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TORSION THEORY IN GIRAUD SUBCATEGORIES OF THE FUNCTOR CATEGORY OF A RING

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Introduction

Category theory plays a central role in the study of ring and module theory; many properties of modules over a ring R are easier to be understood if viewed in terms of results in the abelian category Mod-R (or R-Mod). In the last years a new technique in module theory started to be widely studied to solve or characterize problems that were previously addressed in a classical way without appreciable results. Moreover, given any ring R (we assume that R is a ring with unity 1_R and that $1_R \neq 0_R$) one may consider the abelian category $((R\text{-mod}^{op}), \mathcal{A}b)$ (the **functor category**) of contravariant additive functors from the category of finitely presented left R-modules to the category of abelian groups. Indeed, the study of functor categories, which extend Mod-R, represents a further development in the study of ring theory. An example of the progress in this field is the one of Auslander who, in 1966, proposed to study representation theory of R in terms of the category $((R\text{-mod}^{op}), \mathcal{A}b)$, obtaining for example that if R-mod is abelian (so that R is left-coherent) then also $\text{fp}((R\text{-mod}^{op}), \mathcal{A}b)$ (the category of finitely presented functors in $((R\text{-mod}^{op}), \mathcal{A}b)$) is abelian; Herzog, in [6], describes the contravariant Gabriel spectrum of R, the set of indecomposable injective objects of the functor category $((R\text{-mod}^{op}), \mathcal{A}b)$, as in bijection with the set of pure-injective indecomposable left R-modules. In particular, the set of pure-injective indecomposable left R-modules is in bijection with the points of the covariant Gabriel spectrum of R, too.
Another important result obtained in the study of the functor category involves, this time, the covariant functor category $(R\text{-mod}, \mathcal{A}b)$; Gruson and Jensen in [13] described the injective functors of $(R\text{-mod}, \mathcal{A}b)$ as those isomorphic to $(M \otimes -)$ for some $M \in \text{Mod-}R$ pure-injective module, giving in this way an important characterization of pure-injective modules in Mod-R; this application is explained in detail in Section 2.3.

The growing attention given to the functor category motivates this thesis, whose main objective is to study in detail some aspects of the inclusion of

Mod-R both in the covariant and in the contravariant functor category. More precisely, there exists an embedding of Mod-R in $((R\text{-mod}^{op}), \mathcal{A}b)$ given by the functor

$$M \longmapsto (-, M)$$

which sends M to the contravariant Hom functor $(-, M)$. In Section 2.1, it is proved that this functor has an exact left adjoint, and Mod-R is a Giraud subcategory of $((R\text{-mod}^{op}), \mathcal{A}b)$, which gives more information about the behaviour of the inclusion of the module category in the functor category. On the other hand, in Section 2.2 a dual result is proved: Mod-R turns out to be a co-Giraud subcategory of $(R\text{-mod}, \mathcal{A}b)$, since the inclusion (called *tensor embedding*)

$$M \to (M \otimes -)$$

has an exact right adjoint functor.
So far, the fact that Mod-R embeds as a Giraud (resp. co-Giraud) subcategory in $((R\text{-mod}^{op}), \mathcal{A}b)$ (resp. $(R\text{-mod}, \mathcal{A}b)$) has not been considered as a main subject to focus upon, and it is very difficult to find any reference to it in the state-of-the-art literature.
Chapter 2 is then devoted to introduce functor categories with the purpose of investigating the embedding of the module category, both in the covariant (Section 2.2) and in the contravariant (Section 2.1) case; Section 2.3 is about pure-injective modules and the aforementioned characterization, given as a historical motivation to the introduction of functor categories.

In full generality, given a Giraud (resp. co-Giraud) subcategory of any abelian category, a way to transfer torsion pairs from the ambient category to the Giraud (resp. co-Giraud) subcategory and vice versa was developed in [12]. In Section 3.1, Chapter 3, the functioning of this transfer is explained in full detail, in order to proceed, in Section 3.2, to apply this tool to the case of the functor categories with their Giraud or co-Giraud subcategory Mod-R. In Section 3.1, a condition to have a bijective correspondence between torsion pairs in the ambient category and those in the Giraud (resp. co-Giraud) subcategory is given; only a part of this condition has been fully understood when applied to the case of functor categories. In Section 3.2.1 a full explanation of the few results obtained in the attempt of completing the characterization of the bijection is given; further developments of the theory, possibly succeeding in obtaining such a characterization, may be motivated by the study of (co)tilting torsion pairs. Indeed, tilting torsion pairs, and in general tilting modules, are widely studied in module theory. It may be possible that, once gathered information about the bijection between torsion pairs in the module category and in the functor category, one could be able to understand if the transfer keeps the property of being (co)tilting for a torsion pair. In that case, it would be relevant to understand whether studying (co)tilting torsion pairs in the functor category gives more appreciable results rather than studying them in the module category or not.

In order to address functor categories, some basic concepts are introduced in Chapter 1. Namely, in Section 1.1 the concept of Grothendieck category, which is a property that all the categories studied in the thesis share, is introduced, together with the definition of finitely generated objects and injective envelopes of objects in an abelian category. In Section 1.2 the definition of torsion pair is given and its characterizations are presented; in Section 1.4 Giraud subcategories, the main object of interest in the thesis, are defined and studied in detail; finally in Section 1.5 localizing subcategories are described, following the work of N. Popescu (see [2]), as they are strictly linked to Giraud subcategories and torsion pairs (see the characterization of localizing subcategories in terms of torsion classes given in Theorem 1.5.20). Section 1.3 contains the definition and main properties of Gabriel topologies, which play a central role in the proof of the Gabriel-Popescu Theorem given in Section 1.6. The theorem has been chosen to be included in the thesis since it establishes an inclusion, as a Giraud subcategory, of any Grothendieck category in a suitable module category, giving a powerful and clear example of Giraud subcategory. In detail, any Grothendieck category $\mathcal{G}$ with a generator G can be embedded as a Giraud subcategory in the category Mod-R with $R = \text{End}(G)$. A remarkable corollary of the Gabriel-Popescu Theorem is that any Grothendieck category is quotient of a module category modulo a localizing subcategory, and an original proof for both this corollary and the Gabriel-Popescu theorem is provided. This proof gives, in particular, an explicit description of the Gabriel filter on $R = \text{End}(G)$ associated to the Grothendieck category $\mathcal{G}$.

Chapter 1

Preliminaries

Basic elements of abelian categories theory are developed in the first sections of this chapter, such as the fundamental concept of torsion pair and that of Giraud subcategory. In the last section a proof of the Gabriel-Popescu Theorem is given, together with some remarkable corollaries.

1.1 Grothendieck categories, injective envelopes and finitely generated objects

Grothendieck categories are introduced since they are the object of the Gabriel-Popescu Theorem, by which they can be embedded in module categories.

Definition 1.1.1. *Given a category $\mathcal{A}$, a set $\{U_i : i \in I\}$ of objects of $\mathcal{A}$ is a* ***family of generators*** *if, for every pair of morphisms α, $\beta : A \to B$, with $\alpha \neq \beta$, there exists an index $i \in \mathcal{I}$ and a morphism $u : U_i \to A$ such that $\alpha u \neq \beta u$.*

Proposition 1.1.2. *If $\mathcal{C}$ is cocomplete and has a family of generators, then $\mathcal{C}$ has a generator.*

Definition 1.1.3. *$\mathcal{C}$ is a Grothendieck category if:*

- *$\mathcal{C}$ is abelian;*
- *$\mathcal{C}$ is cocomplete;*
- *$\varinjlim : (I, \mathcal{C}) \to \mathcal{C}$ is exact for any small category I;*
- *$\mathcal{C}$ has a generator.*

If $\mathcal{C}$ is cocomplete, direct limits are exact if and only if they preserve monomorphisms; indeed, the direct limit functor $\varinjlim : (I, \mathcal{C}) \to \mathcal{C}$ (for a small and filtrant category I) is right exact; then it is enough that it preserves monomorphisms for it to achieve left exactness.
Let us consider, given $C \in Ob(\mathcal{C})$, a direct system of subobjects $(C_i)_{i\in I}$ of $\mathcal{C}$. If direct limits are exact in $\mathcal{C}$, then $\varinjlim C_i$ is a subobject of C which coincides with $\sum_I C_i$ and it is called **direct union** of the subobjects C_i. The exactness of limits implies that direct unions preserve finite intersections, so for any subobject B of C one has:

$$AB5: \quad (\sum_I C_i) \cap B = \sum_I (C_i \cap B).$$

Proposition 1.1.4. *The following assertions are equivalent for $\mathcal{C}$ cocomplete and abelian:*

1. *direct limits are exact in $\mathcal{C}$.*
2. *$\mathcal{C}$ satisfies $AB5$.*
3. *$\forall \alpha : B \to C$ and direct system $(C_i)_i \in I$ of subobjects of C, one has $\alpha^{-1}(\sum_I C_i) = \sum_I \alpha^{-1}(C_i)$.*

Proof. See ([1], Chapter V, Proposition 1.1). □

Proposition 1.1.5. *Let I be a small category and $\mathcal{C}$ be an abelian category. Then Fun$(I, \mathcal{C})$ is an abelian category.*

Proof. See ([1], Chapter IV, Proposition 7.1). □

Next result is a well-known result, usually called Yoneda lemma:

Lemma 1.1.6. *Let I be a small preadditive category. For every object $i \in I$ and every additive functor $T : I^{op} \to \mathcal{A}b$ there is an isomorphism:*

$$\mathrm{Nat}(h_i, T) \simeq T(i)$$

which is natural both in i and T (here h_i is contravariant Hom functor $\mathrm{Hom}(-, i)$*).*

Proof. See ([1], Chapter IV, Proposition 7.3). □

Corollary 1.1.7. *A small preadditive category I is equivalent to the full subcategory of* $\mathrm{Hom}(I^{op}, \mathcal{A}b)$ *consisting of the representable functors.*

Since a sequence $0 \to T' \to T \to T'' \to 0$ in $\mathrm{Hom}(I^{op}, \mathcal{A}b)$ is exact if and only if $0 \to T'(i) \to T(i) \to T''(i) \to 0$ is exact for each $i \in I$, we also get:

Corollary 1.1.8. *The family $(h_i)_{i \in I}$ is a family of projective generators for* $\mathrm{Hom}(I^{op}, \mathcal{A}b)$.

Proposition 1.1.9. *If I is a small preadditive category, then the functor category $\mathcal{C} = \mathrm{Hom}(I^{op}, \mathcal{A}b)$ is a Grothendieck category.*

Proof. The category $\mathcal{C}$ is abelian thanks to Proposition 1.1.5. Furthermore, for colimits in $\mathcal{C}$ are computed point-wise, the exactness of direct limits in $\mathcal{A}b$ implies their exactness in $\mathcal{C}$. Finally, the objects in I form a family of generators for $\mathcal{C}$ as proved in Corollary 1.1.7 and Corollary 1.1.8. □

The following few results are used to develop some useful tools about injective envelopes in abelian categories.

Definition 1.1.10. *A category $\mathcal{C}$ is:*

- ***locally small*** *if the class of subobjects of any object is a set;*
- ***pseudo-complemented*** *if the lattice L of subobjects of any object is pseudo-complemented, namely if for any $a \in L$ there exists $c \in L$ such that $c \neq a$ and $a \wedge c = 0$, with c maximal for this property.*

From now on we assume that $\mathcal{C}$ is abelian, locally small and pseudo-complemented.

Definition 1.1.11. *We say that a subobject $B \hookrightarrow C$ is **essential** in C if $B \cap C' \neq 0$ $\forall 0 \neq C' \subseteq C$. Slightly more generally, we call a monomorphism $\alpha : B \to C$ **essential** if $\mathrm{Im}\,\alpha$ is an essential subobject of C.*

Lemma 1.1.12. *If $\alpha : B \to C$ and $\beta : C \to D$ are monomorphisms, $\beta\alpha$ is essential if and only if both α and β are essential.*

Lemma 1.1.13. *If $\alpha : C \to E$ is a monomorphism and E is an injective object, then for every essential monomorphism $\beta : C \to C'$ there exists a monomorphism $\gamma : C' \to E$ such that $\gamma\beta = \alpha$.*

Proof. E is injective $\Rightarrow \exists \gamma : C' \to E$ with $\gamma\beta = \alpha$. Therefore $\mathrm{Ker}\,\gamma \cap \mathrm{Im}\,\beta = \mathrm{Ker}(\gamma\beta) = \mathrm{Ker}\,\alpha = 0$; being β essential, $\mathrm{Ker}\,\gamma = 0$ so γ is a monomorphism. □

Definition 1.1.14. *An* ***injective envelope*** *of an object C is an essential monomorphism $C \hookrightarrow E$ where E is injective.*

Proposition 1.1.15. *If $\alpha : C \to E$ and $\alpha' : C \to E'$ are two injective envelopes of C, then there is an isomorphisms $\gamma : E \to E'$ such that $\gamma\alpha = \alpha'$.*

Proof. By Lemma 1.1.13, there exists a monomorphism $\gamma : E \to E'$ such that $\gamma\alpha = \alpha'$. Furthermore γ is essential by Lemma 1.1.12, and being E injective $\operatorname{Im}\gamma$ splits off as a direct summand of E', so $\operatorname{Im}\gamma = E'$. □

This proposition implies that the injective envelope of a given object is unique up to isomorphisms.

Proposition 1.1.16. *An object E is injective if and only if any essential extension of E is and isomorphism.*

Proof. Let E be injective. Every monomorphism $\alpha : E \to C$ splits; if α is essential, it must be an isomorphism.
Vice versa, if every essential extension of E is an isomorphism, given a monomorphism $\alpha : C \to C'$, and a morphism $\varphi : C \to E$, we have the following push-out diagram:

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ E & \xrightarrow[\beta]{} & P \end{array}$$

β is a monomorphism since α is a monomorphism. If now we denote with K a pseudo-complement of E in P, and with $\pi : P \to P/K$ the canonical morphism, since $K \cap E = 0$ we obtain that $\pi\beta : E \to P \to P/K$ is a monomorphism. Moreover it is essential, since every non-zero subobject of P/K can be written as L/K with $K \subset L$, and being K maximal with respect to $K \cap E = 0$, it follows that $L \cap E \neq 0$, so $\operatorname{Im}(\pi\beta) \cap L/K \neq 0$. By hypothesis, $\pi\beta$ is an isomorphism. So the morphism $(\pi\beta)^{-1}\pi\varphi' : C' \to E$ realizes an extension of φ to C'. Therefore E is injective. □

Proposition 1.1.17. *If C is a subobject of some injective object, then C has an injective envelope.*

Proof. Suppose that C is a subobject of E, with E injective, and that C' is a maximal essential extension of C in E. Every essential extension of C' can be embedded in E by Lemma 1.1.13, and the inclusion is an isomorphism due to the maximality of C'. It follows that C' is injective, so it is the injective envelope of C. □

Proposition 1.1.18. *The monomorphism* $C_1 \oplus ... \oplus C_n \to E(C_1) \oplus ... \oplus E(C_n)$ *induces an isomorphism:*

$$E(C_1 \oplus ... \oplus C_n) \xrightarrow{\sim} E(C_1) \oplus ... \oplus E(C_n).$$

This proposition is an immediate consequence of the following lemma:

Lemma 1.1.19. *If* $C_i \to C_i'$ $(i = 1, ..., n)$ *are essential monomorphisms, then* $C_1 \oplus ... \oplus C_n \to C_1' \oplus ... \oplus C_n'$ *is essential.*

Proof. See ([1], Chapter V, Lemma 2.7). □

Definition 1.1.20.
- *An object* B *of* $\mathcal{C}$ *is called* ***indecomposable*** *if it cannot be written as a direct sum of two non-zero subobjects.*
- *A subobject* B *of* C *in* $\mathcal{C}$ *is called* ***irreducible*** *if it cannot be written as the intersection of two subobjects of* C *in which* B *is properly contained.*
- *A subobject* B *of* C *in* $\mathcal{C}$ *is irreducible if* C/B *is* ***coirreducible****, i.e. any pair of non-zero subobjects of* C/B *has non-zero intersection.*

Proposition 1.1.21. *The following assertions are equivalent for an injective object* E*:*

1. *E is indecomposable;*
2. *any subobject of E is coirreducible;*
3. *E is the injective envelope of a coirreducible object;*
4. *E is the injective envelope of each one of its non-zero subobjects.*

The following proposition extends Baer's criterion for module categories to any Grothendieck category.

Proposition 1.1.22. *If* $\mathcal{C}$ *is a Grothendieck category with a family of generators* $(U_i)_{i \in I}$*, an object* E *is injective if and only if for every monomorphism* $\alpha : C \to U_i$ *and morphism* $\varphi : C \to E$ *there exists a morphism* $\varphi' : U_i \to E$ *such that* $\varphi' \alpha = \varphi$*.*

Proof. See ([1], Chapter V, Proposition 2.9). □

The last part of the section is a brief exposition of results about finitely generated objects in Grothendieck categories.

Definition 1.1.23. *Let $\mathcal{C}$ be a Grothendieck category. An object $C \in Ob(\mathcal{C})$ is called **finitely generated** if the lattice $L(C)$ of its subobjects is compact, i.e. if $\forall\ C = \sum_I C_i$ with $(C_i)_{i\in I}$ direct family of subobjects of C, there exists $i_0 \in I$ such that $C = C_{i_0}$.*

Lemma 1.1.24. *Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence in $\mathcal{C}$.*

1. *If C is finitely generated, then C'' is finitely generated;*
2. *if both C' and C'' are finitely generated, then C is finitely generated.*

Proof. 1) Let $C'' = \sum C_i''$. Every C_i'' can be written as $C_i'' = C_i/C'$ for a unique subobject C_i of C with $C' \subseteq C_i$. The family $(C_i)_{i\in I}$ is also directed, so $C = C_{i_0}$ and then $C'' = C_{i_0}/C' = C''_{i_0}$.
2) If $C = \sum C_i$, being C' finitely generated one has that $C' \subseteq C_{i_0}$ for some i_0. It follows that $C'' = \sum_{i\geq i_0} C_i/C'$, and since C'' is finitely generated, one has that $C'' = C_{i_1}/C'$ for some i_1. It follows, by the five lemma, that $C = C_{i_1}$. □

Proposition 1.1.25. *An object C is finitely generated if and only if the functor $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserves direct unions, namely if $\Phi : \varinjlim \mathrm{Hom}(C, D_i) \to \mathrm{Hom}(C, \sum_I D_i)$ is an isomorphism for any direct system $(D_i)_{i\in I}$ of subobjects of an object D.*

Proof. Of course such morphism is a monomorphism. If C is finitely generated, and $\alpha : C \to \sum_I D_i$, $\mathrm{Im}\,\alpha$ is finitely generated by Lemma 1.1.24. Therefore $\mathrm{Im}\,\alpha \subseteq D_{i_0}$ for some i_0. It follows that Φ is an epimorphism. Vice versa is straightforward. □

Definition 1.1.26. *A category $\mathcal{C}$ is called **locally finitely generated** if it has a family of generators which are finitely generated.*

Lemma 1.1.27. *Let $\mathcal{C}$ be locally finitely generated. If $\alpha : B \to C$ is an epimorphism, and C is finitely generated, then there exists a finitely generated subobject $B' \subseteq B$ such that $\alpha(B') = C$.*

Definition 1.1.28. *An object C is called **finitely presented** if it is finitely generated and every epimorphism $B \to C$ (with B finitely generated) has a finitely generated kernel.*

Proposition 1.1.29. *Let $\mathcal{C}$ be locally finitely generated. An object C is finitely presented if and only if $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserves direct limits, i.e. the morphism $\Phi : \varinjlim \mathrm{Hom}(C, D_i) \to \mathrm{Hom}(C, \varinjlim D_i)$ is an isomorphism for any direct system $(D_i)_{i\in I}$ in $\mathcal{C}$.*

Proof. Let us assume that Φ is an isomorphism; then C is finitely generated by Proposition 1.1.25.
If now $0 \to K \to B \to C \to 0$ is an exact sequence with B finitely generated, and $(K_i)_{i\in I}$ is a family of finitely generated subobjects of K, then $C \simeq \varinjlim B/K_i$. By hypothesis, this isomorphism factors over some B/K_{i_0}, i.e. the sequence $0 \to K/K_{i_0} \to B/K_{i_0} \to C \to 0$ splits. Therefore K/K_{i_0} is finitely generated, so K is finitely generated by Lemma 1.1.24, and C is finitely presented.
Let now C be finitely presented. The morphism Φ is a monomorphism: let $C \to D_i$ such that $C \to D_i \to \varinjlim D_i$ is zero. Then, being C finitely generated, $C \to D_i \to D_j$ is zero for some $j \geq i$, and this implies that Φ is a monomorphism.
It is also an epimorphism: let $\alpha : C \to \varinjlim D_i$. Let D'_i be the image of D_i in $\varinjlim D_i$, so that $\varinjlim D_i = \sum D'_i$. Being C finitely generated, α factors over some D'_i with $\alpha' : C \to D'_i$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow & & \downarrow \alpha' & & \\ 0 & \longrightarrow & K & \longrightarrow & D_i & \longrightarrow & D'_i & \longrightarrow & 0. \end{array}$$

By Lemma 1.1.27, there exists a subobject $P' \subseteq P$ finitely generated, such that $P' \to P \to C$ is an epimorphism. Its kernel $K \cap P$ is finitely generated since C is finitely presented. By Lemma 1.1.30, it follows that $K \cap P' \subseteq \mathrm{Ker}(D_i \to D_j)$ for some $j \geq i$. Moreover $P' \to P \to D_i \to D_j$ factors over C giving the morphism $C \to D_j$, that shows that Φ is an epimorphism. □

Lemma 1.1.30. *Let $\mathcal{C}$ be a category which satisfies $AB5$, and $(C_i)_{i\in I}$ a direct system. Then $\forall k \in I$ one has*

$$\mathrm{Ker}(C_k \to \varinjlim C_i) = \sum_{j\geq k} \mathrm{Ker}(C_k \to C_j).$$

Proof. See ([1], Chapter V, Lemma 1.2). □

1.2 Torsion theory

In this section we assume that $\mathcal{C}$ is a complete, cocomplete, locally small and abelian category.

Definition 1.2.1. *A* ***preradical*** *r of $\mathcal{C}$ assigns to each object C a subobject $r(C)$ in such a way that every morphism $C \to D$ induces a morphism*

$r(C) \to r(D)$ by restriction. In other words, a preradical is a subfunctor of the identity functor on $\mathcal{C}$.

