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Noetherian Modules over Combinatorial Categories

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Meine Herren,

I do not see that the sex of the candidate [Emmy Noether] is an argument against her admission as a Privatdozent. After all, the Senate is not a bathhouse.

- David Hilbert

Quoted in C. Reid, *Hilbert*, 1996.

Abstract

In this project, we analyse subcategories $\mathcal{C}$ of the category of finite sets and functions. A $\mathcal{C}$-module over a ring k is a functor from $\mathcal{C}$ to the category of left k-modules. We investigate whether the category of $\mathcal{C}$-modules is Noetherian whenever the ring k is left-Noetherian.

We present the Gröbner method introduced by Sam and Snowden [SS16], which stipulates combinatorial criteria on $\mathcal{C}$ that guarantee this implication. Following the treatment of Proudfoot and Ramos [PR19a], we apply this method to the opposite of the category of finite graphs of fixed genus and contractions, $\mathcal{G}_g^{\mathrm{op}}$. Moreover, we consider the specific $\mathcal{G}_g^{\mathrm{op}}$-module assigning the homology group of its unordered configuration space to a graph. By means of the Gröbner method, we derive stability results on these homology groups.

We finish with a couple of results of our own. We prove that the Gröbner Method is applicable to several subcategories of the category of finitely generated modules over a fixed finite ring. On the other hand, we prove a partial converse to this method, which yields examples of categories for which the category of $\mathcal{C}$-modules over any ring k is not Noetherian.

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Contents

Introduction

The inspiration for this master thesis are two papers published last year by Nicholas Proudfoot and Eric Ramos [PR19b] and [PR19a]. The first one considers modules over (or representations of) the opposite of the category of trees (acyclic connected graphs) and edge contractions, $\mathcal{T}$. The second one, generalises the methods to the category of connected graphs of any fixed genus g (number of cycles), $\mathcal{G}_g$.

For an essentially small category $\mathcal{C}$, a $\mathcal{C}$-module over a ring k is a functor from $\mathcal{C}$ to the category of left k-modules. Most ring theoretic notions on k-modules have a natural analogue for $\mathcal{C}$-modules. In particular, the category of $\mathcal{C}$-modules over k, $\mathrm{Rep}_k(\mathcal{C})$, is said to be Noetherian if each submodule of a finitely generated module is itself finitely generated.

The framework of modules over categories is a recent discovery that yields a strong tool to prove representation theoretic result in different algebraic contexts. The articles of Thomas Church, Jordan S. Ellenberg, Benson Farb, [CEF15] and Rohit Nagpal [CEFN14] may be considered as the origin. They study modules over the category of finite sets and injective maps, $\mathbf{FI}$, and prove in particular that the category $\mathrm{Rep}_k(\mathbf{FI})$ is Noetherian for any left-Noetherian ring k. The main application, which they deduce, are so called "stability results" when evaluating a finitely generated $\mathbf{FI}$-module in sets of increasing size.

The first chapter of this project is devoted to the generalisation of these results given by Steven V. Sam and Andrew Snowden [SS16]. A combinatorial category is a subcategory of the category of finite sets and functions, $\mathbf{FSet}$. The fundamental question reads, "How do combinatorial properties of $\mathcal{C}$ affect algebraic properties of representations of $\mathcal{C}$?" Based on the concept of Gröbner bases in ring theory, we[1] formulate some criteria for $\mathcal{C}$ that guarantee that $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian for any left-Noetherian ring k. We show that this holds for the category $\mathbf{FSet}$ and, following Andrew Putman and Steven V. Sam [PS14], the category of free modules of finite rank over a finite ring. Moreover, we show that when k is a field and the category has some extra structure, the dimension of a finitely generated $\mathcal{C}$-module evaluated in a set eventually grows polynomially with respect to the size of the set.

In the second chapter, we follow the treatment of Proudfoot and Ramos [PR19b] and [PR19a]. We show that the category $\mathcal{G}_g$ satisfies the Gröbner criteria. Afterwards, by

[1] "we" refers to the author under guidance of the supervisor.

means of similar techniques as in the first chapter, we show that the dimension of a finitely generated $\mathcal{G}_g^{\mathrm{op}}$-module evaluated in a graph is eventually bounded polynomially in the number of edges of the graph. Finally, we deduce that this bound turns into an equality when adding edges in such a way that the structure of the original graph is preserved.

In the third chapter, we continue to follow [PR19a] and study one particular kind of $\mathcal{G}_g^{\mathrm{op}}$-modules. We introduce the n-stranded (un)ordered configuration space of a graph, which is a topological space that characterises the possible positioning and movements of n (indistinguishable) particles on the graph. The homology of these spaces contains topological information about the original graph.

Byung Hee An, Gabriel C. Drummond-Cole and Ben Knudsen [ADCK17] investigate on the computation of the homology of configuration spaces of a finite cell complex. In particular, they introduce a cellular chain model originally due to Jacek Świątkowski, which gives an algorithmic tool to compute these homology groups. By means of this complex, we prove that the assignment of the i-th homology group of the n-stranded unordered configuration space to a graph is a finitely generated $\mathcal{G}_g^{\mathrm{op}}$-module. Applying the theory of the previous chapters, we obtain polynomial bounds on the growth and torsion of these homology groups when increasing the number of edges of a graph.

Finally, in the fourth chapter we present some results that, to our knowledge, are not stated anywhere else yet. Conversely to the above, we search for examples of categories inducing non-Noetherian module categories. After some remarks on the role of the size of the category, we consider several examples like the category of finite groups and the category of finite partially ordered sets. We further generalise our argument to some categories consisting of $\mathbb{F}_\infty$-modules (structures which we introduce along the way). Meanwhile, we realise that our argument is in fact a partial converse to the main theorem of Sam and Snowden. On the other hand, we generalise the positive result of Putman and Sam [SS16] mentioned above to the category of finite projective modules over a finite ring and the category of all finite modules over a finite principal ideal ring.

We finish by discussing some of the questions that we were not able to solve.

Conventions

- The symbol $\mathbb{N}$ denotes the set of positive integers. When we include 0 we write $\mathbb{N}_0$.
- For a category $\mathcal{C}$ we abuse the element notation. When we write $x \in \mathcal{C}$, we mean that x is an object of $\mathcal{C}$ and when we write $f\colon x \to y \in \mathcal{C}$ we that $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.
- Any ring is assumed to have a multiplicative identity element, denoted by 1.
- For a map f and a subset of the domain X, the symbol $f\restriction_X$ denotes the restriction of f to X.

Chapter 1

Gröbner Categories

In this chapter, we analyse modules over a category over a ring k, following the treatment given by Sam and Snowden in [SS16]. In section 1.1, we introduce these objects and remark that many properties of usual k-modules have a direct analogue in this new setting. In particular, we introduce the notions of finitely generated and Noetherian modules. We arrive to the fundamental question, for which categories every finitely generated module is Noetherian. We restrict our attention to the so-called combinatorial categories that are contained in the category of finite sets.

The partial answer to the question above presented in section 1.2 forms the core of this chapter. This method gives some criteria on combinatorial categories that guarantee a positive answer. It is inspired by a classical proof of Hilbert's basis Theorem using Gröbner bases. We work out the details by emphasising the analogy between both settings.

In section 1.3, we prove that the category **OI**, of totally ordered finite sets and order embeddings, satisfies the criteria mentioned above. Moreover, we deduce that the method works for several other categories consisting of all finite sets and categories of free modules of finite rank over a finite ring.

In section 1.4, we present an application of this result. We show that the dimension of a finitely generated **OI**-module over a field evaluated in a set, eventually grows polynomially with respect to the size of the set.

1.1 Modules over a category

In this project we consider essentially small categories.

Definition 1.1. A category $\mathcal{C}$ is locally small if for each pair of objects x, y in $\mathcal{C}$ the morphism class $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a set. A category is small if it is locally small and its class of objects is a set. A category is essentially small if it is equivalent to a small category.

Example 1.2. Let k be a ring, then the category of left k-modules and k-linear maps, $\mathbf{Mod}_k$, is locally small. However, it is not essentially small, since any set S induces a free k-module $k[S]$. The full subcategory of finitely generated modules, $\mathbf{FgMod}_k$, is essentially small category.

From now on, let $\mathcal{C}$ be an arbitrary essentially small category. A usual practice in algebra, is to analyse such abstract object by its 'action' in a context with more structure.

Definition 1.3. Let k be a ring. A $\mathcal{C}$-module (or representation of $\mathcal{C}$) over k is a covariant functor $M \colon \mathcal{C} \to \mathbf{Mod}_k$. The category $\mathrm{Rep}_k(\mathcal{C})$ consists of $\mathcal{C}$-modules over k and natural transformations between them.

Remark 1.4. Notice that to be complete we should call this notion a *left* $\mathcal{C}$-module and define right $\mathcal{C}$-modules accordingly as covariant functors to the category of right k-modules. However, we only consider left $\mathcal{C}$-modules and hence drop the notation. When restricting our attention to commutative rings both notions become equivalent.

Remark 1.5. By definition, a morphism $\phi \in \mathrm{Hom}_{\mathrm{Rep}_k(\mathcal{C})}(M, N)$ consists of k-linear map, $\phi_x \in \mathrm{Hom}_{\mathbf{Mod}_k}(M(x), N(x))$ for each $x \in \mathcal{C}$, such that for each $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ the diagram

$$\begin{tikzcd} M(x) \arrow[r, "\phi_x"] \arrow[d, "M(f)"] & N(x) \arrow[d, "N(f)"] \\ M(y) \arrow[r, "\phi_y"] & N(y) \end{tikzcd}$$

commutes. In particular, by Example 1.2, this implies that $\mathrm{Rep}_k(\mathcal{C})$ is locally small.

Before we continue let us give a first example motivating the name "representation".

Example 1.6. Let G be a group. The corresponding category $\mathbf{G}$ consists of a unique object $\star$ and morphisms given by the group elements of G, where the composition law is the group operation. By definition, the category $\mathbf{G}$ is small. A representation M of $\mathbf{G}$ over a field k consists of a k-vector space $M(\star)$ and a group representation $M \colon G \to \mathrm{Hom}_k(M(\star), M(\star))$.

The category $\mathrm{Rep}_k(\mathcal{C})$ has several similarities to the category $\mathbf{Mod}_k$.

Proposition 1.7. *A natural transformation $\phi \colon M \to N \in \mathrm{Rep}_k(\mathcal{C})$ is an epimorphism (monomorphism) if and only if the corresponding k-linear map $\phi_x \colon M(x) \to N(x)$ is surjective (injective) for each $x \in \mathcal{C}$.*

Proof. We recall that an epimorphism in $\mathbf{Mod}_k$ is precisely a k-linear surjections. First, assume that ϕ_x is surjective for all $x \in \mathcal{C}$. Suppose that there are natural transformations $\psi_1, \psi_2 \colon N \to L \in \mathrm{Rep}_k(\mathcal{C})$ such that $\psi_1 \circ \phi = \psi_2 \circ \phi$. Then, for each $x \in \mathcal{C}$ this implies that $\psi_{1x} \circ \phi_x = \psi_{2x} \circ \phi_x \colon M(x) \to L(x)$. Since ϕ_x is epimorphic, this implies that $\psi_{1x} = \psi_{2x}$ for all $x \in \mathcal{C}$ and hence that $\psi_1 = \psi_2$.

Conversely, assume that ϕ is epimorphic. Let $\tilde{M}$ be the composition of the $\mathcal{C}$-module $M \colon \mathcal{C} \to \mathbf{Mod}_k$ and the forgetful functor $\mathbf{Mod}_k \to \mathbf{Set}$ and define $\tilde{N}$ similarly. Yoneda's Lemma yields an isomorphism

$$\Theta \colon \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(x, \_), \tilde{M}) \xrightarrow{\sim} \tilde{M}(x),$$

which is functorial in $\tilde{M}$. As the left functor is a hom-functor, it preserves epimorphisms. In particular, it follows that the map

$$\phi_x = \Theta \circ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(x, \_), \phi) \circ \Theta^{-1}$$

is an epimorphism in the category $\mathbf{Set}$. This means exactly that ϕ_x is surjective. The argument for monomorphisms is formally dual to the above. □

Consequently, we can define most relations between $\mathcal{C}$-modules over k, analogously to the notions for (left) k-modules.

Definition 1.8. Let N and M be $\mathcal{C}$-modules over a ring k.

- We say that N is a submodule of M if $N(x)$ is a k-submodule of $M(x)$ for each $x \in \mathcal{C}$ and $N(f)$ is the restriction $M(f) \restriction_{N(x)}$ for each morphism f in $\mathcal{C}$ with source x. We denote this by $N \subseteq M$.
- The product module of N and M is given by mapping an object x to the direct product of the k-modules $N(x)$ and $M(x)$ and mapping a morphism f to the component wise k-linear map $(N(f), M(f))$. We denote it by $N \oplus M$.
- The quotient module of M by L is given by mapping an object x to the quotient of the k-modules $M(x)$ by $L(x)$ and mapping a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ to the quotient map $\alpha \mapsto M(f)(\alpha) \pmod{L}(y)$. We denote it by M/L.
- We say that N is a quotient of M if there exists a $\mathcal{C}$-module L such that $M = N/L$. We say that N is a subquotient of M if it is a submodule of a quotient of M (or equivalently a quotient of a submodule of M).

Remark 1.9. It is an easy exercise to see that the product defined above is indeed the categorical product in $\mathrm{Rep}_k(\mathcal{C})$. The module N being a quotient of M is equivalent to the existence of an epimorphism $\phi \colon M \to N \in \mathrm{Rep}_k(\mathcal{C})$.

A category intrinsically provides some modules over itself. They are of great importance in the rest of this project.

Definition 1.10. For an object x in a category $\mathcal{C}$ the corresponding principal projective $\mathcal{C}$-module P_x is given by

$$y \mapsto k[\mathrm{Hom}_{\mathcal{C}}(x,y)] = \bigoplus_{f\colon x\to y} k\cdot e_f,$$
$$(g\colon y\to z)\mapsto \Big(\mathrm{P}_x(g)\colon k[\mathrm{Hom}_{\mathcal{C}}(x,y)]\to k[\mathrm{Hom}_{\mathcal{C}}(x,z)]\colon e_f\mapsto e_{gf}\Big).$$

Remark 1.11. Using the universal property of a free k-module and Yoneda's Lemma, yields bijections

$$\mathrm{Hom}_{\mathrm{Rep}_k(\mathcal{C})}(\mathrm{P}_x, M)\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(x,\_), M)\cong M(x).$$

By Proposition 1.7, the functor $\mathrm{Hom}_{\mathrm{Rep}_k(\mathcal{C})}(\mathrm{P}_x,\_)\colon \mathrm{Rep}_k(\mathcal{C})\to\mathbf{Set}$ preserves epimorphisms. Recall that, since the category $\mathrm{Rep}_k(\mathcal{C})$ is essentially small, this does indeed mean that P_x is a projective object.

As in the context of k-modules, there is a notion of finite generation.

Definition 1.12. Let M be a $\mathcal{C}$-module over k. A subset S of $\coprod_{x\in\mathcal{C}} M(x)$ generates M if S is not contained in $\coprod_{x\in\mathcal{C}} N(x)$ for any strict $\mathcal{C}$-submodule $N\subsetneq M$. In particular, we say that M is finitely generated, if there exists a finite subset S generating it.

Remark 1.13. The notation here is suboptimal. However, $\coprod$ is supposed to recall that we consider a disjoint union. Each $s\in S$ has a unique corresponding object $x\in\mathcal{C}$ such that $s\in M(x)$.

We deduce a more descriptive reformulation for being finitely generated.

Proposition 1.14. *Let M be a $\mathcal{C}$-module. The following statements are equivalent.*

1. *The module M is finitely generated by $\{\alpha_1,\dots,\alpha_\ell\}$ for some $\alpha_i\in M(x_i)$, where $x_i\in\mathcal{C}$.*
2. *For each $x\in\mathcal{C}$ the k-module $M(x)$ consists of all elements of the form*

$$\sum_{i=1}^{\ell}\sum_{f\in H_i}\lambda_f\cdot M(f)(\alpha_i),$$

   where, H_i is a finite subset of $\mathrm{Hom}_{\mathcal{C}}(x_i,x)$ and each λ_f is a constant in k.

3. *The module M is a quotient of $\bigoplus_{i=1}^{\ell}\mathrm{P}_{x_i}$ in $\mathrm{Rep}_k(\mathcal{C})$.*

Remark 1.15. Note that the objects $x_1,\dots,x_n$ are not assumed to be distinct. In particular, the third statement demonstrates that if M and N are finitely generated $\mathcal{C}$-modules, their product $M\oplus N$ is also finitely generated.

Proof. Assume the first statement and pick an object $x\in\mathcal{C}$. By definition, the elements of the form written down in the second statement belong to $M(x)$. On the other hand, these objects define a (finitely generated) k-module $N(x)$ for each $x \in X$. Moreover, for any $g\colon x \to y \in \mathcal{C}$ it follows that

$$M(g)\Big(\sum_{i=1}^{\ell}\sum_{f\in H_i}\lambda_f\cdot M(f)(\alpha_i)\Big)=\sum_{i=1}^{\ell}\sum_{f\in H_i}\lambda_f\cdot M(gf)(\alpha_i)=\sum_{i=1}^{\ell}\sum_{f\in gH_i}\lambda_f\cdot M(f)(\alpha_i),$$

where gH_i is the finite subset of $\mathrm{Hom}_{\mathcal{C}}(x_i, y)$ given by $\{gf \mid f \in H_i\}$. In other words, the image of $N(x)$ under $M(g)$ is contained in $N(y)$. Hence, this defines a $\mathcal{C}$-submodule $N \subseteq M$. As $\alpha_i = M(\mathrm{id}_{x_i})(\alpha_i)$ is an object of $N(x_i)$ for each $i \in \{1, \dots, \ell\}$ we conclude that $N = M$.

Assume the second statement and define the natural transformation $\phi\colon \bigoplus_{i=1}^{\ell} \mathrm{P}_{x_i} \to M$ at level $x \in \mathcal{C}$ by k-linearity and

$$\phi_x\colon \bigoplus_{i=1}^{\ell} k[\mathrm{Hom}_{\mathcal{C}}(x_i,x)] \to M(x)\colon e_f \mapsto M(f)(\alpha_i),$$

for each morphism $f : x_i \to x$. Functoriality follows by noticing that for any $g\colon x \to y \in \mathcal{C}$, it holds that

$$\phi_y\Big(\mathrm{P}_{x_i}(g)(e_f)\Big)=\phi_y(e_{gf})=M(gf)(s_i)=M(g)\Big(M(f)(s_i)\Big)=M(g)\Big(\phi_x(e_f)\Big).$$

By Proposition 1.7, it is enough to prove that ϕ_x is surjective for each $x \in \mathcal{C}$ to conclude that ϕ is an epimorphism. Hence, pick arbitrary $x \in \mathcal{C}$ and $\sum_{i=1}^{\ell}\sum_{f\colon x_i\to x}\lambda_f\cdot M(f)(\alpha_i) \in M(x)$. By the definition of ϕ_x, this is mapped onto by

$$\sum_{i=1}^{\ell}\sum_{f\colon x_i\to x}\lambda_f\cdot e_f=\sum_{i=1}^{\ell}\sum_{f\colon x_i\to x}\lambda_f\cdot \mathrm{P}_{x_i}(f)(e_{\mathrm{id}_{x_i}})\in\bigoplus_{i=1}^{n}k[\mathrm{Hom}_{\mathcal{C}}(x_i,x)].$$

We conclude that M is a quotient module of $\bigoplus_{i=1}^{\ell} \mathrm{P}_{x_i}$.

Finally, let us assume the third statement and let $\phi\colon \bigoplus_{i=1}^{\ell} \mathrm{P}_{x_i} \to M \in \mathrm{Rep}_k(\mathcal{C})$ be the corresponding epimorphism. Set $S = \{\phi_{x_i}(e_{\mathrm{id}_{x_i}}) \mid i \in \{1, \dots, n\}\}$ and suppose that $N \subseteq M$ satisfies $S \subseteq \coprod_{i=1}^{\ell} N(x_i)$. Take arbitrary $x \in \mathcal{C}$ and $\alpha \in M(x)$. By surjectivity of ϕ_x, there is a $\beta \in \bigoplus_{i=1}^{\ell} k[\mathrm{Hom}_{\mathcal{C}}(x_i, x)]$ such that $\phi_x(\beta) = \alpha$. It is of the from

$$\beta=\sum_{i=1}^{n}\sum_{f\colon x_i\to x}\lambda_f\cdot e_f=\sum_{i=1}^{n}\sum_{f\colon x_i\to x}\mathrm{P}_{x_i}(f)(\lambda_f\cdot e_{\mathrm{id}_{x_i}}),$$

where each λ_f is an element of k. Since ϕ_x is k-linear, it follows that

$$\alpha=\sum_{i=1}^{\ell}\sum_{f\colon x_i\to x}\Big(\phi_x\circ \mathrm{P}_{x_i}(f)\Big)(\lambda_f\cdot e_{\mathrm{id}_{x_i}})=\sum_{i=1}^{\ell}\sum_{f\colon x_i\to x}M(f)\Big(\lambda_f\cdot\phi_{x_i}(e_{\mathrm{id}_{x_i}})\Big).$$

In the expression on the right each $M(f)$ is evaluated in an element inside $\coprod_{i=1}^{\ell} N(x_i)$. As $N(f) = M(f) \restriction_{N(x_i)}$ it follows that $\alpha \in N(x)$. We conclude that $N = M$ and hence that M is finitely generated by S. $\square$

We take a step back and consider modules over different categories. Notice that any functor $\Phi\colon \mathcal{C} \to \mathcal{C}'$ induces a functor between the corresponding categories of modules

$$\Phi^*\colon \operatorname{Rep}_k(\mathcal{C}') \to \operatorname{Rep}_k(\mathcal{C})\colon M \mapsto M \circ \Phi,$$

which maps a natural transformation $(\psi\colon M \to N)$ to the natural transformation $\Phi^*(\psi)$ defined at every $x \in \mathcal{C}$ by $\Phi^*(\psi)_x = \psi_{\Phi(x)}\colon M(\Phi(x)) \to N(\Phi(x))$.

In particular, notice that by definition Φ^* preserves all relation defined in 1.8. Moreover, it also preserves mono- and epimorphisms, by Proposition 1.7. We can characterise the functors that preserve finite generation of modules as follows.

Definition 1.16. A functor between essentially small categories, $\Phi\colon \mathcal{C} \to \mathcal{C}'$, is said to satisfy property (F) if for each $x' \in \mathcal{C}'$ there exist finitely many objects $y_1, \ldots, y_n \in \mathcal{C}$ and morphisms $f_i\colon x' \to \Phi(y_i) \in \mathcal{C}'$ satisfying the following condition: for any morphism $f\colon x' \to \Phi(z) \in \mathcal{C}'$, where $z \in \mathcal{C}$, there exists a morphism $g\colon y_i \to z \in \mathcal{C}$ such that $f = \Phi(g) \circ f_i$.

Proposition 1.17. *A functor $\Phi\colon \mathcal{C} \to \mathcal{C}'$ satisfies property (F) if and only if the functor $\Phi^*\colon \operatorname{Rep}_k(\mathcal{C}') \to \operatorname{Rep}_k(\mathcal{C})$ maps finitely generated $\mathcal{C}'$-modules to finitely generated $\mathcal{C}$-modules.*

Proof. Assume that Φ satisfies (F) and consider any principal projective module $\mathrm{P}_{x'} \in \operatorname{Rep}_k(\mathcal{C}')$. Let $y_1, \ldots, y_n \in \mathcal{C}$ and $f_i\colon x' \to \Phi(y_i) \in \mathcal{C}'$ be as in Definition 1.16 and set $S = \{e_{f_1}, \ldots, e_{f_n}\} \subseteq \coprod_{i=1}^n \Phi^*(\mathrm{P}_{x'})(y_i)$. Take arbitrary $y \in \mathcal{C}$ and $\alpha \in \Phi^*(\mathrm{P}_{x'})(y)$, then α is of the form $\alpha = \sum_{f\colon x' \to \Phi(y)} \lambda_f \cdot e_f$. By property (F), each of the above f factors as $\Phi(g) \circ f_i$. This means that $\alpha = \sum_{f\colon x' \to \Phi(y)} \lambda_f \cdot e_{\Phi(g)f_i} = \sum_{f\colon x' \to \Phi(y)} \mathrm{P}_{x_i}\Phi(g)(\lambda_f \cdot e_{f_i})$ is inside the submodule generated by S. Since α was arbitrary, this implies that $\Phi^*(\mathrm{P}_{x'})$ is finitely generated.

More generally, let M be any finitely generated $\mathcal{C}'$-module. By Proposition 1.14, we know there exists objects $x'_1, \ldots, x'_n \in \mathcal{C}$ and an epimorphism $\psi\colon \bigoplus_{i=1}^n \mathrm{P}_{x'_i} \to M \in \operatorname{Rep}_k(\mathcal{C}')$. As mentioned above, the map $\Phi^*(\psi)\colon \bigoplus_{i=1}^n \Phi^*(\mathrm{P}_{x'_i}) \to \Phi^*(M) \in \operatorname{Rep}_k(\mathcal{C})$ is again epimorphic. By the first part of the proof, each $\Phi^*(\mathrm{P}_{x'_i})$ is finitely generated and therefore so is their direct sum by Remark 1.15. Hence, there exist $x_1, \ldots, x_m \in \mathcal{C}$ and an epimorphism $\phi\colon \bigoplus_{j=1}^m \mathrm{P}_{x_j} \to \bigoplus_{i=1}^n \mathrm{P}_{x'_i}$. Composition of $\Phi^*(\psi)$ with ϕ witnesses that $\Phi^*(M)$ is a finitely generated $\mathcal{C}$-module.

Conversely, assume Φ^* preserves finite generation and take any $x' \in \mathcal{C}'$. By assumption, $\Phi^*(\mathrm{P}_{x'})$ is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_i \in \mathrm{P}_{x'}(\Phi(y_i))$ for some $y_i \in \mathcal{C}$. For each i let

$$H_i = \{f_i \in \operatorname{Hom}_{\mathcal{C}}(x', \Phi(y_i)) \mid \text{ the coefficient of } f_i \text{ in } \alpha_i \text{ is not zero}\}.$$

This implies that $\alpha_i = \sum_{f_i \in H_i} \lambda_{f_i} e_{f_i}$ for some $\lambda_i \in k \setminus \{0\}$. We duplicate each $\tilde{y}_i$, $|H_i|$ times to get finitely many object $y_i \in \mathcal{C}$ with unique corresponding map $f_i\colon x' \to \Phi(y_i)$, where $f_i \in \bigcup_{i=1}^n H_i$. Now take any object $z \in \mathcal{C}$ and any morphism $f\colon x' \to \Phi(z) \in \mathcal{C}'$. By Proposition 1.7, there exist finitely many maps $g_i\colon \tilde{y}_i \to z$ for each i and corresponding constants $\lambda_{g_i} \in k$ such that

$$e_f = \sum_{i=1}^{\ell} \sum_{g_i} \lambda_{g_i} (\Phi^*(\mathrm{P}_{x'})(g_i))(\alpha_i) = \sum_{i=1}^{n} \sum_{g_i} \sum_{f_i \in H_i} (\lambda_{g_i} \lambda_{f_i}) e_{\Phi(g_i) f_i}.$$

Take any one of these g_i. In particular, the equality above implies that $\Phi(g_i) f_i = f$. $\square$

Finally, in ring theory a fundamental property of modules is that of Noetherianity. The analogue for modules over categories is the main topic of this thesis.

Definition 1.18. A $\mathcal{C}$-module over k is Noetherian if all its $\mathcal{C}$-submodules are finitely generated. The whole category $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian if every finitely generated $\mathcal{C}$-module over k is Noetherian.

Remark 1.19. Finite generation is preserved by products, by Remark 1.15. Moreover, a set of generators of a $\mathcal{C}$-module descends to a set of generators on any of its quotients. Hence, by Proposition 1.14, it is enough to see that every principal projective module P_x is Noetherian, to conclude that $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian.

There is again an interaction with property (F) from Definition 1.16.

