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Evolution of self-gravitating circumstellar disks

described by Schrödinger equation

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Introduction

An understanding of the physical processes involved during the evolution of circumstellar disks is one of the most important issues of astrophysics in recent years, as the study of planet formation has been revolutionized by recent observational breakthroughs and since the branch which deals with exoplanets has been flourishing for the last two decades, since the first confirmation of a giant exoplanet orbiting a main-sequence star, 51 Pegasi, in 1995.

One of the aims of the study of the phenomena that shape the formation of planetary systems is to elaborate an effective and succinct formulation of the processes involved in the evolution of the circumstellar disk.

This dissertation demonstrates how the temporal evolution of a protoplanetary disk, under specific conditions, can be described by Schrödinger's equation.

In particular, Chapter 1 consists in an introductory overview of the characteristics of a circumstellar disk, focusing on the description of the type of disk concerned by the subsequent analysis and introducing its general framework. Chapter 2 includes a brief interlude explaining the formulation of Schrödinger's equation and its connection with Classical Mechanics. Chapter 3 illustrates the most important analysis methods in use to solve the described problem, characterizes a specific type of disk and presents the actual derivation of the solution. Lastly, Chapter 4 deals with the discussion of some approximations made during the analysis, in the attempt to generalize the model as much as possible.

Chapter 1

Introduction to circumstellar disks

The initial purpose of astronomic studies was to try to find our position or the lack of it in the universe. The first step to do so, obviously, was to observe and explore what surrounds us, the Solar System, in the attempt to understand it. The observed objects were numerous and extremely diversified: planets, which could be clearly distinguished between terrestrial and gas giants, microplanets and nanoplanets, asteroids gathered in the Asteroid Belt, satellites such as Earth's Moon, comets and so on. It was necessary to try to explain their origins, their evolution and ultimately understand their dynamics in order to predict their trajectories.

Since the Copernican revolution astronomers have observed that the planets of the Solar System are orbiting around the Sun in the same sense, and approximately in the same plane. In the eighteenth century Swedenborg, Kant, and Laplace, considering that this instance could not have arisen by chance, proposed roughly what is still today the model of planet formation: the planets "condensed" out of a flattened cloud of gas rotating around the Sun. This model introduced the concept of the protoplanetary disk (also called for short 'PP disk') which, though abandoned during a short period in the early twentieth century, is still at the heart of modern theories of star and planet formation.

Nowadays there is no doubt that the shaping of protostellar disks plays a central role in the context of star and planet formation. Protostars are defined as young stars that gather mass from their parent cloud, growing. The circumstellar disk surrounding the protostar becomes at later stages the natural progenitor of possible planets.

Most astrophysical disks, but not all of them, feed a central mass: by facilitating the transfer of angular momentum, they allow the accretion of material and are therefore essential to star, planet, and satellite formation. With the same principle, they can, for example, regulate the growth of super-massive black holes at the center of a galaxy, therefore indirectly influencing galactic structure.

Although the characteristics of astrophysical disks can range over an enormous variety of length scales, physical properties, compositions and age, they all share the same basic dynamics besides many physical phenomena.

In this chapter we offer an overview of some of the features of disks: section 1.1 has the purpose to present a brief characterization of these objects, explaining how they are defined and observed. Section 1.2 deals with the so-called self-gravitating disks, which are the ones interested by the studies presented later on in this dissertation. Section 1.3 finally gives a preview of the description of the evolution of self-gravitating circumstellar disks by Schrödinger's equation, showing the peculiarity of this type of study.

1.1 Disk description

As previously mentioned, protostellar disks play a central role in the context of star and planet formation. More specifically, according to the nebular hypothesis, disks form because young stars are born from clouds of diffuse gas (approximately 10^5 particles per cm^3) that are characterized by a high angular momentum which prevents the cloud itself to collapse directly to stellar densities (10^{24} particles per cm^3).

Circumstellar disks are therefore an inevitable consequence of the conservation of the angular momentum of the cloud during the phase of formation of the central star through gravitational collapse. Initially disks rapidly channel material towards and ultimately onto the star, little by little the accretion rate decreases: some material is drawn by the star, some matter is dispersed through different mechanisms and a certain amount of material persists in the disk.

Typical values of the disk mass are usually between 0.003 and $0.3 M_{\odot}$, with M_{\odot} being the mass of the Sun. For example, it is estimated that the Minimum Mass Solar Nebula (MMSN), which was the protoplanetary disk that contained the minimum amount of matter necessary to build the planets of the Solar System, had a mass of around $0.02 M_{\odot}$. About the typical dimensions, disks usually extend radially up to $\sim 1000AU$, and are characterized by aspect ratios of about $h/r \sim 0.05$, where h is the disk's vertical scale-height and r is its radius.

Stars in the main sequence are primarily characterized by the proton-proton chain, that means that they burn forming helium through the process of fusion of hydrogen atoms: protoplanetary disks are as a matter of fact made up of relatively cool gas, mostly H_2 , scattered with dust.

The disk exhibits a range of temperatures: the inner parts of the disk are hot ($800 - 900K$), whilst the heating per unit area drops off along increasing distance from the star ($20 - 30K$ for the outer part), even though if the disks extend to a large enough radius, heating due to other surrounding stars can become an important factor. However, the higher temperature of the central part is caused not so much by the presence of the star as by viscosity: internal friction and density are significantly higher towards the inner regions.

While the star emits electromagnetic radiation as black-body radiation, the dust surrounding it reflects this light and re-emits it at a different temperature, contributing to the spectral energy distribution with an infrared excess. Therefore disks can be observed with infrared and radio telescopes. Moreover, since each specific portion of the disk gives a particular black-body signal depending on its temperature and density, it is possible to deduce these two quantities of the specific areas by analyzing said signal. We can therefore determine a temperature and a density profile for the observed disk.

For this reason disks were originally inferred from infrared excesses, directly imaged first in the sub-millimeter, and only later on in the optical, with the discovery of numerous examples in the Orion Nebula operated by the Hubble Space Telescope. The Infrared Astronomical Satellite (IRAS) was the first to allow statistical studies regarding disks, managing to open up the infrared sky. Shortly after the first sensitive detectors at millimeter wavelengths showed that many disks contained large dust grains with enough material to form planetary systems on the scale of our own.

The Hubble Space Telescope managed to give unequivocal evidence for the flattened morphology of disks, thanks to clean images of disk shadows against a bright nebular background. Numerous discoveries were made, in particular, during the last decade, due to improvements in sensitivity, resolution, and wavelength coverage. The Infrared Space Observatory (ISO) and the Spitzer Space Telescope (Spitzer) have been fundamental to expand the catalogue of known disks, while the Submillimeter Array (SMA) and new facilities including the Herschel Space Observatory (Herschel) and Atacama Large Millimeter/-submillimeter Array (ALMA) have been essential to include the sub-millimeter regime.

These observations have highlighted the unevenness of disks: this is due to the different phases of their

evolution, since these objects are studied at different times from their formation and therefore they can be characterized by gaps, asymmetries, spirals, planets.

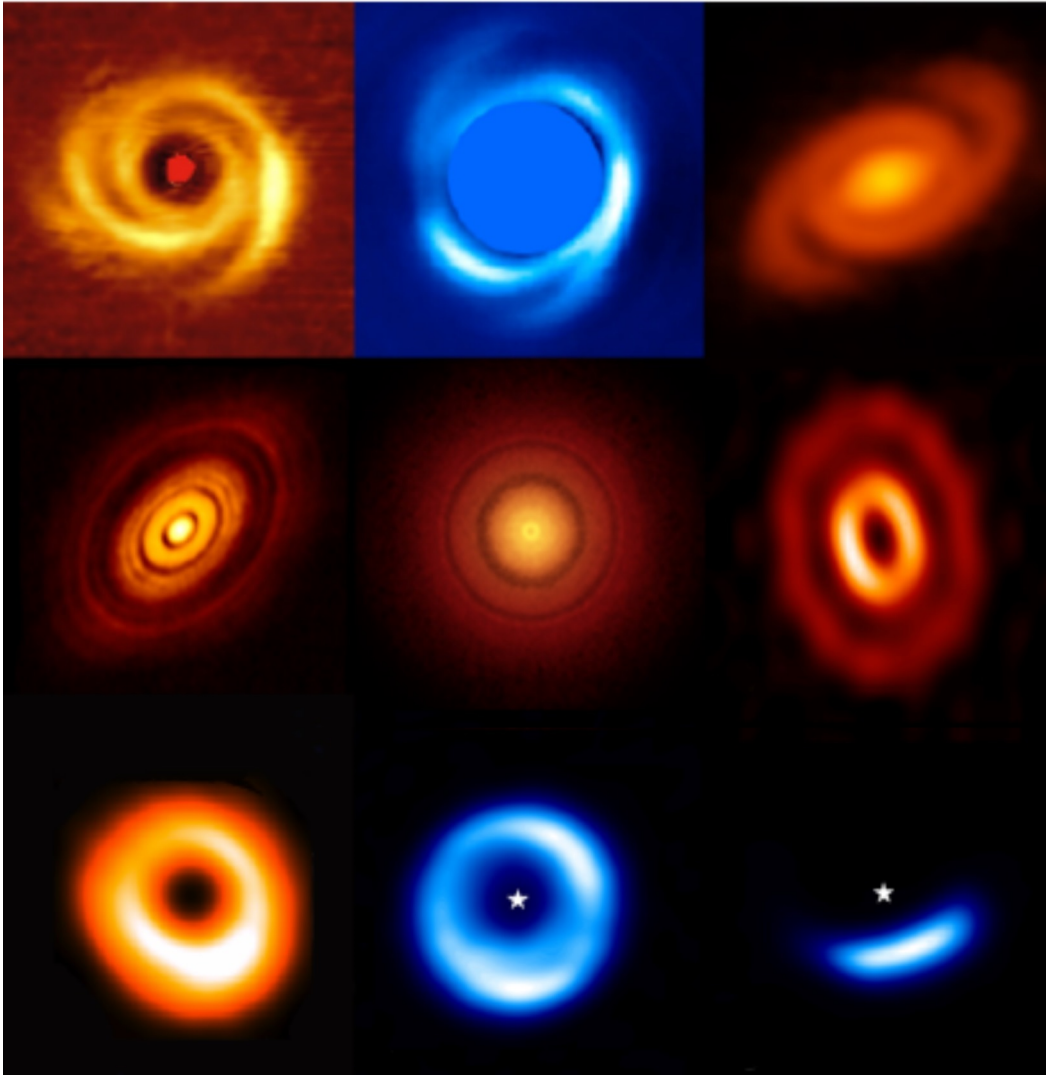


Figura 1: Some examples of the possible substructures of circumstellar disks. From left to right and from top to bottom the images show the disks orbiting around: HD 135344B (Garufi et al. 2013), HD 100453 (Wagner et al. 2015), Elias 2-27 (Perez et al. 2016), HL Tau (Alma Partnership et al. 2015), TW Hya (Andrews et al. 2016), HD 163296 (van der Plas et al. 2016), HD 135344B (Perez et al. 2014), SR 21 (Pinilla et al. 2015), Oph IRS 48 (van der Marel et al. 2013). The first row shows a spiral substructure, the second one the presence of gaps and the third one horseshoe-shaped disks.

