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SUPERSPACE METHODS FOR STUDYING SUPERSYMMETRIC FIELD THEORIES ON CURVED BACKGROUNDS

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Chapter 1

Introduction

Many of the major developments in fundamental physics of the past century arose from identifying and overcoming contradictions between existing ideas. For example, the incompatibility of Maxwell's equations and Galilean invariance led Einstein to propose the special theory of relativity. Similarly, the inconsistency of special relativity with Newtonian gravity led him to develop the general theory of relativity. More recently, the reconciliation of special relativity with quantum mechanics led to the development of quantum field theory.

Another issue of the same character is a tension between general relativity and QFT. Any straightforward attempt to "quantize" general relativity leads to a non-renormalizable theory. Such a theory needs to be modified at short distances or high energies. Hence, it seems quite natural, or we can better say that there is a great belief, that there should exist a more general quantum theory beyond the Standard Model which integrates also General Relativity into a so called Unified Theory. One of such theories is *String Theory*.

A way that string theory does this, is by giving up one of the basic assumptions of quantum field theory, namely that elementary particles are mathematical points and allowing one-dimensional extended objects, called strings.

It has been shown in the $70s - 90s$, that one basic concept employed in the construction of string theory is *Supersymmetry*, namely a symmetry relating particles and fields of integer spin (bosons) and particles and fields of half integer spin (fermions). Since its discovery, supersymmetry managed to retain continuous interest as a basis for the construction of unified theories such as for example, supergravities in four dimension and in $D = 10$ and $D = 11$ which are low energy limits of string theory.

But there are also other reasons why an elementary particle physicist wants to consider supersymmetric theories. One of the main reasons is that quantum behaviour of supersymmetric theories is much better than that of quantum field theories, due to cancellations of divergences in fermion and boson loops. In particular, supersymmetry provides a possibility of protecting the Higgs mass and keeping it at an electro-weak breaking scale, which would otherwise receive huge quantum corrections *(known as Hierarchy problem)*

Other arguments for supersymmetry come from the success of certain simple grand unified theories (GUTs) in which three running coupling constants of electro-weak and strong interactions meet at a single point around $10^{16} GeV$ []. On the contrary, in non-supersymmetric grand unified theories, the couplings for the $U(1), SU(2)$, and $SU(3)$ interactions do not unify at the GUT scale, in the simplest models.[].

Other open fundamental issues regard cosmology. Indeed the nature of Dark Matter and Dark Energy is unclear. One of the hypothesis is that the supersymmetric partners of the observed particles may be possible constituents of Dark matter.

For these reasons, understanding how supersymmetry is realized in Nature is one of the most important challenges theoretical high energy physics has to confront with.

To show schematically the basic idea of supersymmetry, let us take an infinitesimal rigid (or global) supersymmetric transformation

$$\delta_\epsilon B = \bar{\epsilon} F, \quad \delta_\epsilon F = \epsilon \gamma^m \partial_m B, \tag{1.1}$$

B and F denote respectively bosonic and fermionic fields, ϵ denotes the spinorial infinitesimal supersymmetry parameter and ∂_m stands for a space-time derivative while γ^m are Dirac matrices. The commutator of supersymmetric transformations of the bosonic (and fermionic) fields is

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] B = (\bar{\epsilon}_1 \gamma^m \epsilon_2) \partial_m B, \tag{1.2}$$

where the partial derivative tells us that supersymmetry is related to space-time translations ($P_m = -i\partial_m$). This tells us that supersymmetry is a non-trivial part of space-time symmetry.

The general motivation for this thesis is the study of main features of supersymmetric field theories formulated on flat and curved backgrounds, such as on anti De Sitter spaces. We will study in particular, the application of superspace and superfield methods to supersymmetric field theories on curved backgrounds, which can simplify the study of problems related, for example, to the AdS/CFT correspondence and the possibility of obtaining exact quantum results in supersymmetric quantum field theories using the so called *localization techniques*.

To begin with, in chapter 2 we will give a review of four-dimensional space-time rigid supersymmetric theories following [1],[2]. In particular, we will study properties of the superPoincaré algebra and its irreducible representations, in order to describe physical supermultiplets.

Then, we will extend $D = 4$ bosonic space-time to the $\mathcal{N} = 1$, $D = 4$ *superspace*, which is made of both bosonic and fermionic coordinates. By constructing functions on this superspace, called superfields, we will build invariant supersymmetric field theories in a straightforward way. In this framework we will review how to construct matter and Yang-Mills supersymmetric actions. The discussion remains classical and renormalisation issues are not discussed here.

In chapter 3 we will consider local supersymmetric theories [6] by making the supersymmetric parameter depend on space-time coordinates. This leads to supersymmetric generalization of gravity. We will study the component field formulation of the minimal supergravity model in $D = 4$, namely the theory which combines the

Hilbert-Einstein action associated to the gravitational field of spin 2 with the action associated to the fermionic Rarita-Schwinger field of spin 3/2. This theory is constructed with the use of the vielbein formalism, because only in this framework it becomes possible to describe the coupling of spinor fields to gravity. The full $\mathcal{N} = 1$ supergravity Lagrangian will be given and we will show the local supersymmetry transformations that leave this Lagrangian invariant.

The above construction was for the components of the supergravity multiplet. Our aim will also be to study the description of supergravity and its coupling to matter fields with the use of superfield methods. In the following chapters, to simplify this study, we will pass from four to two dimensions in which the superfield construction of the theory is much simpler, but the main conceptual points remain the same.

In chapter 4 we will consider $D = 2$ rigid supersymmetric theories [7]. The importance of 1+1 dimensional theories is also related to the worldsheet description of string theory and potential applications to the study of AdS_2/CFT_1 correspondence.

Then, in chapter 5 we will investigate $D = 2$ supergravity in the framework of superspace and superfields [8]. We shall give the basics of supergeometry, i.e. a curved superspace generalization of the notion of differential geometry of curved manifolds. To simplify the consideration, we will focus on the study of the so called $\mathcal{N} = (1,1)$-supersymmetry, whose parameter is a Majorana spinor.

We will follow the work of Howe [9], who imposed some natural "kinematic" constraints on the supertorsion (analogous to those adopted in four dimensions [10]) in order to reduce the number of the redundant components of the supervielbein. We will find that the supervielbein and superconnection are expressed in terms of the zwei-bein, the Rarita-Schwinger field and one auxiliary scalar field. We then obtain also the other supergeometry objects, such as supertorsion and supercuvature, as functions of one scalar and one vector superfield.

In chapter 6 we fix the superbackground to be an AdS_2 superspace and compute explicitly all its supergeometrical objects (supervierbien, superconnection, supertorsion and supercurvature). Finally, we will couple the scalar (matter) superfield introduced in chapter 4, to the AdS_2 superbackground and construct an appropriate superfield Lagrangian containing a kinetic and mass term as well as a potential interaction term. We then expand the superfield Lagrangian in terms of the component matter fields and observe an interesting phenomenon that in contrast to the flat space, in AdS_2 a massless scalar field may have either massless fermionic superpartner or a massive one with the mass inversely proportional to the AdS_2 radius. This reflects the influence of the curvature of the background. Lastly, we derive the supersymmetric transformations for component fields, under which the Lagrangian is invariant. These transformations form the supergroup $Osp(1|2)$ containing the isometry group $SO(2,1)$ of the AdS_2 space as a subgroup. In conclusion we summarize the results of this thesis.

Chapter 2

Rigid supersymmetry in $D = 4$ dimensions

2.1 The supersymmetry algebra

Supersymmetry algebra (or superalgebra) is the basic ground on which all the supersymmetric invariant theories, both rigid or local ones, are constructed. It can be considered as an extension of the ordinary bosonic Lie algebras.

In addition to the bosonic generators which constitute the basis of the embodied Lie algebras, superalgebra consists of extra fermionic generators Q_α, called *supercharges*, which by definition transform as spinors under the Lorentz group and obey anti-commutation relations. Anticommutation of the supersymmetry generators is a characteristic of any superalgebra, which is important for the validity of the *Haag-Lopuszanski-Sohnius theorem* [4], which will permit to build supersymmetric theories, as we will show.

To build some supersymmetry algebra, it will be important to specify the dimension of the spacetime and what kind of bosonic Lie subalgebra is used as a starting point. Examples of Lie subalgebra are the Poincaré algebra, the *anti de-Sitter algebra* or the *conformal algebra*.

In different space-time dimensions various types of spinors (e.g. Majorana or Weyl) can exist or not [5]. Different spinors choices give rise to diverse supersymmetry algebras.

In this chapter we will fix the dimension to four and the Lie algebra to be the Poincaré algebra. As regarding notation, in this chapter and throughout, we will associate the lower latin indices $(a, b ... m, n ...)$ to vector components, while lower greek indices $(\alpha, \beta ... \mu, \nu ..)$ to spinorial components.

2.1.1 Two-component formalism

Spinors in our formalism will be defined as those objects who carry the *fundamental* representation of $SL(2, C)$, the group of the unimodular 2×2 complex matrices. This is usually referred as the *two-component Weyl formalism.*

Let $M \in SL(2,\mathbb{C})$, then M acts on $\psi \in \mathbb{C}^2$ as

$$\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta, \tag{2.1}$$

where $\alpha = \beta = 1,2$ and ψ is defined as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_\alpha, \tag{2.2}$$

with ψ_1, ψ_2 being complex anticommuting Grassmann numbers. This is the *left-handed Weyl* spinor representation $(\frac{1}{2},0)$. A distinct representation is provided by M^*, which is the *complex conjugate* of the fundamental representation of $SL(2,C)$

$$\bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = M^{*\dot\beta}_{\dot\alpha} \bar\psi_{\dot\beta}, \tag{2.3}$$

with $\dot\alpha, \dot\beta = 1,2$, which is called also *right-handed Weyl* spinor representation $(0,\frac{1}{2})$. To distinguish them from left-handed Weyl spinors, right-handed Weyl spinors are denoted by dotted indices. This two representations are not equivalent, namely no matrix C exists such that $M = CM^*C^{-1}$. We can also define the contravariant spinor and the dotted contravariant spinor as

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}. \tag{2.4}$$

In what follows will be important to introduce the following tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.5}$$

We can show that these are invariant tensors under $SL(2,C)$. As a tensor, $\epsilon_{\alpha\beta}$ transforms like

$$\epsilon'_{\alpha\gamma} = M_\alpha{}^\beta M_\gamma{}^\delta \epsilon_{\beta\delta}, \tag{2.6}$$

and due to the following identity

$$1 = detM = -\frac{1}{2} M_\alpha{}^\beta M_\gamma{}^\delta \epsilon_{\beta\delta}\epsilon^{\alpha\gamma}, \tag{2.7}$$

then

$$\epsilon'_{\alpha\gamma}\epsilon^{\alpha\gamma}\epsilon_{\alpha\gamma} = -2\epsilon_{\alpha\gamma}, \tag{2.8}$$

$$\epsilon'_{\alpha\gamma} = \epsilon_{\alpha\gamma}. \tag{2.9}$$

Spinor contractions

For two anticommuting spinors ψ and χ we have

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \tag{2.10}$$

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\bar\chi^{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\alpha}\bar\chi^{\dot\beta} = -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi. \tag{2.11}$$

Pauli matrices and properties

Any 2×2 complex matrices in the $SL(2, \mathbb{C})$ group can be written as a linear combination of elements of the basis $\sigma^m = (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i)$, of the σ matrices, or Pauli matrices, defined as

$$\sigma^0 = \mathbb{I}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.12}$$

which transform under the $SL(2, \mathcal{C})$ group as follows

$$(\bar{\sigma}^m)^{\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta}(\sigma^m)_{\delta\dot{\gamma}}, \tag{2.13}$$

$$(\sigma^m)_{\alpha\dot{\beta}} = \epsilon_{\dot{\delta}\dot{\beta}}\epsilon_{\gamma\alpha}(\bar{\sigma}^m)^{\dot{\delta}\gamma}, \tag{2.14}$$

and satisfy the following properties

$$tr(\sigma^m \bar{\sigma}^n) = 2\eta^{mn}, \tag{2.15}$$

$$\sigma^m_{\alpha\dot{\beta}}\bar{\sigma}^{\dot{\gamma}\delta}_m = 2\delta^\delta_\alpha \delta^{\dot{\gamma}}_{\dot{\beta}}. \tag{2.16}$$

2.1.2 The full superPoincaré algebra

In the 1960's, with the growing awareness of the significance of internal symmetries, physicists attempted to find a symmetry which would combine in a non-trivial way the space-time Poincaré group with an internal symmetry group. Such an attempt however was impossible within the context of a Lie group because of the Coleman-Mandula theorem.

In 1967, Coleman and Mandula provided the theorem [3] that under certain physical assumptions regarding the S-matrix describing particle scattering, e.g. finite number of scattering particles and certain dependence of scattering on energy and angles, the generators of the symmetry group G of the S-matrix consist only of generators which correspond to

(a) Poincaré invariance which is the $ISO^\dagger(3,1)$ group characterized by the semi-direct product of translations and Lorentz rotations, whose generators P_m and M_{mn} satisfy the commutation relations

$$[P_m, P_n] = 0, \tag{2.17}$$

$$[P_m, M_{bc}] = \eta_{mb}P_c - \eta_{mc}P_b, \tag{2.18}$$

$$[M_{ab}, M_{cd}] = -\eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad}, \tag{2.19}$$

where η_{ab} is the Minkowski space metric

$$\eta_{ab} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{2.20}$$

(b) Internal global symmetries, related to conserved quantum numbers such as electric charge, isospin etc. The symmetry generators are Lorentz scalars and generate the Lie algebra

$$[T_i, T_j] = i f^k{}_{ij} T_k, \tag{2.21}$$

where $f^k{}_{ij}$ are the structure functions.

(c) Discrete symmetries: C, P, and T, and their products.

It is important to note that the space-time and the internal symmetries form the *direct* product of the Poincaré group and the internal group. This last condition however can be circumvented by weakening the assumption of the Coleman-Mandula theorem that the algebra of the S-matrix symmetries involves only commutators, by allowing anticommutating generators in the algebra.

This is the way we enter the realm of the *graded Lie superalgebras* which are defined as follows

Definition 1 (**Graded Lie superalgebra**)**.** A graded Lie superalgebra over a field k consists of a graded vector space E over k, along with a bilinear bracket operation

$$[-,-] : E \bigotimes_k E \to E, \tag{2.22}$$

which satisfies the following axioms

(a) $[-,-]$ respects the gradation of E

$$[E_i, E_j] \subseteq E_{i+j}. \tag{2.23}$$

(b) (Symmetry) If $x \in E_i$ and $y \in E_j$ then

$$[x, y] = -(-1)^{ij}[y, x]. \tag{2.24}$$

(c) (Jacobi identity) If $x \in E_i$, $y \in E_j$, and $z \in E_k$, then

$$(-1)^{ik}[x,[y,z]] + (-1)^{ij}[y,[z,x]] + (-1)^{jk}[z,[x,y]] = 0. \tag{2.25}$$

In our case the supersymmetry algebra is taken to be a Z_2 **graded Lie superalgebra** of order one $(i, j = 0, 1)$. It is a semidirect sum between the following two subalgebras

$$L = L_0 \bigotimes L_1, \tag{2.26}$$

where L_0 denotes the Poincaré algebra and its generators are called "*even*" whereas $L_1 = (Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$. The index $\alpha = 1, 2$ denotes the femionic components of each supercharge, while I count the number of supercharges needed to build the algebra. L_1 is hence a set of N anticommuting chiral and N antichiral spinor generators, called also "*odd*" generators, which transform in the representations $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Lorentz group.

Allowing the anticommuting generators in the symmetry algebra, in 1975 Haag, Lopuszański and Sohnius [4] showed that supersymmetry is the only possible extension of the Poincaré algebra allowed by S-matrix, by taking into account also the other "physical" assumptions, as causality, locality, positivity of energy, finiteness of number of particles, one would not like to relax of the Coleman-Mandula theorem. What is a particular fundamental feature in supersymmetry is the interplay of spacetime symmetry with internal symmetry via non-trivial internal symmetry transformation of Q^I. Furthermore, there is a non trivial connection between Lorentz generators of Poincaré algebra and the generators Q and this means that supersymmetry is not an internal symmetry rather it can be considered as a space-time symmetry.

Definition 2. The **full superPoincaré algebra** in $\mathcal{N} = N, D = 4$ is then given by the following relations

$$\begin{aligned}
[P_m, P_n] &= 0, && (2.27)\\
[M_{mn}, M_{rs}] &= -\eta_{mr}M_{ns} - \eta_{ns}M_{mr} + \eta_{ms}M_{ns} + \eta_{ns}M_{ms}, && (2.28)\\
[M_{mn}, P_r] &= -\eta_{rm}P_n + \eta_{rn}P_m, && (2.29)\\
[P_m, Q^I_\alpha] &= 0, && (2.30)\\
[P_m, \bar{Q}^I_{\dot{\alpha}}] &= 0, && (2.31)\\
[M_{mn}, \bar{Q}^I_\alpha] &= (\sigma_{mn})^\beta_\alpha Q^I_\beta, && (2.32)\\
[M_{mn}, \bar{Q}^{I\dot{\alpha}}] &= (\bar{\sigma}_{mn})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}, && (2.33)\\
\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} &= 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^{IJ}, && (2.34)\\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ}, \qquad Z^{IJ} = -Z^{JI}, && (2.35)\\
\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*, && (2.36)
\end{aligned}$$

where the 2-index Pauli matrices σ^{mn} are defined as

$$(\sigma^{mn})_\alpha{}^\beta = \frac{1}{4}(\sigma^m{}_{\alpha\dot{\gamma}}(\bar{\sigma}^n)^{\dot{\gamma}\beta} - (m \leftrightarrow n)), \quad (\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}((\bar{\sigma}^m)^{\dot{\alpha}\gamma}\sigma^n{}_{\gamma\dot{\beta}} - (m \leftrightarrow n)). \tag{2.37}$$

The relation (2.34)

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^{IJ},$$

is expected because of

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

It is worthwhile to note from (2.30) and (2.31) that Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ commute with translation operators P_m.

2.1.3 Comments

Central charges

Z^{IJ} generators, called *central charges*, are antisymmetric objects which commute with all generators of the algebra and may be associated to a mass of supermultiplets. For this reason, in the massless irreducible representations they vanish. In addition, they are always absent in the case of $I = 1$ and this is called the simple supersymmetry algebra.

R-symmetry

In general, fermionic generators of the superalgebra commute with the internal generators except from those associated with a particular symmetry, called **"R-symmetry"**, which in the case of the N-extended superPoincaré algebra is a $U(N)$ group. Therefore, we have to add also the following commutation relations to (2.27)-(2.36)

$$\begin{aligned} [T_i, T_j] &= i f_{ij}^k T_k, && (2.38)\\ [P_m, T_i] &= 0, && (2.39)\\ [M_{mn}, T_i] &= 0, && (2.40)\\ [Q^I{}_\alpha, T_i] &= t_{iJ}{}^I Q^J{}_\alpha, && (2.41) \end{aligned}$$

where Q's transform under the fundamental representation of $U(N)$.

$\mathcal{N} = 1$ superPoincaré algebra

In what follows we will mainly restrict to the consideration of the $\mathcal{N} = 1$ superPoincaré algebra in $D = 4$, with no central charges

$$\begin{aligned} [P_m, Q_\alpha] &= 0, && (2.42)\\ [P_m, \bar{Q}_{\dot\alpha}] &= 0, && (2.43)\\ [M_{mn}, \bar{Q}_\alpha] &= (\sigma_{mn})_\alpha^\beta Q_\beta, && (2.44)\\ [M_{mn}, \bar{Q}^{\dot\alpha}] &= (\bar\sigma_{mn})^{\dot\alpha}_{\dot\beta} \bar{Q}^{\dot\beta}, && (2.45)\\ \{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2\sigma^m_{\alpha\dot\beta} P_m, && (2.46)\\ \{Q_\alpha, Q_\beta\} &= 0, && (2.47)\\ \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} &= 0. && (2.48) \end{aligned}$$

For $\mathcal{N} = 1$ the R-symmetry group is just $U(1)$. In this case the hermitian matrices t_i are just real numbers and by defining the only $U(1)$ generator as R, one gets

$$[R, Q_\alpha] = -Q_\alpha, \quad [R, \bar{Q}_{\dot\alpha}] = +\bar{Q}_{\dot\alpha}. \quad (2.49)$$

This implies that supersymmetric partners (which are indeed related by the action of the Q's) have different R-charge.

Finally, we can introduce the concept of *superPoincaré group* which can be defined straightforwardly by the exponentiation of the superPoincaré algebra elements

$$G(x,\theta,\bar{\theta},\omega) = exp(ia_m P^m + i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}} + \frac{1}{2}i\lambda^{ab}M_{ab}), \tag{2.50}$$

where a^m is the parameter associated to the translational operator, λ_{ab} is associated to the Lorentz operator, while $\epsilon^\alpha, \bar{\epsilon}_{\dot{\beta}}$ are associated to the fermionic generators.

2.2 Representations of the superPoincaré algebra

In supersymmetry, single-particle states fall into irreducible representations of the supersymmetry algebra, called *supermultiplets*. Each supermultiplet contains both fermionic and bosonic states, which are *superpartners* of each other.

Any irreducible representation of the supersymmetry algebra is a representation of the Poincaré algebra, because the Poincaré algebra is a subalgebra of the superPoincaré algebra. Thus it follows that in general Poincaré representations will be *reducible*. A generic supermultiplet therefore will be a collection of particles of different spin related by supercharges Q_α.

The irreducible representations of the Poincaré superalgebra are classified by the eigenvalues of the *Casimir operators*. Their main property is that they commute with all generators of the algebra and, by employing the *Schur's lemma, they must be proportional to the identity if we consider an irreducible representation.* By this property we can label different irreducible representations with different Casimirs associated to different states, or **particles**.

We have two such operators in the Poincaré algebra given by

$$P_m P^m, \quad W_m W^m, \tag{2.51}$$

where P_m is the translation operator while W_m is the *Pauli-Ljubanski* operator defined as

$$W_m = \frac{1}{2}\epsilon_{mnpq} P^n M^{pq}. \tag{2.52}$$

where $\epsilon_{0123} = -\epsilon^{0123} = +1$. The Pauli-Ljubanski vector satisfies the following commutation relations

$$[W_m, P_n] = 0, \tag{2.53}$$

$$[W_m, M_{ab}] = \eta_{ma} W_b - \eta_{mb} W_a, \tag{2.54}$$

$$[W_m, W_n] = -\epsilon_{mnpq} W^p P^q. \tag{2.55}$$

$P^2 = m^2$ operator is related to mass while W^2 is related to spin.

In supersymmetric extensions of the Standard Model, the supersymmetry generators $Q, Q^\dagger$ also commute with the generators of the $SU(3) \times SU(2) \times U(1)$ gauge group. Therefore, particles in the same supermultiplet must also be in the same representation of the gauge group, and so must have the same electric charges, weak isospin, and color degrees of freedom.

2.2.1 Properties of superPoincaré representations

Before discussing different types of representations in supersymmetry algebra, we will present first some their common properties, which we will explicitly show in the particular $\mathcal{N} = 1$ supersymmetry.

