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Dipartimento di Matematica "Tullio Levi-Civita"
Corso di Laurea Magistrale in Matematica

Noetherianity of representation categories with applications to configuration spaces of graphs

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15 Luglio 2020

Come ad osservare, immobili,
l'immobilitá di spazi vuoti.

Acknowledgements

First of all, I would like to express my full gratitude to my supervisor Prof. Márton Hablicsek for being always available for everything I needed even in the very unusual situation that we end up writing this thesis. I would also like to thank my friend Maël as working together really pushed me and i believe that this collaboration made this whole work more complete (not mentioning that made it less stressful!).
I also have to thank all the ALGaNT Consortium for the opportunity they gave me: these two years were great for a multitude of aspects that go from the academic to the more personal ones.
Speaking of people I have met, a special thanks goes to everybody that made this experience so amazing. Thanks to all the friends that I met during these two years both in Padova and in Leiden, you are too many to mention all, but you made the best of this experience. Un grazie di cuore va poi a tutti i miei "amici di sempre" che comunque sia (distanti e non) non hanno mai fatto mancare la loro presenza.
Last but not least (non trovo un' espressione che renda cosí bene anche in italiano), ringrazio chiaramente tutta la mia famiglia per il continuo e incessante supporto ricevuto durante tutti questi anni e che mi ha permesso di raggiungere questo importante traguardo.

Introduction

In this work we want to present the theory of representations of categories as in [11] and apply it to the study of some topological properties of configuration spaces of graphs. In the last chapter this theory is also applied when studying configuration spaces of trees with a group action.
This theory is quite new and traceback (around 2010) to the work about the so-called 'representation stability' due to Church, Ellenberg and Farb, whose context was essentially topological. Later S. Sam and A. Snowden (cfr. [11]) developed this big categorical machinery, namely the Gröbner theory for representations of categories, that N. Proudfoot and E.Ramos applied to study some representation stability problems related to the topology of configuration spaces of graphs (which is strongly related to the original work of Church, Ellenberg and Farb).
Essentially this is also our approach to the subject:

Chapter 1: In this chapter we introduce the concept of representation of a category and the category that such objects define. Of particular interest is the notion of Noetherianity for such representations, so the Gröbner theory for categories is introduced, as it gives sufficient conditions (of combinatorial nature) on the base category so that its representation category is Noetherian. This chapter is surrounded by several examples of both Noetherian and non-Noetherian categories of representations.

Chapter 2: The base setting of this chapter are graph categories, that are introduced at the beginning. We then apply the theory of chapter 1 to these categories in order to study some topological properties of configuration spaces of graphs. The other main tool used to pursue this study is the theory developed by J. Swiatkowksi and B.H. An, G. Drummond-Cole and B. Knudsen (cfr. [13],[1]) whose main result states that there exists an abstract bigraded differential module (called Swiatkowski complex, $\tilde{S}^{\bullet}(\_)$) built from the graph itself that gives the following (functorial) isomorphism:

$$H_{\star}(S^{\bullet}(G)) \cong H_{\star}(UConf_{\bullet}(G)).$$

Glueing together these two tools (Gröbner and Swiatkowski theories) we are able to give some interesting results about the topological nature of such configuration spaces. For example, we will show that often (really always, but we can't deduce it from this theory) homology groups of configuration spaces of (finite) trees are torsion free.

Chapter 3 In this last chapter we focus on trees with a group action. We will show that we can define a category out of this setting and that its representation category turns out to be Noetherian. A group action on a tree induces a group action also on the configuration spaces: we introduce the definition of orbit configuration space and we study some relations (in homology) between such spaces and their quotients. Again here the Swiatkowski theory will play a central role.

We conclude this work by leaving some open questions.

Contents

Chapter 1

Representations of categories

In this first chapter we introduce the concept of representation of a category (and consequently define the category of such representations) and the related notions of Noetherianity (both for a single representation and for the whole category of representations) as presented in [11]. Always following [11], we then introduce the Gröbner theory for categories which permits to give some sufficient conditions (of combinatorial nature) on a category $\mathcal{C}$ so that the representation category $Rep_k(\mathcal{C})$ is Noetherian. Several examples of both Noetherian and not Noetherian categories of representations are also presented in this chapter.

1.1 Generalities

Notation 1. We will use the following notation:

- Let $\mathcal{C}$ be a category. When writing $x \in \mathcal{C}$ we mean $x \in Ob(\mathcal{C})$ and analogously for morphisms (usually denoted by the letters f or g).

- Let $F : \mathcal{C} \to \mathcal{D}$ a functor. If $f : x_1 \to x_2$ is a morphism in $\mathcal{C}$, we call $F(f) : F(x_1) \to F(x_2)$ the image morphism in $\mathcal{D}$ via F.

- If not specified differently, all the rings (usually denoted by k) over which we construct the categories of representations are Noetherian.

- Let M be a representation of a category $\mathcal{C}$. We call such an M: representation, $\mathcal{C}$-module and often (when it's clear from the context) just module.

- If not specified differently, when writing $H_i(X)$ (for a space X) we always intend singular homology with coefficients in $\mathbb{Z}$.

Definition 1.1. Let $\mathcal{C}$ be a category.

- Let $|\mathcal{C}|$ be the set of isomorphism classes in $\mathcal{C}$, ie $|\mathcal{C}| = \{[x]_\sim : x \in \mathcal{C}\}$ where $x \sim y$ if there is an isomorphism $x \to y$.
- For any $x \in \mathcal{C}$, define $\mathcal{C}_x$ to be the category whose objects are morphisms $x \to y$ for some $y \in \mathcal{C}$ and morphisms the obvious commutative triangles:

$$\begin{tikzcd} x \arrow[r] \arrow[dr] & y \arrow[d] \\ & y' \end{tikzcd}$$

- A category $\mathcal{C}$ is called directed if there are no non-trivial self maps.

Lemma 1.1.1. *If the category $\mathcal{C}$ is directed then $|\mathcal{C}|$ is a poset. The partial order is given by: $[x] \leq [y]$ if there is a morphism $x \to y$.*

Proof. We have to check that this defines an order relation.

i) $[x] \leq [x]$, as there is always a unique map (the identity) $x \to x$;

ii) $[x] \leq [y]$ and $[y] \leq [x]$ means that there are maps $x \to y$ and $y \to x$. Their composition gives a map $x \to x$ that has to be the identity, hence $x \sim y$ and so $[x] = [y]$;

iii) $[x] \leq [y]$ and $[y] \leq [z]$ means that there are maps $x \to y \to z$, their composition gives a map $x \to z$, hence $[x] \leq [z]$. □

- Let k be a ring. A representation of $\mathcal{C}$ is a functor $M : \mathcal{C} \to Mod_k$. $Rep_k(\mathcal{C})$ is the category whose objects are such functors and morphisms are the natural transformations;
- Let $M \in Rep_k(\mathcal{C})$. An element of M means an element of $M(x)$ for some $x \in \mathcal{C}$. So an element of a $\mathcal{C}$-module is a k-module in the usual sense.
- For any $x \in \mathcal{C}$ let $P_x \in Rep_k(\mathcal{C})$ be the representation defined by: $P_x(y) := k[Hom_{\mathcal{C}}(x, y)]$. Here $k[Hom_{\mathcal{C}}(x, y)]$ is the free k-module generated by the elements of $Hom_{\mathcal{C}}(x, y)$. Such modules P_x are called pricipal projectives.

Remark 1. When considering principal projectives thank to Yoneda's lemma we have that: $Hom_{Rep_k(\mathcal{C})}(P_x, M) \cong M(x)$, for every other module M. This implies that the functor $Hom_{Rep_k(\mathcal{C})}(P_x, -)$ is exact, hence $P_x \in Rep_k(\mathcal{C})$ is a projective object. From here its name.

Definition 1.2. Let M be a representation and let S be a set of elements of M. If no subrepresentation of M contains S we say that M is generated by the set S. A representation is called finitely generated if it is generated by a finite set.

Definition 1.3 (Equivalent definition of finitely generated representation)**.** $M \in Rep_k(\mathcal{C})$ is finitely generated if $\exists\, x_1, .., x_n \in \mathcal{C}$ and $v_i \in M(x_i)$ such that $\forall x \in \mathcal{C}$ we have that $M(x)$ is spanned over k by the images of the v_i along morphisms $f_i : x_i \to x$.

Remark 2. The two definitions are really equivalent:

($\Downarrow$) Let M be the smallest representation that contains the set S, where $S = \{v_i \in M(x_i)\}_{i=1,...,n}$. Define a subrepresentation of M as:

$$M'(x) := k[\{M(f_{ij})(v_i)\}_{i=1,...,n}],$$

where $\{f_{ij}\}_j$ are all the possible morphims $x_i \to x$. Notice that if $v_i \in M(x_i)$ is an element of S, we have that $v_i \in M'(x_i)$ as $v_i = M(id_{x_i})(v_i)$. So, $S \subseteq M'$ and by the minimality of M we conclude $M = M'$.

($\Uparrow$) Let $S := \{v_i \in M(x_i)\}_{i=1,...,n}$. Suppose that there exists a subrepresentation N of M, such that $N \supseteq S$. This implies that $\forall\, x \in \mathcal{C}$, $N(x) \supseteq k[\{M(f_{ij})(v_i)\}_{i=1,...,n}]$, where $\{f_{i,j}\}_j$ are all the possible morphims $x_i \to x$. Hence we have $N(x) \supseteq M(x)$, contradiction.

Notice that P_x is a finitely generated representation, generated by one element: $id_x : x \to x$. For any morphism $f : x \to y$ let e_f be the corrisponding element in $k[Hom(x, y)]$.

Remark 3. A $\mathcal{C}$-module M is finitely generated if and only if it is a quotient of a finite direct sum of principal projectives. Indeed:

($\Rightarrow$) Consider the map $\pi : \bigoplus_{i=1}^n P_{x_i} \to M$, such that for $x \in \mathcal{C}$

$$\pi \colon \bigoplus_{i=1}^{n} P_{x_i}(x) \to M(x)$$
$$e_f \mapsto M(f)(v_i)$$

where $f : x_i \to x$. By the second definition of finitely generated representation this map is clearly surjective, realizing M as a subquotient of the sum.

($\Leftarrow$) Suppose we have the following epimorphism $\pi : \bigoplus_{i=1}^n P_{x_i} \twoheadrightarrow M$. Let $v_i \in M(x_i)$ the image via π of the identity morphism $x_i \to x_i$. By the surjectivity of π we have that $M(x)$ is generated by the images of v_i along the maps induced by all possible $\{x_i \to x\}_i$.

Notice that when considering functors $F, G : \mathcal{C} \to Mod_k$, to check that $f : F \to G$ is an epimorphism of functors it's enough to prove that $f(x) : F(x) \to G(x)$ are epimorphisms for any $x \in \mathcal{C}$. Viceversa, if $f : F \to G$ is an epimorphism of functors then so are $f(x) : F(x) \to G(x)$ for any x.

Definition 1.4. $M \in Rep_k(\mathcal{C})$ is Noetherian if every ascending chain of subobjects stabilizes (if and only if every subrepresentation is finitely generated). $Rep_k(\mathcal{C})$ is Noetherian if every finitely generated object in it is Noetherian itself.

Proposition 1.1.2. *$Rep_k(\mathcal{C})$ is Noetherian if and only if every principal projective P_x is Noetherian.*

Proof. If $Rep_k(\mathcal{C})$ is Noetherian so it is P_x, for every x. Conversely, let P_x be Noetherian for every x, M is finitely generated if and only if it is a subquotient of a finite direct sum of principal projectives. As this sum is finite, and all the principal projectives are Noetherian, so it is the sum itself. Noetherianity then descends to quotients. $\square$

Now we would like to introduce a particular property of a functor between two categories $\mathcal{C} \to \mathcal{C}'$ that links the Noetheriantiy of $Rep_k(\mathcal{C})$ and $Rep_k(\mathcal{C}')$.

Definition 1.5. Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a functor. We can define a pull-back functor of representations as follows:

$$\begin{aligned} \Phi^* \colon Rep_k(\mathcal{C}') &\to Rep_k(\mathcal{C}) \\ M &\longmapsto M \circ \Phi \end{aligned}$$

Definition 1.6. A functor $\Phi : \mathcal{C} \to \mathcal{C}'$ satisfies property (F) if for any $x \in \mathcal{C}'$ there are $y_1, .., y_n \in \mathcal{C}$ and maps $f_i : x \to \phi(y_i)$ such that for all $y \in \mathcal{C}$ and any $f : x \to \phi(y)$ there exists a map $g : y_i \to y$ such that $f = \Phi(g) \circ f_i$.

This definition may seem too abstract, but here a proposition to clarify what this property is telling us.

Proposition 1.1.3. *Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a functor that satisfies property (F). Then Φ^* sends finitely generated objects to finitely generated objects.*

Proof. It suffices to show that Φ^* pulls back principal projectives to finitely generated representations. Let $P_x \in Rep_k(\mathcal{C}')$, we need to show that $\Phi^*(P_x)$ is finitely generated, ie that there are $y_1, \ldots, y_n \in \mathcal{C}$ and $v_i \in \Phi(P_x)(y_i)$ such that for every $y \in \mathcal{C}$, $\Phi(P_x)(y)$ is generated by the images of v_i along the maps induced from all the possible $y_i \to y$. But this is true when choosing $\{y_i\}_{i=1}^n$ the same of the definition of property (F) and $\{v_i : x \to \Phi(y_i)\}_{i=1}^n$. Again from the definition of property (F): $\Phi(P_x)(y)$ is generated by all the $x \to \Phi(y)$, and we have that each such morphism is the image of a v_i along a $\Phi(P_x)(y_i) \to \Phi(P_x)(y)$. $\square$

Proposition 1.1.4. *Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be essentially surjective. If $M \in Rep_k(\mathcal{C}')$ is such that $\Phi^*(M) \in Rep_k(\mathcal{C})$ is finitely generated (resp. Noetherian), then M is finitely generated (resp. Noetherian)*

Proof. Let S be the finite set that generates $\Phi^*(M)$, and call S' the corrispondent set of elements of M. Let N be the subrepresentation of M generated by S', then $\Phi^*(N)$ is a subrepresentation of $\Phi^*(M)$ containing S. Hence: $\Phi^*(M) \subset \Phi^*(N)$, and from here the equality $\Phi^*(M) = \Phi^*(N)$. As Φ is essentially surjective then $M = N$, and so M is finitely generated. Now let $\Phi^*(M)$ be Noetherian, and let N be a subrepresentation of M. $\Phi^*(N)$ must be finitely generated, and so must be N. From here the Noetherianity. $\square$

Theorem 1.1.5. *Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be an essentially surjective functor that satisfies property (F). Then if $Rep_k(\mathcal{C})$ is Noetherian, so it is $Rep_k(\mathcal{C}')$.*

Proof. Let $M \in Rep_k(\mathcal{C}')$ be finitely generated. Then $\Phi^*(M)$ is finitely generated as well, and by the hypothesis also Noetherian. But this implies that M is Noetherian itself by the pevious proposition. From here the Noetherianity of $Rep_k(\mathcal{C}')$. $\square$

1.1.1 Examples of non Noetherianity

Now some examples of categories whose representation category is not Noetherian.