The class of preradicals on $\mathcal{C}$ forms a complete lattice, with partial order defined by the following relation: $r_1 \leq r_2$ if $r_1(C) \subseteq r_2(C)$ for each $C \in OB(\mathcal{C})$. Moreover any family $(r_I)_{i\in I}$ of preradicals has a least upper bound, $\sum_I r_i$, and a greatest lower bound, $\bigcap_I r_i$. If r_1 and r_2 are preradicals, one defines the following operations:

$$r_1 r_2(C) = r_1(r_2(C))$$

$$r_1 : r_2(C)/r_1(C) = r_2(C/r_1(C)).$$

*Moreover one says that r is **idempotent** if $rr = r$, and it is a **radical** if $r : r = r$, i.e. if $r(C/r(C)) = 0$ for each $C \in \mathcal{C}$.*

Lemma 1.2.2. *If r is a radical, $D \subseteq r(C) \Rightarrow D \subseteq r(C/D) = r(C)/D$.*

Proof. The morphism $C \to C/D$ induces $r(C) \to r(C/D)$ with kernel D, so $r(C)/D \subseteq r(C/D)$.
On the other hand, $\alpha : C/D \to C/r(C)$, again defined in the canonical way, induces the zero morphism on $r(C/D)$, so $r(C/D) \subseteq \operatorname{Ker}\alpha = r(C)/D$. □

If r is a preradical of $\mathcal{C}$, we define r^{-1} of $\mathcal{C}^{op}$ by setting $r^{-1}(M) = M/r(M)$. It is clear that r is idempotent if and only if r^{-1} is a radical, and vice versa. Given a preradical r one can associate two classes of objects of $\mathcal{C}$:

$$\mathcal{T}_r = \{C : r(C) = C\}$$

$$\mathcal{F}_r = \{C : r(C) = 0\}.$$

Note that $\mathcal{F}_r = \mathcal{T}_{r^{-1}}$.

Proposition 1.2.3. *The class $\mathcal{T}_r$ is closed under quotients and coproducts; the class $\mathcal{F}_r$ is closed under subobjects and products.*

Proof. Let us consider a family $(C_i)_{i\in I}$ of objects in $\mathcal{T}_r$. Since $r(C_i) = C_i$, one has that the image of the morphisms $C_i \to \bigoplus_I C_i$ is contained in $r(\bigoplus_I C_i)$, and using the definition of coproduct it is clear that $r(\bigoplus_I C_i) = \bigoplus_I C_i$. Let now $\alpha : C \to D$ be an epimorphism, so that $\operatorname{Im}\alpha = D$. Moreover $\operatorname{Im}\alpha \subseteq r(D)$ since C is contained in $\mathcal{T}_r$, and clearly $r(D) \subseteq D = \operatorname{Im}\alpha$. An analogous proof can be shown for the class $\mathcal{F}_r$, using the duality mentioned above. □

Corollary 1.2.4. *If $C \in \mathcal{T}_r$ and $D \in \mathcal{F}_r$ then $\operatorname{Hom}_{\mathcal{C}}(C, D) = 0$.*

Any class of objects of $\mathcal{C}$ which is closed under quotients and coproduct is called **pretorsion class**; if the class is instead closed under subobjects and products, it is called **pretorsion-free class**.
Given a pretorsion class $\mathcal{A}$, if $C \in OB(\mathcal{C})$, one can denote with $t(C)$ the sum of all the subobject of C contained in $\mathcal{A}$. In particular, $t(C) \in \mathcal{A}$.
Therefore each object C contains a subobject $t(C)$ contained in $\mathcal{A}$, and $t(C)$ is maximal with respect to this property. Combining this information with the definition of $\mathcal{T}_r$, and considering r idempotent, one obtains:

Proposition 1.2.5. *There exists a bijection between idempotent preradicals of $\mathcal{C}$ and pretorsion classes of $\mathcal{C}$. Dually, there exists a bijection between radicals of $\mathcal{C}$ and pretorsion-free classes of $\mathcal{C}$.*

In particular, if r is a preradical of $\mathcal{C}$, and $\hat{r}$ is the preradical corresponding to $\mathcal{T}_r$ (i.e. $\hat{r}(C)$ is the largest subobject D of C such that $r(D) = D$), then $\hat{r}$ is the largest idempotent less or equal to r).

Proposition 1.2.6. *For every preradical r, there exists a largest idempotent preradical $\hat{r} \leq r$, and there exists a smallest radical $\bar{r} \geq r$.*

Note that $\bar{r}(C)$ is the smallest subobject D of C such that $r(C/D) = 0$.

Proposition 1.2.7. *1. If r is idempotent, then so is also $\bar{r}$.*

2. If r is a radical, then so is also $\hat{r}$.

Proof. It suffices to prove (2); we attempt to prove that $\hat{r}(C/\hat{r}(C)) = 0\,\forall C$, i.e. that $C/\hat{r}(C)$ has no non-zero subobjects in $\mathcal{T}_r$. Let $D \supseteq \hat{r}(C)$ be a subobject of C, $r(D/\hat{r}(C)) = D/\hat{r}(C)$. Then $\hat{r}(C) \subseteq r(D) = D$, so $D \subseteq \hat{r}(C)$ and $D/\hat{r}(C) = 0$. □

Proposition 1.2.8. *The following assertions are equivalent for a preradical r:*

a) r is a left exact functor;

b) if $D \subseteq C$, then $r(D) = r(C) \cap D$;

c) r is idempotent and $\mathcal{T}_r$ is closed under subobjects.

Proof.

$(a \Leftrightarrow b)$ Since the kernel of the morphism $r(C) \to r(C/D)$ induced by $C \to C/D$ is equal to $r(C) \cap D$, b is equivalent to left exactness of r;

$(b \Rightarrow c)$ $r(C) \subseteq C$, and using b we obtain $r(r(C)) = r(C) \cap r(C) = r(C)$, i.e. r is idempotent. If furthermore $C \in \mathcal{T}_r$, and $D \subseteq C$, then $r(D) = C \cap D = D$, so $D \in \mathcal{T}_r$;

$(c \Rightarrow b)$ $r(D) \subseteq r(C) \cap D \subseteq D$ is obvious. Moreover $r(C) \cap D$ belongs to $\mathcal{T}_r$ as a subobject of $r(C)$; r is idempotent, so $r(C) \cap D = r(D)$.

□

Definition 1.2.9. *A pretorsion class is called* ***hereditary*** *if it is closed under subobjects.*

Corollary 1.2.10. *There is a bijection between left exact preradicals and hereditary pretorsion classes.*

Definition 1.2.11. *A* ***torsion pair*** *in $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ made by classes of objects of $\mathcal{C}$ such that:*

- $\mathrm{Hom}_{\mathcal{C}}(T, F) = 0 \ \forall T \in \mathcal{T},\ F \in \mathcal{F}$;
- *if* $\mathrm{Hom}_{\mathcal{C}}(C, F) = 0 \ \forall F \in \mathcal{F}$, *then* $C \in \mathcal{T}$;
- *if* $\mathrm{Hom}_{\mathcal{C}}(T, C) = 0 \ \forall T \in \mathcal{T}$, *then* $C \in \mathcal{F}$.

If this occurs, $\mathcal{T}$ is called ***torsion class****, and its objects are called* ***torsion objects****; $\mathcal{F}$ is called* ***torsion-free class****, and its objects are called* ***torsion-free objects****. In this case, any object $C \in \mathcal{C}$ is the middle term of a short exact sequence $0 \to T \to C \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.*

A class of objects $\mathcal{A}$ **generates** a torsion pair in the following way:

$$\mathcal{F} = \{F : \mathrm{Hom}(C, F) = 0 \, \forall C \in \mathcal{A}\}$$

$$\mathcal{T} = \{T : \mathrm{Hom}(T, F) = 0 \, \forall F \in \mathcal{F}\}$$

in this case $(\mathcal{T}, \mathcal{F})$ is a torsion pair, and $\mathcal{T}$ is the smallest torsion class containing $\mathcal{A}$. Dually, a class $\mathcal{A}$ **cogenerates** a torsion pair in the following way:

$$\mathcal{T} = \{T : \mathrm{Hom}(T, C) = 0 \, \forall C \in \mathcal{A}\}$$

$$\mathcal{F} = \{F : \mathrm{Hom}(T, F) = 0 \, \forall T \in \mathcal{T}\}$$

and $\mathcal{F}$ is the smallest torsion-free class containing $\mathcal{A}$.

Proposition 1.2.12. *The following assertions are equivalent for a class $\mathcal{T}$:*

a) $\mathcal{T}$ is a torsion class for some torsion pair;

b) $\mathcal{T}$ is closed under quotients, coproducts and extensions.

Proof.

$(a \Rightarrow b)$ Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. $\mathcal{T}$ is closed under quotients, and it is closed under coproducts since $\mathrm{Hom}(\bigoplus_I T_i, F) \simeq \prod_I \mathrm{Hom}(T_i, F)$.

Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence with C' and C'' in $\mathcal{T}$. If F is torsion-free, and there exists $\alpha : C \to F$, then α is zero on C', so α factors over C'', but there exist no non-zero morphisms from C'' to F; then $\alpha = 0$ and $C \in \mathcal{T}$.

$(b \Rightarrow a)$ Vice versa, if $\mathcal{T}$ is closed under quotients, coproducts and extensions, let us denote with $(\mathcal{T}', \mathcal{F})$ the torsion pair generated by $\mathcal{T}$. To show that $\mathcal{T} = \mathcal{T}'$, consider C such that $\mathrm{Hom}(C, F) = 0 \, \forall F \in \mathcal{F}$. Since $\mathcal{T}$ is a pretorsion class, there exists a subobject T of C belonging to $\mathcal{T}$, maximal with respect to this property. Actually, $C/T \in \mathcal{F}$; in fact, if $\alpha : T'' \to C/T$ for $T'' \in \mathcal{T}$, $\mathrm{Im}\, \alpha \in \mathcal{T}$, and if $\alpha \neq 0$, one would get a subobject of C strictly larger than T and contained in $\mathcal{T}$, in contradiction with what was defined above. Therefore $\alpha = 0$ and $C/T \in \mathcal{F}$, from which one has $C = T$.

□

By duality one has:

Proposition 1.2.13. *The following assertions are equivalent for a class $\mathcal{F}$:*

a) $\mathcal{F}$ is a torsion-free class for some torsion pair;

b) $\mathcal{F}$ is closed under subobjects, products and extensions.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{T}$ is in particular a pretorsion class, so each object C contains a maximal subobject $t(C)$ contained in $\mathcal{T}$ called **torsion subobject** of C. An object C is torsion-free if and only if $t(C) = 0$, since $C \in \mathcal{F}$ if and only if $\mathrm{Hom}(T, C) = 0 \, \forall T \in \mathcal{T}$. The idempotent preradical t is actually a radical, as one can prove from the fact that $\mathcal{T}$ is closed under extensions. Conversely, if t is an idempotent preradical, one achieves a torsion pair $(\mathcal{T}_t, \mathcal{F}_t)$ with

$$\mathcal{T}_t = \{C : t(C) = C\}$$

$$\mathcal{F}_t = \{C : t(C) = 0\}.$$

Proposition 1.2.14. *There exists a bijection between torsion pairs and idempotent radicals.*

Corollary 1.2.15. *If r is an idempotent preradical, then $\bar{r}$ is the idempotent radical corresponding to the torsion pair generated by $\mathcal{T}_r$.*

Proof. Since $\bar{r}$ is the smallest idempotent radical containing r, it must correspond to the smallest torsion class containing $\mathcal{T}_r$. □

Proposition 1.2.16. *Let $\mathcal{A}$ be a class of objects closed under quotients. The torsion class generated by $\mathcal{A}$ consists of the objects C such that every non-zero quotient of C has a non-zero subobject in $\mathcal{A}$.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be the torsion pair generated by $\mathcal{A}$. Since $\mathcal{A}$ is closed under quotients, an object belongs to $\mathcal{F}$ if and only if it has no non-zero subobject in $\mathcal{A}$. Therefore the assertion becomes that C belongs to $\mathcal{T}$ if and only if it has no non-zero quotient in $\mathcal{F}$, which is an obvious property for any torsion pair $(\mathcal{T}, \mathcal{F})$. □

Definition 1.2.17. *A torsion pair $(\mathcal{T}, \mathcal{F})$ is called **hereditary** if $\mathcal{T}$ is hereditary, i.e. if it is closed under subobjects.*

Thus we can state the following proposition, which can be proved directly from Proposition 1.2.14:

Proposition 1.2.18. *There exists a bijection between hereditary torsion pairs and left exact radicals.*

Proposition 1.2.19. *A torsion pair $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.*

Proof. The fact that t is left exact implies that if $F \in \mathcal{F}$, then $t(E(F)) \cap F = t(F) = 0$, which implies that $E(F) \in \mathcal{F}$, since F is essential in $E(F)$.
Conversely, if $\mathcal{F}$ is closed under injective envelopes, $T \in \mathcal{T}$, $C \subseteq T$, then there exists a morphism $\beta : T \to E(C/t(C))$ such that the diagram

$$\begin{array}{ccc} C & \longrightarrow & T \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ C/t(C) & \rightarrowtail & E(C/t(C)) \end{array}$$

is commutative. However, since $E(C/t(C))$ is torsion-free, $\beta = 0$ so $\alpha = 0$ and $C = t(C) \in \mathcal{T}$. □

Proposition 1.2.20. *Let $\mathcal{A}$ be a class which is closed under subobjects and quotients; then the torsion pair generated by $\mathcal{A}$ is hereditary.*

Proof. It can be proved that the torsion-free class is closed under injective envelopes. Let F be a torsion-free object and $\alpha : C \to E(F)$, with $C \in \mathcal{A}$. Therefore $\operatorname{Im}\alpha \in \mathcal{A}$ and $F \cap \operatorname{Im}\alpha \subseteq F$ is a non-zero object belonging to $\mathcal{A}$, in contradiction with the fact that F is torsion-free. □

Corollary 1.2.21. *If r is a left exact radical, then $\bar{r}$ is exact.*

Corollary 1.2.22. *Let r be a left exact preradical and let C be an object; then $r(C)$ is an essential subobject of $\bar{r}(C)$.*

Proof. Let $D \subseteq \bar{r}(C)$ such that $D \cap r(C) = 0$. Then $r(D) = 0$, so $\bar{r}(D) = 0$. However, since $\bar{r}(D) = \bar{r}(C) \cap D$, one deduces that $D = 0$. □

Proposition 1.2.23. *A torsion pair is hereditary if and only if it can be cogenerated by an injective object.*

Proof. Let E be injective and $\mathcal{T} = \{C : \operatorname{Hom}(C, E) = 0\}$. If $C \in \mathcal{T}$, $D \subseteq C$, and $\alpha : D \to E$ is non-zero morphism, then α extends to $C \to E$, but this is in contradiction with the definition of $\mathcal{T}$. Therefore $\mathcal{T}$ is hereditary.
Conversely, let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair, G a generator for the category, and $E = \prod E(G/L)$ ranging over all $L \subseteq G$ such that $G/L \in \mathcal{F}$. Then E is torsion-free, so $\operatorname{Hom}(C, E) = 0\ \forall C \in \mathcal{T}$. On the other hand, if $C \notin \mathcal{T}$, there exists a subobject $D \subseteq C$ of the form G/L together with a non-zero morphism $\alpha : D \to F$ for some $F \in \mathcal{F}$; $\operatorname{Im}\alpha$ is torsion-free, then α induces a morphism $D \to E$ which extends to $C \to E$. Therefore $C \in \mathcal{T}$ if and only if $\operatorname{Hom}(C, E) = 0$, and from this one deduces that E cogenerates the torsion pair. □

From now on, we assume that the category $\mathcal{C}$ is actually a module category. One has the following results.

Proposition 1.2.24. *A hereditary torsion pair is generated by the family of torsion cyclic modules $A/\mathfrak{a}$.*

Proof. A module M is a torsion module if and only if every cyclic submodule of M is so. The rest of the proof follows immediately. □

A hereditary torsion pair is thus uniquely determined by the family of right ideals $\mathfrak{a}$ for which $A/\mathfrak{a}$ is a torsion module.

Lemma 1.2.25. *If L and M are modules, then* $\operatorname{Hom}(L, E(M)) = 0$ *if and only if* $\operatorname{Hom}(C, M) = 0\ \forall C \subseteq L$ *cyclic submodule.*

Proof. Let $C \subseteq L$, $\alpha : C \to M$ e $\alpha \neq 0$. Then one has $\alpha' : C \to M \to E(M)$, which extends to $\alpha'' : L \to E(M)$, causing a contradiction.
Conversely, let $\alpha : L \to E(M)$, and let $\alpha' : C \to E(M)$ be the restriction. Now $\operatorname{Im} \alpha'$ is a submodule of $E(M)$; M is essential in $E(M)$, so $M \cap \operatorname{Im} \alpha \neq 0$ if $\alpha \neq 0$. Then one finds a non-zero morphism $C \to M$, causing a contradiction. □

Proposition 1.2.26. *Let us consider a hereditary torsion pair cogenerated by the injective module E. A module M is torsion-free if and only if it is a submodule of a direct product of copies of E.*

Proof. Every submodule of a product $E^{(I)}$ is torsion free.
Conversely, let M be torsion-free and $0 \neq x \in M$, then xA is not a torsion module, so $\operatorname{Hom}(xA, E) \neq 0$. Since E is injective, $\forall 0 \neq x \in M\ \exists \mu : M \to E$ such that $\mu(x) \neq 0$. One defines $\eta : M \to E^{(I)}$ (where $I = \operatorname{Hom}(M, E)$) as $\eta(x) = (\mu(x))_{\mu \in I}$. Therefore η is a monomorphism. □

1.3 Gabriel topologies

The results which were previously proved can be applied to the category of modules over a ring A, obtaining a correspondence between hereditary torsion pairs of Mod-A and families of ideals $\mathfrak{a}$ of A for which $A/\mathfrak{a}$ is a torsion module. Such families are families of neighbourhoods of 0 in A for some topologies on the ring.

Definition 1.3.1. *A **topological group** is an abelian group together with a topology for which the maps $(a, b) \longmapsto a + b$ and $a \longmapsto -a$ are continuous.*

In this setting, having fixed an $a \in G$, the map $x \longmapsto a+x$ is a homeomorphism, and U is a neighbourhood of a if and only if $U - a$ is a neighbourhood of 0.
Then, to assign a topology on a group it suffices to provide a filter of neighbourhoods $\mathcal{R}$ of 0, such that it satisfies the following two axioms:

N1) $\forall U \in \mathcal{R}\ \exists V \in \mathcal{R}$ such that $V + V \subseteq U$;

N2) $U \in \mathcal{R} \Rightarrow -U \in \mathcal{R}$.

Conversely, given a filter of neighbourhoods that satisfies these axioms, it generates a unique topology on G.

Definition 1.3.2. *A **topological ring** A is an additive topological group, such that moreover the map $(a, b) \longmapsto ab$ is continuous.*

Since $ab - a_0b_0 = (a - a_0)(b - b_0) + (a - a_0)b_0 + a_0(b - b_0)$, the continuity of the multiplication follows from two further axioms:

N3) $\forall a \in A$, $U \in \mathcal{R}$, $\exists V \in \mathcal{R}$ such that $aV \subseteq U$, $Va \subseteq U$;

N4) $\forall U \in \mathcal{R}\, \exists V \in \mathcal{R}$ such that $VV \subseteq U$.

Thus, given a filter of neighbourhoods of 0 which satisfies N1-N4, one obtains a unique ring topology on A.

Definition 1.3.3. *A ring is called* ***right linearly topologized*** *if there exists a basis of neighbourhoods $\mathscr{T}$ of 0 consisting of right ideals.*

The set $\mathscr{T}$ satisfies:

T1) $\mathfrak{a} \in \mathscr{T}$, $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \mathfrak{b} \in \mathscr{T}$;

T2) $\mathfrak{a}$, $\mathfrak{b} \in \mathscr{T} \Rightarrow \mathfrak{a} \cap \mathfrak{b} \in \mathscr{T}$;

T3) $\mathfrak{a} \in \mathscr{T}$, $a \in A \Rightarrow (\mathfrak{a} : a) \in \mathscr{T}$.

Conversely, if a set $\mathscr{T}$ of right ideals of A satisfying T1-T3 is given, then there exists a unique right linear topology on A for which $\mathscr{T}$ is a basis of neighbourhoods.

Definition 1.3.4. *A right topological A-module (i.e. an additive topological group in which the operation $(x, a) \longmapsto xa$ with $x \in M$ and $a \in A$ is continuous) is called* ***linearly topologized*** *if it has a basis of neighbourhoods of 0 consisting of submodules.*

The open submodules of this basis satisfy:

TM1) $L \subseteq L'$ submodules of M, L is open $\Rightarrow L'$ is open;

TM2) L, L' are open submodules $\Rightarrow L \cap L'$ is open;

TM3) L is open, $x \in M \Rightarrow (L : x)$ is open.

Let us denote with $\mathscr{T}$ the class of such neighbourhoods; then one can define a stronger topology on M, for which the set of open submodules is:

$$\mathscr{T}(M) = \{L \subseteq M : (L : x) \in \mathscr{T}\ \forall x \in M\}$$

this topology is called $\mathscr{T}$-topology on M; it is discrete if and only if $\mathrm{rAnn}(x) \in \mathscr{T}\ \forall x \in M$. In such a situation M is called $\mathscr{T}$-discrete.

Lemma 1.3.5. *The class of $\mathscr{T}$-discrete modules is a hereditary pretorsion class.*

Proof. Since M is $\mathscr{T}$-discrete if and only if $\mathrm{rAnn}(x) \in \mathscr{T}$ $\forall x \in M$, it is clear that the class of $\mathscr{T}$-discrete modules is closed under submodules. It is closed under quotients due to T1 and under direct sums due to T2. □

One thus obtains a left exact preradical t, and $t(M) = \{x \in M : \mathrm{rAnn}(x) \in \mathscr{T}\}$; $t(M)$ is called $\mathscr{T}$-pretorsion submodule of M.

Proposition 1.3.6. *There exists a bijection between:*

1. *Right linear topologies on A;*
2. *Hereditary pretorsion classes of A-modules;*
3. *Left exact preradicals of* Mod-A.

