n^{-1/2} R_{nt}^\lambda$ for the rescaled processes.

Proof. Applying the Dynkin's formula to the function u_λ, one defines the martingale m_t^λ as before:

$$m_t^\lambda = u_\lambda(\xi_t) - u_\lambda(\xi_0) - \int_0^t (\mathscr{L} u_\lambda)(\xi_s) ds$$

and gets

$$\int_0^t V(\xi_s) ds = \int_0^t [\lambda u_\lambda(\xi_s) ds - \mathscr{L} u_\lambda(\xi_s)] ds = \int_0^t \lambda u_\lambda(\xi_s) ds + m_t^\lambda - u_\lambda(\xi_t) + u_\lambda(\xi_0).$$

$\square$

Furthermore, with Lemma 3.2 we proved that the conditions in (3.5) hold; combining these two, the solution u_λ to the resolvent equation satisfies

$$\lim_{\lambda \to 0} \left(\sqrt{\lambda} \parallel u_\lambda \parallel_{v_\alpha^*} + \parallel u_\lambda - u \parallel_1 \right) = 0 \qquad (4.14)$$

for some u in $\mathscr{H}_1$. Now we get

Lemma 4.19. *If condition (4.14) holds, then there exist processes $R^n, m^n \in D(\mathbb{R}_+, \mathbb{R}^d)$ such that, for all $T > 0$ and $n \in \mathbb{N}$*

$$\limsup_{\lambda \to 0} \left\{ \mathbb{E}_{v_\alpha^*} \left[\sup_{t \leq T} \mid m_t^n - m_t^{\lambda, n} \mid^2 \right] + \mathbb{E}_{v_\alpha^*} \left[\sup_{t \leq T} \mid R_t^n - R_t^{\lambda, n} \mid^2 \right] \right\} = 0.$$

Moreover, m^n is a martingale with

$$\mathbb{E}_{v_\alpha^*} \left[\langle m^n \rangle_t \right] = 2t \lim_{k \to 0} \int (u_k \otimes (-\mathscr{L}_S) u_k) dv_\alpha^*.$$

Proof. Fix $T > 0$. For $\lambda, \lambda' > 0$, since v_α^* is an invariant state and since $\mathbb{E}_{v_\alpha^*} \left[\langle m, m \rangle_t \right] = 2t \parallel u \parallel_1^2$, where m_t is the Dynkin martingale associated to a function u in the domain of a generator, we have that the expectation of the quadratic variation of the martingale $m_T^{\lambda, n} - m_t^{\lambda', n}$ is equal to

$$2T \langle (u_\lambda - u_\lambda'), (-\mathscr{L})(u_\lambda - u_{\lambda'}) \rangle_{v_\alpha^*} = 2T \parallel u_\lambda - u_{\lambda'} \parallel_1^2.$$

By the conditions (3.5), u_λ converges in $\mathscr{H}_1$. In particular, $m_T^{\lambda, n}$ is a Cauchy sequence in $L^2(\mathbb{P}_{v_\alpha^*})$ and converges to a certain m_T^n. One can choose a version of $\{ m_t^n : t \geq 0 \}$ that is a right continuous, square integrable martingale. This proves that, for all $T > 0$,

$$\lim_{\lambda \to 0} \mathbb{E}_{v_\alpha^*} \left[\sup_{t \leq T} (m_t^{\lambda, n} - m_t^n)^2 \right] = 0.$$

Moreover from this results and from the identity (4.13), it follows that $R_t^{\lambda, n}$ also converges in $L^2(\mathbb{P}_{v_\alpha^*})$ as $\lambda \downarrow 0$. Denoting this limit by $\{ R_t^n : t \geq 0 \}$, we prove the lemma. $\square$

We introduce here two general results that will be help in the proof of the following theorem.

Lemma 4.20. *Let $H \in \mathscr{H}_{-1} \cap L^2(\pi)$ and let A be a continuous process of finite variation. Then*

$$\mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t A_s H(\xi_s) ds \right| \right] <\sim \mathbb{E} \left[\sup_{t \leq T} \mid A_t \mid^2 \right]^{\frac{1}{2}} T^{\frac{1}{2}} \parallel H \parallel_{-1}.$$

In particular, for $A_t = \int_0^t G(\xi_s) ds$, with $G \in \mathscr{H}_{-1} \cap L^2(\pi)$

$$\mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t \int_0^s G(\xi_r) dr H(\xi_s) ds \right| \right] \lesssim T \parallel G \parallel_{-1} \parallel H \parallel_{-1}.$$

Lemma 4.21. *Let $(\xi_t)_{t \geq 0}$ be a stationary process with tracjectories in $D(\mathbb{R}_+, \mathbb{R}^m)$, such that $\mathbb{E} \left[\sup_{t \leq T} \mid \xi_t \mid \right] \leq CT$ for all $T > 0$ and such that $n^{-1} \xi_n \to a$ for some $a \in \mathbb{R}^m$, both a.s. and in L^1. Then we have for all $T > 0$*

$$\lim_{n \to \infty} \mathbb{E} \left[\sup_{t \leq T} \mid n^{-1} \xi_{nt} - at \mid \right] = 0.$$

We are now ready to prove an invariance principle for the additive functional Y_t in (4.11), seen as rough paths.

Theorem 4.22. *Let $p > 2$ and let υ_α^* be ergodic for $\mathscr{L}^*$. Assume $V \in L^2(\mathbb{P}_{\upsilon_\alpha^*}, \mathbb{R}^d) \cap \mathscr{H}_{-1}(\mathbb{R}^d)$ and that the solution u_λ of the resolvent equation satisfies (4.14) for some $u \in \mathscr{H}_1$. Then the process $(Y^n, \mathbb{Y}^n)$ converges in distribution in the uniform p-variation topology on $C(\mathbb{R}_+, \mathbb{R}^d \oplus \mathbb{R}^{d\otimes d})$ to*

$$\left(B_t^Y, \int_0^t B_s^Y \otimes dB_s^Y + \frac{1}{2}\langle B^Y, B^Y \rangle_t + \Gamma t \right)_{t\geq 0}, \tag{4.15}$$

where B^Y is a d-dimensional Brownian motion with covariance

$$2t \lim_{\lambda\to 0} \int [u_\lambda \otimes (-\mathscr{L}_S)u_\lambda]d\upsilon_\alpha^* = 2t \lim_{\lambda\to 0}\langle u_\lambda, \otimes u_\lambda\rangle_1 \tag{4.16}$$

and where

$$\Gamma = \lim_{\lambda\to 0}\Big(\int [u_\lambda \otimes (-\mathscr{L}_S)u_\lambda]d\upsilon_\alpha^* - \int [u_\lambda \otimes (-\mathscr{L})u_\lambda]d\upsilon_\alpha^*\Big). \tag{4.17}$$

Proof. Consider the decomposition $Y^n = m^n + R^n$ as above. Under the same assumptions of this theorem, one can assert that the laws of the processes $\left\{\frac{1}{\sqrt{n}}\int_0^{tn} V(\xi_s)ds : t \geq 0\right\}$ converge weakly in $C(\mathbb{R}_+, \mathbb{R}^d)$ as $n \uparrow \infty$, to the Wiener measure with zero mean and with covariance matrix $\sigma^2(V) = \{\sigma_{k,l}^2(V) : 1 \leq k, l \leq d\}$ such that

$$\sigma_{k,l}^2(V) = 2\lim_{\lambda\to 0}\langle u_{k,\lambda}, u_{l,\lambda}\rangle_1 = 2\langle u_k, u_l\rangle_1;$$

in the proof of this statement (see Theorem 2.32 of [13]) it is shown that both (m^n) and (Y^n) converge in distribution in $C(\mathbb{R}_+, \mathbb{R}^{2d})$ to a Brownian motion B with covariance

$$\langle B \rangle_t = 2t \lim_{\lambda\to 0} \int [u_\lambda \otimes (-\mathscr{L}_S)u_\lambda]d\upsilon_\alpha^*.$$

Therefore both Y^n and m^n are C-tight and so it is also R^n.