The main stages of the evolution can be identified in:

- Protoplanetary disk: It presents large quantities of gas and dust and can have the potential to form planets and other objects.
- Transition disk: The disk dissipated from inside out, creating a central region with no mass (planet formation can have occurred), while gas density is extremely reduced.

- Debris disk: The disk consists exclusively of dust and larger solids. This type of disk can be observed around stars up to approximately 10 Myr of age (the first one was discovered orbiting Vega by the IRAS satellite), since older stars rarely present a disk.

The understanding of the temporal evolution of disks and their response to external perturbations is essential in order to interpret these observations, and is a key factor in the development of a planet formation theory. Modern theories of disk dissipation and evolution take into account several factors and mechanisms, such as viscosity and magnetic instability, hydrodynamical evolution, photoevaporation, dust settling, dynamical influences and formation of pebbles, planetesimals and eventually planets and similar objects.

While disks composed predominantly of gas (especially hydrogen and helium) can be described with some approximations as fluid disks, therefore obeying to magneto-hydrodynamic laws, planetesimal disks (or alternatively disks of stars orbiting a supermassive black hole) are characterized by a nearly purely dynamical evolution. In this dissertation we analyze the longterm evolution of self-gravitating disks.

1.2 Self-gravitating disks

Having briefly described circumstellar disks in the previous section, it is now necessary to mention and illustrate the nature of self-gravitating disks, since their evolution is discussed in Chapter 3.

Self-gravity is essentially the process by which the individual constituents of a body are held together by the combined effect of the gravitational force of the object as a whole. Given that gravity is much weaker than the other forces, it becomes particularly relevant when we deal with massive bodies.

A simple example is given by the interaction between two protons: if we suppose to position them with a spacing of 1 cm, the gravitational force will be obviously given by Newton's law of universal gravitation:

$$F_{gravitational} = \mathcal{G} \frac{m_1 m_2}{r^2} \quad (1.1)$$

with the gravitational constant $\mathcal{G} \simeq 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ and the masses being $m_1 = m_2 \simeq 1.67 \times 10^{-27} \text{ kg}$. The force hence results $F_{gravitational} \approx 1.87 \times 10^{-60} \text{ N}$, while the electric repulsion is given by Coulomb's law

$$F_{Coulomb} = k_e \frac{q_1 q_2}{r^2} \quad (1.2)$$

where Coulomb's constant is $k_e \simeq 8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$ and the electric charges are $q_1 = q_2 \simeq 1.60 \times 10^{-19} \text{ C}$, therefore $F_{Coulomb} \approx 2.30 \times 10^{-24} \text{ N}$.

This means that self-gravity has an important impact in relation to the physical behavior of large scale objects, such as those who are being dealt with in astronomy: planets, stars and, clearly, disks.

It is important to note that if self-gravity no longer existed, objects such as stars or galaxies would all expand and dissipate. However, if self-gravity was unopposed, particles would clump together. Therefore self-gravitational forces are fundamental to understand planetesimal formation and thus planet formation, in addition to providing an explanation for how accretion disks form and how they stabilize.

The condition for a differentially rotating accretion disk to be marginally unstable to gravitational instability is given by

$$\mathcal{Q} = \frac{c_s \kappa}{\pi \mathcal{G} \Sigma} < 1 \quad (1.3)$$

where c_s is the sound speed, κ is the epicyclic frequency, \mathcal{G} is Newton's gravitational constant and Σ is the surface density of the considered disk.

\mathcal{Q} is habitually referred to as "Toomre \mathcal{Q} " parameter after Alar Toomre's 1964 paper entitled *On the gravitational stability of a disk of stars* ([7]), which deals with the issue of the large-scale gravitational

stability of a highly flattened stellar system. Therefore, the relation that expresses the condition under which the disk is stable, is given by

$$\mathcal{Q} > 1 \quad (1.4)$$

which is called Toomre's stability criterion and establishes the circumstances needed for a protoplanetary flat disk to be stable to its own gravity, that is to its own collapse .

For example, considering a disk orbiting a Solar mass star, 1.3 yields:

$$\Sigma \gtrsim 3.8 \times 10^3 \left(\frac{\mathcal{Q}_{crit}}{1.5} \right)^{-1} \left(\frac{c_s/\Omega r}{0.05} \right) \left(\frac{r}{5AU} \right)^{-2} g \text{ cm}^{-2} \quad (1.5)$$

where Ω is the angular velocity of the particles of the disk ([27]).

The surface densities required to satisfy the condition given by 1.3, which is the gravitational instability, are far higher than actually computed. In this case we indeed obtain a value more than an order of magnitude larger than the estimated Minimum Mass Solar Nebula value at 5 AU. These high densities are however likely at the early stages of a disk's life, when the accretion rate is probably large and gravitational instabilities develop, resulting possibly in the formation of bound clumps of particles, which can rapidly agglomerate to form planetesimals.

Getting back to Toomre's criterion, it is important to highlight that additional factors arise if the analysis is generalized to include potential nonaxisymmetric instabilities within the disk, which are not taken into account in the previous criterion. In this case the control parameter is still \mathcal{Q} , but instability can catch on more easily. However, in most circumstances the subtleties arisen by the introduction of nonaxisymmetric modes are substantially of moderate importance.

This means that the condition under which the disk is stable is now given by:

$$\mathcal{Q} > \mathcal{Q}_{crit} \quad (1.6)$$

where \mathcal{Q} is therefore a dimensionless measure of the threshold below which instability sets in and lies in the range $1 < \mathcal{Q}_{crit} < 2$, generally being of the order of, but slightly larger than, unity.

Analogously, the criterion for the stability of a disk galaxy, which can be considered as a disk of stars, which was the condition originally studied by Toomre in his paper, is given by:

$$\mathcal{Q} = \frac{\sigma_R \kappa}{3.369 \Sigma} > 1 \quad (1.7)$$

where σ_R is the radial velocity dispersion.

1.3 A qualitative introduction to Batygin's study

The core of this dissertation is very much based on the study published by Konstantin Batygin on the 4th issue of the 475th volume of *Monthly Notices of the Royal Astronomical Society* on April 21th of this year.

In his work Batygin, a Russian-American astronomer working as a Researcher and Assistant Professor of Planetary Sciences at Caltech, hardly more than thirty years old, described how the time-dependent Schrödinger equation can describe the long-term evolution of self-gravitating disks. Deciding to abandon commonly used methods to analyze this problem, such as the ones described in section 3.1, Batygin managed to demonstrate that a well-known formula, Schrödinger's equation, which is usually known because it describes quantum mechanical behavior, governs a vastly different realm: disks of matter that surround stars as well as, in extension, planetary rings or star disks which surround a supermassive black hole at the center of a galaxy.

The peculiarity of these studies is highlighted by Batygin himself: interviewed by "Cosmos" magazine

([28]), he affirmed “This discovery is surprising because the Schrödinger equation is an unlikely formula to arise when looking at distances on the order of light-years. The equations that are relevant to subatomic physics are generally not relevant to massive, astronomical phenomena. Thus, I was fascinated to find a situation in which an equation that is typically used only for very small systems also works in describing very large systems.”

Batygin’s approach, discussed in detail in Sections 3.3 and 3.4, consists in applying Gauss’s method to a thin and self-gravitating disk, splitting it into a series of concentric wires. This way, imagining the system as an infinite number of infinitesimally thin rings interacting gravitationally, it is possible to describe said interaction using the perturbation theory and choosing the specific form of the surface density profile. Approximating the number of wires in the disk to be infinite, allows to mathematically blur them together into a continuum: extrapolating the continuum limit automatically brings us to Schrödinger’s equation. The description offered by Batygin, although concerning a simplified version of a circumstellar disk (some of the approximations used are analyzed in Chapter 4), is of fundamental importance, since it is essentially a mathematically simple explanation of a very complex and diverse problem.

The importance of this study is stressed by Yale University astronomy and astrophysics professor Greg Laughlin, who, although not involved in the development of the paper, affirms “The identification of the disk phenomena that can be described by Schrödinger’s equation means we have a lot of insight about it. You get all the lore and understanding, now immediately applicable to a new physical situation.” ([29]). To be thorough, Barker and Ogilvie carried out a similar study publishing in *Monthly Notices of the Royal Astronomical Society*, Volume 458, Issue 4, on March 11th 2016, a paper that demonstrated the analogy between the dynamics of a fluid disk affected by pressure forces and a nonlinear variant of Schrödinger’s equation. However in said work, entitled *Non linear hydrodynamical evolution of eccentric Keplerian discs in two dimensions: validation of secular theory*, the authors highlight that the final form of the new-found equation results to be “too complicated to be worth writing down”.

On the contrary, one strong value of Batygin’s study consists precisely in the incredibly clear form of the final equation, which can generate remarkable benefits from the computational point of view.

As already mentioned, Batygin’s approach implies several approximations and limitations, but as just evaluated, it also offers advantages, which will be later underlined in the “Final Discussion”.

Chapter 2

From Hamilton's equation to Schrodinger's equation

The early years of the twentieth century saw the birth of the modern formulation of Quantum Mechanics by two separate routes: the matrix mechanics of Heisenberg, Born and Jordan and the wave mechanics of de Broglie and Schrödinger. The two formalisms were soon shown to be mathematically equivalent by Schrödinger, Dirac and Eckart.

First Heisenberg, Born and Jordan developed matrix mechanics, abandoning the concept of electronic orbits and the atomic model to focus on basic principles of Classical Mechanics and key informations obtained through spectroscopy. This resulted in the formulation of a brand new theory, which consisted in connecting each physical quantity to a certain matrix, with the peculiarity that these matrices were characterized by a complex and noncommutative algebra.

A different but equivalent approach was based on a supposition made by de Broglie in 1923 and developed by Schrödinger in 1926. Inspired by Einstein's picture of light, de Broglie made the radical proposal that established the wave-particle duality of matter: if light waves can behave under some circumstances like particles, then by symmetry it is reasonable to suppose that particles such as electrons can behave like waves.

An immediate success of the matter wave approach was that it gave a clear explanation of why only very particular energy states were admissible for electrons bound in atoms circulating around the nucleus, that is Bohr's quantization condition, by establishing that a standing wave was associated to the electron so that the wave could not destructively interfere with itself, granting the stability of the atom.

Shortly after the development of the wave-matter hypothesis, Schrödinger managed to obtain a proper three dimensional wave equation for the electron using the analogy between mechanics and optics, observing that particles trace trajectories similarly to the light rays in geometrical optics. It is indeed possible to derive Schrödinger's equation by using an analog of the principle of least action, which is expressed through Fermat's principle of least time for light rays (Maupertuis's principle for particles). There have been numerous derivations of Schrödinger's equation through the years, some examples use wave mechanics, the Ornstein-Uhlenbeck theory of macroscopic Brownian motion with friction, the dynamical postulate of Feynman's path integral.

In quantum mechanics, the Schrödinger equation describes how a system changes with time by relating variations in the state of the system to the total energy of the system, which is expressed by the Hamiltonian operator, generally denoted by \mathcal{H} . The time-dependent Schrödinger equation is given by:

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{\mathcal{H}} |\Psi(t)\rangle \quad (2.1)$$

where i is the imaginary unit and $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant.

Therefore we can assert that one of the cornerstones of quantum physics is undoubtedly the Schrödinger equation, which describes the evolution of a system of quantum objects such as atoms and subatomic particles, given its current state.