Proof. We have already seen that every linear topology induces a pretorsion class, in the proof of Lemma 1.3.5. Conversely, if $\mathcal{A}$ is a pretorsion class, let us consider the set $\mathscr{T}$ of the right ideals $\mathfrak{a}$ of A for which $A/\mathfrak{a} \in \mathcal{A}$. This family satisfies T1, since $\mathcal{A}$ is closed under quotients, and T2 since $A/(\mathfrak{a} \cap \mathfrak{b})$ is a submodule of $A/\mathfrak{a} \oplus A/\mathfrak{b}$, and T3 because if $\mathfrak{a} \in \mathscr{T}$ and $a \in A$, then the sequence $0 \to (\mathfrak{a} : a) \to A \to A/\mathfrak{a}$ is exact, and so $A/(\mathfrak{a} : a) \subseteq A/\mathfrak{a}$. Therefore $\mathscr{T}$ defines a right linear topology on A. We are left to show that this correspondence is a bijection.
If we start by taking $\mathscr{T}$, a right linear topology, we obtain $\mathcal{A} = \{M : \mathrm{rAnn}(x) \in \mathscr{T}\, \forall x \in M\}$, and then $\{\mathfrak{a} : A/\mathfrak{a} \in \mathcal{A}\} = \{\mathfrak{a} : (\mathfrak{a} : a) \in \mathscr{T}\, \forall a \in A\} = \mathscr{T}$ by T1.
On the other side, starting with $\mathcal{A}$, we attain $\mathscr{T} = \{\mathfrak{a} : A/\mathfrak{a} \in \mathcal{A}\}$ and then $\{M : \mathrm{rAnn}(x) \in \mathscr{T}\, \forall x \in M\} = \{M : \text{every cyclic submodule} \in A\} = \mathcal{A}$ by the closure properties of $\mathcal{A}$. □

Our goal is now to identify a certain type of linear topology, for which a correspondence with hereditary torsion classes will take place.

Definition 1.3.7. *If $\mathfrak{a}$ is a right ideal, and there exists $\mathfrak{b} \in \mathscr{T}$ such that $(\mathfrak{a} : b) \in \mathscr{T}$ $\forall b \in \mathfrak{b}$ then $\mathfrak{a} \in \mathscr{T}$. We will use T4 to refer to this assertion; a linear topology satisfying T1-T4 is called* ***(right) Gabriel topology****.*

Theorem 1.3.8. *There exists a bijection between:*

1. *Right Gabriel topologies on A;*
2. *Hereditary torsion pairs of* Mod-A*;*

3. Left exact radicals in Mod-A.

Proof. The equivalence ((2) $\Leftrightarrow$ (3)) has already been proved.
Let $\mathscr{T}$ be a Gabriel topology, $0 \to L \to M \to N \to 0$ an exact sequence of modules where L and N are $\mathscr{T}$-discrete. For every $x \in M$, let $\mathfrak{b} = \mathrm{rAnn}(\bar{x})$, where $\bar{x}$ denotes the image of x in N. Thereby $\mathfrak{b} \in \mathscr{T}$, and $\forall b \in \mathfrak{b}$ one has $xb \in L$, so $\mathrm{Ann}(xb) \in \mathscr{T}$. Since $\mathrm{Ann}(xb) = (\mathrm{Ann}(x) : b)$, T4 implies that $\mathrm{Ann}(x) \in \mathscr{T}$. Then M is $\mathscr{T}$-discrete, so the class of $\mathscr{T}$-discrete modules is closed under extensions and it is therefore hereditary.
On the other hand, if $\mathcal{T}$ is a hereditary torsion class, the corresponding topology $\mathscr{T} = \{\mathfrak{a} : A/\mathfrak{a} \in \mathcal{T}\}$ satisfies T4. Indeed, if $\mathfrak{a}$ is a right ideal for which $(\mathfrak{a} : b) \in \mathscr{T}$ $\forall b \in \mathfrak{b}$, with $\mathfrak{b} \in \mathscr{T}$, one may consider the sequence $0 \to \mathfrak{b}/\mathfrak{a}{\cap}b \to A/\mathfrak{a} \to A/(\mathfrak{a}{+}\mathfrak{b}) \to 0$ where $A/(\mathfrak{a}{+}\mathfrak{b}) \in \mathcal{T}$ since it is a quotient of $A/\mathfrak{b} \in \mathcal{T}$, and furthermore $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}) \in \mathcal{T}$, since $((\mathfrak{a} \cap \mathfrak{b}) : b) = (\mathfrak{a} : b) \in \mathscr{T}$. Inasmuch as $\mathcal{T}$ is closed under extensions, $A/\mathfrak{a} \in \mathcal{T}$, so $\mathfrak{a} \in \mathscr{T}$. □

Therefore, given any right linear topology $\mathscr{T}$ on A, the corresponding class of torsion modules is made by those modules whose elements are all annihilated by right ideals contained in $\mathscr{T}$. These modules are called **$\mathscr{T}$-torsion modules**.

1.4 Giraud subcategories

In this section, we assume that $\mathcal{A}$ is a complete Grothendieck category.

Definition 1.4.1. *A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is **reflective** if the inclusion functor $i : \mathcal{C} \to \mathcal{A}$ has a left adjoint a.*

Let $\mathcal{C}$ be reflective in $\mathcal{A}$, with adjunction isomorphisms:

$$\eta_{B,C} : \mathrm{Hom}(B, C) \to \mathrm{Hom}(a(B), C)$$

for $B \in \mathcal{A}$ and $C \in \mathcal{C}$, and natural transformations

$$\zeta : ai \to 1_{\mathcal{C}} \quad \text{and} \quad \xi : 1_{\mathcal{A}} \to ia$$

respectively counit and unit of the adjunction.

Lemma 1.4.2. *If there exists a morphism $\alpha : a(B) \to B$ with $\alpha\xi_B = 1_B$ then ξ_B is an isomorphism.*

Proof. The morphism $\xi_B : B \to a(B)$ has two factorizations: $\xi_B = 1_{a(B)}\xi_B = (\xi_B\alpha)\xi_B$ on $a(B)$, and the unicity of the factorization gives $1_{a(B)} = \xi_B\alpha$, so $\xi_B N$ is an isomorphism. Indeed, it is well known that the unit of the adjunction is the solution to a universal problem: $\forall\alpha : B \to C$ there exists a unique $\bar{\alpha} : a(B) \to C$ such that $\bar{\alpha}\xi_B = \alpha$. □

Proposition 1.4.3. *Every reflective subcategory $\mathcal{C}$ is complete and cocomplete.*

Proof. The category $\mathcal{C}$ is preadditive, since it is full in $\mathcal{A}$. If I is a small category, and $G : I \to \mathcal{C}$ is a functor, then there exists $B = \varprojlim iG$, with canonical morphisms $\pi_i : B \to G(i)$. Being $G(i) \in \mathcal{C}$, there exists $\bar{\pi}_i : a(B) \to G(i)$ such that $\bar{\pi}_i \xi_B = \pi_i$. The family $(\bar{\pi}_i)_{i \in I}$ is compatible with the morphisms in I, in fact: given $\lambda : i \to j$ in I, one has $G(\lambda)\bar{\pi}_i \xi_B = G(\lambda)\pi_i = \pi_j = \bar{\pi}_j \xi_B$, so $G(\lambda)\bar{\pi}_i = \bar{\pi}_j$. Therefore there exists an induced morphism $\beta : a(B) \to \varprojlim G = B$ such that $\pi_i \beta = \bar{\pi}_i$ $\forall i$.
Therefore $\pi_i \beta \xi_B = \bar{\pi}_i \xi_B = \pi_i$ $\forall i$, so $\beta \xi_B = 1$. By Lemma 1.4.2, we obtain that ξ_B is an isomorphism, and clearly $a(B)$ is a limit for G in $\mathcal{C}$. Then $i(\varprojlim G) = \varprojlim iG$.
As regarding direct limits, using the fact that a preserves them, one may write $a(\varinjlim iG) = \varinjlim aiG = \varinjlim G$ since $ai \simeq 1_{\mathcal{C}}$. □

In other words, inverse limits in $\mathcal{C}$ are calculated in $\mathcal{A}$, while direct limits in $\mathcal{C}$ are calculated in $\mathcal{A}$, and later reflected in $\mathcal{C}$ by means of the functor a.

Proposition 1.4.4. *If $\mathcal{C}$ is reflective in $\mathcal{A}$, and $a : \mathcal{A} \to \mathcal{C}$ preserves kernels, then $\mathcal{C}$ is abelian and has a generator and exact direct limits.*

Proof. We have already shown that $\mathcal{C}$ is preadditive, and that it has inverse and direct limits. Moreover, $\mathcal{C}$ is abelian: indeed, if α is a morphism in $\mathcal{C}$, we wish that $\bar{\alpha} : \operatorname{Coker}(\operatorname{Ker} \alpha) \to \operatorname{Ker}(\operatorname{Coker} \alpha)$ is an isomorphism. To prove it, we write kernels and cokernels in $\mathcal{A}$ as $\underline{\operatorname{Ker}}$ and $\underline{\operatorname{Coker}}$.
In this way, one has

$$\operatorname{Coker}(\operatorname{Ker} \alpha) = a(\underline{\operatorname{Coker}}(\underline{\operatorname{Ker}} \alpha))$$

and

$$\operatorname{Ker}(\operatorname{Coker} \alpha) = \underline{\operatorname{Ker}}(a(\underline{\operatorname{Coker}} \alpha)) = a(\underline{\operatorname{Ker}}(\underline{\operatorname{Coker}} \alpha))$$

since a preserves kernels. Then $\bar{\alpha}$ is an isomorphism.
Direct limits are exact in $\mathcal{C}$: if I is a small directed category, let G, $G' : I \to \mathcal{C}$ be two functors with a monomorphism $G \to G'$. The induced morphism $\varinjlim iG \to \varinjlim iG'$ is a monomorphism in $\mathcal{A}$, and since a preserves monomorphisms, it follows that $\varinjlim G \to \varinjlim G'$ is a monomorphism in $\mathcal{C}$. Finally, it is easy to prove that if U is a generator for $\mathcal{A}$, then $a(U)$ is a generator for $\mathcal{C}$. □

Definition 1.4.5. *A reflective subcategory of $\mathcal{A}$ is called **Giraud subcategory** if the left adjoint of the inclusion preserves kernels.*

Corollary 1.4.6. *If $\mathcal{C}$ is a Giraud subcategory of $\mathcal{A}$, then $\mathcal{C}$ is a Grothendieck category and a is exact. In general i is not exact.*

Proposition 1.4.7. *Let $\mathcal{C}$ be a Giraud subcategory of $\mathcal{A}$. An object C in $\mathcal{C}$ is injective if and only if $i(C)$ is injective in $\mathcal{A}$.*

Proof. The inclusion functor preserves monomorphisms, so if C is injective in $\mathcal{A}$, it is also in $\mathcal{C}$.
Conversely, let C be injective in $\mathcal{C}$ and let us be given $\beta : B \to B'$ in $\mathcal{A}$ and $\varphi : B \to C$. Then φ induces a morphism $a(B) \to C$ which extends, in $\mathcal{C}$, to a morphism $a(B') \to C$. The composition $B' \to a(B') \to C$ extends φ as wished. □

Consider, still assuming $\mathcal{C}$ to be a Giraud subcategory of $\mathcal{A}$, the class of objects B of $\mathcal{A}$ such that $a(B) = 0$ and denote it with $\mathcal{T}$. Similarly, consider the class of objects B of $\mathcal{A}$ such that $B \to ia(B)$ is a monomorphism, and denote it with $\mathcal{F}$.

Proposition 1.4.8. *The pair $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair for $\mathcal{A}$.*

Proof. From the exactness of a it immediately follows that $\mathcal{T}$ is closed under subobjects, quotients and extensions. From the fact that a has a right adjoint, it follows that a preserves coproducts, and then one obtains that $\mathcal{T}$ is a hereditary torsion class.
Of course one has $\mathrm{Hom}_{\mathcal{A}}(T, C) = 0$ $\forall T \in \mathcal{T}$ e $C \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ $\forall T \in \mathcal{T}$ and $F \in \mathcal{F}$. If B is such that $\mathrm{Hom}_{\mathcal{A}}(T, B) = 0\, \forall T \in \mathcal{T}$, then $B \in \mathcal{F}$, because $\mathrm{Ker}(B \to ia(B)) \in \mathcal{T}$. □

1.5 Localizing subcategories

In this section, following the works of N. Popescu (see [2]), the concept of localizing subcategory of an abelian category $\mathcal{A}$ is studied, with the purpose of finding a correspondence with that of torsion class.
Let $\mathcal{A}$ be an abelian category.

Definition 1.5.1. *A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is a* ***Serre subcategory*** *if for every sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, one has $A \in Ob(\mathcal{C})$ if and only if A', $A'' \in Ob(\mathcal{C})$. In other words, $\mathcal{C}$ is closed under subobjects, quotients and extensions.*

Definition 1.5.2. *Let $\mathcal{C} \subseteq \mathcal{A}$ be a Serre subcategory. Then the* ***quotient*** *$\mathcal{A}/\mathcal{C}$ is a category whose objects are exactly those of $\mathcal{A}$, and given X, $Y \in \mathcal{A}/\mathcal{C}$, one defines $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X, Y) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$, where $X' \subseteq X$ and*

$Y' \subseteq Y$ are such that X/X' and Y' belong to $\mathcal{C}$.
*The functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$, sending objects to themselves and morphisms $f : X \to Y$ to the corresponding element of the direct limit, is called **quotient functor**.*

The following universal property holds: if F is such that $F(C) = 0 \ \forall C \in Ob(\mathcal{C})$, then $\exists! F' : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ such that $F'T = F$, as pictured in the diagram below:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle T}\downarrow & \nearrow_{\exists! F'} & \\ \mathcal{A}/\mathcal{C} & & \end{array}$$

Definition 1.5.3. *Let $\mathcal{A}$ be locally small. A Serre subcategory $\mathcal{C}$ is **localizing** if the quotient functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ has a right adjoint $S : \mathcal{A}/\mathcal{C} \to A$ which is called **section functor**.*

Once given these definition, it is necessary to go more into the details of the construction of the quotient category $\mathcal{A}/\mathcal{C}$, in order to state the definition (Lemma 1.5.16) of $\mathcal{C}$-closed object in $\mathcal{A}$, which will play a central role in the proof of Corollary 1.6.11 of the Theorem 1.6.6 (Gabriel-Popescu).

Definition 1.5.4. *Let Σ be a class of morphisms in a category $\mathcal{A}$; the pair $(T, \mathcal{A}_\Sigma)$ (where $T : \mathcal{A} \to \mathcal{A}_\Sigma$ is a functor) is a **category of fractions** of $\mathcal{A}$ with respect to Σ if for each $s \in \Sigma$ $T(s)$ is an isomorphism in $\mathcal{A}_\Sigma$; furthermore if $F : \mathcal{A} \to \mathcal{A}'$ is a functor for which if $s \in \Sigma$ then $F(s)$ is an isomorphism, then $\exists F' : \mathcal{A}_\Sigma \to \mathcal{A}'$ such that $F'T = F$.*

Definition 1.5.5. *A class Σ of morphisms is called **multiplicative** if for any pair of composable morphisms s, s' in Σ, the composition ss' is again contained in Σ, and Σ contains all the identity morphisms.*

Definition 1.5.6. *A class Σ of morphisms is called right (left)-**permutable** if every angle*

$$\begin{array}{ccc} & & Y' \\ & & \downarrow \\ X & \xrightarrow[s]{} & Y \end{array}$$

(or coangle

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle s}\downarrow & & \\ Y) & & \end{array}$$

with $s \in \Sigma$ can be embedded in a diagram

$$\begin{array}{ccc} X' & \xrightarrow{s'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow[s]{} & Y \end{array}$$

(or

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ Y & \longrightarrow & Y') \end{array}$$

with $s' \in \Sigma$.

Definition 1.5.7. *A class Σ of morphisms is right (left)-**simplifiable** if for any pair of morphisms $f,\ g : X \to Y$ for which there exists $s : Y \to Y'$ $(s : X' \to X)$ in Σ such that $sf = sg$ $(fs = gs)$, then there exists $s' : Z \to X$ $(s' : Y \to Z)$ in Σ such that $fs' = gs'$ $(s'f = s'g)$.*

Definition 1.5.8. *A class Σ of morphisms is right (left)-**calculable** if it is multiplicative and right (left)-permutable and simplifiable.*

If $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$, one defines Σ as the set of those morphisms s in $\mathcal{A}$ such that $\operatorname{Ker} s \in \mathcal{C}$ and $\operatorname{Coker} s \in \mathcal{C}$.

Proposition 1.5.9. *Such class of morphisms is **bicalculable** (that is, it is both left and right-calculable).*

Proof. It suffices to show that it is right-calculable. The class Σ is multiplicative: indeed, if $s : X \to Y$ and $s' : Y \to Z$ are elements of Σ, then one has an exact sequence:

$$0 \to \operatorname{Ker} s \to \operatorname{Ker} s's \to \operatorname{Im} s \cap \operatorname{Ker} s' \to 0$$

so $\operatorname{Ker} s's \in Ob(\mathcal{C})$, since $\mathcal{C}$ is a Serre subcategory. The same holds for $\operatorname{Coker} s's$, so $s's$ is a morphism in Σ.
Let us consider the angle

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow[f]{} & Y \end{array}$$

with s in Σ. The pull-back diagram:

$$\begin{array}{ccc} Z \prod_Y X & \xrightarrow{f'} & X \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow[f]{} & Y \end{array}$$

can be built up with $\operatorname{Ker} s' \simeq \operatorname{Ker} s$ and $\operatorname{Coker} s' \subseteq \operatorname{Coker} s$. Therefore $s' \in \Sigma$, so Σ is permutable.

Finally, let us be given a sequence of morphisms in $\mathcal{A}$: $X \xrightarrow{f} Y \xrightarrow{s} Z$ with s in Σ and $sf = 0$. Then $\operatorname{Im} f \subseteq \operatorname{Ker} s$ and one obtains the exact sequence $0 \to \operatorname{Ker} f \xrightarrow{u} X \xrightarrow{p} \operatorname{Im} f \to 0$ where u is a morphism in Σ and $fu = 0$, from which it follows that Σ is simplifiable. □

From now on, we denote by Σ the class $\Sigma_{\mathcal{C}}$, and by $\mathcal{A}/\mathcal{C}$ the quotient category $\mathcal{A}_{\Sigma_{\mathcal{C}}}$, which has the same objects as $\mathcal{A}$ with morphisms defined above.

Lemma 1.5.10. *If $f : X \to Y$ is a morphism in $\mathcal{A}$, $T(f) = 0$ if and only if* $\operatorname{Im} f \in \mathcal{C}$.

Lemma 1.5.11. *Let $f : X \to Y$ be a morphism in $\mathcal{A}$; then $T(f)$ is a monomorphism (respectively an epimorphism) if and only if* $\operatorname{Ker} f \in Ob(\mathcal{C})$ *(respectively* $\operatorname{Coker} f \in Ob(\mathcal{C})$.

Lemma 1.5.12. *The category $\mathcal{A}/\mathcal{C}$ is preabelian; furthermore if $f : X \to Y$ is a morphism in $\mathcal{A}$, and if (X', i) is its kernel (respectively (Y', p) is its cokernel) then $(T(X'), T(i))$ is the kernel of $T(f)$ (respectively $(T(Y'), T(p))$ is the cokernel of $T(f)$).*

Lemma 1.5.13. *Let $f : X \to Y$ be a morphism in $\mathcal{A}$; then $T(f)$ is an isomorphism if and only if both* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *are objects of $\mathcal{C}$.*

Theorem 1.5.14. *The quotient category $\mathcal{A}/\mathcal{C}$ is abelian and the quotient functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is exact.*

Lemma 1.5.15. *Any morphism in the category $\mathcal{A}/\mathcal{C}$ can be written as a composition of the form $T(s'')^{-1}T(f)T(s')^{-1}$ where s' is a monomorphism and s'' is an epimorphism.*

A proof of these last results can be found in ([2] - pages 166-172).

From now until the end of this section, we assume that $\mathcal{C}$ is a localizing subcategory of the locally small abelian category $\mathcal{A}$.

Lemma 1.5.16. *Given an object $M \in Ob(\mathcal{A})$, the following assertions are equivalent:*

1. *$\forall s : X \to Y$, $s \in \Sigma_{\mathcal{C}}$, the canonical morphism $h_M(s) : \mathrm{Hom}_{\mathcal{A}}(Y, M) \to \mathrm{Hom}_{\mathcal{A}}(X, M)$ is an isomorphism;*
2. *if $f : X \to M$ is a monomorphism and $X \in Ob(\mathcal{C})$, then $f = 0$; moreover every monomorphism $s : M \to X$, with $s \in \Sigma_{\mathcal{C}}$ is a section;*
3. *$\forall X \in Ob(\mathcal{A})$ the group morphism*

$$T(X, M) : \mathrm{Hom}_{\mathcal{A}}(X, M) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), T(M))$$

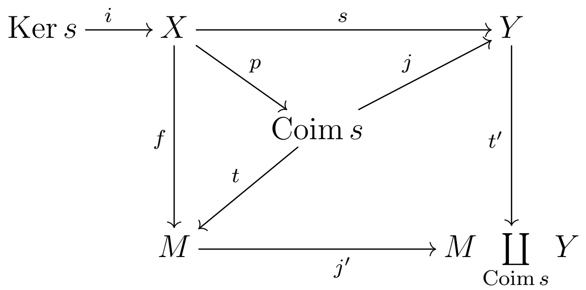
is an isomorphism.

*An object M satisfying one of the above conditions is called **$\mathcal{C}$-closed**.*

Proof.

($1 \Rightarrow 2$) Let (X', p) be the cokernel of $f : X \to M$, and f be a monomorphism; then $\mathrm{Ker}\, p = (X, f)$ and $p \in \Sigma$, so $\exists! t : X \to M$ such that $tp = 1_M$. Therefore p is a monomorphism and $f = 0$.
Moreover, if one has a short exact sequence $0 \to M \xrightarrow{s} X \to X' \to 0$ for which $X' \in Ob(\mathcal{C})$, then $\exists! t : X \to M$ such that $ts = 1_M$.

($2 \Rightarrow 1$) Let $f : X \to M$ be a morphism, and let $i : \mathrm{Ker}\, s \to X$ be the canonical inclusion. Then $fi = 0$, so $\exists! t : \mathrm{Coim}\, s \to M$ such that $tp = f$. One has a commutative diagram:



where $s \in \Sigma$ by hypothesis, and $j \in \Sigma$ since $\mathrm{Ker}\, j = 0$ and $\mathrm{Coker}\, j = \mathrm{Coker}\, s$; moreover j' is a monomorphism and $\mathrm{Coker}\, j' \in Ob(\mathcal{C})$ by the properties of the push-out. Then there exists a section $h : M \coprod_{\mathrm{Coim}\, s} Y \to M$, with $hj' = 1_M$. Therefore $ht's = ht'jp = hj'tp = tp = f$. The uniqueness of ht' is obvious.