In Lemma 3.4 it is shown that the remainder term R_t^n vanishes, as $t \uparrow \infty$, in $L^2(\mathbb{P}_{\upsilon_\alpha^*})$, in particular that $\mathbb{E}_{\upsilon_\alpha^*}\big[\mid R_t^n \mid^2\big] \to 0$ for each $t \geq 0$. This, together with the C-tightness gives the convergence of R^n to zero in distribution in $C(\mathbb{R}_+, \mathbb{R}^d)$. Therefore, we have the joint convergence of (Y^n, m^n, R^n) in distribution in $C(\mathbb{R}_+, \mathbb{R}^{3d})$ to $(B, B, 0)$. Consequently, by Theorem 4.15 we can show the joint convergence

$$\left(Y^n, m^n, \int_0^{\cdot} m_s^n \otimes dY_s^n \right) \to \left(B, B, \int_0^{\cdot} B_s \otimes dB_s + \langle B, B\rangle \right).$$

It remains to study the term $\int_0^{\cdot} R_s^n \otimes dY_s^n$. We claim that for all $T > 0$

$$\lim_{n\to\infty}\mathbb{E}_{\upsilon_\alpha^*}\left[\sup_{t\leq T}\left| \int_0^t (R_s^n + n^{-1/2}u_{n^{-1}}(\xi_{ns})) \otimes dY_s^n \right|\right] = 0. \tag{4.18}$$

To prove the claim we decompose this limit in

$$\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|\int_0^t (R_s^n - R_s^{n^{-1},n})\otimes dY_s^n\right|\right]+$$

$$\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|\int_0^t (R_s^{n^{-1},n} + n^{-1/2}u_{n^{-1}}(\xi_{ns}))\otimes dY_s^n\right|\right]$$

(4.19)

Since $R_s^n - R_s^{n^{-1},n} = m_s^n - m_s^{n^{-1},n}$ and

$$\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}| m_t^n - m_t^{n^{-1},n} |^2\right] \lesssim \| u - u_{n^{-1}}\|_1^2 \to 0,$$

we can apply to the first term in (4.19) integration by parts together with the Burkhölder-Davis-Gundy inequality, see Lemma 4.10, to show that

$$\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|\int_0^t (R_s^n - R_s^{n^{-1},n})\otimes dY_s^n\right|\right]\to 0.$$

The remaining term in (4.19) involves only the continuous finite variation process $R_s^{n^{-1},n}+n^{-1/2}u_{n^{-1}}(\xi_{ns})$, so that we can apply Lemma 4.20 and obtain

$$\limsup_{n\to\infty}\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|\int_0^t (R_s^{n^{-1},n} + n^{-1/2}u_{n^{-1}}(\xi_{ns}))\otimes dY_s^n\right|\right]=$$

$$\limsup_{n\to\infty}\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|\frac{1}{\sqrt{n}}\int_0^{nt}(R_{n^{-1}s}^{n^{-1},n}+n^{-1/2}u_{n^{-1}}(\xi_s))\otimes V(\xi_s)ds\right|\right]\lesssim$$

$$\limsup_{n\to\infty}\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq nT}| R_{n^{-1}t}^{n^{-1},n}+n^{-1/2}u_{n^{-1}}(\xi_t)ds |^2\right]^{\frac{1}{2}}T^{\frac{1}{2}}\| V\|_{-1}.$$

To bound the expected value on the right hand side note that

$$\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}| R_t^{n^{-1},n}-n^{-1/2}u_{n^{-1}}(\xi_{nt}) |^2\right]\lesssim\mathbb{E}_{v_\alpha^*}\left[| n^{-1/2}u_{n^{-1}}(\xi_0) |^2\right]$$

$$+\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|n^{-1/2}\int_0^{nt}n^{-1}u_{n^{-1}}(\xi_s)ds\right|^2\right]$$

$$\lesssim n^{-1}\| u_{n^{-1}}\|_{L^2(\mathbb{P}_{v_\alpha^*})}^2$$

$$=(1+T^2)n^{-1}\| u_{n^{-1}}\|_{L^2(\mathbb{P}_{v_\alpha^*})},$$

and, according to (4.14), the right hand side vanishes for $n\to\infty$, thus we deduce (4.18).

Therefore, it is sufficient to study the limit of $\int_0^t n^{-1/2}u_{n^{-1}}(\xi_{ns})\otimes dY_s^n = n^{-1}\int_0^{nt} u_{n^{-1}}(\xi_{ns})\otimes V(\xi_{ns})ds$. Let $\lambda > 0$, then

$$\mathbb{E}_{v_\alpha^*}\left[\sup_{t\leq T}\left|n^{-1}\int_0^{nt}(u_{n^{-1}}(\xi_{ns}) - u_\lambda(\xi_{ns}))\otimes V(\xi_{ns})ds\right|\right]\leq T\int| (u_{n^{-1}}-u_\lambda)\otimes V | v_\alpha^*(d\xi)$$

$$\leq T\| u_{n^{-1}}-u_\lambda\|_1\| V\|_{-1}$$

and by assumption the right hand side converges to $T \parallel u - u_\lambda \parallel_1 \parallel V \parallel_{-1}$, which goes to zero for $\lambda \to 0$. Moreover by the ergodic theorem the term $n^{-1} \int_0^{nt} u_\lambda(\xi_{ns}) \otimes V(\xi_{ns}) ds$ converges almost surely and in $L^1(\mathbb{P})$ to $t \int [u_\lambda \otimes V] v_\alpha^*(d\xi)$. Through Lemma 4.21 one can prove that this convergence is even uniform in $t \in [0, T]$.

Now it suffices to send $\lambda \to 0$ to deduce that $\int_0^t R_s^n \otimes dY_s^n$ converges to the deterministic limit $-t \lim_{\lambda \to 0} \int [u_\lambda \otimes V] v_\alpha^*(d\xi)$ in $C(\mathbb{R}_+, \mathbb{R}^d)$. Consequently,

$$(Y^n, \mathbb{Y}^n) \to \left(B_t^Y, \int_0^t B_s^Y \otimes dB_s^Y + \langle B^Y, B^Y \rangle_t - \lim_{\lambda \to 0} \int [u_\lambda \otimes V] v_\alpha^*(d\xi) \right),$$

and finally, since $\lambda u_\lambda \to 0$ in $L^2(\mathbb{P}_{v_\alpha^*})$, we have

$$\lim_{\lambda \to 0} \int [u_\lambda \otimes V] v_\alpha^*(d\xi) = \lim_{\lambda \to 0} \int [u_\lambda \otimes (\lambda - \mathscr{L}) u_\lambda] v_\alpha^*(d\xi) = \lim_{\lambda \to 0} \int [u_\lambda \otimes (-\mathscr{L}) u_\lambda] v_\alpha^*(d\xi).$$

$\square$

We introduce, now, some useful definitions, that will lead us to prove the convergence result.

