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Generalized Symmetries and Symmetry Topological Field Theory of 3-dimensional ABJ Theories from String Theory and Branes

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Understanding symmetries in quantum field theory (QFT) is crucial for unraveling the fundamental principles that govern physical systems. Though less explored than ordinary symmetry, the recently introduced notion of generalized symmetry opens up new avenues for understanding complex phenomena. The goal of this thesis is to investigate these symmetries in 3-dimensional QFTs through the derivation of a bulk topological theory in one dimension higher that describes the discrete generalized symmetries of strongly coupled 3d ortho-symplectic ABJ (Aharony-Bergman-Jafferis) theories.
In order to achieve this, the research will rely on holographic duality, established through AdS/CFT correspondence, which eases the investigation of non-perturbative phenomena by mapping a given strongly coupled d-dimensional conformal field theory (CFT) to a weakly coupled gravitational theory in (d+1) dimensions. String theory, and in particular its low-energy limit (in our case IIA supergravity), does in fact offer a useful framework for capturing symmetries through a topological limit or, better said, truncation.
With a particular focus on the flux sector of supergravity and by using the discrete torsions of the holographic IIA geometry, i.e $AdS_4 \times CP^3/\mathbb{Z}_2$, we construct the 4-dimensional "Symmetry topological field theory (TFT)" via compactification on the $CP^3/\mathbb{Z}_2$ space. The resulting TFT then allows to identify generalized symmetries and anomalies of the original QFT in terms of branes, that are the fundamental extended dynamical objects of string theory, and coupling thereof.
This thesis contributes to a deeper understanding of the nature of generalized symmetries in 3-dimensional QFTs and their connections to holography and string theory.

Contents

1 Introduction

One of the most insightful features of a physical theory is the set of symmetries it bears. The most commonly encountered kind of symmetry are 0-form continuous symmetries, that correspond to Lie groups of transformations that leave the theory's action unchanged. However, the definition of a symmetry can be extended to encompass a wider array of invariances: one could give up the continuity of the Lie group, obtaining a discrete symmetry, or go a step further and abandon the group structure altogether, entering the world of category theory and non-invertible symmetries. Another possibility is to have the symmetry act on operators extended in space-time rather than point-like ones, resulting in an higher-form symmetry. Symmetries of these types are collectively known as "generalized symmetries".

The purpose of this research is to apply to a 3-dimensional ABJ (Aharony-Bergman-Jafferis) theory a method developed in [1] [2] [3] aimed at shedding light on the generalized global symmetries of a given conformal theory through an holographic construction. More precisely, starting from the 10-dimensional supergravity associated to the original QFT via AdS/CFT correspondence, the method consists in isolating the Ramond-Ramond and Neveu-Schwarz sector (later referred to as flux sector) and proceeding with the compactification of the internal part of the supergravity background. Before doing so, the flux sector will be modified to encode the Bianchi identities for the fields, which requires an eleventh auxilliary dimension. Once the reduced 5-dimensional action is obtained, it will be possible to select the topological sector by means of a simple limit. At this point one removes the auxilliary dimension by applying Stokes theorem, obtaining a topological theory on AdS_4 that contains BF-terms and, most importantly, Chern-Simons couplings. These can be traced back to the original QFT by means of the "anomaly inflow" paradigm. Such theory is known as "Symmetry Topological Field Theory", or SymTFT for short, and its derivation is the main goal of this research. From the SymTFT one can reconstruct the defect operators associated with the symmetries of the underlying CFT, given that appropriate boundary conditions are set on the TFT fields: different boundary conditions lead to the symmetry structure of different variants of ABJ theory.

Beside providing the means to reconstruct the defect operators, this method of probing a theory's symmetry structure also has the advantage of not relying on a lagrangian description of the initial conformal theory. Moreover, the Symmetry TFT's relation to the anomaly inflow paradigm can contribute to understanding the origin of anomalies.

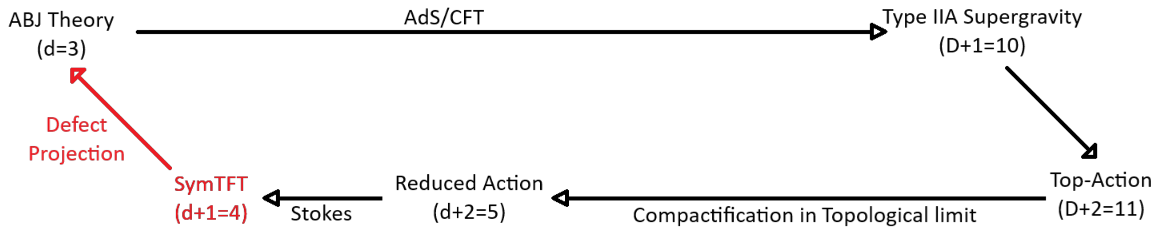

Figure 1.1: Schematic representation of the process utilized in this work to reconstruct the symmetry structure of the ABJ theory.

As mentioned before, the subject of our analysis is a 3-dimensional superconformal ABJ-type theory constructed in [4], that has gauge group $O(2N)_{2k} \times USp(2N)_{-k}$, where N and k are integers, with the latter denoting the Chern-Simons level. Some specifications of this kind of theory include gauge groups such as $O(2N)_{2k} \times USp(2N)_{-k}/\mathbb{Z}_2$ and $SO(2N)_{2k} \times USp(2N)_{-k}$. The holographic dual of these theories can be derived from a type IIA supergravity theory with background manifold $AdS_4 \times CP^3/\mathbb{Z}_2$.
The choice theory was driven by the fact that the symmetry content of these theories has already been probed by the authors of [5] with a completely different approach, relying on the superconformal index. Their results set a concrete expectation of the terms that should appear in the SymTFT. In particular, the $SO(2N)_{2k} \times USp(2N)_{-k}$ gauge theory is expected to bear a 1-form and two 0-form discrete symmetries, whereas the $O(2N)_{2k} \times USp(2N)_{-k}/\mathbb{Z}_2$ variant should turn out to have a 0-form non-invertible symmetry. Our final action will indeed be compatible with these results and one could reconstruct the actual symmetries by choosing the right boundary conditions. Note that since both theories are associated with the same SymTFT, their different symmetry structures are yielded by different boundary conditions.
The thesis is organized as follows: in chapter 2 we provide a review on generalized symmetries, introduced by describing ordinary symmetries in terms of topological operators and then relaxing various part of the definition to obtain different generalizations; we then examine how anomalies translate to the generalized framework while also introducing the notion of "anomaly inflow". We will finally see how this concept gives us the footing needed to devise a way to reconstruct the global symmetries of a theory by means of a topological theory in one dimension higher: the SymTFT.
Since we will make frequent use of string theory concepts and terminology, chapter 3 will be dedicated to a brief review of such theory, focusing on the nature of branes and the low energy limits of string theory, in including supergravity theories (SuGra). We will also discuss the topic of Kaluza-Klein compactifications, seeing as dimensional reduction is an important step in the derivation of the SymTFT.
The following chapter (chapter 4) is instead dedicated to explain the tenets of AdS/CFT correspondence, which establishes a duality relation between d-dimensional conformal

field theories and (compactified) supergravity theories living in AdS_{d+1}. This chapter also introduces the ABJ theory and its holographic dual, providing a concrete example of an AdS_4/CFT_3 duality pair.
We then elaborate on how the SymTFT is constructed starting from the SuGra action, which is the focus of chapter 5. In particular we will discuss how one can extract the topological sector of the dimensionally reduced action and operate a series of manipulations on the flux sector action that ease later computations.
Chapter 6 is the actual computation of the topological action. We will initially restrict to a self-consistent subset of the terms appearing in the action and then, once the path to the final result is clear, include the remaining parts. We conclude the chapter with a discussion of the symmetries and defect operators one can expect to reconstruct from the newly found 4-dimensional action, comparing the results with the expectations set by [5].

2 Generalized Symmetries

Symmetries, both local and global, are fundamental in physics: redundancies are pivotal in the formulation of gauge theories, whereas global symmetries provide powerful constraints to quantum field theories. In particular, obstructions to gauge a global invariance, known as anomalies, are RG-invariant quantities that allow to peek into strongly coupled systems that would be difficult to examine from a strictly perturbative point of view.

Over the last decade, physicists started contemplating the idea of symmetries being part of a wider landscape of properties, which materialized in the concept of "generalized symmetry" [6] [7]. More precisely, by relaxing various elements of the definition of symmetry one encounters higher-form symmetries, non-invertible ones and the more familiar discrete symmetries. In principle, if a generalized symmetry does not give birth to anomalies, it can be gauged just like an ordinary one.

Beside reviewing what a generalized symmetry is, this chapter also explores how these properties can be probed by means of the symmetry TFT, which will be the main subject of our study.

2.1 Extending the Concept of Symmetry

Before introducing generalized symmetries, let us review the definition and properties of an ordinary symmetry, presenting them in terms of topological operators, making the generalization easier.

