Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA
"TULLIO LEVI–CIVITA"

Corso di Laurea Magistrale in Matematica

AIRY EQUATION: A TOPOLOGICAL APPROACH OF ITS STOKES PHENOMENON

RELATORE
Prof. Andrea D'Agnolo

LAUREANDO
Giada Volpato
1157161

19 Aprile 2019

Anno Accademico 2018/2019

Contents

Introduction

The Airy equation is one of the first cases that Stokes studied in his work [9], as it is a simple case in which one can have a clear view of the Stokes phenomenon.

Let $\mathbb{V}$ be a complex line with complex coordinate z, whose dual is $\mathbb{V}^*$ with complex coordinate w. In $\mathbb{V}^*$, the Airy equation $Q\psi = 0$, with $Q = \partial_w^2 - w$, has two entire solutions that have an integral representation given by

$$\psi(w) = \int_\gamma \exp\left(\frac{z^3}{3} - zw\right) dz, \tag{1}$$

with γ an appropriate integration path.

For large value of $|w|$, this is asymptotic to a linear combination of

$$v_\pm = w^{-1/4} e^{g_\pm} \left(1 + \mathcal{O}(w^{-1})\right), \text{ where } g_\pm = \pm\frac{2}{3}w^{3/2},$$

that are multivalued functions of the complex variable w.

The integral above is the Fourier-Laplace transform of $e^{f(z)}$, where $f(z) = z^3/3$.
This transform has a realization at the Weyl algebra level that is the isomorphism

$$P \mapsto {}^L P,$$

given by $z \mapsto -\partial_w,\ \partial_z \mapsto w$.

This induces an equivalence between holonomic algebraic $\mathcal{D}$-modules on $\mathbb{V}$ and on $\mathbb{V}^*$, that we still denote by $\mathcal{M} \mapsto {}^L\mathcal{M}$.

So, following the notation in [2], let $\mathcal{N} = \mathcal{D}_{\mathbb{V}^*}/\mathcal{D}_{\mathbb{V}^*}Q$ be the $\mathcal{D}_{\mathbb{V}^*}$-module associated to the Airy operator Q.
By the observations above, one knows that $\mathcal{N}$ has two exponential factors at ∞, that are g_+ and g_-.

At the level of Weyl algebra, one has $\mathcal{N} \simeq {}^L\mathcal{M}$, where $\mathcal{M} = \mathcal{D}_{\mathbb{V}}/\mathcal{D}_{\mathbb{V}}P$ is the $\mathcal{D}_{\mathbb{V}}$-module associated to $P = z^2 - \partial_z$.
The equation $P\phi = 0$ has the entire solution $\phi(z) = e^{f(z)}$, so f is the only exponential factor of $\mathcal{M}$.

At level of $\mathbb{R}$-constructible enhanced ind-sheaves, if $\mathcal{S}ol^E$ denotes the enhanced solution functor, what we just said is that

$$\mathcal{S}ol^E(\mathcal{N})|_{S\times\mathbb{R}} = \left(\mathbb{E}^{g_+} \oplus \mathbb{E}^{g_-}\right)|_{S\times\mathbb{R}},$$

in an appropriate sector S, and

$$\mathcal{S}ol^E(\mathcal{M}) = \mathbb{E}^{z^3/3}.$$

Moreover,

$$\mathcal{S}ol^E(\mathcal{N}) \simeq \left(\mathbb{E}^{z^3/3}\right)^{\curlywedge},$$

where ${}^{\curlywedge}$ is the enhanced Fourier-Sato transform.

Considering only the case of enhanced sheaves, we will establish explicit isomorphisms

$$\left(E^{g_+} \oplus E^{g_-}\right)|_{S_i\times\mathbb{R}} \simeq \left(E^{z^3/3}\right)^{\curlywedge}|_{S_i\times\mathbb{R}},$$

with S_i a sector bounded by two consecutive Stokes lines.
This will allow us to reconstruct the sheaf $\left(E^{z^3/3}\right)^{\curlywedge}$.

The contents of this thesis are both the mathemathical notions just mentioned and the explicit computation of what we need to study the Airy equation.

In *chapter* 1 we construct the category of enhanced sheaves by adding a real variable to the sheaves of k-vector spaces on M, where k is a field and M is a good topological space.
First of all we define the functors of *convolution* in $D^b(k_{M\times\mathbb{R}})$ in the new variable $t \in \mathbb{R}$, and we prove some properties of these new functors.
Then we define the category of *enhanced sheaves* by a quotient of $D^b(k_{M\times\mathbb{R}})$ with the subcategory $\pi^{-1}D^b(k_M)$, where $\pi : M \times \mathbb{R} \to M$ is the projection. We define also the Grothendieck operations of this new category, and we prove that they satisfy similar properties to those of classical sheaves.

Then we talk about ind-sheaves, bordered spaces and enhanced ind-sheaves. These are the consequent generalizations of enhanced sheaves, so we decide to mention them even if they are not necessary in the computation of the Airy case.

Finally, we define the $\mathbb{R}$-constructible sheaves, that we will use in the following.

In *chapter* 2 we give some basic notions on $\mathcal{D}$-modules and the definition of the enhanced solution functor, since we talked about them in this introduction, and to recall us that there is another equivalent way to study the Airy equation, even if we do not handle it.

The notions instead that will be used in the following is that of exponential $\mathcal{D}$-modules and exponential enhanced sheaves.
Here in fact we concentrate on the definitions of them and in some of their properties.

In *chapter* 3 we find the analogues of the classical Fourier-Laplace transform at the level of $\mathcal{D}$-modules and enhanced sheaves. Since originally the Fourier transform was introduced as an integral transformation with a fixed kernel, first of all we define the integral transformations $\overset{D}{\circ}$ and $\overset{*}{\circ}$. Then we use them to define the Fourier-Laplace transform for $\mathcal{D}$-modules and the enhanced Fourier-Sato transform for enhanced sheaves.

In *chapter* 4 we study the Airy equation.

In the first section we find explicitly the integral representation of the solutions of the Airy equation (1). Then, using Morse theory, we built the cycles that make this integral converges.

In the following, we explain well the Stokes phenomenon and we translate the explicit computations we made before at level of $\mathbb{R}$-constructible enhanced sheaves. We study $E^{g_+} \oplus E^{g_-}$, that comes from the asymptoticity of the solutions of the Airy equation, and $(E^{z^3/3})^{\curlywedge}$, that comes from the integral form of the solutions, reaching to show how to connect them.

Chapter 1

More general sheaves

1.1 Sheaves

Recall that a topological space is *good* if it is Hausdorff, locally compact, countable at infinity and has finite soft-dimension.

Let M be a good topological space and let k be a field. Following the notation used in [7], denote by $D^b(k_M)$ the bounded derived category of sheaves of k-vector spaces on M.
For $S \subset M$ a locally closed subset, denote by k_S the zero extension to M of the constant sheaf on S with stalk k.
For $f : M \to N$ a morphism of good topological spaces, denote by $\otimes$, f^{-1}, $Rf_!$, $R\mathcal{H}om$, Rf_* and $f^!$ the six Grothendieck operations for sheaves.

Define the *duality functor* of $D^b(k_M)$, by

$$D_M F = R\mathcal{H}om(F, \omega_M),$$

for $F \in D^b(k_M)$ and $\omega_M := a_M^! k_{\{pt\}}$ the dualizing complex, where $a_M : M \to \{pt\}$.

Note 1.1. If M is a $\mathscr{C}^0$-manifold of dimension d_M and $or_M := H^{-d_M}(\omega_M)$ is the orientation sheaf, we have

$$\omega_M \simeq or_M[d_M].$$

1.2 Enhanced sheaves

1.2.1 Convolution

Consider the maps

$$\mu, q_1, q_2 : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$$

given by $q_1(x, t_1, t_2) = (x, t_1)$, $q_2(x, t_1, t_2) = (x, t_2)$ and $\mu(x, t_1, t_2) = (x, t_1 + t_2)$. The functors of *convolution* in $D^b(k_{M\times\mathbb{R}})$ in the variable $t \in \mathbb{R}$ are defined by

$$\begin{aligned} F_1 \overset{*}{\otimes} F_2 &= R\mu_!(q_1^{-1}F_1 \otimes q_2^{-1}F_2), \\ R\mathcal{H}om^*(F_1, F_2) &= Rq_{1*}R\mathcal{H}om(q_2^{-1}F_1, \mu^! F_2). \end{aligned} \tag{1.1}$$

We'll write $k_{\{t=0\}}$ (resp. $k_{\{t\leq 0\}}$, $k_{\{t\geq 0\}}$) instead of $k_{M\times\{0\}}$ (resp. $k_{M\times\{t\in\mathbb{R}\colon \leq 0\}}$, $k_{M\times\{t\in\mathbb{R}\colon t\geq 0\}}$).

The convolution product $\overset{*}{\otimes}$ makes $D^b(k_{M\times\mathbb{R}})$ into a commutative tensor category, with $k_{\{t=0\}}$ as unit object.

Notice that $k_{\{t\geq 0\}}$ is idempotent for $\overset{*}{\otimes}$, i.e.

$$k_{\{t\geq 0\}} \overset{*}{\otimes} k_{\{t\geq 0\}} \simeq k_{\{t\geq 0\}}.$$

Moreover, also $k_{\{t\leq 0\}}$ is idempotent for $\overset{*}{\otimes}$, and one has

$$k_{\{t\geq 0\}} \overset{*}{\otimes} k_{\{t\leq 0\}} \simeq 0.$$

Lemma 1.2. *Let $K \in D^b(k_{M\times\mathbb{R}})$. The functors of convolution form an adjoint pair $\left(\cdot \overset{*}{\otimes} K, R\mathcal{H}om^*(K, \cdot)\right)$, i.e.*

$$Hom_{D^b(k_{M\times\mathbb{R}})}(K_1 \overset{*}{\otimes} K, K_2) \simeq Hom_{D^b(k_{M\times\mathbb{R}})}(K_1, R\mathcal{H}om^*(K, K_2)),$$

for $K_1, K_2 \in D^b(k_{M\times\mathbb{R}})$.

Proof. Let $Hom = Hom_{D^b(k_{M\times\mathbb{R}})}$. One has

$$\begin{aligned} Hom(K_1 \overset{*}{\otimes} K, K_2) &= Hom(R\mu_!(q_1^{-1}K_1 \otimes q_2^{-1}K, K_2)) \\ &\simeq Hom(q_1^{-1}K_1 \otimes q_2^{-1}K, \mu^! K_2) \\ &\simeq Hom(q_1^{-1}K_1, R\mathcal{H}om(q_2^{-1}K, \mu^! K_2)) \\ &\simeq Hom(K_1, Rq_{1*}R\mathcal{H}om(q_2^{-1}K, \mu^! K_2)) \\ &= Hom(K_1, R\mathcal{H}om^*(K, K_2)). \end{aligned}$$

□

Remark 1.3. *Since $k_{\{t=0\}}$ is the unit object for $\overset{*}{\otimes}$, by adjunction one has*

$$R\mathcal{H}om^*(k_{\{t=0\}}, H) \simeq H, \quad \forall H \in D^b(k_{M\times\mathbb{R}}).$$

Properties of the convolution functors

Note 1.4. Notice that $\pi \circ \mu = \pi \circ q_1 = \pi \circ q_2$ and that the following diagrams are Cartesian:

$$\begin{array}{ccc} M\times\mathbb{R}^2 & \xrightarrow{q_1} & M\times\mathbb{R}\,, \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\pi} \\ M\times\mathbb{R} & \xrightarrow{\pi} & M \end{array} \qquad \begin{array}{ccc} M\times\mathbb{R}^2 & \xrightarrow{q_2} & M\times\mathbb{R}\,, \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\pi} \\ M\times\mathbb{R} & \xrightarrow{\pi} & M \end{array}$$

$$\begin{array}{ccc} M\times\mathbb{R}^2 & \xrightarrow{q_1} & M\times\mathbb{R}\,. \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle\pi} \\ M\times\mathbb{R} & \xrightarrow{\pi} & M \end{array}$$

This means that one has

$$\begin{aligned} \pi^{-1}R\pi_! &\simeq R\mu_! q_1^{-1} \simeq R\mu_! q_2^{-1} \simeq Rq_{1!}q_2^{-1}, \\ \pi^! R\pi_* &\simeq Rq_{1*}\mu^! \simeq Rq_{2*}\mu^!. \end{aligned}$$

For a proof of these facts one can see for example [7].

Now we'll use this and some classical adjunctions and equivalences for sheaves, to prove some properties of the convolution functors in $D^b(k_{M\times\mathbb{R}})$.

Proposition 1.5. *Let* $K_1, K_2, K_3 \in D^b(k_{M\times\mathbb{R}})$. *One has*

$$(K_1 \overset{*}{\otimes} K_2) \overset{*}{\otimes} K_3 \simeq K_1 \overset{*}{\otimes} (K_2 \overset{*}{\otimes} K_3),$$
$$R\mathcal{H}om^*(K_1 \overset{*}{\otimes} K_2, K_3) \simeq R\mathcal{H}om^*(K_1, R\mathcal{H}om^*(K_2, K_3)).$$

Proof. Consider

$$\mu', q_1', q_2', q_3' : M \times \mathbb{R}^3 \to M \times \mathbb{R},$$

where $\mu'(x, t_1, t_2, t_3) = (x, t_1 + t_2 + t_3)$ and q_1', q_2', q_3' are the projections.
The first equivalence follows since both $(K_1 \overset{*}{\otimes} K_2) \overset{*}{\otimes} K_3$ and $K_1 \overset{*}{\otimes} (K_2 \overset{*}{\otimes} K_3)$ are isomorphic to

$$R\mu'_!({q'}_1^{-1}K_1 \otimes {q'}_2^{-1}K_2 \otimes {q'}_3^{-1}K_3).$$

Let $Hom = Hom_{D^b(k_{M\times\mathbb{R}})}$ and $K \in D^b(k_{M\times\mathbb{R}})$. One has

$$\begin{aligned}
&Hom(K, R\mathcal{H}om^*(K_1 \overset{*}{\otimes} K_2, K_3)) \\
&\quad \simeq Hom(K \overset{*}{\otimes} (K_1 \overset{*}{\otimes} K_2), K_3) \\
&\quad \simeq Hom((K \overset{*}{\otimes} K_1) \overset{*}{\otimes} K_2, K_3), \quad \text{by the first isomorphism} \\
&\quad \simeq Hom(K \overset{*}{\otimes} K_1, R\mathcal{H}om^*(K_2, K_3)) \\
&\quad \simeq Hom(K, R\mathcal{H}om^*(K_1, R\mathcal{H}om^*(K_2, K_3))).
\end{aligned}$$

So we have the second equivalence by Yoneda. □

Proposition 1.6. *Let* $K, K_1, K_2 \in D^b(k_{M\times\mathbb{R}})$ *and* $L \in D^b(k_M)$. *One has*

$$\pi^{-1}L \otimes (K_1 \overset{*}{\otimes} K_2) \simeq (\pi^{-1}L \otimes K_1) \overset{*}{\otimes} K_2,$$
$$R\mathcal{H}om(\pi^{-1}L, R\mathcal{H}om^*(K_1, K_2)) \simeq R\mathcal{H}om^*(\pi^{-1}L \otimes K_1, K_2).$$

In particular,

$$\pi^{-1}L \otimes K \simeq (\pi^{-1}L \otimes k_{\{t=0\}}) \overset{*}{\otimes} K,$$
$$R\mathcal{H}om(\pi^{-1}L, H) \simeq R\mathcal{H}om^*(\pi^{-1}L \otimes k_{\{t=0\}}, K).$$

Proof. For the first equivalence:

$$\begin{aligned}(\pi^{-1}L \otimes K_1) \overset{*}{\otimes} K_2 &= R\mu_!(q_1^{-1}(\pi^{-1}L \otimes K_1) \otimes q_2^{-1}K_2)\\ &\simeq R\mu_!(q_1^{-1}\pi^{-1}L \otimes q_1^{-1}K_1 \otimes q_2^{-1}K_2)\\ &\simeq R\mu_!(\mu^{-1}\pi^{-1}L \otimes q_1^{-1}K_1 \otimes q_2^{-1}K_2)\\ &\simeq \pi^{-1}L \otimes R\mu_!(q_1^{-1}K_1 \otimes q_2^{-1}K_2)\\ &= \pi^{-1}L \otimes (K_1 \overset{*}{\otimes} K_2).\end{aligned}$$