Proposition 1.1.6. *Let* **IFGrp** *be the category whose objects are finite groups and morphisms are injective group homomorphisms. Let k be a ring, then the category $Rep_k(\mathbf{IFGrp})$ is not Noetherian.*

Proof. Let $X_0 := \mathbb{Z}/2\mathbb{Z}$ and for any odd prime p let $X_p := \mathbb{Z}/2p\mathbb{Z}$, the cyclic group of order $2p$. Notice that we always have injective morphisms $X_0 \to X_p$ for any p, whereas there are no injective morphisms $X_{p_1} \to X_{p_2}$, for $p_1 < p_2$. Let M be the subrepresentation of P_{X_0} generated by the set $\bigcup_{p\ prime} P_{X_0}(X_p)$ and suppose by contradiction that it is finitely generated by $\{\alpha_j \in M(X_{p_j})\}_{j=1}^n$. As a consequence, any α_j is of the form:

$$\sum_{i=1}^{n} \sum_{f_{ij}:X_{p_i}\to X_{p_j}} M(f_{ij})(\beta_{f_{ij}}),$$

for $\beta_{f_{ij}} \in P_{X_0}(X_{p_i})$. This means that actually M is generated by the set $\bigcup_{i=1}^n P_{X_0}(X_{p_i})$. Now, let $\bar{p}$ be any prime bigger than $max\{p_j : j = 1, \dots, n\}$ and let $e_g \in M(X_{\bar{p}})$, for $g : X_0 \to X_{\bar{p}}$. As there are no injective morphisms $X_{p_j} \to X_{\bar{p}}$ for any $j = 1, \dots, n$, this element can't be in M, contradiction. So M is not finitely generated and $Rep_k(\mathbf{FGrp})$ is not Noetherian. $\square$

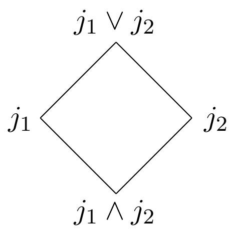
Proposition 1.1.7. *Let* **IFRng** *be the category whose objects are finite rings and morphisms are injective ring homomorphisms. Let k be a ring, then the category $Rep_k(\mathbf{IFRng})$ is not Noetherian.*

Proof. Let $X_0 := \mathbb{F}_2$ and for any odd prime p let $X_p := \mathbb{F}_{2^p}$, the finite field of order 2^p. Notice that we always have (injective) morphisms $X_0 \to X_p$ for any p, whereas there are no morphisms $X_{p_1} \to X_{p_2}$, for $p_1 < p_2$. Again, let M be the subrepresentation of P_{X_0} generated by the set $\bigcup_{p\ prime} P_{X_0}(X_p)$ and suppose it is finitely generated by some $\{\alpha_j \in M(X_{p_j})\}_{j=1}^{n}$. Analogously, we can say that M is generated also by the set $\bigcup_{i=1}^{n} P_{X_0}(X_{p_i})$ and choosing any $e_g \in M(X_{\bar{p}})$, for $g : X_0 \to X_{\bar{p}}$ and $\bar{p} > max\{p_j : j = 1, \dots, n\}$, we reach a contradiction. So, $Rep_k(\mathbf{IFRng})$ is not Noetherian. $\square$

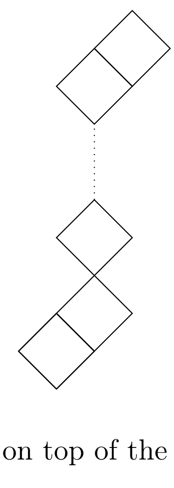
Definition 1.7. A distributive lattice is a triple $(L, \vee, \wedge)$ where L is a set and $\vee, \wedge$ are two binary operations that are commutative, associative, distributive one respect to the other and such that $a \vee a = a$ and $a \wedge a = a$, for any $a \in L$. A morphisms of distributive lattices is a set theoretic map that respects the two operations.

Consider the following distributive lattices (cfr. [5]):

- Let J be the lattice generated by two elements j_1, j_2:



- For any $n \geq 1$ let J_n be the lattice:



  with n vertical squares one on top of the other.

Remark 4. Notice that we can always embed J inside any J_n, but there are no possible embeddings $J_n \hookrightarrow J_m$, for $n < m$.

Proposition 1.1.8. *Let* **FDLat** *be the category whose objects are finite distributive lattices and morphisms are injective morphisms of lattices. Let k be a ring, then the category Rep_k(**FDLat**) is not Noetherian.*

Proof. Let J and J_n as above. Let M be the subrepresentaion of the principal projective P_J generated by $\bigcup_{n\in\mathbb{N}} P_J(J_n)$ and suppose it is finitely generated by $\{\alpha_j \in M(J_j)\}_{j=1}^N$. As in the previous two propositions, we can say that M is also generated by the set $\bigcup_{i=1}^N P_J(J_i)$ and choosing any $e_g \in M(J_k)$ for $k > N$, by remark 4 we reach a contradiction. So, Rep_k(**DFLat**) is not Noetherian. $\square$

1.2 Gröbner theory for categories

In this section we expose the Gröbner theory for categories as presented in [11]. Let $\mathcal{C}$ be a category, the idea is to find some sufficient conditions on the category $\mathcal{C}$ itself to ensure the Noetherianity of the representation category $Rep_k(\mathcal{C})$. As we will see these conditions are of combinatorial nature, hence in general not too hard to test. The main results we will obtain by means of this Gröbner theory are related to the Noetherianity of the representation category of graphs (or trees) which will be presented in the next chapter.

1.2.1 Brief digression on posets and orders

Here we state some basic definitions and results regarding partially ordered sets that will be useful later on.

Definition 1.8. Let X be a poset.

- X satisfies the ascending chain condition (ACC) if every ascending chain in X stabilizes, ie for every $x_1 \leq x_2 \leq \dots$ there is $\tilde{i} > 0$ such that $x_i = x_{i+1}$ for every $i \geq \tilde{i}$. (analogously we can define a descending chain condition, DCC)
- An ideal of X is a subset $I \subseteq X$ such that if $x \in I$ and $x \leq y$ then $y \in I$. Let $(\mathcal{I}(X), \subseteq)$ be the poset of ideals of X. For $x \in X$ the principal ideal $(x) := \{y \in X : y \geq x\}$. An ideal is finitely generated if it is a finite union of principal ideals.
- An antichain (resp. bad sequence) is a sequence $x_1, x_2, \dots$ in X, such that $x_i \not\leq x_j$ for every $i \neq j$ (resp. $i < j$).
- X is Noetherian if one of the following (equivalent) conditions holds:
    1. X satisfies the DCC property and has no infinite bad sequences.
    2. Given a sequence $(x_i)_i \in X$ there exist $i < j$ such that $x_i \leq x_j$.
    3. The poset $\mathcal{I}(X)$ satisfies the ACC.

4. Every ideal of X is finitely generated.

Lemma 1.2.1. *Let X be a Noetherian poset, then there is an infinite sequence of indices $i_1 < i_2 < \ldots$ such that $x_{i_1} \le x_{i_2} \le \ldots$.*

Proof. Let I be the set of indices such that for any $i \in I$ we have that $j > i \Rightarrow x_i \not\le x_j$. Suppose this set is infinite, then this labels an infinite sequence of elements of X and since X is Noetherian this implies that there are $i < j \in I$ with $x_i \le x_j$, but this contradicts the definition of I. So I is finite, choose i_1 any number bigger than all elements of I. There must be a $i_2 > i_1$ with $x_{i_1} \le x_{i_2}$ (as $i_1 \notin I$), but the same is true again for $i_2 < i_3$ (for some i_3) and so on and we recover $x_{i_1} \le x_{i_2} \le \ldots$. ◻

Definition 1.9. Let S be a set. A well-order on S is a total order with the property that every non empty subset of S has a least element with respect to this ordering.

Let X be a poset and define X^* be the set of finite words $x_1 \ldots x_n$ with $x_i \in X$ for every i. Let $x_1 \ldots x_n \le x'_1 \ldots x'_m$ if there are $1 \le i_1 \le \cdots \le i_n \le m$ such that $x_j \le x'_{i_j}$, for every $j = 1 \ldots, n$. In this setting we can prove one last lemma that will be used just in the next chapter when dealing with the example related to categories of injections.

Lemma 1.2.2 (Higman's Lemma)**.** *Let X be a poset. If X is Noetherian then so it is X^*.*

Proof. Suppose by contradiction that X^* is not Noetherian. Construct a minimal bad sequence in this way: for every $i \ge 1$ among all bad sequences of words starting with $w_1, \ldots, w_{i-1}$ choose w_i such that its length is minimal. Let $x_i \in X$ be its first letter and let v_i be the complement. The Noetherianity of X implies that there is a sequence $i_1 < i_1 < \ldots$ such that $x_{i_1} \le x_{i_2} \le \ldots$. Then $w_1, \ldots, w_{i_1-1}, v_{i_1}, \ldots$ is a bad sequence, as $v_{i_j} \le w_{i_j}$ for every j and if $v_{i_j} \le v_{i_{j'}}$ then $w_{i_j} \le w_{i_{j'}}$, that can't be. But this new sequence is smaller than the minimal one chosen before, and this gives a contradiction. ◻

1.2.2 Monomial representations and Gröbner bases

Fix a functor $S : \mathcal{C} \to Set$ and let P be the functor defined as:

$$\begin{aligned} P \colon \mathcal{C} &\to Mod_k \\ x &\longmapsto k[S(x)] \end{aligned}$$

where $k[S(x)]$ is the free k-module generated by the elements of the set $S(x)$. To S we can associate a poset $|S|$. Start with $\tilde{S} := \bigcup_{x \in \mathcal{C}} S(x)$. Given two elements of $\tilde{S}$, namely $f \in S(x)$ and $g \in S(y)$, we say that $f \le g$ if there is a morphism $h : x \to y$ such that $S(h)(f) = g$. Define then an equivalence relation on $\tilde{S}$ as follows: $f \sim g$ if both $f \leq g$ and $g \leq f$ holds. We call $|S|$ the poset defined by $(\tilde{S}/\sim, \leq)$.

Definition 1.10. An element of P is monomial if it is of the form λe_f for some $f \in S(x)$ (for $f \in S(x)$ we call e_f the corrisponding basis element of $k[S(x)]$). Moreover, a subrepresentation M of P is monomial if $M(x)$ is spanned by the monomials it contains.

The representation P itself, seen as a trivial subrepresentation, is clearly monomial. An important feature of these monomial representations is that they can be linked to the poset $|S|$: for any $f \in \tilde{S}$ let $I_M(f) := \{\lambda \in k : \lambda e_f \in M\}$ then $I_M(f) \in \mathcal{I}(k)$, the poset of ideals of k. Notice that $I_M(f) = I_M(g)$ if $f \sim g$, so this gives a well-defined order preserving map $I_M : |S| \to \mathcal{I}(k)$.

Proposition 1.2.3. *Let $\mathcal{M}(P)$ be the poset of monomial subrepresentations of P with the natural partial odering and $\mathcal{F}(|S|, \mathcal{I}(k))$ be the poset of order preserving maps between the respective posets. Then $I : \mathcal{M}(P) \to \mathcal{F}(|S|, \mathcal{I}(k))$ is an isomorphism of posets.*

Proof. We show it is an isomorphism constructing an inverse. Suppose to have a function $H : |S| \to \mathcal{I}(k)$ order preserving, ie $f \leq g$ in $|S|$ then $H(f) \subseteq H(g)$ in $\mathcal{I}(k)$. From this we can construct a monomial subrepresentation M of P by: $M(x) := \sum_{f \in S(x)} H(f)e_f$. Both H and I are order preserving and one inverse to the other. $\square$

Corollary 1.2.4. *The following are equivalent:*

i) Every monomial subrepresentation of P is finitely generated;

ii) The poset $\mathcal{M}(P)$ satisfies the ascending chain condition;

iii) The poset $|S|$ is Noetherian and k is also Noetherian.

Proof. The proof that *i)* is equivalent to *ii)* is standard (essentially the same argument used when dealing with rings and the two (adapted) notions of finitely generation of ideals and ACC). The proof that *ii)* is equivalent to *iii)* is a direct consequence of this lemma:

Lemma 1.2.5. *Let X, Y be posets and call $\mathcal{F}$ the poset of order preserving morphisms between X, Y (for any $f, g : X \to Y$ set $f \leq g$ if $f(x) \leq g(x)$, for any x). If $\mathcal{F}$ respect the ACC and Y has two distinct comparable elements, then X is Noetherian.*

Proof. Let $y_1 < y_2$ in Y. Given any ideal I of X, let:

$$\chi_I : X \to Y$$
$$x \mapsto \begin{cases} y_2 & x \in I \\ y_1 & x \notin I \end{cases} .$$

This is an order preserving function, hence $I \to \chi_I$ embeds $I(X)$ into $\mathcal{F}$. $I(X)$ then has to respect the ACC as well, hence X is Noetherian. $\square$

$I(k)$ has always two distinct comparable elements (namely $(0) \subset k$) and this concludes the proof. $\square$

Definition 1.11. An ordering on a functor $S : \mathcal{C} \to Set$ is a well-order on $S(x)$ (for all $x \in \mathcal{C}$) such that for any morphism $x \to y$ we want $S(x) \to S(y)$ to be strictly order preserving.

Given $\leq$ an ordering on S, let $\alpha := \sum_{f \in S(x)} \lambda_f e_f \in P(x)$ and:

- $Init(\alpha) := \lambda_g e_g$, where $g := max_{\leq}\{f : \lambda_f \neq 0\}$;
- $Init_0(\alpha) := g$.

Now, given any M subrepresentation of P let:

$$Init(M)(x) := k[Init(\alpha) : \alpha \in M(x),\ \alpha \neq 0].$$

Proposition 1.2.6. *$Init(M)$ is a monomial subrepresentation of P.*

Proof. Once shown that $Init(M)$ is actually a representation, the fact that it is monomial is clear from its definition. Let $\alpha := \sum_{i=i}^{n} \lambda_i e_{f_i} \in M(x)$, with the $\lambda_i \neq 0$ and ordered such that $f_i \prec f_1$, for every $i > 1$. $Init(\alpha) = \lambda_1 e_{f_1}$. Given $g : x \to y$ a morphism we have that $M(g)(\alpha) = \sum_{i=1}^{n} \lambda_i e_{M(g)(f_i)}$, but since $M(g)$ must be order preserving we have that still $M(g)(f_i) \prec M(g)(f_1)$ which implies $Init(M(g)(\alpha)) = \lambda_1 e_{M(g)(f_1)} = M(g)(Init(\alpha))$. But so $Init(M)$ is really a representation. $\square$

Proposition 1.2.7. *If $N \subseteq M$ are subrepresentations of P with $Init(M) = Init(N)$, then $M = N$.*

Proof. Suppose by contradiction that $M \neq N$, and let $N(x) \subset M(x)$, for some x. Let $K := \{f \in S(x) : f = Init_0(\alpha),\ \exists \alpha \in M(x) \setminus N(x)\}$; by the assumption $K \neq \emptyset$, so let $g \in K$ be a minimal element with respect to $\preceq$. Let $\alpha \in M(x) \setminus N(x)$ such that $Init_0(\alpha) = g$, by the hypothesis we have that $\exists\ \beta \in N(x)$ such that $Init(\alpha) = Init(\beta)$. $\alpha - \beta \in M(x) \setminus N(x)$, but $Init_0(\alpha - \beta) \prec Init_0(\alpha)$, contradiction $\square$

Definition 1.12. Let M be a subrepresentation of P. A set of elements $\mathcal{G}$ of M is a Gröbner basis of M if $\{Init(\alpha) : \alpha \in \mathcal{G}\}$ generates $Init(M)$.

Proposition 1.2.8. *Let $\mathcal{G}$ be a Gröbner basis of M, then $\mathcal{G}$ generates M.*

Proof. Let N be the subrepresentation of M generated by $\mathcal{G}$. We have that $Init(N) \supseteq \{Init(\alpha) : \alpha \in \mathcal{G}\}$, hence $Init(N) = Init(M)$. But by the previous proposition this implies that $M = N$, hence $\mathcal{G}$ generates M. $\square$

Here a first theorem which really relates the combinatorial properties of the poset $|S|$ to the Noetherianity of the representation P.

Theorem 1.2.9. *Let k be a Noetherian ring, S an orderable functor such that the poset $|S|$ is Noetherian. Then every subrepresentation of P has a finite Gröbner basis. In particular P is a Noetherian object of $Rep_k(\mathcal{C})$.*

Proof. Let M be any subrepresentation of P, we need to show it is finitely generated. We have that $|S|$ and k are Noetherian (one as poset and the other as a ring) so $Init(M)$ is finitely generated. This means that M admits a finite Gröbner basis, but so M itself is finitely generated. From here the Noetherianity of P. $\square$

Now we can introduce the main definition and state the main theorem of this section:

Definition 1.13. A category $\mathcal{C}$ is Gröbner if for any $x \in \mathcal{C}$ the functor

$$\begin{aligned} S_x \colon \mathcal{C} &\to Set \\ y &\longmapsto Hom(x, y) \end{aligned}$$

is orderable and the poset $|S_x|$ is Noetherian. $\mathcal{C}$ is said quasi-Gröbner if there is a functor $\phi : \mathcal{C}' \to \mathcal{C}$ which is essentially surjective and satisfies property (F) and $\mathcal{C}'$ is a Gröbner category.