Definition 4.23. For $f \in D(\mathbb{R}_+, \mathbb{R}^d)$ and $r, T > 0$ we define the modulus of continuity

$$\omega_T(f, r) := \sup_{\substack{s, t \in [0, T]: \\ |s - t| \leq r}} | f(t) - f(s) |.$$

Definition 4.24. A sequence of processes $(X^n)_{n \in \mathbb{N}}$ in $D(\mathbb{R}_+, \mathbb{R}^d)$ is called C-tight if it is tight in the Skorokhod topology and all limit points are continuous processes.

We will use the following lemma [12]

Lemma 4.25. *The sequence* (X^n) *is* C-*tight if and only if the following two conditions hold:*

1. *for all* $T > 0$ *we have*

$$\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{t \in [0, T]} | X_t^n | > K \right) = 0;$$

2. *for all* $\epsilon, T > 0$ *we have*

$$\lim_{r \to 0} \limsup_{n \to \infty} \mathbb{P}(\omega_T(X^n, r) > \epsilon) = 0.$$

If (X^n) *is a sequence of processes in* $C(\mathbb{R}_+, \mathbb{R}^d)$, *then these two conditions are equivalent to tightness in the uniform topology.*

The uniform modulus of continuity is subadditive, indeed

$$\omega_T(f+g,r) \leq \omega_T(f,r) + \omega_T(g,r).$$

From this reason, it follows from the previous Lemma that the sum of two C-tight sequences is again C-tight. In general, this is not true for sequences that are tight in the Skorokhod topology on $D(\mathbb{R}_+, \mathbb{R}^d)$.

Using the arguments from the proof of Theorem 4.22, taking into account also the quadratic covariation between M^n and m_n^λ computed in (4.10), we can obtain the joint convergence

$$(M^n, Y^n, \mathbb{Y}_{0,\cdot}^n) \to \left(B^M, B^Y, \int_0^\cdot B_s^Y \otimes dB_s^Y + \frac{1}{2}\langle B^Y, B^Y \rangle\right).$$

Since the limit is continuous, the triple is even C-tight. So, by Lemma 4.25, also $X^{\lambda,n} = M^n + Y^n$ converges in distribution in the Skorokhod topology to $B = B^M + B^Y$, and the convergence holds jointly for $(M^n, Y^n, \mathbb{Y}_{0,\cdot}^n)$.

For the convergence of $\mathbb{X}^{\lambda,n}$ let us consider the iterated integrals of $X^{\lambda,n}$ given by

$$\mathbb{X}_{0,t}^{\lambda,n} = \int_0^t X_{s-}^n \otimes dM_s^n + \int_0^t M_{s-}^n \otimes dY_s^n + \mathbb{Y}_{0,t}^n. \tag{4.20}$$

We can show that M^n satisfies the UCV property in 4.14 and since Y^n is continuous and of finite variation, from the UCV Theorem 4.15, we get the joint convergence:

$$\left(M^n, Y^n, \mathbb{Y}_{0,\cdot}^n, X^{\lambda,n}, \int_0^\cdot X_{s-}^n \otimes dM_s^n, \int_0^\cdot M_{s-}^n \otimes dY_s^n\right)$$
$$\to \left(B^M, B^Y, \int_0^\cdot B_s^Y \otimes dB_s^Y + \frac{1}{2}\langle B^Y, B^Y \rangle, B, \int_0^\cdot B_s \otimes dB_s^M, \int_0^\cdot B_s^M \otimes dB_s^Y + \langle B^M, B^Y \rangle\right).$$

Since all the limiting processes are continuous the t-nuple is C-tight and the joint convergence extends to sums of the entries, so from (4.20) we get

$$(X^{\lambda,n}, \mathbb{X}_{0,\cdot}^{\lambda,n}) \to \left(B, \int_0^\cdot B_s \otimes dB_s + \frac{1}{2}\langle B^Y, B^Y \rangle + \langle B^M, B^Y \rangle\right)$$
$$= \left(B, \int_0^\cdot B_s \otimes dB_s + \frac{1}{2}\langle B, B \rangle - \frac{1}{2}\langle B^M, B^M \rangle\right),$$

where the equality simply follows considering $B = B^M + B^Y$. From the covariance of the Browninan motion B^M we derive the correction term on the right hand side:

$$t\Gamma = -\frac{1}{2}\langle B^M, B^M \rangle_t = -\frac{t}{2} \sum_{z \in \mathbb{Z}_*^d} \int_0^t z^2 p(z)[1-\xi_s(z)]ds = -\frac{t}{2} \sum_{z \in \mathbb{Z}_*^d} \int \left[z^2 p(z)[1-\xi(z)]\right] \upsilon_\alpha^*(d\xi). \tag{4.21}$$

To complete the proof it only remains to show tightness in p-variation.

4.2.2 Tightness in p-variation

We first want to show tightness of the additive functional $(Y^n, \mathbb{Y}^n)$ in the p-variation Skorokhod topology on $C(\mathbb{R}_+, \mathbb{R}^d \oplus \mathbb{R}^{d \otimes d})$. This holds by the following result.

Lemma 4.26. *If $(Y^n, \mathbb{Y}^n)_n$ is tight in $C(\mathbb{R}_+, \mathbb{R}^d \oplus \mathbb{R}^{d \otimes d})$ and if both the sequences of real valued random variables, $(\| Y^n \|_{p,[0,T]})_n$ and $(\| \mathbb{Y}^n \|_{p/2,[0,T]})$, are tight for every $T > 0$. Then, $(Y^n, \mathbb{Y}^n)_n$ is tight in the p-variation Skorokhod topology on $C(\mathbb{R}_+, \mathbb{R}^d, \mathbb{R}^{d \otimes d})$.*

For our purpose we will often use the following representation of additive functionals:

Lemma 4.27. *Let $H \in L^2(v_\alpha^*, \mathbb{R}^m)$ and let $\Psi \in C(\mathbb{R}^m)$. Then, for $T > 0$, we have*

$$\int_0^t H(\xi_s)ds = \int_0^t (H(\xi_s) - \mathscr{L}_S\Psi(\xi_s))ds + \frac{1}{2}(M_t^\Psi + \hat{M}_t^\Psi - \hat{M}_{T-t}^\Psi), \quad t \in [0,T] \quad (4.22)$$

where M^Ψ is a martingale and $\hat{M}^\Psi$ is a martingale with respect to the backward filtration $\hat{\mathcal{F}}_t = \sigma(\xi_{T-s} : s \leq t)$, such that, for $t \in [0,T]$,

$$\mathbb{E}_{v_\alpha^*}\big[\langle M^\Psi\rangle_t\big] = \mathbb{E}_{v_\alpha^*}\big[\langle \hat{M}^\Psi\rangle_t\big] = 2t \int [\Psi \otimes (-\mathscr{L}_S)\Psi] v_\alpha^*(d\xi) = 2t(\langle \Psi_k, \Psi_l\rangle_1)_{1 \leq k,l \leq m}. \quad (4.23)$$

