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The effect of neutrinos on the stochastic gravitational-wave background

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Introduction

In 2015 the LIGO/Virgo collaboration detected a gravitational wave (GW) signal from the merging of two black-holes [1], achieving the first direct detection of gravitational waves and the probe of the existence of binary stellar-mass black hole systems. After that, other successful results arrived, like the observation of a gravitational wave signal from the merging of two neutron stars [2], which opened the era of multi-messenger astronomy. In the next years other detectors are planned to be built, both on the earth as Einstein Telescope [3] and Cosmic Explorer [4], and on the space, as DECIGO [5] and LISA [6]. Another earth-based detector, KAGRA [7], just started to be active very recently. Gravitational wave detections do not give only the possibility of increasing the knowledge about astrophysical objects, but they allow to explore many different fields. We can get important information about particle physics, because many mechanisms that generated gravitational waves in the early Universe are based on theories beyond the Standard Model of particle physics. Moreover we can have information about the physics at energy scales much larger than the ones that we can reach at colliders nowadays. We can also test different aspects of General Relativity, through the estimations of the speed of propagation and of the polarization of the gravitational waves. There are also many implications for cosmology, as for instance a precise estimation of the Hubble constant [8] using the GW signal as standard sirens, in analogy with the astronomical standard candles. Beside the detection of gravitational waves signal from the merging of astrophysical sources, future GW detectors are expected to increase their sensitivity at the level to eventually detect a stochastic gravitational wave background. A background corresponds to a random signal that can be described only in terms of its statistical properties. There are two main sources of such background: astrophysical and cosmological. The astrophysical background is given by the superposition of signals from unresolved sources, which can be black holes, supernovae and pulsars [9]; in particular, the compact binary coalescences are expected to produce a very loud background [10]. On the other hand, there are also many different cosmological mechanisms that can produce a stochastic background: for instance the preheating at the end of the inflation, topological defects or first order phase transitions [11]. However, in this thesis we will focus on the gravitational waves produced by the quantum vacuum fluctuations during inflation [12]. In general, accordingly to Quantum Field Theory, each field can have small deviations from its classical value, precisely the quantum fluctuations. During inflation, the accelerated expansion of the Universe amplifies these quantum fluctuations, giving rise to a consistent background of gravitational waves. Future detectors, as LISA, will hopefully allow to directly detect such stochastic backgrounds [13]. If a background of cosmological origin would be measured, it would have important implications both on cosmology and on particle physics: it would be a confirm that inflation really happened and that the gravitational field is quantized [14].

A fundamental equation to study the evolution of stochastic variables is the Boltzmann equation, which describes the variations in time of the distribution function related to the observables involved. Such equation has been fundamental to quantify many properties of the cosmic microwave background radiation (CMB) [15, 16, 17]. The CMB spectrum is almost perfectly isotropic, presenting the same features in all the sky (the same temperature T) [18], with some small fluctuations δT which depend on the direction of observation; the Boltzmann equation has been used to study how these perturbations have evolved from early times until now [19, 20, 21]. In full analogy with the CMB, we expect that also the stochastic gravitational wave background (SGWB) present anisotropies with respect to an homogeneous and isotropic background. The purpose of this thesis is to quantify the effect of neutrinos on the anisotropies of the SGWB, underlying the main effects that contribute to amplify or to damp such anisotropies. We have considered in particular the case in which we have three neutrino species, and the case where no neutrino are present. We have approached the problem in analogy with what has been done for the CMB [22]: we started by defining a distribution function f for the gravitons, the quantum corrispective of the gravitational waves, which are spin 2 massless particles, and then we have written down the Boltzmann equation for f in a perturbed spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, assuming that the graviton trajectories in the universe are null geodesics defined by the background metric; in all these computations we have considered gravitons as collisionless particles, under the qualitative argument that they decoupled at early times, around $T_{\rm Pl} \approx 10^{19}$GeV. We have found that the anisotropies of the SGWB are generated by two different causes: by the anisotropies at the moment of their production, in other words by the initial conditions, and by the free streaming of the waves through the perturbed universe; different paths mean different perturbations on the trajectories, thus anisotropies [23, 24]. Due to the anisotropy dependence on the past history (the trajectories) of the gravitons, to characterize them completely we need to know also the evolution of the metric perturbations, so it has been necessary to compute precisely the evolution of the metric perturbations. To do that, we have solved the Einstein equations for the metric perturbations combined with the Boltzmann equation for various particle species in the universe. We have studied the solutions in the cases in which in the Universe there are no neutrinos, $N_\nu = 0$, and the one in which there are three neutrino generations. The result is that switching from $N_\nu = 0$ to $N_\nu = 3$ determines a damping in the amplitudes of the cosmological perturbations, in some specific cases, e.g. for the tensor modes on small scales, the damping of the squared amplitude is reduced up to 35% [26]. We have concluded the work by computing the values of the angular power spectra $\widetilde{C}_\ell$ for the gravitational waves, to study the effect of neutrinos on such a spectra; we have used the Cosmic Linear Anisotropy Solving System (CLASS) [27], an accurate Boltzmann code widely used to investigate many features of the CMB, adapted for this project to the analysis of the SGWB.

The structure of the thesis is the following: in Chapter 1 we have made an overview of gravitational waves and of their cosmological origin, introducing the short-wave formalism that allows to describe the propagation through a generic curved background and we have listed the main features of the spectra (statistical properties, etc.); in Chapter 2 we have solved the Boltzmann equation for gravitons, discussing the characteristics of their distribution function, and expanding the solution in multipoles, finding some integral expressions as function of the metric perturbations; in Chapters 3 and 4 we have discussed the evolution of the tensor and of the scalar metric perturbations respectively; in Chapter 5 we have listed all the results obtained by numerical computations for the stochastic gravitational waves background anisotropies, comparing the two cases $N_\nu = 0$ and $N_\nu = 3$.

Capitolo 1

Cosmological Background of Gravitational Waves

1.1 The need to introduce gravitational waves

Einstein's General Theory of Relativity provides a covariant theory of gravity, in the sense that all the equations from which we can derive the predictions for the observables, are invariant under diffeomorphisms [28].
This means that, giving a transformation of the coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, which maps a manifold to another, such that both the function and its inverse are smooth, the equations do not change under such a change of coordinates, using the fact that scalar, vector and tensor quantities transform in the following way:

$$\begin{aligned} \tilde{\phi}(\tilde{x}) =& \phi(x), \\ \tilde{V}^\mu(\tilde{x}) =& \frac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu(x), & \tilde{V}_\mu(\tilde{x}) =& \frac{\partial x^\nu}{\partial \tilde{x}^\mu} V_\nu(x), \\ \tilde{T}^{\mu\nu}(\tilde{x}) =& \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta} T^{\alpha\beta}(x), & \tilde{T}_{\mu\nu}(\tilde{x}) =& \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} T_{\alpha\beta}(x). \end{aligned} \tag{1.1.1}$$

To be consistent with causality, any information about modifications of the gravitational field has to propagate at maximum at the speed of light c; this is predicted by Special Relativity, which has to be compatible with General Relativity, because, for the equivalence principle, in any arbitrary gravitational field it is possible to restrict to a sufficiently small region in which the physical laws have the same form as in an unaccelerated coordinate system without gravitation [29, 30, 31].
This fact, combined with the analogies between gravitation and electromagnetism[1], makes

[1] We refer to the similarity between the gauge-invariant equations under the symmetry $U(1)$, describing electromagnetic interactions in Quantum Field Theory, and the invariance under diffeomorphisms of General Relativity [32].
Intuitively, we could also think to the resemblances between the Newton's law of universal gravitation and the Coulomb's law.

reasonable expecting radiative solutions also for the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1.2}$$

The left-hand side of the equation, the Einstein tensor $G_{\mu\nu}$, represents the geometric structure of the spacetime, it contains all the information we need to know to describe trajectories of objects in free fall in General Relativity. It is indeed defined in terms of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, related to the Riemann tensor $R^{\rho}_{\mu\sigma\nu}$, which encodes all the information of a given curved manifold. The right-hand side term corresponds to the energy-density of the system: the stress-energy tensor $T_{\mu\nu}$ describes the energy densities and the momenta of all the particles involved. Therefore the Einstein equations represent how the matter content ($T_{\mu\nu}$), curves and determine the structure of the spacetime ($G_{\mu\nu}$). The underlying idea is that, as in electromagnetism changes in the electromagnetic field produce the emission of electromagnetic waves, which modify locally the electromagnetic field themselves, variations in the gravitational field generate gravitational waves which perturb the spacetime geometry as they pass.
The physical effect we can observe is analogue to the electromagnetic's one too: as the presence of an electromagnetic wave at a certain time accelerates an electrically charged test particle, the arrive of a gravitational wave alters the geodesic separation between two test particles if their relative position vector is parallel to the polarization direction of the gravitational wave; we see a tidal acceleration due to the gradient in the gravitational field, which is non-null exactly because the gravitational wave perturbs the metric.
Non-linearity of Einstein's equations make the study of the gravitational waves more complicated with respect to the electromagnetic ones, because this prevents us from finding general radiative solutions. In order to overcome this problem, we will use in this section the so-called weak-field approach, in which we consider only gravitational radiation of very low intensity, basically for two reasons.
The first one is that gravity is a very weak force, thus it is unlikely to find gravitational waves with large amplitudes. In addition, we will consider tensor perturbations to the metric (which correspond to gravitational waves), very small by definition, and so we are not interested in large amplitudes.
The second reason is that we are concerned about the behaviour of the elementary particle associated to gravitation in theories of quantum gravity: the spin-2 graviton. This is possible only if we use the weak-field approximation, otherwise, for large gravitational fields, we are not able to attach a precise meaning of quantum particle to ensemble of particles not enough separated.
This subsection is organized as follows: we will examine the instructive example of the propagation of gravitational waves on a flat space, with a background Minkowski metric and no stress-energy sources; after that we will generalize the result to a non-null energy-momentum tensor; at the end we will discuss the propagation of gravitational waves in the most general scenario, i.e. in a non-flat spacetime, described by a generic background metric $g^{(B)}_{\mu\nu}$.

1.2 Gravitational wave propagation in flat space

The natural starting point is the propagation of GW in a flat space, where $T_{\mu\nu} = 0$ and the background metric is the Minkowski one. This is the simplest case and it is quite easy to find out that the small perturbations to such a metric obey to a wave-like equation.
We define the gravitational waves as the difference between the total metric $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$:

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}, \qquad |h_{\mu\nu}(x)| \ll 1. \tag{1.2.1}$$

As introduced in the previous section, the Einstein equations are invariant under coordinate transformations, thus we consider a generic infinitesimal transformation of the form

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu, \tag{1.2.2}$$

with $\xi^\mu(x)$ an infinitesimal vector field. Under such a map the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = \left(\delta^\alpha_\mu - \frac{\partial \xi^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu - \frac{\partial \xi^\beta}{\partial x^\nu}\right) g_{\alpha\beta}(x) = g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{1.2.3}$$

and, if we write the transformed metric as the Minkowski metric plus a transformed perturbation, i.e.

$$g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x'), \tag{1.2.4}$$

it is clear that the gravitational waves under these coordinate transformations transform as

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{1.2.5}$$

with the additional condition

$$|\partial_\alpha \xi(x)| \leq |h_{\mu\nu}(x)|, \tag{1.2.6}$$

in order to preserve the "weak gravitational fields condition" which requires $|h'_{\mu\nu}(x')| \ll 1$.
Now, we are ready to evaluate the Einstein equations, starting from the Christoffel symbols,

$$\begin{aligned}\Gamma^\mu_{\nu\rho} \equiv& \frac{1}{2} g^{\mu\alpha}(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}) = \frac{1}{2}\left(\eta^{\mu\alpha} + h^{\mu\alpha}\right)(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}) = \\ =& \frac{1}{2}(\partial_\nu h^\mu_\rho + \partial_\rho h^\mu_\nu - \partial^\mu h_{\alpha\rho})\,.\end{aligned} \tag{1.2.7}$$

Then we can calculate the Riemann tensor, evaluating all the quantities at the first order in the perturbation $h_{\mu\nu}$,

$$\begin{aligned}R^\rho_{\mu\sigma\nu} =& \partial_\sigma \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\sigma\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} = \partial_\sigma \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\sigma\mu} = \\ =& \frac{1}{2}\left[\partial_\sigma\partial_\nu h^\rho_\mu + \partial_\sigma\partial_\mu h^\rho_\nu - \partial_\sigma\partial^\rho h_{\nu\mu} - \left(\partial_\nu\partial_\sigma h^\rho_\mu + \partial_\nu\partial_\mu h^\rho_\sigma - \partial_\nu\partial^\rho h_{\sigma\mu}\right)\right] = \\ =& \frac{1}{2}(\partial_\sigma\partial_\mu h^\rho_\nu + \partial_\nu\partial^\rho h_{\rho\mu} - \partial_\nu\partial_\mu h^\rho_\rho - \Box h_{\nu\mu}),\end{aligned} \tag{1.2.8}$$

the Ricci tensor

$$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu} = \frac{1}{2}(\partial_\mu\partial_\rho h^{\rho}_{\nu} + \partial_\nu\partial^{\rho} h_{\rho\mu} - \partial_\nu\partial_\mu h^{\rho}_{\rho} - \Box h_{\mu\nu}), \tag{1.2.9}$$

the Ricci scalar

$$R \equiv R^{\mu}_{\mu} = \partial^{\mu}\partial_\rho h^{\rho}_{\mu} - \Box h^{\mu}_{\mu}, \tag{1.2.10}$$

and finally the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\big[\partial_\mu\partial_\rho h^{\rho}_{\nu} + \partial_\nu\partial^{\rho} h_{\rho\mu} - \partial_\nu\partial_\mu h^{\rho}_{\rho} - \Box h_{\mu\nu} - \eta_{\mu\nu}(\partial^{\mu}\partial_\rho h^{\rho}_{\mu} - \Box h^{\mu}_{\mu})\big]. \tag{1.2.11}$$

We would like to remove the dependence of the last term on the trace h^{μ}_{μ}; to do this we introduce the *trace-reversed* metric perturbation $\bar{h}_{\mu\nu}$, which is equivalent to $h_{\mu\nu}$, but with an opposite sign of the trace:

$$\bar{h}_{\mu\nu}(x) \equiv h_{\mu\nu}(x) - \frac{1}{2}\eta_{\mu\nu}h^{\rho}_{\rho}(x), \qquad \bar{h}^{\mu}_{\mu} = -h^{\mu}_{\mu}. \tag{1.2.12}$$

Using this new form, the Einstein tensor has the simpler form

$$G_{\mu\nu} = \frac{1}{2}\big(\partial_\rho\partial_\nu\bar{h}^{\rho}_{\mu} + \partial^{\rho}\partial_\mu\bar{h}_{\nu\rho} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial^{\beta}\bar{h}^{\alpha}_{\beta}\big). \tag{1.2.13}$$

The Einstein equations in this general form present a large number of degrees of freedom; the symmetric tensor $h_{\mu\nu}$ has 10 independent components, but we have seen that the coordinates of the system are not completely specified: we can perform a coordinate transformation (1.2.2) in such a way the metric can still be written as the Minkowski metric plus a perturbation which depends on the perturbation $h_{\mu\nu}$ through (1.2.5). Therefore we will try to simplify the equations by using proper diffeomorphisms transformation, in order to find explicitly the physical degrees of freedom of the system. A common gauge used to study radiation is the Lorentz gauge:

$$\partial^{\mu}h_{\mu\nu} = 0. \tag{1.2.14}$$

We want to show that, using a coordinate transformation of the form of Eq. (1.2.2), is always possible to achieve the Lorentz gauge: the map which relates the perturbation in a generic gauge $h_{\mu\nu}(x)$ to the perturbation in the Lorentz gauge $\bar{h}'_{\mu\nu}(x')$ is [2]

$$\begin{aligned}\partial^{\mu}\bar{h}'_{\mu\nu}(x') =&0 = \partial^{\mu}\big[h_{\mu\nu}(x) - \partial_\mu\xi_\nu(x) - \partial_\nu\xi_\mu(x) - \frac{1}{2}\eta_{\mu\nu}\big(h^{\rho}_{\rho}(x) - 2\partial^{\alpha}\xi_\alpha(x)\big)\big]= \\ =&\partial^{\mu}\bar{h}_{\mu\nu}(x) - \Box\xi_\nu(x),\end{aligned} \tag{1.2.15}$$

hence the infinitesimal vector field that we need is the solution of

$$\Box\xi_\nu(x) = \partial^{\mu}\bar{h}_{\mu\nu}(x), \tag{1.2.16}$$

[2] We identify $h'_{\mu\nu}(x')$ as the transformed field, but, after the following two equations, we automatically use the redefinition, with a little abuse of notation, $h_{\mu\nu}(x) \equiv h'_{\mu\nu}(x')$.

which always exists, because the d'Alambertian operator $\Box$ is invertible.
In this coordinate system the Einstein tensor has the simple form

$$G_{\mu\nu}(x) = -\frac{1}{2}\bar{h}_{\mu\nu}(x), \tag{1.2.17}$$

this means that in the vacuum the perturbations are solutions of the wave equation:

$$\Box\bar{h}_{\mu\nu} = 0. \tag{1.2.18}$$

The Lorentz gauge imposes 4 additional conditions on the symmetric tensor $h_{\mu\nu}(x)$, therefore the number of physical degrees of freedom becomes $10 - 4 = 6$; neverthless it is immediate to see that the Lorentz gauge is sensitive to some residual gauge transformations: if we apply any coordinate transformation with $\Box\xi_\mu(x) = 0$, the transformed perturbation is still in the Lorentz gauge, hence we need 4 other constraints, one for each component of $\xi_\mu(x)$, to remove completely this redundance. The first condition we impose is that the perturbation has to be traceless,

$$\bar{h}^{\mu\prime}_{\mu} = 0 = \left(h^\mu_\mu - 2\partial^\mu\xi_\mu\right) - \frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}\left(h^\rho_\rho - 2\partial^\rho\xi_\rho\right) = \bar{h}^\mu_\mu + 2\partial^\mu\xi_\mu \to \partial^\mu\xi_\mu = -\frac{\bar{h}^\mu_\mu}{2}, \tag{1.2.19}$$

and other three requires that all the $(0, i)$ components are null,

$$\bar{h}'_{0i} = 0 = \bar{h}_{0i} - \partial_0\xi_i - \partial_i\xi_0 \to \partial_0\xi_i + \partial_i\xi_0 = \bar{h}_{0i}. \tag{1.2.20}$$

In this gauge, called transerve-traceless (TT) gauge, the trace-reversed perturbations, $\bar{h}_{\mu\nu}$, coincide with the original ones, $h_{\mu\nu}$, so the metric perturbations $h_{\mu\nu}$ follow the wave equation (1.2.18) as the trace-reversed ones $\bar{h}_{\mu\nu}$.
When we consider the gravitational waves we can neglect the $(0, 0)$ component of the tensor perturbations: from the Lorentz condition we have

$$\partial^\mu h_{\mu\nu} = 0 \to \partial^0 h_{0\nu} + \partial^i h_{i\nu} = 0 \to \partial^0 h_{00} = 0\,. \tag{1.2.21}$$

This means that the $(0, 0)$ component is time independent, $h_{00}(t, \vec{x}) = h_{00}(\vec{x})$. A time independent term is not related to the gravitational waves, but to the Newtonian potential of the source of the waves, therefore when we are considering only the propagation of the waves (the time dependent part of the metric perturbation) in a vacuum spacetime, $\partial^0 h_{00}$ implies $h_{00} = 0$ [33].
Once we have saturated the gauge freedom, we see that the only physical (measurable) quantities are $6 - 4 = 2$; if we adopt the TT gauge then these degrees of freedom can be related immediately to the two polarizations of the gravitational wave. In order to show that, we can decompose a general gravitational wave into a superposition of plane waves,

$$h_{\mu\nu}(x) = \int d^3k\left(h_{\mu\nu}(\vec{k})e^{ik^\mu x_\mu} + h^*_{\mu\nu}(\vec{k})e^{-ik^\mu x_\mu}\right), \tag{1.2.22}$$

where $|k^0| = |\vec{k}|$ from the wave equation; if we have written the perturbation in the transverse-traceless gauge then the only non-null quantities are the components h_{ij}, which satisfy, from the Lorentz condition in the Fourier space, $k^i h_{ij}(\vec{k}) = 0$. By choosing a proper reference frame in which $\vec{k} = k\hat{n}_z$, we see that $h_{3j} = 0$, so we are only left with h_{11}, h_{12}, h_{21} and h_{22}, which satisfy $h_{12} = h_{21}$ (symmetric tensor) and $h_{22} = -h_{11}$ (traceless). To conclude, the final form for the tensor is

$$h_{\mu\nu}(\vec{k}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(\vec{k}) & h_\times(\vec{k}) & 0 \\ 0 & h_\times(\vec{k}) & h_+(\vec{k}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.2.23}$$

which stresses the fact that the polarizations of the gravitational waves are perpendicular to the direction propagation of the waves, represented by the versor

$$\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{1.2.24}$$

in full analogy with the electromagnetism.

1.3 Linearized theory of gravitational waves propagation in matter

In this subsection we generalize the discussion of the previous one: we consider the more realistic situation of an asymptotically flat spacetime,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \qquad |h_{\mu\nu}(x)| \ll 1, \qquad |h_{\mu\nu}(t, |\vec{x}| \to \infty)| = 0, \tag{1.3.1}$$

with a non-null stress-energy tensor, $T_{\mu\nu}(x) \neq 0$, of order one in perturbations [3].
Also in this case we will evaluate everything up to first order in perturbation theory. An immediate consequence is that the stress-energy tensor is invariant under coordinate transformations thanks to the Stewart Walker lemma [31], which states that a tensor with a null background value does not transform under the coordinate transformations at any order in perturbation theory.
This increases the number of degrees of freedom of the system: the stress-energy tensor is a rank two symmetric tensor, which is conserved,

$$\partial^\mu T_{\mu\nu} = 0, \tag{1.3.2}$$

therefore it has six independent components. These terms cannot be eliminated by coordinate transformations, because the tensor is gauge invariant, therefore we require six corrispective independent elements in $G_{\mu\nu}$, i.e. in the metric perturbation $h_{\mu\nu}$. This is equivalent to state that we cannot write the metric perturbation $h_{\mu\nu}$ only as a transverse-traceless tensor h_{ij}, because such a term would have only two degrees of freedom. In this

[3] If it was not a term of order one in the perturbations, then the unperturbed metric would not be the Minkowski one.

case we have indeed also non-radiative degrees of freedom, such as the $h_{00}(\vec{x})$ term, tied to the matter sources[4]. Notice that it is still possible moving to the Lorentz gauge, because it leaves the metric perturbation with exactly six degrees of freedom, as the stress-energy tensor. The problem is that in this gauge the Einstein equations are misleading, because they can be written as, using Eq. (1.2.18) with a non-null stress-energy tensor,

$$\Box \bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu}. \tag{1.3.3}$$

The problem lies in the fact that, in this gauge, all the six independent components of $h_{\mu\nu}$ seems to be radiative degrees of freedom[5]. Actually, only two of them are effectively radiative, while the other four are not[6]. The fact that six fields are described by Eq. (1.3.3) is due to the coordinate choice, they are are mere sinuosities in the coordinate system [34]. Because of this we will use a gauge invariant approach [35, 36], writing the Einstein equations in terms of quantities which are independent of the coordinates used. We will see that there are some gauge invariant variables such that the Einstein equation have the form of Eq. (1.2.18).
We begin by decomposing the metric perturbation into irreducible parts with respect to spatial
transformations [37]:

- the $(0,0)$ component does not transform since it is a scalar, and then we can set

$$h_{00} = -2\phi; \tag{1.3.4}$$

- the $(0,i)$ component transforms as a three-vector, therefore we write

$$h_{0i} = \beta_i + \partial_i \gamma, \tag{1.3.5}$$

  with $\partial_i \beta_i = 0$.
  In fact, thanks to the Helmoltz's theorem, it is possible to decompose a vector field into an irrotational (curl-free, longitudinal) component $\partial_i \gamma$ and into a solenoidal (divergence-free, transverse) component $\beta_i^{\perp}$. This can be justified by taking the divergence of h_{0i} and noticing that a solution for γ always exists, because the Laplace operator is always invertible, the solution for β_i comes immediately from the difference between h_{0i} and γ;

- the (i,j) component has a similiar structure with respect to the vector's one, it transforms as a 3×3 symmetric tensor, hence we use two suitable functions H and λ, a solenoidal vector field ϵ_i and a rank 2 transverse-traceless symmetric tensor h_{ij}^{TT}:

$$h_{ij} = h_{ij}^{TT} + \frac{1}{3} H \delta_{ij} + \partial_i \epsilon_j + \partial_j \epsilon_i + \Big(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \Big) \lambda, \tag{1.3.6}$$

[4] In the previous section we have neglected such a term because we were in the vacuum, where $T_{\mu\nu} = 0$ precisely.
[5] Radiative means that the degrees of freedom obey wave-like equations.
[6] We will see that they satisfy Poisson-like equations.

with $\partial_i \epsilon_i = 0$, $h^{TT}_{ii} = \partial_i h^{TT}_{ij} = 0$. Also in this case, to justify the uniqueness of the decomposition, we should evaluate h_{ii}, $\partial_i h_{ij}$ and $\partial_i \partial_j h_{ij}$, proving the existence of the solutions for H, ϵ_i and λ, finding at the end the solution for h^{TT}_{ij} by subtraction.

If we consider the map defined in Eq. (1.2.2), using the form $\xi^\mu = (A, B_i + \partial_i C)$ for the infinitesimal vector field, with B_i solenoidal vector, the metric perturbation transforms according to Eq. (1.2.5). Thus we can find the transformation rules for the irreducible parts:

$$
\begin{aligned}
h_{00}:\quad & 2\phi'(x') = 2\phi(x) - 2\partial_0 A \rightarrow \phi'(x') = \phi(x) - \partial_0 A(x);\\
h_{0i}:\quad & \beta_i'(x') + \partial_i' \gamma'(x') = \beta_i(x) + \partial_i \gamma(x) - \partial_i A(x) - \partial_0 B_i(x) - \partial_i \partial_0 C(x),\\
& \beta_i'(x') = \beta_i(x) - \partial_0 B_i(x),\\
& \gamma'(x') = \gamma(x) - A(x) - \partial_0 C(x);\\
h_{ij}:\quad & h_{ij}'(x') = h_{ij}(x) - \partial_{(i} B_{j)}(x) - 2\partial_i \partial_j C(x) =\\
& \qquad = h_{ij} - \partial_{(i} B_{j)} - 2\Big(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\Big) C(x) + \frac{2}{3}\delta_{ij}\nabla^2 C(x),\\
& H'(x') = H(x) - 2\nabla^2 C(x),\\
& \epsilon_i'(x') = \epsilon_i(x) - B_i(x),\\
& \lambda'(x') = \lambda(x) - 2C(x),\\
& h^{TT}_{ij}{}'(x') = h^{TT}_{ij}(x).
\end{aligned}
\tag{1.3.7}
$$

The first thing we notice is that the transverse-traceless perturbation is automatically gauge invariant, i.e. it is not affected by gauge modes which can drive us to incorrect conclusions. For this reason we would like to find other gauge invariant quantities, constructed by linear combinations of the metric perturbation's components (and their derivatives) and express the Einstein equations in terms of only these new variables.
The three functions we construct from the geometrical irreducible parts are

$$
\begin{aligned}
\Phi \equiv & -\phi + \partial_0 \gamma - \frac{1}{2}\partial_0^2 \gamma,\\
\Theta \equiv & \frac{1}{3}\Big(H - \nabla^2 \lambda\Big),\\
\Xi \equiv & \beta_i + \frac{1}{2}\partial_0 \epsilon_i.
\end{aligned}
\tag{1.3.8}
$$

We can decompose the stress-energy tensor in an analogous way:

$$
\begin{aligned}
T_{00} =& \rho,\\
T_{0i} =& S_i + \partial_i S, \qquad\qquad \partial_i S_i = 0,\\
T_{ij} =& P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \Big(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\Big)\sigma, \qquad\qquad \partial_i \sigma_i = \sigma_{ii} = \partial_i \sigma_{ij} = 0.
\end{aligned}
\tag{1.3.9}
$$

We have to write down six independent Einstein equations to include all the degrees of freedom, ten from the symmetry of the metric tensor minus four from the conservation of the stress-energy tensor. It has been shown [38, 39] that the Einstein equations in terms of these gauge invariant variables are

$$\begin{aligned}\nabla^2\Theta =& -8\pi G\rho,\\ \nabla^2\Phi =& 4\pi G\Big(\rho+3P-3\partial_0 S\Big),\\ \nabla^2\Xi =& -16\pi G S_i,\\ \Box h_{ij}^{TT} =& -16\pi G\sigma_{ij}.\end{aligned} \tag{1.3.10}$$

These equations contain the most important result of this subsection: only the transverse-traceless part of the perturbations contain the radiative degrees of freedom. Its evolution is determined by a wave equation with a source, instead the other three are determined by Poisson-like equations (or Laplace equations in the vacuum). We stress again that, in some gauges, like the Lorentz one, all the metric perturbations obey a wave-type equation, but this is an artifacts due to some specific coordinates.
This result will be very important when we will discuss the production mechanism for the gravitational waves and the distribution function of gravitons in the Universe.

1.4 Gravitational waves in a curved background

In this subsection we want to discuss the gravitational waves in the full theory of General Gelativity, perturbing a background which has a non-null curvature. The curvature of the background depends on the gravitational waves and, in principle, on the matter content of the Universe too, if present; through this discussion we will determine the form of the stress-energy tensor for the gravitational waves, which contributes to the curvature and to other geometrical quantities. Another important remark is that we are interested in knowing how the gravitational waves interact with the background curvature (independently from the kind of source which gives the larger contribution to the curvature): as a gravitational wave propagates through a curved background, in fact, its wave fronts change shape [7], the wavelenght changes, and it backscatters off the curvatures to some extent [9]. We will develop a tool, the shortwave formalism [40, 41], under the assumptions that the amplitudes of the gravitational waves are very small and that the ratio between the wavelength of the wave and the background curvature is small too, which allows to prove that all these mentioned effects are extremely small locally. So, locally, linearized theory is still highly accurate.
In principle, there could be some issues in defining the gravitational waves in a curved background: if they are a fluctuation of the background, we should be able to distinguish them from the background itself. To do this we need that the reduced wavelength of the gravitational waves $\bar{\lambda} = \lambda/(2\pi)$, is much smaller than the typical scale $\mathcal{R}$ over which the background varies [8]. In other words, whenever we impose the condition $\bar{\lambda} \ll \mathcal{R}$ we are ensuring that the gravitational waves are small-scale ripples propagating in a background

[7] This effect is similiar to the refraction of the light when it encounters obstacles in its path.
[8] It corresponds to the typical magnitude of the components of the background Riemann tensor $R^{(B)}_{\alpha\beta\gamma\delta}$.

of large-scale curvature $\mathcal{R}$; this limit is also called *geometric optics* regime.
As an example we consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, that will be described in Eq. (2.3.1), which has as typical scale $\mathcal{R} = \frac{1}{aH}$, and gravitational waves which crossed the horizon[9] during the radiation or matter dominated eras: because of the expansion of the Universe the comoving Hubble horizon at the horizon crossing, $a(t_{h.c.})H(t_{h.c.})$, will be much bigger than the comoving Hubble horizon at the present epoch, $a_0 H_0$, hence $\bar{\lambda} \gg \mathcal{R}$ and such gravitational waves are well defined in the cosmological context. Under this hypothesis, the metric can be written as a background term plus a small perturbation due to the gravitational waves:

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu}. \tag{1.4.1}$$

The formalization of what we have assumed above can be resumed in the following relations:

- the gravitational wave $h_{\mu\nu}$ can be expanded with respect to the background using the parameter $\mathcal{P}$, much smaller than 1, which corresponds to the amplitude of the waves[10],

$$h_{\mu\nu} \lesssim O(g^{(B)}_{\mu\nu}) \cdot \mathcal{P}; \tag{1.4.2}$$

- the scale on which the background metric varies is equivalent or bigger than $\mathcal{R}$[11],

$$g^{(B)}_{\mu\nu,\alpha} \lesssim \frac{O(g^{(B)}_{\mu\nu})}{\mathcal{R}}; \tag{1.4.3}$$

- the scale on which the gravitational wave varies goes as $\sim \bar{\lambda}$,

$$h_{\mu\nu,\alpha} \sim \frac{O(h_{\mu\nu})}{\bar{\lambda}}. \tag{1.4.4}$$

A general method to evaluate the Ricci tensor in this case is the following [38, 9]: first of all we use a local Lorentz frame of $g^{(B)}_{\mu\nu}$, in which the background connection coefficients vanish, then we evaluate the Ricci tensor at the second order in such coordinates, and at the end we transform back to the original frame using general covariance. This last step can be done by the simple substitutions $\eta_{\mu\nu} \to g^{(B)}_{\mu\nu}$ and $\partial_\alpha \to D_\alpha$, where D_α is the covariant derivative which acts in the following way

$$D_\alpha T^\mu_\nu = \partial_\alpha T^\mu_\nu + \Gamma^\mu_{\lambda\alpha} T^\lambda_\nu - \Gamma^\lambda_{\alpha\nu} T^\mu_\lambda. \tag{1.4.5}$$

The computation is the generalization at second order in $h_{\mu\nu}$ of the one seen in Section 1.2. The result is

$$R_{\mu\nu} = R^{(B)}_{\mu\nu} + R^{(1)}_{\mu\nu}(h) + R^{(2)}_{\mu\nu}(h) + O(h^3), \tag{1.4.6}$$

[9]This condition corresponds to $k \equiv \frac{1}{\bar{\lambda}} = a(t_{h.c.})H(t_{h.c.})$, physically it means that the mode of the perturbation considered is coming in/going out the comoving Hubble horizon aH. This will be very important when we will consider gravitational waves from inflation in the further sections.

[10]Notice that we have used the "less or equivalent symbol" because $h_{\mu\nu}$ can be affected by corrections of order $\mathcal{P}^2$ or higher, as we will see in this subsection.

[11]The variation is $\Delta g^{(B)}_{\mu\nu} \approx g^{(B)}_{\mu\nu,\alpha} \cdot \Delta x^\alpha$, therefore, for having a significant variation, $\Delta x^\alpha \gtrsim \mathcal{R}$.

with the two terms which perturb the background defined as[12]

$$\begin{aligned}
R^{(1)}_{\mu\nu}(h) &\equiv \frac{1}{2}\Big(-D_\nu D_\mu h^\alpha_\alpha - D^\alpha D_\alpha h_{\mu\nu} + D^\alpha D_\nu h_{\mu\alpha} + D^\alpha D_\mu h_{\alpha\nu}\Big) \sim \frac{\mathcal{P}}{\bar{\lambda}^2},\\
R^{(2)}_{\mu\nu}(h) &\equiv \frac{1}{2}\Big[\frac{1}{2}D_\mu h_{\alpha\beta}D_\nu h^{\alpha\beta} + h^{\alpha\beta}\Big(D_\mu D_\nu h_{\alpha\beta} + D_\alpha D_\beta h_{\mu\nu} - D_\nu D_\beta h_{\alpha\mu} - D_\mu D_\beta h_{\alpha\nu}\Big)\\
&+ D^\beta h^\alpha_\nu\Big(D_\beta h_{\alpha\mu} - D_\alpha h_{\beta\mu}\Big) + \Big(\frac{1}{2}D^\alpha h^\beta_\beta - D\beta h^{\alpha\beta}\Big)\big(D_\nu h_{\alpha\mu} + D_\mu h_{\alpha\nu} - D_\alpha h_{\mu\nu}\big)\Big]\\
&\sim \frac{\mathcal{P}^2}{\bar{\lambda}^2}. \qquad (1.4.7)
\end{aligned}$$

The central part of the shortwave formalism is the solution "order by order" of the Einstein field equations in vacuum:

$$G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \rightarrow R_{\mu\nu} = R^{(B)}_{\mu\nu} + R^{(1)}_{\mu\nu}(h) + R^{(2)}_{\mu\nu}(h) + O(h^3) = 0. \qquad (1.4.8)$$

We have already stressed that $g^{(B)}_{\mu\nu}$ varies only over a range much bigger than the wavelenghts of the gravitational waves we are considering, therefore it is reasonable to evaluate the effect of the gravitational waves by averaging over several wavelenghts λ, this is equivalent to $R^{(B)}_{\mu\nu} = \langle R^{(B)}_{\mu\nu}\rangle$. This means that $R^{(1)}_{\mu\nu}(h)$ gives no contribution to $\langle R^{(B)}_{\mu\nu}\rangle$, because the waves have an oscillatory nature, hence their average is zero. To sum up, we reach the conclusion that the terms of order $\mathcal{P}$ satisfy, independently from the others, the Einstein equations. Thus we are allowed to write

$$R^{(1)}_{\mu\nu}(h) = 0. \qquad (1.4.9)$$

This is the equation for the propagation of the wave $h_{\mu\nu}$ in a curved spacetime, where the coupling of the waves with the background curvature is encoded in the affine connections in the covariant derivatives. This coupling causes gradual changes in the properties of the waves, which can be well described by the formalism of geometric optics for gravitational waves propagating along null geodesics. We will see this in Chapter 2.
Now, we can split Eq. (1.4.8) in two contributions: the first one is the term free of ripples, in the sense that it does not vary on scales of the order of $\bar{\lambda}$,

$$R^{(B)}_{\mu\nu} + \langle R^{(2)}_{\mu\nu}(h)\rangle + O(h^3) = 0, \qquad (1.4.10)$$

and another contribution which considers only terms which vary on the scale $\bar{\lambda}$,

$$R^{(1)}_{\mu\nu}(j) + R^{(2)}_{\mu\nu}(h) - \langle R^{(2)}_{\mu\nu}(h)\rangle + O(h^3) = 0. \qquad (1.4.11)$$

Notice that we have included an $R^{(1)}_{\mu\nu}(j)$ term which comes from the fact that in the relation (1.4.2) we have considered the possibility that $h_{\mu\nu}$ depends on a term linear in $\mathcal{P}$, which

[12] Each derivative of $h_{\mu\nu}$ goes as $\frac{1}{\bar{\lambda}}$, hence $\partial^n_\alpha h_{\mu\nu} \sim \frac{\mathcal{P}}{\bar{\lambda}^n}$.

is the solution of Eq. (1.4.9), plus other terms of higher orders in $\mathcal{P}$. In other words, we are saying that[13]

$$h_{\mu\nu} = h^{lin}_{\mu\nu} + j_{\mu\nu}, \tag{1.4.12}$$

where $j_{\mu\nu}$ is a term of at least of the second order in $\mathcal{P}$ which varies on the scale $\bar{\lambda}$. We can say that it is a non-linear correction of the metric perturbation.
The first equation, the "coarse-grain" and smooth part, describes the background curvature $R^{(B)}_{\mu\nu}$ in terms of quadratic terms in $\mathcal{P}$, so by the knowledge that the stress-energy tensor creates the curvature we deduce that

$$T^{(GW)}_{\mu\nu} \equiv -\frac{1}{8\pi G}\Big(\langle R^{(2)}_{\mu\nu}(h)\rangle - \frac{1}{2}g^{(B)}_{\mu\nu}\langle R^{(2)}_{\mu\nu}(h)\rangle\Big). \tag{1.4.13}$$

After long computations from this formula it can be proved that the stress-energy tensor for the gravitational waves in the TT gauge can be written as [38]

$$T^{(GW)}_{\mu\nu} = \frac{1}{32\pi G}\langle D_\mu h_{\alpha\beta} D_\nu h^{\alpha\beta}\rangle. \tag{1.4.14}$$

The second term, the "fluctuational corrections", ripply on scale $\bar{\lambda}$, shows higher-order phenomena, like wave-wave scattering, in practice the gravitational waves h_{ij} generates corrections $j_{\mu\nu}$ on themselves. One of the most important result of this section is that, if we stop at the first order in $\mathcal{P}$ in perturbation theory, we can neglect any nonlinear interaction of the gravitational waves with themselves and with the background curvature they produce.
The shortwave approximation, to conclude, is no more valid when $\bar{\lambda}/\mathcal{R} \gtrsim 1$, in this case we have to consider the back-scatter of the gravitational waves with the curvature. This is not however a case of physical interest, because we know that

$$\begin{cases} T^{(GW)}_{\mu\nu} \approx \frac{\mathcal{P}^2}{\bar{\lambda}^2} \\ \mathcal{R}^{-2} \approx T^{(GW)}_{\mu\nu} + T^{other}_{\mu\nu} \end{cases} \to \mathcal{R}^{-2} \lesssim \frac{\mathcal{P}^2}{\bar{\lambda}^2}, \tag{1.4.15}$$

therefore such a condition has as immediate consequence that $\mathcal{P} \gtrsim 1$. In this chapter we have considered gravitational waves as small fluctuations of the background, imposing the condition $|h_{\mu\nu}| \ll 1$, thence $\mathcal{P} \gtrsim 1$ certainly violates that.

1.5 Geometric optics and gravitons

This subsection describes in detail the main tools that we will use in Chapter 2: first of all we want to stress why we can use the geometric optics for gravitational waves propagating in a curved background, introducing the main features of such an approximation, and then we want to justify the correspondence between the gravitational waves and spin-2 massless particles, the gravitons.
When we speak about geometric optics we are essentially assuming 3 fundamental laws:

[13] Eq. (1.4.9) is an equation at the first order in $\mathcal{P}$, hence there is no need to introduce $j_{\mu\nu}$, while Eq. (1.4.11) contains terms of the second order in $h_{\mu\nu}$, therefore we need to take into account also $j_{\mu\nu}$.

- the trajectories of the waves are null geodesics, i.e. if we define the wave vector as p_α, then we require the condition $p_\alpha p^\alpha = 0$;
- the polarization tensor is perpendicular to the wave vector and it propagates along the trajectories, i.e. the waves direction of propagation is always parallel to p^α;
- there is an adiabatic invariant associated to the waves amplitude, this corresponds, we will see, to the conservation of the photons number.

We begin with the following parametrization of a gravitational wave $h_{\mu\nu}$ in the Lorentz gauge, $D^\mu h_{\mu\nu} = 0$:

$$h_{\mu\nu}(t, \vec{x}) = h^{(0)}_{\mu\nu}(t, \vec{x}) e^{i\theta(t,\vec{x})}, \tag{1.5.1}$$

where we have encoded the dependence on $\bar{\lambda}$ in θ, while $h^0_{\mu\nu}$ is a factor which varies only on large scales. Notice that in this subsection we do not consider the higher order corrections introduced in Section 1.4. We define the amplitude of the gravitational waves as

$$h \equiv h^*_{\mu\nu} h^{\mu\nu} = h^{(0)*}_{\mu\nu} h^{(0)\mu\nu}, \tag{1.5.2}$$

because this term is insensitive to variations on scales of order $\bar{\lambda}$; the wave vector can be defined as

$$p_\alpha \equiv \partial_\alpha \theta. \tag{1.5.3}$$

Using the Lorentz gauge condition it is immediate to find out that the wave vector is perpendicular to the polarization tensor:

$$D^\mu h_{\mu\nu} = D^\mu h^{(0)}_{\mu\nu} e^{i\theta} + i p^\mu h^{(0)}_{\mu\nu} e^{i\theta} = 0, \tag{1.5.4}$$

which has to be true both at the first and the second order in $\bar{\lambda}$, which means that

$$D^\mu h^{(0)}_{\mu\nu} = 0, \tag{1.5.5}$$

and that

$$p^\mu h^{(0)}_{\mu\nu} = 0. \tag{1.5.6}$$

The explicit form of Eq. (1.4.9) in the Lorentz gauge is [9]

$$D_\alpha D^\alpha h_{\mu\nu} + 2R^{(B)}_{\alpha\mu\beta\nu} h^{\alpha\beta} = 0, \tag{1.5.7}$$

which corresponds, in our parametrization, to[14]

$$e^{i\theta}\Big[D_\alpha D^\alpha h^{(0)}_{\mu\nu} + R^{(B)}_{\alpha\mu\beta\nu} h^{(0)\alpha\beta}\Big] + ie^{i\theta}\Big[D_\alpha\Big(h^{(0)}_{\mu\nu}\partial^\alpha\theta\Big) + D_\alpha h^{(0)}_{\mu\nu}\partial^\alpha\theta\Big] - h^{(0)}_{\mu\nu} e^{i\theta}\partial_\alpha\theta\partial^\alpha\theta = 0. \tag{1.5.8}$$

This equation has to be satisfied at any order in $\bar{\lambda}$ too.
At zero order we have

$$D_\alpha D^\alpha h^{(0)}_{\mu\nu} + R^{(B)}_{\alpha\mu\beta\nu} h^{(0)\alpha\beta} = 0, \tag{1.5.9}$$

[14] The action of the covariant derivative on a scalar quantity like θ is equivalent to the action of a partial derivative.

which is the evolution of the amplitude of the waves on a curved spacetime. This will be shown explicitly in Chapter 3. By looking at the solution at the first order in $\bar{\lambda}$ we have

$$2D_\alpha h^{(0)}_{\mu\nu}p^\alpha + h^{(0)}_{\mu\nu}D_\alpha p^\alpha = 0 \to 2h^{(0)\mu\nu*}D_\alpha h^{(0)}_{\mu\nu}p^\alpha + h^{(0)\mu\nu*}h^{(0)}_{\mu\nu}D_\alpha p^\alpha = 0 \to D_\alpha(hp^\alpha) = 0, \tag{1.5.10}$$

which corresponds to the conservation of the quantity

$$N = \int d^3x hp^0, \tag{1.5.11}$$

which can be thought as the number of gravitational waves, with hp^0 the gravitational waves number density.
At the second order in $\bar{\lambda}$ we have

$$p_\alpha p^\alpha = 0, \tag{1.5.12}$$

which is precisely the condition of null geodesics.
We are now free to think to gravitational waves as a spin-2 particles, the gravitons, with momentum p^μ, moving along null geodesics, $D_\mu p^\mu = 0$ which parallel transports with itself a transverse traceless tensor $h^{(0)}_{\mu\nu}$, $p^\mu h^{(0)}_{\mu\nu} = 0$ and $h^{(0)\mu}_\mu = 0$ [42]. Geometric optics is nothing but the theory which describes the trajectories of such gravitons through spacetime. This is the starting point to discuss the graviton distribution function in Chapter 2.