We can easily affirm that classical analogies can be found: Newton's second law and Hamiltonian mechanics predict the evolution of a classical system starting from its current configuration. Schrödinger's first published derivation of his time-independent equation, from which he obtained the correct energy levels for the hydrogen atom, uses as a starting point the Hamilton-Jacobi equation of classical mechanics ([2]).

In this chapter we describe two different approaches used to derive the wave equation: in Section 2.1 it is shown how said equation can be directly derived from the classical Hamilton-Jacobi equation, if a basic uncertainty is assumed to be present in the momentum, while in Section 2.2 we develop another method which can be applied to our problem, which in analogy with the first one is also based on a classical Hamiltonian describing the interaction between harmonically coupled oscillators. This last approach is later on extended to the interacting rings of material in which a circumstellar disk can be divided.

2.1 Hamilton-Jacobi

As previously touched upon, the wave equation can be directly derived from the classical Hamilton-Jacobi equation.

Firstly we show how it is possible to derive Schrödinger's equation from the Hamilton-Jacobi equation of Classical Mechanics if the momentum and energy are defined as operators. Secondly we show how the same result can be achieved if a specific uncertainty is assumed to be characterizing the momentum. These derivations are based on the articles *Direct derivation of Schrödinger equation from Hamilton-Jacobi equation using uncertainty principle* by P. R. Sarma ([3]) and *Deriving time dependent Schrödinger equation from Wave-Mechanics, Schrödinger time independent equation, Classical and Hamilton-Jacobi equations* by N. P. Barde, D. S. Patil, P. M. Kokne and P. P. Bardapurkar ([4]).

As already mentioned, this is just one of the several methods which are used to express the similarity between Schrödinger's equation and the Hamilton-Jacobi equation.

We now consider a particle of mass m and momentum p , which can be described, in a one-dimensional space, by:

$$-\frac{\partial S}{\partial t} = \frac{p^2}{2m} + V(x,t) = \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V(x,t) \quad (2.2)$$

where $S \equiv S(x,t)$ is the generating function and the momentum p is defined by:

$$p = \frac{\partial S}{\partial x} . \quad (2.3)$$

Substituting

$$\psi(x,t) = \psi_0 \exp(iS/\hbar) \quad \text{or} \quad S = -i\hbar(\ln\psi - \ln\psi_0) \quad (2.4)$$

where ψ_0 is a constant, in the Hamilton-Jacobi equation, we get:

$$p = \frac{\partial S}{\partial x} = -i\hbar \frac{1}{\psi} \frac{\partial \psi}{\partial x} \quad (2.5)$$

and

$$\frac{\partial S}{\partial t} = -i\hbar \frac{1}{\psi} \frac{\partial \psi}{\partial t} . \quad (2.6)$$

Substituting equations 2.5 and 2.6 in 2.2, we obtain:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{1}{\psi} \left(\frac{\partial \psi}{\partial x} \right)^2 + V(x,t) \psi . \quad (2.7)$$

In order to obtain Schrödinger's equation, we need to eliminate the presence of the term $\left(\frac{\partial \psi}{\partial x} \right)^2 \frac{1}{\psi}$ and find a way to put, in its place, the term $\frac{\partial^2 \psi}{\partial x^2}$.

We can proceed writing 2.5 as follows

$$p\psi = -i\hbar \frac{\partial \psi}{\partial x} \quad (2.8)$$

and defining p as the momentum operator

$$p = -i\hbar \frac{\partial}{\partial x} . \quad (2.9)$$

Substituting this equation in 2.2 we finally obtain the fundamental Schrödinger equation.

However, it is important to note that declaring p as an operator, we abruptly go from the classical description to the quantum mechanical description.

An alternative way to derive Schrödinger's equation without using operators can be obtained using the uncertainty principle.

Being Δp the root-mean square uncertainty in p , we can write the average value of the squared momentum as

$$\langle p^2 \rangle = \langle p \rangle^2 + (\Delta p)^2 = p^2 + (\Delta p)^2 \quad (2.10)$$

If a particle has an average momentum p , the average kinetic energy is now given by $[p^2 + (\Delta p)^2]/2m$. Since $(\Delta p)^2$ is related through the uncertainty principle to $(\Delta x)^2$, it can be written as

$$(\Delta p)^2 = \frac{\Delta p}{\Delta x} \Delta p \Delta x . \quad (2.11)$$

Assuming that $\left(\frac{\Delta p}{\Delta x}\right)^2$ can be replaced by $\left(\frac{\partial p}{\partial x}\right)\left(\frac{\partial p}{\partial x}\right)^*$, we can now express $\frac{\Delta p}{\Delta x}$ from $\frac{\partial p}{\partial x}$. From 2.5 we can write

$$\frac{\partial p}{\partial x} = -i\hbar \frac{\partial}{\partial x} \left(\frac{1}{\psi} \frac{\partial \psi}{\partial x} \right) = i\hbar \left[\frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{1}{\psi} \frac{\partial^2 \psi}{\partial x^2} \right) \right] \quad (2.12)$$

and therefore

$$\frac{\Delta p}{\Delta x} = \hbar \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial x} \right)^2 - \hbar \left(\frac{1}{\psi} \right) \left(\frac{\partial^2 \psi}{\partial x^2} \right) . \quad (2.13)$$

In order to find an expression for $\Delta p \Delta x$, we observe that the minimum value of $\Delta p \Delta x$ is given by $\hbar/2$, while its the average value considering a Gaussian error function is

$$\Delta p \Delta x = \hbar . \quad (2.14)$$

From 2.11, 2.13 and 2.14 the average value of the squared momentum is given by

$$\langle p^2 \rangle + (\Delta p)^2 = -\hbar^2 \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\hbar \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial x} \right)^2 - \hbar \frac{1}{\psi} \left(\frac{\partial^2 \psi}{\partial x^2} \right) \right) \hbar = -\hbar^2 \frac{1}{\psi} \frac{\partial^2 \psi}{\partial x^2} . \quad (2.15)$$

Plugging the root-mean square uncertainty in p in the Hamilton-Jacobi equation, that is replacing p^2 by $p^2 + (\Delta p)^2$, and substituting it in 2.15 we get the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial x} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} \psi \right) + V(x,t) \psi . \quad (2.16)$$

This alternative method of derivation of Schrödinger's equation is based on the uncertainty in the momentum: if p is large, Δp is very small compared to $\langle p \rangle$, and we can therefore get back to the Hamilton-Jacobi equation starting from Schrödinger's proceeding as follows: using 2.5 we can write

$$\frac{\partial \psi}{\partial x} = \frac{i}{\hbar} \psi \frac{\partial S}{\partial x} \quad (2.17)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{\hbar^2} \psi \left(\frac{\partial S}{\partial x} \right)^2 + \frac{i}{\hbar} \psi \frac{\partial^2 S}{\partial x^2} = -\frac{1}{\hbar^2} \psi \left[\left(\frac{\partial S}{\partial x} \right)^2 - i\hbar \frac{\partial^2 S}{\partial x^2} \right] . \quad (2.18)$$

Therefore extrapolating the limit for $\hbar \rightarrow 0$, since the second term of the previous equation becomes negligible, we get

$$\left(\frac{\partial^2 \psi}{\partial x^2} \right) \rightarrow -\frac{1}{\hbar^2} \psi \left(\frac{\partial S}{\partial x} \right)^2 = \frac{1}{\psi} \left(\frac{i}{\hbar} \psi \frac{\partial S}{\partial x} \right)^2 = \frac{1}{\psi} \left(\frac{\partial \psi}{\partial x} \right)^2 \quad (2.19)$$

obtaining, as foreseen, 2.7.

2.2 Chain of harmonically coupled oscillators

Another method used to derive Schrödinger's equation is based on the description of an infinite chain of harmonically coupled oscillators.

This derivation is based on the description, illustrated by A. Animalu in *Intermediate quantum theory of crystalline solids*, of lattice waves and vibrations in one-dimensional crystals, in which we consider a linear chain of atoms connected by elastic springs each of a certain spring constant that represent a good estimate of the coupling in the lattice ([25]).

Therefore we now consider an infinite sequence of coupled objects lying, for example, on the x -axis, and we assume that the dynamics of every single object is described by the following action-angle variables:

$$(\Phi_{j-1}, \phi_{j-1}), (\Phi_j, \phi_j), (\Phi_{j+1}, \phi_{j+1}) . \quad (2.20)$$

We then assume that the objects are equidistant, and we call the spacing between each of them δx .

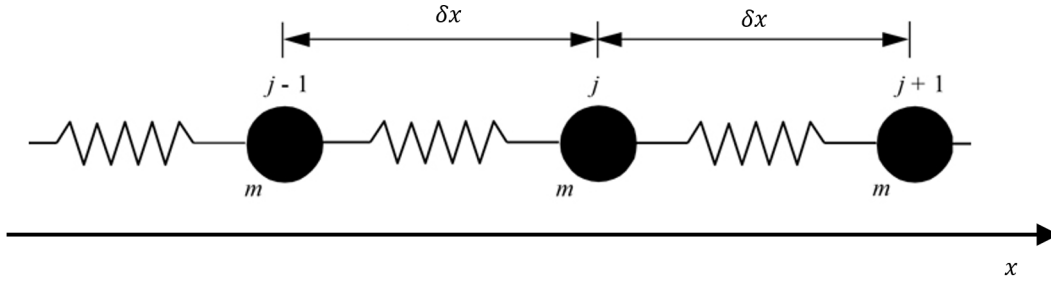


Figure 2: An infinite chain of harmonically coupled oscillators. The positions of the objects are fixed on the x -axis. The objects are equidistant, with a spacing of δx , and they are labeled by an index $j-1, j, j+1, \dots$

Considering exclusively the interaction between the nearest neighbors, the evolution of the object labeled with the index j is described by the Hamiltonian:

$$\mathcal{H} = 2c_1 \Phi_j + 2c_2 \sqrt{\Phi_j \Phi_{j+1}} \cos(\phi_j - \phi_{j+1}) + 2c_2 \sqrt{\Phi_j \Phi_{j-1}} \cos(\phi_j - \phi_{j-1}) \quad (2.21)$$

in which c_1 and c_2 denote variables that can depend on x and t . We now define variables (ξ, ζ) , canonical Cartesian analogs to (Φ, ϕ) , and proceed to include them in a single complex coordinate:

$$\xi = \sqrt{2\Phi} \cos(\phi) \quad \zeta = \sqrt{2\Phi} \sin(\phi) \quad \rightarrow \quad \Psi = \frac{\xi + i\zeta}{\sqrt{2}} = \sqrt{\Phi} \exp(i\phi) . \quad (2.22)$$

Substituting the new coordinates in 2.21, we obtain:

$$\mathcal{H} = 2c_1 \Psi_j \Psi_j^* + c_2 (\Psi_j \Psi_{j+1}^* + \Psi_j^* \Psi_{j+1} + \Psi_j \Psi_{j-1}^* + \Psi_j^* \Psi_{j-1}) \quad (2.23)$$

where Ψ_j^* is the complex conjugate to Ψ_j . We now proceed by deriving Ψ_j with respect to time, obtaining:

$$\frac{d\Psi_j}{dt} = i \frac{\partial \mathcal{H}}{\partial \Psi_j^*} = i(2c_1 \Psi_j + c_2 (\Psi_{j+1} + \Psi_{j-1})) . \quad (2.24)$$