Moreover, assume that v_α^* is ergodic for $\mathscr{L}^*$. Then under the rescaling $T \to nT$ and $M_t^{\Psi,n} = n^{-1/2}M_{nt}^\Psi$ and similarly for $\hat{M}^{\Psi,n}$ both processes converge in distribution in $D([0,T], \mathbb{R}^m)$ to a Wiener process, and by (4.23) they satisfy the UCV condition. If $G \in L^2(v_\alpha^*, \mathbb{R}^m)$ and $A_{s,t} = \int_s^t \int_s^{r_1} G(\xi_{r_2})dr_2 \otimes H(\xi_{r_1})dr_1$ for $0 \leq s < t \leq T$, then

$$A_{s,t} = \frac{1}{2}\int_s^t \int_s^{r_1} G(\xi_{r_2})dr_2 \otimes dM_{r_1}^\Psi - \frac{1}{2}\int_{T-t}^{T-s} \int_{T-r_1}^t G(\xi_{r_2})dr_2 \otimes \hat{M}_{r_1}^\Psi$$

$$+ \frac{1}{2}\int_s^t G(\xi_r)dr \otimes (\hat{M}_{T-s}^\Psi - \hat{M}_{T-t}^\Psi) + \int_s^t \int_s^{r_1} G(\xi_{r_2})dr_2 \otimes (H(\xi_{r_1}) - \mathscr{L}_S\Psi(\xi_{r_1}))dr_1.$$

$$(4.24)$$

Proof. The representation (4.22) is obtained by applying Dynkin's formula to $\Psi(\xi)$ on $[0,t]$ and to $\hat{\Psi}(\xi)$ on $[T-t,T]$, from where we also get (4.23). For the convergence of $M^{\Psi,n}$ and $\hat{M}^{\Psi,n}$ see the proof of Theorems 2.32 and 2.33 of [13]. The representation for $A_{s,t}$ follows by writing the integral against $\hat{M}_{T-}^\Psi$ as a limit of Riemann sums. Note that $\int_0^\cdot G(\xi_r)dr$ is continuous and of finite variation, so the integral is defined pathwise and we do not need to worry about quadratic covariations or the difference between forward and backward integral. $\square$

Under our assumptions, we can prove the first necessary condition to apply Lemma 4.26.

Lemma 4.28. *Let v_α^* be ergodic for $\mathscr{L}^*$ and let $V \in L^2(v_\alpha^*, \mathbb{R}^d) \cap \mathscr{H}_{-1}(\mathbb{R}^d)$. Assume that the solution of the resolvent equation, for some $u \in \mathscr{H}_1$ satisfies*

$$\lim_{\lambda \to 0} \big(\sqrt{\lambda} \| u_\lambda \|_{v_\alpha^*} + \| u_\lambda - u \|_1 \big) = 0.$$

Then the sequence $(Y^n, \mathbb{Y}^n)$ is tight in $C(\mathbb{R}_+, \mathbb{R}^d \oplus \mathbb{R}^{d \otimes d})$.

Proof. With these hypotheses one can generalize the central limit theorem for additive functionals, showing that $(Y^n)_n$ converges in distribution in $C(\mathbb{R}_+, \mathbb{R}^d)$ to a Wiener measure, [13, Thm 2.33]. Furthermore, recall from Observation 4.1 that in order to obtain the convergence of $(Y^n, \mathbb{Y}^n)$ is sufficient to prove the convergence of the processes $(Y^n_{0,\cdot}, \mathbb{Y}^n_{0,\cdot})$. So it is enough to show that $\mathbb{Y}^n_{0,\cdot}$ is tight in $C(\mathbb{R}_+, \mathbb{R}^{d\otimes d})$.

Let $u^V \in C(\mathbb{R}^d)$, then from (4.24) of Lemma 4.27, with V in place of F, we get

$$
\mathbb{Y}^n_{0,t} = \frac{1}{2}\int_0^t Y^n_s \otimes dM^{V,n}_s - \frac{1}{2}\int_{T-t}^T (Y^n_T - Y^n_{T-s}) \otimes d\hat{M}^V_s + \frac{1}{2}Y^n_T \otimes (\hat{M}^V_T - \hat{M}^V_{T-t})
$$
$$
+ \int_0^t Y^n_s \otimes \sqrt{n}(V(\xi_{ns}) - \mathscr{L}_S u^V(\xi_{ns}))ds.
$$

Note that $Y^n_T - Y^n_{T-s}$ is adapted to $\hat{F}_s$ and that the two stochastic integrals ar C-tight in $D([0,T], \mathbb{R}^d)$. The third term on the right hand side is C-tight by the characterization of Lemma 4.25. It remains to treat the last term. From the decomposition (4.11)

$$
\int_0^t Y^n_s \otimes \sqrt{n}(V(\xi_{ns}) - \mathscr{L}_S u^V(\xi_{ns}))ds
$$
$$
= \int_0^t \frac{1}{2}(M^{V,n}_s + \hat{M}^{V,n}_T - \hat{M}^{V,n}_{T-s}) \otimes \sqrt{n}(V(\xi_{ns}) - \mathscr{L}_S u^V(\xi_{ns}))ds
$$
$$
+ \int_0^t \frac{1}{\sqrt{n}} \int_0^{ns} (V(\xi_r) - \mathscr{L}_S u^V(\xi_r))dr \otimes \sqrt{n}(V(\xi_{ns}) - \mathscr{L}_S u^V(\xi_{ns}))ds.
$$

The integral against the two martingales can be handled as before, while the remaining term satisfies

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \left| \frac{1}{n}\int_0^{nt}\int_0^s (V(\xi_r) - \mathscr{L}_S u^V(\xi_r))dr \otimes (V(\xi_s) - \mathscr{L}_S u^V(\xi_s))ds \right| \right] \lesssim T \parallel V - \mathscr{L}_S u^V \parallel^2_{-1}
$$

by Lemma 4.20. By Chebyshev's inequality, we get

$$
\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(\sup_{t\in[0,T]} \mid \mathbb{Y}^n_{0,t} \mid > K \big)
$$
$$
\leq \lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| \frac{1}{n}\int_0^{nt}\int_0^s (V(\xi_r) - \mathscr{L}_S u^V(\xi_r))dr \otimes (V(\xi_s) - \mathscr{L}_S u^V(\xi_s))ds \right| > \frac{K}{2} \right) = 0
$$

and similarly

$$
\lim_{r\to 0} \limsup_{n\to\infty} \mathbb{P}(\omega_T(\mathbb{Y}^n_{0,\cdot}, r) > \epsilon)
$$
$$
\leq \lim_{r\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\omega_T\left(\frac{1}{n}\int_0^{n\cdot}\int_0^s (V(\xi_r) - \mathscr{L}_S u^V(\xi_r))dr \otimes (V(\xi_s) - \mathscr{L}_S u^V(\xi_s))ds, r \right) > \frac{\epsilon}{2} \right)
$$
$$
\lesssim \frac{2}{\epsilon}T \parallel V - \mathscr{L}_S u^V \parallel^2_{-1}.
$$

These arguments provide the conditions of Lemma 4.25, which are indeed equivalent to the tightness. But the set $\mathscr{L}_S C$ is dense in $\mathscr{H}_{-1}$, as seen in Claim E of Section 1.1, and

therefore we can make the right hand side arbitrarily small, i.e. it must be equal to zero. Hence $\mathbb{Y}_{0,\cdot}^n$ satisfies the assumptions of Lemma 4.25 and so it is tight in $C(\mathbb{R}_+, \mathbb{R}^{d \otimes d})$ and the proof is complete. $\qquad\square$

Now we recall the following estimate [11]:

Lemma 4.29. *Let $G \in L^2(\mathbb{P}_{v_\alpha^*}) \cap \mathscr{H}_{-1}$, $T > 0$ and $p > 2$. Then*

$$\mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t G(\xi_s)ds \right|^2 + \left\| \int_0^\cdot G(\xi_s)ds \right\|_{p,[0,T]}^2 \right] \lesssim T \parallel G \parallel_{-1}^2 .$$

In particular, this implies

$$\mathbb{E}\big[\parallel Y^n \parallel_{p,[0,T]}^2 \big] \lesssim T \parallel V \parallel_{-1}^2 . \tag{4.25}$$

To conclude and apply Lemma 4.26, we only need to bound $\parallel \mathbb{Y}^n \parallel_{p,[0,T]}$. For this purpose we use the following Proposition, whose proof is presented in the Appendix.