Consider a field theory for the field $\phi(x)$ in a d-dimensional space-time, described by the action $S = \int d^dx\mathcal{L}[\phi(x)]$, and a continuous group G under which the field transforms (at first order) as in

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\delta\phi(x) \quad . \tag{2.1}$$

These transformations are said to be 'internal' if they only affect the field content of the theory, while they are called 'space-time transformations' if they act on the space-time coordinates and, as a conseuqence, on the fields.

Definition 1 (Symmetry) *The Lie group of transformations G parameterized by α is a symmetry of a given theory if it doesn't affect physics of the system. In other words, if it leaves the action unchanged or, equivalently, if it alters the Lagrangian by a total derivative:*

$$\delta_\alpha S = 0 \qquad \Longleftrightarrow \qquad \delta_\alpha \mathcal{L}[\phi] = \partial_\mu \alpha \Lambda^\mu \quad \textit{for some} \quad \Lambda^\mu \quad . \tag{2.2}$$

According to Nöther theorem, each continuous symmetry is associated with a conserved current. This can be easily seen by variating the Lagrangian and imposing the result to be a total derivative, i.e. assuming the transformation to be a symmetry. Indeed the variation is

$$\delta_\alpha \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \alpha \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\alpha(\partial_\mu \phi) \equiv \partial_\mu \alpha \Lambda^\mu \quad ; \tag{2.3}$$

from the transformation of the field one gets $\delta_\alpha \partial_\mu \phi = \alpha \partial_\mu \delta \phi$. Applying Leibniz rule to the term $\propto \partial_\mu \delta\phi$ in the variation allows to write it as

$$\frac{\partial \mathcal{L}}{\partial \phi} \alpha \delta \phi + \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \alpha \delta \phi \equiv \partial_\mu \alpha \Lambda^\mu \quad . \tag{2.4}$$

In classical settings, one then recognizes the Euler-Lagrange equations, finally leading to the conservation law of the quantity

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi - \Lambda^\mu \qquad \partial_\mu j^\mu = 0 \quad , \tag{2.5}$$

which is referred to as Nöther current. The conservation law can be expressed in differential geometry formalism as the closure of a $(d-1)$-form:

$$d * j_1 = 0 \quad , \tag{2.6}$$

where j_1 denotes the 1-form current of components j^μ and $*$ is the Hodge star operator. Nöther current can be coupled to a background 1-form gauge field $A_1(x)$, that canonically shifts by $\delta_\alpha A_1 = d\alpha$, by including in the action the term

$$S \supset i \int d^d x A_\mu j^\mu = i \int A_1 \wedge *j_1 \qquad \delta_\alpha S = i \int (d\alpha) \wedge *j_1 \quad . \tag{2.7}$$

This is known as a source term and will be an important building block going forward: it is for instance at the root of the anomaly inflow paradigm that we will examine in section 2.2 while in chapter 3 gauge-current couplings are used to associate charged branes to the background fields appearing in string theory.

Having associated a conserved current to each symmetry, it is natural to also define a conserved charge, obtained by integrating the former over the a $(d-1)$-dimensional space-like slice M_{d-1}:

$$Q(M_{d-1}) = \int_M d^{d-1} x \hat{n}_\mu j^\mu = \int_M *j_1 \tag{2.8}$$

The charge operator then acts as the generator of the unitary operators associated to the symmetry transformations by Wigner's theorem that eventually act on the Hilbert space of states[1]. In an Euclidean metric we define such operators as follows

Definition 2 (Symmetry defect operator) *Given a generic manifold* Σ_{d-1} *of codimension* 1 *with no boundary, the Symmetry Defect Operator (SDO) is generated by exponentiation of the conserved charge:*

$$U_\alpha(\Sigma_{d-1}) \;=\; e^{i\alpha Q(\Sigma_{d-1})} \tag{2.9}$$

The term 'defect' highlights that, due to the Euclidean setting, the integration domain of the charge can also be extended in the time-direction. Notice also that, while the charge is by construction a global object, the nature of the defect operator is determined by that of α.

SDOs can be shown to be topological in nature thanks to the conservation of the Nöther current; consider two codimension 1 manifolds $\Sigma \sim \Sigma'$. Here the symbol "$\sim$" indicates homotopy, that is the property of a pair of manifolds that can be smoothly deformed into one another, which is formally defined by the existence of a continuous function $L(t)$ such that

$$\begin{cases} L(0) = \Sigma \\ L(1) = \Sigma' \end{cases} , t \in [0,1]. \tag{2.10}$$

The product of U_α on Σ with its inverse on Σ' then becomes:

$$U_\alpha(\Sigma) \cdot U_{-\alpha}(\Sigma') = e^{i\alpha \int_\Sigma *j_1} e^{-i\alpha \int_{\Sigma'} *j_1} = e^{i\alpha \int_{\Xi_d} d*j_1} \quad , \tag{2.11}$$

where Ξ_d is the d-dimensional manifold swept by the deformation of Σ into Σ'. The conservation law then kills the exponent, confirming that the multiplied operators are indeed one the inverse of the other, despite being constructed on different manifolds. Uniqueness of the inverse then grants identity between the U_α. The upshot of this calculation is that U_α is determined solely by the choice of Σ up to homotopy, which is the hallmark of a topological object.
Let us know include in our theory space a charged point-like object, which is realized as a local operator $\mathcal{O}(x)$. Going back for a moment to the time slice M_{d-1}, the symmetry operator acts on the charged object as

$$U_\alpha(M_{d-1})\mathcal{O}(x)U_{-\alpha}(M_{d-1}) = R_G(\alpha)\mathcal{O}(x) \quad , \tag{2.12}$$

where $R_G(\alpha)$ denotes a representation of the Lie group G and the non-commutativity of $U_\alpha(M)$ and $O(x)$ is granted by the fact that this is an equal time relation, so x surely lies

[1]The unitary operators will act on states as $U_\alpha|*\rangle$ and as $U_\alpha \mathcal{O} U_\alpha^\dagger$ on operators. Therefore, transforming ϕ under U_α one gets the variation $\delta\phi = i[Q,\phi]$.

on M_{d-1}. To understand how defects act on the charged operator, one needs to make use of Ward identity, which is the operatorial counterpart to Nöther's conservation law:

$$\partial_\mu j^\mu(y)\mathcal{O}(x) = \delta^{(d)}(x-y)R(T^a)\mathcal{O}(x) \quad , \tag{2.13}$$

where $R(T^a)$ is a representation of G generated by T^a.
Consider now the expression

$$U_\alpha(\Sigma_{d-1})\mathcal{O}(x)U_{-\alpha}(\Sigma'_{d-1}) \quad , \tag{2.14}$$

assuming that $x \notin \Sigma_{d-1}, \Sigma'_{d-1}$. Since $\mathcal{O}(x)$ and Σ' are space separated we can commute them, reproducing the conditions of (2.11). Nevertheless this time around we are not granted closure of the current, and we need to use (2.13) to reach the final result:

$$\begin{aligned} U_\alpha(\Sigma_{d-1})\mathcal{O}(x)U_{-\alpha}(\Sigma'_{d-1}) &= e^{i\alpha \int_{\Xi_d} d^d y \partial_\mu j^\mu} \mathcal{O}(x) = \\ &= e^{i\alpha R(T^a) \int_{\Xi_d} d^d y \delta(x-y)} \mathcal{O}(x) \quad , \end{aligned} \tag{2.15}$$

meaning that if Ξ_d sweeps through x (so that the integral of the Dirac δ gives 1), the action of the deformation is

$$U_\alpha(\Sigma_{d-1})\mathcal{O}(x) = e^{i\alpha R(T^a)}\mathcal{O}(x)U_\alpha(\Sigma'_{d-1}) = R(\alpha)\mathcal{O}(x)U_\alpha(\Sigma'_{d-1}) \quad , \tag{2.16}$$

while it trivializes if x is not crossed. We will see that this behaviour, represented in Figure 2.1, carries over nicely to the case of higher-form symmetries, where the charged object isn't point-like anymore.

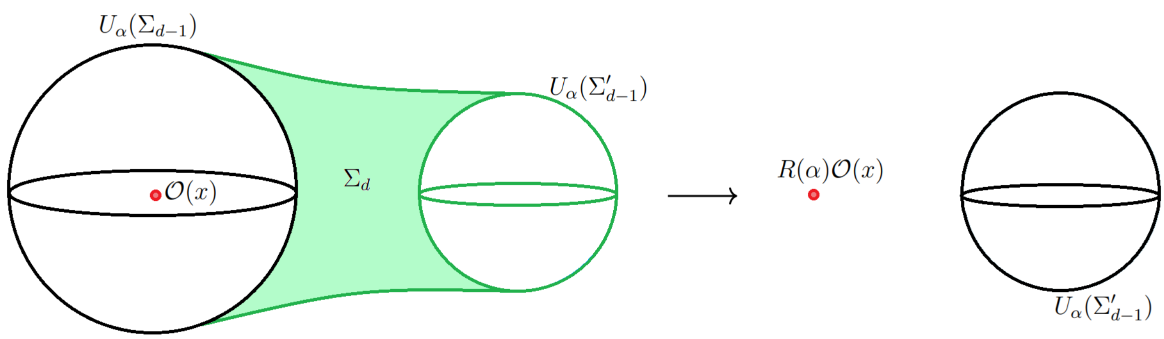

Figure 2.1: A smooth deformation (marked in green) of the symmetry defect operator (U_α) that crosses a local charged operator ($\mathcal{O}(x)$) acts on it via a representation of the symmetry group.