For the second one:

$$\begin{aligned}&R\mathcal{H}om(\pi^{-1}L, R\mathcal{H}om^*(K_1, K_2))\\ &\quad= R\mathcal{H}om(\pi^{-1}L, Rq_{1*}R\mathcal{H}om(q_2^{-1}K_1, \mu^!K_2))\\ &\quad\simeq Rq_{1*}R\mathcal{H}om(q_1^{-1}\pi^{-1}L, R\mathcal{H}om(q_2^{-1}K_1, \mu^!K_2))\\ &\quad\simeq Rq_{1*}R\mathcal{H}om(q_2^{-1}\pi^{-1}L, R\mathcal{H}om(q_2^{-1}K_1, \mu^!K_2))\\ &\quad\simeq Rq_{1*}R\mathcal{H}om(q_2^{-1}\pi^{-1}L \otimes q_2^{-1}K_1, \mu^!K_2)\\ &\quad\simeq Rq_{1*}R\mathcal{H}om(q_2^{-1}(\pi^{-1}L \otimes K_1), \mu^!K_2)\\ &\quad= R\mathcal{H}om^*(\pi^{-1}L \otimes K_1, K_2).\end{aligned}$$

The last isomorphisms follows from the first ones and Remark (1.3), with $K_1 = k_{\{t=0\}}$ and $K_2 = K$. □

Proposition 1.7. *Let* $K, K_1, K_2, K_3 \in D^b(k_{M\times\mathbb{R}})$ *and* $L \in D^b(k_M)$. *One has*

$$R\pi_* R\mathcal{H}om(K_1 \overset{*}{\otimes} K_2, K_3) \simeq R\pi_* R\mathcal{H}om(K_1, R\mathcal{H}om^*(K_2, K_3)),$$
$$R\mathcal{H}om^*(K, \pi^! L) \simeq \pi^! R\mathcal{H}om(R\pi_! K, L).$$

Proof. For the first equivalence:

$$\begin{aligned} &R\pi_* R\mathcal{H}om(K_1 \overset{*}{\otimes} K_2, K_3) \\ &\quad = R\pi_* R\mathcal{H}om(R\mu_!(q_1^{-1}K_1 \otimes q_2^{-1}K_2), K_3) \\ &\quad \simeq R\pi_* R\mu_* R\mathcal{H}om(q_1^{-1}K_1 \otimes q_2^{-1}K_2, \mu^! K_3) \\ &\quad \simeq R\pi_* Rq_{1*} R\mathcal{H}om(q_1^{-1}K_1, R\mathcal{H}om(q_2^{-1}K_2, \mu^! K_3)) \\ &\quad \simeq R\pi_* R\mathcal{H}om(K_1, Rq_{1*} R\mathcal{H}om(q_2^{-1}K_2, \mu^! K_3)) \\ &\quad = R\pi_* R\mathcal{H}om(K_1, R\mathcal{H}om^*(K_2, K_3)). \end{aligned}$$

Instead, for the second one:

$$\begin{aligned} R\mathcal{H}om^*(K, \pi^! L) &= Rq_{1*} R\mathcal{H}om(q_2^{-1}K, \mu^! \pi^! L) \\ &\simeq Rq_{1*} R\mathcal{H}om(q_2^{-1}K, q_1^! \pi^! L) \\ &\simeq R\mathcal{H}om(Rq_{1!} q_2^{-1}K, \pi^! L) \\ &\simeq R\mathcal{H}om(\pi^{-1} R\pi_! K, \pi^! L) \\ &\simeq \pi^! R\mathcal{H}om(R\pi_! K, L). \end{aligned}$$

□

Proposition 1.8. *Let* $K_1, K_2 \in D^b(k_{M\times\mathbb{R}})$. *One has*

$$R\pi_!(K_1 \overset{*}{\otimes} K_2) \simeq R\pi_! K_1 \otimes R\pi_! K_2,$$
$$R\pi_* R\mathcal{H}om^*(K_1, K_2) \simeq R\mathcal{H}om(R\pi_! K_1, R\pi_* K_2).$$

Proof. For the first equivalence:

$$\begin{aligned} R\pi_!(K_1 \overset{*}{\otimes} K_2) &= R\pi_! R\mu_!(q_1^{-1}K_1 \otimes q_2^{-1}K_2) \\ &\simeq R\pi_! Rq_{1!}(q_1^{-1}K_1 \otimes q_2^{-1}K_2) \\ &\simeq R\pi_!(K_1 \otimes Rq_{1!}q_2^{-1}K_2) \\ &\simeq R\pi_!(K_1 \otimes \pi^{-1}R\pi_!K_2) \\ &\simeq R\pi_!K_1 \otimes R\pi_!K_2. \end{aligned}$$

Instead, for the second one:

$$\begin{aligned} R\pi_* R\mathcal{H}om^*(K_1, K_2) &= R\pi_* Rq_{1*} R\mathcal{H}om(q_2^{-1}K_1, \mu^! K_2) \\ &\simeq R\pi_* R\mu_* R\mathcal{H}om(q_2^{-1}K_1, \mu^! K_2) \\ &\simeq R\pi_* R\mathcal{H}om(R\mu_! q_2^{-1}K_1, K_2) \\ &\simeq R\pi_* R\mathcal{H}om(\pi^{-1}R\pi_!K_1, K_2) \\ &\simeq R\mathcal{H}om(R\pi_!K_1, R\pi_*K_2). \end{aligned}$$

□

Proposition 1.9. *Let $L \in D^b(k_M)$ and $H \in D^b(k_{M\times\mathbb{R}})$. Consider $a : M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $a(x,t) = (x,-t)$. One has*

$$R\mathcal{H}om^*(H, \pi^{-1}L \otimes k_{\{t=0\}}) \simeq Ra_* R\mathcal{H}om(H, \pi^! L).$$

Proof. Consider

$$M \xrightarrow{i_0} M \times \mathbb{R} \xrightarrow{\delta^a} M \times \mathbb{R}^2,$$

defined by $i_0(x) = (x,0)$ and $\delta^a(x,t) = (x,-t,t)$. Then the following diagram is Cartesian:

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{\delta^a} & M \times \mathbb{R}^2 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mu} \\ M & \xrightarrow{i_0} & M \times \mathbb{R} \end{array}.$$

Notice that $k_{\{t=0\}} \simeq Ri_{0*}k_M$. Then:

$$\begin{aligned} R\mathcal{H}om^*(H, \pi^{-1}L \otimes k_{\{t=0\}}) &\simeq R\mathcal{H}om^*(H, Ri_{0*}L) \\ &\simeq Rq_{1*}R\mathcal{H}om(q_2^{-1}H, \mu^! Ri_{0*}L) \\ &\simeq Rq_{1*}R\mathcal{H}om(q_2^{-1}H, R\delta^a_* \pi^! L) \\ &\simeq Rq_{1*}R\delta^a_* R\mathcal{H}om(\delta^{a-1} q_2^{-1}H, \pi^! L). \end{aligned}$$

One can conclude since $q_1 \circ \delta^a = a$ and $q_2 \circ \delta^a = id_{M\times\mathbb{R}}$. □

1.2.2 Enhanced sheaves

Recall that if $\mathcal{P}$ is a full triagulated subcategory of a triangulated category $\mathcal{Q}$, the *quotient* category $\mathcal{Q}/\mathcal{P}$ is defined as the localization $\mathcal{Q}_\Sigma$ of $\mathcal{Q}$ with respect to the multiplicative system Σ of morphism u fitting into a distinguished triangle $X \xrightarrow{u} Y \to Z \xrightarrow{+1}$, with $Z \in \mathcal{P}$.

Let $\pi : M \times \mathbb{R} \to M$ be the projection.
Consider the full subcategories of $D^b(k_{M\times\mathbb{R}})$

$$\begin{aligned} \mathcal{N}_\pm &:= \{K \in D^b(k_{M\times\mathbb{R}}) : k_{\{\mp t \geq 0\}} \overset{*}{\otimes} K \simeq 0\} \\ &= \{K \in D^b(k_{M\times\mathbb{R}}) : R\mathcal{H}om^*(k_{\{\mp t\geq 0\}}, K) \simeq 0\}, \\ \mathcal{N} &:= \mathcal{N}_+ \cap \mathcal{N}_- = \pi^{-1}D^b(k_M). \end{aligned}$$

Definition 1.10. The categories of *enhanced sheaves* are defined by

$$\begin{aligned} E^b_\pm(k_M) &:= D^b(k_{M\times\mathbb{R}})/\mathcal{N}_\mp, \\ E^b(k_M) &:= D^b(k_{M\times\mathbb{R}})/\mathcal{N}. \end{aligned}$$

Recall that if $\mathcal{P}$ is a triangulated subcategory of $\mathcal{Q}$, the *right orthogonal* $\mathcal{P}^\perp$ and the *left orthogonal* ${}^\perp\mathcal{P}$ are the full subcategories of $\mathcal{Q}$ defined by

$$\begin{aligned}\mathcal{P}^{\perp} &= \{X \in \mathcal{Q} \colon Hom_{\mathcal{Q}}(Y, X) \simeq 0 \quad \forall Y \in \mathcal{P}\},\\ {}^{\perp}\mathcal{P} &= \{X \in \mathcal{Q} \colon Hom_{\mathcal{Q}}(X, Y) \simeq 0 \quad \forall Y \in \mathcal{P}\}.\end{aligned}$$

One has

$$\begin{aligned}\tilde{E}^b_{\pm}(k_M) &:= {}^{\perp}\mathcal{N}_{\mp} = \{H \colon k_{\{\pm t \geq 0\}} \overset{*}{\otimes} H \xrightarrow{\sim} H\},\\ \tilde{E}^b(k_M) &:= {}^{\perp}\mathcal{N} = \{H \colon \left(k_{\{t \geq 0\}} \oplus k_{\{t \leq 0\}}\right) \overset{*}{\otimes} H \xrightarrow{\sim} H\}.\end{aligned}$$

The same equalities hold for right orthogonals, replacing $\overset{*}{\otimes}$ with $R\mathcal{H}om^*$.

The quotient functor

$$Q : D^b(k_{M\times\mathbb{R}}) \to E^b(k_M),$$

has fully faithful left and right adjoints, respectively given by

$$\begin{aligned}L^E(QF) &= (k_{\{t\geq 0\}} \oplus k_{\{t\leq 0\}}) \overset{*}{\otimes} F \quad \in {}^{\perp}\mathcal{N},\\ R^E(QF) &= R\mathcal{H}om^*\left(k_{\{t\geq 0\}} \oplus k_{\{t\leq 0\}}, F\right) \quad \in \mathcal{N}^{\perp},\end{aligned} \tag{1.2}$$

for $F \in D^b(k_{M\times\mathbb{R}})$.

One has ${}^{\perp}\mathcal{N}_+ \oplus {}^{\perp}\mathcal{N}_- \simeq {}^{\perp}\mathcal{N}$, so there are natural equivalences

$$\begin{aligned}E^b_{\pm}(k_M) &\simeq \mathcal{N}_{\mp}/\mathcal{N} \simeq {}^{\perp}\mathcal{N}_{\mp} \simeq \tilde{E}^b_{\pm}(k_M),\\ E^b(k_M) &\simeq {}^{\perp}\mathcal{N} \simeq E^b_+(k_M) \oplus E^b_-(k_M).\end{aligned} \tag{1.3}$$

The same equivalences hold when replacing left with right orthogonals.

Notice that these results say us that the objects $F \in E^b_{\pm}(k_M)$ are such that

$$k_{\{\pm t\geq 0\}} \overset{*}{\otimes} F \simeq F \text{ and } k_{\{\mp t\geq 0\}} \overset{*}{\otimes} F \simeq 0,$$

and the objects $F \in E^b(k_M)$ are such that

$$(k_{\{t\geq 0\}} \oplus k_{\{t\leq 0\}}) \overset{*}{\otimes} F \simeq F.$$

(1.3) also shows that the categories of enhanced sheaves defined, are triangulated categories.

Operations on enhanced sheaves

Let $f : M \to N$ be a morphism of good topological spaces. Enhanced sheaves are endowed with the six operations $\overset{*}{\otimes}$, $R\mathcal{H}om^*$, Ef^{-1}, Ef_*, $Ef_!$, $Ef^!$. Here, $\overset{*}{\otimes}$ and $R\mathcal{H}om^*$ descend from $D^b(k_{M\times\mathbb{R}})$, and the exterior operations are defined by $Ef_!(QF) = Q(R\tilde{f}_!F)$, $Ef^{-1}(QG) = Q(\tilde{f}^{-1}G)$, ..., where we set $\tilde{f} = f \times id_{\mathbb{R}}$.

There is a natural embedding

$$\begin{aligned} \epsilon : D^b(k_M) &\to E^b_+(k_M) \subset E^b(k_M) \\ F &\mapsto k_{\{t\geq 0\}} \otimes \pi^{-1}F, \end{aligned} \tag{1.4}$$

that is well defined since, by Proposition 1.6 and by the idempotency of $k_{\{t\geq 0\}}$, one has

$$k_{\{t\geq 0\}} \overset{*}{\otimes} (k_{\{t\geq 0\}} \otimes \pi^{-1}F) \simeq (k_{\{t\geq 0\}} \overset{*}{\otimes} k_{\{t\geq 0\}}) \otimes \pi^{-1}F \simeq k_{\{t\geq 0\}} \otimes \pi^{-1}F.$$

Proposition 1.11. *If $f : M \to N$ is a morphism of good topological spaces, then ϵ interchanges the operations $\otimes$, f^{-1}, and $Rf_!$ with $\overset{*}{\otimes}$, $\tilde{f}^{-1}$, and $R\tilde{f}_!$, respectively.*

Proof. Consider the diagram

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{\pi} & M \\ \big\downarrow{\scriptstyle \tilde{f}} & & \big\downarrow{\scriptstyle f} \\ N \times \mathbb{R} & \xrightarrow{\pi} & N. \end{array}$$

This is a Cartesian square, so one has

$$f^{-1}R\pi_! \simeq R\pi_!\tilde{f}^{-1}, \quad \pi^{-1}Rf_! \simeq R\tilde{f}_!\pi^{-1},$$
$$R\tilde{f}_*\pi^! \simeq \pi^! Rf_*, \quad R\pi_*\tilde{f}^! \simeq f^! R\pi_*.$$

Let $F, G \in D^b(k_M)$.

- By Proposition 1.6, for the product one has

$$\begin{aligned}\epsilon(F) \overset{*}{\otimes} \epsilon(G) &= (k_{\{t\geq 0\}} \otimes \pi^{-1}F) \overset{*}{\otimes} (k_{\{t\geq 0\}} \otimes \pi^{-1}G) \\ &\simeq (k_{\{t\geq 0\}} \overset{*}{\otimes} k_{\{t\geq 0\}}) \otimes \pi^{-1}F \otimes \pi^{-1}G \\ &\simeq k_{\{t\geq 0\}} \otimes \pi^{-1}(F \otimes G) = \epsilon(F \otimes G).\end{aligned}$$

- Since $\tilde{f}$ doesn't act on $t \in \mathbb{R}$, then $k_{\{t\geq 0\}} \simeq \tilde{f}^{-1}k_{\{t\geq 0\}}$ (denoting in the same way $k_{M\times\{t\geq 0\}}$ and $k_{N\times\{t\geq 0\}}$). So,

$$\begin{aligned}\epsilon(Rf_!F) &= k_{\{t\geq 0\}} \otimes \pi^{-1}Rf_!F \\ &\simeq k_{\{t\geq 0\}} \otimes R\tilde{f}_!\pi^{-1}F \simeq \tilde{f}_!(\epsilon(F)), \quad \text{and} \\ \epsilon(f^{-1}F) &= k_{\{t\geq 0\}} \otimes \pi^{-1}f^{-1}F \\ &\simeq \tilde{f}^{-1}k_{\{t\geq 0\}} \otimes \tilde{f}^{-1}\pi^{-1}F \simeq \tilde{f}^{-1}(\epsilon(F)).\end{aligned}$$

□

Notice that, since the external operations on enhanced sheaves are induced by the ones in $D^b(k_{M\times\mathbb{R}})$, the compatibility with composition of morphisms and the equivalences between the morphisms that compose a Cartesian square are satisfied also at level of enhanced sheaves.

There are also properties that connect the different Grothendieck operations.