1.6 Stochastic background of gravitational waves

1.6.1 Stochastic variables

In thesis we focus only on the cosmological background of gravitational waves. Cosmological means that the sources which generated the waves are early Universe mechanisms which we will discuss properly in section 1.6.2. Background means that the signal we measure is a random (stochastic) variable, which can be characterized only statistically, so which does not come from a single deterministic event.
As a first step we give a formal definition of a random field, like $h_{ij}(\eta, \vec{x})$ for the gravitational waves case. Let us consider a space of functions $\mathcal{F}$, characterized by some precise requirements, we can say that an n-dimensional random field, $\delta(\vec{x})$, is a set of random variables, one for each point in a 3-dimensional real space, $\mathcal{T}$ such that $\vec{x} \in \mathcal{T}$, defined by a probability functional, $\mathcal{P}[\hat{\delta}(\vec{x})]$, which specifies the probability for the occurrence of a particular realization of the field (i.e. of the function $\hat{\delta}(\vec{x}) \in \mathcal{F}$) over the ensemble. Roughly speaking, a random field is a infinite and continuous collection of random variables each of which is associated with a point of some space.The maximum degree of information we know about random fields are ensemble averages, which are defined as

$$\langle A \rangle \equiv \frac{\int A\, p\, d\tau}{\int p\, d\tau}, \tag{1.6.1}$$

with A any combination of random variables (like δ, δ^2 etc.), p the probability of the microsystem (like the single particles of a gas) and $d\tau$ the volume element in the phase space;

the denominator represents the normalization in this case.
In statistical mechanics, we can speak about ensemble average because we have different copies of the system available, and so we can repeat any measurement and any prediction is meaningful; on the contrary, in cosmology, we have only one observable Universe, thence we need some kind of justification to explain why we can talk about stochastic variables. Because of this we recall the ergodic hypotesis [54]: the configuration in the phase space of a system comes arbitrarily close to any point all the phase space; this is equivalent to an ensemble whose probability of the microsystem in Eq. (1.6.1) is constant over the accessible phase space. In other words, for long enough time the system realizes many times all the possible configuration, thus we can substitute the ensemble average with temporal and/or spatial averages.
It is not immediate that this requirements are fullfilled in our treatment. First of all we have to request that the Universe is almost homogeneous and isotropic; in this way, in each point in the space at the time of gravitational waves production, the initial conditions, from a statistical point of view, are the same. Conversely we would have had some regions were the production would be favored/underdog. Secondly, we need that any gravitational waves source respects causality, and operates at a time where the Hubble horizon was smaller than nowadays: in this way the signals we observe from different part of the Universe are uncorrelated and takes the form of a stochastic background; if we suppose that the source operated at the time t_p, at which the Hubble factor was H_p, then the correlation scale of the emitted waves ℓ_p must satisfy $\ell_p \leq H_p^{-1}$ and it has been shown [12] that the ratio between the redshift correlation scale nowadays, $l_p^0 \equiv l_p \frac{a_0}{a_p}$, and the Hubble factor today H_0, l_p^0/H_0, is of the order of 10^{-11}, therefore the correlation scale is tiny comparable with respect to the Hubble scale.
Possible sources for a stochastic gravitational wave background are inhomogeneities in the preheating after inflation [43], cosmic strings [44], phase transitions in the early universe [12] and primordial black holes [45]. Hower, in this thesis we will not consider this kind of sources and we will focus only on the gravitational waves which have origin as quantum fluctuations of the metric field during the inflation, for which, even if the horizon grows exponentially, we can still speak about a stochastic background.

1.6.2 Inflationary background of gravitational waves

Inflation is an hypothetical period of accelerated expansion of the Universe, it occurred at very early times, before the beginning of the era dominated by radiation. It was introduced to solve two problems of the hot Big-Bang model: the horizon problem and the flatness problem [46, 47].
To explain these problems we start with the definition of the particle horizon. In a Friedmann-Lemaître-Robertson-Walker (FLRW) Universe, described by the line element

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2, \tag{1.6.2}$$

we define the comoving particle horizon at the time t as

$$R^c(t) \equiv \int_0^t \frac{dt'}{a(t')}. \tag{1.6.3}$$

It represents the maximum distance light could have traveled to the observer from the beginning of the Universe, thus it corresponds also to the radius of the sphere that has been in causal contact with the observer. It is useful to introduce another kind of horizon, the comoving Hubble radius r_H^c, defined as

$$r_H^c(t) \equiv \frac{1}{a(t)H(t)}. \tag{1.6.4}$$

It represents the maximum comoving distance which can be traveled in one expansion time[15]. The Hubble radius represents a volume in which events are in causal contact as the particle horizon, but with a little difference: it represents the causal contact between two points at a specific time, so, if two particles are separated by a comoving distance larger than the Hubble horizon they cannot comunicate currently, but maybe in the past they could have, because the Hubble horizon can both decrease and increase. For the particle horizon it is not so: if a distance between two particles is larger than the particle horizon, these particles do not have ever communicated, because this kind of horizon keeps into account all the past history of the particles. However, it can be proved that $R^c \approx r_H^c$, therefore from now on we will consider only the Hubble horizon to describe regions causally connected.

We can study the evolution of the Hubble horizon by recalling that, if the dominant contribution to the energy density of the Universe is given by a particle species which has the pressure and the energy density related by the equation of state $p = w\rho$, the scale factor evolves as

$$a = a_i t^{\frac{2}{3(1+w)}} \to aH = \frac{2a_i}{3(1+w)} t^{-\frac{3w+1}{3(1+w)}}. \tag{1.6.5}$$

This means that the Hubble radius increased during the radiation dominated era ($w = 1/3$) and during the matter epoch ($w = 0$). This automatically generates the so-called horizon problem: we cannot explain indeed how regions which have been outside each other's particle horizon present very similiar features. This is the case of the cosmic microwave background radiation (CMB): we observe an almost isotropic spectrum, with only little small fluctuations [48], but the spectrum describes light which comes from region that are entered in causal contact only recently, this means that in the past they did not communicate and therefore they should not have had the opportunity to reach the same temperature.

The other problem which arises is the flatness problem. If we consider a non-null curvature in a Friedmann-Lemaître-Robertson-Walker (FLRW) Universe, the line element assumes the form

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-Kr^2} + r^2 d\Omega\Big), \tag{1.6.6}$$

[15] An expansion time means a sufficient time to allow to the scale factor a to vary appreaciably.

where (r, Ω) represent spherical coordinates. The first Friedmann equation in this case is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{K}{a^2}, \tag{1.6.7}$$

which leads to an equation for the evolution of the total energy density of the Universe $\Omega(t)$:

$$\Omega(t) - 1 = \frac{K}{a^2(t)H^2(t)}. \tag{1.6.8}$$

If $\Omega(t) = 1$, then the Universe is flat ($K = 0$) and it will be flat forever. On the contrary, if K slightly differs from zero, then the flatness problem emerges. Experimental observations set in fact the limit to the present value of $\Omega(t)$ to $|\Omega_0 - 1| < 0.005$ with 95% of confidence level, but this means that in the past this value had to be much smaller, because the denominator in the right-hand side of Eq. (1.6.8) has been always a growing function. To be precise we require that the initial conditions on the total energy density are [49]

$$|\Omega(10^{-43}s) - 1| \leq O(10^{-60}). \tag{1.6.9}$$

In principle, we could solve these two problems by imposing at early times an almost total isotropy of the CMB and a value very near to one for Ω, but this fine-tuning of the initial conditions seems unnatural.
An accelerated expansion would solve these two problems, because for having an accelerated expansion we require that the parameter which regulates the equation of state between the energy density and the pressure, w, is smaller than $-1/3$, and this leads to an Hubble horizon which decreases in time. Thence the horizon problem is solved if we suppose that the inflation lasted enough to allow to reduce the Hubble radius so much that regions that now are not in causal contact were separated by a distance smaller than the Hubble horizon at the end of the inflation. The flatness problem is solved in the same way if we look at Eq. (1.6.8): if the Hubble horizon decreased a lot during the inflation, automatically for each initial condition before the inflation on $\Omega(t)$, the behaviour of $a(t)H(t)$ during an accelerated phase makes the value of Ω at the end of the inflation so small that it is compatible with the constraints on Ω_0. The evolution of the Hubble horizon is depicted in Figure 1.6.1.

In this section, we consider a very simple inflationary model: we consider a Universe whose energy density is dominated by the one provided by a scalar field, the inflaton ϕ. We will introduce a mechanism in which the inflaton slow rolls to the minimum of its potential[16]. In this way we will see that the contribution to the stress-energy tensor of the inflaton reproduces the equation of state $p = -\rho$, which is the requirement for having an accelerated expansion [51].
The starting point is the action of a scalar field on an unperturbed Friedmann-Lemaître-Robertson-Walker Universe:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big(R + \mathcal{L}_\phi[\phi, g_{\mu\nu}] + \mathcal{L}_{matt}\Big), \tag{1.6.10}$$

[16] We will specify later on the meaning of slow roll, but intuitively it means that the field goes to the minimum of its potential not too fast.

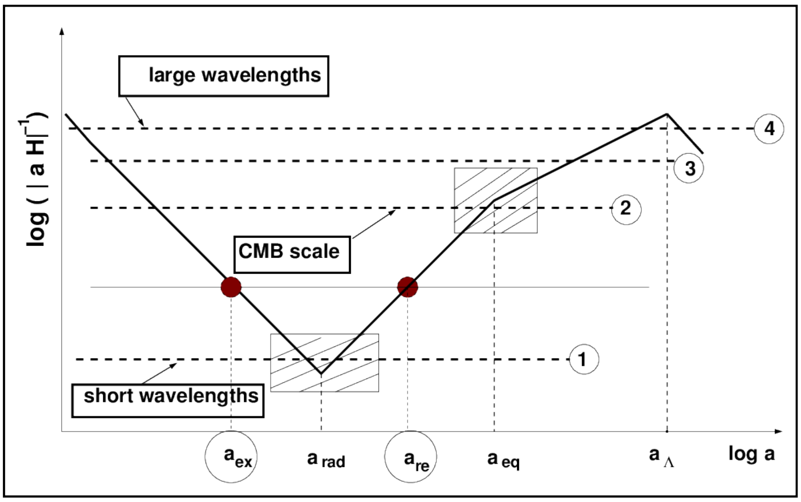

Figura 1.6.1: Hubble radius evolution during the inflation, the radiation-dominated era, the matter-dominated era and the dark energy-dominated era [50]. Horizontal lines represent comoving scales between different regions in the Universe. When a scale (a dashed line) stays above the Hubble horizon (the continuous line), we speak about super-horizon scales, and two regions separated by such a comoving distance cannot be in causal contact at that time. Inflation grants that regions that are entering the horizon at a specific time (for instance at a_{EQ}), have been in causal contact at early times (you can see in the picture that for the line 2 were inside the horizon in the left-hand side of the picture): this is due to the accelerated expansion of the Universe that extremely reduced the Hubble horizon.

where the first term is nothing but the Einstein-Hilbert action, the second term is the lagrangian for the inflaton, while the last one is the lagrangian for the matter content of the Universe, both including radiation and non-relativistic matter, which is subdominant during the inflationary period and we neglect it.
The simplest model we can write down is described by the following Lagrangian density for the inflaton:

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - V(\phi), \tag{1.6.11}$$

where $g^{\mu\nu}$ is the inverse of the metric [17],

$$g_{\mu\nu} = diag(-1, a^2, a^2, a^2) \rightarrow g^{\mu\nu} = diag(-1, a^{-2}, a^{-2}, a^{-2}). \tag{1.6.12}$$

By combining Quantum Field Theory and General Relativity, we know that the equations of motion for the fields are found by imposing that the variations of the action with respects to the fields are null, while the Einstein equations can be found by imposing that the variation of the action with respect to the metric is null too[18].

[17] We have used as coordinates $(t, \vec{x})$, where t is the cosmic time.
[18] The only things we need to know about functional derivatives are the equations

$$\frac{\delta\phi(y)}{\delta\phi(x)} = \delta^{(4)}(x-y) \qquad and \qquad \frac{\delta g^{\mu\nu}(y)}{\delta g^{\alpha\beta}(x)} = \delta^\mu_\alpha \delta^\nu_\beta \delta^{(4)}(x-y).$$

The equation of motion for the field ϕ is then

$$
\begin{aligned}
0 =&\frac{\delta S_\phi[\phi, g_{\mu\nu}]}{\delta\phi(x)} = g^{\mu\nu}(x)D_\nu\Big(\partial_\mu\phi(x)\Big) - \frac{\delta V(\phi)}{\delta\phi(x)} = \\
=&g^{\mu\nu}(x)\Big(\partial_\mu\partial_\nu\phi(x) - \Gamma^\lambda_{\nu\mu}\partial_\lambda\phi(x)\Big) - \frac{\partial V\big(\phi(x)\big)}{\partial\phi} = \\
=&g^{\mu\nu}(x)\Big[\partial_\mu\partial_\nu\phi(x) - \frac{1}{2}g^{\lambda\alpha}\Big(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\Big)\partial_\lambda\phi(x)\Big] - \frac{\partial V\big(\phi(x)\big)}{\partial\phi} = \\
=&g^{00}\Big[\partial_0\partial_0\phi(x) - \frac{1}{2}g^{\lambda\alpha}\Big(\partial_0 g_{\alpha 0} + \partial_\nu g_{\alpha 0} - \partial_\alpha g_{00}\Big)\partial_\lambda\phi(x)\Big] + \\
&+ g^{ij}\Big[\partial_i\partial_j\phi(x) - \frac{1}{2}g^{\lambda\alpha}\Big(\partial_i g_{\alpha j} + \partial_j g_{\alpha i} - \partial_\alpha g_{ij}\Big)\partial_\lambda\phi(x)\Big] - \frac{\partial V\big(\phi(x)\big)}{\partial\phi} = \\
=&-\ddot{\phi}(x) - \frac{1}{a^2}\nabla^2\phi(x) + \frac{3}{2a^2}\Big(-2a\dot{a}\dot{\phi}(x)\Big) - \frac{\partial V\big(\phi(x)\big)}{\partial\phi} \\
&\rightarrow \ddot{\phi}(x) + 3\frac{\dot{a}}{a}\dot{\phi}(x) - \frac{1}{a^2}\nabla^2\phi(x) + \frac{\partial V\big(\phi(x)\big)}{\partial\phi} = 0.
\end{aligned}
\tag{1.6.13}
$$

It is straightforward that, if $\delta S[\phi, g_{\mu\nu}]/\delta g^{\mu\nu} = 0$ corresponds to the Einstein equations, with the geometrical part defined by the variation of the Einstein-Hilbert action, the energy tensor for the field ϕ has to be defined as

$$
\begin{aligned}
T^\phi_{\mu\nu}(x) \equiv& -\frac{2}{\sqrt{-g(x)}}\frac{\delta S_\phi[\phi, g_{\mu\nu}]}{\delta g^{\mu\nu}(x)} = \\
=& -\frac{2}{\sqrt{-g(x)}}\int d^4y\Big(\frac{\delta(\sqrt{-g(y)})}{\delta g^{\mu\nu}(x)}\mathcal{L}_\phi(y) + \sqrt{-g(y)}\frac{\delta\mathcal{L}_\phi(y)}{\delta g^{\mu\nu}(x)}\Big) = \\
=& -\frac{2}{\sqrt{-g(x)}}\Big(-\frac{1}{2}\sqrt{-g(x)}g_{\mu\nu}(x)\mathcal{L}_\phi(x) - \frac{1}{2}\sqrt{-g(x)}\partial_\mu\phi(x)\partial_\nu\phi(x)\Big) = \\
=&\partial_\mu\phi(x)\partial_\nu\phi(x) + g_{\mu\nu}(x)\mathcal{L}_\phi(x) = \\
=&\partial_\mu\phi(x)\partial_\nu\phi(x) - g_{\mu\nu}(x)\Big[\frac{1}{2}g^{\alpha\beta}(x)\partial_\alpha\phi(x)\partial_\beta\phi(x) + V\big(\phi(x)\big)\Big].
\end{aligned}
\tag{1.6.14}
$$

We have seen that to solve the horizon and the flatness problems we require that $p = -\rho$, in this way we have a quasi-exponential accelerated expansion, i.e. of the form $a(t) = e^{Ht}$, with H almost constant (in the case of dark energy it is exactly constant, in this case we will see that we have some small corrections). To do that, it is necessary that the parameter w, which regulates the equation of state of the inflaton, $p = w\rho$, where ρ and p are the energy density and the pressure of the inflaton field, is $w = -1$. To understand which conditions on the motion of the inflaton we have to require, we recall the definition of the energy density and the pressure from the energy-momentum tensor:

$$T^0_0 = -\rho \qquad \textit{and} \qquad T^i_j = p\delta^i_j, \tag{1.6.15}$$

which corresponds to, through explicit computations [19],

$$\begin{aligned}\rho =& - T^0_0 = -\partial^0\phi\partial_0\phi + \left[\frac{1}{2}\Big(-\dot{\phi}\dot{\phi} + \frac{1}{a^2}\nabla^2\phi\Big) + V(\phi)\right] = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}\nabla^2\phi + V(\phi),\\ p =& \frac{1}{3}\delta^i_j T^j_i = \frac{1}{3}\partial^i\phi\partial_i\phi - \left[\frac{1}{2}\Big(-\dot{\phi}\dot{\phi} + \frac{1}{a^2}\nabla^2\phi\Big) + V(\phi)\right] = \frac{1}{2}\dot{\phi}^2 - \frac{1}{3a^2}\nabla^2\phi - V(\phi).\end{aligned} \tag{1.6.16}$$

At this point we consider the quantum nature of the scalar field ϕ: we know that each quantum field can be decomposed into a leading term, called the classical value of the field, which corresponds to the vacuum expectation value (VEV) of the field,

$$\phi_0(x) \equiv \langle 0|\, \phi(x)\, |0\rangle\,, \tag{1.6.17}$$

plus a quantum fluctuation $\delta\phi(x)$, which has, by definition, a null expectation value,

$$\delta\phi(x) \equiv \phi(x) - \phi_0(x) \to \langle 0|\, \delta\phi(x)\, |0\rangle = \phi_0(x) - \langle 0|\, \phi_0(x)\, |0\rangle = 0. \tag{1.6.18}$$

For the cosmological principle, the Universe, except for small perturbations, is spatially homogeneous and isotropic, so we can assume that the classical value of the field depends only on the temporal coordinate, $\phi_0(t, \vec{x}) = \phi_0(t)$, in this way its stress-energy tensor (and so the energy density) will be spatially homogeneous. Under this assumption the energy density and the pressure become

$$\begin{aligned}\rho =& \frac{1}{2}\dot{\phi_0}^2 + V(\phi_0),\\ p =& \frac{1}{2}\dot{\phi_0}^2 - V(\phi_0).\end{aligned} \tag{1.6.19}$$

Therefore, for having $p = -\rho$, we require that

$$V(\phi_0) \gg \frac{1}{2}\dot{\phi_0}^2, \tag{1.6.20}$$

and this is called slow-roll condition, because the kinetic energy of the field is much smaller than its potential $V(\phi_0)$. To require that the inflaton leds to an accelerated expansion of the Universe, we need that this phase lasted enough to solve the horizon and the flatness problem; in practical terms we require that the acceleration of the inflaton field is subdominant with respect to its velocity, in this way the expansion phase can last for a sufficient time. This condition can be seen as

$$|\ddot{\phi_0}| \ll |3H\dot{\phi_0}|. \tag{1.6.21}$$

Our aim is to express these two conditions in terms of the form of the potential for the inflaton $V(\phi_0)$, in order to define some parameters which quantify the slow roll and to

[19] Notice that $\dot{\phi} = \partial_0\phi \neq \partial^0\phi$, in fact the contravariant derivative is defined as

$$\partial^0\phi = g^{0\alpha}\partial_\alpha\phi = -\partial_0\phi.$$

understand better which kind of physics could have produced inflation. We see that this additional condition on ϕ_0, combined with the equation of motion for the scalar field ϕ_0, leads to

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \frac{\partial V(\phi_0)}{\partial \phi_0} = 0 \rightarrow 3H\dot{\phi}_0 = -\frac{\partial V(\phi_0)}{\partial \phi_0}. \tag{1.6.22}$$

We can combine this expression squared with the first Friedmann equation, Eq. (1.6.7) (with $K = 0$),

$$\begin{cases} 9H^2\dot{\phi}_0^{\,2} = -\left(\frac{\partial V(\phi_0)}{\partial \phi_0}\right)^2 \\ H^2 = \frac{8}{3}\pi G V(\phi_0) \end{cases} \rightarrow \frac{1}{2}\dot{\phi}_0^{\,2} = \frac{1}{3}\frac{1}{16\pi G V(\phi_0)}\left(\frac{\partial V(\phi_0)}{\partial \phi_0}\right)^2, \tag{1.6.23}$$

which leads, when included in the first slow roll condition, into a condition on the scalar potential $V(\phi_0)$:

$$\frac{1}{2}\dot{\phi}_0^{\,2} \ll V(\phi_0) \rightarrow \frac{1}{3}\frac{1}{16\pi G}\left(\frac{\frac{\partial V(\phi_0)}{\partial \phi_0}}{V(\phi_0)}\right)^2 \ll 1. \tag{1.6.24}$$

We can now introduce a parameter which keeps into account the slow roll of the inflaton field, it is a function which depends on the form of the potential and it is

$$\epsilon_V \equiv \frac{1}{16\pi G}\left(\frac{\frac{\partial V(\phi_0)}{\partial \phi_0}}{V(\phi_0)}\right)^2, \tag{1.6.25}$$

with the requirement that $\epsilon_V \ll 1$; this quantity is particularly useful because it determines the variations of the Hubble factor:

$$\begin{aligned} \dot{H}^2 = 2H\dot{H} =& \frac{8}{3}\pi G\left(\frac{\partial V}{\partial \phi_0}\dot{\phi}_0 + \dot{\phi}_0\ddot{\phi}_0\right) = \frac{8}{3}\pi G\left[\frac{\partial V}{\partial \phi_0}\dot{\phi}_0 + \dot{\phi}_0\left(-3H\dot{\phi}_0 - \frac{\partial V}{\partial \phi_0}\right)\right], \\ \dot{H} =& -\frac{8}{3}\pi G 3\dot{\phi}_0^{\,2} = -\frac{8\pi G}{9H^2}\left(\frac{\partial V}{\partial \phi_0}\right)^2 = -\frac{8\pi G H^2}{9H^4}\left(\frac{\partial V}{\partial \phi_0}\right)^2, \\ \dot{H} =& \frac{8\pi G H^2}{9\left(\frac{8\pi G}{3}V\right)^2}\left(\frac{\partial V}{\partial \phi_0}\right)^2, \\ \frac{\dot{H}}{H^2} =& -\epsilon_V. \end{aligned} \tag{1.6.26}$$

We want to express also the second slow roll condition, Eq. (1.6.22), in function of the potential $V(\phi_0)$ and its derivatives. What we want to do essentially is to write $\ddot{\phi}_0$ in terms of the potential, to do that we derive with respect to t Eq. (1.6.23):

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}\dot{\phi}_0^{\,2}\right] =& \frac{d}{dt}\left\{\frac{1}{3}\left[\frac{1}{16\pi G V(\phi_0)}\left(\frac{\partial V(\phi_0)}{\partial \phi_0}\right)^2\right]\right\}, \\ \dot{\phi}_0\ddot{\phi}_0 =& \frac{1}{3}\frac{1}{16\pi G V(\phi_0)}\left[2\frac{\partial V(\phi_0)}{\partial \phi_0}\frac{\partial^2 V(\phi_0)}{\partial \phi_0^2}\dot{\phi}_0 - \frac{\left(\frac{\partial V(\phi_0)}{\partial \phi_0}\right)^2}{V(\phi_0)}\dot{\phi}_0\right], \\ \ddot{\phi}_0 =& \frac{1}{3}\frac{1}{16\pi G V(\phi_0)}\frac{\partial V(\phi_0)}{\partial \phi_0}\left[2\frac{\partial^2 V(\phi_0)}{\partial \phi_0^2} - \frac{\frac{\partial V(\phi_0)}{\partial \phi_0}}{V(\phi_0)}\right] \approx \frac{1}{3}\frac{1}{8\pi G}\frac{\partial V(\phi_0)}{\partial \phi_0}\frac{\frac{\partial^2 V(\phi_0)}{\partial \phi_0^2}}{V(\phi_0)}, \end{aligned} \tag{1.6.27}$$

where in the last step we have neglected the second term in the square brackets, because it represents a term of the second order in the slow-roll parameter ϵ_V defined above.
If we now use the second condition for the slow roll we have

$$|\ddot{\phi}_0| \ll \left|3H\dot{\phi}_0\right| \to \left|\frac{1}{3}\frac{1}{8\pi G}\frac{\partial V(\phi_0)}{\partial\phi_0}\frac{\frac{\partial^2 V(\phi_0)}{\partial\phi_0^2}}{V(\phi_0)}\right| \ll \left|-\frac{\partial V(\phi_0)}{\partial\phi_0}\right| \to \frac{1}{3}\frac{\frac{\partial^2 V(\phi_0)}{\partial\phi_0^2}}{8\pi G V(\phi_0)} \ll 1, \tag{1.6.28}$$

because of this we can define a second slow roll parameter, η_V, which has still to be much smaller than one,

$$\eta_V = \frac{1}{8\pi G}\frac{\frac{\partial^2 V(\phi_0)}{\partial\phi_0^2}}{V(\phi_0)}. \tag{1.6.29}$$

We have built a mechanism which justifies a large accelerated expansion of the Universe in a very early epoch, solving the horizon and the flatness problems. Now we can go beyond: we have seen that the inflaton ϕ is a quantum field, thence it is described by its vacuum expectation value plus small fluctuations $\delta\phi(t,\vec{x})$, which are null if averaged over a sufficient large time. In the same way, we can consider the metric $g_{\mu\nu}$ as a quantum field, recogizing that its vacuum expectation value is equivalent to the background metric, defined in Eq. (1.6.12), plus quantum fluctuations $h_{\mu\nu}$.
We will focus in this section on the gravitational waves production, therefore we restrict to the tensor degrees of freedom, i.e. we consider the perturbed metric

$$g_{\mu\nu} = a^2(\eta) diag(-1, \delta_{ij} + h_{ij}), \tag{1.6.30}$$

where we have used the conformal time η. If we substitute such a metric in the action (1.6.10) we obtain the following equation for h_{ij}:

$$h''_{ij}(\eta,\vec{x}) + 2\frac{a'}{a}h'_{ij}(\eta,\vec{x}) - \frac{1}{a^2}\nabla^2 h_{ij}(\eta,\vec{x}) = 0. \tag{1.6.31}$$

In order to provide canonical quantization, we write the metric perturbation using a plane-wave expansion

$$h_{ij}(\eta,\vec{x}) = \sum_{\lambda=\pm 2}\int\frac{d^3k}{(2\pi)^3} e_{ij,\lambda}(\hat{k}) h_\lambda(\eta,\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \tag{1.6.32}$$

with $e_{ij,\lambda}(\hat{k})$ the polarization tensor along the $\hat{k}$ direction which satisfies the condition $e_{ij,\lambda}(-\hat{k}) = e_{ij,\lambda}(\hat{k})$, and $h_\lambda(\eta,-\vec{k}) = h^*_\lambda(\eta,\vec{k})$. If we parametrize the amplitude $h_\lambda(\eta,\vec{k})$ as

$$h_\lambda(\eta,\vec{k}) = \frac{4\sqrt{\pi G}}{a} v_\lambda(\eta,\vec{k}), \tag{1.6.33}$$

with

$$\begin{aligned} h'_\lambda(\eta,\vec{k}) =& \frac{4\sqrt{\pi G}}{a}\left(v'_\lambda(\eta,\vec{k}) - \frac{a'}{a}v_\lambda(\eta,\vec{k})\right), \\ h''_\lambda(\eta,\vec{k}) =& \frac{4\sqrt{\pi G}}{a}\left(v''_\lambda(\eta,\vec{k}) - 2\frac{a'}{a}v'_\lambda(\eta,\vec{k}) + 2\frac{a'^2}{a^2}v_\lambda(\eta,\vec{k}) - \frac{a''}{a}v_\lambda(\eta,\vec{k})\right), \end{aligned} \tag{1.6.34}$$

we see [12] that the Einstein-Hilbert action is written as the action of two real scalar fields $v_\lambda(\eta,\vec{x})$ in Minkowski, with potential which depends on the scale factor a and on the scale k considered. This means that we can quantize the fields $v_\lambda(\eta,\vec{x})$ using the canonical quantization known from quantum field theory, i.e.[20]

$$\hat{v}_\lambda(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\Big(v_k(\eta)e^{-i\vec{k}\cdot\vec{x}}a_{\vec{k},\lambda} + v_k^*(\eta)e^{+i\vec{k}\cdot\vec{x}}a^\dagger_{\vec{k},\lambda}\Big), \tag{1.6.35}$$

which satisfies the canonical commutation relation[21]

$$[v_\lambda(\eta,\vec{x}),\partial_0 v^*_{\lambda'}(\eta,\vec{x}')] = i\delta_{\lambda\lambda'}\delta^{(3)}(\vec{x}-\vec{x}') \rightarrow v_k v_k'^* - v_k^* v_k' = i, \tag{1.6.36}$$

with $a_{\vec{k},\lambda}$ $\big(a^\dagger_{\vec{k},\lambda}\big)$ the annihilation (creation) operator which annihilates (creates) a particle with momentum $\vec{k}$ and polarization λ, remember that for them the only non null commutation relations is

$$[a_{\vec{k},\lambda}, a^\dagger_{\vec{k}',\lambda'}] = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}'). \tag{1.6.37}$$

The equation for the fluctuations (1.6.31), using the expansion (1.6.32) becomes

$$h''_\lambda(\eta,\vec{k}) + 2\frac{a'}{a}h'_\lambda(\eta,\vec{k}) + \frac{k^2}{a^2}h_\lambda(\eta,\vec{k}) = 0, \tag{1.6.38}$$

which, in terms of the field v, corresponds to

$$v''_k(\eta) + \Big(k^2 - \frac{a''}{a}\Big)v_k(\eta) = 0. \tag{1.6.39}$$

We want to find the solution in function of η, hence we write the explicit form of the scale factor in terms of η:

$$\begin{aligned}
\frac{\dot{H}}{H^2} =& -\epsilon_V \rightarrow H = H_i(1-\epsilon_V t) \rightarrow a = a_i e^{H_i t}\Big(1 - \frac{\epsilon_V}{2}H_i^2 t^2\Big),\\
\eta =& \int_0^t dx \frac{1}{a(x)} = \int_0^t dx \frac{1}{a_i}e^{-H_i x}\Big(1 + \frac{\epsilon_V}{2}H_i^2 x^2\Big) =\\
=& -\frac{1}{H_i}\frac{1}{a_i e^{H_i t}\Big(1 - \frac{\epsilon_V}{2}H_i^2 t^2\Big) - \epsilon_V H_i e^{H_i t}(1+H_i t)} =\\
=& -\frac{1}{a(t)H_i}\frac{1}{1-\epsilon_V\Big(1+\frac{\epsilon_V}{2}H_i^2 t^2\Big)(1+H_i t)} =\\
=& -\frac{1}{a(t)}\frac{1}{H_i}\frac{1}{1-\epsilon_V H_i t - \epsilon + O(\epsilon^2)} = -\frac{1}{a(t)H(t)}\frac{1}{1-\frac{H_i}{H(t)}\epsilon_V} = -\frac{1}{a(t)H(t)(1-\epsilon_V)}.
\end{aligned} \tag{1.6.40}$$

[20] We use the "*hat*" to identify the quantized field, which is now an operator which acts on an Hilbert space.

[21] Each other independent commutator involving $v_\lambda(\eta,\vec{x})$ and its temporal derivative is null.

Now we can find the form for a''/a using this relation:

$$\dot{H} = -\epsilon_V H^2 \rightarrow \frac{1}{a}\Big(\frac{a''}{a^2} - 2H^2 a\Big) = -H^2\epsilon_V \rightarrow \frac{a''}{a} = -a^2H^2(2-\epsilon_V) = \frac{2-\epsilon_V}{\eta^2(1-2\epsilon_V)} = \\ = \frac{2}{\eta^2}\Big(1+\frac{3}{2}\epsilon_V\Big), \tag{1.6.41}$$

in this way the equation for the gravitational waves becomes

$$v_k''(\eta) + \Big[k^2 - \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big)\Big] v_k(\eta) = 0, \tag{1.6.42}$$

where $\nu^2 = \frac{9}{4} + 3\epsilon_V$; this equation is a Bessel equation which has as solutions the Hankel functions of the first and of the second kind:

$$v_\lambda(\eta, \vec{k}) = \sqrt{-\eta}\left[c_1(k)H_\nu^{(1)}(-k\eta) + c_2(k)H_\nu^{(2)}(-k\eta)\right]. \tag{1.6.43}$$

If we consider the regime $k\eta \gg 1$, i.e. when the momenta of the gravitons give the dominant contribution to the fluctuations with respect to the background curvature, in other words when we can consider the situation analogue to a flat spacetime, the solution of Eq. (1.6.42) is

$$v_k(\eta) = v_k^{(-)}e^{ik\eta} + v_k^{(+)}e^{-ik\eta}, \tag{1.6.44}$$

where $v_k^{(\pm)}$ are the modes associated to the positive ($v_k^{(+)}$) and to the negative ($v_k^{(-)}$) frequency modes[22]. We do not consider the negative frequency modes, therefore we impose as initial condition $c_k^{(-)} = 0$ as initial condition. Using Eq. (1.6.37) we find that

$$2ik|v_k^{(+)}|^2 = i \rightarrow v_k^{(+)} = \frac{1}{\sqrt{2k}} \rightarrow v_k(\eta) = \frac{1}{\sqrt{2k}}e^{ik\eta}. \tag{1.6.45}$$

Now we impose as constraint that the solution (1.6.43) matches the one found for $k\eta \gg 1$, therefore, by knowing that

$$H_\nu^{(1)}(-k\eta) \approx \begin{cases} \sqrt{\frac{2}{\pi}}e^{-i\pi/2}2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\eta)^{-\nu} & -k\eta \ll 1 \\ \sqrt{\frac{2}{-\pi k\eta}}e^{i(-k\eta-\nu\pi/2-\pi/4)} & -k\eta \gg 1 \end{cases} \\ H_\nu^{(2)}(-k\eta) \approx \sqrt{\frac{2}{-\pi k\eta}}e^{-i(-k\eta-\nu-\pi/4)} \qquad -k\eta \gg 1 \tag{1.6.46}$$

we can set $c_2(k) = 0$ and then our full solution is

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}e^{i(\nu+1/2)\pi/2}\sqrt{-\eta}H_\nu^{(1)}(-k\eta), \tag{1.6.47}$$

[22] They are eigenfunctions of the Hamiltonian operator $\hat{H} = i\partial_=$ with positive and negative eigenvalues respectively.

which has, as interesting limit,

$$
\begin{aligned}
v_k(\eta) =&\frac{\sqrt{\pi}}{2}e^{i(\nu+1/2)\pi/2}\sqrt{\frac{2}{\pi}}e^{-i\pi/2}2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\eta)^{-\nu} = \\
=&e^{i(\nu-1/2)\pi/2}2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\eta)^{1/2-\nu}, \qquad -k\eta \ll 1.
\end{aligned}
\tag{1.6.48}
$$

Before proceeding, we need to give a physical interpretation of what we are doing: Eq. (1.6.39) is nothing but the harmonic oscillator equation with a time dependent frequency, $\omega_k(\eta) \equiv k^2 - a''/a$, thus we are describing a quantum (because of the operators $a_\lambda(\vec{k})$ and $a^\dagger_\lambda(\vec{k})$) harmonic oscillator, which has as total energy for a given mode k the one given by the Hamiltonian formalism, i.e.

$$
E_k = \frac{1}{2}(|v'_k(\eta)|^2 + \omega_k^2(\eta)|v_k(\eta)|^2); \tag{1.6.49}
$$

if we then consider the limit for which $\omega_k^2(\eta)$ is positive and constant, we can think to these oscillations as quantum modes, the gravitons, of frequency (thus energy) ω_k, therefore the number of gravitons for a given k mode is nothing but the total energy divided by the energy carried by a single graviton,

$$
N_k + \frac{1}{2} \equiv \frac{E_k}{\omega_k} = \frac{1}{2\omega_k}(|v'_k(\eta)|^2 + \omega_k^2(\eta)|v_k(\eta)|^2). \tag{1.6.50}
$$

During the inflation the graviton number density is not well defined, it is not conserved and it grows exponentially, thus we need to make a model to predict the graviton abundance at the end of the accelerated expansion. We idealize the end of the inflation as an immediate transition to the radiation dominated era, where $a = \eta$, which occurs at the time η_e, and we evaluate $v_k(\eta_e)$; after that we suppose that all the modes of physical interest enetered the horizon long after the end of the inflation, therefore we evaluate $v_k(\eta_e)$ in the super-horizon limit; after that we insert it in the expression for N_k, which is properly defined now and the number of gravitons can be seen as a graviton density number in function of k, this will be the initial condition for the gravitons distribution function.
Therefore, if we evaluate Eq. (1.6.48) at the first order (in the slow roll parameter ϵ_V), i.e. for $\nu = 3/2$ we have that

$$
\begin{aligned}
|v_k(\eta)| =&\frac{1}{\sqrt{2k}}(-k\eta)^{-1} \approx \frac{1}{k\sqrt{2k}}H_k a(\eta), \\
|v'_k(\eta)| =&\frac{1}{\sqrt{2k}}k(-k\eta)^{-2} \approx \frac{1}{k\sqrt{2k}}H_k^2 a^2(\eta),
\end{aligned}
\tag{1.6.51}
$$

where we have used the fact that $a(\eta)H(\eta)\eta = 1$ at the first order and that $H(\eta)$ is approximately constant during inflation, thus we can take its value at the horizon crossing H_k, defined as $H_k a(\eta_{h.c.}) = k$. This expression tells us that super-Hubble modes $h_k(\eta)$ are constant, because they depend indeed on the ratio $v_k(\eta)/a(\eta)$.

If we evaluate now the number density at η_e we immediately obtain that, using the fact that in the radiation era $\omega_k = k$

$$N_k(\eta_e) + \frac{1}{2} = \frac{1}{4k^4} H_k^2 a_e^2 (H_k^2 a_e^2 + k^2) = \frac{1}{4}\left(\frac{H_k}{H_e}\right)^4 \left(\frac{H_e a_e}{k}\right)^4. \quad (1.6.52)$$

Because of the out-of-horizon condition, $k \ll a_e H_e$, and by the constance of the Hubble parameter during the inflation, $H_k \approx H_e$, it is clear that inflation produced an huge number of gravitons from the vacuum; in non-inflationary models, where an accelerated expansion is missing, this effect is exceedingly small and it produces no observables. A qualitative physical explanation for the huge production of gravitons can be the following: usually in quantum field theory there are continuously productions of pairs of particle and antiparticle, which annihilate immediately after their creation, but, when we consider inflation, the expansion of the universe is so big that they cannot annihilate and a net number of particles is created.
The original quantum behaviour of the variables is lost when the modes exit the horizon and there is a quantum-to-classic transition that allows us to treat the quantities described until now as statistical variables, in the sense that the VEV can be associated to ensemble averages [52].
By using Eqs. (1.6.32), (1.6.33) and (1.6.35) we can quantize the original field we started from, i.e. the metric perturbation:

$$\hat{h}_{ij}(\eta, \vec{x}) = \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \left[h_k(\eta) e^{i\vec{k}\cdot\vec{x}} a_{\vec{k},\lambda} + h_k^*(\eta) e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k},\lambda}^\dagger \right] e_{ij,\lambda}(\hat{k}), \quad (1.6.53)$$

for which we can define the power spectrum as

$$\langle 0 | \hat{h}_{ij,\lambda}(\eta, \vec{k}) \hat{h}_{ij,\lambda'}^*(\eta, \vec{k}') | 0 \rangle \equiv \frac{2\pi^2}{k^3} \delta_{\lambda\lambda'} \delta^{(3)}(\vec{k} - \vec{k}') P^{(\lambda)}(\eta, k), \quad (1.6.54)$$

where we have used as definition[23]

$$\hat{h}_{ij,\lambda}(\eta, k) = \int \frac{d^3x}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \hat{h}_{ij,\lambda}(\eta, \vec{x}) = h_k(\eta) a_{-\vec{k},\lambda} e_{ij,\lambda}(-\hat{k}) + h_k^*(\eta) a_{\vec{k},\lambda}^\dagger e_{ij,\lambda}(\hat{k}). \quad (1.6.55)$$

[23]In practical terms, $\hat{h}_{ij,\lambda}(\eta, \vec{k})$ is the inverse Fourier transform of the quantized metric perturbation.

The computation of the tensor power spectrum is trivial, once we know Eqs. (1.6.40) and (1.6.48)[24]

$$
\begin{aligned}
\langle 0|\,\hat{h}_{ij,\lambda}(\eta,\vec{k})\hat{h}^*_{ij,\lambda'}(\eta,\vec{k}')\,|0\rangle =& e_{ij,\lambda}(\hat{k})e^*_{ij,\lambda'}(\hat{k}')\Big[h_k(\eta)h^*_{k'}\,\langle 0|\,a_{-\vec{k},\lambda}a^{\dagger}_{-\vec{k}',\lambda'}\,|0\rangle + \\
&+ h^*_k(\eta)h_{k'}(\eta)\,\langle 0|\,a^{\dagger}_{\vec{k},\lambda}a_{\vec{k}',\lambda'}\,|0\rangle\Big] = \\
=& 2\delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')|h_k(\eta)|^2 = \\
=& 2\delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')\Big|\frac{4\sqrt{\pi G}}{a}v_k(\eta)\Big|^2 \\
=& = \frac{32\pi G}{a^2}2^{-2\epsilon_V}\Big(\frac{\Gamma(\frac{3}{2}+\epsilon_V)}{\Gamma(\frac{3}{2})}\Big)^2\frac{1}{2k}(-k\eta)^{-2-\epsilon_V} = \\
=& \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')16\pi G 2^{-2\epsilon_V}\Big(\frac{\Gamma(\frac{3}{2}+\epsilon_V)}{\Gamma(\frac{3}{2})}\Big)^2\frac{1-2\epsilon_V}{(1-\epsilon_V)^{-2\epsilon_V}}\times \\
&\times\frac{H^2}{k^3}\Big(\frac{aH}{k}\Big)^{2\epsilon_V},
\end{aligned}
\tag{1.6.56}
$$

therefore the required power spectrum for the tensor modes is

$$
P^{(\lambda)}(\eta,k) = \frac{8G}{\pi}2^{-2\epsilon_V}\Big(\frac{\Gamma(\frac{3}{2}+\epsilon_V)}{\Gamma(\frac{3}{2})}\Big)^2\frac{1-2\epsilon_V}{(1-\epsilon_V)^{-2\epsilon_V}}H^2\Big(\frac{aH}{k}\Big)^{2\epsilon_V}. \tag{1.6.57}
$$

The modes assume constant value once they exit the horizon during inflation, thus the initial condition on the power spectrum we obtain will be the one when the modes re-entered the horizon, which is nothing but the above expression evaluated at $\eta = \eta_{h.c.}$:

$$
P^{(\lambda)}(k) = \frac{8G}{\pi}f(\epsilon_V)H_k^2, \tag{1.6.58}
$$

where $f(\epsilon_V)$ is a function very close to one.
The evaluation of the scalar power spectrum, which comes from the fluctuations of the inflaton field, is identical, except for the fact that we have a dependence on $V(\phi_0)$ for the frequence $\omega_k(\eta)$ of the harmonic oscillator; the result can be written as [12]

$$
P^{(0)}(k) = \frac{G}{\pi\epsilon_V}H_k^2, \tag{1.6.59}
$$

and we can define the tensor to scalar ratio $r = \big(P^{+2}(k) + P^{-2}(k)\big)/P^{(0)}(k)$ which gives $r = 16\epsilon_V$, i.e. the tensor power spectrum is suppressed with respect to the scalar one.