This discussion showed how any "ordinary" symmetry can be described in terms of topological defect operators, yet the inverse is not true: there exists a well of defect operators that cannot be associated with an ordinarily defined symmetry but will instead correspond to a generalized symmetry.

2.1.1 Higher Form Symmetries

Now that we laid out a formal description of ordinary symmetries as topological operators, we can start to gradually stray from it. In particular, in this section we set out to construct a symmetry that acts on extended operators rather than local ones, that is $\mathcal{O}(\Gamma_p)$ as opposed to $\mathcal{O}(x)$, with Γ_p being a p-dimensional manifold within the theory space. Upon acting on the vacuum, operators of this kind generate p-branes rather than the localized particles that $\mathcal{O}(x) = \phi(x)$ would produce. The generalized symmetries that will emerge are called higher-form symmetries, for reasons that will soon be clear.
We saw that symmetries are associated with a closed $(d-1)$-form current or, equivalently, to a conserved vector current (j^μ). The linchpin in constructing higher-form symmetries is to remove the constraint on the current's form degree; in other words, we associate a p-form continuous symmetry with a closed $(d-p-1)$-form, dual to a $(p+1)$ conserved current:

$$d * j_{p+1} = 0 \qquad \Longleftrightarrow \qquad \partial_\mu j^{[\mu_1,\cdots,\mu_{p+1}]} = 0 \quad ; \tag{2.17}$$

in this notation, what we previously referred to as an ordinary symmetry is then deemed a "0-form continuous symmetry".
We can then couple this current to a $(p+1)$-form gauge field B_{p+1} by adding

$$S \supset i \int B_{p+1} \wedge *j_{p+1} \tag{2.18}$$

to the action. Like in the case of 0-form symmetries, (2.17) ensures that any symmetry transformation of B_{p+1} reduces to a boundary term. However the transformation itself is different, as it must be driven by a p-form parameter α_p in order for the form degree to be consistent:

$$B_{p+1} \to B_{p+1} + d\alpha_p \quad . \tag{2.19}$$

Now that the current and transformation parameter are known, one can construct the SDO of the p-form symmetry. The natural extension of (2.9) would be to replace the charge with the $(d-1)$-form given by the product of current and parameter, that is

$$e^{i\alpha Q(\Sigma_{d-1})} \to e^{i\int_{\Sigma_{d-1}} \alpha_p \wedge * j_{p+1}} \quad . \tag{2.20}$$

However, since we are now taking into account the presence of extended charged objects, the total charge (obtained setting $\Sigma_{d-1} = M_{d-1}$) is bound to diverge thus making the operator ill-defined. Therefore it is more interesting to consider defects operators of codimension greater than 1, that are defined as

$$U_\alpha(\Sigma_{d-p-1}) = e^{i\alpha \int_\Sigma *j_{p+1}} \qquad \text{with } \alpha \mid \alpha_p = \alpha\Omega_p \quad ; \tag{2.21}$$

Ω_p is known in mathematics as the global angular form of the normal bundle of Σ_{d-p-1} [7]. If Σ_{d-p-1} is embedded in Σ_{d-1}, $U_\alpha(\Sigma_{d-p-1})$ and the operator constructed in (2.20) should coincide.

As foretold at the beginning of the section, a p-form symmetry acts on p-dimensional operators $\mathcal{O}(\Gamma_p)$. In order to compute such action we will again need the Ward identity, which reads very similarly to (2.13), with the scalar point-supported Dirac delta being replaced by a $(d-p)$-form supported on Γ_p:

$$d * j_{p+1}(x)\mathcal{O}(\Gamma_p) = \delta^{(d-p)}(x \in \Gamma_p)R(T^a)\mathcal{O}(\Gamma_p) \quad . \tag{2.22}$$

A piece of insight one can draw from the ordinary symmetry discussion is that $U_\alpha(\Sigma_{d-p-1})$ will act non-trivially on the operator if the manifold onto which the SDO is wrapped is deformed in a way that changes its intersection with Γ_p. Unlike the 0-form case, where due to the particle-like nature of the charged object the crossing was simply understood x belonging in Ξ_d, here we need a rigorous way to count the junctions between charged operator and defect before and after the deformation.

Definition 3 (Linking number) *Let U_p and V_{d-p-1} be two oriented manifolds of dimension p and $(d-p-1)$ respectively. Let them also be disjoint and homotopically trivial, so that one can introduce a manifold W_{d-p} such that $\partial W_{d-p} = V_{d-p-1}$. $U_p \cap W_{d-p}$ is then a finite set of points $\{p_i\}$. Since $dim(U) + dim(W) = d$, at each intersection their tangent spaces' direct sum generates the whole tangent space of the underlying d-dimensional manifold. Since both U_p and W_{d-p} are oriented, they induce an orientation on each $T_{p_i}U \oplus T_{p_i}W$, that can be aligned or opposite to the orientation of the theory space. Based on this we then define the sign of an intersection as $sign(p_i) = \pm 1$ (respectively) and the Linking number between U_p and V_{d-p-1} as*

$$Link(U_p, V_{d-p-1}) = \sum_i sign(p_i) \quad . \tag{2.23}$$

Notice that while the number of intersections depends on the choice of the 'filling' W_{d-p}, $Link(U_p, V_{d-p-1})$ is fixed.

Consider now two homotopic manifolds Σ_{d-p-1} and Σ'_{d-p-1}, the former being linked to $\mathcal{O}(\Gamma_p)$ and the latter being unlinked. The codimension p manifold swept by the deformation $\Sigma_{d-p-1} \to \Sigma'_{d-p-1}$, that we denote as Ξ_{d-p}, will then cross the charged object. The defect's action on $\mathcal{O}(\Gamma_p)$ is then given by

$$U_\alpha(\Sigma_{d-p-1})\mathcal{O}(\Gamma_p)U_{-\alpha}(\Sigma'_{d-p-1}) \quad . \tag{2.24}$$

Like in the previous case, space separation between Σ' and Γ allows to commute the two operators, which in turn allows to apply Ward identity (2.22), resulting in

$$e^{i\alpha\int_\Xi d*j_{p+1}}\mathcal{O}(\Gamma_p) = e^{i\alpha\int_\Xi \delta^{(d-p)}(\Gamma_p)}\mathcal{O}(\Gamma_p) \quad . \tag{2.25}$$

Seeing as Ξ_{d-p} is, by construction, bounded by Σ_{d-p-1}, the integral can be read as the linking between the codimension $(p+1)$ manifold and the charged object. Therefore, the action of the defect operator going through a charged p-dimensional operator is ultimately given by

$$U_\alpha(\Sigma_{d-p-1})\mathcal{O}(\Gamma_p) = e^{i\alpha Link(\Gamma,\Sigma)}\mathcal{O}(\Gamma_p)U_\alpha(\Sigma'_{d-p-1}) \quad . \tag{2.26}$$

Hitherto we focused on generalizing the formalism of topological symmetry operators to the case of higher-form symmetry. We now dedicate some time to remark some interesting properties about these symmetries.

An important thing to notice about ($p \geq 1$)-form symmetries is that they are always Abelian: in the case of a 0-form symmetry the U_α operators, being related to manifolds of codimension 1, must act according to time ordering and can thus display non-abelian behaviour when swapped. The greater codimensionality of an higher-form SDOs allows to continuously deform the manifold between different times, meaning that an operator acting at $t+\varepsilon$ can be deformed into one at $t-\varepsilon$. This makes time-ordering ill defined, because the operators can be freely swapped exploiting the extra p dimensions, ultimately leading to trivial commutation relations. Non-abelian higher-form symmetries can still appear, but only in theories where the underlying space-time is topologically non-trivial, so that the deformation of the manifolds can be hindered.

Another interesting remark comes from Coleman-Mandula theorem[2], stating that

Theorem 1 (Coleman-Mandula) *Given a relativistic theory ($d > 2$) describing a finite number of massive particle types at any cutoff energy, the symmetry group G associated with said theory is isomorphic to the direct product of the Poincaré group $\mathcal{P}$ and an internal symmetry group E:*

$$G \simeq \mathcal{P} \otimes E \tag{2.27}$$

In other words the space-time symmetry of a relativistic theory is limited to the Poincaré group.