If we consider $\Psi_{j-1}, \Psi_j, \Psi_{j+1}$ to be a discrete representation of a continuous, complex field Ψ_j , we can use the central difference approximation (that is, by definition, $\frac{\partial^2 u}{\partial r^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \mathcal{O}((\Delta r)^2)$), to express

its second derivative, which now reads:

$$\frac{\partial^2 \Psi_j}{\partial x^2} \simeq \frac{\Psi_{j+1} - 2\Psi_j + \Psi_{j-1}}{(\delta x)^2} . \quad (2.25)$$

Inserting equation 2.25 into equation 2.24 and noting that we can assume $\frac{\partial \Psi}{\partial t} = \frac{d\Psi}{dt}$ since we are considering objects fixed to the x-axis we obtain:

$$\frac{\partial \Psi_j}{\partial t} = i \left(2(c_1 + c_2)\Psi_j + c_2(\delta x)^2 \frac{\partial^2 \Psi_j}{\partial x^2} \right) . \quad (2.26)$$

Multiplying both sides by $-\frac{\hbar}{i}$, we now get:

$$i\hbar \frac{\partial \Psi_j}{\partial t} = -\hbar \left(2(c_1 + c_2)\Psi_j + c_2(\delta x)^2 \frac{\partial^2 \Psi_j}{\partial x^2} \right) . \quad (2.27)$$

We can now proceed by making explicit c_1 and c_2 :

$$c_1 = -\frac{V(x,t)}{2} - \frac{\hbar}{2\mu(\delta x)^2} \quad c_2 = \frac{\hbar}{2\mu(\delta x)^2} . \quad (2.28)$$

By making 2.25 exact, that is extrapolating the limit $\delta x \rightarrow 0$, and by substituting 2.28 into 2.27, we get:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + V(x,t) \right] \Psi(x,t) \quad (2.29)$$

which is simply the time-dependent Schrödinger equation. This derivation is easy to extend to the three dimensional case or to the non linear case, which is achieved by adding a nonlinear action term to the Hamiltonian 2.21, for example:

$$\mathcal{H}' = \mathcal{H} + \frac{\kappa}{2} \Phi_j^2 = \mathcal{H} + \frac{\kappa}{2} (\Psi_j \Psi_j^*)^2 \quad (2.30)$$

gives

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + V(x,t) \right] \Psi(x,t) + \kappa |\Psi(x,t)|^2 \Psi(x,t) . \quad (2.31)$$

As formerly explained, this is just one of the many methods to derive Schrödinger's equation through purely Classical Mechanics. This derivation is indeed self-consistent, but by no means it is general, since it applies to a particular Hamiltonian chosen ad hoc with specific interaction coefficients also chosen ad hoc.

However, this derivation is particularly important in the description of the temporal evolution of a disk, since the equations of Lagrange-Laplace governing the secular theory of celestial mechanics can be easily cast into this form.

Chapter 3

Application to a self-gravitating disk

In order to proceed with the actual mathematical analysis of the description of the evolution of a self-gravitating razor-thin disk it is necessary to specify some fundamental characteristics of said disk.

First of all it is important to note that the evolution of disks may easily be affected by external perturbations. In order to identify the procedure that best describes our case, we need to specify the nature of said perturbations, which means that we have to establish their timescale. Since the perturbations arise on secular timescale, which exceeds the orbital period and at the same time is notably shorter than the lifetime of the considered system, whose dynamics can only be poorly represented by the N-body method or a diffusive model.

However, for the sake of completeness Section 3.1 is entirely dedicated to briefly present three of the main methods used to analyze the dynamics of self-gravitating disks. The first two procedures, which are based respectively on the N-body problem and the collision-less Boltzmann equation, will not be employed during our dissertation, while the third method, that goes under the name of Gauss's averaging method of celestial mechanics, is actually fundamental for our derivation. Gauss's technique, adopted essentially in orbital mechanics, consists in imagining to fragment every mass orbiting around the main central mass, in such a way that allows us to think to smear the fragments obtaining an homogeneous disk which can therefore be divided into concentric annuli. Individual bodies are substituted by massive wires that, since we are dealing with a self-gravitating disk, interact gravitationally with one another. Considering the wires one at a time it is possible to analyze the interactions of a single annulus significantly simplifying the problem.

Section 3.2 deals with the description of the disk: in order to carry out the mathematical analysis it is necessary to specify certain characteristics of the considered object. A razor-thin disk is therefore defined, its boundaries are established and the surface density profile is chosen.

Sections 3.3 and 3.4 deal with the determination of the equations that govern the evolution of the described system and their solution given some boundary conditions.

3.1 Most important analysis methods for a self-gravitating disk

In order to describe the evolution of a dynamically cold self-gravitating disk it is necessary to quantify the responses of said disk to external perturbation or stresses. This is obviously crucial to the interpretation modern observations, as well as to the development of a planet formation theory. These perturbations constitute the disturbing function which can be expanded in an infinite series where the individual term can be classified as secular, resonant or short period, according to the given physical problem.

For example, in regards to the 3-body problem, if we aim to study the motion of a third body under the gravitational effects of two other bodies for arbitrary initial conditions, we need to find an alternative path to bypass the fact that this problem is non integrable and we can only make some progress by analyzing the accelerations experienced by the three bodies. If their motions are dominated by a central or primary body, then the orbits of secondary objects are conic sections with small deviations due to their mutual gravitational perturbations. These deviations can be calculated by defining and analyzing the disturbing function: the accelerations of secondary bodies relative to the primary mass can be indeed obtained from the gradient of the perturbing potential. Expanding the disturbing function using Legendre polynomials and expressing it in terms of standard orbital elements as an infinite series, the need to decide which of the terms are important to the analysis and which ones can be ignored arises.

First of all, it is important to note that luckily for most problems we can keep just one or two terms from the disturbing function using the averaging principle, which states that most terms average to zero over a few orbital periods and so they can be ignored by using the averaged disturbing function, $\langle \mathcal{R} \rangle$.

As mentioned before, it is possible to operate a classification of the terms by considering the frequencies or periods associated to the arguments of the terms. Dividing the terms in three categories, secular, resonant and short-period, we note that in our case, considering a self-gravitating razor-thin disk, interesting perturbations arise only in secular timescales, for example $10^5 - 10^6$ years.

To be more precise, we note that the warped stellar disk located at the center of our galaxy supposedly owes its origin to an extremely complex interaction between self-gravitational effects and torques exerted by nearby stellar clusters. In the same way young circumstellar disks may present warps and spiral morphology generally ascribed to interplays between the disks themselves and surrounding stellar or planetary companions. These phenomena are characterized, fortunately, by secular timescales which allow us to adopt one of the three primary means developed to analyze the dynamical evolution of self-gravitating disks.

The first and most straight-forward method is represented by the N-body problem, which is unfortunately non-integrable. To avoid this issue we can operate some approximations which can make it possible to find an analytical solution to a particular form of the N-body problem: for example we can find an approximated version that can be applied to Solar System bodies considering only the effect of purely secular terms in the disturbing function of the system. This was one of the first application of the Laplace-Lagrange secular theory, which constitutes one of the earliest, and best-known results of perturbation theory in celestial mechanics and consists in expanding the phase-averaged gravitational potential of the interacting bodies as a Fourier series in the orbital angles and as a power-series of eccentricities and inclinations.

If we consider N point masses, the purely gravitational interaction exerted among them, once the masses $m_i (i = 1, N)$, positions \mathbf{r}_i , and velocities \mathbf{v}_i are given at some reference time, is given by Newton's Laws: the force per unit mass felt by the body i due to the gravitational interaction with the remaining bodies is thus

$$\dot{\mathbf{r}}_i = \mathbf{v}_i \quad (3.1)$$

$$\dot{\mathbf{v}}_i = \mathbf{F}_i = -G \sum_{j=1; j \neq i}^N \frac{m_j (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \quad (3.2)$$

where the dots denote time derivatives.

At this point two problems arise: the first one focuses on calculating \mathbf{F}_i and presents the difficulty of the prohibitiveness of the direct summation for large N , the second one focuses on the update of the state of the system to a time $(t + \delta t)$, once positions, velocities, time t and an algorithm to compute the forces are given. The problem consists in resolving differential equations at $\delta t \rightarrow 0$.

Analytic or approximated numerical treatments can be useful, but in many cases the solution of an N -body problem requires the explicit numerical integration of the trajectories traced by the bodies. Although a large body of technical literature is devoted to the N -body problem and in spite of the remarkable advances in computational techniques that have been developed during the last decade, this method is still too computationally expensive and specialized for most problems of interest.

A second, more compact mean of analyzing the dynamics of self-gravitating disks is based on the collision-less Boltzmann equation, which yields the system distribution function evolution in phase-space.

Stellar systems may be considered to be collision-less: it has been demonstrated that we can get a satisfying approximation to the orbit of whatever star by calculating the orbit that it would have if the masses characterizing the system were smoothly distributed in space instead of being concentrated into point-like objects. Actually this model's orbit eventually ends up deviating significantly from the true orbit but, if we consider systems characterized by more than a few thousand stars, said deviation remains small during a period of time called t_{relax} . More specifically, if we consider a galaxy, t_{relax} usually tends to be considerably larger than the age of the universe, which grants us that this approximation provides a description which is accurate enough. Another example of a system suitable for this technique is given by globular clusters: once again t_{relax} is much smaller than the cluster's age.

This model requires to consider systems which are perfectly in equilibrium for large t_{relax} and to assume the considered systems as composed by N identical point masses (stars or dark-matter particles or dust particles) orbiting around a central massive body.

This technique consists in calculating the probability of finding a point mass in the six-dimensional phase-space volume $d^3\mathbf{x}d^3\mathbf{v}$ around the position \mathbf{x} and velocity \mathbf{v} . If we call the distribution function f , we have that $f(\mathbf{x};\mathbf{v};t)d^3\mathbf{x}d^3\mathbf{v}$ is the probability that whichever mass, labeled for example with 1, is characterized by phase-space coordinates belonging to the chosen range, at the time t . Since one of the premises of the model is to assume the system composed by N identical masses, the probability has to result the same for masses labeled with 2; 3; : : : ; N .

Obviously f needs to be normalized as follows

$$\int d^3\mathbf{x}d^3\mathbf{v}f(\mathbf{x};\mathbf{v};t) = 1 \quad (3.3)$$

where the integral is calculated over the entirety of the phase space.

It is important to note that the probability evolves with time, and this evolution can be described by a partial differential equation which needs to be satisfied by f as a function of six phase-space coordinates and time.

This equation, called the collision less Boltzmann equation, is obtain by using the continuity equation, given that as f evolves, the probability must be conserved, and is expressed by:

$$\frac{\partial f}{\partial t} + \mathbf{q} \frac{\partial f}{\partial \mathbf{q}} + \mathbf{p} \frac{\partial f}{\partial \mathbf{p}} = 0 \quad (3.4)$$

where we used the canonical phase-space coordinates used in the Hamiltonian formalism: q are the Cartesian coordinates while p are the components of momentum.

This method presents two main problems: finding a correct and satisfying distribution function and managing to resolve the often complicated integrals, which can lead to heavily numerical approaches or excessive approximations.