Proposition 4.30. *Let $(Y_t)_{t \in [0,T]}$ be a predictable r.c.l.l. process with $Y_0 = 0$ and such that $\mathbb{E}\big[\parallel Y \parallel_p^2 \big] < \infty$ for $p > 2$ and let $(N_t)_{t \in [0,T]}$ be a r.c.l.l. local martingale with $\mathbb{E}\big[\langle N \rangle_T\big] < \infty$. Define $A_{s,t} := \int_s^t Y_{s,r} dN_r$. Then for any $q > p$ and for all sufficiently small $\epsilon > 0$*

$$\mathbb{E}\big[\parallel A \parallel_{q/2,[0,T]}^{1-\epsilon} \big] \overset{\lesssim}{\sim} \big(\mathbb{E}\big[\parallel Y \parallel_{p,[0,T]}^2 \big]^{(1-\epsilon)^2/2} + \mathbb{E}\big[\parallel Y \parallel_{p,[0,T]}^2 \big]^{1/2} \big) \mathbb{E}\big[\langle N \rangle_T\big]^{(1-\epsilon)/2}$$
$$\overset{\lesssim}{\sim} \big(1 + \mathbb{E}\big[\parallel Y \parallel_{p,[0,T]}^2 \big]^{1/2} \big) \big(1 + \mathbb{E}\big[\langle N \rangle_T\big]^{1/2} \big).$$

Using this result, we are now ready to show the next Theorem, which give us the bound to obtain tightness of $(\parallel \mathbb{Z}^n \parallel_{p/2,[0,T]})_n$.

Theorem 4.31. *Let $G, H \in \mathscr{H}_{-1} \cap L^2(\mathbb{P}_{v_\alpha^*})$ and set $A_{s,t} = \int_s^t \int_s^{r_1} G(\xi_{r_2}) dr_2 H(\xi_{r_1}) dr_1$. Then we have for all $p > 2$, $T > 0$ and $\epsilon > 0$*

$$\mathbb{E}\big[\parallel A \parallel_{p/2,[0,T]}^{1-\epsilon} \big] \lesssim \big(1 + T^{1/2} \parallel G \parallel_{-1} \big) \big(1 + T^{1/2} \parallel H \parallel_{-1} \big).$$

As a corollary, setting $G = H = n^{-1/2}V$ and replacing T with nT, one obtains

$$\mathbb{E}\big[\parallel \mathbb{Y}^n \parallel_{p/2,[0,T]}^{1-\epsilon} \big] \lesssim \big(1 + (nT)^{1/2} \parallel n^{-1/2}V \parallel_{-1} \big)\big(1 + (nT)^{1/2} \parallel n^{-1/2}V \parallel_{-1} \big)$$
$$= \big(1 + T^{1/2} \parallel V \parallel_{-1} \big)^2. \tag{4.26}$$

So, for all n, there exists C such that

$$\mathbb{E}\big[\parallel \mathbb{Y}^n \parallel_{p/2,[0,T]}^{1-\epsilon} \big] \lesssim C. \tag{4.27}$$

and this shows that $(\parallel \mathbb{Y}^n \parallel_{p/2,[0,T]})_n$ is tight for all $T > 0$.

Proof. (of Theorem 4.31)
Lemma 4.27 shows that

$$A_{s,t} = \frac{1}{2} \int_s^t \int_s^{r_1} G(\xi_{r_2}) dr_2 dM_{r_1}^\Psi - \frac{1}{2} \int_{T-s}^{T-t} \int_{T_{r_1}}^t G(\xi_{r_2}) dr_2 d\hat{M}_{r_1}^\Psi$$

$$+ \frac{1}{2} \int_s^t G(\xi_r) dr (\hat{M}_{T-s}^\Psi - \hat{M}_{T-t}^\Psi) + \int_s^t \int_s^{r_1} G(\xi_{r_2}) dr_2 (H(\xi_{r_1}) - \mathscr{L}_S \Psi(\xi_{r_1})) dr_1.$$

The first two terms on the right hand side can be controlled applying Proposition 4.30, while the third term is bounded as follows

$$\left| \frac{1}{2} \int_s^t G(\xi_r) dr (\hat{M}_{T-s}^\Psi - \hat{M}_{T-t}^\Psi) \right| \lesssim \left\| \int_0^\cdot G(\xi_r) dr \right\|_{p,[s,t]} \| \hat{M}\Psi \|_{p,[T-t,T-s]},$$

and the fourth term is bounded by

$$\left| \int_s^t \int_s^{r_1} G(\xi_{r_2}) dr_2 (H(\xi_{r_1}) - \mathscr{L}_S \Psi(\xi_{r_1})) dr_1 \right|$$

$$\lesssim \sup_{r \in [s,t]} \left| \int_0^r G(\xi_{r_2}) dr_2 \right| \int_s^t | H(\xi_{r_1}) - \mathscr{L}_S \Psi(\xi_{r_1}) | dr_1.$$

So, by Proposition 4.30 together with the bounds just shown, we get

$$\mathbb{E}\big[\, \| A \|_{p/2,[0,T]}^{1-\epsilon} \, \big] \lesssim \left(1 + \mathbb{E}\Big[\Big\| \int_0^\cdot G(\xi_r) dr \Big\|_{p,[0,T]}^2\Big]^{1/2} \right) \left(1 + \mathbb{E}\big[\langle M^\Psi \rangle_T\big]^{1/2} + \mathbb{E}\big[\langle \hat{M}^\Psi \rangle_T\big]^{1/2} \right)$$

$$+ \mathbb{E}\Big[\Big\| \int_0^\cdot G(\xi_r) dr \Big\|_{p,[0,T]}^2\Big]^{(1-\epsilon)/2} \mathbb{E}\big[\, \| \hat{M}^\Psi \|_{p,[0,T]}^2 \, \big]^{(1-\epsilon)/2}$$

$$+ \mathbb{E}\Big[\sup_{r \in [0,T]} \Big| \int_0^r G(\xi_{r_2}) dr_2 \Big|^2 \Big]^{(1-\epsilon)/2} \mathbb{E}\Big[\Big(\int_0^T | H(\xi_{r_1}) - \mathscr{L}_S \Psi(\xi_{r_1}) | dr_1 \Big)^2 \Big]^{(1-\epsilon)/2}$$

$$\lesssim \left(1 + \mathbb{E}\Big[\Big\| \int_0^\cdot G(\xi_r) dr \Big\|_{p,[0,T]}^2\Big]^{1/2} \right) \left(1 + T^{1/2} \| \Psi \|_1 + T \| H - \mathscr{L}_S \Psi \|_{L^2(\mathbb{P}_{v_\alpha^*})} \right)$$

$$\lesssim \left(1 + T^{1/2} \| G \|_{-1} \right) \left(1 + T^{1/2} \| \Psi \|_1 + T \| H - \mathscr{L}_S \Psi \|_{L^2(\mathbb{P}_{v_\alpha^*})} \right),$$

where the last step follows from Lemma 4.27. Now we take $\Psi = u_\lambda^H$ as the solution to the Poisson equation $(\lambda - \mathscr{L}_S) u_\lambda^H = -H$, noting that in general $u_\lambda^H \notin \mathcal{C}$. But we can approximate u_λ^H with functions in $\mathcal{C}$ and get the same estimate. Then standard estimates for the solution of the resolvent equation, see equation (2.15) of [13], give

$$\| u_\lambda^H \|_1 + \sqrt{\lambda} \, \| u_\lambda^H \|_{L^2(\mathbb{P}_{v_\alpha^*})} \lesssim \| H \|_{-1},$$

and since $H - \mathscr{L}_S u_\lambda^H = \lambda u_\lambda^H$, we can send $\lambda \to 0$ to deduce the claimed estimate. $\square$

