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The formal loop space approach to classical and quantum Hamiltonian systems

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Abstract

In this thesis we focus on the study of the formal loop space and its applications to classical and quantum Hamiltonian systems. In particular in the first part of this work we generalize the basic tools of finite dimensional differential geometry to the formal loop space defining in this environment the notions of function, Poisson bracket between functions, coordinate transformation, multivector and Poisson cohomology. The Second part is spent on the exposition and the proof of two fundamental theorems on Poisson geometry of the formal loop space: the Dubrovin and Getzler Theorems. These results allow to simplify the form of Poisson brackets of a particular type (called hydrodynamic) by means of an appro- priate change of coordinates on the formal loop space. In particular the Getzler theorem can be viewed as a generalization of the Weinstein theorem in finite dimensional Poisson geometry.

Contents

Introduction

In this thesis we will discuss the goemetry of a particular infinite dimensional manifold, the loop space. This object can be constructed considering all the loops on a given finite dimensional manifold M, i.e. the maps of the type $u : S_1 \to M$ (one can choose a degree of regularity for loops). We will not discuss structural aspects of the loop space (as the topology) but we are interested in a algebraic formal treatment in order to describe a Poisson structure on it. In this work the manifold will consists in a finite dimensional vector space. In the first chapter there will be a simple recap of the finite dimensional Poisson geometry. The second chapter will be dedicated to the introduction of the basic notions characterazing the formal loop space. The basic object is the following ring of polynomials:

$$\widehat{\mathcal{A}} = \mathbb{C}[[u^*]][u^*_{k>0}][[\epsilon]],$$

where $u^*_{k>0}$ denotes formally the k-derivative of the loop map. Given this ring, called the differential polynomials ring, one can define the notion of function (called local functional) on the formal loop space. The space of local functionals will be denoted with $\widehat{\Lambda}$, while its elements will be suggestively denoted as

$$\int f(u^*, u^*_*)dx,$$

and the reason of this notation will be clarified in details. After giving the concept of function, we will be ready to define the Poisson bracket structure on the formal loop space: as in the finite dimensional case this will be a map from $\widehat{\Lambda} \times \widehat{\Lambda}$ to $\widehat{\Lambda}$ and it will be given by the following formula:

$$\{f; g\} = \int dx \frac{\delta f}{\delta u^\mu} K^{\mu\nu} \frac{\delta g}{\delta u^\nu} \quad f, g \in \widehat{\Lambda},$$

where $\frac{\delta}{\delta u^\mu}$ is the so called variational derivative and

$$K^{\mu\nu} = \sum_{n \geq 0} K^{\mu\nu}_n \partial^n_x$$

is called Hamiltonian operator (the $K^{\mu\nu}_n$ coefficients are differential polynomials). We will impose a fixed degree for the $K^{\mu\nu}_n$ coefficients and the Hamiltonian operators satisfying this degree constraint are called of Hydrodynamic type. In particular the $\epsilon \to 0$ limit of an Hamiltonian operator of hydrodynamic type is simply given by the following relation:

$$K^{\mu\nu}_{|_{\epsilon=0}} = g^{\mu\nu}(u^*)\partial_x + b^{\mu\nu}_\gamma(u^*)u^\gamma_x,$$

where $b^{\mu\nu}_\gamma(u^*) = -g^{\mu\alpha}\Gamma^\nu_{\alpha\gamma}$. The notation for the two coefficients reminds the one used for a metric tensor and an affine connection. This is not accidental as we will see. Another important concept in the study of differential geometry is the change of coordinates transformation: we will be able to extend this notion also to the formal loop space. The ϵ parameter is introduced to allow the correct definition of this type of transformation. The group of change of coordinates transformations is called Miura group. The last part of the second chapter will be dedicated to extend the notions of k-form and k-multivector to the formal loop space. The last chapter is the main chapter. Indeed the last part of this work will be dedicated to the detailed discussion of two important theorems regarding the structure of Hamiltonian operators of hydrodynamic type. Let us start from the first one. This result is known as Dubrovin-Novikov theorem ([6]) and, in a certain sense, allows us to associate some finite dimensional differential objects to the Hamiltonian operators of hydrodynamic type. In particular we will consider the $\epsilon = 0$ limit of Hamiltonian operators of hydrodynamic type written above. First of all we will show that $g^{\mu\nu}$ and $\Gamma^{\mu\nu}_{\gamma}$ transform respectively as a (2,0) tensor and an affine connection (as we have said before the notation was not accidental). After that, we will be ready to prove the theorem. The theorem states that an hydrodynamic type Poisson bracket (with the Hamiltonian operator in the $\epsilon = 0$ limit and $g^{\mu\nu}$ non degenerate) satisfies the antisymmetry and the Jacobi conditions if and only if

- $g^{\mu\nu} = g^{\nu\mu}$, i.e. $g_{\mu\nu}$ is a metric on the target space V.
- $\Gamma^{\gamma}_{\mu\ \nu}$ are the Christoffel symbols corresponding to the Levi-Civita connection of the metric $g_{\mu\nu}$.
- The curvature tensor associated to $\Gamma^{\gamma}_{\mu\nu}$ vanishes.

Therefore an Hamiltonian operator of hydrodynamic type associated to a Poisson bracket satisfying the antisymmetry and Jacobi conditions can be transformed (using the flat coordinates) in such a way that the zero order of its ϵ expansion is of the form

$$\eta^{\mu\nu}\partial_x,$$

where $\eta^{\mu\nu}$ is a non degenerate constant symmetric matrix. We will also comment another similar result, known as Gringberg conditions ([9]), regarding the case in which $g^{\mu\nu}$ can be degenerate. The second theorem, known as Getzler theorem ([8]), can be viewed as a generalization of a famous result of the finite dimensional Poisson geometry, the Weinstein theorem. It states that there exists a Miura transformation bringing any Hamiltonian operator of hydrodynamic type to the canonical form

$$K^{\mu\nu} = \eta^{\mu\nu}\partial_x,$$

where $\eta^{\mu\nu}$ is a non degenerate constant symmetric matrix. We will prove it in all the details. Therefore the Dubrovin-Novikov theorem allows us to transform the zero order of the ϵ expansion of an hydrodynamic type Hamiltonian operator in the canonical form written above, while the Getzler theorem allows us to get rid of all the other orders of the expansion. We will prove this result in all the details. At the end of this work we will have proved two powerful tools for the study of this formal enviroment. In the conclusion we will comment some application of this formalism.

Chapter 1

Summary of finite-dimensional Poisson geometry

In this first section we give a brief recap of the Poisson manifold in the finite dimension case.

Definition. *A* ***Poisson algebra*** $(P, \{\cdot, \cdot\})$ *is a commutative associative algebra* P *with a Lie nracket* $\{\cdot, \cdot\}$ *satisfying the Leibniz rule:*

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{1.0.1}$$

for any $f, g, h \in P$.

Definition. *A* ***Poisson Manifold*** $(M, \{\cdot, \cdot\})$ *is a smooth manifold with a structure of Poisson algebra* $\{\cdot, \cdot\}$ *on the commutative associative algebra* $C^\infty(M)$.

It can be proven (due to the Leibniz rule) that for any Poisson bracket it exists a unique tensor field $\Pi : TM \wedge TM$ such that $\{f, g\} = \Pi(df, dg)$ for any $f, g \in P$. In a local system of coordinates $x^1, ..., x^n$ the Poisson bracket can be written in the following way:

$$\Pi(x) = \Pi^{ij}(x) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \Leftrightarrow \Pi^{ij}(x) = \{x^i, x^j\}(x) \Leftrightarrow \{f, g\}(x) = \Pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \tag{1.0.2}$$

where the derivates are calculated in x and the Jacobi identity reads

$$\frac{\partial \Pi^{ij}}{\partial x^k} \Pi^{kl} + \frac{\partial \Pi^{jl}}{\partial x^k} \Pi^{ki} + \frac{\partial \Pi^{li}}{\partial x^k} \Pi^{kj} = 0 \tag{1.0.3}$$

Definition. *A* ***Casimir function*** *on a Poisson manifold* $(M, \{,\})$ *is a function* $f \in C^\infty(M)$ *satisfying* $\{f, g\} = 0$ *for any* $g \in C^\infty(M)$.

Now we present a theorem that it's important in the study of the Poisson geometry.

Theorem (Weinstein). *Let* (M, Π) *a Poisson manifold and* $p \in M$. *There exists a chart* $(U, x^1, ..., x^n, \zeta_1, ..., \zeta_n, y^1, ..., y^k)$ *with* $p \in U$ *such that*

$$\pi(x) = \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \zeta_i} + \frac{1}{2} c^{ij}(x) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}, \tag{1.0.4}$$

with $c^{ij}(p) = 0$. *Moreover if* Π *has constant rank* $2n$ *and corank* k *we can choose a Weinstein chart on* M $(U, x^1, ..., x^n, \zeta_1, ..., \zeta_n, y^1, ..., y^k)$ *such that* $c^{ij} = 0$ *for any element of* U.

Another important tool in the study of the Poisson manifold is the **Schouten-Nijenhuis bracket**.

Definition. *Let $\Lambda^k := \Gamma(\Lambda^k TM)$ be the space of multivectors. The **Schouten-Nijenhuis bracket** is the unique bilinear pairing*

$$[\cdot,\cdot] : \Lambda^n \times \Lambda^m \to \Lambda^{n+m-1} \tag{1.0.5}$$

that satisfies the following properties:

- $[f, g] = 0 \quad \forall f, g \in C^\infty$.
- *The restriction of $[\cdot,\cdot]$ on $\Lambda^1 \times \Lambda^1$ coincides with the Lie bracket of vector fields (the SN bracket is an extention of the Lie bracket of vector fields to $\Lambda^n \times \Lambda^m$).*
- $[B, A] = (-1)^{nm}[A, B] \quad$ *with* $\quad A \in \Lambda^n, B \in \Lambda^m$ *(graded antisymmetry condition).*
- $[A, B \wedge C] = [A, B] \wedge C + (-1)^{m(n+1)} B \wedge [A, C] \quad$ *with* $\quad A \in \Lambda^n, B \in \Lambda^m$ *(graded Leibniz rule).*

Remark. *One can show that Π antysimmetric is Poisson if and only if $[\Pi, \Pi] = 0$ and that the following relation holds (called **graded Jacobi identity**):*

$$(-1)^{km}[[A, B], C] + (-1)^{lm}[[C, A], B] + (-1)^{kl}[[B, C], A] = 0 \quad \textit{with} \quad A \in \Lambda^k, B \in \Lambda^l, C \in \Lambda^m. \tag{1.0.6}$$

This definitions allow us to define the **Poisson cohomology**. Indeed one can be prove the following result:

Theorem. *Let $\Pi \in \Lambda^2$. Then*

$$[\Pi, [\Pi, A]] = 0 \quad \forall A \in \Lambda^n \tag{1.0.7}$$

So if we consider the following sequence

$$.... \xrightarrow{d_\Pi} \Lambda^{n-1} \xrightarrow{d_\Pi} \Lambda^n \xrightarrow{d_\Pi} \Lambda^{n+1} \xrightarrow{d_\Pi} .... \tag{1.0.8}$$

with $d_\Pi = [\Pi, \cdot]$, we obtain a cochain complex and it's natural to define the **Poisson cohomology** as

$$H^n_\Pi = \frac{Ker(d_\Pi : \Lambda^n \to \Lambda^{n+1})}{Im(d_\Pi : \Lambda^{n-1} \to \Lambda^n)}. \tag{1.0.9}$$

Remark.

$$H^0_\Pi = Cas(M) \tag{1.0.10}$$

and

$$H^2_\Pi = \frac{\{\Delta \in \Lambda^2 | \quad [\Pi, \Delta] = 0\}}{\{\Delta \in \Lambda^2 | \quad \Delta = \mathcal{L}_X \Pi \quad for \quad some \quad X \in \Lambda^1\}}. \tag{1.0.11}$$

As last step of this chapter we recall the flat coordinates theorem, useful in the following:

Theorem. *Let M be a manifold (of dimension n) and ∇ a connection on TM. Then a local system of coordinates $(y_1, ..., y_n)$ such that $\nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha} = 0 \quad \forall \alpha, \beta = 1, ..., n$ exists if and only if the curvature tensor R and the torsion tensor T vanish.*

As a consequence of this theorem, we have the following corollary:

Corollary. *Let M be a (pseudo-)Riemannian manifold . If the tensors T and R vanish, there exists a system of coordinates in which the metric tensor $g_{\mu\nu}$ is constant.*

Chapter 2

Formal loop space

The loop space is the set of functions $u : S^1 \to V$ where V is a N-dimensional vector space with a basis $e_1, ..., e_N$ and x is the coordinate on S^1, so that $u^\alpha = u^\alpha(x)$ is the component along e_α of such loop. One can describe this kind of space intruducing some structures on it. For example one can try to define a topology on it. However we are not interested in this kind of problems but instead in a formal algebraic treatment. So in the following we introduce the objects that it allows us to carry on this kind of anlysis. In this chapter the reference works are [12] and [7].

2.1 Differential polynomials and local functional

Definition. *Let K be a commutative ring. A formal power series is a sequence $\{a_n\}_{n\in\mathbb{N}} \subset K$ which we write as $a = \sum_{n=0}^{\infty} a_n u^n$. We denote the set of formal series as $K[[u]]$ and it inherits the structure of a ring from K.*

The constituent elements of the formal loop space are the so called **differential polynomials**.

Definition. *The ring of **differential polynomials** is*

$$\widehat{\mathcal{A}} = \mathbb{C}[[u^*]][u^*_{k>0}][[\epsilon]], \tag{2.1.1}$$

i.e the set of formal series with respect to the ϵ parameter where the commutative ring is the polynomials ring in $u^*_{k>0}$ with coefficients consisting in formal series in u^* (with $\mathbb{C}$ as commutative ring).

We will denote in some circumstances u_k with $u\underbrace{x....x}_{k-times}$. The role of the ϵ parameter will be clarified below. We endow $\widehat{\mathcal{A}}$ with the grading

$$deg(\epsilon) = -1 \quad and \quad deg(u^\alpha_{k>0}) = k, \tag{2.1.2}$$

and we denote by $\widehat{\mathcal{A}}^{[d]}$ the set of homogeneous differential polynomials of degree d. An essential object in the study of the formal loop space is the natural extention to $\widehat{\mathcal{A}}$ of the x-derivative, i.e. the operator $\partial_x : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ defined as

$$\partial_x := \sum_{k\geq 0} u^\alpha_{k+1} \frac{\partial}{\partial u^\alpha_k}, \tag{2.1.3}$$

where we adopt the Einstein convention for the Greek indeces. This operator has some important properties that we could use in the following: it satisfies the Leibniz rule for the product of two elements of $\widehat{\mathcal{A}}$ and therefore

$$\partial_x^n(f\cdot g)=\sum_{k=0}^{n}\binom{n}{k}\partial_x^k f\cdot\partial_x^{n-k}g \quad with \quad f,g\in\widehat{\mathcal{A}}. \tag{2.1.4}$$

Lemma. *It holds the following identity in $\widehat{\mathcal{A}}$:*

$$\left[\frac{\partial}{\partial u_k^\alpha};\partial_x\right]=\frac{\partial}{\partial u_{k-1}^\alpha} \quad with \quad \frac{\partial}{\partial u_{-1}^\alpha}\equiv 0. \tag{2.1.5}$$

Proof.

$$\begin{aligned}\left[\frac{\partial}{\partial u_k^\alpha};\partial_x\right]f &= \sum_{n\geq 0}\frac{\partial}{\partial u_k^\alpha}\left[u_{n+1}^\beta\frac{\partial f}{\partial u_n^\beta}\right]-\sum_{n\geq 0}u_{n+1}^\beta\frac{\partial}{\partial u_n^\beta}\left[\frac{\partial f}{\partial u_k^\alpha}\right]= \\ &=\sum_{n\geq 0}\frac{\partial f}{\partial u_n^\beta}\delta^{\alpha\beta}\delta_{n+1,k}=\frac{\partial f}{\partial u_{k-1}^\alpha}\end{aligned} \tag{2.1.6}$$

□

As a consequence of this result we have the following corollary.

Corollary (Exchange property)**.** *For any $f,g\in\widehat{\mathcal{A}}$ we have:*

$$\sum_{n\geq 0}\frac{\partial(\partial_x f)}{\partial u_n^\alpha}\partial_x^n g=\partial_x\left(\sum_{n\geq 0}\frac{\partial f}{\partial u_n^\alpha}\partial_x^n g\right) \tag{2.1.7}$$

Proof.

$$\begin{aligned}\sum_{n\geq 0}\frac{\partial(\partial_x f)}{\partial u_n^\alpha}\partial_x^n g &= \sum_{n\geq 0}\partial_x\left(\frac{\partial f}{\partial u_n^\alpha}\right)\partial_x^n g+\sum_{n\geq 1}\frac{\partial f}{\partial u_{n-1}^\alpha}\partial_x^n g= \\ &=\sum_{n\geq 0}\partial_x\left(\frac{\partial f}{\partial u_n^\alpha}\right)\partial_x^n g+\sum_{n\geq 0}\frac{\partial f}{\partial u_n^\alpha}\partial_x^{n+1}g= \\ &=\sum_{n\geq 0}\partial_x\left(\frac{\partial f}{\partial u_n^\alpha}\partial_x^n g\right)= \\ &=\partial_x\left(\sum_{n\geq 0}\frac{\partial f}{\partial u_n^\alpha}\partial_x^n g\right).\end{aligned} \tag{2.1.8}$$

□

Now we can define the set whose elements can be interpreted as functionals defined through S_1-integration over $\widehat{\mathcal{A}}$:

Definition. *The space*

$$\widehat{\Lambda}=\widehat{\mathcal{A}}/(im\partial_x\oplus\mathbb{C}[[\epsilon]]). \tag{2.1.9}$$

*is called **space of local functional** and its elemnts are called **local functionals**.*

$\widehat{\Lambda}^{[d]}$ will denote the d degree part of $\widehat{\Lambda}$. We can interpret the map $[\cdot] : f \in \widehat{\mathcal{A}} \longrightarrow [f] \in \widehat{\Lambda}$ as a formal integral functional defined over $\widehat{\mathcal{A}}$ because it's the simplest map satisfying the most basic property of the integral defined on a space of loops, i.e. the linearity and the fact that $[\partial \widehat{\mathcal{A}}] = 0$ (i.e. $\int (\partial_x f) dx = 0$) (see [5]). According to this intepretation, the equivalence class of $f(u^*_*, \epsilon) \in \widehat{\mathcal{A}}$ will be denoted as $\bar{f} = \int f(u^*_*, \epsilon) dx$. As a consequence of this definition, we have that $\int f' g dx = -\int f g' dx$ since $\partial_x(f)g + \partial_x(g)f = \partial_x(fg)$. We will call this property **integration by parts**. As last step we introduce the concept of change of coordinates in the formal loop space.

Definition. *A **change of coordinates** transformation is a differential polynomials of the form*

$$\tilde{u}^\alpha = \tilde{u}^\alpha(u^*_*, \epsilon) \in \widehat{\mathcal{A}}^{[0]} \quad \textit{with} \quad \textit{det}\left(\frac{\partial \tilde{u}^*|_{\epsilon=0}}{\partial u^*}\right). \tag{2.1.10}$$

Now it's clear why we have introduced the parameter ϵ: its importance lies in the fact that it allows us to invert the change of coordinates transformation solving the ODE $\tilde{u}^\alpha = \tilde{u}^\alpha(u^*_*, \epsilon)$ order by order in ϵ through the formal Frobenius method. What we obtain is the differential polynomial $u^\alpha = u^\alpha(\tilde{u}^*_*, \epsilon)$. So, introducing the ϵ parameter, we are able to invert every change of coordinates transformation defined above and then the set of this transformation is a group that we denote with $\mathcal{M}$. This group is called **Miura group**.

2.2 Poisson bracket

In this section we want to introduce a Poisson structure on the space of local functional $\widehat{\Lambda}$. In order to do that we have to define a new object, the so called **variational derivative**.

Definition. *The* ***variational derivative*** *is the operator* $\frac{\delta}{\delta u^\alpha}:\widehat{\mathcal{A}}\to\widehat{\mathcal{A}}$ *defined as*

$$\frac{\delta}{\delta u^\alpha} := \sum_{k\geq 0}(-\partial_x)^k \circ \frac{\partial}{\partial u^\alpha_k}. \tag{2.2.1}$$

Proposition. $\frac{\delta}{\delta u^\alpha}(Im\partial_x \oplus \mathbb{C}) = 0$ *for any* $\alpha = 1, ..., N$.

Proof.

$$\begin{aligned}\frac{\delta}{\delta u^\alpha}\circ\partial_x &= \sum_{k\geq 0}(-\partial_x)^k \circ \frac{\partial}{\partial u^\alpha_k}\circ\partial_x = \\ &= \sum_{k\geq 0}(-\partial_x)^k \circ \left(\partial_x\circ\frac{\partial}{\partial u^\alpha_k} + \frac{\partial}{\partial u^\alpha_{k-1}}\right) = \\ &= \sum_{k\geq 1}(-\partial_x)^k \circ \frac{\partial}{\partial u^\alpha_{k-1}} + \sum_{k\geq 0}(-1)^k(\partial_x)^{k+1}\circ\frac{\partial}{\partial u^\alpha_k} = \\ &= \sum_{k\geq 0}(-\partial_x)^{k+1} \circ \frac{\partial}{\partial u^\alpha_k} - \sum_{k\geq 0}(-\partial_x)^{k+1}\circ\frac{\partial}{\partial u^\alpha_k} = 0\end{aligned} \tag{2.2.2}$$

□

This result is important because it says us that $\frac{\delta}{\delta u^\alpha}$ is well defined also on $\widehat{\Lambda}$. Therefore $\frac{\delta}{\delta u^\alpha}:\widehat{\Lambda}\to\widehat{\mathcal{A}}$. Another important property for the following parts is the Leibniz rule for the variational derivative (see [2]).

Proposition (Leibniz rule)**.** *For any* $f, g \in \widehat{\mathcal{A}}$ *we have*

$$\frac{\delta}{\delta u^\alpha}(f\cdot g) = \sum_{k\geq 0}\mathcal{T}_{\alpha,k}f\cdot(-\partial_x)^k g + \sum_{k\geq 0}(-\partial_x)^k f\cdot\mathcal{T}_{\alpha,k}g, \tag{2.2.3}$$

where

$$\mathcal{T}_{\alpha,k} := \sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\circ\frac{\partial}{\partial u^\alpha_n}. \tag{2.2.4}$$

Proof.

$$\begin{aligned}\frac{\delta}{\delta u^\alpha}(f\cdot g) &= \sum_{n\geq 0}(-\partial_x)^k\left(\frac{\partial f}{\partial w^\alpha_n}\cdot g + f\cdot\frac{\partial g}{\partial w^\alpha_n}\right) = \\ &= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}(-\partial_x)^{n-k}\frac{\partial f}{\partial w^\alpha_n}\cdot(-\partial_x)^k g + \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}(-\partial_x)^{n-k}\frac{\partial g}{\partial w^\alpha_n}\cdot(-\partial_x)^k f = \\ &= \sum_{k\geq 0}\mathcal{T}_{\alpha,k}f\cdot(-\partial_x)^k g + \sum_{k\geq 0}(-\partial_x)^k f\cdot\mathcal{T}_{\alpha,k}g,\end{aligned} \tag{2.2.5}$$

since

$$\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}.... = \sum_{k\geq 0}\sum_{n\geq k}\binom{n}{k}.... \tag{2.2.6}$$

□

Remark.

$$\mathcal{T}_{\alpha,0} = \frac{\delta}{\delta u^\alpha}. \tag{2.2.7}$$

Lemma.

$$\mathcal{T}_{\alpha,k}\circ\partial_x = \mathcal{T}_{\alpha,k-1} \quad \forall k\in\mathbb{N}, \tag{2.2.8}$$

where $\mathcal{T}_{\alpha,k-1} := \mathcal{T}_{\alpha,0}\circ\partial_x = 0$.

Proof. If $k\geq 1$

$$\begin{aligned}
\mathcal{T}_{\alpha,k}\circ\partial_x &= \sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\circ\frac{\partial}{\partial u_n^\alpha}\circ\partial_x = \\
&= \sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k+1}\circ\frac{\partial}{\partial u_n^\alpha} + \sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\circ\frac{\partial}{\partial u_{n-1}^\alpha} = \\
&= \sum_{n\geq k}\left(\binom{n+1}{k}-\binom{n}{k}\right)(-\partial_x)^{n-k+1}\circ\frac{\partial}{\partial u_n^\alpha} + \frac{\partial}{\partial u_{k-1}^\alpha} = \\
&= \sum_{n\geq k}\binom{n}{k-1}(-\partial_x)^{n-k+1}\circ\frac{\partial}{\partial u_n^\alpha} + \frac{\partial}{\partial u_{k-1}^\alpha} = \\
&= \sum_{n\geq k-1}\binom{n}{k-1}(-\partial_x)^{n-k+1}\circ\frac{\partial}{\partial u_n^\alpha} = \\
&= \mathcal{T}_{\alpha,k-1}.
\end{aligned} \tag{2.2.9}$$

If $k=0$

$$\mathcal{T}_{\alpha,-1} = \mathcal{T}_{\alpha,k}\circ\partial_x = \sum_{n\geq 0}(-\partial_x)^{n+1}\circ\frac{\partial}{\partial u_n^\alpha} + \sum_{n\geq 1}(-\partial_x)^{n}\circ\frac{\partial}{\partial u_{n-1}^\alpha} = 0. \tag{2.2.10}$$

□

Now we're ready to define the Poisson structure on $\widehat{\Lambda}$.