[24]We are using that $e_{ij}(\hat{k}) = e^*_{ij}(-\hat{k})$ and the normalization condition $e_{ij}(\hat{k})e^*_{ij}(\hat{k}) = 2\delta_{\lambda\lambda'}$.

Capitolo 2

Cosmological Gravitational Wave Background Anisotropies

2.1 Boltzmann equation

The most powerful tool to study non-equilibrium processes [53] is the Boltzmann equation. Everytime we want to study a many-body system we define a distribution function $f(t,\vec{x},\vec{p})$ in such a way that the quantity $f(t,\vec{x},\vec{p})d^3xd^3p$ corresponds to the number of particles which, at the time t, stay in a volume d^3x around $\vec{x}$ and which have momenta with values in the volume d^3p around $\vec{p}$. It is appropriate to specify that we have chosen the coordinates in such a way that $\vec{x}$ and $\vec{p}$ are the canonical coordinates and the corrispective conjugate momenta, related by the Hamilton's equations. The Boltzmann equation describes the variation in the abundance of a particle species in terms of its distribution function, of the scattering processes (with particles of the same or of different species) and of external sources:

$$\hat{\mathcal{L}}[f] = \mathcal{C}[f] + \mathcal{J}[f], \tag{2.1.1}$$

where $\hat{\mathcal{L}}$ is the Liouville operator, which evaluates the total derivative of its argument with respect to time, $\mathcal{C}$ is the collision operator, which encodes the information of the creation/annihilation of particles of species described by the function f through scattering processes, and $\mathcal{J}[f]$ is the source operator, which includes all the production phenomena different from elementary particle collisions.
We want to find out the explicit form of the operators described above, by using an argument analogous to the one seen in [54], but using the formalism of general relativity.
We begin with the evaluation of the Liouville operator, considering a system of collisionless particles without external sources: the equation of motion for f is given only by the requirement that, locally, the number of particle is conserved, in other words we can match the number of particles at the time t, at the position $\vec{x}$ and with momentum $\vec{p}$, with the number of particles at the time immediately after t, t', with the position and the momenta in the neighbourhood of $(\vec{x},\vec{p})$, named $(\vec{x}',\vec{p}')$,

$$f(t',\vec{x}',\vec{p}')d^3x'd^3p' = f(t,\vec{x},\vec{p})d^3xd^3p. \tag{2.1.2}$$

The first important thing to stress about using general relativity is that we will use a generic affine parameter λ to parametrize the trajectories (i.e. the geodesics) of the particles, in other words we write $t = t(\lambda)$, $\vec{x} = \vec{x}(\lambda)$ or, equivalently, $x^\mu = x^\mu(\lambda)$; secondly we just recall the form of the equations of motion in general relativity:

$$\begin{aligned} \frac{dx^\mu}{d\lambda} =& p^\mu, \\ \frac{dp^\mu}{d\lambda} =& -\Gamma^\mu_{\nu\rho} p^\nu p^\rho. \end{aligned} \tag{2.1.3}$$

The second thing we observe is that, after an affine parameter variation $\delta\lambda$, all the quantities from whom the distribution function depends on, become

$$\begin{aligned} x^\mu(\lambda+\delta\lambda) =& x^\mu(\lambda) + \frac{dx^\mu}{d\lambda}\delta\lambda = x^\mu(\lambda) + p^\mu\delta\lambda, \\ p^\mu(\lambda+\delta\lambda) =& p^\mu(\lambda) + \frac{dp^\mu}{d\lambda}\delta\lambda = p^\mu(\lambda) - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \delta\lambda, \end{aligned} \tag{2.1.4}$$

with $d^3x'd^3p' = d^3xd^3p$, if $(\vec{x}, \vec{p})$ are Hamiltonian conjugate variables. By substituting all these relations in Eq. (2.1.2) we have

$$\begin{aligned} 0 =& f\Big(x^\mu + \frac{dx^\mu}{d\lambda}\delta\lambda, p^\mu + \frac{dp^\mu}{d\lambda}\delta\lambda\Big) - f(x^\mu, p^\mu) = \frac{df(x^\mu, p^\mu)}{d\lambda} = \\ =& p^\mu \partial_\mu f(x^\alpha, p^\alpha) - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \frac{\partial f(x^\alpha, p^\alpha)}{\partial p^\mu}, \end{aligned} \tag{2.1.5}$$

hence the Liouville operator is

$$\hat{\mathcal{L}} = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \frac{\partial}{\partial p^\mu}. \tag{2.1.6}$$

In absence of collisions $df/d\lambda = 0$, but this does not mean that the number of particles in a given phase space element is constant: the phase space evolves in a nontrivial way in general relativity and we need to keep into account the effects of the metric. As an example we can think to an expanding universe described by the metric $g_{\mu\nu} = a^2(t) diag(-1, +1, +1, +1)$: the physical volume associated to d^3x is proportional also to a^3, this means that if $da/dt > 0$, i.e. the Universe is expanding, the physical volume grows in time, and the particle density is diluted by a factor a^{-3}, even if $df/d\lambda = 0$.
To find the collision operator we think that the contribution to $df(t, \vec{x}, \vec{p})/d\lambda$ due to scatterings is given by the rate of particles which, after the collisions, are in a volume d^3xd^3p around the point $(\vec{x}, \vec{p})$ in the phase space, minus the rate of particles which were in a volume d^3xd^3p around the point $(\vec{x}, \vec{p})$ in the initial state. This subtraction represents the net number of particles which reached the configuration $(\vec{x}, \vec{p})$ in a certain small amount of time. To the most general scattering process $\psi + a + b + ... \leftrightarrow 1 + 2 + ...$ corresponds

the following form for the collision term:

$$\mathcal{C}[f_\psi] = -\int \prod_{\alpha=a,b...} \prod_{\beta=1,2...} d\Pi_\alpha d\Pi_\beta (2\pi)^4 \times \delta^{(4)}(p_\psi + p_a + p_b + ... - p_1 - p_2 - ...)$$
$$\times [|M|^2_{\psi+a+b+..\to 1+2+...} f_\psi f_a f_b ...(1 \pm f_1)(1 \pm f_2)...$$
$$- |M|^2_{1+2+...\to\psi+a+b+..} f_1 f_2 ...(1 \pm f_\psi)(1 \pm f_1)(1 \pm f_2)...], \tag{2.1.7}$$

where $d\Pi_i = g_i \frac{d^3 p_i}{(2\pi)^3 2E_i}$ is the phase space element of the particle of species i, with g_i its internal degrees of freedom i and $E_i = \sqrt{m_i^2 + \vec{p}_i^{\,2}}$ its energy, while $|M|^2$ are the squared matrix elements for the two processes, which represent the transition probabilities from the initial to the final states. Basically, we are writing down that the particle species ψ is created (annihilated) by processes which contains it in the final (initial) states; the probabilities of this processes to occur have to be multiplied by the product of the distribution functions of the initial species because the higher the number of the particle in the initial states, the higher the probability of having the required final state; in addition we put also some factors $1 \pm f_{final}$ which represent the Bose enhancement factors (with the plus) or the Pauli blocking terms (with the minus). These are given by the fact that the rate of producing a certain particle species is proportional to the occupation numbers of the species and, for the Pauli principle, fermions are less favoured to be produced.
To conclude this section, we briefly mention the source operator, $\mathcal{J}[f]$: it represents the external sources of the particle species considered and, for instance, in [55] is parametrized using an emissivity rate $j(t)$, the number of particles produced per unit time and comoving volume, in the form

$$\mathcal{J}[f(t, \vec{x}, \vec{p})] = j(t) f(t, \vec{x}, \vec{p}). \tag{2.1.8}$$

This introduction to the Boltzmann equation is very general, and can be applied to many situations, like the ones mentioned at the beginning of this section. Now, we want to specify all this discussion to the gravitational wave case, trying to define a distribution function for the graviton and to write down their Boltzmann equation.

2.2 Graviton distribution function

In Section 1.6.2 we have seen that the theory of inflation predicts a stochastic background of gravitational waves, which can be described using the distribution function $f(t, \vec{x}, \vec{p})$, where $\vec{p}$ is the momentum of the gravitons propagating in the Universe; this can be done even if the background curvature is large, if we are in the limit in which the amplitude of the waves is small (the weak field limit), because we have proved that it exits an adiabatic invariant which corresponds to the conservation of the graviton number density.
In analogy with the cosmic microwave background radiation (CMB), we suppose that this background is not exactly homogeneous and isotropic, but it has small anisotropies due to the quantum fluctuations of the metric during the inflation, as discussed in Section 1.6.2;

these anisotropies are supposed to be very small, of the order 10^{-5}, because of their origin from cosmological perturbation [56]. We can divide the distribution function in two terms, a leading one, $\bar{f}(t,p)$ (where p is the modulus of the three-momentum), homogeneous and isotropic, plus a perturbation $\delta f(t,\vec{x},\vec{p})$,

$$f(t,\vec{x},\vec{p}) = \bar{f}(t,p) + \delta f(t,\vec{x},\vec{p}). \tag{2.2.1}$$

The main purpose of this chapter is studying the evolution of this distribution function from the early Universe until now, using the Boltzmann equation for gravitational waves. A key difference with respect to the CMB case is in the initial conditions: the leading term of the distribution function for the gravitons is not thermal, it follows a power-law with respect to p, as it is shown in Eq. (1.6.52).

Another important difference is that for the gravitons the collision operator is very close to zero. To prove that we have to show that there are very few collisions between gravitons and other different particle species, in this way we can neglect the collision term.

When we implement quantum field theory with gravity, we can indeed consider in principle scattering processes between the gravitons and other fields, as photons [57]. To demonstrate that these processeses are negligible we introduce the concept of decoupling. The interaction rate $\Gamma_{interaction}$ is defined as the number of interactions for a species per unit time, $\Gamma_{interaction} = n\sigma v$, where n is the number density of the particle species, σ is the total cross section of the scatterings in which such a species is involved, and v is the modulus of the relative velocity of the particles. We say that a particle species decouples when the interaction rate becomes equal to the Hubble expansion rate H; a condition for a particle species to be decoupled is then that $\Gamma_{interaction} \lesssim H$. We can think to this result as the fact that the mean free path of the particle is approximately equal to the horizon size or, from another interpretative point of view, that the time required for an interaction is larger than the characteristic time of the Universe, $\tau_H = H^{-1}$, this means that the particle will have less than one interaction for the rest of its life. If we prove that gravitons decoupled at early times, then we are legitimated to neglect the scatterings with other particle species in the Boltzmann equation.

For massless gauge bosons the cross section goes as $\sigma \sim T^2\alpha^2$, where α is the interaction constant of the interaction associated to the bosons. For gravity the interaction constant goes as $\alpha \sim 1/M_{Pl}^2$, hence for this massless gauge boson the cross section goes as $\sigma \sim T^2/M_{Pl}^4$. Then the particle number density is given by the statistics of relativistic species, $n \sim T^3$, therefore the interaction rate is $\Gamma_{int} \sim T^5/M_{Pl}^4$. If we suppose that gravitons decoupled at very early times, when the radiation gives the dominant contribution to the energy density of the Universe, then the Hubble factor is equal to $H = \frac{T^2}{M_{Pl}}$. By imposing the decoupling condition $\Gamma_{int} = H$ we have that

$$(cost.)\frac{T_{dec}^5}{M_{Pl}^4} = \frac{T_{dec}^2}{M_{Pl}} \rightarrow T_{dec} \approx M_{Pl}, \tag{2.2.2}$$

which means that the gravitons decoupled at the Planck temperature, $T_{dec} \approx 10^{19} GeV$. We have studied the gravitational waves production during the inflation in Section 1.6.2, thus we we can choose as initial time for analyze the system the end of the inflation, when

no more gravitational waves are produced. Inflation ends for sure at temperatures much smaller than $10^{19} GeV$, therefore we can neglect interactions with other particle species. Through the short wavelenght approximation described in Section 1.4, we have seen that we can also neglect gravitational waves self-interactions and scatterings with the background metric at the first order in perturbation theory. Therefore we can consider the collision operator of Eq. (2.1.1) null, $\mathcal{C}[f(\lambda)] = 0$.
The source operator is also zero because we want to consider only the gravitational waves of cosmological origin, which were produced during the inflation, and so we can neglect any astrophysical source as, for example, the mergers of compact objects such as black hole collisions, and so $\mathcal{J}[f(\lambda)] = 0$ [55].
To conclude, the Boltzmann equation for the gravitons is the one for free-streaming particles in a perturbed Friedmann-Lemaître-Robertson-Walker Universe:

$$\frac{df(\lambda)}{d\lambda} = 0. \tag{2.2.3}$$

2.3 The perturbed metric

The starting point for our computations is the most general form for a perturbed spatially flat FLRW metric, given by [35]:

$$ds^2 = -e^{2\phi}dt^2 + 2a(t)\omega_i dx^i dt + a^2(t)(e^{-2\psi}\delta_{ij} + \chi_{ij})dx^i dx^j, \tag{2.3.1}$$

where we have classified the perturbations in accordance with their trasformation properties under spatial coordinate transformations: ϕ and ψ transform as scalars, ω_i as a vector and χ_{ij} as a rank 2 symmetric tensor. Thanks to the Helmoltz's theorem, it is possible to decompose the vector perturbation into an irrotational (curl-free) component $\partial_i\omega^{\parallel}$ and into a solenoidal (divergence-free) component $\omega_i^{\perp}$:

$$\omega_i = \partial_i\omega^{\parallel} + \omega_i^{\perp}, \tag{2.3.2}$$

with $\vec{\nabla} \times \vec{\nabla}\omega^{\parallel} = 0$ and $\vec{\nabla} \cdot \vec{\omega}^{\perp} = 0$.
In an analogous way, the 3×3 symmetric traceless tensor field χ_{ij} can be decomposed using a suitable function $\chi^{\parallel}$, a solenoidal vector field $\chi_i^{\perp}$ and a rank 2 transverse traceless symmetric tensor h_{ij}:

$$\chi_{ij} = \Big(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Big)\chi^{\parallel} + \partial_i\chi_j + \partial_j\chi_i + h_{ij}, \tag{2.3.3}$$

with $\partial_i h_{ij} = 0$, $h_{ii} = 0$ and $\partial_i\chi_i = 0$.
The next step consists in changing the temporal variable x^0, passing from the cosmic time t to the conformal time η, defined by the following relation:

$$\eta \equiv \int_0^t \frac{dt'}{a(t')}. \tag{2.3.4}$$

The conformal time represents the total comoving distance the light could have traveled from the initial time 0 to the time t, therefore at the time t no information could have

propagated further, thus regions separated by a distance greater than η are causally disconnected: η defines the so-called comoving horizon.
The main reason why we have introduced such a quantity is that the physical quantities evolve in very different ways if their scales are bigger or smaller than the comoving horizon, and so it is very useful to study the equations we end up with in these two regimes.
Using this new parametrization, we can write in a simpler way the line element:

$$ds^2 = a^2(\eta)\Big\{-e^{2\phi}d\eta^2 + 2\omega_i dx^i d\eta + \Big[e^{-2\psi}\delta_{ij} + \Big(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Big)\chi^{\parallel} + \partial_i\chi_j + \partial_j\chi_i + h_{ij}\Big]dx^i dx^j\Big\}. \tag{2.3.5}$$

From now on we will work in the Poisson gauge, described in [59], which is defined by setting $\omega_i^{\parallel} = 0$, $\chi^{\parallel} = 0$ and $\chi_i^{\perp} = 0$. The solenoidal field $\omega_i^{\perp}$ is not set to zero in this gauge, but through the dynamical statement that any initial vorticity decays away due to the expansion of the Universe, as shown for example in [58], we can neglect it.
After all these assumptions, we end up with the following metric, which is the one we will use for the rest of this thesis:

$$ds^2 = a^2(\eta)[-e^{2\phi}d\eta^2 + (e^{-2\psi}\delta_{ij} + h_{ij})dx^i dx^j]. \tag{2.3.6}$$

We are interested only in the results at the first order in perturbation theory and so, by expanding the exponentials, we can see that the line element can be written also as

$$ds^2 = a^2(\eta)\{-(1+2\phi)d\eta^2 + [(1-2\psi)\delta_{ij} + h_{ij}]dx^i dx^j\}. \tag{2.3.7}$$

2.4 Graviton geodesics

In general, the distribution function depends only on few variables: the spacetime coordinates, $(\eta, \vec{x})$, the three-momentum of the particles $\vec{p}$ and the mass of the particles m, which determines univocally the zero component of the four-momentum through the relation $m^2 = (p^0)^2 - \vec{p}^2$; notice that we consider gravitons as radiation, thence we set their mass to zero, $m = 0$. In this section we would like to find some useful relations between the momenta of the gravitons p^μ and the metric perturbations ϕ, ψ and h_{ij}, in order to relate the evolution of the graviton distribution function f to these known geometrical quantities, for whom we can determine the evolutions using the Einstein equations. This can be done by using the equations of motion for gravitons in a curved spacetime: thanks to geometric optics discussed in Section 1.5, we know that gravitons are moving along null geodesics determined by the background metric, thus we can write

$$\frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\nu\rho}p^\nu p^\rho. \tag{2.4.1}$$

The first thing we observe is that we can write the physical three-momentum $\vec{p}$ by using the versor $\hat{n}$ which corresponds to its direction of propagation, with the normalization condition $\delta_{ij}n^i n^j = 1$, and its magnitude p, defined by

$$p^2 \equiv g_{ij}p^i p^j. \tag{2.4.2}$$

If we write the three momentum as $p^i = A(p, \hat{n}, \phi, \psi, h_{lm})n^i$ and we insert this parametrization in the above expression we obtain immediately that

$$p^i = \frac{p}{a} e^{\psi} \Big(1 - \frac{1}{2} h_{jk} n^j n^k\Big) n^i. \tag{2.4.3}$$

Everytime we speak about distribution functions, we are implicitly assuming that there is a phase space over which such functions are defined; in our case the phase space is given by the canonically conjugate variables x^i and p_i [59], therefore the volume element is $dx^1 dx^2 dx^3 dp_1 dp_2 dp_3$. In many cases we need to integrate over such variables, for instance we integrate the distribution function over the momenta to obtain quantities like the particle number density or the average energy density per unit volume, and if we do not pay attention we could end up with misleading results.
Instead of using the physical three-momentum, from now on we will use the comoving one, $q^i = qn^i$, with a magnitude q defined as $q \equiv ap$, in this way the metric perturbations are removed from the definition of this new momentum. The physical three momentum can then be expressed in terms of the magnitude of the comoving one:

$$p^i = \frac{q}{a^2} e^{\psi} \Big(1 - \frac{1}{2} h_{jk} n^j n^k\Big) n^i. \tag{2.4.4}$$

We are interested in studying the anisotropies for the cosmological gravitational waves background, and so the particles involved in the Boltzmann equation are massless gravitons, and then we can find an explicit expression also for p^0:

$$0 = g_{\mu\nu} p^\mu p^\nu = -a^2 e^{2\phi} \big(p^0\big)^2 + \frac{a^2}{a^2} g_{ij} p^i p^j = -a^2 e^{2\phi} \big(p^0\big)^2 + \frac{q^2}{a^2} \to p^0 = \frac{q}{a^2} e^{-\phi}. \tag{2.4.5}$$

Under these assumptions, the Boltzmann equation for massless, collisionless particles with no sources is

$$\frac{df}{d\lambda} = 0 = \frac{1}{p^0} \frac{df}{d\lambda} = \frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial f}{\partial q} \frac{dq}{d\eta} + \frac{\partial f}{\partial n^i} \frac{dn^i}{d\eta}. \tag{2.4.6}$$

The first two terms in the equation represent the free-streaming part, which keeps into account the propagation of the perturbation on all scales; if integrated with respect to the physical momenta, we obtain the familiar hydrodynamics continuity equation, which describes the conservation of the density of a fluid in terms of the flux of such a fluid, and the Euler equations, which express the conservation of the momentum of the fluid. The third term is the red-shift contribution, it describes how the energy of the gravitons changes during the Universe evolution, while the fourth one is the gravitational lensing term, which encodes modifications of the direction of the gravitons $\hat{n}$ due to gravity.
By using the definition of the momentum in General Relativity,

$$p^\mu \equiv \frac{dx^\mu}{d\lambda}, \tag{2.4.7}$$

we find a simplification for the free-streaming term:

$$\frac{dx^i}{d\eta} = \frac{d\lambda}{d\eta} \frac{dx^i}{d\lambda} = \frac{p^i}{p^0} = e^{\phi + \Psi} \Big(1 - \frac{1}{2} h_{jk} n^j n^k\Big) n^i. \tag{2.4.8}$$

To express the other elements of the equation in terms of q, n^i and the metric perturbations we need to use the geodesic equations:

$$\frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\nu\rho} p^\nu p^\rho, \tag{2.4.9}$$

with $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbol defined as

$$\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2} g^{\mu\alpha} (\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}). \tag{2.4.10}$$

Before writing the geodesic equations explicitly, we recall which are the metric and the inverse of the metric in this case, in order to calculate the Christoffel symbols:

$$g_{\mu\nu} = a^2 diag\Big(-e^{2\phi}, e^{-2\psi}\delta_{ij} + h_{ij}\Big), \qquad g^{\mu\nu} = \frac{1}{a^2} diag\Big(-e^{-2\phi}, e^{2\psi}\delta^{ij} - h^{ij}\Big). \tag{2.4.11}$$

In order to obtain an analytical form for $\frac{dq}{d\eta}$ we use the expression for $\frac{dp^0}{d\lambda}$, which can be written in function of q and n^i using the definition of p^0

$$\begin{aligned} \frac{dp^0}{d\lambda} =& p^0 \frac{dp^0}{d\eta} = \frac{q}{a^2} e^{-\phi} \Big[-\frac{q}{a^2} e^{-\phi} \frac{d\phi}{d\eta} - \frac{2q}{a^3} e^{-\phi} \frac{da}{d\eta} + \frac{1}{a^2} e^{-\phi} \frac{dq}{d\eta} \Big] = \\ =& \frac{q^2}{a^4} e^{-2\phi} \Big[\frac{1}{q}\frac{dq}{d\eta} - 2\mathcal{H} - \frac{\partial\phi}{\partial\eta} - \frac{\partial\phi}{\partial x^i} n^i \Big], \end{aligned} \tag{2.4.12}$$

or using the explicit form of the geodesic equation

$$\begin{aligned} \frac{dp^0}{d\lambda} =& -\Gamma^0_{\alpha\beta} p^\alpha p^\beta = -\frac{1}{2} g^{0\rho} \Big(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta} \Big) P^\alpha P^\beta = \\ =& -\frac{1}{2} g^{0\rho} \Big(2\partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta} \Big) P^\alpha P^\beta = \\ =& -\frac{1}{2} g^{00} \Big[\partial_0 g_{00} \Big(P^0\Big)^2 + 2\partial_i g_{00} P^i P^0 - \partial_0 g_{ij} P^i P^j \Big] = \\ =& \frac{e^{-2\phi}}{2a^2} \Big[\frac{q^2}{a^4} e^{-2\phi} \Big(-2a \frac{da}{d\eta} e^{2\phi} - 2a^2 \frac{\partial\phi}{\partial\eta} \Big) - 2a^2 2 \frac{\partial\phi}{\partial x^i} \frac{q^2}{a^4} n^i + \\ & + (-1) \frac{q^2}{a^4} e^{2\psi} n^i n^j \Big(1 - h_{lm} n^l n^m \Big) \Big(2a \frac{da}{d\eta} e^{-2\psi} \big(\delta_{ij} + h_{ij}\big) - 2a^2 \frac{\partial\psi}{\partial\eta} \delta_{ij} + \frac{\partial h_{ij}}{\partial\eta} \Big) \Big] = \\ =& \frac{q^2}{a^4} e^{-2\phi} \Big[-\mathcal{H} - \frac{\partial\phi}{\partial\eta} - 2\frac{\partial\phi}{\partial x^i} n^i - \mathcal{H} + \frac{\partial\psi}{\partial\eta} - \frac{1}{2} \frac{\partial h_{lm}}{\partial\eta} n^l n^m \Big]. \end{aligned} \tag{2.4.13}$$

In the calculations we have introduced the parameter $\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta}$, which is related to the Hubble parameter by $\mathcal{H} = aH$.
As already stressed at the end of section 2.3, we are interested only in the equations up to the first-order in the perturbations and so in the above calculations we have to consider only the product between $\frac{\partial\phi}{\partial x^i}$, which is a first-order term in the perturbations, and the

0-order component in the perturbations of the quantity which multiplies it. By comparing the two expressions we end up with

$$\frac{dq}{d\eta} = q\Big(\frac{\partial \psi}{\partial \eta} - \frac{\partial \phi}{\partial x^i}n^i - \frac{1}{2}\frac{\partial h_{lm}}{\partial \eta}n^l n^m\Big). \tag{2.4.14}$$

The Boltzmann equation then assumes the form

$$\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i}n^i + q\frac{\partial f}{\partial q}\Big(\frac{\partial \psi}{\partial \eta} - \frac{\partial \phi}{\partial x^i}n^i - \frac{1}{2}\frac{\partial h_{lm}}{\partial \eta}n^l n^m\Big) = 0. \tag{2.4.15}$$

In the above expression, we have neglected the last term of Eq. (2.4.6). In fact, we are studying the Boltzmann equation at the first order in perturbation theory, while such a term represents a term of the second order. This is due to the fact that we have assumed that the leading term of the graviton distribution function is homogeneous and isotropic, Eq. (2.2.1), therefore each non-null derivative of f with respect to the gravitons direction of propagation will be a term of order one in the perturbations. In addition, if we consider an unperturbed FLRW metric, the gravitons trajectories are "straight lines", so the first non null contribution of $d\hat{n}/d\eta$ is a term of the order one in the perturbations: gravitons change their directions only in presence of the metric perturbations.
At zero order the Boltzmann equation reflects the fact that the graviton number density is diluted by the expansion of the Universe, keeping in mind that when we evaluate each physical quantity we integrate over the physical momentum $\vec{p}$ and not over the comoving one[1]

$$\frac{\partial \bar{f}}{\partial \eta} = 0 \rightarrow \bar{f}(\eta, q) = \bar{f}(q) \rightarrow \bar{n}(\eta, q) = \int dp_1 dp_2 dp_3 \bar{f}(q) = \int \frac{d^3q}{a^3}\bar{f}(q) \sim \frac{1}{a^3}. \tag{2.4.16}$$

In other words, the distribution function at the leading order, $\bar{f}(\eta, p)$ depends on the momentum amplitude p and the conformal time only through the combination $q = a(\eta)p$, the momenta redshift when the Universe expands; this is another motivation for having used the comoving momentum, instead of the physical one. At first-order the equation is

$$\frac{\partial \delta f}{\partial \eta} + \frac{\partial \delta f}{\partial x^i}n^i + q\frac{\partial \bar{f}}{\partial q}\Big(\frac{\partial \psi}{\partial \eta} - \frac{\partial \phi}{\partial x^i}n^i - \frac{1}{2}\frac{\partial h_{lm}}{\partial \eta}n^l n^m\Big) = 0. \tag{2.4.17}$$

We see that there are two "sources" of anisotropies for the primordial gravitational waves: the first one is given by the fact that the initial conditions on the distribution function perturbation δf could depend on $\hat{n}$, and the second one is given by the propagation of the isotropic component $\bar{f}$ along the perturbed background.
In order to underline the similarities with the CMB we will introduce another variable $\Gamma(\eta, \vec{x}, q, \hat{n})$, which is the analogous to $\frac{\delta T}{T}$, using the following redefinition:

$$\delta f(\eta, \vec{x}, q, \hat{n}) = -q\frac{\partial \bar{f}(q)}{\partial q}\Gamma(\eta, \vec{x}, q, \hat{n}). \tag{2.4.18}$$

[1] At zero order in the perturbations $dp_1 dp_2 dp_3 = d^3q/a^3$.

So, in terms of $\Gamma(\eta, \vec{x}, q, \hat{n})$, the equation becomes:

$$\frac{\partial \Gamma}{\partial \eta} + \frac{\partial \Gamma}{\partial x^i} n^i - \Big(\frac{\partial \psi}{\partial \eta} - \frac{\partial \phi}{\partial x^i} n^i - \frac{1}{2} \frac{\partial h_{lm}}{\partial \eta} n^l n^m \Big) = 0. \tag{2.4.19}$$

Moving to Fourier space [2], denoting the partial derivative respect to η with a " $'$ ", and introducting the variable $\mu = \vec{k} \cdot \vec{q}$, we have the final expression for the Boltzmann equation:

$$\Gamma'(\eta, \vec{k}, q, \hat{n}) + ik\mu\Gamma(\eta, \vec{k}, q, \hat{n}) = \psi'(\eta, \vec{k}) - ik\mu\phi(\eta, \vec{k}) - \frac{1}{2} n^i n^j h'_{ij}(\eta, \vec{k}) \equiv S(\eta, \vec{k}). \tag{2.4.21}$$

2.5 Gravitons vs tensor perturbations

Before going on, we would like to stress the important difference between the gravitons, whose evolution is given by the distribution function through a Boltzmann equation, and the tensor perturbations of the FLRW metric. Some confusion could arise from the fact that these two quantities are described by the "same" mathematical object, the rank-2 transverse-traceless tensor $h_{ij}(\eta, \vec{k})$. As already stated in Section 1.4, every time we speak about the propagation of gravitational waves in a curved background, we need to be able to distinguish them from the background, otherwise we cannot say which part of the metric that is changing in time is part of the background (generated by the incoherent superpositions of various perturbations) and which one is part of the gravitational waves. We have also seen that there is a possible separation between the waves and the background only when there is a large scale separation: if the gravitational waves have a typical reduced lenght $\bar{\lambda}$ and the background has a typical lenght scale $\mathcal{R}$, it has to be true that $\bar{\lambda} \ll \mathcal{R}$. In our specific case, this translates into the fact that the comoving momentum of the gravitons considered in the Boltzmann equation, q, is many orders of magnitude bigger than the wavenumber of the perturbations, k. Moreover, the k mode of the distribution function has nothing to do with the momentum q of the gravitons, which is also an argument of f: it represents only the component of the distribution function $f(\eta, \vec{x}, \vec{q})$ which varies on a scale $1/k$, but it still represents the distribution of gravitons with momentum q (and, again, k has to be much greater than q, otherwise we will consider variations of a certain number of particles in a range smaller than the dimension of the particles themselves). To sum up, we are considering small perturbations which are propagating on a large-scale background, $q \gg k$ [24].

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[2] To define the Fourier transform we use the "+" convention:

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \hat{f}(\vec{k}). \tag{2.4.20}$$

For the sake of simplicity we will omit the "hat" in the further equations, but we will write explicitly the dependence on $\vec{x}$ or on $\vec{k}$ when there could be some ambiguities.

2.6 Connection with observables

It is useful to understand how the quantities introduced until now, the distribution function, with its unperturbed and perturbed parts, and the comoving momentum of the gravitons can be related to the physical observables, intended as quantities that can be measured by the detectors. The most important quantity we have to consider is the gravitational waves energy density $\rho_{GW}(t,\vec{x})$: if a single graviton carries an energy p, equal to the frequency of the waves for the wave-particle duality, then the energy of the waves can be written as

$$\rho_{GW}(\eta,\vec{x}) = \int dp_1 dp_2 dp_3 p f(\eta,\vec{x},q,\hat{n}) = \int dq \int d\Omega \frac{q^3}{a^4} f(\eta,\vec{x},q,\hat{n}) = \int d(\mathrm{logq}) \Omega_{\mathrm{GW}}(\eta,\tilde{\mathrm{x}},\mathrm{q}) \rho_{\mathrm{cr}}(\eta_0), \tag{2.6.1}$$

where $\rho_{crit}(\eta_0) = \frac{3H_0^2}{(\pi G)}$ is the critical energy density at the present time η_0, and Ω_{GW} is the spectral energy density of the gravitational waves, defined as the integral over the logarithm of q instead of q itself.
At the zero order the spectral energy density is nothing but

$$\bar{\Omega}_{GW} = \frac{4\pi}{\rho_{crit}} \left(\frac{q}{a_0}\right)^4 \bar{f}(q). \tag{2.6.2}$$

We can write the spectral energy density in terms of an integral over $\hat{n}$ of another variable $\omega(\eta,\vec{x},\vec{q},\hat{n})$, which is the angular part of the spectral energy density:

$$\Omega_{GW}(\eta,\vec{x},q,\hat{n}) \equiv \frac{1}{4\pi} \int d^2\hat{n}\, \omega_{GW}(\eta,\vec{x},q,\hat{n}), \tag{2.6.3}$$

where $d^2\hat{n}$ denotes the surface area element on the unit sphere relative to the variable n^i. The we can define the gravitational waves density contrast by the following ratio

$$\delta_{GW}(\eta,x^i,q,n^i) \equiv \frac{\delta\omega_{GW}(\eta,x^i,q,n^i)}{\bar{\omega}(\eta,q)} = \frac{\frac{q^4}{a^4}\frac{1}{\rho_{cr}}\delta f}{\frac{q^4}{a^4}\frac{1}{\rho_{cr}}\bar{f}} = -\frac{q}{\bar{f}}\frac{\partial \bar{f}}{\partial q}\Gamma. \tag{2.6.4}$$

Then we write the derivative of $\bar{f}$ in terms of the isotropic part of the fractional contribution of the gravitational waves to the energy density $\bar{\Omega}_{GW}$:

$$q\frac{\partial \bar{\Omega}_{GW}}{\partial q} = q\frac{\partial}{\partial q}\int d\Omega \frac{q^4}{a^4}\bar{f}(q) = \int d\Omega \frac{q^4}{a^4}\Big(4\bar{f} + q\frac{\partial \bar{f}}{\partial q}\Big) = 4\bar{\Omega}_{GW} + 4\pi \frac{q^5}{a^4}\frac{\partial \bar{f}}{\partial q}. \tag{2.6.5}$$

At the end we have that the gravitational waves density contrast is

$$\begin{aligned}\delta_{GW}(\eta,\vec{x},q,\hat{n}) = &-\frac{q}{\bar{f}}\frac{1}{4\pi\frac{q^5}{a^4}}\Big(-4\bar{\Omega}_{GW} + q\frac{\partial \bar{\Omega}_{GW}}{\partial q}\Big)\Gamma = \frac{1}{4\pi\frac{q^4}{a^4}\bar{f}}\Omega_{GW}\Big[4 - \frac{\partial(\mathrm{log}\bar{\Omega}_{\mathrm{GW}})}{\partial(\mathrm{logq})}\Big]\Gamma = \\ =&\Big[4 - \frac{\partial(\mathrm{log}\bar{\Omega}_{\mathrm{GW}})}{\partial(\log q)}\Big]\Gamma.\end{aligned} \tag{2.6.6}$$

It is important to stress that, with respect to the photon case, the initial gravitational waves distribution is not thermal, and so, in principle, Γ can depend on q through an arbitrarily complicated dependence. Moreover, because of graviton decoupling at early times, any anisotropy in the initial conditions is not canceled by the scatterings (as it occurs for photons and baryons); we have a kind of "memory" of the initial state. In addition, the dependence of Γ on q nowadays is determined precisely only by the initial conditions, and not by other terms. We will see this better in the next section.
The result that emerges from this section is that once we measure the full spectral energy density, so both the homogeneous component $\bar{\Omega}_{GW}$ and the density contrast δ_{GW}, we have obtained some values to compare with $\bar{f}$ and Γ.

2.7 Solution of the Boltzmann equation for gravitons

In Chapter 3 and in Chapter 4 we will see in detail the evolution of the metric perturbations ϕ, ψ and h_{ij}, for the moment we just want to introduce a possible decomposition of these fields in terms of a transfer function $T(\eta, \vec{k})$, which describes the temporal evolution of the perturbations from early times until the present epoch, multiplied by a stochastic variable which depends only on $\zeta(\vec{k})$ and $h(\vec{k})$, which are related to the initial conditions of the perturbations when they were produced (remember that the scalar and the tensor perturbations were produced during inflation, therefore they have a stochastic nature) and which are constant on large scales:

$$\begin{cases} \phi(\eta, \vec{k}) = T_\phi(\eta, k)\zeta(\vec{k}) \\ \psi(\eta, \vec{k}) = T_\psi(\eta, k)\zeta(\vec{k}) \end{cases} \qquad h_{ij}(\eta, \vec{k}) = \sum_{\lambda=\pm 2} e_{ij,\lambda}(\hat{k}) h(\eta, k) \xi_\lambda(\vec{k}). \tag{2.7.1}$$

The solution of the Equation 2.4.21 is

$$\begin{aligned} \Gamma(\eta_0, \vec{k}, q, \hat{n}) =& e^{ik\mu(\eta_i - \eta_0)}\Gamma(\eta_i, \vec{k}, q, \hat{n}) + \\ &+ \int_{\eta_i}^{\eta_0} d\eta \Big[\psi'(\eta, \vec{k}) + \phi'(\eta, \vec{k}) - \frac{1}{2} h'_{ij}(\eta, \vec{k}) n^i n^j\Big] e^{ik\mu(\eta - \eta_0)} + \\ &- \int_{\eta_i}^{\eta_0} d\eta \frac{d}{d\eta}\Big[\phi(\eta, \vec{k}) e^{ik\mu(\eta-\eta_0)}\Big] = \\ =& e^{ik\mu(\eta_i - \eta_0)}\Gamma(\eta_i, \vec{k}, q, \hat{n}) - \phi(\eta_0, \vec{k}) + \phi(\eta_i, \vec{k}) e^{ik\mu(\eta_i - \eta_0)} + \\ &+ \int_{\eta_i}^{\eta_0} d\eta \Big[\psi'(\eta, \vec{k}) + \phi'(\eta, \vec{k}) - \frac{1}{2} h'_{ij}(\eta, \vec{k}) n^i n^j\Big] e^{ik\mu(\eta - \eta_0)}, \end{aligned} \tag{2.7.2}$$

where we have identified with η_0 the conformal time at which we are making the measurements nowadays and with η_i the conformal time of neutrino decoupling. We have supposed that the gravitational waves were produced during the inflation, which occurs at times smaller than η_i, but we can justify this choice for the initial time by thinking that all the modes of physical interest entered the horizon after η_i and that they were conserved for modes outside the horizon, thence all the contributions to the integrals are non-zero only for $\eta > \eta_i$; moreover $\eta_0 \gg \eta_{inflation}$, then the difference $\eta_0 - \eta_{inflation}$ in the exponential

is approximately equal to $\eta_0 - \eta_i$, thus we can choose η_i as initial time.
Notice that we are not interested in the $\phi(\eta_0, \vec{k})$ term, because it does not depend on μ and so it is not a source of anisotropies.
It is also important to stress which are the main differences with respect to the CMB case analyzed, for instance, in [22]. First of all for the photons we define the optical depth $\tau(\eta)$ as

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a, \tag{2.7.3}$$

where n_e is the electron number density and σ_T the total cross-section for the Compton scattering, which occurs between photons and electrons; this quantity represents the difficulty that a photon encounters in reaching us by starting its path at the time η. Photons started their free streaming after the recombination (corresponding to the conformal time η_*), until that moment they were tightly coupled to electrons and baryons, therefore it is clear that, because of $\eta_i \ll \eta_*$, $\tau(\eta_i) \gg 1$, it is very likely that a photon has scattered from very early times until now. The fact we want to stress is that in the CMB case we do not have terms analogous to $\Gamma(\eta_i, \vec{k}, q, \hat{n})$ and $\phi(\eta_i, \vec{k})$, because they were multiplied by $e^{-\tau(\eta_i)}$, which tends to zero, and then they are suppressed by the scatterings, while for the gravitons we do not have this suppression because they decoupled at very early times, as we have stated at the end of Section 2.2. To conclude, in the gravitons case we have a memory of the initial state, $\Gamma(\eta_i, \vec{k}, q, \hat{n})$, which was canceled by the scatterings for the CMB, and this initial condition term provides the only dependence on q at the first order, while for electromagnetic radiation a q-dependence arises at higher orders only.
We are interested in studying the SGWB anisotropies, i.e. the differences we see between the observables by varying the direction of detection. It is clear that to observe anisotropies we need that the distribution function depends on the direction of propagation $\hat{n}$. Experimentally, what we have when we detect anisotropies, is a map of the values measured in function of the direction of observation. The problem is that it is quite complicate to compare quantitatively the predictions given by a certain model for the anisotropies by keeping the explicit dependence on $\hat{n}$. The problem arises because in this way we are not able to separate large-scale to small-scale contributions to the anisotropies, for instance. Therefore, for angular variables, i.e. defined on the surface of a sphere, as $\hat{n}$, we introduce the spherical harmonics decomposition. We can expand the function Γ in the following way:

$$\Gamma(\eta_0, \vec{x}, q, \hat{n}) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} \Gamma_{\ell m}(\eta_0, \vec{x}, q) Y_{lm}(\hat{n}), \tag{2.7.4}$$

where $Y_{\ell,m}(\theta, \phi)$ are the spherical harmonics, which will be defined in Eq. (2.7.16). The index ℓ in the spherical harmonics expansion is related to the angular scale we are observing, the bigger the ℓ the smaller the angular scale, while the m index represents the direction we are looking at.
We are interested in finding the coefficients of the spherical harmonics expansion of such a function, $\Gamma_{\ell m}(\eta, q)$, because they contain all the information about the distribution function perturbation, and so we multiply the distribution function perturbation by $Y^*_{\ell m}(\hat{n})$ and we

perform an integration over the comoving momentum direction, obtaining the different coefficients.
In the following calculations we will use the following expansion for the complex exponential in terms of the spherical Bessel functions and spherical harmonics:

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{m=+\ell} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r}). \tag{2.7.5}$$

The first contribution arises from the initial conditions:

$$\begin{aligned} \Gamma^{(I)}_{\ell m}(\eta_0, \vec{x}, q) = \int d^2\hat{n} Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} 4\pi \sum_{\ell'=0}^{+\infty} \sum_{m'=-\ell'}^{m'=+\ell'} i^{\ell'} j_{\ell'}[k(\eta_i - \eta_0)] \times \\ \times Y^*_{\ell' m'}(\hat{k}) Y_{\ell' m'}(\hat{n}) \Gamma(\eta_i, k^i, q) = \\ = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} Y^*_{\ell m}(\hat{k}) 4\pi (-i)^\ell \Gamma(\eta_i, k^i, q) j_\ell[k(\eta_0 - \eta_i)], \end{aligned} \tag{2.7.6}$$

where we have used the normalization condition of the spherical harmonics and the fact that under reflections the spherical Bessel functions transform as $j_\ell(x) = (-1)^\ell j_\ell(-x)$.
From now on we will impose$\vec{x} = 0$, which is our position nowadays in the comoving coordinates, as η_0 is our temporal coordinate. These represent the points in the spacetime at which the experiments can take place.
The second term, generated by the propagation of the gravitational waves through the scalar perturbations of the Universe, is similar to the previous one. We have a cancellation of the $\phi(\eta_0, k^i)$ term due to the integral over a single spherical harmonic, which is null except for the trivial case $\ell = m = 0$, and so we reduce to

$$\begin{aligned} \Gamma^{(S)}_{\ell m}(\eta) = \int \frac{d^3k}{(2\pi)^3} 4\pi (-i)^\ell \zeta(k^i) Y^*_{lm}(\hat{k}) \Big\{ T_\phi(\eta_i, k) j_\ell[k(\eta_0 - \eta_i)] + \\ + \int_{\eta_i}^{\eta_0} d\eta \Big[T'_\phi(\eta, k) + T'_\psi(\eta, k) \Big] j_\ell[k(\eta_0 - \eta)] \Big\}. \end{aligned} \tag{2.7.7}$$

We can rename the term in the brace parenthesis with $T^{(0)}_\ell$, which is usually called scalar transfer function. The situation is a little bit more technical for the last contribution, the tensor one, because it involves the spin-weighted spherical harmonics, ${}_sY_{\ell m}$. We can see in fact that we have decomposed the tensor mode using two spin-2 tensors, $e_{ij,\pm 2}$, which represent the two possible polarizations of the gravitational waves.
To understand their meaning it is useful to define a proper orthonormal basis containing the $\hat{k}$ vector.
We call it $(\hat{u}, \hat{v}, \hat{k})$ and it can be constructed in the following way, by choosing a certain

direction $\hat{z}$:

$$\begin{aligned}
\hat{u} =& \frac{(\hat{k}\cdot\hat{z})\hat{k}-\hat{z}}{|\hat{k}\times\hat{z}|} = \frac{1}{\left|\begin{pmatrix} sin\theta_k\phi_k \\ sin\theta_k sin\phi_k \\ cos\theta_k \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}\left[cos\theta_k \begin{pmatrix} sin\theta_k cos\phi_k \\ sin\theta_k sin\phi_k \\ cos\theta_k \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \\
=& \begin{pmatrix} cos\theta_k cos\phi_k \\ cos\theta_k sin\phi_k \\ -sin\theta_k \end{pmatrix}, \\
\hat{v} =& \hat{k}\times\hat{u} = \begin{pmatrix} -sin\phi_k \\ cos\phi_k \\ 0 \end{pmatrix}.
\end{aligned} \tag{2.7.8}$$