(2 ⇒ 3) Let $f : X \to M$ be a morphism such that $T(f) = 0$. Then $\operatorname{Im} f \in Ob(\mathcal{C})$ and $f = 0$; thereby, $T(X, M)$ is a group monomorphism.
Let now $u : T(X) \to T(M)$ be in $\mathcal{A}/\mathcal{C}$. By Lemma 1.5.15, $u = T(s'')^{-1}T(f)T(s')^{-1}$ with $s' : P \to X$, $s'' : M \to Q$, $f : P \to Q$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle s'}\downarrow & & \uparrow{\scriptstyle s''} \\ X & & M \end{array}$$

Moreover $\operatorname{Ker} s'' \in Ob(\mathcal{C})$, from which it follows that s'' is an isomorphism; $\operatorname{Coker} s'' \in Ob(\mathcal{C})$ and $\exists! t : X \to M$ with $ts' = (s'')^{-1}f$; finally $f = s''ts'$, so $T(f) = T(s'')T(t)T(s')$ and $u = T(s'')^{-1}T(f)T(s')^{-1} = T(t)$. It follows that $T(X, M)$ is an epimorphism.

(3 ⇒ 1) Let $s : X \to Y$, $s \in \Sigma$. One has:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(X, M) & \xrightarrow{T(X,M)} & \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), T(M)) \\ {\scriptstyle h_M(s)}\downarrow & & \downarrow{\scriptstyle h_{T(M)}(T(s))} \\ \operatorname{Hom}_{\mathcal{A}}(Y, M) & \xrightarrow[T(Y,M)]{} & \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(T(Y), T(M)) \end{array}$$

The horizontal lines and the second vertical line are isomorphisms, so the first vertical line is an isomorphism, too.

□

Corollary 1.5.17. *Let $\mathcal{A}$, $\mathcal{C}$, T and S be defined as above. One has that $\forall Z \in \mathcal{A}/\mathcal{C}$, $S(Z)$ is a $\mathcal{C}$-closed object.*

Proof. Let us consider the following diagram, with $s : X \to Y$ morphism in Σ, and with horizontal arrows given by the adjunction isomorphisms:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(X, S(Z)) & \xrightarrow{\Phi_{X,Z}} & \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), Z) \\ {\scriptstyle h_{S(Z)}(s)}\downarrow & & \downarrow{\scriptstyle h_Z(T(s))} \\ \operatorname{Hom}_{\mathcal{A}}(Y, S(Z)) & \xrightarrow[\Phi_{Y,Z}]{} & \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(T(Y), Z) \end{array}$$

Clearly $h_{S(Z)}(s)$ is an isomorphism, since $T(s)$ is an isomorphism in $\mathcal{A}/\mathcal{C}$. □

Proposition 1.5.18. *Let $\mathcal{A}$, $\mathcal{C}$, T and S be defined as above. Let $u : 1_{\mathcal{A}} \to ST$ and $\nu : TS \to 1_{\mathcal{A}/\mathcal{C}}$ be respectively the unit and the counit of the adjunction. The following assertions are true:*

1. *ν is a natural isomorphism;*

2. *$\forall X \in Ob(\mathcal{A})$, $u_X \in \Sigma$.*

Proof. 1. Let $Y \in Ob(\mathcal{A}/\mathcal{C})$:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(X, S(Y)) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), Y) \\ {\scriptstyle T(X,S(Y))} \searrow & & \nearrow {\scriptstyle h_{T(X)}(\nu_Y)} \\ & \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(T(X), TS(Y)) & \end{array}$$

$T(X, S(Y))$ is an isomorphism due to Corollary 1.5.17. Being every object of $\mathcal{A}/\mathcal{C}$ of the form $T(X)$ for some $X \in Ob(\mathcal{A})$, the fact that $h_{T(X)}(\nu_Y)$ is an isomorphism implies that ν_Y is an isomorphism for each $Y \in Ob(\mathcal{A}/\mathcal{C})$ by Yoneda.

2. $T(u_X)$ is the inverse of $\nu_{T(X)}$, so it is an isomorphism; then $u_X \in \Sigma$.

□

Corollary 1.5.19. *Let M be an object of $\mathcal{A}$. The following assertions are equivalent:*

1. *M is a $\mathcal{C}$-closed object;*

2. *u_M is an isomorphism.*

Proof.

(1 ⇒ 2) Let M be a $\mathcal{C}$-closed object. Then M has no non-zero subobjects in $\mathcal{C}$, so $\mathrm{Ker}\, u_M = 0$.
Moreover one may remark that $u_M : M \to ST(M)$ is split, so $\mathrm{Coker}\, u_M$ is isomorphic to a subobject of $ST(M)$, then also $\mathrm{Coker}\, u_M = 0$. It follows that u_M is an isomorphism.

(2 ⇒ 1) It follows from Corollary 1.5.17.

□

The following theorem establishes a key point for the study of localizing subcategories; indeed, an immediate consequence of this assertion is that localizing subcategories of a Grothendieck category $\mathcal{A}$ are in bijection with hereditary torsion classes of the category.

Theorem 1.5.20. *If $\mathcal{A}$ is a Grothendieck category, a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$ is localizing if and only if it is closed under arbitrary coproducts.*

Proof. Let $\mathcal{C}$ be a localizing subcategory of $\mathcal{A}$, and let $(C_i)_{i\in I}$ be a family of objects of $\mathcal{C}$. It is well known that $0 = T(\coprod_{i\in I} C_i) = \coprod_{i\in I} T(C_i)$, since if $C_i \in \mathcal{C}$, then $T(C_i) \simeq 0 \in \mathcal{A}/\mathcal{C}$. Now, if we have an object $C \in Ob(\mathcal{A})$ for which $T(C) = 0$, the canonical morphism $0 \xrightarrow{f} C$ is in particular such that $T(f)$ is an isomorphism; by Lemma 1.5.13 this implies that $C = \operatorname{Coker} f \in Ob(\mathcal{C})$. Therefore, since $T(\coprod_{i\in I} C_i) = 0$ by what we have seen before, $\coprod_{i\in I} C_i$ is an object of $\mathcal{C}$, which then is closed under arbitrary coproducts.
Conversely, let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$ and let it be closed under arbitrary coproducts. We need to show that T admits a right adjoint. By Freyd's adjoint functor theorem ([4] - Theorem 5.50), being $\mathcal{A}$ a Grothendieck category, it suffices to prove that T preserves arbitrary coproducts. Indeed, if we consider a family $(A_i)_{i\in I}$ of objects of $\mathcal{A}$, and the canonical morphism $f : \coprod_{i\in I} T(A_i) \to T(\coprod_{i\in I} A_i)$, we obtain that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are objects of $\mathcal{C}$. Therefore f is an isomorphism in the quotient, and T preserves arbitrary coproducts as wished. □

1.6 The Gabriel-Popescu Theorem

The purpose of this section is, given a Gabriel filter $\mathscr{T}$, to associate to it a localizing functor (that is, the left adjoint to the inclusion of a Giraud subcategory):

$$a : \text{Mod-}R \to \text{Mod-}(R, \mathscr{T})$$

where we may obtain, considering $\mathcal{T}$ as the class of $\mathscr{T}$-torsion modules, that $\mathcal{T} = \operatorname{Ker} a$ and the Giraud subcategory Mod-$(R, \mathscr{T})$ is defined as $\mathcal{F} \cap \operatorname{Ker}(\operatorname{Ext}^1_R(\mathcal{T}, -)) = \mathcal{T}^{\perp_0} \cap \mathcal{T}^{\perp_1}$, according to the following definition:

Definition 1.6.1. *Given a class $\mathcal{T}$ of objects of* Mod-R*, one sets:*

$$\mathcal{T}^{\perp_i} := \{M \in \text{Mod-}R : \operatorname{Ext}^i_R(X, M) = 0 \ \forall X \in \mathcal{T}\} \ \ \forall i = 0, 1, 2...$$

Such subcategory is called subcategory of the **$\mathcal{F}$-closed modules**. Moreover, the functor a induces an equivalence of categories Mod-$R/\mathcal{T} \simeq$ Mod-$(R, \mathscr{T})$. Let us consider, for each $M \in$ Mod-R, and each ideal $J \in \mathscr{T}$, the restriction map

$$a_J : M \simeq \operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(J, M)$$

Lemma 1.6.2. *For each $J \in \mathscr{T}$, the morphisms a_J are injective if and only if $M \in \mathcal{T}^{\perp_0} := \mathcal{F}$, and are surjective if and only if $M \in \mathcal{T}^{\perp_1} :=$* $\operatorname{Ker}(\operatorname{Ext}^1_R(\mathcal{T}, -))$

One obtains a well defined morphism $\varphi_M : M \to M_{(\mathscr{T})} := \varinjlim_{J\in\mathscr{T}} \mathrm{Hom}_R(J, M)$.

Lemma 1.6.3. *Let $\mathcal{T}$ be the hereditary torsion class corresponding to $\mathscr{T}$, and let r be the corresponding radical. The morphism $M \to M_{(\mathscr{T})}$ defines an endofunctor $-_{(\mathscr{T})}$ of* Mod-R*, together with a natural transformation $\varphi : 1_{\text{Mod-}R} \to -_{(\mathscr{T})}$ such that, for each $M \in$ Mod-R one has:*

- $\mathrm{Ker}(\varphi_M) = r(M)$*;*
- $\mathrm{Coker}(\varphi_M) \in \mathcal{T}$*;*
- *$M_{(\mathscr{T})} = 0$ if and only if $M \in \mathcal{T}$.*

Theorem 1.6.4. *Denote by a the subfunctor of the identity $a = (-)^2_{(\mathscr{T})}$. Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair in* Mod-R *and r its associated radical, and let $\mathscr{T}$ be the associated Gabriel filter.*
Let $T := \bigoplus_{J\in\mathscr{T}} R/J$. Then a is a localizing functor $a :$ Mod-$R \to$ Mod-$(R, \mathscr{T})$ *associated with* Mod-$(R, \mathscr{T}) := T^{\perp_0} \cap T^{\perp_1}$*. Furthermore:*

- $\mathrm{Ker}\, a = \mathcal{T} = Gen(T)$*;*
- *$\sigma = (\sigma_M)_{M\in\text{Mod-}R}$ is the unit of the adjunction $\langle a, i\rangle$, defined by the natural morphisms $\sigma_M : M \to a(M) = M_{(\mathscr{T})} := \varinjlim_{J\in\mathscr{T}} \mathrm{Hom}_R(J, M/r(M))$, induced by the canonical projection $M \to M/r(M)$ and by the restriction maps $J \hookrightarrow R$, for each $J \in \mathscr{T}$;*
- $\mathrm{Ker}(\sigma_M) = r(M)$ *e* $\mathrm{Coker}(\sigma_M) \in \mathcal{T}$*;*
- *a induces an equivalence* Mod-$R/\mathcal{T} \simeq$ Mod-$(R, \mathscr{T})$*.*

Using this theorem one may deduce that, by fixing a Gabriel filter, it is possible to produce a Giraud subcategory of Mod-R, denoted by Mod-$(R, \mathscr{T})$. In particular, given a Gabriel filter $\mathscr{T}$, one may build the localizing functor a associated with the category $T^{\perp_0} \cap T^{\perp_1}$, where $T = \bigoplus_{J\in\mathscr{T}} R/J$ as in the theorem; conversely, given a Giraud subcategory $\mathcal{C}$ of Mod-R together with a localizing functor a, the kernel of a is a hereditary torsion class $\mathrm{Ker}\, a = \mathcal{T}$, and to $\mathcal{T}$ one may associate the Gabriel filter $\mathscr{T} = \{J \leq R_R : R/J \in \mathcal{T}\}$. The following theorem can thereby be proved:

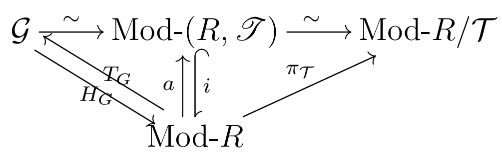
Theorem 1.6.5. *There exists a bijection between Gabriel topologies on R and Giraud subcategories of* Mod-R*.*

Proof. See ([1], Chapter X, Theorem 2.1). □

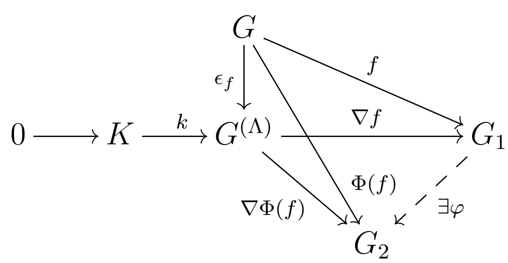
Let now $\mathcal{G}$ be a Grothendieck category with a generator G, let $R = \mathrm{End}(G)$; one has a well defined and faithful functor $H_G := \mathrm{Hom}_{\mathcal{G}}(G, -) : \mathcal{G} \to \text{Mod-}R$. Such functor is full, and induces an equivalence between $\mathcal{G}$ and a Giraud subcategory Mod-R. This is proved in:

Theorem 1.6.6 (Gabriel-Popescu)**.** *Let $\mathcal{G}$ be a Grothendieck category, G a generator, $R = \mathrm{End}_G(G)$, $H_{\mathcal{G}} = \mathrm{Hom}_{\mathcal{G}}(G, -)$. Then:*

- *H_G is full and faithful, so it induces an equivalence between $\mathcal{G}$ and $\mathrm{Im}\, H_G$;*
- *H_G admits a left adjoint exact functor $T_G : \text{Mod-}R \to \mathcal{G}$ such that $T_G(R_R) = G$;*
- *$\mathcal{T} = \mathrm{Ker}(T_G)$ is a hereditary torsion class in Mod-R which is associated to the Gabriel filter $\mathscr{T} = \{I = \langle r_\lambda : \lambda \in \Lambda\rangle \leq R_R : \coprod r_\lambda : G^{(\Lambda)} \twoheadrightarrow G$ is an epimorphism$\}$ of R, and $\mathrm{Im}\, H_G = \text{Mod-}(R, \mathscr{T})$;*
- *there exists a commutative diagram*



Proof. Being G a generator of $\mathcal{G}$, H_G is faithful. We show it is also full: given G_1 and G_2 in $Ob(\mathcal{G})$, and fixed $\Phi \in \mathrm{Hom}_R(H(G_1), H(G_2))$, we look for $\varphi \in \mathrm{Hom}_{\mathcal{G}}(G_1, G_2)$ such that $\forall f \in H(G_1)$ it will be true that $\Phi(f) = \varphi f$. Let $\Lambda = H(G_1)$; for each $f \in \Lambda$ we denote by $\epsilon_f : G \to G^{(\Lambda)}$ the corresponding monomorphism. There exists a unique morphism $\nabla f : G^{(\Lambda)} \to G_1$ such that $\nabla f \epsilon = f$, and ∇f is an epimorphism by the fact that G is a generator. In the same way, there exists a unique morphism $\nabla\Phi(f) : G^{(\Lambda)} \to G_2$ such that $\nabla\Phi(f)\epsilon_f = \Phi(f)$ for all $f \in \Lambda$. Let $k : K \to G^{(\Lambda)}$ be the kernel of ∇f. We obtain the commutative diagram:



We wish that $\nabla\Phi(f)k = 0$; if this is true, we attain the factorization $\nabla\Phi(f) = \varphi\nabla f$ for some $\varphi : G_1 \to G_2$, by the properties of the kernel. Moreover, for any $f : G \to G_1$ we have $\Phi(f) = \nabla\Phi(f)\epsilon_f = \varphi\nabla(f)\epsilon_f = \varphi f$, and this fact shows that the functor H_G is full.
It is left to prove that $\nabla\Phi(f)k = 0$. Let us consider $J \subseteq \Lambda$ finite, $f \in J$; there exist morphisms $\pi'_f : G^{(J)} \to G$, $\epsilon'_f : G \to G^{(J)}$ and $\epsilon_J : G^{(J)} \to G^{(\Lambda)}$. Let $k_J : K_J \to G^{(J)}$ be the kernel of the composition $\nabla f\epsilon_J : G^{(J)} \to G_1$. Being K the direct limit of the kernels K_J over the finite subsets $J \subseteq \Lambda$, it suffices to show that $\nabla\Phi(f)\epsilon_J k_J = 0$. We fix $\beta : G \to K_J$; using the fact that Φ is R-linear, we obtain:
$\nabla\Phi(f)\epsilon_J k_J\beta = \nabla\Phi(f)\epsilon_J(\sum_{f\in J}\epsilon'_f\pi'_f)k_J\beta = \sum_{f\in J}\nabla\Phi(f)\epsilon_f\pi'_f k_J\beta = \sum_{f\in J}\Phi(f)\pi'_f k_J\beta = \sum_{f\in J}\Phi(f\pi'_f k_J\beta) = \Phi(\sum_{f\in J}\nabla f\epsilon_J\epsilon'_f\pi'_f k_J\beta) = \Phi(\nabla f\epsilon_J k_J\beta) = 0$ since $\nabla f\epsilon_J k_J = 0$. The fact that β has arbitrarily been chosen, proves what we wished, that is $\nabla\Phi(f)k = 0$.

We move forward to prove the existence of the left adjoint to H_G and its exactness.
First of all, if T_G exists, it will have to fulfil the equality $T_G(R_R) = H^{-1}(R_R) = G$, and moreover it will have to preserve coproducts, due to right exactness. These remarks suggest the construction of T_G: for each $M \in \text{Mod-}R$ we fix a presentation

$$(*) : \quad R^{(\beta)} \xrightarrow{\psi} R^{(\alpha)} \xrightarrow{\varphi} M \to 0$$

using our remarks, a posteriori we should have:

$$T(*) : \quad G^{(\beta)} \xrightarrow{T(\psi)} G^{(\alpha)} \xrightarrow{T(\varphi)} T(M) \to 0$$

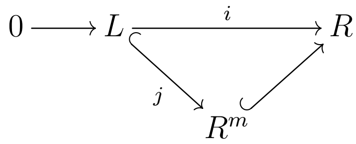
such that $\text{Hom}_R(M, H(-)) \simeq \text{Hom}_{\mathcal{G}}(T(M), -)$. Therefore, we apply the functor $\text{Hom}_R(-, H(-))$ to the sequence $(*)$ obtaining:

$$\begin{array}{ccccccc} 0 \longrightarrow \text{Hom}_R(M, H(-)) & \xrightarrow{\text{Hom}_R(\varphi,H(-))} & \text{Hom}_R(R^{(\alpha)}, H(-)) & \xrightarrow{\text{Hom}_R(\psi,H(-))} & \text{Hom}_R(R^{(\beta)}, H(-)) \\ & & \downarrow\simeq & & \downarrow\simeq \\ & & \text{Hom}_{\mathcal{G}}(G^{(\alpha)}, -) & \xrightarrow[\text{Hom}_{\mathcal{G}}(T(\psi),-)]{} & \text{Hom}_{\mathcal{G}}(G^{(\beta)}, -) \end{array}$$

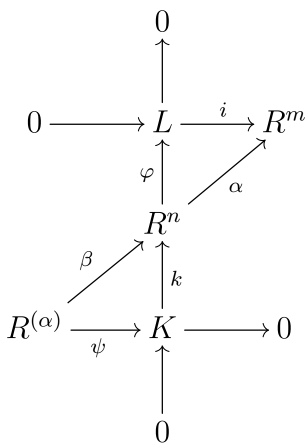
where the square is closed by $\text{Hom}_{\mathcal{G}}(T(\psi), -)$ coming from $T(\psi) : G^{(\beta)} \to G^{(\alpha)}$, and $T(\varphi) = \text{Coker}\, T(\psi)$; $T(\psi)$ exists by Yoneda lemma. Then T defines a functor $T_G : \text{Mod-}R \to \mathcal{G}$ such that $T_G(R_R) = G$. From the diagram and the sequence $T(*)$ we obtain, moreover, that $\text{Hom}_R(M, H(-)) \simeq$

$\mathrm{Ker}(\mathrm{Hom}_{\mathcal{G}}(T(\psi), -)) \simeq \mathrm{Hom}_{\mathcal{G}}(T(M), -)$ in a natural way, that is (T_G, H_G) is and adjoint pair.
We proceed to prove that T_G is an exact functor; of course it is right exact since it is a left adjoint; we are left to show that T_G preserves monomorphisms.

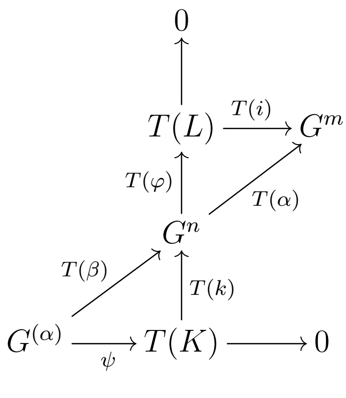
1^{st} case: let $0 \to L \xrightarrow{i} R^{(\alpha)}$ be a monomorphism, and let L be finitely generated. Then i factors:



so that we may assume that $\alpha = m \in \mathbb{N}$. We build a diagram $(D1)$:



with exact rows, column and diagonal. Applying the fact that T is right exact, we can build the diagram $(D2)$:



with exact column and bottom row. We wish the diagonal in $(D2)$ to be exact; from that the injectivity of $T(i)$ will follow using the commutativity of the diagram. For this purpose, we consider the following diagram:

$$\begin{array}{ccccc}
R^{(\alpha)} & \xrightarrow{\ \beta\ } & R^n & \xrightarrow{\ \alpha\ } & R^m \\
\Big\downarrow{\scriptstyle \eta_{R^{(\alpha)}}} & & \Big\downarrow{\scriptstyle \eta_{R^n}\ \simeq} & & \Big\downarrow{\scriptstyle \eta_{R^m}\ \simeq} \\
HT(R^{(\alpha)}) & \xrightarrow[HT(\beta)]{} & HT(R^n) & \xrightarrow[HT(\alpha)]{} & HT(R^m)
\end{array}$$

where η is the unit of the adjunction, and η_{R^n} and η_{R^m} are isomorphisms because H is fully faithful and $R^m = H(G^m)$ and $R^n = H(G^n)$; the upper row is exact. From that the exactness of the bottom row follows as well, and its exactness reflects to that of the sequence $T(R^{(\alpha)}) \to T(R^n) \to T(R^m)$.

2^{nd} case: We recall the every module can be viewed as the direct limit of its finitely generated submodules, that T preserves direct limits (being a left adjoint), and that direct limits are exact: we obtain, then, that T preserves monomorphisms of the form $0 \to L \xrightarrow{R^{(\alpha)}}$ where α is not necessarily finite.

3^{rd} case: Let now $0 \to L \xrightarrow{i} M$ be any monomorphism. Let us consider the pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & K & \xrightarrow[k]{} & R^{(\alpha)} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Since $T(k)$ and $T(i')$ are again monomorphisms, applying T to the diagram we obtain the commutative and exact diagram:

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & T(K) & \longrightarrow & T(P) & \longrightarrow & T(L) & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle T(i')} & & \Big\downarrow{\scriptstyle T(i)} & & \\
0 & \longrightarrow & T(K) & \xrightarrow[T(k)]{} & G^{(\alpha)} & \longrightarrow & T(M) & \longrightarrow & 0
\end{array}$$

which proves that $T(i)$ is a monomorphism.