Definition. *The* ***Poisson bracket*** *on the space of local functional* $\widehat{\Lambda}$ *is defined as*

$$\begin{aligned}
&\{\cdot;\cdot\}_K\colon \widehat{\Lambda}\times\widehat{\Lambda}\to\widehat{\Lambda} \\
&\bar{f},\bar{g}\in\widehat{\Lambda}\mapsto\{\bar{f};\bar{g}\}_K := \int dx\frac{\delta\bar{f}}{\delta u^\mu}K^{\mu\nu}\frac{\delta\bar{g}}{\delta u^\nu}\in\widehat{\Lambda}
\end{aligned} \tag{2.2.11}$$

where

$$K^{\mu\nu} = \sum_{j\geq 0}K_j^{\mu\nu}\partial_x^j \quad \textit{with} \quad K_j^{\mu\nu}\in\widehat{\mathcal{A}}^{[-j+1]}. \tag{2.2.12}$$

The differential operator K is called **Hamiltonian operator**. We will see in the following sections that imposing a fixed degree for the $K_j^{\mu\nu}$ coefficients allows us to avoid convergence problems. We can lift the bracket $\{\cdot;\cdot\}_K$ to a new map defined in the following way:

$$\begin{aligned} \{\cdot;\cdot\}: \quad & \widehat{\mathcal{A}} \times \widehat{\Lambda} \to \widehat{\mathcal{A}} \\ f \in \widehat{\mathcal{A}}, \bar{g} \in \widehat{\Lambda} \mapsto \{f;\bar{g}\} = & \sum_{s\geq 0} \frac{\partial f}{\partial u_s^\mu} \partial_x^s \left(K^{\mu\nu} \frac{\delta \bar{g}}{\delta u^\nu} \right) \in \widehat{\mathcal{A}}, \end{aligned} \tag{2.2.13}$$

compatible with $\{\cdot;\cdot\}_K$ since $\int \{f;\bar{g}\}dx = \{\bar{f};\bar{g}\}_K$ (where $\bar{f} = \int dx f$). Indeed

$$\begin{aligned} \int \{f;\bar{g}\}dx &= \int dx \sum_{s\geq 0} \frac{\partial f}{\partial u_s^\mu} \partial_x^s \left(K^{\mu\nu} \frac{\delta \bar{g}}{\delta u^\nu} \right) = \\ &= \int dx \sum_{s\geq 0} (-\partial_x)^s \left(\frac{\partial f}{\partial u_s^\mu} \right) K^{\mu\nu} \frac{\delta \bar{g}}{\delta u^\nu} = \\ &= \int dx \frac{\delta f}{\delta u^\mu} K^{\mu\nu} \frac{\delta \bar{g}}{\delta u^\nu} = \{\bar{f};\bar{g}\}_K, \end{aligned} \tag{2.2.14}$$

where we've integrated by part iteratively and we've used the fact that the variational derivative is well defined on $\widehat{\Lambda}$. The operator in $\widehat{\mathcal{A}}$

$$\{\cdot;\bar{g}\}, \tag{2.2.15}$$

defined $\forall \bar{g} \in \widehat{\Lambda}$, will be indicated as $D_{\bar{g}}\cdot$.

Remark. *It's obvious from the definition given above that*

$$\int dx D_{\bar{g}} f = \{\bar{f};\bar{g}\}_K \quad \textit{with} \quad f \in \widehat{\mathcal{A}} \quad \textit{and} \quad \bar{f} = \int dx f. \tag{2.2.16}$$

Therefore one can define $D_{\bar{g}}\bar{f} := \int dx D_{\bar{g}} f$.

The fact the we've chosen the $K_j^{\mu\nu} \in \widehat{\mathcal{A}}^{[-j+1]}$ implies that

$$K^{\mu\nu}|_{\epsilon=0} = g^{\mu\nu}(u)\partial_x + b_\gamma^{\mu\nu}(u)u_x^\gamma, \tag{2.2.17}$$

as simple consenquence of the degree counting. We will always assume that $g^{\mu\nu}$ is non-degenerate. This kind of Poisson brackets are called brackets of **hydrodynamic type**. We will see in the next chapter that requiring the antisymmetry condition and the Jacobi identity for the Poisson bracket imposes some interesting conditions on $g^{\mu\nu}(u)$ and $b_\gamma^{\mu\nu}(u)$. This result is known as **Dubrovin-Novikov theorem**. Let us report an useful lemma for the next discussion.

Lemma. *For a Poisson bracket on the formal loop space* $\{\cdot;\cdot\}_K$*, the Jacobi condition is equivalent to*

$$[D_{\bar{g}}; D_{\bar{h}}]\, \bar{f} = D_{\{\bar{g};\bar{h}\}_K} \bar{f}, \tag{2.2.18}$$

where $[D_{\bar{g}}; D_{\bar{h}}] \cdot = D_{\bar{g}} (D_{\bar{h}} \cdot) - D_{\bar{h}} (D_{\bar{g}} \cdot)$.

Proof.

$$\begin{aligned} 0 &= D_{\bar{g}} \left(D_{\bar{h}} \bar{f} \right) - D_{\bar{h}} \left(D_{\bar{g}} \bar{f} \right) - D_{\{\bar{h};\bar{g}\}_K} \bar{f} = \\ &= \{\{\bar{f};\bar{h}\}_K;\bar{g}\}_K - \{\{\bar{f};\bar{g}\}_K;\bar{h}\}_K - \{\bar{f};\{\bar{h};\bar{g}\}_K\}_K = \\ &= \{\{\bar{f};\bar{h}\}_K;\bar{g}\}_K + \{\{\bar{g};\bar{f}\}_K;\bar{h}\}_K + \{\{\bar{h};\bar{g}\}_K;\bar{f}\}_K. \end{aligned} \tag{2.2.19}$$

□

Now we will define how an Hamiltonian operator transforms under a Miura transformation: if $K^{\mu\nu}$ is the Hamiltonian operator associated to a certain Poisson bracket, its Miura transformation under $\tilde{u} \in \mathcal{M}$ is

$$K^{\mu\nu}_{\mathcal{M}} = (L^*)^{\mu}_{\alpha} \circ K^{\alpha\beta} \circ L^{\nu}_{\beta}, \tag{2.2.20}$$

where

$$L^{\nu}_{\beta} = \sum_{s\geq 0} (-\partial_x)^s \circ \frac{\partial \tilde{u}^{\nu}}{\partial u^{\beta}_s} \quad \textit{and} \quad (L^*)^{\mu}_{\alpha} = \sum_{s\geq 0} \frac{\partial \tilde{u}^{\mu}}{\partial u^{\alpha}_s} \partial_x{}^s. \tag{2.2.21}$$

As last step of the section concerning the Poisson brackets we will prove some important theorems regarding the connection between local functional and differential polynomials (see [11]). The first one is the following:

Theorem (First variational principle)**.** *Let $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$ be a differential polynomials. If*

$$\int dx f g = 0 \quad \forall g \in \mathbb{C}[[u^*]][u^*_{k>0}], \tag{2.2.22}$$

then $f = 0$.

Proof. Since $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$, it has the structure $f = f(u^*, u^*_1, ..., u^*_n)$, where n is the highest value reached by k in f. Choosing $g = 1$, we have that

$$\int dx f = 0 \Rightarrow f = \partial_x h, \tag{2.2.23}$$

for some $h \in \mathbb{C}[[u^*]][u^*_{k>0}]$. This implies that f depends on u^*_n linearly. Indeed if f will depend on $(u^*_n)^l$ with $l > 1$, h would depend on $(u^*_n)^{l+1}$ and therefore f should depend on u^*_{n+1}. But n is the highest value reached by k in f. So u^*_n can appear in f only linearly. Moreover f (if it isn't vanishing) has to depend at least on u^*_1 since it's the image of the ∂_x operator. So let us assume that f is non vanishing ($n \geq 1$). Now, choosing $g = f$, we obtain

$$\int dx f^2 = 0 \Rightarrow \frac{\delta}{\delta u^{\alpha}}(f^2) = 0 \quad \forall \alpha \in \{1, ..., N\}, \tag{2.2.24}$$

since the image of the ∂_x operator is contained in the nucleous of the variational derivative. Defining $f^{(n)}_{\alpha} = f^{(n)}_{\alpha}(u^*, u^*_1, ..., u^*_n)$ the coefficient of u^{α}_n, we have (due to the linear dependence of f on u^{α}_n)

$$0 = \frac{\partial}{\partial u^{\alpha}_{2n}} \left(\frac{\delta}{\delta u^{\alpha}}(f^2) \right) = (-1)^n \frac{\partial^2}{\partial u^{\alpha\,2}_n}(f^2) = 2(-1)^n (f^{(n)}_{\alpha})^2. \tag{2.2.25}$$

So $f^{(n)}_{\alpha} = 0 \quad \forall \alpha \in \{1, ..., N\}$. But this is a contradiction since we are assuming non trivial dependence of f on $u^{\alpha}_{n\geq 1}$. Therefore $f = 0$. □

We can generalize this theorem to $\widehat{\mathcal{A}}$.

Theorem. *Let $f \in \widehat{\mathcal{A}}$ be a differential polynomials. If*

$$\int dx f g = 0 \quad \forall g \in \widehat{\mathcal{A}}, \tag{2.2.26}$$

then $f = 0$.

Proof. Since $f \in \widehat{\mathcal{A}}$, it has the following structure:

$$f = \sum_{k\geq 0} f_k \epsilon^k \quad with \quad f_k \in \mathbb{C}[[u^*]][u^*_{k>0}]. \tag{2.2.27}$$

Therefore, restricting the choice of g to $\in \mathbb{C}[[u^*]][u^*_{k>0}]$, from the hypothesis condition we get:

$$\begin{aligned} & fg = \partial_x h_f \quad with \quad \widehat{\mathcal{A}} \ni h_f = \sum_{k\geq 0} h_k \epsilon^k \qquad (h_k \in \mathbb{C}[[u^*]][u^*_{k>0}]) \Rightarrow \\ & \Rightarrow \sum_{k\geq 0} (f_k g)\, \epsilon^k = \sum_{k\geq} (\partial_x h_k)\, \epsilon^k \Rightarrow f_k g = \partial_x h_k \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.2.28}$$

Now we can apply the previous theorem to f_k obtaining that $f_k = 0 \quad k \in \mathbb{N}$. Therefore $f = 0$. □

The second theorem is very important for the following parts: we will use it in the proof of the Dubrovin-Novikov theorem. We call it **second variational principle**.

Theorem (Second variational principle)**.** *Let $X^\mu \in \mathbb{C}[[u^*]][u^*_{k>0}]$ with $\forall \mu \in \{1, ..., N\}$. If for any $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$ we have*

$$\int dx X^\mu \frac{\delta f}{\delta u^\mu} = 0, \tag{2.2.29}$$

then $X^\mu = c u_1^\mu$ with $c \in \mathbb{C}$.

Proof. Let us denote $Z_\nu(f) = \frac{\delta}{\delta u^\nu}\left(X^\mu \frac{\delta f}{\delta u^\mu}\right)$. In our hypothesis $Z_\nu(f) = 0$ for any $\nu \in \{1, ..., N\}$ and $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$. Calling P the highest value reached by the derivative loop index of X^μ, if it's not vanishing, we know that X^μ depends linearly on u^*_P since $\int dx X^\mu = 0$ (choose $f = u^\mu$) and that $P \geq 1$, as discussed in the previous proof. Therefore X^μ has the following form:

$$X^\mu = X^\mu_\nu\left(u^*, ..., u^*_{P-1}\right) u^\nu_P + X^\mu_0\left(u^*, ..., u^*_{P-1}\right). \tag{2.2.30}$$

We are now ready to prove the statement. The relations that we will use in the following part will be proven in the appendix. From now on the Einstein convention isn't at work. Let us suppose that X^μ isn't vanishing and that $P = 2p$ is a even number. Then, choosing $f = \frac{(-1)^p}{2}(u^\mu_p)^2$, we get

$$\frac{\partial Z_\mu(f)}{\partial u^\nu_{2P}} = X^\mu_\nu + \delta^{\mu\nu} X^\mu_\mu = 0. \tag{2.2.31}$$

Therefore $X^\mu_\nu = 0$ for all the values of the indices, in contradiction with our assumptions. So let us consider $P = 2p+1$ an odd number. Choosing $f = \frac{(-1)^{p+1}}{2}(u^\mu_{p+1})^2$, we obtain the following relations (they hold for $P \geq 1$):

$$\begin{cases} 0 = \dfrac{\partial Z_\mu(f)}{\partial u^\nu_{2P+1}} = X^\mu_\nu - \delta^{\mu\nu} X^\mu_\mu \\ 0 = \dfrac{\partial^2 Z_\mu(f)}{\partial u^\mu_{2P+1} \partial u^\nu_P} = (1 + 2\delta^{\mu\nu}) \dfrac{\partial X^\mu_\mu}{\partial u^\nu_{P-1}} \\ 0 = \dfrac{\partial Z_\mu(f)}{\partial u^\mu_{2P}} = 2 \dfrac{\partial X^\mu_0}{\partial u^\mu_{P-1}} + \partial_x X^\mu_\mu. \end{cases} \tag{2.2.32}$$

The first relation of (2.2.32) implies that $X^{\mu}_{\nu} = 0$ for any $\nu \neq \mu$, while the second one of (2.2.32) says us that X^{μ}_{μ} doesn't depend on u^*_{P-1}. Moreover, choosing $f = \frac{(-1)^{p+1}}{6}(u^{\mu}_{p+1})^3$, we get for $P \geq 3$

$$\begin{aligned}\frac{\partial Z_{\mu}(f)}{\partial u^{\mu}_{2P}} &= \left(2\frac{\partial X^{\mu}_{0}}{\partial u^{\mu}_{P-1}} + \partial_x X^{\mu}_{\mu}\right) u^{\mu}_{p+1} - P X^{\mu}_{\mu} u^{\mu}_{p+2} = \\ &= -P X^{\mu}_{\mu} u^{\mu}_{p+2} = 0,\end{aligned} \tag{2.2.33}$$

where we've used the third relation of (2.2.32) found before. It follows that $X^{\mu}_{\mu} = 0$ for any value of the indices, in contradiction with our assumption. Up to this point we've proved that $P \notin \mathbb{N} - \{1\}$. Regarding the $P = 1$ case, firstly we can prove that $\sum_{\mu=1}^{N} c u^{\mu}_{1} \frac{\delta f}{\delta u^{\mu}} = \partial_x h_f$ for any $c \in \mathbb{C}$ and $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$, with $h_f \in \mathbb{C}[[u^*]][u^*_{k>0}]$ (we will prove this fact in the appendix). Therefore, for any $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$, we have

$$\int dx \sum_{\mu=1}^{N} c u^{\mu}_{1} \frac{\delta f}{\delta u^{\mu}} = 0. \tag{2.2.34}$$

So $\sum_{\mu=1}^{N} c u^{\mu}_{1} \frac{\delta f}{\delta u^{\mu}}$ can be a possible form of X^{μ}. Now we will prove that this form is the most general. According to the three relations found above (they are true for $P \geq 1$), we know that the most general form could be $X^{\mu} = c^{\mu} u^{\mu}_{1} + X^{\mu}_{0}(u^*)$, with $\frac{\partial X^{\mu}_{0}}{\partial u^{\mu}} = 0$ and $c^{\mu} \in \mathbb{C}$. Choosing $f = \frac{1}{2}(u^{\mu})^2$, we obtain that

$$\begin{aligned}0 = Z_{\mu}(f) &= \frac{\delta}{\delta u^{\mu}} \left(c^{\mu} u^{\mu}_{1} u^{\mu} + X^{\mu}_{0} u^{\mu}\right) = \\ &= c^{\mu} u^{\mu}_{1} - \partial_x (c^{\mu} u^{\mu}) + \frac{\partial X^{\mu}_{0}}{\partial u^{\mu}} u^{\mu} + X^{\mu}_{0} = X^{\mu}_{0}.\end{aligned} \tag{2.2.35}$$

This means that $X^{\mu} = c^{\mu} u^{\mu}_{1}$. Finally, choosing $f = f(u^*)$, we have

$$\begin{aligned}0 = Z_{\nu}(f) &= \frac{\delta}{\delta u^{\nu}} \left(\sum_{\mu=1}^{N} c^{\mu} u^{\mu}_{1} \frac{\partial f}{\partial u^{\mu}}\right) = \sum_{\mu=1}^{N} c^{\mu} \left(u^{\mu}_{1} \frac{\partial^2 f}{\partial u^{\nu} \partial u^{\mu}} - \delta^{\mu\nu} \partial_x \left(\frac{\partial f}{\partial u^{\mu}}\right)\right) = \\ &= \sum_{\mu=1}^{N} (c^{\mu} - c^{\nu}) u^{\mu}_{1} \frac{\partial^2 f}{\partial u^{\nu} \partial u^{\mu}}.\end{aligned} \tag{2.2.36}$$

Since this relation holds for any $f = f(u^*) \in \mathbb{C}[[u^*]][u^*_{k>0}]$, we must have $c^{\mu} = c^{\nu}$ for any value of the indices. So we have proved that $c u^{\mu}_{1}$ with $c \in \mathbb{C}$ is the most general form for X^{μ}. This ends the proof. □

Remark. *This result holds almost in the same way in* $\widehat{\mathcal{A}}$. *Indeed, writing explicitly* X^{μ} *as series of powers of* ϵ *with coefficients in* $\mathbb{C}[[u^*]][u^*_{k>0}]$ *and choosing* $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$, *we obtain:*

$$\sum_{k\geq 0} \epsilon^k \int dx X^{\mu}_{k} \frac{\delta f}{\delta u^{\mu}} = 0 \iff \int dx X^{\mu}_{k} \frac{\delta f}{\delta u^{\mu}} = 0 \quad \forall k \geq 0. \tag{2.2.37}$$

Finally, applying the previous theorem to X^{μ}_{k} *for any* $k \geq 0$, *we get:*

$$X^{\mu} = \sum_{k\geq 0} c_k u^{\mu}_{1} \epsilon^k = \underbrace{\left(\sum_{k\geq 0} c_k \epsilon^k\right)}_{c(\epsilon)} u^{\mu}_{1} = c(\epsilon) u^{\mu}_{1}. \tag{2.2.38}$$

2.3 Hamiltonian integrable system

Definition. *An **Hamiltonian evulotionary PDE** is a formal partial differential equations of the form*

$$\partial_t u^\alpha = \{u^\alpha; \bar{h}\} = K^{\alpha\nu}\frac{\delta \bar{h}}{\delta u^\nu} \quad \textit{with} \quad \bar{h} \in \widehat{\Lambda}^{[0]} \quad \textit{and} \quad \alpha = 1, ..., N. \tag{2.3.1}$$

$\bar{h}$ is called Hamiltonian of the system. The solution of this equations is a formal power series of the form $u^\alpha(x, t^*, \epsilon) \in \mathbb{C}[[x, t^*, \epsilon]]$.

Given this notion, we can define the concept of integrable system on the formal loop space.

Definition. *An **integrable system**, or an **integrable hierarchy**, is an infinite system of Hamiltonian evulotionary PDEs*

$$\partial_{t^\beta_d} u^\alpha = \{u^\alpha; \bar{h}_{\beta,d}\} = K^{\alpha\nu}\frac{\delta \bar{h}_{\beta,d}}{\delta u^\nu}, \tag{2.3.2}$$

where the generating Hamiltonians of the system $\bar{h}_{\beta,d} \in \widehat{\Lambda}^{[0]}$ with $\beta = 1, ..., N, d \geq 0$ satisfy

$$\{\bar{h}_{\alpha,i}; \bar{h}_{\beta,j}\} = 0 \quad \forall \alpha, \beta, i, j. \tag{2.3.3}$$

The solution of this system is a formal power series of the form $u^\alpha(x, t^*_*, \epsilon) \in \mathbb{C}[[x, t^*_*, \epsilon]]$.

2.4 Forms on formal loop space

In this section we will introduce the notion of **k-forms** on the formal loop space. Let us start by giving the definition. In first place we denote with δu_s^α ($\alpha \in \{1, ..., N\}, s \geq 0$) the generators of the forms space. The formal wedge product between the generators is introduced imposing the standard exchange property, i.e.

$$\delta u_{s_1}^{\alpha_1} \wedge .... \wedge u_{s_k}^{\alpha_k} = (-1)^{N_\sigma} \delta u_{s_{\sigma(1)}}^{\alpha_{\sigma(1)}} \wedge .... \wedge u_{s_{\sigma(k)}}^{\alpha_{\sigma(k)}}, \tag{2.4.1}$$

with $\sigma \in P_k$ and N_σ denoting the number of exchanges associated to σ. Therefore a **k-form** is defined as

$$\omega := \frac{1}{k!} \sum_{s_1,....,s_k \geq 0} \omega_{\alpha_1 s_1;....;\alpha_k s_k} \delta u_{s_1}^{\alpha_1} \wedge .... \wedge \delta u_{s_k}^{\alpha_k}, \tag{2.4.2}$$

where $\omega_{\alpha_1, s_1, ..., \alpha_k, s_k} \in \widehat{\mathcal{A}}$ are differential polynomials antisymmetric w.r.t the simoultaneous permutations

$$\alpha_p, s_p \leftrightarrow \alpha_q, s_q. \tag{2.4.3}$$

For the previous formula we require that only a finite number of coefficients are non vanishing. The reason of this choice will be clarified later. We will indicate the space of the k-forms with $\widehat{\mathcal{A}}_k$ and with $\mathcal{F} = \bigoplus_{k\geq 0} \widehat{\mathcal{A}}_k$ the space of formal forms. It's clear that $\widehat{\mathcal{A}}_0 = \widehat{\mathcal{A}}$. The exterior product between forms is introduced tracing the finite dimensional case: indeed, if $\omega \in \widehat{\mathcal{A}}_k$ and $\zeta \in \widehat{\mathcal{A}}_l$, then $\omega \wedge \zeta \in \widehat{\mathcal{A}}_{k+l}$ and it is given by

$$(\omega \wedge \zeta)_{\alpha_1 s_1, ..., \alpha_{k+l} s_{k+l}} = \frac{1}{k!l!} \sum_{\sigma \in P_{k+l}} (-1)^{N_\sigma} \omega_{\alpha_{\sigma(1)} s_{\sigma(1)}; ....; \alpha_{\sigma(k)} s_{\sigma(k)}} \zeta_{\alpha_{\sigma(k+1)} s_{\sigma(k+1)}; ....; \alpha_{\sigma(k+l)} s_{\sigma(k+l)}}. \tag{2.4.4}$$

This wedge product has the same property of the one defined in the finite dimensional case. We can extend the action of ∂_x to $\mathcal{F}$ implementing the following rules:

$$\begin{aligned} \partial_x \delta u_s^\alpha &= \delta u_{s+1}^\alpha \\ \partial_x(\omega_1 \wedge \omega_2) &= \partial_x \omega_1 \wedge \omega_2 + \omega_1 \wedge \partial_x \omega_2. \end{aligned} \tag{2.4.5}$$

This allows us to define the space of $\widehat{\Lambda}_k$ as

$$\widehat{\Lambda}_k = \widehat{\mathcal{A}}_k / (im\partial_x \oplus \mathbb{C}[[\epsilon]]). \tag{2.4.6}$$

Also here $\widehat{\Lambda}_0 = \widehat{\Lambda}$. Another important tool in the study of k-forms on the formal loop space is the differential $\delta : \widehat{\mathcal{A}}_k \to \widehat{\mathcal{A}}_{k+1}$ ($\forall k \geq 0$), defined in the same way of the finite dimensional case, i.e.

$$\delta\omega = \frac{1}{k!} \sum_{s_1,....,s_k \geq 0} \left(\sum_{t \geq 0} \frac{\partial \omega_{\alpha_1 s_1;....;\alpha_k s_k}}{\partial u_t^{\alpha_0}} \delta u_t^{\alpha_0} \right) \wedge \delta u_{s_1}^{\alpha_1} \wedge .... \wedge \delta u_{s_k}^{\alpha_k}. \tag{2.4.7}$$