Higher-form symmetries evade the hypothesis of this theorem because, acting on extended operators, they appear in theories that do not comply to the assumption on the massive

[2] A more accurate statement of the theorem can be found, for instance, in [8].

particle types seeing as there are infinite possible Γ_p [6]. This means that higher-form symmetries can be both internal or space-time symmetries.

Example: Maxwell theory in $4d$

A notable example of higher-form symmetry hides in 4-dimensional $U(1)$ pure gauge theory, i.e. Maxwell theory. Such theory is described by the well-known action

$$S_M = -\frac{1}{4g^2}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2g^2}\int F\wedge *F \quad , \tag{2.28}$$

where $F = dA_1$ is the 2-form field strength of the 1-form gauge field A_1 and g is a coupling. The equations of motion for A yielded by this action are:

$$d*F = 0 \quad , \tag{2.29}$$

which can be read as the conservation law of a 2-form current $J_2^e = g^{-2}F$. This is the sign of a $U(1)$ 1-form symmetry, that will act on line operators through codimension 2 SDOs of the form

$$U_\alpha^e(\Sigma_2) = e^{i\alpha g^{-2}\int_{\Sigma_2} *F} \quad . \tag{2.30}$$

The $U(1)$ character of the symmetry is testified by the fact that $g^{-2}\int_{\Sigma_2} *F = 2\pi\mathbb{Z}$, because it measures the quantized charge enclosed in Σ_2.
The previously described symmetry is often referred to as "electric 1-form symmetry" in order to distinguish it from the "magnetic" one, stemming from the Bianchi identity

$$dF = 0 \quad , \tag{2.31}$$

which correspond to the conservation of $J_2^m = \frac{1}{2\pi} * F$. The codimension 2 SDO is then

$$U_\alpha^m(\Sigma_2) = e^{\frac{i\alpha}{2\pi}\int_{\Sigma_2} F} \quad . \tag{2.32}$$

Each of these 1-form symmetries can be individually gauged, yet they can't be gauged simultaneously due to the anomaly that arises between them. We will see this in more detail in section 2.2.

2.1.2 Discrete Symmetries

Another possible generalization of the definition of symmetry is represented by discrete symmetries. We will mostly discuss symmetries of the $\mathbb{Z}_N$ ($N \in \mathbb{N}$) type, as it is the most common in physics as theories often bear parity (P) and time inversion (T) invariance, both being $\mathbb{Z}_2$ symmetries. What is lost in this case is the continuity of the group of transformations, causing the symmetry to fall outside the hypotheses of Nöther theorem.

This implies that there is no conserved current that we can use to construct the defect operators associated with a discrete symmetry.
It is however still possible to describe discrete symmetries in terms of topological defect operators. We will discuss how this is done in the case of BF theories, also known as p-form $\mathbb{Z}_N$ gauge theories. The construction of these theories is based on embedding the discrete group in a larger, continuous one. The larger group is then used to write an action in such a way that only a discrete subset of gauge configurations actually contributes to the path integral. This results in a BF action of the form

$$S_{BF} = \frac{iN}{2\pi} \int B_{d-p-1} \wedge dA_p \qquad N \in \mathbb{Z} \quad , \tag{2.33}$$

where A_p and B_{d-p-1} are gauge fields, respectively transforming under the embedding group $U(1) \supset \mathbb{Z}_N$ as

$$A_p \to A_p + d\alpha_{p-1} \qquad B_{d-p-1} \to B_p + d\alpha_{d-p-2} \quad . \tag{2.34}$$

The d-dimensional p-form BF action (2.33) is actually recognizable as the low energy limit of a charge N Abelian Higgs model of gauge group $U(1)$, which undergoes a spontaneous symmetry breaking $U(1) \xrightarrow{SSB} \mathbb{Z}_N$ [7]. Therefore the remnant BF action will be associated with a discrete symmetry group, namely $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p-1)}$ (the superscripts denote the form degrees of the symmetries). This can be seen by computing the equations of motion of (2.33):

$$d\frac{NA_p}{2\pi} = 0 \qquad d\frac{NB_{d-p-1}}{2\pi} = 0 \quad . \tag{2.35}$$

One can interpret these as conservation laws for the gauge dependent "currents"

$$*j_{d-p} = \frac{NA_p}{2\pi} \qquad \text{and} \qquad *\, j_{p+1} = \frac{NB_{d-p-1}}{2\pi} \quad , \tag{2.36}$$

which can in turn be used to formally define the SDOs associated to an element $e^{2\pi ik/N} \in \mathbb{Z}_N$:

$$U^{(p)}_{2\pi k/N}(\Sigma^{d-p-1}) = e^{\frac{2\pi ik}{N}\int_\Sigma *j_{p+1}} \tag{2.37}$$

$$U^{(d-p-1)}_{2\pi k/N}(\tilde{\Sigma}^p) = e^{\frac{2\pi ik}{N}\int_{\tilde{\Sigma}} *j_{d-p}} \quad . \tag{2.38}$$

According to the definitions given in the previous sections, these correspond to a p-form and a $(d-p-1)$-form symmetry. As a consequence of (2.35) one has that $N\oint dA_p = 2\pi\mathbb{Z}$ and $N\oint dB_{d-p-1} = 2\pi\mathbb{Z}$, which in turn imply that the integrals appearing in the SDOs (2.37) are also bound to be discrete, restricting the symmetry group from $U(1)$ to $\mathbb{Z}_N$. These observations reproduce the advertised discrete symmetry group $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p-1)}$.

2.1.3 Non-Invertible Symmetries

All the symmetries examined so far were introduced as groups of transformations, with all the special properties that come with that structure. Yet our former description was founded on the idea of constructing topological operators to act on the theory rather than the group structure itself. Therefore one may further generalize the definition by stripping the symmetry of its group-like structure too. As a matter of fact one can construct topological operators in the more general setting of "categorical symmetry", that do not require the symmetry to fit in a group description. While category theory is a expansive and interesting topic, we will not discuss it here, instead we will limit ourselves to introducing the concept of a "non-invertible" symmetry as a set of topological defect operators U_α parameterized by α and obeying a generic, non-group-like fusion relation $U_{\alpha_1} \times U_{\alpha_2}$. This categorical fusion relation can be arbitrarily complex:

Group-like fusion:	**Categorical fusions:**
$U_{\alpha_1} \times U_{\alpha_2} \; = \; U_{\alpha_1 \alpha_2}$	$U_{\alpha_1} \times U_{\alpha_2} \; = \; \sum_i U_{\alpha_i}$
	$U_{\alpha_1} \times U_{\alpha_2} \; = \; \mathcal{Z} \otimes U_{\alpha_1 \alpha_2}$
	$\vdots$

with $\mathcal{Z}$ being the partition function of a topological theory.

It is worth mentioning that a major hindrance to the study of non-invertible symmetries through category theory lies in the fact that many theorems in that framework are proven only in low dimensions ($d \leq 2$). The approach followed by our research is instead founded on a more physical characterization of the symmetries as they appear in the theory, allowing to extract information on them despite the fact that we are examining a 3-dimensional QFT.

The construction of theories bearing a non-invertible symmetry begins with non-topological defect operator, characterized by non-conservation of the corresponding current j:

$$d * j = \frac{1}{4\pi^2} dAdA \quad , \tag{2.39}$$

with A denoting the gauge field. One can then construct a topological operator using the so-called Page current [9], defined as

$$j_{Page} = j - \frac{1}{4\pi^2} * (AdA) \qquad \Longrightarrow \qquad d * j_{Page} = d(*j - \frac{1}{4\pi^2} AdA) = 0 \quad . \tag{2.40}$$

It is clear from this definition that the resulting operator will be gauge-dependent, therefore it needs to be dressed to be made gauge invariant and thus associated to a generalized symmetry. The 'dressing' process is what ultimately strips the operators of their group-like product. We will now elaborate on this idea through the case of chiral symmetry in Dirac theory, following the discussion by Shao in [9].