The third and last approach relies on Gauss's averaging method of celestial mechanics, which consists in calculating the effect of perturbations from a wire ring made up by spreading the perturbing object around its orbit.

With this technique the resulting perturbations are exactly the same as those derived using the disturbing function.

The main requirement in any numerical method for secular dynamics is an efficient evaluation of the average interaction exerted between two fixed Kepler orbits, each characterized by their parameters: the semi-major axis a and the shape parameters e (eccentricity), ω (argument of periapsis), I (inclination), and Ω (longitude of node).

The expansion of the disturbing function and its consequent average describe the motion for secular dynamics, but this averaging can be thought of as smearing each planet into a ring, that means replacing the individual bodies with a series of massive concentric wires that exert on one-another a gravitational force characterized by time scales significantly longer than the orbital period.

To be thorough, we present two methods to compute the interaction energy between two rings, which we label by α and β .

The first path consists in choosing K points equally spaced in mean anomaly on each ring and then calculate

$$- \sum_{i,j=1}^K \frac{Gm_\alpha m_\beta}{K^2 \Delta_{ij}} \quad (3.5)$$

where m_α and m_β are the masses of the two bodies and Δ_{ij} is the distance between point i on ring α and point j on ring β .

Although this may seem the simplest method, it presents some problems: it indeed requires $\mathcal{O}(K^2)$ computations per ring pair and additionally it tends to converge slowly. For this reasons the best approach consists in computing the potential from the first ring, α , at a point of choice indicated by r , by integrating over the mass elements of said ring, and consequentially in integrating this potential over the locus of points r that lie on ring β .

The value of Gauss's method consists in its efficiency: it represents a more physical approach that, giving the same results as the secular theory, provides a confirmation of the averaging principle.

This method has several applications: for example it can be used to analyze Kozai oscillations, which usually can arise due to the influence of a distant companion. One of the most famous case was the study of Kozai oscillations of planet 16 Cygni Bb, discovered in 1996.

However, the main flaw is that it neglects perturbations that are second-order or higher in the masses of perturbing bodies. In some cases, these second-order terms can be near-resonant and, considering the Solar System, they can be almost as strong as the first-order secular terms.

Summarizing, this approach, further discussed in Sections 3.2 and 3.3., consists in assuming that the disruptive effect of an external body is equivalent to spreading its mass around its orbit so that the internal body moves under the force exerted by a "ring" of material.

3.2 Disk profile

We now consider a circumstellar disk, essentially a set of point-like masses orbiting around a single massive central body.

In order to describe the dynamic evolution of such object it is necessary to identify certain characteristics such as the mean motion, the aspect ratio, the inner and outer radii and the surface density. These physical quantities constitute the profile of the disk.

It is possible to assume that the gravitational potential of the central body, with mass M , dominates in such a way that single bodies follow the trajectories defined by Kepler's third law.

Given:

$$T^2 = ka^3 \quad (3.6)$$

in which T is the orbital period of the body in days and a is the distance from the considered body to the main central object, or the semi-major axis of its elliptical orbit.

Also, in this case, k indicates a gravitational parameter:

$$k = \mathcal{G}M \quad (3.7)$$

The mean motion, n , is defined as the average orbital angular velocity:

$$n = \frac{2\pi}{T} \quad \text{or} \quad n = \frac{360^\circ}{T} \quad \text{or} \quad n = \frac{1}{T} \quad (3.8)$$

depending on the notation adopted.

Plugging in 3.6 and 3.7 into 3.9 we obtain:

$$n = \sqrt{\frac{\mathcal{G}M}{a^3}} \quad (3.9)$$

We now proceed making some approximations:

- orbital eccentricities are assumed not to be excessively large
- mutual inclinations between neighboring orbits are assumed not to be excessively large
- the intrinsic velocity dispersion of a population of objects occupying a certain semi-major axis range is modest in regards to the Keplerian (orbital) velocity

We define the aspect ratio of the disk, given that regarding a geometric shape it indicates the ratio of its sizes in different dimensions, we have:

$$\beta = \frac{h}{a} = \text{constant} \ll 1 \quad (3.10)$$

where h is the scale-height of the disk.

Assuming β as an intrinsic small parameter of the problem, we state that the considered disks are razor-thin.

Observations made through the years have made clear that there is a certain prevalence of flattened disks, due to the combination of dissipation and rotation. For example, if we consider spiral galaxies, the majority of stars lie in a thin disk. As already mentioned in the description presented in Section 1.1, the cloud composed by gas and dust in orbit around a massive body conserves its angular momentum but does not conserve its energy because of several mechanisms such as emission and collision processes. This causes the contraction of the cloud into a flat disk, which constitutes the lowest energy state available.

The temperature profile and spectral energy distribution of the disk are determined by its shape, which can be flat, flared or warped. In this particular case we analyze a flat razor-thin disk, that according to the aforementioned condition, is characterized by a small value of the normalized thickness β .

We now have to deal with a type of problem where the number of objects orbiting around the main central body in the physical system is far too large to represent on a one to one basis in an N-body

simulation. It is indeed impossible to simulate every point-like mass present in the disk. This problem has numerous analogs in the astrophysical field, for example it is similarly impossible to simulate galaxy formation using N-body particles that have the same mass as dark matter-particles. Therefore the objects should not be thought of as real bodies within the physical system, but rather as tracers whose trajectories constitute a statistical sample that represent the dynamics of the real system.

In the described regime, it is necessary to suppress some effects such as, for example, in the case of the study of a galaxy's dynamics, the formation of binaries which is a phenomenon that would occur nonphysically in the N-body system. This can be achieved by replacing the classic Newtonian potential $\Phi \propto \frac{1}{r}$ with a softened potential, for example

$$\Phi \propto \frac{1}{(r^2 + \beta^2)^{1/2}} \quad (3.11)$$

where β is called gravitational softening length and is chosen ad hoc to suppress two-body interactions but at the same time to maintain the large-scale dynamics unaltered. We can then state that this softening is a numerical trick used to prevent numerical divergences when a particle comes too close to another particle. This is obtained by modifying the gravitational potential of each particle.

We now proceed implementing Gauss's averaging method:

- the point-like masses of the orbiting objects are spread around their orbits so that the internal central body moves under the force exerted by a "ring" of material
- we consider the disk of smeared material divided into a series of N nested elliptical wires with normalized thickness β
- we assume that the wires do not cross
- we assume that the epicyclic motion which characterizes the particles is completely described by the softening parameter
- we assume the spacing of the wires to be geometric, that means that we assume the ratio of the semi-major axis of neighboring wires, α , to be constant

$$\alpha \equiv \frac{\alpha_{j-1}}{\alpha_j} = \frac{\alpha_j}{\alpha_{j+1}} = \frac{1}{\beta + 1} \approx 1 \quad . \quad (3.12)$$

We then introduce two dimensionless logarithmic radial coordinates, defined as:

$$\rho = \log \frac{a}{a_{in}} \quad \text{and} \quad \mathcal{L} = \log \frac{a_{out}}{a_{in}} \quad (3.13)$$

with a_{in} which indicates the truncation radius, that is the radius of the wire nearest to the central body, and a_{out} which is the radius of the most external wire: they denote the inner and outer boundaries of the disk respectively.

In terms of these two new coordinates, the wires are now equidistant from each other and the boundaries are now given by $\rho \in [0, \mathcal{L}]$.

Typical values of \mathcal{L} are generally under 10, for example if we consider a protoplanetary nebula we usually have $\mathcal{L} \sim 2\pi$, with $a_{in} \sim 0.05AU$ and $a_{out} \sim 50AU$.

We now need to describe the surface density profile of the disk, $\Sigma(r, t)$, where r is the orbital radius.

By assuming the radial velocity $v_r(r, t)$ of gas in the disk small and having defined the disk as geometrically thin ($\beta \ll 1$), the predominant forces involved are rotational support and gravity.

The evolution of $\Sigma(r, t)$ can be derived by using two fundamental equations that express two important characteristics of the disk: the conservation of mass, through the continuity equation, and the conservation of angular momentum, expressed by the equation of the azimuthal component of the momentum.

We can adopt one of the conventional parametrizations of the surface density of the disk. According to Armitage ([27]), which follows the procedure previously mentioned, $\Sigma(r, t)$ scales as a certain negative power of the orbital radius. Since the employment of Armitage's profile simplifies some of the following calculations, we choose:

$$\Sigma = \Sigma_0 \left(\frac{a_0}{a} \right)^{\frac{1}{2}} \quad (3.14)$$

where Σ_0 is the surface density at a reference semi-major axis a_0 .

Another profile commonly used is for example Mestels's profile, in which $\Sigma \propto \frac{1}{a}$, and the exponential profile, in which $\Sigma \propto \Sigma_0 \exp(a)$.

Using the definitions previously presented, we can express the mass of an individual wire:

$$m_j = \oint \int_{a(1-\beta/2)}^{a(1+\beta/2)} \Sigma a \, da \, d\phi \approx 2\pi\beta\Sigma_0 \sqrt{a_0 a_j^3} + \mathcal{O}(\beta^3) \quad (3.15)$$

where ϕ is the azimuthal coordinate.

Assuming that $a_{in} \ll a_{out}$, we obtain the total disk mass:

$$m_{disk} = \oint \int_{a_{in}}^{a_{out}} \Sigma a \, da \, d\phi = \frac{4\pi}{3} \Sigma_0 \sqrt{a_0 (a_{out}^3 - a_{in}^3)} \approx \frac{4\pi}{3} \Sigma_0 \sqrt{a_0 a_{out}^3} . \quad (3.16)$$

The approximations employed remain valid if the dynamical evolution keeps on being keplerian, which quantitatively implies that the gravitational stability must be assured ([1]):

$$\mathcal{Q} = \frac{hn^2}{\pi\mathcal{G}\Sigma} \gtrsim 1 \quad (3.17)$$

which translates to an upper-limit on the disk mass:

$$m_{disk} \lesssim \beta \frac{M}{2} \quad (3.18)$$

since, as already mentioned,

$$n = \sqrt{\frac{\mathcal{G}M}{a^3}} \quad (3.19)$$

we get

$$\mathcal{Q} = \frac{h}{\pi\mathcal{G}\Sigma} \frac{\mathcal{G}M}{a^3} = \frac{hM}{\pi\Sigma a^3} = \frac{hMa^{1/2}}{\pi\Sigma_0 a^3 a_0^{1/2}} . \quad (3.20)$$

Assuming an average value of a , we finally get:

$$\mathcal{Q} \approx \frac{M\beta}{2m_{disk}} \gtrsim 1 . \quad (3.21)$$

3.3 Governing equations of disk evolution

Having defined the density profile of the disk and having implemented Gauss's method, we can now proceed to find the equations that govern the evolution of our disk.