As in the finite dimensional case, the differential δ satisfies the relation $\delta^2 = 0$ in $\mathcal{F}$. Indeed, for any $\omega \in \mathcal{F}$, we have

$$\begin{aligned} \delta^2 \omega &= \sum_{s_1,....,s_k,t,p \geq 0} \frac{\partial^2}{\partial u_t^\alpha \partial u_p^\beta} \delta u_t^\alpha \wedge \delta_p^\beta \wedge \delta u_{s_1}^{\alpha_1} \wedge .... \wedge \delta u_{s_k}^{\alpha_k} = \\ &= - \sum_{s_1,....,s_k,t,p \geq 0} \frac{\partial^2}{\partial u_t^\alpha \partial u_p^\beta} (\omega_{\alpha_1 s_1;....;\alpha_k s_k}) \delta u_p^\beta \wedge \delta_t^\alpha \wedge \delta u_{s_1}^{\alpha_1} \wedge .... \wedge \delta u_{s_k}^{\alpha_k} = \\ &= -\delta^2 \omega. \end{aligned} \tag{2.4.8}$$

This implies that $\delta^2\omega = 0$ for any $\omega \in \mathcal{F}$. Since

$$\delta \circ \partial_x w = \partial_x \circ \delta w \quad \forall w \in \mathcal{F}, \tag{2.4.9}$$

the differential is also well defined as operator $\delta : \widehat{\Lambda}_k \to \widehat{\Lambda}_{k+1}$ ($\forall k \geq 0$). This is explained by the following lemma:

Lemma.

$$[\delta; \partial_x] = 0 \quad in \quad \mathcal{F}. \tag{2.4.10}$$

Proof.

$$\begin{aligned}
\delta \circ \partial_x w = {} & \frac{1}{k!}\delta\left(\sum_{s_1,\ldots,s_2\geq 0} \partial_x(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k})\delta u^{\alpha_1}_{s_1}\wedge \ldots \wedge \delta u^{\alpha_k}_{s_k} + \sum_{s_1,\ldots,s_k\geq 0} \omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\partial_x(\delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k})\right) = \\
= {} & \frac{1}{k!}\sum_{s_1,\ldots,s_k,t\geq 0} \frac{\partial}{\partial u^\alpha_t}\left(\partial_x\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\delta u^\alpha_t \wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k} + \frac{1}{k!}\sum_{s_1,\ldots,s_k,t\geq 0} \frac{\partial}{\partial u^\alpha_t}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\cdot \\
& \cdot \delta u^\alpha_t \wedge \partial_x(\delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k}) = \\
= {} & \frac{1}{k!}\sum_{s_1,\ldots,s_k,t\geq 0} \partial_x\left(\frac{\partial}{\partial u^\alpha_t}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\right)\delta u^\alpha_t \wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k} + \frac{1}{k!}\sum_{s_1,\ldots,s_k\geq 0,t\geq 1} \frac{\partial}{\partial u^\alpha_{t-1}}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\cdot \\
& \cdot \delta u^\alpha_t \wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k} - \frac{1}{k!}\sum_{s_1,\ldots,s_k\geq 0,t\geq 0} \frac{\partial}{\partial u^\alpha_t}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\delta u^\alpha_{t+1}\wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k} + \\
& + \frac{1}{k!}\sum_{s_1,\ldots,s_k,t\geq 0} \frac{\partial}{\partial u^\alpha_t}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\partial_x(\delta u^\alpha_t \wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k}) = \\
= {} & \frac{1}{k!}\sum_{s_1,\ldots,s_k,t\geq 0} \partial_x\left(\frac{\partial}{\partial u^\alpha_t}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\right)\delta u^\alpha_t \wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k} + \frac{1}{k!}\sum_{s_1,\ldots,s_k,t\geq 0} \frac{\partial}{\partial u^\alpha_t}\left(\omega_{\alpha_1 s_1;\ldots;\alpha_k s_k}\right)\cdot \\
& \cdot \partial_x(\delta u^\alpha_t \wedge \delta u^{\alpha_1}_{s_1}\wedge\ldots\wedge\delta u^{\alpha_k}_{s_k}) = \partial_x \circ \delta w
\end{aligned} \tag{2.4.11}$$

□

An useful theorem regarding the differential δ is the following one (we will not prove this result):

Theorem. *The differential δ produces an exact sequence, i.e. $\delta\omega = 0$ for $\omega \in \widehat{\mathcal{A}}_k$ iff $\omega = \delta\omega'$ for a $\omega' \in \widehat{\mathcal{A}}_{k-1}$ (for $k \geq 1$).*

Another important observation is that, for every $[\omega] \in \widehat{\Lambda}_k$, there exists a representative of the form

$$\tilde{\omega} = \frac{1}{(k-1)!}\sum_{s_1,\ldots,s_k} \tilde{\omega}_{\alpha_1;\alpha_2 s_2;\ldots;\alpha_k s_k} \wedge \delta u^{\alpha_1} \wedge \delta u^{\alpha_2}_{s_2} \wedge \ldots \wedge \delta u^{\alpha_k}_{s_k}, \tag{2.4.12}$$

where $\tilde{\omega}_{\alpha_1;\alpha_2 s_2;\ldots;\alpha_k s_k}$ is obtained moving away the derivatives from $u^{\alpha_1}_{s_1} = \partial^s_x \delta^{\alpha_1}$ (for any $s \geq 1$) in ω through integration by parts and dividing the result by k. The coefficients $\tilde{\omega}_{\alpha_1;\alpha_2 s_2;\ldots;\alpha_k s_k}$ will be called **reduced component** of $[\omega]$. The reduced components are still antysimmetric under the simoultaneous exchange of the pairs $(\alpha_p s_p)$ with $p > 1$. We report only the explicit formula of the reduced components for the 2-forms since it will be useful in the following parts. The coefficients are given by the formula

$$\tilde{\omega}_{\alpha_1;\alpha_2 s} = \frac{1}{2}\sum_{r=0}^{s}\sum_{t\geq s-r}(-1)^t \binom{t}{s-r}\partial_x^{t+r-s}\omega_{\alpha_1 t;\alpha_2 r}. \tag{2.4.13}$$

One can prove that the reduced components have also another exchange property, i.e.

$$\tilde{\omega}_{\alpha_2,\alpha_1 s} = \sum_{t\geq s}(-1)^{t+1}\partial_x^{t-s}\tilde{\omega}_{\alpha_1,\alpha_2 t}. \tag{2.4.14}$$

Finally it is useful for the following parts to write down explicitly the condition $\delta\omega = 0$ in the case of a 2-forms whose components are of the reduced type. This condition, for a $\omega \in \widehat{\mathcal{A}}_2$ whose components are of the reduced type, reads

$$\left(\sum_{m=s}^{t+s}\sum_{r=0}^{m-s} + \sum_{m\geq t+s+1}\sum_{r=0}^{t}\right)(-1)^m \binom{m}{r \quad s}\partial_x^{m-r-s}\left(\frac{\partial\tilde{\omega}_{\alpha;\beta t-r}}{\partial u_m^\gamma}\right) + \frac{\partial\tilde{\omega}_{\gamma;\alpha s}}{\partial u_t^\beta} - \frac{\partial\tilde{\omega}_{\gamma;\beta t}}{\partial u_s^\alpha} = 0, \tag{2.4.15}$$

for any $\alpha, \beta, \gamma \in \{1, ..., N\}$ and $s, t \geq 0$ and where $\binom{m}{r \quad s}$ is the multinomial coeffient defined in the following way:

$$\binom{s}{t_1....t_k} = \frac{s!}{t_1!....t_k!(s - t_1 - .... - t_k)!} \quad with \quad s \geq \sum_{s=1}^{k} t_s. \tag{2.4.16}$$

2.5 Multivectors on formal loop space

In this section we will introduce the notion of **multivectors** on the formal loop space with special emphasis for the concept of **local bivectors**, necessary for the proof of the Getzler theorem. Let us start introducing the space of the differential polynomials depending on multiple loop indeces:

$$\widehat{\mathcal{A}}^{tot} = \bigoplus_{k\geq 0} \widehat{\mathcal{A}}^k, \tag{2.5.1}$$

where

$$\widehat{\mathcal{A}}^k = [[u(x_1)^*_*, ..., u(x_k)^*_*]] \left[u(x_1)^*_{p_1\geq 0}, ..., u(x_k)^*_{p_k\geq 0}\right] [[\epsilon]] \tag{2.5.2}$$

and $x_1, ..., x_k \in S_1$. We can also generalize the space of local funtionals in the following way:

$$\widehat{\Lambda}^{tot} = \bigoplus_{k\geq 0} \widehat{\Lambda}^k, \tag{2.5.3}$$

where

$$\widehat{\Lambda}^k = \widehat{\mathcal{A}}^k / (\mathbb{C} \oplus Im\partial_{x_1} \oplus .... \oplus Im\partial_{x_k}). \tag{2.5.4}$$

An elements $f(u(x_1), ..., u(x_k), ...) \in \widehat{\Lambda}^k$ will be denoted as

$$\int f(u(x_1), ..., u(x_k), ...) dx_1 .... dx_k. \tag{2.5.5}$$

The gradation of these spaces is a simple generalization of the one given for $\widehat{\mathcal{A}}$.

Remark. *It's obvious that*

$$\widehat{\mathcal{A}}_0 = \widehat{\mathcal{A}}^1 = \widehat{\mathcal{A}} \quad and \quad \widehat{\Lambda}_0 = \widehat{\Lambda}^1 = \widehat{\Lambda}. \tag{2.5.6}$$

Here $\widehat{\Lambda}^0 = \widehat{\mathcal{A}}^0 = \mathbb{C}$.

Let us introducethe generators of the space of multivectors. They are denoted with $\frac{\partial}{\partial u^\alpha_s}$ ($\alpha \in \{1, ..., N\}, s \geq 0$). The formal wedge product between the generators is introduced imposing the standard exchange property, i.e.

$$\frac{\partial}{\partial u^{\alpha_1}_{s_1}(x_1)} \wedge .... \wedge \frac{\partial}{\partial u^{\alpha_k}_{s_k}(x_k)} = (-1)^{N_\sigma} \frac{\partial}{\partial u^{\alpha_{\sigma(1)}}_{s_{\sigma(1)}}(x_{\sigma(1)})} \wedge .... \wedge \frac{\partial}{\partial u^{\alpha_{\sigma(k)}}_{s_{\sigma(k)}}(x_{\sigma(k)})}, \tag{2.5.7}$$

with $\sigma \in P_k$ and N_σ denoting the number of exchanges associated to σ. Then a **k-vector** on the formal loop space is defined as

$$\alpha = \sum_{s_1,....,s_2\geq 0} \alpha^{\beta_1,s_1,....,\beta_k,s_k} \left(u(x_1), ..., u(x_k), u_x(x_1), ..., u_x(x_k)\right) \frac{\partial}{\partial u^{\beta_1}_{s_1}(x_1)} \wedge .... \wedge \frac{\partial}{\partial u^{\beta_k}_{s_k}(x_k)}, \tag{2.5.8}$$

where $\alpha^{\beta_1,s_1,....,\beta_k,s_k} \left(u(x_1), ..., u(x_k), u_x(x_1), ..., u_x(x_k)\right) \in \widehat{\mathcal{A}}^k$ are differential polynomials antisymmetric w.r.t the simoultaneous permutations

$$\beta_p, s_p, x_p \leftrightarrow \beta_q, s_q, x_q. \tag{2.5.9}$$

We will indicate the space of the k-vector with $\widehat{\mathcal{V}}^k$ and with $\mathcal{V} = \bigoplus_{k\geq 0} \widehat{\mathcal{V}}^k$ the space of the multivectors. As for forms space $\mathcal{F}$, we can endow $\mathcal{V}$ with the natural exterior product: if

$\alpha \in \widehat{\mathcal{V}}^k$ and $\beta \in \widehat{\mathcal{V}}^k$, then $\alpha \wedge \beta \in \widehat{\mathcal{V}}^{k+l}$ and its coordinates are defines as follows:

$$\begin{aligned}(\alpha \wedge \gamma)^{\beta_1 s_1,....,\beta_{k+l} s_{k+l}} = &\sum_{\sigma \in P_{k+l}} (-1)^{N_\sigma} \alpha^{\beta_{\sigma(1)} s_{\sigma(1)},....,\beta_{\sigma(k)} s_{\sigma(k)}}(x_{\sigma(1)},....,x_{\sigma(k)},u(x_{\sigma(1)}),....,u(x_{\sigma(k)}),....)\cdot \\ &\cdot \gamma^{\beta_{\sigma(k+1)} s_{\sigma(k+1)},....,\beta_{\sigma(k+l)} s_{\sigma(k+l)}}(x_{\sigma(k+1)},....,x_{\sigma(k+l)},u(x_{\sigma(k+1)}),....,u(x_{\sigma(k+l)}),....).\end{aligned} \tag{2.5.10}$$

We can now define the contraction of a k-vector $\alpha \in \widehat{\mathcal{V}}^k$ with k 1-forms $\omega^1,....,\omega^k \in \widehat{\mathcal{A}}_1$. This is an element of $\widehat{\Lambda}$ and it reads

$$\begin{aligned}\alpha(\omega_1,....,\omega_k) =& \frac{1}{k!}\int \sum_{s_1,....,s_k \geq 0} \sum_{\sigma \in P_k} (-1)^{N_\sigma} \omega^{\sigma(1)}_{s_1 \beta_1}(u(x_1),....)....\omega^{\sigma(k)}_{s_k \beta_k}(u(x_k),....)\cdot \\ &\cdot \alpha^{s_1 \beta_1;....;s_k \beta_k}(u(x_1),....,u(x_k),....)dx_1....dx_k.\end{aligned} \tag{2.5.11}$$

From this definition, we can understand why we have to define the forms as finite linear combinations (while the multivectors don't have this constraint). If we had defined both forms and multivectors as infinite linear combiination of base elements, the contraction between them could have contained an infinite number of similar polynomials and therefore presented convergence problems. Also other objects that we will consider in the following parts would be affected by this kind of divergence behavior. The choice of considering only finite sums on the definition of the forms allows us to avoid these convergence problems. This constraint will be relaxed imposing a gradation on the forms and multivectors spaces. Let us define the **Lie derivative** along a vector field of a multivector.

Definition. *Let $\xi \in \widehat{\mathcal{V}}^1$ and $\alpha \in \widehat{\mathcal{V}}^k$ be a vector field and a k-vector in $\mathcal{V}$. Then the* ***Lie derivative*** *of α along ξ, denoted by $Lie_\xi \alpha$, is a k-vector defined as*

$$(Lie_\xi \alpha)^{\beta_1 s_1,....,\beta_k s_k} = \sum_{j=1}^k \sum_{t \geq 0} \xi^{\gamma t}(u(x_j)) \frac{\partial}{\partial u^\gamma_t(x_j)} \alpha^{\beta_1 s_1,....,\beta_k s_k} - \sum_{j=1}^k \sum_{t \geq 0} \frac{\partial \xi^{\beta_j s_j}(x_j)}{\partial u^\gamma_t(x_j)} \alpha^{\beta_1 s_1,..,\beta_{j-1} s_{j-1},\gamma t,..,\beta_k s_k}. \tag{2.5.12}$$

Definition. *A multivector $\alpha \in \mathcal{V}$ is* ***translation invariant*** *iff*

$$Lie_{\partial_x \alpha = 0}. \tag{2.5.13}$$

The translation invariant maltivectors have some interesting properties.

Lemma. *The components of a translation invariant $\alpha \in \mathcal{V}^k$ read*

$$\begin{aligned}&\alpha^{\beta_1,s_1,....,\beta_k,s_k}(u(x_1),....,u(x_k),u_x(x_1),....,u_x(x_k)) \\ &\quad = \partial^{s_1}_{x_1} \circ .... \circ \partial^{s_k}_{x_k} A^{\beta_1,....,\beta_k}(u(x_1),....,u(x_k),u_x(x_1),....,u_x(x_k)),\end{aligned} \tag{2.5.14}$$

where $A^{\alpha_1,....,\alpha_k}(u(x_1),....,u(x_k,u_x(x_1),....,u_x(x_k))$ are differential polynomials antisymmetric w.r.t. simoultaneous permutations

$$\beta_p, x_p \leftrightarrow \beta_q, x_q. \tag{2.5.15}$$

Proof. To give an idea of the general proof we discuss the $k=1$, i.e. the vector field case. From the condition of translation invariance $Lie_{\partial_x X=0}$ for a vector field X, we get (using the fact that $\partial_x = \sum_{n \geq 0} u^\alpha_{n+1} \frac{\partial}{\partial u^\alpha_n}$)

$$0 = (Lie_{\partial_x} X)^{\alpha s} = \sum_{n \geq 0} u^\beta_{n+1} \frac{\partial X^{\alpha s}}{\partial u^\beta_n} - \sum_{n \geq 0} \delta^\alpha_\beta \delta_{s+1 n} X^{\beta n} = \partial_x X^{\alpha s} - X^{\alpha s+1}. \tag{2.5.16}$$

Therefore $X^{\alpha s+1} = \partial_x X^{\alpha s}$ for any value of $\alpha \in \{1, ..., N\}$ and $S \geq 0$. Applying iteratively this relation, we obtain

$$X^{\alpha s} = \partial_x^s X^{\alpha 0} \quad \alpha \in \{1, ..., N\}, s \geq 0. \tag{2.5.17}$$

So we have $A^\alpha = X^{\alpha 0}$. □

Lemma. *The contraction (2.5.11) for a translation invariant k-vector $\alpha \in \mathcal{V}^k$ is a well defined map $\alpha(\cdot) : \widehat{\Lambda}_1^{\times k} \to \widehat{\Lambda}^k$.*

Proof. Let's prove it in the case $k = 1$ (the general case is a simple generalization). Then if $\omega = \partial_x h$ with $\omega, h \in \widehat{\mathcal{A}}_1$(i.e. $\omega_\beta^s = \partial_x h_\beta^s + h_\beta^{s-1}$ with $h_\beta^{-1} = 0$ for every $\beta \in \{1, ..., N\}$), we have

$$\begin{aligned}\alpha(\omega) &= \int \sum_{s \geq 0} \alpha^{\beta,s} \omega_\beta^s dx = \int \sum_{s \geq 0} \partial_x^s A^\beta \left(\partial_x h_\beta^s + h_\beta^{s-1}\right) dx = \\ &= \int \sum_{s \geq 0} \partial_x h_\beta^s \partial_x^s A^\beta dx + \int \sum_{s \geq 1} h_\beta^{s-1} \partial_x \circ \partial_x^{s-1} A^\beta dx = \int \sum_{s \geq 0} \left(\partial_x h_\beta^s \partial_x^s A^\beta + h_\beta^s \partial_x \circ \partial_x^s A^\beta\right) dx = \\ &= \int \partial_x \left(h_\beta^s \partial_x^s A^\beta\right) dx = [0] .\end{aligned} \tag{2.5.18}$$

Therefore if $\omega'' = \omega' + \partial_x h$ with $\omega', \omega'' \in \widehat{\mathcal{A}}_1$, then $\alpha(\omega') = \alpha(\omega'')$. □

Now that the exterior algebra and the Lie derivative on $\mathcal{V}$ have been defined, one can define the **Schouten - Nijenhuis bracket** on $\mathcal{V}$, i.e. the unique bilinear pairing

$$[\cdot;\cdot] : \widehat{\mathcal{V}}^k \times \widehat{\mathcal{V}}^l \to \widehat{\mathcal{V}}^{k+l-1} \quad with \quad k + l \geq 1, \tag{2.5.19}$$

satisfying the property listed in the first chapter. We will report explicitly (in the last part of this chapter) the expressions of the Schouten - Nijenhuis bracket only for the multivectors interesting for the proof of the Getzler theorem. The following step consists in considering a particular extension of the notion of translational invariant multivector , i.e we want to include the possibility for the coefficients to be of the form

$$\widehat{O} = \sum_{s_1, ..., s_2 \geq 0} o_{s_2, s_3, ..., s_k} \left(u(x_1), u_x(x_1), ...\right) \delta^{(s_2)}(x_1 - x_2) \delta^{(s_3)}(x_1 - x_3) .... \delta^{(s_k)}(x_1 - x_k), \tag{2.5.20}$$

where $o_{s_2, s_3, ..., s_k} \left(u(x_1), u_x(x_1), ...\right) \in \widehat{\mathcal{A}}$ and the linear operator

$$\delta^{(s_2)}(x_1 - x_2) \delta^{(s_3)}(x_1 - x_3) .... \delta^{(s_k)}(x_1 - x_k) : \widehat{\mathcal{A}}^k \to \widehat{\mathcal{A}} \tag{2.5.21}$$

(with $s_1, ..., s_2 \geq 0$) acts on $f(x_1, ..., x_k) = f(x_1, ..., x_k, u(x_1), ..., u(x_k), ...) \in \widehat{\mathcal{A}}^k$ in the following way (we will denote the image of the delta operator with the integral notation):

$$\int f(x_1, ..., x_k) \delta^{(s_2)}(x_1 - x_2) .... \delta^{(s_k)}(x_1 - x_k) dx_2 .... dx_k = \partial_{x_2}^{s_1} \circ .... \circ \partial_{x_k}^{s_k} f(x_1, ..., x_k)_{|x_2 = x_1, ..., x_k = x_1} \in \widehat{\mathcal{A}}. \tag{2.5.22}$$

Remark. *Here there is an abuse of notation: indeed the integral denotes both the elements of $\widehat{\Lambda}^{tot}$ and the image of the delta operators. However this ambiguity will not cause problems in the understanding of the formulas. The meaning of the integral will be clear from the context.*

Definition. *A* ***local k-vector*** *is a translation invariant k-vector whose components* $A^{\alpha_1,\ldots,\alpha_k}$ *have the form described by the formula (2.5.20), i.e.*

$$A^{\alpha_1,\ldots,\alpha_k} = \sum_{p_2,\ldots,p_k\geq 0} B^{\alpha_1,\ldots,\alpha_k}_{p_2,\ldots,p_k}(u(x),u_x(x),\ldots)\delta^{(s_2)}(x_1-x_2)\delta^{(s_3)}(x_1-x_3)\ldots\delta^{(s_k)}(x_1-x_k). \tag{2.5.23}$$

At the moment we assume that only a finite number of coefficients $o_{s_2,s_3,\ldots,s_k}$ in (2.5.20) are non vanishing in order to avoid the convergence problems described above. We will denote the space of local k-vectors with $\widehat{\Lambda}^k_{loc}$ and with $\widehat{\Lambda}_{loc} = \bigoplus_{k\geq 0}\widehat{\Lambda}^k_{loc}$ the space of local vectors. $\widehat{\Lambda}_{loc}$ is not closed w.r.t. to the exterior product but the following result holds.

Lemma. $\widehat{\Lambda}_{loc}$ *is closed w.r.t. the Schouten-Nijenhuis.*

The next formula gives us the explicit form of the contraction in the case of a local k-vector.

Lemma. *For a local k-vector* $\alpha\in\widehat{\Lambda}^k_{loc}$*, the formula (2.5.11) for the contraction (as a map* $\alpha(\cdot):\widehat{\Lambda}_1^{\times k}\to\widehat{\Lambda}$*) becomes*

$$\alpha(\omega_1,\ldots,\omega_k) = \int B^{\alpha_1,\ldots,\alpha_k}_{p_2,\ldots,p_k}(u(x),u_x(x),\ldots)\omega^1_{\alpha_1}(u(x),u_x(x),\ldots)\partial_x^{p_2}\omega^2_{\alpha_2}(u(x),u_x(x),\ldots) \\ \ldots\partial_x^{p_2}\omega^k_{\alpha_k}(u(x),u_x(x),\ldots)dx, \tag{2.5.24}$$

where $\omega^p_{\alpha_p} = \sum_{s\geq 0}(-1)^s\partial_x^s\omega^{p,s}_{\alpha_p}$ *is the reduced form of* $\omega^p\in\widehat{\Lambda}_1$ *(*$p\leq k$*).*

The delta operators satisfy some remarkable properties.