Now we are interested in the relation between the expressions for $\hat{n}$ in the canonical basis $\{\hat{x},\hat{y},\hat{z}\}$ and in this new basis $\{\hat{u},\hat{v},\hat{k}\}$, defined from the angles $\{\theta_n,\phi_n\}$ and $\{\theta_\mu,\phi_\mu\}$ respectively, which means nothing but that the graviton direction in the two basis has the form

$$\hat{n} = \begin{pmatrix} \hat{n}\cdot\hat{x} \\ \hat{n}\cdot\hat{y} \\ \hat{n}\cdot\hat{z} \end{pmatrix} = \begin{pmatrix} sin\theta_n cos\phi_n \\ sin\theta_n sin\phi_n \\ cos\theta_n \end{pmatrix}, \qquad \hat{n}_\mu = \begin{pmatrix} \hat{n}_\mu\cdot\hat{u} \\ \hat{n}_\mu\cdot\hat{v} \\ \hat{n}_\mu\cdot\hat{k} \end{pmatrix} = \begin{pmatrix} sin\theta_\mu cos\phi_\mu \\ sin\theta_\mu sin\phi_\mu \\ cos\theta_\mu \end{pmatrix}. \tag{2.7.9}$$

To change the basis we use the relation $\hat{n} = S(\Omega_k)\hat{n}_\mu$, where $S(\Omega_k)$ is the rotation matrix of the angles $\{\theta_k,\phi_k\}$, which has as columns the vectors $\hat{u}$, $\hat{v}$, $\hat{k}$ in the canonical basis. This change of basis is quite important because it allows us to simplify the product between the graviton direction and the polarization tensor.
We recall that the circular polarizations are defined as

$$e_{ij,+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_{ij,\times} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.7.10}$$

From this we can define the left-handed and right-handed chirality basis we will use:

$$\begin{aligned}
e_{ij,R} =& e_{ij,+2} = \frac{e_{ij,+}+ie_{ij,\times}}{\sqrt{2}} = \frac{1}{2}[(u_iu_j - v_iv_j) + i(u_iv_j + u_jv_i)], \\
e_{ij,L} =& e_{ij,-2} = \frac{e_{ij,+}-ie_{ij,\times}}{\sqrt{2}} = \frac{1}{2}[(u_iu_j - v_iv_j) - i(u_iv_j + u_jv_i)].
\end{aligned} \tag{2.7.11}$$

It is clear that, by the tensor modes decomposition already seen, we can write

$$\begin{aligned}
\frac{n^in^j}{2}h'_{ij}(\eta,\vec{k}) =& \frac{n^in^j}{2}h'(\eta,k)\sum_{\lambda=R,l} e_{ij,\lambda}(\hat{k})\hat{\xi}_\lambda(\vec{k}) = \\
=& \frac{h'(\eta,k)}{4}(1-\mu^2)\Big(e^{2i\phi_\mu}\hat{\xi}_R(\vec{k}) + e^{-2i\phi_\mu}\hat{\xi}_L(\vec{k})\Big).
\end{aligned} \tag{2.7.12}$$

After that we need to take into account the fact that the spherical harmonics involved in the integral are modified by the changing of the integration angle, from Ω_n to Ω_μ, and we know that they transform in a simple form given by the Wigner D-matrix:

$$Y^*_{\ell m}(\Omega_n) = \sum_{m'=-\ell}^{+\ell} D^{(\ell)}_{mm'}(\Omega_k) Y^*_{\ell m}(\Omega_\mu). \tag{2.7.13}$$

We also know that the Wigner matrices are related to the spin-weighted spherical harmonics through the relation

$$D^{(\ell)}_{ms}[S(\Omega_k)] = \sqrt{\frac{4\pi}{2\ell+1}}(-1)^s_{-s}Y^*_{\ell m}(\Omega_k). \tag{2.7.14}$$

The last steps before evaluating this very complicated integral is to consider the fact that we will end up with a certain integration over the anomaly angle ϕ_μ which has the form

$$\int_0^{2\pi} d\phi_\mu e^{i\alpha\phi_\mu} \tag{2.7.15}$$

and it is different from 0 only for $\alpha = 0$, which leads to the conditions $\delta_{\lambda,2}\zeta_R + \delta_{\lambda,-2}\zeta_L$. We recall also the spherical harmonics expansion in terms of the associated Legendre polynomials:

$$Y^*_{\ell m}(\theta,\phi) = (-1)^m \Big[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\Big]^{\frac{1}{2}} \mathcal{P}_{\ell m}(cos\theta)e^{-im\phi}. \tag{2.7.16}$$

The associated Legendre polynomials are defined from the Legendre polynomials:

$$\begin{aligned} \mathcal{P}_{\ell|m|}(x) =& (-1)^{|m|}(1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}\mathcal{P}_\ell(x)}{dx^{|m|}}, \\ \mathcal{P}_{\ell-|m|}(x) =& (-1)^{|m|}\frac{(\ell-|m|)!}{(\ell+|m|)!}\mathcal{P}_{\ell|m|}(x). \end{aligned} \tag{2.7.17}$$

We just recall that the Legendre polynomials are defined by the differential equation

$$\frac{d}{dx}\Big[(1-x^2)\frac{d\mathcal{P}_\ell(x)}{dx}\Big] = -\ell(\ell+1)\mathcal{P}_\ell(x). \tag{2.7.18}$$

They are very useful because we know the integral of an explicit expression involving the Legendre polynomials

$$\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_\ell(\mu)e^{ik\mu(\eta-\eta_0)} = \frac{1}{(-i)^\ell}j_\ell[k(\eta-\eta_0)] \tag{2.7.19}$$

and we will use it to evaluate the following integral, which appears in the tensor contribution:

$$
\begin{aligned}
I =& \int_{-1}^{+1} \frac{d\mu}{4} e^{ik\mu(\eta-\eta_0)} (1-\mu^2)^2 \frac{d^2\mathcal{P}_\ell}{d\mu^2} = \int_{-1}^{+1} \frac{d\mu}{4} e^{ik\mu(\eta-\eta_0)} (1-\mu^2) \Big[2\mu \frac{d\mathcal{P}_\ell}{d\mu} - \ell(\ell+1)\mathcal{P}_\ell \Big] = \\
=& \int_{-1}^{+1} \frac{d\mu}{4} e^{ik\mu(\eta-\eta_0)} \Big\{ -\frac{2}{ik(\eta-\eta_0)} \Big[\mu \frac{d}{d\mu} \Big((1-\mu^2) \frac{d\mathcal{P}_\ell}{d\mu} \Big) + 2(1-\mu^2) \frac{d\mathcal{P}_\ell}{d\mu} \Big] \\
& \qquad - \ell(\ell+1)(1-\mu^2)\mathcal{P}_\ell \Big\} = \\
=& \int_{-1}^{+1} \frac{d\mu}{4} e^{ik\mu(\eta-\eta_0)} \Big\{ + \frac{2\mu}{ik(\eta-\eta_0)} \ell(\ell+1)\mathcal{P}_\ell + \frac{2}{k^2(\eta-\eta_0)^2} \ell(\ell+1)\mathcal{P}_\ell - \frac{2\ell(\ell+1)}{ik(\eta-\eta_0)} \mu \mathcal{P}_\ell + \\
& \qquad + \frac{\ell(\ell+1)}{ik(\eta-\eta_0)} (1-\mu^2) \frac{d\mathcal{P}_\ell}{d\mu} \Big\} = \\
=& \int_{-1}^{+1} \frac{d\mu}{4} \frac{e^{ik\mu(\eta-\eta_0)}}{k^2(\eta-\eta_0)^2} \mathcal{P}_\ell \ell(\ell+1)(2-\ell^2-\ell) = \\
=& -\frac{1}{2}(-i)^\ell \frac{j_\ell[k(\eta_0-\eta)]}{k^2(\eta_0-\eta)^2} \ell(\ell+1)(\ell+2)(\ell-1).
\end{aligned}
\tag{2.7.20}
$$

Now we are ready to calculate the contribution from the tensor part:

$$
\begin{aligned}
\Gamma^{(T)}_{\ell m}(\eta_0) =& -\frac{1}{2} \int d^2\hat{n} Y^*_{\ell m}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{ik\mu(\eta-\eta_0)} \sum_{\lambda=\pm 2} e_{ij,\lambda}(\hat{k}) \xi_\lambda(\vec{k}) n^i n^j h'(\eta,k) = \\
&= -\int \frac{d^3k}{(2\pi)^3} \sum_\lambda \sqrt{\frac{4\pi}{2\ell+1}} (-1)^\lambda_{-\lambda} Y^*_{lm}(\Omega_k) \times \\
&\times \int d\Omega_\mu Y^*_{l\lambda}(\Omega_\mu) \frac{1-\mu^2}{4} \Big[e^{2i\phi_\mu}\xi_R + e^{-2i\phi_\mu}\xi_L \Big] \int_{\eta_i}^{\eta_0} d\eta e^{ik\mu(\eta-\eta_0)} h'(\eta,k) = \\
&= \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \sqrt{\frac{(\ell-\lambda)!}{(\ell+\lambda)!}} {}_{-\lambda}Y^*_{\ell m}(\Omega_k) 2\pi (\delta_{\lambda,2}\xi_R + \delta_{\lambda,-2}\xi_L) \frac{1}{2}(-i)^l \times \\
&\times \int_{\eta_i}^{\eta_0} d\eta h'(\eta,k) \frac{j_\ell[k(\eta_0-\eta)]}{k^2(\eta_0-\eta)2} (\ell-1)\ell(\ell+1)(\ell+2) \Big[\delta_{2,\lambda} + \frac{(\ell-|\lambda|)!}{(\ell+|\lambda|)!} \delta_{\lambda,-2} \Big] = \\
&= \int \frac{d^3k}{(2\pi)^3} 4\pi(-i)^\ell \sum_{\lambda=\pm 2} \xi_\lambda(\vec{k}) {}_{-\lambda}Y^*_{\ell m}(\Omega_k) \int_{\eta_i}^{\eta_0} d\eta h'(\eta,k) \frac{j_\ell[k(\eta_0-\eta)]}{k^2(\eta_0-\eta)^2} \frac{1}{4} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}}.
\end{aligned}
\tag{2.7.21}
$$

We cannot use the mean value of the coefficients $\Gamma_{\ell m}$ to compare theory and experiments, because their origin is the quantum fluctuation of the metric, which has a null expectation value, and so we will focus to the $\Gamma_{\ell m}$ variance $\tilde{C}_\ell$ defined as

$$
\langle \Gamma_{\ell m} \Gamma_{\ell' m'} \rangle \equiv \delta_{\ell\ell'} \delta_{mm'} \tilde{C}_\ell, \tag{2.7.22}
$$

which is the analogous of the angular power spectrum C_ℓ used for the CMB.
The physical meaning of these coefficients is the following: suppose having measured the distribution function perturbation $\Gamma(\eta_0, q, \hat{n})$, then by multiplying the experimental values by $Y^*_{\ell m}(\hat{n})$ and integrating over the direction of detection, we find the experimental values for the $\Gamma_{\ell m}$'s.
If we assume that the variance is insensitive to m[3], thus for a fixed ℓ the coefficients of the spherical harmonics expansion follow the same distribution and we have a statistic sample of $2\ell+1$ data. Then we can show that the mean value of such a sample $\bar{\Gamma}_{\ell m}$ is null:

$$\bar{\Gamma}_{\ell m} = \frac{1}{2\ell+1}\sum_{m'} \Gamma_{\ell m'} \approx 0, \tag{2.7.23}$$

and we can define a new sample of $2\ell+1$ variances $\tilde{C}_{\ell,m}$[4], one for each m.
We define the cosmic variance as the ratio between the standard deviation[5] of the $\tilde{C}_{\ell m}$, that we call $\Delta\tilde{C}_\ell$, and the expectation value of $\tilde{C}_\ell$. We can easily see that it goes qualitatively as

$$\left(\frac{\Delta\tilde{C}_\ell}{\tilde{C}_\ell}\right)_{cosmic\ \ variance} = \sqrt{\frac{2}{2\ell+1}}. \tag{2.7.24}$$

Hence the bigger the ℓ the lower the uncertainty in knowing the variance.
Now, we assume that the primordial power spectra have the form [6]

$$\begin{aligned}
\langle\Gamma(\eta_i, \vec{k}, q)\Gamma^*(\eta_i, \vec{k}', q)\rangle =& (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')P^{(I)}(k,q)\frac{2\pi^2}{k^3},\\
\langle\zeta(\vec{k})\zeta^*(\vec{k}')\rangle =& (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')P^{(0)}(k)\frac{2\pi^2}{k^3},\\
\langle\xi_\lambda(\vec{k})\xi^*_{\lambda'}(\vec{k}')\rangle =& (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')P^{(\lambda)}(k)\delta_{\lambda\lambda'}\frac{2\pi^2}{k^3}.
\end{aligned} \tag{2.7.25}$$

It is reasonable to suppose that the different contributions to Γ are uncorrelated, because they come from different origins, and so using Eqs. (2.7.6), (2.7.7) and (2.7.21) we obtain

---

[3]This is the case for the correlators of $\Gamma^{(S)}_{\ell m}$ and $\Gamma^{(T)}_{\ell m}$: if we look at Eq. (2.7.2) we see that the scalar and the tensor terms depend on $\hat{n}$ only through the combination $\hat{n}\cdot\hat{k}$, thus we can conclude that they have statistically isotropic angular correlators [24].

[4]In this case the m index does not mean that the variance depends on the direction in the sky we are looking, it identifies only the m-element of the sample.

[5]For a sample of x_i values we define the standard deviation as $\sigma = \sqrt{\frac{\sum_i(x_i^2-\bar{x}^2)}{N}}$, and we stress that $\sigma \sim \sqrt{\frac{1}{N}}$, where N is the number of data in the sample. This fact is important to understand the behaviour of the cosmic variance.

[6]The second one and the third one are defined by Eqs. (1.6.59) and (1.6.58) respectively.

three different correlators:

$$\begin{aligned}
\frac{\tilde{C}_{\ell,I}(\eta_0,q)}{4\pi} =& \frac{1}{4\pi}\langle \Gamma^{(I)}_{\ell m}(\eta_0,q)\Gamma^{(I)^*}_{\ell m}(\eta_0,q)\rangle = \\
=& \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Gamma(\eta_i,\vec{k},q)\Gamma^*(\eta_i,\vec{k}',q)\rangle j_\ell[k(\eta_0-\eta_i)]j_\ell[k'(\eta_0-\eta_i)]\times \\
& \times (4\pi)(-i)^\ell[(-i)^\ell]^* Y^*_{\ell m}(\hat{k})Y_{\ell m}(\hat{k}') = \\
=& \int \frac{d^3k}{(2\pi)^3}|Y_{\ell m}(\hat{k})|^2 P^{(I)}(k,q)\frac{2\pi^2}{k^3}4\pi j^2_\ell[k(\eta_0-\eta_i)] = \\
=& \int \frac{dk}{k}P^{(I)}(k,q)j^2_\ell[k(\eta_0-\eta_i)], \\
\frac{\tilde{C}_{\ell,S}(\eta_0)}{4\pi} =& \int \frac{dk}{k}P^{(0)}(k)\Big\{T_\phi(\eta_i,k)j_\ell[k(\eta_0-\eta_i)]+ \\
& + \int_{\eta_i}^{\eta_0} d\eta\Big[T'_\phi(\eta,k)+T'_\psi(\eta,k)\Big]j_\ell[k(\eta_0-\eta)]\Big\}^2 = \\
=& \int \frac{dk}{k}P^{(0)}(k)T^{(0)^2}_\ell(\eta_i,\eta_0,k), \\
\frac{\tilde{C}_{\ell,T}(\eta_0)}{4\pi} =& \sum_{\alpha=\pm 2}\int \frac{dk}{k}P^{(\alpha)}(k)\left[\int_{\eta_i}^{\eta_0} d\eta h'(\eta,k)\frac{j_\ell[k(\eta_0-\eta)]}{k^2(\eta_0-\eta)^2}\frac{1}{4}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\right]^2 = \\
=& \sum_{\alpha=\pm 2}\int \frac{dk}{k}P^{(\alpha)}(k)T^{(\alpha)^2}_\ell(\eta_i,\eta_0,k).
\end{aligned} \tag{2.7.26}$$

The main purpose of this thesis is to evaluate these correlators for a wide range of values of ℓ. In addition, we would like to understand the role neutrinos play in these angular power spectra, evaluating the anisotropies in both the cases in which there were no neutrinos in the Universe, and in the physical case of three neutrino generations.
The angular power spectra $\tilde{C}_\ell$'s are described by equations analogues to ones found for the angular power spectra of the CMB [60, 61, 62]. We expect then a priori to observe similiar features in the case of the SGWB. In the following chapters we will study the evolutions for the scalar and the tensor metric perturbations, in order to understand better their contribution to the power spectra. Qualitatively, we will expect for the scalar contribution large values of the angular power spectra for small ℓ, determined by effect analogues to the ISW effect, and a central peak due to modes that crossed the horizon around the time of equality between matter and radiation. For the tensor contribution, on the other hand, we expect constant values for small ℓ, and smaller and decreasing values for larger ℓ.

Capitolo 3

Effect of neutrinos on tensor modes evolution

3.1 Evolution of the tensor perturbations

In the previous chapter we have seen how to describe the contribution to the gravitational waves anisotropies by tensor perturbations along the path of the gravitons during their propagation; to evaluate such a contribution we need to know the function $h(\eta, k)$ from the epoch of neutrino decoupling, which we define as η_i, until the present epoch, η_0. In this chapter we evaluate such a contribution, observing how the presence of neutrinos in the Universe can affect the dynamics. We will see specifically that three neutrino generations cause a damping on the amplitude of the tensor modes, underlying the regimes in which such a damping is negligible and the ones in which it can be up to 35%. The general idea is that, for the decomposition theorem, discussed in A.1, the scalar, the vector and the tensor modes are decoupled at linear order in perturbation theory, so we can consider the Einstein equations for each mode independently. Since here we will focus on the tensor evolution, we can take only the transverse-traceless part of the Einstein tensor $G_{\mu\nu}$ and of the stress-energy tensor $T_{\mu\nu}$. Under these prescriptions, it can be shown that the only non null contribution to $T_{\mu\nu}$ is provided by decoupled radiation (all relativistic particles which freely stream), therefore the contribution of both neutrinos and photons should be included in principle, but electromagnetic radiation decouples during the matter dominated epoch, then it starts giving a contribution only when its energy density fraction is negligible with respect to the total one. So we can take into account only the neutrino contribution which will mainly contribute during the radiation era. The procedure consists in finding a solution of the Boltzmann equation for the neutrinos, obtaining an expression for their distribution function in terms of the tensor perturbations of the FLRW metric. Such a solution will be then used to write an explicit expression (in terms of the h_{ij}) for the neutrino stress-energy tensor, which will lead to an integro-differential Einstein equation for the transverse-traceless tensor. After this, we will solve such an equation in two regimes: first of all we will keep into account only the small scales, i.e. the modes which entered the horizon during the radiation era, and then we will try to find out a solution for general wavelengths. What we will find is an analytical solution for the evolution of small scales and a numerical solution for general wavelengths. After a discussion about which scales (multipoles ℓ) are more affected by this effect, we will see that the small scales are the ones that perceive mostly the presence of neutrinos: in particular they will feel a damping in their squared amplitude and a tiny phase shift in their oscillations.

3.2 Damping effects in the wave equation

Because of the decomposition theorem, we know that the tensor modes evolve independently from the scalar and the vector ones. In this chapter we focus only on the tensor perturbation of the full perturbed FLRW metric

$$g_{\mu\nu} = a^2(\eta)\begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} + h_{ij}(\eta,\vec{x}) \end{pmatrix}, \qquad g^{\mu\nu} = \frac{1}{a^2(\eta)}\begin{pmatrix} -1 & 0 \\ 0 & \delta^{ij} - h^{ij}(\eta,\vec{x}) \end{pmatrix}. \tag{3.2.1}$$

The stress-energy tensor can be written it in terms of an isotropic part, the leading order term, plus an anisotropic perturbation term:

$$\begin{aligned} T_{ij}(\eta,\vec{x}) =& \bar{p}(\eta) g_{ij}(\eta,\vec{x}) + a^2\pi_{ij}(\eta,\vec{x}), \\ T^i_j(\eta,\vec{x}) =& g^{ik}T_{kj}(\eta,\vec{x}) = \bar{p}(\eta)\delta^i_j + \frac{1}{a^2}\delta^{ik}a^2\pi_{kj}(\eta,\vec{x}) = \bar{p}(\eta)\delta_{ij} + \pi_{ij}(\eta,\vec{x}). \end{aligned} \tag{3.2.2}$$

We recall that when we speak about transverse-traceless perturbations, we mean that $h_{ii} = \partial_i h_{ij} = 0$ and $\pi_{ii} = \partial_i \pi_{ij} = 0$.
In A.3, we have seen that if we have a non null contribution from the anisotropic part of the stress-energy tensor, π_{ij}, the Einstein equations lead to the following equation for the tensor fluctuations:

$$h''_{ij}(\eta,\vec{x}) + 2\frac{a'}{a}h'_{ij}(\eta,\vec{x}) - \frac{1}{a^2}\nabla^2 h_{ij}(\eta,\vec{x}) = 16\pi G\pi_{ij}(\eta,\vec{x}). \tag{3.2.3}$$

As already stated, the only important contribution to the anisotropic part of the stress-energy tensor is given by the neutrinos (at the end of this section we will explicitly show why it is negligible for the photons) and, in order to calculate that, we introduce the neutrino distribution function $F(\eta,\vec{x},\vec{q})$, for whom we will study the evolution through a Boltzmann equation. We will follow the same approach used for the gravitons in Section 2.4, working under the two following assumptions:

- we are assuming that neutrinos are massless, $m_\nu = 0$, neglecting their small masses of the order of the eV, which would have caused some modifications in the definitions of the momenta with respect to the gravitons, modifying a little the structure of the Boltzmann equation;

- we are assuming that all the modes of physical interest re-entered the horizon after neutrino decoupling, so when neutrinos started freely-streaming, the modes with smaller wavelenghts will simply follow a transverse-traceless Einstein equation with a null anisotropic stress and it will be shown later on that they are damped by the expansion of the Universe in a very short time. An important remark is that we know that h_{ij} remains constant when it crosses the horizon and it remains time-independent until it re-enters the horizon [26]. Therefore we are legitimated to not care about the behaviour of the solutions for times very far from the horizon crossing and we can take as initial time to integrate the solutions the conformal time corresponding to neutrino decoupling, η_i.

Thus we can write down the Boltzmann equation for collisionless and massless particles, obtaining an equation identical to Eq. (2.4.17), without keeping into account the scalar perturbations for the decomposition theorem:

$$\frac{\partial F(\eta, \vec{x}, q, \hat{m})}{\partial \eta} + \frac{\partial F(\eta, \vec{x}, q, \hat{m})}{\partial x^i} m^i - \frac{1}{2} h'_{jk} m^j m^k q \frac{\partial F(\eta, \vec{x}, q, \hat{m})}{\partial q} = 0, \tag{3.2.4}$$

where $\hat{m}$ is the momentum direction for neutrinos and q is the neutrinos comoving three-momentum. Now we are ready to discuss the main features of the neutrino phase-space distribution in order to find out the correct order-by-order expressions for the Boltzmann equation.
We notice that if neutrinos started freely-streaming at η_i, then $F(\eta_i, \vec{x}, \vec{q})$ is a Fermi-Dirac distribution, because neutrinos were at thermal equilibrium with electrons and positrons[1]

$$F(\eta_i, \vec{q}) = F_0(\eta_i, q) = \frac{1}{(2\pi)^3} \frac{1}{e^{\frac{\sqrt{g^{ij} p_i p_j}}{k_B T}} + 1} = \frac{1}{(2\pi)^3} \frac{1}{e^{\frac{q}{k_B T a}} + 1}, \tag{3.2.5}$$

where $F_0(\eta_i, q)$ means a term of order 0 in the perturbations.
The distribution function at the conformal time η can be written as the sum of a leading homogeneous and isotropic term, $\bar{F}(\eta, q)$, plus a first order term in perturbation, $\delta F(\eta, \vec{x}, \vec{q})$:

$$F(\eta, \vec{x}, \vec{q}) = \bar{F}(\eta, q) + \delta F(\eta, \vec{x}, \vec{q}), \tag{3.2.6}$$

with $\delta F(\eta_i, \vec{x}, \vec{q}) = 0$. So the equation at the zero-order becomes

$$\frac{\partial \bar{F}(\eta, q)}{\partial \eta} = 0 \rightarrow \bar{F}(\eta, q) = F_0(\eta_i, q) \equiv F_0(q), \tag{3.2.7}$$

and this means that the leading term of the distribution does not change implying that the zero-order neutrino density scales as a^{-3}:

$$n_\nu = \int dp_1 dp_2 dp_3 \bar{F}(\eta, q) = \int \frac{d^3 q}{a^3} F_0(q) = \frac{1}{a^3} \int d^3 q F_0(q) \sim \frac{1}{a^3}, \tag{3.2.8}$$

where $\vec{p}$ is the physical momentum, the one over which we integrate when we evaluate the quantities of physical interest. Moreover, by observing that $\bar{F}(\eta, q)$ is time-independent, we can deduce an explicit relation between a and T: because every time dependence in

[1]The definitions of q, as already stressed, is the one given by Eqs. (2.4.2), $q \equiv ap$.

the Fermi-Dirac distribution written above is embedded in $a(\eta)T(\eta)$, $a \sim T^{-1}$ in order to have a constant distribution[2].
An useful tool to solve the Boltzmann equation is to move to the Fourier space[3], where the equation assumes the form

$$\delta F'(\eta, \vec{k}, \vec{q}) + ik\mu\delta F(\eta, \vec{k}, \vec{q}) = \frac{1}{2}q\frac{dF_0(q)}{dq}h'_{jk}(\eta, \vec{k})m^j m^k, \tag{3.2.9}$$

which solution reads:

$$\delta F(\eta, k, \mu, \vec{q}) = \frac{1}{2}q\frac{dF_0(q)}{dq}m^j m^k \int_{\eta_i}^{\eta} d\eta' e^{ik\mu(\eta'-\eta)}h'_{jk}(\eta', \vec{k}), \tag{3.2.10}$$

where all the dependence on $\hat{k}$ is embedded in $\mu = \hat{k}\cdot\hat{m}$. This solution is important because it allows to write down the explicit form for the stress-energy tensor, which is, according to the definition given in Eq. (A.2.15),

$$T^i_{j,\nu}(\eta, \vec{x}) \equiv \frac{g_\nu}{\sqrt{-g}} \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} F(\eta, \vec{x}, \vec{q}) \frac{p^i p_j}{p^0}, \tag{3.2.11}$$

where the quantity g represents the number of internal degrees of freedom of the particle species considered, as the helicity stases.
The computation is quite complicate, because for almost each of the various terms we need to take into account the zero order and the first order in the perturbations. We compute the various terms separately starting from the inverse of the square root of the determinant of the metric:

$$\frac{1}{\sqrt{-g}} = \frac{1}{\sqrt{-Det(g_{\mu\nu})}} = \frac{1}{\sqrt{-a^8 Det\begin{pmatrix}-1 & 0 \\ 0 & \delta_{ij}+h_{ij}\end{pmatrix}}} = \frac{1}{a^4}, \tag{3.2.12}$$

where we have used the following expansion for the determinant at the first order:

$$\begin{aligned} Det\left(\delta_{ij}+h_{ij}\right) =& Det\begin{pmatrix}1+h_{11} & h_{12} & h_{13} \\ h_{21} & 1+h_{22} & h_{23} \\ h_{31} & h_{32} & 1+h_{33}\end{pmatrix} = \\ =&(1+h_{11})Det\begin{pmatrix}1+h_{22} & h_{23} \\ h32 & 1+h_{33}\end{pmatrix} + (h_{21}+h_{31})O(h) = \\ =&Det(\delta_{ij}) + h_{ii} = Det(\delta_{ij}) = 1, \end{aligned} \tag{3.2.13}$$

where $O(h)$ is a term at least of the first order in the perturbations and h_{ii} is null because, by definition, h_{ij} is a traceless tensor. The remaining terms to compute are the expansion

[2]Another way to see that is recalling that the number density of relativistic particles, as neutrinos, scales as T^3 and then it is immediate to se that $T \sim a^{-1}$.

[3]For the Fourier transform we use the convention introduced in Eq. (2.4.21).

at the first order of the product of the momenta, for which we can use Eqs. (2.4.4) and (2.4.5),

$$\begin{aligned}\frac{p^i p_j}{p^0} =&\Big(1-\frac{1}{2}h_{ln}m^l m^n\Big)m^i q\Big(1-\frac{1}{2}h_{ln}m^l m^n\Big)\Big(m^j+h_{jk}m^k\Big)=\\ =&q\Big(1-h_{ln}m^l m^n\Big)m^i\Big(m^j+h_{jk}m^k\Big),\end{aligned} \tag{3.2.14}$$

and the form of the distribution function in the real space (because the definition of the stress-energy tensor is given in such a space, we will select only later its $\vec{k}$ mode in the Fourier space)

$$\begin{aligned}F(\eta,\vec{x},\vec{q}) =&F_0(q)+\delta F(\eta,\vec{x},\vec{q}) =\\ =&F_0(q)+\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\frac{1}{2}q\frac{dF_0(q)}{dq}m^j m^k\int_{\eta_i}^{\eta}d\eta' e^{ik\mu(\eta'-\eta)}h'_{jk}(\eta',\vec{k}).\end{aligned} \tag{3.2.15}$$

The last step consists in determining the change of variables of integration, from $dp_1dp_2dp_3$ to $dq^1dq^2dq^3$, recalling that for $\vec{p}=f(\vec{q})\vec{q}$ the change of variables is given by $dp_1dp_2dp_3 = |Det\big(f(\vec{q})\big)|dq^1dq^2dq^3$.
By lowering the spatial index of the physical three-momentum p^i described by Eq. (2.4.4) we have

$$p_i = g_{ij}p^j = a^2(\delta_{ij}+h_{ij})\frac{q}{a^2}\Big(1-\frac{1}{2}h_{jk}m^j m^k\Big)m^i = (\delta_{ij}+h_{ij})\Big(1-\frac{1}{2}h_{ln}m^l m^n\Big)q^i \tag{3.2.16}$$

and so the explicit change of variables of integration is

$$\begin{aligned}dp_1dp_2dp_3 =&\left|Det\left((\delta_{ij}+h_{ij})\Big(1-\frac{1}{2}h_{ln}m^l m^n\Big)\right)\right|d^3q =\\ =&Det(\delta_{ij}+h_{ij})\Big(1-\frac{1}{2}h_{ln}m^l m^n\Big)^3 d^3q = \Big(1-\frac{3}{2}h_{ln}m^l m^n\Big)d^3q.\end{aligned} \tag{3.2.17}$$

By combining all the terms together we get the final expression for Fourier transform of the stress-energy tensor:

$$\begin{aligned}T^i_{j,\nu}(\eta,\vec{k}) =&\frac{g_\nu}{a^4}\int\frac{d^3q}{(2\pi)^3}\Big(1-\frac{3}{2}h_{ln}(\eta,\vec{k})m^l m^n\Big)\times\\ &\times q\big(m^i m^j+m^i h_{jk}(\eta,\vec{k})m^k\big)\big(1-h_{ln}(\eta,\vec{k})m^l m^n\big)\times\\ &\times\Big[F_0(q)+\frac{1}{2}q\frac{dF_0(q)}{dq}m^j m^k\int_{\eta_i}^{\eta}d\eta' e^{ik\mu(\eta'-\eta)}h'_{jk}(\eta',\vec{k})\Big]=\\ =&\frac{g_\nu}{a^4}\int\frac{d^3q}{(2\pi)^3}\Big(1-\frac{3}{2}h_{ln}(\eta,\vec{k})m^l m^n\Big)q\big(m^i m^j+m^i h_{jk}(\eta,\vec{k})m^k\big)\\ &\times\Big[F_0(q)-F_0(q)h_{ln}(\eta,\vec{k})m^l m^n+\\ &+\frac{1}{2}q\frac{dF_0(q)}{dq}m^l m^n\int_{\eta_i}^{\eta}d\eta' e^{ik\mu(\eta'-\eta)}h'_{ln}(\eta',\vec{k})\Big].\end{aligned} \tag{3.2.18}$$

Before going on with the explicit computation we should define the projectors that isolate the transverse and traceless part of T_{ij}, i.e. the anisotropic stress π_{ij} which is involved in the damping of the gravitational waves. The usual definition is given in the coordinate space and the projectors are

$$P^{is}_{rj} \equiv \delta^i_r\delta^s_j - \delta^i_r(\nabla^2)^{-1}\partial^s\partial_j - \delta^s_j(\nabla^2)^{-1}\partial^i\partial_r + \frac{1}{2}[(\nabla^2)^{-1}]^2\partial^i\partial_r\partial^s\partial_j - \frac{1}{2}\delta^i_j\delta^s_r + \frac{1}{2}\delta^i_j(\nabla^2)^{-1}\partial^s\partial_r + \\ + \frac{1}{2}\delta^s_r(\nabla^2)^{-1}\partial^i\partial_j, \tag{3.2.19}$$

while in the Fourier space they can be written as

$$\tilde{P}^{is}_{rj} \equiv \delta^i_r\delta^s_j - \delta^i_r\hat{k}^s\hat{k}_j - \delta^s_j\hat{k}^i\hat{k}_r + \frac{1}{2}\hat{k}^i\hat{k}_r\hat{k}^s\hat{k}_j - \frac{1}{2}\delta^i_j\delta^s_r + \frac{1}{2}\delta^i_j\hat{k}^s\hat{k}_r + \frac{1}{2}\delta^s_r\hat{k}^i\hat{k}_j. \tag{3.2.20}$$

As we have seen in the previous chapter we consider the following two polarization tensors:

$$\epsilon_{ij,R}(\hat{k}) = \frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_{ij,L}(\hat{k}) = \frac{1}{2}\begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.2.21}$$

they are transverse-traceless tensors with respect to a vector $\hat{k}$ aligned with the z-axis. In our further integration we will always consider the $\{\hat{u}, \hat{v}, \hat{k}\}$ basis defined in Section 2.7.
We would like to remove from the Boltzmann equation the dependence on the spatial indices, focusing only on the amplitude of h_{ij}, in order to simplify the dependences on m^i, which we could not be canceled in other ways. To do that we use the relations

$$h_{ij} = h_R e_{ij,R} + h_L e_{ij,L} \rightarrow h_{ij}e_{ij,R/L} = \frac{1}{4}h_{L/R}Tr\begin{pmatrix} 2 & \mp 2i & 0 \\ \pm 2i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow h_{L/R} = h_{ij}e_{ij,R/L}, \tag{3.2.22}$$

so the only part of the stress-energy tensor which contributes to the variation of $h_{ij,\lambda}$ will be
$\pi_\lambda = \pi_{ij}e_{ij,-\lambda}$, thus we will solve the equation for π_λ and not for π_{ij}, we can pass from the one to another simply by multiplying for the corrispective polarization tensor.
In practice, after we have found the transverse and traceless part of $T^i_{j,\nu}$ we should also want to express the result in function of the component $\epsilon_{ij,\lambda}$, so we are looking for the total projection

$$e_{ij,\lambda}\tilde{P}^{is}_{rj}T^r_{s,\nu} = e_{ij}\delta^i_r\delta^s_j T^r_{s,\nu}.; \tag{3.2.23}$$

this choice has simplified a lot the calculations: we are left only with this term because each δ^i_j term is cancelled by the fact that $e_{ij,\lambda}$ is traceless and each $\hat{k}^i$ or $\hat{k}_j$ term is cancelled by the fact that $e_{ij,\lambda}$ is transverse. We end up with the expression

$$\begin{aligned}\pi(\eta,\vec{k})_{-\lambda} =& e_{ij,\lambda}\tilde{P}^{is}_{rj}T^{r}_{s,\nu} = \frac{g_\nu}{a^4}\int\frac{d^3q}{(2\pi)^3}q\big(e_{ij,\lambda}m^im^j + m^ie_{ij,\lambda}e_{jk,-\lambda}h_{-\lambda}(\eta,\vec{k})m^k\big)\times\\ &\times\Big[F_0(q) - \sum_{\lambda'}\frac{5}{2}F_0(q)h_{\lambda'}(\eta,\vec{k})e_{jk,\lambda'}m^jm^k+\\ &+\sum_{\lambda'}\frac{1}{2}q\frac{dF_0(q)}{dq}m^jm^ke_{jk,\lambda'}\int_{\eta_i}^{\eta}d\eta' e^{ik\mu(\eta'-\eta)}h'_{\lambda'}(\eta,\vec{k})\Big]=\\ =&\frac{g_\nu}{a^4}\int\frac{d^3q}{(2\pi)^3}q\Big[F_0(q)e_{ij,\lambda}m^im^j - \frac{5}{2}F_0(q)h_{-\lambda}(\eta,\vec{k})(e_{jk,\lambda}m^jm^k)(e_{jk,-\lambda}m^jm^k)+\\ &+\frac{1}{2}q\frac{dF_0(q)}{dq}(\eta,\vec{k})(e_{jk,\lambda}m^jm^k)(e_{jk,-\lambda}m^jm^k)\times\\ &\times\int_{\eta_i}^{\eta}d\eta' e^{ik\mu(\eta'-\eta)}h'_{-\lambda}(\eta,\vec{k})+\\ &+F_0(q)m^ie_{ij,\lambda}e_{jk,-\lambda}h(\eta,\vec{k})m^k\Big].\end{aligned} \tag{3.2.24}$$

In the basis we have chosen we can see that the direction of $\vec{q}$ has the simple form

$$\hat{m} = (sin\theta\, cos\phi,\, sin\theta\, sin\phi,\, cos\theta), \tag{3.2.25}$$

and from this we immediately obtain that

$$e_{ij,\lambda}m^im^j = \frac{1}{2}sin^2\theta e^{-\lambda i\phi}. \tag{3.2.26}$$

This is the reason why when we consider the term "quadratic" in $e_{ij,\lambda}$ we consider the two tensors with different polarizations, in the other cases we would have had a null result after the integration over ϕ, while in this way we make the complex exponential null. For the same reason the first term in the integral is null, because we have an angular integration over a periodic function for two times the period and then that contribute cancels.
Before computing the final expression for the tensor we calculate separately the integral over the spherical coordinates:

$$\begin{aligned}\int_{-1}^{+1}d\mu\int_0^{2\pi}d\phi(\eta,\vec{k})(e_{jk,\lambda}m^jm^k)(e_{jk,-\lambda}m^jm^k) =& \int_{-1}^{+1}d\mu\int_0^{2\pi}d\phi\frac{1}{4}(1-\mu^2)^2 =\\ =&\frac{1}{4}\int_{-1}^{+1}d\mu(1-2\mu^2+\mu^4)(2\pi) =\\ =&\frac{1}{4}\frac{16}{15}(2\pi).\end{aligned} \tag{3.2.27}$$

There is another non-trivial term to evaluate, which depends on the non trivial matrix product

$$\begin{aligned}
e_{ik,\lambda}e_{kj,-\lambda} =&\frac{1}{4}\left[\begin{pmatrix} 1 & -(-1)^{\delta_{\lambda,2}}i & 0\\ -(-1)^{\delta_{\lambda,2}}i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} 1 & -(-1)^{\delta_{-\lambda,2}}i & 0\\ -(-1)^{\delta_{-\lambda,2}}i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\right]_{ij} =\\
=&\frac{1}{2}\begin{pmatrix} 1 & (-1)^{\delta_{\lambda,2}}i & 0\\ -(-1)^{\delta_{\lambda,2}}i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij},
\end{aligned} \tag{3.2.28}$$

therefore the last non-trivial term to calculate is

$$\int_{-1}^{+1} d\mu \int_0^{2\pi} d\phi e_{ik,\lambda}e_{ij,-\lambda}m^k m^j = \int_{-1}^{+1} d\mu \int_0^{2\pi} d\phi \frac{1}{2}(1-\mu^2) = \frac{1}{2}\frac{4}{3}(2\pi). \tag{3.2.29}$$

Now we see that the total expression is (using the differentiation by parts for $\frac{dF_0(q)}{dq}$):

$$\begin{aligned}
\pi_\lambda(\eta,\vec{k}) =&\frac{g_\nu}{a^4}\int \frac{d^3q}{(2\pi)^3}q^2\frac{1}{4}(1-\mu^2)^2\frac{1}{2}\frac{dF_0(q)}{dq}\int_{\eta_i}^{\eta} d\eta' e^{ik\mu(\eta'-\eta)}h'_\lambda(\eta',\vec{k})+\\
&+\frac{1}{a^4}\int \frac{dq}{(2\pi)^3}q^3 h_\lambda(\eta,\vec{k})F_0(q)(2\pi)(\frac{5}{2}\frac{1}{4}\frac{16}{15}-\frac{1}{2}\frac{4}{3})=\\
=&-\frac{g_\nu}{2}\frac{2\pi}{(4\pi)(2\pi)^3}\Big[\int dq q^4 4\pi F_0(q)\Big]\int_{-1}^{+1} d\mu(1-\mu^2)^2\int_{\eta_i}^{\eta} d\eta' e^{ik\mu(\eta'-\eta)}h'_\lambda(\eta',\vec{k}) =\\
=&-4\rho_\nu^{(0)}(\eta)\int_{\eta_i}^{\eta} d\eta'\left[\frac{1}{16}\int_{-1}^{+1} d\mu(1-\mu^2)^2 e^{ik\mu(\eta'-\eta)}\right]h'_\lambda(\eta',\vec{k}) =\\
=&-4\rho_\nu^{(0)}(\eta)\int_{\eta_i}^{\eta} d\eta' K(\eta-\eta')h'_\lambda(\eta',\vec{k}),
\end{aligned} \tag{3.2.30}$$

where we have defined the Kernel

$$K[k(\eta-\eta')] \equiv \frac{1}{16}\int_{-1}^{+1} d\mu(1-\mu^2)^2 e^{ik\mu(\eta'-\eta)}, \tag{3.2.31}$$

and the unperturbed neutrino energy density as

$$\rho_\nu^{(0)}(\eta) \equiv g_\nu \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} pF_0\big(q(p)\big) = \frac{g_\nu}{a^4}\int \frac{d^3q}{(2\pi)^3}qF_0(q). \tag{3.2.32}$$

The important result is that for the tensor modes we end up with the following equation (coming back to h_{ij}):

$$h''_{ij}(\eta) + 2\frac{a'(\eta)}{a(\eta)}h'_{ij}(\eta) + k^2 h_{ij}(\eta) = -24 f_\nu(\eta)\Big(\frac{a'(\eta)}{a(\eta)}\Big)^2\int_{\eta_i}^{\eta} d\eta' K[k(\eta-\eta')]h'_{ij}(\eta'), \tag{3.2.33}$$

where $f_\nu = \frac{\rho_\nu^{(0)}}{\bar{\rho}}$, it is the energy density fraction of neutrinos with respect to the total energy density. Remember also that the initial condition is $h'_{ij}(\eta_i) = 0$, because the tensor modes are constant outside the horizon and all the modes of interest entered after neutrino decoupling.
In the rest of the chapter we will use the following parametrization (which is possible because there is no mixing-term between the various elements of the tensor):

$$\begin{aligned} u(\eta) =& k\eta, \\ h_{ij}(u) =& h_{ij}(0)\chi(u). \end{aligned} \tag{3.2.34}$$

If all the modes were outside the horizon at the initial time, then $k\eta_i \ll 1$ and, for the purpose of the following sections, we can assume that $u = 0$, and so the initial conditions are $\chi(0) = 1$ and $\chi'(0) = 0$. The equation we want to solve is then

$$\chi''(u) + 2\frac{a'(u)}{a(u)}\chi'(u) + \chi(u) = -24 f_\nu(u)\Big(\frac{a'(u)}{a(u)}\Big)^2 \int_0^U dU K(u-U)\chi'(U). \tag{3.2.35}$$

Notice that we are not considering the specific polarization mode h_λ, but a generic one, χ, because, even if the two polarization modes start with different initial conditions (encoded in $h_{ij}(0)$), the equations for the damping are the same for the two modes, as we have seen, so we can proceed with the calculation of the damping factor in general.