Proposition 1.12. *Let $f : M \to N$ be a morphism of good topological spaces. Let $F \in D^b(k_{M\times\mathbb{R}})$ and $G, H \in D^b(k_{N\times\mathbb{R}})$. One has*

$$\begin{aligned}
Ef_!(Ef^{-1}(QG) \overset{*}{\otimes} (QF)) &\simeq (QG) \overset{*}{\otimes} Ef_!(QF),\\
Ef^{-1}((QG) \overset{*}{\otimes} (QH)) &\simeq Ef^{-1}(QG) \overset{*}{\otimes} Ef^{-1}(QH),\\
R\mathcal{H}om^*((QG), Ef_*(QF)) &\simeq Ef_* R\mathcal{H}om^*(Ef^{-1}(QG), (QF)),\\
R\mathcal{H}om^*(Ef_!(QF), (QG)) &\simeq Ef_* R\mathcal{H}om^*((QF), Ef^!(QG)),\\
Ef^! R\mathcal{H}om^*((QG), (QH)) &\simeq R\mathcal{H}om^*(Ef^{-1}(QG), Ef^!(QH)).
\end{aligned}$$

Proof. To avoid confusion, denote by $q_1, q_2, \mu : M \times \mathbb{R}^2 \to M \times \mathbb{R}$ the usual operations and by ${q'}_1, {q'}_2, \mu' : N \times \mathbb{R}^2 \to N \times \mathbb{R}$ the same in N. Let Q be the quotient functor both in M and in N, for semplicity. Let $\bar{f} = f \times id_{\mathbb{R}} \times id_{\mathbb{R}}$.
One has the Cartesian square

$$\begin{array}{ccc}
M \times \mathbb{R}^2 & \xrightarrow{\mu} & M \times \mathbb{R} \\
\downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \tilde{f}} \\
N \times \mathbb{R}^2 & \xrightarrow{\mu'} & N \times \mathbb{R}.
\end{array}$$

It remains Cartesian also is we replace μ, μ' with either $q_1, {q'}_1$ or $q_2, {q'}_2$.
Now we use these facts and Note 1.4 to prove the proposition.

(i)

$$\begin{aligned}
(QG) \overset{*}{\otimes} Ef_!(QF) &\simeq Q(R{\mu'}_!({q'}_1^{-1} G \otimes {q'}_2^{-1} R\tilde{f}_! F))\\
&\simeq Q(R{\mu'}_!({q'}_1^{-1} G \otimes R\bar{f}_! q_2^{-1} F))\\
&\simeq Q(R{\mu'}_! R\bar{f}_!(\bar{f}^{-1} {q'}_1^{-1} G \otimes q_2^{-1} F))\\
&\simeq Q(R\tilde{f}_! R\mu_!(q_1^{-1} \tilde{f}^{-1} G \otimes q_2^{-1} F))\\
&\simeq Ef_!(Ef^{-1}(QG) \overset{*}{\otimes} (QF)).
\end{aligned}$$

(ii)

$$
\begin{aligned}
Ef^{-1}((QG) \overset{*}{\otimes} (QH)) &\simeq Q(\tilde{f}^{-1} R\mu'_!(q_1'^{-1} G \otimes q_2'^{-1} H)) \\
&\simeq Q(R\mu_! \bar{f}^{-1}(q_1'^{-1} G \otimes q_2'^{-1} H)) \\
&\simeq Q(R\mu_!(\bar{f}^{-1} q_1'^{-1} G \otimes \bar{f}^{-1} q_2'^{-1} H)) \\
&\simeq Q(R\mu_!(q_1^{-1} \tilde{f}^{-1} G \otimes q_2^{-1} \tilde{f}^{-1} H)) \\
&= Ef^{-1}(QG) \overset{*}{\otimes} Ef^{-1}(QH).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
R\mathcal{H}om^*((QG), Ef_*(QF)) &\simeq Q(Rq'_{1*} R\mathcal{H}om(q_2'^{-1} G, \mu'^! R\tilde{f}_* F)) \\
&\simeq Q(Rq'_{1*} R\mathcal{H}om(q_2'^{-1} G, R\bar{f}_* \mu^! F)) \\
&\simeq Q(Rq'_{1*} R\bar{f}_* R\mathcal{H}om(\bar{f}^{-1} q_2'^{-1} G, \mu^! F)) \\
&\simeq Q(R\tilde{f}_* Rq_{1*} R\mathcal{H}om(q_2^{-1} \tilde{f}^{-1} G, \mu^! F)) \\
&\simeq Ef_* R\mathcal{H}om^*(Ef^{-1}(QG), (QF)).
\end{aligned}
$$

(iv)

$$
\begin{aligned}
R\mathcal{H}om^*(Ef_!(QF), (QG)) &\simeq Q(Rq'_{1*} R\mathcal{H}om(q_2'^{-1} R\tilde{f}_! F, \mu'^! G)) \\
&\simeq Q(Rq'_{1*} R\mathcal{H}om(R\bar{f}_! q_2^{-1} F, \mu'^! G)) \\
&\simeq Q(Rq'_{1*} R\bar{f}_* R\mathcal{H}om(q_2^{-1} F, \bar{f}^! \mu'^! G)) \\
&\simeq Q(R\tilde{f}_* Rq_{1*} R\mathcal{H}om(q_2^{-1} F, \mu^! \tilde{f}^! G)) \\
&\simeq Ef_* R\mathcal{H}om^*((QF), Ef^!(QG)).
\end{aligned}
$$

(v)

$$\begin{aligned} Ef^! R\mathcal{H}om^*((QG),(QH)) &\simeq Q(\tilde{f}^! R{q'}_{1*} R\mathcal{H}om({q'}_2^{-1}G, {\mu'}^! F)) \\ &\simeq Q(Rq_{1*}\bar{f}^! R\mathcal{H}om({q'}_2^{-1}G, {\mu'}^! F)) \\ &\simeq Q(Rq_{1*} R\mathcal{H}om(\bar{f}^{-1}{q'}_2^{-1}G, \bar{f}^! {\mu'}^! F)) \\ &\simeq Q(Rq_{1*} R\mathcal{H}om(q_2^{-1}\tilde{f}^{-1}G, \mu^! \tilde{f}^! F)) \\ &\simeq R\mathcal{H}om^*(Ef^{-1}(QG), Ef^!(QG)). \end{aligned}$$

□

1.3 Enhanced ind-sheaves

Let M be a good topological space. The derived category of enhanced ind-sheaves on M is defined as a quotient of the derived category of ind-sheaves on the bordered space $M \times \mathbb{R}_\infty$. Let's mention here these notions even if they are beyond the scope of this thesis.

References are made to [5] for ind-sheaves, and to [3], [4] and [6] for bordered spaces and enhanced ind-sheaves.

1.3.1 Ind-sheaves

Let $\mathcal{C}$ be a category and $\mathcal{C}^\vee = Fct(\mathcal{C}^{op}, \mathcal{S}et)$ be the category of contravariant functors from $\mathcal{C}$ to the category of sets. Denote $h : \mathscr{C} \to \mathscr{C}^\vee$ the Yoneda embedding given by $X \mapsto Hom_{\mathscr{C}}(*, X)$.

We denote by $"\varinjlim"$ the inductive limit in $\mathscr{C}^\vee$, i.e. if I is a small category and $\alpha : I \to \mathscr{C}$ is a functor, we set $"\varinjlim"\alpha = \varinjlim(h \circ \alpha)$. In other words, $"\varinjlim"\alpha$ is the object of $\mathscr{C}^\vee$ defined by:

$$"\varinjlim"\alpha : \mathscr{C} \ni X \mapsto \varinjlim_{i \in I} Hom_{\mathscr{C}}(X, \alpha(i)).$$

Definition 1.13. An object F of $\mathcal{C}^\vee$ is an *ind-object* if there exists a small filtrant category $\mathcal{J}$ and an inductive system $F' : \mathcal{J} \to \mathcal{C}$ such that $F \simeq \text{"}\varinjlim_{\mathcal{J}}\text{"} F'$.

Denote by $Ind(\mathcal{C})$ the full subcategory of $\mathcal{C}^\vee$ consisting of ind-objects in $\mathcal{C}$.

Definition 1.14. An *ind-sheaf* is an ind-object in the category of sheaves with compact support.

Denote by $I(k_M) = Ind(Mod_c(k_M))$ the category of ind-sheaves, where $Mod_c(k_M)$ is the category of sheaves with compact support. It is an abelian tensor category with $\otimes$ as a tensor product and k_M as a unit object.

There is a natural embedding of sheaves into ind-sheaves:

$$\begin{aligned} \iota : Mod(k_M) &\to I(k_M) \\ F &\mapsto \text{"}\varinjlim_{U}\text{"}(k_U \otimes F), \end{aligned}$$

for U running over the relatively compact open subsets of M. It is an exact and fully faithful functor and it has an exact left adjoint α given by

$$\begin{aligned} \alpha : I(k_M) &\to Mod(k_M) \\ \text{"}\varinjlim\text{"} F_i &\mapsto \varinjlim F_i. \end{aligned} \tag{1.5}$$

The functor α has an exact fully faithful left adjoint, denoted β.

Denote by $D^b(Ik_M)$ the bounded derived category of ind-sheaves and, for a morphism $f : M \to N$ of good topological spaces, denote by $\otimes$, $R\mathcal{I}hom$, f^{-1}, Rf_*, $Rf_{!!}$, $f^!$ the six Grothendieck operations for ind-sheaves.

Remark 1.15. *In* $I(k_M)$ *there is a sheaf-valued hom-functor*

$$\mathcal{H}om := \alpha \circ \mathcal{I}hom,$$

such that $\Gamma(M; \mathcal{H}om(F, G)) \simeq Hom_{I(k_M)}(F, G)$.

1.3.2 Ind-sheaves on bordered spaces

Definition 1.16. A *bordered space* is a pair $M_\infty = (M, \check{M})$ of a good topological space $\check{M}$ and an open subset $M \subset \check{M}$.

A morphism $f : M_\infty \to N_\infty$ is a continuous map $f : M \to N$ such that the first projection $\check{M} \times \check{N} \to \check{M}$ is proper on the closure $\bar{\Gamma}_f$ of the graph Γ_f of f. This assumption is satisfied in particular if either $M = \check{M}$ or $\check{N}$ is compact.
Moreover the composition of two morphism is the composition of the underlying continuous maps.
In this way, we have constructed the category of bordered spaces.

The category of topological spaces embeds into that of bordered spaces by the identification $M = (M, M)$.

Definition 1.17. A morphism $f : M_\infty \to N_\infty$ is called *semi-proper*, if the second projection $\bar{\Gamma}_f \to \check{N}$ is proper.
f is *proper* if $f : M \to N$ is proper.

Definition 1.18. A subset S of a bordered space $M_\infty = (M, \check{M})$ is a subset of M. We say that S is open (resp. closed, locally closed) if it is so in M.
We say that S is *relatively compact* if it is contained in a compact subset of $\check{M}$.

The triangulated category of ind-sheaves on M_∞ is defined by

$$D^b(Ik_{M_\infty}) = D^b(Ind(Mod_c(k_{M_\infty}))),$$

where $Mod_c(k_{M_\infty}) \subset Mod(k_M)$ is the full subcategory of sheaves on M whose support is relatively compact in M_∞.

There is a natural equivalence of triangulated categories

$$D^b(Ik_{M_\infty}) \simeq D^b(Ik_{\check{M}})/D^b(Ik_{\check{M}\setminus M}).$$

The quotient functor $\mathsf{q} : D^b(Ik_{\check{M}}) \to D^b(Ik_{M_\infty})$ has a left adjoint l and a right adjoint r, both fully faithful, given by

$$\mathsf{l}(\mathsf{q}F) = k_M \otimes F, \qquad \mathsf{r}(\mathsf{q}F) = R\mathcal{I}hom(k_M, F).$$

For $f : M_\infty \to N_\infty$ a morphism of bordered spaces, the six operations for ind-sheaves on bordered spaces are defined by

$$\begin{aligned} \mathsf{q}F_1 \otimes \mathsf{q}F_2 &= \mathsf{q}(F_1 \otimes F_2), \quad R\mathcal{I}hom(\mathsf{q}F_1, \mathsf{q}F_2) = \mathsf{q}R\mathcal{I}hom(F_1, F_2), \\ Rf_{!!}(\mathsf{q}F) &= \mathsf{q}Rq_{!!}(k_{\Gamma_f} \otimes p_1^{-1}F), \quad Rf_*(\mathsf{q}F) = \mathsf{q}Rq_*R\mathcal{I}hom(k_{\Gamma_f}, p_1^!F), \\ f^{-1}(\mathsf{q}G) &= \mathsf{q}Rp_{!!}(k_{\Gamma_f} \otimes p_2^{-1}G), \quad f^!(\mathsf{q}G) = \mathsf{q}Rp_{1*}R\mathcal{I}hom(k_{\Gamma_f}, p_2^!G), \end{aligned}$$

where $p_1 : \check{M} \times \check{N} \to \check{M}$ and $p_2 : \check{M} \times \check{N} \to \check{N}$ are the projections.

There is a natural exact embedding

$$\begin{aligned} \iota_M : D^b(k_M) &\to D^b(Ik_{M_\infty}) \\ F &\mapsto \text{“}\varinjlim_{U}\text{”}(k_U \otimes F), \end{aligned}$$

for U running over the family of relatively compact open subsets of M_∞.

1.3.3 Ind-sheaves with an extra variables

Notation 1.19. *Let* $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ *be the two-point compactification of the affine line. The* bordered line *is*

$$\mathbb{R}_\infty := (\mathbb{R}, \bar{\mathbb{R}}).$$

Note 1.20. Let $P = \mathbb{R} \sqcup \{\infty\}$ be the real projective line. $\mathbb{R}_\infty$ is isomorphic to $(\mathbb{R}, P)$ as a bordered space.

Let M_∞ be a bordered space. Consider the morphisms

$$\mu, q_1, q_2 : M_\infty \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty,$$

where $\mu(x, t_1, t_2) = (x, t_1 + t_2)$ and q_1, q_2 are the natural projections.

Definition 1.21. The convolution functors are defined as follows:

$$\overset{+}{\otimes} : D^b(Ik_{M_\infty \times \mathbb{R}_\infty}) \times D^b(Ik_{M_\infty \times \mathbb{R}_\infty}) \to D^b(Ik_{M_\infty \times \mathbb{R}_\infty})$$
$$(F_1, F_2) \mapsto R\mu_{!!}(q_1^{-1} F_1 \otimes q_2^{-1} F_2),$$

$$\mathcal{I}hom^+ : D^b(Ik_{M_\infty \times \mathbb{R}_\infty})^{op} \times D^b(Ik_{M_\infty \times \mathbb{R}_\infty}) \to D^b(Ik_{M_\infty \times \mathbb{R}_\infty})$$
$$(F_1, F_2) \mapsto Rq_{1*} R\mathcal{I}hom(q_2^{-1} F_1, \mu^! F_2).$$

The category $D^b(Ik_{M_\infty \times \mathbb{R}_\infty})$ has a structure of commutative tensor category with $\overset{+}{\otimes}$ as a tensor product and $k_{\{t=0\}}$ as unit object.

1.3.4 Enhanced ind-sheaves

Denote by $\pi_\infty : M_\infty \times \mathbb{R}_\infty \to M_\infty$ the projection. As we did for enhanced sheaves, consider the full subcategories of $D^b\left(Ik_{M_\infty \times \mathbb{R}_\infty}\right)$

$$\begin{aligned} \mathcal{N}_\pm &:= \{K \in D^b\left(Ik_{M_\infty \times \mathbb{R}_\infty}\right) : k_{\{\mp t \geq 0\}} \overset{+}{\otimes} K \simeq 0\} \\ &= \{K \in D^b\left(Ik_{M_\infty \times \mathbb{R}_\infty}\right) : \mathcal{I}hom^+(k_{\{\mp t \geq 0\}}, K) \simeq 0\}, \\ \mathcal{N} &:= \mathcal{N}_+ \cap \mathcal{N}_- = \pi_\infty^{-1} D^b(Ik_{M_\infty}). \end{aligned}$$

Definition 1.22. The categories of *enhanced ind-sheaves* are defined by

$$\begin{aligned} E^b_\pm(Ik_{M_\infty}) &:= D^b\left(Ik_{M_\infty \times \mathbb{R}_\infty}\right) / \mathcal{N}_\mp, \\ E^b(Ik_{M_\infty}) &:= D^b\left(Ik_{M_\infty \times \mathbb{R}_\infty}\right) / \mathcal{N}. \end{aligned}$$

One has

$$
\begin{aligned}
{}^{\perp}\mathcal{N}_{\pm} &= \{H \colon k_{\{\mp t \geq 0\}} \overset{+}{\otimes} H \xrightarrow{\sim} H\},\\
{}^{\perp}\mathcal{N} &= \{H \colon \left(k_{\{t \geq 0\}} \oplus k_{\{t \leq 0\}}\right) \overset{+}{\otimes} H \xrightarrow{\sim} H\},
\end{aligned}
$$

and the same equalities hold for right orthogonals, replacing $\overset{+}{\otimes}$ with $\mathcal{I}hom^{+}$.