Lemma.

$$\int\delta(x_1-x_2)\ldots\delta(x_1-x_k)f dx_2\ldots dx_k = \int\delta(x_p-x_1)\ldots\delta(x_p-x_{p-1})\delta(x_p-x_{p+1})\ldots\delta(x_p-x_k)f dx_2\ldots dx_k, \tag{2.5.25}$$

for any $p\leq k$ *and* $f\in\widehat{\mathcal{A}}^k$*.*

Lemma. *Let* $f\in\widehat{\mathcal{A}}^2$ *be a differential polyniamial depending only on one loop variable. Then*

$$\delta^{(p)}(x-y)f(y) = \sum_{q=0}^{p}\binom{p}{q}\delta^{(p-q)}(x-y)f^{(q)}(x). \tag{2.5.26}$$

Proof. Let $g\in\widehat{\mathcal{A}}^2$. Then

$$\int\delta^{(p)}(x-y)f(y)g(x,y)dy = \partial_y^p\left(f(y)g(x,y)\right)_{|x=y} = \sum_{q=0}^{p}\binom{p}{q}f^{(q)}(x)\partial_y^{p-q}g(x,y)_{|x=y} = \\ = \sum_{q=0}^{p}\binom{p}{q}f^{(q)}(x)\int\delta^{(p-q)}(x-y)g(x,y)dy = \int\sum_{q=0}^{p}\binom{p}{q}f^{(q)}(x)\delta^{(p-q)}(x-y)g(x,y)dy.$$

□

Lemma. *Let* $f\in\widehat{\mathcal{A}}^2$*. Then*

$$\int\delta^{(s)}(y-x)f(x,y)dy = \int(-1)^s\delta^{(s)}(x-y)f(x,y)dy. \tag{2.5.27}$$

Proof.

$$\int \delta^{(s)}(y-x)f(x,y)dy = \int (-1)^s\delta(y-x)\partial_y^s f(x,y)dy = \int (-1)^s\delta(x-y)\partial_y^s f(x,y)dy =$$
$$= (-1)^s\partial_y^s f(x,y)_{|x=y} = (-1)^s\int \delta^{(s)}(x-y)f(x,y)dy.$$

□

Let's now discuss how we can choose a gradation the space of the local multivector. In order to do that, we have to assign a degree to the delta operators. In particular

$$deg\delta^{(s)}(x-y) = s+1 \quad with \quad s \geq 0. \tag{2.5.28}$$

Then we can choose a subset of the local k-vector space (for any $k \geq 0$) whose elements $\alpha \in \mathcal{V}^k$ are such that

$$degA^{\alpha_1,\dots,\alpha_k} = k \quad \forall\alpha_1,\dots,\alpha_k \in \{1,\dots,N\}. \tag{2.5.29}$$

From now on we will intend with $\widehat{\Lambda}^k_{loc}$ (for any $k \geq 0$) the subset of the local k-vectors whose elements have degree k and with $\widehat{\Lambda}_{loc}$ the direct sum of these subsets. $\widehat{\Lambda}^0_{loc}$ coincides $\widehat{\mathcal{A}}^{[0]}$. The coefficients of an element of $\widehat{\Lambda}^1_{loc}$ can be decomposed in the following way:

$$A^\alpha = \sum_{k\geq 1}\epsilon^{k-1}A^\alpha_k(u(x),\dots,u^k(x)), \tag{2.5.30}$$

with $A^\alpha_k(u(x),\dots,u^k(x)) \in \widehat{\mathcal{A}}$ and $degA^\alpha_k = k$ $(k \geq 0)$, while the coefficients of an element of $\widehat{\Lambda}^2_{loc}$ read as

$$A^{\alpha\beta} = \sum_{k\geq 0}\epsilon^k A^{\alpha\beta}_{[k]}, \tag{2.5.31}$$

where

$$A^{\alpha\beta}_{[k]} = \sum_{s\leq k+1} A^{\alpha\beta}_{k,s}(u(x),\dots,u^s(x))\delta^{(k+1-s)}(x-y), \tag{2.5.32}$$

with $A^{\alpha\beta}_{k,s}(u(x),\dots,u^s(x)) \in \widehat{\mathcal{A}}$ and $degA^{\alpha\beta}_{k,s} = s$ $(k \geq 0, 0 \leq s \leq k+1)$. Gradating the space of multivectors allows us to relax the constraints given in the previous parts. For example the constraint that we have put on $B^{\alpha_1,\dots,\alpha_k}_{p_2,\dots,p_k}$ (non vanishing only in a finite number) is no more satisfied if we choose a gradated local k-vector. However this choice doesn't lead to convergence problems for the objects that we have defined before since the fixed degree of the multivectors allows the presence of a finite number of terms proportional to a fixed power of ϵ. One can gradate also the space of the forms and relax the constraint that we have put on them. In the last step of this section we will study some property of the element of Λ^2_{loc} since the Poisson structures are particular contractions of these objects (as in the finite dimensional case). Let's start writing explicitly the form of the coefficients of a generic element of Λ^2_{loc}. The first property to investigate is the antisymmetry condition.

Lemma. *The antisymmetry condition,i.e.* $A^{\alpha\beta}(x,y) = -A^{\beta\alpha}(y,x)$, *reads*

$$A^{\beta\alpha}_t(u(x),u_x(x),\dots) = \sum_{s\geq t}(-1)^{s+1}\binom{s}{t}\partial_x^{s-t}A^{\alpha\beta}_t(u(x),u_x(x),\dots). \tag{2.5.33}$$

Proof. Let $f \in \widehat{\mathcal{A}}^2$. Then

$$\begin{aligned}\int A^{\beta\alpha}(u(y), u_y(y), ...) f dy &= \int \sum_{s\geq 0} A_s^{\beta\alpha}(u(y), u_y(y), ...)\delta^{(s)}(y-x) f dy = \\ &= \int \sum_{s\geq 0} (-1)^s A_s^{\beta\alpha}(u(y), u_y(y), ...)\delta^{(s)}(x-y) f dy = \\ &= \sum_{s\geq 0} (-1)^s \int A_s^{\beta\alpha}(u(y), u_y(y), ...)\delta^{(s)}(x-y) f dy = \\ &= \int \sum_{s\geq 0} \sum_{t=0}^{s} (-1)^t \binom{s}{t} \partial_x^{s-t} A_s^{\beta\alpha}(u(x), u_x(x), ...)\delta^{(t)}(x-y) f dy = \\ &= \int \sum_{t\geq 0} \sum_{s\geq t} (-1)^t \binom{s}{t} \partial_x^{s-t} A_s^{\beta\alpha}(u(x), u_x(x), ...)\delta^{(t)}(x-y) f dy,\end{aligned}$$

where we've used (2.5.26) and (2.5.27). Then, using $A^{\alpha\beta}(x, y) = -A^{\beta\alpha}(y, x)$, we obtain the thesis. □

Remark. *Suppose that* $A_s^{\mu\nu} \neq 0$ *iff* $s = 0$ *and* $A_0^{\mu\nu}$ *is a costant matrix. Then, from the previous result, we can deduce that* $A_0^{\mu\nu}$ *is an antisymmetric constant matrix. In the case that the only non vanishing term is* $A_1^{\mu\nu}$ *and this is a constant matrix, we obtain that* $A_1^{\mu\nu}$ *is a symmetric constant matrix.*

Let us investigate the connection between the definition of Poisson bracket that we have given in the previous sections and the bivectors. Let $f, g \in \widehat{\Lambda}$. Then

$$\{f; g\} = \int \frac{\delta f}{\delta u^\mu} K_s^{\mu\nu} \partial_x^s \left(\frac{\delta g}{\delta u^\nu}\right) dx = \int\int \frac{\delta f}{\delta u^\mu(x)} K_s^{\mu\nu} \delta^{(s)}(x-y) \left(\frac{\delta g}{\delta u^\nu(y)}\right) dxdy = \omega(\delta\,[f]\,, \delta\,[g]), \tag{2.5.34}$$

where $\omega^{\mu\nu} = \sum_{s\geq 0} K_s^{\mu\nu}\delta^{(s)}(x-y) \in \widehat{\Lambda}^2_{loc}$ (it's evident that the gradation of the elements of $\widehat{\Lambda}^2_{loc}$ coincides with the coefficients gradation of the Hamiltonian operators of hydrodynamic type). Therefore the Poisson brackets of hydrodynamic type on $\widehat{\Lambda}$ can be written as contractions of bivectors, as in the finite dimensional case. This is incomplete since we haven't discussed how the antisymmetry condition and the validity of the Jacobi identity characterize the associated bivector. This is explained by the following result (as in the finite dimensional case, we will not prove this theorem):

Theorem. *The bracket associated to* $\omega \in \widehat{\Lambda}^2_{loc}$ *is of the Poisson type, i.e. it satisfies the antisymmetry condition and the Jacobi identity, iff* $[\omega; \omega] = 0$.

We are ready to introduce the **Poisson cohomology**. The definitions will be the same of the finite dimensional case. In first place let us define the cohomology differential associated to a $\bar{\omega} \in \widehat{\Lambda}^2_{loc}$ of Poisson type, i.e. satisfying $[\omega; \omega]$. It reads as

$$\begin{aligned}\partial_{\bar{\omega}} : \widehat{\Lambda}^k_{loc} \to \widehat{\Lambda}^{k+1}_{loc} \quad (k \geq 0)\\ \partial_{\bar{\omega}}\alpha = [\bar{\omega}; \alpha] \in \widehat{\Lambda}^{k+1}_{loc} \quad \textit{with} \quad \alpha \in \widehat{\Lambda}^k_{loc}.\end{aligned} \tag{2.5.35}$$

This satisfies the fundamental property characterizing the differentials.

Lemma. $\partial^2_{\bar{\omega}}\alpha = 0$ *for any* $\alpha \in \widehat{\Lambda}_{loc}$.

Proof. From the graded Jacobi identity of the Schouten-Nijenhuis bracket we obtain

$$0 = [\bar{\omega};[\bar{\omega};\alpha]] + \underbrace{[\alpha,[\bar{\omega};\bar{\omega}]]}_{=0} + [\bar{\omega};[\bar{\omega};\alpha]] = 2\,[\bar{\omega};[\bar{\omega};\alpha]] = 2\partial^2_{\bar{\omega}}\alpha, \tag{2.5.36}$$

with $\alpha \in \widehat{\Lambda}_{loc}$. □

Then the complex

$$.... \xrightarrow{\partial_{\bar{\omega}}} \widehat{\Lambda}^{k-1}_{loc} \xrightarrow{\partial_{\bar{\omega}}} \widehat{\Lambda}^{k}_{loc} \xrightarrow{\partial_{\bar{\omega}}} \widehat{\Lambda}^{k+1}_{loc} \xrightarrow{\partial_{\bar{\omega}}} .... \tag{2.5.37}$$

has a natural notion of cohomology.

Definition. *The* ***Poisson cohomology*** *is defined as*

$$H^k_{\bar{\omega}} = \frac{Ker\left(\partial_{\bar{\omega}} : \widehat{\Lambda}^k_{loc} \to \widehat{\Lambda}^{k+1}_{loc}\right)}{Im\left(\partial_{\bar{\omega}} : \widehat{\Lambda}^{k-1}_{loc} \to \widehat{\Lambda}^{k}_{loc}\right)} \quad \forall k \geq 0. \tag{2.5.38}$$

Finally we write down explicitly the form of $\partial_{\bar{\omega}}\alpha = [\bar{\omega};\alpha]$ for a generic $\alpha \in \widehat{\Lambda}^2_{loc}$ and $\widehat{\Lambda}^2_{loc} \ni \bar{\omega}^{\mu\nu} = \eta^{\mu\nu}\delta'(x-y)$ (the coefficients of α are $A^{\mu\nu} = \sum_{s\geq 0} A^{\mu\nu}_s \delta^s(x-y)$). It reads as

$$\widehat{\Lambda}^3_{loc} \ni [\bar{\omega};\alpha]^{\mu\nu\gamma} = \Bigg[-\frac{\partial A^{\mu\nu}_t}{\partial u^\beta_{s-1}}\eta^{\beta\gamma} + \sum_{r\geq 0, q\leq t-1} (-1)^{q+r+s}\binom{q+r+s}{q \quad r}\partial^r_x\left(\frac{\partial A^{\gamma\mu}_{q+r+s}}{\partial u^\beta_{t-q-1}}\right)\eta^{\beta\nu} + \\ + \sum_{q\leq s, q+r+t\geq 1} (-1)^{q+r+t}\binom{q+r+t}{q \quad r}\partial^r_x\left(\frac{\partial A^{\nu\gamma}_{s-q}}{\partial u^\beta_{q+r+t-1}}\right)\eta^{\beta\mu}\Bigg]\delta^{(t)}(x-y)\delta^{(s)}(x-z). \tag{2.5.39}$$

This expression seems problematic. In particular the second summation seems to be divergent due to the the fact that the sum index r is arbitrary big. But it isn't since the degree of α is fixed. Indeed consider the decomposition (2.5.31) of α as power of ϵ. It's not difficult to find the following relation:

$$A^{\mu\nu}_s = \sum_{k\geq 0, k\geq s-1} \epsilon^k A^{\mu\nu}_{k,k+1-s}. \tag{2.5.40}$$

Therefore the term $A^{\mu\nu}_s$ contains ϵ^k for $k \geq s-1$. This means that, varying s, the term ϵ^k (with k fixed) is contained in $A^{\mu\nu}_s$ only for $s \leq k+1$. And this implies that the second summation doesn't diverge since there is only a finite number of terms proportion to a fixed power of ϵ to vary of the sum index. For the sake of completeness, let us report explicitly the form of $\partial_{\bar{\omega}}f = [\bar{\omega};f]$ with $f \in \widehat{\Lambda}^0_{loc}$ and $\partial_{\bar{\omega}}X = [\bar{\omega};X]$ with $X \in \widehat{\Lambda}^1_{loc}$. The first one reads

$$\widehat{\Lambda}^1_{loc} \ni [\bar{\omega};f]^\mu = \eta^{\mu\nu}\partial_x\frac{\delta f}{\delta u^\nu}, \tag{2.5.41}$$

while the second one reads

$$\underbrace{[\bar{\omega};X]^{\mu\nu}}_{\in\widehat{\Lambda}^2_{loc}} = -\eta^{\mu\alpha}\partial_x\frac{\delta X^\nu}{\delta u^\alpha}\delta(x-y) - \sum_{r\geq 0}\left[\frac{\partial X^\mu}{\partial u^\alpha_s}\eta^{\alpha\nu} + \sum_{t\geq s}(-1)^t\binom{t+1}{s+1}\eta^{\mu\alpha}\partial^{t-s}_x\left(\frac{\partial X^\nu}{\partial u^\alpha_t}\right)\right]\delta^{(s+1)}(x-y). \tag{2.5.42}$$

Also for $[\bar{\omega};X]^{\mu\nu}$ the convergence problems are avoided thanks to the fixed degree of X.

Chapter 3

Theorems for Poisson bracket of the hydrodinamic type

In this chapter we will present two important theorems for the study of the Poisson geometry on the formal loop space with Hamiltonian operator of hydrodynamic type. The first one, the **Dubrovin-Novikov theorem**, is related to the antisymmetry of the Poisson bracket and the validity of the Jacobi identity as we have anticipated in the first chapter. The second one, the **Getzler theorem** can be viewed as a generalization of the Weinstein theorem that we have described in the first chapter.

3.1 Dubrovin-Novikov theorem

Let consider the Hamiltonian operator of hydrodynamic type defined in the previous section that, in the limit $\epsilon = 0$, it assumes the form

$$K^{\mu\nu}|_{\epsilon=0} = g^{\mu\nu}(u)\partial_x + b^{\mu\nu}_{\gamma}(u)u^{\gamma}_x. \tag{3.1.1}$$

Let us define $\Gamma^{\gamma}_{\mu\nu} = -g_{\mu\alpha}b^{\alpha\gamma}_{\nu}$ where $g_{\mu\nu} = (g^{\mu\nu})^{-1}$ (we have assumed that $g^{\mu\nu}$ is nondegenerate). As first step we prove a prelimanary proposotion that it specifies how $g^{\mu\nu}$ and $\Gamma^{\gamma}_{\mu\nu}$ transfrom under a change of coordinates on the target space V.

Proposition. *Let $v^{\alpha} = v^{\alpha}(u)$ be a smooth change of coordinates on the space target V. Then*

- $g^{\mu\nu}$ *transforms as a (2;0) tensor.*
- $\Gamma^{\gamma}_{\mu\nu}$ *transforms as an affine connection.*

Proof. Notice that we can regard a change of coordinates on the target space V as a Miura transformation for which $\tilde{u}^{\alpha}(u^*_*;\epsilon) = \tilde{u}^{\alpha}|_{\epsilon=0}$. Then

$$L^{\nu}_{\beta} = \frac{\partial v^{\nu}}{\partial u^{\beta}}, \quad \textit{and} \quad (L^*)^{\mu}_{\alpha} = \frac{\partial v^{\mu}}{\partial u^{\alpha}}, \tag{3.1.2}$$

since the Miura transformation depends only on u. This implies that

$$K^{\mu\nu}_{\mathcal{M}}(u) = \frac{\partial v^{\mu}}{\partial u^{\alpha}}(u)g^{\alpha\beta}(u)\left(\partial_x\left(\frac{\partial v^{\nu}}{\partial u^{\beta}}(u)\right) + \frac{\partial v^{\nu}}{\partial u^{\beta}}(u)\partial_x\right) + \frac{\partial v^{\mu}}{\partial u^{\alpha}}(u)\frac{\partial v^{\nu}}{\partial u^{\beta}}(u)b^{\alpha\beta}_{\gamma}(u)u^{\gamma}_x. \tag{3.1.3}$$

Now, if we rewrite $K_M^{\mu\nu}$ using the inverse coordinate transformation $u^\alpha(v)$, we obtain

$$K'^{\mu\nu}(v) = K_{\mathbb{M}}^{\mu\nu}(u(v)) = \frac{\partial v^\mu}{\partial u^\alpha}(u(v))\frac{\partial v^\nu}{\partial u^\beta}(u(v))g^{\alpha\beta}(u(v))\partial_x + \left(\frac{\partial v^\mu}{\partial u^\alpha}(u(v))\frac{\partial^2 v^\nu}{\partial u^\beta \partial u^\sigma}(u(v))\frac{\partial u^\sigma}{\partial v^\gamma}(v)g^{\alpha\beta}(u(v)) + \frac{\partial v^\mu}{\partial u^\alpha}(u(v))\frac{\partial v^\nu}{\partial u^\beta}(u(v))\frac{\partial u^\theta}{\partial v^\gamma}(v)b_\theta^{\alpha\beta}(u(v))\right) v_x^\gamma. \tag{3.1.4}$$

Therefore we have found that

$$g'^{\mu\nu}(v) = \frac{\partial v^\mu}{\partial u^\alpha}(u(v))\frac{\partial v^\nu}{\partial u^\beta}(u(v))g^{\alpha\beta}(u(v))$$
$$b_\gamma'^{\mu\nu}(v) = \frac{\partial v^\mu}{\partial u^\alpha}(u(v))\frac{\partial^2 v^\nu}{\partial u^\beta \partial u^\sigma}(u(v))\frac{\partial u^\sigma}{\partial v^\gamma}(v)g^{\alpha\beta}(u(v)) + \frac{\partial v^\mu}{\partial u^\alpha}(u(v))\frac{\partial v^\nu}{\partial u^\beta}(u(v))\frac{\partial u^\theta}{\partial v^\gamma}(v)b_\theta^{\alpha\beta}(u(v)) \tag{3.1.5}$$

and this proves that $g^{\mu\nu}$ transforms as a (2;0) tensor under a change of coordinates. Regarding $\Gamma_{\mu\nu}^\gamma$, using the identities $g^{\mu\alpha}(u(v))g_{\alpha\nu}(u(v)) = \frac{\partial v^\alpha}{\partial u^\nu}(u(v))\frac{\partial u^\mu}{\partial v^\alpha}(v) = \delta_\nu^\mu$ and the tensor transformation relation for $g'_{\mu\nu}(v) = \frac{\partial u^\alpha}{\partial v^\mu}(v)\frac{\partial u^\beta}{\partial v^\nu}(v)g_{\alpha\beta}(u(v))$, we obtain

$$\Gamma'^\gamma_{\mu\nu}(v) = -g'_{\mu\alpha}(v)b'^{\alpha\gamma}_\nu(v) = -\frac{\partial u^\alpha}{\partial v^\mu}(v)\frac{\partial u^\beta}{\partial v^\nu}(v)\frac{\partial^2 v^\gamma}{\partial u^\alpha \partial u^\beta}(u(v)) + \frac{\partial u^\alpha}{\partial v^\mu}(v)\frac{\partial u^\beta}{\partial v^\nu}(v)\frac{\partial v^\gamma}{\partial u^\theta}(u(v))\Gamma^\theta_{\alpha\beta}(u(v)). \tag{3.1.6}$$

Applying $\frac{\partial v^\mu}{\partial u^\lambda}(u(v))\frac{\partial v^\nu}{\partial u^\eta}(u(v))\frac{\partial u^\zeta}{\partial v^\gamma}(v)$ to both side of (3.1.6) and moving the first term of the second side to the first one we get

$$\frac{\partial v^\mu}{\partial u^\lambda}(u(v))\frac{\partial v^\nu}{\partial u^\eta}(u(v))\frac{\partial u^\zeta}{\partial v^\gamma}(v)\Gamma'^\gamma_{\mu\nu}(v) + \frac{\partial u^\zeta}{v^\gamma}(v)\frac{\partial^2 v^\gamma}{\partial u^\lambda \partial u^\eta}(u(v)) = \Gamma^\zeta_{\lambda\eta}(u(v)), \tag{3.1.7}$$

that it's the transformation under change coordinates of an affine connection. This ends the proof. □

Now we're ready to present the Dubrovin-Novikov theorem, proved in 1983 by the two mathematicians from which it takes its name.

Theorem (Dubrovin-Novikov, [6])**.** *Let $K^{\mu\nu}$ be an Hamiltonian operator of hydrodynamic type (with $g^{\mu\nu}$ nondegenerate) associated with the Poisson bracket $\{\cdot;\cdot\}_K$. Then the Poisson bracket $\{\cdot;\cdot\}_K|_{\epsilon=0}$ is antisymmetric and satisfies the Jacobi identity if and only if these conditions are satisfied:*

- *$g^{\mu\nu} = g^{\nu\mu}$, i.e. $g_{\mu\nu}$ is a metric on the target space V.*
- *$\Gamma_{\mu\ \nu}^{\gamma}$ are the Christoffel symbols corresponding to the Levi-Civita connection of the metric $g_{\mu\nu}$.*
- *The curvature tensor associated to $\Gamma_{\mu\nu}^\gamma$ vanishes.*

Proof. We will follow the proof contained in [3]. We start proving the direct implication. Imposing the antisymmetry of the Poisson bracket and the validity of the Jacobi identity, we will prove the following relations

$$\begin{cases} g^{\mu\nu} = g^{\nu\mu} \\ b_\gamma^{\mu\nu} + b_\gamma^{\nu\mu} = \dfrac{\partial g^{\mu\nu}}{\partial u^\gamma} \\ b_\gamma^{\mu\nu} g^{\gamma\alpha} = b_\gamma^{\alpha\nu} g^{\gamma\mu} \\ b_\gamma^{\mu\nu} b_\beta^{\gamma\alpha} - b_\gamma^{\mu\alpha} b_\beta^{\gamma\nu} = g^{\mu\gamma}\left(\dfrac{\partial b_\beta^{\nu\alpha}}{\partial u^\gamma} - \dfrac{\partial b_\gamma^{\nu\alpha}}{\partial u^\beta}\right). \end{cases} \tag{3.1.8}$$

Once we will prove these relations, we will be able to prove the thesis. Let us assume the antisymmetry condition (valid for $\forall f, g \in \widehat{\Lambda}$)

$$\{\bar{f}; \bar{g}\}_{K|_{\epsilon=0}} + \{\bar{g}; \bar{f}\}_{K|_{\epsilon=0}} = 0. \tag{3.1.9}$$

Therefore, if we consider two differential polynomyals $f, g \in \widehat{\mathcal{A}}$, we have

$$\int dx \left[\frac{\delta f}{\delta u^\mu} g^{\mu\nu} \partial_x \left(\frac{\delta g}{\delta u^\nu} \right) + \frac{\delta g}{\delta u^\nu} g^{\nu\mu} \partial_x \left(\frac{\delta f}{\delta u^\mu} \right) + \frac{\delta f}{\delta u^\mu} (b^{\mu\nu}_\gamma + b^{\nu\mu}_\gamma) u^\gamma_x \frac{\delta g}{\delta u^\nu} \right] = 0. \tag{3.1.10}$$

Using the relation (integration by parts)

$$\int dx \frac{\delta g}{\delta u^\nu} g^{\nu\mu} \partial_x \left(\frac{\delta f}{\delta u^\mu} \right) = \int dx \left[-\partial_x \left(\frac{\delta g}{\delta u^\nu} \right) g^{\nu\mu} \frac{\delta f}{\delta u^\mu} - \frac{\delta f}{\delta u^\nu} \frac{\partial g^{\nu\mu}}{\partial u^\gamma} u^\gamma_x \frac{\delta g}{\delta u^\mu} \right], \tag{3.1.11}$$

we obtain

$$\int dx \frac{\delta f}{\delta u^\mu} \left[(g^{\mu\nu} - g^{\nu\mu})\, \partial_x \left(\frac{\delta g}{\delta u^\nu} \right) + \left(b^{\mu\nu}_\gamma + b^{\nu\mu}_\gamma - \frac{\partial g^{\nu\mu}}{\partial u^\gamma} \right) \frac{\delta g}{\delta u^\nu} \right] = 0. \tag{3.1.12}$$