Proposition 1.20. *Let $\Phi \colon \mathcal{C} \to \mathcal{C}'$ be an essentially surjective functor satisfying property (F). If $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian, then $\mathrm{Rep}_k(\mathcal{C}')$ is Noetherian.*

Proof. Take an object $x' \in \mathcal{C}'$ and consider the corresponding principal projective $\mathcal{C}'$-module, $\mathrm{P}_{x'}$. By Proposition 1.17, $\Phi^*(\mathrm{P}_{x'})$ is finitely generated. Take an arbitrary submodule $N \subseteq \mathrm{P}_{x'}$. By assumption, $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian. Hence, $\Phi^*(\mathrm{P}_{x'})$ is Noetherian, which implies that its submodule $\Phi^*(N)$ is finitely generated. Therefore, take a finite set of generators $S \subseteq \coprod_{y \in \mathcal{C}} \Phi^*(N)(y)$. We can also see S as a subset of $\coprod_{y \in \mathcal{C}} N(\Phi(y))$ and consider the $\mathcal{C}$-submodule $L \subseteq N$ generated by S. It follows that $\Phi^*(L) = \Phi^*(N)$, but since Φ is essentially surjective, it follows that $L = N$. Therefore, N is finitely generated and as it was arbitrary, this implies that $\mathrm{P}_{x'}$ is Noetherian. As x' was also arbitrary, we conclude that each principal projective $\mathcal{C}'$-module is Noetherian. By Remark 1.19, it follows that $\mathrm{Rep}_k(\mathcal{C}')$ is Noetherian. $\square$

The fundamental question we are interested in is the following.

Question 1.21. Given a left-Noetherian ring k and a category $\mathcal{C}$ does it hold that $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian?

In fact, the analysis is usually restricted to categories having finitely many morphisms between any two objects inside it. We motivate this further in section 4.1. Therefore, from now on we focus on the next class of categories.

Definition 1.22. We call a category, $\mathcal{C}$, combinatorial if there exist a faithful functor from $\mathcal{C}$ to the category of finite sets and set theoretical functions, $\mathbf{FSet}$.

The following sections of this chapter are dedicated to the introduction of some criteria on combinatorial categories that imply a positive answer to Question 1.21. Some examples of a negative answer are deduced in section 4.2.

1.2 Gröbner methods

In search for criteria on a combinatorial category $\mathcal{C}$ that ensure that $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian, it is natural to continue the analogy with ring theory. In particular, the proof of the following theorem is the inspiration for what follows.

Theorem 1.23 (Hilbert's basis Theorem)**.** *If k is a left-Noetherian ring, then for each $n \in \mathbb{N}$ the polynomial ring $k[x_1, \dots, x_n]$ is also left-Noetherian.*

We start by defining the notions needed in the proof.

Definition 1.24. Consider the polynomial ring $R = k[x_1, \dots, x_n]$.

- A monomial in R is a polynomial of the form $\lambda x_1^{m_1} \cdots x_n^{m_n}$ where $\lambda \in k$ and $m_i \in \mathbb{N}_0$ for each i. Every polynomial is a finite sum of monomials.
- We denote by $\mathfrak{M}$ the set of all monomials in R with coefficient $\lambda = 1$.
- An admissible order $\preccurlyeq$ on $\mathfrak{M}$ is a well-order that is compatible with multiplication, which means that $LP \preccurlyeq QP$ whenever $L \preccurlyeq Q$ for all $P, Q, L \in \mathfrak{M}$.
- Given an admissible order $\preccurlyeq$ on $\mathfrak{M}$, the initial term, $\mathrm{init}(f)$, of a nonzero polynomial $f \in R$ is the monomial summand $\lambda \in P$ in f for which $P \in \mathfrak{M}$ is $\preccurlyeq$-maximal.
- A monomial ideal $I \subseteq R$ is a left ideal generated by monomials. We denote by $\mathcal{M}(R)$ the set of all monomial ideals in R.
- For any left ideal $I \subseteq R$ the corresponding initial ideal, $\mathrm{init}(I)$, is the monomial ideal generated by $\{\mathrm{init}(f) \mid f \in I\}$.
- A Gröbner basis of a left ideal I is a subset $G \subseteq I$ such that the ideal $\mathrm{init}(I)$ is generated by $\{\mathrm{init}(f) \mid f \in G\}$.

Example 1.25. Consider the polynomial ring $\mathbb{Z}[x_1, x_2]$. The lexicographic order is an admissible order on $\mathfrak{M}$ given by

$$x_1^{m_1} x_2^{m_2} \preccurlyeq x_1^{m_1'} x_2^{m_2'} \text{ if } m_1 < m_1', \text{or if } m_1 = m_1' \text{ and } m_2 \leqslant m_2'.$$

For any admissible order, it holds that if I is a monomial ideal, then $I = \mathrm{init}(I)$ and a set of generators is a Gröbner basis.

We will see that Hilbert's basis Theorem can be reduced to a statement about Noetherianity of partially ordered sets (posets). Therefore, recall the following standard material.

Definition 1.26. Let $(X, \leqslant)$ be a poset.

- An ideal I is a subset of X such that, if $x \in I$ and $x \leqslant y$, then $y \in I$. We denote by $\mathcal{I}_X$, the set of ideals in X.
- the principal ideal corresponding to $x \in X$ is given by $I_x = \{y \in X \mid y \geqslant x\}$. An ideal is finitely generated if it is the union of finitely many principal ideals.

- A poset $(X, \leqslant)$ is Noetherian if one (and hence each) of the following conditions holds.
  - Given an infinite sequence $x_1, x_2, \ldots$ of elements of X, there exist $i < j \in \mathbb{N}$ such that $x_i \leqslant x_j$.
  - The set $\mathcal{I}_X$ satisfies the ascending chain condition (ACC), that is for any ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals in X there is an $N \in \mathbb{N}$ such that $I_j = I_N$ for all $j \geqslant N$.
  - Every ideal in X is finitely generated.

The following result about posets is of particular importance to us, so we recall its proof. Let X and $\mathcal{I}$ be two posets. We denote by $\mathcal{F}(X, \mathcal{I})$ the set of order preserving maps from X to $\mathcal{I}$. It is itself partially ordered by the relation $\phi \leqslant \psi$ if $\phi(x) \leqslant_{\mathcal{I}} \psi(x)$ for all $x \in X$.

Proposition 1.27. *If X is Noetherian and $\mathcal{I}$ satisfies the ascending chain condition, then $\mathcal{F}(X, \mathcal{I})$ satisfies the ascending chain condition as well.*

Proof. Suppose that there is an infinite strictly increasing sequence $I_1 \lneq I_2 \lneq \ldots \in \mathcal{F}(X, \mathcal{I})$. This means that for each $n \in \mathbb{N}$ we can find $x_n \in X$ such that $I_n(x_n) \subsetneq I_{n+1}(x_n)$. Since we assumed that X is Noetherian, there is an infinite non-decreasing subsequence of $(x_j)_{j \in J}$. By construction, $I_j(x_j) \subsetneq I_{j+1}(x_j) \subseteq I_{j+1}(x_{j+1})$ for each $j \in J$. Hence, there is an infinite sequence $(I_j(x_j))_{j \in J}$ of ideals of k that is strictly increasing. This contradicts the assumption that $\mathcal{I}$ satisfies the ascending chain condition. We conclude that $\mathcal{F}(|\mathcal{C}_x|, \mathcal{I}_k)$ must satisfy the ascending chain condition. □

The proof of Hilbert's basis Theorem now follows completely from analysing the pointwise partial order $\leqslant$ on $\mathbb{N}^n$, which is given by $(m_1, \ldots, m_n) \leqslant (m'_1, \ldots, m'_n)$ if $m_i \leqslant m'_i \in \mathbb{N}$ for each $i \in \{1, \ldots, n\}$.

Lemma 1.28 (Dickson's Lemma)**.** *The poset $(\mathbb{N}^n, \leqslant)$ is Noetherian.*

Proof. We show that for every infinite sequence $\underline{m}^1, \underline{m}^2, \ldots \in N^n$ there exists some $i < j \in \mathbb{N}$ such that $\underline{m}^i \leqslant \underline{m}^j$. We work by induction on n.

For the base case $n = 1$, we immediately notice the "stronger" result that any infinite sequence of natural numbers has an infinite non-decreasing subsequence (since only finitely many elements can be smaller than m_1 or infinitely many of them are equal and so on...). For $n \geqslant 1$ let $\underline{m}^1, \underline{m}^2, \ldots$ be an infinite sequence in N^{n+1}. Split each vector as a tuple $\underline{m}^i = (\underline{\tilde{m}}^i, m^i_{n+1})$ in $\mathbb{N}^n \times \mathbb{N}$ by separating the last coordinate. By the base case, there exists a infinite non-decreasing subsequence $(m^l_{n+1})_{l \in L} \in \mathbb{N}$. By the induction hypotheses on the corresponding subsequence, $(\underline{\tilde{m}}^l)_{l \in L} \in \mathbb{N}^n$, there is some $i < j$ such that $\underline{\tilde{m}}^i \leqslant \underline{\tilde{m}}^j$. By choice of our subsequence, it follows that $\underline{m}^i \leqslant \underline{m}^j$ in $\mathbb{N}^{n+1}$. □

To translate this result back to our setting, notice that the set of monomials with trivial coefficient $\mathfrak{M}$ comes equipped with a canonical partial order of division, that is $P \leqslant Q \in \mathfrak{M}$ if there exists $L \in \mathfrak{M}$ such that $Q = LP$. Clearly this induces an isomorphism of posets $\mathbb{N}^n \xrightarrow{\sim} \mathfrak{M}\colon (m_1, \ldots, m_n) \mapsto x_1^{m_1} \cdots x_n^{m_n}$, where $\mathbb{N}^n$ is ordered under the point-wise partial order.

Lemma 1.29. *Let k be a left-Noetherian ring, then any monomial ideal in $k[x_1, \ldots, x_n]$ is finitely generated.*

Proof. Let $R = k[x_1, \ldots, x_n]$ and consider the set of monomial ideals $\mathcal{M}(R)$ ordered partially by inclusion. We give another description of $\mathcal{M}(R)$. Let $\mathcal{I}_k$ be the set of left ideals in k partially ordered by inclusion and consider the canonical order on $\mathfrak{M}$. We claim that the following map is an isomorphism.

$$\Theta\colon \mathcal{M}(R) \to \mathcal{F}(\mathfrak{M}, \mathcal{I}_k)\colon I \mapsto \Big(\Theta(I)\colon P \mapsto \{\lambda \in k \mid \lambda P \in I\}\Big)$$

There are several things to check here. Firstly, since I is a left ideal of R, it follows that $\Theta(M)(I)$ is a left ideal of k. To see that any $\Theta(I)$ is order preserving, let $Q = LP$ for $P, Q, L \in \mathfrak{M}$. The fact that λP is an element of I implies that $\lambda Q = L \cdot \lambda P$ is an element of I as well. Therefore, $P \leqslant Q$ gives $\Theta(I)(P) \subseteq \Theta(I)(Q)$.

Next, we want to see that Θ itself is order preserving and injective. Let $I \subsetneq J \in \mathcal{M}(R)$, this implies that $\Theta(I)(P) \subseteq \Theta(J)(P)$ for all $P \in \mathfrak{M}$ and that the inclusion is strict for at least one P. This means exactly that $\Theta(I) \lneq \Theta(J)$. Finally, for surjectivity, choose $F \in \mathcal{F}(\mathfrak{M}, \mathcal{I}_k)$ arbitrary. We define I_F to be the left ideal of R generated by $\{\lambda Q \mid Q \in \mathfrak{M}$ and $\lambda \in F(Q)\}$. By definition, it is monomial and satisfies $\Theta(I_F) = F$.

By Dickson's Lemma 1.28, $\mathfrak{M}$ is Noetherian and by assumption $\mathcal{I}_k$ satisfies the ascending chain condition. By Proposition 1.27, it follows that $\mathcal{M}(R)$ satisfies the ascending chain condition. Suppose that $I \subseteq k[x_1, \ldots, x_n]$ is a non-finitely generated monomial ideal. Pick any monomial $P_1 \in I$. As I is not finitely generated, we can pick some monomial $P_2 \in I$ that is not an element of the left ideal generated by P_1 (denoted $\langle P_1 \rangle$). Inductively we can continue picking a monomial $P_i \in I \setminus \langle P_1, \ldots, P_{i-1} \rangle$ for each $i \geqslant 2$. This gives an infinite ascending sequence of monomial ideals

$$\langle P_1 \rangle \subsetneq \langle P_1, P_2 \rangle \subsetneq \langle P_1, P_2, P_3 \rangle \subsetneq \ldots \quad \subsetneq I.$$

This yields a contradiction, implying that I had to be finitely generated. □

Proof Hilbert's basis Theorem. Consider any left ideal $I \in k[x_1, \ldots, x_n]$. Fix any admissible order on $k[x_1, \ldots, x_n]$ (for example the lexicographic one), which allows us to define $\operatorname{init}(I)$. By Lemma 1.29, there exists a finite Gröbner basis G of I. Now let J be the subideal of I generated by G. Suppose that there exists $f \in I \setminus J$, then take such f with $\operatorname{init}(f)$ $\preccurlyeq$-minimal. Since $\operatorname{init}(f)$ is an element of $\operatorname{init}(I)$, there exists a $g \in J$ such that $\operatorname{init}(f) = \operatorname{init}(g)$. This leads to a contradiction since $f - g \in I \setminus J$ and $\operatorname{init}(f - g) \preccurlyeq \operatorname{init}(f)$. This means that I is finitely generated by G. As I was arbitrary we conclude that $k[x_1, \ldots, x_n]$ is Noetherian. □

The powerful idea of Sam and Snowden [SS16] is to transfer this whole proof to the setting of $\mathcal{C}$-modules. The categories $\mathcal{C}$ for which this can be done are the so called Gröbner categories.

The first assumption we make is that $\mathcal{C}$ has no nontrivial endomorphisms, that is $\mathrm{Hom}_{\mathcal{C}}(x,x) = \{\mathrm{id}_x\}$ for each $x \in \mathcal{C}$. This may seem like a rather big requirement and it is indeed not strictly needed to develop the general theory as Sam and Snowden do. However, it makes the argument much more transparent and we will see later why it is the only case in which this argument is used in practice.

Let us begin by generalising the definitions made in 1.24. Whereas before the goal was to see that all submodules (= ideals) of $k[x_1, \dots, x_n]$ are finitely generated, now we want this for all submodules of an arbitrary principal projective $\mathcal{C}$-module P_x (see Remark 1.19).

Definition 1.30. Consider a principal projective $\mathcal{C}$-module P_x.

- A monomial in $\mathrm{P}_x(y)$, for an object $y \in \mathcal{C}$, is an element of the form $\lambda \cdot e_f$, where $\lambda \in k$ and $f\colon x \to y \in \mathcal{C}$. Every element in $\mathrm{P}_x(y)$ is a finite sum of monomials.
- Set $\mathcal{C}_x = \bigcup_{y \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x,y)$ and let $|\mathcal{C}_x|$ be the quotient set where isomorphic morphisms are identified.
- An admissible order $\preccurlyeq$ on $|\mathcal{C}_x|$ is a well-order that is compatible with post-composition, which means that $hf \preccurlyeq hg$ whenever $f \preccurlyeq g$ for all $f, g \in \mathrm{Hom}_{\mathcal{C}}(x,y)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(y,z)$.
- Given an admissible order $\preccurlyeq$ on $|\mathcal{C}_x|$ the initial term, $\mathrm{init}(\alpha)$, of a nonzero object $\alpha \in \mathrm{P}_x(y)$ is the monomial summand $\lambda \cdot e_f$ of α for which $[f] \in |\mathcal{C}_x|$ is $\preccurlyeq$-maximal.
- A monomial submodule $M \subset \mathrm{P}_x$ is a submodule such that for each $y \in \mathcal{C}$ the k-module $M(y)$ is spanned by the monomials it contains.
- For any submodule $M \subseteq \mathrm{P}_x$ the corresponding initial module, $\mathrm{init}(M)$, is the monomial submodule defined by $\mathrm{init}(M)(y) = \mathrm{span}_k\{\mathrm{init}(\alpha) \mid \alpha \in M(y) \setminus \{0\}\}$. We denote by $\mathcal{M}(\mathrm{P}_x)$ the set of all monomial submodules of P_x.
- A Gröbner basis of a submodule $M \subseteq \mathrm{P}_x$ is a subset $G \subseteq \bigcup_{y \in \mathbb{C}} M(y)$, such that the initial module, $\mathrm{init}(M)$, is generated by $\{\mathrm{init}(\alpha) \mid \alpha \in G\}$.

Remark 1.31. The analogy should be clear. The only difference is that in this case everything happens in a collection of k-modules. For example, instead of being defined on the set of trivial coefficient monomials $\mathfrak{M}$, the admissible order is now defined on the set of morphisms $|\mathcal{C}_x|$ indexing all monomials in $\bigcup_{y \in \mathcal{C}} \mathrm{P}_x(y)$.

In the proof of Hilbert's basis Theorem, we did not only use an admissible order on $\mathfrak{M}$, but also the canonical partial order of division. For $|\mathcal{C}_x|$, we can define the analogue canonical order by $f \leqslant g$ if there exists an $h \in |\mathcal{C}_x|$ such that $g = h \circ f$. The ingredients used in the proof of Hilbert's basis Theorem naturally reveal the following criteria.

Definition 1.32. An essentially small category $\mathcal{C}$ is called Gröbner if it has no nontrivial endomorphisms and satisfies the following conditions for each $x \in \mathcal{C}$.

(G1) There exist an admissible order $\preccurlyeq$ on $|\mathcal{C}_x|$.

(G2) The poset $(|\mathcal{C}_x|, \leqslant)$ under the canonical order is Noetherian.

It is useful to notice the following direct consequence of this definition.

Lemma 1.33. *If the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are Gröbner, so is the product category $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$.*

Proof. Clearly, $\mathcal{C}$ has no nontrivial endomorphisms. Furthermore, fixing $x_1 \in \mathcal{C}_1$ and $x_2 \in \mathcal{C}_2$, there is a canonical identification $|\mathcal{C}_{(x,y)}| \cong |\mathcal{C}_{1x}| \times |\mathcal{C}_{2y}|$. Therefore, if $\preccurlyeq_1$ and $\preccurlyeq_2$ are admissible orders on the components, then the lexicographic order, given by $(f_1, f_2) \preccurlyeq (g_1, g_2)$ if $f_1 \prec_1 g_1$ or if $f_1 = g_1$ and $f_2 \preccurlyeq_2 g_2$, is admissible on $|\mathcal{C}_{(x,y)}|$. On the other hand, the canonical order on $|\mathcal{C}_{(x,y)}|$ is simply the product order of he canonical orders on both components. As the product of two Noetherian posets is again Noetherian, (G2) holds as well. □

This definition enables us to state and prove our first main theorem, which gives a partial answer to Question 1.21.

Theorem 1.34. *If $\mathcal{C}$ is a Gröbner category and k is a left-Noetherian ring, then the category $\operatorname{Rep}_k(\mathcal{C})$ is Noetherian.*

The definitions were made in such a way that the proof is completely analogue to that of Hilbert's basis Theorem. To notice the adaptations and to be complete, we repeat the steps.

Lemma 1.35. *If (G2) holds for $x \in \mathcal{C}$ and k is a left-Noetherian ring, then every monomial submodule $M \subseteq \mathrm{P}_x$ is finitely generated.*

Proof. Consider the set of monomial submodules $\mathcal{M}(\mathrm{P}_x)$ partially ordered by the submodule relation. Let $\mathcal{I}_k$ be the set of left ideals of k ordered by inclusion and consider the canonical order $\leqslant$ on $|\mathcal{C}_x|$. We claim that the following map is an isomorphism.

$$\Theta\colon \mathcal{M}(\mathrm{P}_x) \to \mathcal{F}(|\mathcal{C}_x|, \mathcal{I}_k)\colon M \mapsto \Big(\Theta(M)\colon (f\colon x \to y) \mapsto \{\lambda \in k \mid \lambda \cdot e_f \in M(y)\}\Big)$$

There are several things to check here. Firstly, since $M(y)$ is a k-module it follows that $\Theta(M)(f)$ is a left ideal of k. To see that any $\Theta(M)$ is order preserving, suppose that $g = h \circ f$, where $g \in \operatorname{Hom}_{\mathcal{C}}(x, z)$ and $h \in \operatorname{Hom}_{\mathcal{C}}(y, z)$. The fact that $\lambda \cdot e_f$ is an element of $M(y)$ implies that $\lambda \cdot e_g = \lambda \cdot e_{hf} = \mathrm{P}_x(f)(\lambda \cdot e_f) = M(f)(\lambda \cdot e_f)$ is an element of $M(Z)$. Therefore, $f \leqslant g$ gives $\Theta(M)(f) \subseteq \Theta(M)(g)$.

To see that Θ itself is order preserving and injective, notice that $N \subsetneq M$ implies that $N(y) \subseteq M(y)$ for all $y \in \mathcal{C}$ and that the inclusion is strict for at least one y. This means exactly that $\Theta(N) \nleq \Theta(M)$. For surjectivity, take $F \in \mathcal{F}(|\mathcal{C}_x|, \mathcal{I}_k)$ arbitrary. By definition, the assignment $M_F(y) = \bigoplus_{f \in \operatorname{Hom}_{\mathcal{C}}(x,y)} F(f) \cdot e_f$ is functorial. Hence, it defines a submodule $M_F \subseteq \mathrm{P}_x$ that satisfies $\Theta(M_F) = F$.

By assumption, $|\mathcal{C}_x|$ is Noetherian and $\mathcal{I}_k$ satisfies the ascending chain condition. By Proposition 1.27, it follows that $\mathcal{M}(\mathrm{P}_x)$ also satisfies the ascending chain condition. Suppose that a monomial submodule $M \subseteq \mathrm{P}_x$ is non-finitely generated. This means that there exists an infinite sequence of monomials, $\alpha_i \in M(x_i)$ for some $x_i \in \mathcal{C}$, such that α_i is not inside the submodule generated by $\{\alpha_1, \ldots, \alpha_{i-1}\}$. This yields a strictly ascending chain of monomial submodules of P_x. This is a contradiction, hence M was finitely generated. $\square$

Proof Theorem 1.34. Fix any object $x \in \mathcal{C}$ and any $\mathcal{C}$-submodule M of P_x. By (G1), there exists an admissible order $\preccurlyeq$ on $|\mathcal{C}_x|$, which allows us to define $\mathrm{init}(M)$. By (G2) and Lemma 1.35, there exists a Gröbner basis G of M. Let N be the submodule of M generated by G. Suppose that there exists an $\alpha \in M \setminus N$, then take such α with $\mathrm{init}(\alpha)$ $\preccurlyeq$-minimal. Since $\mathrm{init}(\alpha) \in \mathrm{init}(M)$, there exists a $\beta \in N$ such that $\mathrm{init}(\alpha) = \mathrm{init}(\beta)$. This leads to a contradiction, since $\alpha - \beta \in M \setminus N$ and $\mathrm{init}(\alpha - \beta) \preccurlyeq \mathrm{init}(\alpha)$. This means that M is finitely generated by G. As M was arbitrary, we conclude that P_x is Noetherian and as x was also arbitrary, so is $\mathrm{Rep}_k(\mathcal{C})$. $\square$

Now that we translated the whole Gröbner method to our setting, we want to apply the theorem above. In practice however, because of the assumption that there are no nontrivial endomorphisms, very few categories are Gröbner. The way around this is to recall Proposition 1.20.

Definition 1.36. A category $\mathcal{C}'$ is quasi-Gröbner if there exist a Gröbner category $\mathcal{C}$ and an essentially surjective functor $\Phi\colon \mathcal{C} \to \mathcal{C}'$ satisfying property (F).

Remark 1.37. It is a simple exercise to show that the composition of two functors satisfying property (F), satisfies property (F) again. As the same holds for essential surjectivity, it is (by recursion) enough to know that $\mathcal{C}$ is quasi-Gröbner to deduce that $\mathcal{C}'$ is quasi-Gröbner.

Similarly, if $\Phi\colon \mathcal{C} \to \mathcal{C}'$ and $\Psi\colon \mathcal{D} \to \mathcal{D}'$ are essentially surjective functors satisfying property (F), then so is $\Phi \times \Psi\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$. By Lemma 1.33, it follows that the product category of two quasi-Gröbner categories is again quasi-Gröbner.

Corollary 1.38. *If $\mathcal{C}$ is a quasi-Gröbner category and k is a left-Noetherian ring, then $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian.*

Proof. Combine Theorem 1.34 and Proposition 1.20. $\square$

1.3 Finite sets and free modules

After the development of the general Gröbner method ensuring Noetherianity of $\mathrm{Rep}_k(\mathcal{C})$, let us see some concrete examples. First of all we consider categories consisting of all finite sets.

Definition 1.39. The categories **FI** and **FS** consist of all finite sets and have respectively injections and surjections as morphisms. The categories **OI** and **OS** consist of ordered finite sets and have respectively order-preserving injections and surjections as morphisms.

Remark 1.40. In particular, the study of **FI**-modules has a rich history. See [Far14] for a survey and many references.

By definition, these categories are combinatorial. The aim of putting a total order on the sets should be clear by now. It is namely to rule out nontrivial endomorphisms.

Theorem 1.41. *The category* **OI** *is Gröbner.*

As in the case of Hilbert's basis Theorem, the proof boils down to a classical result on Noetherian posets.

Definition 1.42. Let $(X, \leqslant)$ be a poset. A finite word in X is an object of the form $x_1 \cdots x_n$, where $n \in \mathbb{N}$ and $x_i \in X$, or the empty set. We denote by X^* the set of finite words in X, partially ordered by $x_1 \cdots x_n \leqslant y_1 \cdots y_m$ if there exists $1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant m \in \mathbb{N}$ such that $x_j \leqslant x_{i_j}$ for each $j \in \{1, \ldots n\}$. The length function, $\ell \colon X^* \to \mathbb{N}_0$, is given by $\ell(x_1 \cdots x_n) = n$ and $\ell(\emptyset) = 0$.

Remark 1.43. In particular, X^* has the structure of a monoid, where the operation is given by concatenation, that is $(x_1 \cdots x_n, y_1 \cdots y_m) \mapsto x_1 \cdots x_n y_1 \cdots y_m$.

Lemma 1.44 (Higman's Lemma)**.** *If $(X, \leqslant)$ is a Noetherian poset, then $(X^*, \leqslant)$ is a Noetherian poset.*

Proof. Suppose that the statement is false, this means that there exists an infinite sequence $w_1, w_2, \ldots \in X^*$, such that $w_i \not\leqslant w_j$ for all $i < j$. We say that such a sequence is "bad". Take the bad sequence that is minimal in the following sense. Among all bad sequences beginning with $w_1, \ldots, w_{i-1}$, in this sequence $\ell(w_i)$ is minimal. Let x_i be the first element of w_i for each $i \in \mathbb{N}$. Since X is Noetherian, we can find an infinite increasing subsequence $(x_i)_{i \in I}$. We consider a new sequence $w_1, \ldots, w_{i_1 - 1}, v_{i_1}, v_{i_2}, \ldots \in X^*$. As $v_{i_k} \leqslant w_{i_k}$ for any $i_k \in I$, it follows that $w_j \not\leqslant v_{i_k}$ for all $j \in \{1, \ldots, i_1 - 1\}$. Moreover, $w_{i_k} \not\leqslant w_{i_l}$ implies that $v_{i_k} \not\leqslant v_{i_l}$ for all $k \leqslant l \in \mathbb{N}$. We conclude that the new sequence is bad. However, the fact that $\ell(v_{i_1}) = \ell(w_{i_1}) - 1$ contradicts the presumed minimality of the original sequence. We conclude that no bad sequence exists. $\square$

Proof Theorem 1.41. For $n \in \mathbb{N}$, let $[n]$ denote the poset $\{1, \ldots, n\}$ with the usual order and set $[0] = \emptyset$. We notice directly that any element of **OI** is isomorphic to a unique $[n]$, where $n \in \mathbb{N}_0$. Moreover, any order-preserving injection from an ordered finite set to itself is the identity. It remains to show that $|\mathbf{OI}_{[n]}|$ satisfies (G1) and (G2) for any $n \in \mathbb{N}_0$. One easily checks that the following order is admissible

$$\mathrm{Hom}_{\mathbf{OI}}(x, y) \ni f \preccurlyeq g \in \mathrm{Hom}_{\mathbf{OI}}(x, z) \begin{cases} \text{if } f(1) < g(1), \\ \text{or } f(1) = g(1) \text{ and } f(2) < g(2), \\ \vdots \\ \text{or } f\restriction_{[n-1]} = g\restriction_{[n-1]} \text{ and } f(n) < g(n), \\ \text{or } f = g \text{ and } y \subseteq z. \end{cases}$$

Next, we must consider the canonical order on $|\mathbf{OI}_{[n]}|$. Set $X = [2]$, then the next map is an embedding of partial orders

$$\psi \colon |\mathbf{OI}_{[n]}| \to X^* \colon (f \colon [n] \mapsto [m]) \mapsto w_1 \cdots w_m \text{ where } w_i = \begin{cases} 1 & \text{if } i \in f([n]), \\ 2 & \text{else.} \end{cases}$$

Since X is finite and hence Noetherian, it follows by Higman's Lemma 1.44, that condition (G2) holds. □

However, **OI** is a very simple category and not of real interest in itself.