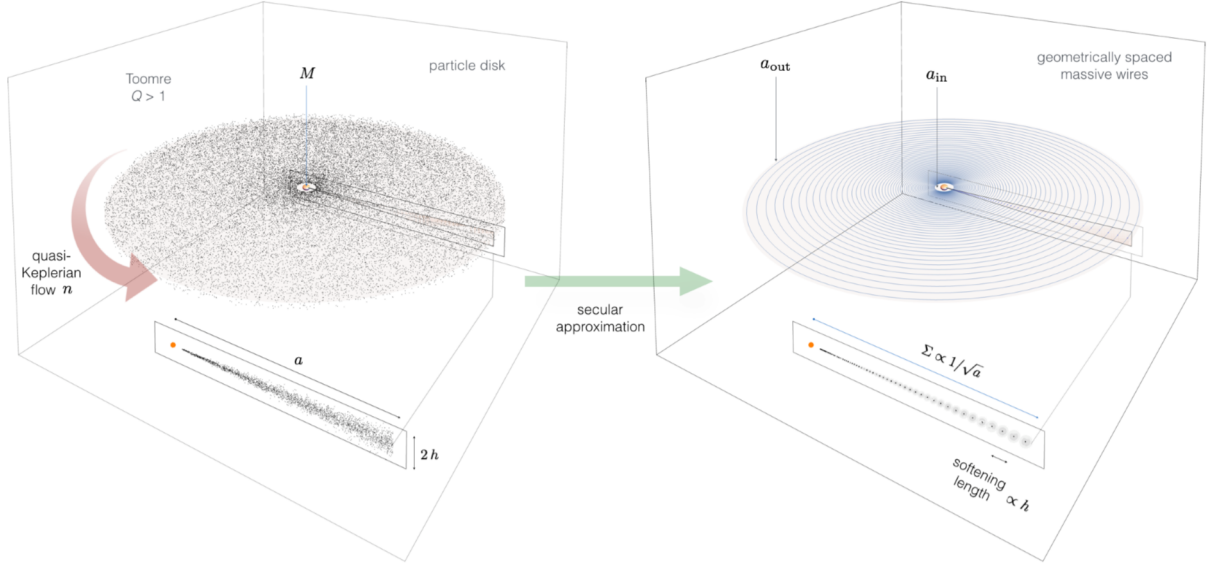


Figure 3: Structure of the disk. According to Gauss's averaging method the quasi-Keplerian disk composed of $N \gg 1$ particles is modeled as a sequence of geometrically spaced massive wires. The image shows the inner and outer boundaries of the disk, respectively a_{in} and a_{out} , and the surface density profile Σ . From Batygin, Konstantin, *Schrödinger Evolution of Self-Gravitating Disks*, published in *Monthly Notices of the Royal Astronomical Society*, Volume 475, Issue 4, Pages 5070–5084, 21 April 2018.

We consider one of the wires that constitute the disk, and we label it with an index $j \neq 1, N$. Restricting its range of interaction so that we can consider just its nearest neighbors, labeled by $j + 1$ and $j - 1$, we can describe its inclination dynamics through the disturbing function, scaled for simplicity by a characteristic angular momentum $m\sqrt{GMa}$, that regulates the exchange of angular momentum:

$$\mathcal{R}_j = \frac{1}{2}\mathcal{B}_{j,j}i_j^2 + \mathcal{B}_{j,j-1}i_ji_{j-1}\cos(\Omega_j - \Omega_{j-1}) + \mathcal{B}_{j,j+1}i_ji_{j+1}\cos(\Omega_j - \Omega_{j+1}) \quad (3.22)$$

where i indicates the orbital inclination, Ω the longitude of ascending node, and \mathcal{B} 's are interaction coefficients depending on the semi major axis ratios and the masses.

We now define a new set of canonically conjugated variables, in order to substitute i and Ω :

$$p = i \sin(\Omega) \quad (3.23)$$

$$q = i \cos(\Omega) \quad (3.24)$$

where p is the Cartesian coordinate, and q is the momentum.

We can collect these variables into a single complex one just like we did in equation 2.22:

$$\eta = \frac{q + ip}{\sqrt{2}} = \frac{i}{\sqrt{2}} \exp(i\Omega) \quad (3.25)$$

so that we can rewrite equation 3.22

$$\mathcal{R}_j = \mathcal{B}_{j,j} \eta_j \eta_j^* + \mathcal{B}_{j,j-1} (\eta_j \eta_{j-1}^* + \eta_{j-1} \eta_j^*) + \mathcal{B}_{j,j+1} (\eta_j \eta_{j+1}^* + \eta_{j+1} \eta_j^*) . \quad (3.26)$$

It is clear how this equation is similar to the Hamiltonian 2.23, but to understand deeply the analogy we need to define a relationship between the coefficients $\mathcal{B}_{j,j}, \mathcal{B}_{j,j-1}, \mathcal{B}_{j,j+1}$. To do so, we need to specify their form using the assumption that $m_j \ll M$:

$$\mathcal{B}_{j,j} = -(\mathcal{B}_{j,j-1} + \mathcal{B}_{j,j+1}) \quad \mathcal{B}_{j,j-1} = \frac{n_j m_{j-1}}{4 M} \alpha \tilde{b}_{3/2} \{\alpha\} \quad \mathcal{B}_{j,j+1} = \frac{n_j m_{j+1}}{4 M} \alpha^2 \tilde{b}_{3/2} \{\alpha\} \quad (3.27)$$

where $\tilde{b}_{3/2}^{(1)} \{\alpha\}$ is the Laplace coefficient of the first kind.

Since we established in equation 3.12 that $\alpha \approx 1 - \beta$, expressions 3.27 yields

$$\mathcal{B}_{j,j-1} \approx \mathcal{B}_{j,j+1} . \quad (3.28)$$

We therefore define the quantity

$$B = \frac{n_j m_j}{4 M} \frac{\alpha}{2} \tilde{b}_{3/2}^{(1)} \{\alpha\} \left[\frac{m_{j-1}}{m_j} + \frac{m_{j+1}}{m_j} \frac{1}{1+\beta} \right] = \frac{n_j m_j}{4 M} \alpha \tilde{b}_{3/2}^{(1)} \{\alpha\} \left[\frac{2+2\beta+\beta^2}{2(1+\beta)^{3/2}} \right] \quad (3.29)$$

so that we can affirm

$$|\mathcal{B}_{j,j-1} - B| = |\mathcal{B}_{j,j+1} - B| \approx \frac{\beta}{4} + \mathcal{O}(\beta^2) \ll 1 \quad (3.30)$$

$$\mathcal{B}_{j,j} = -2B . \quad (3.31)$$

Recalling the choice of the surface density profile 3.14 and the resulting relations $m_j \propto \sqrt{a^3}$ and $n_j \propto 1/\sqrt{a^3}$, we have that B results constant throughout the disk, since the dependencies compensate themselves, whereas if we had operated a different choice for the density profile, B would have depended from the semi-major axis.

So, deriving η_j with respect to time, we have:

$$\frac{d\eta_j}{dt} = i \frac{\partial \mathcal{R}_j}{\partial \eta_j^*} \approx iB (\eta_{j-1} - 2\eta_j + \eta_{j+1}) . \quad (3.32)$$

As already explained (equation 3.13), the wires are equidistant in ρ , so that:

$$\delta\rho = \rho_{j+1} - \rho_j = \rho_j - \rho_{j-1} = \log(1 + \beta) . \quad (3.33)$$

Adopting the central difference approximation (equation 2.25) and since the semi-major axes of the wires are considered secularly invariant, we can re-propose 3.32 in the form:

$$\frac{\partial \eta}{\partial t} = iB (\log(1 + \beta))^2 \frac{\partial^2 \eta}{\partial \rho^2} . \quad (3.34)$$

Recalling that the wires are influenced by their nearest neighbors, we can extrapolate the limit $\alpha \rightarrow 1$ from equation 3.29. By doing so we come across a problem: Laplace coefficients are as a matter of fact singular at $\alpha = 1$, which can be classically interpreted with the fact that the gravitational potential becomes infinite at null separations. We can bypass this problem by softening the coefficients by the disk aspect ratio as follows:

$$\tilde{b}_{3/2}^{(j)} \{\alpha\} = \frac{2}{\pi} \int_0^\pi \frac{\cos(j\psi)}{(1 - 2\alpha \cos(\psi) + \alpha^2 + \beta^2)^l} d\psi , \quad (3.35)$$

so that these new defined softened coefficients make it possible to extrapolate the $\alpha \rightarrow 1$ limit whereas the unsoftened form diverges, and the Laplace coefficient of the first kind $\tilde{b}_{3/2}^{(1)}$ can now be expressed as the following approximation:

$$\alpha \tilde{b}_{3/2}^{(1)} \{ \alpha \} \approx \frac{1}{\pi \beta^2} + \mathcal{O}(\beta^{-1}) \quad (3.36)$$

which can be implemented into equation 3.29.

$$B = \frac{n_j m_j}{4 M \pi \beta^2} \left[\frac{2 + 2\beta + \beta^2}{2(1 + \beta)^{3/2}} \right] . \quad (3.37)$$

Approximating $\log(1 + \beta) \approx \beta$:

$$\frac{\partial \eta}{\partial t} = i \frac{n_j m_j}{4 M \pi \beta^2} \left[\frac{2 + 2\beta + \beta^2}{2(1 + \beta)^{3/2}} \right] (\log(1 + \beta))^2 \frac{\partial^2 \eta}{\partial \rho^2} = i \frac{n_j m_j}{4 M \pi} \left[\frac{2 + 2\beta + \beta^2}{2(1 + \beta)^{3/2}} \right] \frac{\partial^2 \eta}{\partial \rho^2} . \quad (3.38)$$

Proceeding by multiplying both sides by $i\omega_i$ we obtain:

$$i\omega_i \frac{\partial \eta}{\partial t} = -\omega_i \frac{n_j m_j}{4 M \pi} \left[\frac{2 + 2\beta + \beta^2}{2(1 + \beta)^{3/2}} \right] \frac{\partial^2 \eta}{\partial \rho^2} , \quad (3.39)$$

but given that

$$\omega_i = \frac{n}{4\pi} \frac{m}{M} = \frac{\beta \Sigma_0 \sqrt{GM a_0}}{2M} , \quad (3.40)$$

we can write:

$$i\omega_i \frac{\partial \eta}{\partial t} = -\omega_i^2 \left[\frac{2 + 2\beta + \beta^2}{2(1 + \beta)^{3/2}} \right] \frac{\partial^2 \eta}{\partial \rho^2} \quad (3.41)$$

and finally expanding to leading order in β and keeping in mind that $\log(1 + \beta) \approx \beta$, we obtain the potential-free Schrödinger's equation of inclination dynamics within the disk

$$i\omega_i \frac{\partial \eta}{\partial t} = -\omega_i^2 \frac{\partial^2 \eta}{\partial \rho^2} . \quad (3.42)$$

It is important to note that for a quantum particle confined to an infinite square potential well so that the potential is expressed by:

$$V(x) = \begin{cases} 0 & \text{if } 0 < \rho < \mathcal{L} \\ \infty & \text{elsewhere} \end{cases} \quad (3.43)$$

the general solution to Schrödinger's one-dimensional, time-dependent equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x) \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t) \quad (3.44)$$

is given by

$$\psi(x, t) = \psi(x) \exp(-i\omega t) \quad (3.45)$$

where $\omega = E/\hbar$ and where $\psi(x)$ satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + V(x) \psi(x) = E \psi(x) . \quad (3.46)$$

Considering the infinite square well as the potential, Schrödinger's equation for the area between the boundaries from $\rho = 0$ to $\rho = \mathcal{L}$, from the general form 3.44, is:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) . \quad (3.47)$$

It is clear how it is possible to trace a parallel between this case and our description of secular angular momentum exchange within self-gravitating disks.