SuperPoincaré Casimirs

In supersymmetry algebra, $P^2 = P_m P^m$ operator is still a Casimir, however W^2 is not a Casimir anymore, because it does not commute with the supersymmetric generators $(Q, \bar{Q})$, as it can be seen explicitly by taking the relation

$$[W^m, Q_\alpha] = -P_n (\sigma^{mn})_\alpha{}^\beta Q_\beta, \tag{2.56}$$

In the $\mathcal{N} = 1$ superPoincaré algebra we can however define another Casimir operator by generalizing the spin concept and introducing the following vector field

$$B_m = W_m - \frac{1}{4} \bar{Q}_{\dot{\alpha}} (\bar{\sigma}_m)^{\dot{\alpha}\beta} Q_\beta, \tag{2.57}$$

We introduce also a tensor C_{mn} defined as follows

$$C_{mn} = B_m P_n - B_n P_m, \tag{2.58}$$

and finally we can write the new Casimir of the supersymmetry algebra defined by

$$\tilde{C}_2 \equiv C_{mn} C^{mn}, \tag{2.59}$$

which is called *superspin* operator. Its eigenvalue is the same for all the fields in a given supermultiplet.

The mass degeneracy between bosons and fermions in the same irreducible representation is something we do not observe in known particle spectra; this implies that supersymmetry, if it at all is realized in Nature, must be broken at higher energies than the order of electro-weak scale.

Positivity of energy

As is known, the Poincaré group has irreducible unitary representations of two types: positive-energy representations and negative-energy representations. Only the positive-definite representations are physically admissible whereas those of negative-energy are discarded.

One remarkable property of supersymmetric theories is that the energy in a unitary representation of the superPoincaré algebra is always non-negative definite.

In a mathematical language [11], this property may be restated as follows; given a momentum eigenstate $|p\rangle$, the hermitian operator P_m acts as $-i\partial_m$ on a given field, thus on a plane wave $\phi = exp(ip \cdot x)$, P_m acts as $P_m \cdot \phi = p_m \phi$. Therefore on a momentum eigenstate $|p\rangle$, the eigenvalue of P_m is p_m. Thus the supersymmetry algebra becomes

$$[Q_\alpha, \bar{Q}_{\dot{\beta}}] \, |p\rangle = 2 \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} |p\rangle \,, \tag{2.60}$$

where we used (2.34) and p^0 denotes the energy of a generic state $|\phi\rangle$. Recalling that $p^0 = p_0$, we obtain

$$p^0 \, |p\rangle = \frac{1}{4} \left([Q_1, Q_1{}^\dagger] + [Q_2, Q_2{}^\dagger] \right) |p\rangle \,. \tag{2.61}$$

In other words, the Hamiltonian can be written in the following way

$$H = \frac{1}{4}\left(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2\right). \tag{2.62}$$

Its expectation value computed in a certain state $|\psi\rangle$

$$\langle\psi| H |\psi\rangle = \frac{1}{4}\left(||Q_1 |\psi\rangle ||^2 + ||Q_1^\dagger |\psi\rangle ||^2 + ||Q_2 |\psi\rangle ||^2 + ||Q_2^\dagger |\psi\rangle ||^2\right) \geq 0. \tag{2.63}$$

is given by a sum of squares which leads to a positive value unless the state is annihilated by all the supercharges, in which case it is zero.

Degrees of freedom

A supermultiplet contains an equal number of bosonic n_B and fermionic n_F degrees of freedom ($n_B = n_F$). To prove this statement we introduce the *fermion number operator*

$$(-1)^{N_F} = \begin{cases} +1 & \text{Bosons,} \\ -1 & \text{Fermions,} \end{cases}$$

or equivalently

$$(-)^{N_F} |B\rangle = |B\rangle\,, \tag{2.64}$$

$$(-1)^{N_F} |F\rangle = -|F\rangle\,, \tag{2.65}$$

and then

$$(-)^{N_F} Q_\alpha = -Q_\alpha (-)^{N_F}. \tag{2.66}$$

Multiplying (2.34) with the fermionic number operator and taking the trace we get the following identity

$$tr\left(((-)^{N_f}\{Q_\alpha, \bar{Q}_{\dot\beta}\}\right) = tr\left(-Q_\alpha(-)^{N_f}\bar{Q}_{\dot\beta} + Q_\alpha(-)^{N_f}\bar{Q}_{\dot\beta}\right) = 0, \tag{2.67}$$

$$= 2\sigma^m{}_{\alpha\dot\beta} tr((-)^{N_F} P_m) = 0. \tag{2.68}$$

For fixed non-zero momentum P_m,

$$tr(-1)^{N_F} = 0, \tag{2.69}$$

Therefore, any supersymmetry representation contains an equal number of fermionic and bosonic states.

2.2.2 Massless representations

Irreducible supersymmetric representations will be distinguished in massive and massless supermultiplets. They will possess different values of the mass, but also of the spin of the fields of the supermultiplet.

We will first consider massless representations in the general $\mathcal{N}$-extended case and then reduce to the simpler $\mathcal{N} = 1$. In all massless representations, central charges vanish, namely $Z^{IJ} = 0$ and thus all Q's and all $\bar{Q}$'s commute among themselves.

To build irreducible massless representations of supersymmetry we will follow the following strategy. Let us begin by going into a frame where the translation generator is $P_m = (E, 0, 0, E)$ (E is the energy) because $C_1 = P^m P_m = 0$. It can also be shown in [12], that in the $\mathcal{N} = 1$ case the second Casimir $\tilde{C}_2 = C_{mn} C^{mn} = 0$.

We then proceed by considering the relation (2.34) and figuring out the components of the matrix form

$$\{Q^I{}_\alpha, \bar{Q}^J{}_{\dot\beta}\} = 2(\sigma^m)_{\alpha\dot\beta} P_m \delta^{IJ} = \tag{2.70}$$

$$= 2E(\sigma^0 - \sigma^3)_{\alpha\dot\beta}\delta^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ}, \tag{2.71}$$

$$\rightarrow \{Q^I{}_1, \bar{Q}^J{}_{\dot 1}\} = 0, \tag{2.72}$$

where the product

$$\sigma^m P_m = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}. \tag{2.73}$$

(2.72) implies that both generators $Q^I{}_1$ and $\bar{Q}^J{}_{\dot 1}$ are trivially realized, indeed by taking a generic state $|\phi\rangle$, the expectation value of (2.72) on that state gives us

$$0 = \langle\phi| \{Q_1{}^I, \bar{Q}_{\dot 1}{}^J\} |\phi\rangle = ||Q_1{}^I |\phi\rangle ||^2 + ||\bar{Q}_{\dot 1}{}^J |\phi\rangle ||^2 = 0, \tag{2.74}$$

which is solved by $Q_1 |\phi\rangle = \bar{Q}_{\dot 1} |\phi\rangle = 0$. The remaining non trivial generators are Q_2 and $\bar{Q}_{\dot 2}$, namely only N of the original $2N$ generators.

It is convenient to redefine the non trivial generators as

$$a_I \equiv \frac{1}{\sqrt{4E}} Q^I_2, \quad a_I^\dagger \equiv \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot 2}, \tag{2.75}$$

which satisfy the following anticommutation relations

$$\{a_I, a_J^\dagger\} = \delta^{IJ}, \quad \{a_I, a_J\} = 0, \quad \{a_I^\dagger, a_J^\dagger\} = 0. \tag{2.76}$$

(2.75) operators are analogous to the creation and annihilation *ladder operators* in the classical harmonic oscillators.

With these operators in hand we may proceed to the next step of the construction of physical states of the supermultiplet. The starting fundamental state is the *Clifford vacuum*, annihilated by all the $a_I s$. It will carry a mass $m = 0$ with helicity λ_0 and we will denote it in the bracket formalism by $|E, \lambda_0\rangle$ ($|\lambda_0\rangle$, for short). So we impose that

$$a_I |\lambda_0\rangle = 0. \tag{2.77}$$

This state can be either bosonic or fermionic and it has not to be confused with the actual vacuum of the theory. By the Clifford vacuum we can obtain the entire

$\mathcal{N}$-extended massless supermultiplet by acting on $|\lambda_0\rangle$ with creation operators $a_I^\dagger$ and employing the (2.76) relations, as follows

$$
\begin{aligned}
|\lambda_0\rangle\,, \qquad &: \; 1 state \\
a_I^\dagger |\lambda_0\rangle \equiv \left|\lambda_0 + \frac{1}{2}\right\rangle_I , \qquad &: \; N states \\
a_I^\dagger a_J^\dagger |\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ}\,, \qquad &: \; \frac{1}{2}N(N-1) states \\
&\cdots \\
a_1^\dagger a_2^\dagger ... a_k^\dagger |\lambda_0\rangle \equiv \left|\lambda_0 + \frac{k}{2}\right\rangle , \qquad &: \; \binom{N}{k} = \frac{N!}{k!(N-k)!} states \\
&\cdots \\
a_1^\dagger a_2^\dagger ... a_N^\dagger |\lambda_0\rangle \equiv \left|\lambda_0 + \frac{N}{2}\right\rangle , \qquad &: \; \binom{N}{N} = 1 state
\end{aligned}
$$

and then the total number of states in the irreducible representation is given by

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N = (2^{N-1})_B + (2^{N-1})_F. \tag{2.78}$$

The construction of a massless supermultiplet is finished after adding a multiplet CPT conjugate to the former in order to make the whole multiplet CPT-invariant. If the supermultiplet is self-conjugate under CPT it is no needed to add anything.

$\mathcal{N} = 1$ Massless supermultiplet

We now consider the simplest case of $\mathcal{N} = 1$, where only one chiral and one antichiral Weyl spinor generators are added to form the superPoincaré algebra.

Definition 3. *Matter (chiral) supermultiplet.* We start from the Clifford vacuum of helicity $\lambda_0 = 0$. Acting on it with the creation operator $a^\dagger$ we obtain the multiplet

$$\lambda_0 = 0 \rightarrow \left(0, \frac{1}{2}\right) \oplus_{CPT} \left(-\frac{1}{2}, 0\right), \tag{2.79}$$

where we introduced the CPT conjugate of the multiplet, for the CPT invariance of the representation. The degrees of freedom of this representation are those of one Weyl fermion and one complex scalar. In a $\mathcal{N} = 1$ supersymmetric theory this is the representation where matter sits.

Definition 4. *Gauge multiplet (or vector multiplet).* We start from the Clifford vacuum with helicity $\lambda_0 = \frac{1}{2}$.

$$\lambda_0 = \frac{1}{2} \rightarrow \left(\frac{1}{2}, +1\right) \oplus_{CPT} \left(-1, -\frac{1}{2}\right). \tag{2.80}$$

The degrees of freedom are those of one vector and one Weyl fermion. This is the representation one needs to describe gauge fields in a supersymmetric theory.

Definition 5. *Gravitino multiplet.* This multiplet is made of one vector field and one gravitino field of spin $\frac{3}{2}$ as follows

$$\lambda_0 = 1 \to \left(1, +\frac{3}{2}\right) \oplus_{CPT} \left(-\frac{3}{2}, -1\right) \tag{2.81}$$

A consistent interaction of this multiplet requires coupling to the supergravity multiplet, which in $\mathcal{N} = 1$ case consists of a gravitino and a graviton.

Definition 6. *Graviton multiplet (supergravity multiplet)*

$$\lambda_0 = \frac{3}{2} \to \left(+\frac{3}{2}, +2\right) \oplus_{CPT} \left(-2, -\frac{3}{2}\right). \tag{2.82}$$

Massless supermultiplets of $\mathcal{N} = 2$ supersymmetry

Definition 7. *Matter multiplet (hypermultiplet)* is now bigger in number of states in comparison to the simplest $\mathcal{N} = 1$ case because we have two pairs of supersymmetry generators. Starting now with a Clifford vacuum of helicity $-\frac{1}{2}$ we get

$$\lambda_0 = -\frac{1}{2} \to \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \oplus_{CPT} \left(+\frac{1}{2}, 0, 0, -\frac{1}{2}\right), \tag{2.83}$$

which corresponds to the degrees of freedom of two Weyl fermions and two complex scalars.

Definition 8. *Gauge(vector) multiplet*

$$\lambda_0 = 0 \to \left(0, +\frac{1}{2}, +\frac{1}{2}, +1\right) \oplus_{CPT} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right). \tag{2.84}$$

The degrees of freedom are those of one vector, two Weyl fermions and one complex scalar. In $\mathcal{N} = 1$ language this is the sum of a vector and a matter multiplet, both transforming in the same adjoint representation of the gauge group.

Definition 9. *Gravitino multiplet*

$$\lambda_0 = -\frac{3}{2} \to \left(-\frac{3}{2}, -1, -1, -\frac{1}{2}\right) \oplus_{CPT} \left(+\frac{1}{2}, +1, +1, +\frac{3}{2}\right). \tag{2.85}$$

The degrees of freedom are those of a spin 3/2 particle, two vectors and one Weyl fermion.

Definition 10. *Graviton multiplet*

$$\lambda_0 = -2 \to \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \oplus_{CPT} \left(+1, +\frac{3}{2}, +\frac{3}{2}, +2\right). \tag{2.86}$$

The degrees of freedom are those of a graviton, two gravitinos and a vector, which is usually called *graviphoton* in the supergravity literature.

In the case of $\mathcal{N} > 4$ it is not possible to avoid gravity since any representation must have at least one state with helicity $\frac{3}{2}$ that requires coupling to gravity. Hence, theories with $\mathcal{N} > 4$ are all supergravity theories. It is interesting to note that $\mathcal{N} = 8$ supergravity allows only one possible representation with highest helicity 2, and for higher N one cannot avoid higher spin states, with helicity $\frac{5}{2}$ or higher. Therefore, $\mathcal{N} = 8$ is an upper bound on the number of supersymmetries in $D = 4$ space-time which do not involve higher spins.[1]

If we consider other space-time dimensions we have to remake the statement of the upper bound indeed in ten space-time dimensions the maximum allowed supersymmetry is $\mathcal{N} = 2$.

A dimension-independent statement can be made counting the number of single component supersymmetry generators. The maximum allowed number of generators for non-gravitational theories is 16 (which is $\mathcal{N} = 4$ in four dimensions) and 32 for theories with gravity (which is $\mathcal{N} = 8$ in four dimensions).

2.2.3 Massive Representations

We will follow the same logical procedure used in building massless representations, however we must point out some critical differences between massive and massless supermultiplets. The most relevant is that states now are massive, so we can choose the preferred frame to be in the rest frame

$$P_m = (m, 0, 0, 0). \tag{2.87}$$

In the massive case we also better refer to the spin concept rather than helicity and the parameters that label states in a particular representation are denoted by $|m, j\rangle$, where *"m"* denotes mass and *"j"* the spin.

Another big difference from the massless case is that the number of non-trivial generators gets not diminished, indeed there remain the full set of $2N$ creation and annihilation operators as can be derived from the anticommutation relations

$$\{Q^I{}_\alpha, \bar{Q}^J{}_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}}\delta^{IJ}. \tag{2.88}$$

We better redefine supersymmetry generators $Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}$ by introducing the following operators

$$a_{1,2}{}^I \equiv \frac{1}{\sqrt{2m}} Q^I_{1,2}, \quad a^{\dagger I}_{1,2} \equiv \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}}{}^I, \tag{2.89}$$

which again this describes an algebra analogous to that of the harmonic algebra with ladder generators.

The building of massive representations starts by defining a Clifford vacuum state $|\Omega\rangle$, with mass m and spin λ_0, annihilated by both the set of $a^I{}_1$ and $a^I{}_2$. Then, acting on this Clifford vacuum with the creation operators and using the anticommutation relations, we obtain the remaining states of the generic $\mathcal{N}$-extended massive supermultiplet.

[1]The higher spin fields are known to have problems to interact with each other and with gravity. This issue is the subject of the higher spin theory.

$\mathcal{N}=1$ **Massive supermultiplets**

Definition 11. *Matter (or chiral) multiplet.* We start by assigning to the Clifford vacuum mass "m" and spin $j=0$ and then we act on this state with all creation operators

$$j=0 \to \left(\ -\tfrac{1}{2}, \quad 0, \quad 0', \quad +\tfrac{1}{2} \ \right).$$

In this multiplet there is no need to add any CPT conjugate and furthermore we may see that the second scalar state (0') has opposite parity with respect to 0, so it's a pseudoscalar.

Definition 12. *Massive vector multiplet*

$$j=\frac{1}{2} \to \left(\ -1, \quad 2\times -\tfrac{1}{2}, \quad 2\times 0, \quad 2\times +\tfrac{1}{2}, \quad 1 \ \right). \tag{2.90}$$

The degrees of freedom are those of one massive vector, one massive Dirac fermion and one massive scalar and these are the same of those of a massless vector multiplet plus one massless matter multiplet.

2.3 $\mathcal{N}=1$ Superspace

Until now we have pointed out the explicit structure of the superPoincaré algebra in the case $\mathcal{N}=1$ and $D=4$ and how to build massless and massive representations of this superalgebra.

Here we want to continue the development of tools needed in the construction of supersymmetry Lagrangians. If we construct supersymmetric actions in the ordinary Minkowski space-time, the supersymmetry invariance of the theory is not evident. To prove the invariance one has to take length computations, by explicitly varying the action under a given supersymmetric transformation that acts on all fields concerned. However a clever treatment can be followed in order to simplify computations and to make explicit the supersymmetric nature of the action.

Superspace is the key. It is a "special" framework, which extends the Minkowski space-time, and leads to theories where supersymmetry invariance becomes *manifest*, in the sense that an action made up of functions (*superfields*) of *superspace*, is automatically supersymmetric invariant without any computation.

Superspace and superfields will be our building blocks in order to derive the supersymmetric dynamics of fields and their interactions.

In the case of $\mathcal{N}=1, D=4$ a point in superspace is parametrized by 4 bosonic coordinates (commuting coordinates) and 4 fermionic (anticommuting coordinates) and it is denoted by

$$z^M=(x^m,\theta_\alpha,\bar{\theta}_{\dot{\alpha}}), \tag{2.91}$$

with M the superindex which stands either for bosonic or fermionic indices, and we introduced a set of constant Grassmann numbers which form a right-handed Weyl spinor θ_α and its complex conjugate left-handed Weyl spinor $\bar{\theta}_{\dot{\alpha}}$. These coordinates anticommute with everything fermionic and commute with everything bosonic

$$\{\theta^\alpha, \theta^\beta\} = 0, \tag{2.92}$$
$$\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0, \tag{2.93}$$
$$\{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = 0, \tag{2.94}$$
$$\theta_\alpha\theta_\beta\theta_\gamma = 0. \tag{2.95}$$

Therefore we can resume that superspace coordinates obey the following property

$$z^M z^N = z^N z^M (-1)^{MN}. \tag{2.96}$$

where we follow the rule that M, N are 0 for bosonic coordinates and 1 for fermionic coordinates.

Supersymmetry transformations on $(x, \theta, \bar{\theta})$ read as follows

$$x'^m = \Lambda^m{}_n x^n + a^m + i\theta\sigma^m\bar{\epsilon} - i\epsilon\sigma^m\bar{\theta}, \tag{2.97}$$
$$\theta_\alpha = n_\alpha{}^\beta\theta_\beta + \epsilon_\alpha, \tag{2.98}$$
$$\bar{\theta}_{\dot{\alpha}} = \bar{n}_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\theta}_{\dot{\beta}} + \bar{\epsilon}_{\dot{\alpha}}, \tag{2.99}$$

where ϵ_α are constant spinorial parameters, $\Lambda_n{}^m$ are Lorentz generators in *vectorial representation* while $n_\alpha{}^\beta$ are Lorentz generators in *spinorial representation* which are related by the following formula

$$\Lambda^{-1}{}_m{}^n\sigma^m_{\alpha\dot{\alpha}} = n_\alpha{}^\beta\sigma^n{}_{\beta\dot{\beta}}\bar{n}_{\dot{\alpha}}{}^{\dot{\beta}}. \tag{2.100}$$

To complete our development of superspace let us define how derivation and integration operators act on Grassmann numbers.

Definition 13. *Derivation* in superspace is defined as

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}, \quad \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \tag{2.101}$$

and

$$\partial_\alpha\theta^\beta = \delta^\beta_\alpha, \quad \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}, \quad \partial_\alpha\bar{\theta}_{\dot{\beta}} = 0, \quad \bar{\partial}^{\dot{\alpha}}\theta^\beta = 0. \tag{2.102}$$

where $\epsilon^{\alpha\beta}$ is defined in (2.5).

Definition 14. *Integration* instead is defined as

$$\int d\theta = 0, \quad \int d\theta\theta = 1, \tag{2.103}$$

so for a generic function $f(\theta) = f_0 + \theta f_1$ we can have the following results

$$\int d\theta f(\theta) = f_1, \quad \int d\theta\delta(\theta) f(\theta) = f_0, \tag{2.104}$$

which means that integration in Grassmann coordinates is equivalent to a derivation.

If we introduce also

$$d^2\theta \equiv \frac{1}{2}d\theta^1 d\theta^2, \quad d^2\bar\theta \equiv \frac{1}{2}d\bar\theta^{\dot 2} d\bar\theta^{\dot 1}, \tag{2.105}$$

then we have other properties as

$$\int d^2\theta\ \theta\theta = \int d^2\bar\theta\ \bar\theta\bar\theta = 1, \int d^2\theta d^2\bar\theta\ \theta\theta\bar\theta\bar\theta = 1, \tag{2.106}$$

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta, \quad \int d^2\bar\theta = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\alpha}\bar\partial_{\dot\beta}. \tag{2.107}$$

2.4 $\mathcal{N}=1$ Superfields

Now we introduce supersymmetric objects which live in superspace called superfields. They are functions of the superspace coordinates and are composed of many ordinary bosonic fields.

The most general scalar superfield we can write is

$$\begin{aligned} Y(x,\theta,\bar\theta) &= f(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) \\ &+ \theta\sigma^m\bar\theta v_m(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta d(x), \end{aligned} \tag{2.108}$$

where we made a taylor-like expansion in $\theta, \bar\theta$ coordinates and we stopped at the second order in θ and $\bar\theta$ because $\theta_\alpha\theta_\beta\theta_\gamma = 0$.

In what follows, it will be very useful to consider the differential form of the supercharges $Q_\alpha, \bar Q_{\dot\alpha}$. So let us first define *infinitesimal translations* of a scalar superfield in superspace as

$$\delta_{\epsilon,\bar\epsilon} Y(x,\theta,\bar\theta) \equiv Y(x+\delta x; \theta+\delta\theta, \bar\theta+\delta\bar\theta) - Y(x,\theta,\bar\theta) \tag{2.109}$$

$$\equiv i\epsilon^\alpha Q_\alpha Y + i\bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha} Y. \tag{2.110}$$

and taylor-expand the right-hand side of (2.109) which becomes

$$\begin{aligned} \delta_{\epsilon,\bar\epsilon} Y(x,\theta,\bar\theta) &= Y(x,\theta,\bar\theta) + i(\theta\sigma^m\bar\epsilon - \epsilon\sigma^m\bar\epsilon)\partial_m Y(x,\theta,\bar\theta) \\ &+ \epsilon^\alpha\partial_\alpha Y(x,\theta,\bar\theta) + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} Y(x,\theta,\bar\theta) - Y(x,\theta,\bar\theta) \\ \delta_{\epsilon,\bar\epsilon} Y(x,\theta,\bar\theta) &= [\epsilon^\alpha\partial_\alpha + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha} + i(\theta\sigma^m\bar\epsilon - \epsilon\sigma^m\bar\theta)\partial_m] Y(x,\theta,\bar\theta), \end{aligned} \tag{2.111}$$

and thus we get the following expression for the $Q_\alpha, \bar Q_{\dot\alpha}$ generators in the differential form

$$Q_\alpha = -i\partial_\alpha - \sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m, \tag{2.112}$$

$$\bar Q_{\dot\alpha} = +i\bar\partial_{\dot\alpha} + \theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m. \tag{2.113}$$

We can introduce also the useful covariant derivatives $D_\alpha, \bar D_{\dot\alpha}$ as

$$D_\alpha = \partial_\alpha + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m, \tag{2.114}$$

$$\bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m. \tag{2.115}$$

These derivatives anticommute with the supersymmetric generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ and satisfy the following relations

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^m_{\alpha\dot{\beta}}\partial_m = -2\sigma^m_{\alpha\dot{\beta}}P_m, \tag{2.116}$$
$$\{D_\alpha, D_\beta\} = 0, \tag{2.117}$$
$$\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \tag{2.118}$$

which imply the following

$$\delta_{\epsilon,\bar{\epsilon}}(D_\alpha Y) = D_\alpha(\delta_{\epsilon,\bar{\epsilon}}Y). \tag{2.119}$$

This means that $D_\alpha Y$ is a superfield if Y is a superfield.