3.3 Short wavelenght modes

We start considering wavelenghts so short that they re-entered the horizon, which corresponds to the condition $k > aH$, during the radiation-dominated era, though long after neutrino decoupled.
As stated at the end of the previous section, we can set $\eta_i \approx 0$, therefore also the cosmic time introduced in Eq. (2.3.1), related to the conformal time through Eq. (2.3.4), is almost zero. The zero of the time defined in such a way that $a(t) = \alpha t^{\frac{1}{2}}$, so it is quite simple to rescale the Hubble term in Eq. (3.2.35) in terms of u:

$$u = k\int_0^t dt' \frac{1}{a(t')} = k\int_0^t dt' \frac{t^{-\frac{1}{2}}}{\alpha} = \frac{2k}{\alpha} t^{\frac{1}{2}} = \frac{2ka}{\alpha^2} \rightarrow a = \frac{\alpha^2}{2k} u \rightarrow \frac{a'(u)}{a(u)} = \frac{1}{u}. \tag{3.3.1}$$

For three neutrino flavors the fraction of energy density remains constant respect to u, in fact $f_\nu(u) = f_\nu(0) = 0.40523$, during the radiation era, and so the equation becomes:

$$\chi''(u) + \frac{2}{u}\chi'(u) + \chi(u) = -\frac{24 f_\nu(0)}{u^2}\int_0^U dU K(u-U)\chi'(U). \tag{3.3.2}$$

It is easy to verify that the solution for the homogeneous equation is simply (just by substituting it in the equation and applying the initial conditions)

$$\chi(u) = \frac{sin(u)}{u}. \tag{3.3.3}$$

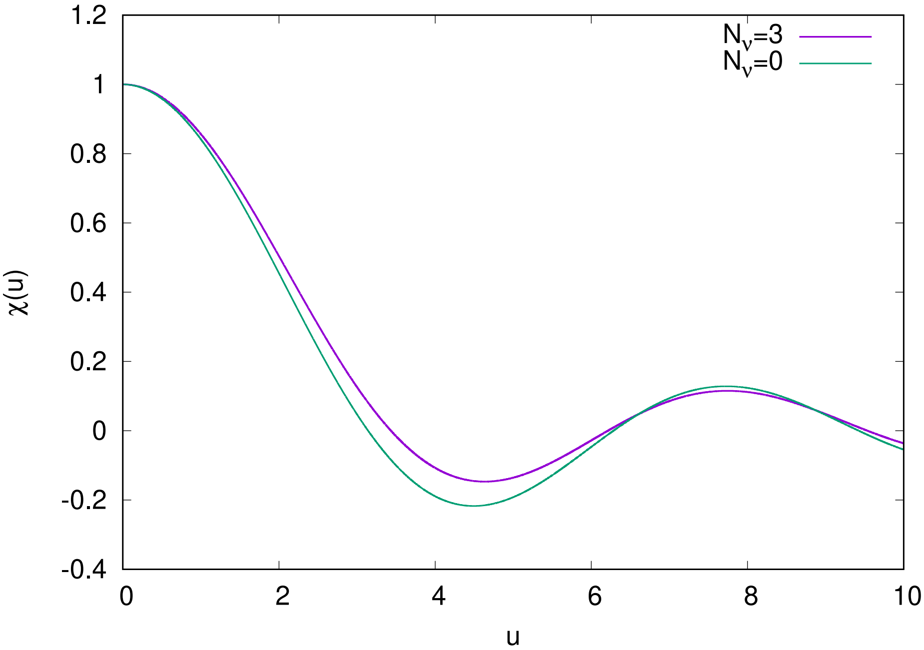

Figura 3.3.1: Plot of the numerical solution of Eq. (3.3.2) for $N_\nu = 3$ neutrino generations, which leads to a fractional energy density $f_\nu(0) = 0.40523$, and the homogeneous solution (3.3.3) for comparison. We have zoomed the $u \leq 10$ range, in order to show that the total solution approaches the homogeneous one for $u \ll 1$, and that it crosses a transient regime for $1 \leq u \leq 10$.

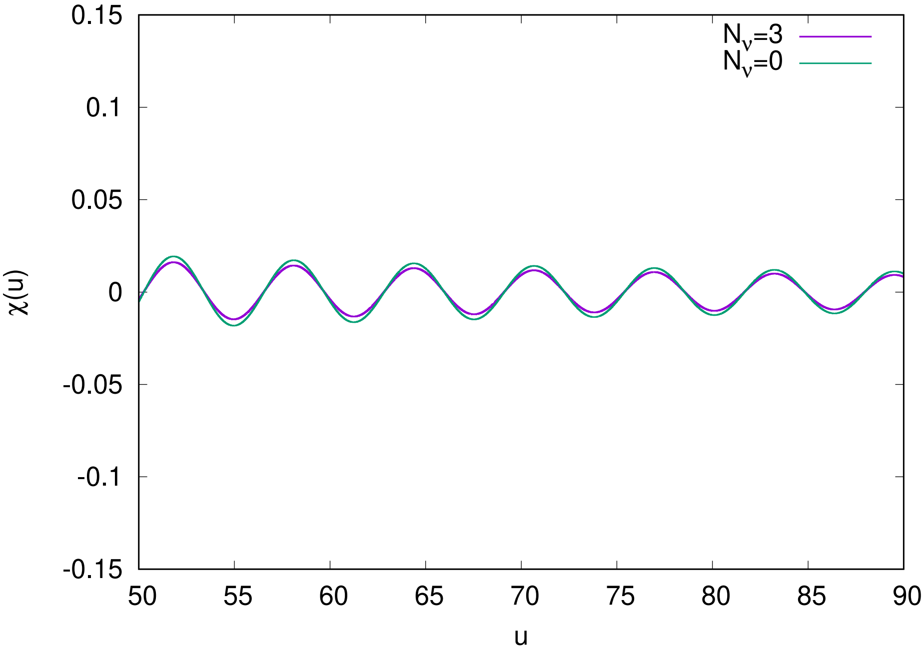

Figura 3.3.2: Behaviour of the solution of Eq. 3.3.2 for later times. In general both solutions follow a regime described by $\frac{sin u}{u}$: we have two oscillating solutions that reduce more and more their amplitudes (for $u \approx 10$ the amplitudes are of order 10^{-1}, as shown in the previous plot, while now, for $u \approx 50$, the amplitudes are of order 10^{-2}). It is clear that the tensor modes are damped in presence of neutrinos: the $N_\nu = 3$ solution is always smaller than the homogeneous one and, even if by eye we cannot observe that, there is a very tiny phase shift between the two. Doing a fit we can find the parameters of Eq. (3.3.4), which are $\delta = 0.007$ and $A = 0.80$.

The full solution is found through a numerical computation and the results are reported in Figures 3.3.1 and 3.3.2. We can describe the solution in an analytical way for mainly two regimes: for $u \ll 1$ we see that it pretty follows the homogeneous solution written above, while for $u \gg 1$ there is a very small phase shift δ in the solution, due to the presence of an approximately constant source term, and there is a change in the amplitude of the modes; the new amplitude is $A = 0.8026$, i.e. the solution at large u can be written as

$$\chi(u) = A\frac{sin(u+\delta)}{u}. \tag{3.3.4}$$

We can give a rough estimate of the effect that we should observe in the $\tilde{C}_\ell$ spectra, introduced in Eq. (2.7.26): the tensor correlator $\tilde{C}_{\ell,T}$ depends on the integral over η of $\left|h'(\eta,k)\right|^2 \approx |A|^2 \left|h'_{N_\nu=0}(\eta,k)\right|^2$ (we have to consider also the other regimes of the full solution, but quantitatively this is what happens for most of the integration time which goes from $\eta_i \approx 0$ to $\eta_0 \gg 1$), hence we have a damping in the correlators which is equal to $1 - |A|^2 \approx 32\%$. Such a damping occurs in the CMB temperature too, we have a complete analogy between the photons and the gravitons case, therefore the C_ℓ coefficients are reduced by the same factor in presence of three neutrino generations.
The analogous effect due to the photons is negligible because they decoupled when their fractional energy density was much smaller than one, therefore they could not give a significative contribution to the anisotropic stress. Photons decoupled in fact at the so called last scattering surface, which occurred at a redshift $z_{l.s.} \approx 1090$. Knowing that the radiation-matter equality happened at $z_{EQ} \approx 3600$, photons started free-streaming when they were a subdominant contribution to the energy density of the Universe, i.e. $f_\gamma(\eta_{l.s.}) \ll f_\nu(\eta_i)$, therefore we can neglect their contribution.

3.4 General wavelenghts

In this last section, we want to provide a general solution for the tensor modes equation, valid for each k and that is equivalent to the one described in Eq. (3.3.4) in the large k regime. To deal with the general wavelenght case, we will make some rescalings in order to make the equations simpler.
First of all we will introduce the scale factor at the radiation-matter equality time, $a_{EQ} = a(t_{EQ})$, from which we can define the new temporal variable of the equation:

$$y(t) \equiv \frac{a(t)}{a_{EQ}}, \tag{3.4.1}$$

with the initial condition $y(0) = 0$. Recalling that, from (3.2.34), $du = k\frac{dt}{a}$, we can write the derivative of y with respect to u:

$$\frac{dy}{du} = \frac{dt}{du}\frac{da}{dt} = \frac{1}{a_{EQ}}\frac{a(t)}{k}\frac{da(t)}{dt} = \frac{a(t)}{ka_{EQ}}a(t)H(t) = \frac{a^2(t)}{ka_{EQ}}\sqrt{\frac{8\pi G}{3}[\rho_m(t) + \rho_r(t) + \rho_\lambda(t)]} =$$
$$= \frac{a^2(t)H_0}{ka_{EQ}}\sqrt{\frac{1}{\rho_c}\Big[\rho_{m,0}\Big(\frac{a_0}{a(t)}\Big)^3 + (\rho_{\nu,0} + \rho_{\gamma,0})\Big(\frac{a_0}{a(t)}\Big)^4 + \rho_\lambda\Big]}, \tag{3.4.2}$$

where with the "0" index we identify the quantities evaluated nowadays. We know that for non-relativistic matter the energy density scales as $\rho_m \sim a^{-3}$, while for relativistic matter it scales as $\rho_r \sim a^{-4}$. Now we want to find how to derive the redshift at which the matter-dominated era began, which is

$$\Omega_{m,EQ} = \Omega_{m,0}\Big(\frac{a_0}{a_{EQ}}\Big)^3 = \Omega_{r,EQ} = \Omega_{r,0}\Big(\frac{a_0}{a_{EQ}}\Big)^4 \to 1 + z_{EQ} = \frac{a_0}{a_{EQ}} = \frac{\Omega_{m,0}}{\Omega_{\gamma,0} + \Omega_{\nu,0}}, \tag{3.4.3}$$

and the redshift at which the dark energy gave the same contribution of the radiation (we identify it with the DE index):

$$\Omega_{r,DE} = \Omega_{r,0}\Big(\frac{a_0}{a_{DE}}\Big)^4 = \Omega_{\Lambda,DE} = \Omega_{\Lambda,0} \to 1 + z_{DE} = \frac{a_0}{a_{DE}} = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{\gamma,0} + \Omega_{\nu,0}}\right)^{\frac{1}{4}}. \tag{3.4.4}$$

From experimental observations we see that the redshift of the equivalence between matter and radiation is $z_{EQ} \approx 3600$, while the redshift at which the dark energy started to give bigger contributions to the total energy density respect to the radiation is $z_{DE} \approx 0.9$ and the redshift at which the photons decoupled is $z_{l.s.} \approx 1090$.
This legitimates us to neglect $\Omega_{\Lambda,0}$ from the previous calculations, because we are interested in $y(t)$ during the radiation-era, well before the contribution of the dark energy becomes dominant and when neutrinos give still some effects.

$$\begin{aligned}\frac{du}{dy} &= \frac{ka_{EQ}}{a^2H_0}\frac{1}{\sqrt{\Omega_{m,0}}\sqrt{\Big(\frac{a_0}{a_{EQ}}\Big)^3\big(\frac{a_{EQ}}{a}\Big)^3 + \frac{\Omega_{\gamma,0}+\Omega_{\nu,0}}{\Omega_{m,0}}\Big(\frac{a_0}{a_{EQ}}\Big)^4\big(\frac{a_{EQ}}{a}\Big)^4}} = \\ &= \frac{k}{a_0H_0}\frac{a_{EQ}^2}{a^2}\frac{a_0}{a_{EQ}}\frac{1}{\sqrt{\Omega_{m,0}}\sqrt{(1+z_{EQ})^3\big(\frac{1}{y}\big)^3 + (1+z_{EQ})^3\big(\frac{1}{y}\big)^4}} = \\ &= \frac{k(1+z_{EQ})}{y^2H_0a_0}\frac{y^2}{(1+z_{EQ})^{\frac{3}{2}}\sqrt{\Omega_{m,0}(1+y)}} = \\ &= \frac{k}{a_0H_0\sqrt{\Omega_{m,0}(1+z_{EQ})}}\frac{1}{\sqrt{1+y}} = \frac{Q}{\sqrt{1+y}},\end{aligned} \tag{3.4.5}$$

where we have defined parameter Q as

$$Q \equiv \frac{k}{a_0H_0\sqrt{\Omega_m(1+z_{EQ})}} = \sqrt{2}\frac{k}{k_{EQ}}, \tag{3.4.6}$$

using the definition of the wavelenght which crosses the horizon at the matter-radiation equivalence, $k_{EQ} \equiv a_{EQ} H_{EQ}$, with H_{EQ}, a_{EQ} the Hubble parameter and the scale factor at the matter-radiation equivalence respectively. The parameter Q is particularly important because it quantifies the scale we are looking at in this general wavelength discussion: to be more precise, the $Q \gg 1$ limit represents the small scales studied in the previous chapter, while now we are interested in the $Q \lesssim 1$ case.
In this way we can express, recalling that $y(0) = u(0) = 0$ (using Eqs. (3.2.34) and (3.4.1)), by solving a simple differential equation, u in term of y:

$$u(y) = \int_0^y d\tilde{y} \frac{Q}{\sqrt{1+\tilde{y}}} + u(0) = 2Q\sqrt{1+\tilde{y}}|_0^y = 2Q(\sqrt{1+y} - 1). \tag{3.4.7}$$

We can also write the fractional energy density of neutrinos for a generic epoch, using the fact that
$\rho_\nu(y_{EQ}) + \rho_\gamma(y_{EQ}) = \rho_m(y_{EQ})$:

$$\begin{aligned} f_\nu(y) =& \frac{\rho_\nu(y)}{\rho_\nu(y) + \rho_\gamma(y) + \rho_m(y)} = \frac{\rho_\nu(y_{EQ})\left(\frac{a_{EQ}}{a}\right)^4}{\rho_m(y_{EQ})\left(\frac{a_{EQ}}{a}\right)^3 + \left[\rho_\nu(y_{EQ}) + \rho_\gamma(y_{EQ})\right]\left(\frac{a_{EQ}}{a}\right)^4} = \\ =& \frac{\rho_\nu(y_{EQ})}{\rho_\nu(y_{EQ}) + \rho_\gamma(y_{EQ})} \frac{\frac{1}{y^4}}{\frac{1}{y^3} + \frac{1}{y^4}} = \frac{\rho_\nu(0)}{\rho_\nu(0) + \rho_\gamma(0)} \frac{1}{1+y} = \frac{f_\nu(0)}{1+y}, \end{aligned} \tag{3.4.8}$$

where in the last step we have assumed that $f_\nu(y) = cost.$ during the radiation era, because ρ_ν scales as a^{-4} as ρ_γ.
Now we can compute the various derivatives we have in the equation for χ:

$$\begin{aligned} \chi'(y) =& \frac{d\chi\big(u(y)\big)}{du} = \frac{\sqrt{1+y}}{Q} \frac{d\chi(y)}{dy}, \\ \chi''(y) =& \frac{\sqrt{1+y}}{Q} \frac{d}{dy}\left(\frac{\sqrt{1+y}}{Q} \frac{d\chi(y)}{dy}\right) = \frac{1}{Q^2}\left[(1+y)\frac{d^2\chi}{dy^2} + \frac{1}{2}\frac{d\chi}{dy}\right], \\ \frac{a'(y)}{a(y)} =& \frac{1}{a(y)} \frac{\sqrt{1+y}}{Q} \frac{da(y)}{dy} = \frac{\sqrt{1+y}}{Q} \frac{1}{a(y)} a_{EQ} = \frac{\sqrt{1+y}}{Qy}. \end{aligned} \tag{3.4.9}$$

After defining a new form for the Kernel introduced in Eq. (3.2.31) [4], as $K(y, y') = K[2kQ(\sqrt{1+y} - \sqrt{1+y'})]$, the equation takes the form

$$(1+y)\frac{d^2\chi}{dy^2} + \left(2\frac{1+y}{y} + \frac{1}{2}\right)\frac{d\chi}{dy} + Q^2\chi = -24\frac{f_\nu(0)}{y^2} \int_0^y dy' K(y, y') \frac{d\chi(y')}{dy'}. \tag{3.4.10}$$

To understand the behaviour of the function χ, in figure 3.4.1 we have plotted the solution of the above equation for $Q = 0.55$, which corresponds to $k/k_{EQ} = 0.389$.

[4] We say redefine because the dependence on the variables of $K(y, y')$ is different with respect to $K[k(\eta - \eta')]$, even if the function is the same, we identify both of them with K, with a little abuse of notation.

An important quantity which determines different features in the angular power spectrum for the CMB is χ', therefore to give an approximatively estimation of neutrinos effects we evaluate $|\chi'(y_{l.s.})|^2$, we compute it at the last scattering because the physical effect for photons started when they decoupled. This quantity in general is highly non trivial, because for a general Q the net effect will not be a simple damping as for the short wavelenght case.
When we consider, for instance, a multipole coefficient of order ℓ, the dominant contribution comes from the wave number $k \approx \frac{a_{l.s.}\ell}{d_{l.s.}}$, where $d_{l.s.}$ is the angular diameter distance of the surface at the last scattering:

$$\begin{aligned} d_{l.s.} =& a_{l.s.}\int_{t_{l.s.}}^{t_0} dt \frac{1}{a(t)} = a_{l.s.}\int_{a_{l.s.}}^{a_0} da \frac{1}{a^2 H} = \\ =& a_{l.s.}\int_{a_{l.s.}}^{a_0} da \frac{1}{a^2 H_0 \sqrt{\Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_r \left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda}} = \\ =& \frac{a_{l.s.}}{H_0}\int_{\frac{1}{1+z_{l.s.}}}^{1} dx \frac{a_0}{a_0^2 x^2} \frac{1}{\sqrt{\frac{\Omega_m}{x^3} + (1-\Omega_m)}} = \\ =& \frac{1}{H_0(1+z_{l.s.})}\int_{\frac{1}{1+z_{l.s.}}}^{1} dx \frac{1}{\sqrt{\Omega_m x + (1-\Omega_m)x^4}}, \end{aligned} \tag{3.4.11}$$

the radiation contribution to the total density is negligible with respect to matter and dark energy, while the dark energy contribution is the dominant one, even for a small period of time only.
We have also used the change of variable $x = \frac{a}{a_0} = \frac{1}{1+z}$.
Thus the multipole that receives the main contribution from wavelenghts that are just coming into the horizon at the matter-radiation equality is

$$\begin{aligned} \ell_{EQ} \equiv \frac{d_{l.s.} k_{EQ}}{a_{l.s.}} =& \frac{1}{H_0(1+z_{l.s.})}\int_{\frac{1}{1+z_{l.s.}}}^{1} dx \frac{1}{\sqrt{\Omega_m x + (1-\Omega_m)x^4}} \frac{a_{EQ}}{a_{l.s.}} H_0 \sqrt{2\Omega_m (1+z_{EQ})^3} = \\ =& \sqrt{2\Omega_m(1+z_{EQ})}\int_{\frac{1}{1+z_{l.s.}}}^{1} dx \frac{1}{\sqrt{\Omega_m x + (1-\Omega_m)x^4}} \approx 149, \end{aligned} \tag{3.4.12}$$

where we have used $\Omega_m = 0.3$ and $z_L = 1090$. Using these cosmological parameters in Eq. (3.4.6), we see that

$$Q = \sqrt{2}\frac{k}{k_{EQ}} = \sqrt{2}\frac{\frac{a_{l.s.}}{d_{l.s.}}\ell}{k_{EQ}} = \sqrt{2}\frac{\ell}{\ell_{EQ}} \approx \frac{\ell}{105}. \tag{3.4.13}$$

We want to express the evolution for the tensor modes χ in terms of y and of the parameter Q, therefore we need to know which is the physical meaning of Q to understand properly the equations. From the previous relation we understand that for $k \ll k_{EQ}$ we also have

that $\ell \ll \ell_{EQ}$, and thus $Q \ll 1$ (this is the large-scale case) when the modes re-entered the horizon during the matter-dominated era. Now we will focus on this particular regime.
If we look at the precise form of the Kernel introduced in Eq. (3.2.31) we have that

$$K(y,y') = \frac{1}{16}\int_{-1}^{+1} d\mu(1-\mu^2)^2 e^{i\mu 2Q(\sqrt{1+y}-\sqrt{1+y'})}, \tag{3.4.14}$$

where the exponential is equal to 1 for large-scales, because $Q \ll 1$, and so the Kernel assumes the constant value

$$K(y,y') = \frac{1}{16}\int_{-1}^{+1} d\mu(1+\mu^4-2\mu^2) = \frac{1}{15}. \tag{3.4.15}$$

Thus the differential equation (3.4.10) assumes a simpler form:

$$(1+y)\frac{d^2\chi}{dy^2} + \left[\frac{1}{2} + \frac{2(1+y)}{y}\right]\frac{d\chi}{dy} + Q^2\chi = -\frac{24 f_\nu(0)}{15y^2}\chi(y). \tag{3.4.16}$$

Because of $Q \ll 1$, we can expand the solution in powers of Q and it is immediate to see that $\chi = 1 = cost.$ is the solution for the zero-order term (the initial conditions given immediately after Eq. (3.2.34) are precisely $\chi(0) = 1$ and $\chi'(0) = 0$) and we define the higher-order terms as $\chi = 1 - Q^2 g$, and the equation becomes

$$(1+y)\frac{d^2 g}{dy^2} + \left[\frac{1}{2} + \frac{2(1+y)}{y}\right]\frac{dg}{dy} + \frac{8 f_\nu(0)}{5y^2} g = 1, \tag{3.4.17}$$

where we have neglected only a $Q^4 g$ term coming from $Q^2\chi$, because the other terms in the equation were multiplied only for Q^2 and so such a term is negligible. The initial conditions on the equation are $g(0) = g'(0) = 0$.
As already stress at the end of Section 3.3, we are interested in the ratio between the squared amplitude of $h'(\eta,k)$ in both $N_\nu = 3$ and $N_\nu = 0$, in order to find out the explicit form for the damping. In this case we see that the damping can be written in terms of $g(y)$:

$$\left|\frac{h'(\eta,k)}{h'_{N_\nu=0}(\eta,k)}\right|^2 = \left|\frac{\chi'(\eta,k)}{\chi'_{N_\nu=0}(\eta,k)}\right|^2 = \left|\frac{g'(y)}{g'_{N_\nu=0}(y)}\right|^2. \tag{3.4.18}$$

We can numerically solve Eq. (3.4.17), finding out for example that this ratio is equal to 0.90 when we evaluate it at the last scattering, for $\Omega_m = 0.3$; this is important for evaluating the CMB anisotropies for large scales; by changing Q from 0.1 to 0.8 we see that the ratio is reduced only by the 2% (see for example the plot in Figure 3.4.1), thus we can conclude that for small values of Q the damping is quite insensitive to k. While, for larger Q, we observe a modification also to the phase of χ and then the ratio between the damped and the undamped function represents the ratio between functions oscillating with different phases:

$$\left|\frac{h'(\eta_{l.s.},k)}{h'_{N_\nu=0}(\eta_{l.s.},k)}\right|^2 = |A|^2\left|\frac{sin(y_{l.s.}+\delta)}{sin(y_{l.s.})}\right|^2, \tag{3.4.19}$$

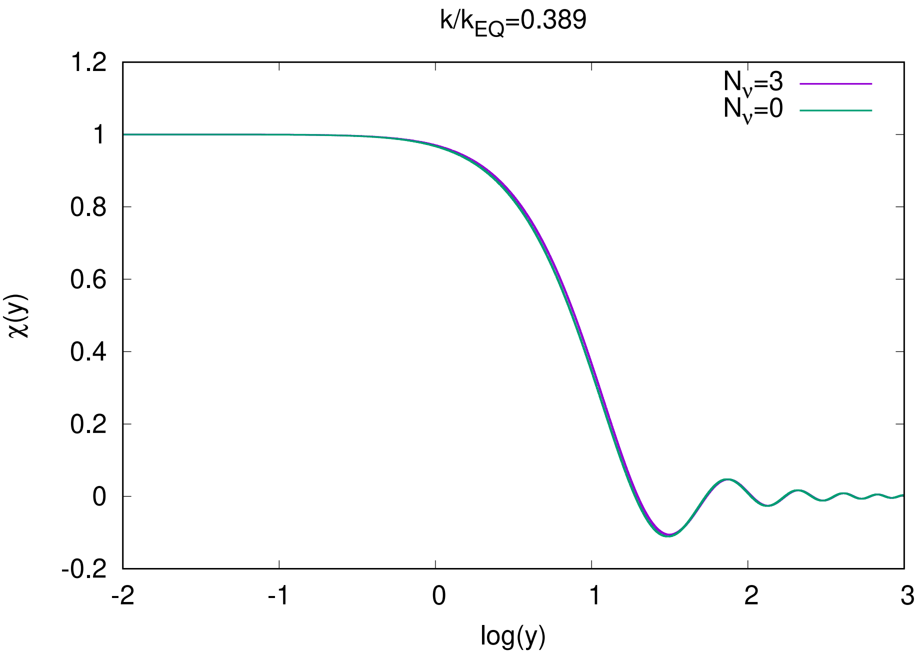

Figura 3.4.1: Plot of the solutions of Eq. (3.4.10) for $N_\nu = 3$ and $N_\nu = 0$. We observe that the two solutions are almost constant until the mode enters the horizon, after that we have on oscillating and decreasing mode. The two solutions does not differ so much, but, by the values of their derivatives at the last scattering, $y_{l.s.} \approx 3.3$, we find that $\left|\frac{\chi'(y_{l.s.})}{\chi'_{N_\nu=0}(y_{l.s.})}\right|^2 = 0.911$. By choosing for example $k/k_{EQ} = 0.566$, we find out that the ratio is about 0.924. So we can state that it is quite insensitive to k, and so to Q.

with δ very small, and we can study we behaviour of this ratio in function of Q, whom dependence is contained in δ. When $y_{l.s.} + \delta \approx n\pi$, with n integer number, we have two oscillating functions with a small phase difference, which means that the denominator reaches the zero at another time with respect to the numerator, hence we observe narrow spikes, from which we cannot say anything about $|A|^2$.
When the denominator is far away from zero however, we are quite insensitive to the phase difference, which is very small, and then the ratio corresponds exactly to the squared amplitude. For example, for $Q \approx 10$, far away from the spikes we find flat regions (a demonstration that we are insensitive to the phase difference) and we find $|A|^2 = 0.64$, according to the results of Section 3.3.

Capitolo 4

Effect of neutrinos on scalar modes evolution

4.1 Evolution of the scalar perturbations

In Chapter 3, we have seen the evolution of the tensor modes and the damping effect due to neutrinos. In this chapter we are interested in evaluating how the scalar perturbations evolve in presence of neutrinos, in order to find the effect that they have on the $\tilde{C}_{\ell,S}$ defined in Eq. (2.7.26). In this case, however, the analysis will be more difficult than the one for the tensor modes. In fact, when we consider scalar modes, we have an higher number of geometrical quantities involved: the scalar potentials ϕ and ψ, which depend on an higher number of particle species, because in this case the physical quantity involved is the diagonal part of the stress energy tensor. as seen in A.2. This means that they receives contributions from radiation, in this case from photons too, and from non-relativistic matter, i.e. from cold dark matter and baryons (which are not considered in this discussion however, because they are subdominant). In this chapter we will pay particular attention on the different cosmological epochs. In fact, in this case, the equations for the evolution of ϕ and ψ depend on the perturbations of the energy density. This contribution will be determined by the distribution function perturbations of the particle species which give the dominant contribution during a certain era: during the radiation-dominated era it will be due to photons and neutrinos only, while during the radiation dominated era it will depend only on non-relativistic matter. The Boltzmann equations for radiation and non-relativistic matter are very different, and so the evolution for ϕ and ψ. In addition, the transition between the two regimes, which happens around η_{EQ}, is very complicated, and it can be studied analytically only in the limit of large and small scales. While, for the horizon crossing regime, we can only provide numerical solutions. The main role of neutrinos also in this case is played in the "scalar" part of the anisotropic stress

$$k^2(\phi - \psi) = -32\pi G a^2 \rho_r \mathcal{N}_2, \tag{4.1.1}$$

where we have neglected from Eq. (A.2.27) the photons contribution, for the same reason of the previous chapter. This relation leads to $\phi = \psi$ for zero neutrino generations, while for three neutrino generations, $N_\nu = 3$, it changes the discussion on three levels: first of all it changes, obviously, the dynamics of the whole system, because we have an additional equation coupled to many others, secondly, it changes the initial conditions between ϕ and ψ, which now will differ from a certain quantity related to the neutrino quadrupole $\mathcal{N}_2$ evaluated at the neutrino decoupling time η_i, and as last effect it changes the initial condition of ϕ and ψ with respect to the perturbations generated by the quantum fluctuations of the inflaton field, discussed in Section 1.6.2, by a certain factor which depends on the neutrino energy density fraction $f_\nu(\eta_i)$. The effects we have found in this case is that ψ is enhanced with respect to zero neutrinos, while ϕ is reduced, both for small and large scales, moreover we have also analyzed the net effect on $(\phi + \psi)/2$, which is the variable involved in the $\tilde{C}_{\ell,S}$.

4.2 Effect on the initial conditions

As already stated, the first effect we can find out, when we have neutrinos in the Universe, is related to the initial conditions. We recall again that we are studying modes which enter the horizon long enough after neutrino decoupling and that such modes are conserved when they are far outside the horizon. Thence, in this section we are looking for the values of the quantities θ, $\mathcal{N}$, δ, v, ϕ and ψ, defined in A.2, at the initial time η_i[1], immediately after neutrino decoupling.

Before going further, we will derive the explicit value of neutrino decoupling. It is known that neutrinos decoupled at $T_{\nu,d} \approx 2MeV$ [63, 64], i.e. during the radiation era. By knowing that the CMB temperature is nowadays $T_{\gamma,0} = 2.73K = 2.3 \times 10^{-10} MeV$, we find immediately that the redshift at which neutrinos decoupled is[2]

$$z_{\nu,dec} = \frac{a_0}{a_i} - 1 \approx \frac{T_{\nu,dec}}{T_{\gamma,0}} \approx 10^{10}. \tag{4.2.1}$$

During radiation era, a scales as η, as seen in Eq. (3.3.1), therefore $aH = \eta$ This means that if we consider all the modes of physical interest to be larger than the Hubble radius defined in Section 1.6.2, we can assume that $k\eta \ll 1$. We start by considering the Boltzmann equation for neutrinos (A.2.40), neglecting tensor perturbations, because they cannot affect the evolution of the scalar ones for the decomposition theorem[3]

$$\mathcal{N}'(\eta, k, q, \mu) + ik\mu\mathcal{N}(\eta, k, q, \mu) = \psi'(\eta, k) - ik\mu\phi(\eta, k). \tag{4.2.2}$$

From this we can find the equation for the corrispective multipole of order ℓ of $\mathcal{N}$, by using the following identity for the Legendre polynomials:

$$(\ell + 1)\mathcal{P}_{\ell+1}(\mu) = (2\ell + 1)\mu\mathcal{P}_\ell(\mu) - \ell\mathcal{P}_{\ell-1}. \tag{4.2.3}$$

[1]All the quantities considered in this section are evaluated at η_i, but we do not write the explicit dependence on the initial time in the intermediate computations for clarity reason. We will write it explicitly only in the final results.

[2]We are also using the fact that during the radiation dominated era $a \sim T^{-1}$.

[3]In chapter 3 we have defined the neutrino distribution function perturbation as $\delta F(\eta, \vec{k})$, while in this case we will use $\mathcal{N}$ exactly for distinguishing the "scalar" and the "tensor" part of such a distribution.

For $\ell \geq 2$, if we integrate over $\frac{d\mu}{2}\frac{1}{(-i)^l}\mathcal{P}_\ell(\mu)$ we obtain the following equation (the right hand side of the equation is trivially null for the orthonormality of the Legendre polynomials):

$$\mathcal{N}'_\ell - \frac{k\ell}{2\ell+1}\mathcal{N}_{\ell-1} + \frac{k(\ell+1)}{2\ell+1}\mathcal{N}_{\ell+1} = 0. \tag{4.2.4}$$

If we consider also the equations for the first two multipoles we end up with the following system:

$$\begin{aligned} \mathcal{N}'_0 + k\mathcal{N}_1 =& \psi', \\ \mathcal{N}'_1 - \frac{k}{3}\mathcal{N}_0 + \frac{2k}{3}\mathcal{N}_2 =& \frac{k}{3}\phi, \\ \mathcal{N}'_2 - \frac{2k}{5}\mathcal{N}_1 + \frac{3k}{5}\mathcal{N}_3 =& 0, \\ \mathcal{N}'_3 - \frac{3k}{7}\mathcal{N}_2 + \frac{4k}{7}\mathcal{N}_4 =& 0 \rightarrow \mathcal{N}_3 \sim (k\eta)(a\mathcal{N}_2 + b\mathcal{N}_3) \rightarrow \mathcal{N}_3 \ll \mathcal{N}_2 \,. \end{aligned} \tag{4.2.5}$$

Recalling that $k\eta \ll 1$ we can neglect $\mathcal{N}_3$ because it is much smaller than $\mathcal{N}_2$. We will use also the longitudinal traceless part of the (i,j) Einstein equation, Eq. (A.2.27), making clear the relation between the scale factor and the proper time in the radiation era:

$$\begin{aligned} k^2(\phi - \psi) =& -32\pi G a^2 \rho_\nu^{(0)}\mathcal{N}_2 = -12\frac{\rho_\nu^{(0)}}{\rho_{tot}}\frac{8\pi G a^2 \rho_{tot}}{3}\mathcal{N}_2 = -12 f_\nu(\eta_i)\frac{(a')^2}{a^2}\mathcal{N}_2 = \\ =& -\frac{12 f_\nu(\eta_i)}{\eta^2}\mathcal{N}_2. \end{aligned} \tag{4.2.6}$$

The idea is to differentiate twice with respect to the conformal time in order to compare the two possible equations for $\mathcal{N}_2$. Writing Eq. (A.2.41) for the initial conditions we have

$$\begin{aligned} k^2\psi + 3\mathcal{H}\Big(\psi' + \mathcal{H}\phi\Big) =& -16\pi G a^2 \rho_r \theta_r \rightarrow \frac{\psi'}{\eta} + \frac{\phi}{\eta^2} = -2\frac{8\pi G \rho_r a^2}{3}\Big(\frac{\rho_\gamma}{\rho_r}\theta_0 + f_\nu \mathcal{N}_0\Big) = \\ =& -\frac{2}{\eta^2}\big[(1-f_\nu)\theta_0 + f_\nu\mathcal{N}_0\big], \end{aligned} \tag{4.2.7}$$

where $\rho_r = \rho_\gamma + \rho_\nu$. Using the monopole part of Eqs. (A.2.37) and (A.2.40) we have that

$$\begin{cases} \theta'_0 + k\theta_1 = \psi' \\ \mathcal{N}_0 + k\mathcal{N}_1 = \psi' \end{cases} \rightarrow \begin{cases} \theta'_0 = \psi' \\ \mathcal{N}_0 = \psi' \end{cases} \tag{4.2.8}$$

Substituting these two expressions in the $(0,0)$ component of the Einstein equations written above and deriving with respect to the conformal time η we find

$$\eta\psi'' + \psi' + \phi' = -2\big[(1-f_\nu) + f_\nu\big]\psi' \rightarrow \eta\psi'' + 3\psi' + \phi' = 0. \tag{4.2.9}$$

This equation has a constant solution, $\phi' = \psi' = 0$ and, for $\psi' \approx \phi'$, a decaying solution which goes as η^{-4}. The constant mode is the one we are interested in and using this result in the $(0,0)$ component of the Einstein equations we find out that

$$\phi = -2\big[(1-f_\nu)\theta_0 + f_\nu\mathcal{N}_0\big]. \tag{4.2.10}$$

In addition $\theta_0 = \mathcal{N}_0$, because the perturbations in the early Universe did not distinguish between photons and neutrinos, and so they produce the same perturbations to the distribution function, and then we find

$$\theta_0(\eta_i, k) = \mathcal{N}_0(\eta_i, k) = -\frac{\phi(\eta_i, k)}{2}. \tag{4.2.11}$$

We can then derive twice with respect to the proper time Eq. (4.2.6), neglecting the temporal derivatives of the scalar perturbations, and we end up with

$$\mathcal{N}_2'' = -\frac{k^2}{6 f_\nu(\eta_i)}(\phi - \psi). \tag{4.2.12}$$

By putting this equation together with the ones for the $\mathcal{N}$ multipoles, Eqs. (4.2.5), we obtain the final relation about the initial conditions:

$$\begin{aligned} 0 =& \mathcal{N}_2'' - \frac{2k}{5}\mathcal{N}_1' = -\frac{k^2}{6 f_\nu(\eta_i)}(\phi - \psi) - \frac{2k^2}{5}\frac{1}{3}\Big[\mathcal{N}_0 - 2\mathcal{N}_2 + \phi\Big] = \\ =& -\frac{k^2}{6 f_\nu(\eta_i)}(\phi - \psi) - \frac{2k^2}{5}\frac{1}{3}\Big[-\frac{\phi}{2} + 2\frac{k^2\eta_i^2}{12 f_\nu(\eta_i)}(\phi - \psi) + \phi\Big] = \\ =& \frac{k^2}{6 f_\nu(\eta_i)}\psi - \frac{k^2}{6 f_\nu(\eta_i)}\phi\Big[1 + \frac{2}{5} f_\nu(\eta_i)\Big] \end{aligned} \tag{4.2.13}$$

So in presence of neutrinos not only the scalar perturbations ϕ and ψ follow two different evolutions, but also their initial conditions show a difference described by

$$\psi_i = \Big[1 + \frac{2}{5} f_\nu(\eta_i)\Big]\phi_i, \tag{4.2.14}$$

where $\psi_i = \psi(\eta_i, k)$ and $\phi_i = \phi(\eta_i, k)$.

In order to detect any possible effect due to neutrinos, we should also investigate if the initial condition ϕ_i depends on $f_\nu(\eta_i)$ itself. We know that in the very early Universe, the quantum fluctuations of the inflaton field [22] produced the inhomogeneities and anisotropies we observe nowadays.

First of all we define the gauge-invariant quantity

$$\zeta(\eta, k) \equiv -\Big(\psi(\eta, k) + \frac{1}{6}k^2\chi^{\|}(\eta, k)\Big) - \mathcal{H}\frac{\delta\rho(\eta, k)}{\rho^{(0)\prime}(\eta)}, \tag{4.2.15}$$

where we have not specified the gauge. This quantity is gauge-invariant by definition, and it is proved that it is conserved when the mode k is out of the horizon; we can see that it corresponds to the curvature of space-like surfaces of constant time in the comoving gauge. Such a curvature crossed the horizon during the inflation, assuming a constant value ζ_I, and, because all the modes of interest were outisde the horizon at the "initial time" η_i, it re-entered the horizon after η_i. Hence we can relate the initial perturbations to ζ_I, defined as $\zeta_I \equiv -\zeta(\eta_i, k)$[4].

[4]We have chosen the minus sign as a convention in order to have the initial condition on ϕ and ψ, of the same sign of ζ_I.

In Section 2.3 we have chosen to work in the Poisson gauge, where $\chi^{\parallel} = 0$, and we know that in the radiation-dominated era the Friedmann equation for the unperturbed energy density is:

$$\rho^{(0)\prime} = -3\mathcal{H}\big(\rho^{(0)} + p^{(0)}\big) = -\frac{4}{\eta}\rho^{(0)}, \tag{4.2.16}$$

and we can write the total unperturbed energy density $\rho^{(0)}$ as the sum of the unperturbed neutrinos and photons energy densities. The final expression we find is then

$$\begin{aligned} -\zeta_I =& -\psi - \frac{1}{\eta}\frac{\delta\rho_\gamma + \delta\rho_\nu}{-4(\rho_\gamma^{(0)} + \rho_\nu^{(0)})} = -\psi + \frac{4(1-f_\nu)\theta_0 + 4f_\nu\mathcal{N}_0}{4} = -\psi + \theta_0 = -\psi - \frac{1}{2}\phi = \\ =& -\phi\Big(1 + \frac{2}{5}f_\nu\Big) - \phi = -\frac{3}{2}\phi\Big(1 + \frac{4}{15}f_\nu\Big), \end{aligned} \tag{4.2.17}$$

And so we find

$$\phi(\eta_i, k) = \frac{2}{3}\left(1 + \frac{4}{15}f_\nu(\eta_i)\right)^{-1}\zeta_I. \tag{4.2.18}$$

This is very important, because when we will study the evolution of ϕ and we will make a comparison between the $N_\nu = 0$ case and the $N_\nu = 3$, we will keep into account that the initial amplitude for ϕ considering neutrinos is about $\big(1 + \frac{4}{15}f_\nu(\eta_i)\big)^{-1} \approx \frac{9}{10}$ of the amplitude without considering them.
To conclude this section, we will find the initial conditions for the fields δ and v, because they will play a role for the calculations of the small scale modes in Section 4.5.
Assuming that we are dealing with adiabatic perturbations, we have that $\delta = -\frac{3}{2}\phi_i$ [22] and using Eq. (A.2.43) we find

$$k^2\psi_i = -\frac{3}{2\eta_i^2}\Big(\delta + \frac{3v}{k\eta_i}\Big) \to -\frac{2}{3}(k\eta_i)^2\psi_i = -\frac{3}{2}\phi_i + \frac{3v}{k\eta} \to v = \frac{k\eta_i}{2}\phi_i, \tag{4.2.19}$$

where we have neglected the ψ_i term, because it is a quadratic term in $k\eta_i$.