Denote $Q : D^b(Ik_{M_\infty \times \mathbb{R}_\infty}) \to E^b(Ik_M)$ the quotient functor. It has a left adjoint L^E and a right adjoint R^E, given by

$$
\begin{aligned}
L^E(H) &= \left(k_{\{t \geq 0\}} \oplus k_{\{t \leq 0\}}\right) \overset{+}{\otimes} H \quad \in {}^{\perp}\mathcal{N},\\
R^E(H) &= \mathcal{I}hom^{+}\left(k_{\{t \geq 0\}} \oplus k_{\{t \leq 0\}}, H\right) \quad \in \mathcal{N}^{\perp},
\end{aligned}
$$

for $H \in E^b(Ik_M)$.

One has ${}^{\perp}\mathcal{N}_{+} \oplus {}^{\perp}\mathcal{N}_{-} \simeq {}^{\perp}\mathcal{N}$, so there are natural equivalences

$$
\begin{aligned}
E^b_{\pm}(Ik_{M_\infty}) &\simeq \mathcal{N}_{\mp}/\mathcal{N} \simeq {}^{\perp}\mathcal{N}_{\mp},\\
E^b(Ik_{M_\infty}) &\simeq {}^{\perp}\mathcal{N} \simeq E^b_{+}(Ik_{M_\infty}) \oplus E^b_{-}(Ik_{M_\infty}).
\end{aligned}
$$

The same equivalences hold when replacing left with right orthogonals.

The category $E^b(Ik_M)$ is endowed with an analogue of the convolution functors, denoted again $\overset{+}{\otimes}$ and $\mathcal{I}hom^{+}$. Moreover, if $f : M_\infty \to N_\infty$ is a morphism of bordered spaces, we have Ef^{-1}, Ef_*, $Ef_{!!}$, $Ef^{!}$ that are induced by $\tilde{f}$ at the level of ind-sheaves.
Moreover, $E^b(Ik_M)$ has a natural hom-functor $\mathcal{H}om^E$ with values in $D^b(k_M)$, given by

$$
\begin{aligned}
\mathcal{H}om^E : E^b(Ik_M)^{op} \times E^b(Ik_M) &\to D^b(k_M)\\
(K_1, K_2) &\mapsto \alpha R\pi_* R\mathcal{I}hom(L^E K_1, L^E K_2),
\end{aligned}
$$

where α induced by (1.5).

Set

$$k_M^E = \text{“} \varinjlim_{a \to +\infty} \text{”} k_{\{t \geq a\}} \in E^b(Ik_M).$$

The duality functor for enhanced ind-sheaves is defined by

$$\begin{aligned} D_M^E : E^b(Ik_M) &\to E^b(Ik_M)^{op} \\ K &\mapsto \mathcal{I}hom^+ \left(K, \omega_M^E\right), \end{aligned}$$

where we set $\omega_M^E := k_M^E \otimes \pi^{-1}\omega_M$.

1.4 ℝ-construibility

Proofs of the propositions of this section can be found in [7], Chapter VIII.

1.4.1 Subanalytic sets

Let Z be a subset of X, real analytic manifold.

Definition 1.23. One says Z is *subanalytic* at $x \in X$ if there exist an open neighborhood U of x, compact manifolds $Y_j^i (i = 1, 2, 1 \leq j \leq N)$ and morphisms $f_j^i : Y_j^i \to X$ such that:

$$Z \cap U = U \cap \cup_{j=1}^{N} (f_j^1(Y_j^1) \setminus f_j^2(Y_j^2)).$$

If Z is subanalytic at each $x \in X$, one says Z is subanalytic in X.

Subanalytic sets inherit the following properties.

Proposition 1.24. *(i) Assume Z is subanalytic in X. Then $\bar{Z}$ and $Int(Z)$ are subanalytic in X. Moreover the connected components of Z are locally finite and subanalytic.*

(ii) Assume Z_1 and Z_2 are subanalytic in X. Then $Z_1 \cup Z_2, Z_1 \setminus Z_2, Z_1 \cap Z_2$ are subanalytic.

(iii) Let $f : Y \to X$ be a morphism of manifolds. If $Z \subset X$ is subanalytic in X, then $f^{-1}(Z)$ is subanalytic in Y.
If $W \subset Y$ is subanalytic in Y and f is proper on $\bar{W}$, then $f(W)$ is subanalytic in X.

(iv) Let Z be a closed subanalytic subset of X. Then there exist a manifold Y and a proper morphism $f : Y \to X$ such that $f(Y) = Z$.

1.4.2 $\mathbb{R}$-constructible sheaves

Let k be a field and M be a real analytic manifold.

Definition 1.25. Let $F \in \mathrm{Ob}(D^b(M))$.
One says that F is $\mathbb{R}$-*constructible* if there exists a locally finite covering $M = \cup_{i\in I} X_i$ by subanalytic subsets such that for all $j \in \mathbb{Z}$ and all $i \in I$, both $F|_{X_i}$ and $H^j(F)|_{X_i}$ are locally constant of finite rank.

Example 1.26. If Z is a locally closed subanalytic subset of M, then the sheaf k_Z is $\mathbb{R}$-constructible.

Denote by $D^b_{\mathbb{R}\text{-}c}(k_M)$ the full subcategory of $D^b(k_M)$ whose objects have $\mathbb{R}$-constructible cohomologies.
This is a triangulated category that is closed under $\otimes$, $R\mathcal{H}om$ and the duality functor D_M.

Proposition 1.27. *Let $f : M \to N$ be a morphism of real analytic manifolds.*

(i) The functors f^{-1} and $f^!$ send $D^b_{\mathbb{R}\text{-}c}(k_N)$ to $D^b_{\mathbb{R}\text{-}c}(k_M)$.

(ii) If f is semiproper, then the functors $Rf_{!!}$ and Rf_* send $D^b_{\mathbb{R}\text{-}c}(k_M)$ to $D^b_{\mathbb{R}\text{-}c}(k_N)$.

Let $P = \mathbb{R} \cup \{\infty\}$ be the real projective line.
Denote by $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty})$ the full subcategory of $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}})$ whose objects F are such that $Rj_!F$ is $\mathbb{R}$-constructible in $M \times P$, where $j : M \times \mathbb{R} \to N \times \mathbb{R}$ is the embedding. We can also consider $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty})$ as a full subcategory of $D^b_{\mathbb{R}\text{-}c}(k_{M\times P})$, since $Rj_!$ is fully faithfull.

Definition 1.28. The triangulated category $\tilde{E}^b_{\mathbb{R}\text{-}c}(k_M)$ of $\mathbb{R}$-*constructible enhanced sheaves* is the full subcategory of $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty})$ whose objects F satisfy $F \simeq k_{\{t\geq 0\}} \overset{*}{\otimes} F$.

If $M_\infty = (M, \check{M})$ is a bordered space, the category $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_{M_\infty})$ of ℝ-*constructible enhanced ind-sheaves* is defined as the full subcategory of $E^b(\mathrm{I}k_{M_\infty})$ whose objects K satisfy the following property: for any relatively compact open subset $U \subset M$ there exists $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_M)$ such that

$$\pi^{-1}k_U \otimes K \simeq k^E_M \overset{+}{\otimes} QF.$$

Proposition 1.29. *Let $f : M \to N$ be a morphism of real analytic manifolds.*

(i) $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ is a triangulated subcategory of $E^b(\mathrm{I}k_M)$.

(ii) The duality functor D^E_M gives an equivalence

$$E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)^{op} \xrightarrow{\sim} E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M),$$

and there is a canonical isomorphism of functors $id_{E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)} \xrightarrow{\sim} D^E_M \circ D^E_M$.

(iii) The functors Ef^{-1} and $Ef^!$ send $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_N)$ to $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$, and

$$D^E_M \circ Ef^{-1} \simeq Ef^! \circ D^E_M,$$
$$D^E_M \circ Ef^! \simeq Ef^{-1} \circ D^E_M.$$

(iv) Assume that f is semi-proper. Then the functors Ef_* and $Ef_{!!}$ send $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ to $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_N)$, and

$$D^E_N \circ Ef_* \simeq Ef_{!!} \circ D^E_M,$$
$$D^E_N \circ Ef_{!!} \simeq Ef_* \circ D^E_M.$$

Proposition 1.30. *Let $K, K' \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$. Then both $K \overset{+}{\otimes} K'$ and $\mathcal{I}hom^+(K, K')$ are ℝ-constructible, and one has isomorphisms*

(i) $D^E_M\left(K \overset{+}{\otimes} K'\right) \simeq \mathcal{I}hom^+(K, D^E_M K')$;

(ii) $D^E_M \mathcal{I}hom^+(K, K') \simeq K \overset{+}{\otimes} D^E_M K'$;

(iii) $\mathcal{I}hom^+(K, K') \simeq \mathcal{I}hom^+(D^E_M K', D^E_M K)$;

(iv) $\mathcal{H}om^E(K, K') \simeq \mathcal{H}om^E(D^E_M K', D^E_M K)$.

Chapter 2

Riemann-Hilbert correspondence

Most of the results of this chapter will not be used in the following. Anyway, since these are classical and interesting notions, it was decided to insert them to have a more complete view of the treatment.

2.1 $\mathcal{D}$-modules

References for this section are made to [7] and [8].

Let X be a complex manifold with (complex) dimension d_X. Denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the rings of holomorphic functions and of differential operators, respectively. Denote by Ω_X the sheaf of differential forms of top degree d_X with coefficients in $\mathcal{O}_X$.

Denote by $D^b(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules. For $f : X \to Y$ a morphism of complex manifolds, denote by $\overset{D}{\otimes}$, Df^*, Df_* the operations for $\mathcal{D}$-modules.

There is an equivalence of categories

$$\begin{aligned} r : D^b(\mathcal{D}_X) &\xrightarrow{\sim} D^b(\mathcal{D}_X^{op}) \\ \mathcal{M} &\mapsto \mathcal{M}^r = \Omega_X \otimes^L_{\mathcal{O}_X} \mathcal{M}. \end{aligned}$$

Let $\mathcal{M} \in D^b(\mathcal{D}_X)$. Its *dual* is

$$\mathbb{D}_X\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}\right)[d_X],$$

where $\Omega_X^{\otimes -1} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ and the shift is chosen so that $\mathbb{D}_X\mathcal{O}_X \simeq \mathcal{O}_X$.

Recall that a $\mathcal{D}_X$-module $\mathcal{M}$ is locally of finite type (resp. presentation) if locally on X, there exists an exact sequence $\mathcal{D}_X^n \to \mathcal{M} \to 0$ (resp. $\mathcal{D}_X^{n_0} \to \mathcal{D}_X^{n_1} \to \mathcal{M} \to 0$), for some $n \in \mathbb{N}$ (resp. $n_0, n_1 \in \mathbb{N}$).
A $\mathcal{D}_X$-module $\mathcal{M}$ is called *coherent* if it is locally of finite type and if, for any open subset U, any sub-$\mathcal{D}_U$-module locally of finite type is locally of finite presentation.

Definition 2.1. A $\mathcal{D}_X$-module $\mathcal{M}$ is *quasi-good* if, for any relatively compact open subset $U \subset X$, $\mathcal{M}|_U$ is the sum of a filtrant family of coherent $(\mathcal{O}_X|_U)$-submodules.
A $\mathcal{D}_X$-module is *good* if it is quasi-good and coherent.

To a coherent $\mathcal{D}_X$-module $\mathcal{M}$ we can associate its characteristic variety $char(\mathcal{M})$, that is a closed conic involutive subset of the cotangent bundle T^*X. When $dim_{\mathbb{C}}(char(\mathcal{M})) \leq d_X$, $\mathcal{M}$ is called *holonomic*.

Denote by $D^b_{hol}(\mathcal{D}_X)$ the full subcategory of $D^b(\mathcal{D}_X)$ of objects with holonomic cohomologies, and by $D^b_{g\text{-}hol}(\mathcal{D}_X)$ the full subcategory of objects with good and holonomic cohomologies. Both are triangulated categories.

2.1.1 Exponential $\mathcal{D}$-modules

Let $D \subset X$ be a complex analytic hypersurface and denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic functions with poles along D. It is a holonomic $\mathcal{D}_X$-module.

Let $\mathcal{M} \in D^b(\mathcal{D}_X)$, set

$$\mathcal{M}(*D) = \mathcal{M} \overset{D}{\otimes} \mathcal{O}_X(*D).$$

Set $U = X \setminus D$. For $\varphi \in \mathcal{O}_X(*D)$, set

$$\mathcal{D}_X e^{\varphi} = \mathcal{D}_X / \{P | P e^{\varphi} = 0 \text{ on } U\},$$

$$\mathcal{E}^{\varphi}_{U|X} = \mathcal{D}_X e^{\varphi} \overset{D}{\otimes} \mathcal{O}_X(*D).$$

Hence $\mathcal{D}_X e^{\varphi}$ is a $\mathcal{D}_X$-submodule of $\mathcal{E}^{\varphi}_{U|X}$.
Note that $\mathcal{E}^{\varphi}_{U|X}$ is a holonomic $\mathcal{D}_X$-module which satisfies

$$\mathcal{E}^{\varphi}_{U|X} \simeq \mathcal{E}^{\varphi}_{U|X}(*D).$$

Lemma 2.2. *For $\varphi \in \mathcal{O}_X(*D)$ one has*

$$\left(\mathbb{D}_X \mathcal{E}^{\varphi}_{U|X}\right)(*D) \simeq \mathcal{E}^{-\varphi}_{U|X}.$$

2.2 Tempered solutions

Definition 2.3. The *solution functor* is

$$\begin{aligned} \mathcal{S}ol_X : D^b(\mathcal{D}_X)^{op} &\to D^b(\mathbb{C}_X) \\ \mathcal{M} &\mapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

If $\mathcal{M} = \dfrac{\mathcal{D}_X}{\mathcal{D}_X \mathcal{P}}$, with $\mathcal{P} \in \mathcal{D}_X$, then

$$\mathcal{S}ol_X(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = (\mathcal{O}_X \xrightarrow{\mathcal{P}} \mathcal{O}_X) \in D^b(\mathbb{C}_X).$$

By the functor β (the left adjoint to α in (1.5)), there is a natural notion of $\mathcal{D}_X$-module in the category of ind-sheaves. We denote by $D^b(I\mathcal{D}_X)$ the corresponding derived category.

Let $U \subset X$ be an open subset of the real analytic manifold X.

Definition 2.4. A function $\varphi : U \to \mathbb{C}$ has *polynomial growth* at $x_0 \in X \backslash U$ if there exist a sufficiently small compact neighborhood K of x_0 and constants $C > 0$, $r \in \mathbb{Z}_{>0}$ such that

$$|\varphi(x)| \leq C \; dist(K \setminus U, x_0)^{-r} \; \forall x \in K \cap U,$$

where *dist* is an Euclidean distance with respect to a local coordinate system.
A smooth function $\varphi \in \mathscr{C}^\infty_X(U)$ is *tempered* at $x_0 \in X \setminus U$ if all of its derivatives have polynomial growth at x_0.

For an open subanalytic subset U in X, denote by $\mathscr{C}^{\infty,t}_X(U)$ the subspace of $\mathscr{C}^\infty_X(U)$ consisting of tempered $\mathscr{C}^\infty$-functions.
The presheaf $\mathscr{C}^{\infty,t}_X : U \mapsto \mathscr{C}^{\infty,t}_X(U)$ is an ind-sheaf on X.