2.4.1 Fields and irreducible representations of supersymmetry

In general, superfields correspond to reducible representations of supersymmetry. To relate a superfield to an irreducible supersymmetry representation we need to impose on the superfield certain constraints. We will now show what kind of superfield corresponds to the chiral supermultiplet of (2.79). To start with, let us first relate the chiral supermultiplet to the corresponding Matter fields multiplet.

In $\mathcal{N} = 1$ case, the starting point is the commutation relation between one antichiral Weyl supercharge and the complex scalar field $\phi(x)$

$$[\bar{Q}_{\dot{\alpha}}, \phi(x)] = 0. \tag{2.120}$$

The complex scalar field plays the analogous role of the Clifford vacuum in irreducible representations. As for the construction of supermultiplets, if we act on this field with Q_α we obtain a new field on the same representation.

The new fermionic field of the Matter representation can be defined as

$$[Q_\alpha, \phi(x)] \equiv \psi_\alpha(x) \tag{2.121}$$

Reacting again with Q_α on the ψ spinor we can in principle have a new bosonic field

$$\{Q_\alpha, \psi_\beta(x)\} = F_{\alpha\beta}(x), \tag{2.122}$$
$$\{\bar{Q}_{\dot{\alpha}}, \psi_\beta(x)\} = X_{\dot{\alpha}\beta}(x). \tag{2.123}$$

It can be shown that $X_{\dot{\alpha}\beta}$ is not a new field but just a time derivative of the scalar field ϕ, but if we apply the Jacobi identity on the $(\phi, Q, \bar{Q})$ then we have

$$\{Q_\alpha, [Q_\beta, \phi]\} - \{Q_\beta, [\phi, Q_\alpha]\} = 0 \rightarrow F_{\alpha\beta} + F_{\beta\alpha} = 0, \tag{2.124}$$

where the field $F_{\alpha\beta}$ is obviously antisymmetric (on $\alpha \leftrightarrow \beta$) and this implies that

$$F_{\alpha\beta}(x) = \epsilon_{\alpha\beta}F(x), \tag{2.125}$$

where F is a new scalar field.

If we repeat the previous steps acting with Q and $\bar{Q}$ on this new field, in principle we can write

$$[Q_\alpha, F] = \lambda_\alpha,$$
$$[\bar{Q}_{\dot{\alpha}}, F] = \bar{X}_{\dot{\alpha}}.$$

This time, after applying the Jacobi identity, it can be proven that λ_α is a vanishing and $\bar{X}_{\dot{\alpha}}$ is proportional to the spacetime derivative of the field ψ, so no new fields are added. We have to stop here and the final multiplet of fields is then given by the following multiplet

$$(\phi, \psi, F). \tag{2.126}$$

This is also called the *Wess-Zumino multiplet* and it is the field counterpart of the chiral multiplet of states. Notice that the equality of the number of fermionic and bosonic states for a given representation still holds: we are now off-shell, and the spinor ψ_α has four degrees of freedom; this is the same number of bosonic degrees of freedom, two coming from the complex scalar field ϕ and two from the complex scalar field F. Going on-shell instead, the 4 fermionic degrees of freedom reduce to just 2 propagating degrees of freedom, due to Dirac equation. The reduction for the bosonic degrees of freedom, comes out by the fact that on the mass-shell, F field is an auxiliary field. In conclusion, there remain $2B + 2F$ on-shell conditions, related to those of the massless state representation.

This strategy can be generalized in the construction of other kind of field multiplets, like for example vector multiplet etc.

2.4.2 Chiral superfields

The *chiral superfield* can be obtained by imposing on the general superfield expression (2.108) the following constraint

$$\bar{D}_{\dot{\alpha}} \Phi = 0, \tag{2.127}$$

while an anti-chiral $\bar{\Phi}$ satisfies

$$D_\alpha \bar{\Phi} = 0. \tag{2.128}$$

Let us introduce a useful set of coordinates, denoting the so called *chiral superspace* parametrized by

$$y^m = x^m + i\theta\sigma^m\bar{\theta}. \tag{2.129}$$

Coordinates transform under supersymmetry as

$$\delta\theta^\alpha = \epsilon^\alpha, \tag{2.130}$$
$$\delta y^m = 2i\theta\sigma^m\bar{\epsilon}. \tag{2.131}$$

In this reduced superspace the chiral superfield Φ can be expanded in the following form

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y). \tag{2.132}$$

Following (2.110) the chiral (or anti-chiral) superfield transforms under supersymmetric transformations as

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) = (i\epsilon Q + i\bar{\epsilon}\bar{Q})\Phi(y,\theta). \tag{2.133}$$

The differential operators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ in the $(y^m, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ coordinate system result in

$$Q_\alpha = -i\partial_\alpha, \tag{2.134}$$

$$\bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^m}. \tag{2.135}$$

Plugging these equations in (2.133) one gets

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) &= \left(\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^m_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\frac{\partial}{\partial y^m}\right)\Phi(y,\theta), \qquad (2.136)\\ &= \sqrt{2}\epsilon\psi - 2\epsilon\theta F + 2i\theta\sigma^m\bar{\epsilon}\left(\frac{\partial}{\partial y^m}\phi + \sqrt{2}\theta\frac{\partial}{\partial y^m}\psi\right),\\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta\left(-\sqrt{2}\epsilon F + \sqrt{2}i\sigma^m\bar{\epsilon}\frac{\partial}{\partial y^m}\phi\right) - \theta\theta\left(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^m\frac{\partial}{\partial y^m}\psi\right). \end{aligned}$$

The final expression for the supersymmetry variation of the different field components of the chiral superfield Φ reads

$$\delta\phi = \sqrt{2}\epsilon\psi, \tag{2.137}$$

$$\delta\psi_\alpha = \sqrt{2}i(\sigma^m\bar{\epsilon})_\alpha\partial_m\phi - \sqrt{2}\epsilon_\alpha F, \tag{2.138}$$

$$\delta F = i\sqrt{2}\partial_m\psi\sigma^m\bar{\epsilon}. \tag{2.139}$$

We can show for consistency that

$$\bar{D}_{\dot{\alpha}}\theta_\beta = \bar{D}_{\dot{\alpha}}y^m = 0, \qquad D_\alpha\bar{\theta}_{\dot{\beta}} = D_\alpha\bar{y}^m = 0. \tag{2.140}$$

as wanted.

In the chiral basis, the expressions for the covariant derivatives are the following

$$D_\alpha = \partial_\alpha + 2i\sigma^m_{\alpha b\dot{e}ta}\bar{\theta}^{\dot{\beta}}\partial_m, \tag{2.141}$$

$$\bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}}. \tag{2.142}$$

Taylor-expanding (2.132) around x^m we get

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^m\bar{\theta}\partial_m\phi(x) - \theta\theta F(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x), \tag{2.143}$$

and analogously for $\bar{\Phi}(x,\theta,\bar{\theta})$ we have

$$\bar{\Phi}(x,\theta,\bar{\theta}) = \bar{\phi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - i\theta\sigma^m\bar{\theta}\partial_m\bar{\phi}(x) - \bar{\theta}\bar{\theta}\bar{F}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\psi}(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\bar{\phi}(x). \tag{2.144}$$

This chiral superfield has the degrees of freedom of the chiral multiplet of fields (2.126) and on-shell it corresponds to a $\mathcal{N} = 1$ multiplet of states, carrying an irreducible representation of the $\mathcal{N} = 1$ supersymmetry algebra.

2.4.3 Real Superfields

Real supermultiplets containing the bosonic vector field can be described by *real superfields*, defined by imposing the following constraint

$$V = \overline{V}, \tag{2.145}$$

on the generic superfield (2.108). This constraint leads to the following expression for V

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \theta\sigma^m\bar\theta v_m + \frac{i}{2}\theta\theta(M(x)+iN(x)) \\ &- \frac{1}{2}\bar\theta\bar\theta(M(x)-iN(x)) + i\theta\theta\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\bar\sigma^m\partial_m\chi(x)\right) + \\ &- i\bar\theta\bar\theta\theta\left(\lambda(x) - \frac{i}{2}\sigma^m\partial_m\bar\chi(x)\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D(x) - \frac{1}{2}\partial^2 C(x)\right), \end{aligned} \tag{2.146}$$

with eight fermionic fields (like $\chi, \bar\chi, \lambda, \bar\lambda$) and eight bosonic fields (C, M, N, D, v_m).

We also would like to introduce the supersymmetric extension of the analogous *gauge transformation* of gauge theory in QFT, because it will reduce the number of the on-shell degrees of freedom of the superfield V to $2_F + 2_B$ (those for a massless $\left(\frac{1}{2}, 1\right)$ vector representation of supersymmetry).

It can be shown that under the transformation

$$V \to V + \Phi + \bar\Phi, \tag{2.147}$$

where Φ is the chiral superfield (2.132), the vector field v_m in V transforms as an (abelian) gauge field

$$v_m \to v_m + \partial_m(2Im(\phi)). \tag{2.148}$$

We note that $\Phi + \bar\Phi$ is a real superfield. Under (2.147) the component fields of V transform as

$$C \to C + 2Re\phi, \tag{2.149}$$
$$\chi \to \chi - i\sqrt{2}\psi, \tag{2.150}$$
$$M \to M - 2ImF, \tag{2.151}$$
$$N \to N + 2ReF, \tag{2.152}$$
$$D \to D, \tag{2.153}$$
$$\lambda \to \lambda, \tag{2.154}$$
$$v^m \to v^m + 2\partial^m(Im\phi). \tag{2.155}$$

One can see that properly fixing the gauge, namely by choosing field components of Φ to be

$$Re\phi = -\frac{C}{2}, \quad \psi = -\frac{i}{\sqrt{2}}\chi, \quad ReF = -\frac{N}{2}, \quad ImF = \frac{M}{2}, \tag{2.156}$$

one can gauge away C, M, N, χ. This choice is called the *Wess-Zumino gauge*, defined as a gauge where no restrictions are putted on v^m, to leave the freedom of ordinary gauge transformations, in which the real superfield can be written as

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{2.157}$$

Therefore, taking into account the gauge transformations of v_m (2.155), we end-up with only $4_B + 4_F$ degrees of freeedom off-shell. As we shall see, D will turn out to be an auxiliary field; therefore, by imposing the equations of motion for D, λ and the vector v^m, one will end up with $2_B + 2_F$ degrees of freedom on-shell, the right number of a massless vector supermultiplet.

Let us end this section with two important comments. The first comment is that

$$V^2{}_{WZ} = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}v_m v^m, \tag{2.158}$$

because each term in the expansion of V_{WZ} contains at least one θ, and this implies

$$V^n{}_{WZ} = 0 \quad n \geq 3, \tag{2.159}$$

Second, we can notice that in the Wess-Zumino gauge, the real superfield is not automatically supersymmetric, because acting with a supersymmetry transformation one obtains a new superfield which is not in the WZ gauge. Hence one has to do a compensating supersymmetric gauge transformation, by choosing certain Φ, to come back to the WZ gauge.

2.5 Supersymmetric actions

In this section we will employ the concepts of superfields and superspace in $\mathcal{N}=1$, $D=4$ superspace, developed in the previous sections, and show how to construct rigid supersymmetric actions. We will construct in particular the supersymmetric matter Lagrangian and pure superYang-Mills.

But first we will prove a fundamental property of superspace and superfield formalism.

Theorem 2.5.1. *Superspace actions are supersymmetric invariant if they are functionals of superfields or their derivatives in superspace.*

Let us begin demonstration by noting that the measure is invariant under supertranslations

$$\int d\theta\theta = \int d(\theta+\zeta)(\theta+\zeta) = 1, \tag{2.160}$$

therefore

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x d^2\theta d^2\bar{\theta} Y(x,\theta,\bar{\theta}) = \int d^4x d^2\theta d^2\bar{\theta} \delta_{\epsilon,\bar{\epsilon}} Y(x,\theta,\bar{\theta}). \tag{2.161}$$

Recalling (2.111)

$$\delta_{\epsilon,\bar{\epsilon}} Y = \epsilon^\alpha \partial_\alpha Y + \bar{\epsilon}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} Y + \partial_m[-i(\epsilon\sigma^m\bar{\theta} - \theta\sigma^m\bar{\epsilon})Y],$$

we see that only its last term is not trivial under integration on $d^2\theta d^2\bar{\theta}$, however it does not contribute to the variation of the action because it is a total space-time derivative. In other words, the integrand of (2.161), is supersymmetric invariant under supersymmetry transformations, namely

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x d^2\theta d^2\bar{\theta} Y(x,\theta,\bar{\theta}) = 0. \tag{2.162}$$

□

Therefore, supersymmetric invariant actions are constructed by integrating in superspace a certain superfield which must have the property that after we integrate Grassmann coordinates, it gives rise to a real Lagrangian density of dimension four, transforming as a scalar density under Poincaré transformations like the following example

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{A}(x,\theta,\bar{\theta}) = \int d^4x \mathcal{L}(\phi(x),\psi(x),A_m(x),...). \tag{2.163}$$

2.5.1 $\mathcal{N}=1$ Matter actions

To contruct the simplest supersymmetric action (the so called *Wess-Zumino model*), let us take the product of a chiral superfield Φ with its complex conjugate anti-chiral superfield $\bar{\Phi}$

$$\mathcal{L}_{kin}{}^{matter} = \int d^2\theta d^2\bar{\theta} \bar{\Phi}\Phi. \tag{2.164}$$

This integral satisfies the following properties

(a) It is supersymmetric invariant since it is built of superfields in superspace.

(b) It is real because $(\theta^2\bar{\theta}^2)^\dagger = \theta^2\bar{\theta}^2$ and of course $\bar{\Phi}\Phi$ is a real term.

(c) It has the right physical dimension ([M]=4) for a *renormalizable theory*, indeed ϕ has dimension one in four dimension space-time, then $[\theta] = -1/2$, as can be deduced by comparing dimensions of $\phi(x)$ and $\theta\psi(x)$, with ψ of dimension $\frac{3}{2}$. We must have then $[d\theta] = +1/2$ because the differential is equivalent to a derivative for Grassmann variables. Finally, since $[\bar{\Phi}\Phi] = 2$, the $\theta^2\bar{\theta}^2$ component of $\Phi\bar{\Phi}$ has dimension [M]=4.

Lagrangian (2.164) can be expanded in component fields in $(x, \theta, \bar{\theta})$ coordinates as

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\bar{\Phi}\Phi = \partial_m\bar{\phi}\partial^m\phi + \frac{i}{2}(\partial_m\psi\sigma^m\bar{\psi} - \psi\sigma^m\partial_m\bar{\psi}) + \bar{F}F + ... \qquad (2.165)$$

where dots stand for total derivative terms which do not contribute to the action. This is the kinetic matter Lagrangian describing the degrees of freedom of a free massless chiral supermultiplet.

The $F, \bar{F}$ are *auxiliary fields*, with no propagating degrees of freedom, as it can be noticed by the absence of any derivative in F. Integrating them out (their equations of motion are $F = 0, \bar{F} = 0$) one gets supersymmetry which only closes on-shell.

To find equations of motion from the action (2.164) we cannot perform directly the variation with respect to Φ or $\bar{\Phi}$, because these superfields are constrained. To take into account the chiral condition (2.127) (or the anti-chiral (2.128)), we must rewrite (2.164) in the equivalent form

$$\int d^2\theta d^2\bar{\theta}\bar{\Phi}\Phi = -\frac{1}{4}\int d^2\theta\bar{\Phi}D^2\Phi, \qquad (2.166)$$

by using (2.128) and the relations (2.107) up to total space-time derivative

$$\int d^2\theta = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}D_{\dot{\alpha}}D_{\dot{\beta}}. \qquad (2.167)$$

Now we can vary the action with respect to $\bar{\Phi}$ getting

$$D^2\Phi = 0, \qquad (2.168)$$

which gives the right equations of motion of massless fields ϕ, ψ, F

$$\Box\phi = 0, \qquad (2.169)$$

$$i\sigma^m\partial_m\psi = 0, \qquad (2.170)$$

$$F = 0. \qquad (2.171)$$

We can generalize this matter Lagrangian by introducing a function of Φ and $\bar{\Phi}$ as

$$\int d^2\theta d^2\bar{\theta}K(\bar{\Phi}, \Phi). \qquad (2.172)$$

The function $K(\Phi, \bar{\Phi})$ should satisfy the following properties

(a) it should be a superfield, to ensure supersymmetry invariance of the integral.

(b) It should be a real and scalar function.

The most general expression of $K(\Phi, \bar{\Phi})$ compatible with all these properties is

$$K(\Phi, \bar{\Phi}) = \sum_{m,n=1}^{\infty} c_{mn} \bar{\Phi}^m \Phi^n, \tag{2.173}$$

where the reality condition is ensured by the relation $c_{mn} = c^*_{nm}$.

All coefficients c_{mn} with either m or n greater than one have negative mass dimension, because chiral superfields have mass dimension equal to one. c_{11} instead is dimensionless. In general, a contribution like (2.172) will describe a supersymmetric invariant theory but *non-renormalizable*. Indeed, the coefficients c_{mn} will be proportional to a scale parameter Λ as follows

$$c_{mn} \sim \Lambda^{2-(m+n)}. \tag{2.174}$$

The function $K(\Phi, \bar{\Phi})$, called *Kahler potential*, is related to a complex manifold, called *Kahler manifold*, whose complex coordinates are the chiral and antichiral superfields $\Phi^i, \bar{\Phi}_j$, while the Kahler metric is defined as the second derivative term

$$g_{ij} = \frac{\partial^2 K(\Phi, \bar{\Phi})}{\partial \Phi^i \partial \bar{\Phi}_{\bar{j}}}. \tag{2.175}$$

Eq(2.165) is constructed with *kinetic-like terms* between fields of the chiral supermultiplet. In order to describe also *potential-like terms* between component fields, we will integrate in chiral superspace a generic chiral superfield $\Sigma(\Phi)$

$$\int d^4y d^2\theta \Sigma(\Phi). \tag{2.176}$$

Eq.(2.176) is more general than (2.172) in the following sense. Any integral in superspace can be rewritten as an integral in chiral superspace, indeed for any superfield $Y(x, \theta, \bar{\theta})$ we can find a covariant derivative

$$\int d^4x d^2\theta d^2\bar{\theta} Y = \int d^4x d^2\theta \bar{D}^2 Y, \tag{2.177}$$

due to the relation (2.167). Furthermore $\bar{D}^2 Y$ is manifestly chiral since $\bar{D}^3 = 0$. The converse however is not true in general, indeed if we consider the integral

$$\int d^4x d^2\theta \Phi^n, \tag{2.178}$$

this cannot be converted into an integral in full superspace because there are no present any covariant derivatives like before.

The chiral superfield $\Sigma(\Phi)$ will be substituted from the holomorphic superfield $W(\Phi)$, called *superpotential*, defined by

$$W(\Phi) = \sum_{n=1}^{\infty} a_n \Phi^n. \tag{2.179}$$

Potential-like terms are present in the following Lagrangian

$$\mathcal{L}_{int} = \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}), \tag{2.180}$$

where the second term is added to ensure reality condition.

To preserve renormalizability, the superpotential should have at most the dimension $[M] = 3$ or equivalently to be at most cubic in Φ.

To obtain the expansion of $\mathcal{L}_{int}$ in component fields we have to make the Taylor-expansion of the superpotential in powers of θ around $\Phi|_{\theta=0} = \phi$

$$W(\Phi) = W(\phi) + \sqrt{2}\frac{\partial W}{\partial \phi}\theta\psi - \theta\theta\left(\frac{\partial W}{\partial \psi}F + \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi\right). \tag{2.181}$$

Thus the most general chiral matter superfield Lagrangian with kinetic-like and potential-like terms has the following form

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta} K(\bar{\Phi}, \Phi) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}). \tag{2.182}$$

For renormalizable theories the Kähler potential is just $\bar{\Phi}\Phi$ and the superpotential is at most cubic, so in component fields we have

$$\mathcal{L}_{matter} = \partial_m\bar{\phi}\partial^m\phi + \frac{i}{2}(\partial_m\psi\sigma^m\bar{\psi} - \psi\sigma^m\partial_m\bar{\psi}) + \bar{F}F - \frac{\partial W}{\partial \phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi + \frac{1}{2}\frac{\partial^2 \bar{W}}{\partial\bar{\phi}\partial\bar{\phi}}. \tag{2.183}$$

The auxiliary fields F and $\bar{F}$ can be integrated out by using their equations of motion

$$\bar{F} = \frac{\partial W}{\partial \phi}, \quad F = \frac{\partial \bar{W}}{\partial \bar{\phi}}, \tag{2.184}$$

and the remaining on-shell Lagrangian is

$$\mathcal{L}_{on-shell}{}^{matter} = \partial_m(\bar{\phi})\partial^m\phi + \frac{i}{2}(\partial_m\psi\sigma^m\bar{\psi} - \psi\sigma^m\partial_m\bar{\psi}) - |\frac{\partial W}{\partial \phi}|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi + \frac{1}{2}\frac{\partial^2 \bar{W}}{\partial\bar{\phi}\partial\bar{\phi}}. \tag{2.185}$$

We can define the *scalar potential* as

$$V(\phi, \bar{\phi}) = |\frac{\partial W}{\partial \phi}|^2 = \bar{F}F. \tag{2.186}$$

Lagrangian (2.164) can be generalized by introducing a set of chiral superfields Φ^i with i=1,2,...,n as follows

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta} K(\bar{\Phi}_i, \Phi^i) + \int d^2\theta W(\Phi^i) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}_i), \tag{2.187}$$

and for renormalizable theories we have

$$K(\Phi^i, \bar{\Phi}_i) = \bar{\Phi}_i \Phi^i, \qquad W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k. \tag{2.188}$$

In this case the scalar potential reads

$$V(\phi^i, \bar{\phi}_i) = \sum_{i=1}^{n} |\frac{\partial W}{\partial \phi^i}|^2 = \bar{F}_i F^i, \tag{2.189}$$

where

$$\bar{F}_i = \frac{\partial W}{\partial \phi^i}, \quad F^i = \frac{\partial \bar{W}}{\partial \bar{\phi}_i}. \tag{2.190}$$

We also may generalize our formalism by dealing with *effective* supersymmetry field theories. We then allow generalizations of Kähler potential and superpotential and these models are called supersymmetric *σ-models*. They reveal deep relation between supersymmetry and geometry because as we mentioned above $K(\Phi, \bar{\Phi})$ defines a metric of a complex manifold parametrized by $\Phi^i, \bar{\Phi}_{\bar{j}}$.