Now we can apply the second variational principle to (3.1.12) obtaining:

$$(g^{\mu\nu} - g^{\nu\mu})\, \partial_x \left(\frac{\delta g}{\delta u^\nu} \right) + \left(b^{\mu\nu}_\gamma + b^{\nu\mu}_\gamma - \frac{\partial g^{\nu\mu}}{\partial u^\gamma} \right) \frac{\delta g}{\delta u^\nu} = c_g(\epsilon) u^\mu_1 \quad \forall g \in \widehat{\mathcal{A}}. \tag{3.1.13}$$

Choosing $g = \frac{(-1)^p}{2} (u^\nu_p)^2$ with $p \geq 1$ and $\nu \in \{1, ..., N\}$, one gets

$$(g^{\mu\nu} - g^{\nu\mu})\, u^\nu_{2p+1} + \left(b^{\mu\nu}_\gamma + b^{\nu\mu}_\gamma - \frac{\partial g^{\nu\mu}}{\partial u^\gamma} \right) u^\nu_{2p} = c_g(\epsilon) u^\mu_1. \tag{3.1.14}$$

Since $g^{\mu\nu}$ and $b^{\mu\nu}_\gamma$ depend only on u^*, the identity holds if and only if $c_g(\epsilon) = 0$ and

$$\begin{cases} g^{\mu\nu} - g^{\nu\mu} = 0 \\ b^{\mu\nu}_\gamma + b^{\nu\mu}_\gamma - \dfrac{\partial g^{\nu\mu}}{\partial u^\gamma} = 0, \end{cases} \tag{3.1.15}$$

for any $\mu, \nu \in \{1, ..., N\}$. We have proved the first two relations of (3.1.8). It's easy to see (simply using the identity derived in the previous steps) that also (3.1.15) implies the antisymmetry condition of the bracket. So we have proved that a bracket is antisymmetric iff the relations(3.1.15) hold. In order to prove the last two conditions, we will exploit the Jacobi identity. Indeed let us assume the Jacobi identity in $\widehat{\Lambda}$, i.e.

$$\{\{\bar{f}; \bar{g}\}_{K|_{\epsilon=0}}; \bar{h}\}_{K|_{\epsilon=0}} + \{\{\bar{f}; \bar{g}\}_{K|_{\epsilon=0}}; \bar{h}\}_{K|_{\epsilon=0}} + \{\{\bar{f}; \bar{g}\}_{K|_{\epsilon=0}}; \bar{h}\}_{K|_{\epsilon=0}} = 0 \quad \forall \bar{f}, \bar{g}, \bar{h} \in \widehat{\Lambda}, \tag{3.1.16}$$

equivalent to (as we've seen before)

$$[D_{\bar{g}}; D_{\bar{h}}]\, \bar{f} = D_{\{\bar{g}; \bar{h}\}_K} \bar{f}. \tag{3.1.17}$$

This means that, for any $f, g, h \in \widehat{\mathcal{A}}$, we have

$$0 = P = \int dx \sum_{m\geq 0} \frac{\partial}{\partial u^\alpha_m} \left(\sum_{n\geq 0} \frac{\partial f}{\partial u^\mu_n} \partial_x^n K^{\mu\nu} \left(\frac{\delta g}{\delta u^\nu} \right) \right) \partial_x^m K^{\alpha\beta} \left(\frac{\delta h}{\delta u^\beta} \right) - \int dx \sum_{m\geq 0} \frac{\partial}{\partial u^\alpha_m} \left(\sum_{n\geq 0} \frac{\partial f}{\partial u^\mu_n} \cdot \right.$$
$$\left. \partial_x^n K^{\mu\beta} \left(\frac{\delta h}{\delta u^\beta} \right) \right) \partial_x^m K^{\alpha\nu} \left(\frac{\delta g}{\delta u^\nu} \right) - \int dx \frac{\partial f}{\partial u^\mu_n} \partial_x^n K^{\mu\alpha} \left(\frac{\delta}{\delta u^\alpha} \left(\frac{\delta g}{\delta u^\nu} K^{\nu\beta} \frac{\delta h}{\delta u^\beta} \right) \right) =$$
$$= \int dx \sum_{n\geq 0} \frac{\partial f}{\partial u^\mu_n} \left[\sum_{m\geq 0} \frac{\partial}{\partial u^\alpha_m} \left(\partial_x^n K^{\mu\nu} \left(\frac{\delta g}{\delta u^\nu} \right) \right) \partial_x^m K^{\alpha\beta} \left(\frac{\delta h}{\delta u^\beta} \right) - \sum_{m\geq 0} \frac{\partial}{\partial u^\alpha_m} \left(\partial_x^n K^{\mu\beta} \left(\frac{\delta h}{\delta u^\beta} \right) \right) \partial_x^m K^{\alpha\nu} \left(\frac{\delta g}{\delta u^\nu} \right) \right.$$
$$\left. - \partial_x^n K^{\mu\alpha} \left(\frac{\delta}{\delta u^\alpha} \left(\frac{\delta g}{\delta u^\nu} K^{\nu\beta} \frac{\delta h}{\delta u^\beta} \right) \right) \right] =$$
$$= \int dx \sum_{n\geq 0} \frac{\partial f}{\partial u^\mu_n} \partial_x^n \left[\sum_{m\geq 0} \frac{\partial}{\partial u^\alpha_m} \left(K^{\mu\nu} \left(\frac{\delta g}{\delta u^\nu} \right) \right) \partial_x^m K^{\alpha\beta} \left(\frac{\delta h}{\delta u^\beta} \right) - \sum_{m\geq 0} \frac{\partial}{\partial u^\alpha_m} \left(K^{\mu\beta} \left(\frac{\delta h}{\delta u^\beta} \right) \right) \partial_x^m K^{\alpha\nu} \left(\frac{\delta g}{\delta u^\nu} \right) \right.$$
$$\left. - K^{\mu\alpha} \left(\frac{\delta}{\delta u^\alpha} \left(\frac{\delta g}{\delta u^\nu} K^{\nu\beta} \frac{\delta h}{\delta u^\beta} \right) \right) \right], \tag{3.1.18}$$

where we've used the exchange property (2.1.7) in the last step. Now, applying the Leibniz rule for the variational derivatives (2.2.3) on the last member of the previous relation, i.e.

$$\frac{\delta}{\delta u^\alpha} \left(\frac{\delta g}{\delta u^\nu} K^{\nu\beta} \frac{\delta h}{\delta u^\beta} \right) = \sum_{k\geq 0} \mathcal{T}_{\alpha,k} \frac{\delta g}{\delta u^\nu} \cdot (-\partial_x)^k \left(K^{\nu\beta} \left(\frac{\delta h}{\delta u^\beta} \right) \right) + \sum_{k\geq 0} (-\partial_x)^k \frac{\delta g}{\delta u^\nu} \cdot \mathcal{T}_{\alpha,k} \left(K^{\nu\beta} \left(\frac{\delta h}{\delta u^\beta} \right) \right), \tag{3.1.19}$$

and the Leibniz rule for the x-derivatives and the partial ones, we obtain

$$P = \int dx \sum_{n\geq 0} \frac{\partial f}{\partial u^\mu_n} \partial_x^n \left[A^\mu_{(1)} + A^\mu_{(2)} + A^\mu_{(3)} + B^\mu \right] = \int dx \frac{\delta f}{\delta u^\mu} \left[A^\mu_{(1)} + A^\mu_{(2)} + A^\mu_{(3)} + B^\mu \right], \tag{3.1.20}$$

where

$$\begin{aligned}
A^\mu_{(1)} &:= \frac{\partial g^{\mu\nu}}{\partial u^\alpha} \partial_x \left(\frac{\delta g}{\delta u^\nu} \right) K^{\alpha\beta} \left(\frac{\delta h}{\delta u^\beta} \right) + \sum_{m=0,1} \frac{\partial (b^{\mu\nu}_\gamma u^\gamma_x)}{\partial u^\alpha_m} \left(\frac{\delta g}{\delta u^\nu} \right) \partial_x^m K^{\alpha\beta} \left(\frac{\delta h}{\delta u^\beta} \right) \\
A^\mu_{(2)} &:= -\frac{\partial g^{\mu\beta}}{\partial u^\alpha} \partial_x \left(\frac{\delta h}{\delta u^\beta} \right) K^{\alpha\nu} \left(\frac{\delta g}{\delta u^\nu} \right) - \sum_{m=0,1} \frac{\partial (b^{\mu\beta}_\gamma u^\gamma_x)}{\partial u^\alpha_m} \left(\frac{\delta h}{\delta u^\beta} \right) \partial_x^m K^{\alpha\nu} \left(\frac{\delta g}{\delta u^\nu} \right) \\
A^\mu_{(3)} &:= -K^{\mu\alpha} \left[\frac{\delta g}{\delta u^\nu} \frac{\partial g^{\nu\beta}}{\partial u^\alpha} \partial_x \left(\frac{\delta h}{\delta u^\beta} \right) + \sum_{l=0,1} \frac{\delta g}{\delta u^\nu} (-\partial_x)^l \left(\frac{\partial \left(b^{\nu\beta}_\gamma u^\gamma_x \right)}{\partial u^\alpha_l} \frac{\delta h}{\delta u^\beta} \right) \right. \\
&\left. -\partial_x \left(\frac{\delta g}{\delta u^\nu} \right) \frac{\partial (b^{\nu\beta}_\gamma u^\gamma_x)}{\partial u^\alpha_1} \frac{\delta h}{\delta u^\beta} \right]
\end{aligned} \tag{3.1.21}$$

and

$$B^{\mu} := K^{\mu\nu}\left(\sum_{m\geq 0}\frac{\partial}{\partial u_m^{\alpha}}\left(\frac{\delta g}{\delta u^{\nu}}\right)\cdot\partial_x^m\left(K^{\alpha\beta}\frac{\delta h}{\delta u^{\beta}}\right)\right) - K^{\mu\beta}\left(\sum_{m\geq 0}\frac{\partial}{\partial u_m^{\alpha}}\left(\frac{\delta h}{\delta u^{\beta}}\right)\cdot\partial_x^m\left(K^{\alpha\nu}\frac{\delta g}{\delta u^{\nu}}\right)\right)$$
$$- K^{\mu\alpha}\left[\sum_{k\geq 0}\mathcal{T}_{\alpha,k}\frac{\delta g}{\delta u^{\nu}}(-\partial_x)^k\left(K^{\nu\beta}\frac{\delta h}{\delta u^{\beta}}\right) + \sum_{k\geq 0}(-\partial_x)^k\left(\frac{\delta g}{\delta u^{\nu}}\right)\sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\left(g^{\nu\beta}\frac{\partial}{\partial u_n^{\alpha}}\left(\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)+\right.\right.$$
$$\left.\left.+b_{\gamma}^{\nu\beta}u_x^{\gamma}\frac{\partial}{\partial u_n^{\alpha}}\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right]. \tag{3.1.22}$$

In the previous steps we've used the exchange property in B^{μ} in order to factorize the hamiltonian operator $K^{\mu\alpha}$ and we've exploited the following properties:

$$\mathcal{T}_{\alpha,k}\left(g^{\nu\beta}\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right) = \sum_{n\geq k}\binom{n}{k}\partial_x^{n-k}\left[\frac{\partial g^{\nu\beta}}{\partial u_n^{\alpha}}\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right) + g^{\nu\beta}\frac{\partial}{\partial u_n^{\alpha}}\left(\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right] =$$
$$= \frac{\partial g^{\nu\beta}}{\partial u^{\alpha}}\partial_x\frac{\delta h}{\delta u^{\beta}}\delta_{0k} + \sum_{n\geq k}\binom{n}{k}\partial_x^{n-k}\left[g^{\nu\beta}\frac{\partial}{\partial u_n^{\alpha}}\left(\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right], \tag{3.1.23}$$

since $g^{\nu\beta}$ depends only on u^* and

$$\mathcal{T}_{\alpha,k}\left(b_{\gamma}^{\nu\beta}u_x^{\gamma}\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right) = \sum_{n\geq k}\binom{n}{k}\partial_x^{n-k}\left[\frac{\partial(b_{\gamma}^{\nu\beta}u_x^{\gamma})}{\partial u_n^{\alpha}}\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right) + b_{\gamma}^{\nu\beta}u_x^{\gamma}\frac{\partial}{\partial u_n^{\alpha}}\left(\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right] =$$
$$= \sum_{l=0,1}\delta_{0k}(-\partial_x)^l\left(\frac{\partial\left(b_{\gamma}^{\nu\beta}u_x^{\gamma}\right)}{\partial u_l^{\alpha}}\frac{\delta h}{\delta u^{\beta}}\right) + \frac{\partial(b_{\gamma}^{\nu\beta}u_x^{\gamma})}{\partial u_1^{\alpha}}\frac{\delta h}{\delta u^{\beta}}\delta_{1k} + \sum_{n\geq k}\binom{n}{k}\partial_x^{n-k}\left[b_{\gamma}^{\nu\beta}u_x^{\gamma}\frac{\partial}{\partial u_n^{\alpha}}\left(\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right], \tag{3.1.24}$$

since $b_{\gamma}^{\nu\beta}u_x^{\gamma}$ depends only on u^* and u_1^*. One can prove that

$$B^{\mu} = 0 \quad \forall f, g, h \in \widehat{\mathcal{A}} \tag{3.1.25}$$

and we will prove this fact in the appendix. It's easy to see that the vanishing of the B^{μ} terms implies that

$$P = \int dx \sum_{j,k=0,1,2} C_{jk}^{\mu\nu\beta}\frac{\delta f}{\delta u^{\mu}}\frac{\delta g}{\delta u^{\nu}}^{(j)}\frac{\delta h}{\delta u^{\beta}}^{(k)}, \tag{3.1.26}$$

where $\left(\frac{\delta *}{\delta u^*}\right)^{(i)} := \partial_x^i\left(\frac{\delta *}{\delta u^*}\right)$ and $C_{jk}^{\mu\nu\beta}$ are differential polynomials depending on $u_{k\leq 2}^*$ (we will explicitly see this fact in the following parts). This is due to the fact that the $A_{(*)}^{\mu}$ terms contain $\left(\frac{\delta *}{\delta u^*}\right)^{(i)}$ with $i \leq 2$ and they do not contain terms of the form $\frac{\partial}{\partial u_i^*}\left[\left(\frac{\delta *}{\delta u^*}\right)^{(j)}\right]$ (with $i, j \in \mathbb{N}$). Now, applying the second variational principle, we obtain that

$$\sum_{j,k=0,1,2} C_{jk}^{\mu\nu\beta}\frac{\delta g}{\delta u^{\nu}}^{(j)}\frac{\delta h}{\delta u^{\beta}}^{(k)} = c_{gh}(\epsilon)u_1^{\mu}. \tag{3.1.27}$$

Choosing $g = \frac{(-1)^l}{2}(u_l^{\nu})$ and $h = \frac{(-1)^p}{2}(u_p^{\beta})$ with $l, p \geq 2$ and $\nu, \beta \in \{1, ..., N\}$, we get

$$\sum_{j,k=0,1,2} C_{jk}^{\mu\nu\beta}u_{2l+j}^{\nu}u_{2p+k}^{\beta} = c_{gh}(\epsilon)u_1^{\mu}. \tag{3.1.28}$$

Since $C_{jk}^{\mu\nu\beta}$ depend on $u^*_{k\leq 2}$ and $2l+j, 2p+k \geq 4$, the previous formula is a sum of independent terms and therefore the relation is verified if and only if $c_{gh}(\epsilon)=0$ and

$$C_{jk}^{\mu\nu\beta}=0, \tag{3.1.29}$$

for any value of the indices. If we choose $\nu=\beta$, the previous argument holds if one takes $2p-2k>2$ in order to avoid the presence of dependent terms, i.e terms of the type $u^\nu_m u^\nu_n$ and $u^\nu_r u^\nu_s$ with $m=s$ and $n=r$. Indeed the necessary condition to the presence of this terms is $2l+j=2p+k$ for some $j,k\in\{0,1,2\}$, equivalent to $2l-2p=k-j$. Since $k-j\leq 2$, taking $2l-2p>2$, we avoid the undesired presence of the dependent contributions. Let us calculate the $C_{jk}^{\mu\nu\beta}$ terms. It's easy to see that the contributions to the coefficients $C_{jk}^{\mu\nu\beta}$ coming from $A^\mu_{(1)}$ are (denoted by $C_{(1)jk}^{\mu\nu\beta}$)

$$\begin{cases} C_{(1)00}^{\mu\nu\beta}=\left(\dfrac{\partial b_\gamma^{\mu\nu}}{\partial u^\alpha}b_\theta^{\alpha\beta}+b_\alpha^{\mu\nu}\dfrac{\partial b_\theta^{\alpha\beta}}{\partial u^\gamma}\right)u^\gamma_x u^\theta_x+b_\alpha^{\mu\nu}b_\theta^{\alpha\beta}u^\theta_{xx}\\ C_{(1)01}^{\mu\nu\beta}=\left(\dfrac{\partial b_\gamma^{\mu\nu}}{\partial u^\alpha}g^{\alpha\beta}+b_\alpha^{\mu\nu}\dfrac{\partial g^{\alpha\beta}}{\partial u^\gamma}+b_\alpha^{\mu\nu}b_\gamma^{\alpha\beta}\right)u^\gamma_x\\ C_{(1)10}^{\mu\nu\beta}=\dfrac{\partial g^{\mu\nu}}{\partial u^\alpha}b_\gamma^{\alpha\beta}u^\gamma_x\\ C_{(1)11}^{\mu\nu\beta}=\dfrac{\partial g^{\mu\nu}}{\partial u^\alpha}g^{\alpha\beta}\\ C_{(1)02}^{\mu\nu\beta}=b_\alpha^{\mu\nu}g^{\alpha\beta}\\ C_{(1)2k}^{\mu\nu\beta}=C_{(1)12}^{\mu\nu\beta}=0\quad\forall k\in\{0,1,2\}. \end{cases} \tag{3.1.30}$$

The $C_{(2)jk}^{\mu\nu\beta}$ can be obtained from $C_{(1)jk}^{\mu\nu\beta}$ simply exchanging the indices and the functions in the following way

$$C_{(2)jk}^{\mu\nu\beta}(g,h)=C_{(1)jk}^{\mu,\nu\to\beta,\beta\to\nu}(g\to h,h\to g). \tag{3.1.31}$$

As regards $C_{(3)jk}^{\mu\nu\beta}$, we will explicitly calculate them in the appendix since it's a long calculation. Having the explicit form of the $C_{jk}^{\mu\nu\beta}$ as function of $g^{\mu\nu}$ and $b_\gamma^{\mu\nu}$, we are able to write down the equations associated to the vanishing of these coefficients. Between these equations, the interesting relations are

$$\begin{cases} C_{00}^{\mu\nu\beta}=\underbrace{\left(b_\alpha^{\mu\nu}b_\gamma^{\alpha\beta}-b_\alpha^{\mu\beta}b^{\alpha\nu}+g^{\mu\alpha}\dfrac{\partial b_\alpha^{\nu\beta}}{\partial u^\gamma}-g^{\mu\alpha}\dfrac{\partial b_\gamma^{\nu\beta}}{\partial u^\alpha}\right)}_{D_\gamma^{\mu\nu\beta}}u^\gamma_{xx}+E_{\gamma\theta}^{\mu\nu\beta}(u^*)u^\gamma_x u^\theta_x\\ C_{02}^{\mu\nu\beta}=b_\alpha^{\mu\nu}g^{\alpha\beta}+g^{\mu\alpha}\left(b_\alpha^{\nu\beta}-\dfrac{\partial g^{\nu\beta}}{u^\alpha}\right), \end{cases} \tag{3.1.32}$$

where we don't write down explicitly $E_{\gamma\theta}^{\mu\nu\beta}(u^*)$ since it isn't useful at this point. Since the coefficients $D_\gamma^{\mu\nu\beta}$ and $E_{\gamma\theta}^{\mu\nu\beta}$ depend only on u^*, the vanishing of $C_{00}^{\mu\nu\beta}$ occurs if and only if $D_\gamma^{\mu\nu\beta}$ and $E_{\gamma\theta}^{\mu\nu\beta}$ vanish individually, i.e.

$$C_{00}^{\mu\nu\beta}=0\iff D_\gamma^{\mu\nu\beta}=E_{\gamma\theta}^{\mu\nu\beta}+E_{\theta\gamma}^{\mu\nu\beta}=0\quad\forall\mu,\nu,\beta,\gamma,\theta\in\{1,....,N\}, \tag{3.1.33}$$

where for $E^{\mu\nu\beta}_{\gamma\theta}$ we've considered the fact that $u^\gamma_x u^\theta_x$ is invariant under $\gamma \leftrightarrow \theta$. The vanishing equations leading to desired relations are $D^{\mu\nu\beta}_\gamma = 0$ and $C^{\mu\nu\beta}_{02} = 0$. Indeed from $C^{\mu\nu\beta}_{02} = 0$ we obtain

$$\begin{aligned} 0 &= b^{\mu\nu}_\alpha g^{\alpha\beta} + b^{\nu\beta}_\alpha g^{\alpha\mu} - \frac{\partial g^{\nu\beta}}{\partial u^\alpha} g^{\alpha\mu} = \\ &= b^{\mu\nu}_\alpha g^{\alpha\beta} + b^{\nu\beta}_\alpha g^{\alpha\mu} - \left(b^{\nu\beta}_\alpha + b^{\beta\nu}_\alpha \right) g^{\alpha\mu} = \\ &= b^{\mu\nu}_\alpha g^{\alpha\beta} - b^{\beta\nu}_\alpha g^{\alpha\mu}, \end{aligned} \tag{3.1.34}$$

where we've used the symmetry and the compatibility conditions (3.1.15). This is exactly the third relation of (3.1.8). From $D^{\mu\nu\beta}_\gamma = 0$ we have

$$0 = b^{\mu\nu}_\alpha b^{\alpha\beta}_\gamma - b^{\mu\beta}_\alpha b^{\alpha\nu} + g^{\mu\alpha} \frac{\partial b^{\nu\beta}_\alpha}{\partial u^\gamma} - g^{\mu\alpha} \frac{\partial b^{\nu\beta}_\gamma}{\partial u^\alpha} \tag{3.1.35}$$

that it's exactly the fourth relation of (3.1.8). We have proved the four relations (3.1.8). We're ready to conclude the proof of the direct implication. The first relation is clearly the symmetry condition of the tensor $g^{\mu\nu}$. So $g_{\mu\nu}$ (i.e. its inverse) defines a metric on the target space V. Inserting $b^{\mu\nu}_\gamma = -g^{\mu\alpha}\Gamma^\nu_{\alpha\gamma}$ in the second relation we obtain

$$\frac{\partial g^{\mu\nu}}{\partial u^\gamma} + g^{\mu\alpha}\Gamma^\nu_{\alpha\gamma} + g^{\nu\alpha}\Gamma^\mu_{\alpha\gamma} = 0, \tag{3.1.36}$$

that it's the compatibility condition of the Christoffel symbols $\Gamma^\nu_{\alpha\gamma}$ with the metric $g_{\mu\nu}$. Adopting the same substitution in the third relation, we obtain

$$g^{\mu\alpha} g^{\nu\beta} \Gamma^\beta_{\alpha\beta} = g^{\mu\alpha} g^{\nu\beta} \Gamma^\beta_{\beta\alpha} \Rightarrow \Gamma^\beta_{\alpha\beta} = \Gamma^\beta_{\beta\alpha}, \tag{3.1.37}$$

i.e. the torsionless condition of the connection represented by $\Gamma^\nu_{\alpha\gamma}$. Therefore $\Gamma^\nu_{\alpha\gamma}$ are the Christoffel symbols of the Levi-Civita connection of the metric $g_{\mu\nu}$. Lastly the fourth relation leads to

$$g^{\mu\tau} g^{\gamma\eta} (\Gamma^\nu_{\gamma\tau}\Gamma^\alpha_{\beta\eta} - \Gamma^\alpha_{\gamma\tau}\Gamma^\nu_{\beta\eta}) = -g^{\mu\gamma} \frac{\partial}{\partial u^\gamma}(g^{\nu\eta}\Gamma^\alpha_{\eta\beta}) + g^{\mu\gamma} \frac{\partial}{\partial u^\beta}(g^{\nu\eta}\Gamma^\alpha_{\eta\gamma}). \tag{3.1.38}$$

Using the Leibniz rule and the compatibility condition for the terms containing the derivative of $g^{\nu\eta}$ (i.e $\frac{\partial g^{\mu\nu}}{\partial u^\gamma} = -g^{\mu\alpha}\Gamma^\nu_{\alpha\gamma} - g^{\nu\alpha}\Gamma^\mu_{\alpha\gamma}$), we obtain

$$\begin{aligned} g^{\mu\tau} g^{\gamma\eta} (\Gamma^\nu_{\gamma\tau}\Gamma^\alpha_{\beta\eta} - \Gamma^\alpha_{\gamma\tau}\Gamma^\nu_{\beta\eta}) = -g^{\mu\gamma} \frac{\partial \Gamma^\alpha_{\eta\beta}}{\partial u^\gamma} g^{\nu\eta} + g^{\mu\nu} \frac{\partial \Gamma^\alpha_{\eta\gamma}}{\partial u^\beta} g^{\nu\eta} + g^{\mu\gamma}(g^{\nu\tau}\Gamma^\eta_{\tau\gamma} + g^{\eta\tau}\Gamma^\nu_{\tau\gamma})\Gamma^\alpha_{\eta\beta} \\ - g^{\mu\gamma}(g^{\nu\tau}\Gamma^\eta_{\tau\beta} + g^{\eta\tau}\Gamma^\nu_{\tau\beta})\Gamma^\alpha_{\eta\gamma}, \end{aligned} \tag{3.1.39}$$

from which it follows

$$g^{\mu\gamma} g^{\nu\tau} \underbrace{\left(\Gamma^\eta_{\tau\gamma}\Gamma^\alpha_{\eta\beta} - \Gamma^\eta_{\tau\beta}\Gamma^\alpha_{\eta\gamma} + \frac{\partial \Gamma^\alpha_{\tau\gamma}}{\partial u^\beta} - \frac{\partial \Gamma^\alpha_{\tau\beta}}{\partial u^\gamma} \right)}_{R_{\tau\gamma}{}^\alpha{}_\beta} = 0 \Rightarrow R_{\tau\gamma}{}^\alpha{}_\beta = 0. \tag{3.1.40}$$

$R_{\tau\gamma}{}^\alpha{}_\beta$ are the components of the curvature tensor associated to $\Gamma^\gamma_{\mu\nu}$. Therefore we have proved that the curvature tensor vanishes. We have concluded the proof of the direct implication. The inverse implication is quite simple. Indeed the hypothesis allow us to use the flat coordinanates (we remind you that in the first part of this section we have proved that $g^{\mu\nu}$ and $\Gamma^{\mu\nu}_{\beta}$ transform respectevely as a (2,0) tensor and an affine connection). In these coordinates the Hamiltonian operator becomes

$$K'^{\mu\nu} = \eta^{\mu\nu}\partial_x, \tag{3.1.41}$$

where $\eta^{\mu\nu}$ is a constant non degerate symmetric matrix. Finally it is easy to prove that $K'^{\mu\nu}$ satisfies the jacobi identity and antisymmetry condition. The proof of this result is similar to the one that we've done for the direct implication. The details are present in [2]. This ends the proof of this theorem. □

As last step of this section we want to underline another result that can be deduced from what we've done in the proof of the Dubrovin-Novikov theorem. Indeed The Dubrovin-Novikov theorem holds if the $g^{\mu\nu}$ tensor is non degenerate. But from its proof is simply derivable another result, valid in the degenerate case.