The fact that $\mathcal{T} = \operatorname{Ker} T_G$ is a hereditary torsion class is immediately verified due to the properties of T_G which we just proved. We go on proving the

description of the Gabriel filter given in the statement of the theorem, and the fact that $\mathcal{G} \simeq \text{Mod-}(R, \mathscr{T})$ will easily follow, because $\mathcal{G}$ (or, better, its image by H_G in Mod-R) is the Giraud subcategory corresponding to the hereditary torsion class $\mathcal{T}$; once the description of $\mathscr{T}$ is proved, then, it will follow that $H_G(\mathcal{G})$ is the Giraud subcategory corresponding to $\mathscr{T}$, indeed that for which $M \in H_G(\mathcal{G})$ if and only if $\varinjlim_{J \in \mathscr{T}} \text{Hom}_R(J, M) \simeq M$.
An ideal $I \leq R_R$ is an element of the Gabriel filter if and only if $R/I \in \mathcal{T} = \text{Ker}\, T_G$. Now consider the diagram:

$$\begin{array}{ccccccccc} & & R^{(\Lambda)} & & & & & & \\ & & \downarrow \varphi & \searrow & & & & & \\ 0 & \longrightarrow & I & \longrightarrow & R_R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

where $I = \langle r_\lambda : \lambda \in \Lambda \rangle$ and $\varphi(j_\lambda) = r_\lambda$, with j_λ the element of $R^{(\Lambda)}$ with 0 in every entry except for a 1 in the λ^{th} one.
The row, the column and the sequence from $R^{(\Lambda)}$ to R_R to R/I to 0 are exact. Since the functor T_G is a left adjoint, it is right exact, and calculating it on the diagram we get:

$$\begin{array}{ccccccc} T_G(R^{(\Lambda)}) & \longrightarrow & T_G(R_R) & \longrightarrow & T_G(R/I) & \longrightarrow & 0 \\ \big| \wr & & \big| \wr & & & & \\ G^{(\Lambda)} & \xrightarrow[\coprod r_\lambda]{} & G & & & & \end{array}$$

so $I \in \mathscr{T}$ if and only if $R/I \in \mathcal{T}$ if and only if $T_G(R/I) = 0$ if and only if $\coprod r_\lambda$ is an epimorphism.

The last assertion of the theorem is easily proved using the forthcoming Corollary 1.6.11 (for what concerns the second isomorphism in the upper row) and the following diagram (for the remaining morphisms):

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{H_{G0}} & \mathcal{G}_0 \\ & {\scriptstyle H} \searrow \nwarrow {\scriptstyle T} & {\scriptstyle i} \downarrow \uparrow {\scriptstyle a} \\ & & \text{Mod-}R \end{array}$$

- H_{G0} is the fully faithful restriction of H_G to its essential image, so it is an equivalence of categories $\mathcal{G}_0 = \text{Im}\, H_G = \text{Mod-}(R, \mathscr{T})$;

- (a, i) is an adjunction, because
$$\mathrm{Hom}_R(a(M), H_{G0}(X)) = \mathrm{Hom}_R(H_{G0}T(M), H_{G0}(X)) \simeq$$
$$\simeq \mathrm{Hom}_{\mathcal{G}}(T(M), X) \simeq \mathrm{Hom}_R(M, H_G(X))$$
with $M \in$ Mod-R e $X \in Ob(\mathcal{G})$;
- a is exact, indeed both T and H_{G0} are exact, (H_{G0} is an equivalence between Grothendieck categories), so $\mathcal{G}_0$ is the Giraud subcategory of Mod-R which is equivalent to the Grothendieck category $\mathcal{G}$.

□

In the last part of this section, some considerable corollaries of the Gabriel-Popescu Theorem are stated and proved, as, for example, the Morita theorem (1958).

Corollary 1.6.7. *Every Grothendieck category $\mathcal{G}$ is complete, has injective envelopes and has an injective cogeneretor.*

Proof. The category $\mathcal{G}$ is equivalent to the subcategory Mod-$(R, \mathscr{T})$. In particular, $\mathcal{G}$ is complete.
Moreover, let $M \in \mathcal{G} \simeq$ Mod-$(R, \mathscr{T})$; $E(M)$ in Mod-R belongs both to $\mathcal{T}^{\perp_0}$, since $(\mathcal{T}, \mathcal{T}^{\perp_0})$ is hereditary, and to $\mathcal{T}^{\perp_1}$, being $E(M)$ injective. Therefore $E(M) \in Ob(\mathcal{G})$.
Finally, $\mathcal{T}^{\perp_0} = \mathrm{Cogen}(E)$ for some E injective in $\mathcal{T}^{\perp_0} \cap \mathcal{T}^{\perp_1}$, which then is an injective object of $\mathcal{G}$ and cogenerates it, since it cogenerates $\mathcal{T}^{\perp_0}$ which contains $\mathcal{G}$ as a full subcategory. □

Definition 1.6.8. *An object $G \in \mathcal{G}$ is **self-small** if for any cardinal α,* $\mathrm{Hom}_{\mathcal{G}}(G, G^{(\alpha)}) \simeq \mathrm{Hom}_{\mathcal{G}}(G, G)^{(\alpha)}$.

Corollary 1.6.9. *A Grothendieck category $\mathcal{G}$ is equivalent to a module category* Mod-R *if and only if it has a self-small projective generator G.*

Proof. If $\mathcal{G}$ is equivalent to a module category, then R_R is self-small and it is a projective generator.
Conversely, if G is self-small, and it is a projective generator, the functor H_G preserves arbitrary coproducts of copies of G, and it is exact. Therefore, using the Gabriel-Popescu Theorem, $\forall I \trianglelefteq R$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
R^{(\Lambda)} & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\
\downarrow{\scriptstyle \sigma_{R^{(\Lambda)}}} & & \downarrow{\scriptstyle \sigma_R} & & \downarrow{\scriptstyle \sigma_{R/I}} & & \\
H_GT_G(R^{(\Lambda)} & \longrightarrow & H_GT_G(R) & \longrightarrow & H_GT_G(R/I) & \longrightarrow & 0
\end{array}$$

which shows that, since the first two vertical arrows are isomorphisms, also $\sigma_{R/I}$ is an isomorphism. Then $I \in \mathscr{T}$ if and only if $R/I \in \mathcal{T}$ if and only if $T_G(R/I) = 0$ if and only if $H_G T_G(R/I) = 0$ if and only if $R/I = 0$ if and only if $I = R$. Therefore the Gabriel filter is trivial and $\mathcal{G} \simeq \text{Mod-}(R, \mathscr{T}) \simeq$ Mod-R. □

Theorem 1.6.10 (Morita, 1958). *Let R and S be two rings. The categories S-*Mod *and R-*Mod *are equivalent if and only if there exists a projective generator ${}_SU$ of S-*Mod *such that $R \simeq \text{End}({}_SU)$. In that case the functors giving the equivalence are identified, up to natural isomorphism, by the functors $H = \text{Hom}_S(U, -) : S\text{-Mod} \to R\text{-Mod}$ e $T = -\bigotimes_R U : R\text{-Mod} \to S\text{-Mod}$.*

Proof. Using Corollary 1.6.9, $R \simeq \text{End}({}_SU)$, and we obtain the equivalence; the functor H is that of the Gabriel-Popescu Theorem and T is one of its adjoint functors; it follows from the fact that all the adjoint functors to the same functor are isomorphic that T is the localizing functor built in the Gabriel-Popescu Theorem.
Conversely, given an equivalence of categories $F : S\text{-Mod} \to R\text{-Mod}$ with quasi-inverse G, and fixed ${}_SU = G(R_R)$, we obtain natural isomorphisms $R \simeq \text{Hom}_R(R, R) \simeq \text{Hom}_S(G(R), G(R)) = \text{End}({}_SU)$; moreover, for all $M \in S$-Mod, one has $F(M) \simeq \text{Hom}_R(R, F(M)) \simeq \text{Hom}_S(G(R), M) = H(M)$. Finally, $G \simeq -\bigotimes_R U$ because they are both left adjoint to $H \simeq F$. □

Corollary 1.6.11. *From the Gabriel-Popescu Theorem one deduces that every Grothendieck category is equivalent to the quotient of a module category modulo a localizing subcategory.*

To prove the assertion, the following lemma is needed:

Lemma 1.6.12. *Let $\mathcal{C} \subseteq \mathcal{A}$ be a localizing subcategory. The section functor $S : \mathcal{A}/\mathcal{C} \to \mathcal{A}$ is fully faithful and induces and equivalence between $\mathcal{A}/\mathcal{C}$ and the Giraud subcategory $\mathcal{B}$ of $\mathcal{A}$ made by the $\mathcal{C}$-closed objects.*

Proof. The functor S is fully faithful if and only if the counit of the adjunction is a natural isomorphism, and this fact has been already proved in Proposition 1.5.18.
The essential image of S is made by $\mathcal{C}$-closed objects, for what has been proved in Corollary 1.5.17; therefore one has a fully faithful functor $S : \mathcal{A}/\mathcal{C} \to \mathcal{B}$; it is essentially surjective, because if X is a $\mathcal{C}$-closed object, one can find $T(X)$ in $\mathcal{A}/\mathcal{C}$ such that $ST(X) \simeq X$.
It is left to show that the $\mathcal{C}$-closed objects actually form a Giraud subcategory of $\mathcal{A}$. Consider the functors: $i : \mathcal{B} \to \mathcal{A}$ and $ST : \mathcal{A} \to \mathcal{B}$; T is exact

and S is left exact, so ST preserves cokernels. It is left to prove that the pair (ST, i) is adjoint. Our goal is to prove that $h_M(u_X) : \mathrm{Hom}_{\mathcal{B}}(ST(X), M) \to \mathrm{Hom}_{\mathcal{A}}(X, i(M))$ is an isomorphism. Consider the diagram:

$$\begin{array}{ccc} X & \longrightarrow & M \\ {\scriptstyle u_X}\downarrow & \nearrow & \\ ST(X) & & \end{array}$$

Being M a $\mathcal{C}$-closed object, $h_M(u_X)$ is clearly an isomorphism due to the characterization of the $\mathcal{C}$-closed objects given in Lemma 1.5.16, since u_X is in Σ for any X, due to Proposition 1.5.18. □

Proof (Corollary). In the Gabriel-Popescu Theorem, the kernel of the functor T is the hereditary torsion class $\mathcal{T}$, which, by Theorem 1.5.20 is equivalent to a localizing subcategory. We foresee, then, that $\mathcal{G} \simeq$ Mod-$R/\mathcal{T}$. More precisely, it is required to prove that $\mathcal{G}$ is exactly the category of $\mathcal{T}$-closed objects. Let us consider $X \in \mathcal{T}$ and $M \in Ob(\mathcal{G})$; then $\mathrm{Hom}(X, M) = \mathrm{Hom}(X, i(M)) \simeq \mathrm{Hom}(a(X), M) = 0$ since X belongs to the kernel of a (we are identifying $\mathcal{G}$ with the equivalent Giraud subcategory of Mod-R). One obtains then that there exists no non-zero morphism from an object of $\mathcal{T}$ to one of $\mathcal{G}$; in particular there exist no monomorphism, so any object of $\mathcal{G}$ is a $\mathcal{T}$-closed object.
Conversely, if M is a $\mathcal{T}$-closed object, we assume by contradiction that it does not belong to $\mathcal{G}$, which has been characterized as Mod-$(R, \mathscr{T}) = \{M \in$ Mod-$R : \varinjlim_{J \in \mathscr{T}} \mathrm{Hom}_R(J, M) = M\}$ in the Gabriel-Popescu Theorem. Therefore $\varinjlim_{J \in \mathscr{T}} \mathrm{Hom}_R(J, M) \neq M$, so there exists a morphism $f : R \to M$ that cannot be expressed in terms of the direct limit. This means that there exists an ideal $J \in \mathscr{T}$ such that $\bar{f} : R/J \to M$ is a non-zero morphism; this leads us to a contradiction with the definition of $\mathcal{T}$-closed object, since $R/J \in \mathcal{T}$. Thus the category $\mathcal{G}$ is precisely the category of $\mathcal{T}$-closed objects in Mod-R, so it is equivalent to Mod-$R/\mathcal{T}$.

□

Chapter 2

The functor category

In this chapter we will deal with the category of additive functors from the category R-mod of finitely presented left R modules to the category of abelian groups. Here we assume R to be a ring with identity and such that $1_R \neq 0_R$. Both the category of *covariant* functors and the category of *contravariant* functors are useful to study and describe in a complete way some particular classes of modules (i.e. pure-injective modules), since the category Mod-R can be embedded in the functor category; to achieve such a goal, it is necessary to describe this immersion.

2.1 Contravariant functors

In this section we denote by $((R\text{-mod})^{op}, \mathcal{A}b)$ the category of contravariant additive functors from the category of finitely presented left R-modules to the category of abelian groups. In general, given a category $\mathcal{C}$ and two objects C and D, we denote by $(C, D)_{\mathcal{C}}$ the set $\mathrm{Hom}_{\mathcal{C}}(C, D)$, and if there is no ambiguity about the category in which we consider the morphisms, we simply use the notation (C, D).
We also recall that in a general functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, subfunctors, quotient functors and exact sequences of functors are defined point-wise. From Proposition 1.1.5 we obtain immediately that the category $((R\text{-mod})^{op}, \mathcal{A}b)$ is abelian.

Definition 2.1.1. *A functor $F \in ((R\text{-mod})^{op}, \mathcal{A}b)$ is called **representable** if it is isomorphic to a functor of the form $(-, M)$, where M is a finitely presented module.*

Lemma 2.1.2 (Yoneda)**.** *Let M be a finitely presented left R-module and $F \in ((R\text{-mod})^{op}, \mathcal{A}b)$. There is an isomorphism $\Theta_{M,F} : [(-, M), F] \to F(M)$ which is natural both in M and F.*

Lemma 2.1.3. *Every representable functor in* $((R\text{-mod})^{op}, \mathcal{A}b)$ *is finitely generated and projective. In particular, it is finitely presented.*

Proof. Let $(-, M) = \sum_\lambda F_\lambda$ a direct sum of functors in $((R\text{-mod})^{op}, \mathcal{A}b)$; then $1_M \in (M, M) = \sum_\lambda F_\lambda(M)$, so $1_M \in F_\lambda(M)$ for some λ. By Yoneda, this gives a morphism $(-, M) \to F_\lambda$, which, composed with the inclusion $F_\lambda \hookrightarrow (-, M)$ gives the identity $1_{(-,M)}$. Indeed, it is the isomorphism corresponding to the identity 1_M; therefore the inclusion is an epimorphism, so $(-, M)$ is finitely generated as wished.
To prove projectivity, let us consider an epimorphism $\pi : F \to G$ and a morphism $\mu : (-, M) \to G$, which, by Yoneda, corresponds to some $m \in G(M)$. Being $\pi_M : F(M) \to G(M)$ an epimorphism, there exists $n \in F(M)$ going to m. Let $\nu : (-, M) \to F$ be the morphism which corresponds to n via Yoneda. Then $\pi\nu = \mu$. □

Proposition 2.1.4. *The representable functors of* $((R\text{-mod})^{op}, \mathcal{A}b)$ *generate the category. Indeed, for every functor* $F : (R\text{-mod})^{op} \to \mathcal{A}b$ *there is an epimorphism* $\bigoplus_i (-, M_i) \to F$ *for some* M_i *finitely presented left* R*-modules. Also,* F *is finitely generated if and only if this direct sum may be taken to be finite.*

Proof. The category R-mod is skeletally small, i.e. the class of isomorphism classes of its objects is a set. Thus we may consider, for any module in the category, a representative M of its isomorphism class. Define the morphism $\bigoplus_M (-, M)^{(F(M))} \to F$ to have component at $m \in F(M)$ the morphism $f_m : (-, M) \to F$ which corresponds to m via Yoneda. This map is surjective by definition.
For the second part of the statement, we have $F = \sum_i \mathrm{Im}(f_i)$ where f_i is the i-th component map of $\bigoplus_i (, M_i) \to F$, so F finitely generated implies F is a sum of finitely many of these, so finitely many of the direct summands will be sufficient. The converse follows since, by Lemma 2.1.3, each functor $(-, M_i)$ is finitely generated (and every image of a finitely generated object is finitely generated). □

We say that *idempotents split* in a preadditive category $\mathcal{C}$ if, for every $M \in \mathcal{C}$, each idempotent $e = e^2 \in \mathrm{End}(M)$ has a kernel, and the canonical map $\mathrm{Ker}(e) \oplus \mathrm{Ker}(1-e) \to M$ is an isomorphism. In particular, the category R-mod has split idempotents.

Proposition 2.1.5. *The finitely generated projective objects of* $((R\text{-mod})^{op}, \mathcal{A}b)$ *are the direct summands of finite direct sums of representable functors. Since* R-mod *has split idempotents and finite direct sums, then these are precisely the representable functors.*

Proof. The first statement is a direct corollary of Lemma 2.1.3 and Proposition 2.1.4. For the second statement, since the category R-mod has finite direct sums, then $(-, M_i) \oplus (-, M_j) \simeq (-, M_i \oplus M_j)$, so if F is finitely generated and projective, F is, without loss of generality, a direct summand of a functor of the form $(-, M)$. Let $\pi : (-, M) \to F$ split the inclusion, and let $f \in \text{End}(-, M)$ be π followed by the inclusion, so that $f = f^2$. The Yoneda embedding is fully faithful, so there exists and $e \in \text{End}(M)$ corresponding to f via Yoneda, hence with $e = e^2$. Then we have $M = \text{Im}(e) \oplus \text{Ker}(e)$ and then it follows quickly that $F \simeq (-, \text{Im}(e))$, so F is representable. □

With these last results we have established that representable functors correspond to finitely generated projective ones, and that the finitely generated projective functors are a family of generators for the category of functors. We have almost proved the following:

Proposition 2.1.6. *The Yoneda embedding*

$$Y : R\text{-mod} \to ((R\text{-mod})^{op}, \mathcal{A}b)$$

$$M \to (-, M)$$

is fully faithful and it is left exact. It is an equivalence between the category R-mod *of finitely presented left R-modules and the category Proj*$((R\text{-mod})^{op}, \mathcal{A}b)$ *of finitely generated projective objects of* $((R\text{-mod})^{op}, \mathcal{A}b)$.

A detailed proof may be found in ([1], Chapter IV, Corollary 7.4). Directly from Proposition 1.1.9, one gets that the category $((R\text{-mod})^{op}, \mathcal{A}b)$ is a Grothendieck category; using the fact that representable functors are a family of generators for the category, a generator for this category is $\bigoplus_{M \in R\text{-mod}} (-, M)$ (the coproduct is well defined since R-mod is a small category).

Limits in the category $((R\text{-mod})^{op}, \mathcal{A}b)$ are calculated object-wise; thus we may give the following definition:

Definition 2.1.7. *An object* $G \in ((R\text{-mod})^{op}, \mathcal{A}b)$ *is called* ***flat*** *if it is isomorphic to a direct limit of finitely generated projective functors*

$$G \simeq \varinjlim (-, A_i)$$

The full subcategory of flat functors is denoted by Flat$((R\text{-mod})^{op}, \mathcal{A}b)$. Since the Yoneda embedding ensures us the equivalence stated above, we may consider the corresponding direct limit $M \simeq \varinjlim A_i$ in R-Mod. Using a well-known characterization of finitely presented modules ([1] Proposition V.3.4), we find

$$G \simeq \varinjlim(-, A_i) \simeq (-, \varinjlim A_i) \simeq (-, M)$$

Thus the following may be proved:

Proposition 2.1.8. *The functor $Y : R\text{-Mod} \to ((R\text{-mod})^{op}, \mathcal{A}b)$ yields an equivalence of categories between R-Mod and Flat$((R\text{-mod})^{op}, \mathcal{A}b)$, and it is left exact and fully faithful.*

A proof of this proposition may be found in ([7], Theorem 1.4). According to the last result, the module category R-Mod can be embedded in $((R\text{-mod})^{op}, \mathcal{A}b)$, and can be seen as the full subcategory of flat functors. Actually, the full subcategory of flat functors corresponds to the full subcategory of left exact functors, as we prove in the following lemma. Thus, the category of left R-modules can be seen as the full subcategory of left exact functors in $((R\text{-mod})^{op}, \mathcal{A}b)$.

Lemma 2.1.9. *A functor $F \in ((R\text{-mod})^{op}, \mathcal{A}b)$ is flat if and only if it is left exact.*

Proof. According to what was shown above, we already know that if F is a flat functor, then $F \simeq (-, M)$, hence it is left exact. Conversely, let us assume that F is left exact; we want to show that $F \simeq (-, F(R))$, which is a flat functor. Consider the obvious isomorphism of abelian groups $\alpha(R) : F(R) \to (R, F(R))$; it induces, for every finitely generated free module R^n, an isomorphism $\alpha(R^n) : F(R^n) \to (R^n, F(R))$.
Now, given $A \in R$-Mod, we can consider a free presentation of A:

$$R^m \to R^n \to A \to 0$$

and then we may apply F and $(-, F(R))$, obtaining the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(A) & \longrightarrow & F(R^n) & \longrightarrow & F(R^m) \\ & & & & \big\downarrow \wr & & \big\downarrow \wr \\ 0 & \longrightarrow & (A, F(R)) & \longrightarrow & (R^n, F(R)) & \longrightarrow & (R^m, F(R)) \end{array}$$

from which it is clear that a morphism $\alpha(A) : F(A) \to (A, F(R))$ is induced, and one can prove that it is an isomorphism using the five lemma. $\square$

This far, we have established an embedding of the module category into the functor category. According to what we dealt with in section 1.4, the natural question which arises from this situation is whether this functor has a left adjoint or not. Furthermore, properties of this adjoint should be studied in detail. The last part of this section is devoted to this topic.
Let us consider the functor $-_R : ((R\text{-mod})^{op}, \mathcal{A}b) \to R\text{-Mod}$, called R-**valuation**, which acts on functor by "evaluating" them on the left R-module ${}_RR$.