Example: Non-Invertible Chiral Symmetry in $4d$ Dirac Theory

Consider 4-dimensional $U(1)$ gauge theory with a unit charge fermion ψ, described by the QED action

$$S = \int \left[\frac{1}{2g^2} F \wedge *F + i\bar{\psi}\not{D}\psi\right] \qquad F = dA\ . \tag{2.41}$$

As well known, this theory has a global $U(1)_A$ chiral symmetry that acts on the fermion as $\psi \to e^{i\alpha\gamma^5}\psi$ associated with the current

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \quad , \tag{2.42}$$

which displays an ABJ anomaly given by

$$d * j_A = \frac{1}{4\pi^2} F \wedge F \tag{2.43}$$

We have yet to discuss anomalies and their role in QFT, and we will do so in the next section; for now it is enough to notice that due to (2.43), the naïve form of the 0-form chiral defect operator, i.e.

$$\mathcal{U}_\alpha^A(\Sigma_3) = e^{i\alpha \int_\Sigma *j_A} \quad , \tag{2.44}$$

is not a topological operator. In fact, if we consider a continuous deformation $\Sigma_3 \to \Sigma_3'$ sweeping Ξ_4, one gets

$$\mathcal{U}_\alpha^A(\Sigma_3)\mathcal{U}_{-\alpha}^A(\Sigma_3') = e^{i\alpha \int_\Xi d*j_A} \qquad \Longleftrightarrow \qquad \mathcal{U}_\alpha^A(\Sigma_3) = \mathcal{U}_\alpha^A(\Sigma_3') e^{\frac{i\alpha}{4\pi^2}\int_\Xi F\wedge F} \quad . \tag{2.45}$$

The upshot of this discussion is that $\mathcal{U}_\alpha^A$ does not lead to a generalized symmetry as it does not correspond to a topolgical defect.
By means of a Page current defined as in (2.40) one can indeed define a gauge-dependent topological operator:

$$\mathcal{U}_\alpha^{Page}(\Sigma_3) = e^{i\alpha \int_\Sigma *j_{Page}} \qquad \xrightarrow{A \to A + d\Lambda} \qquad \mathcal{U}_\alpha^{Page}(\Sigma_3) e^{\frac{i\alpha}{4\pi^2}\int *(d\Lambda\wedge F)} \quad , \tag{2.46}$$

where we assume the chiral angle to be fractional: $\alpha = 2\pi/N$. In order to fix this operator's gauge invariance we need to borrow from condensed matter physics and in particular from the tridimensional fractional quantum Hall effect, described by the gauge invariant Lagrangian

$$\mathcal{L}_{FQH} = \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA \tag{2.47}$$

where we introduced an additional $U(1)$ dynamical gauge field a, whose classical equation of motion is easily solved by $a = -A/N$ up to a total derivative.
We can then go back to 4-dimensional QED and include the gauge-invariant topological operator by dressing (2.46) with the Hall gauge field a, which only lives on the defect Σ_3 and does not affect the physics elsewhere. The final form of the dressed operator is

$$U_{\frac{1}{N}}(\Sigma_3) = \int [Da]_M e^{\frac{2\pi i}{N} *j_A + \mathcal{L}_{FQH}} \quad , \tag{2.48}$$

which can be interpreted as an actual SDO and thus associated to a symmetry. This, however, comes at the cost of the operator's invertibility, which is broken by the $[Da]_M$ integration; hence $U_{\frac{1}{N}}(\Sigma_3)$ corresponds to a non-invertible symmetry.
The relation between (2.48) and (2.46) can be seen by integrating out a using its equation of motion. This is an illegal manipulation that causes the gauge-invariance to break, but it serves as an heuristic connection between $U_{\frac{1}{N}}$ and the Page operator.

2.2 Anomalies and Generalized Symmetries

As mentioned before, anomalies play a pivotal role in our search for generalized symmetry as they act as a lasting imprint of a broken global symmetry. For this reason we will now review the concept of anomaly and examine how they can be interpreted in terms of an higher dimensional theory. We then generalize the formalism to higher-form and discrete symmetry.
An anomaly is most simply defined as follows:

Definition 4 (Anomaly) *A 0-form continuous symmetry of a given theory is said to be anomalous if its gauging is obstructed by the emergence of an extra term. The obstruction is quantified by the non-conservation of the current*

$$\partial_\mu J^\mu(x) = -\mathcal{A}[B_\mu] \quad , \tag{2.49}$$

where $\mathcal{A}$ is known as anomaly function or simply anomaly and it is generally a function of the background gauge fields associated with the symmetry.

We will focus on two types of anomaly: 't Hooft and Adler–Bell–Jackiw (ABJ) anomalies. 't Hooft anomalies are obstructions that only become visible when gauging a symmetry. ABJ anomalies instead emerge as a direct breaking of a global symmetry at quantum level, that is when loop contributions are included. Notice that this does not contradict Nöther's theorem as its proof made use of the classical equations of motion of the theory.
In order to see how an ABJ anomaly can emerge, consider the theory $S[\phi, A_\mu]$ with a gauge symmetry (associated to the dynamical field A_μ) and a global 0-form symmetry transforming the matter fields like $\phi' = \phi + \alpha\delta\phi$; at classical level we defined the system

as symmetric under $\phi \to \phi'$ if such transformation yields $\delta\mathcal{L}[\phi, A_\mu] = \alpha\partial_\mu J^\mu$, with J coinciding with the Nöther current in the case of a global symmetry. Moving to path integral formalism, i.e. a fully quantum setting, the focus shifts from the action to the partition function, causing the field measures to come into play:

$$\mathcal{Z} = \int D\phi DA e^{iS[\phi,A]} \quad . \tag{2.50}$$

The reasoning then proceeds analogously to the proof of Slavnov-Taylor identities.
By means of a simple renaming of the variables, one can move to (ϕ', A') without affecting the path integral; then by substituting the α-transformed forms of the fields one gets

$$\mathcal{Z} = \int D(\phi + \alpha\delta\phi) DA_\mu e^{iS[\phi,A]} e^{+i\int d^dx \alpha\partial J} \quad . \tag{2.51}$$

It remains to examine the variation of the field measure. While the bosonic measure DA_μ is invariant by construction, the matter field's $(D\phi)$ can be impacted by the transformation. Let us define the variation of the measure as

$$D\phi \to D\phi' = D\phi e^{i\alpha\int d^dx\mathcal{A}} \quad . \tag{2.52}$$

Substituting this back into (2.51) one clearly sees that, in order for the partition function to remain unchanged, it must be

$$i\alpha\left(\mathcal{A} + \partial_\mu J^\mu\right) = 0 \quad , \tag{2.53}$$

reproducing the non-conservation of the current as stated in the definition.
Since it relies on the existence of a current, the definition provided above is only valid for continuous symmetry and is therefore not well suited for a generalized description. Because of this we look for a more general way to define anomalies.
To do so, consider the effective action obtained by coupling the Nöther current of the anomalous symmetry to a background gauge field B_μ as in (2.7). The 'background' denomination signals that B_μ obeys the classical equations of motion and is thus not integrated:

$$W[B_\mu] = -i\log\int D\phi DA e^{iS[\phi,A] + i\int B\wedge *J} \quad . \tag{2.54}$$

From this perspective a symmetry is deemed anomalous if the corresponding $W[B_\mu]$ is gauge dependent. In fact, by transforming B_μ under an illegally gauged α-transformation, one can use (2.53) to easily deduce the variation of the effective action from

$$
\begin{aligned}
e^{iW[B']} &= \int D\phi DA e^{iS[\phi,A]+i\int B'\wedge J} = \int D\phi DA e^{iS[\phi,A]+i\int d^dx\alpha\partial J} = e^{iW[B]+i\int d^dx\alpha\partial J} \\
\delta W[B_\mu] &= \int d^dx\alpha\partial_\mu J^\mu(x) = -\int d^dx\alpha\mathcal{A}[B_\mu] \quad .
\end{aligned}
\tag{2.55}
$$

We can then more generally define an anomaly in terms of the variation of the effective action as follows:

Definition 5 (Anomaly) *A symmetry is said to be anomalous if the partition function is not invariant under a symmetry transformation of the background fields*

$$
\mathcal{Z}[B] \xrightarrow{B+\delta_\alpha B} e^{\mathcal{A}[B,\alpha]}\mathcal{Z}[B] \quad . \tag{2.56}
$$

2.2.1 The Anomaly Inflow Paradigm

The anomalous function $\mathcal{A}[A_\mu]$ has a nice interpretation as the remnant of the coupling of the theory to an higher dimensional topological theory (TFT). In fact, considering a theory on a manifold $N^{(d+1)} \mid \partial N^{(d+1)} = M^d$ where the background field is decoupled, one can express the anomalous phase in terms of the variation under a gauge transformation of some functional $\hat{\mathcal{A}}[A_\mu]$ in the bulk theory:

$$
\delta_\alpha \hat{\mathcal{A}}[A_\mu] \;=\; \partial_\mu(\alpha\mathcal{A}[A_\mu])\hat{n}^\mu \tag{2.57}
$$

Interpreting $\hat{\mathcal{A}}$ as the TFT action, it's then straightforward to see that the full $N^{(d+1)} \cup M^d$ system is anomaly-free by virtue of the gauge invariance of the effective action $W[A_\mu]$. The variation of the $N^{(d+1)}$ action can in fact be lowered to a d-dimensional boundary term that cancels the one emerging from the original theory:

$$
\hat{W}[A_\mu] \;=\; W[A_\mu]e^{i\int_N \hat{\mathcal{A}}[A]} \quad \xrightarrow{A+\partial\alpha} \quad \hat{W}[A_\mu]e^{-i\int_M \mathcal{A}[A,\alpha]}e^{i\int_N d^{d+1}x\partial_\mu(\alpha\mathcal{A})\hat{n}^\mu} \;=\; \hat{W}[A_\mu] \quad . \tag{2.58}
$$

This interpretation of anomalies as the remnant of an higher-dimensional anomaly-free theory is known as "anomaly inflow" paradigm.