3.4 Solution

In order to determine the solution to equation 3.42 we need to establish the boundary conditions and impose them.

Resuming the case of an infinite square well, we need to impose the conditions $\Psi(0) = \Psi(\mathcal{L}) = 0$, which are called the Dirichlet boundary conditions, since the wave function Ψ has to vanish at the boundaries. However, in our case the orbital inclination is not required to cancel itself at the boundaries, that is at the margins of the disk. To deduce its behavior in such regions, we need to analyze the discrete system in proximity of the disk's inner and outer edges:

- outer edge: the disturbing function for the wire labeled by $j = N$ is:

$$\mathcal{R}_N = -B\eta_N\eta_N^* + B(\eta_N\eta_{N-1}^* + \eta_{N-1}\eta_N^*) \quad (3.48)$$

with the adoption of the same approximations used in section 3.3.

Thus, the resulting equation takes the form:

$$\frac{d\eta_N}{dt} = i\frac{\partial\mathcal{R}_N}{\partial\eta_N^*} = -iB(\eta_N - \eta_{N-1}) . \quad (3.49)$$

We note that the backward difference approximation

$$\frac{\partial\eta_N}{\partial\rho} = \frac{\eta_N - \eta_{N-1}}{\delta\rho} + \mathcal{O}(\delta\rho)^2 \quad (3.50)$$

recalls the right hand side of equation 3.49.

Correspondingly, since $B(\delta\rho) \approx \omega_i/\beta$, we obtain the condition for $\rho = \mathcal{L}$:

$$i\frac{\partial\eta}{\partial\rho} = i\frac{\eta_N - \eta_{N-1}}{\delta\rho} = -\frac{\partial\eta}{\partial t} \frac{1}{B\delta\rho} \quad (3.51)$$

$$i\frac{\partial\eta}{\partial\rho} = -\frac{\beta}{\omega_i} \frac{\partial\eta}{\partial t} . \quad (3.52)$$

- inner edge: we proceed as for the case of the outer edge, with the carefulness to operate a substitution of indexes. The disturbing function is therefore given by:

$$\mathcal{R}_1 = -B\eta_1\eta_1^* + B(\eta_1\eta_2^* + \eta_2\eta_1^*) \quad (3.53)$$

since we are now considering the nearest rings to the central body, which are labeled by 1 and 2. Given that ring 1 interacts with its nearest outer neighbor, whereas ring N interacts with its nearest inner neighbor, in this case the derivative appears with the opposite sign. The boundary condition at $\rho = 0$ is:

$$i\frac{\partial\eta}{\partial\rho} = \frac{\beta}{\omega_i} \frac{\partial\eta}{\partial t} \quad (3.54)$$

Having specified the boundary conditions, we can proceed to find the solution to equation 3.42, a potential-free Schrödinger's equation, that is a diffusion equation in imaginary time. From the physical point of view, this means that the process of diffusion must conserve the volume of the underlying distribution function in the phase-space, which means that the time evolution must be unitary. This does not give any problem, since in deriving Schrödinger's equation from Hamilton's equations as we have done in Chapter 2 we assured this condition, since Hamilton's equations are rooted in Liouville's theorem, which as a matter of fact is a key theorem in classical statistical and Hamiltonian mechanics and

one of its formulations states that the volume of a physic system in the phase-space must be conserved, or that the phase-space distribution function is constant along the trajectories of the system.

Equation 3.42 must therefore be satisfied by standing waves, the normal modes of a disk, which are labeled with l and are characterized by frequency ω_l .

By separation of variables equation 3.42 gives:

$$\eta_l = c_l \exp(-i\omega_l t) \mathcal{J}_l \quad (3.55)$$

where c_l is a constant, we get the quantum harmonic oscillator equation which is expressed as follows

$$\omega_l \mathcal{J}_l + \omega_i \frac{\partial^2 \mathcal{J}_l}{\partial \rho^2} = 0 \quad (3.56)$$

and the previously defined boundary conditions yield:

$$\frac{\partial \mathcal{J}_l}{\partial \rho} = -\beta \left(\frac{\omega_l}{\omega_i} \right) \mathcal{J}_l \Big|_{\rho=0} \quad (3.57)$$

and

$$\frac{\partial \mathcal{J}_l}{\partial \rho} = \beta \left(\frac{\omega_l}{\omega_i} \right) \mathcal{J}_l \Big|_{\rho=\mathcal{L}} . \quad (3.58)$$

Equation 3.56 finds a solution in

$$\mathcal{J}_l = \cos \left(\sqrt{\frac{\omega_l}{\omega_i}} \rho \right) - \beta \sqrt{\frac{\omega_l}{\omega_i}} \sin \left(\sqrt{\frac{\omega_l}{\omega_i}} \rho \right) . \quad (3.59)$$

So that we have:

$$\frac{\partial \mathcal{J}_l}{\partial \rho} = -\sqrt{\frac{\omega_l}{\omega_i}} \sin \left(\sqrt{\frac{\omega_l}{\omega_i}} \rho \right) - \beta \sqrt{\frac{\omega_l}{\omega_i}} \sqrt{\frac{\omega_l}{\omega_i}} \cos \left(\sqrt{\frac{\omega_l}{\omega_i}} \rho \right) \quad (3.60)$$

and for $\rho = \mathcal{L}$ we get:

$$\begin{aligned} & -\sqrt{\frac{\omega_l}{\omega_i}} \sin \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) - \beta \sqrt{\frac{\omega_l}{\omega_i}} \sqrt{\frac{\omega_l}{\omega_i}} \cos \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) = \\ & = \beta \left(\frac{\omega_l}{\omega_i} \right) \left[\cos \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) - \beta \sqrt{\frac{\omega_l}{\omega_i}} \sin \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) \right] \end{aligned} \quad (3.61)$$

which yields:

$$\left[\frac{\beta \omega_l}{2\omega_i} - \frac{1}{2\beta} \right] \sin \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) = \sqrt{\frac{\omega_l}{\omega_i}} \cos \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) \quad (3.62)$$

which unfortunately does not admit a simple solution, but can be noticeably simplified imposing the limit of a razor-thin disk, that means extrapolating the limit for $\beta \rightarrow 0$, which gives us :

$$\sin \left(\sqrt{\frac{\omega_l}{\omega_i}} \mathcal{L} \right) = 0 \quad (3.63)$$

which is easily solvable.

Adopting the razor-thin limit as a leading-order approximation, equation 3.62 results in:

$$\frac{\omega_l}{\omega_i} = \left(\frac{l\pi}{\mathcal{L}} \right)^2 \left(1 - 4 \frac{\beta}{\mathcal{L}} \right) + \mathcal{O}(\beta^2) . \quad (3.64)$$

Therefore the normal inclination modes are:

$$\eta_l = c_l \exp \left[-i \left(\frac{l\pi}{\mathcal{L}} \right)^2 \left(1 - 4 \frac{\beta}{\mathcal{L}} \right) \omega_l t \right] \times \left[\cos \left(\frac{l\pi\rho}{\mathcal{L}} \sqrt{1 - 4 \frac{\beta}{\mathcal{L}}} \right) - \beta \frac{l\pi}{\mathcal{L}} \sqrt{1 - 4 \frac{\beta}{\mathcal{L}}} \sin \left(\frac{l\pi\rho}{\mathcal{L}} \sqrt{1 - 4 \frac{\beta}{\mathcal{L}}} \right) \right]. \quad (3.65)$$

Where we remind that the fundamental ω_l is not a function of a since we chose a specific surface density profile (equation 3.14).

Equation 3.65 is further simplified if we employ the razor-thin approximation $\beta \rightarrow 0$, and is expressed as follows:

$$\eta_l = c_l \exp \left[-i \left(\frac{l\pi}{\mathcal{L}} \right)^2 \omega_l t \right] \cos \left(\frac{l\pi\rho}{\mathcal{L}} \right) \quad (3.66)$$

Where the coefficients c_l are determined from Fourier decomposition of the initial conditions.

The evolution of a self-gravitating razor-thin disk is therefore fully described by a superposition of the eigenstates determined in 3.66, which as we already mentioned indicate standing waves characterized by frequency ω_l .

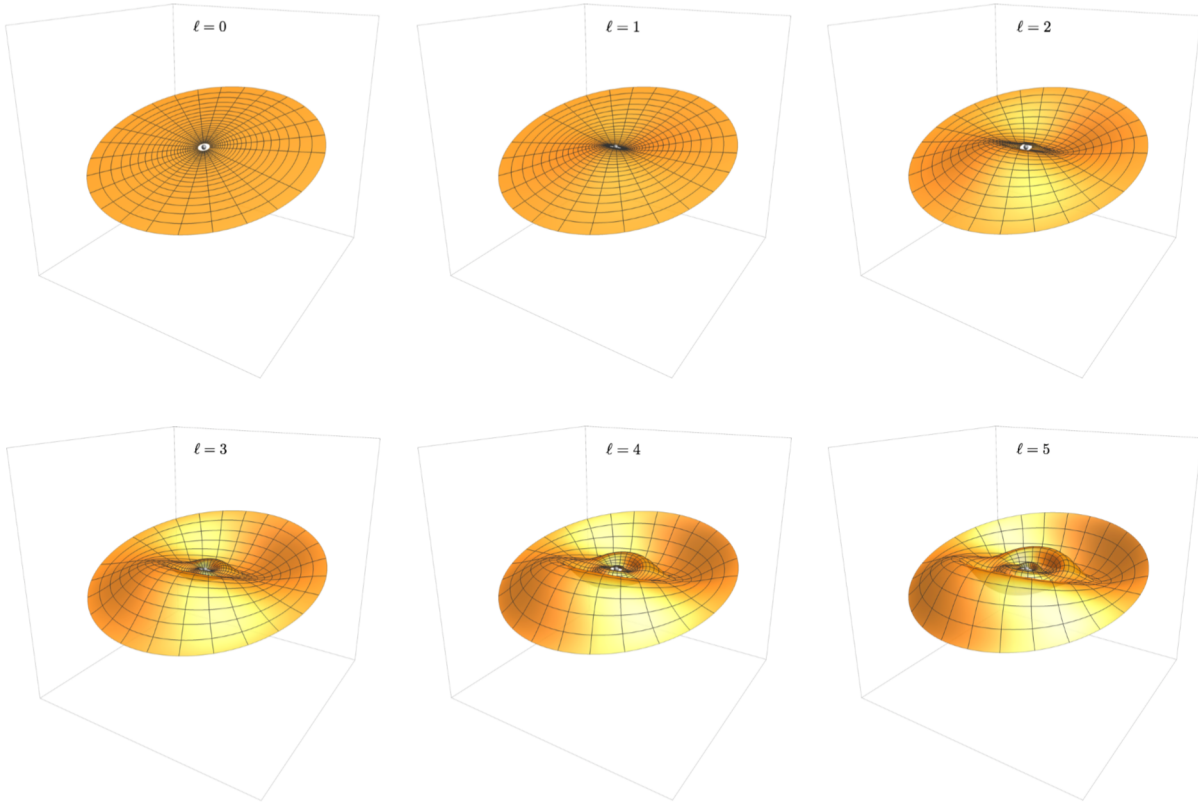


Figure 4: Six low-frequency inclination normal modes of a razor-thin disk. These images are obtained assuming an amplitude of $c_l = \pi/10$. From Batygin, Konstantin, *Schrödinger Evolution of Self-Gravitating Disks*, published in *Monthly Notices of the Royal Astronomical Society*, Volume 475, Issue 4, Pages 5070–5084, 21 April 2018.