With this argument we have shown that $(\parallel Y^n \parallel_{p,[0,T]} + \parallel \mathbb{Y}^n \parallel_{p/2,[0,T]})_n$ is tight. Moreover, from the construction of the path, we have

$$\parallel X^{\lambda,n} \parallel_{p,[0,T]} \leq \parallel M^n \parallel_{p,[0,T]} + \parallel Y^n \parallel_{p,[0,T]} \tag{4.28}$$

and, by Lepingle p-variation BDG (4.13), we get $\mathbb{E}\big[\parallel M^n \parallel^2_{p,[0,T]} \big] \lesssim \mathbb{E}\big[\langle M^n \rangle_T\big] \lesssim 1$, while $\mathbb{E}\big[\parallel Y^n \parallel^2_{p,[0,T]} \big] \lesssim T \parallel V \parallel^2_{-1}$ by equation (4.25). For what concerns the sequence $(\mathbb{X}^{\lambda,n})_n$, we have

$$
\begin{aligned}
\parallel \mathbb{X}^{\lambda,n} \parallel_{p/2,[0,T]} &\leq \left\| \left(\int_s^t X^n_{r-,s-} \otimes dM^n_r \right)_{0 \leq s \leq t \leq T} \right\|_{p/2,[0,T]} \\
&+ \left\| \left(\int_s^t M^n_{r,s} \otimes dY^n_r \right)_{0 \leq s \leq t \leq T} \right\|_{p/2,[0,T]} + \parallel \mathbb{Y}^n \parallel_{p/2,[0,T]} .
\end{aligned} \tag{4.29}
$$

For the first term on the right hand side we apply Proposition 4.30:

$$
\begin{aligned}
\mathbb{E}\Big[\Big| \Big\| \Big(\int_s^t X^n_{r-,s-} \otimes dM^n_r \Big)_{0 \leq s \leq t \leq T} \Big\|^{1-\epsilon}_{p/2,[0,T]} \Big| \Big] &\lesssim \\
\big(1 + \mathbb{E}\big[\parallel X^n \parallel^2_{p',[0,T]} \big]^{1/2} \big) \big(1 + \mathbb{E}\big[\langle M^n \rangle_T\big]^{1/2} \big) &\lesssim 1,
\end{aligned}
$$

where $p' \in (2,p)$. The second term on the right hand side of (4.29) can be similarly controlled by Proposition 4.30 and via integration by parts. While for the third one, we already knew that $(\parallel \mathbb{Y}^n \parallel_{p/2,[0,T]})_n$ is tight. Hence, we get the tightness of $(\parallel \mathbb{X}^{\lambda,n} \parallel_{p/2,[0,T]})$.

With this argument we have completed the proof of Theorem 4.16, stating that under the p-variation Skorokhod topology our process $(X^n_{s,t}, \mathbb{X}^n_{s,t})$ converges to

$$\left(B, \left(\int_0^t B_s \otimes dB_s + \frac{1}{2}\langle B,B \rangle_t + \Gamma t \right)_{t \geq 0} \right).$$

So, the convergence of $\mathbb{X}^n_{s,t}$ to the Itô integral is corrected by a term of the form $\frac{1}{2}\langle B,B \rangle_t + \Gamma t$. This reflects the fact that we actually obtain a correction Γt to the Stratonovich integral

$$\int_0^t B_s \otimes \circ dB_s = \int_0^t B_s \otimes dB_s + \frac{1}{2}\langle B,B \rangle_t.$$

The Stratonovich integral is an alternative the Itô integral, defined such that the chain rule of ordinary calculus holds.

One should also note that in Theorem 4.16

$$\Gamma = -\frac{1}{2} \sum_{z \in \mathbb{Z}^d_*} \int \big[z^2 p(z)[1 - \xi(z)] \big] dv^*_\alpha,$$

so we get $\Gamma = 0$ if $p(z) = 0$ for all $z \in \mathbb{Z}^d_*$, but this represents one of the degenerate cases that we excluded, in order to be able to study the process.

Chapter 5

Appendix

We present here the proof of some results applied in the previous chapter.

Lemma 4.20 Let $H \in \mathscr{H}_{-1} \cap L^2(\pi)$ and let A be a continuous process of finite variation. Then

$$\mathbb{E}\left[\sup_{t \leq T}\left|\int_0^t A_s H(\xi_s)ds\right|\right] \lesssim \mathbb{E}\left[\sup_{t \leq T} | A_t |^2\right]^{\frac{1}{2}} T^{\frac{1}{2}} \parallel H \parallel_{-1} .$$

In particular, for $A_t = \int_0^t G(\xi_s)ds$, with $G \in \mathscr{H}_{-1} \cap L^2(\pi)$

$$\mathbb{E}\left[\sup_{t \leq T}\left|\int_0^t \int_0^s G(\xi_r)dr H(\xi_s)ds\right|\right] \lesssim T \parallel G \parallel_{-1} \parallel H \parallel_{-1} .$$

Proof. The second inequality follows from the first one together with the usual estimate of Lemma 4.29.

To show the first inequality, let $\Psi \in \mathcal{C}$ and apply Lemma 4.27:

$$\int_0^t A_s H(\xi_s)ds = \frac{1}{2}\int_0^t A_s dM_s^\Psi - \frac{1}{2}\int_{T-t}^T (A_T - A_{T-s})dr d\hat{M}_s^\Psi$$
$$+ \frac{1}{2}A_T(\hat{M}_T^\Psi - \hat{M}_{T-t}^\Psi) + \int_0^t A_s(H(\xi_s) - \mathscr{L}_S\Psi(\xi_s))ds,$$

since A in continuous and of finite variation, we can interpret the integrals against $\hat{M}^\Psi$ in a pathwise sense. Moreover, from the BDG and the Cauchy-Schwartz inequalities together with Lemma 4.29

$$\mathbb{E}\left[\sup_{t \leq T}\left|\int_0^t A_s H(\xi_s)ds\right|\right] \lesssim \mathbb{E}\left[\sup_{t \leq T} | A_t |^2\right]^{1/2} T^{1/2} \parallel \Psi \parallel_1$$
$$+ \mathbb{E}\left[\sup_{t \leq T}\left|\int_0^t A_s(H(\xi_s) - \mathscr{L}_S\Psi(\xi_s))ds\right|\right]$$
$$\lesssim \mathbb{E}\left[\sup_{t \leq T} | A_t |^2\right]^{1/2} (T^{1/2} \parallel \Psi \parallel_1 + T \parallel H - \mathscr{L}_S\Psi \parallel_{L^2(\pi)}).$$