Besides this insights on the origin of anomalies, the construction behind anomaly inflow offers an interesting avenue when treaded backwards: could we, starting from a TFT known to couple to a QFT living on the boundary of the underlying space-time, be able to reconstruct the anomalies and thus the symmetries of the boundary theory? The scope of this thesis is doing just that, and we will elaborate on how this can be achieved in section (2.3).

2.2.2 Anomalies for p-form and Discrete Symmetries

Now that we introduced anomalies for ordinary symmetries, we can generalize the results to higher-form and discrete ones.

The generalization to higher-form anomalies is quite simple: the gauge field becomes a $(p+1)$-form instead of a 1-form. In terms of gauge variation of the effective action, an anomalous p-form symmetry is characterized by

$$\delta_\alpha W[A_{p+1}] = -\int d^d x \mathcal{A}[A_{p+1}, \alpha_p] \neq 0 \quad , \tag{2.59}$$

where we included the dependence on the p-form symmetry parameter in the anomaly function.

For discrete symmetries we cannot describe anomalies as non-conservation of currents, seeing as the current is not well defined in this case. We can instead detect anomalies in BF theory: recalling that the action (2.33) corresponds to a $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p-1)}$, one can couple the BF theory to background fields C_{p+1} and D_{d-p} (that, like A_p and B_{d-p-1} only contribute to the dynamics in discrete configurations) by modifying the action as follows

$$S_{BF} = \frac{iN}{2\pi} \int \left[B_{d-p-1} \wedge dA_p - B_{d-p-1} \wedge C_{p+1} + A_p \wedge D_{d-p}\right] \quad . \tag{2.60}$$

The related $\mathbb{Z}_N$ gauge transformations are

$$\begin{aligned} C_{p+1} &\xrightarrow{\mathbb{Z}_N^p} C_{p+1} + d\alpha_p \qquad & D_{d-p} &\xrightarrow{\mathbb{Z}_N^{(d-p-1)}} D_{d-p} + d\alpha_{d-p-1} \\ A_p &\xrightarrow{\mathbb{Z}_N^p} A_p + \alpha_p \qquad & B_{d-p-1} &\xrightarrow{\mathbb{Z}_N^{(d-p-1)}} B_{d-p-1} + \alpha_{d-p-1} \end{aligned} \tag{2.61}$$

Variating the action under the full symmetry group then gives

$$\delta S = \frac{iN}{2\pi} \int \left[-\alpha_{d-p-1} \wedge C_{p+1} + \alpha_p \wedge D_{d-p}\right] \quad , \tag{2.62}$$

which can only be cancelled by inflow, signaling the presence of a mixed anomaly between the two discrete symmetries of BF theory. The anomaly and its counterpart in the $(d+1)$ bulk theory are, respectively:

$$\mathcal{A} = \frac{iN}{2\pi} \int_{M_d} D_{d-p} \wedge C_{p+1} \qquad \delta_\alpha \hat{\mathcal{A}} = \frac{iN}{2\pi} \int_{M_{d+1}} d(D_{d-p} \wedge C_{p+1}) \quad . \tag{2.63}$$

Example: Maxwell theory in $4d$

Let us go back to Maxwell theory on 4-dimensional Minkowski space. We previously saw that such theory bears two 1-form symmetries: the electric $U(1)_E$ associated with the 2-form current F and the magnetic $U(1)_M$ associated with $*F$. As foretold, there is a mixed 't Hooft anomaly between these two symmetries, which can be probed by trying to gauge both $U(1)_E$ and $U(1)_M$ at the same time. To do so, we couple both currents to background 2-form gauge fields (B_2^e, B_2^m) by modifying the action as follows

$$S = \frac{1}{2g^2}\int F \wedge *F \xrightarrow{gauging} \frac{1}{2g^2}\int (F - B_2^e) \wedge *(F - B_2^e) + \frac{i}{2\pi}\int B_2^m \wedge F \tag{2.64}$$

$$B_2^e \xrightarrow{U(1)_E} B_2^e + d\alpha_1^e \qquad\qquad B_2^m \xrightarrow{U(1)_E} B_2^m + d\alpha_1^m \tag{2.65}$$

Note that the B_2^e-coupling is of a rather peculiar form compared to (2.7). This is done to include an additional counter-term meant to ensure the kinetic term's invariance under $U(1)_E$ since the dynamical field A_1 is also affected by such symmetry, precisely as $A_1' = A_1 + \alpha_1^e$. On the other hand $U(1)_M$ is associated with monopole operators, therefore it doesn't affect the electric field A_1, allowing us to couple the current as per usual [7].
Transforming now the action under $U(1)_M$ one gets

$$\delta_M S \propto \int d\alpha_1^m \wedge F = -\int \alpha_1^m \wedge dF = 0 \quad , \tag{2.66}$$

meaning that the magnetic 1-form symmetry survived the gauging. If we instead transform under $U(1)_E$ we find, from the $B_2^m \wedge F$ coupling, that

$$\delta_E S \propto \int d\alpha_1^e \wedge *F + \int B_2^m \wedge d\alpha_1^e = \int B_2^m \wedge d\alpha_1^e \quad . \tag{2.67}$$

The gauging has then broken $U(1)_E$ invariance. One could think of adding a counter term (only related to the background gauge fields, so that it doesn't affect the dynamics) to cancel $\delta_E S$, however doing this will always result in the breaking of $U(1)_M$. We have then found a 't Hooft anomaly between the higher form symmetries of Maxwell theory.
The anomaly can be expressed through inflow from a theory in $N_5 \mid \partial N_5 = Mink_4$ of action

$$S_{inflow} = -\frac{i}{2\pi}\int_{N_5} B_2^e \wedge dB_2^e \quad . \tag{2.68}$$

Indeed, variating this action along a $U(1)_E$ transformation reproduces exactly the breaking (2.67).

2.3 A Theory for Capturing Symmetries

As foretold, the goal of our research is to construct a theory that encodes the symmetries of an otherwise difficult to examine QFT, the $3d$ ABJ theory. As it focuses on the symmetries, that we went to great lengths to describe in terms of topological operators, the theory we look for will be a topological theory (TFT).
In the previous section we also saw that, through anomaly inflow, the symmetry structure

of a d-dimensional theory is intertwined with that of a $(d+1)$-dimensional one that has the former's space-time as boundary. In order to be used to reconstruct the symmetries of the original system, the higher dimensional theory must be built with a specific set of properties [2][10]:

Definition 6 (Symmetry TFT) *Given a d-dimensional theory $\mathcal{T}$ with symmetry $\mathcal{S}$, the Symmetry Topological Field Theory, or SymTFT, is defined as a $(d+1)$-dimensional topological theory admitting two d-dimensional boundaries:*

- *A topological, gapped boundary $\mathcal{B}_{\mathcal{S}}^{Sym}$ whose topological defects realize the structure of $\mathcal{S}$;*
- *A not necessarily topological boundary $\mathcal{B}_{\mathcal{T}}^{Ph}$ reproducing the physical conditions of the theory $\mathcal{T}$.*

Via compactification of the interval separating them, one can collapse these boundaries together, resulting in the original theory $\mathcal{T}$ with symmetry $\mathcal{S}$.

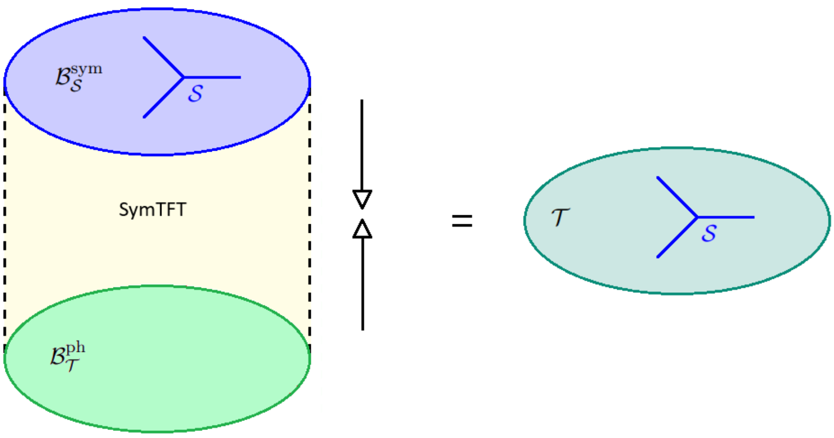

Figure 2.2: Starting from the SymTFT, living in $(d+1)$ dimensions, the corresponding lower dimensional QFT is obtained by collapsing the symmetry boundary onto the physical one.