Lemma 2.1.10. *The R-valuation functor $-_R : ((R\text{-mod})^{op}, \mathcal{A}b) \to R\text{-Mod}$ is exact.*

Proof. The proof is straightforward: exact sequences in $((R\text{-mod})^{op}, \mathcal{A}b)$ are defined object-wise, so if we have an exact sequence of functors $0 \to F \to G \to H \to 0$, then the sequence $0 \to F(R) \to G(R) \to H(R) \to 0$ is still exact, by definition. □

Proposition 2.1.11. *The pair $\langle -_R, Y \rangle$ is and adjoint pair of functors.*

Proof. Let $F \in ((R\text{-mod})^{op}, \mathcal{A}b)$ be a functor, and $M \in R\text{-Mod}$ a left R-module. Our goal is to show that we have an isomorphism $\mathrm{Hom}_R(F(R), M) \simeq \mathrm{Hom}_{((R\text{-mod})^{op}, \mathcal{A}b)}(F, (-, M))$ which is natural both in M and in F. Let us consider the morphism:

$$\begin{aligned} \mathrm{Hom}_{((R\text{-mod})^{op}, \mathcal{A}b)}(F, (-,M)) &\to \mathrm{Hom}_R(F(R), M) \\ \eta &\longmapsto (\eta(R) : F(R) \to (R, M) \simeq M) \end{aligned}$$

which sends a natural transformation on the left to its R-component. The fact that this is an isomorphism is equivalent to having that any natural transformation η as above may be completely (and uniquely) determined by its R-component.
Let us assume we have a morphism $\eta(R) : F(R) \to M$; it clearly induces a unique morphism $\eta(R^n) : F(R^n) \to (R^n, M) \simeq M^n$ since F is an additive functor (it preserves direct sums). So our R-component determines in a unique way the R^n-components for a finite n. Let us consider now any finitely presented module A, and a free presentation:

$$R^m \xrightarrow{f} R^n \xrightarrow{p} A \to 0$$

let us apply both F and $(-, M)$ to this exact sequence, in order to get the following commutative diagram:

$$\begin{array}{ccccccc} & & F(A) & \xrightarrow{F(p)} & F(R^n) & \xrightarrow{F(f)} & F(R^m) \\ & & \downarrow & & \downarrow{\scriptstyle \eta(R^n)} & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_R(A, M) & \xrightarrow{p_*} & M^n & \xrightarrow{f_*} & M^m \end{array}$$

the dotted vertical arrow is the morphism we are looking for, and with a little exercise in diagram chasing, we may prove it is completely determined by the second vertical arrow, thus it is unique as we wanted. Consider an element $a \in F(A)$, and take $a' = \eta(R^n)F(p)(a) \in M^n$. This element belongs to the kernel of f_*, since $f_*\eta(R^n)F(p)(a) = \eta(R^m)F(f)F(p)(a) = 0$ by commutativity of the diagram and exactness of the upper row. Then a' belongs to the image of p_*, and since p_* is a monomorphism, there exists a unique element $a'' \in \mathrm{Hom}_R(A, M)$ such that $p_*(a'') = a'$. Finally, define, for every $a \in F(A)$, $\eta(A)(a) = a''$. This is the required morphism. □

Corollary 2.1.12. *The category* R-Mod *is a Giraud subcategory of* $((R\text{-mod})^{op}, \mathcal{A}b)$.

The proof follows immediately from Lemma 2.1.10 and Proposition 2.1.11.

2.2 Covariant functors

In this section we will consider the functor category $(R\text{-mod}, \mathcal{A}b)$ consisting of the additive covariant functors from the category of finitely presented left R-modules to the category of abelian groups. We will study again an embedding of the category Mod-R into $(R\text{-mod}, \mathcal{A}b)$, with a focus on possible adjoint functors.

Remark 2.2.1. *In the category* $(R\text{-mod}, \mathcal{A}b)$*, we define subfunctors, quotient functors, exact sequences and direct limits object-wise, as we did for* $((R\text{-mod})^{op}, \mathcal{A}b)$.
Moreover, the results about generators, representable functors and projective functors stated in Lemma 2.1.3, Proposition 2.1.4 and Proposition 2.1.5 are still true passing to our category $(R\text{-mod}, \mathcal{A}b)$*, as can be easily proved.*

Let us consider the functor

$$\begin{aligned} T : \text{Mod-}R &\to (R\text{-mod}, \mathcal{A}b) \\ M &\longmapsto (M \otimes -) : R\text{-mod} \to \mathcal{A}b \end{aligned}$$

which is called **tensor embedding**.

Lemma 2.2.2. *Let F and F' be two functors in* $(R\text{-mod}, \mathcal{A}b)$*, with F right exact, and let τ, $\tau' : F \to F'$. Then $\tau(R) = \tau'(R) \Rightarrow \tau = \tau'$.*

Proof. Let L a finitely presented left R-module, and $R^m \to R^n \xrightarrow{\pi} L \to 0$ a free presentation. Applying F and F' to this presentation, we obtain the following commutative diagram:

$$\begin{array}{ccccc} F(R^n) & \xrightarrow{F(\pi)} & F(L) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tau(R^n)} & & {\scriptstyle \tau(L)}\downarrow\downarrow{\scriptstyle \tau'(L)} & & \\ F'(R^n) & \xrightarrow[F'(\pi)]{} & F'(L) & \longrightarrow & 0 \end{array}$$

one has that $\tau(R^n) = \tau(R)^n = \tau'(R)^n = \tau'(R^n)$; hence one has that $\tau(L)F(\pi) = F'(\pi)\tau(R^n) = F'(\pi)\tau'(R^n) = \tau'(L)F(\pi)$. Since $F(\pi)$ is an epimorphism, it follows that $\tau = \tau'$. □

Theorem 2.2.3. *The tensor embedding* $T : \text{Mod-}R \to (R\text{-mod}, \mathcal{A}b)$ *is a full embedding and it is left adjoint to the* R*-valuation functor* $-_R : (R\text{-mod}, \mathcal{A}b) \to \text{Mod-}R$.

Proof. If $(M\otimes -) \simeq (N\otimes -)$, evaluating them at R, we obtain $M \simeq M\otimes R \simeq N \otimes R \simeq N$. If $\tau : (M \otimes -) \to (N \otimes -)$ is a natural transformation, its R-component is $\tau(R) : M\otimes R \to N\otimes R$, that is a map $\tau(R) : M \to N$. Thus τ and $(\tau(R) \otimes -)$ are natural transformation with the same R-component, and since $(M \otimes -)$ and $(N \otimes -)$ are right exact functors, by Lemma 2.2.2 we obtain that $\tau = (\tau(R) \otimes -)$. Hence T is full.
To prove the adjunction, first note that if $F \in (R\text{-mod}, \mathcal{A}b)$, then $F({}_RR)$ has a right R-module structure, since $\text{End}({}_RR) = R$: if $a \in F(R)$ and $s \in R$, set $as := F(- \times s)a$ and $a(st) = F(- \times st)a = F((- \times t)(- \times s))a = (as)t$.
The natural isomorphism $((M\otimes -), F) \simeq (M, F(R))$ sends $\tau : (M\otimes -) \to F$ to its R-component. By Lemma 2.2.2 it is a monomorphism. To define the inverse map: let $g : M \to F(R)$, let $\tau_g : (M \otimes -) :\to F$ whose component at $L \in R$-mod is defined by $\tau_g(m \otimes l) = F(l)g(m)$, where by $F(l)$ we denote the value of F at the morphism $R \to L$ sending 1_R to l. The collection of morphisms τ_g is indeed a natural transformation, and its R-component is exactly g. These processes define the adjunction. □

Proposition 2.2.4. *The tensor embedding* $T : \text{Mod-}R \to (R\text{-mod}, \mathcal{A}b)$ *is a fully faithful functor, and yields an equivalence of categories between* Mod-R *and the full subcategory* $Rex(R\text{-mod}, \mathcal{A}b)$ *of* $(R\text{-mod}, \mathcal{A}b)$ *consisting of the right exact functors.*

Proof. The fact that T is full has been proved in Theorem 2.2.3. The tensor embedding acts on morphisms sending $f : M \to N$ to $(f \otimes -) : (M \otimes -) \to (N \otimes -)$ whose L-component (for a finitely presented left R-module L) is $f \otimes 1_L : M \otimes L \to N \otimes L$. It is clear that if $(f \otimes -) = (g \otimes -)$, then their R-components must be the same, so $f = g$, and this proves that T is fully

faithful.
For the second statement, note that the functor $(M \otimes -)$ is right exact for any right R-module M; it is left to prove that, for any right exact functor $F \in (R\text{-mod}, \mathcal{A}b)$, there exists $M \in \text{Mod-}R$ such that $F \simeq (M \otimes -)$.
Consider the right R-module $F(R)$; we claim that $F \simeq (F(R) \otimes -)$. Of course we have a canonical isomorphism $\eta(R) : F(R) \to F(R) \otimes R$, which induces an isomorphism $\eta(R^n) : F(R^n) \to F(R) \otimes R^n$ since F is additive. Let us consider now $L \in R$-mod and a free presentation $R^m \to R^n \to L \to 0$. We apply the right exact functors F and $(F(R) \otimes -)$ to this sequence to get a commutative diagram:

$$\begin{array}{ccccccc} F(R^m) & \longrightarrow & F(R^n) & \longrightarrow & F(L) & \longrightarrow & 0 \\ \big\downarrow \wr & & \big\downarrow \wr & & \downarrow & & \\ F(R) \otimes R^m & \longrightarrow & F(R) \otimes R^n & \longrightarrow & F(R) \otimes L & \longrightarrow & 0 \end{array}$$

with a little exercise in diagram chasing, it is clear that there is a unique induced map corresponding to the dotted arrow; using the five lemma, it is easy to show that it is also an isomorphism. □

The last results provide us a situation that is dual to the one we had in the case of contravariant functors; in particular by Proposition 2.2.4 a correspondence between right R-modules and right exact functors in the category $(R\text{-mod}, \mathcal{A}b)$ is established, while we had a correspondence with left exact functors in the contravariant case; by Theorem 2.2.3, moreover, we have an adjunction $\langle T, -_R \rangle$ in which the right adjoint is an exact functor (the proof of this fact is exactly the same as that of Lemma 2.1.10). This configuration appears very similar to that of Giraud subcategories, except for the fact that we have a *right* adjoint to the inclusion functor which is exact, instead of having a left adjoint. This motivates a brief digression on Co-Giraud subcategories.

2.2.1 Co-Giraud subcategories

Recalling what we dealt with in Section 1.4, we may dualize those results to deal with Co-Giraud subcategories. In this section we use $\mathcal{C}$ to denote a category and $\mathcal{A} \subseteq \mathcal{C}$ a full subcategory.

Definition 2.2.5. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{C}$. An object $C \in \mathcal{C}$ is said to be* ***codivisible*** *(respectively divisible) if the functor* $\text{Hom}_{\mathcal{C}}(C, -)$ *(respectively* $\text{Hom}_{\mathcal{C}}(-, C)$*) is exact on all the exact sequences $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{F}$ (respectively $X'' \in \mathcal{T}$).*

A morphism $f : D \to C$ (respectively $g : C \to D$) is a ***colocalization*** *(respectively a localization) if* $\mathrm{Ker}(f)$, $\mathrm{Coker}(f) \in \mathcal{F}$ *and* $D \in \mathcal{T}$ *is codivisible (respectively* $\mathrm{Ker}(g)$, $\mathrm{Coker}(g) \in \mathcal{T}$ *and* $D \in \mathcal{F}$ *is divisible).*

Proposition 2.2.6. *Let $f_1 : D_1 \to C_1$, $f_2 : D_2 \to C_2$ two colocalizations with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. Let $g : C_1 \to C_2$ be a morphism. There exists a unique $h : D_1 \to D_2$ such that $f_2h = gf_1$.*

Proof. Let t be the idempotent radical associated to the torsion pair. Then it is clear that $\mathrm{Im}(f_i) = t(C_i)$. Hence one has

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(f_1) & \longrightarrow & D_1 & \longrightarrow & t(C_1) & \longrightarrow & 0 \\ & & & & & & \big\downarrow {\scriptstyle t(g)} & & \\ 0 & \longrightarrow & \mathrm{Ker}(f_2) & \longrightarrow & D_2 & \longrightarrow & t(C_2) & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Ker}(f_2)$ is torsion free and D_1 is codivisible, there exists $h : B_1 \to B_2$ making the diagram commute. Furthermore, h is unique since $D_1 \in \mathcal{T}$. □

Next corollary shows that colocalization is unique up to isomorphism:

Corollary 2.2.7. *Let $f_1 : D_1 \to C$, $f_2 : D_2 \to C$ be two colocalizations of C. Then $D_1 \simeq D_2$.*

Corollary 2.2.8. *If every object $C \in \mathcal{C}$ has its colocalization $\varphi_C : L(C) \to C$, then L is an additive endofunctor of $\mathcal{C}$, and $\varphi : L \to 1_{\mathcal{C}}$ is a natural transformation. L is called* ***colocalization functor****.*

Proof. We must show that L is additive. Let $f, g : C \to D$ be two morphisms. Then we have a commutative diagram

$$\begin{array}{ccc} L(C) & \xrightarrow{\varphi_C} & t(C) \\ {\scriptstyle L(f+g)} \big\downarrow & & \big\downarrow {\scriptstyle t(f+g)} \\ C(D) & \xrightarrow[\varphi_D]{} & t(D) \end{array}$$

Thus

$$\begin{aligned} \varphi_D L(f+g) = t(f+g)\varphi_C = \\ = (t(f)+t(g))\varphi_C = t(f)\varphi_C + t(g)\varphi_C = \\ = \varphi_D L(f) + \varphi_D L(g) = \varphi_D(L(f)+L(g)) \end{aligned}$$

This shows that $L(f)+L(g)-L(f+g)$ maps $L(C)$ into $\mathrm{Ker}(\varphi_D)$. On the other hand, $L(C) \in \mathcal{T}$ and $\mathrm{Ker}(\varphi_D) \in \mathcal{F}$. Therefore $L(f) + L(g) - L(f + g) = 0$. This proves that L is additive. □

Of course, everything can be naturally dualized for localizations.

Definition 2.2.9. *A torsion pair $(\mathcal{T}, \mathcal{F})$ is called* ***cohereditary*** *if $\mathcal{F}$ is closed under quotients, and it is called* ***strongly cohereditary*** *if every object in $\mathcal{C}$ has a colocalization with respect to $(\mathcal{T}, \mathcal{F})$.*

Definition 2.2.10. *Let $r : \mathcal{C} \to \mathcal{C}$ be a covariant functor. Then one says that:*

- *r is a* ***pre-coradical*** *if $r(C)$ is a quotient of C for every object C in the category;*

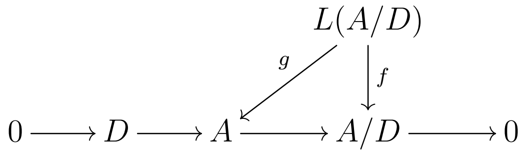
- *r is* ***idempotent*** *if $r(r(C)) = r(C)$ for every object C in the category;*

- *r is a* ***coradical*** *if $r(\mathrm{Ker}(\psi_C)) = 0$ for every object C in the category, where ψ_C is the C-component of the obvious natural transformation $\psi : 1_{\mathcal{C}} \to r$;*

The bijective correspondence between torsion pairs and idempotent radicals established in Proposition 1.2.14 can be extended dualizing it to idempotent coradicals.

Proposition 2.2.11. *Every strongly cohereditary torsion pair is cohereditary.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be strongly cohereditary with associated idempotent radical t. Let $C \in \mathcal{F}$ and let D be a subobject of C. There exists a subobject $A \subseteq C$ such that $t(C/D) = A/D$. Let $f : L(A/D) \to A/D$ be its colocalization; then f is an epimorphism since $A/D \in \mathcal{T}$. Because $L(A/D)$ is codivisible and $D \in \mathcal{F}$, there exists $g : L(A/D) \to A$ such that the following diagram:



is commutative. But we know that $A \in \mathcal{F}$, hence $g = 0$, so $f = 0$. This implies that $A/D = 0$ because f is an epimorphism. Thus $\mathcal{F}$ is closed under quotient objects. □

Theorem 2.2.12. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair, and suppose that every object of $\mathcal{T}$ is a quotient of a projective object. Then $(\mathcal{T}, \mathcal{F})$ is strongly cohereditary if and only if it is cohereditary.*

Proof. See ([11], Theorem 1.6). □

Both last two result can be dualized.

Proposition 2.2.13. *The following assertions are equivalent for a torsion pair* $(\mathcal{T}, \mathcal{F})$*:*

1. $(\mathcal{T}, \mathcal{F})$ *is cohereditary;*
2. t *preserves epimorphisms;*
3. r *is right exact.*

Proof. See [1]. □

Lemma 2.2.14. *Suppose* $(\mathcal{T}, \mathcal{F})$ *is a cohereditary torsion pair. Let* $f : C \to D$ *be an epimorphism such that* $C \in \mathcal{T}$ *and* $\mathrm{Ker}(f) \in \mathcal{F}$*. Then* f *is a minimal epimorphism.*

Proof. Let X be a subobject of C such that $X \hookrightarrow C \xrightarrow{f} D$ is an epimorphism. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & X+K & \longrightarrow & D & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & C & \xrightarrow[f]{} & D & \longrightarrow & 0
\end{array}$$

where $X + K = C$, since $K = \mathrm{Ker}(f)$. Thus $C/X \simeq X + K/X \simeq K/X \cap K$. On the other hand, $C/X \in \mathcal{T}$ since $C \in \mathcal{T}$ and $K/X \cap K \in \mathcal{F}$ since $K \in \mathcal{F}$ and $\mathcal{F}$ is closed under quotients. Therefore $X = C$, and this implies that f is minimal. □

Theorem 2.2.15. *Let* $(\mathcal{T}, \mathcal{F})$ *be a strongly cohereditary torsion pair with colocalization functor* L*. Then* L *is right exact.*

Proof. Let $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ be a short exact sequence. Then one has:

$$\begin{array}{ccccccc}
L(C') & \xrightarrow{L(f)} & L(C) & \xrightarrow{L(g)} & L(C'') & & \\
\downarrow h' & & \downarrow h & & \downarrow h'' & & \\
t(C') & \xrightarrow[t(f)]{} & t(C) & \xrightarrow[t(g)]{} & t(C'') & \longrightarrow & 0
\end{array}$$

where $t(g)$ and columns are epimorphisms. Since h'' is a minimal epimorphism, $L(g)$ is an epimorphism. Indeed, if by contradiction $L(g)(L(C)) \subsetneq$

$L(C'')$, one still has that $h''L(g)$ is an epimorphism, since it is equal to $t(g)h$ which are both epimorphisms, so one would have a subobject of $L(C'')$ which would cover all $t(C'')$, but then h'' would not be minimal anymore. Now, put $K = \mathrm{Ker}(L(g))$. One must prove that $K = \mathrm{Im}(L(f))$; $K \in \mathcal{T}$ since $0 \to K \to L(C) \to L(C'') \to 0$ is exact, $L(C) \in \mathcal{T}$ and $L(C'')$ is codivisible. Clearly, $h(K) \subseteq \mathrm{Ker}(t(g)) \subseteq \mathrm{Ker}(g) = \mathrm{Im}(f)$; hence $h(K) \subseteq \mathrm{Im}(t(f))$, and $\mathrm{Im}(hL(f)) = h(K)$. On the other hand, $\mathrm{Ker}(h_{|K}) \subseteq \mathrm{Ker}(h) \in \mathcal{F}$. Thus $h_{|K} : K \to h(K)$ is minimal, and this implies that $\mathrm{Im}(L(f)) = K$, therefore L is right exact. □

Definition 2.2.16. *Let $\mathcal{A} \subseteq \mathcal{C}$ be a full subcategory. Then $\mathcal{A}$ is said:*

- ***coreflective*** *if there exists a right adjoint **a** to the inclusion functor $i : \mathcal{A} \to \mathcal{C}$;*

- ***co-Giraud*** *if it is coreflective and **a** is exact.*

Theorem 2.2.17. *Let $\mathcal{A} \subseteq \mathcal{C}$ be a full subcategory of $\mathcal{C}$ whose objects consist of torsion and codivisible objects of $\mathcal{C}$. Let $i : \mathcal{A} \to \mathcal{C}$ be the inclusion, denote $L = a$. Then a is a right adjoint of i.*

Proof. Let $C \in \mathcal{A}$, $D \in \mathcal{C}$. Since C is codivisible, the exact sequence $0 \to \mathrm{Ker}(L(D) \to D) \to D \to t(D) \to 0$ induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(C, L(D)) \simeq \mathrm{Hom}_{\mathcal{C}}(C, t(D)$. Also, the exact sequence $0 \to t(D) \to D \to r(D) \to 0$ induces $\mathrm{Hom}_{\mathcal{C}}(C, D) \simeq \mathrm{Hom}_{\mathcal{C}}(C, t(D)$. Since $\mathcal{A}$ is full in $\mathcal{C}$, the above induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(i(C), D) \simeq \mathrm{Hom}_{\mathcal{A}}(C, a(D))$. □

Corollary 2.2.18. *The inclusion functor i preserves cokernels and a preserves kernels and cokernels.*

Corollary 2.2.19. *The full subcategory $\mathcal{A} \subseteq \mathcal{C}$ is an abelian category.*

Proof. See ([10], Corollary 2.6). □

Theorem 2.2.20. *Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$ and $i : \mathcal{A} \to \mathcal{C}$ the inclusion functor. Then $\mathcal{A}$ is a category consisting of objects which are torsion and codivisible with respect to some strongly cohereditary torsion pair if and only if $\mathcal{A}$ is a co-Giraud subcategory of $\mathcal{C}$.*

Proof. See ([10], Theorem 2.7). □

Corollary 2.2.21. *If $\mathcal{A}$ is a co-Giraud subcategory of $\mathcal{C}$, then it is abelian.*

The following corollary establishes a fundamental correspondence:

Corollary 2.2.22. *There is a bijective correspondence between co-Giraud subcategories of $\mathcal{C}$ and strongly cohereditary torsion pairs of $\mathcal{C}$.*

2.3 Pure-injective modules

In this section we will address a particular class of modules, namely the pure-injective modules. In particular we will prove how this class is equivalent to the subclass of injective functors in the covariant functor category.