By approximation one can take $\Psi = \Phi_\lambda^H$ as the solution to the Poisson equation $(\lambda - \mathscr{L}_S)\Phi_\lambda^H = -H$ and as in the proof of Theorem 4.31 we use that $\| \Phi_\lambda^H \|_1 \leq \| H \|_{-1}$ for all $\lambda > 0$ and that $\| H - \mathscr{L}_S\Phi_\lambda^H \|_{L^2(\pi)} \to 0$ as $\lambda \to 0$ to deduce the claimed estimate. $\square$

Lemma 4.21 Let $(\xi_t)_{t\geq 0}$ be a stationary process with tracjectories in $D(\mathbb{R}_+, \mathbb{R}^m)$, such that $\mathbb{E}\big[\sup_{t\leq T} |\xi_t|\big] \leq CT$ for all $T > 0$ and such that $n^{-1}\xi_n \to a$ for some $a \in \mathbb{R}^m$, both a.s. and in L^1. Then we have for all $T > 0$

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{t\leq T} | n^{-1}\xi_{nt} - at |\big] = 0.$$

Proof. This follows from an adaptation of the proof of Theorem 2.29 in [13]: we decompose

$$| n^{-1}\xi_{nt} - at | \leq \sup_{s\in[0,1]} \frac{|\xi_{\lfloor nt\rfloor + s} - \xi_{\lfloor nt\rfloor}|}{n} + \frac{\lfloor nt\rfloor}{n} \left|\frac{\xi_{\lfloor nt\rfloor}}{\lfloor nt\rfloor} - a\right| + | a | \left(t - \frac{\lfloor nt\rfloor}{n}\right).$$

The last term on the right hand side is bounded by $\frac{|a|}{n}$, while the first term on the right hand side is bounded for all $t \in [0, T]$ by

$$\sup_{s\in[0,1]} \frac{|\xi_{\lfloor nt\rfloor + s} - \xi_{\lfloor nt\rfloor}|}{n} \leq T \max_{k\leq\lfloor nT\rfloor} \frac{\sup_{s\in[0,1]} | Y_{k+s} - Y_k |}{\lfloor nT\rfloor},$$

and by Lemma 2.30 in [13] the right hand side vanishes as $n \to \infty$, both a.s. and in L^1. The last remaining term can be handled analogously to the proof of Theorem 2.29 in [13]. $\square$

Proposition 4.30 Let $(Y_t)_{t\in[0,T]}$ be a predictable r.c.l.l. process with $Y_0 = 0$ and such that $\mathbb{E}\big[\| Y \|_p^2\big] < \infty$ for $p > 2$ and let $(N_t)_{t\in[0,T]}$ be a r.c.l.l. local martingale with $\mathbb{E}\big[\langle N\rangle_T\big] < \infty$. Define $A_{s,t} := \int_s^t Y_{s,r} dN_r$. Then for any $q > p$ and for all sufficiently small $\epsilon > 0$

$$\mathbb{E}\big[\| A \|_{q/2,[0,T]}^{1-\epsilon}\big] \lesssim \big(\mathbb{E}\big[\| Y \|_{p,[0,T]}^2\big]^{(1-\epsilon)^2/2} + \mathbb{E}\big[\| Y \|_{p,[0,T]}^2\big]^{1/2}\big)\mathbb{E}\big[\langle N\rangle_T\big]^{(1-\epsilon)/2}$$
$$\lesssim \big(1 + \mathbb{E}\big[\| Y \|_{p,[0,T]}^2\big]^{1/2}\big)\big(1 + \mathbb{E}\big[\langle N\rangle_T\big]^{1/2}\big).$$

Proof. Let $n \in \mathbb{Z}$ and define the dyadic stopping times $(\tau_k^n)_{n,k\in\mathbb{N}_0}$, by $\tau_0^n := 0$ and

$$\tau_{k+1}^n := \inf\{t \leq \tau_k^n :| Y_t - Y_{\tau_k^n} |\leq 2^{-n}\},$$

and set $Y_t^n := \sum_k Y_{\tau_k^n} \mathbf{1}_{[\tau_k^n, \tau_{k+1}^n)}(t)$, so that $\| Y^n - Y \|_\infty \leq 2^{-n}$. We have

$$| A_{s,t} |\leq \int_s^t (Y_r - Y_r^n) dN_r + \left|\int_s^t Y_r^n dN_r - Y_s^n N_{s,t}\right| + | (Y_s^n - Y_s)N_{s,t} | . \qquad (5.1)$$

Now we can find an estimation for each term on the right hand side.

To bound the first term, let $\epsilon > 0$ and set

$$K = \sum_{n \in \mathbb{Z}} \min\{2^{2n(1-\epsilon)}, 2^{2n/(1-\epsilon)}\} \sup_{\tau \leq T} \left| \int_0^\tau (Y_r - Y_r^n) dN_r \right|^2.$$

Then

$$\left| \int_s^t (Y_r - Y_r^n) dN_r \right| \lesssim (2^{-n(1-\epsilon)} + 2^{-n/(1-\epsilon)}) K^{1/2}.$$

For the second term in (5.1), we can note that if there are no τ_k^n in (s,t), then $\left| \int_s^t Y_r^n dN_r - Y_s^n N_{s,t} \right| = 0$ and we are done. Otherwise, let $\tau_{k_0}^n, \ldots, \tau_{k_0+m}^n$ with $m \geq 0$ be those $(\tau_k^n)_k$ which are in (s,t), for $N \geq 2$, and note that

$$\left| \int_s^t Y_r^n dN_r - Y_s^n N_{s,t} \right| \leq \left| Y_{s,\tau_{k_0}^n}^n N_{\tau_{k_0}^n,\tau_{k_0+m}^n} \right|$$
$$+ \left| \int_{\tau_{k_0}^n}^{\tau_{k_0+m}^n} Y_r^n dN_r - Y_{\tau_{k_0}^n}^n N_{\tau_{k_0}^n,\tau_{k_0+m}^n} \right| + \left| Y_{s,\tau_{k_0+m}^n}^n N_{\tau_{k_0+m}^n,t} \right|$$
$$\leq c(s,t)^{1/p+1/r} + \left| \int_{\tau_{k_0}^n}^{\tau_{k_0+m}^n} Y_r^n dN_r - Y_s^n N_{\tau_{k_0}^n,\tau_{k_0+m}^n} \right| + c(s,t)^{1/p+1/r},$$

where c is a control function that controls both the p-variation of Y and the r-variation of N. We claim that

$$\left| \int_{\tau_{k_0}^n}^{\tau_{k_0+m}^n} Y_r^n dN_r - Y_{\tau_{k_0}^n}^n N_{\tau_{k_0}^n,\tau_{k_0+m}^n} \right| \lesssim m^{1-1/p-1/r} c(s,t)^{1/p+1/r}.$$