Theorem 1.2.10. *Let k be a Noetherian ring. If $\mathcal{C}$ is a quasi-Gröbner category, then $Rep_k(\mathcal{C})$ is Noetherian.*

Proof. Let $\mathcal{C}$ be Gröbner. This implies that all the principal projectives P_x are Noetherian, hence $Rep_k(\mathcal{C})$ is Noetherian as well. Let $\mathcal{C}$ be quasi-Gröbner: there is an essentially surjective functor $\mathcal{C}' \to \mathcal{C}$ with property (F), where $\mathcal{C}'$ is Gröbner. Hence $Rep_k(\mathcal{C}')$ is Noetherian, but by theorem 1.1.5 $Rep_k(\mathcal{C})$ must be Noetherian as well. $\square$

As announced in the beginning, this theorem gives a sufficient condition of combinatorial nature, to be tested directly on the category $\mathcal{C}$, that ensures the Noetherianity of $Rep_k(\mathcal{C})$.
There is also another equivalent definition for a directed category $\mathcal{C}$ to be Gröbner. In this particular case this alternative definition is more easy to test, for example later on we will show that the category $\mathcal{G}_g^{op}$ of graphs of fixed genus g is quasi-Gröbner using this definition.

Remark 5. An admissible order $\leq$ on $|\mathcal{C}_x|$ is a well-order with the additional property that given $f, f' \in Hom(x, y)$ with $f \leq f'$ and $g \in Hom(y, z)$ then: $gf \leq gf'$ must hold as well.

Definition 1.14. Let $\mathcal{C}$ be a directed category. $\mathcal{C}$ is Gröbner if and only if $|\mathcal{C}_x|$ has an admissible order (G1) and is Noetherian as a poset (G2), for any $x \in \mathcal{C}$.

Remark 6. Let $\mathcal{C}$ be directed. The two definitions are really equivalent:

Step 1: Let $x \in \mathcal{C}$ then $|\mathcal{C}_x| \cong |S_x|$. Indeed, the two sets $Ob(\mathcal{C}_x)$ and $\tilde{S}_x$ are equal. In $|\mathcal{C}_x|$ two morphisms f, g are identified if there is an isomorphism h such that $g = hf$, whereas two morphisms f, g in $|S_x|$ are identified if there are h, h' such that $g = hf$ and $f = h'g$. Since $\mathcal{C}$ is directed h, h' must be isomorphisms, so $|\mathcal{C}_x|$ and $|S_x|$ are the same quotient of $Ob(\mathcal{C}_x)$. The order is defined in the same way, so they are isomorphic posets. Hence $|\mathcal{C}_x|$ is Noetherian if and only if $|S_x|$ is Noetherian.

Step 2: An admissible order on $|\mathcal{C}_x|$ induces an ordering on S_x. Let $\preceq$ be an admissible order on $|\mathcal{C}_x|$. Since $\mathcal{C}$ is directed, the natural map $S_x(y) \to |\mathcal{C}_x|$ is an injection (every element of $S_x(y)$ is its own entire class under the equivalence relation in $|\mathcal{C}_x|$). We define an order on $S_x(y)$ by restricting $\preceq$ to it, this defines an ordering of S_x.

Step 3: An ordering of S_x ($\preceq$) induces an admissible order on $|\mathcal{C}_x|$. Let $\mathcal{C}_0$ be a set of representatives of isomorphism classes in $\mathcal{C}$. Choose a well-order on $\mathcal{C}_0$, say $\preceq'$. Since $\mathcal{C}$ is directed, the natural map $\amalg_{y \in \mathcal{C}_0} S_x(y) \to |\mathcal{C}_x|$ is bijective. We define a well-order $\preceq''$ in $\amalg_{y \in \mathcal{C}_0} S_x(y)$: given any two morphisms $f : x \to y$, $g : x \to z$ set $f \preceq'' g$ if $y \preceq' z$ or, if $y = z$, $f \preceq g$ in $S_x(y)$. This defines an admissible order on $|\mathcal{C}_x|$: it is a well order, and if $f, f' : x \to y$ such that $f \preceq'' f'$ then for any $g : y \to z$ we have that $g \circ f = S(g)(f)$. By definition of ordering of a functor, $S(g)$ must be order preserving hence $g \circ f \preceq'' g \circ f'$.

Remark 7. Property (G2) can sometimes be useful, especially in the proceeding of this work when dealing with graph/trees categories, to be restated as: $|\mathcal{C}_x|$ admits no bad sequences.

We need now to introduce another property of a functor $\Phi : \mathcal{C}' \to \mathcal{C}$ that if satisfied gives condition for Gröbnerianity in the style of property (F). We will use this particular property only to prove theorem 3.1.1 in the last section.

Definition 1.15. A functor $\Phi : \mathcal{C}' \to \mathcal{C}$ satisfies property (S) (for Sub) if the following condition holds: if $x \to y$ and $x \to z$ are morphisms in $\mathcal{C}'$ and there exists a morphism $\tilde{h} : \Phi(y) \to \Phi(z)$ in $\mathcal{C}$ such that $\Phi(g) = \tilde{h} \circ \Phi(f)$, then there is $h : y \to z$ such that $g = h \circ f$.

Proposition 1.2.11. *Let $\Phi : \mathcal{C}' \to \mathcal{C}$ be a faithful functor satisfing property (S) and suppose $\mathcal{C}$ is Gröbner. Then $\mathcal{C}'$ is Gröbner.*

Proof. Let x be an object of $\mathcal{C}'$. We first claim that the natural map $i : |S_x| \to |S_{\Phi(x)}|$ induced by Φ is strictly order-preserving. Indeed, let $f : x \to y$ and $g : x \to z$ be elements of $|S_x|$ such that $i(f) \leq i(g)$. Then there exists $\tilde{h} : \Phi(y) \to \Phi(z)$ such that $\tilde{h} \circ \Phi(f) = \Phi(g)$. By property (S), there exists

$h : y \to z$ such that $h \circ f = g$. Thus $f \leq g$, establishing the claim. It follows from this and the Noetherianity of $|S_{\Phi(x)}|$, that $|S_x|$ is Noetherian. Finally, an ordering on $S_{\Phi(x)}$ clearly induces one on $S_{\Phi(x)}|_{\mathcal{C}'}$ and this restricts to one on S_x, and we can conclude. □

Remark 8. Let $\mathcal{C}$ be a category. From this last proposition we can see how much stronger is obtaining the Noetherianity for the representation category of $\mathcal{C}$ when $\mathcal{C}$ is Gröbner than when it is not. This last proposition tells us that basically if we are dealing with $\mathcal{C}$ a Gröbner category and we consider a subcategory $\mathcal{C}'$ (respecting a little additional property), this is going to be Gröbner as well. So in this case we have that Noetherianity is kept when passing to the category of representations of (some) subcategories. Instead in general, if we don't know anything about the Gröbnerianity of the categories we are dealing with, this is far from being true.

1.2.3 Examples of Noetherianity

Here some examples of categories whose representation category is Noetherian. These proofs relies on the Gröbner theory just presented.

Categories of injections

Two first interesting examples of (quasi-) Gröbner categories are the categories of injections: **OI** and **FI**. The objects of **FI** are finite sets and morphisms are injections between these sets. **OI** is the ordered version: its objects are totally ordered finite set and morphisms are order preserving injections. Notice that there is a natural forgetful functor $\Phi : \mathbf{OI} \to \mathbf{FI}$, so the idea is to show that **OI** is Gröbner and Φ satisfies property (F).

- Fix $n \in \mathbb{N}$. Let $S := \{0, 1\}$ and let $\mathcal{L}_n$ be the language on S in which every word has exactly n letters equal to 0. Partially order $\mathcal{L}_n$ saying that: $w_1...w_k \leq v_1...v_r$ if there is $1 \leq i_1...i_k \leq r$ such that $v_{i_1}...v_{i_k} = w_1...w_k$. By Higmans's lemma $\mathcal{L}_n$ is Noetherian.
- **OI** is clearly directed, so to prove its Gröbnerianity we have to show properties $(G1)$ and $(G2)$. Let $x \in \mathbf{OI}$ be a finite set of size n and consider $f \in Hom_{\mathbf{OI}}(x, [m])$, where $[m] = \{1, \ldots, m\}$. Define a function $h : [m] \to S$ such that it is 0 on the image of f and 1 elsewhere, this gives a word in $\mathcal{L}_n$. Since f must be order preserving ad injective we can actually recover it from h, so this defines an isomorphism of posets between $|\mathbf{OI}_x|$ and $\mathcal{L}_n$. Hence $|\mathbf{OI}_x|$ is Noetherian (G2). The classic lexicographic order on $\mathcal{L}_n$ restricts to an admissible order on $|\mathbf{OI}_x|$, so property $(G1)$. **OI** is Gröbner.
- Let $x \in \mathbf{FI}$ be a set of size n. We need to find $y_1, \ldots, y_k \in \mathbf{OI}$ and $f_i : x \to \Phi(y_i)$ such that for any $y \in \mathbf{OI}$ totally ordered fnite set and

any $f : x \to \Phi(y)$ this last factors as:

$$\begin{tikzcd} & \Phi(y_i) \arrow[dr, "\Phi(g_i)"] & \\ x \arrow[ur, "f_i"] \arrow[rr, "f"] & & \Phi(y) \end{tikzcd}$$

for some $g_i : y_i \to y$. Now notice that any $f : x \to y$ can be factored as

$$\begin{tikzcd} & x \arrow[dr, "f'"] & \\ x \arrow[ur, "\sigma"] \arrow[rr, "f"] & & y \end{tikzcd}$$

where σ is a permutation of S_n and f' an order preserving map. By choosing $k = n!$, $y_i = [n]$ for every $i = 1, \dots, n!$, $f_i : x \to \Phi(y_i)$ as the map sending x to its i^{th}-permutation of S_n and g_i the order preserving map just defined, we can conclude that Φ has property (F). Hence **FI** is quasi-Gröbner.

So, the representation categories $Rep_k(\mathbf{OI})$ and $Rep_k(\mathbf{FI})$ are Noetherian for any fixed Noetherian ring k.

Linear categories

A linear map between free modules (from here the name linear categories) is called splittable when its image is a direct summand. Consider the categories $\mathbf{VI}_R$ and $\mathbf{VA}_R$ of free modules of finite rank over a finite ring R, with morphisms respectively injective splittable maps and splittable maps. More in detail, fix R a finite ring:

$\mathbf{VI}_R$: Is the category whose objects are free R-modules of finite rank and morphisms are injective splittable R-linear maps.

$\mathbf{VA}_R$: Is the category whose objects are free R-modules of finite rank and morphisms are splittable R-linear maps.

Lemma 1.2.12. *Let R be a finite ring. The categories $\mathbf{VI}_R$ and $\mathbf{VA}_R$ are quasi-Gröbner.*

Proof. Consider the following functor:

$$\begin{aligned} \Phi \colon \mathbf{FS}^{op} &\to \mathbf{VI}_R \\ S &\longmapsto Hom_R(R[S], R) = R[S]^* \end{aligned}$$

where $R[S]$ is the free R-module generated by the elements of S. As any free R-module of finite rank n is isomorphic to $\bigoplus_{i=1}^n R$, it's clear the essential surjectivity of Φ. To conclude (for $\mathbf{VI}_R$) we only need to prove that it has property (F). Take $U \in \mathbf{VI}_R$. We have to show that for any $S \in \mathbf{FS}^{op}$ and any splittable injection $f : U \to R[S]^*$, there are finitely many $S_i \in \mathbf{FS}^{op}$ together with splittable inections $f_i : U \to R[S_i]^*$ and surjections $g_i : S \to S_i$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & R[S_i]^* \arrow[dr, "\Phi(g_i)"] & \\
U \arrow[ur, "f_i"] \arrow[rr, "f"] & & R[S]^*
\end{tikzcd}$$

The commutativity of this diagram is equvalent to the commutativity of the dual one:

$$\begin{tikzcd}
 & R[S_i] \arrow[dr, "f_i^*"] & \\
R[S] \arrow[ur, "\Phi(g_i)^*"] \arrow[rr, "f^*"] & & U^*
\end{tikzcd}$$

Consider $f^* : R[S] \to U^*$ and let $T \in U^*$ be the image of S. The map f^* factorizes as $R[S] \to R[T] \to U^*$, where the first map actually come from a surjection $S \to T$. By this observation we can choose the S_i to be all the subsets of U^* that span it as an R-module and f_i^* to be the natural maps $R[S_i] \to U^*$. This giver the quasi-Gröbnerianity of $\mathbf{VI}_R$. Now, the fact that the inclusion functor $i : \mathbf{VI}_R \to \mathbf{VA}_R$ has property (F) is a direct consequence of the fact that any splittable morphism between finite rank R-modules factorize through a surjection followed by a splittable injection. Composition of essentially surjective functors which satisfy property (F), gives again a functor of the same type, hence also $\mathbf{VA}_R$ is quasi-Gröbner. □

As before, this lemma tells us that $Rep_k(\mathbf{VI}_R)$ and $Rep_k(\mathbf{VA}_R)$ are Noetherian for any fixed Noetherian ring k.

Remark 9. We want now also to point out that the finiteness condition on the ring R is necessary to reach the Noetherianity of the representation category $Rep_k(\mathbf{VI}_R)$. First consider the following lemma:

Lemma 1.2.13. *Let R be a non zero ring and Γ be a group that contains a non finitely generated subgroup. Then the group ring $R[\Gamma]$ is not Noetherian.*

Proof. Given any subgroup H of Γ, we have that the kernel of the surjective morphism $R[\Gamma] \to R[\Gamma/H]$ is an ideal (call it I_H) of $R[\Gamma]$. For any $H' \gneqq H$ we have that $I_{H'} \subsetneqq I_H$ and so, since Γ has a non-finitely generated subgroup, there is an infinite sequence of subgroups $H_1 \lneqq H_2 \lneqq \dots$ giving an infinite ascending chain of ideals in $R[\Gamma]$: $I_{H_1} \subsetneqq I_{H_2} \subsetneqq \dots$. In conclusion, $R[\Gamma]$ is not Noetherian. □

The idea now is to prove that the group $Aut_{\mathbf{VI}_R}(R^2)$ contains a subgroup which is not finitely generated. In fact, once proven that, we can construct the following module $M \in Rep_k(\mathbf{VI}_R)$:

$$M(X) := \begin{cases} k[Aut_{\mathbf{VI}_R}(X)] & X \cong R^2 \\ 0 & \text{Otherwise} \end{cases}.$$

This module is fintely generated as it is a quotient of the principal projective module P_{R^2}. On the other hand, now $k[Aut_{\mathbf{VI}_R}(R^2)]$ is not Noetherian by the previous lemma, so let N be a non finitley generated submodule of it. Let M' be the subrepresentation of M generated by $N \subseteq M(R^2)$ and suppose it is finitely generated. There are $X_1, \ldots, X_n \in \mathbf{VI}_R$ such that $M'(R^2) = N$ is generated by images of maps $M'(X_i) \to M'(R^2)$ induced by injective morphisms $X_i \to R^2$. Such maps and the $\{X_i\}_{i=1}^n$ are finitely many which implies the finitely generation of N (as an R-module), so we reach a contradiction.
We are left to prove that $Aut_{\mathbf{VI}_R}(R^2)$ contains a subgroup which is not finitely generated. Notice first that always $Aut_{\mathbf{VI}_R}(R^2)$ contains $SL_2(R)$.

$\mathbb{Z} \subseteq R$: Then $SL_2(R)$ contains a rank 2 free subgroup, consider the one generated by the matrices:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

This group contains a non finitely generated subgroup.