Theorem 1.45. *The category* $\mathbf{FI}$ *is quasi-Gröbner.*

Proof. By Theorem 1.41, **OI** is Gröbner. We consider the forgetful functor $\Phi\colon \mathbf{OI} \to \mathbf{FI}$, which is of course essentially surjective. To check property (F), take an arbitrary $x' \in \mathbf{FI}$ and let n be its cardinality. Let the posets $y_1, \dots, y_{n!}$ all be equal to $[n] \in \mathbf{OI}$ and let the morphisms $f_i\colon x \mapsto y_i \in \mathbf{FI}$ be all permutations in $\mathrm{Sym}(x') = \mathrm{Sym}(n)$. It follows that any injection, $f\colon x' \to [m]$ for some $[m] \in \mathbf{OI}$, factors as one of the permutations f_i followed by an order preserving map $g\colon y_i = [n] \to [m]$. □

Theorem 1.46. *The category* $\mathbf{FSet}$ *is quasi-Gröbner.*

Proof. By Theorem 1.45, **FI** is quasi-Gröbner. We consider the inclusion functor, $\Phi\colon \mathbf{FI} \to \mathbf{FSet}$, which is of course essentially surjective. Pick $x' \in \mathbf{FA}$ and let $f_i\colon x' \to y_i$ be representatives of all classes of surjections with domain x. Notice that these are finitely many morphisms. It follows that any map $f\colon x \to z \in \mathbf{FSet}$, factors as a surjection f_i (by restricting the codomain) followed by an inclusion $g\colon y_i \to y$. Hence, Φ satisfies property (F) and by Remark 1.37, the statement follows. □

This provides a first couple of examples of a positive answer to Question 1.21.

Corollary 1.47. *For any left-Noetherian ring k, the categories* $\mathrm{Rep}_k(\mathbf{OI})$, $\mathrm{Rep}_k(\mathbf{FI})$ *and* $\mathrm{Rep}_k(\mathbf{FSet})$ *are Noetherian.*

Proof. Combine Theorems 1.41, 1.45 and 1.46 and Corollary 1.38. □

The proof illustrates how to use the methods for a general category $\mathcal{C}$. The first difficulty is to create another category $\mathcal{C}'$ with no nontrivial endomorphisms and a essentially surjective functor, $\mathcal{C}' \to \mathcal{C}$, satisfying property (F). Afterwards, whereas there may be many orders satisfying (G1), the real challenge is to check property (G2). An argument analogue to the above, essentially coming down to Higman's Lemma again, yields the following.

Theorem 1.48. *The category* $\mathbf{OS}^{\mathrm{op}}$ *is Gröbner and the category* $\mathbf{FS}^{\mathrm{op}}$ *is quasi-Gröbner.*

Proof. Theorems 8.1.1 and 8.1.2 in [SS16]. □

Let us see a more exotic example of the method at work.

Definition 1.49. Let R be a commutative ring. The category $\mathbf{FMod}_R$ consists of all finitely generated R-modules and R-linear maps. The full subcategory of free R-modules of finite rank is denoted $\mathbf{V}_R$.

A linear map between free R-modules is called splittable if its image is a direct summand of the codomain. The subcategories $\mathbf{VS}_R$ and $\mathbf{VSI}_R$ consist of the same objects as $\mathbf{V}_R$ but contain respectively only the splittable maps or splittable injections.

Remark 1.50. Notice that for a finite ring R, an R-module is finitely generated if and only if it is finite. For a field K, any finitely generated module is a free vector space and any linear map is splittable. Hence, the categories $\mathbf{FMod}_K$, $\mathbf{V}_K$, and $\mathbf{VS}_K$ are all equal and we denote them by $\mathbf{FVec}_K$.

The next theorem is a combination of Theorem 8.3.1. in [SS16] and Theorem C in a follow-up paper by Sam and Putman [PS14].

Theorem 1.51. *Let R be a finite commutative ring. The categories $\mathbf{VSI}_R$, $\mathbf{VS}_R$ and $\mathbf{V}_R$ are quasi-Gröbner.*

Remark 1.52. We demand the ring R to be finite to work with a combinatorial category. See section 4.1 for the problems that occur when R is infinite.

Proof. By Theorem 1.48, $\mathbf{FS}^{\mathrm{op}}$ is Gröbner. For an R-module M, we denote the dual module $\mathrm{Hom}_R(M, R)$ by M^*. We consider the functor $\Phi\colon \mathbf{FS}^{\mathrm{op}} \to \mathbf{VSI}_R\colon S \mapsto R[S]^*$, given on the level of morphisms by mapping the surjection $f\colon T \to S$ to the map

$$\Phi(f)\colon R[S]^* \to R[T]^*\colon g \mapsto (g \circ R[f]), \text{ where } R[f]\colon R[T] \to R[S]\colon e_t \mapsto e_{f(t)}.$$

Since any free R-module of finite rank is congruent to $R^n \cong \Phi(\{1, \ldots, n\})$ for some $n \in \mathbb{N}$, Φ is essentially surjective. We are left to check property (F) for each $R^n \in \mathbf{VSI}_R$. Since R is finite, there exist only finitely many subsets $T_1, \ldots, T_n \subseteq (R^n)^*$ that span $(R^n)^*$ as an R-module. For each T_i, let $f_i\colon R^n \to R[T_i]^*$ be the dual of the natural maps $R[T_i] \to (R^n)^*$. If we pick any finite set S and a splittable injection $f\colon R^n \to R[S]^*$, this map induces a dual surjection $f^*\colon R[S] \to (R^n)^*$. The image of $\{e_s \mid s \in S\}$ under f^* generates $(R^n)^*$ and is thus equal to some T_i. Hence, f^* factorizes as

$$R[S] \xrightarrow{\Phi(g)^*} R[T_i] \xrightarrow{f_i^*} (R^n)^*,$$

where the first map comes from a surjection $g\colon S \to T$. The dual of this composition shows that property (F) holds and we conclude that $\mathbf{VSI}_R$ is quasi-Gröbner.

Next, consider the inclusion functor $\mathbf{VSI}_R \to \mathbf{VS}_R$, which is clearly essentially surjective. As in the proof of Theorem 1.46, it satisfies property (F) because any splittable map can be factored as a splittable surjection to its image (of which there are finitely many) followed by a splittable injection. By Remark 1.37, it follows that $\mathbf{VS}_R$ is quasi-Gröbner.

Finally, consider the inclusion functor $\iota\colon \mathbf{VSI}_R \to \mathbf{V}_R$, which is clearly essentially surjective. To check property (F), fix $R^n \in \mathbf{V}_R$ and let N be the cardinality of the set $|R|^n$ (recall R is finite). Notice that there are only finitely many R-linear maps, $f_i\colon R^n \to R^{(n_{f_i})}$, satisfying $n_{f_i} \leqslant N$. Now consider any m greater than N and any map $f\colon R^n \to R^m \in V(R)$. By fixing (standard) bases for R^n and R^m, we may regard f as an $m \times n$ matrix with coefficients in R. Since $R^{1\times n}$ contains exactly N elements, at least

$m - N$ rows of this matrix are equal. Changing the bases (or equivalently the matrix by elementary row operations), f is thus represented by a matrix with $m - N$ rows equal to zero. Hence, we can factor f as $R^n \xrightarrow{f_i} R^{(n_{f_i})} \xrightarrow{\iota(g)} R^m$, where f_i is the R-linear map given by the submatrix of nonzero rows (hence $n_{f_i} \leqslant N$) and g is a splittable injection. We conclude that $\mathbf{V}_R$ is quasi-Gröbner. □

Example 1.53. In particular, if we let R and k both be equal to a finite field $\mathbb{F}_q$, the theorem above gives that the category $\mathrm{Rep}_{\mathbb{F}_q}(\mathbf{FVec}_{\mathbb{F}_q})$, consisting of functors $\mathbf{FVec}_{\mathbb{F}_q} \to \mathbf{Vec}_{\mathbb{F}_q}$, is Noetherian. This statement is in fact equivalent (dual) to the celebrated *Artinian conjecture*, formulated by Lionel Schwartz, Jean Lannes and Nicholas Kuhn in the late 1980's (B.12 in [Kuh94]).

A natural question to ask is whether the full category $\mathbf{FMod}_R$ also gives a positive answer to Question 1.21 for any finite ring R. We investigate this further in section 4.3.

1.4 Growth

In this last section we review a further consequence of the Gröbner methods as introduced by Sam an Snowden in [SS16].

Suppose that $\mathcal{C}$ is a combinatorial category and M a finitely generated $\mathcal{C}$-module. We consider a question of another kind than thus far, namely how does M behave when it is evaluated in objects of increasing cardinality. More precisely let k be a field this time and M a $\mathcal{C}$-module over k. Consider the function

$$\dim_k^M : \mathcal{C} \to \mathbb{N}_0 : x \mapsto \dim_k M(x).$$

As $\mathcal{C}$ consists (after some identification) of finite sets, one could wonder how $\dim_k^M$ grows when increasing the cardinality of x. A priory, it is not clear why there should be any regular behaviour at all. Even if x and x' have the same cardinality, they might lead to completely different k-vector spaces $M(x)$ and $M(x')$. However, Sam and Snowden demonstrate that certain categories have a special structure that guarantees some control over $\dim_k^M$.

We do not want to get into details about this so called $\mathcal{O}$-lingual structure as it requires many more concepts (formal language). Anyhow, they show that all the finite set categories from Definition 1.39 have this structure and that the structure is closed under the categorical product. Let $\mathbf{OI}^r$ be the r-fold product of the category $\mathbf{OI}$. In particular, they arrive at the following result.

Proposition 1.54. *Let k be a field and M a finitely generated $\mathbf{OI}^r$-module. There exists a multivariate polynomial $P(t) \in \mathbb{Z}[t_1, \ldots, t_r]$, such that $\dim_k^M(\underline{m}) = P(\underline{m})$ when $\underline{m}$ is sufficiently large in each coordinate.*

Proof. Follows from Theorem 6.3.2, Proposition 6.3.3, and Theorem 7.1.2 in [SS16]. □

To give at least some clue on the reason this result is true, we present the proof of the following related result in the special case that $r = 1$. Our treatment follows that of Theorem B in [CEFN14] about the category **FI**.

Definition 1.55. Let M be an **OI**-module. We say that M is finitely generated in degree $\leqslant d$, if there exists an epimorphism $\bigoplus_{i=1}^{n} \mathrm{P}_{[m_i]} \to M \in \mathrm{Rep}_k(\mathbf{OI})$, where $n \in \mathbb{N}$ and $m_i \in \{1, \ldots, d\}$ (repetition is allowed).

Proposition 1.56. *Let k be a field and M a* **OI***-module that is finitely generated in degree $\leqslant d$. There exists a polynomial $P(t) \in \mathbb{Q}[t]$ of degree at most d, such that $\dim_k^M(m) = P(m)$ when $m \in \mathbb{N}$ is sufficiently large.*

Remark 1.57. We adopt the (usual) convention that the zero polynomial has degree -1.

We need the next algebraic fact.

Lemma 1.58. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. Suppose that there exists a polynomial $Q(t) \in \mathbb{Q}[t]$ of degree $d-1$, such that the difference function $\Delta(f)(m) = f(m+1) - f(m)$ is equal to $Q(m)$ for sufficiently large m. Then, there exists a polynomial $P(t) \in \mathbb{Q}[t]$ of degree d such that $f(m) = P(m)$ for sufficiently large m.*

Proof. Proposition 1.7.3 in the book of Robin Hartshorne [Har77]. □

To proceed to the proof of Proposition 1.56 we want to reduce the statement to so-called torsion free **OI**-modules. For non-negative integers $m \leqslant m'$, consider the canonical inclusion $\iota_m^{m'} : [m] \to [m'] : x \mapsto x \in \mathbf{OI}$.

Definition 1.59. Let M be a **OI**-module. We say that an element $\alpha \in M([m])$ is torsion if there exists an integer $m' \geqslant m$ such that $M(\iota_m^{m'})(\alpha) = 0$. We say that M is torsion free if 0 is the only torsion element in $M([m])$ for each $m \in \mathbb{N}_0$.

Lemma 1.60. *Let M be a finitely generated* **OI***-module.*

- *The torsion elements $\mathrm{Tor}_M([m]) = \{\alpha \in M([m]) \mid \alpha \text{ is torsion}\}$ form an* **OI***-submodule $\mathrm{Tor}_M \subseteq M$.*
- *For $[m]$ large enough $\mathrm{Tor}_M([m]) = 0$.*
- *The quotient module $M' = M/\mathrm{Tor}_M$ is torsion free.*

Proof. As the maps $M(\iota_m^{m'})$ are k-linear, $\mathrm{Tor}_M([m])$ is a k-submodule of $M([m])$.

- Further, we need to show torsion is functorial. Let $\alpha \in \mathrm{Tor}_M([m])$ and consider any $g : [m] \to [\tilde{m}] \in \mathbf{OI}$. Set

$$g' : [m'] \to [\tilde{m} - m + m'] : x \mapsto \begin{cases} g(x) & \text{if } x \leqslant m, \\ (\tilde{m} - m) + x & \text{esle.} \end{cases}$$

Notice that this leads to a commutative diagram

$$\begin{tikzcd}
[m] \arrow[rr, "\iota_m^{m'}"] \arrow[d, "g"] & & {[m']} \arrow[d, "g'"] \\
{[\tilde{m}]} \arrow[rr, "\iota_{\tilde{m}}^{\tilde{m}-m+m'}"] & & {[\tilde{m}-m+m'].}
\end{tikzcd}$$

In particular, this shows that $M(\iota_{\tilde{m}}^{\tilde{m}-m+m'})(M(g)(\alpha)) = M(g')(0) = 0$. In other words, $M(g(\alpha))$ is torsion.

- Since M is finitely generated and **OI** is Noetherian, it follows that Tor_M is finitely generated by some set $\{\alpha_1, \dots, \alpha_\ell\}$, where $\alpha_i \in Tor_M([m_i])$ for some $m_i \in \mathbb{N}_0$. For each $i \in \{1, \dots, \ell\}$ let m_i' be an integer such that the inclusion $\iota_{m_i}^{m_i'}$ witnesses that α_i is torsion. Let m be any integer greater or equal to $\max\{m_i' \mid i \in \{1, \dots \ell\}\}$. Notice that any morphism $f_i\colon m_i \to m$ factors as $m_i \xrightarrow{\iota_{m_i}} m_i' \to m$. By definition, it follows that $\mathrm{Tor}_M(f_i)(\alpha_i) = 0$ for any of these f_i. By definition of the α_i it follows that $\mathrm{Tor}_M([m]) = \{0\}$.
- Suppose that there is some $(\alpha \bmod \mathrm{Tor}_M([m]) \in M'([m])$ and $m' \geqslant m$ such that $M'(\iota_m^{m'})(\alpha \bmod \mathrm{Tor}_M([m])) = 0$. By definition, this means that $M(\iota_m^{m'})(\alpha) \in \mathrm{Tor}_M([m'])$, implying that there is an $m'' \geqslant m'$ satisfying $M(\iota_{m'}^{m''})(M(\iota_m^{m'})(\alpha)) = 0$. But this is nothing else than $M(\iota_m^{m''})(\alpha)$, which means that α is torsion. We conclude that $\alpha \equiv 0 \bmod \mathrm{Tor}_M([m])$.

□

Proof of Proposition 1.56. We work by induction on d. The base case $d = 0$ implies that $M = \bigoplus_{i=1}^n \mathrm{P}_{[0]}$, which is the **OI**-module given by $[m] \mapsto \{0\}$ for all $m \in \mathbb{N}_0$ and mapping all morphism to $\mathrm{id}_{\{0\}}$. In particular, it follows that $\dim_k^M$ is equal to the 0 polynomial, whose degree is less than 0 by convention. Next, assume d is at least 1 and let $\psi\colon N \to M \in \mathrm{Rep}_k(\mathbf{OI})$ be an epimorphism, where $N = \bigoplus_{i=1}^n \mathrm{P}_{[m_i]}$ for some $1 \leqslant m_1, \dots, m_n \leqslant d$.

Consider the quotient module $M' = M/\mathrm{Tor}_M$. By composing ψ with the natural quotient map $M \to M'$, it follows that M' is also finitely generated in degree $\leqslant d$. Furthermore, by the second part of Lemma 1.60, it holds that $\dim_k(M([m])) = \dim_k(M'([m]))$ for sufficiently large m. Therefore, we may assume without loss of generality that M is torsion free. For $g\colon [m] \to [m'] \in \mathbf{OI}$ let

$$g_{+1}\colon [m+1] \to [m'+1]\colon x \mapsto \begin{cases} g(x) & \text{if } x \leqslant m, \\ m'+1 & \text{if } x = m+1. \end{cases}$$

We consider the **OI**-module M_{+1}, given by $M_{+1}([m]) = M([m+1])$ and $M_{+1}(g) = M(g_{+1})$. Notice that there is a natural transformation, $\phi_M\colon M \to M_{+1}$, given at level $[m]$ by $M(\iota_m^{m+1})$. By the assumption that M is torsion free, ϕ_M is injective at each level and so by Proposition 1.7 ϕ is a monomorphism. Let $\Delta M \in \mathrm{Rep}_k(\mathbf{OI})$ be the quotient module $M_{+1}/\phi_M(M)$.

In particular, consider the principal projective module $\mathrm{P}_{[j]}$ for some $j \leqslant d$. Set

$$H_j = \{f \in \mathrm{Hom}_{\mathbf{OI}}([j],[m+1]) \mid m+1 \text{ is not contained in the image of } f\}.$$

Notice the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathbf{OI}}([j],[m]) \xrightarrow{\sim} H_j \colon f \mapsto \iota_m^{m+1} \circ f,\\ \mathrm{Hom}_{\mathbf{OI}}([j],[m+1]) \setminus H_j \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{OI}}([j-1],[m]) \colon f \mapsto f \upharpoonright [j-1].\end{aligned}$$

In particular, this shows that for each $m \in \mathbb{N}_0$

$$\begin{aligned}\Delta \mathrm{P}_{[j]}([m]) &\cong k[\mathrm{Hom}_{\mathbf{OI}}([j],[m+1])] / \Big(\iota_m^{m+1}(k[\mathrm{Hom}_{\mathbf{OI}}([j],[m])]\Big)\\ &\cong k[\mathrm{Hom}_{\mathbf{OI}}([j],[m+1]) \setminus H_j]\\ &\cong k[\mathrm{Hom}_{\mathbf{OI}}([j-1],[m-1])] \cong \mathrm{P}_{[j-1]}([m]).\end{aligned}$$

Clearly, the construction Δ preserves direct products. Combined with the identifications made above, it follows that

$$\Delta N \cong \Delta \bigoplus_{i=1}^{n} \mathrm{P}_{[m_i]} \cong \bigoplus_{i=1}^{n} \Delta \mathrm{P}_{[m_i]} \cong \bigoplus_{i=1}^{n} \mathrm{P}_{[m_i - 1]}.$$

Now ψ induces a shifted natural transformation $\psi_{+1} \colon N_{+1} \to M_{+1} \in \mathrm{Rep}_k(\mathbf{OI})$ given at the level $[m]$ by the map $\psi_{[m+1]}$. By Proposition 1.7, this is an epimorphism. Similarly, we get a well-defined epimorphism, $\Delta\psi \colon \Delta N \to \Delta M$, given at level $[m]$ by

$$\Delta N([m]) \to \Delta M([m]) \colon \alpha \bmod \phi_N(N) \mapsto \psi_{+1}(\alpha) \bmod \phi_M(M).$$

This witnesses that ΔM is finitely generated in degree $\leqslant (d-1)$. By the induction hypotheses, it follows that

$$\dim_k(\Delta M([m])) = \dim_k(M([m+1])) - \dim_k(M([m])) = (\Delta \dim_k^M)([m])$$

is equal to a polynomial of degree at most $d-1$ for sufficiently large m. By Lemma 1.58, we conclude that $\dim_k^M$ is eventually equal to a polynomial of degree at most d. $\square$

Chapter 2

Graphs and contractions

In the first chapter, we have displayed the background needed to analyse the Noetherianity of modules over combinatorial categories. In this chapter, following the treatment of Proudfoot and Ramos [PR19a] and [PR19a], we turn to a specific class of categories consisting of graphs and contractions. In section 2.1, we introduce these notions and show that contractions preserve the genus (number of holes) of a graph. Afterwards, we introduce planar rooted trees and S-labellings of (the vertices of) a graph. Adding these structures to our objects we get auxiliary categories without nontrivial endomorphisms (as in the case of **OI**).

Section 2.2, is dedicated to the proof that the Gröbner method of the first chapter is applicable to these graph categories. The main step in the proof is (a variant of) Kruskall's tree Theorem, which asserts that for a finite set S the poset of S-labelled trees and label preserving embeddings is Noetherian.

In section 2.3, we consider the k-dimensional growth of a finitely generated module when evaluating it in graphs with an increasing number of edges, similarly as in section 1.4. We introduce the concepts of d-small(ish) modules and demonstrate that their growth is bounded by a polynomial of degree at most d. Moreover, we introduce sprouting and subdivision, two ways to increase the number of edges while preserving the structure of the original graph. Finally, we illustrate that over these constructions, the polynomial bound given above can be turned into an equality.

2.1 Contraction categories

Let us begin by introducing the objects of central interest to us in this chapter.

Definition 2.1. A graph G is a finite CW-complex of dimension at most 1. The set of 0-cells is denoted $V(G)$, since the are called the vertices of G. The set of 1-cells is denoted $E(G)$, since they are called the edges of G.

Moreover, in this project we always assume that a graph is non-empty and connected.

A loop is an edge connecting a vertex to itself. The degree of a vertex v is the number of edges that are incident to v, where a loop counts double.

Remark 2.2. Notice that this definition allows a graph to contain loops and parallel edges between the same two vertices. In the literature, this is sometimes called a multigraph or pseudograph.

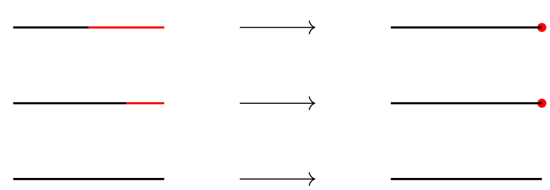
In particular, a graph is a topological space and thus morphisms should be continuous maps.

Definition 2.3. Let G and G' be graphs.

- A very cellular map is a continuous map, $f\colon G \to G'$, such that each vertex of G is mapped to a vertex of G' and each edge of G to a vertex or an edge of G'. If an edge maps to a vertex, we say that it is contracted by f.
- Two very cellular maps are called equivalent if they are homotopic through very cellular maps. A graph morphism, $f\colon G \to G'$, is an equivalence class of very cellular maps.
- A contraction is a surjective graph morphism, whose fibers are connected and acyclic.
- A proper contraction is a contraction where at least one edge of G is contracted. A simple contraction is a contraction where exactly one edge of G is contracted.

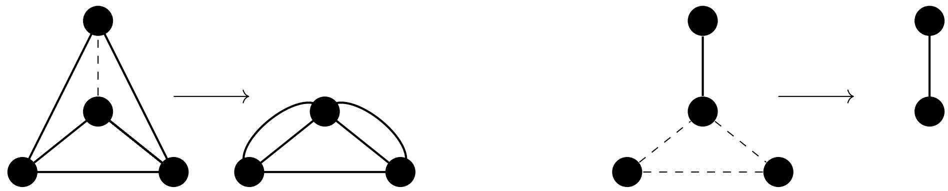
These definitions may sound more abstract than they are.

Example 2.4. The equivalence of very cellular maps states that it does not matter precisely how an edge is mapped to an edge. For example, consider the graph I of two points and one edge between them. Consider the following very cellular maps $f\colon I \to I$, acting like the identity on the vertices and shrinking the red part of the edge



They are all homotopic and hence define the same morphisms of graphs, namely the identity id_I. To picture contractions, we make the contracted edges dashed.

In the illustration below, the left map is a contraction. The right one is not because it contains a cyclic fibre.



In particular, the examples illustrate that contractions preserve the number of cycles in a graph. Let us make this precise.

Definition 2.5. The genus g of a graph is defined as the dimension of the first (singular) homology group $H_1(G;\mathbb{Q})$. Equivalently, $g = |E(G)| - |V(G)| + 1$.

Lemma 2.6. *Let $f\colon G \to G'$ be a contraction, then the genus of G and G' is equal.*

Proof. First, suppose that $f\colon G \to G'$ is a simple contraction. This means that $|E(G)| = |E(G')| + 1$. Let $e = (v_0, v_1)$ be the unique contracted edge in G. Suppose that v and v' are distinct vertices in G satisfying $f(v) = f(v')$. Since fibers are connected, this would imply that v and v' are connected by a path in G, whose edges are all contracted to $f(v)$. We conclude that the only case in which this happens is $\{v, v'\} = \{v_0, v_1\}$. Hence, it also follows that $|V(G)| = |V(G')| + 1$, implying that the genus of both graphs is equal. To conclude, we notice that any contraction can be written as a finite composition of simple contractions. $\square$

In this chapter and the next one, we analyse the following combinatorial categories.

Definition 2.7. For $g \in \mathbb{N}$, the category $\mathcal{G}_g$ consist of graphs of genus g and contractions.

Remark 2.8. In particular, notice that a graph of genus 0 cannot have any cycles, parallel edges or loops. Since we already assumed graphs to be connected, we thus recover the usual definition of a tree. Therefore, we write $\mathcal{T}$ instead of $\mathcal{G}_0$.

We are interested in modules over the opposite categories $\mathcal{G}_g^{\mathrm{op}}$. We aim to apply the Gröbner method from chapter 1. Therefore, the first step is to somehow eliminate any nontrivial endomorphisms in these category. Let us first do so for trees.

Definition 2.9. A rooted tree, (T, v_0), consists of a tree T and a fixed vertex $v_0 \in V(T)$, called the root. A contraction of rooted trees is a root preserving contractions of trees. We let $\mathcal{RT}$ denote the corresponding category.

The root order $\leqslant_0$ is a partial order on the vertices of T, given by $v \leqslant_0 v'$ if the unique path from v to the root v_0 passes through v'. In particular, the root is the maximal vertex with respect to $\leqslant_0$.

Remark 2.10. For a contraction, $f\colon (T, v) \to (T', v')$, it is trivial but useful to observe that

1. $v \leqslant_0 w \in T$ implies that $f(v) \leqslant_0 f(w) \in T'$ and
2. $v' \leqslant_0 w' \in T'$ implies that $\max_{\leqslant_0} f^{-1}(v') \leqslant_0 \max_{\leqslant_0} f^{-1}(w')$.

The category $\mathcal{RT}$ still contains nontrivial endomorphisms.

Definition 2.11. Let (T, v_0) be a rooted tree. For a vertex $v \in V(T)$, let $\mathrm{in}(v)$ be the set of all edges at v except the one in the unique path from v to the root v_0.

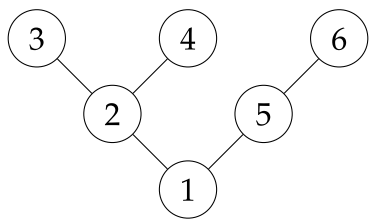
A planar rooted tree, $(T, v_0, (\leqslant_v))$, consists of a rooted tree and of a total-order $\leqslant_v$ on the set $\mathrm{in}(v)$, for each vertex v. Generally, we suppress the orders $(\leqslant_v)$ from the notation.

The corresponding depth-first order $\leqslant_T$ is the unique refinement of the reversed root order $\geqslant_0$ that respects the order $\leqslant_v$ for each $v \in V(T)$.

A contraction of planar rooted trees is a contraction of the corresponding rooted trees, $f\colon (T, v_0) \to (T', v_0')$, such that $\min_{\leqslant_T} f^{-1}(v') <_T \min_{\leqslant_T} f^{-1}(w')$, whenever $v' <_{T'} w'$. Let $\mathcal{PT}$ denote the corresponding category.

Visualising these objects greatly clarifies the definitions.

Example 2.12. When drawing a rooted tree, we put the root at the bottom. When dealing with a planar rooted tree, we draw the edges in $\mathrm{in}(v)$ from left to right with respect to the order $\leqslant_v$. In the picture below, the vertices are numbered with respect to the depth-first order.