Definition 2.3.1. *Let R be a ring, and let $0 \to A \to B \to C \to 0$ be an exact sequence of right R-modules; it is called* ***pure*** *if the induced sequence of abelian groups $0 \to \mathrm{Hom}_R(E, A) \to \mathrm{Hom}_R(E, B) \to \mathrm{Hom}_R(E, C) \to 0$ is exact for every finitely presented right R-module E.*

Definition 2.3.2. *A submodule A of a right R-module B is a* ***pure submodule*** *if $0 \to A \to B \to B/A \to 0$ is pure.*
A ***pure monomorphism*** *is a monomorphism $A \hookrightarrow B$ whose image is a pure submodule of B.*

Definition 2.3.3. *An R-module M is* ***pure-injective*** *if the sequence $0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact for every pure exact sequence $0 \to A \to B \to C \to 0$.*

Proposition 2.3.4. *For every right R-module M, there exists a pure exact sequence $0 \to A \to B \to M \to 0$ where B is a direct sum of finitely presented modules.*

Proof. Let S be a set of finitely presented right R-modules such that every finitely presented right R module is isomorphic to an element of S. For every $N \in S$ let $N^{\mathrm{Hom}(N,M)}$ be the direct sum of copies of N indexed in $\mathrm{Hom}(N, M)$. Define $B = \bigoplus_{N \in S} N^{\mathrm{Hom}(N,M)}$. Let $g : B \to M$ be the morphism whose restriction to the copy of N indexed by $\varphi \in \mathrm{Hom}(N, M)$ is φ for every $N \in S$ and $\varphi \in \mathrm{Hom}(N, M)$. Since R is finitely presented, there is a module in S which is isomorphic to R. Hence g is surjective.
Let $A = \mathrm{Ker}(g)$ and $f : A \to B$ be the inclusion so that $0 \to A \to B \to M \to 0$ is exact. To prove this is actually a pure sequence, one needs that for every finitely presented module E and every morphism $\psi : E \to M$, there is a morphism $\psi' : E \to B$ such that $\psi = g\psi'$. Now, there exists $N \in S$ which is isomorphic to E. Let $\alpha : N \to E$ be such an isomorphism, and $\epsilon : N \to B$ the embedding into the direct summand of B indexed by $\psi\alpha \in \mathrm{Hom}(N, M)$. Then $\psi' = \epsilon\alpha^{-1}$ has the required property. □

To study pure-injective modules, we need a definition:

Definition 2.3.5. *Let $M = (r_{ij})$ be an $n \times m$ matrix with entries in a ring R, let A be a right R-module. Consider the n-tuples $X = (x_1, ..., x_n) \in A^n$*

such that $XM = 0$. The set $S_M = \{X \in A^n : XM = 0\}$ is an additive subgroup of A^n. Hence if $\pi_1 : A^n \to A$ is the canonical projection onto the first summand, the image $\pi_1(S_M)$ of S_M is an additive subgroup of A. The additive subgroups of A arising in this fashion, are called ***finitely definable subgroups*** *of A.*
Any additive subgroup G of A is ***definable*** *if it is an intersection of finitely definable subgroups.*

Proposition 2.3.6. *Let A be a right R-module, and let $r \in R$. The following statements are true:*

a) the groups Gr and $(G :_A r) = \{a \in A : ar \in G\}$ are finitely definable subgroups of A for every finitely definable subgroup G of A;

b) there is a bijective correspondence between the set of all finitely definable subgroups G of A that contain $(0 :_A r)$ and the set of all finitely definable subgroups G' of A defined by $G \longmapsto G' = Gr$.

Proof. Given G finitely definable subgroup of A, there is an $n \times m$ matrix M with entries in R such that $G = \pi_1(S)$ where $S = \{X \in A^n : XM = 0\}$ and $\pi_1 : A^n \to A$ is the usual projection. It is easy to see that if $\pi_1' : A^{n+1} \to A$ is the canonical projection onto the first summand, then $Gr = \pi_1'(S')$ where $S' = \{Y \in A^{n+1} : Y \begin{pmatrix} 1 & 0 & \dots & 0 \\ -r & & & \\ . & & & \\ . & & A & \\ 0 & & & \end{pmatrix} = 0\}$ and $(G :_A r) = \pi_1'(S'')$ where $S'' = \{Y \in A^{n+1} : Y \begin{pmatrix} r & 0 & \dots & 0 \\ -1 & & & \\ . & & & \\ . & & A & \\ 0 & & & \end{pmatrix} = 0\}$ This proves a); b) is an easy consequence. □

Lemma 2.3.7. *Sum and intersection of two finitely definable subgroups of A are finitely definable subgroups of A.*

This lemma proves that if A is an R-module, and F is a finite subset of R, the additive subgroup $(0 :_A F)$ is a finitely definable subgroup of A.

Lemma 2.3.8. *If A is a pure submodule of B, then for every finitely definable subgroup G of A there exists a finitely definable subgroup H of B such that $G = H \cap A$.*

Proof. If G is a finitely definable subgroup of A, there exists an $n \times m$ matrix M with entries in R such that $G = \pi_1(S)$ where $S = \{X \in A^n : XM = 0\}$ and π_1 is the usual projection. Define $S' = \{Y \in B^n : YM = 0\}$ and $\pi_1' : B^n \to B$ the usual projection; write $H = \pi_1'(S')$, so that H is a finitely definable subgroup of B. Then $G \subseteq H \cap A$. The other inclusion follows from ([9], Theorem 1.27). □

Definition 2.3.9. *Let R be a ring. A* ***compact topological module*** *$(N, \mathscr{T})$ is a right R-module N together with a compact Hausdorff topology $\mathscr{T}$ on the set N with the property that addition and multiplication are continuous maps.*

Theorem 2.3.10. *The following assertions are equivalent for a right module M over an arbitrary ring R:*

a) M is pure-injective;

b) every pure exact sequence $0 \to M \to B \to C \to 0$ of right R-modules is split;

c) there exists a compact topological R-module $(N, \mathscr{T})$ such that M is isomorphic to a direct summand of N;

d) a system $\sum_{i \in I} x_i r_{ij} = m_j$ with $j \in J$ of linear equations in M is soluble in M whenever it is finitely soluble in M (here $r_{ij} \in R$, $m_j \in M$ $\forall i, j$ and for every $j \in J$ there are finitely many $i \in I$ with $r_{ij} \neq 0$);

e) if $m_i \in M$, G_i is a finitely definable subgroup of M for every $i \in I$, and the family $\{m_i + G_i : i \in I\}$ has the finite intersection property, then $\bigcap_{i \in I} m_i + G_i \neq \emptyset$.

Any module satisfying *d*) is called **algebraically compact**.

Corollary 2.3.11. *If $\varphi : R \to S$ is a ring homomorphism, then every pure-injective right S-module is also pure-injective as an R-module.*

Proposition 2.3.12. *Let e be an idempotent element in R. Let M, N be two R-modules, $m \in Me$ and $n \in Ne$. Suppose that N is pure-injective. The following are equivalent:*

a) for every finitely presented R-module F, every element $a \in Fe$ and every homomorphism $f : F \to M$ such that $f(a) = m$, there exists a homomorphism $f' : F \to N$ such that $f'(a) = n$;

b) there exists a homomorphism $g : M \to N$ such that $g(m) = n$.

Proof. See ([9], Proposition 1.37). □

Now, let us consider again the category of covariant additive functors $(R\text{-mod}, \mathcal{A}b)$; recall that we defined a functor $\Psi : \text{Mod-}R \to (R\text{-mod}, \mathcal{A}b)$ by assigning $\Psi(M) = (M \otimes -)$. This functor is fully faithful, hence Mod-R is equivalent to a full subcategory of $(R\text{-mod}, \mathcal{A}b)$ (we actually proved that this full subcategory is precisely that of right exact functors).

Lemma 2.3.13. *A sequence $0 \to A \to B \to C \to 0$ of right R-modules is exact and pure in* Mod-R *if and only if the sequence $0 \to (A \otimes -) \to (B \otimes -) \to (C \otimes -) \to 0$ is exact in $(R\text{-mod}, \mathcal{A}b)$.*

Proof. It follows from ([9], Theorem 1.27). □

Proposition 2.3.14. *An object $F \in (R\text{-mod}, \mathcal{A}b)$ is injective if and only if it is isomorphic to $\Psi(M) = (M \otimes -)$ for some pure-injective module M. Hence the full subcategory of pure-injective modules in* Mod-R *is equivalent to the full subcategory of injective objects of $(R\text{-mod}, \mathcal{A}b)$.*

Proof. Let F be an injective object of $(R\text{-mod}, \mathcal{A}b)$. We claim that if X, Y and Z are finitely presented left R-modules, and $X \to Y \to Z \to 0$ is an exact sequence in R-mod, then $F(X) \to F(Y) \to F(Z) \to 0$ is exact.
To prove this claim, let X, Y and Z be finitely presented left R-modules, and let $X \to Y \to Z \to 0$ be exact. For every $N \in R\text{-mod}$, the sequence $0 \to \text{Hom}(Z, N) \to \text{Hom}(Y, N) \to \text{Hom}(X, N)$ is exact, that is the sequence $0 \to \text{Hom}(Z, -) \to \text{Hom}(Y, -) \to \text{Hom}(X, -)$ is exact in $(R\text{-mod}, \mathcal{A}b)$. Since F is injective, the functor $\text{Nat}(-, F)$ is exact; hence the sequence $\text{Nat}(\text{Hom}(X, -), F) \to \text{Nat}(\text{Hom}(Y, -), F) \to \text{Nat}(\text{Hom}(Z, -), F) \to 0$ is exact. By Yoneda, it follows that $F(X) \to F(Y) \to F(Z) \to 0$ is exact.
By what was proved in Proposition 2.2.4 we know that F is a functor of the form $(F(R) \otimes -)$. We call $M = F(R)$, so that $F = \Psi(M)$. We need to prove that M is pure-injective. Let A be a pure submodule of B, $\epsilon : A \hookrightarrow B$ be the inclusion, and let $f : A \to M$ be a homomorphism. Then $(\epsilon \otimes -) : (A \otimes -) \to (B \otimes -)$ is a monomorphism in $(R\text{-mod}, \mathcal{A}b)$ by Lemma 2.3.13, hence $(f \otimes -) : (A \otimes -) \to (M \otimes -)$ extends to $u : (B \otimes -) \to (M \otimes -)$ with $u(\epsilon \otimes -) = (f \otimes -)$. Since Ψ is fully faithful, there exists a unique $g : B \to M$ such that $u = (g \otimes -)$; in particular, $(g \otimes 1_R)(\epsilon \otimes 1_R) = f \otimes 1_R$, that is $g\epsilon = f$, therefore M is pure-injective.
Conversely, let M be a pure-injective module; then $(M \otimes -) \in (R\text{-mod}, \mathcal{A}b)$, hence $(M \otimes -)$ is a subobject of an injective object $G \in (R\text{-mod}, \mathcal{A}b)$ (since

the functor category is a Grothendieck category). We know that there exists an E pure-injective such that $G \simeq (E \otimes -)$. Hence there exists a monomorphism $(M \otimes -) \to (E \otimes -)$, which must be of the form $(f \otimes -)$ with $f : M \to E$. By Lemma 2.3.13, f is a pure monomorphism. But M is pure-injective, so M is isomorphic to a direct summand of E due to $b)$ in Theorem 2.3.10, hence $(M \otimes -)$ is isomorphic to a direct summand of $G \simeq (E \otimes -)$. In particular, $(M \otimes -)$ is injective. □

2.3.1 Sigma-pure-injective modules

Definition 2.3.15. *A right R-module A is said to be* ***Σ-pure-injective*** *(or* ***Σ-algebraically compact****) if for any index set I the direct sum $A^{(I)}$ is a pure-injective module A over R.*

Theorem 2.3.16. *The following assertions are equivalent for a right R-module A:*

a) A is Σ-pure-injective;

b) for every I, the submodule $A^{(I)} \subseteq A^I$ is a direct summand of A^I;

c) the submodule $A^{(\mathbb{N})} \subseteq A^{\mathbb{N}}$ is a direct summand of $A^{\mathbb{N}}$;

d) A satisfies the descending chain condition on its finitely definable subgroups.

Proof. See ([9], Theorem 1.40). □

Example 2.3.17. *If R is an arbitrary ring, then:*

- *if A is a right R-module that is Artinian as a left* End(A)*-module, A is Σ-pure-injective;*

- *if A is an Artinian right R-module, then it is Σ-pure-injective as a left* End(A)*-module.*

This happens because if ${}_SA_R$ is an S-R-bimodule, every finitely definable subgroup of A_R is in fact an S-submodule of ${}_SA$.

Corollary 2.3.18. *Every pure submodule A of a Σ-pure-injective module B is a direct summand of B.*

Proof. See ([9], Corollary 1.42). □

Corollary 2.3.19. *If R is a ring, A is a Σ-pure-injective module, $r \in R$ such that the right multiplication by r is an injective endomorphism of A, then this morphism is an automorphism of A (as an abelian group).*

Proof. See ([9], Corollary 1.43). □

Corollary 2.3.20. *Let $0 \neq e$ be an idempotent element in R (an arbitrary ring). If A is a Σ-pure-injective module then A is a Σ-pure-injective eRe-module.*

Proof. See ([9], Corollary 1.44). □

Chapter 3

Moving torsion pairs through Giraud subcategories

In this chapter, referring to [12], we will analyse the process of moving a torsion pair in an abelian category $\mathcal{C}$ to a Giraud subcategory $\mathcal{D}$ of $\mathcal{C}$, and vice versa; the case of $\mathcal{D}$ being a co-Giraud subcategory of $\mathcal{C}$ will be studied as well.
In the last section the techniques developed will be applied to the case of the functor category (both in the covariant and contravariant case).

3.1 General setting

The aim of this section is to define a way to move torsion classes through exact functors and subsequently through a Giraud (resp. co-Giraud) subcategory $\mathcal{D}$ of $\mathcal{C}$. From now on, $\mathcal{C}$ is an abelian category, $\mathcal{D}$ is a Giraud (resp. co-Giraud) subcategory of $\mathcal{C}$, i (resp. j) : $\mathcal{D} \to \mathcal{C}$ is the inclusion functor which is right (resp. left) adjoint to the localization (resp. colocalization) functor l (resp r): $\mathcal{C} \to \mathcal{D}$. Moreover $\mathcal{S}$ denotes the kernel of the localization (resp. colocalization) functor. Before studying the transfer of torsion pairs, we need to characterize the class $\mathcal{S}^{\perp}$ in the Giraud case, and the class ${}^{\perp}\mathcal{S}$ in the co-Giraud case. In the Giraud case, we denote by $\eta : 1_{\mathcal{D}} \to il$ (resp. $\epsilon : li \to 1_{\mathcal{C}}$) the unit (resp. the counit) of the adjunction $\langle l, i \rangle$ and by $\mathcal{S}^{\perp}$ the class of object defined below:

$$\mathcal{S}^{\perp} = \{D \in \mathcal{D} : (S, D) = 0, \ \forall S \in \mathcal{S}\}$$

It is easily seen that since i is fully faithful the counit of the adjunction is an isomorphism of functors. In particular, for any $D \in \mathcal{D}$, one has that

$l(\eta(D)) = \epsilon_{l(D)}^{-1}$ is an isomorphism. One obtains that:

$$\mathcal{S}^{\perp} = \{D \in \mathcal{D} : \eta_D : D \to il(D) \text{ is a monomorphism}\}$$

Indeed, for any $D \in \mathcal{D}$ we have that $\mathrm{Ker}(\eta_D) \in \mathcal{S}$ (since $l(\mathrm{Ker}(\eta_D)) = \mathrm{Ker}(l(\eta_D)) = 0$ because $l(\eta_D)$ is an isomorphism), hence given $X \in \mathcal{S}^{\perp}$ the kernel map $\mathrm{Ker}(\eta_X) \hookrightarrow X$ is zero and so η_X is a monomorphism.
On the other hand if we consider $D \in \mathcal{D}$ such that η_D is a monomorphism, then for any object S in $\mathcal{S}$ we have $(S, D) \subseteq (S, il(D)) \simeq (l(S), l(D)) = 0$ so $D \in \mathcal{S}^{\perp}$.
Dually, in the co-Giraud case, denoting by $\epsilon : jr \to 1_D$ the counit of the adjunction $\langle j, r\rangle$, we have that (given $S = \mathrm{Ker}(r)$):

$$\begin{aligned} {}^{\perp}\mathcal{S} &= \{D \in \mathcal{D} : (D, S) = 0,\ \forall S \in \mathcal{S}\} \\ &= \{D \in \mathcal{D} : \epsilon_D : jr(D) \to D \text{ is an epimorphism}\}. \end{aligned}$$

Since torsion classes (resp. torsion-free classes) are closed under coproducts and quotients (resp. products and subobjects), it is natural to use the left (resp. right) adjoint functor l (resp. i) in order to move torsion classes (resp. torsion-free classes) from $\mathcal{D}$ to $\mathcal{C}$ (resp. from $\mathcal{C}$ to $\mathcal{D}$).

Lemma 3.1.1. *Let $\mathcal{D}$ be an abelian category and $\mathcal{T}$ a torsion class on $\mathcal{D}$. Let $l : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories which respects arbitrary colimits. Then the class*

$$l^{\leftarrow}(\mathcal{T}) = \{C \in \mathcal{C} : l(C) \in \mathcal{T}\}$$

is a torsion class in $\mathcal{D}$.

Proof. Clearly, the class $l^{\leftarrow}(\mathcal{T})$ is closed under taking coproducts and quotients, because so is $\mathcal{T}$ and l respects arbitrary colimits. Let us show that $l^{\leftarrow}(\mathcal{T})$ is closed under extensions. Consider a short exact sequence in $\mathcal{C}$

$$0 \to X_1 \to C \to X_2 \to 0$$

with $X_1, X_2 \in l^{\leftarrow}(\mathcal{T})$. By applying the functor l to this sequence, one obtains an exact sequence in $\mathcal{D}$:

$$l(X_1) \to l(C) \to l(X_2) \to 0$$

since l is right exact; here $l(X_1)$, $l(X_2) \in \mathcal{T}$. Taking the kernel K of the morphism $l(C) \to l(X_2)$, we see that K is an epimorphic image of $l(X_1)$, so $K \in \mathcal{T}$, therefore $l(C) \in \mathcal{T}$ as extension of objects in a torsion class. We conclude that $C \in l^{\leftarrow}(\mathcal{T})$. □

Lemma 3.1.2. *Let $\mathcal{D}$ be an abelian category and $\mathcal{F}$ a torsion-free class on $\mathcal{D}$. Let $r : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories which respects arbitrary limits. Then the class*

$$r^{\leftarrow}(\mathcal{F}) = \{C \in \mathcal{C} : r(C) \in \mathcal{F}\}$$

is a torsion-free class in $\mathcal{C}$.

The proof is straightforward since this lemma is the dual of the previous one.

Corollary 3.1.3. *Let $\mathcal{C}$ be an abelian category with a Giraud subcategory $\mathcal{D}$. Suppose that $\mathcal{C}$ is endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$. Then the class $i^{\leftarrow}(\mathcal{Y}) = \{D \in \mathcal{D} : i(D) \in \mathcal{Y}\}$ is a torsion-free class on $\mathcal{D}$.*

Proposition 3.1.4. *Let $\mathcal{C}$ be an abelian category with a Giraud subcategory $\mathcal{D}$. Suppose that $\mathcal{D}$ is endowed with a torsion pair $(\mathcal{T}, \mathcal{F})$. Then the classes $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$:*

$$\hat{\mathcal{T}} = l^{\leftarrow}(\mathcal{T}) = \{X \in \mathcal{C} : l(X) \in \mathcal{T}\}$$

$$\hat{\mathcal{F}} = l^{\leftarrow}(\mathcal{F}) \cap \mathcal{S}^{\perp} = \{Y \in \mathcal{C} : l(Y) \in \mathcal{F} \text{ and } Y \in \mathcal{S}^{\perp}\}$$

define a torsion pair on $\mathcal{C}$ such that $i(\mathcal{T}) \subseteq \hat{\mathcal{T}}$, $i(\mathcal{F}) \subseteq \hat{\mathcal{F}}$, $l(\hat{\mathcal{T}}) = \mathcal{T}$ and $l(\hat{\mathcal{F}}) = \mathcal{F}$.

Proof. For any $T \in \mathcal{T}$ we have $li(T) \simeq T$, which proves that $i(\mathcal{T}) \subseteq \hat{\mathcal{T}}$. Moreover, given $F \in \mathcal{F}$ it is clear that $i(F) \in \mathcal{S}^{\perp}$ and $li(F) \simeq F \in \mathcal{F}$, hence $i(\mathcal{F}) \subseteq \hat{\mathcal{F}}$. We deduce that $\mathcal{T} = li(\mathcal{T}) \subseteq l(\hat{\mathcal{T}}) \subseteq \mathcal{T}$ and $\mathcal{F} = li(\mathcal{F}) \subseteq l(\hat{\mathcal{F}}) \subseteq \mathcal{F}$, which prove that $l(\hat{\mathcal{T}}) = \mathcal{T}$ and $l(\hat{\mathcal{F}}) = \mathcal{F}$. Let us show that $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$ is a torsion pair on $\mathcal{C}$.
Given $X \in \hat{\mathcal{T}}$ and $Y \in \hat{F}$,

$$(X, Y) \hookrightarrow (X, il(Y)) \simeq (l(X), l(Y)) = 0$$

where the first inclusion holds since $Y \in \hat{\mathcal{F}} \subseteq \mathcal{S}^{\perp}$ and $\mathcal{S}^{\perp} = \{D \in \mathcal{D} : \eta_D : D \to il(D) \text{ is a monomorphism}\}$. It remains to prove that for any $C \in \mathcal{C}$ there exists a short exact sequence

$$0 \to X \to C \to Y \to 0$$

with $X \in \hat{\mathcal{T}}$ and $Y \in \hat{\mathcal{F}}$.
Given $C \in \mathcal{C}$ there exists $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that the sequence

$$0 \to T \to l(C) \to F \to 0$$

is exact in $\mathcal{D}$. Let us define $X = i(T) \times_{il(C)} C$; then we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i(T) & \longrightarrow & il(C) & \longrightarrow & i(F) & & \\ & & \uparrow & & \uparrow{\scriptstyle \eta_C} & & \uparrow & & \\ 0 & \longrightarrow & X & \longrightarrow & C & \longrightarrow & C/X & \longrightarrow & 0 \end{array}$$

with exact rows (the second by definition, while the first because i is left exact) and the map $C/X \hookrightarrow i(F)$ is a monomorphism since the first square is Cartesian.
Let us apply the functor l to the diagram (remembering that it is exact, hence it preserves pullbacks and exact sequences, and that $li \simeq 1_{\mathcal{D}}$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & l(C) & \longrightarrow & F & \longrightarrow & 0 \\ & & \simeq\uparrow & & 1_{l(C)}\uparrow & & \simeq\uparrow & & \\ 0 & \longrightarrow & l(X) & \longrightarrow & l(C) & \longrightarrow & l(C/X) & \longrightarrow & 0 \end{array}$$

Here the first row coincides with the second, which is exact, $l(X) \simeq T \times_{l(C)} l(C) \simeq T \in \mathcal{T}$, which proves that $X \in \hat{\mathcal{T}}$ and so $l(C/X) \simeq F \in \mathcal{F}$, and the third vertical arrow of the previous diagram proves that $C/X \in \mathcal{S}^{\perp}$, thus $C/X \in \hat{\mathcal{F}}$. □

Proposition 3.1.5. *Let $\mathcal{C}$ be an abelian category with a Giraud subcategory $\mathcal{D}$. Suppose that $\mathcal{C}$ is endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$, and let*

$$l(\mathcal{X}) = \{T \in \mathcal{D} : T \simeq l(X),\ \exists X \in \mathcal{X}\}$$

$$l(\mathcal{Y}) = \{F \in \mathcal{D} : F \simeq l(Y),\ \exists Y \in \mathcal{Y}\}$$

Then $(l(\mathcal{X}), l(\mathcal{Y}))$ defines a torsion pair on $\mathcal{C}$ if and only if $il(\mathcal{Y}) \subseteq \mathcal{Y}$. In this case, $i^{\leftarrow}(\mathcal{Y}) = l(\mathcal{Y})$.