2.5.2 $\mathcal{N} = 1$ SuperYang-Mills in $D = 4$

Abelian case

Let us now consider a supersymmetric invariant action that describe the dynamics of a real (vector) superfield, as a generalisation of the known Yang-Mills theory.

We begin from the abelian case. The first step is to define a superfield which contains the *field strength* when expanded in θ coordinates to this aim we write

$$W_\alpha = -\frac{1}{4} \bar{D}\bar{D} D_\alpha V, \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD\bar{D}_{\dot{\alpha}} V, \tag{2.191}$$

W_α is a chiral superfield due to the property $\bar{D}^3 = 0$ and it is invariant under the gauge transformation (2.147), indeed

$$W_\alpha \;\rightarrow\; W'_\alpha = W_\alpha - \frac{1}{4} \bar{D}_{\dot{\beta}} \bar{D}^{\dot{\beta}} D_\alpha (\Phi + \bar{\Phi}) = W_\alpha + \frac{1}{4} \bar{D}^{\dot{\beta}} \bar{D}_{\dot{\beta}} D_\alpha \Phi, \tag{2.192}$$

$$= W_\alpha + \frac{1}{4} \bar{D}^{\dot{\beta}} \{\bar{D}_{\dot{\beta}}, D_\alpha\} \Phi = W_\alpha + \frac{i}{2} \sigma^m_{\alpha\dot{\beta}} \partial_m \bar{D}^{\dot{\beta}} \Phi = W_\alpha, \tag{2.193}$$

where we used $\bar{D}_{\dot{\beta}} \bar{D}^{\dot{\beta}} = -\bar{D}^{\dot{\beta}} \bar{D}_{\dot{\beta}}$ in (2.192) and the condition $\bar{D}_{\dot{\beta}} \Phi = 0$ in (2.193).

To expand in component fields (2.191) it is convenient to work in the Wess-Zumino gauge (2.156), thus the real superfield V in y-coordinates (2.129) is

$$V_{WZ}(y,\theta,\bar\theta) = \theta\sigma^m\bar\theta v_m(y) + i\theta\theta\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\theta\lambda(y) + \frac{1}{2}\theta\theta\bar\theta\bar\theta(D(y) - i\partial_m v^m(y)), \quad (2.194)$$

which when expanded in $(x,\theta,\bar\theta)$ components reduces to the expression (2.157). The computation of covariant derivatives on the real superfield is straightforward and, helped by the following identity

$$\sigma^n\bar\sigma^m - \eta^{mn} = 2\sigma^{mn}, \quad (2.195)$$

we obtained

$$D_\alpha V_{WZ} = \sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}v_m + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta\bar\theta\lambda_\alpha + \theta_\alpha\bar\theta\bar\theta D + 2i(\sigma^{mn})_\alpha^{\ \beta}\theta_\beta\bar\theta\bar\theta\partial_m v_n + \theta\theta\bar\theta\bar\theta\sigma^m_{\alpha\dot\beta}\partial_m\bar\lambda^{\dot\beta}. \quad (2.196)$$

By noting that

$$\bar D\bar D\theta\theta = -4, \quad (2.197)$$

we finally obtain the expansion of the W_α superfield

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{mn}\theta)_\alpha F_{mn} + \theta\theta(\sigma^m\partial_m\bar\lambda)_\alpha. \quad (2.198)$$

It is invariant under gauge transformations and also contains the Maxwell tensor $F_{mn} = \partial_m v_n - \partial_n v_m$ among its components. It is also called *gaugino superfield* because it is an instance of a chiral superfield which starts with a Weyl fermion, the superpartner of the $v_m(x)$ gauge field. We focus now in the construction of a Lagrangian that is gauge invariant. A first attempt may be the term

$$\int d^2\theta W^\alpha W_\alpha. \quad (2.199)$$

Substituting (2.198) into (2.199) and retaining only θ^2 terms we have

$$\int d^2\theta W^\alpha W_\alpha = -2i\lambda\sigma^m\partial_m\bar\lambda + D^2 - \frac{1}{2}(\sigma^{mn})^{\alpha\beta}(\sigma^{rs})_{\alpha\beta}F_{mn}F_{rs}, \quad (2.200)$$

where $(\sigma^{mn})_\alpha^{\ \beta} = tr\sigma^{mn} = 0$.

To obtain the familiar Maxwell term in the supersymmetric Yang-Mills action, let us use the following identity

$$(\sigma^{mn})^{\alpha\beta}(\sigma^{rs})_{\alpha\beta} = \frac{1}{2}(g^{mr}g^{ns} - g^{ms}g^{nr}) - \frac{i}{2}\epsilon^{mnrs}, \quad (2.201)$$

(with $\epsilon^{0123} = +1$) then we have

$$\int d^2\theta W^\alpha W_\alpha = -\frac{1}{2}F_{mn}F^{mn} - 2i\lambda\sigma^m\partial_m\bar\lambda + D^2 + \frac{i}{4}\epsilon^{mnrs}F_{mn}F_{rs}. \quad (2.202)$$

and by adding its hermitian conjugate the whole Lagrangian become real, as follows

$$\mathcal{L}_{gauge} = \int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta}\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}} = -F_{mn}F^{mn} - 4i\lambda\sigma^m\partial_m\bar{\lambda} + 2D^2. \quad (2.203)$$

We used the identity $\chi\sigma^m\bar{\psi} = -\bar{\psi}\bar{\sigma}^m\chi$. Notice that the last term in (2.202) is a total derivative so (2.202) is actually real modulo a total derivative. This is the supersymmetric version of the Abelian gauge Lagrangian.

The Lagrangian (2.203) can be re-written as an integral in full superspace

$$\int d^2\theta W^\alpha W_\alpha = \int d^2\theta d^2\bar{\theta} D^\alpha V W_\alpha. \quad (2.204)$$

Non Abelian case

In the non-abelian case we deal with a non abelian gauge group G and therefore we must promote the vector superfield to

$$V = V_a T^a \quad a = 1, ..., dimG, \quad (2.205)$$

where T^a are hermitian generators and V_a are vector superfields. Moreover, we must define the finite version of the gauge transformation (2.147) as

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda}, \quad (2.206)$$

which reduces to (2.147) at first order in the gauge parameter Λ, upon the identification of $\Phi = -i\Lambda$. Therefore the gaugino superfield can be generalized as follows

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}(e^{-V}D_\alpha e^V), \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4}DD(e^V\bar{D}_{\dot{\alpha}}e^{-V}), \quad (2.207)$$

which can be reduced to (2.191) to first order in V. We can show that under the finite gauge transformations (2.206), W_α transforms convariantly

$$\begin{aligned} W_\alpha \to W'_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\left[e^{i\Lambda}e^{-V}e^{-i\bar{\Lambda}}D_\alpha\left(e^{i\bar{\Lambda}}e^V e^{-i\Lambda}\right)\right], & (2.208)\\ &= -\frac{1}{4}\bar{D}\bar{D}[e^{i\Lambda}e^{-V}((D_\alpha e^V)e^{-i\Lambda} + e^V D_\alpha e^{-i\Lambda})], & (2.209)\\ &= -\frac{1}{4}e^{i\Lambda}\bar{D}\bar{D}(e^{-V}D_\alpha e^V)e^{-i\Lambda} = e^{i\Lambda}W_\alpha e^{-i\Lambda}. & (2.210)\end{aligned}$$

In (2.208) we used the identities $\bar{D}_{\dot{\alpha}}e^{-i\Lambda} = 0, D_\alpha e^{i\bar{\Lambda}} = 0$ and also $\bar{D}\bar{D}D_\alpha e^{-i\Lambda} = 0$ because Λ is a chiral superfield. We have already noted that in the Wess-Zumino gauge each element of V_{WZ} is first order in θ coordinate and hence we have $e^V = 1 + V + \frac{1}{2}V^2$ and

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\left[\left(1 - V + \frac{1}{2}V^2\right) D_\alpha \left(1 + V + \frac{1}{2}V^2\right)\right], \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{8}\bar{D}\bar{D}D_\alpha V^2 + \frac{1}{4}\bar{D}\bar{D}V D_\alpha V, \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{8}\bar{D}\bar{D}V D_\alpha V - \frac{1}{8}\bar{D}\bar{D}D_\alpha V V + \frac{1}{4}\bar{D}\bar{D}V D_\alpha V, \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V]. \end{aligned}$$

The second term is new, induced by the non-abelian case, which leads to

$$\frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] = \frac{1}{2}(\sigma^{mn}\theta)_\alpha[v_m, v_n] - \frac{i}{2}\theta\theta\sigma^m_{\alpha\dot{\beta}}\left[v_m, \bar{\lambda}^{\dot{\beta}}\right]. \tag{2.211}$$

Finally, the non-abelian case of the generalized supersymmetry field strength may be written as

$$W_\alpha = -i\lambda(y) + \theta_\alpha D(y) + i(\sigma^{mn}\theta)_\alpha F_{mn} + \theta\theta(\sigma^m D_m\bar{\lambda}(y))_\alpha, \tag{2.212}$$

where now also the ordinary field strength and covariant derivative acquire non-Abelian contributions

$$F_{mn} = \partial_m v_n - \partial_n v_m - \frac{i}{2}[v_m, v_n], \quad D_m = \partial_m - \frac{i}{2}[v_m,]. \tag{2.213}$$

It is convenient to explicite the coupling constant g like

$$V \to 2gV \Leftrightarrow v_m \to 2gv_m, \quad \lambda \to 2g\lambda, \quad D \to 2gD, \tag{2.214}$$

which implies

$$F_{mn} = \partial_m v_n - \partial_n v_m - ig[v_m, v_n], \quad D_m = \partial_m - ig[v_m,]. \tag{2.215}$$

The final result of the non abelian case of the super-YangMills Lagrangian reads

$$\begin{aligned} \mathcal{L}_{SYM} &= \frac{1}{32\pi}Im\left(\tau\int d^2\theta Tr W^\alpha W_\alpha\right), \\ &= Tr\left[-\frac{1}{4}F_{mn}F^{mn} - i\lambda\sigma^m D_m\bar{\lambda} + \frac{1}{2}D^2\right] + \frac{\theta_{YM}}{32\pi^2}g^2 Tr F_{mn}\tilde{F}^{mn}, \end{aligned}$$

where we introduced the complexified gauge coupling

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}, \tag{2.216}$$

and the dual field strength

$$\tilde{F}^{mn} = \frac{1}{2}\epsilon^{mnrs}F_{rs}, \tag{2.217}$$

while the generators are normalized as $Tr[T^\alpha T^\beta] = \delta^{\alpha\beta}$. We now turn to the consideration of supergravity.

Chapter 3

$\mathcal{N} = 1$ Supergravity in $D = 4$ dimensions

3.1 Introduction

Supergravity theories are the local supersymmetric theories whose parameter depends on space-time coordinates. A local supersymmetric transformation has the following schematic form

$$\delta_\epsilon B = \epsilon^\alpha(x) F_\alpha, \tag{3.1}$$
$$\delta_\epsilon F_\alpha = \gamma^m{}_\alpha{}^\beta \bar{\epsilon}_\beta(x)\partial_m B, \tag{3.2}$$
$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]B = 2\epsilon_2\gamma^m\bar{\epsilon}_1(x)\partial_m B, \tag{3.3}$$
$$[\delta_{\epsilon 1}, \delta_{\epsilon 2}]F = 2\epsilon_2\gamma^m\bar{\epsilon}_1(x)\partial_m F. \tag{3.4}$$

Equations (3.3)-(3.4) tell us that the infinitesimal parameter $(2\epsilon_2\gamma^m\bar{\epsilon}_1)(x)$ is associated to operators which act as local diffeomorphism transformations. Hence any local supersymmetric theory should be diffeomorphism invariant and thus include gravity. The simplest gravity theory is the Einstein's theory of general relativity. Our aim in this chapter will be the construction of a local supersymmetry generalisation of the ordinary general relativity.

Supergravities are basically built of a *graviton multiplet* and, in addition may also contain other matter multiplets of the underlying rigid supersymmetry algebra. The graviton multiplet consists of the vielbein $e_m{}^a(x)$ describing the graviton and a number N of vector-spinor fields $\chi^i_m{}^\alpha(x)$ with $i = 1, ..., \mathcal{N}$, called Rarita-Schwinger fields or gravitinos, and their lower spin supersymmetric partners. In the basic case of $\mathcal{N} = 1$ supergravity, in $D = 4$ space-time dimensions, the graviton multiplet (2.82) consists entirely of the graviton and only one Majorana spinor gravitino. In all other cases, both $\mathcal{N} \geq 2$ in $D = 4$ dimensions and $\mathcal{N} \geq 1$ for $D \geq 5$, additional fields are required in the graviton multiplet.

Supergravity theories exist for space-time dimensions $D \leq 11$. For $D = 4$, theories exist upon $\mathcal{N} = 8$. Beyond these limits conventional local supersymmetry fails to underlie a consistent interacting theory.

In this chapter we will consider the *minimal* $\mathcal{N} = 1$ *supergravity* in four space-time dimensions, minimal in the sense that it is the smallest possible supersymmetric extension of Einstein's theory of general relativity. We will discuss the form of the action and supersymmetry transformation rules.

We are interested in considering *supergravity* theories for several reasons. One is that supergravities with extended $\mathcal{N} > 1$ supersymmetry may unify gauge interactions with gravity.

Another reason is that many of the ultraviolet divergences expected in a field theory containing gravity are known to cancel in the maximal $\mathcal{N} = 8$ theory, but it still has to be proved that it is ultraviolet finite to all orders in perturbation theory.

Finally, 10-dimensional supergravity theories, such as the Type IIA and Type IIB theories, are related to the superstring theories. Supergravity appears as the low-energy limit of superstring theory. This means that the dynamics of the massless modes of the superstring are described by supergravity.

3.2 General Relativity tools

Some geometrical objects, useful in building general relativity, are required to be introduced as a basic blocks of the formalism we will use. The **metric** of a curved space-time is denoted by the symmetric tensor g_{mn} that satisfies the following *covariant constancy condition*

$$\nabla_l g_{mn} = \partial_l g_{mn} - \Gamma^r_{ln} g_{rm} - \Gamma^r_{lm} g_{rn} = 0. \tag{3.5}$$

Given a freely falling coordinate system ξ^α and any other coordinate system x^m, we can write down, according to the Principle of Equivalence, the equation of motion of a particle moving freely under the influence of purely gravitational force

$$0 = \frac{d^2 x^l}{d\tau^2} + \Gamma^l{}_{mn} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} \tag{3.6}$$

where $\Gamma^r{}_{lm}$ is called the *affine connection*, or Christoffel symbol, defined as

$$\Gamma_{mn}{}^l = \frac{\partial x^l}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^m \partial x^n}. \tag{3.7}$$

where τ is the proper time of the special relativity. ∇_m is the **covariant derivative** defined by

$$\nabla_m V_n = \partial_m V_n - \Gamma^l_{mn} V_l. \tag{3.8}$$

Vectors like V^m transform as

$$V'^m = \frac{\partial x'^m}{\partial x^n} V^n. \tag{3.9}$$

It can be shown in [13] that (3.8) transforms as a tensor under diffeomorphisms $x \to x'(x)$, namely

$$\nabla'_m V'_n = \frac{\partial x^s}{\partial x'^m} \frac{\partial x^r}{\partial x'^n} \nabla_s V_r. \tag{3.10}$$

The *Christoffel symbol*, alternatively from (3.7), can also be related to the metric and its derivative, by permuting indices (lmn) in (3.5) and subtracting the resulting equations

$$\Gamma^l_{mn} = \frac{1}{2} g^{lr}(\partial_m g_{nr} + \partial_n g_{mr} - \partial_r g_{mn}) + K^l_{mn}. \tag{3.11}$$

K^l_{mn} is the so-called *contorsion*, defined in terms of the torsion field $T^l{}_{mn}$ which in general is not zero

$$K^l_{mn} = \frac{1}{2}(T_n{}^l{}_m + T_m{}^l{}_n + T^l{}_{mn}). \tag{3.12}$$

General relativity however is based on geometry with zero torsion. In supergravity theory instead the torsion is nontrivial, as we will see below.

Another object that defines the geometry of a curved space is the *Riemann tensor*, constructed in function of the Christoffel symbols

$$R_{mn}{}^r{}_s = 2\partial_{[m}\Gamma^r_{n]s} + 2\Gamma^r_{[m|l}\Gamma^l_{|n]}, \tag{3.13}$$

where in the second term $A_{[m|n}B_{|s]} = \frac{1}{2}(A_{mn}B_s - A_{sn}B_m)$. By construction, it can be proved that the Riemann tensor transforms covariantly as a tensor field.

We can define from it the *Ricci tensor*, by contracting two indices as follows

$$R_{mn} = R_{rm}{}^r{}_n. \tag{3.14}$$

We can also define the *Ricci scalar* like

$$R = g^{mn}R_{mn}. \tag{3.15}$$

Finally the *Einstein-Hilbert action*, given in [13] has the following form

$$I_G \equiv -\frac{1}{16\pi G}\int \sqrt{g(x)}R(x)d^4x, \tag{3.16}$$

where $g(x)$ denotes the determinant of the metric ($g = det g_{mn}$).

Varying the action with respect to g_{mn}

$$\delta I_G = \frac{1}{16\pi G}\int \sqrt{g}[R^{mn} - \frac{1}{2}g^{mn}R]\delta g_{mn}d^4x, \tag{3.17}$$

we obtain the *vacuum Einstein equations*

$$R_{mn} - \frac{1}{2}g_{mn}R = 0. \tag{3.18}$$

3.3 Vielbein formalism

As we told in the section 3.1, the supergravity multiplet is made also of the gravitino $\frac{3}{2}$ spinor-field.

In order to deal with spinors (like the gravitino) in a curved space, it is necessary to introduce a more appropriate formalism to formulate supergravity models because the spinors transform under a representation of the Lorentz group but they behaves as scalars under diffeomorphisms. We can rewrite all terms in the *vielbein formalism* which is based on the idea to search for a locally inertial basis, in which one can apply the usual Lorentz transformations on spinors.

The **vielbein** is a matrix ${e_m}^a$ which relates different local reference frames. A natural curved frame in each point of the curved space is associated with the differential dx^m. Using ${e_m}^a$ we get a different frame

$$e^a(x) = {e_m}^a(x)dx^m \tag{3.19}$$

which is orthogonal in the sense that

$${e_m}^a {e_n}^b g^{mn} = \eta^{ab}. \tag{3.20}$$

where $\eta_{ab} = diag(1, -1, -1, -1)$.

When we change the local coordinates from x to $x'(x)$, the vielbein transforms according to

$${e'_m}^a(x') = \frac{\partial x^n}{\partial x'^m}{e_n}^a(x), \tag{3.21}$$

while eq.(3.20) remains invariant under the local Lorentz transformations

$${e'_m}^a(x) = {e_m}^b(x){\Lambda_b}^a(x). \tag{3.22}$$

The fundamental relation between the curved spacetime metric and the vielbein

$$g_{mn}(x) = {e_m}^a(x){e_n}^b(x)\eta_{ab}, \tag{3.23}$$

states that the vielbein transforms lower Lorentz indices a, b to lower indices in the curved coordinates m, n.

The vielbein acts on Lorentz vectors in the following way

$$X_m = {e_m}^a X_a, \tag{3.24}$$

while the *inverse vielbein* acts on Einstein vectors like

$$X_a = {e_a}^m X_m, \tag{3.25}$$

and on the contravariant vectors

$$X^m = X^a {e_a}^m, \tag{3.26}$$

$$X^a = X^m {e_m}^a. \tag{3.27}$$

Therefore, using the constant γ-matrices we can also define the matrix

$$\gamma_m(x) = {e_m}^a(x)\gamma_a, \tag{3.28}$$

which depend on x^m and satisfy the following anticommutations relations

$$\{\gamma_m(x), \gamma_n(x)\} = 2g_{mn}(x). \tag{3.29}$$

Spinors $\chi_\alpha(x)$ transform as scalars under the general space-time coordinate transformations, and under a spinorial representation $\mathcal{R}$ of the local Lorentz group

$$\chi'_\alpha(x) = \mathcal{R}(\Lambda(x))_\alpha{}^\beta \chi_\beta(x). \tag{3.30}$$

In order to couple spinors to gravity we introduce the *Lorentz covariant derivative*

$$D_m \equiv \partial_m + \frac{1}{2}\omega_m{}^{ab} M_{ab}, \tag{3.31}$$

where $\omega_m{}^{ab}$ are a set of objects called the *spin connection* ω which play the role of gauge fields of the local Lorentz symmetry. They are antisymmetric in Lorentz indices a, b, because M_{ab} are the $SO(1,3)$ antisymmetric generators already introduced in (2.19). M_{ab} act on vectors and spinors as follows

$$M_{ab} X^c = 2\delta^c_{[a} X_{b]}, \tag{3.32}$$

and

$$M_{ab}\chi = \frac{1}{2}\gamma_{ab}\chi, \tag{3.33}$$

where $\gamma_{ab} = \gamma_{[a}\gamma_{b]} = \frac{1}{2}[\gamma_a, \gamma_b]$.

The form of the the spin connection is fixed (modulo Lorentz transformations) by imposing the following covariant constancy condition

$$D_m e_n{}^a - \Gamma^l_{mn} e_l{}^a = \partial_m e_n{}^a + \omega_m{}^a{}_b e_n{}^b - \Gamma^l_{mn} e_l{}^a = 0, \tag{3.34}$$

which upon the antisymmetrization and using (3.11) takes the form

$$D_{[m} e_{n]}{}^a = \frac{1}{2} T^a{}_{mn}, \tag{3.35}$$

In the case of vanishing torsion, (3.35) reduce to

$$D_{[m} e_{n]}{}^a = (\partial_{[m} + \frac{1}{2}\omega_{[m}{}^{ab} M_{ab}) e_{n]}{}^a = 0, \tag{3.36}$$

which leads to

$$\omega_m{}^{ab}[e] = \frac{1}{2} e_{cm}(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}), \tag{3.37}$$

where $\Omega_{abc} = e_a{}^m e_b{}^n (\partial_m e_{nc} - \partial_n e_{mc})$ are the so called *objects of anholonomicity*. In the case of non-trivial torsion the spin connection must be rewrited as

$$\omega_m{}^{ab} = \omega_m{}^{ab}[e] + K^a{}_m{}^b, \tag{3.38}$$

where $K^a{}_m{}^b = e_l{}^a e^{nb} K^l{}_{mn}$, related to (3.12).

We can rewrite the *curvature tensor* in terms of the spin connection in the following way

$$R_{mn}{}^{ab}[\omega] \equiv 2\partial_{[m}\omega_{n]}{}^{ab} + 2\omega_{[m}{}^{ac}\omega_{n]c}{}^{b}, \tag{3.39}$$

where

$$R_{mn}{}^{ab} = R_{mn}{}^{r}{}_{s}e_{r}{}^{a}e^{bs}, \tag{3.40}$$

relates the curvature (3.39) with (3.13). In the vielbein formalism the Einstein-Hilbert action has the following form

$$\mathcal{L}_{EH}[e_m{}^a] = -\frac{1}{4}\sqrt{|g|}R = -\frac{1}{4}|e|e_a{}^m e_b{}^n R_{mn}{}^{ab}, \tag{3.41}$$

where $|e|$ denotes the determinant of the tetrad ($e = det(e_m{}^a) = \sqrt{|g|}$).