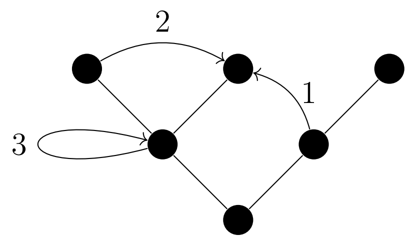
Since a contraction is a surjective map, it follows that any endomorphisms in $\mathcal{PT}$ is bijective. As it must also preserve the depth-first order, the only possible map is the identity. Hence, the opposite category $\mathcal{PT}^{\mathrm{op}}$ has no nontrivial endomorphisms either. We move back to the general case of graphs of genus g. To achieve the same result, we consider them as trees with g extra edges.

Definition 2.13. Let G be a graph of genus g. A spanning (planar rooted) tree in G, is a CW-subcomplex of G that is a (planar rooted) tree containing all vertices of G.

A rigidified graph, (G, T, v, τ), is a graph G of genus g along with a choice of a spanning planar rooted tree (T, v) in it and a total order and orientation τ of the g edges in $E(G) \setminus E(T)$.

A contraction of rigidified graphs is a contraction of the corresponding graphs that restricts to a contraction of the corresponding spanning planar rooted trees and preserves the order and orientation of the g extra edges. Let $\mathcal{PG}_g$ denote the corresponding category.

Example 2.14. For a rigidified graph, we draw the spanning planar rooted tree following the conventions in 2.12 and enumerate and orient the g extra edges according to τ. We do not emphasize the depth-first order anymore, since it is intrinsic in the drawing. Below we depict an example of an element in $\mathcal{PG}_3$.



Remark 2.15. Consider a contraction of rigidified graphs. In particular, note that none of the g extra edges can get contracted, since this would not lead to a contraction at the level of spanning trees (see Example 2.4). As the contraction preserves the order and orientation of the extra edges, it is completely defined by its restriction to the spanning tree.

Like in the case of trees, an endomorphism in $\mathcal{PG}_g$ is a bijective map. By our analysis on $\mathcal{PT}$, it must restrict to the identity on the spanning tree. By Remark 2.15, it is therefore the identity on the whole graph. Hence, $\mathcal{PG}_g^{\mathrm{op}}$ has no nontrivial endomorphisms.

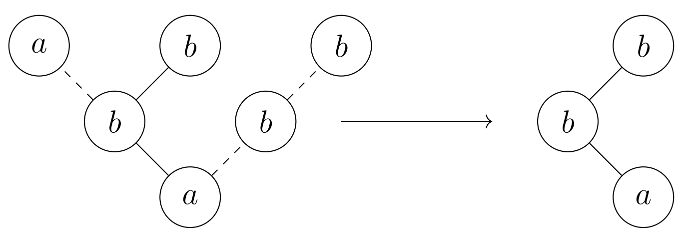
To conclude this section, we introduce one more type of structure on a graph. We will need this in the next section to be able to reduce contractions of graphs of any genus to contractions of trees.

Definition 2.16. Let (T, v) be a planar rooted tree and let S be a finite set. An S-labelling of T is a map $\ell \colon V(T) \to S$. An S-labeled planar rooted tree is a triple (T, v, ℓ).

A contraction of S-labeled planar rooted trees, $f \colon (T, v, \ell) \to (T', v', \ell')$, is a contraction of planar rooted trees such that $l'(w') = l(\max_{\leqslant_0} \phi^{-1}(w'))$. Let $\mathcal{PT}_S$ denote the corresponding category.

Remark 2.17. A vertex $w \in V(T)$ is called f-maximal if it satisfies $u \leqslant_0 w$ whenever $f(u) = f(w)$. It follows that f is a contraction of S-labelled planar rooted trees if and only if $l'(f(w)) = l(w)$ for each f-maximal vertex w of T.

Example 2.18. Let $S = \{a, b\}$. See below an example of a contraction of S-labelled.



As we added more structure to planar rooted trees, the category $\mathcal{PT}_S^{\mathrm{op}}$ does not contain any nontrivial endomorphisms. We are all set to apply the theory of chapter 1.

2.2 Applying the Gröbner method

This section is dedicated to the proof of the following theorem.

Theorem 2.19. *The category $\mathcal{G}_g^{\mathrm{op}}$ is quasi-Gröbner*

We follow the method explained in chapter 1 and start by considering $\mathcal{PG}_g^{\mathrm{op}}$. In the previous section, we already saw that this category does not have any nontrivial endomorphisms. We check the two Gröbner criteria.

Proposition 2.20. *The category $\mathcal{PG}_g^{\mathrm{op}}$ satisfies (G1).*

Proof. Fix a rigidified graph (G, T, v, τ) of genus g. Recall that the task is to supply the set $|{\mathcal{PG}_g^{\mathrm{op}}}_{(G,T,v,\tau)}|$, of equivalence classes of contractions with target (G, T, v, τ), with an admissible order. By Remark 2.15, such a contraction is completely determined by its restriction on the level of the spanning trees. Hence, it is enough to give an admissible order on $|{\mathcal{PT}^{\mathrm{op}}}_{(T,v)}|$.

So let $f\colon T_f \to T$ and $g\colon T_g \to T$ be contractions. First of all, we set $f \prec g$ if T_f has fewer edges than T_g. Note there are only finitely many trees with fixed number of edges and finitely many contractions between two fixed graphs. Hence, in an infinite decreasing sequence of contractions the number of edges of the source must eventually decrease to that of T itself. This guarantees that the order is well-founded with global minimum id_T.

The next case is that T_f and T_g have the same number of edges but are not equal. Since these graphs can not be part of the same composition of maps, the order we choose does not really matter. To make some choice, take the first vertex v_1 after the root, with respect to the depth-first order $\leqslant_T$. Consider the subtree above v_1, that is the full subtree on the vertices v satisfying $v \leqslant_0 v_1$. If these subtrees of T_f and T_g have the same number of edges again, proceed with the next vertex v_2 with respect to the depth-first order and so on. This procedure must eventually lead to a choice.

The last case to consider is $T_f = T_g$. Here, the choice of the order does matter to make it admissible. We consider the vertices v of T following the depth-first order $\leqslant_T$ in T. If the $\leqslant_{T_f}$-minimal vertex in $f^{-1}(v)$ is smaller than the one in $g^{-1}(v)$, then we stipulate that $f \prec g$. If they happen to be equal, we consider the next vertex in the $\leqslant_T$ order and apply the same convention. This eventually leads to a choice and it is a direct consequence of the construction that $\preccurlyeq$ is compatible with precomposition of any contraction $h\colon T' \to T_f$. $\square$

To check condition (G2), we want to pass again from general graphs of genus g to trees. Therefore, we use the trick mentioned at the end of the last section, namely to translate the extra information into a labelling as defined in 2.16.

Definition 2.21. Let (G, T, v, τ) be a rigidified graph of genus g and set $S_g = \{0, 1\}^{2g}$. For each $i \in \{1, \ldots, g\}$, let w_{2i-1} and w_{2i} be the vertices at which the i'th extra oriented edge respectively originates and terminates. The S-labelling on (T, v) corresponding to the order and orientation τ is defined by

$$\begin{aligned}\ell_\tau \colon V(T) \to S \colon w \mapsto \left(\ell_\tau^j(w)\right)_{j\in\{1,\dots,2g\}}, \text{ where}\\ \ell_\tau^j \colon V(T) \to \{0,1\} \colon w \mapsto \begin{cases} 1 & \text{if } w \geqslant_0 w_j, \\ 0 & \text{else.}\end{cases}\end{aligned}$$

Lemma 2.22. *The assignment*

$$\Phi \colon \mathcal{PG}_g \to \mathcal{PT}_{S_g} \colon (G,T,v,\tau) \mapsto (T,v,\ell_\tau)$$

that sends a morphisms $f\colon G \to G'$ to $f\restriction_T \colon T \to T'$, is a fully faithful functor.

Proof. First, let us check that Φ is well-defined. Functoriality of the restriction operation is clear. It remains to see that for any contraction of rigidified graphs, $f\colon G \to G' \in \mathcal{PG}_g$, the restriction $f\restriction_T$ preserves the S-labeling. Notice that f preserves the order and orientation τ if and only if $f(w_j) = w'_j$ for all $j \in \{1,\dots,2g\}$.

Let w be an f-maximal vertex. If $w \geqslant_0 w_j$, then $f(w) \geqslant_0 f(w_j) = w'_j$, by Remark 2.10(1). Conversely, if $f(w) \geqslant_0 f(w_j)$, let $\tilde{w}$ be the unique f-maximal vertex in $f^{-1}(w'_j)$ such that $\tilde{w} \geqslant_0 w_j$. By Remark 2.10(2), it follows that $w \geqslant_0 \tilde{w} \geqslant_0 w_j$, implying that $\ell_\tau^j(w) = 1$. Therefore, it also holds that $\ell_{\tau'}^j(f(w)) = 1$ and thus $\ell_\tau(w) = \ell_{\tau'}(f(w))$.

Remark 2.15 immediately implies faithfulness of Φ and tells us there is only one way to enlarge a morphism of planar rooted trees to one of the full graphs. Since $w_j \geqslant_0 w_j$, it follows that $\ell_{\tau'}^j(f(w_j)) = 1$, which means that $f(w_j) \geqslant_0 w'_j$. Again, let $\tilde{w}$ be an f-maximal vertex in $f^{-1}(w'_j)$. It follows that $1 = \ell_{\tau'}^j(f(\tilde{w})) = \ell_\tau^j(\tilde{w})$, which implies that $\tilde{w} \geqslant_0 w_j$ and hence $w'_j = f(\tilde{w}) \geqslant_0 f(w_j)$. We conclude that $f(w_j) = w'_j$ for all $j \in \{1,\dots,2g\}$, which shows that f is indeed a contraction of rigidified graphs. This proves fullness of F. $\square$

We take further advantage of the notion of labellings to also encode the codomain of a contraction.

Definition 2.23. Let S be a finite set and let $f\colon (T',v',\ell') \to (T,v,\ell)$ be a contraction of S-labelled planar rooted trees. Let $U_{(S,T)}$ denote the set $S \times (V(T) \cup \{0\})$. The $U_{(S,T)}$-labelling on T' corresponding to f is given by

$$\ell_f \colon T' \to U_{(S,T)} \colon w' \mapsto \begin{cases} (\ell'(w'), f(w')) & \text{if } w' \text{ is } f\text{-maximal}, \\ (\ell'(w'), 0) & \text{else.}\end{cases}$$

For a finite set U, we define the preorder $\leqslant_U$ on $\mathcal{PT}_U^{\mathrm{op}}$ by $(T,v,\ell) \geqslant_U (T',v',\ell')$ if $\mathrm{Hom}_{\mathcal{PT}_U}((T,v,\ell),(T',v',\ell')) \neq \emptyset$.

We consider the canonical order of composition $\leqslant$ (see Definition 1.32) on $\left|\left(\mathcal{PT}_S^{\mathrm{op}}\right)_{(T,v,\ell)}\right|$ and compare it to the preorder $\leqslant_{U_{(S,T)}}$ on $\mathcal{PT}^{\mathrm{op}}_{U_{(S,T)}}$.

Lemma 2.24. *Let S be a finite set and let (T, v, ℓ) be an S-labelled planar rooted tree. The map*

$$F\colon \left|\left(\mathcal{PT}_S^{\mathrm{op}}\right)_{(T,v,\ell)}\right| \to \mathcal{PT}^{\mathrm{op}}_{U_{(S,T)}} \colon \Big(f\colon (T', v', \ell') \to (T, v, \ell)\Big) \mapsto (T', v', \ell_f)$$

is an order embedding with respect to the orders mentioned above.

Proof. First, let $f_i\colon (T_i, v_i, \ell_i) \to (T, v, \ell)$ for $i \in \{1, 2\}$ be contractions of S-labelled planar rooted trees such that $f_1 \leqslant f_2$. This means that there is a contraction $g\colon (T_2, v_2, \ell_2) \to (T_1, v_1, \ell_1) \in \mathcal{PT}_S$, satisfying $f_2 = f_1 \circ g$. We want to show that g also induces a contraction in $\mathcal{PT}_{U_{(S,T)}}$. Hence, let w be an g-maximal vertex in T_2 and consider its label $l_{f_2}(w) = (l_2(w), x)$. Since g is a contraction in $\mathcal{PT}_S$, we already know that $l_2(w) = l_1(g(w))$. As for the second component of the labelling, what is left to show is that (for w, which is g-maximal)

$$w \text{ is } f_2\text{-maximal if and only if } g(w) \text{ is } f_1\text{-maximal.} \tag{$\star$}$$

Suppose that w is f_2-maximal and take a vertex v' in T_1 satisfying $f_1(v') = f_1 \circ g(w)$. Any of the vertices $w' \in g^{-1}(v')$ satisfies $f_2(w') = f_1(v') = f_2(w)$, implying that $w' \leqslant_0 w$. In particular, this implies that $v' \leqslant_0 g(w)$ and as v' was arbitrary that $g(w)$ is f_1-maximal. Conversely, assuming that $g(w)$ is f_1-maximal, it follows that $v' \leqslant_0 g(w)$ for all $v' \in f_1^{-1}(f_2(w))$. Since each $w' \in f_2^{-1}(f_2(w))$ gets mapped to such a v' by g and w is g-maximal, we conclude that $w' \leqslant_0 w$.

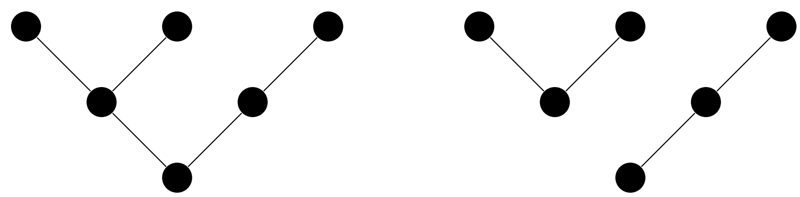
The other way around, suppose that there is a contraction $g\colon (T_2, v_2, \ell_{f_2}) \to (T_1, v_1, \ell_{f_1}) \in \mathcal{PT}_{U_{(S,T)}}$. By considering the first component of the labelling, g is also a contraction in $\mathcal{PT}_S$. Moreover, by the second component, $f_2(w) = f_1 \circ g(w)$ for all w in T_2 that are g and f_2-maximal. But by $(\star)$, (T_1 being a fixed graph) g is automatically fixed on the rest of T_2 and this yields $f_2 = f_1 \circ g$ everywhere. □

After these identifications, we are finally able to work towards (G2). The following lemma plays the role of Higman's Lemma 1.44 in the case of finite sets. The proof we present is based on the ones of Lemma 10 in [Bar15] and Theorem 1.2 in [Dra14]. In fact, the argument was originally introduced in [NW65].

Lemma 2.25 (Kruskall's tree Theorem)**.** *Let U be a finite set. The poset $(\mathcal{PT}_U^{\mathrm{op}}, \leqslant_U)$ is Noetherian.*

Proof. For convenience, we shorten the notation of an U-labelled planar rooted tree, from (T, v, ℓ) to T, in this proof. As in the proof of Higman's Lemma 1.44, an infinite sequence $T_1, T_2, \ldots$ is called "bad" if $T_i \not\leqslant T_j$ for all $i < j$. We need to show there exist no bad sequence.

If there exists a bad sequence, then, since S is finite, there exists an infinite subsequence in which the labels of the roots $\ell_i(v_i) \in U$ are all equal. Let $\mathbb{T} = T_1, T_2, \ldots \in \mathcal{PT}_U^{\mathrm{op}}$ be the infinite bad sequence with equal labeling of the root that is minimal in the following sense. Among all bad sequences starting with $T_1, \ldots, T_{i-1}$, the tree T_i in $\mathbb{T}$ has a minimal number of vertices. For each tree T_i, let v_i^1 be the first neighbour of the root v_i with respect to the depth-first order. Let (T_i^1, v_i^1, ℓ_i) be the first branch of T. That is the full planar subtree above v_i^1, meaning that $v \in T_i^1$ if and only if $v \leqslant_0 v_i^1$. Let (T_i^2, v_i, ℓ_i) be the remaining part of T_i after removing T_i^1 and the edge between v_i and v_i^1.



The tree T_i on the left decomposes as the trees T_i^1 and T_i^2 on the right.

Next, let $\boldsymbol{T}^2$ be the set $\{T_i^2 \mid i \in \mathbb{N}\}$. We claim that $(\boldsymbol{T}^2, \leqslant_U)$ is Noetherian. If not, it contains a bad sequence and therefore also a bad (sub)sequence, $T_{i_1}^2, T_{i_2}^2, \ldots$, with increasing labels $i_j < i_l$ for $j < l$. Consider the sequence

$$\mathbb{T}^2 = T_1,\ \ldots,\ T_{i_1-1},\ T_{i_1}^2,\ T_{i_2}^2,\ \ldots \in \mathcal{PT}_U^{\mathrm{op}}.$$

By assumption, there cannot be any increment for $i, j \in \{i_1, i_2, \ldots\}$. There cannot be any increment for $i, j \in \{1, \ldots, i_1 - 1\}$ either, because the sequence $\mathbb{T}$ is bad. The last case is $T_q \leqslant_U T_{i_l}^2$ for $q \in \{1, \ldots, i_1 - 1\}$ and any l. This means that there is a contraction $f\colon T_{i_l}^2 \to T_q$. However, there also is a natural contraction $T_{i_l} \to T_{i_l}^2$ given by contracting the first branch to the root. The composition of these contractions shows that $T_q \leqslant_U T_{i_l}$. As $q \leqslant i_1 \leqslant i_l$, this contradicts the fact that $\mathbb{T}$ is bad. We conclude that $\mathbb{T}^2$ is bad. However, as $T_{i_1}^2$ has fewer vertices than T_{i_1} this contradicts the minimality of $\mathbb{T}$. Therefore, $(\boldsymbol{T}^2, \leqslant_U)$ is indeed Noetherian.

In particular, this means that there exists a sequence $T_{j_1}^2, \leqslant_U T_{j_1}^2 \leqslant_U \cdots$, where $j_1 < j_2 < \cdots$. Consider the corresponding sequence $T_{j_1}^1, T_{j_2}^1, \ldots$ of first branches. We play the same game again. If it is bad, we could consider the sequence

$$\mathbb{T}^1 = T_1,\ \ldots,\ T_{j_1-1},\ T_{j_1}^1,\ T_{j_2}^1,\ \ldots \in \mathcal{PT}_U^{\mathrm{op}}.$$

As before, there cannot be any increment in the first or last part of the sequence by assumption. Suppose there is a contraction $f\colon T_{i_l}^1 \to T_q$. In particular, notice that this means that the label of v_i^1 is equal to label of v_q and therefore to that of the roots of all trees in $\mathbb{T}$ by assumption. Therefore, there is a natural contraction $T_{i_l} \to T_{i_l}^1$, contracting all of $T_{i_l}^2$ and the edge (v_i, v_i^1). Composed with f this witnesses $T_q \leqslant_U T_{i_l}$. Therefore, $\mathbb{T}^1$ is a bad sequence, contradicting again the minimality of $\mathbb{T}$.

Finally, we conclude that there is $j_l < j_m$ such that $T_{j_l}^1 \leqslant_U T_{j_m}^1$. As we also had $T_{j_l}^2 \leqslant_U T_{j_m}^2$, we can combine the two contractions to get a contraction of the full graphs witnessing $T_{j_l} \leqslant_U T_{j_m}$. This contradicts the fact that $\mathbb{T}$ was bad. We conclude that there cannot be any bad sequence at all. □

We are ready to get back to the Gröbner method.

Theorem 2.26. *The category $\mathcal{PG}_g^{\mathrm{op}}$ is Gröbner.*

Proof. We already saw that $\mathcal{PG}_g^{\mathrm{op}}$ has no nontrivial endomorphisms and satisfies (G1) (Proposition 2.20). Now for (G2), fix a rigidified graph (G,T,v,τ) of genus g. Suppose that there exist contractions $f_i\colon (G_i,T_i,v_i,\tau_i)\to(G,T,v,\tau)\in|\mathcal{PG}_g^{\mathrm{op}}{}_{(G,T,v,\tau)}|$, forming a bad sequence $f_1,f_2\ldots$, with respect to the canonical order. By Lemma 2.22, passing the restrictions of each f_i to the spanning tree yields a bad sequence in $|\mathcal{PT}^{\mathrm{op}}_{S_g(T,v,\ell_\tau)}|$. By Lemma 2.24, this leads to a bad sequence $(T_1,v_1,\ell_{f_1}),(T_2,v_2,\ell_{f_2}),\ldots\in\left(\mathcal{PT}^{\mathrm{op}}_{U_{(S,T)}}\right)$, by passing through the map F. However, such sequence does not exist by Lemma 2.25. We conclude that $\mathcal{PG}_g^{\mathrm{op}}$ satisfies (G2) and therefore that it is Gröbner. $\square$

The last step is to reintroduce the nontrivial endomorphisms in $\mathcal{G}_g^{\mathrm{op}}$.

Proof Theorem 2.19. We consider the forgetful functor $\Phi\colon\mathcal{PG}_g^{\mathrm{op}}\to\mathcal{G}_g^{\mathrm{op}}$, which is essentially surjective. To check property (F), fix a graph G of genus g. Consider all possible contractions, $f_i\colon G_i\to G$, from all planar rooted trees (G_i,T_i,v_i,τ_i) satisfying $|E(G_i)|\leqslant|E(G)|+g$. Notice that there are only finitely many such graphs, finitely many ways to turn them into planar rooted trees and finitely many isomorphism classes of contractions between two fixed graphs.

Consider any rigidified graph (G',T',v',τ') of genus g and any contraction $f\colon G'\to G\in\mathcal{G}_g$. To translate this to a contraction in $\mathcal{PG}_g$, one must be careful that f might be contracting some of the extra edges (the ones in $G'\setminus T'$). Hence, let $g\colon(G',T',v',\tau')\to(G_i,T_i,v_i,\tau_i)$ be the restriction of f to T' that acts like the identity on the g extra edges of G'. In particular, this means that G_i has at most g more edges than G (the extra edges that were not contracted). Therefore, f factors as $G'\xrightarrow{\Phi(g_i)}G_i\xrightarrow{f_i}G$, where f_i is one of the maps that we fixed in advance. We conclude that $\mathcal{G}_g^{\mathrm{op}}$ is Gröbner. $\square$

2.3 Smallness and polynomial growth

We now know that the category $\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is Noetherian for any left-Noetherian ring k. In this section, we want to translate the results of section 1.4 to the graph categories.

Definition 2.27. Let M be a $\mathcal{G}_g^{\mathrm{op}}$-module.

- A filtration of M is a sequence of submodules $\{0\}=M_m\subseteq M_{m-1}\subseteq\cdots\subseteq M_0=M$.
- The associated graded module to such filtration is $\mathrm{gr}(M)=\bigoplus_{j=0}^{m-1}M_j/M_{j+1}$.

We characterise the "size" of a $\mathcal{G}_g^{\mathrm{op}}$-module on different levels.

Definition 2.28. Let M be a $\mathcal{G}_g^{\mathrm{op}}$-module.

- We say that M is finitely generated in degree $\leqslant d$ if there exists an epimorphism $\bigoplus_{i=1}^n\mathrm{P}_{G_i}\to M\in\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$, where each G_i has at most d edges.

- We say that M is d-small if it is a subquotient of a module that is finitely generated in degree $\leqslant d$.
- We say that M is d-smallish if there is a filtration $\{0\} \subseteq M_{m-1} \subseteq \cdots \subseteq M$ such that the associated graded module $\mathrm{gr}(M)$ is d-small.

Notice that the notion of d-smallishness is weaker than that of d-smallness (trivial filtration $\{0\} \subseteq M$), which is again weaker than that of being finitely generated in degree $\leqslant d$. However, it is still strong enough for the following.

Proposition 2.29. *A d-smallish module is finitely generated.*

Proof. Firstly, consider a d-small module M. This means that M is a subquotient of a finitely generated (in degree $\leqslant d$) module N. Any of the quotient modules of N is finitely generated. As M is a submodule of such quotient and $\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is Noetherian by Theorem 2.19, it follows that M is finitely generated as well.

Next, consider any d-smallish module M. Let $M_m \subseteq \cdots \subseteq M_1 \subseteq M$ be a filtration, such that $\mathrm{gr}(M)$ is d-small. By the above, $\mathrm{gr}(M)$ is finitely generated. Since $\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is Noetherian, the submodules M_{m-1} and $M_{m-2}/M_{m-1} \subseteq \mathrm{gr}(M)$ are finitely generated. Hence, so is the module $M_{m-2} \cong M_{m-1} \oplus M_{m-2}/M_{m-1}$. Repeating this argument $m-2$ times along the filtration, we conclude that $M = M_0 = M_1 \oplus M_1/M_0$ is finitely generated as well. □

The extra information we get from specifying d is that we can bound the "growth" of the module.

Proposition 2.30. *Let k be a field and M a d-smallish $\mathcal{G}_g^{\mathrm{op}}$-module, then there exists a polynomial $f_M(t) \in \mathbb{Z}[t]$ of degree at most d, such that $\dim_k M(G) \leqslant f_M(|E(G)|)$ for all $G \in \mathcal{G}_g$.*

Remark 2.31. Notice that the assumption that k is a field is needed to ensure that all finitely generated modules are in fact free and it makes sense to consider $\dim_k M(G)$.

Proof. First, we consider the case of a principal projective module $\mathrm{P}_{G'}$ corresponding to a graph G' with d edges. Up to automorphisms of G', a contraction $f\colon G \to G'$ is fixed by the choice of $|E(G)| - d$ edges to contract. Of course not all choices are allowed (see Example 2.4). It follows that

$$\dim_k \mathrm{P}_{G'}(G) = |\mathrm{Hom}_{\mathcal{G}_g}(G, G')| \leqslant |\operatorname{Aut}_{\mathcal{G}_g}(G')| \binom{|E(G)|}{d}.$$

Observe that for fixed d, the binomial coefficient $\binom{t}{d}$ is a polynomial of degree d. Hence, in this case the statement holds for $f_{\mathrm{P}_{G'}}(t) = |\operatorname{Aut}_{\mathcal{G}_g}(G')|\binom{t}{d}$.

If N is a quotient module of M, then $\dim_k N(G) \leqslant \dim_k M(G)$. Moreover, $\dim_k(M_1 \oplus M_2(G)) = \dim_k M_1(G) + \dim_k M_2(G)$. Therefore, the statement also holds for all modules that are finitely generated in degree $\leqslant d$. If N is a subquotient of M, then $\dim_k N(G) \leqslant \dim_k M(G)$ yielding the result for all d-small modules. Finally, for d-smallish modules, notice that $\dim_k(\mathrm{gr}(M)(G)) = \dim_k(M(G))$. □

The theorem above only gives an inequality. We cite Proudfoot and Ramos [PR19b]: *"We cannot possibly expect equality, since the dimension of $M(G)$ usually depends on the structure of G, not just on the number of edges."* Therefore, we introduce two ways to create new graphs out of a fixed one, while preserving much of the structure.

Definition 2.32. Let G be a graph.

- Let $e_1, \dots, e_r$ be distinct oriented non-loop edges of G and set $\underline{e} = (e_1, \dots, e_r)$. Further, let $\underline{m} = (m_1, \dots, m_r)$ be a tuple of non-negative integers. The graph $G(\underline{e}, \underline{m})$ is obtained by subdividing edge e_i in m_i new edges. In particular, $m_i = 0$ means contracting the edge e_i. The new vertices on e_i are labelled $v_i^0, \dots, v_i^{m_i}$ in the order of the orientation.
- Let $v_1, \dots, v_r$ be distinct vertices of G and set $\underline{v} = (v_1, \dots, v_r)$. Further, let $\underline{m} = (m_1, \dots, m_r)$ be a tuple of non-negative integers. The graph $G(\underline{v}, \underline{m})$ is obtained by attaching m_i new edges (with a leaf) to v_i. This is called sprouting m_i leaves at v_i. These leaves are labelled $v_i^1, \dots, v_i^{m_i}$ (in any order).

Example 2.33. A graph is a tree if and only if it can be obtained out of the graph $\bullet$ (unique vertex, no edges) by repeatedly sprouting vertices and subdividing edges.

Recall the category **OI** from Definition 1.39. We already mentioned that any of its objects is isomorphic to one of the posets $[n] = \{1, \dots, n\}$, where $n \in \mathbb{N}_0$. Similarly, for $\underline{m} = (m_1, \dots, m_r) \in \mathbb{N}_0^r$, let $[\underline{m}]$ be the tuple of posets $([m_1], \dots, [m_n])$. It follows that any object in the r-product category $\mathbf{OI}^r$ is isomorphic to some $[\underline{m}]$.

Definition 2.34. Fix a graph G of genus g.