4.3 The $\psi(\phi)$ relation

In this section we want to combine the longitudinal traceless part of the Einstein equations with the Boltzmann equation for neutrino to find out a precise relation between ϕ and ψ that can be inserted in the remaining dynamical equations, essentially to remove one degree of freedom.
We start by the solution of the Boltzmann equation for neutrinos in the Fourier space,

$$\mathcal{N}(\eta, \vec{k}, \vec{q})' + ik\mu\mathcal{N}(\eta, \vec{k}, \vec{q}) = \psi'(\eta, k) - ik\mu\phi(\eta, k), \tag{4.3.1}$$

which has a solution analogue to the one defined in Eq. (2.7.2) for the gravitons,

$$\begin{aligned}\mathcal{N}(\eta,\vec{k}) =&\mathcal{N}_i e^{ik\mu(\eta_i-\eta)} + \int_{\eta_i}^{\eta} d\eta' \Big[\psi'(\eta',k^i) + \phi'(\eta',k^i)\Big] e^{ik\mu(\eta'-\eta)} - \phi(\eta,k^i) + \\ &+ \phi(\eta_i,k^i) e^{ik\mu(\eta_i-\eta)} = \\ =& \int_{\eta_i}^{\eta} d\eta' \Big[\psi'(\eta',k^i) + \phi'(\eta',k^i)\Big] e^{ik\mu(\eta'-\eta)} - \phi(\eta,k^i) + \frac{1}{2}\phi(\eta_i,k^i) e^{ik\mu(\eta_i-\eta)}.\end{aligned} \tag{4.3.2}$$

Using the definition of the neutrino quadrupole we immediately find out that

$$\begin{aligned}\mathcal{N}_2 =& \frac{1}{(-i)^2}\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_2(\mu)\mathcal{N} = \\ =& -\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_2(\mu)\bigg\{\int_{\eta_i}^{\eta} d\eta' \Big[\psi'(\eta',k^i) + \phi'(\eta',k^i)\Big] e^{ik\mu(\eta'-\eta)} - \phi(\eta,k^i) + \\ &+ \frac{1}{2}\phi(\eta_i,k^i) e^{ik\mu(\eta_i-\eta)}\bigg\} = \\ =& \int_{\eta_i}^{\eta} d\eta' \Big[\psi'(\eta',k^i) + \phi'(\eta',k^i)\Big] j_2[k(\eta-\eta')] + \frac{1}{2}\phi(\eta_i,k^i) j_2[k(\eta-\eta_i)].\end{aligned} \tag{4.3.3}$$

We can insert this in the longitudinal, traceless component of the (i,j) Einstein equation, Eq. (A.2.27), removing the neutrino dependence and obtaining an expression which depends only on the scalar perturbations:

$$k^2(\phi-\psi) = -32\pi G a^2 \rho_\nu^{(0)} \bigg[\int_{\eta_i}^{\eta} d\eta' \Big[\psi'(\eta',k^i) + \phi'(\eta',k^i)\Big] j_2[k(\eta-\eta')] + \frac{1}{2}\phi(\eta_i,k^i) j_2[k(\eta-\eta_i)]\bigg]. \tag{4.3.4}$$

It is always true that for relativistic particles $\rho_r \sim a^{-4}$ and then the term $\rho_r a^4$ is time independent, thus we will multiply the equation by a factor a^2 and we differentiate with respect to the conformal time:

$$\begin{aligned}a^2k^2(\phi-\psi) =& -32\pi G a^4 \rho_\nu^{(0)} \bigg[\int_{\eta_i}^{\eta} d\eta' \Big[\psi'(\eta',k^i) + \phi'(\eta',k^i)\Big] j_2[k(\eta-\eta')] + \\ &+ \frac{1}{2}\phi(\eta_i,k^i) j_2[k(\eta-\eta_i)]\bigg] \\ \frac{d}{d\eta}\Big[a^2k^2(\phi-\psi)\Big] =& -32\pi G a^4 \rho_\nu^{(0)} \bigg[\Big(\psi'(\eta,k^i) + \phi'(\eta,k^i)\Big) j_2[k(\eta-\eta)] + \\ &+ \frac{1}{2}\phi(\eta_i,k^i) \frac{d}{d\eta} j_2[k(\eta-\eta_i)]\bigg] \\ \frac{d}{d\eta}\Big[a^2k^2(\phi-\psi)\Big] =& -16\pi G a^4 \rho_\nu^{(0)} \phi(\eta_i,k^i) \frac{d}{d\eta} j_2[k(\eta-\eta_i)],\end{aligned} \tag{4.3.5}$$

where we have used the following expansion for the spherical Bessel function around 0[5]

$$
\begin{aligned}
j_2(x) &\equiv \Big(\frac{3}{x^2}-1\Big)\frac{sinx}{x}-\frac{3cosx}{x^2}\\
&\simeq \Big(\frac{3}{x^2}-1\Big)\Big(1-\frac{x^2}{6}+\frac{x^4}{120}\Big)-3\Big(\frac{1}{x^2}-\frac{1}{2}+\frac{x^2}{24}\Big)\\
&\simeq \frac{1}{15}x^2 \simeq 0.
\end{aligned} \tag{4.3.6}
$$

By using Eq. (4.2.14) we can integrate the above differential equation with respect to the conformal time using the proper initial conditions and we have that

$$
k^2a^2(\phi-\psi)+\frac{2}{5}f_\nu(\eta_i)\phi_i k^2 a_i^2 = -16\pi G a^4 \rho_\nu^{(0)}\phi_i j_2[k(\eta-\eta_i)], \tag{4.3.7}
$$

so we end up with the following expressions in terms of the only variable ψ:

$$
\phi = \psi - \frac{2}{5}f_\nu(\eta_i)\frac{a_i^2}{a^2}\phi_i - \frac{16\pi G a^4\rho_\nu^{(0)}\phi_i j_2[k(\eta-\eta_i)]}{k^2a^2} = \psi+\beta. \tag{4.3.8}
$$

where β represents the differences with respect to the zero neutrinos case ($\beta=0$). By using the explicit form of the order two spherical Bessel function we see that it can be written as

$$
\begin{aligned}
\beta =& -\phi_i\frac{f_\nu(\eta_i)a_i^2}{a^2}\Big[\frac{2}{5}+\frac{6\mathcal{H}^2(\eta_i)j_2[k(\eta-\eta_i)]}{k^2}\Big]=\\
=& -\phi_i\frac{f_\nu(\eta_i)a_i^2}{a^2}\Bigg\{\frac{2}{5}+6\frac{1}{(k\eta_i)^2}\Bigg[\Bigg(\frac{3}{[k(\eta-\eta_i)]^2}-1\Bigg)\frac{sin[k(\eta-\eta_i)]}{k(\eta-\eta_i)}-\frac{3cos[k(\eta-\eta_i)]}{[k(\eta-\eta_i)]^2}\Bigg]\Bigg\},
\end{aligned} \tag{4.3.9}
$$

where we have used the fact that η_i is in the radiation era and so $\mathcal{H}(\eta_i)=\frac{1}{\eta_i}$. In the next sections, we will discuss the Einstein equations in different regimes: outside and inside the cosmological horizon, and in the matter and in radiation eras, therefore we need to consider, in principle, in order to check if the result is reasonable, the limits of such a function in all these regimes.
Before doing that it is quite important to establish a relation between the conformal time η and another variable we will use for many discussions, $y=\frac{a}{a_{EQ}}$. From the first Friedmann equation we have

$$
\begin{aligned}
\dot{a}^2 =& \frac{8}{3}\pi G a^2\rho = \frac{8}{3}\pi G a_{EQ}^2\Big(\frac{a}{a_{EQ}}\Big)^2\Big[\rho_m(\eta_{EQ})\Big(\frac{a_{EQ}}{a}\Big)^3+\rho_r(\eta_{EQ})\Big(\frac{a_{EQ}}{a}\Big)^4\Big]=\\
=& \frac{8}{3}\pi G\rho_r(\eta_{EQ})a_{EQ}^2\frac{1}{y^2}(1+y),
\end{aligned} \tag{4.3.10}
$$

but the left-hand side of the equation can be written also as

$$
(\dot{a})^2 = \Big(\frac{a'}{a}\Big)^2 = \Big(\frac{y'}{y}\Big)^2 \tag{4.3.11}
$$

[5]In the above expressione we have $j_2[k(\eta-\eta)]$ to evaluate.

and by a comparison of the two equations we have

$$\frac{y'}{\sqrt{1+y}} =\sqrt{\frac{8}{3}\pi G\rho_r(\eta_{EQ})a_{EQ}^2} \to 2(\sqrt{1+y}-\sqrt{1+y_i}) = \sqrt{\frac{8}{3}\pi G\rho_r(\eta_{EQ})a_{EQ}^2}(\eta-\eta_i). \tag{4.3.12}$$

From Section 4.2 we know that the redsfhit at which neutrino decoupled is about, 10^{10}, so the correspondent y_i will be

$$y_i = \frac{a_i}{a_{EQ}} = \frac{a_i}{a_0}\frac{a_0}{a_{EQ}} = \frac{1+z_{EQ}}{1+z_{\nu,d}} \approx 10^{-7}. \tag{4.3.13}$$

For very small y we can expand at the first order the square root

$$y - y_i = \sqrt{\frac{8}{3}\pi G\rho_r(\eta_{EQ})a_{EQ}^2}(\eta-\eta_i), \tag{4.3.14}$$

and then we can set $y_i = \sqrt{\frac{8}{3}\pi G\rho_r(\eta_{EQ})a_{EQ}^2}\eta_i$. As we will see this convention simplifies a lot the further expressions.
As already stressed, in the next sections we will deal with the β variable and almost always we will solve the evolution equations numerically, hence we write down the explicit forms of β and its derivatives in terms of the parameters we will use in the numerical computations:

- we start with the super-horizon case, i.e. $k\eta \ll 1$. In this case we can expand around zero the spherical Bessel function, which assumes the value $\frac{1}{15}k^2(\eta-\eta_i)^2$, according to Eq. (4.3.6):

$$\begin{aligned}
\beta(y) =& -\phi\Big(\frac{y_i}{y}\Big)^2(\eta_i,\vec{k})f_\nu(\eta_i)\Big[\frac{2}{5}+\frac{6k^2(\eta-\eta_i)^2}{15k^2\eta_i^2}\Big]=\\
=& -\frac{2}{5}\phi(\eta_i,\vec{k})f_\nu(\eta_i)\Big[\Big(\frac{y_i}{y}\Big)^2+4\Big(\frac{\sqrt{1+y}-1-\frac{y_i}{2}}{y}\Big)^2\Big]=\\
=& -\frac{2}{5}\phi(\eta_i,\vec{k})f_\nu(\eta_i)\frac{1}{y^2}\Big[2y_i^2+8+4y+4y_i-8\Big(1+\frac{y_i}{2}\Big)\sqrt{1+y}\Big],\\
\frac{d\beta(y)}{dy} =& \frac{2}{5}\phi(\eta_i,\vec{k})f_\nu(\eta_i)\frac{1}{y^2}\Big\{\frac{2}{y}\Big[2y_i^2+8+4y+4y_i-4\Big(1+\frac{y_i}{2}\Big)\sqrt{1+y}\Big]+\\
&-4+4\frac{1+\frac{y_i}{2}}{\sqrt{1+y}}\Big\},\\
\frac{d^2\beta(y)}{dy^2} =& \frac{2}{5}\phi(\eta_i,\vec{k})f_\nu(\eta_i)\frac{1}{y^2}\Big\{-\frac{6}{y^2}\Big[2y_i^2+8+4y+4y_i-4\Big(1+\frac{y_i}{2}\Big)\sqrt{1+y}\Big]+\frac{16}{y}+\\
&-\frac{8}{y}\frac{1+\frac{y_i}{2}}{\sqrt{1+y}}-\frac{1+\frac{y_i}{2}}{(1+y)^{3/2}}\Big\}.
\end{aligned} \tag{4.3.15}$$

Now we can divide the results into the epochs we are considering:

- The deep radiation-era case, for $\eta_i \ll \eta \ll \eta_{EQ}$, which corresponds to the $y \ll 1$ case, and then we immediately see that β assumes a constant value,

$$\beta = -\frac{2}{5}\phi(\eta_i, \vec{k}) f_\nu(\eta_i), \tag{4.3.16}$$

  this agrees indeed with what we have seen in Section 4.2;

- The matter epoch case, for which $\eta \gg \eta_{EQ}$, which corresponds to $y \gg 1$ and then we immediately see that β vanishes, and so the two scalar perturbations are equal, exactly as in the case in which $f_\nu(\eta) = 0$. This is in agreement with the fact that the neutrino quadrupole is related to the fraction of energy density of neutrinos with respect to the total one, which is negligible in the matter epoch, thus we expect that the longitudinal traceless Einstein equation corresponds to the condition $\phi = \psi$;

- the second case consists in considering the modes which are deeply inside the horizon, i.e. $k\eta \gg 1$.
  It is immediate to see that now the spherical Bessel function goes to zero very rapidly and then we need to consider only the first term in the β expression:

$$\begin{aligned} \beta(y) &= -\frac{2}{5}\phi(\eta_i, \vec{k}) f_\nu(\eta_i)\left(\frac{y_i}{y}\right)^2 \\ \frac{d\beta(y)}{dy} &= \frac{4}{5}\phi(\eta_i, \vec{k}) f_\nu(\eta_i)\frac{y_i^2}{y^3} = -\frac{2}{y}\beta \end{aligned} \tag{4.3.17}$$

  Also in this case we consider the previous two interesting regimes:

  - during the radiation era $\frac{y_i}{y}$ is not necessarily negligible, and then we kept the above expression for β, observing that it goes as $\frac{1}{y^2}$ for small scales when they enter the horizon, hence we expect that there is still a contribution given by neutrinos even if the modes have crossed the horizon;
  - during the matter-dominated era the ratio is necessarily negligible, since $y_i \sim 10^{-7}$. It is very small because it represents the neutrino decoupling redshift, while $y \geq 1$, so β vanishes and we can take $\beta = 0$, accordingly to the fact that during the epoch dominated by non-relativistic matter we should be insensitive to the effects of radiation (as neutrinos) on the potentials.

4.4 Large Scale Evolution

The evolution on large scales is the simplest case to study: it involves modes that crossed the horizon well after the equivalence between matter and radiation, therefore for large part of the discussion we will not consider the k terms in the set of equations and this will lead to many simplifications. The basic idea of the large scales evolution is that during the radiation era the modes are far away from the horizon, thus they are constant. They become sensitive only to the transition between the radiation and the matter epoch,

during which they decrease their amplitudes until they are far away from y_{EQ}. We will see that even for the sub-horizon evolution during the matter epoch the modes are constant, therefore the only relevant changes occur at $y \approx y_{EQ}$. In this discussion we have neglected the dark energy epoch, which starts recently, because we have seen that the main effects for neutrinos are present in the radiation dominated era.
The general method we have adopted consists in solving the full set of equations introduced in A.2.3, dividing the discussion into super-horizon and sub-horizon evolution: in the first case we study the evolution of all the particle species (photons, neutrinos and cold dark matter), while in the second one we will focus only on the non-relativistic ones, i.e. only cold dark matter, because we are deep in the matter epoch.

4.4.1 Super-horizon dynamics

When we consider super-horizon modes we require the condition that $k \ll aH$. During radiation dominated era $aH = 1/\eta$, while in the matter epoch $aH = 2/\eta$, this depends on the fact that a goes as η during the radiation era and as η^2 during the matter era. Thus we infer that when we require that the modes are out of the horizon $k\eta \ll 1$.
The idea is to express $\phi(\eta, k)$ in terms of $a(\eta)$ generically, to avoid troubles generated by the fact that the scale factor does not have a simple dependence on η at the radiation-matter equality.
From the dark matter velocity equation, Eq. (A.2.39), we find that v is suppressed by the expansion of the Universe, and so we can neglect it in further calculations:

$$v' + \mathcal{H}v = -ik\phi \rightarrow v' + \mathcal{H}v = 0 \rightarrow v \sim \frac{1}{a}. \tag{4.4.1}$$

Notice that we can neglect the θ multipoles of order higher than one in Eq. (A.2.37): if the Compton scattering is very efficient (as they are during the radiation era), it drives θ to θ_0, and θ_1 is generated only by non-null bulk velocities of the electrons involved in the Compton scattering, thus the photons anisotropies in this stage are fully characterized by the monopole and by the dipole [22]. This is precisely the reason why we have not taken them into account in Chapter 3 and in Eq. (A.2.43). Therefore we see that the equations for θ_0 and θ_1 are identical to the ones for $\mathcal{N}_0$ and $\mathcal{N}_2$ and we decide to use a simpler notation, studying the evolution for ρ_r and θ_r, using

$$\rho_r\theta_{r,0} = \rho_{\gamma,0}\theta + \rho_{\nu,0}\mathcal{N}_0. \tag{4.4.2}$$

Now using the monopole part of Eq. (A.2.37) (equivalent to Eq. (A.2.40)), written in terms of θ_r, and using also Eq. (A.2.38), we have

$$\begin{aligned} \theta'_{r,0} + k\theta_{r,1} =& \psi' \rightarrow \theta'_{r,0} = \psi', \\ \delta' + ikv =& 3\psi' \rightarrow \delta' = 3\psi' \rightarrow \theta_{r,0} = \frac{\delta}{3}. \end{aligned} \tag{4.4.3}$$

Notice that in all this discussion we will not consider baryons, because they account for the 4% of the total energy density of the Universe, therefore they are negligible with respect to cold dark matter. The last step consists in substituting these relations in Eq. (A.2.41) obtaining

$$\begin{aligned} k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\phi) =& - 4\pi G a^2(\rho_{dm}\delta + 4\rho_r\theta_{r,0}), \\ 3\mathcal{H}\psi' + 3\mathcal{H}^2\psi + 3\mathcal{H}^2\beta =& - 4\pi G a^2\rho_{dm}\delta\Big(1 + \frac{4}{3}\frac{\rho_r}{\rho_{dm}}\Big). \end{aligned} \tag{4.4.4}$$

Now, we are ready to shift to the y variable, through which the transition between the radiation and the matter epochs is automatically taken into account, without specifying the dependence of a on η:

$$y = \frac{a}{a_{EQ}} = \frac{\rho_{dm}}{\rho_r} \to \frac{d}{d\eta} = \frac{dy}{d\eta}\frac{d}{dy} = \frac{a'}{a}\frac{a}{a_{EQ}}\frac{d}{dy} = \mathcal{H}y\frac{d}{dy}\,. \tag{4.4.5}$$

In this way our equation becomes

$$\begin{aligned} +3\mathcal{H}^2 y\frac{d}{dy}\psi + 3\mathcal{H}^2\psi + 3\mathcal{H}^2\beta =& - \frac{3}{2}\frac{8\pi G a^2\rho_{dm}}{3}\delta\Big(1 + \frac{4}{3y}\Big) = -\frac{3}{2}\frac{(a')^2}{a^2}\frac{\rho_d m}{\rho_{tot}}\delta\Big(1 + \frac{4}{3y}\Big) = \\ =& - \frac{3}{2}\mathcal{H}^2\frac{1}{\frac{1}{y}+1}\delta\Big(1 + \frac{4}{3y}\Big) \\ y\frac{d\psi}{dy} + \psi + \beta =& - \frac{y}{2(y+1)}\delta\Big(1 + \frac{4}{3y}\Big). \end{aligned} \tag{4.4.6}$$

By deriving with respect to y and using Eq. (4.4.3) (in order to express δ in function of ψ) we have

$$\begin{aligned} 0 =& \frac{d}{dy}\Big[\frac{6(y+1)}{3y+4}\Big(y\frac{d\psi}{dy} + \psi + \beta\Big)\Big] + 3\frac{d}{dy}\psi = \\ =& \frac{6(y+1)}{3y+4}\Big[2\frac{d\psi}{dy} + y\frac{d^2\psi}{dy^2} + \frac{d\beta}{dy}\Big] + \frac{6}{(3y+4)^2}\Big(y\frac{d\psi}{dy} + \psi + \beta\Big) + 3\frac{d\psi}{dy} = \\ =& \frac{6(y+1)y}{3y+4}\frac{d^2\psi}{dy^2} + \Big[2(y+1)(3y+4) + \frac{(3y+4)^2}{2} + y\Big]\frac{6}{(3y+4)^2}\frac{d\psi}{dy} + 6\psi + \\ &+ 6\frac{\beta + (y+1)(3y+4)\frac{d\beta}{dy}}{(3y+4)^2}. \end{aligned} \tag{4.4.7}$$

The final result is then

$$\frac{d^2\psi}{dy^2} + \frac{21y^2 + 54y + 32 + 8}{2y(y+1)(3y+4)}\frac{d\psi}{dy} + \frac{1}{y(y+1)(3y+4)}\psi + \frac{\beta + (y+1)(3y+4)\frac{d\beta}{dy}}{y(y+1)(3y+4)} = 0. \tag{4.4.8}$$

By setting $f_\nu(\eta_i) = 0$, which corresponds to $\beta = 0$ we end up with the well-known "neutrinoless" equation [22]

$$0 = \frac{d^2\psi}{dy^2} + \frac{21y^2 + 54y + 32}{2y(3y+4)(y+1)}\frac{d\psi}{dy} + \frac{1}{y(y+1)(3y+4)}\psi, \tag{4.4.9}$$

which has as solution (recall that in this case $\psi_i = \phi_i$ because we have no neutrinos)

$$\psi = \frac{1}{10y^3}\Big(16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\Big)\psi_i, \tag{4.4.10}$$

The idea is that the solution considering the neutrinos contribution is the one without neutrinos plus a particular solution given by the source term which depends on β. The system of equations to solve is

$$\begin{cases} 0 = & \frac{d^2\psi}{dy^2} + \frac{21y^2+54y+32}{2y(y+1)(3y+4)}\frac{d\psi}{dy} + \frac{1}{y(y+1)(3y+4)}\psi + \frac{\beta+(y+1)(3y+4)\frac{d\beta}{dy}}{y(y+1)(3y+4)} \\ \phi = & \psi + \beta \\ \beta = & -\frac{2}{5}\phi(\eta_i,\vec{k})f_\nu(\eta_i)\frac{1}{y^2}\Big[2y_i^2 + 8 + 4y + 4y_i - 4\Big(1+\frac{y_i}{2}\Big)\sqrt{1+y}\Big] \end{cases} \tag{4.4.11}$$

We have solved it numerically for $y_i \sim 10^{-7}$, the initial time at which neutrinos decoupled (corresponding to a redshift of order 10^{10}), and we have used also the value $f_\nu(\eta_i) = 0.4$, seen also in Chapter 3. We have normalized the result with respect to the initial value of the scalar perturbations for $N_\nu = 0$, i.e. $\phi_i = \frac{2}{3}\zeta_I$. The numerical results are given in Fig. 4.4.1.
We have found a modification of $(\phi+\psi)/2$ with respect to ϕ for zero neutrino generations which is in accordance with the one discussed in [65]. Another important fact to stress is that once the modes have crossed the horizon the solutions for $N_\nu = 0$ and $N_\nu = 3$ become identical.

4.4.2 Sub-horizon dynamics

Now we consider modes which entered the horizon deeply in the matter-dominated era, so for redshifts $z \geq 1000$.
At these times from the previous analysis we can see that the potential ϕ and ψ are constant and they are almost equal to the potentials obtained by non-considering neutrinos. Essentially we are going to study the Boltzmann and the Einstein equations for sub-horizon scales, and we want to understand if the parameter β (and so the neutrinos) plays a significant role in the dynamics during this new regime.
After the horizon crossing for these large scale modes, we are not considering the radiation anymore, except, at least in principle, for the Einstein equation which depends on the neutrino quadrupole, because there we have only ρ_ν, which is not summed to ρ_{dm} and so it cannot be neglected.
In this section we will use Eqs. (A.2.38), (A.2.39) and (A.2.43):

$$\begin{aligned} \delta' + kv =& 3\psi', \\ v' + \mathcal{H}v =& k\phi \to (av)' = ka\phi, \\ k^2\psi =& -\frac{3}{2}\mathcal{H}^2\Big(\delta + \frac{3\mathcal{H}v}{k}\Big). \end{aligned} \tag{4.4.12}$$

The last equation is found by combining in a proper way the $(0,i)$ and the (i,j) Einstein equations. The solution can be found by following different steps (η_* is the initial conformal

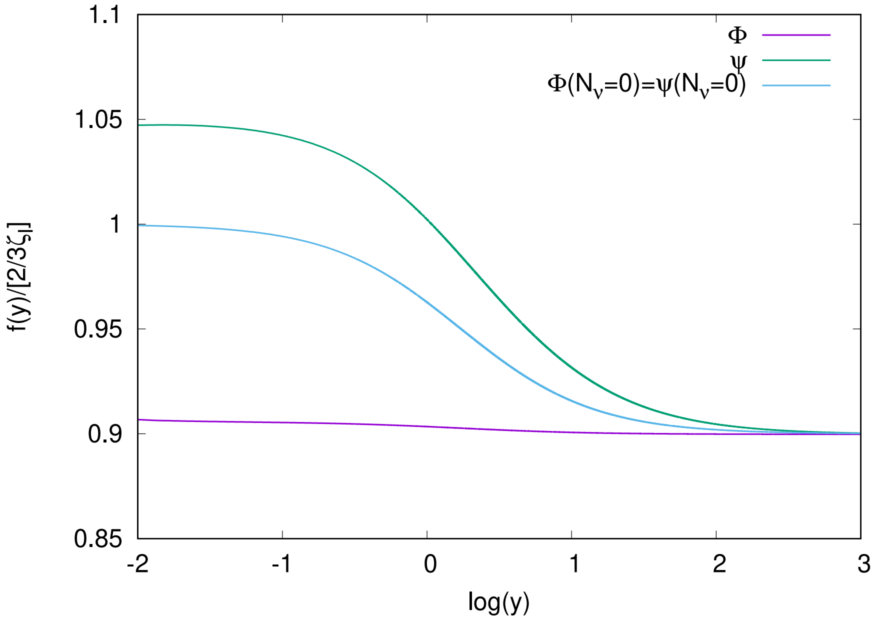

Figura 4.4.1: Numerical evolution of the cosmological potentials ϕ, ψ and $\phi(N_\nu = 0) = \psi(N_\nu = 0)$. We can see that before the radiation-matter equality we have the biggest differences: ϕ assumes an almost constant value, while ψ changes a lot; after $y = 30$ the three functions tend to the asymptotic value $\frac{9}{10}\phi_i$.

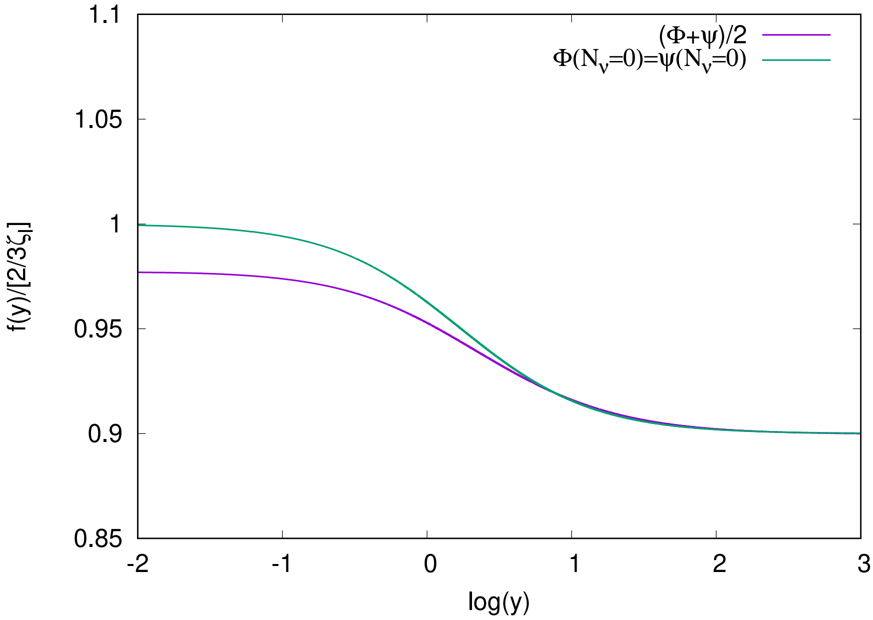

Figura 4.4.2: Plot of $(\phi + \psi)/2$ for $N_\nu = 0$ and $N_\nu = 3$. We are interested in the behaviour of the sum of ϕ and ψ, because it gives an important contribution to the gravitational waves anisotropies in Eq. (2.7.26), through an effect similiar in form to the ISW effect for the CMB. We see that the half sum of the potentials is smaller with respect to non-considering neutrinos, and we see that this difference is non-negligible only before the radiation-matter equality.

time, by definition it is deeply in the matter era and it is the one which corresponds to the final time of the previous section):

$$\begin{aligned}
a(t) =& bt^{\frac{2}{3}} \to \eta = \int dt \frac{t^{-\frac{2}{3}}}{b} = \frac{3a^{\frac{1}{2}}}{b^2} \to a = \eta^2 \to \mathcal{H} = \frac{2}{\eta}, \\
v(\eta) =& v_* \frac{a_*}{a(\eta)} + \frac{k}{a(\eta)} \int_{\eta_*}^{\eta} d\eta' a(\eta')\phi(\eta') = v_* \frac{\eta_*^2}{\eta^2} + \frac{k}{\eta^2} \int_{\eta_*}^{\eta} d\eta' \eta'^2 \phi(\eta'), \\
\delta(\eta) =& -\frac{2k^2\psi(\eta)}{3\mathcal{H}^2(\eta)} - \frac{3\mathcal{H}(\eta)v(\eta)}{k} = -\frac{6}{\eta^3}\left(\frac{v_*\eta_*^2}{k} + \int_{\eta_*}^{\eta} d\eta' \eta'^2 \phi(\eta')\right) - \frac{k^2\eta^2}{6}\psi(\eta), \\
0 =& \delta' + kv - 3\psi' = \\
=& \frac{18 + k^2\eta^2}{\eta^4}\left\{\int_{\eta_*}^{\eta} d\eta' \eta'^2 \Big[\psi(\eta') + \beta(\eta')\Big] + \frac{v_*\eta_*^2}{k}\right\} - \frac{k^2\eta^2 + 18}{6}\psi' - \frac{k^2\eta^2 + 18}{3\eta}\psi - \frac{6}{\eta}\beta.
\end{aligned} \tag{4.4.13}$$

We can derive with respect to the conformal time this expression and we obtain a second-order differential equation for ψ:

$$\begin{aligned}
0 =& -\eta^2\psi - \eta^2\beta + \left\{\frac{\eta^4}{6}\psi' + \frac{\eta^3}{3}\psi + \frac{6\eta^3}{18 + k^2\eta^2}\beta\right\}' \\
0 =& \frac{\eta^4}{6}\psi'' + \eta^3\psi' + \left[6\frac{(\beta'\eta^3 + 3\beta\eta^2)(18 + k^2\eta^2) - 2\beta k^2\eta^4}{(18 + k^2\eta^2)^2} - \eta^2\beta\right]
\end{aligned} \tag{4.4.14}$$

In Section 4.3 we have seen that $\beta = 0$ during the matter epoch, and so the equation we end up with is

$$\frac{\eta^4}{6}\psi'' + \eta^3\psi' = 0. \tag{4.4.15}$$

This has a solution that is constant, since $\psi'' = \psi' = 0$, and it leads to the final solution (with no differences with respect to the neutrinoless case)

$$\phi(\eta, \vec{k}) = \psi(\eta, \vec{k}) = \frac{9}{10}\phi(\eta_i, \vec{k}). \tag{4.4.16}$$

4.5 Small Scales Evolution

In this section we will discuss small scales modes, which crossed the horizon during the radiation-dominated era and far away from the radiation-matter equality, i.e. the horizon crossing happened at $y_{h.c.} \ll y_{EQ}$; in terms of k we are studying only modes with $k \geq 10\,k_{EQ}$, with k_{EQ} defined immediately after Eq. (3.4.6). The solutions for modes with $k \approx k_{EQ}$ can be found only numerically, while in this case we want to provide an analytical solution to understand the relations between the quantities involved. Crossing the horizon during the radiation epoch means that in the first stages of the evolution we cannot neglect the k terms in the Einstein and in the Boltzmann equations and this will lead, as we will see, to an osccilating and damped behaviours of the potential ϕ and ψ which go as $j_1(k\eta/\sqrt{3})$, where j_1 is the first spherical Bessel function. The main complication in this case is given by the fact that in principle we need to study the evolutions not only of θ_r during the radiation-dominated era, but also of δ and v which characterize the cold dark matter, because now it is not true that $\delta' = 3\psi'$ or that $v = 0$ as for large scales, and their values at the end of the age dominated by radiation will be important because they will influence the evolution of ϕ and ψ during the matter epoch. In addition, even if $\rho_{dm} \leq \rho_r$ during the radiation-dominated era, we cannot state that $\rho_{dm}\delta \leq \rho_r\theta_{r,0}$, because the dark matter density contrast grows quite fast even before the equality, therefore we should find the time, during the radiation-dominated era, at which the matter contribution at the first order in perturbation theory becomes more important than the radiation's one. So we have to study the evolution before and after such time and matching the two solutions obtained at this $y_{matching}$. Another assumption we make is that the matching between this two solutions has to be done at a time far away from the equivalence and from the horizon crossing, $y_{h.c.} \ll y_{matching} \ll 1$. Even if all the method discussed above is necessary to determine the full solutions for ϕ and ψ from η_i until η_0, for evaluating the $\tilde{C}_{\ell,S}$ coefficients defined in Eq. (2.7.26) we need to know only the scalar potentials before $y_{matching}$, because they decay very fast after the horizon crossing and the contributions at large times to the $\tilde{C}_\ell$ is negligible, because they are almost zero. Hence what we have done is considering only the sub-horizon evolution of the potentials during the radiation-dominated era, until they assume a value very close to zero. As we will see, also in this case neutrinos play a significative role: their presence generates a difference in $(\phi + \psi)/2$ with respect to ϕ and ψ in the $N_\nu = 0$ case. However this damping will be negligible once the modes are very close to zero.'

4.5.1 Horizon-crossing

We have already seen that the modes out of the horizon are constant, with the initial conditions defined in Section 4.2. If the scalar perturbations cross the horizon during the radiation-dominated era they influence and are influenced by the quantities that dominate the contribution to the energy density of the Universe, i.e. the perturbation to the photons and to the neutrinos distribution functions, θ and $\mathcal{N}$, so the system of equations we need to consider is given by the Boltzmann equations for these two relativistic species, Eqs. (A.2.37) and (A.2.40), the Einstein equation (A.2.43) and the relation between the scalar perturbations, Eq. (4.3.8), which comes from the longitudinal traceless Einstein equation, provided by a non-null neutrino quadrupole $\mathcal{N}_2$.
We use Eqs (A.2.43) and (A.2.37) (with the same discussion about neglecting higher order

multipoles used in Section 4.4):

$$\begin{cases} k^2\psi = -\frac{3}{2}\mathcal{H}^2\frac{1}{\rho_{tot}}\Big[\rho_{dm}\delta + 4\rho_r\theta_{r,0} + \frac{3\mathcal{H}}{k}\Big((\rho_{dm}v + 4\rho_r\theta_{r,1}\Big)\Big] = -6\mathcal{H}^2\Big[\theta_{r,0} + \frac{3\mathcal{H}}{k}\theta_{r,1}\Big], \\ \theta'_{r,0} + k\theta_{r,1} = \psi', \\ \theta'_{r,1} - \frac{k}{3}\theta_{r,0} = \frac{k}{3}\phi = \frac{k}{3}\psi + \frac{k}{3}\beta. \end{cases} \tag{4.5.1}$$

Using the fact that $\mathcal{H} = \frac{1}{\eta}$, when radiation dominates, we find

$$\begin{cases} \theta_{r,0} = -\frac{k^2\eta^2}{6}\psi - \frac{3}{k\eta}\theta_{r,1} \\ 0 = -\frac{k^2\eta}{3}\psi - \frac{k^2\eta^2}{6}\psi' + \frac{3}{k\eta^2}\theta_{r,1} - \frac{3}{k\eta}\theta'_{r,1} + k\theta_{r,1} - \psi' \\ 0 = \theta'_{r,1} + \frac{1}{\eta}\theta_{r,1} - \frac{k}{3}\psi\Big(1 - \frac{k^2\eta^2}{6}\Big) - \frac{k}{3}\beta \end{cases}$$

$$\begin{cases} \theta_{r,0} = -\frac{k^2\eta^2}{6}\psi - \frac{3}{k\eta}\theta_{r,1} \\ \theta_{r,1} = \frac{k\eta^2}{6}\Big(\psi' + \frac{1}{\eta}\psi\Big) + \frac{k\eta}{\Big(6+k^2\eta^2\Big)}\beta \\ 0 = -\frac{3}{k\eta}\Big[\frac{k\eta}{3}\psi'\Big(1 + \frac{k^2\eta^2}{6}\Big) + \frac{k}{3}\frac{k^2\eta^2}{3}\psi + \theta'_{r,1} - \theta_{r,1}\frac{1}{\eta}\Big(1 + \frac{k^2\eta^2}{3}\Big)\Big] \end{cases}$$

$$\begin{cases} \theta_{r,0} = -\frac{k^2\eta^2}{6}\psi - \frac{3}{k\eta}\theta_{r,1} \\ \theta_{r,1} = \frac{k\eta^2}{6}\Big(\psi' + \frac{1}{\eta}\psi\Big) + \frac{k\eta}{\Big(6+k^2\eta^2\Big)}\beta \\ 0 = \frac{\eta}{2}\Big(\psi' + \frac{\psi}{\eta}\Big) + \frac{\eta^2}{6}\Big(\psi'' + \frac{\psi'}{\eta} - \frac{\psi}{\eta^2}\Big) + \frac{6-k^2\eta^2}{(6+k^2\eta^2)^2}\beta + \frac{\eta}{6+k^2\eta^2}\beta' + \frac{1}{6+k^2\eta^2}\beta - \frac{\psi}{3}\Big(1 - \frac{k^2\eta^2}{6}\Big) - \frac{\beta}{3} \end{cases} \tag{4.5.2}$$

The equation we want to solve is the last one that can be written as

$$0 = \psi'' + \frac{4}{\eta}\psi' + \frac{k^2}{3}\psi + \frac{6}{\eta^2}\Big\{-\beta\frac{k^4\eta^4 + 12k^2\eta^2}{3(6+k^2\eta^2)^2} + \beta'\frac{\eta}{6+k^2\eta^2}\Big\}. \tag{4.5.3}$$

The solution of the associated homogeneous equation ($\beta = 0$) is well-known [22] and it is given in terms of the spherical Bessel function:

$$\psi = 3\frac{sin\frac{k\eta}{\sqrt{3}} - \frac{k\eta}{\sqrt{3}}cos\frac{k\eta}{\sqrt{3}}}{\Big(\frac{k\eta}{\sqrt{3}}\Big)^3}\psi_i. \tag{4.5.4}$$

The full system of equations we need to solve is

$$\begin{cases} 0 = \psi'' + \frac{4}{\eta}\psi' + \frac{k^2}{3}\psi + \frac{6}{\eta^2}\Big\{-\beta\frac{k^4\eta^4+12k^2\eta^2}{3(6+k^2\eta^2)^2} + \beta'\frac{\eta}{6+k^2\eta^2}\Big\} \\ \phi = \psi + \beta \\ \beta = -\phi_i\frac{f_\nu(\eta_i)a_i^2}{a^2}\Big\{\frac{2}{5} + 6\frac{1}{(k\eta_i)^2}\Big[\Big(\frac{3}{[k(\eta-\eta_i)]^2} - 1\Big)\frac{sin[k(\eta-\eta_i)]}{k(\eta-\eta_i)} - \frac{3cos[k(\eta-\eta_i)]}{[k(\eta-\eta_i)]^2}\Big]\Big\} \end{cases} \tag{4.5.5}$$

The solutions are computed numerically and reported in Figure 4.5.1, while in Figure 4.5.2 we have shown explicitly the effect due to neutrinos on very small scales, making a comparison with the scalar perturbations in the case without neutrino, $N_\nu = 0$. We can see that when a mode crosses the horizon deep in the radiation-dominated era, the scalar perturbations decay, in analogy with the form given by Eq. (4.5.4). Once we have seen that the solutions approach zero, we can stop here the discussion, because we know that during the matter dominated era the solutions are constant, thus they are still zero for later times.

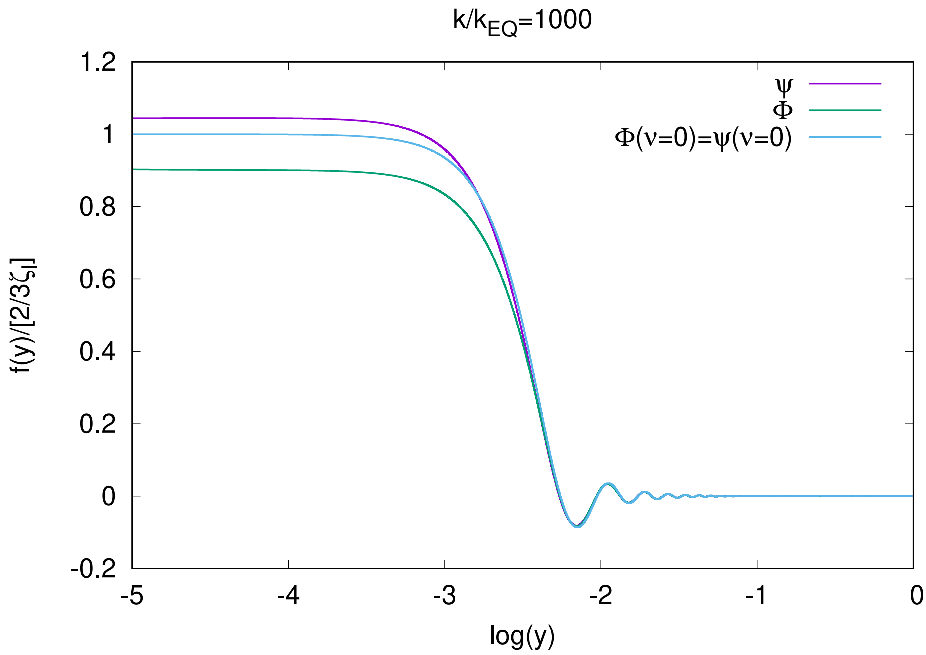

Figura 4.5.1: Numerical evolution of the perturbations ϕ and ψ cfor $N_\nu = 3$ and $N_\nu = 0$. Before crossing the horizon, the solutions are constant, as seen in Section 4.4, while once they enter the horizon in the radiation era they start decaying and oscillating according to [22], with some little differences given by the initial conditions and by slightly modified equations of motion. We have used the coordinate y instead of η used in the analytic form of the differential equations coherently with Figures 4.4.1 and 4.4.2.

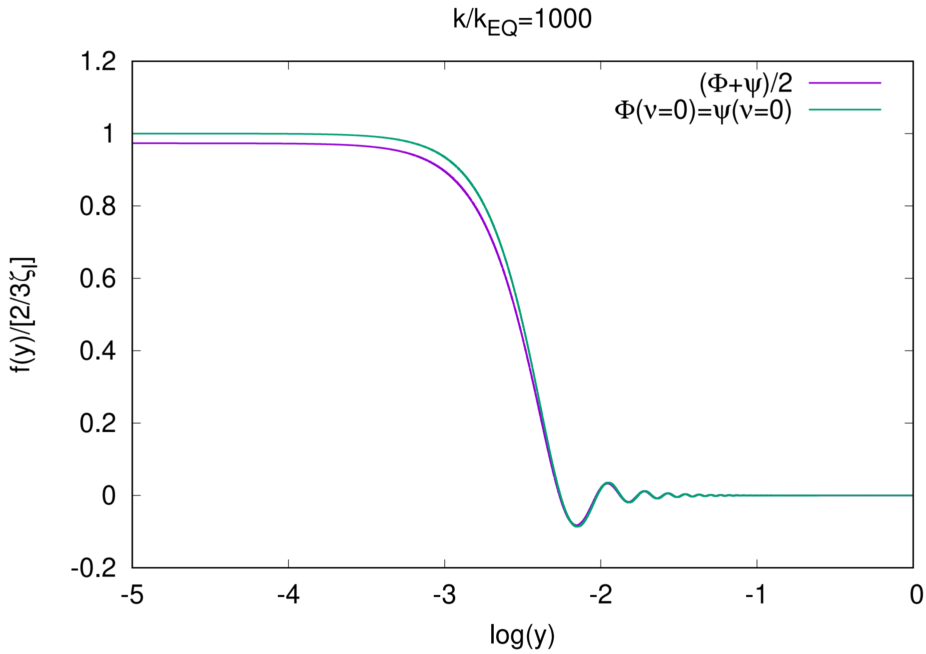

Figura 4.5.2: Plot of $(\phi + \psi)/2$ for $N_\nu = 0$ and $N_\nu = 3$. With respect to the large scales case, we see that the two solutions go to zero with a decreasing mode once they cross the horizon during the radiation dominated era. Also in this case we can still see a difference between the two functions by varying the number of neutrino generations considered.

Capitolo 5

Effect of neutrinos on angular power spectra of the SGWB

5.1 CLASS

In this chapter we will evaluate explicitly the impact of neutrinos on the anisotropies of the gravitational waves background of cosmological origin, through the effects of neutrinos on the cosmological perturbations studied in the previous chapters. We will quantify numerically the impact on the SGWB angular spectra $\tilde{C}_{\ell,S}$ and $\tilde{C}_{\ell,T}$[1] using the Cosmic Linear Anisotropy Solving System (CLASS) [27].
CLASS is a Boltzmann code, written in C, that computes many cosmological quantities of interest, as the angular power spectrum of the CMB or the matter power spectrum, with high precision. Essentially, through the simulation of linear perturbations dynamics, the code is able to determine CMB and large scale structure observables, with an accuracy that can be set by the user by changing some few parameters. CLASS is divided in different modules, each of which plays a different role, for instance the "background module" evaluates the evolution of the background quantites in a FLRW universe, like the scale factor a or the unperturbed energy densities of the different particle species. In order to compute the observables in different cosmological models, CLASS accepts a various range of inputs: it is possible to vary the values of some parameters as the Hubble constant or the baryon density in some specific intervals, in this way the user can understand the effects of their changings. That is not all: it is also possible to include additional particle species in the code, for example non-cold dark matter relics or decaying cold dark matter, or for instance it is possible to switch the value of the curvature of the universe from a null to a positive or to a negative value. In addition, CLASS admits the possibility to choose between two different gauges, the newtonian one we have used until now, and the synchronous one, defined, for instance, in [59], therefore the code easily adapts to different formalisms. One of the most important features of CLASS is its flexibility: there is no hard coding, in the sense that all the equations written explicitly in the code are true in

[1]We have neglected in this discussion the anisotropies which derive from the initial conditions on the perturbation of the graviton distribution function, $\tilde{C}_{\ell,I}$.

all cosmologies, as the Friedmann equations or the Boltzmann equations for the different particle species. Moreover, because of its clear structure, determined by the subdivision in modules, it is very easy to find out where the various equations are written, making simpler to modify them.
This is particularly important for us, because we are interesting in evaluating the following two angular power spectra,

$$\frac{\tilde{C}_{\ell,S}(\eta_0)}{4\pi} = \int \frac{dk}{k} P^{(0)}(k) \Big\{ T_\phi(\eta_i, k) j_\ell[k(\eta_0 - \eta_i)] + \\ + \int_{\eta_i}^{\eta_0} d\eta \Big[T'_\phi(\eta, k) + T'_\psi(\eta, k) \Big] j_\ell[k(\eta_0 - \eta)] \Big\}^2, \tag{5.1.1}$$

$$\frac{\tilde{C}_{\ell,T}(\eta_0)}{4\pi} = \sum_{\alpha=\pm 2} \int \frac{dk}{k} P^{(\alpha)}(k) \Bigg[\int_{\eta_i}^{\eta_0} d\eta h'(\eta, k) \frac{j_\ell[k(\eta_0 - \eta)]}{k^2(\eta_0 - \eta)^2} \frac{1}{4} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \Bigg]^2,$$

which slightly differs from the ones known for the CMB [66], which are the ones computed by CLASS. We have proceeded then by modifying a little bit the code, changing the source functions which determine the anisotropy spectra, i.e. we have changed the corrispective of (2.7.2) for the CMB. To do that it has been necessary to lower the CLASS default intial time of integration (set to the recomibation time), and we had to change the arguments of the integrals of the analogue of (2.7.2). This procedure is discussed in detail in Appendix B.
We have divided also in this case the discussion between the tensor and the scalar modes, analyzing them separately in Sections 5.2 and 5.3. In Section 5.4 we have made a brief comparison between the tensor and the scalar modes, and we have discussed possible measurements of stochastic gravitational wave background anisotropies.