3.4 The Palatini action

The action (3.41) leads to the second order equations of motion for $e_m{}^a$. There is however a possibility of modifying the action in such a way that it produces first order equations of motion. This action, called Palatini action, will be useful for finding complete form of the locally supersymmetric action.

In the Palatini action the spin connection $\omega_m{}^{ab}$ is considered to be a priori independent of the vielbein and the corresponding Lagrangian has the following form

$$\mathcal{L}_P[e,\omega] = -\frac{1}{4}|e|e_a{}^m e_b{}^n R_{mn}{}^{ab}[\omega]. \tag{3.42}$$

The field equations are derived from this Lagrangian by varying with respect to both the connection and the vielbein fields:

$$\delta\mathcal{L}_P = -\frac{1}{2}|e|\left(R_m{}^a(\omega) - \frac{1}{2}e_m{}^a R(\omega)\right)\delta e_a{}^m - \frac{3}{2}|e|(D_m e_n{}^a)e^m_{[a}e_b{}^n.e^r_{c]}\delta\omega_r{}^{bc}. \tag{3.43}$$

The second term in (3.43) leads to the equations of motion (3.36) which are solved by (3.37). Upon inserting this solution $\omega = \omega[e]$ into the first equation, we recover the ordinary Einstein equations of motion. Therefore the Palatini formulation is equivalent to the standard second order formulation of general relativity, at least at the classical level.

The relation which connects the two Lagrangians is

$$\mathcal{L}_{EH}[e] = \mathcal{L}_P[e,\omega]|_{\omega=\omega[e]}. \tag{3.44}$$

3.5 The Minimal Supergravity action

To construct the minimal supergravity action, in addition to the gravitation action we must introduce the kinetic term associated to the gravitino field. This term was first constructed by Rarita-Schwinger [14] in 1940's in the free non gravitational case. In the gravity background, the Rarita-Schwinger Lagrangian has the following form

$$\mathcal{L}_{RS} = \frac{1}{2}\epsilon^{mnrs}\bar{\chi}_m\gamma_n\gamma_5 D_r\chi_s. \tag{3.45}$$

where D_r is the covariant derivative (3.31).

Adding this term to the Palatini Lagrangian (3.42), we have

$$\mathcal{L}_0[e,\chi,\omega] = -\frac{1}{4}|e|e_\alpha{}^m e_b{}^n R_{mn}{}^{ab}[\omega] + \frac{1}{2}\epsilon^{mnrs}\bar{\chi}_m\gamma_n\gamma_5 D_r\chi_s. \tag{3.46}$$

which is almost the final form of the full Lagrangian.

Now we want to find those local supersymmetry transformations which leave invariant the Lagrangian. We first must find the supersymmetric transformations on the vielbein and the gravitino and then add to the action extra terms which ensure its supersymmetry invariance. A natural ansatz we can take for the supersymmetric transformation is the following

$$\begin{aligned} \delta_\epsilon e_m{}^a &= -i\bar{\epsilon}\gamma^a\chi_m, \\ \delta_\epsilon \chi^\alpha{}_m &= D_m\epsilon^\alpha. \end{aligned} \tag{3.47}$$

Heuristically, they have a desired form because the bosonic vielbein transforms into its presumed superpartner and because if χ_m is supposed to play the role of a "gauge fields of local supersymmetry" it transforms as $D_m\epsilon$. This transformation however is not yet complete.

As already said above, we have to add to (3.46) extra terms with higher powers in the fermionic fields $\mathcal{L}_{\chi^4}$, without changing its general form but only properly adapting the spin connection.

Consider a general variation of (3.46)

$$\delta\mathcal{L}_0 = \frac{\delta\mathcal{L}_0}{\delta e_m{}^a}\delta e_m{}^a + \frac{\delta\mathcal{L}_0}{\delta\chi_m{}^\alpha}\delta\chi_m{}^\alpha + \frac{\delta\mathcal{L}_0}{\delta\omega_n{}^{bc}}\delta\omega_n{}^{bc} \tag{3.48}$$

where $\delta\mathcal{L}_0/\delta\omega$ leads to

$$D_{[m}e_{n]}{}^a = -\frac{i}{2}\bar{\chi}_m\gamma^a\chi_n, \tag{3.49}$$

The term in the right hand side in (3.49) is due to the contribution of the Rarita-Schwinger term. This equation can be solved by the *modified spin connection*

$$\hat{\omega}_m{}^{ab} = \hat{\omega}_m{}^{ab}[e,\psi] = \omega_m{}^{ab}[e] + K^a{}_m{}^b, \tag{3.50}$$

with $K^a{}_m{}^b = -i(\bar{\chi}^{[a}\gamma^{b]}\chi_m + \frac{1}{2}\bar{\chi}^a\gamma_m\chi^b)$.

As a result, substituting (3.50) into (3.46) we get the Lagrangian

$$\mathcal{L}_0[e,\chi,\hat{\omega}[e]] = -\frac{1}{4}|e|e_\alpha{}^m e_b{}^n R_{mn}{}^{ab}[\hat{\omega}[e]] + \frac{1}{2}\epsilon^{mnrs}\bar{\chi}_m\gamma_n\gamma_5\hat{D}_r\chi_s, \tag{3.51}$$

which is locally supersymmetric invariant under the locally supersymmetric transformations of the form

$$\begin{aligned} \delta_\epsilon e_m{}^a &= -i\bar{\epsilon}\gamma^a\chi_m, \\ \delta_\epsilon \chi_m &= \hat{D}_m\epsilon, \end{aligned} \tag{3.52}$$

where we considered $\hat{D}_m = D(\hat{\omega})$.

This can be checked by direct computations [6]. We have thus reviewed the form of the component action of $\mathcal{N}=1$ supergravity in $D=4$.

Our aim is to study the description of supergravity and its couplings to matter fields with the use of superspace methods and the superfield formalism. To simplify this study, we will pass in the next chapters from four to two dimensions, in which the superfield construction of the theory is much simpler but the main conceptual points remain the same.

Chapter 4

Rigid supersymmetry in $D = 2$ dimensions

The road map followed in the previous two chapters was based on the criterion of generalizing more and more Lagrangians of supersymmetry theories in $D = 4$. We found in the second chapter the general expression for the Wess-Zumino model and superYang-Mills rigid theories without gravity and then, in the third chapter we discussed the Lagrangian of $\mathcal{N} = 1$, $D = 4$ pure supergravity.

But what happens if we couple pure supergravity to matter and/or YM Lagrangians? In $D = 4$ the resultant theory is a bit complicated so we have chosen a simpler case where to do computations. We will go to $D = 2$ space-time dimensions and use the superfield formalism and in chapters $5, 6$ we will obtain superfield Lagrangians describing the coupling of a matter scalar field to $\mathcal{N} = (1, 1)$, $D = 2$ supergravity.

A general motivation in exploring supersymmetric field theories in two space-time dimensions is related also to superstrings because a string is a one dimensional object which sweeps a two-dimensional surface (called worldsheet) and its action is invariant under local $D = 2$ supersymmetry. In this chapter we will construct rigid supersymmetries in $D = 2$ flat superspace by using some results achieved in the previous chapters. As we will see, the main differences between $D = 4$ and $D = 2$ are related to the properties of spinors.

4.1 Two-dimensional fermions formalism

In $D = 2$ the flat space-time metric η^{mn} is defined by the 2×2 matrix (with $m, n = 0, 1$)

$$\eta^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

We introduce also the 2×2 matrices, denoted γ^m, which satisfy to the anticommutation property

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}, \tag{4.2}$$

where γ^m are chosen to be

$$\gamma^0{}_\alpha{}^\beta = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1{}_\alpha{}^\beta = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.3}$$

where σ_i are the Pauli matrices. A Dirac spinor may be written as a two component complex row vector

$$\psi^\alpha = \begin{pmatrix} \psi^1, \psi^2 \end{pmatrix}. \tag{4.4}$$

In $D = 2$ the Lorentz group corresponds to an abelian $SO(1,1)$ group and its algebra is composed by only the single generator M^{01} and we can represent it in the *spinorial representation* as

$$M^{01} = -\frac{1}{4}[\gamma^0, \gamma^1] = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.5}$$

This form of M^{01} implies that each spinorial component of the Dirac spinor transforms independently under Lorentz transformations, as follows

$$\psi'_1 = e^{-\frac{\theta}{2}}\psi_1, \tag{4.6}$$
$$\psi'_2 = e^{+\frac{\theta}{2}}\psi_2. \tag{4.7}$$

We can define operators that project Dirac spinors into independent spinors called *Weyl spinors*. The projectors can be expressed by

$$P_R = \frac{1}{2}(1 + \gamma_3), \tag{4.8}$$
$$P_L = \frac{1}{2}(1 - \gamma_3), \tag{4.9}$$

where γ_3 is defined as

$$\gamma_3 \equiv -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.10}$$

with the following properties

$$(\gamma^3)^2 = \mathbb{I}, \tag{4.11}$$
$$\{\gamma^a, \gamma^3\} = 0, \tag{4.12}$$
$$\gamma^a\gamma^3 = \gamma^b\epsilon_b{}^a. \tag{4.13}$$

If we apply these operators to the Dirac spinors they lead to the right-handed and left-handed Weyl spinors respectively

$$\psi_1 \equiv \psi_R = P_R\psi, \tag{4.14}$$
$$\psi_2 \equiv \psi_L = P_L\psi. \tag{4.15}$$

We can introduce also *Majorana spinors* ψ_M which are defined by the condition

$$\bar{\psi}_{M\alpha} = C^{-1}_{\alpha\beta}\psi^\beta, \tag{4.16}$$

where the Dirac conjugate $\bar{\psi}$ is defined as

$$\bar{\psi}_\alpha = \psi^{\beta *} g_{\beta\alpha} = \begin{pmatrix} \psi^{2*} \\ \psi^{1*} \end{pmatrix}, \qquad g_{\beta\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.17}$$

and C is the charge conjugation matrix which satisfy the relation

$$C^{-1}_{\alpha\beta} = C^{\dagger\gamma\delta} g_{\delta\alpha} g_{\gamma\beta}. \tag{4.18}$$

Requiring

$$\psi_c \bar{\psi}_c = \psi\bar{\psi}, \tag{4.19}$$
$$\psi_c \gamma^a \bar{\psi}_c = -\psi\gamma^a\bar{\psi}, \tag{4.20}$$

then we can define the charge conjugation matrix up to an arbitrary number of unit modulus which we fix to unity,

$$C^{\alpha\beta} = \epsilon^{\alpha\beta}, \qquad C^{-1}_{\alpha\beta} = -\epsilon_{\alpha\beta} \tag{4.21}$$

where

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.22}$$

Therefore Majorana spinors satisfy the pseudo-reality condition

$$\psi^{*\alpha} = \psi^\beta {\gamma^3}_\beta{}^\alpha, \tag{4.23}$$

It has no definite parity because the definition of a Majorana spinor implies the γ^3 matrix. The first component is real while the second is purely imaginary

$$\psi^{1*} = \psi^1, \qquad \psi^{2*} = -\psi^2. \tag{4.24}$$

The bilinear combinations between two different Majorana spinors are

$$\psi\bar{\chi} = \chi\bar{\psi} = \psi^\alpha \chi^\beta \epsilon_{\beta\alpha} = -\psi_R\chi_L + \psi_L\chi_R, \qquad \textit{real} \tag{4.25}$$
$$\psi\gamma^3\bar{\chi} = -\chi\gamma^3\bar{\psi} = -\psi_R\chi_L - \psi_L\chi_R, \qquad \textit{real} \tag{4.26}$$
$$\psi\gamma^0\bar{\chi} = -\chi\gamma^0\bar{\psi} = \psi_R\chi_R - \psi_L\chi_L, \qquad \textit{imaginary} \tag{4.27}$$
$$\psi\gamma^1\bar{\chi} = -\chi\gamma^1\bar{\psi} = \psi_R\chi_R + \psi_L\chi_L, \qquad \textit{imaginary} \tag{4.28}$$

whereas for a single Majorana spinor there is only one non trivial bilinear

$$\theta\bar{\theta} = 2\theta_L\theta_R. \tag{4.29}$$
$$\theta\gamma^a\bar{\theta} = 0, \tag{4.30}$$
$$\theta\gamma^3\bar{\theta} = 0. \tag{4.31}$$

In two dimensions, the Weyl and Majorana conditions can be imposed independently and simultaneously, because they are not equivalent representations. We note that Dirac spinors have two complex components, Weyl spinors have one complex component only, while Majorana spinors have two real components ψ_1 and ψ_2, and Majorana-Weyl spinors have a single real component. The possibility of imposing simultaneously Weyl and Majorana conditions exists in fact in space-time dimensions 2(mod8).

The massless Dirac equation for spinors and the corresponding Lagrangian in $D = 2$ are better formulated in the *light cone coordinates* because as we will show, they allow to obtain a simple form of the Dirac field equation and its solutions.

The light-cone coordinates are defined as

$$x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^1), \quad x^- \equiv \frac{1}{\sqrt{2}}(x^0 - x^1), \tag{4.32}$$

which lead to

$$x^m y_m = x^0 y^0 - x^1 y^1 = x^+ y^- + x^- y^+ = \begin{pmatrix} x^+ & x^- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y^+ \\ y^- \end{pmatrix}, \tag{4.33}$$

where the flat metric is then

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.34}$$

Indices are lowered or raised with this metric so that

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = \begin{pmatrix} x_- \\ x_+ \end{pmatrix}, \tag{4.35}$$

and hence we can write

$$x_+ = \frac{1}{\sqrt{2}}(x_0 + x_1) = x^-, \quad x_- = \frac{1}{\sqrt{2}}(x_0 - x_1) = x^+. \tag{4.36}$$

An infinitesimal Lorentz transformation is given by $\delta P^m = -\theta (M^{01})^m{}_n P^n$ where

$$(M^{01})^m{}_n = (\eta^{0m} \delta^1{}_n - \eta^{1m} \delta^0{}_n) \tag{4.37}$$

This leads to

$$\begin{aligned} \delta P^0 = -\theta P^1, \quad &\rightarrow \quad \delta P^+ = -\theta P^+ \rightarrow P^+ = P^+ e^{-\theta}, \\ \delta P^1 = -\theta P^0, \quad &\rightarrow \quad \delta P^- = +\theta P^- \rightarrow P^- = P^- e^{+\theta}. \end{aligned}$$

and the corresponding transformations of $x^\pm$. The light-cone derivatives are defined by

$$\partial_+ = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1), \quad \partial_- = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1), \tag{4.38}$$

which lead to

$$\partial^m \partial_m = 2\partial_+ \partial_-, \quad \partial_+ x^+ = \partial_- x^- = 1, \quad \partial_+ x^- = \partial_- x^+ = 0. \tag{4.39}$$

Therefore the Dirac equation reads

$$i(\gamma^+\partial_+ + \gamma^-\partial_-)\psi = 0, \tag{4.40}$$

which in matrix form becomes

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\partial_+ + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\partial_-\right)\psi = 0, \tag{4.41}$$

and by using

$$\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^1), \quad \gamma^- = \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^1), \tag{4.42}$$

it has the following solution

$$\left\{ \begin{array}{l} \partial_+\psi_2 = 0 \rightarrow \psi_2 = \psi_-(x^-) \\ \partial_-\psi_1 = 0 \rightarrow \psi_1 = \psi_+(x^+) \end{array} \right\}.$$

This means that ψ_1, ψ_2 are right and left moving fermions respectively, namely a function of x^+ represents a right-moving wave for which x^1 increases for increasing time x^0. A Weyl fermion then is either right-moving or left-moving.

As regarding the bosons we mention the Klein-Gordon equation of motion for massless bosons

$$\partial^m\partial_m\phi = 2\partial_+\partial_-\phi = 0, \tag{4.43}$$

whose general solution is

$$\phi = \phi_-(x^-) + \phi_+(x^+), \tag{4.44}$$

hence a free massless scalar field is a superposition of a right-mover $\phi_-(x^-)$ and a left-mover $\phi_+(x^+)$.

4.2 Rigid superPoincaré algebra in $D = 2$

N-extended superPoincaré algebra in $D = 2$ has the following relations

$$\begin{aligned}
[M^{01}, M^{01}] &= 0, && (4.45)\\
[P_r, M_{mn}] &= \eta_{rm}P_n - \eta_{rn}P_m, && (4.46)\\
[P^m, P^n] &= 0, && (4.47)\\
[M^{01}, {Q^i}_\alpha] &= \frac{1}{2}(\gamma^{01}Q^i)_\alpha = +\sigma_3 Q^i_\alpha, && (4.48)\\
[P^m, {Q^i}_\alpha] &= 0, && (4.49)\\
\{Q^i_\alpha, Q^j_\beta\} &= -2(\gamma^m C^{-1})_{\alpha\beta}P_m\delta^{ij}, && (4.50)\\
&= -2[P^0(\sigma_3)_{\alpha\beta} - P^1\delta_{\alpha\beta}]\delta^{ij}, && (4.51)
\end{aligned}$$

with $\alpha, \beta = 1, 2$ and $i = 1, ..., N$. Its relations are almost identical to those of the supersymmetry algebra in $D = 4$.

In light-cone coordinates we have

$$\{Q_1^i, Q_1^j\} = -2\sqrt{2}P^-\delta^{ij} = -2\sqrt{2}P_+\delta^{ij}, \tag{4.52}$$
$$\{Q_2^i, Q_2^j\} = +2\sqrt{2}P^+\delta^{ij} = 2\sqrt{2}P_-\delta^{ij}, \tag{4.53}$$
$$\{Q_1^i, Q_2^j\} = 0. \tag{4.54}$$

It is transparent the fact that Q_1 and Q_2 define separately closed algebra because they are related via their anticommutators to the right and to the left translation generators respectively and then their numbers need not to be the same. We will discern them by calling Q_1^i and Q_2^j right and left generators associated to right an left algebra respectively and hence the following expressions $Q_R^i \equiv Q_1^i, i = 1, ..., p$, and $Q_L^j \equiv Q_2^j, j = 1, ..., q$ mean that p and q indicate the numbers of right and left supersymmetries respectively, with the complete number being $\mathcal{N} = (p, q)$.

In this section we consider only the two simplest cases, directly relevant to superstring theories:$(1,0)$ supersymmetry and $(1,1)$ supersymmetry, which is analogous to the four-dimensional $\mathcal{N} = 1$ case.

4.2.1 $\mathcal{N} = (1,0)$ Rigid Supersymmetry in $D = 2$

The nomenclature $\mathcal{N} = (1,0)$ means that the superPoincaré algebra must be built by the introduction of only one extra fermionic generator. This generator must be a right-handed Majorana-Weyl spinor (or left-handed). From (4.52) we obtain

$$\{Q_R, Q_R\} = -2\sqrt{2}P^- = -2\sqrt{2}P_+, \tag{4.55}$$

and this implies

$$Q_R^2 = -\sqrt{2}P^- = -\sqrt{2}P_+. \tag{4.56}$$

In order to write down supersymmetric Lagrangians written in the superfield formalism, we have to introduce the superspace in $D = 2$ and $\mathcal{N} = (1,0)$. A point in this superspace has the coordinates

$$z^M = (x^m, \theta_L), \tag{4.57}$$

or in light-cone coordinates

$$(x^+, x^-, \theta_L), \tag{4.58}$$

where θ_L is a Majorana-Weyl coordinate, with chirality opposite to that of Q_R in order to preserve invariance under Lorentz transformations of the superfield $\phi(x^m, \theta_L)$ in the supersymmetry transformations

$$\delta\phi = i\epsilon_L Q_R \phi. \tag{4.59}$$

θ_L satisfies to

$$\theta_L^2 = 0, \tag{4.60}$$
$$\{\theta_L, anyspinor\} = 0. \tag{4.61}$$

The supersymmetric generator reads

$$Q_R = -i\left(\frac{\partial}{\partial\theta_L} - i\sqrt{2}\theta_L\partial_+\right). \tag{4.62}$$

while the corresponding covariant derivative

$$D_R = -\frac{\partial}{\partial\theta_L} - i\sqrt{2}\theta_L\partial^-, \tag{4.63}$$

which anticommutes with the supersymmetry generator

$$\{D_R, Q_R\} = 0, \tag{4.64}$$

In this framework we can introduce a Lorentz scalar superfield

$$\Phi(x, \theta_L) = \phi(x) + \theta_L\psi_R(x), \tag{4.65}$$

with a real scalar field $\phi(x)$ and a right-handed Majorana-Weyl spinor ψ_R. Furthermore auxiliary fields do not appear.

To construct a free Lagrangian for the superfield Φ, we combine derivatives $D_R, \partial^+, \partial^-$ of Φ. Since the Lagrangian must have dimension 2 while θ_L has dimension $-\frac{1}{2}$, the superfield must have dimension $\frac{3}{2}$. It must also contain two derivatives. In this case, the θ_L component is a Lorentz invariant Lagrangian.

Therefore we have

$$\begin{aligned}\mathcal{L} &= \int d\theta_L[i\partial^+\Phi D_R\Phi], \\ &= +i\psi_R\partial_+\psi_R + \sqrt{2}(\partial_+\phi)(\partial_-\phi),\end{aligned} \tag{4.66}$$

where the derivatives are

$$D_R\Phi = -\psi_R - i\sqrt{2}\theta_L\partial^-\phi, \tag{4.67}$$

$$\partial^+\Phi = \partial^+\phi + \theta_L\partial^+\psi_R. \tag{4.68}$$

We can finally find the rigid supersymmetry transformation under which the Lagrangian is invariant. By employing (4.59) we have

$$\delta\Phi = i\epsilon_L Q_R\Phi = \epsilon_L\psi_R - i\sqrt{2}\epsilon_L\theta_L\partial^-\phi,$$

which corresponds to the component transformations

$$\delta\phi = \epsilon_L\psi_R, \tag{4.69}$$

$$\delta\psi_R = i\sqrt{2}\epsilon_L\partial^-\phi. \tag{4.70}$$

which satisfy the supersymmetry algebra

$$[\delta_1, \delta_2]\phi = -2\sqrt{2}\epsilon_L^2\epsilon_L^1(P^-\phi).$$

The theory describes a free, right-handed Weyl-Majorana spinor and a real scalar field. The supersymmetry transformations do not act on the left-moving part of ϕ, for which $\partial_-\phi = \partial^+\phi = 0$, hence (1,0) supersymmetry only affects the right-moving waves of the superfield.

4.2.2 $\mathcal{N} = (1,1)$ Rigid Supersymmetry in $D = 2$

The $\mathcal{N} = (1,1)$ case will be more interesting for us, because we will generalize in the future chapters the Lagrangian which we construct in this section, to a local supersymmetry.

In order to build the superPoincaré algebra, we must introduce one more extra fermionic generator in addition to that of the $\mathcal{N} = (1,0)$ theory, in such a way that the supersymmetric charges form a Majorana, two-component real spinor (Q)

$$Q = \begin{pmatrix} Q_R \\ Q_L \end{pmatrix}. \tag{4.71}$$

The superalgebra is given by the anticommutation relations (4.52) and (4.53) where $Q_1 = Q_R$, $Q_2 = Q_L$.