Physically, the eigenstates expressed by 3.66 describe stationary nodal bending waves, characterized by a regression frequency given by ω_l . Figure 4 shows six low frequency modes in physical space. The lowest index allowed is $l = 0$, which corresponds to an uniformly inclined static disk.

Chapter 4

Approximations discussion

The analysis carried out in Sections 3.3 and 3.4 is based on numerous approximations, which are fundamental in order to guarantee the succinct final form of the equations describing the evolution of the disk. Some of the first assumptions made are for example the geometry of the disk, considered to be flat, and more specifically razor-thin. Proceeding with our analysis, we assumed the gravitational influences to be exclusively exerted between adjacent wires, neglecting the possibility of the mutual interaction between wires that are not in direct contact with one another. Moreover, we have always considered our system to be isolated, not affected by possible interplays with other bodies that reside in the surrounding environment. Considering that the majority of astrophysical disks are not completely cloistered systems, it is necessary to extend our model in order to describe the possibility of an interaction with, for example, a companion such as a massive planet.

The following sections are dedicated to the discussion of some of the approximations made during our analysis, with a view to make it as general as possible. We therefore try to explain, in some cases with a rigorous analytical proceeding and in some cases with a more qualitative description, how our approximations influence the correct evolution of the disk.

4.1 Non adjacent wires

One of the most important approximations considered in the previous analysis is that the influence which is established between rings actually exists exclusively between the nearest neighbors. This approximation implicates that the disturbing function of the considered wire j is only built with the terms regarding the adjacent wires, $j+1$ and $j-1$, but neglects the interactions that occur between j and other surrounding wires $j+2, j+3, \dots, N$ and $j-2, j-3, \dots, 1$.

We proceed now by expanding the method to the consideration of the wires labeled with $j+2$ and $j-2$. The disturbing function is therefore expressed as follows:

$$\begin{aligned} \mathcal{R}_j = & \mathcal{B}_{j,j} \eta_j \eta_j^* + \mathcal{B}_{j,j-1} (\eta_j \eta_{j-1}^* + \eta_{j-1} \eta_j^*) + \mathcal{B}_{j,j+1} (\eta_j \eta_{j+1}^* + \eta_{j+1} \eta_j^*) + \\ & + \mathcal{B}_{j,j-2} (\eta_j \eta_{j-2}^* + \eta_{j-2} \eta_j^*) + \mathcal{B}_{j,j+2} (\eta_j \eta_{j+2}^* + \eta_{j+2} \eta_j^*) . \end{aligned} \quad (4.1)$$

Keeping in mind that $\beta \ll 1$, we assume that all the wires are still close enough to each other to allow us to apply the limit $\alpha \rightarrow 1$, moreover we assume that the coefficients \mathcal{B} vary mainly depending on the value of the Laplace coefficient calculated for different α .

Considering a $j : j \pm \nu$ coupling, where $\nu \in \mathbb{Z}$, and defining the interaction length $\alpha = 1/(1+\beta)^\nu$, we have:

$$\alpha \tilde{b}_{3/2}^{(1)} \{ \alpha \} \approx \frac{2}{(1-\nu^2)\pi\beta^2} + \mathcal{O}(\beta^{-1}) \quad (4.2)$$

which yields

$$\mathcal{B}_{j,j-2} = \mathcal{B}_{j,j+2} \approx \frac{2}{5} \mathcal{B}_{j,j+1} = \frac{2}{5} \mathcal{B}_{j,j-1} \quad (4.3)$$

Proceeding as previously done in Section 3.3 we obtain the equation which describes the motion of wire j :

$$\frac{d\eta_j}{dt} \approx i \left(\frac{n_j m_j}{4\pi M \beta^2} \right) \left[- \left(2 + \frac{4}{5} \right) \eta_j + \eta_{j+1} + \eta_{j-1} + \frac{2}{5} (\eta_{j-2} + \eta_{j+2}) \right] . \quad (4.4)$$

Adopting the central difference approximation

$$\frac{\partial^4 \eta}{\partial \rho^4} = \frac{+\eta_{j-2} - 4\eta_{j-1} + 6\eta_j - 4\eta_{j+1} + \eta_{j+2}}{(\Delta\rho)^4} + \mathcal{O}((\Delta\rho)^4) \quad (4.5)$$

the continuum limit, $\alpha \rightarrow 1$, gives

$$\frac{\partial \eta}{\partial t} \approx i \omega_i \left[\frac{13}{5} \frac{\partial^2 \eta}{\partial \rho^2} + \beta^2 \frac{2}{5} \frac{\partial^4 \eta}{\partial \rho^4} \right] . \quad (4.6)$$

Likewise, if we adopt the same procedure extending it to the case $j \pm 3$, we obtain

$$\frac{d\eta}{dt} \approx i \omega_i \left[\frac{22}{5} \frac{\partial^2 \eta}{\partial \rho^2} + \beta^2 \frac{8}{5} \frac{\partial^4 \eta}{\partial \rho^4} + \beta^4 \frac{1}{5} \frac{\partial^6 \eta}{\partial \rho^6} \right] \quad (4.7)$$

and so on.

As mentioned before, keeping in mind that $\beta \ll 1$, we note that since the interaction length α largely exceeds the gravitational softening length, we can consider the contributions of the higher-order derivatives to be negligible. As a matter of fact, even though these terms are surely important to ensure the comprehensiveness of the model described, they are not decisive in quantitatively describing it: we conclude that the approximation which restricts the interactions to the nearest-neighbor results to be adequate.

4.2 Perturbed disks

Even considering the formerly discussed approximations, the analysis conducted in Chapter 3 neglects the substantial possibility of an interaction of some kind with the environment that surrounds the considered disk. Although cases of cloistered systems surely exist in nature, the majority of disks usually are located in a dynamically eventful space, characterized by an assorted range of phenomena which can easily perturb our disk, thus modifying its formerly described long-term evolution.

These occurrences can be classified either as ascribed to an external gravitational forcing or as extrinsic effects of some sort, such as processes of radiative nature and turbulence phenomena.

Some secular perturbations caused by an external gravitational forcing that can affect the evolution of a circumstellar disk can be for example caused by the presence of a bound companion such as a binary star or a massive planet, or the influence of the ambient potential of a stellar birth cluster.

In the attempt to describe how these perturbations affect the evolution of the considered disk, it is possible to obtain quantitative measures of its tendency towards deformation.

We now consider a perturbing companion of mass m' , located on an orbit characterized by an eccentricity e' , an inclination i' and a semi-major axis a' which has to be larger than the outer boundary of the disk, a_{out} . If the angular momentum of said companion exceeds considerably the angular momentum budget of the disk, we have that the influence exerted by m' on the disk is relevant, while the back-reaction of the disk can be easily overlooked. With these considerations we can choose to operate in the frame of reference of the companion, orienting the coordinate system to coincide with its plane of orbit such that $i' = 0$ and expressing the gravitational potential as a power series of (a/a') . While the Lagrange-Laplace theory requires the assumption of having to deal with small inclinations and eccentricities, in this case the disturbing function which describes the orbit-averaged gravitational potential of a companion needs no assumptions on the eccentricities and the inclinations of the orbit, but requires $a/a' \ll 1$. With these considerations we therefore obtain the disturbing function illustrated in *The determination of planetary structure in tidally relaxed inclined systems*, by R. A. Mardling ([30]):

$$\mathcal{R} = \frac{n m'}{4 M} \left(\frac{a}{a'}\right)^3 \frac{1}{(\sqrt{1-e'^2})^3} \left[\left(1 + \frac{3}{2}e'^2\right) \left(\frac{3\cos^2(i) - 2}{2}\right) + \frac{15}{4}e'^2 \sin^2(i) \cos(2(\bar{\omega} - \Omega)) \right] \quad (4.8)$$

in which the term

$$\frac{15}{4}e'^2 \sin^2(i) \cos(2(\bar{\omega} - \Omega)) \quad (4.9)$$

governs the Kozai-Lidov effect, which is a secular mechanism that causes a libration of the orbit's argument of pericenter, that determines a periodic exchange between the orbit's eccentricity and inclination. To prevent this, it is possible to bypass its occurrence by imposing a restriction on the mutual inclination between the disk and the companion and by approximating the mutual inclination so that we have $\cos(i) \approx 1$. Once this assumptions have been made, we can derive the new nodal regression rate $d\Omega/dt$, with $\Omega = \left(\frac{p}{q}\right)$ (an in-depth derivation is shown in [1]) finding eventually

$$\frac{d\Omega}{dt} \approx -\frac{3 m'}{4 M} \left(\frac{a}{a'}\right)^3 \frac{n}{(\sqrt{1-e'^2})^3} \quad (4.10)$$

that describes a rotation of the phase.

This effect can be included in Schrödinger's equation through a potential exponential term

$$i\omega_i \frac{\partial \eta}{\partial t} = -\omega_i^2 \frac{\partial^2 \eta}{\partial \rho^2} + \omega_i \omega' \exp(3\rho/2) \eta \quad (4.11)$$

with ω' scaling the perturbation constant ω in order to describe the effects given by the newly described interactions.

One version of a Schrödinger's equation characterized by an exponential potential is employed in the description of molecular interaction. Once again the evolution of a macroscopic object such as a circumstellar astrophysical self-gravitating disk, this time affected by some kind external perturbations, is described analogously to a phenomenon belonging to an apparently diametrically opposed field.

Final discussion

Schrödinger's equation is generally thought of as the fundamental mathematical postulate of quantum mechanics. However, a non linear form of this equation can describe several phenomena and has numerous applications, such as Langmuir plasma oscillations ([22]) and nonlinear optics ([23]).

In this dissertation we have demonstrated that a certain form of Schrödinger's linear equation can describe the secular evolution of a self-gravitating razor-thin circumstellar disk, under some specific conditions. The derivation relies on the Lagrange-Laplace perturbation theory, studying the dynamical evolution of an astrophysical disk influenced by perturbations on the secular timescale, which greatly exceeds the period of the orbit.

The first remarkable aspect of this analysis consists in the fact that the evolution of a protoplanetary disk can be described by a wave equation. Although other means of analyzing the dynamics of self-gravitating disks exist, such as the N-body approach or the Boltzmann collisionless method, this study is of key importance since it offers a succinct and well-known equation that describes the considered problem. While other methods of resolution may result computationally expensive and suitable more for specialized circumstances than for generalized cases or may require particular approximations, Schrödinger's evolution constitutes an immediate and simple model.

The second important characteristic of the results of this discussion is that since this wave equation is exactly solvable, it offers a fascinating simplification of an otherwise complex problem.

Another merit of this model is its extendability: we can generalize the derived equations for different superficial density profiles, alternative aspect ratios that don't have to be assumed constant, several geometric characteristics. Although these modifications would possibly introduce a dependence of the fundamental frequency upon the semi-major axis, depriving the model of its considerable practical use, they are indeed applicable, in order to describe in the most comprehensive way the long term evolution of disks.

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