- For a tuple $\underline{e}$ of (distinct, oriented, non-loop) edges in G, the corresponding subdivision functor is defined on objects by $\Phi_{G,\underline{e}} \colon \mathbf{OI}^r \to \mathcal{G}_g^{\mathrm{op}} \colon [\underline{m}] \mapsto G(\underline{e}, \underline{m})$. It maps the morphism $f = (f_i)_{i \in \{1,\dots,r\}} \colon [\underline{m}] \to [\underline{n}]$ to

$$\Phi_{G,\underline{e}}(f) \colon G(\underline{e}, \underline{n}) \to G(\underline{e}, \underline{m}) \colon v \mapsto \begin{cases} v_i^s & \text{if } v = v_i^t, \\ v & \text{else,} \end{cases}$$

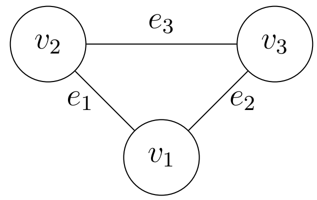
  where $s = \max\{j \mid f_i(j) \leqslant t\}$ under the convention that $\max \emptyset = 0$.

- For a tuple $\underline{v}$ of vertices in G, the corresponding sprouting functor is defined on objects by $\Phi_{G,\underline{v}} \colon \mathbf{OI}^r \to \mathcal{G}_g^{\mathrm{op}} \colon [\underline{m}] \mapsto G(\underline{v}, \underline{m})$. It maps the morphism $f = (f_i)_{i \in \{1,\dots,r\}} \colon [\underline{m}] \to [\underline{n}]$ to

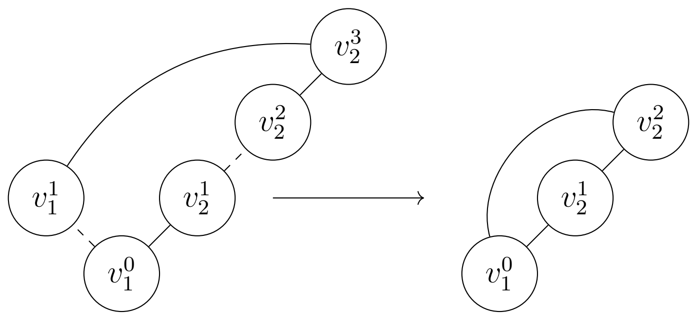
$$\Phi_{G,\underline{v}}(f) \colon G(\underline{v}, \underline{n}) \to G(\underline{v}, \underline{m}) \colon v \mapsto \begin{cases} v & \text{if } v \text{ is a vertex of } G, \\ v_i^s & \text{if } v = v_i^t \text{ and } f_i(s) = t, \\ v_i & \text{if } v = v_i^t \text{ and } f_i^{-1}(t) = \emptyset. \end{cases}$$

We believe that an example suffices to see that the above assignments are indeed well-defined functors.

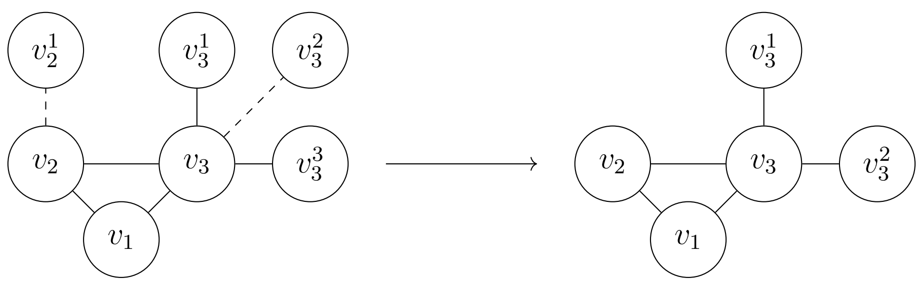
Example 2.35. Let G be the genus 1 graph depicted below.



Set $\underline{n} = ([0], [2])$, $\underline{m} = ([1], [3])$. Moreover, let $f = (f_1, f_2)$, where f_1 is the empty map $[0] = \emptyset \to [1]$ and f_2 is the ordered injection $[2] \to [3]\colon 1 \mapsto 1, 2 \mapsto 3$. Set $\underline{e} = (e_1, e_2)$, where both edges are oriented upwards. The contraction $\Phi_{G,\underline{e}}(f)$ is depicted below.



Set $\underline{v} = (v_2, v_3)$. The contraction $\Phi_{G,\underline{e}}(f)$ is depicted below.



These functors fit in our framework because they preserve finite generation.

Lemma 2.36. *For any graph G and any choice of*

- *edges $\underline{e}$, the subdivision functor $\Phi_{G,\underline{e}}$ satisfies property (F).*
- *vertices $\underline{v}$, the sprouting functor $\Phi_{G,\underline{v}}$ satisfies property (F).*

Proof. Fix $G' \in \mathcal{G}_g$. For subdivision, consider first the finitely many $\underline{m} \in \mathbb{N}_0^r$ for which $|\underline{m}| = \sum_{i=1}^r m_i$ is at most $|E(G')| + r$. Let $f_i\colon G(\underline{e}, \underline{m_i}) \to G'$ be the finitely many representatives of all isomorphism classes of contractions of such $\underline{m}$. Any contraction $f\colon G(\underline{e}, \underline{m}) \to G'$ contracts $|E(G(\underline{e}, \underline{m}))| - |E(G')| = |E(G)| + |\underline{m}| - r - |E(G')|$ edges. Hence, at least $|\underline{m}| - r - |E(G')|$ of these contracted edges are subdivided ones. Contracting first only those subdivided edges and afterwards the others, yields a decomposition $G(\underline{e}, \underline{m}) \xrightarrow{\Phi_{G,\underline{e}}(g)} G(\underline{e}, \underline{m_i}) \xrightarrow{f_i} G'$, for some $g\colon \underline{m_i} \to \underline{m} \in \mathbf{OI}^r$ and some f_i as above.

For sprouting, the argument is similar. In this case, we only need to take representatives of maps from $G(\underline{v}, \underline{m})$, such that $|\underline{m}| \leqslant |E(G')|$. Any map will factor through one of these, because $|E(G(\underline{v}, \underline{m}))| = |E(G)| + |\underline{m}|$. $\square$

Subdivision and sprouting preserve enough of the structure of the original graph to turn the upper bound of 2.30 into an equality.

Theorem 2.37. *Let k be a field, let M be a d-smallish $\mathcal{G}_g^{\mathrm{op}}$-module and let G be a graph of genus g.*

- *For each tuple of (distinct, oriented, non-loop) edges $\underline{e}$, there exists a polynomial $f_{M,G,\underline{e}} \in \mathbb{Z}[t_1, \ldots, t_r]$ of total degree at most d, such that $\dim_k M(G(\underline{e}, \underline{m})) = f_{M,G,\underline{e}}(m_1, \ldots, m_r)$ when $\underline{m}$ is sufficiently large in each coordinate.*
- *For each tuple of vertices $\underline{v}$, there exists a polynomial $f_{M,G,\underline{v}} \in \mathbb{Z}[t_1, \ldots, t_r]$ of total degree at most d, such that $\dim_k M(G(\underline{v}, \underline{m})) = f_{M,G,\underline{v}}(m_1, \ldots, m_r)$ when $\underline{m}$ is sufficiently large in each coordinate.*

Proof. We prove the statement for subdivision, the argument for sprouting is identical. By Proposition 2.29, M is finitely generated and so by Lemma 2.36, $\Phi^*_{G,\underline{e}}(M) \in \mathrm{Rep}_k(\mathbf{OI}^r)$ is also finitely generated (see Proposition 1.17). By Proposition 1.54, there exists a polynomial $f_{M,G,\underline{e}} \in \mathbb{Z}[t_1, \ldots, t_r]$ such that

$$\dim_k M(G(\underline{e}, \underline{m})) = \dim_k \Phi^*_{G,\underline{e}}(M)([\underline{m}]) = f_{M,G,\underline{e}}(m_1, \ldots, m_r),$$

when $\underline{m}$ is sufficiently large in each coordinate. By Proposition 2.30, $\dim_k M(G(\underline{e}, \underline{m}))$ is also bounded above by a polynomial of degree d in the variable $|E(G(\underline{e}, \underline{m}))| = |E(G)| - r + |\underline{m}|$. Since G and r are fixed, it follows that the total degree of $f_{M,G,\underline{e}}$ is at most d. $\square$

Chapter 3

Homology of the configuration space

The first chapter treated the general theory of modules over categories. The second chapter focused on all modules over the specific category $\mathcal{G}_g^{\text{op}}$. In this third chapter, we specify even further to one type of modules over this category, following [PR19a] and [PR19a]. In section 3.1, we define the topological spaces associated to a graph that describe the movements of a fixed number of particles over it. Afterwards, we consider the homology of these so-called configuration space. However, we realise that the assignment of a configuration space to its graph does not seem functorial with respect to contractions.

Świątkowski introduced a cubical complex that is a deformation retract of the configuration space of a graph. In section 3.2, we introduce the corresponding reduced Świątkowski chain complex, following the treatment of An, Drummond-Cole, and Knudsen in [ADCK17]. As the complex turns out to be functorial with respect to contractions, this enables us to conclude that the assignment of a homology group of a configuration space to a graph is a $\mathcal{G}_g^{\text{op}}$-module after all. Section 3.3 consists of some concrete computations, which illustrate how the complex offers an algorithmic tool to compute these homology groups. Moreover, we see how torsion can appear in the first homology group.

In section 3.4, we show that the previously introduced modules are in fact d-small, where d is the sum of the order of the homology group, the number of particles and the genus of the graphs considered. By the results in section 1.4, this implies that under sprouting the ranks of the homology groups grow polynomially. Moreover, it demonstrates the existence of a bound on the exponent of the torsion appearing in the homology groups of any graph of fixed genus.

3.1 Configuration spaces

Recall that the graphs that we consider are connected topological spaces.

Definition 3.1. Let X be a topological space and n a positive integer.

- The n-stranded ordered configuration space, $\mathrm{Conf}_n(X)$, is the set $\{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}$ endowed with the subspace topology of the product space X^n.
- The n-stranded unordered configuration space, $\mathrm{UConf}_n(X)$, is the orbit space $\mathrm{Conf}_n(X)/S_n$ of the permutation action $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.
- We adopt the convention that $\mathrm{Conf}_0(X) = \mathrm{UConf}_0(X) = \emptyset$.

Remark 3.2. Notice that an embedding of topological spaces, $i\colon X \to Y$, induces an embedding of the n-stranded unordered configuration spaces,

$$\tilde{i}\colon \mathrm{UConf}_n(X) \to \mathrm{UConf}_n(Y)\colon (x_1, \ldots, x_n) \bmod S_n \qquad \mapsto (i(x_1), \ldots, i(x_n)) \bmod S_n.$$

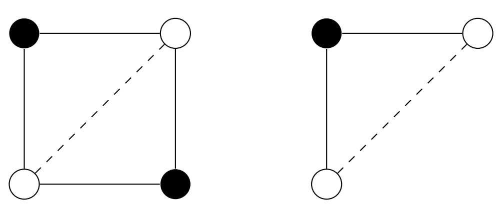
Moreover, if i is an homeomorphism so is $\tilde{i}$.

An interpretation of $\mathrm{Conf}_n(X)$ is that it models the ways to position n distinct "particles" in X without overlap. Such a positioning is what we call a configuration. A path in $\mathrm{Conf}_n(X)$ then describes a way to move the particles around without collision. In the unordered version $\mathrm{UConf}_n(X)$, we let the particles be indistinguishable. Hence, swapping the position of any two of them yields the same configuration.

In this project, we restrict our attention to the unordered configuration spaces of graphs. The master thesis of Daniel Lütgehetmann [LRV16] gives a (much more complete) introduction for the case of ordered configuration spaces.

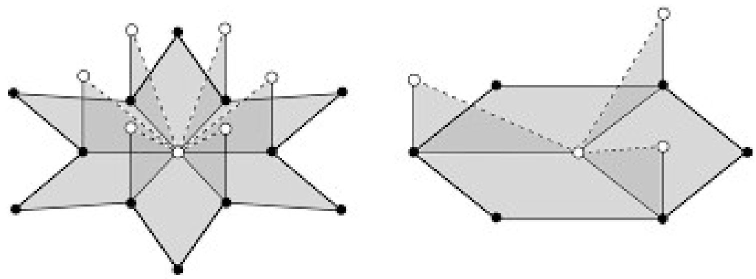
Example 3.3. Dimensions grow quickly but we can visualise a few examples.

- Let I be the unique tree consisting of two vertices. By definition, $\mathrm{Conf}_2(I)$ is the set $I^2 \setminus \{(x, x) \mid x \in I\}$ and in $\mathrm{UConf}_2(I)$ we identify all pairs $(x, y) \sim (y, x)$. The natural quotient map $\mathrm{Conf}_2(I) \to \mathrm{UConf}_2(I)$ can be visualised as a fold along the diagonal.

  Generally, $\mathrm{Conf}_n(I)$ is the n-dimensional hypercube I^n, where the $\binom{n}{2}$ hypersurfaces of dimension $n-1$ defined by $x_i = x_j$ for $i \neq j$ are removed. In $\mathrm{UConf}_n(I)$ we "fold" along all these hypersurfaces and end up with an n-simplex with some open faces.

- Consider the 3-star, that is the tree having one vertex of valence 3 and three of valence 1. According to its shape we denote it by Y. The 2-stranded configuration spaces of Y have the following shape.



This picture was copied from chapter II in [MS17]. We follow the explanation of Example 1 in [AG02]. The left image depicts $\mathrm{Conf}_2(Y)$. The hole in the center is the unallowed assignment of the two particles in the central vertex of Y. The six (solid) spokes, going out of the hole, correspond to the movement of one of the particles along one of the three edges in Y. The vertical open triangle attached to the spoke corresponds to the second (previously fixed) particle moving along the same edge as the first, but always staying strictly behind it. If we go clockwise from one of the spokes to the next, both the moving particle and the chosen edge change. The (floor) rhombus, between the two spokes, corresponds to both particles moving freely in their respective edge. Its outer point is the configuration where both particles are in the leaf of their edge.

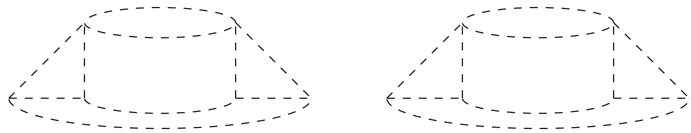
The right image depicts $\mathrm{UConf}_2(Y)$. It is obtained out of the left image by identifying each point with its reflection through a vertical line through the hole in the middle. Notice in particular that $\mathrm{UConf}_2(Y)$ is homotopic to a circle

- The most simple example of a graph that is not a tree is the graph O, consisting of a unique vertex with a loop. The configuration spaces of O can be obtained out of those of I by making the identifications $0 \sim 1$ in any coordinate. This means gluing opposite $n-1$ dimensional hypersurfaces in the n dimensional hypercube and identifying all its corners. However, we can also analyse them directly[1].

  We begin with $\mathrm{Conf}_2(O)$. Fixing the position of one particle and an orientation (say clockwise), the position of the other particle is parametrised by an open interval (say $(0,1)$). Hence, $\mathrm{Conf}_2(O)$ is a cylinder with open top and bottom. In $\mathrm{UConf}_2(O)$, any point gets identified with its image after mirroring through the point in the middle of the cylinder. This yields an open Möbius strip.

  For general $\mathrm{Conf}_n(O)$ where $n \geqslant 2$, there exist $(n-1)!$ possible circular orders of the particles $x_1, \ldots, x_n$. Since there is no way to move between configurations having different orders, $\mathrm{Conf}_n(O)$ has $(n-1)!$ connected components. We continue inside such a component, X_i. Fixing the position of the first particle, the position of the other $n-1$ particles is now parametrised by the choice of $n-1$ positive real numbers summing up to less than 1. Thus, it is given by the $(n-1)$-simplex with open sides, Δ_{n-1}. Hence, X_i is homeomorphic to $O \times \Delta_{n-1}$ and in particular it is homotopic to a circle.

[1]We found this description on https://mathoverflow.net/questions/206546.



An illustration of the space $\mathrm{Conf}_3(O)$.

> For the unordered configuration space, note that the transposition (i, j) identifies (among other things) each point in the component X_i with a point in the component X_j. In particular, it follows that $\mathrm{UConf}_n(O)$ is connected. Inside a component X_i, each point is identified with $n-1$ others, corresponding to subsequent rotations of the position of the particles in O. As the particle i is in another position each time, $\mathrm{UConf}_n(O)$ is still of the form $O \times X$ for some contractible topological space X. In particular, $\mathrm{UConf}_n(O)$ is homotopic to O itself.

To get back to the setting of the previous chapters, we would like to translate the information in these topological spaces to some kind of modules. Natural candidates are the (singular) homology groups $H_i(\mathrm{UConf}_n(G); \mathbb{Z})$ for $i \in \mathbb{N}$.

Example 3.4. Consider the 3-star Y. In Example 3.3, we saw that $\mathrm{UConf}_2(Y)$ is homotopic to a circle. Hence, we know that the only nontrivial homology group of Y are $H_0(\mathrm{UConf}_2(Y); \mathbb{Z}) \cong H_1(\mathrm{UConf}_2(Y); \mathbb{Z}) \cong \mathbb{Z}$.

By Remark 3.2, the n-stranded unordered configuration space is invariant under subdivision (Definition 2.32). Therefore, vertices having exactly two incoming edges, might as well be deleted. Moreover, looking closely at the example of Y we see that the homology depends only on the movements of particles around the central vertex. Essentially, this is because leaves are topologically indistinguishable from any point on their corresponding edge.

Definition 3.5. A vertex in a graph is called essential, if it has degree at least 3.

This allows us to treat the zeroth homology group in a global way.

Proposition 3.6. *For any (connected) graph G and any positive integer n, $\mathrm{UConf}_n(G)$ is path-connected. Hence, $H_0(\mathrm{UConf}_n(G); \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Let us first assume that G has no essential vertex. Note that the only graphs satisfying this are obtained by subdivision of the graphs I and O. In Example 3.3, we observed that $\mathrm{UConf}_n(I)$ is an n-simplex and that $\mathrm{UConf}_n(O)$ is homotopic to O. In particular, these spaces are path-connected.

On the other hand, if G does contain an essential vertex v, then it is possible to shuffle the position of the n particles around in any way, by moving them one by one to v. [2] Hence, there is a path between any two configurations in $\mathrm{Conf}_n(G)$. As $\mathrm{UConf}_n(G)$ is a quotient of this space, it is also path-connected. □

[2] To visualize this shuffling, see animation "Media, Star 4" on the website of Lütgehetmann https://userpage.fu-berlin.de/luetge/.

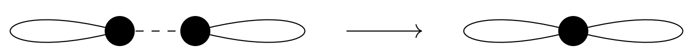
To illustrate that higher homology groups do contain information about the original graph, we mention the next result from Ki Hyoung Ko and Hyo Won Park [KP12].

Proposition 3.7. *A graph G is planar if and only if $H_1(\mathrm{UConf}_n(G);\mathbb{Z})$ is torsion-free.*

Proof. This is a consequence of Theorem 3.16 in [KP12]. □

Finally, we would like to conclude that $H_i(\mathrm{UConf}_n(\_);\mathbb{Z})$ is a $\mathcal{G}_g^{\mathrm{op}}$-module. As stated in Remark 3.2, the construction of the unordered configuration space (and hence of its homology groups) is functorial with respect to graph-embeddings. However, $\mathrm{UConf}_n(\_)$ is not functorial with respect to contractions.

Example 3.8. Let $f\colon G \to G' \in \mathcal{G}_2$ be the contraction depicted below.



There exists no topological embedding $i\colon G' \to G$. Hence, there seems to be no natural way in which f induces a map $\tilde{f}\colon \mathrm{UConf}_n(G') \to \mathrm{UConf}_n(G)$.

However, it turns out that at the level of homology, functoriality can still be achieved. To understand this, we first need to introduce a more adapted way to compute these homology groups.

3.2 The Świątkowski complex

In this section, we introduce a bigraded complex that computes the homology groups of the unordered configuration space of a graph. We begin by recalling the relevant ring theoretic notions.

Definition 3.9. Let G be a group.

- A ring R is called G-graded if there exist an R-submodule $R_g \subseteq R$ for each $g \in G$, such that, as R-modules, the relations $R = \bigoplus_{g\in G} R_g$ and $R_g \cdot R_h \subseteq R_{gh}$ hold.
- Let R be a G-graded ring. An R-module M is G-graded if there exists subgroups $M_g \subseteq M$ for all $g \in G$ such that, as Abelian groups, the relations $M = \bigoplus_{g\in G} M_g$ and $R_g \cdot M_h \subseteq M_{gh}$ hold.
- Graded means $(\mathbb{Z},+)$-graded. Likewise, bigraded stands for $(\mathbb{Z}^2,+)$-graded, where $+$ is the component-wise addition.

For any ring R and any group G, there is a trivial G-grading defined by setting $R_0 = R$ and $R_g = 0$ for all $g \neq 0 \in G$. Under this G-grading of R, any direct decomposition of a module $M = \bigoplus M_g$ is a G-grading. In what follows we use this bigrading on the ring $\mathbb{Z}$.

Definition 3.10. The homology groups of a graph G form the bigraded Abelian group

$$H_\bullet(\mathrm{UConf}_\star(G)) = \bigoplus_{(i,n)\in\mathbb{N}^2} H_i(\mathrm{UConf}_n(G)).$$

Recall how homology is algebraically introduced by means of a chain complex with differential maps ($\partial_i\partial_{i+1} = 0$),

$$(A_\bullet, \partial) = 0 \xleftarrow{\partial_0} A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \xleftarrow{\partial_3} \cdots ,$$

by setting $H_i(A_\bullet) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$ for all $i \in \mathbb{N}_0$. Moreover, recall that a morphism of chain complexes, $f\colon (A_\bullet, \partial) \to (A'_\bullet, \partial')$, is a sequence of maps $f_i\colon A_i \to B_i$ for $i \in \mathbb{N}_0$ such that $f_{i+1}\partial_i = \partial'_{i+1} f_i$.

Definition 3.11. We call a bigraded R-module $A = \bigoplus_{(i,n)} A_{i,n}$ a bigraded complex if $A_{i,n} = 0$ whenever i or n is negative and if it is accompanied by differential maps $\partial_{i,n}\colon A_{i,n} \to A_{i-1,n} \in \mathbf{Mod}_R$ for $(i, n) \in \mathbb{N} \times \mathbb{N}_0$ such that each row $A_{*,n}$ forms a chain complex.

$$\begin{array}{l}
\vdots \\
0 \xleftarrow{\partial_{0,2}} A_{0,2} \xleftarrow{\partial_{1,2}} A_{1,2} \xleftarrow{\partial_{2,2}} A_{2,2} \xleftarrow{\partial_{3,2}} A_{3,2} \xleftarrow{\partial_{4,2}} \cdots \\
0 \xleftarrow{\partial_{0,1}} A_{0,1} \xleftarrow{\partial_{1,1}} A_{1,1} \xleftarrow{\partial_{2,1}} A_{2,1} \xleftarrow{\partial_{3,1}} A_{3,1} \xleftarrow{\partial_{4,1}} \cdots \\
0 \xleftarrow{\partial_{0,0}} A_{0,0} \xleftarrow{\partial_{1,0}} A_{1,0} \xleftarrow{\partial_{2,0}} A_{2,0} \xleftarrow{\partial_{3,0}} A_{3,0} \xleftarrow{\partial_{4,0}} \cdots
\end{array}$$

A map of bigraded complexes, $f\colon A \to A'$, consists of a morphism of chain complexes $f_n\colon A_{*,n} \to A'_{*,n}$ for each $n \in \mathbb{N}_0$. We denote the corresponding category $\mathbf{BiC}_\bullet(R)$.

The homology of a bigraded complex A is the bigraded Abelian group

$$H_\bullet(A) = \bigoplus_{(i,n)} H_i(A_{*,n}).$$

We get to the construction of the bigraded complex of interest. It owes its name to Jacek Świątkowski, who introduced it in the form of a cubical complex in [Ś01]. However, we follow the approach given in [ADCK17].

Definition 3.12. Let $G = (V, E)$ be a graph.

- A half-edge h consists of a vertex $v(h)$ and an edge $e(h)$ incident to it. For a loop, we distinguish the corresponding clockwise and counterclockwise half-edge. The symbol $H(v)$ denotes the set of half-edges containing fixed vertex v.
- Let $\mathbb{Z}[E]$ be the integral polynomial ring, where the variables are the edges of G. We introduce a bigrading on $\mathbb{Z}[E]$ by setting $\operatorname{grade}(e) = (0, 1)$ for all $e \in E$. That means that

$$\mathbb{Z}[E]_{i,n} = \begin{cases} \{f \in \mathbb{Z}[E] \mid f \text{ is homogeneous of degree } n\} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

- For each vertex v, let $S(v)$ be the free Abelian group generated by the set $H(v) \cup \{\emptyset, v\}$. It is bigraded by setting $\operatorname{grade}(\emptyset) = (0, 0)$, $\operatorname{grade}(v) = (0, 1)$ and $\operatorname{grade}(h) = (1, 1)$ for all $h \in H(v)$.

- The Świątkowski complex of G is the $\mathbb{Z}[E]$-module

$$S(G) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S(v),$$

where the tensorproducts are taken over the ring $\mathbb{Z}$. Its bigrading is given by

$$\mathrm{grade}(f \otimes \bigotimes_{v \in V} s_v) = \mathrm{grade}(f) + \sum_{v \in V} \mathrm{grade}(s_v),$$

where $f \in \mathbb{Z}[E]$ and $s_v \in S(v)$. The differentials $\partial_{i,n} \colon S(G)_{i,n} \to S(G)_{i-1,n}$ are defined by $\mathbb{Z}[E]$-linearity and the rules

$$\begin{aligned} v, \emptyset &\mapsto 0, \\ h &\mapsto e(h) \otimes \emptyset - 1 \otimes v(h), \\ x_1 \otimes x_2 &\mapsto \partial_{i,n}(x_1) \otimes x_2 + (-1)^{\ell} \partial_{i,n}(x_2) \otimes x_1, \end{aligned}$$

where $x_i \in S(v_i)$ for $v_1 \neq v_2$ and ℓ is the first component of $\mathrm{grade}(x_2)$.

- For any vertex v, let $\tilde{S}(v)$ be the submodule of $S(v)$ generated by $\emptyset$ and the differences $h - h'$ for all $h, h' \in H(v)$.
- The reduced Świątkowski complex of G is the $\mathbb{Z}[E]$-submodule of $S(G)$ given by

$$\tilde{S}(G) = \mathbb{Z}[E] \otimes_{\mathbb{Z}} \bigotimes_{v \in V} \tilde{S}(v),$$

with the same bigrading and differential as before. In particular, this means that $\partial(h - h') = (e(h) - e(h')) \otimes \emptyset$.

Remark 3.13. As an Abelian group $\tilde{S}(G)_{i,n}$ is generated by elements of the form

$$e_1 \cdots e_{n-i} \bigotimes_{j=1}^{i} (h_{j0} - h_{j1}) \bigotimes_{v \notin \{v_1, \dots, v_i\}} \emptyset,$$

where $v(h_{j0}) = v(h_{j1}) = v_j \in V$ for $j \in \{1, \dots, i\}$ and these vertices are distinct (whereas the edges $e_1, \dots, e_{n-i}$ are not necessarily). In particular, $\tilde{S}(G)_{i,n} = 0$ if i exceeds n.

We can visualise some parts of the reduced Świątkowski complex directly.

Proposition 3.14. *Let $G = (E, V)$ be any graph.*

1. *If i exceeds the number of non-leaves in G, then $\tilde{S}(G)_{i,n} = 0$ for all n.*
2. *The zeroth row $\tilde{S}(G)_{\star,0}$ equals $0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$*
3. *As an Abelian group, any component $\tilde{S}(G)_{i,n}$ is free. Moreover, the ranks satisfy*

$$\begin{aligned} \mathrm{rank}(\tilde{S}(G)_{0,n}) &= \binom{n + |E| - 1}{|E| - 1}, \\ \mathrm{rank}(\tilde{S}(G)_{i,n}) &= \mathrm{rank}(\tilde{S}(G)_{i,i}) \cdot \mathrm{rank}(\tilde{S}(G)_{0,n-i}) \ \textit{for } n > i. \end{aligned}$$

4. *For $n \geqslant 1$, the start of row $\tilde{S}(G)_{\star,n}$ is*

$$0 \longleftarrow \mathbb{Z}^{\binom{n+|E|-1}{|E|-1}} \xleftarrow{\partial 1,n} \mathbb{Z}^{(2|E|-|V|)\cdot\binom{n+|E|-2}{|E|-1}},$$

where the image of $\partial_{1,n}$ is $L_n = \{(x_k) \in \mathbb{Z}^{\binom{n+|E|-1}{|E|-1}} \mid \sum_k x_k = 0\}$.