Superspace is more extended than the $\mathcal{N} = (1,0)$ case due to the presence of one more Grassmann coordinate (θ_R). Its points hence are parametrized by the following coordinates

$$z^M = (x^m, \theta_R, \theta_L), \tag{4.72}$$

where the two Grassmann coordinates satisfy the following relations

$$\{\theta_L, \theta_R\} = 0, \tag{4.73}$$
$$\theta_L^2 = \theta_R^2 = 0. \tag{4.74}$$

We are ready to introduce the basic component of superfield formalism, namely the real scalar superfield $\Phi(x^m, \theta_L, \theta_R)$ which is given as follows

$$\Phi(x, \theta_L, \theta_R) = \phi(x) + \theta\bar{\psi}(x) - \frac{1}{2}\theta\bar{\theta}F(x), \tag{4.75}$$
$$= \phi(x) + \theta_L\psi_R(x) - \theta_R\psi_L(x) + \theta_R\theta_L F(x), \tag{4.76}$$

where we used the bilinear combinations for a single Majorana spinor (4.25) and (4.29).

The fields $\phi(x)$ and $F(x)$ are two real scalar fields and ψ_L and ψ_R are two Majorana-Weyl spinors, hence there are two bosonic and two fermionic degrees of freedom.

The realisation of the supersymmetry charges Q on the superfield Φ is obtained by the supersymmetric transformation

$$\delta\Phi = i\epsilon\bar{Q}\Phi. \tag{4.77}$$

which in light-cone coordinates are

$$Q_R = -i\frac{\partial}{\partial\theta_L} - \sqrt{2}\theta_L\partial^-,$$
$$Q_L = +i\frac{\partial}{\partial\theta_R} - \sqrt{2}\theta_R\partial^+,$$

and the covariant derivatives satisfy the following properties

$$D_R = +i\frac{\partial}{\partial\theta_L} - \sqrt{2}\theta_L\partial^-, \tag{4.78}$$

$$D_L = -i\frac{\partial}{\partial\theta_R} - \sqrt{2}\theta_R\partial^+, \tag{4.79}$$

where we can note that

$$\{D_L, Q_L\} = \{D_L, Q_R\} = \{D_R, Q_R\} = \{D_R, Q_L\} = 0. \tag{4.80}$$

The variation of field components of Φ have the following form

$$\delta\phi = \epsilon_L\psi_R - \epsilon_R\psi_L = \epsilon\bar{\psi}, \tag{4.81}$$

$$\delta\psi = -F\epsilon - \gamma^m\partial_m\phi\epsilon, \tag{4.82}$$

$$\delta F = -\sqrt{2}i\epsilon_L\partial^-\psi_L + \sqrt{2}\epsilon_R\partial^+\psi_R \tag{4.83}$$

We can construct a free Lagrangian for the free Majorana spinor $\psi(x)$, the real scalar $\phi(x)$ and the auxiliary real field $F(x)$ as

$$\begin{aligned}\mathcal{L} &= \int d^2\theta[-(D_R\Phi)(D_L\Phi)], \\ &= 2(\partial^+\phi)(\partial^-\phi) + i\sqrt{2}\psi_R\partial^+\psi_R - i\sqrt{2}\psi_L\partial^-\psi_L + F^2, \\ &= (\partial^m\phi)(\partial_m\phi) + i\bar{\psi}\gamma^m\partial_m\psi + F^2.\end{aligned} \tag{4.84}$$

where covariant derivatives have each dimension $[M] = \frac{1}{2}$, and then the component $\theta_L\theta_R$ of the superfield Lagrangian which has dimension $[M] = 2$, transforms as a total derivative under supersymmetry. The action is then supersymmetric invariant, with the right dimension.

Chapter 5

Superfield Supergravity in curved $D = 2$ superspace

5.1 Introduction

In this chapter we will generalize the $\mathcal{N} = (1,1)$ rigid supersymmetry to the local supersymmetry in the curved superspace.

Our aim will be the derivation of all the geometrical apparatus in curved superspace, in order to apply it in the following chapter to describe matter superfields in the AdS_2 superbackground. We will develop first the supervielbein and superconnection Cartan variables, and then the supercurvature and supertorsion objects which will complete our treatement.

Let us begin by noting that pure supergravity in $D = 2$ is *not dynamical*, because it does not have propagating physical degrees of freedom, like the bosonic $D = 2$ gravitational theory, whose Einstein-Hilbert action is *purely topological* (*Euler characteristic* of the $D = 2$ topology). Gravity however changes the superspace from flat to a generic curved superspace and hence it affects the dynamics of matter fields. The Rarita-Schwinger Lagrangian term written in $D = 2$ as $\mathcal{L} = \epsilon^{ml}\psi_m\gamma^n\partial_n\psi_l$, is a total derivative and then it does not contribute to the action.

To generalize ordinary formulae of differential geometry to superspace we will follow a simple working rule. The rule is that the summation over repeated indices is always performed from the upper left corner to the lower right one with no indices in between.

5.2 Supervielbein and Superconnection

To describe curved superspace geometry we have to generalize the concept of vielbein by the introduction of the corresponding *supervielbein* $E_M{}^A$, where M are curved indices while A are Lorentzian indices. If we define the inverse supervielbein $E_A{}^M$, the following identities hold

$$E_A{}^M E_M{}^B = \delta_A{}^B, \quad E_M{}^A E_A{}^N = \delta_M{}^N. \tag{5.1}$$

The supervielbein is a superfield containing the vielbein $e_m{}^a$ and the gravitino field $\chi_m{}^\alpha$, as we will see below. It is used to perform the transformation of anholonomic indices $(A, B, ...)$ into holonomic ones $(N, M, ...)$ as

$$V^A = V^M E_M{}^A, \qquad V^M = V^A E_A{}^M. \tag{5.2}$$

We will also introduce the analogue of connection in superspace, called *super-connection* $\Omega_{MA}{}^B$. It takes values in the $SO(1,1)$ algebra and obey the symmetry property

$$\Omega_{MAB} + \Omega_{MBA}(-1)^{AB} = 0, \tag{5.3}$$

Since the $SO(1,1)$ is an abelian symmetry, the Lorentz connection in two dimensions reduces to one vector index superfield Ω_M. We can write

$$\Omega_{MA}{}^B = \Omega_M L_A{}^B, \tag{5.4}$$

where $L_A{}^B$ is the matrix of Lorentz rotations defined as

$$L_A{}^B = \begin{pmatrix} \epsilon_a{}^b & 0 \\ 0 & -\frac{1}{2}\gamma^{3\beta}_\alpha \end{pmatrix}, \tag{5.5}$$

with the following properties

$$L_{AB} = -L_{BA}(-1)^A, \quad L_A{}^B L_B{}^C = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & \frac{1}{4}\delta_\alpha{}^\gamma \end{pmatrix}. \tag{5.6}$$

The connection (5.4) is used to define covariant derivatives of a Lorentz supervector as follows

$$\nabla_M V^A = \partial_M V^A + \Omega_M V^B L_B{}^A, \tag{5.7}$$

$$\nabla_M V_A = \partial_M V_A - \Omega_M L_A{}^B V_B. \tag{5.8}$$

The (anti)commutator of the covariant derivatives applied to a Lorentz supervector defines the supercurvature and supertorsion as

$$[\nabla_M, \nabla_N\} V_A = -R_{MNA}{}^B V_B - T_{MN}{}^P \nabla_P V_A, \tag{5.9}$$

where $R_{MNA}{}^B$ is the supercurvature while $T_{MN}{}^P$ is the supertorsion which in Cartan variables become

$$R_{MNA}{}^B = \partial_M \Omega_{NA}{}^B - \Omega_{MA}{}^C \Omega_{NC}{}^B (-1)^{N(A+C)} - (M \leftrightarrow N)(-1)^{MN}, \tag{5.10}$$

$$T_{MN}{}^A = \partial_M E_N{}^A + E_N{}^B \Omega_{MB}{}^A (-1)^{M(B+N)} - (M \leftrightarrow N)(-1)^{MN}. \tag{5.11}$$

In terms of the Lorentz connection (5.4) the curvature may be rewritten as follows

$$R_{MNA}{}^{B} = (\partial_M \Omega_N - \partial_N \Omega_M (-1)^{MN}) L_A{}^B \equiv F_{MN} L_A{}^B, \tag{5.12}$$

where

$$F_{MN} = \partial_M \Omega_N - \partial_N \Omega_M (-1)^{MN}. \tag{5.13}$$

It is useful also to introduce the following derivative for the future computations

$$D_A = E_A{}^M \partial_M. \tag{5.14}$$

which acts on the scalar superfields. The most general covariant derivative however will be

$$\nabla_A = D_A + L_B{}^A E_A{}^M \Omega_M. \tag{5.15}$$

5.3 Supergravity constraints and gauge fixing

5.3.1 Gauge fixing

We will now analyse the component structure of the supervielbein and superconnection and show that upon partial Wess-Zumino gauge fixing and imposing constraints of supertorsion, we will reduce the component content of $E_M{}^A$ and Ω_M to that of the $\mathcal{N} = (1,1)$, $D = 2$ supergravity multiplet.

Supergravity theories are invariant under bosonic diffeomorphisms $\xi^m(x)$, local Lorentz boosts $W^{(0)}(x)$ and local supersymmetry transformations $\xi^\mu(x)$, where all parameters of the transformations depend only on the bosonic coordinates x^m but these transformations appear as subgroups of the full superspace diffeomorphisms $z'^M = \xi^M(z)$, which when expanded in θ coordinates become

$$z'^M = \xi^M(z) = \xi^{(0)M}(x) + \theta^\nu \xi^{(1)}{}_\nu{}^M(x) + \frac{1}{2}\theta\bar{\theta}\xi^{(2)M}(x). \tag{5.16}$$

Note that $\xi^m(x)$ and $\xi^\mu(x)$ are associated to $\xi^{(0)M}(x)$.

Under the full superspace diffeomorphisms (5.16) the components $E_\alpha{}^M$ transform as

$$E'^{\,M}_{\alpha} = \frac{\partial z'^M}{\partial z^N} E_\alpha{}^N, \tag{5.17}$$

$$E'^{\,m}_{\alpha} = E_\alpha{}^n \partial_n \xi^m + E_\alpha{}^\nu (\xi_\nu^{(1)m} + \theta_\nu \xi^{(2)m}), \tag{5.18}$$

$$E'^{\,\mu}_{\alpha} = E_\alpha{}^n \partial_n \xi^\mu + E_\alpha{}^\nu (\xi_\nu^{(1)\mu} + \theta_\nu \xi^{(2)\mu}). \tag{5.19}$$

The large local symmetry contained in (5.16) allows us to gauge away some components of the supervielbein by fixing part of the superdiffeomorphisms and leaving only those corresponding to $\xi^{(0)M}(x)$. Expanding the inverse supervielbein in θ components

$$E_A{}^M = E_A^{(0)M} + \theta^\nu E_{\nu A}^{(1)M} + \frac{1}{2}\theta\bar{\theta} E_A^{(2)M}, \tag{5.20}$$

we can always fix a gauge using the function ${\xi^{(1)}}_\nu{}^M$ in such a way that

$$E^{(0)}{}_\alpha{}^m = 0, \qquad E^{(0)}{}_\alpha{}^\mu = \delta_\alpha{}^\mu. \tag{5.21}$$

Furthermore we can use the $\xi^{(2)m}$ and $\xi^{(2)\mu}$ to get rid off the antysimmetric parts of the first order components in θ of $E_A{}^M$,

$$E^{(1)}{}_{\nu\alpha}{}^m = E^{(1)}{}_{\alpha\nu}{}^m, \qquad E^{(1)}{}_{\nu\alpha}{}^\mu = E^{(1)}{}_{\alpha\nu}{}^\mu. \tag{5.22}$$

In an analogous way we may eliminate some components of Ω_M using the Lorentz boost transforms

$$\Omega'{}_A = S^{-1}{}_A{}^B(\Omega_B - D_B W), \tag{5.23}$$

where $S^{-1}{}_A{}^B$ is the inverse Lorentz transformation matrix corresponding to the boost parameter

$$W(z) = W^{(0)}(x) + \theta^\nu W^{(1)}(x)_\nu + \frac{1}{2}\theta\bar{\theta}W^{(2)}(x). \tag{5.24}$$

$W^0(x)$ is associated with the conventional Lorentz boost, while $W^{(1)}$ and $W^{(2)}$ can be freely gauge fixed e.g. by setting

$$\Omega^{(0)}_\alpha = 0, \qquad \Omega^{(1)}{}_{\nu\alpha} = \Omega^{(1)}{}_{\alpha\nu}, \tag{5.25}$$

Therefore the free parameters which remain, namely the $\xi^m, \xi^\mu, W^{(0)}$, describe bosonic diffeomorphisms, local supersymmetry transformations and local Lorentz boosts respectively.

Expanding now the inverse supervielbein and superconnection in θ we have

$$E_a{}^m = e_a{}^m + \theta^\nu f_{\nu a}{}^m + \frac{1}{2}\theta\bar{\theta}g_a{}^m, \tag{5.26}$$

$$E_a{}^\mu = \chi_a{}^\mu + \theta^\nu f_{\nu a}{}^\mu + \frac{1}{2}\theta\bar{\theta}g_a{}^\mu, \tag{5.27}$$

$$E_\alpha{}^m = \theta^\nu f_{\nu\alpha}{}^m + \frac{1}{2}\theta\bar{\theta}g_\alpha{}^m, \qquad f_{\nu\alpha}{}^m = f_{\alpha\nu}{}^m, \tag{5.28}$$

$$E_\alpha{}^\mu = \delta_\alpha{}^\mu + \theta^\nu F_{\nu\alpha}{}^\mu + \frac{1}{2}\theta\bar{\theta}g_\alpha{}^\mu, \qquad f_{\nu\alpha}{}^\mu = f_{\alpha\nu}{}^\mu \tag{5.29}$$

$$\Omega_a = \omega_a + \theta^\nu \bar{u}_{\nu\alpha} + \frac{1}{2}\theta\bar{\theta}v_\alpha, \tag{5.30}$$

$$\Omega_\alpha = \theta^\nu \rho_{\nu\alpha} + \frac{1}{2}\theta\bar{\theta}v_\alpha, \qquad \rho_{\nu\alpha} = \rho_{\alpha\nu}. \tag{5.31}$$

Among the fields contained in (5.26)-(5.31) there are the graviton $e_a{}^m$, the gravitino and spin connection ω_a but there are many other redundant fields some of which do not have physical meaning, so we need to get rid off them. The component structure of the supervielbein $E_M{}^A$ will be given in the next section.

5.3.2 Supergravity constraints

To reduce still redundant number of component fields of $E_M{}^A$ and Ω_M and thus to express them only in terms of the supergravity multiplet, we may impose constraints on the supertorsion, in a similar way done by Howe in [9] and used by Ertl in [8]

$$T_{ab}{}^c = 0, \tag{5.32}$$
$$T_{\alpha\beta}{}^a = 2i(\gamma^a\epsilon)_{\alpha\beta}, \tag{5.33}$$
$$T_{\alpha\beta}{}^\gamma = 0, \tag{5.34}$$

where (5.33) can be rewritten as

$$T_{\alpha\beta}{}^M = 2i(\gamma^a\epsilon)_{\alpha\beta}E_a{}^M, \tag{5.35}$$

which may be contracted with suitable γ-matrices as

$$T_{\alpha\beta}{}^M(\epsilon\gamma^3)^{\beta\alpha} = 0, \tag{5.36}$$
$$T_{\alpha\beta}{}^M(\epsilon\gamma_a)^{\beta\alpha} = -4iE_a{}^M. \tag{5.37}$$

We can use (5.11) to solve separately eq.(5.36) and (5.37), and referring to [8], we can find out that

$$E_a{}^m = e_a{}^m + i(\theta\gamma^m\bar{\chi}_a) - \theta\bar{\theta}(\lambda^m\bar{\chi}_a), \tag{5.38}$$
$$E_a{}^\mu = \chi_a{}^\mu + i(\theta\gamma^b\bar{\chi}_a)\chi_b{}^\mu - \frac{i}{2}A(\theta\gamma_a)^\mu - \frac{1}{2}\rho_a(\theta\gamma^5)^\mu \tag{5.39}$$
$$+\frac{1}{2}\theta\bar{\theta}[i\epsilon^{mn}(\tilde{\nabla}_m\chi_n\gamma_a\gamma^5)^\mu + \frac{i}{2}c_b(\chi_a\gamma^b)^\mu + A(\chi\gamma_a)^\mu], \tag{5.40}$$
$$E_\alpha{}^m = i(\gamma^m\bar{\theta})_\alpha + \theta\bar{\theta}\bar{\lambda}^m{}_\alpha, \tag{5.41}$$
$$E_\alpha{}^\mu = \delta_\alpha{}^\mu + i(\gamma^b\bar{\theta})_\alpha\chi_b{}^\mu - \frac{1}{2}\theta\bar{\theta}[\lambda^2\delta_\alpha{}^\mu + \frac{1}{2}A\delta_\alpha{}^\mu + \frac{i}{2}c_b\gamma^{b\,\mu}_\alpha], \tag{5.42}$$
$$\Omega_a = \omega_a + (\theta\bar{u}_a) + \frac{1}{2}\theta\bar{\theta}v_a, \tag{5.43}$$
$$\Omega_\alpha = A(\theta\gamma^5\epsilon)_\alpha + i\rho_b(\theta\gamma^b\epsilon)_\alpha \tag{5.44}$$
$$+\frac{1}{2}\theta\bar{\theta}[4\epsilon^{mn}(\tilde{\nabla}_m\bar{\chi}_n)_\alpha + 2c_b(\gamma^5\bar{\lambda}^b)_\alpha - 2iA(\gamma^5\bar{\chi})_\alpha], \tag{5.45}$$

where

$$\rho_a = -\epsilon_a{}^b c_b - 4i(\lambda_a\gamma^5\bar{\chi}), \tag{5.46}$$
$$v_a = 4\epsilon^{mn}(\tilde{\nabla}_m\bar{\chi}_n)_a + 2c_b(\gamma^5\bar{\lambda}^b)_\alpha - 2iA(\gamma^5\bar{\chi})_\alpha, \tag{5.47}$$
$$\chi_m = \chi\gamma_m + \lambda_m. \tag{5.48}$$

We can put in evidence the fact that we are left with two supermultiplets, the supergravity multiplet $\Xi = \{e_a{}^m, \chi_a{}^\mu, A\}$ and the Lorentz connection supermultiplet $\Omega_a = \{\omega_a, u_a{}^\nu, v_a\}$. The supergravity multiplet is the only independent one while the connection is a function of it.

Once obtained the components of the inverse of supervierbein and superconnection, we can find all the geometrical objects of superspace, namely the supercurvature and supertorsion, but in doing this we need the supervierbein. Solving the eq(5.1), the supervierbein is expressed as

$$E_m{}^a = e_m{}^a - 2i(\theta\gamma^a\bar{\chi}_m) + \frac{1}{2}\theta\bar{\theta}Ae_m{}^a, \tag{5.49}$$

$$E_m{}^\alpha = -\chi_m{}^\alpha + \frac{i}{2}A(\theta\gamma_m)^\alpha + \frac{1}{2}\rho_m(\theta\gamma^3)^\alpha, \tag{5.50}$$

$$E_\mu{}^a = i(\theta\gamma^a\epsilon)_\mu, \tag{5.51}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\alpha(1 - \frac{1}{4}\theta\bar{\theta}A). \tag{5.52}$$

So we finally are able to compute all the components of the supertorsion and supercurvature which can be compactly expressed using the Bianchi identities for T^A and $R_A{}^B$ (for explicit computations we refer to [8]), in terms of a scalar S and vector superfield T_a. For the supertorsion we have

$$T_{\alpha\beta}{}^\gamma = 0, \tag{5.53}$$

$$T_{\alpha\beta}{}^a = 2i(\gamma^a\epsilon)_{\alpha\beta}, \tag{5.54}$$

$$T_{\alpha a}{}^\beta = \frac{i}{2}S\gamma^\beta_{a\alpha} + \frac{1}{2}\epsilon_a{}^bT_b\gamma^{3\,\beta}_\alpha, \tag{5.55}$$

$$T_{\alpha a}{}^b = -T_{a\alpha} = 0, \tag{5.56}$$

$$T_{ab}{}^\alpha = \frac{1}{2}\epsilon_{ab}(\epsilon\gamma^5)^{\alpha\beta}\nabla_\beta S, \tag{5.57}$$

$$T_{ab}{}^c = \delta_a{}^cT_b - \delta_b{}^cT_a. \tag{5.58}$$

and for the supercurvature we have

$$F_{\alpha\beta} = 2S(\gamma^5\epsilon)_{\alpha\beta} + 2iT_a(\gamma^a\gamma^5\epsilon)_{\alpha\beta}, \tag{5.59}$$

$$F_{\alpha a} = -F_{a\alpha} = i(\gamma_a\gamma^5)_\alpha{}^\beta\nabla_\beta S + \epsilon_a{}^b\nabla_\alpha T_b, \tag{5.60}$$

$$F_{ab} = \epsilon_{ab}[-\frac{1}{2}\epsilon^{\alpha\beta}\nabla_\beta\nabla_\alpha S + S^2 - \nabla_cT^c + T_cT^c]. \tag{5.61}$$

The scalar superfield turns out to be

$$S = A + 2\epsilon^{mn}(\theta\gamma^5\tilde{\nabla}_m\bar{\chi}_n) + 2i(\theta\bar{\chi})\lambda^2 - 2iA(\theta\bar{\chi}) \tag{5.62}$$

$$- \frac{1}{2}\theta\bar{\theta}\left[\frac{1}{2}\tilde{R} - 4i\epsilon^{mn}(\chi\gamma^5\tilde{\nabla}_m\bar{\chi}_n) + 4i\tilde{\nabla}_a(\lambda^a\bar{\chi}) + 4\chi^2\lambda^2 + A(\chi^a\bar{\chi}_a) + A^2\right] \tag{5.63}$$

where $\tilde{R}$ is the scalar curvature for vanishing (bosonic) torsion.

The vector superfield is nothing other than the trace of the supertorsion

$$T_a = t_a + (\theta\bar{\tau}_a) + \frac{1}{2}\theta\bar{\theta}s_a, \tag{5.64}$$

and

$$t_a = \hat{t}_a - 4i(\lambda_a\bar{\chi}), \quad (5.65)$$
$$(\theta\bar{\tau}_a) = 2i\epsilon^{mn}(\theta\gamma_a\gamma^5\tilde{\nabla}_m\bar{\chi}_n) - i\epsilon_a{}^b c_c(\theta\gamma^c\gamma^5\bar{\chi}_b) \quad (5.66)$$
$$+ 4\chi^2(\theta\bar{\lambda}_a) - A(\theta\bar{\chi}_a) + \epsilon_a{}^b(\theta\bar{u}_b), \quad (5.67)$$
$$s_a = -\partial_a A + 4\epsilon^{mn}(\tilde{\nabla}_m\chi_n\gamma_a\gamma^5\bar{\chi}) - 2\epsilon_a{}^b c_c(\lambda^c\gamma^5\bar{\chi}_b) + 2iA(\chi\bar{\lambda}_a) + \epsilon_a{}^b v_b \quad (5.68)$$

5.4 Symmetry transformations

The Wess-Zumino gauge with the supertorsion constraint was useful for fixing almost all the components of the geometric objects, however several parameters remain free: the zeroth components of superspace diffeomorphisms $\xi^{(0)M}$ and the zeroth component of the Lorentz rotation $W^{(0)}$.