The ind-sheaf of *tempered holomorphic functions* $\mathcal{O}^t_X$ is defined as the Dolbeault complex with coefficients in $\mathscr{C}^{\infty,t}_X$.
More precisely, denoting by X^c the complex conjugate manifold to X, and by $X_\mathbb{R}$ the underlying real analytic manifold, we set:

$$\mathcal{O}^t_X = R\mathcal{H}om_{\mathcal{D}_{X^c}}\left(\mathcal{O}_{X^c}, \mathscr{C}^{\infty,t}_{X_\mathbb{R}}\right).$$

Definition 2.5. Define the *tempered solution functor* by

$$\begin{aligned} \mathcal{S}ol^t_X : D^b\left(\mathcal{D}_X\right)^{op} &\to D^b(I\mathbb{C}_X) \\ \mathcal{M} &\mapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^t_X). \end{aligned}$$

One has $\mathcal{S}ol_X \simeq \alpha \circ \mathcal{S}ol^t_X$.

2.3 Enhanced solutions

Let $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ be the complex projective line and $i : X \times P \to X \times \mathbb{P}$ be the closed embedding.
Denote by $\tau \in \mathbb{P}$ the affine coordinate, so that $\tau \in \mathcal{O}_\mathbb{P}(*\infty)$.

Definition 2.6. The *enhanced solution functor* is given by

$$\begin{aligned}\mathcal{S}ol_X^E : D^b(\mathcal{D}_X)^{op} &\to E^b(I\mathbb{C}_X)\\ \mathcal{M} &\mapsto i^! \mathcal{S}ol^t_{X\times\mathbb{P}}(\mathcal{M} \boxtimes^D \mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}})[2],\end{aligned}$$

where $\boxtimes^D$ denotes the exterior product for $\mathcal{D}$-modules.

The functorial properties of $\mathcal{S}ol^E$ are summarized in the next theorem:

Theorem 2.7. *Let $f : X \to Y$ be a complex analytic map and $d_X = dim_{\mathbb{C}} X$. Let $\mathcal{M} \in D^b_{g\text{-}hol}(\mathcal{D}_X)$, $\mathcal{M}_1, \mathcal{M}_2 \in D^b_{hol}(\mathcal{D}_X)$ and $\mathcal{N} \in D^b_{hol}(\mathcal{D}_Y)$. Assume that supp$\mathcal{M}$ is proper over Y. Then one has*

$$\mathcal{S}ol_X^E(Df^*\mathcal{N}) \simeq Ef^{-1}\mathcal{S}ol_Y^E(\mathcal{N}),$$
$$\mathcal{S}ol_Y^E(Df_*\mathcal{M})[d_Y] \simeq Ef_{!!}\mathcal{S}ol_X^E(\mathcal{M})[d_X],$$
$$\mathcal{S}ol_X^E(\mathcal{M}_1) \overset{+}{\otimes} \mathcal{S}ol_X^E(\mathcal{M}_2) \simeq \mathcal{S}ol_X^E(\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2).$$

Notation 2.8. *Let $D \subset X$ be a closed complex analytic hypersurface and set $U = X \setminus D$. For $\varphi \in \mathcal{O}_X(*D)$, we set*

$$E^{\varphi} := k_{\{t+\Re\varphi(x)\geq 0\}} \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_X),$$
$$\mathbb{E}^{\varphi} := k_M^E \overset{+}{\otimes} QE^{\varphi} \in E^b_{\mathbb{R}\text{-}c}(Ik_X),$$

where Q is the quotient functor and we set for short

$$\{t + \Re\varphi(x) \geq 0\} = \{(x,t) \in X \times \mathbb{R} | t + \Re\varphi(x) \geq 0\}.$$

Lemma 2.9. *Let $g : Y \to X$ be a morphism of manifolds and $\varphi \in \mathcal{O}_X(*D)$. Then*

$$Eg^{-1}(E^{\varphi}) \simeq E^{\varphi\circ g}.$$

Lemma 2.10. *Let* $\varphi, \psi \in \mathcal{O}_X(*D)$*, then*

$$E^{\varphi} \overset{*}{\otimes} E^{\psi} \simeq E^{\varphi+\psi}.$$

Proof. Recall that, in (1.1), to define the convolution product $\overset{*}{\otimes}$ we used $q_1, q_2, \mu : X \times \mathbb{R}^2 \to X \times \mathbb{R}$, that do not act on X.
So proving this lemma is equivalent to proving that

$$k_{\{t \geq a\}} \overset{*}{\otimes} k_{\{t \geq b\}} \simeq k_{\{t \geq a+b\}},$$

where $a = \Re\varphi(x)$ and $b = \Re\psi(x)$, since we can fix $x \in X$.

First of all one has $k_{\{t \geq a\}} \overset{*}{\otimes} k_{\{t \geq b\}} \simeq R\mu_! k_S$, where we set $S = \{t_1 \geq a,\ t_2 \geq b\} \subset \mathbb{R}^2$.
Let $T \subset S$ be a halfline that starts in (a, b). One has a morphism $k_S \to k_T$ and $R\mu_! k_T \simeq k_{\{t \geq a+b\}}$.
One can conclude since $R\mu_! k_S \simeq R\mu_! k_T$, in fact

$$R\mu_! k_{S \setminus T} \to R\mu_! k_S \to R\mu_! k_T \xrightarrow{+1}$$

is a distinguished triangle and $R\mu_! k_{S \setminus T} = 0$. □

Theorem 2.11. *Using Notations 2.8, one has*

$$\mathcal{S}ol_X^E(\mathcal{E}_{U|X}^{\varphi}) \simeq \mathbb{E}^{\varphi}.$$

2.4 Riemann-Hilbert correspondence

Theorem 2.12. *The enhanced solution functor induces a fully faithful functor*

$$\mathcal{S}ol_X^E : D^b_{hol}(\mathcal{D}_X)^{op} \to E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X).$$

Morever, there is a functorial way of reconstructing $\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$ *from* $\mathcal{S}ol_X^E(\mathcal{M})$.

Denote by $j : U \to X$ the embedding.

Lemma 2.13. *Let* $\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$ *be such that* $\mathcal{M} \simeq \mathcal{M}(*D)$. *Assume that* X *is compact. Then there exists* $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_U)$ *such that* $R\tilde{j}_! F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_X)$ *and*

$$\mathcal{S}ol_X^E(\mathcal{M}) \simeq k_X^E \overset{+}{\otimes} QR\tilde{j}_! F.$$

Proof. Set $S = \mathcal{S}ol_X^E(\mathcal{M})$. Since $\mathcal{M}$ is holonomic, S is $\mathbb{R}$-constructible. Since X is compact, there exists $F' \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_X)$ with $S \simeq k_X^E \overset{+}{\otimes} QF'$. Since $\mathcal{M} \simeq \mathcal{M}(*D)$, one has

$$S \simeq \pi^{-1} k_U \otimes S \simeq k_X^E \overset{+}{\otimes} Q(\pi^{-1} k_U \otimes F').$$

Hence $F = \tilde{j}^{-1} F'$ satisfies the assumptions in the statement. $\square$

Chapter 3

Fourier transform

Originally, the Fourier transform was introduced as an integral transform with kernel associated to $e^{-\langle z,w\rangle}$, where z is a system of coordinates in a complex vector space and w a system of coordinates in the dual vector space. Hence, if we want to generalize the classical definition at level of $\mathcal{D}$-modules and of enhanced sheaves, first of all we have to define an analogue of the integral transform.

For this chapter, we use the same notations introduced in [1].

3.1 Integral transforms

Consider a diagram of complex manifolds

$$\begin{array}{ccc} & S & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & Y. \end{array} \tag{3.1}$$

At the level of $\mathcal{D}$-modules, the integral transform with kernel $\mathcal{L} \in D^b(\mathcal{D}_S)$ is the functor

$$\begin{aligned} \cdot \overset{D}{\circ} \mathcal{L} : D^b(\mathcal{D}_X) &\to D^b(\mathcal{D}_Y) \\ \mathcal{M} &\mapsto \mathcal{M} \overset{D}{\circ} \mathcal{L} = Dq_*(\mathcal{L} \overset{D}{\otimes} Dp^*\mathcal{M}). \end{aligned}$$

At the level of enhanced ind-sheaves, the integral transform with kernel $H \in E^b(Ik_S)$ is the functor

$$\cdot \overset{+}{\circ} H : E^b(Ik_X) \to E^b(Ik_Y)$$
$$K \mapsto K \overset{+}{\circ} H = Eq_{!!}(H \overset{+}{\otimes} Ep^{-1}K).$$

By the isomorphisms in Theorem 2.7, one has

Corollary 3.1. *Let $\mathcal{M} \in D^b_{g\text{-}hol}(\mathcal{D}_X)$ and $\mathcal{L} \in D^b_{g\text{-}hol}(\mathcal{D}_S)$. Assume that $p^{-1}supp(\mathcal{M}) \cap supp(\mathcal{L})$ is proper over Y. Set $K = \mathcal{S}ol^E_X(\mathcal{M})$ and $H = \mathcal{S}ol^E_S(\mathcal{L})$. Then there is a natural isomorphism in $E^b_{\mathbb{R}\text{-}c}(Ik_Y)$:*

$$\mathcal{S}ol^E_Y(\mathcal{M} \overset{D}{\circ} \mathcal{L}) \simeq K \overset{+}{\circ} H[d_S - d_Y].$$

Consider now the diagram of real analytic manifolds induced by (3.1)

$$\begin{array}{ccc} & S \times \mathbb{R} & \\ {}^{\tilde{p}}\swarrow & & \searrow^{\tilde{q}} \\ X \times \mathbb{R} & & Y \times \mathbb{R}, \end{array}$$

where $\tilde{p} = p \times id_{\mathbb{R}}$ and $\tilde{q} = q \times id_{\mathbb{R}}$.
The natural integral transform for $\mathbb{R}$-constructible enhanced sheaves with kernel $L \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_S)$ is the functor

$$\cdot \overset{*}{\circ} L : \tilde{E}^b_{\mathbb{R}\text{-}c}(k_X) \to \tilde{E}^b_{\mathbb{R}\text{-}c}(k_Y)$$
$$F \mapsto F \overset{*}{\circ} L = R\tilde{q}_!(L \overset{*}{\otimes} \tilde{p}^{-1}F).$$

Proposition 3.2. *Let $\mathcal{M} \in D^b_{g\text{-}hol}(\mathcal{D}_X)$, $\mathcal{L} \in D^b_{g\text{-}hol}(\mathcal{D}_S)$, and assume that $p^{-1}supp(\mathcal{M}) \cap supp(\mathcal{L})$ is proper over Y. Let $F \in$*

$\tilde{E}^b_{\mathbb{R}\text{-}c}(k_X)$, $L \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_S)$, and assume that there are isomorphisms

$$\begin{aligned} \mathcal{S}ol^E_X(\mathcal{M}) &\simeq k^E_X \overset{+}{\otimes} QF, \\ \mathcal{S}ol^E_S(\mathcal{L}) &\simeq k^E_S \overset{+}{\otimes} QL. \end{aligned} \tag{3.2}$$

Then there is a natural isomorphism in $E^b_{\mathbb{R}\text{-}c}(Ik_Y)$:

$$\mathcal{S}ol^E_Y(\mathcal{M} \overset{D}{\circ} \mathcal{L}) \simeq k^E_Y \overset{+}{\otimes} Q(F \overset{*}{\circ} L)[d_S - d_Y].$$

Note that if X and S are compact, then for any $\mathcal{M} \in D^b_{hol}(\mathcal{D}_x)$ and $\mathcal{L} \in D^b_{hol}(\mathcal{D}_S)$ there exist $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_X)$ and $L \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_S)$ satisfying (3.2).

3.2 Fourier-Laplace transform

Let $\mathbb{V}$ be a complex vector space with finite dimension $d_{\mathbb{V}}$, and let $\mathbb{P} = ((\mathbb{V} \oplus \mathbb{R}) \setminus \{0\})/\mathbb{R}^+$ be its projective compactification. Denote $j : \mathbb{V} \to \mathbb{P}$ the embedding and $\mathbb{H} = \mathbb{P} \setminus \mathbb{V}$.
Recall that the classical Fourier transform interchanges objects in $\mathbb{V}$ with objects in the dual space $\mathbb{V}^*$.

Definition 3.3. Let $D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ be the full subcategory of $D^b_{hol}(\mathcal{D}_{\mathbb{P}})$ whose objects $\mathcal{M}$ satisfy $\mathcal{M} \simeq \mathcal{M}(*\mathbb{H})$.

The pairing

$$\begin{aligned} \mathbb{V} \times \mathbb{V}^* &\to \mathbb{C} \\ (z, w) &\mapsto \langle z, w \rangle \end{aligned}$$

defines a meromorphic function on $\mathbb{P} \times \mathbb{P}^*$, where $\mathbb{P}^*$ is the projective compactification of the dual vector space of $\mathbb{V}$, and it has poles along $(\mathbb{P} \times \mathbb{P}^*) \setminus (\mathbb{V} \times \mathbb{V}^*)$.

Let $\mathbb{V}_\infty = (\mathbb{V}, \mathbb{P})$ and $\mathbb{V}^*_\infty = (\mathbb{V}^*, \mathbb{P}^*)$. Consider the projections

$$\begin{array}{ccc} & \mathbb{V}_\infty \times \mathbb{V}^*_\infty & \\ \swarrow & & \searrow \\ \mathbb{V}_\infty & & \mathbb{V}^*_\infty. \end{array}$$

Definition 3.4. Set

$$\mathcal{L} = \mathcal{E}^{\langle z,w\rangle}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty},\ \mathcal{L}^a = \mathcal{E}^{-\langle w,z\rangle}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}.$$

The *Fourier-Laplace transform* of $\mathcal{N} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}^*_\infty})$ is given by

$$\mathcal{N}^\wedge = \mathcal{N} \overset{D}{\circ} \mathcal{L}^a \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty}).$$

The *inverse Fourier-Laplace transform* of $\mathcal{M} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ is given by

$$\mathcal{M}^\vee = \mathcal{M} \overset{D}{\circ} \mathcal{L} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}^*_\infty}).$$

Theorem 3.5. *The Fourier-Laplace transform* $^\wedge$ *and the inverse Fourier-Laplace transform* $^\vee$ *are quasi-inverse of each other, and interchange* $D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ *and* $D^b_{hol}(\mathcal{D}_{\mathbb{V}^*_\infty})$*.*

3.3 Enhanced Fourier-Sato transform

Definition 3.6. Define $\tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty})$ to be the full triangulated subcategory of $\tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}})$ whose objects F satisfy $R\tilde{j}_!F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{P}})$.

Consider the projections

$$\begin{array}{ccccc} & & \mathbb{V} \times \mathbb{V}^* \times \mathbb{R} & & \\ & \swarrow^{\tilde{p}} & \downarrow{\scriptstyle \bar{p}} & \searrow^{\tilde{q}} & \\ \mathbb{V} \times \mathbb{R} & & \mathbb{V} & & \mathbb{V}^* \times \mathbb{R} \end{array}.$$

Definition 3.7. Using Notation 2.8, set

$$L = E^{\langle z,w\rangle} \text{ and } L^a = E^{-\langle w,z\rangle}.$$

The *enhanced Fourier-Sato transform* of $G \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}^*_\infty})$ is given by

$$G^\curlywedge = G \overset{*}{\circ} L^a[d_\mathbb{V}] \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty}).$$

The *enhanced inverse Fourier-Sato transform* of $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty})$ is given by

$$F^\curlyvee = F \overset{*}{\circ} L[d_\mathbb{V}] \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}^*_\infty}).$$

Proposition 3.8. *The enhanced Fourier-Sato transform* $^\curlywedge$ *and the enhanced inverse Fourier-Sato transform* $^\curlyvee$ *are quasi-inverse of each other, and they interchange* $\tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty})$ *and* $\tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}^*_\infty})$*, since* $\tilde{p}$ *and* $\tilde{q}$ *are semiproper.*

Using the Definitions 3.7 and let ϵ as in (1.4).
Denote by $u : \mathbb{V} \times \mathbb{V}^* \to \mathbb{V}_\infty \times \mathbb{V}^*_\infty$ the embedding.