Proof. Everything essentially follows from Remark 3.13.

1. For a leaf v, the set $H(v)$ consists of a unique element. Hence, $\tilde{S}(V)$ is isomorphic to $\mathbb{Z}\langle\emptyset\rangle$. In other words, it plays no role (after tensoring) in $\tilde{S}(G)$. Therefore, if i exceeds the number of available non-leaves, there can be no generator of the form in Remark 3.13.

2. The component $\tilde{S}(G)_{0,0}$ consists of the constant functions in $\mathbb{Z}[E]$ tensored with the empty set in each vertex. The rest of the terms is zero since i exceeds $n = 0$.

3. The monomials of degree n in $\mathbb{Z}[E]$ generate $\tilde{S}(G)_{0,n}$. To count them one may consider the partitions of n into $|E|$ subsets that may also be empty. Hence, there are $\binom{n+|E|-1}{|E|-1}$ many.

   Furthermore, suppose that there is a basis for $\tilde{S}(G)_{i,i}$ (consisting of tensorproducts of i differences of half-edges). The tensor products between all elements in this basis and the monomials of degree $n-i$ produces a basis for $\tilde{S}(G)_{i,n}$.

4. Note that $\tilde{S}(G)_{1,1}$ is generated by all the possible differences between two half-edges $h - h'$ with equal vertex v. The differences in $H(v)$ can all be obtained as sums of differences $h_0 - h'$ for one fixed $h_0 \in H(v)$, hence we obtain $H(v) - 1$ independent generators for each vertex v. (Note this even holds for leaves, where we need 0 generators.) It follows that $\tilde{S}(G)_{1,1}$ has rank $\sum_{v \in V}(H(v) - 1) = 2|E| - |V|$. Hence, the stated ranks are correct by the third statement.

   The map $\partial_{1,1}$ acts on the above basis by $h - h' \mapsto e(h) - e(h')$. Hence, its image is definitely contained in the subgroup L_1, defined in the statement. Furthermore, the image is generated by sums of any difference of two adjacent edges (edges incident to the same vertex). Since we work under the assumption that graphs are connected, any difference of edges $e - e' \in E$ can be obtained as a sum of differences of adjacent edges. Hence, they are in the image as well. As these differences generate the full L_1, we are done in the case $n = 1$.

   In general, $\partial_{1,n}$ is given on the basis by $f \otimes (h - h') \mapsto f \cdot e(h) - f \cdot e(h')$ for any monomial $f \in \mathbb{Z}[E]$ of degree n. Hence, its image is contained in L_n. By the same argument as before, the image must contain any difference $fe - fe'$ for two edges. By repeated procedure of this, any difference between two monomials $f - f' \in \mathbb{Z}[E]$ is also inside the image of $\partial_{1,n}$. These differences generate the subgroup L_n.

$\square$

The relevance of these bigraded complexes is the following.

Theorem 3.15. *There are isomorphisms of bigraded Abelian groups*

$$H_\bullet(\mathrm{UConf}_\star(G)) \cong H_\bullet(S(G)) \cong H_\bullet(\tilde{S}(G)).$$

In particular, the i'th homology group of the n-stranded unordered configuration space is isomorphic to $H_i(\tilde{S}(G)_{\star,n})$

Proof. The proof of this result requires Morse theory, which goes beyond the scope of this thesis. It can be found in [ADCK17] as Theorem 4.5 or, after translating back to the setting of cubical complexes, in [Ś01]. □

Let us illustrate that the theorem holds.

Example 3.16. Take any graph G of genus g. Proposition 3.14(2) implies that the only nontrivial homology of the zeroth row is $H_0(\tilde{S}(G)_{\star,0}; \mathbb{Z}) = \mathbb{Z}$. This is in accordance with the convention that $\mathrm{UConf}_0(G) = \emptyset$. For the first row, remember that $\tilde{S}(G)_{i,1} = 0$ whenever $i \geqslant 2$. Proposition 3.14(4) implies that $H_0(\tilde{S}(G)_{\star,1}; \mathbb{Z}) \cong \mathbb{Z}$ and that $H_1(\tilde{S}(G)_{\star,1}; \mathbb{Z})$ is the free Abelian group of rank

$$(2|E| - |V|) \cdot \binom{|E|-1}{|E|-1} - \binom{|E|}{|E|-1} + 1 = 2|E| - |V| - |E| + 1 = g.$$

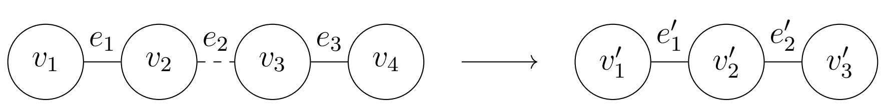
This is in accordance with the fact that $\mathrm{UConf}_1(G) \cong G$ and the very definition of the genus. For any row n, 3.14(4) implies that $H_0(\tilde{S}(G)_{\star,n}; \mathbb{Z}) = \mathbb{Z}$. This is in accordance with Proposition 3.6.

The reduced Świątkowski complex yields an algorithmic tool to compute the homology of the unordered configuration spaces of any graph. We present concrete examples in section 3.3

To investigate functoriality of the homology of the unordered configuration space, we may thus look at the level of the reduced Świątkowski complex.

Proposition 3.17. *The assignment $\tilde{S} \colon \mathcal{G}_g^{\mathrm{op}} \to \mathbf{BiC}_\bullet(\mathbb{Z}[E])$ of the reduced Świątkowski complex to a graph is functorial with respect to contractions.*

Proof. We must define how the functor maps contractions. We illustrate the form of $\tilde{S}(f) \colon \tilde{S}(G') \to \tilde{S}(G)$ by considering a simple example. Let $f \colon G \to G' \in \mathcal{T}$ be the contraction below.



Let h_{ij} be the half-edge (v_i, e_j) and define h'_{ij} similarly. We define $\tilde{f}^* \colon \tilde{S}(G') \to \tilde{S}(G)$ by specifying the image of all generators, namely vertices, edges and half-edges. Away from the edge contraction (vertex v_2' in the image), we straightforwardly set

$$\tilde{f}^*\colon v_1' \mapsto v_1, \quad v_3' \mapsto v_4, \quad e_1' \mapsto e_1, \quad e_2' \mapsto e_2, \quad h_{11}' \mapsto h_{11}, \quad h_{32}' \mapsto h_{43}.$$

To encode the contracted information, we set

$$\tilde{f}^*\colon v_2' \mapsto e_2, \qquad h_{21}' \mapsto h_{21} - h_{22}, \qquad h_{22}' \mapsto h_{33} - h_{32}.$$

It is clear that the bigrading is preserved by this definition. As for the differential, it is trivially preserved for any half-edge not containing v_2' and we compute that

$$\partial \circ \tilde{f}^*(h_{21}') = \partial(h_{21} - h_{22}) = e_1 - v_2 - (e_2 - v_2) = e_1 - e_2 = \tilde{f}^*(e_1' - v_2') = \tilde{f}^* \circ \partial(h_{21}').$$

This construction is easily generalised to any simple contraction. Moreover, for two simple contraction, f and g, it is clear that $\tilde{S}(f) \circ \tilde{S}(g) = \tilde{S}(g) \circ \tilde{S}(f)$. As any contraction can be written (uniquely up to order) as a composition of simple ones, there is a unique functorial continuation of $\tilde{S}$ to the full category $\mathcal{G}_g^{\mathrm{op}}$. □

Passing trough the homology groups of the complex and using the identification of Theorem 3.15, we obtain the maps

$$H_i(\tilde{f}_{i,n}^*)\colon H_i(\mathrm{UConf}_n(G')) \to H_i(\mathrm{UConf}_n(G)).$$

Hence, we derived the requested functoriality and conclude that the assignment of homology groups $H_i(\mathrm{UConf}_n(\_);\mathbb{Z})$ is a $\mathcal{G}_g^{\mathrm{op}}$-module over $\mathbb{Z}$.

3.3 Examples of computations

Before continuing the analysis of the $\mathcal{G}_g^{\mathrm{op}}$-module $H_i(\mathrm{UConf}_n(\_);\mathbb{Z})$, we believe it is useful and interesting to present some examples of computations of these groups. The reduced Świątkowski complex is our main tool.

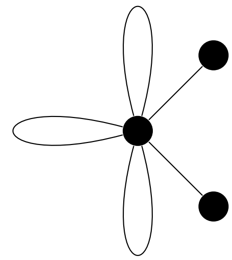
Although our exposition involves quite some notation, this section is intended to make the reader appreciate how some algorithmic computations produce information about a big range of complicated topological spaces.

We shorten the notation by suppressing tensor products with $\emptyset$. Moreover, we denote the tensor product with a polynomial by a blank space. In particular, this means that for any $f \in \mathbb{Z}[E]$ and $h - h' \in H(v_0)$,

$$f(h - h') = f \otimes (h - h') \bigotimes_{v \neq v_0} \emptyset.$$

The combination of 3.14(1) and (4) directly reveals the homology of graphs with only one essential vertex.

Example 3.18. Let S_k^l be the graph with one central vertex v_0 having l loops and k leaves attached to it. For example, S_2^3 looks like this.



As these graphs have only one non-leaf, 3.14(1) implies that $\tilde{S}(G)_{i,n} = 0$ for all $i > 1$. Hence, the second and higher homology groups of $\mathrm{UConf}_n(S_k^l)$ vanish whereas the first one is the free Abelian group of rank

$$\begin{aligned} &(2|E| - |V|) \cdot \binom{n + |E| - 2}{|E| - 1} - \binom{n + |E| - 1}{|E| - 1} + 1 = \\ &(2l + k - 1) \cdot \binom{n + l + k - 2}{l + k - 1} - \binom{n + l + k - 1}{l + k - 1} + 1. \end{aligned}$$

Notice that the graphs of Example 3.3 are all part of this class $I = S_1^0$, $Y = S_3^0$ and $O = S_0^1$. This formula coincides with the description of their configuration spaces given there. We highlight that for Y the element

$$\begin{aligned} \alpha_{123} &= e_1(h_2 - h_3) + e_2(h_3 - h_1) + e_3(h_1 - h_2) \\ &= -e_1(h_1 - h_2) + e_1(h_1 - h_3) - e_2(h_1 - h_3) + e_3(h_1 - h_2) \end{aligned}$$

is (up to sign) the unique generator of $H_1(\tilde{S}(Y)_{\star,2}; \mathbb{Z})$. For general $n \geqslant 3$, a possible choice of the generator of $H_1(\tilde{S}(Y)_{\star,n}; \mathbb{Z})$ is $e_1^{n-2}\alpha_{123}$.

Things get much more complicated when having two or more essential vertices.

Example 3.19. We consider the graph of genus 2 consisting of two vertices and tree edges connecting them. According to its shape, we denote it by Θ. In particular, it contains six half-edges, which we denote $h_{ij} = (v_i, e_j)$. The second row of $\tilde{S}(G)$ looks like

$$0 \leftarrow \mathbb{Z}^6 \xleftarrow{\partial_{1,2}} \mathbb{Z}^{12} \xleftarrow{\partial_{2,2}} \mathbb{Z}^4 \leftarrow 0 \leftarrow \cdots,$$

where we fix the following bases

$$\begin{aligned} \tilde{S}(G)_{0,1} : \alpha =& \{e_1^2,\ e_1e_2,\ e_1e_3,\ e_2^2,\ e_2e_3,\ e_3^2\}, \\ \tilde{S}(G)_{1,1} : \beta =& \{e_1(h_{11} - h_{12}),\ e_1(h_{11} - h_{13}),\ e_1(h_{21} - h_{22}),\ e_1(h_{21} - h_{23}), \\ & e_2(h_{11} - h_{12}),\ e_2(h_{11} - h_{13}),\ e_2(h_{21} - h_{22}),\ e_2(h_{21} - h_{23}), \\ & e_3(h_{11} - h_{12}),\ e_3(h_{11} - h_{13}),\ e_3(h_{21} - h_{22}),\ e_3(h_{21} - h_{23})\}, \\ \tilde{S}(G)_{2,1} : \gamma =& \{(h_{11} - h_{12}) \otimes (h_{21} - h_{22}),\ (h_{11} - h_{12}) \otimes (h_{21} - h_{23}), \\ & (h_{11} - h_{13}) \otimes (h_{21} - h_{22}),\ (h_{11} - h_{13}) \otimes (h_{21} - h_{23})\}. \end{aligned}$$

It follows from Proposition 3.14(4) that the kernel of $\partial_{1,2}$ has rank $12-(6-1)=7$. It clearly contains the linearly independent differences

$$b_1=\beta_1-\beta_3,\ b_2=\beta_2-\beta_4,\ b_3=\beta_5-\beta_7,\ b_4=\beta_6-\beta_8,\ b_5=\beta_9-\beta_{11},\ b_6=\beta_{10}-\beta_{12}.$$

Inspired by the case of the graph Y, we notice that the element

$$\begin{aligned} B=&-e_1(h_{11}-h_{12})+e_1(h_{11}-h_{13})-e_2(h_{11}-h_{13})+e_3(h_{11}-h_{12}) \\ =&-\beta_1+\beta_2-\beta_6+\beta_9. \end{aligned}$$

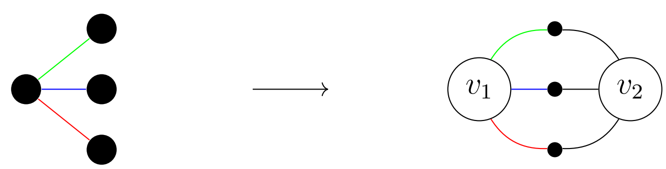
enlarges the above set to a basis of the kernel of $\partial_{1,2}$. For $\partial_{2,2}$ direct computation shows that

$$\begin{aligned} \gamma_1 &\mapsto -b_1+b_3, \\ \gamma_2 &\mapsto b_2-b_4+B, \\ \gamma_3 &\mapsto b_1-b_5-B, \\ \gamma_4 &\mapsto -b_2+b_6. \end{aligned}$$

In particular, (considering the second b_i in each row) it follows that $\partial_{2,2}$ is injective and so we conclude that

$$H_i(\mathrm{UConf}_2(\Theta);\mathbb{Z})\cong\begin{cases}\mathbb{Z} & \text{if } i=0,\\ \mathbb{Z}^7/\mathbb{Z}^4=\mathbb{Z}^3 & \text{if } i=1,\\ 0 & \text{esle.}\end{cases}$$

In fact, we should notice that fixing the choice of a vertex (say v_1) of Θ, there is a natural embedding of topological space $\iota_1\colon Y\to\Theta$ depicted below.



By Remark 3.2, this leads to an embedding $\tilde{\iota_1}\colon \mathrm{UConf}_2(Y)\to\mathrm{UConf}_2(\Theta)$ and, because homology is functorial with respect to continuous maps, to a group homomorphism $H_1(\tilde{\iota_1})\colon H_1(\mathrm{UConf}_2(Y))\to H_1(\mathrm{UConf}_2(\Theta))$. This map is precisely given by

$$\alpha_{123}\mapsto B \bmod \partial_{2,2}(\tilde{S}(\Theta)_{2,2}).$$

Suppose that we now take the other vertex v_2 of Θ and consider the embedding $\iota_2\colon Y\to\Theta$ that is the mirror of the previous image. Then the induced map $H_1(\tilde{i}_2)$ is again given by

$$\begin{aligned} \alpha_{123}\mapsto e_1(h_{21}-h_{22})+e_1(h_{21}-h_{23})-e_2(h_{21}-h_{23})+e_3(h_{21}-h_{22})=& \\ B+b_1-b_2+b_4-b_5=B+\partial_{2,2}(\gamma_3-\gamma_2)=B \bmod \partial_{2,2}(\tilde{S}(\Theta)_{2,2}).& \end{aligned}$$

We now consider the case of a general number $n \geqslant 3$ of particles. By Proposition 3.14(4), the row $\tilde{S}(\Theta)_{\star,n}$ looks like

$$0 \leftarrow \mathbb{Z}^{\binom{n+2}{2}} \xleftarrow{\partial_{1,n}} \mathbb{Z}^{4\cdot\binom{n+1}{2}} \xleftarrow{\partial_{2,2}} \mathbb{Z}^{4\cdot\binom{n}{2}} \leftarrow 0 \leftarrow \cdots,$$

where this time the bases are given by

$$\begin{aligned}
\tilde{S}(G)_{0,n} : \alpha =& \{e_1^n,\ e_1^{n-1}e_2,\ e_1^{n-1}e_3,\ e_1^{n-2}e_2^2,\ e_1^{n-2}e_2e_3,\ e_1^{n-2}e_3^2 \ldots,\ e_3^n\},\\
\tilde{S}(G)_{1,1} : \beta =& \{e_1^{n-1}(h_{11}-h_{12}),\ e_1^{n-1}(h_{11}-h_{13}),\ e_1^{n-1}(h_{21}-h_{22}),\ e_1^{n-1}(h_{21}-h_{23}),\\
& e_1^{n-2}e_2(h_{11}-h_{12}),\ \ldots\\
& \vdots\\
& e_3^{n-1}(h_{11}-h_{12}),\ e_3^{n-1}(h_{11}-h_{13}),\ e_3^{n-1}(h_{21}-h_{22}),\ e_3^{n-1}(h_{21}-h_{23})\},\\
\tilde{S}(G)_{2,1} : \gamma =& \{e_1^{n-2}(h_{11}-h_{12})\otimes(h_{21}-h_{22}),\ e_1^{n-2}(h_{11}-h_{12})\otimes(h_{21}-h_{23}),\\
& e_1^{n-2}(h_{11}-h_{13})\otimes(h_{21}-h_{22}),\ e_1^{n-2}(h_{11}-h_{13})\otimes(h_{21}-h_{23}),\\
& e_1^{n-3}e_2(h_{11}-h_{12})\otimes(h_{21}-h_{22}),\ \ldots\\
& \vdots\\
& e_3^{n-2}(h_{11}-h_{12})\otimes(h_{21}-h_{22}),\ e_3^{n-2}(h_{11}-h_{12})\otimes(h_{21}-h_{23}),\\
& e_3^{n-2}(h_{11}-h_{13})\otimes(h_{21}-h_{22}),\ e_3^{n-2}(h_{11}-h_{13})\otimes(h_{21}-h_{23})\}.
\end{aligned}$$

It follows from Proposition 3.14(4) that the kernel of $\partial_{1,n}$ has rank

$$4\cdot\binom{n+1}{2} - (\binom{n+2}{2} - 1) = \frac{3}{2}n^2 + \frac{1}{2}n = 2\cdot\binom{n+1}{2} + \binom{n}{2}.$$

The last characterisation is useful to recognise the basis

$$\begin{aligned}
&b_1 = \beta_1 - \beta_3, \quad b_2 = \beta_2 - \beta_4, \quad \ldots \quad, \quad b_{2\cdot\binom{n+1}{2}} = \beta_{4\cdot\binom{n+1}{2}-2} - \beta_{4\cdot\binom{n+1}{2}},\\
&b'_1 = e_1^{n-2}\cdot B, \quad b'_2 = e_1^{n-3}e_2\cdot B, \quad \ldots \quad, \quad b'_{\binom{n}{2}} = e_3^{n-2}\cdot B.
\end{aligned}$$

Essentially, the map $\partial_{2,n}$ maps elements of γ in the same way as $\partial_{2,2}$,while respecting the front multiplications by monomials (of degree $n-2$) in $\mathbb{Z}[E]$. Therefore, naively one might think it is always injective. However, notice that for $n \geqslant 6$ it holds that

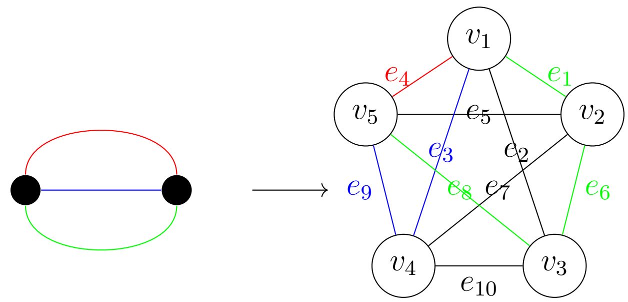
$$\operatorname{rank}(\tilde{S}(G)_{2,n}) = 4\binom{n}{2} = 2n^2 - 2n > \frac{3}{2}n^2 + \frac{1}{2}n = \operatorname{rank}(\ker(\partial_{1,n})).$$

This means that $\partial_{2,n}$ cannot be injective and therefore that $H_2(\mathrm{UConf}_n(\Theta)) \neq 0$ for $n \geqslant 6$.

Finally, we want to present an example of torsion in the homology of the unordered configuration space of a graph. Proposition 3.7 ensures the existence of torsion in the first homology group of the n-stranded unordered configuration space of a non-planar graphs, whenever $n \geqslant 2$.

Example 3.20. We consider the complete graph on five vertices K_5 and the complete bipartite graph with three vertices on each side, $K_{3,3}$. Both graphs are non-planar. The following exposition comes from [ADCK17].

To see the origin of the torsion, consider the following embedding $f_1 \colon \Theta \to K_{5,5}$.



Recall, from the previous example, the embeddings $\iota_1, \iota_2 \colon Y \to \Theta$, which satisfy

$$H_1(\tilde{\iota_1})(\alpha_{123}) = H_1(\tilde{\iota_2})(\alpha_{123}) = -H_1(\tilde{\iota_2})(\alpha_{213}).$$

This amounts to the relation

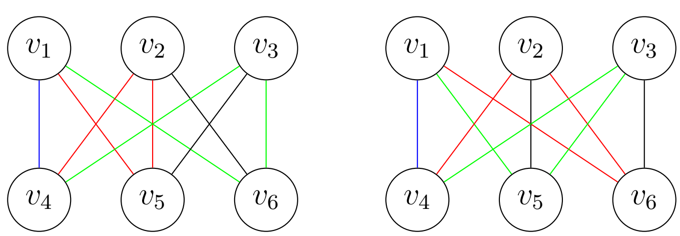
$$\begin{aligned} e_4(h_{13} - h_{11}) + e_3(h_{11} - h_{14}) + e_1(h_{14} - h_{13}) = f_{11}(\alpha_{123}) \\ = -f_{12}(\alpha_{213}) = -e_9(h_{57} - h_{54}) - e_7(h_{54} - h_{59}) - e_4(h_{59} - h_{57}), \end{aligned}$$

where f_{1j} denotes the composition $f_1 \circ \iota_j$ for $j = 1, 2$. Now, notice that by rotating the image above to the left, we get an analogue embedding $f_2 \colon \Theta \hookrightarrow K_5$ that reaches the vertices v_5 and v_4. Moreover, note that by construction $-f_{12}(\alpha_{213})$ is equal to the image of α_{123} under $f_2 \circ \iota_1$ (which we denote f_{21}). By repeating the same argument four more times, we conclude that

$$\begin{aligned} f_{11}(\alpha_{123}) =& f_{21}(\alpha_{123}) = -f_{22}(\alpha_{213}) \\ =& f_{31}(\alpha_{123}) = -f_{32}(\alpha_{213}) \\ =& f_{41}(\alpha_{123}) = -f_{42}(\alpha_{213}) \\ =& f_{51}(\alpha_{123}) = -f_{52}(\alpha_{213}) = -f_{11}(\alpha_{123}). \end{aligned}$$

Hence, we see that $f_{11}(\alpha_{123})$ is a 2-torsion element inside $H_1(\mathrm{UConf}_2(K_5))$.

We treat the case of $K_{3,3}$ similarly by considering the embeddings f_1 and f_2 of Θ below.



Let ι_1 and ι_2 be the same embeddings of Y in Θ as before and set $f_{ij} = f_i \circ \iota_j$. We conclude that $f_{11}(\alpha_{123}) = f_{12}(\alpha_{123}) = f_{22}(\alpha_{123}) = f_{21}(\alpha_{132}) = -f_{11}(\alpha_{123})$.

In fact, the general proof of Proposition 3.7 can also be deduced from the above example, using a theorem of K. Wagner [Wag37] that roughly says that any non-planar graph contains a copy of K_5 or $K_{3,3}$ inside it. For the complete exposition see Appendix C in [ADCK17].

3.4 Smallness

In this section, we analyse the growth of the $\mathcal{G}_g^{\mathrm{op}}$-modules $H_i(\mathrm{UConf}_n(\_);\mathbb{Z})$ in the fashion of section 2.3. The main result is the following.

Theorem 3.21. *Fix $g, i, n \in \mathbb{N}_0$. The $\mathcal{G}_g^{\mathrm{op}}$-module $H_i(\mathrm{UConf}_n(\_), \mathbb{Z})$ is $(g+i+n)$-small.*

Proof. Again we use the reduced Świątkowski complex. By Theorem 3.15, we know that $H_i(\mathrm{UConf}_n(\_);\mathbb{Z}) = H_i(\tilde{S}_{\star,n}(\_))$. Since it is a subquotient of the $\mathcal{G}_g^{\mathrm{op}}$-module $\tilde{S}(\_)_{i,n}$, it is enough to show that this module is finitely generated in degree $\leqslant (g+i+n)$.

We claim that for any graph G with more than $(g+i+n)$ edges, the Abelian group $\tilde{S}(G)_{i,n}$ is generated by the images of all maps $\tilde{S}(f)$ that correspond to simple contractions $f\colon G \to G'$. By composition of simple contractions, this actually implies that $\tilde{S}(G)_{i,n}$ is generated by the images of all maps $\tilde{S}(f)$ that correspond to contractions $f\colon G \to G'$, where G' has at most $(g+i+n)$ edges. There exist only finitely many of these graphs $G_1, \dots, G_k$. Moreover, the Abelian group $\tilde{S}(G_j)_{i,n}$ is finitely generated for each $j \in \{1, \dots, k\}$. Let l_j be the minimal number of generators of $\tilde{S}(G_j)_{i,n}$ (differences of half-edges and $\emptyset$). Hence, we can define an epimorphism $\phi\colon \bigoplus_{j=1}^{k} \mathrm{P}_{G_j}^{\oplus l_j} \to \tilde{S}(\_)_{i,n}$, by Proposition 1.14. It follows that $\tilde{S}(\_)_{i,n}$ is indeed finitely generated in degree $\leqslant (g+i+n)$.

To prove the claim, let G be a graph with more than $(g+i+n)$ edges and, using Remark 3.13, let

$$\sigma = e_1 \cdots e_{n-i} \bigotimes_{j=1}^{i} (h_{j0} - h_{j1}) \bigotimes_{v \notin \{v_1,\dots,v_i\}} \emptyset$$

be a generator of the Abelian group $\tilde{S}(G)_{i,n}$. We call $v_1, \dots, v_i$ the distinguished vertices of σ. After reordering them, we can assume there exist some $r \in \{1, \dots, i\}$ such that

$$(v_j, v_l) \in E(G) \text{ for some } l \in \{1, \dots, i\} \text{ if and only if } j \leqslant r.$$

Furthermore, we may assume that $(v_j, v_l) = e(h_{j1})$, using that the other generators can be obtained by the differences $h_{j0} - h_{j1} = [h_{j0} - (v_j, (v_j, v_l))] - [h_{j1} - (v_j, (v_j, v_l))]$.

Under these assumptions, we say that the distinguished edges of type 1 of σ are the edges connecting two distinguished vertices of σ and the loops in G. The distinguished edges of type 2 are all of the edges

$$e_1,\ \dots,\ e_{n-i},\ e(h_{10}),\ e(h_{11}),\ \dots,\ e(h_{i0}),\ e(h_{i1}).$$

We want to count the number of distinguished edges of any type. Let t be the number of loops at non-distinguished vertices and let H be the induced subgraph of G on

$\{v_1, \ldots, v_r\}$. The graph H does not contain the t loops defined above and thus has genus at most $g-t$. Hence, it has at most $g-t+r$ edges. Since H contains all edges connecting two distinguished vertices (including loops), the number of distinguished edges of type 1 in G is at most $t+g-t+r=g-r$. On the other hand, the distinguished edges of type 2 are at most $(n-i)+i+i=n+i$. However, by our first assumption we did count $e(h_{ji})$ for $j \leqslant r$ both as a type 1 and type 2 distinguished edge. Hence, the total number of distinguished edges actually is at most $g-r+n+i-r=g+i+n$. Hence, G contains an edge e that is not distinguished. Let us consider the simple contraction $f\colon G \to G'$ contracting e. It follows that $\tilde{S}(G')_{i,n}$ contains the object

$$\sigma' = f(e_1)\cdots f(e_{n-i}) \bigotimes_{j=1}^{i} (f(h_{j0}) - f(h_{j1})) \bigotimes_{v \notin \{f(v_1),\ldots,f(v_i)\}} \emptyset \in \tilde{S}(G)_{i,n}.$$

All that is left to see is that $\tilde{S}(f)_{i,n}(\sigma') = \sigma$. Recalling Proposition 3.17, the only objects we should consider are the half-edge $h=(v_j, e)$, where v_j is a distinguished. By definition we see that

$$\tilde{S}(f)(f(h_{j0}) - f(h_{j1})) = (h_{j0} - h) - (h - h_{j1}) = h_{j0} - h_{j1}.$$

It follows that $\tilde{S}(f)_{i,n}(\sigma') = \sigma$, which proves our claim. □

We deduce some consequences of this result. Firstly, we want to apply Theorem 2.37. However, to do so we should consider $\mathcal{G}_g^{\mathrm{op}}$-modules over a field k. Hence, we change the coefficient group of the homology theory to $\mathbb{Q}$ and end up with the $(g+i+n)$-small module $H_i(\mathrm{UConf}_n(\_);\mathbb{Q}) \in \mathrm{Rep}_{\mathbb{Q}}(\mathcal{G}_g^{\mathrm{op}})$.