Knowing the SymTFT of a theory $\mathcal{T}$ one can then recover the symmetry structure of $\mathcal{T}$ by projecting the bulk topological defects onto $\mathcal{B}_{\mathcal{S}}^{Sym}$, at least in the case of Abelian symmetries [2]; compactifying the interval then injects these defects into the physical theory. Whether an SDO $U(\Sigma_{p+1})$ of the SymTFT translates to a defect in d-dimensions (corresponding to a symmetry of $\mathcal{T}$) or to a charged operator depends on the boundary conditions it satisfies. The details of this process are beyond the scope of our research and are more precisely discussed in [2], however let us remark that if the defect is subject to Neumann boundary conditions[3], i.e. if it is free to move on the d-dimensional boundary,

[3]Neumann and Dirichlet conditions are conventional denominations for boundary conditions of extended objects and will be discussed in more detail when talking about branes in section 3.3.

the projection will result in an SDO, whereas if the defect's ends are fixed, meaning that it obeys Dirichlet boundary conditions, the result of compactifiaction is a generalized p-dimensional charge.
If a projected defect links to a generalized charge, it will act on it as described in previous sections, with the charge being related to the linking number.
The question that remains unanswered is how does one actually construct the Symmetry TFT. A glimpse of the path to the SymTFT can be caught by noticing that the properties of a SymTFT closely resembles those of an holographic dual: in both cases we associate a given QFT with a $(d+1)$-dimensional bulk theory that has the QFT's d-dimensional space-time as a boundary. As we will see in more detail in chapter 5, the construction does indeed start like that of an holographic dual gravitational theory, with the difference that instead of focusing on the dynamics in $(d+1)$-dimensions, we will search for a way to isolate the topological part.

3 Fundamentals of String Theory and Supergravity

Before moving on with the construction of the Symmetry TFT, we need to review the core ideas and results of string theories and in particular of supergravity theories. In fact, as hinted at the end of last chapter, the construction of the topological theory will require familiarity with the holographic setup. More precisely we will associate a low energy superstring theory (i.e. a supergravity theory) to our ABJ theory by means of the AdS/CFT correspondence, that will be discussed in detail in chapter 4.
Beside supergravity theories, we will also focus on the nature of Neveu-Schwarz and magnetically charged brane sources, since these objects will play an important role in both describing the ortho-symplectic ABJ theory and in recognizing the anomalies that appear in the symmetry topological field theory.

3.1 The Bosonic String

In its modern incarnation string theory is a candidate "theory of everything", capable of describing all the fundamental forces of nature in a single quantum mechanical formalism. Superstrings, the supersymmetric version of a string, actually do provide a theory of quantum gravity, though it hasn't yet been possible to prove whether it can describe our universe. Regardless, string theory was originally conceived as a way to understand strongly coupled systems and to this day sees a lot of usage in that regard, mainly thanks to holographic correspondences allowing to earn new insights on quantum gauge theories, which is exactly the aim of this thesis.
As the name suggests, the core assumption of string theories is that, instead of point-like particles, the universe is populated by extended objects in the form of tiny loops that can vibrate in different ways giving birth to a whole array of particle species, each with a specific mass and spin; in the case of a superstring, this can include both bosonic and fermionic particles.
As time goes by, point-particles sweep "world-lines" $X^\mu(\tau)$ across a D-dimensional Minkowski space-time, that is curves determined through minimization of the relativistic action

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \quad , \tag{3.1}$$

which one can observe to be proportional to the length of the curve.
Similarly a string sweeps a 2-dimensional surface known as a "world-sheet". Aside from the time-like parameter $\tau \in \mathbb{R}$, the world-sheet is described by a second parameter σ, that identifies the position along the string itself. Most known particles are believed to correspond to closed strings, therefore we can assume σ to be periodic: $\sigma \in [0, 2\pi[$. The two parameters are merged in the notation $\sigma_\alpha = (\tau, \sigma)$.
In analogy with the particle case, one deduces that the string's action should be proportional to the world-sheet area. Such area is computed in terms of the pull-back $\gamma_{\alpha\beta}$ of the flat Minkowski metric onto the generally curved world-sheet $X^\mu(\sigma_\alpha)$. The area element is then the square root of the determinant of this metric, meaning that the action for a relativistic closed string is given by:

$$S_{NG} = -\frac{1}{2\pi l_s^2} \int d^2\sigma \sqrt{-\det \gamma} \quad , \tag{3.2}$$

known as the Nambu-Goto action. The constant parameter l_s is known as "string scale" and plays a role in computing effective theories. The Nambu-Goto action has a global Poincaré invariance inherited from the relativistic framework as well as reparameterization freedom of the world-sheet: $\sigma_\alpha \to \sigma'_\alpha(\sigma)$.
It is possible to derive an action equivalent to Nambu-Goto's by promoting the pull-back metric $\gamma_{\alpha\beta}$ to dynamical field, denoted as $g_{\alpha\beta}$; this action, called Polyakov action, is then

$$S_P = -\frac{1}{4\pi l_s^2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad . \tag{3.3}$$

Note that solving the equations of motion for $g_{\alpha\beta}$ leads to

$$g_{\alpha\beta} = 2f(\sigma) \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \eta_{\mu\nu} \quad , \tag{3.4}$$

whose difference from the pull-back metric $(2f)$ factors out of the action, reproducing the Nambu-Goto expression exactly. This observation leads directly to an important property of S_P: besides the symmetries it shares with S_{NG}, the Polyakov action also enjoys Weyl invariance:

$$g_{\alpha\beta} \to e^{\phi(\sigma)} g_{\alpha\beta} \tag{3.5}$$

This symmetry can be interpreted as a local scale transformation preserving angles, i.e. a conformal transformation.
The two local symmetries provide with a gauge freedom that allows to select a flat Minkowski metric on the world-sheet. Replacing this metric in its own equations of motion, provides some important constraints on the string dynamics: the (01)-component in particular assumes the form $\partial_\tau X \cdot \partial_\sigma X = 0$, meaning that the the physical oscillation modes are tranversal to the string itself.

Strings were hitherto described in a classical setting. The quantization of a string theory is a delicate procedure of which we only provide the main steps; more detailed procedures can be found, for instance, in [11] or [12]. The first step towards a quantum theory of strings is to expand the world-sheet into discrete oscillation modes:

$$X^\mu_{L,R}(\sigma_\pm) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu \sigma_\pm + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha,\tilde{\alpha})^\mu_n e^{-in\sigma_\pm} \quad , \tag{3.6}$$

where $\sigma_\pm = \tau \pm \sigma$ and the left/right decomposition is a consequence of the constraints given by the equations of motion for the metric. The same constraints will translate to binding conditions on the α and $\tilde{\alpha}$ coefficients. In particular, the vanishing of terms proportional to $e^{i0\sigma_\pm}$ provides a constraint on $p^\mu p_\mu$, i.e. on the invariant mass of the string:

$$M^2 = -p^\mu p_\mu = \frac{4}{l_s^2}\sum_{n>0}\alpha_n\cdot\alpha_{-n} = \frac{4}{l_s^2}\sum_{n>0}\tilde{\alpha}_n\cdot\tilde{\alpha}_{-n} \tag{3.7}$$

The equality between the two expressions of the mass is known as 'level matching'.
The quantum description of the string can then be obtained by determining the classical physical solutions to the constraints and then quantizing them. This approach is known as lightcone quantization as it makes use of space-time coordinates combining the time direction with a spatial one, which describes a light-cone. In particular we choose

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \quad . \tag{3.8}$$

Although they make the computation easier, these coordinates introduce a subtle point of concern: because they hide Lorentz invariance, an anomaly could surreptitiously arise in the quantization process, causing Poincaré symmetry to break.
The solutions of the EoM in these coordinates can be expressed in terms of only the transversal modes, i.e. α^i_n with $n \in [1, D-2]$. Together with the 0-modes x^i, p^i, p^+ and x^- that will determine the string's ground state, these transverse oscillators are then promoted to operators, all obeying to well known commutation relations:

$$\left[x^i, p^j\right] = i\delta^{ij} \qquad \left[x^-, p^+\right] = -i \tag{3.9}$$

$$\left[\alpha^i_n, \alpha^j_m\right] = \left[\tilde{\alpha}^i_n, \tilde{\alpha}^j_m\right] = n\delta^{ij}\delta_{n(-m)} \tag{3.10}$$

with every other commutator being vanishing.
One then defines the ground state of the string as $|0;p\rangle$ such that

$$\hat{p}^\mu|0;p\rangle = p^\mu|0;p\rangle \quad , \quad \alpha_n|0;p\rangle = \tilde{\alpha}_n|0;p\rangle = 0 \quad \forall n > 0 \tag{3.11}$$