$\mathbb{Z} \not\subseteq R$: Then all elements of the additive group of R are killed by a positive integer l, so if R is an infinite ring the additive group of it must be non finitely generated. The additive group of any ring embeds as a subgroup of $SL_2(R)$ via the morphism $r \to \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, and so again $SL_2(R)$ contains a non finitely generated subgroup.

Chapter 2

Configuration spaces of graphs

In this chapter we introduce the defintions of configuration spaces of graphs and we study some of their topological properties by means of the theory developed in the previous chapter. In doing this we follow the work presented in [8] and [9]. We will be particularly interested in studying the homology groups of such spaces, for whose calculation the theory of Swiatkowski is presented following the works in [1], [8], [9], [13].

2.1 Categories of graphs

By a graph we mean a finite CW-complex of dimension at most 1. The 0-cells are the vertices and the 1-cells are the edges. If G is a connected non-empty graph we define the genus g as the rank of its first homology group. A map $f : G \to G'$ of CW-complexes is very cellular if it send vertices to vertices and edges to edges or vertices. An edge which is sent to a vertex is called contraced. If G and G' are graphs, define a graph morphism from G to G' to be an equivalence class of very cellular maps, where two very cellular maps are equivalent if and only if they are homotopic through very cellular maps. We define a contraction of graphs to be a surjective graph morphism with contractible fibres. Now a list of the graph categories that we will be dealing with, together with some basic results about them.

- A tree is a graph of genus 0, a rooted tree is a couple (T, v) where T is a tree and v a fixed vertex, called root. Given a rooted tree, we can define a partial order on the set of vertices: $v \leq w$ if there is a downward path from v to w. $\mathbf{T}$ is the category whose objects are rooted trees and morphism are order embeddings that preserve the root.

- $\mathcal{RT}$ is the category whose objects are rooted trees and morphisms are contractions that respect the root.

Proposition 2.1.1. *The categories $\mathcal{RT}^{op}$ and $\mathbf{T}$ are equivalent.*

Proof. Let (T, v) and (T', v') be rooted trees. Given a contraction $f : (T, v) \to (T', v')$ in $\mathcal{RT}$, construct a morphism $f^* : (T', v') \to (T, v)$ in $\mathbf{T}$ by sending each vertex of T' to the maximal vertex in its preimage. Conversely, given a morphism: $g : (T', v') \to (T, v)$ in $\mathbf{T}$, construct a contraction $g^* : (T, v) \to (T', v')$ in $\mathcal{RT}$ that sends each vertex w of $\mathbf{T}$ to the minimal vertex of T' whose image under g lies weakly above w. Let $f : T \to T'$ a contraction, the induced embedding is:

$$\begin{aligned} f^* \colon T' &\to T \\ v &\mapsto max\{f^{-1}(v)\}. \end{aligned}$$

The induced contraction then looks like:

$$\begin{aligned} f^{**} \colon T &\to T' \\ w &\mapsto min\{v \in T' : f^*(v) \text{ lies weakly above } w\}. \end{aligned}$$

As $f^*(v) = max\{f^{-1}(v)\}$ and $w \in f^{-1}(f(w))$ is the smaller vertex lying weakly above w, we conclude $f^{**}(w) = f(w)$ (ie, $f^{**} = f$). Analogously it works for $g^{**} = g$ and we conclude that the two constructions are mutually inverse. □

- A planar rooted tree is a rooted tree with a total ordering on $in(v)$ (the set of edges adjacent to v) for every v. $\mathbf{PT}$ is the category whose objects are planar rooted trees (the total ordering is given by a clockwise depth-first tree walk) and morphisms are order embeddings that preserve the root and the depth-first ordering on vertices.

- $\mathcal{PT}$ is the category whose objects are planar rooted trees and morphisms are contractions of rooted trees with the additional property that, if v comes before w in the depth-first order, then the first vertex in the preimage of v comes before the first vertex in the preimage of w.

A similar result hold also for these last two categories:

Proposition 2.1.2. *The categories $\mathcal{PT}^{op}$ and $\mathbf{PT}$ are equivalent.*

The goal of the work in [2] is to show that $\mathbf{T}$ is a quasi-Gröbner category. To do so it's required quite a lot of technical work and some intermediate steps to arrive to the conclusion. We just mention the most important one as it regards the category just defined above and gives an idea of the strategy of the proof:

Proposition 2.1.3. *The category $\mathbf{PT}$ is Gröbner and the forgetful functor $\mathbf{PT} \to \mathbf{T}$ is essentially surjective and satisfies property (F).*

This ensures also the Gröbnerianity of the category $\mathcal{PT}^{op}$ as they are equivalent.

- $\mathcal{G}_g$ is the category whose objects are graphs of genus g and whose morphisms are graph contractions. Notice that a contraction doesn't change the number of cycles in a graph, ie it doesn't change its genus, so this category is well defined. Notice that $\mathcal{G}_0 \cong \mathcal{T}$.

- A rigidified graph G of genus g is a graph with a choice of a planar rooted spanning tree (a subtree of G containing all the vertices) and of an ordering and an orientation of the g extra edges. We can identify such graphs with quadruples (G, T, v, τ) where: G is a graph of genus g, (T, v) is a planar rooted spanning tree and τ is an isomoprhism from R_g (the graph with one vertex and g loops) to G/T. Let $\mathcal{PG}_g$ the category whose objects are rigidified graphs of genus g and morphisms are contractions that restricts to contractions of planar rooted trees (the spanning tree) and are compatible with the ordering and orientations of the extra edges (notice that no extra edge can be contracted). Notice that: $\mathcal{PG}_0 \cong \mathcal{PT}$.

The goal now is to show that the category $\mathcal{G}_g^{op}$ is quasi-Gröbner, so that we have Noetherianity when considering the representation category $Rep_k(\mathcal{G}_g^{op})$. To do this some techincal work is required. The completeness of it can be found in [9], we will just recall the general idea of the proof and some important steps.

Theorem 2.1.4. *The category $\mathcal{G}_g^{op}$ is quasi-Gröbner.*

Proof. The basic idea of the proof is to show first that the category $\mathcal{PG}_g^{op}$ is Gröbner and then considerig the essentially surjective forgetful functor $for : \mathcal{PG}_g^{op} \to \mathcal{G}_g^{op}$, show that this has property (F). The most techincal part is showing that the category $\mathcal{PG}_g^{op}$ satisfies property (G2), ie that the set $(\mathcal{PG}_g^{op})_{(G,T,v,\tau)}$ doesn't admit any bad sequence, but we can take it for granted here.

Lemma 2.1.5. *For any g, the category $\mathcal{PG}_g^{op}$ satisfies property (G1)*

Proof. Fix a rigidified graph of genus g: (G, T, v, τ). Recall that satisfing property (G1) means that $(\mathcal{PT}_g^{op})_{(G,T,v,\tau)}$ admits a linear order $\preceq$ that is compatible with pre-composition (we are working in the opposite category). By the Gröbnerianity of the category $\mathcal{PT}^{op}$, we have that this category satisfies property (G1). By the fact that contractions of rigidified graphs are defined from their restrictions on the spanning tree ($\in \mathcal{PT}^{op}$), we can conclude. $\square$

$\mathcal{PG}_g^{op}$ satisfies both properties (G1) and (G2), hence it is Gröbner.

Lemma 2.1.6. *The forgetful functor $for : \mathcal{PG}_g^{op} \to \mathcal{G}_g^{op}$ is essentially surjective and has property (F).*

Proof. By definition of property (F) we need to take G a graph of genus g, a finite collection of (G_i, T_i, v_i, τ_i) rigidified graphs of genus g along with contractions $f_i : G_i \to G$ such that for any rigidified graph (G', T', v', τ') and contraction $f : G' \to G$, this last factor as $f = f_i \circ for(\psi)$, for some $\psi : (G', T', v', \tau') \to (G_i, T_i, v_i, \tau_i)$. For our rigidified graphs (G_i, T_i, v_i, τ_i) and contractions f_i, we will choose a representative of every possible isomorphism class of such structures whose number of edges is at most $|E(G)| + g$. There are only finitely many rigidified graphs with at most $|E(G)| + g$ edges, and also finitely many contractions from them to G, hence we are considering a finite collection of objects in $\mathcal{PT}_g^{op}$. So fix a (G', T', v', τ') and a contraction f. Let E be the set of contracted edges og G' and consider the canonical contraction $\psi : (G', T', v', \tau') \to (G'/(E \cap T'), T'/(T' \cap E), v', \tau')$, we have that $f = f_i \circ for(\psi)$, if we show that $|E(G'/(E \cap T'))| \leq |G| + g$. $|E| = |E(G')| - |E(G)|$ and $|T'| = |E(G')| - g$, hence $|E \cap T'| \geq |E(G')| - (|E(G)| + g)$. From this it follows that $|E(G'/(E \cap T'))| = |E(G') - |E \cap T'| \leq |E(G)| + g$. □

And this concludes the proof of the theorem as well. □

All this ensures the Noetherianity of the representation categories we will be dealing with. Now a couple of new definitions that will be used in the next chaper when talking about configuration spaces of graphs.

Definition 2.1. Let $\mathcal{C}$ be any one of the graph/trees categories introduced above:

- $M \in Rep_k(\mathcal{C})$ is generated in degrees $\leq d$ if $\exists\ X_1, .., X_n$ objects in $\mathcal{C}$ with at most d edges (all objects are graphs or trees) such that M is a quotient of the direct sum $\bigoplus_{i=1}^n P_{X_i}$;

- M is d-small if it is a subquotient of a module generated in degrees $\leq d$.

Direct consequence of the definition of smallness is the following proposition. It will be useful later on when deling with homology of configuration spaces.

Proposition 2.1.7. *If M is d-small, then M is finitely generated.*

Proof. Clear from the definition. □

Remark 10. Notice that there is no reason to think that if a module is d-small, this is also generated in degrees $\leq d$. We know it is finitely generated, but we have no control over the generators. This is something to always keep in mind when considering submodules of finitely generated modules: we don't know anything about the generators of such submodules, in the case the category is Noetherian, we just know that there are finitely many of them.

Now a theorem that shows the most important aspect of a d-small module: we may have not any control over its generators, but it has stability.

Proposition 2.1.8. *Let k be a field and $M \in Rep_k(\mathcal{G}_g^{op})$ a d-small module. Then there exists a polynomial $f_M(t) \in \mathbb{Z}[t]$ of degree at most d such that $dim_k(M(G)) \leq f_M(|E(G)|)$, for any G.*

Proof. By the smallness of M (and so its finitely generation) we can just consider the case of $M = P'_G$, for G' a genus g graph with d edges. For any G a contraction $G \to G'$ is determined (up to automorphisms of G') by the choice of $|E(G) - d|$ edges to contract. Hence we have that:

$$dim_k(P_{G'}(G)) \leq |Aut(G')| \cdot \binom{|E(G)|}{d} = f_M(|E(G)|).$$

$\square$

Example 1. In [4], an analogous stability result is proved also for **FI**-modules. Anticipating some concepts that will be introduced in the next section, here an example of how this type of results can actually give counter examples for finitely generated modules.

Theorem 2.1.9. *Let k be a field and V a finitely generated* **FI**-*module over k. Then there exists an integer-valued polynomial $P(t) \in \mathbb{Q}[t]$ such that for all sufficiently large n (ie $|x|$):*

$$dim_k V(x) \leq P(|x|).$$

Now let I be the graph consisting of a single edge and consider the following **FI**-module:

$$H : x \longmapsto H_0(Conf_{|x|}(I); \mathbb{Q}).$$

Notice that any injection $x \to y$ gives a map at the level of configuration spaces (hence then a map in homology) just by fixing the last $|y| - |x|$ particles in $Conf_{|y|}(I)$. As shown later on there is a cubical complex $\tilde{K}_n I$ of dimension 0 that emebeds as a deformation retract of $Conf_n(I)$, for every n. The 0-cells of $\tilde{K}_n I$ are in bijection with the elements of the permutation group S_n. Hence we have that $dim_{\mathbb{Q}} H_0(Conf_n(I); \mathbb{Q}) = n!$. This implies that there is no polynomial that can bound $dim_{\mathbb{Q}} H(-)$ from above, so our **FI**-module is not finitely generated.

2.2 Configuration spaces of graphs and their homology

Definition 2.2. Let G be a graph. The set:

$$Conf_n(G) := \{(x_1, ..., x_n) \in G^n : x_i \neq x_j \; \forall i \neq j\},$$

is called the n^{th}-ordered configuration space of G. It is naturally endowed with the subspace topology induced by the inclusion $Conf_n(G) \subseteq G^n$.

This can be imagined as the space of all the possible configurations of n particles in G, where no two particles are allowed to occupy the same position. The natural action of the permutation group S_n on the corrisponding $Conf_n(G)$ (ie, the order of the particles is not relevant anymore) leads to the following definition.

Definition 2.3. Let G be a graph. The set:

$$UConf_n(G) := \{(x_1, ..., x_n) \in G^n : x_i \neq x_j \, \forall i \neq j\}/S_n,$$

is called the n^{th}-unordered configuration space of G. It is naturally endowed with the quotient topology induced by the projection $Conf_n(G) \rightarrow Conf_n(G)/S_n$.

2.2.1 The deformation retract

Given a graph G let $B(G)$ be the set of branched vertices of G. A vertex is branched if it is adjacent to at least three edges. Define also $b(G) := |B(G)|$, and let E_G be the set of edges of G. Each edge carries two distinct orientations. For an oriented edge s denote by $|s|$ the underlying unoriented edge, by $-s$ the same edge with the opposite orientation, and by v_s the vertex adjacent to $|s|$ which is determined by the orientation of s.

Definition 2.4 (Cubical complex)**.** A cubical complex consists of a finite set V and a collection, say $\square$, of subsets of V such that:

- for each $v \in V$, the set $\{v\}$ is in $\square$;
- if σ and τ are in $\square$, then $\sigma \cap \tau$ is either empty or in $\square$;
- if $\sigma \in \square$ then the collection of elements of $\square$ that are contained in σ is isomorphic as a poset (ordered by inclusion) to the poset of the faces of a cube.

Notice that from this definition of cubical complex we can derive a natural notion of dimension for such structures.

Theorem 2.2.1. *Let G be a graph and n a natural number. There exists a cubical complex $K_n(G)$ (of dimension equal to $min\{n, b(G)\}$) such that it embeds as a deformation retract into the configuration space $UConf_n(G)$.*

The idea to prove this is to introduce an abstract graded poset $P_n(G)$ and then showing that this is actually the face poset of a uniquely determined cubical complex. This last can be explicitely embedded as a deformation retract into $UConf_n(G)$.

Definition 2.5. Let $P_n(G) := (P_n^{(0)}G, \dots, P_n^{(k)}G, \dots)$ where $P_n^{(k)}$ are the k-faces, defined to be couples (f, S) such that:

1. $f : E_G \cup B(G) \to \mathbb{N} \cup \{0\}$;
2. $S = \{s_1, \dots, s_k\}$, oriented edges of G;
3. $v_{s_1}, \dots, v_{s_k}$, all different branched vertices of G;
4. $f(b) \in \{0, 1\}$ for any $b \in B(G)$ and $f(v_{s_i}) = 0$ for any $i = 1, \dots, k$;
5. $\sum_{a \in E_G \cup B(G)} f(a) = n - k$.

Intuitively the kth-face is to be thought as the configuration of k fixed oriented edges on which one particle is moving concorldy with the orientation, approaching its branched vertex. The function f then gives the "positions" of all the other (they are not moving) $n - k$ particles; it says how many other particles every edge has (including the fixed oriented ones). Following the definition of configuration space we have that f on the set of branched vertices can assume just values in $\{0, 1\}$ (no multiple particles are allowed to be at the same time on one vertex), and so always 0 in the case of a vertex being approached by one of the moving particles.