We already noticed that the n-stranded unordered configuration space is invariant under subdivision. However, when sprouting a vertex it is a priori not clear at all how the unordered configuration space of the graph changes.

Corollary 3.22. *Fix non-negative integers g, i and n and let G be a graph of genus g. For each tuple of vertices $\underline{v}$, there exists a polynomial $f_{G,\underline{v}} \in \mathbb{Z}[t_1, \ldots, t_r]$, of total degree at most $(g+i+n)$, such that*

$$\dim_{\mathbb{Q}} H_i(\mathrm{UConf}_n((G(\underline{v}, \underline{m}))) = f_{G,\underline{v}}(m_1, \ldots, m_r),$$

when $\underline{m}$ is sufficiently large in each coordinate

Proof. Combine Theorems 3.21 and 2.37. □

Example 3.23. Consider the tree $G = \bullet$, having one vertex v and no edges. Sprouting m leaves to it yields the m-star $S_m = S_m^0 = \bullet(v, m)$. We use the computation in section 3.3 for general S_k^l and change the homology coefficient to $\mathbb{Q}$. It follows that row n of $\tilde{S}(S_m) \otimes \mathbb{Q}$ is

$$0 \leftarrow \mathbb{Q}^{\binom{n+m-1}{m-1}} \leftarrow \mathbb{Q}^{(m-1)\cdot\binom{n+m-2}{m-1}} \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

and that the first homology group $H_1(\mathrm{UConf}_n(Y);\mathbb{Q}) \cong \ker(\partial_{1,n})$ is of dimension

$$(m-1)\cdot\binom{n+m-2}{n-1} - \binom{n+m-1}{n} + 1.$$

For fixed $n \in \mathbb{N}$ this expression is a polynomial in $\mathbb{Z}[m]$ of degree n.

When changing the homology coefficients to $\mathbb{Q}$, we are in fact eliminating all torsion in the bigraded complex. However, Theorem 3.21 also allows us to deduce something about this eventual torsion as well.

Corollary 3.24. *Fix non-negative integers g, i, n. There exists a constant $d_{g,i,n} \in \mathbb{N}$ such that for any graph G of genus g, the torsion part of $H_i(\mathrm{UConf}_n(G); \mathbb{Z})$ has exponent at most $d_{g,i,n}$.*

Proof. For each graph G of genus g, let $\mathrm{Tor}_{g,i,n}(G)$ be the torsion subgroup of $H_i(\mathrm{UConf}_n(G); \mathbb{Z})$. Recall that any morphism of Abelian groups preserves the torsion subgroups. It follows that $\mathrm{Tor}_{g,i,n}(\_)$ is a $\mathcal{G}_g^{\mathrm{op}}$-submodule of $H_i(\mathrm{UConf}_n(\_); \mathbb{Z})$. As this module is $(g+i+n)$-small by Theorem 3.21, it is in particular finitely generated by Proposition 2.29. The category $\mathrm{Rep}_{\mathbb{Z}}(\mathcal{G}_g^{\mathrm{op}})$ is Noetherian, by Theorem 2.19 and 1.34. Hence, $\mathrm{Tor}_{g,i,n}(\_)$ is also finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_j \in \mathrm{Tor}_{g,i,n}(G_j)$ for some graph G_j of genus g. Let $o_j \in \mathbb{N}$ be the order of α_j (in the corresponding Abelian torsion group) and set $d_{g,i,n} = \mathrm{lcm}(o_1, \ldots, o_k)$. An arbitrary $\alpha \in \mathrm{Tor}_{g,i,n}(G)$ is of the form

$$\alpha = \sum_{j=1}^{\ell} \sum_{f_j \colon G \to G_j} \lambda_{f_j} \cdot \mathrm{Tor}_{g,i,n}(f)(\alpha_j),$$

where each λ_{f_i} is an integer. We conclude that the order of α is at most $d_{g,i,n}$. □

Example 3.25. For $i = 1$, the proof of Proposition 3.7 (or the method illustrated at the end of section 3.3) actually demonstrates that for any graph G the only torsion appearing in $H_1(\mathrm{UConf}_n(G); \mathbb{Z})$ is of exponent 2. Thus, $d_{g,1,n}$ can be taken equal to 2 for all $n, g \in \mathbb{N}_0$.

Chapter 4

More (non-)Noetherian results

In contrast to the previous ones, this chapter is composed almost entirely of ideas we came up with ourselves. Although it is fair to say that most results are direct consequences or slight adaptation of other work.

Instead of continuing the study of graphs and contractions, we return to the fundamental Question 1.21: which categories induce a Noetherian module category? In section 4.1, we make some direct observations on the influence of the size of a category. This motivates why we restrict our attention to combinatorial categories in the rest of this project.

Section 4.2 is devoted to the introduction of combinatorial categories that do not induce a Noetherian category of modules over any ring. We consider the categories of finite (Abelian) groups and finite posets. Afterwards, we introduce the notion of $\mathbb{F}_\infty$-modules in a hands-on way, following the exposition of the supervisor and Máté L. Juhász [HJ17]. We show how the argument for posets can be adapted to this setting. Finally, we realise that these examples can be generalised to the statement that condition (G2) in Definition 1.32 is in fact necessary to induce Noetherian module categories.

To finish, section 4.3 comprises two generalisations on Theorem 1.51. The first one treats the category of finite projective modules over any finite ring. The second one treats the category of all finite modules over a finite principal ideal ring.

4.1 Size of the category

We begin with a trivial but useful observation.

Proposition 4.1. *Let $\mathcal{C}$ be an essentially small category, let $x \in \mathcal{C}$ and let k be a left-Noetherian ring. If x is the source of only finitely many morphisms in $\mathcal{C}$, then the principal projective module $\mathrm{P}_x \in \mathrm{Rep}_k(\mathcal{C})$ is Noetherian.*

Proof. Let $x_1, \ldots, x_n$ be the objects in $\mathcal{C}$, for which $\mathrm{Hom}_{\mathcal{C}}(x, x_i)$ is nonempty. The principal projective module P_x, assigns a free k-module of finite rank to any of these x_i. Since k is Noetherian, it follows that any k-submodule of $\mathrm{P}_x(x_i)$ is finitely generated.

Take any $\mathcal{C}$-submodule $M \subseteq \mathrm{P}_x$. For each i, the k-module $M(x_i)$ is generated by some finite set of element $S_i \subseteq \mathrm{P}_x(x_i)$. As $\mathrm{P}_x(x') = 0$ for all $x' \in \mathcal{C} \setminus \{x_1, \ldots, x_n\}$, it follows that the $\mathcal{C}$-module M is generated by the finite set $\bigcup_{i=1}^n S_i$. □

In particular, this yields a class of categories that trivially induce Noetherian module categories.

Corollary 4.2. *Let $\mathcal{C}$ be a category that contains only finitely many morphisms, then $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian for any left-Noetherian ring k.*

Proof. For any $x \in \mathcal{C}$, there exist only finitely many morphisms in $\mathcal{C}$ having source x. By Proposition 4.1, the corresponding principal projective module P_x is Noetherian. As x was chosen arbitrarily we conclude, by Remark 1.19, that $\mathrm{Rep}_k(\mathcal{C})$ is Noetherian. □

The remaining results in this section, are modelled on Theorem N in [PS14]. We consider modules over the category $\mathbf{G}$ corresponding to a group G, introduced in Example 1.6. Notice that if G is finite, Corollary 4.2 applies.

Proposition 4.3. *The category $\mathrm{Rep}_k(\mathbf{G})$ is Noetherian if and only if the group ring $k[G]$ is left-Noetherian.*

Proof. The category $\mathrm{Rep}_k(\mathbf{G})$ is Noetherian if and only if the principal module $\mathrm{P}_\star$ corresponding to the unique object $\star \in \mathbf{G}$ is Noetherian. The definition of this module is that $\mathrm{P}_\star(\star) = k[\mathrm{Hom}_{\mathbf{G}}(\star, \star)] \cong k[G]$ and $\mathrm{P}_\star(g)\colon k[G] \to k[G]\colon e_f \mapsto e_{g+f}$ for $g, f \in G$. Hence, a submodule $M \subseteq \mathrm{P}_\star$ is determined by the choice of a k-submodule $M(\star) \subseteq k[G]$. Moreover, to be functorial $M(\star)$ must be invariant under the actions of $\mathrm{P}_\star(g)$ for all $g \in G$, that is to say under left multiplication by any element in $k[G]$. Hence, the $\mathbf{G}$-submodules of $\mathrm{P}_\star$ correspond exactly to the left ideals of $k[G]$ and the statement follows. □

As far as we know, no full characterisation of left-Noetherian group rings has been proven so far. However, the following is known by Lemma 2.22 in [PS14].

Definition 4.4. A group is called Noetherian if all its subgroups are finitely generated.

Lemma 4.5. *If the ring $k[G]$ is left-Noetherian, then the ring k is left-Noetherian and the group G is Noetherian.*

Proof. Suppose that I is a left ideal of k that is not finitely generated. Set $I[G] = \{\sum_{g \in G} \lambda_g g \in k[G] \mid \lambda_g \in I \text{ for all } g \in G\}$. This is a left ideal of $k[G]$ that can not be finitely generated either.

On the other hand, suppose that H is a subgroup of G that is not finitely generated. Pick $h_1 \in H$ and let H_1 be the subgroup generated by h_1. Now pick $h_2 \in H \setminus H_1$ and let H_2 be the subgroup generated by h_1 and h_2. By repeating this procedure, we create an ascending sequence of subgroups $H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq \ldots$ all contained in H. For each such subgroup, let I_i be the kernel of the map $\phi_i \colon k[G] \to k[G/H_i] \colon e_f \mapsto e_{f \bmod H_i}$. By definition, $H_i \subsetneq H_j$ implies $I_i \subsetneq I_j$. Therefore, this yields an infinite increasing sequence of left ideals of $k[G]$. □

We can generalise this to the setting of any locally small category $\mathcal{C}$.

Proposition 4.6. *If $\mathcal{C}$ contains an object x such that the monoid ring $k[\mathrm{Hom}_{\mathcal{C}}(x,x)]$ is not left-Noetherian, then $\mathrm{Rep}_k(\mathcal{C})$ is not Noetherian.*

Proof. Let I be a left ideal of $k[\mathrm{Hom}_{\mathcal{C}}(x,x)]$ that is not finitely generated. Consider the $\mathcal{C}$-submodule $M \subseteq \mathrm{P}_x$ generated by $I \subset \mathrm{P}_x(x)$. Any $\beta \in M(x)$ can be obtained as a finite sum

$$\beta = \sum_{f \colon x \to x} M(f)(\beta_f) = \sum_{f \colon x \to x} e_f * \beta_f,$$

where $\beta_f \in I$ and $*$ is the product in the monoid ring. It follows that $\beta \in I$ and hence that $M(x) = I$. Suppose that M is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_i \in M(x_i)$ for some $x_i \in \mathcal{C}$. By Proposition 1.14, each α_i can be written as a finite sum

$$\alpha_i = \sum_{f_i \colon x \to x_i} M(f_i)(\beta_{f_i}),$$

for some $\beta_{f_i} \in I$. By definition of the α_i, this implies that each $\beta \in M(x)$ can be written as a finite sum

$$\begin{aligned}\beta &= \sum_{g_i \colon x_i \to x} \lambda_{g_i} \cdot M(g_i)(\alpha_i) \\ &= \sum_{g_i \colon x_i \to x} \sum_{f_i \colon x \to x_i} (\lambda_g \lambda_{f_i}) M(g_i f_i)(\beta_{f_i}) = \sum_{g_i \colon x_i \to x} \sum_{f_i \colon x \to x_i} (\lambda_g \lambda_{f_i}) e_{g_i f_i} \star \beta_f,\end{aligned}$$

for some $\lambda_g \in k$. To summarise, this would mean that any $\beta \in I$ can be generated inside $k[\mathrm{Hom}_{\mathcal{C}}(x,x)]$ by the finite set of elements $\bigcup_{i=1}^{k} \{\beta_{f_i} \mid \lambda_{f_i} \neq 0\} \subseteq I$. This is in contradiction with the fact that the ideal I is a not finitely generated. Hence, M cannot be finitely generated and therefore P_x is not Noetherian. □

In particular, this result implies that the category of $\mathcal{C}$-modules over a ring k is non-Noetherian whenever the ring k itself is not left-Noetherian. Another consequence is that every natural category containing "too many" morphisms induces non-Noetherian module categories.

Example 4.7. Consider the category $\mathbf{CSet}$ of countable sets and (set theoretical) functions. The set of natural numbers $\mathbb{N}$ has a countable number of endomorphisms. Consider the set

$$I = \{\sum_f \lambda_f e_f \in k[\mathrm{Hom}_{\mathbf{CSet}}(\mathbb{N}, \mathbb{N})] \mid \lambda_f = 0, \text{if the image of } f \text{ is finite}\}.$$

If the image of a map f has cardinality n, then the image of any composition gf has cardinality at most n. Hence, I is in fact a left ideal of the monoid $k[\mathrm{Hom}_{\mathbf{CSet}}(\mathbb{N}, \mathbb{N})]$.

Suppose that I is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_i = \sum_f \lambda_f^i e_f$ with all but finitely many $\lambda_f^i \neq 0$. Let H be the set of endomorphisms f of $\mathbb{N}$ such that λ_f^i is nonzero for at least one $i \in \{1, \ldots, \ell\}$. Let m be the maximum cardinality of the image of a map in H. Now, let f_0 be any endomorphism of $\mathbb{N}$ with finite image of cardinality strictly greater than m and consider the element $1 \cdot e_{f_0} \in I$. By the remark about cardinalities above, f_0 cannot be obtained by post composition with a map in H, which means in particular that e_{f_0} cannot be generated by $\{\alpha_1, \ldots, \alpha_\ell\}$. We conclude that I is not finitely generated. By Proposition 4.6, it follows that $\mathrm{Rep}_k(\mathbf{CSet})$ is not Noetherian.

Notice that Proposition 4.6 does not rule out every category with infinite morphism classes. For example, let k be the ring $\mathbb{Z}$ and G the group $(\mathbb{Z}, +)$. Theorem 0.2 in [Put18] (based on a proof in [Hal54]) demonstrates that the group ring $\mathbb{Z}[(\mathbb{Z}, +)]$ is Noetherian. It follows by Proposition 4.3 that $\mathrm{Rep}_{\mathbb{Z}}(\mathbf{Z})$ is Noetherian, where $\mathbf{Z}$ is the category corresponding to the group $(\mathbb{Z}, +)$.

However, this illustrates why the analysis of Question 1.21 has a different (more set theoretical) flavour, once we allow the existence of infinitely many morphisms between two objects.

4.2 Non-Noetherianity

Thus far, we focused on combinatorial categories that yield a positive answers to Question 1.21. We felt the necessity to also seek some non-Noetherian module categories. The categories in this section, induce such examples over any ring k.

Example 4.8. The category $\mathbf{Head}$ consists of an initial object, x_0 (the head), and countably many other objects, x_n $n \in \mathbb{N}$ (the hairs), with no morphisms between them. It is clearly combinatorial, for example consider the functor that maps x_0 to $\emptyset$ and the x_i to any family of distinct sets.

Unfolding the definition in Example 1.10, we see that the principal projective module at the head P_{x_0} maps each object to a copy of k and each morphism to id_k. We consider the $\mathbf{Head}$-submodule $M \subseteq \mathrm{P}_{x_0}$ generated by $\bigcup_{i \in \mathbb{N}} \mathrm{P}_{x_0}(x_i)$. Concretely, it is given by $M(x_0) = \{0\}$, $M(x_n) = \mathrm{P}_{x_0}(x_n)$ and $M(f \colon x_0 \to x_n) \equiv 0$ for all $n \in \mathbb{N}$.

Since there are no nonzero maps available, for each $n \in \mathbb{N}$, at least one element in $M(x_n)$ is needed to generated M. Hence, M is not finitely generated, meaning that P_{x_0} is not Noetherian.

Remark 4.9. Notice that to see $\mathrm{Rep}_k(\mathbf{Head})$ is non-Noetherian, actually all we need is the fact that $\mathrm{Hom}_{\mathbf{Head}}(x_i, x_j) = \emptyset$ for all $i < j$. This namely implies that we need a generator in x_j for arbitrarily large $j \in \mathbb{N}$.

Although the example above may seem somewhat artificial, it provides a method to prove that $\mathrm{Rep}_k(\mathcal{C})$ is non-Noetherian for a general combinatorial category $\mathcal{C}$. Namely, search for a substructure in $\mathcal{C}$ similar to **Head** and consider an infinitely generated submodule of the principal projective module at the head. The following results illustrate this approach. Let **FGrp** be the category of finite groups and group homomorphisms and **FAb** the full subcategory of finite Abelian groups.

Proposition 4.10. *The category* $\mathrm{Rep}_k(\mathbf{FGrp})$ *is non-Noetherian for any ring* k*. The same holds for* $\mathrm{Rep}_k(\mathbf{FAb})$.

Proof. We present the proof for **FGrp**. The argument for **FAb** is exactly the same by restricting all functors to this subcategory. Let G_0 be the cyclic group of order 2. Notice that a group homomorphism $f\colon G_0 \to G$ is fixed by the image of 1, which must lie in $\mathrm{Tor}_2(G) = \{x \in G \mid 2x = 0\}$ (where $2x$ means $x + x$). Therefore, the corresponding principal projective module is given by

$$\mathrm{P}_{G_0}(G) = k[\mathrm{Hom}_{\mathbf{FGrp}}(G_0, G)] \cong \bigoplus_{x \in \mathrm{Tor}_2(G)} k \cdot e_x \text{ for } G \in \mathbf{FGrp},$$
$$\mathrm{P}_X(f)\colon \mathrm{P}_X(G) \to \mathrm{P}_X(H)\colon e_x \mapsto e_{f(x)}, \text{ for } f\colon G \to H \in \mathbf{FGrp}.$$

Enumerate the prime numbers $p_1 = 2, p_2 = 3, \dots$ and for each $i \in \mathbb{N}$ let G_i be the cyclic group of order $2p_i$. Consider the **FGrp**-submodule $M \subseteq \mathrm{P}_{G_0}$ generated by $\bigcup_{i \in \mathbb{N}} \mathrm{P}_{G_0}(G_i)$.

Concretely, if a map $f\colon G_0 \to G$ factors as $G_0 \xrightarrow{h} G_i \xrightarrow{g} G$ it means that $h \equiv 0 \equiv f$ or $f(1) = g(p_1) = p_i g(1)$. It follows that

$$M(G) = \bigoplus_{x \in A^G} k \cdot e_x,$$

where $A^G = \{x \in G \mid x = py \text{ for } y \in G \text{ and } p \text{ prime}\}$. In particular, notice that $M(\mathbb{Z}/2\mathbb{Z}) = k \cdot e_0$. Suppose that M is finitely generated by some set $\{\alpha_1, \dots, \alpha_\ell\}$, where $\alpha_i \in M(H_i)$ for some finite group H_i. Using Proposition 1.14, let $\phi\colon \bigoplus_{i=1}^k \mathrm{P}_{H_i} \to M \in \mathrm{Rep}_k(\mathbf{FGrp})$ be the corresponding epimorphism. Let $m' = \max\{|H_i| \mid i \in \{1, \dots, k\}\}$ and let p_j be a prime that is strictly greater than m'. Consider $e_{p_j} \in M(G_j)$. Suppose that there exists some group H with $|H| \leqslant n$ and some $f\colon H \to G_j$ that maps some $e_x \in M(H)$ to e_{p_j}. This means that $pf(y) = f(py) = f(x) = p_j$ for some $y \in H$ and some prime p satisfying $p \leqslant m' < p_j$. This contradicts the fact that p_j is prime. In particular, this means that $\phi_{G_p}\colon \bigoplus_{i=1}^k \mathrm{P}_{H_i}(G_p) \to M(G_p)$ is not surjective, contradicting the fact that ϕ is an epimorphism. We conclude that M is not finitely generated. □

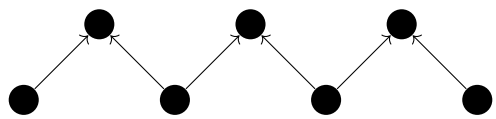
Essentially, the trick in the above proof was to restrict our attention to nonzero maps to recognise the head structure of the G_i. The next category we consider is **FPos**, which consists of finite partially ordered sets and order-preserving maps. This time the trick, is to restrict our attention to maps with what we call a "connected" image. The following objects play a key role.

Definition 4.11. For $n \in \mathbb{N}$, we define the n-sawtooth as a set of $2n+1$ elements $X_n = \{x_1, \ldots, x_{2n+1}\}$ partially ordered by the relations

$$x_{2i} > x_{2i-1} \text{ and } x_{2i} > x_{2i+1} \text{ for } i \in \{1, \ldots, n\}.$$

We call the elements x_1 and x_{2n+1}, the extremities of the set X_n.

We depict a poset as a directed graph where $x \to y$ means $x < y$. The 3-sawtooth X_3 looks like this.



Notice that the sawtooth shape is preserved by any order-preserving map. In particular, for $i < n \in \mathbb{N}$ there exists no order-preserving map $f\colon X_i \to X_n$ such that the image contains both extremities of X_n.

Proposition 4.12. *The category* $\mathrm{Rep}_k(\mathbf{FPos})$ *is non-Noetherian for any ring* k.

Proof. Consider the set $X_0 = \{a, b\}$ with the trivial partial order, that is a and b are unrelated. Notice that any set-theoretic map from X_0 to a partially ordered set $(X, \leqslant)$ is order-preserving. Hence, the corresponding projective module can be written as

$$\mathrm{P}_{X_0}(X) = k[\mathrm{Hom}_{\mathbf{FPos}}(X_0, X)] \cong \bigoplus_{(x_1,x_2)\in X^2} k \cdot e_{x_1,x_2} \text{ for } X \in \mathbf{FPos},$$

$$\mathrm{P}_{X_0}(f)\colon \mathrm{P}_{X_0}(X) \to \mathrm{P}_{X_0}(Y)\colon e_{x_1,x_2} \mapsto e_{f(x_1),f(x_2)}, \text{ for } f\colon X \to Y \in \mathbf{FPos},$$

where we identified a morphism with the ordered tuple of its image. Let M be the $\mathbf{FPos}$-submodule of P_{X_0} generated by $\coprod_{i\in\mathbb{N}} \mathrm{P}_{X_0}(X_i)$, where X_i is the i-sawtooth defined above. Concretely, for an order-preserving map $f\colon X_i \to X$ and any two objects x, x' in the image of f, there exist objects $x_2, x_3, \ldots, x_{2i}$ in the image of f and a "path"

$$x \equiv x_2 \equiv x_3 \equiv \ldots x_{2i} \equiv x',$$

where the symbol $\equiv$ is shorthand for $<, >$, or $=$. Hence, M can be described as

$$M(X) = \bigoplus_{(x,x')\in X^2 \text{ such that } x\sim x'} k \cdot e_{x,x'},$$

where $x \sim x'$ denotes the (equivalence) relation of being connected, that is the existence of a path $x \equiv x_1 \equiv \cdots \equiv x_n \equiv x'$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

Suppose that M is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_j \in M(Y_j)$ for some finite poset Y_j. Let n be the smallest integer such that the cardinality of each Y_j is at most $2n+1$. Now, let N be the **FPos**-submodule of M generated by $\coprod_{i \in \{1,3,5,\dots,n\}} \mathrm{P}_{X_0}(X_i)$. Again, we can describe N concretely by

$$N(X) = \bigoplus_{(x,x') \in X^2 \text{ such that } x \sim_n x'} k \cdot e_{x,x'},$$

where $x \sim_n x'$ denotes the relation of being connected by a path
$x \equiv x_2 \equiv x_3 \equiv \cdots \equiv x_{2n} \equiv x'$ of length at most $2n+1$.

In particular, notice that, since the cardinality of Y_j is at most $2n+1$, $N(Y_j) = M(Y_j)$ for each $j \in \{1, \dots, \ell\}$. By definition of the α_i, it must hold that $N = M$. However, this is clearly not true. For example, consider the n'-sawtooth for any $n' > n$, then the base object corresponding to its extremities satisfies $e_{x_1, x_{2n'+1}} \in M(X_{n'}) \setminus N(X_{n'})$. We conclude that M is not finitely generated and therefore that P_{X_0} is not Noetherian. $\square$

The next objects of interest are the so called $\mathbb{F}_\infty-$modules. The structure $\mathbb{F}_\infty$ was first introduced in [Dur07] as the residue field corresponding to the (usual) Archimedean valuation of the field of fractions $\mathbb{Q}$. However, we follow the hands-on introduction to its modules as done in [HJ17].

Definition 4.13. An $\mathbb{F}_\infty$-module is a set X with

- a transitive and commutative operation $+$, satisfying $x + x = x$,
- an inverse notion $-x$, satisfying $-(-x) = x$ and $-(x + x') = -x + (-x')$,
- a 0 object, satisfying $x + (-x) = 0$,

for each for all $x, x' \in X$. An $\mathbb{F}_\infty$-module homomorphism is a map $f \colon X \to Y$, satisfying $f(x + x') = f(x) + f(x')$ and $f(-x) = -f(x)$.

We let $\mathbf{Mod}_{\mathbb{F}_\infty}$ denote the corresponding category and $\mathbf{FMod}_{\mathbb{F}_\infty}$ the full subcategory of finite objects. The subcategories $\mathbf{FModI}_{\mathbb{F}_\infty}$ and $\mathbf{FModS}_{\mathbb{F}_\infty}$ also consist of all finite $\mathbb{F}_\infty$-modules, but respectively only contain injective or surjective homomorphisms.

Remark 4.14. This definition seems close to that of a group. In particular, $x = -x$ implies that $x = x + x = x + (-x) = 0$ and conversely $-0 = -(0 - 0) = -0 + 0 = 0$. However, the essential difference is that in an $\mathbb{F}_\infty$-module the 0 object acts as a sink by

$$0 + x = 0 + 0 + x = 0 + x + 0 = 0 + x + x + (-x) = 0 + x + (-x) = 0.$$

These new objects carry a natural partial order.

Lemma 4.15. *There is a faithful functor*

$$\Theta \colon \mathbf{FMod}_{\mathbb{F}_\infty} \to \mathbf{FPos} \colon (X, 0, -, +) \mapsto (X, \leqslant),$$

where the relation $x \leqslant x'$ is defined by $x + x' = x$, that sends each morphism to itself.

Proof. First, notice that reflexivity and antisymmetry of $\leqslant$ follow respectively from idempotence and commutativity of the $+$ operation. For transitivity, let $x_1 \leqslant x_2 \leqslant x_3$. This implies that $x_1 + x_3 = (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3) = x_1 + x_2 = x_1$, so that $x_1 \leqslant x_3$. Hence, $(X, \leqslant)$ is a partial order. Furthermore, $x + x' = x$ implies that $f(x) + f(x') = f(x + x') = f(x)$. Hence, Θ is well-defined and faithful by definition. $\square$

The first nontrivial example of an $\mathbb{F}_\infty$-module is $\mathbb{F}_\infty$ itself. It is defined as the set $\{-1, 0, 1\}$, where $x + x' = 0$ except if $x = x' \in \{1, -1\}$. Clearly, there is a bijection

$$\Phi \colon \operatorname{Hom}_{\mathbf{Mod}_{\mathbb{F}_\infty}}(\mathbb{F}_\infty, X) \to X \colon f \mapsto f(1).$$

Hence, $\mathbb{F}_\infty$ is the analogue of the one point set $\{\star\}$ in the category of posets. In search for the analogue of the poset $X_0 = \{a, b\}$, we consider the next class of examples.