Lemma 3.9. *For* $F' \in D^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty})$ *one has*

$$\epsilon(F')^\curlywedge \simeq R\tilde{q}_!(L \otimes \bar{p}^{-1}F'),$$

$$\mathcal{S}ol^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}(\mathcal{L}) = \mathbb{E}^{\langle z,w\rangle}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \simeq k^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{+}{\otimes} QR\tilde{u}_!L.$$

Lemma 3.10. *Denote by* $h : \mathbb{V}^* \to \mathbb{V}^*_\infty$ *the embedding.*
Let $\mathcal{M} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ *and* $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty})$ *satisfy*

$$\mathcal{S}ol^E_{\mathbb{V}_\infty}(\mathcal{M}) \simeq k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} QR\tilde{j}_!F. \tag{3.3}$$

Then, there is an isomorphism

$$\mathcal{S}ol^E_{\mathbb{V}^*_\infty}(\mathcal{M}^\wedge) \simeq k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} QR\tilde{h}_!F^\curlywedge[d_\mathbb{V}].$$

An analogous result holds for $\wedge$ and $\curlywedge$ replaced by $\vee$ and $\curlyvee$, respectively.
Note that the hypothesis (3.3) is not too restrictive, since for any $\mathcal{M} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ there is an $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(k_{\mathbb{V}_\infty})$ satisfying it, by Lemma 2.13.

Chapter 4

The Stokes phenomenon of the Airy equation

4.1 The Airy equation

Let z be the coordinate of a complex line $\mathbb{V}$, and w the dual coordinate on $\mathbb{V}^*$.

The *Airy equation* on $\mathbb{V}^*$ is

$$Q\psi = 0, \text{ where } Q = \partial_w^2 - w. \tag{4.1}$$

4.1.1 Solution by integral representation

We will find solutions of (4.1) in the form

$$\psi(w) = \int_\gamma f(z) e^{-zw} dz, \tag{4.2}$$

where $f(z)$ is an unknown function and γ is a path to be determined.

Note that integrating by parts, one has:

$$
\begin{aligned}
-\int_\gamma w f(z) e^{-zw} dz &= \int_\gamma f(z) \frac{\partial e^{-zw}}{\partial z} dz \\
&= \int_\gamma \left(\frac{\partial \left(f(z) e^{-zw} \right)}{\partial z} - f'(z) e^{-zw} \right) dz \\
&= f(z) e^{-zw}|_\gamma - \int_\gamma f'(z) e^{-zw} dz,
\end{aligned}
$$

where we set $g(z)|_\gamma = g(b) - g(a)$, with a and b the endpoints of the path γ.
So, substituting (4.2) in the Airy equation and differentiating under the integral sign, we obtain:

$$
\begin{aligned}
\psi'' - w\psi &= \int_\gamma z^2 f(z) e^{-zw} dz - \int_\gamma w f(z) e^{-zw} dz \\
&= f(z) e^{-zw}|_\gamma + \int_\gamma \left(z^2 f(z) - f'(z) \right) e^{-zw} dz = 0.
\end{aligned}
$$

First of all, $f(z)$ can be chosen such that

$$z^2 f(z) - f'(z) = 0, \text{ i.e. } f(z) = c e^{\frac{z^3}{3}},$$

where c is a constant. For simplicity, let $c = 1$.

So, we have

$$\psi(w) = \int_\gamma \exp\left(\frac{z^3}{3} - zw \right) dz. \tag{4.3}$$

It remains to choose γ such that

$$f(z) e^{-zw}|_\gamma = e^{\frac{z^3}{3} - zw}|_\gamma = 0.$$

Clearly this is satisfied if γ is a closed curve, but this choice gives the trivial solution of (4.1), $\psi(w) = 0$.

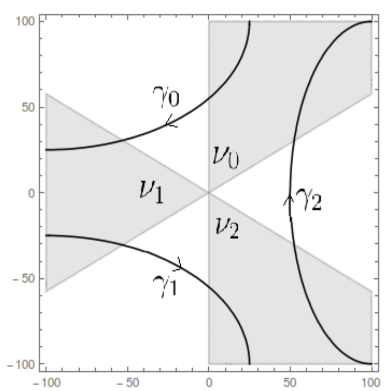
So, since $e^{\frac{z^3}{3}-zw} \neq 0$ for any value of z, the only other possible choice of γ is a path that begins and ends in sectors of $z = \infty$ for which $e^{\frac{z^3}{3}-zw} \to 0$ as $z \to \infty$. This is equivalent to $\Re\left(\frac{z^3}{3} - zw\right) \to 0$ as $z \to \infty$.

Now, for large z, $\frac{z^3}{3} - zw \approx z^3$. So, setting $z = |z|e^{i\theta}$, one has $\Re(z^3) = |z|^3\cos(3\theta)$. Then, to have the right convergence, γ has to begin and end in sectors of $|z| = \infty$ for which

$$\frac{\pi}{6} + \frac{2}{3}n\pi < \theta < \frac{\pi}{2} + \frac{2}{3}n\pi, \; n \in \mathbb{Z}.$$

There are three of such sectors (modulo 2π) defined by the choice of $n = 0, 1, 2$: call them ν_0, ν_1 and ν_2 respectively.

Denote by γ_j the path that begins in ν_j and ends in ν_{j+1}, where j is thought modulo 3.
Notice that the sum of $\gamma_0 + \gamma_1 + \gamma_2$ is homologous to zero, hence the three corresponding solutions sum to zero.



4.1.2 Morse theory

We now follow [10] for a while to find explicitly the integration cycles.

By an appropriate change of variables, all solutions to (4.1) have the form

$$\psi(\lambda) = \int_{\mathcal{C}} \exp(\mathcal{I}) dx, \tag{4.4}$$

where $\mathcal{I} = \lambda\left(\frac{x^3}{3} - x\right)$, and $\mathcal{C}$ is the integration cycle corresponding to γ.

Proof. Using first of all the ramification

$$\begin{aligned} r : \mathbb{C}_v &\to \mathbb{C}_w \\ v &\mapsto w = v^2, \end{aligned}$$

and then the isomorphism

$$\mathbb{C}_x \times \mathbb{C}_v^\times \overset{\sim}{\longleftrightarrow} \mathbb{C}_z \times \mathbb{C}_v^\times, \text{ defined by } \begin{cases} z = xv \\ v = v \end{cases},$$

we obtain:

$$\frac{z^3}{3} - zw = \frac{z^3}{3} - zv^2 = \frac{x^3v^3}{3} - xv^3 = v^3\left(\frac{x^3}{3} - x\right).$$

Finally setting $\lambda = v^3$, we have exactly the exponent $\mathcal{I}$ in (4.4). □

Let X be the complex x-plane and $u \in \mathbb{R}$, denote X_u the part of X where the points are such that $\Re(\mathcal{I}) < u$, fixed λ. Then $\mathcal{C}$ is contained in $X \setminus X_{-T}$, with T very large (i.e. it is contained in the coloured region of Figure (4.1)).

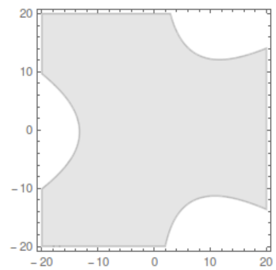


Figure 4.1: $X \setminus X_{-T} = \{x \in X : T + \Re(\mathcal{I}) \geq 0\}$

Moreover, it can be seen that, if $\mathcal{C}_j$ corresponds to γ_j for $j = 0, 1, 2$, they are such that $\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ is the boundary of an inner region, so it's zero at level of homology. In other words, they are linearly dependent.

Definition 4.1. The *critical points* of a function are the points at which all first derivatives vanish.
A critical point is *non-degenerate* if the matrix of second derivatives is invertible at that point.
A *Morse function* is a real-valued function whose critical points are non-degenerate.

Let $f = f(x) = f(a+ib) = u(a,b) + iv(a,b)$ be a holomorphic function. By the Cauchy-Riemann equations:

$$\begin{cases} u_a = v_b \\ u_b = -v_a \end{cases} \Rightarrow \begin{cases} u_{aa} = v_{ab} = -u_{bb} \\ u_{ab} = v_{bb} = -v_{aa} \end{cases} . \tag{4.5}$$

Recall that $\partial_x = \frac{1}{2}(\partial_a - i\partial_b)$. Then one has

$$f' = \frac{1}{2}\left((u_a + v_b) + i(v_a - u_b)\right) = u_a - iu_b$$
$$f'' = \frac{1}{2}\left((u_{aa} - u_{bb}) + i(u_{ab} + u_{ab})\right) = u_{aa} - iu_{ab}.$$

First of all, $f' = 0$ if and only if $du = 0$. It means that the real part of an holomorphic function has the same critical points of the function. Moreover, one has

$$det(Hess(u)) = u_{aa}u_{bb} - u_{ab}^2 = -(u_{aa}^2 + u_{ab}^2) = -|f''|^2. \quad (4.6)$$

So, a point is non-degenerate if $0 \neq det(Hess(u))$, that is equivalent to $f'' \neq 0$.

Consider $h = \Re(\mathcal{I})$. Since $\mathcal{I}$ is a holomorphic function, h has the same critical points of $\mathcal{I}$:

$$\frac{\partial \mathcal{I}}{\partial x} = \lambda(x^2 - 1) = 0 \;\Leftrightarrow\; x = \pm 1.$$

So there are two critical points $p_\pm$ at $x = \pm 1$, and the values of $\mathcal{I}$ and h at them are respectively

$$\mathcal{I}_\pm = \mp \frac{2}{3}\lambda, \quad h_\pm = \mp \frac{2}{3}\Re(\lambda).$$

Moreover,

$$\frac{\partial^2 \mathcal{I}}{\partial x^2}|_{p_\pm} = 2\lambda x|_{p_\pm} = \pm 2\lambda \neq 0,$$

so the critical points are non-degenerate.
Then h is a Morse function.

Definition 4.2. The *index* of a non-degenerate critical point is the number of negative eigenvalues of the matrix of second derivatives at that point.
In other words, it's the number of directions in which the considered Morse function decreases.

If $f(x) = f(a+ib) = u(a,b)+iv(a,b)$ is a holomorphic function and x_0 is a critical point, then it has index 1 by (4.6).

Moreover, one has

$$\begin{aligned} u(x) - u(x_0) &= \Re(f(x) - f(x_0)) \\ &= \Re\left(\frac{1}{2}f''(x_0)(x - x_0)^2 + o((x - x_0)^2)\right) \\ &\sim \frac{1}{2}\Re\left(f''(x_0)(x - x_0)^2\right), \end{aligned}$$

since we are only interested in a neighborhood of x_0.
If $f''(x_0) = re^{i\eta}$ and $x - x_0 = \rho e^{i\theta}$, then the quantity above is equal to $\frac{1}{2}r\rho^2\cos(\eta + 2\theta)$. When $\cos(\eta + 2\theta) = \pm 1$, one has the directions of the positive and negative eigenspace of $Hess(u)$ respectively. These are given by

$$\begin{aligned} \theta = \theta_+ &= -\frac{\eta}{2} + k\pi,\ k \in \mathbb{Z},\ \text{or} \\ \theta = \theta_- &= -\frac{\eta}{2} + \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}. \end{aligned}$$

So, the positive and the negative eigenspaces of $Hess(u)$ are

$$\mathbb{R}e^{i\theta_\pm} = \mathbb{R}\left(\pm f''(x_0)\right)^{-1/2}.$$

In our case, $\mathcal{I}''(p_\pm) = \pm 2\lambda$. So there is only one negative eigenspace of $Hess(h)$ for $p_\pm$, then they both have index 1.

Notice that both the degeneracy and the index of a critical point are independent from the choice of the local coordinate system used. So the above computation makes sense.

Definition 4.3. u is a *perfect* Morse function if the difference between the indices of distinct critical points of u are never equal to ± 1.

In general, if u is a Morse function on a manifold X, the rank of the q-dimensional homology group of X is at most the number of the critical points of u with index q. Moreover, these upper bounds are reached if u is a perfect Morse function.

In our case, it is easily seen that h is a perfect Morse function. If T is very large, then the first homology group of $X \setminus X_{-T}$ has rank 2 while the others homology groups vanish. This is why the cycles $\mathcal{C}_i,\ i = 1, 2, 3$ are linearly dependent.

Morse theory gives also a recipe to construct the generators of the homology. On any manifold X with real coordinates γ^i, we can pick a Riemannian metric g^{ij}. The gradient flow equation is

$$\frac{\partial \gamma^i}{\partial t} = -g^{ij} \frac{\partial u}{\partial \gamma^j}. \tag{4.7}$$

It's also called *downward flow* since u is strictly decreasing along a flow, except for a constant solution that sits at a critical point for every t. Indeed: $\frac{\partial u}{\partial t} = \sum_i \frac{\partial u}{\partial \gamma^i} \frac{\partial \gamma^i}{\partial t} = -\sum_i \left(\frac{\partial u}{\partial \gamma^i} \right)^2 < 0$.
A non-constant flow can reach a critical point of u only at $t = \pm\infty$, in fact if at some t, the flow reaches a critical point, then it has to be constant, by the flow equation.

Let p be a non-degenerate critical point of u, consider (4.7) on $(-\infty, 0]$ with the boundary condition that $\gamma^i(t)$ approaches p for $t \to -\infty$. If p has index k, the space of solutions $\mathcal{J}_p$ is a k-dimensional manifold, since the only possible directions for the flow starting from p are the ones for which u decreases. Since (4.7) is first order in time, a flow is uniquely determined by its value at $t = 0$, so $\mathcal{J}_p$ can be thinked as a submanifold of X by the embedding

$$\begin{array}{c} \mathcal{J}_p \longrightarrow X \\ \gamma^i(t) \mapsto \gamma^i(0) \end{array};$$

equivalently $\mathcal{J}_p$ can be thought as the submanifold of X consisting

of points that can be reached at every t by a flow that start at p at $t=-\infty$.
$\mathcal{J}_p$ define a cycle in $H_1(X,\ X_{-T};\ \mathbb{Z})$ if it's closed, i.e. any sequence of points in $\mathcal{J}_p$ has a subsequence that either converges or tends to $u=-\infty$. This fails only if there exists a complete flow line l (defined in $\mathbb{R}\cup\{\pm\infty\}$) which starts at p at $t=-\infty$ and ends at another critical point q at $t=+\infty$, in fact in this case $l\subset\mathcal{J}_p$ and a sequence of points in l can converge to $q\notin\mathcal{J}_p$.

In our case, we can use the Kahler metric $ds^2=|dx|^2$, so the flow equation becomes

$$\begin{cases}\frac{\partial x}{\partial t}=-\frac{\partial\bar{\mathcal{I}}}{\partial\bar{x}}\\ \frac{\partial\bar{x}}{\partial t}=-\frac{\partial\mathcal{I}}{\partial x}\end{cases}. \tag{4.8}$$

$\Im(\mathcal{I})$ is conserved along every flow, indeed

$$\frac{\partial\Im(\mathcal{I})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2i}(\mathcal{I}-\bar{\mathcal{I}})\right)=\frac{1}{2i}\left(\frac{\partial\mathcal{I}}{\partial x}\frac{\partial x}{\partial t}-\frac{\partial\bar{\mathcal{I}}}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial t}\right)=0,$$

by (4.8). Recall that the value of $\mathcal{I}$ at the critical points of h is

$$\mathcal{I}_\pm=\mp\frac{2}{3}\lambda\ \Rightarrow\ \Im(\mathcal{I}_\pm)=\mp\frac{2}{3}\Im(\lambda).$$

So $\Im(\mathcal{I}_+)=\Im(\mathcal{I}_-)$ if and only if the imaginary part of λ is zero, that is λ is real. Hence, a flow can connect p_+ and p_- only if λ is real. Conversely, if $\lambda\in\mathbb{R}$, the part of the real axis $(-1,1)$ is a flow that connect the two critical points. Since we consider $\lambda\neq 0$, $\Im(\lambda)=0$ consists of two open rays, called *Stokes rays.*

In conclusion: away from the Stokes rays, a downward flow that starts at one critical point cannot end at the other, instead it always flow to $h=-\infty$. Since $p_\pm$ have index 1, we can attach to them $\mathcal{J}_{p_\pm}=\mathcal{J}_\pm$, that are 1-dimensional manifolds.

In any case, if $\lambda \neq 0$, then $\mathcal{I}_+ \neq \mathcal{I}_-$. So, on a Stokes ray it has to be $\Re(\mathcal{I}_+) = h_+ \neq h_- = \Re(\mathcal{I}_-)$. Indeed, if $\Re(\lambda) > 0$ then $h_+ < h_-$, and if $\Re(\lambda) < 0$ then $h_- < h_+$.