Under infinitesimal superdiffeomorphisms parametrized by a vector superfield $\xi^M(z)$, and by an infinitesimal Lorentz (super-)boost with parameter W(z), the inverse supervierbein and the anholonomic components of the Lorentz superconnection obey the transformation formulas

$$\delta E_A{}^M = \xi^N\partial_N E_A{}^M - E_A{}^N\partial_N\xi^M - WL_A{}^B E_B{}^M, \quad (5.69)$$
$$\delta\Omega_A = \xi^N\partial_N\Omega_A - WL_A{}^B\Omega_B - E_A{}^M\partial_M W. \quad (5.70)$$

To find the explicit form of the remaining symmetry transformations after the gauge fixing we decompose

$$\xi^m = \zeta^m + \theta^\nu k_\nu{}^m + \frac{1}{2}\theta\bar{\theta}l^m, \quad (5.71)$$
$$\xi^\mu = \zeta^\mu + \theta^\nu k_\nu{}^\mu + \frac{1}{2}\theta\bar{\theta}l^\mu, \quad (5.72)$$
$$W = \omega + \theta^\nu k_\nu + \frac{1}{2}\theta\bar{\theta}l. \quad (5.73)$$

where $\zeta^m(x), \zeta^\mu(x), \omega(x)$ are the parameters of the bosonic diffeomorphismss, supersymmetry transformations and Lorentz boosts, respectively.

In order to mantain the Wess-Zumino gauge conditions (5.21) and (5.25), we must impose the following conditions

$$\delta E^{(0)}{}_\alpha{}^m = 0, \quad \delta E^{(0)}{}_\alpha{}^\mu = 0, \quad \delta\Omega^{(0)}{}_\alpha = 0. \quad (5.74)$$

For the bosonic diffeomorphisms and Lorentz rotations we have that (setting $\zeta^\mu = 0$)

$$\xi^m = \zeta^m, \quad (5.75)$$
$$\xi^\mu = \frac{1}{2}\omega(\theta\gamma^5)^\mu, \quad (5.76)$$
$$W = \omega, \quad (5.77)$$

and for the local supersymmetry transformations (setting $\zeta^m = 0, \omega = 0$)

$$\xi^m = i(\theta\gamma^m\bar{\zeta}) - \theta\bar{\theta}(\zeta\bar{\lambda}^m), \tag{5.78}$$

$$\xi^\mu = \zeta^\mu + i(\theta\gamma^b\bar{\zeta})\chi_b{}^\mu + \frac{1}{2}\theta\bar{\theta}\left[\lambda^2\zeta^\mu + \frac{1}{2}c_b(\zeta\gamma^b)^\mu\right], \tag{5.79}$$

$$W = -A(\theta\gamma^5\bar{\zeta} - i\rho_b(\theta\gamma^b\bar{\zeta}) - \theta\bar{\theta}\left[\epsilon^{mn}(\zeta\tilde{\nabla}_m\bar{\xi}_n) + c_b(\zeta\gamma^5\bar{\lambda}^b - i\lambda^2(\zeta\gamma^5\bar{\xi})\right]. \tag{5.80}$$

The transformation rules for the supergravity multiplet are obtained by considering the variations $\delta E^{(0)}{}_a{}^m, \delta E^{(0)}{}_a{}^\mu, \delta\Omega^{(1)}{}_a$ under the transformations (5.75)-(5.80). For the bosonic diffeomorphisms and Lorentz boosts one has the transformations

$$\delta e_a{}^m = \zeta^n\partial_n e_a{}^m - e_a{}^n\partial_n\zeta^m - \omega\epsilon_a{}^b e_b{}^m, \tag{5.81}$$

$$\delta\chi_a{}^\mu = \zeta^n\partial_n\chi_a{}^\mu - \omega\epsilon_a{}^b\chi_b{}^\mu - \frac{1}{2}\omega(\chi_a\gamma^5)^\mu, \tag{5.82}$$

$$\delta A = \zeta^m\partial_m A. \tag{5.83}$$

Under the local supersymmetry, the supergravity supermultiplet transforms as follows

$$\delta e_a{}^m = 2i(\zeta\gamma^m\bar{\chi}_a), \tag{5.84}$$

$$\delta\chi_a{}^\mu = -\tilde{\nabla}_a\zeta^\mu - 2i(\chi\bar{\lambda}_a)\zeta^\mu - 2i(\lambda_b\bar{\chi}_a)(\zeta\gamma^b)^\mu - \frac{i}{2}A(\zeta\gamma_a)^\mu, \tag{5.85}$$

$$\delta A = 2\epsilon^{mn}(\zeta\gamma^5\tilde{\nabla}_m\bar{\chi}_n) + 2i\lambda^2(\zeta\bar{\chi}) - 2iA(\zeta\bar{\chi}). \tag{5.86}$$

We can conclude this chapter by noting that now we are ready to use all the formalism in order to restrict our study to a specific superbackground. What we will consider in the next chapter is a supersymmetry field theory of a matter superfield in the AdS_2 superspace.

Chapter 6

Dynamics of a scalar superfield in AdS_2 superspace

6.1 AdS_2 bosonic spacetime

In this chapter we will consider the dynamics of a scalar superfield in AdS_2 superspace which is a particular solution of $\mathcal{N} = (1,1), D = 2$ supergravity. In this specific superspace we will derive all the geometrical objects and construct the *matter superfield Lagrangian*. We will expand it in its field components, which describes the dynamics in AdS_2 of the scalar supermultiplet and finally we will also derive its supersymmetric transformations.

Let us start by imposing some conditions to reduce the generic curved $D = 2$ superspace to the AdS_2 one. The conditions are the following

$$e_m{}^a = e_m{}^a|_{AdS_2}, \qquad \chi_m{}^\alpha = 0, \qquad A = const. \tag{6.1}$$

They are the right conditions because we have noted in section 5.1 that supergravity fields do not propagate in $D = 2$ space-time. Rarita-Schwinger field must vanish in order to preserve local Lorentz invariance, while the vielbein is fixed to the AdS_2 bosonic space-time. Finally, from (5.86) we get $\delta A = 0$ which lead to a constant value for the auxiliary field.

To have a better insight into the structure of this superspace we first define the metric of the bosonic AdS_2 space. The AdS_2 space-time with signature $(1, -1)$ can be isometrically embedded in the Minkowskian-like manifold $R^{(2,1)}$, whose points are parametrized by $X_A = (X_0, X_{0'}, X_1)$ and the spacetime interval is denoted by

$$ds^2 = dX_A dX^A = dX_0{}^2 + dX_{0'}{}^2 - dX_1{}^2, \tag{6.2}$$

leading thus to the Lorentz metric $\eta_{AB} = diag(1, 1, -1)$.

Then, the condition

$$X_0{}^2 + X_{0'}{}^2 - X_1{}^2 = r^2, \tag{6.3}$$

restricts the $D = 3$ space-time onto the embedded manifold in $D = 2$, called AdS_2 manifold and r denotes its radius of curvature.

This manifold is described by two coordinates and its metric could be derived from that of the $R^{(2,1)}$ space, by rewriting each X_i coordinate as a function of the two AdS_2 coordinates and taking into account the (6.3) condition.

In hyper-spherical coordinates we may write

$$X_0 = r\ cosh\theta\ cos\phi, \quad X_{0'} = r\ cosh\theta\ sin\phi, \quad X_1 = r\ sinh\theta, \tag{6.4}$$

which satisfies the (6.3) condition and where $\theta \in (-\infty, \infty), \phi \in [0, 2\pi]$.

The interval of the AdS_2 space-time now reads

$$ds^2 = r^2 cosh^2\theta d\phi^2 - r^2 d\theta^2 = g_{mn} dx^m dx^n, \tag{6.5}$$

where $x_m = (x_0, x_1) = (\phi, \theta)$ and g_{mn} is the metric:

$$g_{mn} = \begin{pmatrix} r^2 cosh^2\theta & 0 \\ 0 & -r^2 \end{pmatrix}, \quad g^{mn} = \begin{pmatrix} \frac{1}{r^2 cosh^2\theta} & 0 \\ 0 & -\frac{1}{r^2} \end{pmatrix}, \quad g \equiv det g_{mn} = -r^4 cosh^2\theta. \tag{6.6}$$

Therefore we can derive using the definition

$$g_{mn} = e_m{}^a e_n{}^b \eta_{ab}, \quad \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.7}$$

the zwei-bein

$$e_m{}^a = \begin{pmatrix} rcosh\theta & 0 \\ 0 & r \end{pmatrix}, \quad e_a{}^m = \begin{pmatrix} \frac{1}{rcosh\theta} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}. \tag{6.8}$$

6.1.1 Isometries of AdS_2

The AdS_D space is a maximally symmetric space-time which is a vacuum solution of Einstein's equation with a negative scalar curvature (or negative cosmological constant) corresponding to a negative vacuum energy density (attractive force) and positive pressure.

It is an analog of the hyperbolic space or a sphere S^D, just as the Minkowski space and De Sitter space are analogous to Euclidean and elliptical spaces.

Symmetries which leave the metric invariant are called isometries.

Definition 15. A metric $g_{mn}(x)$ is said to be form-invariant under a given coordinate transformation $x \to x'$, when

$$g'_{mn}(x') = g_{mn}(x'). \tag{6.9}$$

AdS_2 space-time has $SO(2,1)$ as the group of isometries generated by the following relations

$$[P_m, P_n] = M_{mn}, \tag{6.10}$$

$$[M_{mn}, P_l] = -\eta_{l[m} P_{n]}. \tag{6.11}$$

6.2 AdS_2 superspace formalism

Using the conditions (6.1) on the supergravity multiplet, we get the components of the supervielbein and superconnection introduced in (5.38), for the AdS_2 case

$$E_m{}^a = e_m{}^a + \frac{1}{2}\theta\bar{\theta}Ae_m{}^a, \tag{6.12}$$

$$E_m{}^\alpha = \frac{i}{2}A(\theta\gamma_m)^\alpha + \frac{1}{2}\rho_m(\theta\gamma^5)^\alpha, \tag{6.13}$$

$$E_\mu{}^a = i(\theta\gamma^a\epsilon)_\mu, \tag{6.14}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\alpha(1 - \frac{1}{4}\theta\bar{\theta}A). \tag{6.15}$$

while the inverse supervielbein has the following expression

$$E_a{}^m = e_a{}^m, \tag{6.16}$$

$$E_a{}^\mu = -\frac{i}{2}A(\theta\gamma_a)^\mu - \frac{1}{2}\rho_a(\theta\gamma^5)^\mu, \tag{6.17}$$

$$E_\alpha{}^m = i(\gamma^m\bar{\theta})_\alpha, \tag{6.18}$$

$$E_\alpha{}^\mu = \delta_\alpha{}^\mu - \frac{1}{2}\theta\bar{\theta}[\frac{1}{2}A\delta_\alpha{}^\mu + \frac{i}{2}c_b\gamma^b{}_\alpha{}^\mu] \tag{6.19}$$

where

$$c_b = \epsilon_b{}^a\omega_a. \tag{6.20}$$

The superconnection is expressed by

$$\Omega_a = \omega_a + (\theta\bar{u}_a + \frac{1}{2}\theta\bar{\theta}v_a), \tag{6.21}$$

$$\Omega_\alpha = A(\theta\gamma^5\epsilon)_\alpha + i\rho_b(\theta\gamma^b\epsilon)_\alpha. \tag{6.22}$$

We are now able to compute the anholonomic components of supercurvature and supertorsion (5.53)-(5.61) in our particular case. To this end, we will only substitute the expressions of the scalar (5.62) and the vector (5.64) superfields.

The scalar superfield becomes

$$S = A - \frac{1}{4}\theta\bar{\theta}(\tilde{R} + 2A^2), \tag{6.23}$$

while the vector superfield is

$$t_a = \hat{t}_a, \tag{6.24}$$

$$(\theta\bar{\tau}_a) = \epsilon_a{}^b(\theta\bar{u}_b), \tag{6.25}$$

$$s_a = -\partial_a A + \epsilon_a{}^b v_b. \tag{6.26}$$

Finally we note that isometries of the $\mathcal{N} = (1,1)$ AdS_2 superspace are given by the $Osp(1|2)$ group which contains as the bosonic subgroup the group $Sp(2) \cong SO(2,1)$.

Its algebra has the following form

$$
\begin{aligned}
[P_m, P_n] &= M_{mn}, && (6.27)\\
[M_{mn}, P_l] &= -\eta_{l[m}P_{n]}, && (6.28)\\
\{Q_\alpha, Q_\beta\} &= \gamma^a{}_{\alpha\beta}P_a + \frac{1}{2}\gamma^3_{\alpha\beta}M_{mn}\epsilon^{mn}, && (6.29)\\
[Q_\alpha, P_m] &= (\gamma_m)_\alpha{}^\beta Q_\beta, && (6.30)\\
[Q_\alpha, M_{mn}] &= \gamma_{mn\alpha}{}^\beta Q_\beta. && (6.31)
\end{aligned}
$$

These relations are useful in obtaining the irreducible representations of the AdS_2 superalgebra. We can observe that the $P^2 = P_mP^m$ operator is not a Casimir operator anymore because of the commutation relation (6.42) and therefore P^2 does not commute with all generators. We thus expect that the component fields of a given representation have different mass.

We shall now fix the value of the constant A by considering (5.84)-(5.86) in the AdS_2 background

$$
\begin{aligned}
&\delta e_a{}^m = 0, && (6.32)\\
&\delta\chi_a{}^\mu = -\nabla_a\zeta^\mu - \frac{i}{2}A(\zeta\gamma_a)^\mu = 0, && (6.33)\\
&\delta A = 0. && (6.34)
\end{aligned}
$$

6.2.1 Computations of "A" field

We have to find A from the equation (6.33). We will follow the formalism of Ertl's work and we refer to its results as basic equations. We recall that in the AdS_2 background (6.5) the Rarita-Schwinger field vanishes and this implies that from (6.33) we obtain

$$\nabla_a\zeta^\mu + \frac{i}{2}A(\zeta\gamma_a)^\mu = 0 \qquad (6.35)$$

By applying the derivative once again

$$
\begin{aligned}
\nabla_b\nabla_a\zeta^\mu + \frac{i}{2}A\nabla_b(\zeta\gamma_a)^\mu &= \nabla_b\nabla_a\zeta^\mu + \frac{i}{2}A\nabla_b\zeta^\beta\gamma_{a\beta}{}^\mu = && (6.36)\\
&= \nabla_b\nabla_a\zeta^\mu + \frac{i}{2}A(-\frac{i}{2}A(\zeta\gamma_b\gamma_a)^\mu, && (6.37)
\end{aligned}
$$

we get

$$\nabla_b\nabla_a\zeta^\mu + \frac{1}{4}A^2(\zeta\gamma_b\gamma_a)^\mu = 0, \qquad (6.38)$$

and taking the commutator of the above equation we get

$$[\nabla_b, \nabla_a]\zeta^\mu + \frac{1}{4}A^2\left(\zeta[\gamma_b, \gamma_a]\right)^\mu = 0 \qquad (6.39)$$

We can rewrite this equation as

$$e_b{}^m e_a{}^n e_\alpha{}^\mu \left([\nabla_m, \nabla_n]\zeta^\alpha + \frac{1}{4}A^2 \left(\zeta[\gamma_m, \gamma_n] \right)^\alpha \right) = 0 \tag{6.40}$$

and substituting $\zeta^\alpha = \epsilon^{\alpha\beta}\zeta_\beta$ we obtain

$$e_b{}^m e_a{}^n e_\alpha{}^\mu \left(\epsilon^{\alpha\beta}[\nabla_m, \nabla_n]\zeta_\beta + \frac{1}{4}A^2 \left(\zeta[\gamma_m, \gamma_n] \right)^\alpha \right) = 0 \tag{6.41}$$

We recall that the (anti)commutator of covariant derivatives (in (5.9)) is defined as

$$[\nabla_M, \nabla_N\}V_A = -R_{MNA}{}^B V_B - T_{MN}{}^P \nabla_P V_A,$$

In our case we have

$$[\nabla_m, \nabla_n]\zeta_\beta = -R_{mn\beta}{}^C \zeta_C, \tag{6.42}$$

where the torsion is considered to vanish recalling the constraints (5.32)-(5.34).

To compute the form of the right-hand side of (6.42) we remember that the curvature may be rewritten as in (5.12)

$$R_{MNA}{}^B = (\partial_M \Omega_N - \partial_N \Omega_M (-1)^{MN}) L_A{}^B \equiv F_{MN} L_A{}^B,$$

where $L_A{}^B$ is given in (5.5) as

$$L_A{}^B = \begin{pmatrix} \epsilon_a{}^b & 0 \\ 0 & -\frac{1}{2}\gamma^{3\beta}_\alpha \end{pmatrix},$$

and F_{MN} is given in (5.13) as $F_{MN} = \partial_M \Omega_N - \partial_N \Omega_M (-1)^{MN}$.

Hence, we have

$$-R_{mn\beta}{}^\gamma = -F_{mn} L_\beta{}^\gamma = +\frac{1}{2}\partial_{[m}\omega_{n]}\gamma^{3\gamma}_\beta, \tag{6.43}$$

and

$$R_{mn}{}^{ab} = (\partial_{[m}\omega_{n]})\epsilon^{ab}, \tag{6.44}$$

we then obtain

$$\frac{1}{2}\epsilon_{bc} R_{mn}{}^{cb} = \partial_{[m}\omega_{n]}, \quad \textit{where} \quad \epsilon^{cb}\epsilon_{bc} = 2. \tag{6.45}$$

Substituting this result in (6.43) we get

$$-R_{mn\beta}{}^\gamma = \frac{1}{4}\epsilon_{bc} R_{mn}{}^{cb}\gamma^{3\gamma}_\beta. \tag{6.46}$$

We can now use the curvature of the AdS_2 space-time

$$R_{mn}{}^{ab} = -\frac{1}{r^2}(e_m{}^a e_n{}^b - e_m{}^b e_n{}^a). \tag{6.47}$$

Thus (6.42) becomes

$$[\nabla_m, \nabla_n\}\zeta_\beta = -\frac{1}{4r^2}\epsilon_{bc}(e_m{}^c e_n{}^b - e_m{}^b e_n{}^c)\gamma^{3\gamma}_\beta \zeta_\gamma, \tag{6.48}$$

and then in (6.41) we have

$$e_b{}^m e_a{}^n e_\alpha{}^\mu \left(-\epsilon^{\alpha\beta}\frac{1}{4r^2}\epsilon_{bc}(e_m{}^c e_n{}^b - e_m{}^b e_n{}^c)\gamma^{3\gamma}_\beta \zeta_\gamma + \frac{1}{4}A^2\left(\zeta[\gamma_m, \gamma_n]\right)^\alpha\right) = 0. \tag{6.49}$$

By readjusting the equation in a simpler form we get

$$e_b{}^m e_a{}^n e_\alpha{}^\mu \left(-\frac{1}{2r^2}\epsilon_{mn}(\zeta\gamma^3)^\alpha + \frac{1}{2}A^2\epsilon_{mn}\left(\zeta\gamma^3\right)^\alpha\right) = 0, \tag{6.50}$$

where we used $\frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m) = \epsilon_{mn}\gamma^3$.

Thus we obtain, that for consistency

$$A = \frac{1}{r}, \tag{6.51}$$

and hence, the supersymmetry parameter $\zeta^\mu(x)$ satisfies the Killing spinor equation

$$\nabla_a\zeta^\mu = -\frac{i}{2r}(\zeta\gamma_a)^\mu. \tag{6.52}$$

6.3 Scalar superfield Lagrangian in AdS_2 superspace

We are now ready to consider the dynamics of a scalar superfield Φ in the AdS_2. Take the action

$$S = \frac{1}{2}\int d^2x d^2\theta(E\mathcal{D}_\alpha\Phi\mathcal{D}^\alpha\Phi), \tag{6.53}$$

which describes the dynamics of the scalar superfield in a generic $\mathcal{N} = (1,1)$, $D = 2$ supergravity background.

In (6.53) $\mathcal{D}_\alpha$ is the covariant derivative defined in (5.14) and E is the superdeterminant of $E_M{}^A$ defined in general as

$$E = sdetE_M{}^A = \frac{det(E_m{}^a - E_m{}^\beta E^{-1\nu}_\beta E_\nu{}^a)}{detE_\mu{}^\alpha}, \tag{6.54}$$

which in terms of the supergravity supermultiplet (5.49)-(5.52) we have

$$E = dete_m{}^a[1 + 2i(\theta\bar{\chi} + \frac{1}{2}\theta\bar{\theta}(2\chi^2 + \lambda^2 + A)]. \tag{6.55}$$

For the AdS_2 superspace E reduces to

$$E = e\left(1 + \frac{1}{2r}\theta\bar{\theta}\right). \tag{6.56}$$

From this superdeterminant, it follows that the supervolume of the AdS_2 superspace is non-zero and has the following value

$$\int dx^2 d\theta^2 E = \frac{1}{2r} Vol_{AdS_2}. \tag{6.57}$$

This is in contrast, e.g. to the zero supervolume of the supersphere S^3 considered in [15].