Theorem (Grinberg, 1985)**.** *([9]) Let $K^{\mu\nu}$ be an Hamiltonian operator of hydrodynamic type associated with the Poisson bracket $\{\cdot;\cdot\}_K$. The Poisson bracket $\{\cdot;\cdot\}_K|_{\epsilon=0}$ is antisymmetric and it satisfies the Jacobi identity if and only if the conditions*

$$\begin{cases} g^{\mu\nu} = g^{\nu\mu} \\ b^{\mu\nu}_{\gamma} + b^{\nu\mu}_{\gamma} = \dfrac{\partial g^{\mu\nu}}{\partial u^{\gamma}} \\ b^{\mu\nu}_{\gamma} g^{\gamma\alpha} = b^{\alpha\nu}_{\gamma} g^{\gamma\mu} \\ b^{\mu\nu}_{\gamma} b^{\gamma\alpha}_{\beta} - b^{\mu\alpha}_{\gamma} b^{\gamma\nu}_{\beta} = g^{\mu\gamma}\left(\dfrac{\partial b^{\nu\alpha}_{\beta}}{\partial u^{\gamma}} - \dfrac{\partial b^{\nu\alpha}_{\gamma}}{\partial u^{\beta}}\right) \\ \displaystyle\sum_{cyclic(\mu,\nu,\alpha)} \left[\left(\frac{\partial b^{\mu\nu}_{\gamma}}{\partial u^{\beta}} - \frac{\partial b^{\mu\nu}_{\beta}}{\partial u^{\gamma}}\right) b^{\gamma\alpha}_{\eta} + \left(\frac{\partial b^{\mu\nu}_{\gamma}}{\partial u^{\eta}} - \frac{\partial b^{\mu\nu}_{\eta}}{\partial u^{\gamma}}\right) b^{\gamma\alpha}_{\beta}\right] = 0 \end{cases} \tag{3.1.42}$$

are satisfied for any value of the free indices.

Proof. From the proof of the Dubrovin-Novikov theorem, we have that

$$\begin{cases} g^{\mu\nu} = g^{\nu\mu} \\ b^{\mu\nu}_{\gamma} + b^{\nu\mu}_{\gamma} = \dfrac{\partial g^{\mu\nu}}{\partial u^{\gamma}} \end{cases} \tag{3.1.43}$$

holds iff the bracket is antisymmetric. Concerning the Jacobi identity, we have found that, without using the non degeneracy conditions, the Jacobi identity is identically equivalent to

$$\int dx \sum_{j,k=0,1,2} C^{\mu\nu\beta}_{jk} \frac{\delta f}{\delta u^{\mu}} \frac{\delta g}{\delta u^{\nu}}^{(j)} \frac{\delta h}{\delta u^{\beta}}^{(k)} = 0 \quad \forall \mu,\nu,\beta \in \{1,....,N\}, \tag{3.1.44}$$

with $C^{\mu\nu\beta}_{00} = D^{\mu\nu\beta}_{\gamma}(u^*)u^{\gamma}_{xx} + E^{\mu\nu\beta}_{\gamma\theta}(u^*)u^{\gamma}_{x}u^{\theta}_{x}$. In turn we have seen that (3.1.44) holds iff $C^{\mu\nu\beta}_{jk} = 0$ for any value of the indices. Afterwards we have proved that $D^{\mu\nu\beta}_{\gamma} = 0$ and $C^{\mu\nu\beta}_{02} = 0$ are indentically equivalent to

$$\begin{cases} b^{\mu\nu}_{\gamma} g^{\gamma\alpha} = b^{\alpha\nu}_{\gamma} g^{\gamma\mu} \\ b^{\mu\nu}_{\gamma} b^{\gamma\alpha}_{\beta} - b^{\mu\alpha}_{\gamma} b^{\gamma\nu}_{\beta} = g^{\mu\gamma}\left(\dfrac{\partial b^{\nu\alpha}_{\beta}}{\partial u^{\gamma}} - \dfrac{\partial b^{\nu\alpha}_{\gamma}}{\partial u^{\beta}}\right) \end{cases} \tag{3.1.45}$$

and it's not difficult to show that $E^{\mu\nu\beta}_{\gamma\theta} + E^{\mu\nu\beta}_{\theta\gamma} = 0$ (coming from (3.1.32)) is equivalent to

$$\sum_{cyclic(\mu,\nu,\alpha)} \left[\left(\frac{\partial b^{\mu\nu}_{\gamma}}{\partial u^{\beta}} - \frac{\partial b^{\mu\nu}_{\beta}}{\partial u^{\gamma}} \right) b^{\gamma\alpha}_{\eta} + \left(\frac{\partial b^{\mu\nu}_{\gamma}}{\partial u^{\eta}} - \frac{\partial b^{\mu\nu}_{\eta}}{\partial u^{\gamma}} \right) b^{\gamma\alpha}_{\beta} \right] = 0 \quad \forall \mu, \nu, \alpha, \beta, \eta \in \{1, ..., N\}. \tag{3.1.46}$$

Finally we will prove in the appendix that the other relations (i.e. $C^{\mu\nu\beta}_{jk=} = 0$ with $(j,k) \neq \{(0,0);(0,2)\}$) can be obtained using (3.1.42). This ends the proof since we have proved that $C^{\mu\nu\beta}_{jk} = 0$ for any value of the indices iff (3.1.42) holds. □

The last observation of this section is the following one: in general we will not consider Poisson structures in the limit of $\epsilon = 0$. But we have proved the Dubrovin-Novikov theorem for Hamiltonian operators in this limit. Is the direct implication of the Dubrovin-Novikov true also for the zero order of the general case? The answer is yes since the order division given by the ϵ parameter implies that zero order of the Hamiltonian operator satisfies the antisymmetry and the Jacobi conditions if the Hamiltonian operator satisfies them.

3.2 Getzler theorem

In this section we will prove Getzler theorem, important for the study of the formal loop space. Let us write down the statement of the theorem.

Theorem (Getzler)**.** *([8]) Let $K^{\mu\nu}$ be a Hamiltonian operator of hydrodynamic type, i.e of the type (3.1.1) (with $g^{\mu\nu}$ non degenerate). There exists a Miura transformation bringing any Poisson structure of hydrodynamic type to the canonical form*

$$K^{\mu\nu} = \eta^{\mu\nu}\partial_x, \tag{3.2.1}$$

where $\eta^{\mu\nu}$ is a non degenerate constant symmetric matrix.

We will work adopting the distributionl formalism. Let us introduce an important result useful for the proof of this theorem.

Lemma. *Let $\bar{w} = \eta^{\mu\nu}\delta'(x-y)$ be the canonical form. Then any cocycle in $H^1_{\bar{w}}$ or in $H^2_{\bar{w}}$ is trivial.*

Proof. We will not prove explicitly the triviality of the $\epsilon-$vanishing element of $H^1_{\bar{w}}$ because it's not interesting for the following parts. However we will use this result during this proof. Let us start considering the closure condition $\partial_{\bar{w}}\alpha = 0$ (2.5.39) for $\alpha \in \Lambda^2_{loc}$ $(\alpha^{\mu\nu} = \sum_{s\geq 0} A^{\mu\nu}_s \delta^{(s)}(x-y))$. This leads to

$$\begin{aligned} &-\frac{\partial A^{\mu\nu}_t}{\partial u^\beta_{s-1}}\eta^{\beta\gamma} + \sum_{r\geq 0, q\leq t-1}(-1)^{q+r+s}\binom{q+r+s}{q \quad r}\partial^r_x\left(\frac{\partial A^{\gamma\mu}_{q+r+s}}{\partial u^\beta_{t-q-1}}\right)\eta^{\beta\nu} + \\ &+ \sum_{q\leq s, q+r+t\geq 1}(-1)^{q+r+t}\binom{q+r+t}{q \quad r}\partial^r_x\left(\frac{\partial A^{\nu\gamma}_{s-q}}{\partial u^\beta_{q+r+t-1}}\right)\eta^{\beta\mu} = 0, \end{aligned} \tag{3.2.2}$$

for any $\mu, \nu, \gamma \in \{1, ..., N\}$ and $s, t \in \mathbb{N}$. In the following, the relations will be valid for any possible value of the free indices. It's clear that the first term of the previous formula is defined if $s \geq 1$, while the second one is defined if $t \geq 1$. Now, choosing $s = t = 0$, (3.2.2) leads to

$$\partial_x\left(\frac{\delta A^{\nu\gamma}_0}{\delta u^\beta}\right)\eta^{\beta\mu} = 0 \Rightarrow \frac{\delta A^{\nu\gamma}_0}{\delta u^\theta} = 0 \tag{3.2.3}$$

(only the third term of (3.2.2) is non vanishing). Since $degA^{\nu\gamma}_0 = 1$, the solution of the last equation is

$$A^{\mu\nu}_0 = \partial_x B^{\mu\nu}, \tag{3.2.4}$$

where $B^{\mu\nu} \in \widehat{\mathcal{A}}$. Using $s = 0$ and $t > 0$, (3.2.2) becomes

$$\sum_{r\geq 0, q\leq t-1}(-1)^{q+r}\binom{q+r}{r}\partial^r_x\left(\frac{\partial A^{\gamma\mu}_{q+r}}{\partial u^\beta_{t-q-1}}\right)\eta^{\beta\nu} + \sum_{r+t\geq 1}(-1)^{r+t}\binom{r+t}{r}\partial^r_x\left(\frac{\partial A^{\nu\gamma}_0}{\partial u^\beta_{r+t-1}}\right)\eta^{\beta\mu} = 0, \tag{3.2.5}$$

where we've used the fact that the third term is defined only if $q = 0$. The first term of the previous relation can be obtained as derivative w.r.t. u^β_{t-1} of the antisymmetry condition for $A_0^{\mu\gamma}$. Indeed

$$\begin{aligned}\frac{\partial A_0^{\mu\gamma}}{\partial u_{t-1}^\beta}\eta^{\beta\nu} &= \sum_{r\geq 0}(-1)^{r+1}\frac{\partial}{\partial u_{t-1}^\beta}\left(\partial_x^r(A_r^{\gamma\mu})\right) = \\ &= \sum_{r\geq 0}(-1)^{r+1}\sum_{q\leq \min(r;t-1)}\binom{r}{q}\partial_x^{r-q}\frac{\partial A_r^{\gamma\mu}}{\partial u_{t-1-q}^\beta} = \\ &= \sum_{r\geq q}\sum_{q\leq t-1}(-1)^{r+1}\binom{r}{q}\partial_x^{r-q}\frac{\partial A_r^{\gamma\mu}}{\partial u_{t-1-q}^\beta} = \\ &= -\sum_{r\geq 0}\sum_{q\leq t-1}(-1)^{r+q}\binom{r+q}{q}\partial_x^{r}\frac{\partial A_{r+q}^{\gamma\mu}}{\partial u_{t-1-q}^\beta},\end{aligned} \tag{3.2.6}$$

where in the second step we've used the exchange property (2.1.5). Then we have

$$\begin{aligned}0 &= -\frac{\partial A_0^{\mu\gamma}}{\partial u_{t-1}^\beta}\eta^{\beta\nu} + \sum_{r+t\geq 1}(-1)^{r+t}\binom{r+t}{r}\partial_x^r\left(\frac{\partial A_0^{\nu\gamma}}{\partial u_{r+t-1}^\beta}\right)\eta^{\beta\mu} = \\ &= -\frac{\partial A_0^{\mu\gamma}}{\partial u_{t-1}^\beta}\eta^{\beta\nu} - \sum_{r\geq t-1}(-1)^{r}\binom{r+1}{t}\partial_x^{r-t+1}\left(\frac{\partial A_0^{\nu\gamma}}{\partial u_{r}^\beta}\right)\eta^{\beta\mu} \quad \forall t\geq 1.\end{aligned} \tag{3.2.7}$$

The right hand of the previous relation is clearly the term proportional to $\delta^{(t-1)}(x-y)$ of $\partial_{\bar{w}}a^\gamma$ with $(a^\gamma)^\beta := A_0^{\beta\gamma}$, as one can see from (2.5.42). This implies that $\partial_{\bar{w}}a^\gamma = 0$ (since also the term proportional to $\delta(x-y)$ is vanishing for (3.2.3)) and therefore $(a^\gamma)^\beta$ is a cocycle of $H^1_{\bar{w}}$. Using the first part of the lemma, we can assert that there exist N differential polynomials $q^1, ..., q^N$ such that $a^\gamma = \partial_{\bar{w}}q^\gamma$. Defininig the vector field $z := \sum_{s\geq 0}\partial_x^s q^\gamma \frac{\partial}{\partial u_s^\gamma}$, it's easy to see that the equivalent cocycle to α,

$$\alpha' = \alpha + \partial_{\bar{w}}z, \tag{3.2.8}$$

has the coefficient $A_0^{'\mu\nu}$ equal to zero, i.e. $A_0^{'\mu\nu} = 0$. Indeed the term proportional to $\delta(x-y)$ in (2.5.42) is $-\eta^{\mu\alpha}\partial_x\frac{\delta q^\nu}{\delta u^\alpha} = -\partial_{\bar{\omega}}q^\nu$ that is equal to $-A_0^{\mu\nu}$. Using this fact, we will able to show that $\alpha' = \partial_{\bar{w}}h$ for a vector field $h\in\Lambda^1_{loc}$ and that $\alpha = \partial_{\bar{w}}(h-z)$ as desired. Firstly, let us lower the indices of $A_s^{\gamma\beta}$ throught the inverse of the matrix $\eta^{\mu\nu}$:

$$g_{\mu;\nu s} = \eta_{\mu\gamma}\eta_{\nu\beta}A_s^{\gamma\beta} \quad for \quad s\geq 1. \tag{3.2.9}$$

One can prove that

$$\begin{cases} g_{\mu;\nu 1} = \partial_x w_{\mu;\nu 0} \\ g_{\mu;\nu s} = \partial_x w_{\mu;\nu s-1} + w_{\mu;\nu s-2} \quad for \quad s\geq 2, \end{cases} \tag{3.2.10}$$

for some differential polynomial $w_{\mu;\nu 0}, w_{\mu;\nu 1}, ... \in \widehat{\mathcal{A}}$. The first relation is derivable in the same way of (3.2.3), choosing in (3.2.2) $s=1$ and $t=0$. From these relations, one can prove that $\omega_{\mu;\nu s}$ satisfies the antisymmetry condition (2.4.14) for the reduced 2-form and that $\omega_{\mu;\nu s}$ is closed w.r.t. the δ-differential (see the details in [7]). Since the δ-differential produces an exact sequence, there exists a 1-form $\phi = \phi_\mu\delta u^\mu$ such that $w = \delta\phi$. Finally it's easy to see that the components of the vector field that we're searching are simply given by $h^\mu = \eta^{\mu\nu}\phi_\mu$. □

This result is fundamental for the proof of Getzler theorem. Indeed we will use the vanishing of $\widehat{\Lambda}^2_{loc}$ to prove the following lemma that, basically, ends the proof. It is understood from now on that the Dubrovin-Novikov theorem has been applied, i.e. for a bivector $\alpha \in \widehat{\Lambda}^2_{loc}$ the coefficients $A^{\mu\nu}_{[0]}$ are the elements of a non degenerate constant symmetric matrix.

Lemma. *([4]) Let $\alpha \in \widehat{\Lambda}^2_{loc}$ be a bivector such that $[\alpha;\alpha] = 0$. Considering the expansion of α in powers of ϵ, i.e. $\alpha = \sum_{k\geq 0} \epsilon^k \alpha_k$, there exists a vector field $\widehat{\Lambda}^1_{loc} \ni X = \sum_{k\geq 0} \epsilon^k X_k$ for which holds the following relation :*

$$\alpha_k = T^X_k(\alpha_0) \quad \forall k \geq 0, \tag{3.2.11}$$

where α_0 has coefficients rapresented by a non degenerate constant symmetric matrix (Dubrovin-Novikov theorem) and

$$T^X_k = \sum_{j_1+2j_2+....+kj_k=k} \frac{1}{j_1!....j_k!} Lie^{j_1}_{X_1} \circ .... \circ Lie^{j_k}_{X_k} \tag{3.2.12}$$

is called Schur polynomial operator of order k.

Proof. The proof will be done by induction. The case $k = 1$ is very simple: indeed expanding the relation $[\alpha;\alpha] = 0$ in powers of ϵ, we obtain

$$\sum_{i+j=l} [\alpha_i;\alpha_j] = 0, \tag{3.2.13}$$

valid for any $l \geq 0$. Then for $l = 1$ we get the condition $2\,[\alpha_0;\alpha_1] = 0$, i.e. α_1 satisfies the closure condition of ∂_{α_0}. Therefore, for the previous lemma, there exists a vector field $X_1 \in \widehat{\Lambda}^1_{loc}$ such that $\alpha_1 = \partial_{\alpha_0} X_1 = Lie_{X_1}\alpha_0$. In order to prove the induction step, we have to use the following identity (that we will not prove):

$$T^X_l([\alpha;\alpha]) = \sum_{i+j=l} \left[T^X_i(\alpha); T^X_j(\alpha)\right], \tag{3.2.14}$$

true for any $\alpha \in \widehat{\Lambda}^2_{loc}$ and $X \in \widehat{\Lambda}^1_{loc}$ (remind that the vector field enters in the definition of the Schur operator). Let us assume that (3.2.11) is true for $k \leq n$ and $X_1, ..., X_n \in \widehat{\Lambda}^1_{loc}$ are the associate vector fields. Then consider as $n+1$ field the vanishing one. We will denote with $\hat{T}_k$ the k-order Schur polynomial associated to $(X_1, ..., X_n, 0)$, while T_k will denote the k-order Schur polynomial associated to $(X_1, ..., X_n)$ and T_{n+1} will be the Schur operator for which holds the identity (3.2.11) for $k = n+1$. It's clear that

$$\hat{T}_k = T_k \quad \forall k \leq n \Rightarrow \hat{T}_k(\alpha_0) = T_k(\alpha_0) = \alpha_k. \tag{3.2.15}$$

Using what we've just written and the condition $[\alpha;\alpha] = 0$, the relation (3.2.14) becomes

$$\begin{aligned} 0 = \hat{T}_{n+1}([\alpha_0;\alpha_0]) &= \sum_{i+j=l} \left[\hat{T}^X_i(\alpha_0); \hat{T}^X_j(\alpha_0)\right] = 2\left[\alpha_0; \hat{T}_{n+1}(\alpha_0)\right] + \sum_{i+j=l, i,j\neq 0} [\alpha_i;\alpha_j] \\ &= 2\left[\alpha_0; \hat{T}_{n+1}(\alpha_0)\right] - 2\,[\alpha_0;\alpha_{n+1}], \end{aligned} \tag{3.2.16}$$

where in the last step we have used the relation (3.2.13). Therefore we have proved that

$$\left[\alpha_0; \alpha_{n+1} - \hat{T}_{n+1}(\alpha_0)\right] = \partial_{\alpha_0}(\alpha_{n+1} - \hat{T}_{n+1}(\alpha_0)) = 0 \tag{3.2.17}$$

and that there exists a vector field $X_{n+1} \in \widehat{\Lambda}^1_{loc}$ (for the previous lemma) such that

$$\alpha_{n+1} - \hat{T}_{n+1}(\alpha_0) = Lie_{X_{n+1}}\alpha_0. \tag{3.2.18}$$

So X_{n+1} is the vector field that we are looking for since $T_{n+1} = Lie_{X_{n+1}}\alpha_0 + \hat{T}_{n+1}(\alpha_0) = \alpha_{n+1}$. We have proved the lemma. □

Remark. *Note that the k-order Schur operator is nothing but the terms of the expansion in ϵ powers of the exponential operator*

$$e^{Lie_x} = e^{\sum_{k\geq 0}\epsilon^k X_k}. \tag{3.2.19}$$

Therefore the lemma that we've proved says us that there exists a vector field $X \in \widehat{\Lambda}^1_{loc}$ such that

$$\alpha = e^{Lie_X}(\alpha_0). \tag{3.2.20}$$

Now we can find a Miura transformation that transforms the Poisson bivector α in α_0 through the relation (3.2.20). Indeed let us consider the coefficients A^β of X defined by the previous lemma (A^β are the coefficients that allow us to write down the vector field X in the form $X = \sum_{s\geq 0}\partial_x^s A^\beta \frac{\partial}{\partial u_s^\beta}$) and the associated PDE

$$\frac{\partial \tilde{u}^\beta}{\partial \epsilon}(u_*^*, \epsilon) = A^\beta(u_*^*, \epsilon) \quad \forall \beta \in \{1, ..., N\}, \tag{3.2.21}$$

with the boundary condition $\tilde{u}^\beta(u_*^*, \epsilon)_{|\epsilon=0} = u^\beta$ for any $\beta \in \{1, ..., N\}$ (we want to preserve the $A^{\mu\nu}_{[0]}$ of the Poisson bivector since it has already the correct form due to the Dubrovin-Novikov theorem). The solution is of degree 0 (since its derivative w.r.t. to ϵ has to be of degree 1) and the Jacobian of $\tilde{u}^\beta(u_*^*, \epsilon)_{|\epsilon=0}$ is the identity. Therefore it can be viewed as a Miura transformation. The following lemma is the one that allows us to end the proof (we will not prove this lemma).

Lemma. *Let's consider the Miura transformation obtained as solution of (3.2.21). Then the Miura transformation of the Poisson bivector α can be written as*

$$\alpha^{\mathcal{M}} = e^{Lie_X}(\alpha), \tag{3.2.22}$$

where X is the vector field defining the PDE (3.2.21).

So the Miura transformation that we were searching is the solution of (3.2.21) where X is the vector field found out applying the Schur operator lemma. The proof is completed.