Definition 2.6. Let $(f_1, S) \prec (f_2, S \cup \{s\})$ if one of the two conditions holds:

- $f_1(a) = \begin{cases} f_2(a) + 1 & a = |s| \\ f_2(a) & \text{otherwise} \end{cases}$
- $f_1(a) = \begin{cases} f_2(a) + 1 & a = v_s \\ f_2(a) & \text{otherwise} \end{cases}$

Intuitively, $(f_1, S) \prec (f_2, S \cup \{s\})$ if (f_1, S) can be thought as a photograph of $(f_2, S \cup \{s\})$ along its extra fixed oriented edge. (f_1, S) has one particle less moving, but it must have a particle (fixed) "along the way" of the extra one moving in $(f_2, S \cup \{s\})$. This idea helps in giving some sense to the fact that this is actually the face poset of some cubical complex: for example it gives quite a clear picture about the fact that we can go from k-faces to k-faces moving along a $(k+1)$-face. This relation can be extended in a natural way giving a partial ordering on $P_n(G)$.

Remark 11. Notice that for any $F \in P_n^{(k)}G$, the subposet $\{A : A \prec F\}$ in $P_n(G)$ is actually isomorphic to the face poset of a k-dimensional cube. Hence $P_n(G)$ is the face poset of a unique cubical complex: $K_n(G)$.

Remark 12. The dimension of $K_n(G)$ is equal to the max k such that $P_n^{(k)}G \neq \emptyset$. By conditions 3 and 5 in the definition of $P_n(G)$ it's clear how $dim(K_n(G)) \leq min\{n, b(G)\}$. The opposite direction is true as well as it is possible to construct a couple (f, S) with $|S| = min\{n, b(G)\}$. So:

$$dim(K_n(G)) = min\{n, b(G)\}.$$

Definition 2.7. Let AB be a segment and s be the corresponding oriented segment with $v_s = B$, $v_{-s} = A$ and let $t_s, t_{-s} \in [0,1]$. Define $D_{AB}(n, (t_s)_{|s|=AB}) := \{a_1, \ldots, a_n\}$ such that:

1. $a_i \in AB \; \forall \, i = 1, \ldots, n$;
2. $A \leq a_1 \leq \cdots \leq a_n \leq B$ (when considering an oriented segment it's natural to define a partial order on its points which respects the orientation: $x_1 \leq x_2$ if x_1 comes before x_2 in the orientation of the segment);
3. $|a_1 - a_2| = \cdots = |a_{n-1} - a_n|$;
4. $n \geq 2$, $|A - a_1| = t_{-s} \cdot |a_1 - a_2|$ and $|B - a_n| = t_s \cdot |a_1 - a_2|$, for $(t_s, t_{-s}) \neq (0,0)$.

So for $e \in E_G$, define analogously $D_e(n, (t_s)_{|s|=e})$.

Now, let $F = (f, S)$ be a face of $K_n(G)$. Consider the isomorphism of cubes $\tau : F \to [0,1]^{|S|}$ such that, for every vertex $p = (\psi, \emptyset)$ of F, $\tau(p) : S \to [0,1]$ is given by $\tau(p)(s) = 1 - \psi(v_s)$. For every $x \in F$, $\tau(x)$ can be extended to a function $\tau_0(x) : E'_G \to [0,1]$ (E'_G is the set of all oriented edges of G) such that:

$$\tau_0(x)(e) := \begin{cases} \tau(x)(s) & s \in S \\ 1 \quad \text{otherwise} \end{cases} .$$

For every $F = (f, S) \in K_n(G)$ define:

$$\begin{aligned} i_F \colon F &\to UConf_n(G) \\ x &\mapsto \{b \in B(G) : f(b) = 1\} \cup \bigcup_{e \in E_G} D_e(\tilde{f}(e), (\tau_0(x)(s))_{e=|s|}) \end{aligned}$$

where $\tilde{f}(e) := f(e) + |\{s \in S : |s| = e\}|$. The family of such mappings $\{i_F : F \in K_n(G)\}$ actually defines the embedding $i : K_n(G) \hookrightarrow UConf_n(G)$. Intuitively, this embedding is to be seen in this way: it sends every k-face to a continuous collection of configurations that in a sense depict the moves (taken instant by instant) of the k-particles "moving" along the chosen oriented edges of the face. Again intuitively it can be seen how the retraction $r : UConf_n \to UConf_n(G)$ should be: every configuration should be sent to a particular configuration with the same disposition of particles among the vertices and same number of particles in every edge but disposed in a fixed way such that when considering configurations in $K_n(G)$ this gives the identity.

Now fix a length metric on G such that every edge has length 1. Let $C \in UConf_n(G)$, this subdivides E_G in some segments. For every oriented edge s with vertex v_s call d_s^C the length of the segment of the above subdivision which is contained in the edge $|s|$ and adjacent to v_s from the side determined by the orientation of s. Let $n_e^C := |C \cap e^0|$, where e^0 is an edge without its branched vertex. Define $\delta_s^C := d_s^C \cdot (n_s^C + 1)$ and consequently

$$t_s^C := \begin{cases} 1 & v_s \in C \text{ or } v_s \text{ free} \\ min\{1, \dfrac{\delta_s^C}{min\{\delta_{s'}^C : s' \neq s, v_{s'} = v_s\}}\} & \text{otherwise} \end{cases}.$$

The retraction is then defined as:

$$r(C) := (C \cap B(G)) \cup \bigcup_{e \in E_G} D_e(n_e^C, (t_s^C)_{e=|s|}).$$

To conclude that $K_n(G)$ is a deformation retract of $UConf_n(G)$ it's needed an homotopy between $r : UConf_n(G) \to UConf_n(G)$ and $id_{UConf_n(G)} : UConf_n(G) \to UConf_n(G)$. Notice that for each $C \in UConf_n(G)$, $C \cap B(G) = r(C) \cap B(G)$ and $|C \cap e^0| = |r(C) \cap e^0|$. For every $e \in E_G$ let $\{C_e(t) : t \in [0,1]\}$ be the unique continuous 1-parameter family of configurations in e^0 that connects $C \cap e^0$ and $r(C) \cap e^0$. Finally the homotopy desired is given by:

$$\begin{aligned} H \colon [0,1] \times UConf_n(G) &\to UConf_n(G) \\ (t, C) &\mapsto (C \cap B(G)) \cup \bigcup_{e \in E_G} C_e(t). \end{aligned}$$

Remark 13. As shown by D. Lütgehetmann this result still holds even when considering ordered configuration spaces. The construction of the cubical complex $\tilde{K}_n G$, the retraction and the homotopy is essentially the same. In that proof the result for unordered configuration spaces follows the result for ordered ones: just notice how the action of S_n on $Conf_n(G)$ induces an action on $\tilde{K}_n G$ and that both the inclusion and the retraction are S_n-equivariant maps.

2.2.2 The Swiatkowski complex

Computing the homology groups of configuration spaces of grpahs directly turns out to be quite difficult also when the considered graphs are quite simple. The configuration spaces can be really messy, as it will be shown in some examples at the end of this chapter. One tool which can simplify the work consistently was introduced in [1]. The idea is to compute these homology groups as the homology groups of an abstract bigraded differential module constructed from the graph itself.

Definition 2.8 (Reduced Swiatkowski complex)**.** Let G be a graph. Call $A_G := \mathbb{Z}[E(G)]$ the integral polynomial ring with variables the edges of the graph. An half-edge h is a pair $(e(h), v(h))$ where $e(h)$ is an edge and

$v(h)$ is one of its two vertices. Now, for every $v \in V(G)$ define $\tilde{S}(v)$ as the free A_G-module generated by $\emptyset$ together with all $h - h'$ half-edges having $v(h) = v(h') = v$. Bigrade $\tilde{S}(v)$ imposing:

- grade (1,1) to half-edges;
- grade (0,1) to edges;
- grade (0,0) to $\emptyset$.

Define also ∂_v to be an A_G-linear differential of degree $(-1, 0)$ by setting:

- $\partial_v(h - h') = (e(h) - e(h'))\emptyset$;
- $\partial_v \emptyset = 0$.

We call reduced Swiatkowski complex the following bigraded differential module:

$$\tilde{S}(G) := (\bigotimes_{v \in V(G)} \tilde{S}(v),\ \partial),$$

where the tensor product is taken over the ring $\mathbb{Z}[E(G)]$.

Remark 14. Notice that the reduced Swiatkowski complex permits to forget about all the vertices of G of valence ≤ 2: indeed such vertices are quite ininfluent for the configuration space itself, which instead relies particularly on the branched ones. For example, when considering star graphs the tensor product collapses into just one factor ($\tilde{S}(v)$ of the unique branched vertex) making a lot of computations much more easy.

Theorem 2.2.2. *Let G be an object of one of the graph categories $\mathbf{T}, \mathcal{T}^{op}$ or $\mathcal{G}_g^{op}$ presented before. Then there is a functorial isomorphism:*

$$H_\bullet(UConf_*(G)) \cong H_\bullet(\tilde{S}(G)).$$

This theorem is proved in [1] using Morse theory. The difficult (and very long) part of the proof is proving the functoriality of such an isomorphism and this part is not covered here but taken for granted. We can though point out a couple of facts.

Remark 15. Notice that if G is a graph where every vertex (a part from the leaves) is branched, then the weight k subcomplex of $\tilde{S}(G)$:

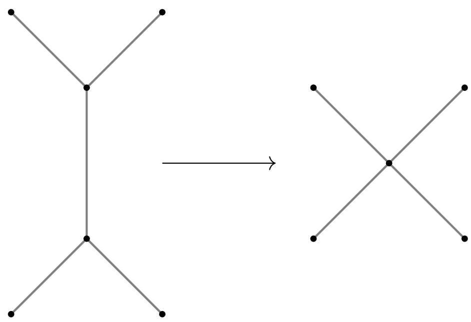
$$\cdots \to \tilde{S}(G)_{i+1,k} \to \tilde{S}(G)_{i,k} \to \tilde{S}(G)_{i-1,k} \to \dots$$

is isomorphic to the cellular chain complex of the cubical complex K_kG which is a defomration retract of $UConf_k(G)$. Indeed: $\tilde{S}_{i,k}$ is (freely) generated by elements of the form:

$$\sigma = e_1 \cdots e_{k-i} \bigotimes_{j=1}^{i} (h_{j_0} - h_{j_1}),$$

where e_i are edges of G and h_{j_0}, h_{j_1} are half edges at the branched vertex v_j for any j. Notice that such an element defines a unique i-cell of the cubical complex $K_k G$ (it fixes $k-i$ particles on the edges $e_1, \ldots, e_{k-i}$ and defines i half edges with a moving particle) and viceversa. So in this particular case theorem 2.2.2 is proved at least at the level of objects.

Remark 16. Notice that the problems in proving functoriality (ie proving that $T \to H_i(UConf_n(T))$ is really a functor) arise when considering contractions. In the case of embeddings is quite straightforward to see how such morphisms of graphs induce a morphism at the level of configuration spaces and then in homology. But consider for example the following contraction $G_1 \to G_2$:



There is no way to define a morphism at the level of configuration spaces ($UConf_n(G_2) \to UConf_n(G_1)$) so that it then induces the desired map in homology $H_i(UConf_n(G_2)) \to H_i(UConf_n(G_1))$. The difficult part of theorem 2.2.2 is to prove that anyway such contractions define a map at the level of Swiatkowski complex (which then induces the map in homology).

Since it is needed in the proof of theorem 2.2.5, we just describe explicitely (taking the existence for granted) such map, say $\tilde{\varphi}_* : \tilde{S}(G') \to \tilde{S}(G)$, at the level of Swiatkowski complexes. First consider the case where the number of edges of G is one greater than the number of edges of G', we call it a simple contraction. Identify the unique edge of G that is contracted by φ with the interval $[0,1]$. Let h_0 (respectively h_1) be the half edge of G consisting of the vertex 0 (respectively 1) and the edge $[0,1]$. Let $w' \in G'$ be the image of the edge $[0,1]$. Each edge of G' is mapped to isomorphically by a unique edge of G, and similarly for half edges. This gives a canonical ring homomorphism $A_{G'} \to A_G$ along with an $A_{G'}$-module homomorphism:

$$\bigotimes_{v' \in V(G') - \{w'\}} \tilde{S}(v') \to \bigotimes_{v \in V(G) - \{0,1\}} \tilde{S}(v).$$

We now need to analyze the case of half-edges of G' with endpoint w'. Let $h' \in G'$ with $v(h') = w'$, call h the unique half-edge of G mapping to h'. We then define an $A_{G'}$-module homomorphism $\tilde{S}(w') \to \tilde{S}(0) \otimes \tilde{S}(1)$ by the formula:

$$\emptyset \longmapsto \emptyset \otimes \emptyset; \quad h' \longmapsto \begin{cases} (h-h_0)\otimes\emptyset, & v(h)=0 \\ \emptyset\otimes(h-h_1), & v(h)=1 \end{cases} .$$

Tensoring these two maps together, we obtain the homomorphism $\tilde{\varphi}_*$. This homomorphism respects the differential. Arbitrary contractions may be obtained as compositions of simple contractions and the induced homomorphism is independent from the choice of factorization into simple contractions.

To summarize (cfr. [9]):

Theorem 2.2.3. *There is a bigraded differential $\mathcal{G}_g^{op}$-module that assigns to each graph G its reduced Swiatkowski complex $\tilde{S}(G)$. The homology of this bigraded complex is again a $\mathcal{G}_g^{op}$ module, it assigns to each graph G the bigraded abelian group: $H_\bullet(UConf_*(G))$.*

Remark 17. From this theorem and the shape of the Swiatkowski complex we can easily get some informations about these homology groups of unordered configuration spaces. Let G be any (finite) graph:

1. For any couple (i,n) such that $i>n$ we have: $\tilde{S}_{i,n}(G)=0$. Indeed, there is no element whose bigrade is $(1,0)$ so no element can have a bigrade with the first coordinate stricltly bigger that the second one. This implies that $H_i(UConf_n(G))=0$ whenever $i>max\{n,b(G)\}$ (recall also theorem 2.2.1).

2. If G has $b(G)$ branched vertices, it's reduced Swiatkowski complex is of the form:

$$\begin{array}{llll} \dots 0 \to \tilde{S}_{b(G),i+1} & \to \tilde{S}_{b(G)-1,i+1} & \to \dots \\ \dots 0 \to \tilde{S}_{b(G),i} & \to \tilde{S}_{b(G)-1,i} & \to \dots \\ \dots 0 \to \tilde{S}_{b(G),i-1} & \to \tilde{S}_{b(G)-1,i-1} & \to \dots \end{array}$$

   So, as computing $H_{b(G)}(UConf_n(G),\mathbb{Z})$ is computing a kernel of a homomorphism between free $\mathbb{Z}$-modules and such kernels are torsion free, $H_{b(G)}(UConf_n(G),\mathbb{Z})$ is torsion free for any n.

Now we are left to prove the smallness result previously anticipated.

Proposition 2.2.4. *For any couple $(i,n)\in\mathbb{N}_{\geq 0}\times\mathbb{N}_{\geq 2}$, the $\mathcal{G}_g^{op}$-module*

$$\begin{aligned} S_{i,n}\colon \mathcal{G}_g^{op} &\to Mod_{\mathbb{Z}} \\ G &\longmapsto \tilde{S}(G)_{i,n} \end{aligned}$$

is generated in degrees $\leq (n+i+g)$.

Proof. The group $\tilde{S}(G)_{i,n}$ is generated by elements of the form:

$$\sigma = e_1 \cdots e_{n-i} \bigotimes_{j=1}^{i} (h_{j0} - h_{j_1}) \otimes \bigotimes_{v \notin \{v_1,\ldots,v_i\}} \emptyset,$$

where $\{e_1, \ldots, e_{n-1}\}$ are edges (not necessarily distinct), $\{v_1, \ldots, v_i\}$ are vertices (distinct), and for each j, h_{j0} and h_{j1} are half-edges at the vertex v_j. For a particular σ of this form, we will call $\{v_1, \ldots, v_i\}$ the set of distinguished vertices. Without loss of generality, we may assume that there is some integer r with $0 \leq r \leq i$ such that v_j is adjacent to some other distinguished vertex (possibly itself) if and only if $j \leq r$. We may also assume that, if $j \leq r$, $e(h_{j1})$ connects v_j to another distinguished vertex (possibly itself). We call an edge e a distinguished edge if one of the following five conditions hold:

- e is a loop,
- e connects two distinguished vertices,
- $e = e_k$ for some $k \leq n - i$,
- $e = e(h_{j0})$ for some $j \leq i$,
- $e = e(h_{j1})$ for some $j \leq i$.