Proof. Let us suppose that $il(\mathcal{Y}) \subseteq \mathcal{Y}$ first. Then, since $li \simeq 1_{\mathcal{D}}$, one has $i^{\leftarrow}(\mathcal{Y}) = l(\mathcal{Y})$ and by Corollary 3.1.3 this is a torsion-free class on $\mathcal{D}$. Given $T \in l(\mathcal{X})$ (i.e. $T \simeq l(X)$, with $X \in \mathcal{X}$), and $F \in i^{\leftarrow}(\mathcal{Y})$, one has $(T, F) = (l(X), F) \simeq (X, i(F)) = 0$ since $i(F) \in \mathcal{Y}$. Now let $D \in \mathcal{D}$. There exist $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and a short exact sequence in $\mathcal{C}$:

$$0 \to X \to i(D) \to Y \to 0.$$

Applying l to this sequence we get a short exact sequence in $\mathcal{D}$:

$$0 \to l(X) \to D \to l(Y) \to 0$$

where $l(X) \in l(\mathcal{X})$ and $l(Y) \in l(\mathcal{Y})$, which proves that $(l(\mathcal{X}), l(\mathcal{Y}))$ is a torsion pair on $\mathcal{D}$.
Conversely, if $(l(\mathcal{X}), l(\mathcal{Y}))$ is a torsion pair on $\mathcal{C}$, then for every $X \in \mathcal{X}$ and every $Y \in \mathcal{Y}$ one has $0 = (l(X), l(Y)) \simeq (X, il(Y))$, therefore $il(Y) \in \mathcal{Y}$. □

Theorem 3.1.6. *Let $\mathcal{C}$ be an abelian category with a Giraud subcategory $\mathcal{D}$. There exists a bijective correspondence between torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{C}$ satisfying $il(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$ and torsion pairs $(\mathcal{T}, \mathcal{F})$ on $\mathcal{D}$.*

Proof. On one hand, taking a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{D}$, the torsion pair $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$ satisfies $il(\hat{\mathcal{F}}) \subseteq \hat{\mathcal{F}}$ and one easily verifies that $(l(\hat{\mathcal{T}}), l(\hat{\mathcal{F}})) = (\mathcal{T}, \mathcal{F})$.
On the other hand, given $(\mathcal{X}, \mathcal{Y})$ a torsion pair on $\mathcal{C}$ satisfying $il(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$, its corresponding torsion pair on $\mathcal{D}$ is $(l(\mathcal{X}), l(\mathcal{Y}))$ for which it is clear that $l(\hat{\mathcal{Y}}) = l^{\leftarrow}(l(\mathcal{Y})) \cap \mathcal{S}^{\perp} = \mathcal{Y}$ and so $(\mathcal{X}, \mathcal{Y}) = (l(\hat{\mathcal{X}}), l(\hat{\mathcal{Y}}))$. □

Dually one obtains:

Theorem 3.1.7. *Let $\mathcal{C}$ be an abelian category with a co-Giraud subcategory $\mathcal{D}$. There exists a bijective correspondence between torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{C}$ satisfying $jr(\mathcal{X}) \subseteq \mathcal{X} \subseteq^{\perp} \mathcal{S}$ and torsion pairs $(\mathcal{T}, \mathcal{F})$ on $\mathcal{D}$.*

3.2 The case of functor categories

In Section 3.1 a condition to have a one-to-one correspondence between torsion pairs on a category and on a Giraud (or co-Giraud) subcategory of its was established. Our next aim is to specialize this condition to the case of the functor category of a ring, namely that studied in Chapter 2.

Let us consider first the category $((R\text{-mod})^{op}, \mathcal{A}b)$ of contravariant additive functors from the category of finitely presented left modules over a ring R to the category of abelian groups.
In Chapter 2 we proved (see Corollary 2.1.12) that Mod-R is a Giraud subcategory of $((R\text{-mod})^{op}, \mathcal{A}b)$. In particular, we have an adjunction $\langle -_R, i \rangle$ where $-_R$ is the R-valuation functor, and i is the inclusion of Mod-R as the subcategory of representable functors in $((R\text{-mod})^{op}, \mathcal{A}b)$.
Looking at what was proved in Theorem 3.1.6, the condition for torsion pairs of the form $(\mathcal{X}, \mathcal{Y})$ in $((R\text{-mod})^{op}, \mathcal{A}b)$ to be in a bijective correspondence with those of Mod-R is the following:

$$i(-_R)(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$$

where $\mathcal{S} = \{F \in ((R\text{-mod})^{op}, \mathcal{A}b) : F(R) = 0\}$.
Our goal is to characterize the torsion pairs $(\mathcal{X}, \mathcal{Y})$ in $((R\text{-mod})^{op}, \mathcal{A}b)$ such that the condition above is satisfied.

Lemma 3.2.1. *Let us consider the abelian category* $((R\text{-mod})^{op}, \mathcal{A}b)$. *Then* $\mathcal{S}^{\perp} = \{F \in ((R\text{-mod})^{op}, \mathcal{A}b) : F$ *sends epimorphisms to monomorphisms*$\}$.

Proof. ($\subseteq$) Let $F \in \mathcal{S}^{\perp}$; then the F-component of the unit of the adjunction is a monomorphism $\eta(F) : F \to i(F(R))$ due to the characterization of $\mathcal{S}^{\perp}$ in the general case at the beginning of the previous section. Then consider $M \in R$-mod, and a presentation $R^n \to R^m \to M \to 0$ of M. We obtain the following commutative diagram by applying F and $i(F(R)) = \mathrm{Hom}(-, F(R))$ to the presentation of M:

$$\begin{array}{ccccccc}
 & & F(M) & \longrightarrow & F(R^m) & \longrightarrow & F(R^n) \\
 & & \big\downarrow & & \simeq\big\downarrow & & \simeq\big\downarrow \\
0 & \longrightarrow & \mathrm{Hom}(M, F(R)) & \longrightarrow & \mathrm{Hom}(R^m, F(R)) & \longrightarrow & \mathrm{Hom}(R^n, F(R))
\end{array}$$

where the second row is exact; in particular, looking at the left square of the diagram, by commutativity one gets that $F(M) \to F(R^m)$ must be a monomorphism.
In the general case, for L, $M \in R$-mod such that $L \to M \to 0$ is exact, one may find two epimorphisms (since L and M are finitely presented) of the form $R^m \to M \to 0$ and $R^l \to L \to 0$. By choosing $n = \max(l, m)$, one gets the diagram

$$\begin{array}{ccccc}
R^n & = & R^n & & \\
\big\downarrow & & \big\downarrow & & \\
L & \longrightarrow & M & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \\
0 & & 0 & &
\end{array}$$

Applying F to the diagram and using the same argument as above, one gets the diagram:

$$\begin{array}{ccc}
F(R^n) & = & F(R^n) \\
\big\uparrow & & \big\uparrow \\
F(M) & \longrightarrow & F(L)
\end{array}$$

by which it is clear that $0 \to F(M) \to F(L)$ is exact.

(⊇) Let F be a functor which sends epimorphisms to monomorphisms. Then for every $M \in R$-mod and presentation $R^n \to R^m \to M \to 0$ one gets the commutative diagram where the second row is exact:

$$\begin{array}{ccccccc} & & F(M) & \hookrightarrow & F(R^m) & \longrightarrow & F(R^n) \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & \mathrm{Hom}(M, F(R)) & \longrightarrow & \mathrm{Hom}(R^m, F(R)) & \longrightarrow & \mathrm{Hom}(R^n, F(R)) \end{array}$$

which proves that $0 \to F(M) \to \mathrm{Hom}(M, F(R))$ is exact, therefore $F \in \mathcal{S}^{\perp}$.

□

The following corollary follows easily from Lemma 3.2.1.

Corollary 3.2.2. *Let us consider the functor category* $((R\text{-mod})^{op}, \mathcal{A}b)$ *and a torsion pair* $(\mathcal{X}, \mathcal{Y})$ *on it. Then* $\mathcal{Y} \subseteq \mathcal{S}^{\perp}$ *if and only if every functor in* $\mathcal{Y}$ *sends epimorphisms to monomorphisms.*

Let us consider now the category $((R\text{-mod}), \mathcal{A}b)$ of covariant additive functors from the category of finitely presented left R-modules to the category of abelian groups. In Chapter 2 we proved that the category Mod-R is a co-Giraud subcategory of $((R\text{-mod}), \mathcal{A}b)$. In particular, we have an adjunction $\langle j, -_R \rangle$ where $-_R$ is the usual R-valuation functor and j is the tensor embedding, described in Section 2.2. In Theorem 3.1.7 a condition to have a bijective correspondence between torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $((R\text{-mod}), \mathcal{A}b)$ and those on Mod-R was found, and it translates in our case in the following way:

$$j(-_R)(\mathcal{X}) \subseteq \mathcal{X} \subseteq^{\perp} \mathcal{S}$$

where $\mathcal{S} = \{F \in ((R\text{-mod}), \mathcal{A}b) : F(R) = 0\}$.
Again, our aim is to characterize the torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $((R\text{-mod}), \mathcal{A}b)$ that satisfy this condition. The next lemma is essentially dual to Lemma 3.2.1.

Lemma 3.2.3. *Let us consider the abelian category* $((R\text{-mod}), \mathcal{A}b)$*. Then* ${}^{\perp}\mathcal{S} = \{F \in ((R\text{-mod}), \mathcal{A}b) : F$ *sends epimorphisms to epimorphisms*$\}$.

Proof. (⊆) Let F be a functor in ${}^{\perp}\mathcal{S}$. Then the F-component of the counit of the adjunction is an epimorphism $\epsilon(F) : j(F(R)) \to F$ due to the characterization of ${}^{\perp}\mathcal{S}$ in the general case at the beginning of the previous section. Then consider $M \in R$-mod, and a presentation $R^n \to R^m \to M \to 0$ of M. We obtain the following commutative diagram by applying F and $j(F(R)) = - \otimes F(R)$ to the presentation of M:

$$\begin{array}{ccccccc}
F(R^n) & \longrightarrow & F(R^m) & \longrightarrow & F(M) & & \\
\uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} & & \twoheaduparrow & & \\
R^n \otimes F(R) & \longrightarrow & R^m \otimes F(R) & \longrightarrow & M \otimes F(R) & \longrightarrow & 0
\end{array}$$

in which the second row is exact; using the same argument employed in Lemma 3.2.1, one obtains that $F(R^m) \to M \to 0$ is exact. In the general case, for L, $M \in R$-mod such that $L \to M \to 0$ is exact, one may find a commutative diagram as below:

$$\begin{array}{ccccc}
R^n & = & R^n & & \\
\downarrow & & \downarrow & & \\
L & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

Applying F to the diagram and using the same argument as above, one gets the diagram:

$$\begin{array}{ccc}
F(R^n) & = & F(R^n) \\
\twoheaddownarrow & & \twoheaddownarrow \\
F(L) & \longrightarrow & F(M)
\end{array}$$

by which it is clear that $F(L) \to F(M) \to 0$ is exact.

($\supseteq$) Let F be a functor which send epimorphisms to epimorphisms. Then for every $M \in R$-mod and presentation $R^n \to R^m \to M \to 0$ one gets the commutative diagram where the second row is exact:

$$\begin{array}{ccccccc}
F(R^n) & \longrightarrow & F(R^m) & \longrightarrow & F(M) & \longrightarrow & 0 \\
\uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} & & \uparrow & & \\
R^n \otimes F(R) & \longrightarrow & R^m \otimes F(R) & \longrightarrow & M \otimes F(R) & \longrightarrow & 0
\end{array}$$

which proves that $M \otimes F(R) \to F(M) \to 0$ is exact, therefore $F \in {}^{\perp}\mathcal{S}$.

□

The following corollary follows easily from Lemma 3.2.3.

Corollary 3.2.4. *Let us consider the functor category* $((R\text{-mod}), \mathcal{A}b)$ *and a torsion pair* $(\mathcal{X}, \mathcal{Y})$ *on it. Then* $\mathcal{X} \subseteq^{\perp} \mathcal{S}$ *if and only if every functor in* $\mathcal{X}$ *sends epimorphisms to epimorphisms.*

3.2.1 The inclusion $il(\mathcal{Y}) \subseteq \mathcal{Y}$ and further developments

In Corollary 3.2.2 and Corollary 3.2.4 a characterization of the inclusion $\mathcal{Y} \subseteq \mathcal{S}^{\perp}$ and of $\mathcal{X} \subseteq^{\perp} \mathcal{S}$, which appear in Theorem 3.1.6 and Theorem 3.1.7, has been given. In order to have a complete and detailed framework of the translation of the formulas given in the aforementioned theorems, we should now focus upon the study of the inclusion $il(\mathcal{Y}) \subseteq \mathcal{Y}$ (in the contravariant case; that of the covariant case would follow easily).
In writing this thesis, much effort has been made to obtain a complete characterization of the inclusion, with no considerable result; only few facts have been understood, and all of them carry only the information that any torsion-free class $\mathcal{Y}$ in $((R\text{-mod}^{op}), \mathcal{A}b)$ that satisfies $il(\mathcal{Y}) \subseteq \mathcal{Y}$ should not have some properties, while none of this facts gives any useful hint of the desirable properties that $\mathcal{Y}$ should satisfy.
In the following lines some of these facts are shown, hoping that they may establish a basis for further developments of the study of the transfer of torsion pairs from the functor category to the underlying module category and vice versa.

Fact 3.2.5. *We can not require that, for any* $Y \in \mathcal{Y}$, $il(Y) \subseteq Y$. *This would imply that* $il(\mathcal{Y}) \subseteq \mathcal{Y}$, *but since the condition we are studying is also* $Y \subseteq \mathcal{S}^{\perp}$, *this implies that the unit of the adjunction* $\langle l, i\rangle$ *is a monomorphism, so that* $Y \subseteq il(Y)$. *Then we would get* $Y \simeq il(Y)$, *and this would allow us to consider only trivial torsion pairs in* $((R\text{-mod}^{op}), \mathcal{A}b)$.

Fact 3.2.6. *Let us consider the short exact sequence:*

$$0 \to Y \xrightarrow{\eta(Y)} il(Y) \to \mathrm{Coker}(\eta(Y)) \to 0$$

where $Y \in \mathcal{Y}$ *and* $\eta(Y)$ *is the unit of the adjunction* $\langle l, i\rangle$.
A way to obtain that $il(Y)$ *belongs to* $\mathcal{Y}$ *may be to assume that* $\mathrm{Coker}(\eta(Y))$ *belongs to* $\mathcal{Y}$, *in order to have an extension. But we know that the cokernel of the unit of the adjunction belongs to the kernel of the left adjoint functor, which is* $\mathcal{S}$; *we know as well that* $\mathcal{Y} \subseteq \mathcal{S}^{\perp}$, *so we would get that* $\mathrm{Coker}(\eta(Y)) \in \mathcal{S} \cap \mathcal{S}^{\perp} = 0$, *obtaining again an isomorphism* $Y \simeq il(Y)$.

Lemma 3.2.7. *In our hypothesis,* $il(\mathcal{Y}) \subseteq \mathcal{S}^{\perp}$ *for any torsion-free class* $\mathcal{Y}$ *in* $((R\text{-mod}^{op}), \mathcal{A}b)$.

Proof. Let us consider $F \in \mathcal{Y}$; then we have to prove that $il(F) = (-, F(R)) \in \mathcal{S}^{\perp}$. Let us fix an $S \in \mathcal{S}$; then we get

$$\mathrm{Nat}(S, (-, F(R)) = \mathrm{Nat}(S, il(F)) \simeq \mathrm{Hom}(l(S), l(F)) = \mathrm{Hom}(S(R), F(R)) = 0$$

using the adjunction isomorphisms and the fact that $S \in \mathrm{Ker}(l)$. Since the choice of S was arbitrary, we can conclude that $il(F) \in \mathcal{S}^{\perp}$. □

Anyway, a simple example of the transfer of torsion pair may be given in the most trivial case:

Example 3.2.8. *Let us consider the trivial torsion pair* $(0, \mathrm{Mod}\text{-}R)$ *in* $\mathrm{Mod}\text{-}R$. *By the correspondence proved in Section 3.1, the corresponding torsion pair in* $((R\text{-mod}^{op}), \mathcal{A}b)$ *is* $(\mathcal{S}, \mathcal{S}^{\perp})$*, because if we call* $\mathcal{T} = 0$ *and* $\mathcal{F} = \mathrm{Mod}\text{-}R$ *in* $\mathrm{Mod}\text{-}R$*, then*

$$\mathcal{X} := l^{\leftarrow}(\mathcal{T}) = \{X \in ((R\text{-mod}^{op}), \mathcal{A}b) : l(X) = X(R) = 0\} = \mathcal{S}$$

and also

$$\mathcal{Y} := l^{\leftarrow}(\mathcal{F}) \cap \mathcal{S}^{\perp} =$$
$$= \{Y \in ((R\text{-mod}^{op}), \mathcal{A}b) : l(Y) = Y(R) \in \mathrm{Mod}\text{-}R \textit{ and } Y \in \mathcal{S}^{\perp}\} =$$
$$= ((R\text{-mod}^{op}), \mathcal{A}b) \cap \mathcal{S}^{\perp} = \mathcal{S}^{\perp}.$$

Furthermore the torsion class just found is unique, since $\mathcal{Y}$ *verifies the condition* $il(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$*, where the inclusion on the right is actually an equality, and that on the left is trivial because of Lemma 3.2.7.*

The study of the transfer of torsion pairs from the category $((R\text{-mod}^{op}), \mathcal{A}b)$ to the underlying module category and vice versa may be used as a tool to understand tilting and cotilting torsion pairs in module categories. First, let us give a definition:

Definition 3.2.9. *Let* $\mathcal{C}$ *be a category. A torsion pair* $(\mathcal{T}, \mathcal{F})$ *on* $\mathcal{C}$ *is* ***tilting*** *if any object* $C \in \mathcal{C}$ *is subobject of some* $T \in \mathcal{T}$*. The torsion pair is* ***cotilting*** *if any object* $C \in \mathcal{C}$ *is quotient of some object* $F \in \mathcal{F}$*.*

A first step further in the study of the transfer of torsion pairs may be to study whether the property of being (co)tilting is preserved or not in the transfer process. While writing this thesis, only one direction of the transfer has been proved to keep the property of being (co)tilting for torsion pairs, and is stated in next proposition:

Proposition 3.2.10. *Let us consider the category* $\mathcal{C} = ((R\text{-mod}^{op}), \mathcal{A}b)$*. Let* $(\mathcal{X}, \mathcal{Y})$ *be a tilting torsion pair in* $\mathcal{C}$*; then the torsion pair* $(l(\mathcal{X}), l(\mathcal{Y}))$ *is tilting in* Mod-R*.*

Proof. Let us consider a module M in Mod-R, and let us consider first the case of $(\mathcal{X}, \mathcal{Y})$ being a tilting torsion pair. Let us consider the functor $(-, M) \in \mathcal{C}$; since $(\mathcal{X}, \mathcal{Y})$ is tilting, there exists $X \in \mathcal{X}$ and a monomorphism $\eta : (-, M) \hookrightarrow X$. In particular, $\eta(R) : (R, M) \to X(R)$ must be a monomorphism, so that (since $(R, M) \simeq M$) M is a subobject of $X(R) \in l(\mathcal{X})$. Then any $M \in$ Mod-R is subobject of an element of $l(\mathcal{X})$, and therefore $(l(\mathcal{X}), l(\mathcal{Y}))$ is a tilting torsion pair in Mod-R. The proof for the cotilting case can be easily derived by this one. □

The other direction of the transfer has yet to be understood; possibly the transfer of a tilting torsion pair of Mod-R to $((R\text{-mod}^{op}), \mathcal{A}b)$ will not guarantee a lifting to a tilting torsion pair, and further conditions should be considered.

Bibliography

[1] Bo Stenstrom, *Rings of quotients - An introduction to methods of ring theory*, Springer-Verlag 1975

[2] N. Popescu, *Abelian categories with applications to rings and modules*, London, Academic Press Inc., 1973

[3] P. Gabriel, *Des catégories abéliennes*, Bulletin de la Société Mathématique de France 90 (1962), 323–448.

[4] Carl Faith, *Algebra: rings, modules categories*, Springer-Verlag Berlin Heidelberg, 1973

[5] Mike Prest, *The functor category*, Categorical Methods in Representetion Theory, Bristol, Sept. 2012

[6] Ivo Herzog, *Contravariant functors on the category of finitely presented modules*, Israel journal of mathematics 167 (2008)

[7] W.W. Crawley-Boevey, *Locally finitely presented additive categories*, Communications in Algebra 22(5), 1994, pages 1641-1674

[8] Jensen, Lenzig, *Model theoretic algebra*, Bull. Amer. Math. Soc. (N.S.), Volume 25, Number 1 (1991)

[9] Alberto Facchini, *Module theory: Endomorphism rings and direct sum decompositions in some classes of modules*, Springer Science and Business Media, 2012

[10] H. Tachikawa, K. Ohtake, *Colocalization and localization in abelian categories*, Journal of algebra, Volume 56 Issue 1, Jan. 1979, pages 1-23

[11] K. Ohtake, *Colocalization and localization*, Journal of Pure and Applied Algebra 11 (1977), 217-241.

[12] R. Colpi, L. Fiorot, F. Mattiello, *On tilted Giraud subcategories*, Journal of Pure and Applied Algebra 220 (2016), 346-363.

[13] L. Gruson, C. U. Jensen, *Dimensions cohomologique reliées aux foncteurs* $\varprojlim^i$, Séminaire, in d'Algèbre Paul Dubreil et Marie-Paule Malliavin, Paris 1980, Lecture Notes in Mathematics, 867, Springer-Verlag, 1981, pp. 234–294.