6.3.1 Lagrangian expansion in component fields

We recall that the expression for the scalar superfield was furnished in (4.75). We can compute the covariant derivative applied on it

$$\mathcal{D}_\alpha \Phi = \left[i(\gamma^m \bar{\theta})_\alpha \partial_m + (\delta_\alpha{}^\mu - \frac{1}{2}\theta\bar{\theta}(\frac{1}{2r}\delta_\alpha{}^\mu + \frac{i}{2}c_b \gamma^b{}_\alpha{}^\mu))\partial_\mu \right] \left(\phi(x) + \theta\bar{\psi}(x) + \frac{1}{2}\theta\bar{\theta}F(x) \right), \tag{6.58}$$

$$\mathcal{D}_\alpha \Phi = i(\gamma^m \bar{\theta})_\alpha \left[\partial_m(\phi + \theta\bar{\psi}) \right] + \psi_\alpha + \theta_\alpha F - \frac{1}{2}\theta\bar{\theta} \left(\frac{1}{2r}\psi_\alpha + \frac{i}{2}c_b \gamma^b{}_\alpha{}^\mu \psi_\mu \right), \tag{6.59}$$

$$\mathcal{D}^\alpha \Phi = -i(\theta\gamma^n)^\alpha [\partial_n \phi + \partial_n(\theta\bar{\psi})] + \psi^\alpha + \theta^\alpha F - \frac{1}{2}\theta\bar{\theta}[\frac{1}{2r}\psi^\alpha + \frac{i}{2}c_b \gamma^{b\alpha\mu}\psi_\mu]. \tag{6.60}$$

By multiplying (6.59) with (6.60) we obtain

$$\mathcal{D}_\alpha \Phi \mathcal{D}^\alpha \Phi = \left[i(\gamma^m \bar{\theta})_\alpha \partial_m(\phi + \theta\bar{\psi}) + \psi_\alpha + \theta_\alpha F - \frac{1}{2}\theta\bar{\theta} \left(\frac{1}{2r}\psi_\alpha + \frac{i}{2}c_b \gamma^b{}_\alpha{}^\mu \psi_\mu \right) \right] \tag{6.61}$$

$$\times \left[-i(\theta\gamma^n)^\alpha [\partial_n \phi + \partial_n(\theta\bar{\psi})] + \psi^\alpha + \theta^\alpha F - \frac{1}{2}\theta\bar{\theta} \left(\frac{1}{2r}\psi^\alpha + \frac{i}{2}c_b \gamma^{b\alpha\mu}\psi_\mu \right) \right] \tag{6.62}$$

$$\tag{6.63}$$

$$= (\gamma^m \bar{\theta})_\alpha (\theta\gamma^n)^\alpha (\partial_m \phi)(\partial_n \phi) + i(\gamma^m \bar{\theta})_\alpha \partial_m(\theta\bar{\psi})\psi^\alpha + i(\gamma^m \bar{\theta})_\alpha (\partial_m \phi)\theta^\alpha F + \tag{6.64}$$

$$+ \ \psi_\alpha(-i(\theta\gamma^n)^\alpha \partial_n(\theta\bar{\psi})) - \psi\bar{\psi} + \frac{1}{4r}\theta\bar{\theta}\psi\bar{\psi} - \psi_\alpha \frac{1}{2}\theta\bar{\theta}(\frac{i}{2}c_b \gamma^{b\alpha\mu}\psi_\mu) + \tag{6.65}$$

$$+ \ \theta_\alpha F(-i(\theta\gamma^n)^\alpha \partial_n \phi) - \theta^2 F^2 + \frac{1}{2}\theta^2(\frac{1}{2r}\psi\bar{\psi}) - \frac{1}{2}\theta\bar{\theta}(\frac{i}{2}c_b \gamma^b{}_\alpha{}^\mu \psi_\mu)\psi^\alpha \tag{6.66}$$

where $i(\gamma^m \bar{\theta})_\alpha (\partial_m \phi)\theta^\alpha F$ and $\theta_\alpha F(-i(\theta\gamma^n)^\alpha \partial_n \phi)$ vanish because of the relation

$$\theta\gamma^m \bar{\theta} = 0. \tag{6.67}$$

The remaining terms are

$$\mathcal{D}_\alpha \Phi \mathcal{D}^\alpha \Phi = -(\gamma^m \bar{\theta})_\alpha (\theta\gamma^n)^\alpha (\partial_m \phi)(\partial_n \phi) - 2i(\psi\gamma^m \theta)\partial_m(\theta\bar{\psi}) - \theta^2 F^2 + \tag{6.68}$$

$$+ \frac{i}{2}\theta\bar{\theta}\psi^\alpha c_b \gamma^b{}_\alpha{}^\mu \psi_\mu + \frac{1}{2r}\theta\bar{\theta}\psi\bar{\psi} - \psi\bar{\psi}. \tag{6.69}$$

We now have

$$-2i(\psi\gamma^m\theta)\partial_m(\theta\bar{\psi}) = +i\theta\bar{\theta}\psi\gamma^m(\partial_m\psi), \tag{6.70}$$

$$+\frac{i}{2}\theta\bar{\theta}\psi^\alpha c_b\gamma^b{}_\alpha{}^\mu\psi_\mu = +\frac{i}{2}\theta\bar{\theta}\psi^\alpha\epsilon_b{}^c\omega_c\gamma^b{}_\alpha{}^\mu\psi_\mu = +\frac{i}{2}\theta\bar{\theta}\psi^\alpha\gamma^c{}_\alpha{}^\beta\gamma^3{}_\beta{}^\mu\omega_c\psi_\mu, \tag{6.71}$$

$$(\gamma^m\bar{\theta})_\alpha(\gamma^n\theta)^\alpha(\partial_m\phi)(\partial_n\phi) = -\theta\bar{\theta}(\partial_m\phi)(\partial^m\phi), \tag{6.72}$$

Recall that the covariant derivative on fermions is

$$D_m\psi = (\partial_m + \frac{1}{2}\gamma^3\omega_m)\psi. \tag{6.73}$$

We can therefore write the Lagrangian components

$$\mathcal{L} = \frac{1}{2}\int d^2\theta E[-\theta\bar{\theta}(\partial_m\phi)(\partial^m\phi) + i\theta\bar{\theta}\psi\gamma^m(D_m\bar{\psi}) - \theta\bar{\theta}F^2 + \frac{1}{2r}\theta\bar{\theta}\psi\bar{\psi} \tag{6.74}$$

$$-\psi\bar{\psi}]. \tag{6.75}$$

Hence, using (6.56), we will have

$$\mathcal{L} = \frac{1}{2}e[-(\partial_m\phi)(\partial^m\phi) + i\psi\gamma^m(D_m\bar{\psi}) - F^2]. \tag{6.76}$$

The equations of motion are given as

$$\Box\phi = 0, \tag{6.77}$$

$$i\gamma^m(D_m\psi) = 0, \tag{6.78}$$

$$F = 0. \tag{6.79}$$

where $\Box = D_m\partial^m$.

Actually, we encounter an ambiguity in the definition of the mass for the fermion field. Even if the Dirac equation is massless, i.e. the "Dirac" mass of the fermion is zero

$$D_m\gamma^m\psi = 0, \tag{6.80}$$

one may find that the "Klein-Gordon" mass of this fermion is not zero. Indeed, acting on the Dirac equation by the Dirac operator again we have

$$D_n\gamma^n D_m\gamma^m\psi = D_m D^m\psi + \gamma^{nm}D_n D_m\psi = +c/r^2\psi = 0, \tag{6.81}$$

where in the passage (6.81) we used $\gamma^n\gamma^m = \frac{1}{2}\{\gamma^n, \gamma^m\} + \gamma^{mn}$, and c is a certain constant (to be computed) which is fixed by the AdS_2 curvature. So we see that the "Klein-Gordon" mass of the fermion is non-zero, while that of the scalar is zero. So, it is more strict to speak about the fact that in the absence of the mass term in the Lagrangian the "Dirac" mass of the fermion and the "Klein-Gordon" mass of the scalar are zero.

6.3.2 Massive Lagrangian term

We see that the Lagrangian (6.76) provides us with the kinetic terms for the massless fields ϕ and ψ propagating in AdS_2. We can make these fields massive by adding the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\int d^2\theta E(M\Phi\Phi), \tag{6.82}$$

where M is an arbitrary mass parameter. Expanding this Lagrangian in components

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\int d^2\theta EM(\phi(x)+\theta\bar{\psi}(x)+\frac{1}{2}\theta\bar{\theta}F(x))(\phi(x)+\theta\bar{\psi}(x)+\frac{1}{2}\theta\bar{\theta}F(x)), && (6.83)\\
&= \frac{1}{2}\int d^2\theta EM(\theta\bar{\theta}\phi F+\theta^\alpha\psi_\alpha\theta^\beta\psi_\beta+\phi\phi), && (6.84)\\
&= \frac{1}{2}\int d^2\theta EM(\theta\bar{\theta}\phi F-\theta^\alpha\theta^\beta\psi_\alpha\psi_\beta+\phi\phi), && (6.85)\\
&= \frac{1}{2}\int d^2\theta EM(\theta\bar{\theta}\phi F+\frac{1}{2}\theta\bar{\theta}\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta+\phi\phi), && (6.86)\\
&= \frac{1}{2}\int d^2\theta EM(\theta\bar{\theta}\phi F-\frac{1}{2}\theta\bar{\theta}\psi\bar{\psi}+\phi\phi), && (6.87)\\
&= \frac{1}{2}\int d^2\theta M(\theta\bar{\theta}\phi F-\frac{1}{2}\theta\bar{\theta}\psi\bar{\psi}+\frac{1}{2r}\theta\bar{\theta}\phi\phi), && (6.88)
\end{aligned}$$

we arrive at

$$\mathcal{L} = \frac{1}{2}M(\phi F-\frac{1}{2}\psi\bar{\psi}+\frac{1}{2r}\phi\phi). \tag{6.89}$$

Thus, the full off-shell Lagrangian will be

$$\mathcal{L} = \frac{1}{2}e[-(\partial_m\phi)(\partial^m\phi)+i\psi\gamma^m(D_m\bar{\psi})-F^2+M(\phi F-\frac{1}{2}\psi\bar{\psi}+\frac{1}{2r}\phi\phi)]. \tag{6.90}$$

We can get rid off the auxiliary field $F(x)$ in the Lagrangian, by solving its equations of motion

$$F = \frac{1}{2}M\phi(x). \tag{6.91}$$

Substituting this solution in (6.90) we get

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}e[-(\partial_m\phi)(\partial^m\phi)+i\psi\gamma^m(D_m\bar{\psi})-\frac{1}{4}M^2\phi^2(x)+\frac{1}{2}M^2\phi^2(x) && (6.92)\\
&\quad -\frac{1}{2}M\psi\bar{\psi}+\frac{1}{2r}M\phi^2], && (6.93)\\
&= \frac{1}{2}e[-(\partial_m\phi)(\partial^m\phi)+i\psi\gamma^m(D_m\bar{\psi})+\frac{1}{2}\mathcal{M}^2\phi^2-\frac{1}{2}M\psi\bar{\psi}], && (6.94)
\end{aligned}$$

where

$$\mathcal{M}^2 = \frac{M}{2}\left(\frac{M}{2}+\frac{1}{r}\right). \tag{6.95}$$

The equations of motion are the following

$$\phi : \mathcal{M}^2\phi + \Box\phi = 0. \tag{6.96}$$

For the spinor field the equation of motion is

$$i\gamma^m(D_m\bar{\psi}) - \frac{M}{2}\psi = 0. \tag{6.97}$$

We can conclude that in AdS_2 background, scalar superfield Lagrangians lead to supermultiplets of a scalar and spinorial fields which may have different mass. In the case of the kinetic Lagrangian only, scalar and spinorial fields have vanishing mass. When we add the massive Lagrangian instead, in general the two fields will have different masses. One particular case is when we take the mass of the fermion to be $\frac{M}{2} = \frac{-1}{r}$ which lead to a vanishing mass of the scalar field.

6.3.3 Scalar superfield self-interaction

To the above Lagrangians describing the propagation of the massless and (partially) massive scalar superfields in the AdS_2 superspace one can add the potential term

$$\int dx^2 d\theta^2 EV(\Phi), \tag{6.98}$$

which describes a general self-interaction of the superfield Φ.

6.3.4 Supersymmetry transformations

Now we wish to find the AdS_2 supersymmetry transformations which leave the whole Lagrangian (6.90) invariant.

We will derive these transformations from the variation of a scalar superfield

$$\delta\Phi = \xi^m\partial_m\Phi + \xi^\mu\partial_\mu\Phi, \tag{6.99}$$

where its parameters are given in the general case in (5.78)-(5.79), but in AdS_2 space-time we have

$$\xi^m = i(\theta\gamma^m\bar{\zeta}), \tag{6.100}$$

$$\xi^\mu = \zeta^\mu + \frac{1}{2}\theta\bar{\theta}[\frac{i}{2}c_b(\zeta\gamma^b)^\mu]. \tag{6.101}$$

Hence we can expand the variation of the scalar superfield

$$\delta\Phi = i\theta^\alpha\gamma^m{}_\alpha{}^\beta\zeta_\beta\partial_m(\phi + \theta\bar{\psi}) + (\zeta^\mu + \frac{1}{2}\theta^2[\frac{i}{2}c_b(\zeta\gamma^b)^\mu])(\psi_\mu + \theta_\mu F). \tag{6.102}$$

And we have the following transformations of the component fields

$$\delta\phi = \zeta^\mu\psi_\mu, \tag{6.103}$$
$$\delta\psi_\alpha = i(\gamma^m\bar{\zeta})\partial_m\phi + \zeta_\alpha F, \tag{6.104}$$
$$\delta F = i\zeta\gamma^m(\partial_m\psi + \frac{1}{2}\gamma^3\omega_m\psi) = i\zeta\gamma^m D_m\psi, \tag{6.105}$$

where in (6.105) we used the identity

$$\gamma^a\gamma^3 = \epsilon_b{}^a\gamma^b. \tag{6.106}$$

Variation of the massive term

We will prove explicitly in what follows that the above transformations leave invariant the massive Lagrangian term. This will allow to cross-check the correctness of the coefficients in the component Lagrangian derived from the superfield

Let us begin by varying (6.89)

$$\delta\mathcal{L} = \frac{1}{r}\phi\delta\phi + \delta\phi F + \phi\delta F - \psi\delta\psi = 0, \tag{6.107}$$

we get

$$\delta\mathcal{L} = \frac{1}{r}\phi(\zeta^\mu\psi_\mu) + \zeta^\mu\psi_\mu F + \phi i\zeta\gamma^m(\partial_m\psi + \frac{1}{2}\gamma^3\omega_m\psi) - \psi(i(\gamma^m\bar{\zeta})\partial_m\phi + \zeta_\alpha F), \tag{6.108}$$
$$= \frac{1}{r}\phi(\zeta\psi) + \underbrace{\zeta\psi F}_{1} + i\phi\zeta\gamma^m\partial_m\psi + \frac{i}{2}\phi\zeta\gamma^m\gamma^3\omega_m\psi - i\psi\gamma^m\bar{\zeta}\partial_m\phi - \underbrace{\psi\zeta F}_{1}. \tag{6.109}$$

Terms underbraced with 1 cancel themselves. Let us now finish computations

$$-i\psi\gamma^m\bar{\zeta}\partial_m\phi = +i\zeta\gamma^m\bar{\psi}\partial_m\phi = -i\partial_m\zeta\gamma^m\bar{\psi}\phi - i\zeta\gamma^m\partial_m\bar{\psi}\phi, \tag{6.110}$$
$$= \frac{-1}{2r}\zeta\gamma_m\gamma^m\bar{\psi}\phi + \frac{i}{2}\omega_m\zeta\gamma^3\gamma^m\bar{\psi}\phi - i\zeta\gamma^m\partial_m\bar{\psi}\phi, \tag{6.111}$$
$$= -\frac{1}{r}\zeta\bar{\psi}\phi - \frac{i}{2}\zeta\gamma^m\gamma^3\bar{\psi}\omega_m\phi - i\zeta\gamma^m\partial_m\bar{\psi}\phi, \tag{6.112}$$

which cancels all remaining terms. In (6.111) we used the Killing spinor equations

$$\nabla_a\zeta^\mu = -\frac{i}{2r}(\zeta\gamma_a)^\mu. \tag{6.113}$$

while in (6.112) the identity $\gamma_m\gamma^m = 2$. Thus all elements vanish and the invariance of the Lagrangian under local supersymmetric transformations is proved.

Chapter 7

Conclusions

In the second chapter we have reviewed the $\mathcal{N}=1$, $D=4$ superPoincaré algebra and its representations. We have introduced the superspace formalism in four dimensions and discussed the main properties of the chiral and real superfields. Then, we used them for constructing rigid supersymmetric theories of matter and pure SYM fields in four dimensions.

In the third chapter we reviewed pure $N=1$ supergravity theory in four dimensions. This theory is based on the supergravity multiplet composed of a graviton and a spinor-vector field called gravitino. The only framework where gravity-spinor coupling becomes possible is the vierbein formalism which introduces a local Lorentz frame at any point in the curved background and thus allows one to deal with the fermionic fields transforming under spinorial representations of the Lorentz group.

In chapter four we passed to the two dimensional space-time where gravitational theories considerably simplify. We settled our formalism of superspace and superfields and also the formalism of the two components spinors in two dimensions. Then, for a reference example of supersymmetric Lagrangian, we constructed rigid supersymmetries of type $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$.

The $\mathcal{N}=(1,1)$ supersymmetry was generalized in the fifth chapter to be the local supersymmetry underlying a supergravity theory in two dimensions. In these dimensions pure gravity does not describe propagating particles of spin 2. This also implies that supergravity does not have dynamical physical degrees of freedom on the mass shell. But off-the mass shell supergravity has a non-trivial supergeometry and is invariant under a group of local superdiffeomorphisms. All geometrical apparatus of Riemann geomtry was generalized to curved superspace by introducing the supervielbein, superconnection, supertorsion and supercurvature.. The method followed to reach this purpose was to impose some constraints on the supertorsion (Howe [9]) and fix the Wess-Zumino gauge to sweep away the redundant fields and leaving only the supergravity multiplet. In the end of the chapter local supersymmetry transformations of the supergravity multiplet were obtained.

In the chapter six we focused on the particular AdS_2 superspace by requiring that the Rarita-Schwinger spinor-vector vanishes and the vielbein is fixed to the Anti De-Sitter vielbein. From the requirement of self-consistency of the supersymmetric transformations, we derived the value of the auxiliary field which was related to the inverse of the radius of the AdS_2 superspace. We then constructed a kinetic Lagrangian for a scalar superfield in this AdS_2 superspace and we found that its expression is similar to the rigid supersymmetric $\mathcal{N} = (1, 1)$ case but with the AdS_2 vielbein used to define the integration measure and contract the vector indices. Adding another (massive) term to the Lagrangian we obtained its field components, and we found that the scalar and spinor fields do not have the same mass. This result would be expected from the structure of the AdS_2 isometry superalgebra, in which the translational generator P_m does not commute with the Q_α (in contrast to the flat case), therefore $P_m P^m$ is not a Casimir of the AdS_2 superalgebra. The mass relations were

$$\mathcal{M}^2 = \frac{M}{2}\left(\frac{M}{2} + \frac{1}{r}\right). \tag{7.1}$$

where M is the spinorial mass, while $\mathcal{M}$ is the scalar mass and it can be seen that the scalar mass depends on the inverse radius of AdS_2 spacetime. If we choose $M = -\frac{2}{r}$ we obtain a vanishing scalar mass and a non-vanishing spinorial mass. As a final exercise we found the supersymmetric transformations which leave the full Lagrangian invariant and cross-checked the result by checking the invariance of the Lagrangian under these transformations. Our expectations for future work is to addressed problems of the AdS_2/CFT_1 correspondence and string problems in general. The superspace formalism used in this thesis can be reused in higher dimensions, where geometrical objects become more complex but the ideas remain the same.

The results of the thesis, i.e. the construction of the Lagrangians for matter fields in AdS_2 provides the basis for further studying of these theories and in particular their quantum properties, which include the calculation of correlation functions etc.

For further developments of the results of this thesis, we can mention the study of supersymmetric theories on the direct product of spaces $AdS_2 \times S_2$, i.e. AdS_2 and the two-sphere, or their Euclidean counterparts, on $S_2 \times S_2$, with the purpose of studying AdS_2/CFT_1 correspondence and other dualities with the use of the so called localization technique which allows one to compute e.g. partition functions in these theories explicitly without resorting to perturbative methods.

Chapter 8

Appendix-Conventions

8.1 The Lorentz and Poincaré groups

In this chapter we will recollect some crucial conventions, used abroad in the text, but we will also give a more complete picture to some arguments left apart. We will follow the so called "West Coast" conventions on the construction of the metric of the Minkowski space-time, namely

$$\eta_{mn} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{8.1}$$

therefore the *spacetime interval* is defined as

$$ds^2 = +(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = dx^m \eta_{mn} dx^m, \tag{8.2}$$

which is an invariant relativistic observable. Opposite signs choices in the metric would lead to differences in sign of some objects.

Minkowski space-time is described by translations which form the group $\mathcal{M}_4$, mathematically expressed as the coset group

$$M_4 = \frac{ISO(3,1)}{SO(3,1)}. \tag{8.3}$$

A point on this space-time is denoted by

$$(x^m) = (x^0, x^1, x^2, x^3), \tag{8.4}$$

where $x^0 = t$ and $x^1, x^2, x^3$are the space components of the four-vector x^m.

The spacetime transformations that leave all the relativistic observables invariant are the Lorentz transformations defined by

$$x^{m\prime} = \Lambda^m{}_n x^n. \tag{8.5}$$

They form a Lie group (Lorentz Group) constrained by the following constraint

$$\Lambda^T \eta \Lambda = \eta, \tag{8.6}$$

required by the invariance of the spacetime interval between two events.

We can express the full Lorentz group by

$$O(1,3) = (\Lambda \in GL(4,R) | \Lambda^T \eta \Lambda = \eta). \tag{8.7}$$

We will refer especially to the so called *proper orthocronous Lorentz subgroup* which preserves both orientation and the direction of time, and is denoted by $SO^{\dagger}(1,3)$,which is expressed as

$$L_+^{\uparrow} = \{\Lambda \in O(1,3) | det\Lambda = +1, \Lambda_0^0 \geq +1\}. \tag{8.8}$$

There is a larger class of transformations in the Minkowski space-time M_4 which leave invariant physical quantities. This class constitutes the so called *Poincaré group* defined as the *semidirect product* between the Lorentz and translation transformations

$$ISO^{\dagger}(1,3) = SO^{\dagger}(1,3) \ltimes R^{1,3}. \tag{8.9}$$

where *translations* are explicitly given by

$$x^m \to x'^m = x^m + a^m, \tag{8.10}$$

and a^m is a constant four-vector. Putting together translations and Lorentz transformations one get the following general transformations

$$x^m \to x'^m = \Lambda^m{}_n x^n + a^m. \tag{8.11}$$

Two Poincaré transformations compose as

$$(\Lambda_2, a_2) \circ (\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2). \tag{8.12}$$

8.1.1 Lorentz and Poincaré algebra

Lie algebras are vector spaces obtained from Lie groups by Taylor-expanding around the identity element. For the $SO^{\dagger}(1,3)$ group it is convenient to determine representations of the algebra before than those of the group. Then one can find the group representations by exponentiation.

The Lorentz algebra is derived by linearizing the condition (8.6) around the identity

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b, \tag{8.13}$$

we have

$$\eta_{mn} = \Lambda^r{}_m \Lambda^s{}_n \eta_{rs} = (\delta^r_m + \omega^r_m)(\delta^s_n + \omega^s_n)\eta_{rs} = \eta_{mn} + \eta_{ms}\omega^s_n + \eta_{ns}\omega^s_m + ... \tag{8.14}$$

where dots stand for higher orders in ω and then the Lorentz algebra can be defined as

$$so(1,3) = \{\omega \in gl(4,R) | \eta\omega = -(\eta\omega)^T, tr\omega = 0\}, \tag{8.15}$$

where ω are represented in the fundamental representations by antisymmetric 4×4 matrices which have six independent components hence they have dimension $dim(so(1,3)) = 6$.

This implies that there are 6 generators (J_i, K_i) with $i = 1,2,3$, satisfying the following non covariant relations

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, & (8.16)\\ [J_i, K_j] &= i\epsilon_{ijk}K_k, & (8.17)\\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. & (8.18) \end{aligned}$$

The Poincaré group elements near the identity can be written as

$$g(\Lambda, a) = \mathbb{I} - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma} + ia_\mu P^\mu, \tag{8.19}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$.

Poincaré algebra is given by the following set of commutators in covariant form

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) & (8.20)\\ [M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) & (8.21)\\ [P_\mu, P_\nu] &= 0 & (8.22) \end{aligned}$$

In order to construct representations of the Lorentz algebra it is useful to introduce complex linear combinations of J_i and K_i

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i). \tag{8.23}$$

We can rewrite the algebra (8.16)-(8.18) as

$$[J_i^\pm, J_j^\pm] = i\epsilon_{ijk}J_k^\pm, \qquad [J_i^\pm, J_j^\mp] = 0. \tag{8.24}$$

From this relations we show that the complex Lorentz algebra is equivalent to two $su(2)_C$ or equivalently to two $sl(2,C)$ algebras. Hence at the level of complex algebras we have:

$$so(1,3)_C \simeq su(2)_C \oplus su(2)_C \tag{8.25}$$

where the Lorentz algebra $so(1,3)$ is a real form of $so(1,3)_C$.

Representations of Lorents group can be given in function of those of SU(2) thanks to the isomorphism ($SO(1,3) \simeq \frac{SU(2)\otimes SU(2)}{Z_2}$) which are labelled by spins, so we have to work with spinors.

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