Let t be the number of loops that are not at distinguished vertices. Let H be the subgraph induced by $\{v_1, \ldots, v_r\}$, which in particular contains all the loops at distinguished vertices. H has genus at most $g - t$, hence it has at most $r + g - t$ edges. Hence, the total number of distinguished edges is $\leq t + (r + g - t) + (n - i) + i + (i - r) = n + i + g$. Let G such that $|G| > n + i + g$. Since there are at most $n + i + g$ distinguished edges, we may choose an edge e which is not distinguished. Let $G' := G - e$ be the graph obtained from G by contracting e, and let $\varphi : G \to G'$ be the canonical simple contraction. Let e'_k be the image of e_k in G', v'_j the image of v_j, h'_{j0} the image of h_{j0}, and h'_{j1} the image of h_{j1}. Let:

$$\sigma' = e'_1 \cdots e'_{n-i} \bigotimes_{j=1}^{i} (h'_{j0} - h'_{j_1}) \otimes \bigotimes_{v' \notin \{v'_1,\ldots,v'_i\}} \emptyset \in \tilde{S}(G')_{i,n}.$$

We claim that $\sigma = \tilde{\varphi}_* \sigma'$. If e is not incident to any vertex v_j , this is obvious. The interesting case occurs when e is incident to one of the distinguished vertices. Assume without loss of generality that it is incident to v_1, and let w be the other end point of e (e can't be a loop). Let h be the half-edge of G with $e(h) = e$ and $v(h) = v_1$. Applying the map $\tilde{\varphi}_*$ it replaces each e'_k with e_k. When $j > 1$, it replaces h'_{j0} with h_{j0} and h'_{j1} with h_{j1}. It replaces h'_{10} with $h_{10} - h$ and h'_{11} with $h_{11} - h$. This means that it replaces $h'_{j0} - h'_{j1}$ with

$h_{j0} - h_{j1}$, and therefore $\sigma = \tilde{\varphi}_* \sigma'$. We thus conclude that every element of $\tilde{S}(G)_{i,n}$ is a linear combination of elements in the images of map associated with simple contractions, and we can conclude. $\square$

From this last proposition and theorem 2.2.3 it follows:

Theorem 2.2.5. *For any couple $(i, n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$, the $\mathcal{G}_g^{op}$-module:*

$$\begin{aligned} M_{i,n} \colon \mathcal{G}_g^{op} &\to Mod_{\mathbb{Z}} \\ G &\longmapsto H_i(UConf_n(G), \mathbb{Z}) \end{aligned}$$

is $(n + i + g)$-small, hence also finitely generated.

Remark 18. We presented theorems 2.2.3 and 2.2.5 just for the category $\mathcal{G}_g^{op}$, but recall that these results hold as well when considering the category $\mathcal{T}^{op}$ (ie in the case of $g = 0$).

We can now show an application of this theorem (which more in general is an application of the Noetherian property for such categories of representations) to the study of the homology groups of configuration spaces of trees.

Corollary 2.2.6. *Let T be a tree and $(i, n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$. Then there is a positive integer $t_{i,n}$ that annihilates the torsion part of $H_i(UConf_n(T), \mathbb{Z})$.*

Proof. Consider the category $\mathcal{T}^{op}$. As already shown, this category is Gröbner hence $Rep_{\mathbb{Z}}(\mathcal{T}^{op})$ is Noetherian. For any fixed couple $(i, n) \in \mathbb{N} \times \mathbb{N}_{\geq 2}$ consider the $\mathcal{T}^{op}$-module:

$$\begin{aligned} M_{i,n} \colon \mathcal{T}^{op} &\to Mod_{\mathbb{Z}} \\ T &\longmapsto H_i(UConf_n(T), \mathbb{Z}). \end{aligned}$$

By theorem 2.2.5 we know that all these $\mathcal{T}^{op}$-modules $M_{i,n}$ are $n + i$-small and so finitely generated. Recall now that for any homomorphism of abelian groups $f : G_1 \to G_2$ there is an induced morphism (the restriction of f on $Tor(G_1)$) between their torsion subgroups: $f|_{Tor(G_1)} : Tor(G_1) \to Tor(G_2)$. This enable us to define (for any (i, n)) the submodule $T_{i,n}$ of $M_{i,n}$ by:

$$\begin{aligned} T_{i,n} \colon \mathcal{T}^{op} &\to Mod_{\mathbb{Z}} \\ T &\longmapsto Tor(H_i(UConf_n(T), \mathbb{Z})), \end{aligned}$$

where with $Tor(H_i(UConf_n(T), \mathbb{Z}))$ we intend the torsion subgroup of the corrisponding homology group. As $Rep_{\mathbb{Z}}(\mathcal{T}^{op})$ is Noetherian and the modules $M_{i,n}$ are finitely generated, then so are the submodules $T_{i,n}$. This means that there are $\{T_j\}_{j=1}^n$ trees such that $T_{i,n}(T)$ is generated by images of maps like $T_{i,n}(T_j) \to T_{i,n}(T)$ induced by contractions $T \to T_j$. We conclude that the exponent of the torsion part of $H_i(UConf_n(T), \mathbb{Z})$ is given by the least common multiple of the exponents of the torsion subgroups $T_{i,n}(T_j)$, for $j = 1, \ldots, n$. $\square$

Actually we can go a bit further in studying torsion of these homology groups.

Proposition 2.2.7. *Let G be a planar graph (a graph that can be embedded in the plane with no self intersections), then $H_1(UConf_n(G))$ is torsion free.*

This result is a corollary of Theorem 3.5 in [6]. A direct consequence of this proposition is the following:

Corollary 2.2.8. *Let T be any tree such that $b(T) \leq 2$. Then, for any couple $(i, n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$, the groups $H_i(UConf_n(T), \mathbb{Z})$ are torsion free.*

Proof. If there are no branched vertices there is nothing to prove. The case $b(T) = 1$ is the case of star graphs and the result is proved in the next section, so we are left to prove the case $b(T) = 2$. Notice that for such trees, their Swiatkowski complex is of the form:

$$\begin{gathered} \dots 0 \to \tilde{S}_{2,3}(T) \to \tilde{S}_{1,3}(T) \to \tilde{S}_{0,3}(T) \to 0 \\ \dots 0 \to \tilde{S}_{2,2}(T) \to \tilde{S}_{1,2}(T) \to \tilde{S}_{0,1}(T) \to 0 \\ \dots 0 \to \tilde{S}_{2,1}(T) \to \tilde{S}_{1,1}(T) \to \tilde{S}_{0,1}(T) \to 0 \end{gathered}$$

so torsion in homology can appear only for $i \in \{0, 1, 2\}$. But, for any $n \geq 2$ we have that:

1. $H_0(UConf_n(T)) = \mathbb{Z}$, straightforward computations similar to the ones for star graphs in the next section.
2. $H_1(UConf_n(T))$ is torsion free by proposition 2.2.7, as trees are clearly planar graphs.
3. $H_2(UConf_n(T))$ is torsion free for any k as it is a kernel of a morphism between free $\mathbb{Z}$-modules.

□

Remark 19. Notice that in these results about torsion in homology we are always considering just trees and not graphs in general. In fact, the only result that can be extended to graphs is corollary 2.2.6:

Corollary 2.2.9. *Let G be a graph of genus g and $(i, n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$. Then there is a positive integer $t_{i,n,g}$ that annihilates the torsion part of $H_i(UConf_n(G), \mathbb{Z})$.*

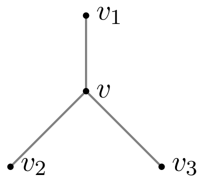
Remark 20. It's fair to point out that such results about torsion in homology for configuration spaces of trees can be all seen as a consequence of a bigger theorem (cfr. [3]) which states:

Theorem 2.2.10. *Let T be a tree. Then $H_i(Uconf_n(T))$ is torsion free for any couple $(i, n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$.*

Anyway, the proof of this theorem relies on different arguments not mentioned here.

2.2.3 Example: Star graphs

Star graphs are the simplest case of graphs with branched vertices: they have just one of them. On the other hand, they also play an important role in studying configuration spaces of trees as they act like building blocks (cfr. corollary 2.2.13). We present here some computations about their configuration spaces as, in this easy case, we can make them explicit. For example, we can write some lines of code to visualize the cubical complex which is a deformation retract of these spaces and confront this graphic result with some computations on their homology groups coming from the Swiatkowski complex itself.
Here a figure representing the star graph Y_3.



Since there is only one branched vertex also the reduced Swiatkowski complex will be not too complicated: no tensor products will appear. For any $i \in \{1,2,3\}$ call e_i the edge vv_i, call h_i the half edge at the vertex v corrisponding to the edge e_i and h_{ij} the difference $h_i - h_j$, as required by the reduced Swiatkowski construction. Here the last three rows of the Swiatkowski complex for Y_3.

$$0 \xrightarrow{\partial_{(2,2)}} \bigoplus_{i=1}^{3} \mathbb{Z}e_i h_{12} \oplus \bigoplus_{j=1}^{3} \mathbb{Z}e_j h_{13} \xrightarrow{\partial_{(1,2)}} \bigoplus_{i,j=1}^{3} \mathbb{Z}e_i e_j \xrightarrow{\partial_{(0,2)}} 0$$

$$0 \xrightarrow{\partial_{(2,1)}} \mathbb{Z}h_{12} \oplus \mathbb{Z}h_{13} \xrightarrow{\partial_{(1,1)}} \bigoplus_{i=1}^{3} \mathbb{Z}e_i \xrightarrow{\partial_{(0,1)}} 0$$

$$0 \xrightarrow{\partial_{(2,0)}} 0 \xrightarrow{\partial_{(1,0)}} \mathbb{Z}\emptyset \xrightarrow{\partial_{(0,0)}} 0$$

Here $\partial_{(i,j)} : \tilde{S}(v)_{i,j} \to \tilde{S}(v)_{i-1,j}$ is the induced differential between the components respectively bigraded as: (i,j) and $(i-1,j)$. So, by Theorem 2.2.2, we have that:

$$H_i(UConf_j(\Gamma)) = \frac{Ker(\partial_{(i,j)})}{Im(\partial_{(i+1,j)})}.$$

In the case of the graph Y_3 it's not so difficult to make some considerations to understand better how these configuration spaces are made.

- The first fact to notice is that we will always have $H_0(UConf_n(Y_3)) \cong \mathbb{Z}$ for all n. Indeed, consider:

$$\frac{Ker(\partial_{(0,n)})}{Im(\partial_{(1,n)})} = \frac{\bigoplus_{1\leq i_1,..,i_n\leq 3}\mathbb{Z}e_{i_1}\cdots e_{i_n}}{\bigoplus_{j=2}^{3}(\bigoplus_{1\leq i_1,..,i_{n-1}\leq 3}\mathbb{Z}e_{i_1}\cdots e_{i_{n-1}}(e_1-e_j))} \cong \mathbb{Z}.$$

  It is easy to see that the relations induced when taking the quotient with the images of the elements of the middle row will always give a free $\mathbb{Z}$-module on one generator. This tells us that at least all these configuration spaces are always connected.

- When considering the first homology group things behave a bit differently. In general we have that $H_1(UConf_n(Y_3)) = Ker(\partial_{(1,n)})$ and, as $\partial_{(1,n)}$ is a linear morphism between two free modules of finite rank over $\mathbb{Z}$, the kernel is again a free $\mathbb{Z}$-module of finite rank. So no torsion can appear in this case. Moreover, when considering $H_1(UConf_2(Y_3))$ we see that $ker(\partial_{(1,2)})$ is generated by the element $a_{123} := e_1h_{23}+e_2h_{31}+e_3h_{12}$, so in particular we have that $H_1(UConf_2(Y_3)) \cong \mathbb{Z}$. Instead, when considering $n \geq 2$ particles it's not true anymore that $ker(\partial_{(1,n)})$ is free of rank 1, but the rank will be higher (for a concrete example of the growth of the rank just consider Figure 2.3 in the following pages). Notice in fact that $ker(\partial_{(1,n)}) \neq 0$ for any n as $\partial_{(1,n)}(e_i^{(n-2)}a_{123}) = 0$.

- The last easy observation is that $H_i(UConf_n(Y_3)) = 0$ for any $i \geq 2$ and any $n \in \mathbb{N}$.

In Figure 2.1 a picture of $UConf_2(Y_3)$ where it is clear that the space is homotopy equivalent to a circle: $H_1(UConf_2(Y_3)) \cong \mathbb{Z}$. The white points are to be considered as holes, and the dashed lines as open borders.

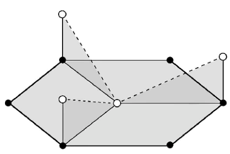
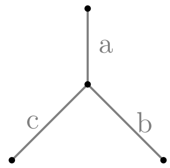
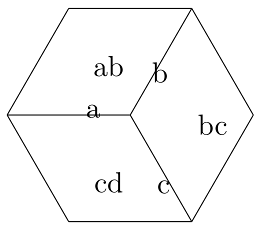


Figure 2.1: The space $UConf_2(Y_3)$.

Remark 21. To explain Figure 2.1, consider the following picture of Y_3 with th edges lableled as:



Notice that when considering configurations of two particles on Y_3 with the condition that they must lie in two different edges (and their order doesn't count), they define the space:



And when adding the configurations of two particles lying on the same edge, we are adding the missing open flags (half squares as the order doesn't count) of figure 2.1.

Remark 22. All the consideration made above for the star graph Y_3 naturally generalize to star graphs Y_k for any $k \geq 3$. In particular, these configuration spaces are always connected, their first homology groups are always torsion free and all higher homology groups vanish. Summarizing:

$$dim_h(UConf_n(Y_k)) = 1, \ \forall n \geq 2, \ k \geq 3.$$

Here with $dim_h(X)$ we intend the homological dimension of the topological space X, defined as the biggest interger n such that $H_n(X, \mathbb{Z}) \neq 0$.

This last result fits with with a more complete version of Theorem 2.2.1 (see [13]), which states:

Theorem 2.2.11. *Let Γ be a graph, $n \in \mathbb{N}$ and let $b(\Gamma)$ be the number of vertices of valence at least 3. Then:*

- *There is a cubical complex $K_n\Gamma$ of dimension $= min(b(\Gamma), n)$ such that it embeds as a deformation retract into $UConf_n(\Gamma)$;*
- *$\pi_1(UConf_n(\Gamma))$ contains a subgroup isomorphic to $\mathbb{Z}^k$, where $k = min(b(\Gamma), \lceil \frac{n}{2} \rceil)$.*

Remark 23. As a consequence we have that if $n \geq 2b(\Gamma)$ then:

$$dim_h(UConf_n(\Gamma)) = b(\Gamma).$$

This is our case as $b(Y_k) = 1$, for all $k \geq 3$. So we can conclude that configuration spaces of star graphs are homotopic to connected buquet of circles, so actually homotopic to graphs themselves (cfr. Figures 2.2 and 2.3).

Notice that when dealing with star graphs we can define the following **FI**-modules for any couple $(i, k) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$:

$$\begin{aligned} W_{i,k} \colon \mathbf{FI} &\to Mod_{\mathbb{Z}} \\ [n] &\mapsto H_i(UConf_k(Y_n)) \ . \end{aligned}$$

Indeed, for any injection $[n] \hookrightarrow [m]$ we always have an inclusion morphism $UConf_k(Y_n) \hookrightarrow UConf_k(Y_m)$ by forgetting $m-n$ edges and this induces a map in homology, for any index $i \geq 0$.