Conclusion

The theorems that we have proved in this work are fundamental results for the study of Hamiltonian systems using the formal loop space approach in a classical sense. For example these methods are suitable to investigate integrable systems, in particular integrable hierarchies. The paradigmatic integrale hierarchy is the Kortewegde Vries hierarchy (see [5],[1]). But it's the quantization of these structures that offers some interesting challenge. We will mention some of them. A very interesting question regards the validity in the case of the formal loop space of the result that M. Kontsevich (1998 Fields medallist) has obtained in the finite dimensional Poisson geoemtry enviroment. More precisely, the results obtained by Kontsevich concern the so called deformation quantization. This is a way of quantizing theories (used in mathematics) done without the explicit representation of the observables algebra through an Hilbert space, but just describing evolution inside an abstrat non-commutative associative algebra. This abstract associative algebra is obtained deforming the commutative algebra of the classical observables, i.e. function belonging to $C^{\infty}(M)$, through the introdution of a parameter denoted with $\hbar$ (since we want that a classical limit exists). The interesting deformations class is the one whose elements are called star products. Between these star products, there is a natural way to define equivent star products. Defined these objects, as one can immagine, it's not difficult to give also a notion of deformation of a Poisson structure and of equivalence between them. The beautiful result obtained by Kontsevich ([10]) gives us a way to construct explicitly a bijective map between the equivalence class of star products and the equivalence class of Poisson deformations. Therefore studying the class of Poisson strucutures deformations allows us to extract informations about the deformations of the observables algebra. An interesting question, for example, is if the quantization of the observables algebra is unique up to isomorphisms. This result is obtained considering finite dimensional manifold. Is the Kontsevich theorem true also when the manifold is infinite dimensional as in the case of the formal loop space? This is still an open question. Another interesting subject is the one regarding the discovery of E. Witten and Kontsevich about the relationship between KdV and the topology of the moduli spaces of stable algebraic curves (referece work [13]). This result is known in the branch of algebraic geometry as Witten Conjecture (even if it is a theorem since it has been proved by Kontsevich). This discovery has opened some interesting directions on the study of the connection between integrable systems, Quantum field theories and String theory. To see some references, read the introduction of [7].

Appendix A

Second variational principle

A.1 Proof of the derative relations

In this section we will prove all the derivative relation used in the proof of the second variational principle. Let us start by P even considering $Z_\mu\left(\frac{(-1)^p}{2}(u_p^\mu)^2\right)$ (Einstein's convention not at work):

$$
\begin{aligned}
Z_\mu\left(\frac{(-1)^p}{2}(u_p^\mu)^2\right) &= \frac{\delta}{\delta u^\mu}\left(\sum_{\nu=1}^{N} X_\nu^\mu u_P^\nu u_P^\mu + X_0^\mu u_P^\mu\right) = \\
&= \partial_x^P\left(\sum_{\nu=1}^{N} X_\nu^\mu u_P^\nu\right) + \partial_x^P(X_\mu^\mu u_P^\mu) + R(u^*, ..., u_{2P-1}^*) = \\
&= \sum_{\nu=1}^{N} X_\nu^\mu u_{2P}^\nu + X_\mu^\mu u_{2P}^\mu + R(u^*, ..., u_{2P-1}^*).
\end{aligned}
\tag{A.1.1}
$$

Then it's evident that

$$
\frac{\partial Z_\mu}{\partial u_{2P}^\beta} = X_\beta^\mu + \delta^{\mu\beta} X_\mu^\mu. \tag{A.1.2}
$$

Passing to the $P = 2p+1$ case, we have

$$
Z_\mu\left(\frac{(-1)^{p+1}}{2}(u_{p+1}^\mu)^2\right) = \frac{\delta}{\delta u^\mu}\left(\sum_{\nu=1}^{N} X_\nu^\mu u_P^\nu u_{P+1}^\mu + X_0^\mu u_{P+1}^\mu\right). \tag{A.1.3}
$$

Isolating the terms proportional to u_{2P+1}^*, one obtain

$$
\begin{aligned}
Z_\mu\left(\frac{(-1)^{p+1}}{2}(u_{p+1}^\mu)^2\right) &= \partial_x^{P+1}\left(\sum_{\nu=1}^{N} X_\nu^\mu u_P^\nu\right) - \partial_x^P(X_\mu^\mu u_{P+1}^\mu) + R(u^*, ..., u_{2P}^*) = \\
&= \sum_{\nu=1}^{N} X_\nu^\mu u_{2P+1}^\nu - X_\mu^\mu u_{2P+1}^\mu + R(u^*, ..., u_{2P}^*),
\end{aligned}
\tag{A.1.4}
$$

that it leads to

$$
\frac{\partial Z_\mu}{\partial u_{2P+1}^\beta} = X_\beta^\mu - \delta^{\mu\beta} X_\mu^\mu. \tag{A.1.5}
$$

Now, from $Z_\mu\left(\frac{(-1)^p}{2}(u_p^\mu)^2\right)$ we have to isolate the terms proportional to $u_{2P}^* u_P^*$ (in this calculation we will use the fact that $X_\nu^\mu = 0$ for any $\nu \neq \mu$):

$$
\begin{aligned}
&Z_\mu\left(\frac{(-1)^{p+1}}{2}(u^\mu_{p+1})^2\right)=\partial_x^{P-1}\left(\frac{\partial X^\mu_\mu}{\partial u^\mu_{P-1}}u^\mu_P u^\mu_{P+1}\right)-\partial_x^P(X^\mu_\mu u^\mu_{P+1})+\partial_x^{P+1}(X^\mu_\mu u^\mu_P)+R=\\
&=\frac{\partial X^\mu_\mu}{\partial u^\mu_{P-1}}u^\mu_P u^\mu_{2P}-P\partial_x(X^\mu_\mu)u^\mu_{2P}+(P+1)\partial_x(X^\mu_\mu)u^\mu_{2P}+\partial_x^{P+1}(X^\mu_\mu)u^\mu_P+R=\\
&=\frac{\partial X^\mu_\mu}{\partial u^\mu_{P-1}}u^\mu_P u^\mu_{2P}+\left(\sum_{\nu=1}^N\sum_{m\le P-1}\frac{\partial X^\mu_\mu}{u^\nu_{m-1}}u^\nu_m\right)u^\mu_{2P}+\partial_x^P\left(\sum_{\nu=1}^N\sum_{m\le P-1}\frac{\partial X^\mu_\mu}{u^\nu_{m-1}}u^\nu_m\right)u^\mu_P+R=\\
&=\frac{\partial X^\mu_\mu}{\partial u^\mu_{P-1}}u^\mu_P u^\mu_{2P}+\left(\sum_{\nu=1}^N\frac{\partial X^\mu_\mu}{u^\nu_{P-1}}u^\nu_P\right)u^\mu_{2P}+\left(\sum_{\nu=1}^N\frac{\partial X^\mu_\mu}{u^\nu_{P-1}}u^\nu_{2P}\right)u^\mu_P+R,
\end{aligned}
\tag{A.1.6}
$$

where R contains the terms not proportional to $u^*_{2P}u^*_P$. Therefore it's evident that

$$
\frac{\partial^2 Z_\mu}{\partial u^\mu_{2P}\partial u^\nu_P}=(1+2\delta^{\mu\nu})\frac{\partial X^\mu_\mu}{\partial u^\nu_{P-1}}. \tag{A.1.7}
$$

Isolating the terms proportional to u^*_{2P}, one finds that (also here we will use the fact that $X^\mu_\nu=0$ for any $\nu\neq\mu$ and the fact tha X^μ_μ doesn't depend on u^*_{P-1})

$$
\begin{aligned}
Z_\mu\left(\frac{(-1)^{p+1}}{2}(u^\mu_{p+1})^2\right)&=\underbrace{\partial_x^{P-1}\left(\frac{\partial X^\mu_\mu}{\partial u^\mu_{P-1}}\right)u^\mu_P u^\mu_{P+1})}_{=0}-\partial_x^P(X^\mu_\mu u^\mu_{P+1})+\partial_x^{P+1}(X^\mu_\mu u^\mu_P)+\partial_x^{P-1}\left(\frac{\partial X^\mu_0}{\partial u^\mu_{P-1}}u^\mu_{P+1}\right)+\\
&+\partial_x^{P+1}(X^\mu_0)+R=\\
&=\underbrace{-P\partial_x X^\mu_\mu u^\mu_{2P}+(P+1)\partial_x X^\mu_\mu u^\mu_{2P}}_{\partial_x X^\mu_\mu u^\mu_{2P}}+\frac{\partial X^\mu_0}{\partial u^\mu_{P-1}}u^\mu_{2P}+\sum_{\nu=1}^N\frac{\partial X^\mu_0}{\partial u^\nu_{P-1}}u^\nu_{2P}+R,
\end{aligned}
\tag{A.1.8}
$$

where R contains the terms not proportional to u^*_{2P}. Clearly this implies that

$$
\frac{\partial Z_\mu}{\partial u^\mu_{2P}}=\partial_x X^\mu_\mu+2\frac{\partial X^\mu_0}{\partial u^\nu_{P-1}}. \tag{A.1.9}
$$

The last relation requires that computations of $Z_\mu\left(\frac{(-1)^{p+1}}{6}(u^\mu_{p+1})^3\right)$ (also here we will use the fact that $X^\mu_\nu=0$ for any $\nu\neq\mu$ and the fact tha X^μ_μ doesn't depend on u^*_{P-1}):

$$
\begin{aligned}
&Z_\mu\left(\frac{(-1)^{p+1}}{6}(u^\mu_{p+1})^3\right)=\frac{\delta}{\delta u^\mu}\left(\sum_{l=0}^p\binom{p}{l}u^\mu_{P-l}u^\mu_{p+l+2}(X^\mu_\mu u^\mu_P+X^\mu_0)\right)=\\
&=\sum_{l=0}^p\sum_{m\le P-2}(-1)^m\binom{p}{l}\partial_x^m\left(\frac{\partial X^\mu_\mu}{\partial u^\mu_m}u^\mu_P u^\mu_{P-l}u^\mu_{p+l+2}\right)-\sum_{l=0}^p\binom{p}{l}\partial_x^P(X^\mu_\mu u^\mu_{P-l}u^\mu_{p+l+2})+\\
&+\sum_{l=0}^p\binom{p}{l}(-\partial_x)^{P-l}(X^\mu_\mu u^\mu_P u^\mu_{p+l+2})+\sum_{l=0}^p\binom{p}{l}(-\partial_x)^{p+l+2}(X^\mu_\mu u^\mu_P u^\mu_{P-l})+
\end{aligned}
\tag{A.1.10}
$$

$$+\sum_{l=0}^{p}\sum_{m\leq P-1}(-1)^m\binom{p}{l}\partial_x^m\left(\frac{\partial X_0^\mu}{\partial u_m^\mu}u_{P-l}^\mu u_{p+l+2}^\mu\right)+\sum_{l=0}^{p}\binom{p}{l}(-\partial_x)^{P-l}(X_0^\mu u_{p+l+2}^\mu)+$$
$$+\sum_{l=0}^{p}\binom{p}{l}(-\partial_x)^{p+l+2}(X_0^\mu u_{P-l}^\mu) \tag{A.1.11}$$

We've to isolate the terms proportional to u_{2P}^μ. From the first terms of the previous formula we don't get any term since $P-l\leq P$ and $p+l+2\leq P+1$ while $m\leq P-2$ (from the derivative of X_μ^μ it's not possible to have terms since it depends at most on u_{2P-2}^μ). Therefore we can get at most terms proportional to u_{2P-1}^μ. From the second term we obtain

$$-\sum_{l=0}^{p}\binom{p}{l}\partial_x^P(X_\mu^\mu u_{P-l}^\mu u_{p+l+2}^\mu)=-X_\mu^\mu u_{2P}^\mu u_{p+2}^\mu-P\partial_x(X_\mu^\mu u_{p+1}^\mu)u_{2P}^\mu-pX_\mu^\mu u_{2P}^\mu u_{p+2}^\mu+R=$$
$$=-(P+p+1)X_\mu^\mu u_{2P}^\mu u_{p+2}^\mu-P\partial_x X_\mu^\mu u_{2P}^\mu u_{p+1}^\mu+R, \tag{A.1.12}$$

since we get a u_{2P}^μ term deriving P times u_{P-l}^μ for $l=0$, deriving P-1 times u_{P+l+2}^* for $l=p$ (so we have to derive 1 time X_μ^μ) and deriving P times u_{P+l+2}^* for $l=p-1$. From the third term one get

$$\sum_{l=0}^{p}\binom{p}{l}(-\partial_x)^{P-l}(X_\mu^\mu u_P^\mu u_{p+l+2}^\mu)=-X_\mu^\mu u_{2P}^\mu u_{p+2}^\mu+R, \tag{A.1.13}$$

since from u_P^μ we can get u_{2P}^μ putting $l=0$. There is another contribution coming from u_{p+l+2}^μ in the case $P\leq 3$. Indeed the maximum values reached by the derivative index of u_{p+l+2}^μ after the application of ∂_x is $p+l+2+P-l=3p+3$ for which the following inequalities holds:

$$3p+3\geq 2P=4p+2 \iff p\leq 1 \iff p=0,1. \tag{A.1.14}$$

In the case $P=3$ ($p=1$) one can verify that the terms coming from u_{p+l+2}^μ vanish: this is why the relation that we are proving holds for $P\geq 3$. Indeed we get the following terms:

$$X_\mu^\mu u_3^\mu(\underbrace{-u_6^\mu}_{p=1,l=0}+\underbrace{u_6^\mu}_{p=1,l=1})=0. \tag{A.1.15}$$

In the $P=1$ case there isn't the elison of terms as in the $N=3$ case. From the fourth term we get

$$\sum_{l=0}^{p}\binom{p}{l}(-\partial_x)^{p+l+2}(X_\mu^\mu u_P^\mu u_{P-l}^\mu)=(P+1)\partial_x(X_\mu^\mu u_{p+1}^\mu)u_{2P}^\mu-pX_\mu^\mu u_{2P}^\mu u_{p+2}^\mu=$$
$$=(P+1-p)X_\mu^\mu u_{p+2}^\mu u_{2P}^\mu+(P+1)\partial_x X_\mu^\mu u_{2P}^\mu u_{p+1}^\mu, \tag{A.1.16}$$

where we'used the same arguments used above (also here there is the $N=3$ elision of the extra-term). From the fifth term we have

$$\sum_{l=0}^{p}\sum_{m\leq P-1}(-1)^m\binom{p}{l}\partial_x^m\left(\frac{\partial X_0^\mu}{\partial u_m^\mu}u_{P-l}^\mu u_{p+l+2}^\mu\right)=\frac{\partial X_0^\mu}{\partial u_{P-1}^\mu}u_{p+1}^\mu u_{2P}^\mu+R, \tag{A.1.17}$$

where we've applied the same arguments used above. The sixth term doesn't have terms proportional to u^μ_{2P} for $N \geq 3$. In the case $N = 3$ one can verify in the same way described above that the terms proportional to u^μ_{2P} vanish. The extra- terms elision for the $N = 3$ case is present also in the seveth term. Moreover, applying ∂_x^{P+1} $(l = p)$ to X^μ_0, we get

$$\sum_{l=0}^{p} \binom{p}{l} (-\partial_x)^{p+l+2} (X^\mu_0 u^\mu_{P-l}) = \frac{\partial X^\mu_0}{\partial u^\mu_{P-1}} u^\mu_{p+1} u^\mu_{2P} + R. \tag{A.1.18}$$

Finally, summing all the terms proportional to u^μ_{2P}, we get the desired relation:

$$\frac{\partial Z_\mu}{\partial u^\mu_{2P}} = \left(2\frac{\partial X^\mu_0}{\partial u^\mu_{P-1}} + \partial_x X^\mu_\mu \right) u^\mu_{p+1} - P X^\mu_\mu u^\mu_{p+2}. \tag{A.1.19}$$

A.1.1 Exactness of $cu^\mu_1 \frac{\delta f}{\delta u^\mu}$

In this section we will prove that

$$\int dx c u^\mu_1 \frac{\delta f}{\delta u^\mu} = 0, \tag{A.1.20}$$

for any $f \in \mathbb{C}[[u^*]][u^*_{k>0}]$ (this is true also in $\widehat{\mathcal{A}}$). In order to prove this fact, let us introduce two operators. The first one is called **generalized momentum** operator.

Definition. *The* ***generalized momentum*** *operator of type (t, s), for $\alpha, s \geq 0$, is defined in $\widehat{\mathcal{A}}$ as*

$$p_{\mu,t,s} := \sum_{k\geq 0} (-1)^k \binom{k+s}{s} \partial_x^k \circ \frac{\partial}{\partial u^\mu_{t+s+k}}, \tag{A.1.21}$$

where $\mu \in \{1, ..., N\}$ is the direction index. Moreover for $s = -1$ we can define the momentum operator as

$$p_{\mu,t,-1} = \frac{\partial}{\partial u^\mu_{t-1}} \tag{A.1.22}$$

For $t = s = 0$ we have that $p_{\mu,0,0} = \frac{\delta}{\delta u^\mu}$

The second one is called **Energy** operator.

Definition. *The* ***Energy*** *operator is defined in $\widehat{\mathcal{A}}$ as (for $s \geq -1$)*

$$E_s = \sum_{t\geq 1} u^\mu_t p_{\mu,t,s}. \tag{A.1.23}$$

We denote with $E = E_0 - \mathbb{1}_{\widehat{\mathcal{A}}}$ (where $\mathbb{1}_{\widehat{\mathcal{A}}}$ is the identity operator in $\widehat{\mathcal{A}}$).

For this two operators the following theorem holds:

Theorem. *In $\widehat{\mathcal{A}}$ the following operatorial identities hold:*

- $p_{\mu,t,s} \circ \partial_x = p_{\mu,t,s-1}$
- $\partial_x \circ p_{\mu,t,s} = p_{\mu,t,s-1} - p_{\mu,t-1,s}$
- $\partial_x \circ E = -u^\mu_1 \frac{\delta}{\delta u^\mu}$.

Proof. For the first identity we have

$$\begin{aligned}
p_{\mu,t,s}\circ\partial_x &= \sum_{k\geq 0}(-1)^k\binom{k+s}{s}\partial_x^k\circ\frac{\partial}{\partial u^\mu_{t+s+k}}\circ\partial_x = \\
&= \sum_{k\geq 0}(-1)^k\binom{k+s}{s}\partial_x^k\circ\left(\frac{\partial}{\partial u^\mu_{t+s+k-1}}+\partial_x\circ\frac{\partial}{\partial u^\mu_{t+s+k}}\right) = \\
&= \sum_{k\geq 0}(-1)^k\binom{k+s}{s}\partial_x^k\circ\frac{\partial}{\partial u^\mu_{t+s+k-1}} - \sum_{k\geq 1}(-1)^k\binom{k+s-1}{s}\partial_x^k\circ\frac{\partial}{\partial u^\mu_{t+s+k-1}} = \\
&= \frac{\partial}{\partial u^\mu_{t+s-1}} + \sum_{k\geq 1}(-1)^k\left(\binom{k+s}{s}-\binom{k+s-1}{s}\right)\partial_x^k\circ\frac{\partial}{\partial u^\mu_{t+s+k-1}} = \\
&= \frac{\partial}{\partial u^\mu_{t+s-1}} + \sum_{k\geq 1}(-1)^k\binom{k+s-1}{s-1}\partial_x^k\circ\frac{\partial}{\partial u^\mu_{t+s+k-1}} = \\
&= p_{\mu,t,s-1}
\end{aligned} \tag{A.1.24}$$

The other two identities are a consequence of the first one. Indeed

$$\begin{aligned}
\partial_x\circ p_{\mu,t,s} &= \sum_{k\geq 0}(-1)^k\binom{k+s}{s}\partial_x^{k+1}\circ\frac{\partial}{\partial u^\mu_{t+s+k}}\circ\partial_x = \\
&= \sum_{k\geq 0}(-1)^k\binom{k+s}{s}\partial_x^k\circ\left(-\frac{\partial}{\partial u^\mu_{t+s+k-1}}+\circ\frac{\partial}{\partial u^\mu_{t+s+k}}\circ\partial_x\right) = \\
&= p_{\mu,t,s}\circ\partial_x - p_{\mu,t-1,s} = p_{\mu,t,s-1}-p_{\mu,t-1,s}
\end{aligned} \tag{A.1.25}$$

and

$$\begin{aligned}
\partial_x\circ E_0 &= \sum_{t\geq 1}\partial_x\circ(u^\mu_t p_{\mu,t,0}) = \\
&= \sum_{t\geq 1}u^\mu_{t+1}p_{\mu,t,0} + \sum_{t\geq 1}u^\mu_t\partial_x\circ p_{\mu,t,0} = \\
&= \sum_{t\geq 2}u^\mu_t p_{\mu,t-1,0} + \sum_{t\geq 1}u^\mu_t(p_{\mu,t,-1}-p_{\mu,t-1,0}) = \\
&= -u^\mu_1 p_{\mu,0,0} + \sum_{t\geq 1}u^\mu_t\frac{\partial}{\partial u^\mu_{t-1}} = \\
&= -u^\mu_1 p_{\mu,0,0} + \sum_{t\geq 0}u^\mu_{t+1}\frac{\partial}{\partial u^\mu_t} = \\
&= -u^\mu_1 p_{\mu,0,0} + \partial_x.
\end{aligned} \tag{A.1.26}$$

□

From (A.1.26) it's clear that any $f \in\in \mathbb{C}[[u^*]][u^*_{k>0}]$ (or $\widehat{\mathcal{A}}$)

$$c u^\mu_1\frac{\delta f}{\delta u^\mu} = \partial_x(-cE(f)), \tag{A.1.27}$$

as we wanted to prove.