Explicitly computation

Since $\Im(\mathcal{I})$ is conserved along a flow, then $\mathcal{J}_+$ and $\mathcal{J}_-$ are respectively contained in the graph of

$$\Im(\mathcal{I}) = \Im(\mathcal{I}_\pm) = \mp\frac{2}{3}\Im(\lambda).$$

Given $p_\pm = \pm 1$, the critical points of $\mathcal{I}(x) = \lambda\left(\frac{x^3}{3} - x\right)$, set $\xi = x - p_\pm$ as the local coordinate. Then,

$$\mathcal{I}(x) - \mathcal{I}(p_\pm) = \frac{\lambda}{3}\xi^2(\xi \pm 3).$$

Set $\lambda = re^{i\eta}$, $\xi = \rho e^{i\vartheta}$. We saw that our cycles $\mathcal{J}_\pm$ are contained in $\Im(\mathcal{I}) = \Im(\mathcal{I}_\pm)$, that is equivalent to:

$$\begin{aligned} 0 &= \Im(\mathcal{I}(x)) - \Im(\mathcal{I}(p_\pm)) = \Im(\mathcal{I}(x) - \mathcal{I}(p_\pm)) \\ &= \frac{r\rho^2}{3}\Im\left(\rho e^{i(3\vartheta+\eta)} \pm 3e^{i(2\vartheta+\eta)}\right) \\ &= \frac{r\rho^2}{3}\left(\rho\sin(3\vartheta+\eta) \pm 3\sin(2\vartheta+\eta)\right). \end{aligned}$$

So,

$$\rho = \rho(\vartheta) = \mp 3\frac{\sin(2\vartheta+\eta)}{\sin(3\vartheta+\eta)}.$$

Then its slopes at $p_\pm$ is given by $\rho = 0$, i.e.

$$\vartheta_0 = -\frac{\eta}{2} + h\frac{\pi}{2}, \quad h \in \mathbb{Z},$$

and slopes at ∞ is given by $\rho = \infty$, i.e.

$$\vartheta_\infty = -\frac{\eta}{3} + k\frac{\pi}{3}, \quad k \in \mathbb{Z},$$

unless one has the case $\frac{0}{0}$, i.e.

$$\eta = (3h - 2k)\pi \quad \exists h, k \in \mathbb{Z} \quad \Leftrightarrow \quad \eta \in \mathbb{Z}\pi.$$

So we have an explicit parametric computation of the graphs that contain our cycles, that is given by:

$$x = p_\pm + \xi = p_\pm + \rho(\vartheta)e^{i\vartheta}, \quad \vartheta \in [0, 2\pi[.$$

These are shown in Figure 4.2.

As we can see, the equations found describe different branches in the complex x-plane X and we can't take all of them into account. There are two possible choices: following Witten's argument as in [10], one can choose the steepest descent curves, i.e. those along which h decreases. In this way, we can find the generators of the relative homology $H_1(X, X_{-T}; \mathbb{Z})$.
But we make a different choice, since we are interesting in the compact support cohomology, that is described by Borel-Moore cycles. We will see why in the following section, precisely in (4.11), when we will talk about enhanced Fourier-Sato transform.

So we choose the steepest ascent curves. In this way if the critical point belongs to the region $t + h(x) \geq 0$, also its corresponding cycles is contained in it.

Making a choice rather than the other means choosing the range of the parameter ϑ in the above computation.
Since $\vartheta_\infty = (-\eta + k\pi)/3, \ k \in \mathbb{Z}$, the range for one of the curve should be

$$\vartheta \in R_k := \left[\frac{-\eta + k\pi}{3}, \frac{-\eta + (k+1)\pi}{3}\right], \text{ fixed } k \in \mathbb{Z}.$$

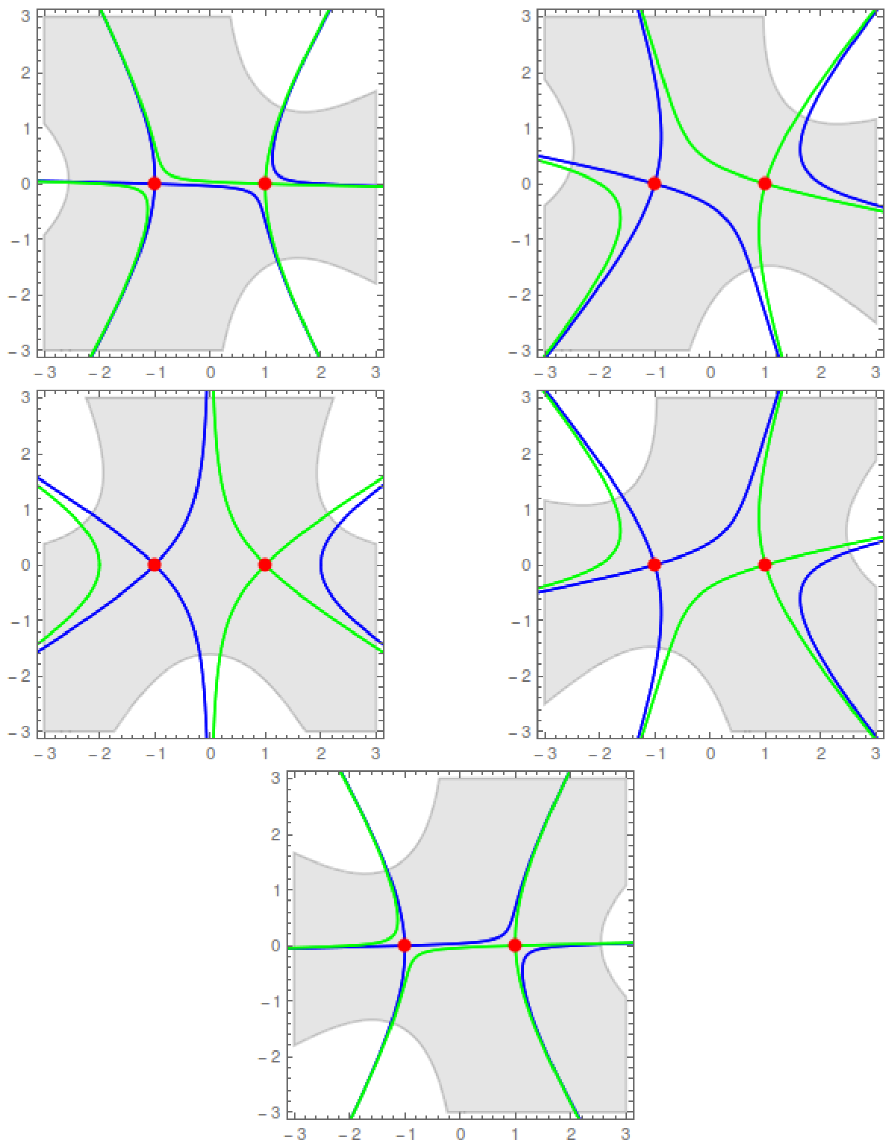


Figure 4.2: The green graph represents $\Im(\mathcal{I}) = \Im(\mathcal{I}_+)$ and the blue one represents $\Im(\mathcal{I}) = \Im(\mathcal{I}_-)$; the red points are $p_\pm$ and the gray region is the one described by $t + h(x) \geq 0$ (in this case with t very large). The five different images show what happens to the graphs at the varying of λ, more precisely as its argument η grows.

In this way we are sure that we have pick up a whole cycle (from ∞ to ∞) and that it passes through the critical point, in fact $\vartheta_0 = (-\eta + k\pi)/2 \in R_k$.

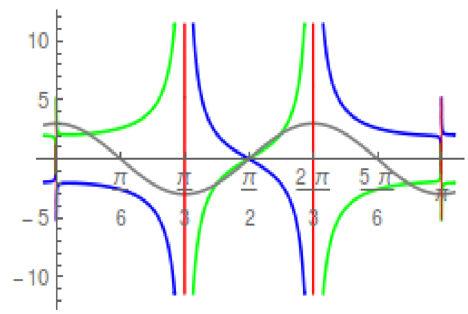


Figure 4.3: The green curves represents the growing of the graph that contains $\mathcal{J}_+$, the blue one represents the growing of the graph that contains $\mathcal{J}_-$ and the gray one represents h at infinity.

To choose the right k, let's look at Figure 4.3: we need the coloured curves to have, near the asymptotes, the same sign as the gray one, which represents h at infinity. So for the curve that pass through p_+ we pick $k = 2$, instead for the one that pass through p_- we pick $k = 1$.

So fixed $\vartheta \in R_2$ for the parametric computation of $\mathcal{J}_+$ and $\vartheta \in R_1$ for $\mathcal{J}_-$, we finally obtain the cycles that we can see in Figure 4.4.

Here we can clearly notice that when λ approaches the Stokes rays (the first and the last images), $\mathcal{J}_\pm$ approaches $p_\mp$.

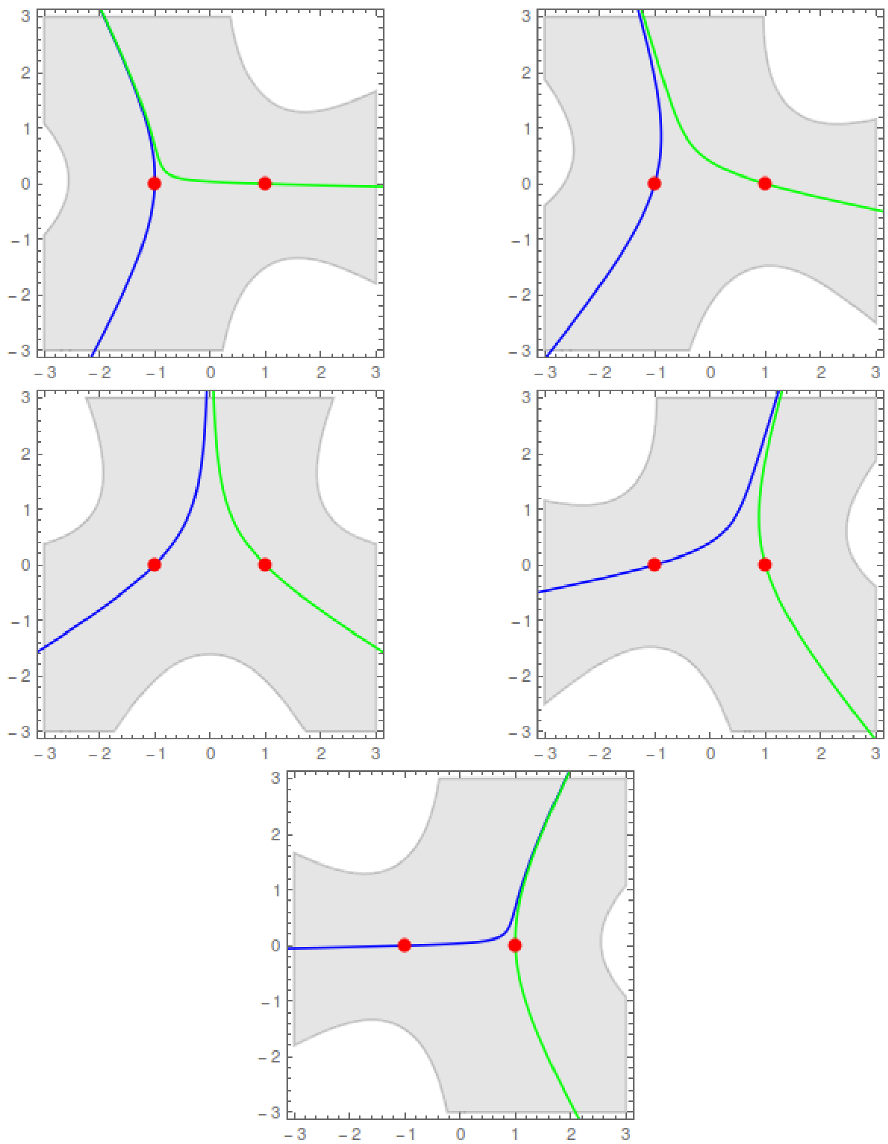


Figure 4.4: The green cycle is $\mathcal{J}_+$ while the blue one is $\mathcal{J}_-$. The various images of this figure represent what happens when λ varies. The gray region is the one described by $t + h(x) \geq 0$, with t very large.

Crossing a Stokes ray

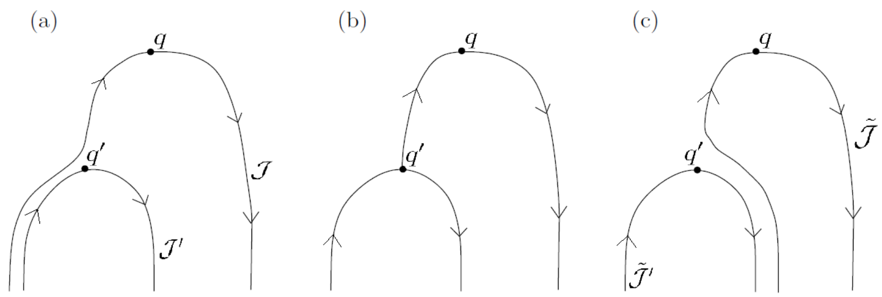


Figure 4.5: Chosen an appropriate orientation for the cycles, this figure illustrates their behavior in crossing a Stokes ray

As we approach a Stokes ray, denote q and q' the critical points with, respectively, larger and smaller value of h. After choosing an appropriate orientation for the cycles attached to the critical points, denote them $\mathcal{J}$ and $\mathcal{J}'$.
Since q' is the critical point with smaller value of h, the downward trajectories that starts at q' can only flow to $h = -\infty$, so even if $\mathcal{J}'$ crosses a Stokes ray, nothing happens. Instead, exists a trajectory that starts at q and flow down to q', as depicted in Figure 4.5(b), so $\mathcal{J}$ jumps in crossing a Stokes ray:

$$\mathcal{J} \to \mathcal{J} \pm \mathcal{J}',$$

where the sign depends on the orientation of the two cycles and the direction in which λ crosses it. In other words, the passage is described by

$$\begin{pmatrix} \mathcal{J} \\ \mathcal{J}' \end{pmatrix} \mapsto \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{J} \\ \mathcal{J}' \end{pmatrix}. \tag{4.9}$$

In (a) and (c) of Figure 4.5, are represented the cycles "before" and "after" the jump, with a fixed choice of orientation for them.

So we can see that going from (a) to (c) (passing through (b)) we have $\tilde{\mathcal{J}} = \mathcal{J} - \mathcal{J}'$. Then the transformation matrix is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Viceversa, i.e. crossing the Stokes ray in the opposite direction, we have $\mathcal{J} = \tilde{\mathcal{J}} + \tilde{\mathcal{J}}'$, so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

4.2 The Stokes phenomenon

The solutions $\psi(w)$ of the Airy equation (4.1) are asymptotic for large value of $|w|$ to a linear combination of

$$v_{\pm} = w^{-\frac{1}{4}} \exp\left(\pm\frac{2}{3}w^{\frac{3}{2}}\right), \tag{4.10}$$

that are multivalued functions of the complex variable w with a branch point at $w = 0$.
The Airy function $Ai(w)$ is an entire solution of (4.1). Therefore, as we go once around the branch point, it will return to its original value, but v_+ and v_- will not. Indeed, in the exponential factors of (4.10), it appears the square root of a complex number. So, we need to pick up a determination of it in order to make sense to the definition of $v_{\pm}$. This choice gives a ramification, i.e. if we go once around the branch point we do not return to the same value, but if we go around twice, we do.

This means that the asymptotic behavior of the function is not the same in the whole complex plane: this is the basic *Stokes phenomenon.*
In fact, we can see that if the sign of $\Re\left(\frac{2}{3}w^{\frac{3}{2}}\right)$ changes, the exponential factors in (4.10) either increase to $+\infty$ or decrease to 0. So, there is a change of the asymptotic behavior when w crosses

the lines (called *Stokes lines*) defined by $\Re\left(\frac{2}{3}w^{\frac{3}{2}}\right) = 0$, i.e. when the argument of w is equal to $\frac{\pi}{3} + 2n\pi, n \in \mathbb{Z}$.

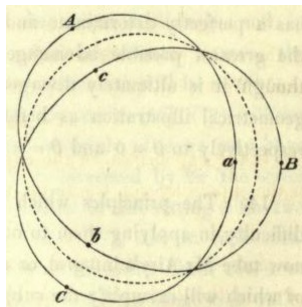


Figure 4.6: In this figure the dotted line represents α, the argument of w, and the curve around it is the one defined by $\cos(\frac{3}{2}\alpha)$.