For $m = 0$ there is nothing to show, so let $m \geq 1$. We apply Young's maximal inequality despite the fact that the regularities of Y and N do not satisfy the compatibility condition that is necessary for the construction of the Young integral. The strategy of Young is to successively delete points $\tau_{k_0+l}^n$ from the partition of stopping times in order to pass from $\int_{\tau_{k_0}^n}^{\tau_{k_0+m}^n} Y_r^n dN_r$ to $Y_{\tau_{k_0}^n}^n N_{\tau_{k_0}^n,\tau_{k_0+m}^n}$. By super-additivity of c, there must exists $l \in \{1, \ldots, m\}$ for which $c(\tau_{k_0+l-1}^n, \tau_{k_0+l+1}^n) \leq \frac{2}{m} c(s,t)$. Deleting $\tau_{k_0+l}^n$ from the partition and subtracting the resulting integral from $\int_{\tau_{k_0}^n}^{\tau_{k_0+m}^n} Y_r^n dN_r$, we get

$$\left| Y_{\tau_{k_0+l-1}^n}^n N_{\tau_{k_0+l-1}^n,\tau_{k_0+l}^n} + Y_{\tau_{k_0+l}^n}^n N_{\tau_{k_0+l}^n,\tau_{k_0+l+1}^n} - Y_{\tau_{k_0+l-1}^n}^n N_{\tau_{k_0+l-1}^n,\tau_{k_0+l+1}^n} \right|$$
$$= \left| Y_{\tau_{k_0+l-1}^n,\tau_{k_0+l}^n,\tau_{k_0+l+1}^n}^n N_{\tau_{k_0+l}^n,\tau_{k_0+l+1}^n} \right|$$
$$\leq c(\tau_{k_0+l-1}^n, \tau_{k_0+l+1}^n)^{1/p+1/r} \leq \left(\frac{2}{m} c(s,t) \right)^{1/p+1/r}.$$

Successively deleting all points except $\tau_{k_0}^n = s$ and $\tau_{k_0+N}^n = t$ from the partition gives

$$\left| \int_{\tau_{k_0}^n}^{\tau_{k_0+m}^n} Y_r^n dN_r - Y_{\tau_{k_0}^n}^n N_{\tau_{k_0}^n,\tau_{k_0+m}^n} \right| \leq \sum_{k=1}^m \left(\frac{2}{k} c(s,t) \right)^{1/p+1/r} \lesssim m^{1-1/p-1/r} c(s,t)^{1/p+1/r},$$

which is the claimed inequality. Next we use the simple bound

$$m = \sharp\{k : \tau_k^n \in (\tau_{k_0}^n, \tau_{k_0+m}^n]\} \leq 2^{np} \parallel Y \parallel_{p,[\tau_{k_0}^n, \tau_{k_0+m}^n]}^p \leq 2^{np} c(s,t),$$

and therefore

$$\left| \int_s^t Y_r^n dN_r - Y_s^n N_{s,t} \right| \lesssim c(s,t)^{1/p+1/r} + 2^{n(p-1-p/r)} c(s,t).$$

Moreover, by replacing Y and N with $Y/ \parallel Y \parallel_{p,[0,T]}$ and $N/ \parallel N \parallel_{r,[0,T]}$ we get also

$$\left| \int_s^t Y_r^n dN_r - Y_s^n N_{s,t} \right| \lesssim c(s,t)^{1/p+1/r} \parallel Y \parallel_{p,[0,T]} \parallel N \parallel_{r,[0,T]}$$
$$+ 2^{n(p-1-p/r)} c(s,t) \parallel Y \parallel_{p,[0,T]}^{p-p/r} \parallel N \parallel_{r,[0,T]},$$

where c is now a control function with $c(0,T) \leq 1$.

For what concerns the last term in (5.1), it is bounded by $| (Y_s^n - Y_s)N_{s,t} | \leq 2^{-n} \parallel N \parallel_{r,[0,T]}$. The combination of the previous steps gives a bound for (5.1):

$$| A_{s,t} | \leq \int_s^t (Y_r - Y_r^n) dN_r + \left| \int_s^t Y_r^n dN_r - Y_s^n N_{s,t} \right| + | (Y_s^n - Y_s)N_{s,t} |$$
$$\lesssim (2^{-n(1-\epsilon)} + 2^{-n/(1-\epsilon)}) K^{1/2} + c(s,t)^{1/p+1/r} \parallel Y \parallel_{p,[0,T]} \parallel N \parallel_{r,[0,T]}$$
$$+ 2^{n(p-1-p/r)} c(s,t) \parallel Y \parallel_{p,[0,T]}^{p-p/r} \parallel N \parallel_{r,[0,T]} + 2^{-n} \parallel N \parallel_{r,[0,T]}$$

for a control function $c(0,T) \leq 1$. Choose now $n \in \mathbb{Z}$ such that $2^{n(p-1-p/r)} c(s,t) \parallel Y \parallel_{p,[0,T]}^{p-p/r}$ and 2^{-n} are of the same order, i.e. such that $2^{-n} \simeq c(s,t)^{1/(p-p/r)} \parallel Y \parallel_{p,[0,T]}$. Then we end up with

$$| A_{s,t} | \lesssim c(s,t)^{(1-\epsilon)/(p-p/r)} (\parallel Y \parallel_{p,[0,T]}^{1-\epsilon} + \parallel Y \parallel_{p,[0,T]}^{1/(1-\epsilon)}) K^{1/2}$$
$$+ c(s,t)^{1/p+1/r} \parallel Y \parallel_{p,[0,T]} \parallel N \parallel_{r,[0,T]}$$
$$+ c(s,t)^{1/(p-p/r)} \parallel Y \parallel_{p,[0,T]} \parallel N \parallel_{r,[0,T]} .$$

Since $q > p > 2$ there exist $r > 2$ and $\epsilon > 0$ such that

$$\frac{1-\epsilon}{p-p/r} = \frac{(1-\epsilon)\frac{r}{r-1}}{p} > \frac{2}{q} \quad \text{and} \quad \frac{1}{p} + \frac{1}{r} > \frac{2}{q},$$

which leads to

$$| A_{s,t} | \lesssim c(s,t)^{2/q} (\parallel Y \parallel_{p,[0,T]}^{1-\epsilon} + \parallel Y \parallel_{p,[0,T]}^{1/(1-\epsilon)}) (K^{1/2} + \parallel N \parallel_{r,[0,T]}), \qquad (5.2)$$

and then

$$\mathbb{E}\big[\parallel A \parallel_{q/2,[0,T]}^{1-\epsilon} \big] \lesssim \mathbb{E}\big[(\parallel Y \parallel_{p,[0,T]}^{(1-\epsilon)^2} + \parallel Y \parallel_{p,[0,T]}) (K^{1/2} + \parallel N \parallel_{r,[0,T]}) \big]$$
$$\lesssim (\mathbb{E}\big[\parallel Y \parallel_{p,[0,T]}^2 \big]^{(1-\epsilon)^2/2} + \mathbb{E}\big[\parallel Y \parallel_p^2 \big]^{1/2}) \mathbb{E}\big[K + \parallel N \parallel_{r,[0,T]}^2 \big]^{1/2}$$
$$\lesssim (\mathbb{E}\big[\parallel Y \parallel_p^2 \big]^{(1-\epsilon)^2/2} + \mathbb{E}\big[\parallel Y \parallel_p^2 \big]^{1/2}) \mathbb{E}\big[\langle N \rangle_T \big]^{(1-\epsilon)/2},$$

where in the last step we applied Lepingle's p-variation BDG inequality and we used that

$$\mathbb{E}[K] \lesssim \sum_{n \in \mathbb{Z}} \min\{2^{2n(1-\epsilon)}, 2^{2n/(1-\epsilon)}\} \mathbb{E}\left[\sup_{\tau \leq T} \int_0^\tau (Y_r - Y_r^n)^2 dN_r\right]$$
$$\leq \sum_{n \in \mathbb{Z}} \min\{2^{2n(1-\epsilon)}, 2^{2n/(1-\epsilon)}\} 2^{-2n} \mathbb{E}\left[\langle N \rangle_T\right] \lesssim \mathbb{E}\left[\langle N \rangle_T\right].$$

$\square$

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