Appendix B

Dubrovin-Novikov theorem

B.1 Vanishing of B^μ coefficients

First, let us remind the definition of B^μ:

$$B^\mu := K^{\mu\nu}\left(\sum_{m\geq 0}\frac{\partial}{\partial u_m^\alpha}\left(\frac{\delta g}{\delta u^\nu}\right)\cdot\partial_x^m\left(K^{\alpha\beta}\frac{\delta h}{\delta u^\beta}\right)\right) - K^{\mu\beta}\left(\sum_{m\geq 0}\frac{\partial}{\partial u_m^\alpha}\left(\frac{\delta h}{\delta u^\beta}\right)\cdot\partial_x^m\left(K^{\alpha\nu}\frac{\delta g}{\delta u^\nu}\right)\right)$$
$$- K^{\mu\alpha}\left[\sum_{k\geq 0}\mathcal{T}_{\alpha,k}\frac{\delta g}{\delta u^\nu}(-\partial_x)^k\left(K^{\nu\beta}\frac{\delta h}{\delta u^\beta}\right) + \sum_{k\geq 0}(-\partial_x)^k\left(\frac{\delta g}{\delta u^\nu}\right)\sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\left(g^{\nu\beta}\frac{\partial}{\partial u_n^\alpha}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right) + \right.\right.$$
$$\left.\left. + b_\gamma^{\nu\beta}u_x^\gamma\frac{\partial}{\partial u_n^\alpha}\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right]. \tag{B.1.1}$$

From the first and the third terms we have

$$K^{\mu\nu}\left(\sum_{m\geq 0}\frac{\partial}{\partial u_m^\alpha}\left(\frac{\delta g}{\delta u^\nu}\right)\cdot\partial_x^m\left(K^{\alpha\beta}\frac{\delta h}{\delta u^\beta}\right)\right) - K^{\mu\alpha}\left(\sum_{k\geq 0}\mathcal{T}_{\alpha,k}\frac{\delta g}{\delta u^\nu}(-\partial_x)^k\left(K^{\nu\beta}\frac{\delta h}{\delta u^\beta}\right)\right) =$$
$$= K^{\mu\nu}\left(\sum_{m,l\geq 0}(-\partial_x)^l\left(\frac{\partial^2 g}{\partial u_m^\alpha\partial u_l^\nu}\cdot\partial_x^m\left(K^{\alpha\beta}\frac{\delta h}{\delta u^\beta}\right)\right)\right) - K^{\mu\alpha}\left(\sum_{k\geq 0, l\leq k}(-1)^{l+k}\mathcal{T}_{\alpha,k-l}\frac{\partial g}{\partial u_l^\nu}(\partial_x)^k\left(K^{\nu\beta}\frac{\delta h}{\delta u^\beta}\right)\right) =$$
$$= K^{\mu\nu}\left(\sum_{m,l\geq 0}\sum_{k=0}^{l}(-1)^l\binom{l}{k}(-\partial_x)^{l-k}\frac{\partial^2 g}{\partial u_m^\alpha\partial u_l^\nu}\partial_x^{m+l}\left(K^{\alpha\beta}\frac{\delta h}{\delta u^\beta}\right)\right) - K^{\mu\alpha}\left(\sum_{l\geq 0, k\geq l}(-1)^{l+k}\mathcal{T}_{\alpha,k-l}\frac{\partial g}{\partial u_l^\nu}\cdot\right.$$
$$\left.\cdot(\partial_x)^k\left(K^{\nu\beta}\frac{\delta h}{\delta u^\beta}\right)\right) =$$
$$= K^{\mu\nu}\left(\sum_{m,k\geq 0}\sum_{l\geq k}(-1)^k\binom{l}{k}(-\partial_x)^{l-k}\frac{\partial^2 g}{\partial u_m^\alpha\partial u_l^\nu}\partial_x^{m+l}\left(K^{\alpha\beta}\frac{\delta h}{\delta u^\beta}\right)\right) - K^{\mu\alpha}\left(\sum_{l\geq 0, k\geq l}(-1)^{l+k}\mathcal{T}_{\alpha,k-l}\frac{\partial g}{\partial u_l^\nu}\cdot\right.$$

$$\cdot(\partial_x)^k\left(K^{\nu\beta}\frac{\delta h}{\delta u^\beta}\right)\Bigg) =$$
$$= K^{\mu\nu}\left(\sum_{m,k\geq 0}(-1)^k \mathcal{T}_{\nu,k}\frac{\partial g}{\partial u^\alpha_m}\partial_x^{m+l}\left(K^{\alpha\beta}\frac{\delta h}{\delta u^\beta}\right)\right) - K^{\mu\alpha}\left(\sum_{l,m\geq 0}(-1)^m \mathcal{T}_{\alpha,m}\frac{\partial g}{\partial u^\nu_l}(\partial_x)^{l+m}\left(K^{\nu\beta}\frac{\delta h}{\delta u^\beta}\right)\right) =$$
$$= 0 \tag{B.1.2}$$

This result will be used to show the vanishing of the other terms. Let us consider the fourth term. We have

$$K^{\mu\alpha}\left[\sum_{k\geq 0}\sum_{n\geq k}\binom{n}{k}(-\partial_x)^k\left(\frac{\delta g}{\delta u^\nu}\right)(-\partial_x)^{n-k}\left(g^{\nu\beta}\frac{\partial}{\partial u^\alpha_n}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}(-\partial_x)^k\left(\frac{\delta g}{\delta u^\nu}\right)(-\partial_x)^{n-k}\left(g^{\nu\beta}\frac{\partial}{\partial u^\alpha_n}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{n\geq 0}(-1)^n(-\partial_x)^n\left(\frac{\delta g}{\delta u^\nu}g^{\nu\beta}\frac{\partial}{\partial u^\alpha_n}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}(-\partial_x)^k\left(g^{\nu\beta}\frac{\delta g}{\delta u^\nu}\right)(-\partial_x)^{n-k}\left(\frac{\partial}{\partial u^\alpha_n}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(g^{\nu\beta}\frac{\delta g}{\delta u^\nu}\right)\sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\left(\frac{\partial}{\partial u^\alpha_n}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(g^{\nu\beta}\frac{\delta g}{\delta u^\nu}\right)\mathcal{T}_{\alpha,k}\left(\partial_x\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^{k+1}\left(g^{\nu\beta}\frac{\delta g}{\delta u^\nu}\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^\beta}\right)\right] =$$
$$= -K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(g^{\nu\beta}\partial_x\left(\frac{\delta g}{\delta u^\nu}\right)\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^\beta}\right)\right] - K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(\frac{\partial g^{\nu\beta}}{\partial u^\gamma}u^\gamma_x\frac{\delta g}{\delta u^\nu}\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^\beta}\right)\right]. \tag{B.1.3}$$

Following the same steps for the fifth term, we obtain the following contributions:

$$K^{\mu\alpha}\left[\sum_{k\geq 0}\sum_{n\geq k}\binom{n}{k}(-\partial_x)^k\left(\frac{\delta g}{\delta u^\nu}\right)(-\partial_x)^{n-k}\left(b^{\nu\beta}_\gamma u^\gamma_x\frac{\partial}{\partial u^\alpha_n}\left(\frac{\delta h}{\delta u^\beta}\right)\right)\right] =$$
$$= K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(b^{\nu\beta}_\gamma u^\gamma_x\frac{\delta g}{\delta u^\nu}\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^\beta}\right)\right] =$$
$$= -K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(b^{\beta\nu}_\gamma u^\gamma_x\frac{\delta g}{\delta u^\nu}\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^\beta}\right)\right] + K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(\frac{\partial g^{\nu\beta}}{\partial u^\gamma}u^\gamma_x\frac{\delta g}{\delta u^\nu}\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^\beta}\right)\right], \tag{B.1.4}$$

where we've used the compatibility condition in the last step (second equation of (3.1.15)). Finally, considering the sum of the second, fourth and fifth terms, we have

$$K^{\mu\beta}\left(\sum_{m\geq 0}\frac{\partial}{\partial u^{\alpha}_{m}}\left(\frac{\delta h}{\delta u^{\beta}}\right)\cdot\partial^{m}_{x}\left(K^{\alpha\nu}\frac{\delta g}{\delta u^{\nu}}\right)\right)+K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(\frac{\delta g}{\delta u^{\nu}}\right)\sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\circ\right.$$
$$\left.\left(g^{\nu\beta}\frac{\partial}{\partial u^{\alpha}_{n}}\left(\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right)\right]+K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(\frac{\delta g}{\delta u^{\nu}}\right)\sum_{n\geq k}\binom{n}{k}(-\partial_x)^{n-k}\left(b^{\nu\beta}_{\gamma}u^{\gamma}_{x}\frac{\partial}{\partial u^{\alpha}_{n}}\left(\frac{\delta h}{\delta u^{\beta}}\right)\right)\right]=$$
$$=K^{\mu\beta}\left(\sum_{m\geq 0}\frac{\partial}{\partial u^{\alpha}_{m}}\left(\frac{\delta h}{\delta u^{\beta}}\right)\cdot\partial^{m}_{x}\left(K^{\alpha\nu}\frac{\delta g}{\delta u^{\nu}}\right)\right)-K^{\mu\alpha}\left[\sum_{k\geq 0}(-\partial_x)^k\left(b^{\beta\nu}_{\gamma}u^{\gamma}_{x}\frac{\delta g}{\delta u^{\nu}}\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^{\beta}}\right)+\sum_{k\geq 0}(-\partial_x)^k\right.$$
$$\left.\left(g^{\nu\beta}\partial_x\left(\frac{\delta g}{\delta u^{\nu}}\right)\right)\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^{\beta}}\right)\right]=$$
$$=K^{\mu\beta}\left(\sum_{m\geq 0}\frac{\partial}{\partial u^{\alpha}_{m}}\left(\frac{\delta h}{\delta u^{\beta}}\right)\cdot\partial^{m}_{x}\left(K^{\alpha\nu}\frac{\delta g}{\delta u^{\nu}}\right)\right)-K^{\mu\alpha}\left(\sum_{k\geq 0}(-1)^k\mathcal{T}_{\alpha,k}\left(\frac{\delta h}{\delta u^{\beta}}\right)\partial^{k}_{x}\left(K^{\beta\nu}\frac{\delta g}{\delta u^{\nu}}\right)\right)=0, \tag{B.1.5}$$

since this has the same structure of (B.1.2), which it's vanishing. This implies that $B^{\mu}=0\quad\forall\mu\in\{1,....,N\}$.

B.1.1 Calculation of $C^{\mu\nu\beta}_{(3)jk}$

First, let us reintroduced the $A^{\mu}_{(3)}$ coefficients:

$$\begin{aligned} A^{\mu}_{(3)} &= -K^{\mu\alpha}\left[\frac{\delta g}{\delta u^{\nu}}\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}}\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right) + \sum_{l=0,1}\frac{\delta g}{\delta u^{\nu}}(-\partial_x)^l\left(\frac{\partial\left(b^{\nu\beta}_{\gamma}u^{\gamma}_x\right)}{\partial u^{\alpha}_l}\frac{\delta h}{\delta u^{\beta}}\right) - \partial_x\left(\frac{\delta g}{\delta u^{\nu}}\right)\frac{\partial(b^{\nu\beta}_{\gamma}u^{\gamma}_x)}{\partial u^{\alpha}_1}\frac{\delta h}{\delta u^{\beta}}\right] = \\ &= -K^{\mu\alpha}\left[\frac{\delta g}{\delta u^{\nu}}\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}}\partial_x\left(\frac{\delta h}{\delta u^{\beta}}\right) + \frac{\delta g}{\delta u^{\nu}}\frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}}u^{\gamma}_x\frac{\delta h}{\delta u^{\beta}} - \frac{\delta g}{\delta u^{\nu}}\partial_x\left(b^{\nu\beta}_{\alpha}\frac{\delta h}{\delta u^{\beta}}\right) - \partial_x\left(\frac{\delta g}{\delta u^{\nu}}\right)b^{\nu\beta}_{\alpha}\frac{\delta h}{\delta u^{\beta}}\right] \end{aligned} \tag{B.1.6}$$

From the last member of the previous relation, it's simple to derive $C^{\mu\nu\beta}_{(3)jk}$ $\forall i, j \in \{0, 1, 2\}$:

$$\begin{cases} C^{\mu\nu\beta}_{(3)00} = g^{\mu\alpha}\left[-\partial_x\left(\frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}}\right)u^{\gamma}_x - \frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}}u^{\gamma}_{xx} + \partial_{xx}(b^{\nu\beta}_{\alpha})\right] + \left[-\frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}}u^{\gamma}_x + \partial_x(b^{\nu\beta}_{\alpha})\right]b^{\mu\alpha}_{\theta}u^{\theta}_x \\ C^{\mu\nu\beta}_{(3)10} = g^{\mu\alpha}\left[-\frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}}u^{\gamma}_x + 2\partial_x(b^{\nu\beta}_{\alpha})\right] + b^{\nu\beta}_{\alpha}b^{\mu\alpha}_{\theta}u^{\theta}_x \\ C^{\mu\nu\beta}_{(3)01} = g^{\mu\alpha}\left[-\partial_x\left(\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}}\right) - \frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}}u^{\gamma}_x + 2\partial_x(b^{\nu\beta}_{\alpha})\right] + b^{\mu\alpha}_{\theta}u^{\theta}_x\left[-\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}} + b^{\nu\beta}_{\alpha}\right] \\ C^{\mu\nu\beta}_{(3)11} = g^{\mu\alpha}\left[-\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}} + 2b^{\nu\beta}_{\alpha}\right] \\ C^{\mu\nu\beta}_{(3)02} = -g^{\mu\alpha}\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}} + g^{\mu\alpha}b^{\nu\beta}_{\alpha} \\ C^{\mu\nu\beta}_{(3)20} = g^{\mu\alpha}b^{\nu\beta}_{\alpha} \\ C^{\mu\nu\beta}_{(3)21} = C^{\mu\nu\beta}_{(3)12} = C^{\mu\nu\beta}_{(3)22} = 0. \end{cases} \tag{B.1.7}$$

B.1.2 Dependence of the vanishing relation

In this section we will prove that, using

$$\begin{cases} g^{\mu\nu} = g^{\nu\mu} \\ b^{\mu\nu}_{\gamma} + b^{\nu\mu}_{\gamma} = \dfrac{\partial g^{\mu\nu}}{\partial u^{\gamma}} \\ b^{\mu\nu}_{\gamma} g^{\gamma\alpha} = b^{\alpha\nu}_{\gamma} g^{\gamma\mu} \quad (C^{\mu\nu\beta}_{02} = 0) \\ b^{\mu\nu}_{\gamma} b^{\gamma\alpha}_{\beta} - b^{\mu\alpha}_{\gamma} b^{\gamma\nu}_{\beta} = g^{\mu\gamma}\left(\dfrac{\partial b^{\nu\alpha}_{\beta}}{\partial u^{\gamma}} - \dfrac{\partial b^{\nu\alpha}_{\gamma}}{\partial u^{\beta}}\right) \quad (D^{\mu\nu\beta}_{\gamma} = 0), \end{cases} \tag{B.1.8}$$

the other coefficients (i.e. $C^{\mu\nu\beta}_{jk}$ with $(j,k) \neq \{(0,0);(0,2)\}$) vanish automatically $\forall \mu,\nu,\beta,\theta,\gamma \in \{1,....,N\}$].

$\diamond$ $C^{\mu\nu\beta}_{21}, C^{\mu\nu\beta}_{12}, C^{\mu\nu\beta}_{22}$

They are automatically vanishing since there aren't terms of the type $\left(\frac{\delta *}{\delta u^*}\right)^{(2)}\left(\frac{\delta *}{\delta u^*}\right)^{i}$ with $i \in \{1,2\}$.

$\diamond$ $C^{\mu\nu\beta}_{20}$

$$C^{\mu\nu\beta}_{20} = b^{\nu\beta}_{\alpha} g^{\alpha\mu} - b^{\mu\beta}_{\alpha} g^{\alpha\nu} \tag{B.1.9}$$

This is clearly the third relation written above.

$\diamond$ $C^{\mu\nu\beta}_{11}$

$$\begin{aligned} C^{\mu\nu\beta}_{11} &= \frac{\partial g^{\mu\nu}}{\partial u^{\alpha}} g^{\alpha\beta} - \frac{\partial g^{\mu\beta}}{\partial u^{\alpha}} g^{\alpha\nu} + g^{\mu\alpha}\left[-\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}} + 2b^{\nu\beta}_{\alpha}\right] = \\ &= \left(b^{\mu\nu}_{\alpha} + b^{\nu\mu}_{\alpha}\right) g^{\alpha\beta} - \left(b^{\mu\beta}_{\alpha} + b^{\beta\mu}_{\alpha}\right) g^{\alpha\nu} + g^{\mu\alpha}\left(-b^{\beta\nu}_{\alpha} + b^{\nu\beta}_{\alpha}\right) = \\ &= b^{\mu\nu}_{\alpha} g^{\alpha\beta} - \underbrace{b^{\beta\nu}_{\alpha} g^{\alpha\mu}}_{b^{\mu\nu}_{\alpha} g^{\alpha\beta}} + b^{\nu\mu}_{\alpha} g^{\alpha\beta} - \underbrace{b^{\beta\mu}_{\alpha} g^{\alpha\nu}}_{b^{\nu\mu}_{\alpha} g^{\alpha\beta}} + b^{\nu\beta}_{\alpha} g^{\alpha\mu} - \underbrace{b^{\mu\beta}_{\alpha} g^{\alpha\nu}}_{b^{\nu\beta}_{\alpha} g^{\alpha\mu}} = 0, \end{aligned} \tag{B.1.10}$$

where in the second step we've used the second relation and in the last step the third one.

$\diamond$ $C^{\mu\nu\beta}_{01}$

$$\begin{aligned} C^{\mu\nu\beta}_{01} &= \left(\frac{\partial b^{\mu\nu}_{\gamma}}{\partial u^{\alpha}} g^{\alpha\beta} + b^{\mu\nu}_{\alpha}\frac{\partial g^{\alpha\beta}}{\partial u^{\gamma}} + b^{\mu\nu}_{\alpha} b^{\alpha\beta}_{\gamma}\right) u^{\gamma}_{x} - \underbrace{\frac{\partial g^{\mu\beta}}{\partial u^{\alpha}}}_{b^{\mu\beta}_{\alpha} + b^{\beta\mu}_{\alpha}} b^{\alpha\nu}_{\gamma} u^{\gamma}_{x} + g^{\mu\alpha}\left[-\partial_x \underbrace{\left(\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}}\right)}_{b^{\nu\beta}_{\alpha} + b^{\beta\nu}_{\alpha}} - \frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}} u^{\gamma}_{x} + 2\partial_x (b^{\nu\beta}_{\alpha})\right] + \\ &+ b^{\mu\alpha}_{\gamma} u^{\gamma}_{x}\left[-\underbrace{\frac{\partial g^{\nu\beta}}{\partial u^{\alpha}}}_{b^{\nu\beta}_{\alpha} + b^{\beta\nu}_{\alpha}} + b^{\nu\beta}_{\alpha}\right] = \left(\frac{\partial b^{\mu\nu}_{\gamma}}{\partial u^{\alpha}} g^{\alpha\beta} + b^{\mu\nu}_{\alpha}\frac{\partial g^{\alpha\beta}}{\partial u^{\gamma}} + b^{\mu\nu}_{\alpha} b^{\alpha\beta}_{\gamma} - b^{\mu\beta}_{\alpha} b^{\alpha\nu}_{\gamma} - b^{\beta\mu}_{\alpha} b^{\alpha\nu}_{\gamma} - \frac{\partial b^{\nu\beta}_{\alpha}}{\partial u^{\gamma}} g^{\mu\alpha} \right. \\ &\left. - \frac{\partial b^{\beta\nu}_{\alpha}}{\partial u^{\gamma}} g^{\mu\alpha} - \frac{\partial b^{\nu\beta}_{\gamma}}{\partial u^{\alpha}} g^{\alpha\mu} + 2\frac{\partial b^{\nu\beta}_{\alpha}}{\partial u^{\gamma}} g^{\alpha\mu} - b^{\mu\alpha}_{\gamma} b^{\beta\nu}_{\alpha}\right) u^{\gamma}_{x} \end{aligned} \tag{B.1.11}$$

Since the parentesis argument depends only on u^*, we have that $C_{01}^{\mu\nu\beta}$ vanishes if and only if the parentesis argument vanishes $\forall\gamma \in \{1, ..., N\}$. So we have

$$\begin{aligned}
&\underbrace{g^{\beta\alpha}\frac{\partial b_\gamma^{\mu\nu}}{\partial u^\alpha} - b_\alpha^{\beta\mu} b_\gamma^{\alpha\nu}}_{g^{\beta\alpha}\frac{\partial b_\alpha^{\mu\nu}}{\partial u^\gamma} - b_\alpha^{\beta\nu} b_\gamma^{\alpha\mu}} - b_\gamma^{\mu\alpha} b_\alpha^{\beta\nu} + b_\alpha^{\mu\nu}\frac{\partial g^{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_\alpha^{\beta\nu}}{\partial u^\gamma} g^{\mu\alpha} + b_\alpha^{\mu\nu} b_\gamma^{\alpha\beta} - b_\alpha^{\mu\beta} b_\gamma^{\alpha\nu} \underbrace{-\frac{\partial b_\gamma^{\nu\beta}}{\partial u^\alpha} g^{\alpha\mu} + \frac{\partial b_\alpha^{\nu\beta}}{\partial u^\gamma} g^{\alpha\mu}}_{-b_\alpha^{\mu\nu} b_\gamma^{\alpha\beta} + b_\alpha^{\mu\beta} b_\gamma^{\alpha\nu}} = \\
&= g^{\beta\alpha}\frac{\partial b_\alpha^{\mu\nu}}{\partial u^\gamma} \underbrace{-b_\alpha^{\beta\nu} b_\gamma^{\alpha\mu} - b_\gamma^{\mu\alpha} b_\alpha^{\beta\nu}}_{-b_\alpha^{\beta\nu}\frac{\partial g^{\alpha\mu}}{\partial u^\gamma}} + b_\alpha^{\mu\nu}\frac{\partial g^{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_\alpha^{\beta\nu}}{\partial u^\gamma} g^{\mu\alpha} = g^{\beta\alpha}\frac{\partial b_\alpha^{\mu\nu}}{\partial u^\gamma} + b_\alpha^{\mu\nu}\frac{\partial g^{\alpha\beta}}{\partial u^\gamma} - \left(b_\alpha^{\beta\nu}\frac{\partial g^{\alpha\mu}}{\partial u^\gamma} + \frac{\partial b_\alpha^{\beta\nu}}{\partial u^\gamma} g^{\mu\alpha} \right) = \\
&= \frac{\partial}{\partial u^\gamma} \underbrace{(b_\alpha^{\mu\nu} g^{\alpha\beta})}_{b_\alpha^{\beta\nu} g^{\alpha\mu}} - \frac{\partial}{\partial u^\gamma}(b_\alpha^{\beta\nu} g^{\alpha\mu}) = 0,
\end{aligned} \tag{B.1.12}$$

where we've used iterately all the relations.

◇ $C_{10}^{\mu\nu\beta}$

$$\begin{aligned}
C_{10}^{\mu\nu\beta} &= \frac{\partial g^{\mu\nu}}{\partial u^\alpha} b_\gamma^{\alpha\beta} u_x^\gamma - \left(\frac{\partial b_\gamma^{\mu\beta}}{\partial u^\alpha} g^{\alpha\nu} + b_\alpha^{\mu\beta}\frac{\partial g^{\alpha\nu}}{\partial u^\gamma} + b_\alpha^{\mu\beta} b_\gamma^{\alpha\nu} \right) u_x^\gamma + g^{\mu\alpha}\left[-\frac{\partial b_\gamma^{\nu\beta}}{\partial u^\alpha} u_x^\gamma + 2\partial_x(b_\alpha^{\nu\beta}) \right] + b_\alpha^{\nu\beta} b_\theta^{\mu\alpha} u_x^\theta \\
&= \left(\frac{\partial g^{\mu\nu}}{\partial u^\alpha} b_\gamma^{\alpha\beta} - \frac{\partial b_\gamma^{\mu\beta}}{\partial u^\alpha} g^{\alpha\nu} - b_\alpha^{\mu\beta}\frac{\partial g^{\alpha\nu}}{\partial u^\gamma} - b_\alpha^{\mu\beta} b_\gamma^{\alpha\nu} - g^{\mu\alpha}\frac{\partial b_\gamma^{\nu\beta}}{\partial u^\alpha} + g^{\mu\alpha}\frac{\partial b_\alpha^{\nu\beta}}{\partial u^\gamma} + b_\alpha^{\nu\beta} b_\gamma^{\mu\alpha} \right) u_x^\gamma
\end{aligned} \tag{B.1.13}$$

Since the parentesis argument depends only on u^*, we have that $C_{01}^{\mu\nu\beta}$ vanishes if and only if the parentesis argument vanishes $\forall\gamma \in \{1, ..., N\}$. So we have

$$\left(\underbrace{\frac{\partial g^{\mu\nu}}{\partial u^\alpha}}_{b_\alpha^{\mu\nu} + b_\alpha^{\nu\mu}} b_\gamma^{\alpha\beta} - \frac{\partial b_\gamma^{\mu\beta}}{\partial u^\alpha} g^{\alpha\nu} - b_\alpha^{\mu\beta}\frac{\partial g^{\alpha\nu}}{\partial u^\gamma} - b_\alpha^{\mu\beta} b_\gamma^{\alpha\nu} \underbrace{-g^{\mu\alpha}\frac{\partial b_\gamma^{\nu\beta}}{\partial u^\alpha} + 2g^{\mu\alpha}\frac{\partial b_\alpha^{\nu\beta}}{\partial u^\gamma}}_{-b_\alpha^{\mu\nu} b_\gamma^{\alpha\beta} + b_\alpha^{\mu\beta} b_\gamma^{\alpha\nu} + g^{\mu\alpha}\frac{\partial b_\alpha^{\nu\beta}}{\partial u^\gamma}} + \underbrace{b_\alpha^{\nu\beta} b_\gamma^{\mu\alpha}}_{-b_\alpha^{\nu\beta} b_\gamma^{\alpha\mu} + b_\alpha^{\nu\beta}\frac{\partial g^{\alpha\mu}}{\partial u^\gamma}} \right) u_x^\gamma = \tag{B.1.14}$$

$$\begin{aligned}
&= \underbrace{g^{\mu\alpha}\frac{\partial b_\alpha^{\nu\beta}}{\partial u^\gamma} + b_\alpha^{\nu\beta}\frac{\partial g^{\alpha\mu}}{\partial u^\gamma}}_{\frac{\partial}{\partial u^\gamma}\left(b_\alpha^{\nu\beta} g^{\alpha\mu}\right)} + b_\alpha^{\nu\mu} b_\gamma^{\alpha\beta} \underbrace{-b_\alpha^{\nu\beta} b_\gamma^{\alpha\mu} - \frac{\partial b_\gamma^{\mu\beta}}{\partial u^\alpha} g^{\alpha\nu}}_{-b_\alpha^{\nu\mu} b^{\alpha\beta} - \frac{\partial b_\alpha^{\mu\beta}}{\partial u^\gamma}} - b_\alpha^{\mu\beta}\frac{\partial g^{\alpha\nu}}{\partial u^\gamma} = \\
&= \frac{\partial}{\partial u^\gamma}\left(b_\alpha^{\nu\beta} g^{\alpha\mu}\right) - \underbrace{\left(\frac{\partial b_\alpha^{\mu\beta}}{\partial u^\gamma} + b_\alpha^{\mu\beta}\frac{\partial g^{\alpha\nu}}{\partial u^\gamma} \right)}_{\frac{\partial}{\partial u^\gamma}\left(b_\alpha^{\mu\beta} g^{\alpha\nu}\right)} \\
&= \frac{\partial}{\partial u^\gamma}\left(b_\alpha^{\nu\beta} g^{\alpha\mu}\right) - \frac{\partial}{\partial u^\gamma}\underbrace{\left(b_\alpha^{\mu\beta} g^{\alpha\nu}\right)}_{b_\alpha^{\nu\beta} g^{\alpha\mu}} = 0,
\end{aligned} \tag{B.1.15}$$

where we've used iterately all the relations.

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