Theorem 2.2.12. *Let $k \in \mathbb{N}_{\geq 2}$. Then the* **FI***-module $W_{1,k}$ is finitely generated in degrees ≤ 3.*

Proof. We have to show that $H_1(UConf_k(Y_n))$ is generated by the images of maps $H_1(UConf_k(Y_3)) \to H_1(UConf_k(Y_n))$, induced by embeddings like $Y_3 \hookrightarrow Y_n$. We proceed by induction on n, noticing that the base cases are already proved (for any $n \leq 3$ the result is trivial). Let a be any element in $S_{1,k}(Y_n)$ such that $\partial(a) = 0$, we can write such elements as $\sum_{i=1}^{n} p_i h_{1i}$ with $p_i \in \mathbb{Z}[E(Y_n)]$ and $h_{1i} = h_1 - h_i$, difference of half edges. The condition $\partial(a) = 0$ translates as:

$$\partial(a) = \sum_{i=1}^{n} p_i(e_1 - e_i) = 0,$$

which is satisfied if and only if:

$$\sum_{i=1}^{n} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} e_i p_i = 0.$$

For each $i \leq n-2$, we write $p_i = (e_n - e_{n-1})p_i' + r_i$, where r_i does not involve the variable e_n. Now define $a_{i,n-1,n} := e_i h_{n-1n} + e_{n-1} h_{ni} + e_n h_{in-1}$ and notice that $\partial(a_{i,n-1,n}) = 0$. We can rewrite a as:

$$a = \sum_{i=1}^{n-2} (p_i' a_{i,n-1,n} + r_i h_{1i}) + q_{n-1} h_{1n-1} + q_n h_{1n},$$

where

$$q_{n-1} := p_{n-1} - \sum_{i+1}^{n-2} (e_i - e_n) p_i' \quad \text{and} \quad q_n := p_n - \sum_{i+1}^{n-2} (e_{n-1} - e_i) p_i'.$$

Now rewrite also:

$$q_{n-1} = e_n q_{n-1}' + r_{n-1} \quad \text{and} \quad q_n = e_n q_n' + r_n,$$

where r_n and r_{n-1} do not involve the variable e_n. Then considering terms involving e_n in the first two equations we have:

$$e_n(q'_{n-1} + q'_n) = 0 \quad \text{and} \quad e_{n-1}q'_{n-1} + e_n q' n + r_n = 0$$

which holds if and only if:

$$(e_n - e_{n-1})q'_n + r_n = 0.$$

Since r_n does not involve the variable e_n, it follows that $q'_n = r_n = 0$, whence $q_n = 0$. We further conclude that $q'_{n-1} = 0$ and so q_{n-1} does not involve thevariable e_n. So, $a - \sum_{i=1}^{n-2} a_{i,n-1,n}$ does not involve e_n or h_{1n} and so must lie in the image of maps induced by inclusions $Y_{n-1} \hookrightarrow Y_n$ that miss the nth edge. Thanks to the inductive step we conclude. □

Corollary 2.2.13. *Let T be any tree and $k \in \mathbb{N}_{\geq 2}$. Then $H_1(UConf_k(T))$ is generated by images of maps $H_1(UConf_k(Y_3)) \to H_1(UConf_k(T))$ induced by embeddings $Y_3 \hookrightarrow T$.*

The proof of this theorem can be found in [1]. It's a proof by induction on $b(T)$, the number of branched vertices of T. The base case ($b(T) = 1$) is proved by the previous theorem, but for the inductive step some more theory (that is not mentioned in this work) is needed.

Remark 24. Notice that corollary 2.2.13 can be restated as: the $\mathcal{T}^{op}$-module $T \mapsto H_1(UConf_k(T))$ is generated in degrees ≤ 3, for any $k \geq 2$. This could suggest that proposition 2.2.7 (when considering trees) can be seen as a direct consequence of this result, indeed consider again the submodule $T \mapsto Tor(H_1(UConf_k(T)))$ and the fact that torsion doesn't appear in $H_1(UConf_k(Y_3))$. But this is wrong: proposition 2.2.7 is proved by other means and such a conclusion would be in contrast with remark 10.

Example 2. As said before, configuration spaces of star graphs are homotopic to graphs themselves (here we will consider only the unordered configuration sapces). The definition of deformation retract, at least in the case of these particularly simple graphs, is quite easy to translate into some lines of code so that we can really visualize how these spaces changes while varing the number of edges or the number of particles considered. Here the software Wolfram Mathematica have been used to draw these graphs.
Consider configurations of n particles on a star graph with e edges. The idea is to consider the 0-cells and the 1-cells described before essentially as lists of numbers. Every edge corrispond to a number in $\{1, \dots, e\}$ and let V be the only branched vertex. We start by constructing all the 1-cells: start from the lists made of all the combinations with repetitions of $n - 1$ elements taken from $\{1, \dots, e\}$ (the fixed particles) and then for each one of them create e lists made by appending each time, as the last element, one number in $\{1, \dots, e\}$ (the "moving" particle).

```
CELL1[e_, n_] := Module[{},
comb[x_, y_] := With[{t = x, k = y},
    Join @@ Table[IntegerPartitions[s, {k}, Range[t]],
    {s, k, t k}]];
f[a_, b_] := Append[a, b];
cell[t_, k_] := Flatten[Table[Map[f[#, i] &, comb[t, k]],
    {i, 1, t}], 1];
cell[e, n - 1]]
```

Now from each one of these 1-cell we want to create the two 0-cells that are its vertices. So for each 1-cell list we create a list made of two 0-cells lists: one is the same list of the 1-cell (the "extra" particle is in the edge) and one is the list of the 1-cell where the last element is replaced by V (the "extra" particle is in the branched vertex).

```
CELL0[e_, n_] := Module[{},
g[x_] := {x, Append[Delete[x, -1], V]};
g1[z_, y_] := Map[g, CELL1[z, y]];
g2[a_, b_] := Map[Sort, Flatten[g1[a, b], 1]];
g3[c_, d_] := Partition[g2[c, d], 2];
g3[e, n]]
```

Finally just some lines of code to rewrite the couples of vertices in the way that Mathematica needs to draw a graph.

```
cub_complex[e_, n_] := Module[{},
edges[{x_, y_}] := UndirectedEdge[x, y];
edges1[a_, b_] := Map[edges, CELL0[a, b]];
cub_complex[c_, d_] := Graph[edges1[c, d]];
cub_complex[e, n]]
```

This last module gives as output a drawing of the cubical complex which embeds as a deformation retract of the unordered configuration space of n particles of the star graph on e edges. Considering the examples presented in the figures we notice that as long as the number of edges is fixed to 3 and we let n grow (Figure 2.2), Mathematica is able to draw the graph so that there are no self intersections between the edges, so for example it's really easy to visualize the "holes" of it. Instead, when the number of edges is bigger than 3 (Figure 2.3), self intersections appear and the graph becomes really messy and almost impossible to read. For the pourposes of this work what we are really interested in studying of these configuration spaces are their homology groups. When dealing with star graphs the only interesting homology group to consider is the first one, for which we already know it is free of finite rank. So, as this rank is equal to the genus of the cubical complex we just need to compute this last quantity (recall that for any graph G the genus is equal to the difference between the number of edges of G and the number of edges of a spanning tree).

```
genus[G_] := Module[{},
h1[x_] := EdgeCount[x] - EdgeCount[FindSpanningTree[x]];
h1[G]]
```

In Figure 2.4 and Figure 2.5 some examples of how the genus of the cubical complex varies when considering an increasing number of particles (n) or of edges (e), ie the growth of the rank of $H_1(UConf_n(Y_e))$.

Remark 25. By theorem 2.2.12 we know that the **FI**-module:

$$[n] \longmapsto H_1(UConf_k(Y_n))$$

is finitely generated, so by theorem 2.1.9 we expect that the growth of the rank of $H_1(UConf_k(Y_n))$ (for any fixed k) is bounded by a polynomial. In figure 2.5 we notice how the rank seems to be bounded by a cubical polynomial when $k = 3$.

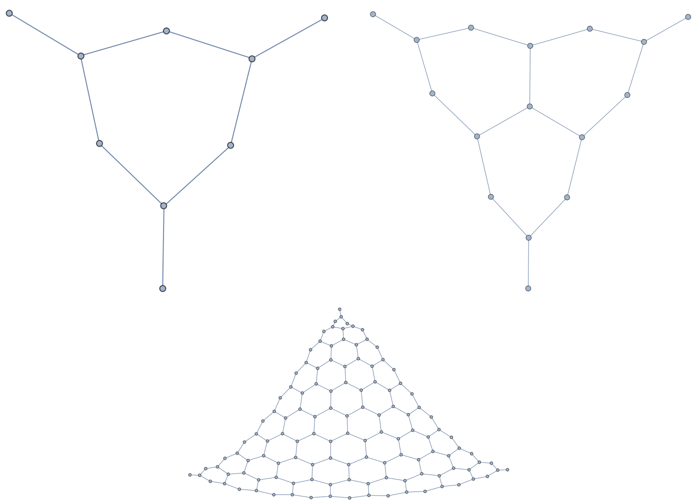


Figure 2.2: In order the cubical complexes: K_2Y_3, K_3Y_3 and $K_{10}Y_3$.

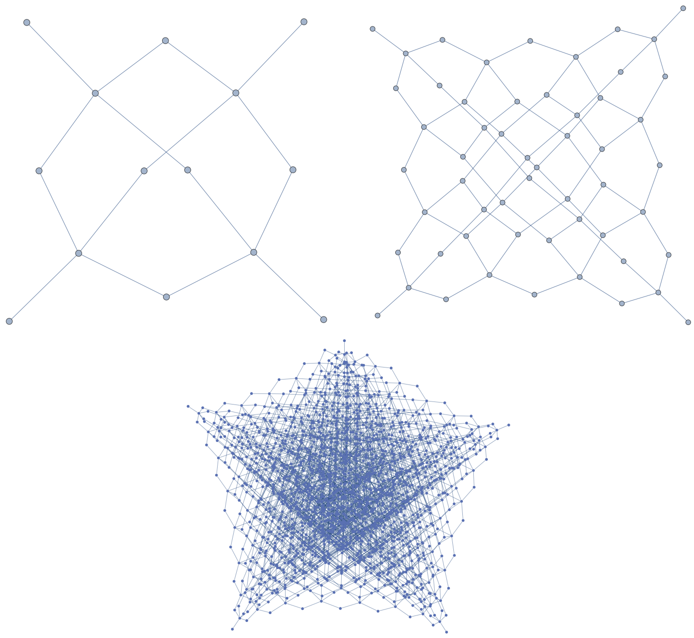


Figure 2.3: In order the cubical complexes: K_2Y_4, K_4Y_4 and K_8Y_6.

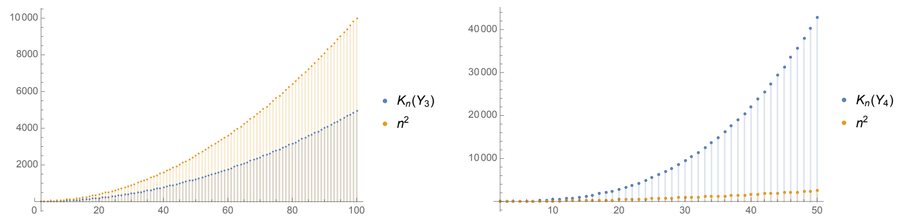


Figure 2.4: Comparison between the growth of the genus of the cubical complexes K_nY_3 and K_nY_4, also related to n^2.

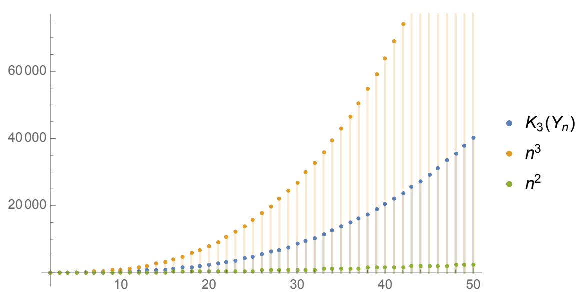


Figure 2.5: Growth of the genus of $K_3(Y_n)$ compared to n^2 and n^3.

Chapter 3

Groups acting on trees and their configuration spaces

In this last chapter we consider actions of (finite) groups on trees. We notice that these actions induce actions also at the level of configuration spaces turning their homology groups into $k[G]$-modules. We show that we can define a category out of this setting for which some Noetherianity related properties are studied. We introduce the concept of orbit configuration space of trees and we prove some relations between quotients of these spaces. We conclude this chapter by showing some examples of such quotients in the simple case of the symmetric group S_3 acting on the star graph Y_3.

3.1 A Noetherianity result

Definition 3.1 (Group action on a tree)**.** Let G be a group and T be a tree. G acts on T if there are maps

$$\begin{aligned} G \times V(T) &\to V(T) \\ (g, v) &\longmapsto g \cdot v \end{aligned}$$

$$\begin{aligned} G \times E(T) &\to E(T) \\ (g, e) &\longmapsto g \cdot e \end{aligned}$$

such that both defines a group action on the sets $V(T)$ and $E(T)$ and these actions are compatible: if $g \cdot v_1 = v_1'$ and $g \cdot v_2 = v_2'$ where v_1, v_2 are vertices of the edge e, then v_1', v_2' are vertices of the edge $g \cdot e$.

Remark 26. Such an action is called free if whenever $g \cdot x = x$ holds, it implies that $g = 1_G$.

One of the most important theorems about groups acting on trees (cfr. [12]) states that an action of a group on a tree is free if and only if the group is free itself. So in particular when considering finite groups acting on trees, these actions will never be free. We are not really interested in this theorem itself, but as the groups we will consider in the rest of the section will always be finite it's fair to point out that all the actions will never be free.

It's now natural to think whether one can define a category out of this setting and in case which kind of combinatorial properties this category could have.

Definition 3.2. Fix a group G and consider the trees T that admits a G-action. A contraction $f : T_1 \to T_2$ between such trees is G-equivariant if $f(g \cdot v) = g \cdot f(v)$ for any $v \in V(T_1)$ (and analogously for the edges). We can so define the category $G\mathcal{T}$ as the category whose objects are trees that admits a G-action and morphisms are G-equivariant contraction of trees. Analogously define $G\mathcal{RT}$ as a full subcategory of $\mathcal{RT}$.

Remark 27. Notice that $G\mathcal{T}$ is naturally a subcategory of the already defined category $\mathcal{T}$.

Theorem 3.1.1. *Let G be a group and k any Noetherian ring. The representation category $Rep_k(G\mathcal{T}^{op})$ is Noetherian.*

Proof. Recall from proposition 2.1.3 that the category **PT** is Gröbner and that the forgetful functor **PT** $\to$ **T** is essentially surjective and respect property (F). Now let G**PT** be the full subcategory of **PT** whose objects are planar rooted trees that adimts a G-action and morphisms are G-equivariant morphisms of planar rooted trees and analogously define G**T** to be the corrispondent full subcategory of **T**. Again we have that the forgetful functor G**PT** $\to G$**T** is essentially surjective and satisfies property (F). We now claim that G**PT** is Gröbner. Indeed, the faithful inclusion functor $i : G\mathbf{PT} \hookrightarrow \mathbf{PT}$ has property (S): let $f : T_1 \to T_2$ and $g : T_1 \to T_3$ be morphisms in G**PT** and let $\tilde{h} : i(T_2) \to i(T_3)$ be such that $i(g) = \tilde{h} \circ i(f)$. Then as both $i(g)$ and $i(f)$ are G-equivariant then so must be $\tilde{h}$, implying that there is $h : T_2 \to T_3$ in G**PT** such that $g = h \circ f$. Now, by proposition 1.2.11 we have that G**PT** is Gröbner and so by what observed before G**T** is quasi-Gröbner. Similarly to proposition 2.1.1 we also have that the categories G**T** and $G\mathcal{RT}^{op}$ are equivalent, so both are quasi-Gröbner. The forgetful functor $G\mathcal{RT}^{op} \to G\mathcal{T}^{op}$ is essentially surjective satisfing property (F), so $G\mathcal{T}^{op}$ is quasi-Gröbner and we can conclude that $Rep_k(G\mathcal{T}^{op})$ is Noetherian.