Definition 4.16. For each integer $n \geqslant 2$, let P_n be the regular $2n$-gon (geometric objects). Denote its vertices by $v_1, \ldots, v_{2n}$, the edge between v_i and v_{i+1} by e_i and the edge between v_1 and v_{2n} by e_{2n}. The corresponding $\mathbb{F}_\infty$-module, X_n, is the set of all faces of P_n

$$\Big\{P_n\Big\} \cup \Big\{v_i, e_i \mid i \in \{1, \ldots, 2n\}\Big\}.$$

As for the operations, we set $P_n = 0$, $-v_i = v_{i \pm n}$ and $-e_i = e_{i \pm n}$ and

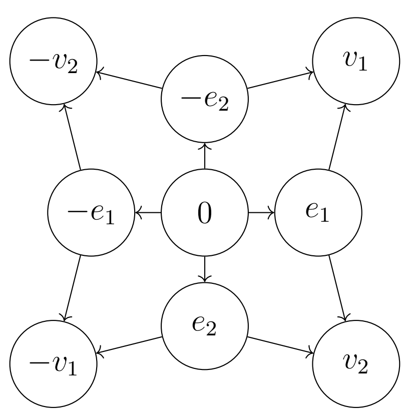
$$x + x' = \begin{cases} e_i & \text{if } \{x, x'\} = \{v_i, v_{i+1}\}, \\ x & \text{if } x = x', \\ 0 & \text{else.} \end{cases}$$

Remark 4.17. An interpretation of the addition is that $x + x'$ is the smallest face containing both x and x'. Hence, the 0 element is P_n itself. Accordingly the partial order on $\Theta(X_n)$ yields, $x \leqslant x'$ when x contains x'. In particular, this means that a part of $\Theta(X_n)$ looks like

$$v_1 > e_1 < v_2 > e_2 < v_3 > \ldots < v_{n-1} > e_{n-1} < v_n.$$

We call v_1 and v_n, the extremities of X_n.

The directed graph of the poset corresponding to the module X_2, arising from the square (4-gone), looks like this.



One should remember that e_1 is nothing but $v_1 + v_2$ and likewise $e_2 = v_2 - v_1$. Hence, in this case there is a bijection

$$\Phi\colon \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{F}_\infty}}(X_2, X) \to X^2\colon f \mapsto (f(v_1), f(v_2)).$$

The problem in reproducing the proof of Proposition 4.12 for $\mathbf{FMod}_{\mathbb{F}_\infty}$, is that by Remark 4.14, the posets $\Theta(X)$ corresponding to $\mathbb{F}_\infty$-modules always have a minimum element 0. In particular, any two points $x, x' \in X$ are connected by the trivial path $x \geqslant 0 \leqslant x'$. Therefore, we cannot adapt the $\sim$ relation in an easy way without losing functoriality. However, essentially the problem is not the existence of these paths but of the morphism that contract nontrivial path down to zero. These morphisms are not injective.

Proposition 4.18. *The category* $\mathrm{Rep}_k(\mathbf{FModI}_{\mathbb{F}_\infty})$ *is non-Noetherian, for any ring* k.

Proof. Let $X_0 = \{0, \pm e_1, \pm e_2\}$ be the $\mathbb{F}_\infty$-submodule of the module X_2 considered above, where we forget about the existence of vertices. For any $\mathbb{F}_\infty$-module, let V_X denote the set $\{(x_1, x_2) \in X^2 \mid x_1 + x_2 = 0 \text{ and } 0 \neq x_1 \neq x_2 \neq 0\}$. Any injective $\mathbb{F}_\infty$-homomorphism, $f\colon X_0 \to X$, is fixed by a pair $(f(e_1), f(e_2)) \in V_X$. Hence, the corresponding principal projective module is

$$\mathrm{P}_{X_0}(X) = k[\mathrm{Hom}_{\mathbf{FModI}_{\mathbb{F}_\infty}}(X_0, X)] \cong \bigoplus_{(x_1,x_2)\in V_X} k \cdot e_{x_1,x_2} \text{ for } X \in \mathbf{FMod}_{\mathbb{F}_\infty},$$

$$\mathrm{P}_{X_0}(f)\colon \mathrm{P}_{X_0}(X) \to \mathrm{P}_{X_0}(Y)\colon e_{x_1,x_2} \mapsto e_{f(x_1),f(x_2)}, \text{ for } f\colon X \to Y \in \mathbf{FModI}_{\mathbb{F}_\infty},$$

where we identified a morphism with the ordered tuple of its image.

Inside any $\mathbb{F}_\infty$-module X, considering the poset $\Theta(X) = (X, \leqslant)$, we introduce the relation of being connected "by above". That is $x \sim x'$ if there exists a path $x < x_1 > x_2 < \cdots < x_n > x'$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. Notice that any injective order-preserving map preserves this relation. This yields a submodule $M \subseteq \mathrm{P}_{X_0}$ defined by

$$M(X) = \bigoplus_{(x_1,x_2)\in V_X \text{ such that } x_1 \sim x_2} k \cdot e_{x_1,x_2}.$$

Suppose that M is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_j \in M(Y_j)$ for some finite $\mathbb{F}_\infty$-module Y_j. Let n be the cardinality of the largest set Y_j. Define a new relation, $x \sim_n x'$ if there exists a path $x < x_2 > x_3 < \cdots < x_{n-1} > x'$ of length n in X. This is still preserved by injective order-preserving maps. Therefore, we can define the submodule $M' \subseteq M$ by

$$M'(X) = \bigoplus_{(x_1,x_2)\in V_X \text{ such that } x_1 \sim_n x_2} k \cdot e_{x_1,x_2}.$$

Since each Y_j has at most n elements the relations $\sim_n$ and $\sim$ coincide in it. Hence, $\alpha_j \in M'(Y_j)$ for all $j \in \{1, \ldots, \ell\}$, which implies that $M' = M$. However, take any integer $n' > \frac{n+1}{2}$ and consider the $2n'$-gone $\mathbb{F}_\infty$-module $X_{n'}$ from Definition 4.16. By Remark 4.17, its extremities satisfy $v_1 \sim v_{n'}$ by a path of length $2n' - 1 > n$ and the tuple $(v_1, v_{n'})$ is an element of $V_{X_{2n'}}$. This means that $e_{v_1,v_{n'}} \in M(X_{2n'}) \setminus M'(X_{2n'})$, which contradicts the assumption that $M = M'$. We conclude that M is not finitely generated and hence that P_{X_0} is not Noetherian.. $\square$

Conversely, one could also consider the case of surjections instead of injections. The category $\mathbf{FModS}_{\mathbb{F}_\infty}$ trivially leads to Noetherian module categories by Proposition 4.1. The more interesting category to consider is the opposite one.

Proposition 4.19. *The category* $\mathrm{Rep}_k(\mathbf{FModS}_{\mathbb{F}_\infty}^{\mathrm{op}})$ *is non-Noetherian, for any ring* k.

Proof. As in the proof of Theorem 4.18, we consider the $\mathbb{F}_\infty$-module $X_0 = \{0, \pm e_1, \pm e_2\}$. In this case, a surjective $\mathbb{F}_\infty$-homomorphism $f\colon X \to X_0$ is fixed by the disjoint sets $U_1 = f^{-1}(e_1)$ and $U_2 = f^{-1}(e_2)$. The necessary and sufficient condition on those sets to define a morphism is that

$$x, x' \in U_i \text{ if and only if } x + x' \in U_i \text{ for both sets } i = 1, 2. \tag{$\star$}$$

Hence, the corresponding principal projective module looks like

$$\mathrm{P}_{X_0}(X) = k[\mathrm{Hom}_{\mathbf{FModS}_{\mathbb{F}_\infty}}(X, X_0)] \cong \bigoplus_{(U_1,U_2)} k \cdot e_{U_1,U_2},$$
$$\mathrm{P}_{X_0}(f)\colon \mathrm{P}_{X_0}(X) \to \mathrm{P}_{X_0}(Y)\colon e_{U_1,U_2} \mapsto e_{f^{-1}(U_1), f^{-1}(U_2)},$$

where $f\colon Y \to X \in \mathbf{FModS}_{\mathbb{F}_\infty}$ and the direct sum is taken over all tuples of disjoint subsets of X satisfying condition ($\star$).

Inside any $\mathbb{F}_\infty$-module X, considering the poset $\Theta(X) = (X, \leqslant)$, we introduce the relation of being connected "by below but above 0". That is $x \sim x'$ if there exists $x_1, \ldots, x_n \in X$ such that

$$\begin{aligned} &x \not\equiv x_i \not\equiv x' \text{ and } x_i \not\equiv x_j \text{ for all } i \neq j \in \{1, \ldots, n\} \text{ and} \\ &x + x_1 \neq 0, \quad x_1 + x_2 \neq 0, \quad \ldots \quad , \quad x_{n-1} + x_n \neq 0, \quad x_n + x' \neq 0, \end{aligned}$$

where the notation $x \equiv x'$ is shorthand for $x < x', x > x'$ or $x = x'$. For any surjective $\mathbb{F}_\infty$-homomorphism $f\colon Y \to X$, and any $x, x' \in X$ there exist $y \in f^{-1}(x)$ and $y' \in f^{-1}(x')$. Moreover, $x \not\equiv x'$ implies that $y \not\equiv y'$ and $x + x' \neq 0$ implies that $y + y' \neq 0$. Hence, the $\sim$ relation is preserved in $\mathbf{FModS}_{\mathbb{F}_\infty}^{\mathrm{op}}$. We say that a pair of disjoint subsets (U_1, U_2) is admissible if ($\star$) is satisfied and there exist $x_1 \in U_1$ and $x_2 \in U_2$ such that $x_1 \sim x_2$. This yields a submodule $M \subseteq \mathrm{P}_{X_0}$ given by

$$M(X) = \bigoplus_{(U_1,U_2) \text{ admissible}} k \cdot e_{U_1,U_2}.$$

Suppose that M is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_j \in M(Y_j)$ for some finite $\mathbb{F}_\infty$-module Y_j. Let n be the first odd integer such that each Y_j has cardinality at most n. Define the new relation $x \sim_n x'$ if there exists $x_2, x_3, \ldots, x_{\frac{n-1}{2}} \in X$ such that

$$\begin{aligned} &x \not\equiv x_i \not\equiv x' \text{ and } x_i \not\equiv x_j \text{ for all } i \neq j \in \left\{2, 3, \ldots, \frac{n-1}{2}\right\} \text{ and} \\ &x + x_2 \neq 0, \quad x_2 + x_3 \neq 0, \quad \ldots \quad , \quad x_{\frac{n-1}{2}} + x' \neq 0. \end{aligned}$$

In other words, we again fix the length of the path to be n. This relation is also preserved in $\mathbf{FModS}_{\mathbb{F}_\infty}^{\mathrm{op}}$. Therefore, we can define the pair of disjoint subsets (U_1, U_2) to be

n-admissible if $(\star)$ is satisfied and there exist $x_1 \in U_1$ and $x_2 \in U_2$ such that $x_1 \sim_n x_2$. This induces a submodule $M' \subseteq M$ given by

$$M'(X) = \bigoplus_{(U_1,U_2)\ n-\text{admissible}} k \cdot e_{U_1,U_2}.$$

The sets Y_j have at most n elements so that the relations $\sim_n$ and $\sim$ coincide in it. It follows that $\alpha_j \in M'(Y_j)$ for all $j \in \{1, \ldots, k\}$, which implies that $M' = M$. However, take any integer $n' > \frac{n+1}{2}$ and consider the $2n'$-gone $\mathbb{F}_\infty$-module $X_{n'}$ from Definition 4.16. By Remark 4.17, the extremities satisfy $v_1 \sim v_{n'}$ by a path of length $2n' - 1 > n$. Moreover, the disjoint subsets $U_1 = \{y_1\}$ and $U_2 = \{y_m\}$ clearly satisfy $(\star)$, which means that $e_{U_1,U_2} \in M(X_m) \setminus M'(X_m)$. Hence, we reach a contradiction and conclude that M is not finitely generated and that P_{X_0} is not Noetherian □

Towards the end of the project, we realised that all results in this section are applications of the following statement, which is a partial converse to the Gröbner method.

Proposition 4.20. *Let k be a ring and let $\mathcal{C}$ be an essentially small category. Assume that $\mathcal{C}$ does not satisfy property (G2) from Definition 1.32, then $\mathrm{Rep}_k(\mathcal{C})$ is not locally Noetherian.*

Proof. By assumption, there is an object x_0 such that $|\mathcal{C}_{x_0}|$ is not Noetherian. Hence, there exists $f_i \colon x_0 \to x_i$ for each $i \in \mathbb{N}$ such that the sequence $f_1, f_2, f_3, \ldots$ satisfies $f_i \not\leq f_j$ whenever $i < j$.

Consider the principal projective module $\mathrm{P}_{x_0} \in \mathrm{Rep}_k(\mathcal{C})$ and the submodule M generated by $\{e_{f_i} \in \mathrm{P}_{x_0}(x_i) \mid i \in \mathbb{Z}_{>0}\}$. Suppose that M is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_j \in M(y_j)$ for some object $y_j \in \mathcal{C}$. By definition of M, each α_j is equal to a finite sum of the form

$$\sum_{i=1}^{N_j} \sum_{g_{ij} \colon x_i \to y_j} M(g_{ij})(e_{f_i}).$$

Hence, we conclude that M is also generated by $\{e_{f_i} \in \mathrm{P}_{x_0}(x_i) \mid i \in \{1, \ldots, N\}\}$, where $N = \max\{N_j \mid j \in \{1, 2, \ldots, \ell\}\}$. However, by assumption $e_{f_{N+1}} \in \mathrm{P}_{x_0}(x_{N+1})$ can not be generated by these elements. We conclude that M is not finitely generated and therefore that P_{x_0} is not Noetherian. □

4.3 Finite modules over a finite ring

Let R be a finite commutative ring. At the end of section 1.3, we raised the question whether or not the category $\mathbf{FMod}_R$, of finite R-modules and R-linear maps, induces Noetherian module categories.

In Theorem 1.51, we saw this holds for the full subcategory of free objects $\mathbf{V}_R$. We are able to enlarge the class of considered modules one first step.

Definition 4.21. Let $\mathbf{FPMod}_R$ be the category of finite (or equivalently finitely generated) projective modules over the finite ring R.

Proposition 4.22. *Let R be a finite commutative ring and k a left-Noetherian ring, then $\mathrm{Rep}_k(\mathbf{FPMod}_R)$ is Noetherian.*

Proof. First, we consider the principal projective module $\mathrm{P}_{R^n} \in \mathrm{Rep}_k(\mathbf{FMod}_R)$, corresponding to the free R-module R^n for some $n \in \mathbb{N}$. Let $M \subseteq \mathrm{P}_{R^n}$ be any submodule. Notice that we can define the restrictions of these $\mathbf{FPMod}_R$-modules to the full subcategory $\mathbf{V}_R$. Clearly, $\mathrm{P}_{R^n} \restriction_{\mathbf{V}_R}$ is nothing else but the principal projective module of R^n in $\mathbf{V}_R$, $\mathrm{P}^{\mathbf{V}_R}_{R^n}$. By Theorem 1.51, $\mathbf{V}_R$ is quasi-Gröbner and thus $\mathrm{Rep}_k(\mathbf{V}_R)$ is Noetherian. Therefore, $\mathrm{P}^{\mathbf{V}_R}_{R^n}$ is Noetherian and in particular $M \restriction_{\mathbf{V}_R}$ is finitely generated by some set $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_i \in M(R^{n_i})$ for some $n_i \in \mathbb{N}$. This generates all elements in $M(R^m)$ for any $m \in \mathbb{N}$.

Let P be any finite projective R-module and pick any $\beta \in M(P)$. As P is finitely generated, there exists a surjective R-linear map $f\colon R^m \to P$ for some $m \in \mathbb{N}$. As P is projective, this map must split, meaning that there exists an R-linear map $g\colon P \to R^m$ such that $f \circ g = \mathrm{id}_P$. As $M(g)(\beta)$ is an element in $M(R^m)$, it is generated by $\{\alpha_1, \ldots, \alpha_\ell\}$. We conclude that $\beta = M(\mathrm{id}_P)(\beta) = M(f)(M(g)(\beta))$ is also generated by $\{\alpha_1, \ldots, \alpha_\ell\}$. As P and β were chosen arbitrarily, this implies that the elements $\{\alpha_1, \ldots, \alpha_\ell\}$ actually generate the full $\mathbf{FPMod}_R$-module M. As M was also arbitrary, we conclude that P_{R^n} is Noetherian.

Next, we consider the principal projective module $\mathrm{P}_P \in \mathrm{Rep}_k(\mathbf{FPMod}_R)$ corresponding to any finite projective R-module P. Let $M \subseteq \mathrm{P}_P$ be any module. Once again, we use the existence of a surjection $f\colon R^m \to P$. It namely leads to the natural transformation, $\mathrm{P}_f\colon \mathrm{P}_P \to \mathrm{P}_{R^m}$, defined at level $L \in \mathbf{FPMod}_R$ by the map

$$\mathrm{P}_{f_L}\colon k[\mathrm{Hom}_R(P, L)] \to k[\mathrm{Hom}_R(R^m, L)]\colon e_h \mapsto e_{hf}.$$

To check functoriality, notice that for $q\colon L \to L' \in \mathbf{FPMod}_R$, it holds that

$$\mathrm{P}_{f_{L'}} \circ \mathrm{P}_N(q)(e_h) = \mathrm{P}_{f_{L'}}(e_{qh}) = e_{qhf} = \mathrm{P}_{R^m}(q)(e_{hf}) = \mathrm{P}_{R^m}(q) \circ \mathrm{P}_{f_L}(e_h).$$

Notice, by Proposition 1.7, that P_f is a monomorphism. Define $\tilde{M}(L) = \mathrm{P}_{f_L}(M(L))$ and notice that if $\tilde{\beta} = \mathrm{P}_{f_L}(\beta) \in \tilde{M}(L)$, then for any $q\colon L \to L'$ it follows that

$$\mathrm{P}_{R^m}(q)(\tilde{\beta}) = \mathrm{P}_{R^m}(q) \circ \mathrm{P}_{f_L}(\beta) = \mathrm{P}_{f_{L'}} \circ \mathrm{P}_N(q)(\beta) = \mathrm{P}_{f_{L'}}(M(\beta)),$$

which is an element in $\tilde{M}(L')$. We conclude that $\tilde{M}$ is a submodule of P_{R^m}. By the first part of the proof, it is finitely generated by some set $\{\tilde{\alpha_1}, \ldots, \tilde{\alpha_\ell}\}$, where $\tilde{\alpha_i} \in \tilde{M}(L_i)$. As before, because P is projective the map f splits. Hence, there is $g\colon P \to R^m$ such that $f \circ g = \mathrm{id}_P$. This leads to a natural transformation, $\mathrm{P}_g\colon \mathrm{P}_{R^m} \to \mathrm{P}_P$, defined analogously to P_f. It follows immediately that $\mathrm{P}_g \circ \mathrm{P}_f = \mathrm{id}_{\mathrm{P}_P}$. Finally, set $\alpha_i = \mathrm{P}_{g_{L_i}}(\tilde{\alpha_i})$ for each $i \in \{1, \ldots, \ell\}$ and pick $L \in \mathbf{FPMod}_R$ and $\beta \in M(L)$ arbitrarily. It follows that

$$\begin{aligned}\beta = \mathrm{P}_{g_L} \circ \mathrm{P}_{f_L}(\beta) &= \mathrm{P}_{g_L}\Big(\sum_{i=1}^{\ell} \sum_{h\colon L_i \to L} \lambda_h \tilde{M}(h)(\tilde{\alpha_i})\Big) \\ &= \sum_{i=1}^{\ell} \sum_{h\colon L_i \to L} \lambda_h M(h) \mathrm{P}_{g_{L'}}(\alpha_i) = \sum_{i=1}^{\ell} \sum_{h\colon L_i \to L} \lambda_h M(h)(\alpha_i).\end{aligned}$$

Hence, β is generated by $\{\alpha_1, \dots, \alpha_\ell\}$. We conclude that M is finitely generated and therefore that P_P is Noetherian. $\square$

Recall that a commutative ring R is called a principal ideal ring (PIR) if each ideal is generated by one element. We denote by (x) the ideal generated by $x \in R$. As in the well known case of principal ideal domains, there exists a structure theorem for finitely generated modules over a principal ideal ring.

Lemma 4.23. *Let R be a principal ideal ring. Each finitely generated R-module is of the form*

$$R^{n_0} \oplus (R/(x_1))^{n_1} \oplus (R/(x_2))^{n_2} \cdots \oplus (R/(x_\ell))^{n_\ell},$$

where $(x_1), \dots, (x_l)$ are distinct nontrivial principal ideals in R and n_i is a non-negative integer for each $i \in \{0, 1, \dots, \ell\}$.

Proof. This result is of purely commutative algebraic nature[1]. As each ideal is generated by one element, a principal ideal ring is in particular a so-called Noetherian ring. T.W. Hungerford proves in Theorem 1 [Hun68], that a principal ideal ring is a so-called elementary divisor ring. It follows from I. Kaplansky's Theorems 9.1 and 9.3 [Kap49], that modules over a Noetherian elementary divisor ring are of the form mentioned above. $\square$

Proposition 4.24. *If R is a finite principal ideal ring, then $\mathbf{FMod}_R$ is quasi-Gröbner.*

Proof. As R is finite, it has finitely many ideals $(x_1), \dots, (x_\ell)$. In Theorem 1.51, we saw that the full subcategory of free R-modules of finite rank $\mathbf{V}_R$ is quasi-Gröbner. For each ideal I, let $\mathbf{V}_R^I$ be the full subcategory of R-modules of the form $(R/I)^n$ for $n \in \mathbb{N}$. Since $\mathrm{Hom}_R((R/I)^n, (R/I)^m) = \mathrm{Hom}_{R/I}((R/I)^n, (R/I)^m)$, notice that this category is isomorphic to the category of free R/I-modules of finite rank, $\mathbf{V}_{R/I}$. It follows, from Remark 1.37, that the category $\mathbf{V}_R \times \mathbf{V}_R^{(x_1)} \times \cdots \times \mathbf{V}_R^{(x_\ell)}$ is quasi-Gröbner as well. We consider the functor

$$\begin{gathered}\Phi\colon \mathbf{V}_R \times \mathbf{V}_R^{(x_1)} \times \cdots \times \mathbf{V}_R^{(x_\ell)} \to \mathbf{FMod}_R \\ \left(R^{n_0}, (R/(x_1))^{n_1}, \dots, (R/(x_\ell))^{n_\ell}\right) \mapsto R^{n_0} \oplus (R/(x_1))^{n_1} \oplus \cdots \oplus (R/(x_\ell))^{n_\ell},\end{gathered}$$

mapping a tuple of morphisms to the corresponding component-wise morphism on the direct sum. By Lemma 4.23, Φ is essentially surjective. Pick any morphism $f\colon N \to L \in \mathbf{FMod}_R$. It is of the form

$$R^{n_0} \oplus (R/(x_1))^{n_1} \oplus \cdots \oplus (R/(x_\ell))^{n_\ell} \to R^{m_0} \oplus (R/(x_1))^{m_1} \oplus \cdots \oplus (R/(x_\ell))^{m_\ell}.$$

For $j \in \{1, \dots, m_0\}$, let $p_j\colon L \to R$ be the projection on the j'th coordinate (the j'th copy of R in R^{m_0}). Notice that $p_j \circ f$ must be one of the finitely many maps in $\mathrm{Hom}_R(N, R)$, which is bijective to

$$(\mathrm{Hom}_R(R, R))^{n_0} \times (\mathrm{Hom}_R(R/(x_1), R))^{n_1} \times \cdots \times (\mathrm{Hom}_R(R/(x_\ell), R))^{n_\ell}.$$

[1]This proof was found in the following discussion https://mathoverflow.net/questions/22722.

Similarly, for any one of the projections $p_j\colon N \to R/(x_i)$, where $j \in \{n_0+1, n_0+2, \dots, \sum_{i=0}^{\ell} n_i\}$, it holds that $p_j \circ f$ corresponds to an element of

$$(\mathrm{Hom}_R(R, R/(x_i)))^{n_0} \times (\mathrm{Hom}_R(R/(x_1), R/(x_i)))^{n_1} \times \cdots \times (\mathrm{Hom}_R(R/(x_\ell), R/(x_i)))^{n_\ell}.$$

Therefore, let $f_i\colon N \to R^{n'} \oplus (R/(x_1))^{n'_1} \oplus \cdots \oplus (R/(x_\ell))^{n'_\ell}$ be all morphisms satisfying

$$n'_i \leqslant \prod_{q=0}^{\ell} |\mathrm{Hom}_R(R/(x_q), R/(x_i))|^{n_q} \text{ for each } i \in \{0, 1, \dots, \ell\}, \tag{$*$}$$

where x_0 is the zero element of R. Note that, since R is finite and all coefficients are bounded, these are finitely many maps. If the morphism $f\colon N \to L$ is not equal to one of these f_i, it means that there is at least one m_i not satisfying $(*)$. This implies that for the m_i projections, $p_j\colon R \to R/(x_i)$, there exist $j_1 \neq j_2 \in \{n_{i-1}+1, n_{i-1}+2, \dots, n_i\}$ such that $p_{j_1} \circ f = p_{j_2} \circ f$. Therefore, we can factor f as

$$N \xrightarrow{f'} \left(R^{m_0} \oplus (R/(x_1))^{m_1} \oplus \cdots \oplus (R/(x_1))^{(m_i-1)} \oplus \cdots \oplus (R/(x_\ell))^{m_\ell}\right) \xrightarrow{\Phi g'} L,$$

where f' is defined as f forgetting about the coordinate j_2 in the target and g is the identity on all summands except for $e_{j_1} \mapsto e_{j_1} + e_{j_2}$. We repeat this procedure until the target of f' satisfies condition $(*)$. At this point, we have factored f as $\Phi(g) \circ f_i$, for an injective morphism g in the category $\mathbf{V}_R \times \mathbf{V}_R^{(x_1)}) \times \cdots \times \mathbf{V}_R^{(x_\ell)}$ and one of the f_i above. As N was arbitrary, we conclude that Φ satisfies property (F) and hence that $\mathbf{FMod}_R$ is quasi-Gröbner. $\square$

One more direct generalisation of this result.

Corollary 4.25. *If R is the product of finitely many finite principal ideal rings, then $\mathbf{FMod}_R$ is quasi-Gröbner.*

Proof. Remember that if R_1 and R_2 are commutative rings, $\mathbf{FMod}_{R_1 \times R_2}$ is isomorphic to the product category $\mathbf{FMod}_{R_1} \times \mathbf{FMod}_{R_2}$. In particular, if $R = \prod_{i=1}^{n} R_i$ it follows that $\mathbf{FMod}_R \cong \prod_{i=1}^{n} \mathbf{FMod}_{R_i}$. Assuming that all R_i are finite principal ideal rings, the result follows from Proposition 4.24 and Remark 1.37. $\square$

Discussion

During this project, we encountered various questions that we were not able to solve yet. We list the ones that are most interesting in our opinion.

- Is any combinatorial category that satisfies property (G2) quasi-Gröbner? If this were true, Corollary 1.38 would turn into an if and only if statement, by combining Proposition 4.20 and Proposition 4.6. This would yield a complete characterisation of our fundamental Question 1.21.
- What more can be said about the existence of torsion in higher homology groups of the unordered configuration space of graphs? Ultimately, one could hope for a characterisation, similar to Proposition 3.7 for general order i. In the (few) computations we realised, the appearance of torsion in the reduced Świątkowski chain complex seems to be bounded to rigid rules.
- How does the configuration space of a graph and its homology groups vary when adding more and more particles ($n >> 0$)? The rows of the reduced Świątkowski chain complex seem to be quite similar, once n exceeds the number of essential vertices in G.
- Is the category $\mathbf{FMod}_R$ quasi-Gröbner for any finite commutative ring? Because a finite ring is Artinian, we can reduce the question to the case of finite local rings, as in Corollary 4.25. However, we lack a description of all finitely generated modules over these rings to proceed as in Proposition 4.24.
- Does the category $\mathbf{FMod}_{\mathbb{F}_\infty}$, containing all morphism, induce Noetherian module categories or not? The statement of Jakob Scholbach [Sch14] that $\mathbb{F}_\infty$ "is badly behaved from a K-theoretic point of view.", leads us to expect that the answer is no. However, we did not find a violation of axiom (G2).

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