5.2 Neutrino effects on the scalar angular power spectra

For the scalar angular power spectrum we have computed only the contribution integrated over η to the anisotropies because we expect that it gives the dominant contribution, thence in this section when we speak about $\tilde{C}_{\ell,S}$ we refer to

$$\frac{\tilde{C}_{\ell,S}(\eta_0)}{4\pi} = \int \frac{dk}{k} P^{(0)}(k) \Big\{ \int_{\eta_i}^{\eta_0} d\eta \Big[T'_\phi(\eta, k) + T'_\psi(\eta, k) \Big] j_\ell[k(\eta_0 - \eta)] \Big\}^2, \tag{5.2.1}$$

where we have neglected the $T_\phi(\eta_i, k)$ term. In this way we have obtained the plot depicted in Figure 5.2.1.
Before examining the differences between the two cases, $N_\nu = 0$ and $N_\nu = 3$, we want to justify the trend of the two functions. In other words, we want to explain why they reach their maximum value at small ℓ, we want also to motivate why they have a characteristic peak near $\ell = 100$ and why they are almost null for large ℓ's.
First of all we evaluate the maximum k that contributes to a given multipole, using an argument analogue to the one seen in Section 3.4. When we consider a multipole of order

ℓ, we are considering contributions on an angular scale $\theta = 2\pi/\ell$, therefore the maximum contribution is given by the scale $\lambda = 2\pi/k$, and $\lambda = \theta d_{com}$, where d_{com} is the comoving distance from us of the emission point, which occurs at t_i (the cosmic time at which neutrino decoupled), which is

$$d_{com} \equiv \int_{t_i}^{t_0} dt' \frac{1}{a(t')} = \eta_0 - \eta_i \approx \eta_0. \tag{5.2.2}$$

In this way we know that the multipole ℓ is sensitive to modes around the wavenumber k which satisfies the relation $\ell = k\eta_0$. We recall briefly that in this section we will use as units for η the Mpc and for k the Mpc^{-1}, with $\eta_0 \approx 1.4 \times 10^4$ Mpc, $\eta_{EQ} \approx 100$ Mpc and $k_{EQ} \approx 8 \times 10^{-3}$ Mpc^{-1}.
After this, we can explain the structure of the spectrum in the following way: from Eq. (5.2.1) we see that $\tilde{C}_{\ell,S}$ is sensitive to variations of the scalar perturbations with respect to proper time, therefore the bigger the variations for a given k, the bigger the anisotropies for the correspondent ℓ. The two important variations of the scalar potentials we consider are the one which occurs during the transition from a radiation dominated era to a matter dominated era, discussed in 4.4, and the evolution during the dark energy epoch, where the scalar potentials decrease [22].
In the first case, the biggest variations are provided by modes with $k \approx k_{EQ}$, while modes with $k \ll k_{EQ}$ change less, and modes with $k \gg k_{EQ}$ are averaged out when integrated [69]. This is the reason why we observe a peak around $\ell = 100$: it represents modes that crossed the horizon around the time of equality.
For analogue reasons, the contributions to the anisotropies from late times integrations decrease with ℓ: the large-scale angular power spectra are the most affected by the variations of the scalar potentials during the dark energy dominated era [70]. The large values of $\tilde{C}_{\ell,S}$ for small ℓ are determined precisely by this effect.

In Chapter 4 we have learnt the role of neutrinos in the evolution of the scalar metric perturbations $\phi(\eta, k)$ and $\psi(\eta, k)$. In order to understand properly neutrinos total effect on the angular power spectrum (5.2.1), we need also to keep into account how neutrinos affect the evolution of background quantities, such as the scale factor a [67].
First of all, when we remove neutrinos from the particle content in the numerical simulation, we need to decide which source of energy density would compensate the lack of neutrinos. We have fixed the energy densities of the photons and of the non-relativistic matter, Ω_γ and Ω_m, varying the dark energy density Ω_Λ, using the balance equation

$$\Omega_\gamma + \Omega_m + \Omega_\nu + \Omega_\Lambda = 1. \tag{5.2.3}$$

Imposing $\Omega_\nu = 0$ has two important consequences:

- the fraction of radiation energy density is reduced by passing from $N_\nu = 3$ to $N_\nu = 0$, this means that we are anticipating the radiation/matter equality. This fact is very important because for three neutrino generations we have more modes which enter the Hubble radius during the radiation dominated era, therefore we have an higher number of modes damped after they crossed the horizon during radiation domination.

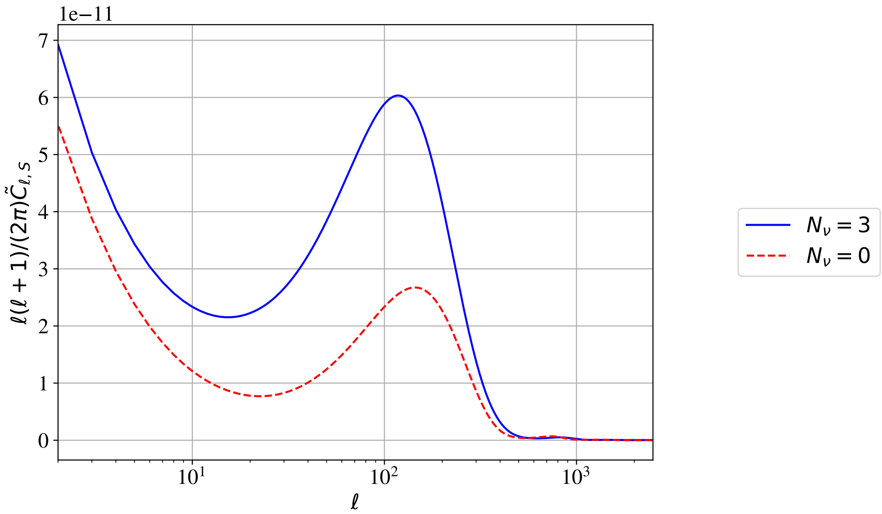

Figura 5.2.1: Plot of the scalar contribution to the gravitons angular power spectra $\tilde{C}_{\ell,S}$ for $N_\nu = 0$ and $N_\nu = 3$, neglecting the initial term given by $T_\phi(\eta_i, k)$. We recognize an almost identical behaviour with respect to the Integrated Sachs Wolfe (ISW) contribution to the CMB scalar angular power spectrum [68], characterized by a peak around $k \approx k_{EQ}$ ($l \approx 110$) and by a growing spectrum for small ℓ, determined by late integrated effects.

From Figures 4.4.2 and 5.2.2 we infer that neutrino damping on the scalar modes is maximum for $k \approx k_{EQ}$. Therefore we expect the maximum difference between the angular power spectra for $N_\nu = 0$ and $N_\nu = 3$ around the $\tilde{C}_{\ell,S}$ peak, i.e. at $\ell = 100$. In addition, we see a shift on the right of the spectrum for $N_\nu = 0$. This is due to the fact that the maximum contribution for $N_\nu = 0$ comes from k_{EQ} for zero neutrino generations, which is bigger with respect to k_{EQ} for $N_\nu = 3$, therefore the peak corresponds to an higher ℓ too;

- the fraction of dark energy density is enhanced with zero neutrinos, this anticipates the equivalence between matter and dark energy. An immediate consequence is that for $N_\nu = 0$ the potentials start decay during the dark energy epoch earlier than for $N_\nu = 3$. This would have generated an enhancement for low ℓ of $\tilde{C}_{\ell,S}$ in the $N_\nu = 0$ case, but the increase of Ω_Λ is so small that this effect is completely negligible, as you can see in Figure 5.2.2.

5.3 Neutrino effects on the tensor angular power spectra

The tensor angular power spectrum is determined by the integral

$$\frac{\tilde{C}_{\ell,T}(\eta_0)}{4\pi} = \sum_{\alpha=\pm 2} \int \frac{dk}{k} P^{(\alpha)}(k) \left[\int_{\eta_i}^{\eta_0} d\eta h'(\eta, k) \frac{j_\ell[k(\eta_0 - \eta)]}{k^2(\eta_0 - \eta)^2} \frac{1}{4} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \right]^2, \tag{5.3.1}$$

and the result we have obtained is depicted in Figure 5.3.1.
It has been proved that for the tensor modes the angular power spectrum is constant for

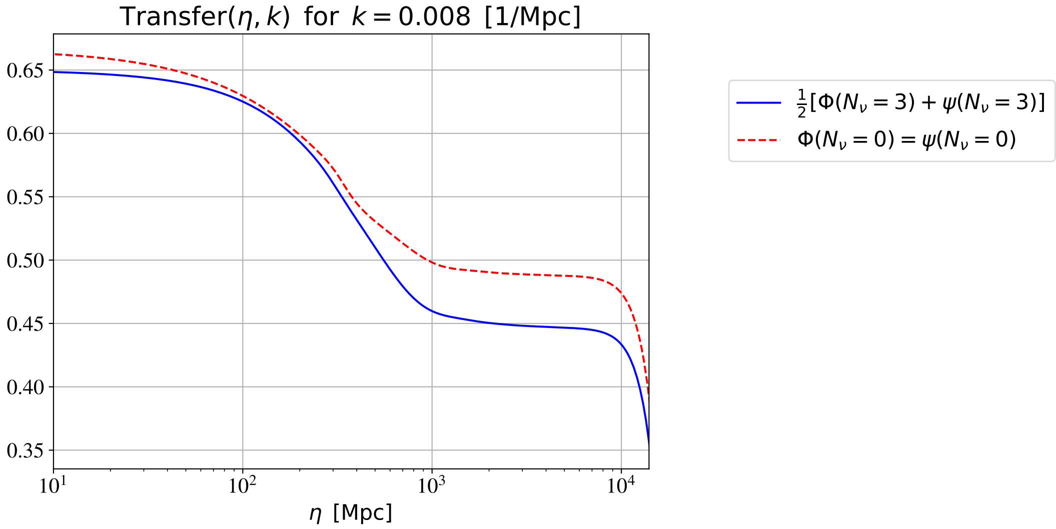

Figura 5.2.2: Plot of ϕ and ψ for $k = k_{EQ}$ and for $N_\nu = 0$ and $N_\nu = 3$. We can see how the variations of the potentials ϕ and ψ for modes with $k \approx k_{EQ}$ are widely larger in the $N_\nu = 3$ case than the $N_\nu = 0$ case. The two values of Ω_Λ do not differ so much, thus we cannot see any difference in the decay of the potentials during the dark energy dominated era.

large scales, until $\ell \approx 100$, then it decreases very fast, as ℓ^{-2} or ℓ^{-4} [71]. This explains completely the spectrum we have found. Also in this case we can justify the results by thinking that $\tilde{C}_{\ell,T}$ is sensitive to the variations with respect to the conformal time of the tensor perturbations of the metric, therefore the bigger the decays, the bigger the anisotropies.
The differences between the two different $\tilde{C}_{\ell,T}$ in the figure can be explained by thinking to the results of Chapter 3. We have seen in fact that neutrinos damp tensor modes by a factor 0.80 for modes that crossed the horizon in the radiation era deeply enough to reach a stable oscillating solution, for instance for $k/k_{EQ} \geq 10$. We have seen also that the analogous effect for large scales is almost negligible. These two aspects have to be combined with the fact that imposing $N_\nu = 0$ anticipates the time of equality between matter and radiation.
The reasons of the differences between the two power spectra are now very clear: the $\tilde{C}_{\ell,T}$'s are equal for $l \leq 30$, because the damping effect of neutrinos on the tensor modes is almost null; around $\ell = 30$ the $N_\nu = 0$ spectrum starts decaying faster, this is due to the fact that the modes enter in the radiation dominated era earlier; at the end we see, for large ℓ's, that the $\tilde{C}_{\ell,T}$ peaks reach higher values for $N_\nu = 0$ with respect to $N_\nu = 3$, this is due to neutrinos damping effect on tensor modes for small scales.

5.4 Future Perspectives

In the two previous sections we have seen the effects of neutrinos on the anisotropies of the stochastic gravitational wave background.

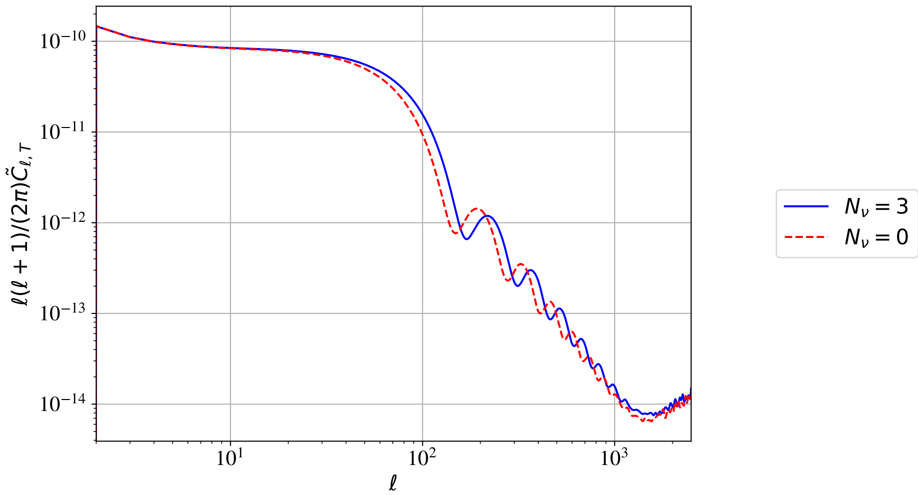

Figura 5.3.1: Plot of the tensor contribution to the graviton angular power spectra $\tilde{C}_{\ell,T}$ for $N_\nu = 0$ and $N_\nu = 3$. For large values of ℓ we can see that the $N_\nu = 0$ solution is enhanced by a factor which depends on the effect studied in Section 3.3. We can distinguish a clear shift in the oscillations between the two spectra, which depends on the different η_{EQ} in the models with zero and three neutrino generations.

For the scalar contribution to the angular power spectrum, neutrinos enhance the amplitudes of $\tilde{C}_{\ell,S}$, giving rise to larger anisotropies with respect to the case with no neutrinos in the universe. This effect is bigger for multipoles around $\ell = 100$, while it becomes negligible for $\ell \geq 500$. We have explained this enhancement using the fact that adding ultra-relativistic particle species in a cosmological model determines a delay in the time of radiation-matter equality, this allows small scale modes to give more significative contributions to the anisotropies.
On the other hand, the tensor contribution to the angular power spectrum is sensitive to neutrinos only for $\ell \geq 30$, while for smaller ℓ, the contributions with zero and three neutrino generations are identical. For $\ell \geq 100$ we have found that neutrinos damp the angular power spectrum. This effect is due to the fact that neutrinos damp the amplitudes of the tensor perturbations of the metric, and such an effect is appreciable only on small scales, thus for high multipoles ℓ.
In Figures 5.4.1 and 5.4.2 we have plotted the metric perturbations evolution for different wavenumbers, in order to check the consistency of the results obtained in Chapters 3 and 4.
From an experimental point of view, nowadays interferometers are setting smaller and smaller bound limits on the energy density of the stochastic gravitational wave background [72]. Such a background could be of astrophysical or of cosmological origin. The main difference between the two signals is the frequency dependence: the cosmological processes that generated the background are characterized by some specific frequencies, therefore, with proper techniques [73], we should be able to separate the two components that generate the background. Future space-based interferometers, as LISA [6], and ground-based detectors, as Einstein Telescope (ET) [74], are expect to reach sensitivities higher enough to measure

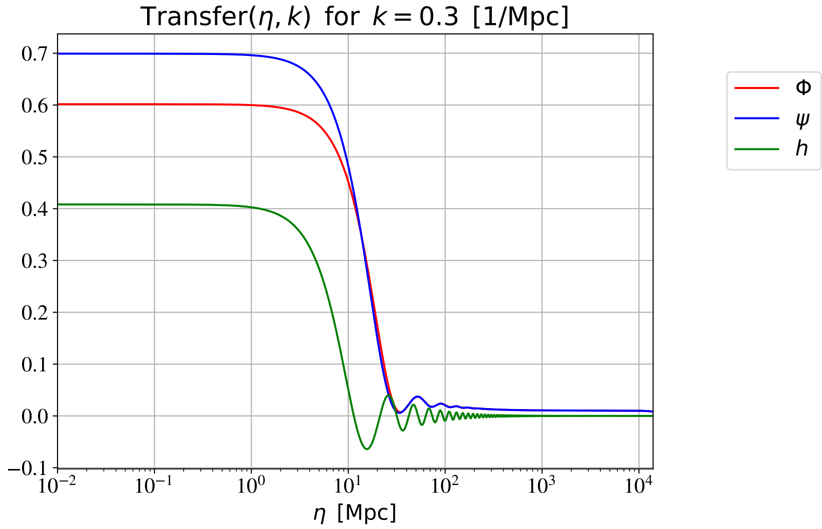

Figura 5.4.1: Plot of the metric perturbations for $N_\nu = 3$ and $k = 0.3$. We have not normalized the potentials with respect to their initial values as in Chapters 3 and 4, in this way we can see explicitly the differences between the initial values of the scalar and the tensor metric perturbations.

the stochastic gravitational waves background of cosmological origin.
We expect then that the subsequent step for future interferometers will be the measurement of the anisotropies of this background, for instance by the correlation of the signals of more detectors [75]. We have also seen that the quantity which characterizes a stochastic gravitational wave background is the energy density per logarithmic frequency,

$$\bar{\Omega}_{GW}(f) \equiv \frac{f}{\rho_{crit}} \frac{d\rho_{GW}}{df}. \tag{5.4.1}$$

The physical observable related to $\bar{\Omega}_{GW}$ will be the density constrast δ_{GW}, defined in Eq. (2.6.4), which represents the perturbation of the quantity $\bar{\Omega}_{GW}$ along a certain direction of observation. In C we have related such a quantity to the $\tilde{C}_{\ell,S}$, $\tilde{C}_{\ell,T}$ power spectra found in the previous section.
To conclude, even if future detectors do not reach angular resolutions higher enough to cover the entire ℓ range of the angular power spectra plotted in the previous two sections, we can still find out important information about the evolution of the gravitons through a perturbed universe.

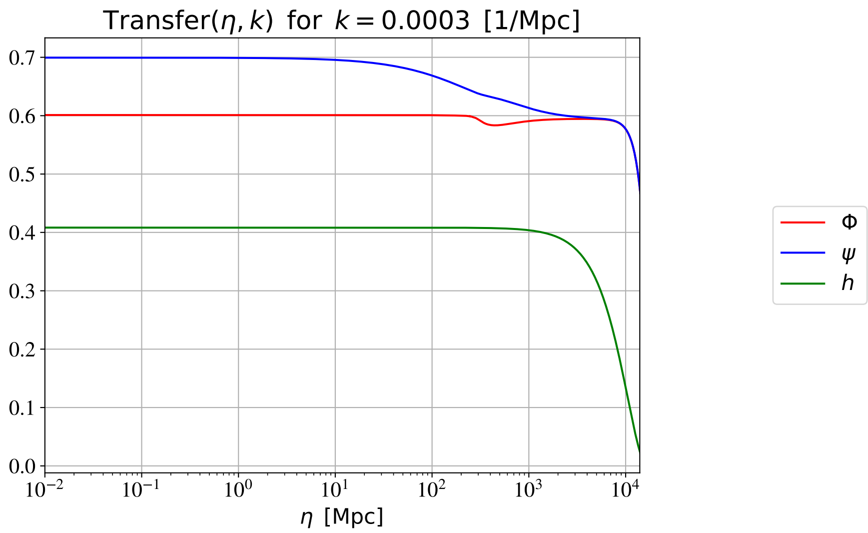

Figura 5.4.2: Plot of the metric perturbations for $N_\nu = 3$ and $k = 0.0003$. In this picture we can see that the tensor perturbations varies more than the scalar ones for large scales. This is due to the fact that the biggest decrease for the scalar perturbations is due to the dark energy dominated era, while the tensor perturbations decay only when they cross the horizon.

Capitolo 6

Conclusions

In this thesis we have performed a complete and consistent treatment on the SGWB anisotropies. We have focused especially on the role played by neutrinos in the enhancement/damping of these anisotropies, trying to interpretate the final results by using the knowledge obtained during the thesis.
In Chapter 1 we have put solid basis for studying gravitational waves, specifically we have seen that we can describe properly the propagation of gravitational waves on a curved background by using the shortwave approximation, assuming that the characteristic lenght of the waves is much smaller than the background's one. This assumption allows us to use geometric optics for the gravitational waves, ensuring the possibility to describe the waves as massless particles that follow null geodesics determined by the background. In this chapter we have also seen the GW production mechanism due to amplification of the quantum fluctuation of the metric during the inflation. This is particularly relevant because it gives the initial condition on the number of gravitons at the end of the inflation, confirming that we can talk about a stochastic background. Chapter 2 provides us an espression for the angular power spectra of the SGWB, Eqs. (2.7.26), through the solution of the Boltzmann equation for the graviton distribution function at the first order in perturbation theory. By looking a the expressions we end up with, it has been possible to distinguish the analogies between the CMB and SGWB and to hypotize a similiar behaviour of the angular power spectrum. Chapter 3 and 4 show the effect of neutrinos in the evolution of the scalar and the tensor perturbations. Figures 3.3.2 show that tensor modes are damped for small scales by adding neutrinos to the particle content of the Universe, while this effect for large scales, Figure 3.4.1, is negligible. For the scalar perturbations we have found two interesting equations, Eqs. 4.4.11 and 4.5.5, which describe the evolutions of the perturbations for large and small scales respectively. The solutions, depicted in Figures 4.4.2 and 4.5.2, are in agreement with the numerical solutions of the full set of equations. For the scalar case we have found significative differences between the case with and without neutrinos, which will play a fundamental role in the angular power spectra of the gravitational waves.
The main and original results of the thesis are given in Chapter 5, where we have studied the effects of neutrinos on the SGWB angular power spectra separately for the scalar and the tensor contributions. The plots, found adapting the CLASS code to the SGWB case,

are depicted in Figures 5.2.1 and 5.3. For the scalar power spectrum, we have evaluated the contribution to the ISW. We can still see clearly the effect of neutrinos: the $N_\nu = 3$ spectrum presents an enhancement with respect to the $N_\nu = 0$ case, this is due to the fact that introducing neutrinos means to retard the equivalence between matter and radiation, therefore more modes will decay by staying longer in the radiation era, giving a larger contribution to the anisotropies. The enhancement reaches its maximum for multipoles of order $\ell \approx 100$, because neutrinos give the larger effects for modes that cross the horizon at the time of radiation/matter equality. This effect is appreaciable also for larger scales, while small scales essentially give no contribution to the anisotropies we found. Therefore we have not found any neutrino effects for large values of ℓ. For the tensor modes the situation is the opposite: we have revealed a damping for small scales (high multipoles), due to neutrino damping of tensor modes discussed in Chapter 3, while for larger scales such effect is negligible and therefore the tensor contributions to the power spectra are identical in the $N_\nu = 0$ and $N_\nu = 3$ cases.
As future perspectives, we would like to improve the modifications to the CLASS code, in order to find the full scalar contribution to the angular power spectrum, making possible to compare the scalar and the tensor contributions with respect to the total anisotropies. The result of this thesis, even being preliminary, shows how future interferometers can have a strong impact, besides on astrophysics and cosmology, also on the particle physics content of the universe.

Appendice A

Einstein equations in the perturbed Universe

A.1 The decomposition theorem

In Section 2.3 we have decomposed the metric perturbations in relation to their transformation rules under spatial transformations, dividing the modes into scalar, vector and tensor. In principle, when we try to evaluate the Einstein tensor for such a generic metric, we need to take into account all the possible combinations in G_{ij} between ϕ, ψ, ω_i and χ_{ij}, finding out how one perturbation can affect the evolution of another one. However, if we stop at linear order in perturbation theory, as we have done for all this thesis, we can prove [76] the decomposition theorem, which states that any linear differential equation, at most of the second order in the derivatives, can be decomposed into mutually decoupled equations, each of which contains only one type of perturbations (e.g. only scalars, only vectors or only tensors) if the spatial part of the background metric has a constant curvature (which is the case for the unperturbed FLRW background, which has, at maximum, $K \neq 0$, which is still constan however). This means that when we write down the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{A.1.1}$$

we can compute the Einstein equations separately for the various modes, imposing that certain modes are zero when we write down the equations for different modes; this is precisely what we will do in this appendix, diving the work in two subsections, the first one for the computations of the scalar part, the latter for the computation of the tensor part, vector modes are set to zero automatically by the gauge choice and by the expansion of the Universe, as already stated. A brief comment about the stress-energy tensor is needed too: clearily each energy-momentum tensor depends both on the tensor and the scalar modes, thus we can state that it has a component whom evolution is determined by the scalar perturbations and another one which is determined by the tensor perturbations, because of the equation for these two parts are decoupled, we can think that the two evolves independently and then we can decompose, at linear order, the total distribution function into a part which corresponds to the tensor modes, and into a component which

is determined by the scalar modes. This is the case of neutrinos, the only particle species which is sensitive to the tensor modes, for this we have used two distinct notations for the distribution function perturbations, $\delta F(\eta, \vec{x}, \vec{q})$ for the tensor modes and $\mathcal{N}(\eta, \vec{x}, \vec{q})$ for the scalar modes.
In the next sections we will compute separately the Einstein equations for the scalar and the tensor modes.

A.2 Scalar perturbations

A.2.1 Geometrical part

The aim of this section is finding the explicit form, in function of the scalar perturbations of the FLRW metric, the Einstein tensor $G_{\mu\nu}$. The starting point is the line element in the Poisson gauge, introduced in Section Eq. (2.3.7):

$$ds^2 = a^2(\eta)[-(1+2\phi)d\eta^2 + (1-2\psi)\delta_{ij}dx^i dx^j], \tag{A.2.1}$$

therefore the metric and its inverse are

$$g_{\mu\nu} = a^2\big(-(1+2\phi), \delta_{ij}(1-2\psi)\big), \qquad g^{\mu\nu} = \frac{1}{a^2}\big(-(1-2\phi), \delta^{ij}(1+2\psi)\big). \tag{A.2.2}$$

We evaluate explicitly all the Christoffel'symbols, defined by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{A.2.3}$$

We start by evaluating the affine connection for $\mu = 0$:

$$\Gamma^0_{\mu\nu} = \frac{1}{2}g^{0\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = -\frac{1-2\phi}{2a^2}(\partial_\mu g_{0\nu} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}) \tag{A.2.4}$$

and we have

$$\begin{aligned}
\Gamma^0_{00} =& -(1-2\phi)[-\mathcal{H}(1+2\phi) - \phi'] = \mathcal{H} + \phi', \\
\Gamma^0_{i0} =& \Gamma^0_{0i} = \partial_i\phi, \\
\Gamma^0_{ij} =& (1-2\phi)[\mathcal{H}(1-2\psi) - \psi']\delta_{ij} = [\mathcal{H}(1-2\psi-2\phi) - \psi']\delta_{ij}.
\end{aligned} \tag{A.2.5}$$

The remaining case is the one for $\mu = i$:

$$\Gamma^i_{\mu\nu} = \frac{1+2\psi}{2a^2}\delta^{ik}\{\partial_\mu[a^2(1-2\psi)]\delta_{k\nu} + \partial_\nu[a^2(1-2\psi)]\delta_{\mu k} - \partial_k g_{\mu\nu}\} \tag{A.2.6}$$

and then we have

$$\begin{aligned}
\Gamma^i_{00} =& \partial^i\phi, \\
\Gamma^i_{j0} =& (1+2\psi)\delta^i_j[\mathcal{H}(1-2\psi) - \psi'] = [\mathcal{H} - \psi']\delta^i_j, \\
\Gamma^i_{jk} =& (\partial^i\psi\delta_{jk} - \partial_j\psi\delta^i_k - \partial_k\psi\delta^i_j).
\end{aligned} \tag{A.2.7}$$

We recall the form of the Riemann tensor,

$$R^{\rho}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma_\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}, \tag{A.2.8}$$

and of the Ricci tensor,

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}. \tag{A.2.9}$$

From this we can find the explicit form of the Ricci tensor, which derives from contraction of the Riemann tensor:

$$\begin{aligned}
R_{00} =& \partial_\sigma \Gamma^{\sigma}_{00} - \partial_0 \Gamma^{\sigma}_{\sigma 0} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{00} - \Gamma^{\sigma}_{0\lambda}\Gamma^{\lambda}_{\sigma 0} = \\
=& [\partial_0 \Gamma^0_{00} + \partial_i \Gamma^i_{00}] - [\partial_0 \Gamma^0_{00} + \partial_0 \Gamma^i_{i0}] + [\Gamma^0_{00}\Gamma^\sigma_{\sigma 0} + \Gamma^i_{00}\Gamma^\sigma_{\sigma i}] - [\Gamma^0_{00}\Gamma^0_{00} + 2\Gamma^i_{00}\Gamma^0_{i0} + \Gamma^i_{0j}\Gamma^j_{0i}] = \\
=& [\mathcal{H}' + \phi'' + \partial_i\partial^i\phi] - [\mathcal{H}' + \phi'' + 3(\mathcal{H}' - \psi'')] + \\
& + [(\mathcal{H} + \phi')(\mathcal{H} + \phi' + 3(\mathcal{H} - \psi')) + \partial^i\phi(\partial_i\phi - 3\partial_i\psi)] + \\
& - [(\mathcal{H} + \phi')^2 + 2\partial^i\phi\partial_i\phi + 3(\mathcal{H} - \psi')^2] = \\
=& \nabla^2\phi + 3\psi'' - 3\mathcal{H}' + 3\mathcal{H}(\phi' - \psi') = \nabla^2\phi + 3\psi'' - 3\frac{a''}{a} + 3\mathcal{H}^2 + 3\mathcal{H}(\phi' + \psi'), \\
R_{0i} =& \partial_\sigma \Gamma^{\sigma}_{0i} - \partial_i \Gamma^{\sigma}_{\sigma 0} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{0i} - \Gamma^{\sigma}_{i\lambda}\Gamma^{\lambda}_{\sigma 0} = \\
=& [\partial_0 \Gamma^0_{0i} + \partial_j \Gamma^j_{0i}] - \partial_i(\Gamma^0_{00} + \Gamma^j_{j0}) + [\Gamma^\sigma_{\sigma 0}\Gamma^0_{0i} + \Gamma^\sigma_{\sigma j}\Gamma^j_{0i}] + \\
& - [\Gamma^0_{i0}\Gamma^0_{00} + \Gamma^j_{i0}\Gamma^0_{0j} + \Gamma^0_{ij}\Gamma^j_{00} + \Gamma^k_{ij}\Gamma^j_{0k}] = \\
=& [\partial_i\phi' - \partial_i\psi'] - [\partial_i\phi' - 3\partial_i\psi'] + \\
& + \{\partial_i\phi[\mathcal{H} + \phi' + 3(\mathcal{H} - \psi')] + \delta^j_i(\mathcal{H} - \psi')(\partial_j\phi - 3\partial_j\psi)\} + \\
& - \{\partial_i\phi(\mathcal{H} + \phi') + \partial_i\phi(\mathcal{H} - \psi') + \partial_i\phi[\mathcal{H}(1 - 2\psi - 2\phi) - \psi'] - 3\mathcal{H}\partial_i\psi\} = \\
=& 2\partial_i\psi' + 2\mathcal{H}\partial_i\phi, \\
R_{ij} =& \partial_\sigma \Gamma^{\sigma}_{ij} - \partial_j \Gamma^{\sigma}_{\sigma i} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{ij} - \Gamma^{\sigma}_{j\lambda}\Gamma^{\lambda}_{\sigma i} = \\
=& \{[(\mathcal{H}'(1 - 2\psi - 2\phi) - 2\mathcal{H}(\phi' + \psi') - \psi'']\delta_{ij} + \partial_k(\partial^k\psi\delta_{ji} - \partial_j\psi\delta^k_i - \partial_i\psi\delta^k_j)\} + \\
& - [\partial_i\partial_j(\phi - 3\psi)] + \\
& + \{[\mathcal{H}(1 - 2\psi - 2\phi) - \psi'][\mathcal{H} + \phi' + 3(\mathcal{H} - \psi')]\delta_{ij} + (\partial_i\phi - 3\partial_i\psi)O(\psi, \phi)\} + \\
& + (-1)\{\partial_i\phi\partial_j\phi + \delta_{jk}[\mathcal{H}(1 - 2\psi - 2\phi) - \psi']\delta^k_i(\mathcal{H} - \psi') + \\
& + \delta^k_j(\mathcal{H} - \psi')\delta_{ik}[\mathcal{H}(1 - 2\psi - 2\phi) - \psi'] + O^2(\psi, \phi)\} = \\
=& \Big[\Big(\frac{a''}{a} + \mathcal{H}^2\Big)(1 - 2\psi - 2\phi) - \mathcal{H}(5\psi' + \phi') - \psi'' + \nabla^2\psi\Big]\delta_{ij} + \partial_i\partial_j(\psi - \phi).
\end{aligned} \tag{A.2.10}$$

The Einstein equations in General Relativity relate the variations of the metric to the form of the stress-energy tensor:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{A.2.11}$$

with the Ricci scalar defined as

$$R \equiv R^{\mu}_{\mu} = g^{\mu\alpha}R_{\alpha\mu}. \tag{A.2.12}$$

We will focus on the right-hand side of the equation in the following section, in this one we are interested only in evaluating the geometrical quantities, i.e. the left-hand side. Because of the covariance of the equation, we can rewrite the equations by rising one of the two indices, for example the μ index, in this way we have a simplification, because $g^{\mu}_{\nu} = g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}$ and then we have a simpler product to calculate at the first order:

$$G^{\mu}_{\alpha} = g^{\mu\alpha} G_{\alpha\nu} = g^{\mu\alpha} R_{\alpha\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\Big(g^{00} R_{00} + g^{i\rho} R_{\rho_i}\Big). \tag{A.2.13}$$

The explicit results are then

$$
\begin{aligned}
G^0_0 =& R^0_0 - \frac{1}{2} g^0_0 (R^0_0 + R^i_i) = \frac{1}{2}(R^0_0 - R^i_i) = \frac{1}{2}(g^{00}R_{00} - g^{ik}R_{ki}) = \\
=& \frac{1}{2a^2}\Big\{ -(1-2\phi)\Big[\nabla^2\phi + 3\psi'' - 3\frac{a''}{a} + 3\mathcal{H}^2 + 3\mathcal{H}(\phi' + \psi')\Big] + \\
& - \Big[\Big(\frac{a''}{a} + \mathcal{H}^2\Big)(1 - 2\psi - 2\phi) - \mathcal{H}(5\psi' + \phi') - \psi'' + \nabla^2\psi\Big]\delta^{ik}\delta_{kj}(1+2\psi) + \\
& - \delta^{ik}\partial_k\partial_j(\psi - \phi)\Big\} = \\
=& \frac{1}{2a^2}\Big[-\nabla^2\phi - 3\psi'' + 3\frac{a''}{a}(1-2\phi) - 3\mathcal{H}^2(1-2\phi) - 3\mathcal{H}(\phi' + \psi') + 3\mathcal{H}(5\psi' + \phi') + \\
& + 3\psi'' - 3\nabla^2\psi - \nabla^2(\psi - \phi) - 3\Big(\frac{a''}{a} + \mathcal{H}^2\Big)(1-2\phi)\Big] = \\
=& \frac{1}{a^2}\Big[\mathcal{H}^2(6\phi - 3) + 6\mathcal{H}\psi' - 2\nabla^2\psi\Big],
\end{aligned}
$$

$$
\begin{aligned}
G^0_i =& R^0_i - \frac{1}{2} g^0_i R = R^0_i = g^{00}R_{0i} = \Big[-\frac{1}{a^2}(1-2\phi)\Big]\Big[2\partial_i\psi' + 2\mathcal{H}\partial_i\phi\Big] = \\
=& -\frac{2}{a^2}\Big[\mathcal{H}\partial_i\phi + \partial_i\psi'\Big],
\end{aligned}
$$

$$
G^i_0 = g^{ij}R_{j0} = \frac{1}{a^2}\delta^{ij}\Big(2\partial_j\psi' + \mathcal{H}\partial_j\phi\Big) = \frac{2}{a^2}\Big(\partial^i\psi' + \mathcal{H}\partial^i\phi\Big)
$$

$$
\begin{aligned}
G^i_j =& R^i_j - \frac{1}{2} g^i_j R = \Big[R^i_j - \frac{1}{2}\delta^i_j\Big(R^k_k + R^0_0\Big)\Big] = \\
=& \frac{1}{a^2}\Big\{\Big[\Big(\frac{a''}{a} + \mathcal{H}^2\Big)(1-2\phi) - \mathcal{H}(5\psi' + \phi') - \psi'' + \nabla^2\psi\Big]\delta^i_j + \partial^i\partial_j(\psi - \phi) + \\
& - \frac{1}{2}\Big[-\nabla^2\phi - 3\psi'' + 3\frac{a''}{a}(1-2\phi) - 3\mathcal{H}^2(1-2\phi) - 3\mathcal{H}(\phi' + \psi') - 3\mathcal{H}(5\psi' + \phi') + \\
& - 3\psi'' + 3\nabla^2\psi + \nabla^2(\psi - \phi) + 3\Big(\frac{a''}{a} + \mathcal{H}^2\Big)(1-2\phi)\Big]\delta^i_j\Big\} = \\
=& \frac{1}{a^2}\Big\{\Big[-2\frac{a''}{a}(1-2\phi) + \mathcal{H}^2(1-2\phi) + 2\mathcal{H}\phi' + 4\mathcal{H}\psi' + 2\psi'' - \nabla^2\psi + \nabla^2\phi\Big]\delta^i_j + \\
& + \partial^i\partial_j(\psi - \phi)\Big\}.
\end{aligned}
\tag{A.2.14}
$$

A.2.2 Stress-energy tensor part

We start by the definition of the energy-momentum tensor given in General Relativity for a system of a certain particle species [22]:

$$
T^\mu_\nu, i = \frac{g_i}{\sqrt{-g}} \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \frac{p^\mu p_\nu}{p^0} f_i(\eta, \vec{x}, \vec{p}), \tag{A.2.15}
$$

where g_i is the degeneracy of the species and it corresponds to the number of the possible helicity states of the particle, while f_i is the distribution function of such a particle, while $g \equiv Det(g_{\mu\nu})$.
We start recalling that in this discussion we are interested only in the neutrino damping on the scalar perturbations, and so we can write, from Eqs (2.4.4) and (2.4.5), the four-momentum as

$$\begin{aligned} p^\mu =& \frac{q}{a^2}(e^{-\phi}, e^{\psi} n^i), \\ p_\mu =& q(-e^{\phi}, e^{-\psi}\delta_{ij} n^j). \end{aligned} \tag{A.2.16}$$

These are in fact completely general expressions for massless particles, so they are valid not only for gravitons, but for photons and for neutrinos too.
The integration is made over the physical momentum P_i and so we can immediately write the change of variable of integration as

$$dp_1 dp_2 dp_3 = d^3 q \left| det \begin{pmatrix} e^{-\psi} & 0 & 0 \\ 0 & e^{-\psi} & 0 \\ 0 & 0 & e^{-\psi} \end{pmatrix} \right| = d^3 q e^{-3\psi}. \tag{A.2.17}$$

The determinant of the metric gives also a non-null contribution at the first order:

$$\frac{1}{\sqrt{-g}} = \frac{1}{\sqrt{a^8 Det \begin{pmatrix} e^{2\phi} & 0 \\ 0 & \delta_{ij} e^{-2\psi} \end{pmatrix}}} = \frac{e^{3\psi-\phi}}{a^4}. \tag{A.2.18}$$

Now we can explicitly compute the expression for the stress-energy tensor for the various components.
We will write the expression for a generic particle specicies i, described by a distribution function
$f_i(\eta, \vec{x}, \vec{q}) = f_i^{(0)}(\eta, \vec{x}, q) + \delta f_i(\eta, \vec{x}, \vec{q})$ (where δf_i represents a perturbation to the distribution) and by an effective degree of freedom g_i, and when we will write the final form for the Einstein equation we will sum all the stress-energy tensors of the various particle species involved in cosmology, which are photons, baryons (which we neglect in our discussion however) and dark matter[1].
We start from the $(0,0)$ component:

$$\begin{aligned} T^0_0 =& -\sum_i g_i \frac{e^{\phi-3\psi}}{a^3} \int \frac{d^3 q e^{-3\psi}}{(2\pi)^3} E_i e^{\phi} f_i = -\sum_i g_i \frac{1}{a^3} \int \frac{d^3 q}{(2\pi)^3} E_i f_i = \\ =& -\sum_i g_i \frac{1}{a^3} \int \frac{d^3 q}{(2\pi)^3} E_i [f_i^{(0)} + \delta f_i] = -\sum_i \left[\rho_i^{(0)} + \frac{g_i}{a^3} \int \frac{d^3 q}{(2\pi)^3} E_i \delta f_i \right], \end{aligned} \tag{A.2.19}$$

[1]The discussion for the dark matter is identical to the massless particles one, except for the fact that the physical four-momentum differs from the massless case for the zero component: $p^0 = e^{-\phi} E$, with $E \equiv \sqrt{m^2 + \frac{q^2}{a^2}}$.
$\rho_i^{(0)} \equiv \frac{1}{a^3} \int \frac{d^3 q}{(2\pi)^3} E_i f_i^{(0)}$ for all the particle species, but E_i can be equal or not to q if the particle is massless or massive.

the minus sign in this equation is justified by the fact that in the signature we have chosen, $(-,+,+,+)$, the $(0,0)$ component of the stress-energy tensor is defined as minus the energy density, i.e. $T^0_0 = -\rho$.
We are interested only in the neutrino contribution, but in the evolution are involved also the densities of all the other particles involved in cosmology[2], i.e. [22]:

$$\begin{aligned} f_\gamma \equiv & f_\gamma^{(0)} - q\frac{\partial f_\gamma^{(0)}}{\partial q}\theta, \\ f_\nu \equiv & f_\nu(\eta_i) - q\frac{\partial f_\nu(\eta_i)}{\partial q}\mathcal{N}, \\ n_{dm} \equiv & \frac{1}{a^3}\int \frac{d^3q}{(2\pi)^3} f_{dm} = n_{dm}^{(0)}(1+\delta), \\ v^i \equiv & \frac{1}{a^4 n_{dm}}\int \frac{d^3q}{(2\pi)^3} f_{dm}\frac{q^i}{E}. \end{aligned} \tag{A.2.20}$$

We want also to stress that $\delta = \frac{\delta\rho_{dm}}{\rho_{dm}}$, it is the fractional overdensity for both the particle number and for the energy, because for non-relativistic particles $\rho_{n.r.} = m n_{n.r.}$.
With these definitions we can write the total energy-momentum tensor, in order to find out, from the Einstein equation, the first equation we will use in the further analysis:

$$\begin{aligned} T^0_0 = & = -\sum_{i=\gamma,\nu}\left[\rho_i^{(0)} + \frac{g_i}{a^3}\int \frac{d^3q}{(2\pi)^3} q\Big(-\frac{\partial f_i^{(0)}}{\partial q}\theta_i\Big)\right] - (\rho_{dm}^{(0)} + \delta\rho_{dm}) = \\ & = -\sum_{i=\gamma,\nu}\left[\rho_i^{(0)} + 4\frac{g_i}{a^3}\int \frac{d^3q}{(2\pi)^3} q f_i^{(0)}\theta_i\right] - \rho_{dm}^{(0)}(1+\delta) = \\ & = -\left[\rho_\gamma^{(0)}(1+4\theta_0) + \rho_\nu^{(0)}(1+4\mathcal{N}_0) + \rho_{dm}^{(0)}(1+\delta)\right], \end{aligned} \tag{A.2.21}$$

where we have used for θ_0 and η_0 the definition of the monopole of a certain function f:

$$f_0 = \frac{1}{4\pi}\int d\Omega f. \tag{A.2.22}$$

It is trivial to check that the $(0,0)$ Einstein equation at the zero-order gives us the first Friedmann equation, then we will focus only on the first-order equation:

$$\nabla^2\psi - 3\mathcal{H}\Big(\psi' + \mathcal{H}\phi\Big) = 4\pi G a^2\Big(\rho_\gamma^{(0)}4\theta_0 + \rho_\nu^{(0)}4\mathcal{N}_0 + \rho_{dm}^{(0)}\delta\Big). \tag{A.2.23}$$

Notice that we are dealing with only two variables, ϕ and ψ, and so in principle we need only the longitudinal traceless part of the (i,j) equation, because the other one will not provide additional conditions on the system, but in this case we will consider also that part because it leads to an expression we prefer to use to study the evolution of the potentials in the dark-energy era.