□

Remark 28. For any couple $(i, n) \in \mathbb{N} \geq 0 \times \mathbb{N}_{\geq 2}$ we can define the following representation:

$$\begin{aligned} GM_{i,n} \colon G\mathcal{T}^{op} &\to Mod_{\mathbb{Z}[G]} \\ T &\mapsto H_i(UConf_n(T), \mathbb{Z}). \end{aligned}$$

Since the group G is finite we have that $\mathbb{Z}[G]$ is a Noetherian ring, so this module is an object of the Noetherian category $Rep_{\mathbb{Z}[G]}(G\mathcal{T}^{op})$.

3.2 Orbit configuration spaces and quotients

Notice that when dealing with graphs in general we can always consider their embedding into a real space, the so called realization ($Real(\cdot)$). A compatible group action on a tree T naturally induces a group action on $Real(T)$. This allow us to introduce the so called orbit configuration spaces, as a group action on $Real(T)$ induces a group action on the configuration space itself.

Definition 3.3. Let G be a group acting on T a tree. We call

$$GConf_n(T) := \{(x_1, .., x_n) \in T^n : Gx_i \cap Gx_j = \emptyset \ \forall i \neq j\}$$

the n^{th}-orbit ordered configuration space. Analogously we call

$$GUConf_n(T) := \{(x_1, .., x_n) \in T^n : Gx_i \cap Gx_j = \emptyset \ \forall i \neq j\}/S_n$$

the n^{th}-orbit unordered configuration space.

Remark 29. Notice that $GConf_n(T)$ (resp. $GUConf_n(T)$) is a subspace of $Conf_n(T)$ (resp. of $UConf_n(T)$), so they are endowed with the natural subspace topology.

When considering actions of groups on spaces is always natural to ask who is the quotient space. In this case we can really consider several ways to construct some quotient spaces out of the group action on T.

- We can consider $Conf_n(T/G)$, the configuration space of the quotient tree T/G;
- Since G acts on $Real(T)$, we can consider the induced action of G on $Conf_n(T)$, and then take the quotient: $Conf_n(T)/G$. The induced actions works like this: $g \cdot (x_1, .., x_n) := (g \cdot x_1, .., g \cdot x_n)$;
- We still have a G-action on $GConf_n(T)$, so we can even consider the quotient $Gconf_n(T)/G$;
- Finally, on $GConf_n(T)$ we also have a natural action of the group G^n, defined in this way: $(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_n) = (g_1 \cdot x_1, \ldots, g_n \cdot x_n)$. Notice first that this action is not defined on $Conf_n(T)$, as it may happen that there are $g, g'; x, x'$ such that $g \cdot x = g' \cdot x'$, which is in contrast with the definition of configuration space.

One first question that naturally arises is wheter some of these spaces are related in some topologically interesting ways.

Example 3. Let T be a tree. The following holds:

$$Conf_n(T/G) \cong (GConf_n(T))/G^n.$$

Indeed, recall that $GConf_n(T) = \{(x_1, \dots, x_n) \in T^n : Gx_i \neq Gx_j \ i \neq j\} = \{(x_1, \dots, x_n) \in T^n : [x_i]_G \neq [x_j]_G \ i \neq j\}$, where by $[x]_G$ we mean the equivalence class of the element x induced by the group action G on T (which is the orbit Gx). On the other hand, we have that $Conf_n(T/G) = \{([x_1]_G, \dots, [x_n]_G) \in (T/G)^n : [x_i]_G \neq [x_j]_G \ i \neq j\}$. Consider the following map:

$$\begin{aligned} f \colon GConf_n(T)/G^n &\to Conf_n(T/G) \\ [(x_1, \dots, x_n)]_{G^n} &\longmapsto ([x_1]_G, \dots, [x_n]_G) \end{aligned}$$

First of all, f is well defined as if $(x_1, \dots, x_n) \sim_{G^n} (y_1, \dots, y_n)$ we have that there exists $(g_1, \dots, g_n \in G^n)$ such that $y_i = g_i x_i$ for any $i = 1, \dots, n$; so $f([(y_1, \dots, y_n)]_{G^n}) = ([y_1]_G, \dots, [y_n]_G) = ([x_1]_G, \dots, [x_n]_G) = f([(x_1, \dots, x_n)]_{G^n})$. Moreover, this map is continuous and has a continuous inverse defined as: $f^{-1}(([y_1]_G, \dots, [y_n]_G)) = [(y_1, \dots, y_n)]_{G^n}$ and from here the homeomorphism.

Remark 30. Notice that the same proof actually works when considering unordered configuration spaces, so:

$$UConf_n(T/G) \cong (GUConf_n(T))/G^n.$$

Now, consider the spaces $UConf_n(T)$, $UConf_n(T)/G$ and the following theorem (cfr. [10]):

Theorem 3.2.1. *Let X be a simplicial complex and G a finite group acting simplicially on it (for all $v \in V(X)$, the map $v \mapsto g \cdot v$ is a simplicial map). Then:*

$$H^k(X/G; \mathbb{Q}) \cong (H^k(X; \mathbb{Q}))^G \quad \textit{and} \quad H_k(X/G; \mathbb{Q}) \cong (H_k(X; \mathbb{Q}))_G.$$

The idea now is to show that we can apply this theorem when considering $X = UConf_n(T)$. Clearly this configuration space is not a simplicial complex, but the theorem holds when instead of $UConf_n(T)$ we consider the cubical complex $K_n(T)$. Hence, if we are able to show that we still have an homotopy equivalence when passing to the quotient ($K_n(T)/G \sim UConf_n(T)/G$) we would conclude that:

$$H^k(UConf_n(T)/G; \mathbb{Q}) \cong (H^k(UConf_n(T); \mathbb{Q}))^G$$

$$H_k(UConf_n(T)/G; \mathbb{Q}) \cong (H_k(UConf_n(T); \mathbb{Q}))_G.$$

Definition 3.4. Let G be a group. A *G-CW* structure on a toplogical space X with an action of G is a *CW* structure on X such that cells are permuted by the group action.

Theorem 3.2.2 (cfr. [7]). *A map $f : X \to Y$ of G-CW complexes is a G-equivariant homotopy equivalence (ie, the quotients X/G and Y/G are still homotopic) if and only if $f^H : X^H \to Y^H$ is an homotopy equivalence for every $H \leq G$.*

We want to apply this theorem with $X = K_n(T)$, $Y = UConf_n(T)$ and f the embedding $i : K_n(T) \to UConf_n(T)$ described in section 2.2.1. To do so we need to do three things: show that both $K_n(T), UConf_n(T)$ have the structure of G-CW complexes, show that the embedding i is G-equivariant and finally that all the i^H are still homotopy equivalences.

1. $K_n(T)$ has naturally a CW structure. We need to show that G permutes its cells. Let $F = (f, S)$ a cell, seeing $K_n(T)$ as embedded inside $UConf_n(T)$ we have that the action of $g \in G$ sends $F = (f, S)$ to the cell $g \cdot F = (f \circ g^{-1}, g(S))$. Here g denotes both the element of the group and its corrispondent automorphism in $Aut(K_n(T))$. So $g \cdot F$ is again a cell of the same dimension of F. Now we need to give a CW-structure on $UConf_n(T)$. Let the 0-cells be all the possible configurations of n points on T, the 1-cells be all the possible configurations of $n-1$ points on T together with the choice of an edge where a particle is moving, and so on similarly to how we defined the poset $P_n(T)$. This gives a CW-structure on $UConf_n(T)$ (notice that this CW-complex is not finite) which, similarly to what just said for $K_n(T)$, is also a G-CW-structure.

2. Recall that the embedding $i : K_n(T) \to UConf_n(T)$ was defined from the collection of embeddings $i_F : F \to UConf_n(T)$, for every face F. We just need to show that any such i_F is G-equivariant, ie that $i_{g \cdot F}(g \cdot x) = g \cdot i_F(x)$. This becomes clear when writing them down:

$$g \cdot i_F(x) = \{g \cdot b : f(b) = 1\} \cup \bigcup_{e \in E_T} D_{g \cdot e}(\tilde{f}(e); (\tau_0(x)(s))_{|s|=e})$$

   and

$$i_{g \cdot F}(g \cdot x) = \{b : f \circ g^{-1}(b) = 1\} \cup \bigcup_{e \in E_T} D_e(\widetilde{f \circ g^{-1}}(e); (\tau_0(g \cdot x)(s))_{|g \cdot s|=e}).$$

3. Consider $i^H : K_n(T)^H \to UConf_n(T)^H$. By adapting the definitions of the retraction and of the homotopy, already used in a previous section, restricting them to $K_n(T)^H$ and $UConf_n(T)^H$ this gives an embedding of $K_n(T)^H$ as a deformation retract into $UConf_n(T)^H$, for any $H \leq G$. From here the homotopy equivalences required by the theorem.

We conclude by proving theorem 3.2.1

Proof of Theorem 3.2.1. As over $\mathbb{Q}$ homolgy and cohomolgy are one the dual to the other, by proving one of the two claims, automatically is proved also the other. Indeed, suppose $H^k(X/G, \mathbb{Q}) \cong (H^k(X, \mathbb{Q}))^G$ then:

$$H_k(X/G, \mathbb{Q})^* \cong H^k(X/G, \mathbb{Q}) \cong (H^k(X, \mathbb{Q}))^G \cong$$

$$\cong (H_k(X, \mathbb{Q})^*)^G \cong (H_k(X, \mathbb{Q})_G)^*,$$

which implies that:

$$H_k(X/G, \mathbb{Q}) \cong (H_k(X, \mathbb{Q}))_G.$$

We implicitely used the fact that given a vector space V and a group G acting on it, then $(V^*)^G \cong (V_G)^*$ (the proof basically relies on the isomorphism $(V/W)^* \cong W^0$). Subdividing X appropriately we can assume that X/G is again a simplicial complex with p-simplices in bijection with G-orbits of p-simplices of X, for any $p \geq 0$. The action of G on X makes the simplicial cochain complex $C^*(X, \mathbb{Q})$ into a cochain complex of $\mathbb{Q}[G]$-modules. By the bijection mentioned above we have:

$$C^*(X/G, \mathbb{Q}) \cong (C^*(X, \mathbb{Q}))^G.$$

Indeed, in general recall that the simplicial chain $C_k(X)$ is a free abelian group with base elements given by the k-simplices, the isomorphism is given by:

$$C_k(X/G; \mathbb{Q}) \to (C_k(X, \mathbb{Q})^G$$
$$[\sigma_k] \longmapsto \sum_{g \in G} g \cdot \sigma_k$$

and then dualized to pass to cochains. The theorem now is a direct consequence of the following lemma.

Lemma 3.2.3. *Let G be a finite group an C^* a cochain complex of $\mathbb{Q}[G]$-modules. Then:*

$$(H^\bullet(C^*))^G \cong H^\bullet((C^*)^G).$$

Proof. Whenever G is a finite group it has finitely many irreducible representations over $\mathbb{Q}$, call them $V_1, \dots, V_p$. For any $\mathbb{Q}[G]$-module we can hence decompose it uniquely as $V = V_1^{k_1} \oplus \cdots \oplus V_p^{k_p}$. So each term of C^* has a decomposition of this type, in which p just depend on the group G, so p is "the same" for all terms. This decomposition is respected by the coboundary map so the component V_i of $H^k(C^*)$ is isomorphic to the k^{th} cohomology module of the cochain complex made by the components V_i of all the terms of C^*. This in particular is true when consdiering the trivial representation, hence when considering $V_i = V^G$, and we conclude. ☐

Summarizing we can state the following theorem.

Theorem 3.2.4. *Let G be a finite group and T a tree. Then for any couple $(k, n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 2}$:*

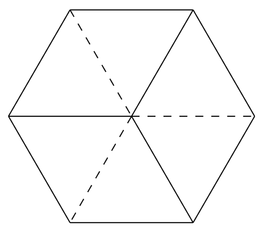
$$H^k(UConf_n(T)/G; \mathbb{Q}) \cong (H^k(UConf_n(T); \mathbb{Q}))^G$$
$$H_k(UConf_n(T)/G; \mathbb{Q}) \cong (H_k(UConf_n(T); \mathbb{Q}))_G.$$

3.3 Example: S_3 acting on Y_3

Now some examples to show how the various quotients mentioned above look like in the simple case of the symmetric group S_3 acting on the star graph Y_3 by permuting the 3 edges (notice that the central vertex is a global fixed point).

- Consider the space: $UConf_2(Y_3/S_3)$. Notice that the quotient tree Y_3/S_3 is the one edge tree, so $UConf_2(Y_3/S_3)$ can be seen as a square minus the diagonal and then folded (identified) along it, hence homotopic to a point.

- Consider the space: $S_3UConf_2(Y_3)$. This is the subspace of $UConf_2(Y_3)$ made by cutting out all and only the configurations of two points lying on two different edges at the same distance from the central vertex. Recall remark 21, from that picture we need to cut along the following dashed diagonals:



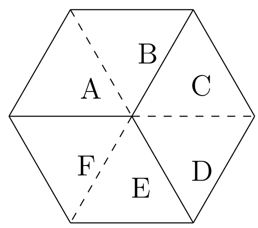
  which gives us a space homotopic to three points.

- Consider the space $S_3UConf_2(Y_3)/S_3^2$. This space is given by folding (identifing) in the previous picture all the 6 triangles to a single one of them, and then folding it together with the 'flag' at one of his sides. This actually is a space homeomorphic to the space obtained by the identification of the two triangles coming from cutting the diagonal of a square. So as in remark 30: $S_3UConf_2(Y_3) \cong UConf_2(Y_3/S_3)$.

- Consider the space $S_3UConf_2(Y_3)/S_3$. This space is given by the following foldings (identifications):

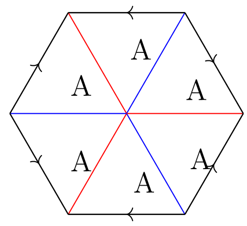
  - Triangle $F \to$ triangle A, along their only common edge;
  - Rhombus $ED \to$ rhombus BC, along the dashed edge they have 'in common';

– The triangle $C \to$ triangle B, along their only common edge.



This give a space homotopic to two points (a square minus the diagonal).

- Consider the space $UConf_2(Y_3)/S_3$. This space is given by the following identifications, where we identify objects of the same colours, arrows (respecting the direction) and triangles with the same letter.



These identifications give a space which is homotopic to a single point.

All the pictures are always missing the 'flags', but their behaviour in these quotient spaces should be clear enough from what said.

Open questions

1. By the end of chapter 1 we introduced the linear categories $\mathbf{VI}_R$ and $\mathbf{VA}_R$ and proved that their representation categories are Noetherian. What can we say about Noetherianity when considering other similar linear categories? Consider for example the category of finitely generated modules over a finite ring R.

2. Recall remark 28. Notice that even though we know that the $\mathcal{T}^{op}$-modules

$$\begin{aligned} M_{i,n} \colon \mathcal{T}^{op} &\to Mod_{\mathbb{Z}} \\ T &\mapsto H_i(UConf_n(T), \mathbb{Z}) \end{aligned}$$

are finitely generated, it's not straightforward to get the same result for the corrispondent $G\mathcal{T}^{op}$-modules $GM_{i,n}$. Notice that a similar approach to the one used for proving the same result for the category $\mathcal{T}^{op}$ (cfr. theorem 2.2.5) may not work. Even if we can define the $G\mathcal{T}^{op}$-module $T \mapsto \tilde{S}(T)_{i,n}$ (the Swiatkowski complex admits naturally a G-action), we can't adapt the proof to our case: that proof relies on the fact that we can consider simple contractions that do not contract a 'distinguished' edge, which is somethiung we don't know in our case. Notice in fact that in the category $G\mathcal{T}^{op}$ we don't have much control over the contractions (not all trees admits a G-action). So, what can we say about the finitely generation of the modules $GM_{i,n}$? (Notice that this would have nice implications as the category $Rep_{\mathbb{Z}[G]}(G\mathcal{T}^{op})$ is Noetherian)

3. As said in remark 16, it's not natural to define a morphism:

$$H_i(UConf_n(T_2)) \to H_i(UConf_n(T_1)),$$

induced by a contraction $T_1 \to T_2$. So, is it possible to define a $G\mathcal{T}^{op}$-module $T \mapsto H_i(GUConf_n(T))$? If yes, is it a submodule of $GM_{i,n}$?

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