Stokes studied his phenomenon in the Airy case in [9]. Here we find Figure 4.6, that gives us a clear view of what we just said. In fact, the dotted circle represents $\alpha = arg(w)$ and it alone is supposed to vary (i.e. $|w|$ is fixed). The curve instead is the one defined by $\cos\left(\frac{3}{2}\alpha\right)$, that describes

$$\Re\left(\frac{2}{3}w^{3/2}\right) = \frac{2}{3}|w|^{3/2}\cos\left(\frac{3}{2}\alpha\right).$$

So, the ramification that we were talking about can be seen following the curve, since it needs two complete revolutions to return to itself.

Moreover, if we cover the figure with successive circular sectors in which we take into account the part outside the curve, then if we going twice aroud it turns out that the central region was never considered, the regions bounded by the different branches of the curve were considered once and the external region twice. This explains how in the next section we choose to study the sheaf which is locally constant along this stratification.

4.2.1 Level of $\mathbb{R}$-constructible enhanced sheaves

The idea now is to translate the asymptotic informations given by (4.10) at level of $\mathbb{R}$-constructible enhanced sheaves.

Let $\psi_{\pm}(w) = \pm\frac{2}{3}w^{\frac{3}{2}}$ and denote by α the argument of w. We want to study $E^{\psi_+} \oplus E^{\psi_-}$. The stalk of this sheaf changes at the change of $t \in \mathbb{R}$ and $\alpha = \arg(w)$:

- if exists $n \in \mathbb{Z}$ such that $\alpha + 2n\pi \in \{\pi, \pm\frac{\pi}{3}\}$ (i.e. if we are on a Stokes line), then $\Re(\psi_+) = \Re(\psi_-)$. So:

$$\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)} = \begin{cases} \mathbb{C}^2 & \text{, if } t + \Re(\psi_+) \geq 0; \\ 0 & \text{, otherwise ;} \end{cases}$$

- if $\alpha + 2n\pi \notin \{\pi, \pm\frac{\pi}{3}\}, \forall n \in \mathbb{Z}$, then $\Re(\psi_+) \neq \Re(\psi_-)$. So:

  (i) if $\frac{\pi}{3} + \frac{4}{3}n\pi < \alpha < \pi + \frac{4}{3}n\pi,\ n \in \mathbb{Z}$, then $\Re(\psi_+) > \Re(\psi_-)$ and we have

$$\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)} = \begin{cases} 0 & \text{, if } t < -\Re(\psi_-); \\ \mathbb{C} & \text{, if } -\Re(\psi_-) \leq t < -\Re(\psi_+); \\ \mathbb{C}^2 & \text{, if } t \geq -\Re(\psi_+); \end{cases}$$

  (ii) if $-\frac{\pi}{3} + \frac{4}{3}n\pi < \alpha < \frac{\pi}{3} + \frac{4}{3}n\pi,\ n \in \mathbb{Z}$, then $\Re(\psi_+) > \Re(\psi_-)$ and we have

$$\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)} = \begin{cases} 0 & \text{, if } t < -\Re(\psi_+); \\ \mathbb{C} & \text{, if } -\Re(\psi_+) \leq t < -\Re(\psi_-); \\ \mathbb{C}^2 & \text{, if } t \geq -\Re(\psi_-). \end{cases}$$

Notice that the rank of $\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)}$ can be seen also by Figure 4.7, indeed if $\alpha = \arg(w)$ is fixed, there are three cases: if

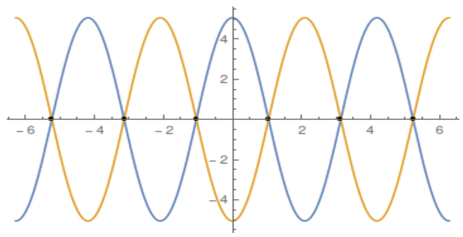


Figure 4.7: Graphics of $\Re(\psi_+)$ and $\Re(\psi_-)$

we are below both the graphics it is zero, if we are between them it is 1 (i.e. the stalk is $\mathbb{C}$) and if we are above both of them it is 2 (i.e. the stalk is $\mathbb{C}^2$).

Define $t_+ := -\Re(\psi_+)$ and $t_- := -\Re(\psi_-)$. When w is fixed, these are well defined real numbers, and notice that the sheaf that we are considering can be rewritten as

$$E^{\psi_+} \oplus E^{\psi_-} = \mathbb{C}_{\{t \geq t_+\}} \oplus \mathbb{C}_{\{t \geq t_-\}}.$$

In this way we can easily see that the rank of $\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)}$, if we are not on a Stokes line, change exactly when $t = t_+$ and $t = t_-$.

4.2.2 The enhanced Fourier-Sato transform

Recall that the classical Fourier-Laplace transform of a function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\hat{f}(w) = \frac{1}{2\pi i} \int_\gamma f(z) e^{-zw} dz,$$

with γ an appropriate path. So, by (4.3), we see that $Ai(w)$ is exactly the transform of $\exp(z^3/3)$.

Recall also that in the previous chapter we defined the enhanced Fourier-Sato transform as an analogue of the above transform for $\mathbb{R}$-constructible enhanced sheaves.

In this case we are working with a one-dimensional complex vector space $\mathbb{V}$ with complex coordinate z, and its dual $\mathbb{V}^*$ with complex coordinate w.
So using the same notations as in Definition 3.7, for $F \in \tilde{E}^b_{\mathbb{R}\text{-}c}(\mathbb{C}_{\mathbb{V}_\infty})$ one has

$$F^\curlywedge = R\tilde{q}_! \left(E^{-zw} \overset{*}{\otimes} \tilde{p}^{-1} F \right) [1],$$

where $\overset{*}{\otimes}$ denotes the convolution functor, $\tilde{p} : \mathbb{V} \times \mathbb{V}^* \times \mathbb{R} \to \mathbb{V} \times \mathbb{R}$ and $\tilde{q} : \mathbb{V} \times \mathbb{V}^* \times \mathbb{R} \to \mathbb{V}^* \times \mathbb{R}$ denote the projections, and where, for ψ a meromorphic function, one sets

$$E^\psi = \mathbb{C}_{\{t + \Re(\psi(z)) \geq 0\}}.$$

As we just saw, the Airy function is the transform of $e^{\varphi(z)}$, with $\varphi(z) = z^3/3$, so we want to study the enhanced Fourier-Sato transform of $F = E^\varphi$:

$$F^\curlywedge = R\tilde{q}_! \left(E^{-zw} \overset{*}{\otimes} \tilde{p}^{-1} E^\varphi \right) [1] \simeq R\tilde{q}_! (E^{\frac{z^3}{3} - zw})[1].$$

In fact, let $\bar{\varphi}(z, w) = \varphi(z)$. By Lemma 2.9, one has $\tilde{p}^{-1} E^\varphi \simeq E^{\bar{\varphi}}$. Moreover, the convolution $E^{-zw} \overset{*}{\otimes} E^{\bar{\varphi}} \simeq E^{-zw + \bar{\varphi}} = E^{-zw + \frac{z^3}{3}}$, by Lemma 2.10.
So, the stalk of $F^\curlywedge$ at $(w, t) \in \mathbb{C} \times \mathbb{R}$ is

$$\begin{aligned} (F^\curlywedge)_{(w,t)} &= R\Gamma_c \left(\tilde{q}^{-1}(w, t);\ E^{\frac{z^3}{3} - zw}|_{\tilde{q}^{-1}(w,t)} \right) [1] \\ &= R\Gamma_c \left(\mathbb{C}_z \times \{w\} \times \{t\};\ E^{\frac{z^3}{3} - zw}|_{\mathbb{C}_z \times \{w\} \times \{t\}} \right) [1]. \end{aligned} \tag{4.11}$$

Therefore we are interested in the compact support first homology group of the region $S = \{t + \Re\left(\frac{z^3}{3} - zw\right) \geq 0\}$.
Fixed $w \in \mathbb{C}$, at the change of $t \in \mathbb{R}$ one has S as in Figure 4.8.

So the homology group that we are studying vanishes in the first case, has rank one in the second case and has rank 2 in the last one.

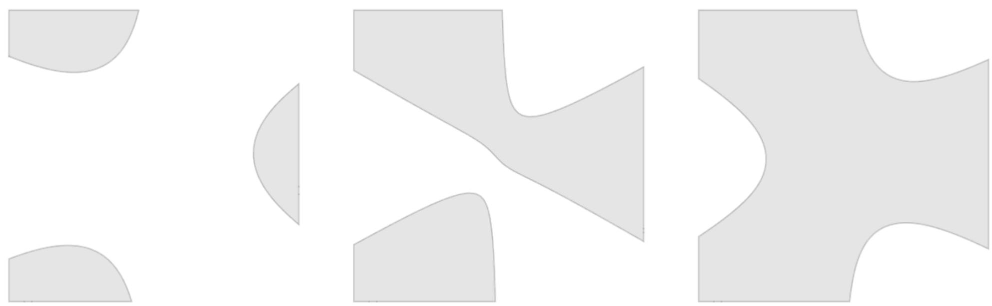


Figure 4.8: The region defined by $t + \Re\left(\frac{z^3}{3} - zw\right) \geq 0$ in the complex z-plane, fixed w and increasing t.

4.2.3 Conclusions

Recall that if S is a subanalytic subset of the complex vector space $\mathbb{V}$, its Borel-Moore homology groups are defined by

$$H_k^{BM}(S;\mathbb{C}) = H^k R\Gamma(S;\omega_S),$$

where $k \in \mathbb{Z}$ and ω_S is the dualizing complex of S.
So, if $S = \{z \colon t + \Re\left(\frac{z^3}{3} - zw\right) \geq 0\}$ as in the previous sections, by Poincaré duality one has

$$H_1^{BM}(S) \simeq H_c^1(S;\mathbb{C}_S)^*.$$

This is exactly what we need to compute the homology of (4.11) of the previous section.

Recall that, using Morse theory, we were able to find the cycles $\mathcal{J}_+$ and $\mathcal{J}_-$, that pass through the critical points p_+ and p_- of $\mathcal{I}$, respectively. Reversing the change of variables used to obtain

(4.4), we have $\mathcal{I} = \frac{z^3}{3} - zw$ and $p_\pm = \pm w^{1/2}$.
$\mathcal{J}_\pm$ were constructed to be those along which we have steepest ascent of $h = \Re(\mathcal{I})$. Since S is defined by $t + h(z) \geq 0$, if $p_\pm \in S$ then also the whole corresponding cycle is contained in the region. This makes them Borel-Moore cycles, hence they are the generators of the dual of the compact support homology in (4.11).

The change of its rank is when $t = t_+$ and $t = t_-$. In fact,

$$p_+ = w^{\frac{1}{2}} \in S, \text{ if } t \geq t_-, \text{ and}$$
$$p_- = -w^{\frac{1}{2}} \in S, \text{ if } t \geq t_+.$$

Suppose now that w is fixed and $t_+ < t_-$ (the opposite case is analogous). One has

(a) if $t < t_+$, neither of the cycles are contained in S, so

$$\operatorname{rank}\left((F^\curlywedge)_{(w,t)}\right) = \operatorname{rank}\left(\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)}\right) = 0;$$

(b) if $t_+ \leq t < t_-$, only $\mathcal{J}_- \subset S$, so

$$\operatorname{rank}\left((F^\curlywedge)_{(w,t)}\right) = \operatorname{rank}\left(\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)}\right) = 1;$$

(c) if $t \geq t_-$, both $\mathcal{J}_+$ and $\mathcal{J}_-$ are contained in S, so

$$\operatorname{rank}\left((F^\curlywedge)_{(w,t)}\right) = \operatorname{rank}\left(\left(E^{\psi_+} \oplus E^{\psi_-}\right)_{(w,t)}\right) = 2.$$

Since we can do the same computation that we did before to find explicit parametric equations for $\mathcal{J}_\pm$, we can draw the cycles in the complex z-plane. In this way we can have a clear view of what we just described by looking at Figure 4.9. Here is shown the case $t_+ < t_-$, as above, so the three images represent the cases (a), (b) and (c) respectively.

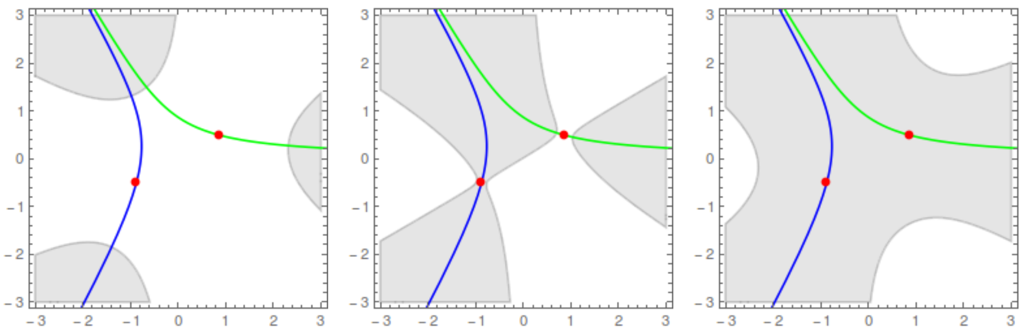


Figure 4.9: Here are represented the cycles J_+ (the green one) and J_- (the blue one), and their position with respect to the region S (the coloured one). This is the case when $t_+ < t_-$, and the three figures show what happens when t is respectively $t < t_+$, $t_+ \leq t < t_-$ and $t \geq t_-$.

Recall that the Stokes lines are defined by $\Re\left(\frac{2}{3}w^{3/2}\right) = 0$.
Let l_i be the Stokes lines cyclically ordered and let S_i be the sector bounded by l_{i-1} and l_i.
The discussion we have done so far in this chapter leads us to conclude that

$$F^{\curlywedge}|_{S_i \times \mathbb{R}} \simeq \left(E^{\psi_+} \oplus E^{\psi_-}\right)|_{S_i \times \mathbb{R}}, \tag{4.12}$$

for every sector S_i.
Moreover, one knows that (4.12) extends to $W_i = S_i \cup l_i \cup S_{i+1}$.

We just saw that the homology that defines $F^{\curlywedge}$ is generated by the cycles denoted by $\mathcal{J}_\pm$, and recall that in (4.9) we computed the transformation matrices that describe what happens to our generator cycles when we cross a Stokes line. In other words, these transforms explain how to pass from an isomorphism to the other in $W_i \cap W_{i+1}$.

In conclusion, we are sure that we can reconstruct the sheaf $F^{\curlywedge}$ completely from $E^{\psi_+} \oplus E^{\psi_-}$.

Bibliography

[1] A. D'Agnolo, M. Hien, G. Morando, C. Sabbah, *Topological computation of some Stokes phenomena on the affine line*, Ann. Inst. Fourier (Grenoble), to appear (2018), pp. 51

[2] A. D'Agnolo, M. Kashiwara, *A microlocal approach to the enhanced Fourier-Sato transform in dimension one*, Adv. Math. **339** (2018), 1-59

[3] A. D'Agnolo, M. Kashiwara, *Enhanced perversities*, J. reine angew. Math. (Crelle's journal), ahead of print (2016), pp.57

[4] A. D'Agnolo, M. Kashiwara, *Riemann-Hilbert correspondence for holonomic $\mathcal{D}$-modules*, Publ. Math. Inst. Hautes Études Sci. **123** no. 1 (2016), 69-197

[5] M. Kashiwara, P. Schapira, *Ind-sheaves*, Astérisque 271 (2001), 136 pp.

[6] M. Kashiwara, P. Schapira, *Irregular holonomic kernels and Laplace transform*, Selecta Mathematica (2015), $\mathsf{doi:10.1007/s00029-015-0185-y}$, 55 pp.

[7] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften 292, Springer-Verlag (1990).

[8] J. Schneiders, *An Introduction to $\mathcal{D}$-modules*, Bulletin de la Société Royale des Sciences de Liège, Vol.63 (1994).

[9] G.G. Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments*, Transaction of the Cambridge Philosophical Society, Vol. 10 (1857), 105-128.

[10] E. Witten, *Analytic continuation of Chern-Simons Theory*, AMS/IP Studies in Advanced Mathematics, Vol. 50 (2010).