[2]Respect to the previous formula, we have used the parametrizations $\delta f_\gamma = -\frac{\partial f_\gamma^{(0)}}{\partial q}\theta$, $\delta f_\nu = -\frac{\partial f_\nu^{(0)}}{\partial q}\mathcal{N}$.

Because of this we apply, in the Fourier space, the projector $\hat{k}^j\hat{k}_i - \frac{1}{3}\delta_i^j$ to the stress-energy momentum tensor T_i^j and to the Einstein tensor G_i^j, and in this way we are left with the longitudinal traceless part (we write the tilde only the first time, to indicate that from now on we are in the Fourier space, but after that we will omit it for the simplicity of the notation):

$$\left(\hat{k}^j\hat{k}_i - \frac{1}{3}\delta_i^j\right)\tilde{G}_j^i = \frac{2k^2}{3a^2}(\phi - \psi). \tag{A.2.24}$$

The calculation for the Fourier transform of the stress- energy tensor is quite simple (here the quantity of which we make the Fourier transform is f):

$$\begin{aligned}\left(\hat{k}^j\hat{k}_i - \frac{1}{3}\delta_i^j\right)\tilde{T}_j^i =&\sum_i \frac{e^{3\psi-\phi}}{a^4}\int \frac{d^3qe^{-3\psi}}{(2\pi)^3}qe^{\phi}\left(\hat{k}^j\hat{k}_i - \frac{1}{3}\delta_i^j\right)n^in_j\tilde{f}_i = \\ =&\sum_i \frac{1}{a^4}\int \frac{d^3q}{(2\pi)^3}q\left(f_i^{(0)} + \delta f_i\right)\left(\mu^2 - \frac{1}{3}\right) = \\ =&\sum_i \frac{1}{a^4}\int \frac{dq}{(2\pi)^3}2\pi q^3\int_{-1}^{+1} d\mu\left(f_i^{(0)} + \delta f_i\right)\frac{2}{3}\left[\frac{1}{2}\left(3\mu^2 - 1\right)\right] = \\ =&-\frac{1}{a^4}\int \frac{dq}{(2\pi)^3}4\pi q^4\int_{-1}^{+1}\frac{d\mu}{2}\left[\frac{\partial f_\gamma^{(0)}}{\partial q}\theta + \frac{\partial f_\nu^{(0)}}{\partial q}\mathcal{N}\right]\frac{2}{3}\mathcal{P}_2 = \\ =&\frac{8}{3}\left[\frac{4\pi}{a^4}\int \frac{dq}{(2\pi)^3}f_\gamma^{(0)}q^4\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_2\theta + \frac{4\pi}{a^4}\int \frac{dq}{(2\pi)^3}f_\nu^{(0)}q^4\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_2\mathcal{N}\right] = \\ =&-\frac{8}{3}\left(\rho_\gamma^{(0)}\theta_2 + \rho_\nu^{(0)}\mathcal{N}_2\right),\end{aligned} \tag{A.2.25}$$

where i identifies a single particle species, then we have used the definition $\mu \equiv \hat{n}\cdot\hat{k}$, and we have used also the fact that all the quantities involved in the integral does not depend on the ϕ angle, defined as the anomaly angle of $\hat{n}$, but only on the azimuthal angle θ, implicitly contained in μ, in fact $\mu = cos\theta$.
Moreover, the integral over μ for the $f_i^{(0)}$ term is null, because $f_i^{(0)}$ represents the isotropic part of the distribution function, and the integral over all the domain of integration for $\mathcal{P}_2(\mu)$ is always null.
We recall also the definition of the ℓ-order multipole for a function f, which is

$$f_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_\ell(\mu)\theta \rightarrow f_2 = -\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_2(\mu)f. \tag{A.2.26}$$

So the second Einstein equation we will use is

$$k^2(\phi - \psi) = -32\pi Ga^2\left(\rho_\gamma^{(0)}\theta_2 + \rho_\nu^{(0)}\mathcal{N}_2\right). \tag{A.2.27}$$

Notice that we have not included higher multipoles for the cold dark matter and for the baryons, because they are fully characterized by the density contrasts and by the velocity

fields.
Another observation to make is that the photon contribution is relatively small, because when the energy density of the photons is quite large, i.e. during the radiation era, the quadrupole contribution is small, and so we need to take only into account the effect of neutrinos.
In order to evaluate also the remaining part of the (i, j) Einstein equation, we will use the projector

$$\begin{aligned}
(\delta_i^j - \hat{k}^j\hat{k}_i)\tilde{G}_j^i =& \frac{2}{a^2}\Big[-2\frac{a''}{a}(1-2\phi) + \mathcal{H}^2(1-2\phi) + 2\mathcal{H}\phi' + 4\mathcal{H}\psi' + 2\psi'' + k^2\psi - k^2\phi\Big] \\
(\delta_i^j - \hat{k}^j\hat{k}_i)T_j^i =& \frac{1}{a^4}\int \frac{d^3q}{(2\pi)^3} q\Big(\delta_i^j - \hat{k}^j\hat{k}_i\Big) n^i n_j \tilde{f}_{\nu,tot} = \\
=& \frac{1}{a^4}\int \frac{d^3q}{(2\pi)^3} q\Big(1-\mu^2\Big)\Big(f_\nu^{(0)} - q\frac{\partial f_\nu^{(0)}}{\partial q}\mathcal{N}\Big) = \\
=& \frac{1}{a^4}\int dq q^3 2\pi \int_{-1}^{+1} d\mu (1-\mu^2) f_\nu(\eta_i) + \frac{1}{a^4}\int dq q^3 4 f_\nu(\eta_i)\int_{-1}^{+1} d\Omega \mathcal{N}\frac{2}{3} + \\
&+ \frac{1}{a^4}\int dq q^3 4 f_\nu(\eta_i)\int_{-1}^{+1}\frac{d\mu}{2} 2\mathcal{N}\frac{2}{3}\frac{1}{2}\mathcal{P}_2(\mu) 2\pi = \\
=& \frac{8}{3}\rho_\nu^{(0)} + \frac{8}{3}\rho_\nu^{(0)}\mathcal{N}_0 + \frac{4}{3}\rho_\nu^{(0)}\mathcal{N}_2.
\end{aligned} \tag{A.2.28}$$

We have also used the fact that for the non-relativistic matter the (i, j) component of the stress-energy tensor is at least a term of the second order in the perturbations, and so we can neglect such a term. The main difference with respect to the previous case is that now we also have, in principle, a monopole term for the photons which is not negligible, because we neglect only multipoles of higher order respect to the dipole, and so we need to take into account also the CMB.
So another Einstein equation we will use is

$$\begin{aligned}
-2\frac{a''}{a}(1-2\phi) + \mathcal{H}^2(1-2\phi) + 2\mathcal{H}\phi' + 4\mathcal{H}\psi' + 2\psi'' + k^2\psi - k^2\phi = \frac{4}{3}a^2\rho_\nu^{(0)}\theta_0 + \frac{4}{3}a^2\rho_\nu^{(0)}\mathcal{N}_0 + \\
+ \frac{2}{3}a^2\rho_\nu^{(0)}\mathcal{N}_2.
\end{aligned} \tag{A.2.29}$$

The last Einstein equation we want to find out involves the $(0, i)$ equation. We start by writing the G_i^0 Einstein tensor described in Eq. (A.3.9) in the Fourier space:

$$G_i^0 = -\frac{2ik_i}{a^2}\Big(\mathcal{H}\phi + \psi'\Big). \tag{A.2.30}$$

The $(0, i)$ component of the Einstein tensor is

$$
\begin{aligned}
T_i^0 =&\sum_j \frac{e^{3\psi-\phi}}{a^4}\int \frac{d^3q e^{-3\psi}}{(2\pi)^3} q e^{-\psi}\delta_{ij}n^j\left(f_j^{(0)}+\delta f_j\right)= \sum_j \frac{e^{-\psi-\phi}}{a^4}\int \frac{d^3q}{(2\pi)^3}q\delta_{ij}n^j\delta f_j = \\
=&\sum_j \frac{1-\psi-\phi}{a^4}\int\frac{d^3q}{(2\pi)^3}q\delta_{ij}n^j\delta f_j = \sum_j\frac{1}{a^4}\int\frac{d^3q}{(2\pi)^3}q\delta_{ij}n^j\delta f_j = \\
=&\sum_{j=\gamma,\nu}\frac{1}{a^4}\int\frac{d^3q}{(2\pi)^3}q\delta_{ij}n^j\left(-\frac{\partial f_j^{(0)}}{\partial q}\theta_j\right)+\frac{1}{a^4}\int\frac{d^3q}{(2\pi)^3}\delta_{ij}\frac{q^j}{E}\delta f_{dm} = \\
=&\rho_{dm}^{(0)}v^i\delta_{ij}+4\rho_r\frac{1}{4\pi}\int d\Omega\delta_{ij}n^j\theta_r = \rho_{dm}^{(0)}v^i\delta_{ij}+4\rho_r\frac{1}{4\pi}\int d(cos\theta)\delta_{ij}n^j\theta_r 2\pi = \\
=&\rho_{dm}^{(0)}v^i\delta_{ij}+4\rho_r\int\frac{d(cos\theta)}{2}\delta_{ij}n^j\theta_r,
\end{aligned}
\tag{A.2.31}
$$

where we have used the definitions given in Eqs. (A.2.20) and the fact that the unperturbed part of the distribution function $f^{(0)}$ does not depend on the angular coordinates and so the $\delta_{ij}n^j$ integral is null for isotropy at zero order.
After that we project the $(0, i)$ Einstein equation along $\hat{k}^i$, obtaining

$$
-\frac{2ik}{a^2}\left(\mathcal{H}\phi+\psi'\right)= 8\pi G\left[\rho_{dm}^{(0)}v_i+4\rho_r^{(0)}\int_{-1}^{+1}\frac{d\mu}{2}\mathcal{P}_1(\mu)\theta_r\right], \tag{A.2.32}
$$

using the fact that $n^i\hat{k}^j\delta_{ij}=\mu$; moreover, considering the definition in Eq. (A.2.26) and the fact that $\mathcal{P}_1=\mu$, we have that[3]

$$
\mathcal{H}\phi+\psi' = \frac{4\pi Ga^2}{k}\left(\rho_{dm}^{(0)}v+4\rho_r\theta_{r,1}\right). \tag{A.2.33}
$$

We can consider also the Fourier transform of Eq. (A.2.23):

$$
-k^2\psi-3\mathcal{H}\left(\psi'+\mathcal{H}\phi\right)= 4\pi Ga^2\left(\rho_\gamma^{(0)}4\theta_0+\rho_\nu^{(0)}4\mathcal{N}_0+\rho^{(0)}\delta\right), \tag{A.2.34}
$$

and by diving this equation by $\mathcal{H}$ and summing it to the other one we obtain

$$
-\frac{k^2\psi}{3\mathcal{H}} = \frac{4\pi Ga^2}{3\mathcal{H}}\left[\frac{3\mathcal{H}}{k}\left(\rho_{dm}^{(0)}v+4\rho_r\theta_{r,1}\right)+\frac{1}{\mathcal{H}}\left(\rho_\gamma^{(0)}4\theta_0+\rho_\nu^{(0)}4\mathcal{N}_0+\rho^{(0)}\delta\right)\right]. \tag{A.2.35}
$$

The last Einstein equation we will use is then

$$
\psi = -\frac{4\pi Ga^2}{k^2}\left[\rho_{dm}\delta+4\rho_r\theta_{r,0}+\frac{3\mathcal{H}}{k}\left(\rho_{dm}v+4\rho_r\theta_{r,1}\right)\right]. \tag{A.2.36}
$$

[3] We should have $\frac{v}{(-i)}$, but, for the further calculations, we will use a new definition of v, which is $v \equiv iv$ (with a little abuse of notation).

A.2.3 The full set of equations

Just to sum up, we will write the Einstein equation and the Boltzmann equations (and the corresponding moments) we will use in the further discussion [22], recalling that we are not considering baryonic matter:

$$\theta' + ik\mu\theta = \psi' - ik\mu\phi - \tau'\big[\theta_0 - \theta - \frac{1}{2}\mathcal{P}_2(\mu)\theta_2\big], \tag{A.2.37}$$

$$\delta' + ikv = 3\psi', \tag{A.2.38}$$

$$v' + \mathcal{H}v = -ik\phi, \tag{A.2.39}$$

$$\mathcal{N}' + ik\mu\mathcal{N} = \psi' - ik\mu\phi, \tag{A.2.40}$$

$$k^2\psi + 3\mathcal{H}\Big(\psi' + \mathcal{H}\phi\Big) = -4\pi Ga^2\big(\rho_{dm}\delta + 4\rho_r\theta_{r,0}\big), \tag{A.2.41}$$

$$k^2(\phi - \psi) = -32\pi Ga^2\rho_r\theta_{r,2}, \tag{A.2.42}$$

$$\psi = -\frac{4\pi Ga^2}{k^2}\Big[\rho_{dm}\delta + 4\rho_r\theta_{r,0} + \frac{3\mathcal{H}}{k}\Big(\rho_{dm}v + 4\rho_r\theta_{r,1}\Big)\Big]. \tag{A.2.43}$$

A.3 Tensor perturbations

In this section we are interested in evaluating the form of the Einstein tensor $G_{\mu\nu}$ for the tensor modes, while the form of the stress-energy tensor for such modes is explicitly computed in Chapter 3. The discussion is analogue to the one seen for the scalar modes in Section A.2, except for the fact that the metric and its inverse we are using now are

$$g_{\mu\nu} = a^2(-1, \delta_{ij} + h_{ij}) \qquad g^{\mu\nu} = \frac{1}{a^2}(-1, \delta^{ij} - h^{ij}). \tag{A.3.1}$$

We evaluate explicitly all the Christoffel'symbols, defined by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{A.3.2}$$

We start by evaluating the affine connection for $\mu = 0$:

$$\Gamma^0_{\mu\nu} = \frac{1}{2}g^{0\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = -\frac{1}{2a^2}(\partial_\mu g_{0\nu} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}) \tag{A.3.3}$$

and we have

$$\begin{aligned}
\Gamma^0_{00} =& \mathcal{H} \\
\Gamma^0_{i0} =& \Gamma^0_{0i} = 0, \\
\Gamma^0_{ij} =& \frac{1}{2a^2}\Big[2aa'(\delta_{ij} + h_{ij}) + a^2 h'_{ij}\Big] = \mathcal{H}\delta_{ij} + \mathcal{H}h_{ij} + \frac{h'_{ij}}{2}.
\end{aligned} \tag{A.3.4}$$

The remaining case is the one for $\mu = i$:

$$\Gamma^i_{\mu\nu} = \frac{\delta^{il} - h^{il}}{2a^2}\{\partial_\mu g_{l\nu} + \partial_\nu g_{\mu l} - \partial_l g_{\mu\nu}\} \tag{A.3.5}$$

and then we have

$$\begin{aligned}
\Gamma^i_{00} =& 0, \\
\Gamma^i_{j0} =& \frac{(\delta^{il} - h^{il})}{2a^2}\partial_0 g_{jl} = \frac{(\delta^{il} - h^{il})}{2a^2}(2aa'\delta_{ij} + 2aa'h_{ij} + h'_{ij}) = \frac{h^{i\,\prime}_j}{2} + \delta^i_j\mathcal{H}, \\
\Gamma^i_{jk} =& \frac{\delta^{il} - h^{il}}{2}(\partial_j h_k l + \partial_k h_{jl} - \partial_l h_{jk}) = \frac{1}{2}(\partial_j h^i_k + \partial_k h^i_j - \partial^i h_{jk}).
\end{aligned} \tag{A.3.6}$$

We recall the form of the Riemann tensor:

$$R^\rho_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\sigma_\mu} + \Gamma^\rho_{\sigma\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}, \tag{A.3.7}$$

from this we can find the explicit form of the Ricci tensor, which derives from contraction of the Riemann tensor:

$$\begin{aligned}
R_{\mu\nu} =& R^\sigma_{\mu\sigma\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}, \\
R_{00} =& \partial_\sigma \Gamma^\sigma_{00} - \partial_0 \Gamma^\sigma_{\sigma 0} + \Gamma^\sigma_{\sigma\lambda}\Gamma^\lambda_{00} - \Gamma^\sigma_{0\lambda}\Gamma^\lambda_{\sigma 0} = \\
=& [\partial_0 \Gamma^0_{00} + \partial_i \Gamma^i_{00}] - [\partial_0 \Gamma^0_{00} + \partial_0 \Gamma^i_{i0}] + [\Gamma^0_{00}\Gamma^\sigma_{\sigma 0} + \Gamma^i_{00}\Gamma^\sigma_{\sigma i}] + \\
& - [\Gamma^0_{00}\Gamma^0_{00} + 2\Gamma^i_{00}\Gamma^0_{i0} + \Gamma^i_{0j}\Gamma^j_{0i}] = \\
=& -3\mathcal{H}' + \mathcal{H}^2 - \mathcal{H}^2 = -3\mathcal{H}' \\
R_{ij} =& \partial_\sigma \Gamma^\sigma_{ij} - \partial_j \Gamma^\sigma_{\sigma i} + \Gamma^\sigma_{\sigma\lambda}\Gamma^\lambda_{ij} - \Gamma^\sigma_{j\lambda}\Gamma^\lambda_{\sigma i} = \\
=& \partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij} - \partial_j \Gamma^0_{0i} - \partial_j \Gamma^k_{ki} + \Gamma^0_{00}\Gamma^0_{ij} + \Gamma^0_{0k}\Gamma^k_{ij} + \\
& + \Gamma^k_{k0}\Gamma^0_{ij} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^0_{j0}\Gamma^0_{0i} - \Gamma^0_{jk}\Gamma^k_{0i} - \Gamma^k_{j0}\Gamma^0_{ki} - \Gamma^k_{jl}\Gamma^l_{ki} = \\
=& \mathcal{H}'(\delta_{ij} + h_{ij}) + \mathcal{H}h'_{ij} + \frac{1}{2}h''_{ij} - \frac{1}{2}\partial_k\partial^k h_{ij} + \mathcal{H}^2(\delta_{ij} + h_{ij}) + \frac{1}{2}\mathcal{H}h'_{ij} + 3\mathcal{H}^2(\delta_{ij} + h_{ij}) + \\
& + \frac{3}{2}\mathcal{H}h'_{ij} - \Big(\frac{h^{k\,\prime}_j}{2} + \delta^k_j\mathcal{H}\Big)\Big(\mathcal{H}\delta_{ik} + \mathcal{H}h_{ik} + \frac{h'_{ik}}{2}\Big) + \\
& - \Big(\frac{h^{k\,\prime}_i}{2} + \delta^k_i\mathcal{H}\Big)\Big(\mathcal{H}\delta_{kj} + \mathcal{H}h_{kj} + \frac{h'_{kj}}{2}\Big) = \\
=& \mathcal{H}'(\delta_{ij} + h_{ij}) + \mathcal{H}h'_{ij} + \frac{1}{2}h''_{ij} - \frac{1}{2}\partial_k\partial^k h_{ij} + \mathcal{H}^2(\delta_{ij} + h_{ij}) + \\
& + \frac{1}{2}\mathcal{H}h'_{ij} + 3\mathcal{H}^2(\delta_{ij} + h_{ij}) + \frac{3}{2}\mathcal{H}h'_{ij} - \Big(2h'_{ij} + 2\mathcal{H}^2\delta_{ij} + 2\mathcal{H}^2 h_{ij}\Big) = \\
=& \frac{1}{2}h''_{ij} + \mathcal{H}h'_{ij} - \frac{1}{2a^2}\nabla^2 h_{ij} + h_{ij}(\mathcal{H}' + 2\mathcal{H}^2) + \delta_{ij}(\mathcal{H}' + 2\mathcal{H}^2)
\end{aligned} \tag{A.3.8}$$

The Einstein tensor for the (i, j) component is then

$$\begin{aligned}G_{ij} =&R_{ij} - \frac{1}{2}g_{ij}(g^{0\alpha}R_{0\alpha} + g^{i\alpha}R_{i\alpha}) = R_{ij} - \frac{1}{2}(\delta_{ij} + h_{ij})[3\mathcal{H}' + 3(\mathcal{H}' + 2\mathcal{H}^2)] = \\ =&\frac{1}{2}h''_{ij} + \mathcal{H}h'_{ij} - \frac{1}{2a^2}\nabla^2 h_{ij} + h_{ij}(\mathcal{H}' + 2\mathcal{H}^2) + \delta_{ij}(\mathcal{H}' + 2\mathcal{H}^2) - (\delta_{ij} + h_{ij})(3\mathcal{H}' + 3\mathcal{H}^2) = \\ =&\frac{1}{2}h''_{ij} + \mathcal{H}h'_{ij} - \frac{1}{2a^2}\nabla^2 h_{ij} - (\delta_{ij} + h_{ij})(2\mathcal{H}' + \mathcal{H}^2) = \frac{1}{2}h''_{ij} + \mathcal{H}h'_{ij} + \\ &- \frac{1}{2a^2}\nabla^2 h_{ij} - \frac{g_{ij}}{a^2}(2\mathcal{H}' + \mathcal{H}^2).\end{aligned} \tag{A.3.9}$$

It is immediate to find that the equation for the tensor modes is

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 16\pi G\pi_{ij}. \tag{A.3.10}$$

Appendice B

CLASS modifications

B.1 General method

The Cosmic Linear Anisotropy Solving System (CLASS) is a code in C that solves in a fast and precise way the combined Friedmann and Boltzmann equations for all the components of the Universe. Using CLASS, it is possible to evaluate easily the CMB angular power spectra C_ℓ's, which present similiar features with their gravitational waves analogues, the $\tilde{C}_\ell$'s we are looking for. Our aim is then to modify a little bit the C code, in order to find these coefficients following the same steps of what has been done for the CMB, using, as input, $N_\nu = 3$ and $N_\nu = 0$, to observe the expected damping effect of neutrinos on the gravitational waves.
From a theoretical point of view, the C_ℓ coefficients for the CMB can be written as

$$C_\ell = 4\pi \int \frac{dk}{k} \big(\theta_\ell(\tau_0, k)\big)^2 P(k), \tag{B.1.1}$$

where $P(k)$ is the primordial power spectrum (we are implicitly summing over scalar and tensor modes, which have different power spectra). It is clear that the only difference with respect to the gravitational waves is given by a different function θ_ℓ, which has to be compared with the transfer functions defined as $T_\ell^{(0)}$ and $T_\ell^{(\pm 2)}$; the comparison is quite immediate, because these functions can be written as[1]

$$\begin{aligned} \theta_\ell^{(0)}(\tau_0, k) &= \int_{\tau_{ini}}^{\tau_0} d\tau S_T^{(0)}(\tau, k) j_\ell[k(\tau_0 - \tau)], \\ \theta_\ell^{(\alpha)}(\tau_0, k) &= \int_{\tau_{ini}}^{\tau_0} d\tau S_T^{(\alpha)}(\tau, k) \frac{1}{4} \sqrt{\frac{(l+2)!}{(l-2)!}} \frac{j_\ell[k(\tau_0 - \tau)]}{[k(\tau_0 - \tau)]^2}, \end{aligned} \tag{B.1.2}$$

where the functions S_T are called source functions and "(0)" and "($\alpha = \pm 2$)" correspond to scalar and tensor modes. In this case we have two differences: a different form of the source

[1] In this section we identify the conformal time η as τ, because it is defined in this way in the CLASS code, thus when we will explicitly write down the modifications of CLASS, it will be immediate to understand whom τ corresponds to.

functions for the gravitational waves (for which we will immediately make a comparison) and the initial time of integration, which is equal to the recombination time for the CMB and to the neutrino decoupling for our discussion about the gravitons. Proceeding with the discussion, we see that the source functions for the CMB are[2]

$$\begin{aligned} S_T^{(0)}(\tau,k) \equiv& g(\frac{\delta_g}{4}+\psi)+(gK^{-2}\theta_b)'+e^{-K}(\phi'+\psi')+pol., \\ S_T^{(\alpha)}(\tau,k) \equiv& -e^{-K}h', \end{aligned} \tag{B.1.3}$$

where $g=-K'e^{-K}$ is the visibility function[3] and K is the optical depth for the photons, while *pol.* corresponds to a polarization term; these source functions as the be compared with the ones for the gravitational waves,

$$\begin{aligned} \tilde{S}_T^{(0)}(\tau,k) \equiv& \psi(\tau,k)\delta(\tau-\tau_i)+\phi'(\tau,k)+\psi'(\tau,k), \\ \tilde{S}_T^{(\alpha)}(\tau,k) \equiv& h'. \end{aligned} \tag{B.1.4}$$

In order to obtain the results we are looking for, two steps are necessary: to modify properly the source functions and to change the initial time of integration; we will discuss each corrispective procedure in a different section.

B.2 Modifying the source functions

In CLASS we are interested in the accuracy of integration and in the speed of computation, because of this we write θ_ℓ in the following form:

$$\begin{aligned} \theta_\ell^{(0)}(\tau_0,k) = \int_{\tau_{ini}}^{\tau_0} d\tau \Big\{ S_{T,0}^{(0)}(\tau,k) j_\ell[k(\tau_0-\tau)] + S_{T,1}^{(0)}(\tau,k)\frac{dj_\ell[k(\tau_0-\tau)]}{d[k(\tau_0-\tau)]} \\ + S_{T,2}^{(0)}\frac{1}{2}\Big[3\frac{d^2 j_\ell[k(\tau_0-\tau)]}{d[k(\tau_0-\tau)]^2}+j_\ell[k(\tau_0-\tau)]\Big]\Big\}, \end{aligned} \tag{B.2.1}$$

and the best choice to do such an integration, i.e. the one imposed by default by CLASS, is

$$\begin{aligned} S_{T,0}^{(0)} =& g\Big(\frac{\delta_g}{4}+\phi\Big)+e^{-K}2\phi'+g'\theta_b+g\theta_b', \\ S_{T,1}^{(0)} =& e^{-K}k(\psi-\phi), \\ S_{T,2}^{(0)} =& ..., \end{aligned} \tag{B.2.2}$$

[2] For the same reason of the previous note, we will use the following convention:

$$\phi_{CLASS}=\psi_{OUR} \qquad \textit{and} \qquad \psi_{CLASS}=\Phi_{OUR},$$

therefore ψ and ϕ assume the definitions given above.

[3] This contribution is equivalent to a Dirac delta, in fact

$$\int_0^{\tau_0} d\tau g\Big(\frac{\delta_g(\tau,k)}{4}+\psi(\tau,k)\Big)=\frac{\delta_g(\tau_*,k)}{4}+\psi(\tau_*,k),$$

where τ_* is the conformal time at the recombination.

where we are not interested in writing explicitly the last term because they involve the CMB polarization, which we do not consider for the gravitational waves case.
The choice we have made to modify the source functions consists in imposing that $g = 0$, $e^{-K} = 1$ and $\delta_g = 0$ in the equations (B.1.3), observing that they give equivalent source functions to the ones of the stochastic gravitational waves background. Notice that by imposing $g = 0$ we are explicitly neglecting the $\psi(\tau, k)\delta(\tau - \tau_i)$ contribution in the source functions, we have done this because we expect that the neglected term is subdominant, improvements on this side should be done in a future work. in The source functions are calculated in the perturbation module, which can be found through the path *source/perturbations.c*, and they can be found by searching for $\_set\_source\_(ppt->index\_tp\_t0)$ (for both the scalar and the tensor modes). Notice that we could work both in the newtonian and in the poisson gauge, but we prefer the first one, because all the results found until now have been found in such a gauge.
It is trivial to check that the correspondences between the variables in the program and the ones named in the equations are the following:

$$\begin{aligned}
_set_source_(ppt->index_tp_ti) &\leftrightarrow S^{(0)}_{T,i}, \\
_set_source_(ppt->index_tp_t2) &\leftrightarrow S^{(\alpha)}_{T,2} \\
y[ppw->pv->index_pt_phi] &\leftrightarrow \phi, \\
pvecmetric[ppw->index_mt_phi_prime] &\leftrightarrow \phi', \\
pvecmetric[ppw->index_mt_psi] &\leftrightarrow \psi, \\
pba->conformal_age &\leftrightarrow \tau_0, \\
delta_g &\leftrightarrow \delta_g, \\
pvecthermo[pth->index_th_g] &\leftrightarrow g, \\
pvecthermo[pth->index_th_dg] &\leftrightarrow g', \\
y[ppw->pv->index_pt_gwdot] &\leftrightarrow h'.
\end{aligned} \tag{B.2.3}$$

To sum up, the modified part of the code will be then

```
if (ppt->gauge == newtonian) {
\_set\_source\_(ppt->index\_tp\_t0) =
switch\_isw * ( 2. * pvecmetric[ppw->index\_mt\_phi\_prime]);

\_set\_source\_(ppt->index\_tp\_t1)=switch\_isw*k*(pvecmetric[ppw->index\_mt\_psi]-
          y[ppw->pv->index\_pt\_phi]);

\_set\_source\_(ppt->index\_tp\_t2) = 0;
}
```

For the tensor part the situation is simpler, because we have that in the CMB case the only non null term is $S^{(\alpha)}_{T,2}$ and it is equal to

$$S^{(\alpha)}_{T,2} = -e^{-K} h', \tag{B.2.4}$$

thus we impose

$$\tilde{S}^{(\alpha)}_{T,2} = h' \tag{B.2.5}$$

and the modified part is quite simple too (we impose also null each tensor polarization):

```
if (ppt->has_source_t == _TRUE_) {
      _set_source_(ppt->index_tp_t2) = y[ppw->pv->index_pt_gwdot];
    }

    /* tensor polarization */
    if (ppt->has_source_p == _TRUE_) {

      /* Note that the correct formula for the polarization source
         should have a minus sign, as shown in Hu & White. We put a
         plus sign to comply with the 'historical convention'
         established in CMBFAST and CAMB. */

      _set_source_(ppt->index_tp_p) = 0;
    }
```

B.3 Modifying the time integration range and time sampling

CLASS samples the source functions from a certain initial conformal time, τ_{ini}, and for a certain number of times until τ_0; it can be seen that for the CMB the sampling starts at around $\tau \approx 200$ Mpc, while we would like to make the sampling begin at times very near to 0, like for instance at $\tau \approx 0.0001$ Mpc, in addition we would like also to be sure that, once we have chosen this new initial time for the sampling, the program takes a sufficient number of points, so we should also check that this is true.
We recognize that the initial sampling time is defined in "*output/perturbations.c*" at the line

```
tau_lower = pth->tau_ini;
```

through an iteration cycle in which the ratio of the Hubble and the thermo scales are compared. We are not particularly interested in it, the only important results are that $\tau_{ini,th} \approx 24$ Mpc and $\tau_{ini} = 227.499466$ Mpc; we can change this result imposing by hand at the beggining of this cycle that the initial time before the iteration is equal to the initial time tabled to evaluate the background quantities, $pba->tau\_table[0]$, i.e. we substitute to the previous expression the following one:

```
tau_lower = pba->tau_table[0];
```

After that, we notice that the condition which regulates the bisection for finding the correct initial time is

```
if (pvecback[pba->index_bg_a]*
          pvecback[pba->index_bg_H]/
          pvecthermo[pth->index_th_dkappa] >
          ppr->start_sources_at_tau_c_over_tau_h)

        tau_upper = tau_mid;
      else
        tau_lower = tau_mid;
```

where *start_sources_at_tau_c_over_tau_h* is a parameter which assumes a value defined in the header file *include/precision.h* that can be modified, and we change it from $8 \cdot 10^{-2}$ to $8 \cdot 10^{-9}$: the lower this value the lower the initial time for sampling the sources,

```
class_precision_parameter(start_sources_at_tau_c_over_tau_h,double,0.000000008)
```

In this way we are able to set $\tau_{ini} = 0.000775$, but another problem arises and it concerns the sampling rates. If we take two consecutive conformal times, the sampling works as it follows:

$$\tau_{i+1} = \tau_i + sampling\_stepsize * timescale\_source, \tag{B.3.1}$$

where *sampling_stepsize* is a quantity set to 0.1 in *output/precision.h*, while *timescale_source* is a quantity that needs to be evaluate by understanding the physics of the system. For instance when we consider late times we have that

$$\begin{cases} rate_{thermo} = \frac{g}{g'} \\ rate_{late} = \left(2\frac{a'}{a} - \frac{a'^2}{a^2}\right)^{1/2} \end{cases} \rightarrow timescale\_source = \frac{1}{\frac{1}{rate_{thermo}} + \frac{1}{rate_{late}}}, \tag{B.3.2}$$

which is defined in the following lines of code:

```
if (ppt->has_cmb == _TRUE_) {

      /* variation rate of thermodynamics variables */
      rate_thermo = pvecthermo[pth->index_th_rate];

      /* variation rate of metric due to late ISW effect
      (important at late times) */
      a_prime_over_a = pvecback[pba->index_bg_H]
      * pvecback[pba->index_bg_a];
      a_primeprime_over_a = pvecback[pba->index_bg_H_prime]
       * pvecback[pba->index_bg_a]
        + 2. * a_prime_over_a * a_prime_over_a;
      rate_isw_squared = fabs(2.*a_primeprime_over_a
                              -a_prime_over_a*a_prime_over_a);

      /* compute rate */
      timescale_source = sqrt(rate_thermo*rate_thermo+rate_isw_squared);
    }
```

Unfortunately, $pvecthermo[pth- > index\_th\_rate]$ is huge when evaluated at neutrino decoupling, therefore if we start the integration at that time, the program requires an infinite amount of time, thus we will use as initial time (at maximum) $\eta_i = 30$ Mpc, which corresponds approximately to

```
class_precision_parameter(start_sources_at_tau_c_over_tau_h,double,0.0004)
```

This will be our definitive value for the initial time.

B.4 Python Notebooks

Here we put some plots to understand better when CLASS starts giving troubles.
First of all we use as input $N_\nu = 10^{-16}$ instead of exactly $N_\nu = 0$, because this gives solutions with a similiar behaviour to $N_\nu = 3$, it seems like that putting $N_\nu = 0$ gives some troubles in certain parts of the code (in addition N_ν is so small that phyisically there are no significant differences with considering zero neutrinos). The code to plot in the Python Notebook the scalar contributions to the anisotropies, $\tilde{C}_{\ell,S}$ is then

```
# import necessary modules
# uncomment to get plots displayed in notebook
%matplotlib inline
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
from classy import Class
from scipy.optimize import fsolve
from scipy.interpolate import interp1d
import math
# esthetic definitions for the plots
font = {'size'   : 16, 'family':'STIXGeneral'}
axislabelfontsize='large'
matplotlib.rc('font', **font)
matplotlib.mathtext.rcParams['legend.fontsize']='medium'
plt.rcParams["figure.figsize"] = [8.0,6.0]
#############################################
#
# Cosmological parameters and other CLASS parameters
#
common_settings = {'output':'tCl',#'l_max_tensors':'2500',
                   'gauge':'newtonian'#,'modes':'t'}
common_settings1 = {output':'tCl',
                   'YHe':0.24532,
                   'gauge':'newtonian','modes':'s',
                   'N_ur':0.0000000000000001,'N_ncdm':0}
```

```
Ma = Class()
Ma.set(common_settings)
Ma.set({'modes':'s'})
Ma.compute()
cl_tota = Ma.raw_cl(2500)
M1 = Class()
M1.set(common_settings1)
M1.compute()
cl_tot1 = M1.raw_cl(2500)
M1.struct_cleanup()  # clean output
M1.empty()           # clean input
plt.xlim([2,2500])
plt.xlabel(r"$\ell$")
plt.ylabel(r"$\tilde{C}_{\ell,S}$")
plt.grid()
ell = cl_tota['ell']
factor = 1.e10*ell*(ell+1.)/2./math.pi
plt.loglog(ell,factor*cl_tota['tt'],'b-',label=r'$N_\nu=3$')
ell = cl_tot1['ell']
factor = 1.e10*ell*(ell+1.)/2./math.pi
plt.loglog(ell,factor*cl_tot1['tt'],'r-',linestyle='dashed',
           label=r'$N_\nu=0$')
plt.legend(loc='right',bbox_to_anchor=(1.4, 0.5))
plt.savefig('fondogw_scalar.pdf',bbox_inches='tight')
```

The Python Notebook used to plot the tensor contribution to the angular power spectrum is:

```
# import necessary modules
# uncomment to get plots displayed in notebook
%matplotlib inline
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
from classy import Class
from scipy.optimize import fsolve
from scipy.interpolate import interp1d
import math
# esthetic definitions for the plots
font = {'size'   : 16, 'family':'STIXGeneral'}
axislabelfontsize='large'
matplotlib.rc('font', **font)
matplotlib.mathtext.rcParams['legend.fontsize']='medium'
plt.rcParams["figure.figsize"] = [8.0,6.0]
```

```
#############################################
#
# Cosmological parameters and other CLASS parameters
#
common_settings = {'output':'tCl',#'l_max_tensors':'2500',
                   'gauge':'newtonian'#,'modes':'t'}
common_settings1 = {'output':'tCl',
                   'YHe':0.24532,
                   'gauge':'newtonian','modes':'t',
                   'N_ur':0.0000000000000001,'N_ncdm':0}

Mb = Class()
Mb.set(common_settings)
Mb.set({'modes':'t','l_max_tensors':'2500'})
Mb.compute()
cl_totb = Mb.raw_cl(2500)
M1 = Class()
M1.set(common_settings1)
M1.set({'l_max_tensors':'2500'})
M1.compute()
cl_tot1 = M1.raw_cl(2500)
#cl_lensed1 = M1.lensed_cl(2500)
M1.struct_cleanup()  # clean output
M1.empty()           # clean input
ratio_oscillating=[]
l_grafico=[]
ratio=[]

plt.xlim([2,2500])
plt.xlabel(r"$\ell$")
plt.ylabel(r"$\tilde{C}_{\ell,T}$")
plt.grid()
#
ell = cl_totb['ell']
factor = 1.e10*ell*(ell+1.)/2./math.pi
plt.loglog(ell,factor*cl_totb['tt'],'b-',label=r'$N_\nu=3$')
ell = cl_tot1['ell']
factor = 1.e10*ell*(ell+1.)/2./math.pi
plt.loglog(ell,factor*cl_tot1['tt'],'r-',linestyle='dashed',
           label=r'$N_\nu=0$')
plt.legend(loc='right',bbox_to_anchor=(1.4, 0.5))
plt.savefig('fondogw_tensor.pdf',bbox_inches='tight')
```

Appendice C

$\bar{f}(q)$ neutrino damping

Until now we have only computed the effects of neutrinos on the gravitational waves anisotropies, considering the role of their anisotropic stress on the evolution of the metric perturbations. We have also seen, in Eq. (1.6.50), that the graviton occupation number for a given comoving momentum q is related to the squared amplitude of the q-mode of the waves, i.e. to $|v_q(\eta)|^2$ or, alternatively, to $|h_q(\eta)|^2$, therefore we see that the damping of the tensor modes h_{ij} has not only the effect of damping the anisotropies which derive from the free-streaming of the gravitons, but it decreases also the graviton number density or, in other words, the unperturbed graviton distribution function $\bar{f}(q)$. In this last section we want to evaluate this effect, relating the abstract concept of graviton distribution function perturbation Γ to an observable quantity, the density contrast δ_{GW} of the gravitational waves.
From Section 2.6, we know that the total energy density for the gravitational waves at the leading order can be written as

$$\rho^{(0)}_{GW}(\eta,\vec{x}) = \int dp_1 dp_2 dp_3 p\bar{f}(q) = \frac{1}{a^4}\int d^3q q\bar{f}(q), \tag{C.0.1}$$

but we know also, from Eq. (1.4.14), that it is equal to

$$\begin{aligned}\rho^{(0)}_{GW}(\eta,\vec{x}) =& T^{00}_{GW}(\eta,\vec{x}) = \frac{1}{32\pi G a^2}\langle h'_{ij}(\eta,\vec{x})h'^{ij}(\eta,\vec{x})\rangle = \\ =&\frac{1}{32\pi G a^2}\sum_{\lambda,\lambda'}\int\frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}e_{ij,\lambda}(\hat{q})e^{ij,\lambda'}(\hat{q}')e^{i\vec{x}\cdot(\vec{q}+\vec{q}')}\langle h'_\lambda(\eta,q)h'_\lambda(\eta,q')\rangle.\end{aligned} \tag{C.0.2}$$

For different reasons, in this thesis we have used three different parametrizations for the gravitational waves:

$$\begin{aligned}h_{ij}(\eta,\vec{q}) =& \sum_\lambda e_{ij,\lambda}h_q(\eta), \\ h_{ij}(\eta,\vec{q}) =& \sum_\lambda e_{ij,\lambda}\xi_\lambda(\vec{q})h(\eta,q), \\ h_{ij}(\eta,\vec{q}) =& h_{ij}(0,\vec{q})\chi(\eta,q),\end{aligned} \tag{C.0.3}$$

defined in Eqs. (1.6.32), (2.7.1) and (3.2.34) respectively. By looking at the results of Chapter 3, we know that $\chi(\eta, q)$ is damped by neutrinos, mainly on small scales, for $k > k_{EQ}$, therefore $\bar{f}(q)$, which is related to $|h_\lambda(\eta, q)|^2$, will be damped by neutrinos too. In fact, by using Eq. (2.7.1), we see that $h_\lambda(\eta, q) = \xi_\lambda(\vec{q})h(\eta, q)$, where $\xi_\lambda(\vec{q})$ corresponds to the initial conditions fixed by the inflation, while $h(\eta, q)$ corresponds to the temporal evolution normalized with respect to the initial conditions. Therefore, by using Eq. (2.7.25), we write the energy density as

$$\rho_{GW}^{(0)} = \frac{1}{32\pi G a^2}\sum_\lambda \int \frac{d^3q}{(2\pi)^3}\frac{4\pi^2}{q^3}P^{(\lambda)}(q)|h'(\eta, q)|^2, \tag{C.0.4}$$

from whom, by a comparison with Eq. (C.0.1), we find out that the unperturbed neutrino distribution function is

$$\bar{f}(q) = \frac{1}{128\pi^2 G}|a(\eta)h'(\eta, q)|^2 \frac{1}{q^4}\sum_\lambda P^{(\lambda)}(q). \tag{C.0.5}$$

One of the most important future experiments which is supposed to be able to detect the cosmological gravitational waves background is LISA, which is sensitive to frequencies ν in the range $[10^{-5}, 0.1]$ Hz, which corresponds to a range of wavelenghts of $[6.4\times10^9, 6.45\times10^{13}]$ Mpc^{-1}, where we have used the relation $\lambda = 2\pi/k$ and $\lambda\nu = c$, with 1 Mpc$= 3\times10^{22}$ m. We are indeed considering small scale modes, which have the squared amplitudes reduced by a factor 35% in presence of neutrinos, this means that the unperturbed graviton distribution function we are considering will be damped by the same factor too.
We can check that the above expression for $\bar{f}(q)$ is time independent, consistently with Eq. (2.4.16): using Eqs. (1.6.33) and (1.6.39), we see that, for super horizon modes, we have

$$v_q'' + q^2 v_q = 0 \to v_q \sim e^{\pm iq\eta} \to h_q \sim \frac{e^{\pm iq\eta}}{a} \to ah' \sim ae^{\pm iq\eta}\left(\frac{\pm iq}{a} - \frac{a'}{a^2}\right) \sim e^{\pm iq\eta}(\pm iq - aH), \tag{C.0.6}$$

therefore, by using $k \gg aH$, we have proved that $\bar{f}(q)$ is constant,

$$|a(\eta)h'(\eta, q)|^2 \sim q^2 \to \bar{f}(q) = \frac{cost.}{q^2}P^{(\lambda)}(k). \tag{C.0.7}$$

To conclude all this thesis work, we want to apply all the things seen until now to the physical observables from interferometers. The quantity that could be measured is the density contrast δ_{GW}, defined in Eq. (2.6.4), which is equal to

$$\delta_{GW} = -\frac{q}{\bar{f}(q)}\frac{d\bar{f}(q)}{dq}\Gamma = 2\Gamma, \tag{C.0.8}$$

where we have used the scale-invariant power spectra defined in Eq. (1.6.58). We can decompose the density constrast in spherical harmonics, finding the analogue two-point functions, which are nothing but the $2\tilde{C}_\ell$ found in Section 5.2.

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