Acting on each string vacuum $|0;p\rangle$, that determines the motion of the string center of mass, the creation (α_{-n}, $\tilde{\alpha}_{-n}$) and annihilation (α_n, $\tilde{\alpha}_n$) operators generate a Fock space with modes restricted to the positive definite part of the metric, i.e. along the $i = \{1, ...D-2\}$ directions. The resulting Hilbert space, made up by the ensemble of all these Fock spaces, is then positive definite as expected from the physical states of a system.
At the end of the quantization process the expression for the invariant mass reads

$$M^2 = \frac{4}{l_2^2}\left(N - \frac{D-2}{24}\right) \tag{3.12}$$

where N is the 'level', defined in analogy with the number operator of harmonic oscillators. An immediate observation one can make is that (3.12) is not a positive definite quantity. The ground state, i.e. $\langle N \rangle = 0$, has in fact a negative mass and is therefore known as a Tachyon state. The treatment of tachyonic states is one of the open problems of string theory, yet it only emerges in bosonic string theory.
Due to the level matching condition, the first excited states ($\langle N \rangle = \langle \tilde{N} \rangle = 1$) are obtained acting on the vacuum with both α^i_{-1} and $\tilde{\alpha}^i_{-1}$ operators,

$$\alpha^i_{-1}\tilde{\alpha}^i_{-1}|0;p\rangle \quad , \tag{3.13}$$

generating $(D-2)^2$ states, each of mass $M^2 \propto 1 - (D-2)/24$. At this point we need to take care of the subtlety mentioned when introducing lightcone gauge: we have to make sure that the states fit in a representation of the Lorentz group $SO(1, D-1)$ so that Poincaré invariance of Polyakov action is inherited by the quantum theory. This is only achieved if the $(D-2)^2$ first excited states are massless, so that they can fit into a representation of $SO(D-2) \subset SO(1, D-1)$. This yields:

$$M^2 = \frac{4}{l_2^2}\left(1 - \frac{D-2}{24}\right) = 0 \qquad \Longleftrightarrow \qquad D = 26 \quad , \tag{3.14}$$

meaning that a consistent (bosonic) quantum string theory must live in 26 dimensions; this value of D is called critical dimension.
Further oscillations of the strings, i.e. $N \geq 2$, will necessarily be massive since D has been fixed and thus need to fit into a representation of $SO(D-1)$.
For the purpose of our research, the most important consequence of string quantization is that the $24 \otimes 24$ representation of $SO(24)$ in which the $N = 1$ states fit can be decomposed in three modes: a singlet (trace) mode, a traceless symmetric mode and an antisymmetric one. Oscillations along any of these modes can be identified with a quantum of a massless field with whom the string interacts. Namely

- Singlet modes are associated with a scalar field $\Phi(X)$, the "Dilaton field", and correspond to scale transformations.
- Antisymmetric oscillations correspond to a 2-form field $B_{\mu\nu}(X)$, known as "Kalb-Ramond field" or "Neveu-Schwarz field" in supersymmetric frameworks, that plays a role akin to that of an electromagnetic field.
- Finally traceless symmetric modes are associated with a spin 2 tensor field $G_{\mu\nu}(X)$. This field is identified with the space-time metric through the Feynman-Weinberg argument, stating that any massless spin 2 field theory is actually Einstein's gravity.

These fields will play an important role later on when we will derive low energy limits of string theories and will become the main subject of supergravity theories.

3.2 Superstring Zoology

Up until now, the string was presented through a purely bosonic lens; however, if string theory aims at describing the full spectrum of physical particles, it needs to include fermions as well. This is achieved by endowing the string with supersymmetric properties, giving birth to the so called 'superstring'. The main feature of a superstring is that its worldsheet can vibrate along both bosonic and fermionic modes. Superstring theories are strictly related to supergravity and are the main kind of bulk theories involved in holographic dualities. As a matter of fact, the 3d ABJ theory we will examine in later sections is dual to a type IIA superstring theory.
The construction of a supersymmetric string involve a number of choices that lead to different theories that fall under two main categories:

Definition 7 (Type II Theory) *A type II superstring theory describes strings in an $\mathcal{N}=2$ supersymmetric framework, with both L- and R-moving solutions oscillating along both bosonic and fermionic modes.*

Definition 8 (Heterotic Theory) *Heterotic superstring theories are instead built with $\mathcal{N}=1$ supersymmetry, and having fermionic modes only in the right-moving solution.*

Regardless of the type of theory, the quantization process of superstrings leads to a critical dimension of $D=10$ and, as anticipated, displays tachyonless spectrum.
On the other hand the field content is not unique: while $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ appear in the quantization of any string, there are some additional field that appear in supersymmetric cases, the nature of which depends on what kind of theory was chosen.
Type II theories include massless bosonic p-form fields known as 'Ramond-Ramond fields', that are to be interpreted as gauge fields. More precisely, type IIA superstrings have a 1-form and a 3-form Ramond-Ramond field, denoted as C_μ and $C_{\mu\nu\rho}$ respectively. Type

IIB theories have instead a scalar, a 2-form and a 4-form RR field: C, $C_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$. Being gauge fields, RR fields are each associated with a $(p+1)$-field strength given by $F = dC$. These field strengths, which we often refer to as fluxes, will be the central subject of the SymTFT derived from the 10-dimensional supergravity.
On the other hand, heterotic string theories include a field associated with a non-Abelian gauge group that can either be $SO(32)$ or $E_8 \times E_8$, resulting in a super-Yang-Mills theory in ten dimensions.
It should be noted that all the different types of superstrings turn out to be specifications of the same 11-dimensional framework known as M-theory[1]. In particular a type II theory would represent the perturbative theory resulting from a small coupling limit of the non-perturbative M-theory, with the dimensional gap being the result of an S^1 compactification (see section 3.5).

3.3 Open Strings and D-Branes

The discussion in section 3.1 only contemplated the existence of closed strings, however the dynamics of an open string's end-points are quite interesting because they lead to the introduction of D-branes, a new dynamical object that string theory will need to include. String theory is local, meaning that any point of the string will move according to physics at that point in space-time, unaware of whether it's part of a closed or open string. The only exception to this statement is represented by the end-points of an open string, since there's no neighborhood of the end-point that can be identified with an internal point's. Hence the open string dynamics are still described by Polyakov action (3.3), aided by some boundary conditions to take care of the end-points. Such boundary conditions are imposed to make up for the fact that the σ parameter on the world-sheet is not periodic anymore ($\sigma \in [0, \pi]$), causing the appearance of a non-trivial boundary term when minimizing the action: if we consider the action within the interval $[\tau_0, \tau_f]$ in conformal gauge the variation takes the form

$$\delta S_P \propto \int d^2\sigma (EoM) \cdot \delta X + \left[\int_0^{\pi} d\sigma \partial_\tau X \cdot \delta X \right]_{\tau_0}^{\tau_f} - \left[\int_{\tau_0}^{\tau_f} d\tau \partial_\sigma X \cdot \delta X \right]_{\sigma=0}^{\sigma=pi}$$

where the first boundary term is clearly vanishing as $\delta X|_{\tau_0,\tau_f} = 0$. The second one instead only vanishes under the condition

$$(\partial_\sigma X^\mu \delta X_\mu)_{\sigma=0,\pi} = 0 \quad , \tag{3.15}$$

which can be satisfied by two kinds of solutions, going under the names of Dirichlet and Neumann boundary conditions, which we already encountered in section 2.3.

[1]More details on the nature of M-theory can be found in [13].

A Dirichlet boundary condition fixes a component of an end-point's world-line by setting its variation to 0:

$$\delta X^\mu|_{\sigma=\{0,\pi\}} = 0 \quad . \tag{3.16}$$

A Neumann condition on the other hand leaves the end-point it's applied on free to move in the μ direction, i.e. it sets

$$\partial_\sigma X^\mu|_{\sigma=\{0,\pi\}} = 0 \quad , \tag{3.17}$$

without any constraint on the variation. Notice that the two ends aren't necessarily subject to the same set of boundary conditions.
Choosing Neumann conditions for $\mu = \{0, ..., p\}$ and Dirichlet ones the remaining $D-p-1$ directions restricts the end-points of a given string to move onto a $(p+1)$-hypersurface, known as a Dp-brane, causing the Lorentz group to break into

$$SO(1, D-1) \xrightarrow{Dp-brane} SO(1,p) \times SO(D-p-1) \quad , \tag{3.18}$$

where $SO(D-p-1)$ corresponds to the rotational symmetry of the brane in space, while $SO(1,p)$ is the residual Lorentz invariance of the end-points' dynamics within the brane itself.
The peculiar thing about D-branes is that, despite originating as constraints on open string dynamics, we will see later on (in section 3.4.3) that they develop dynamics of their own, thus needing to be considered as independent physical objects. In some cases the inclusion of open strings, and thus D-branes, is necessary for a string theory to be consistent. A prime example of this is found in type IIA superstring theory. As a dynamic object, a Dp-brane sweeps a $p+1$-dimensional world-volume, determined by the so called Dirac action. Parameterizing the world-volume as $X(\zeta)$, with $\zeta = (\tau, \zeta_1, ..., \zeta_p)$, the Dirac action realizes in an higher dimensional version of (3.2):

$$S_D = -T_p \int d^{p+1}\zeta \sqrt{-\det\gamma} \qquad\qquad \gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\zeta_\alpha}\frac{\partial X^\nu}{\partial\zeta_\beta}\eta_{\mu\nu} \tag{3.19}$$

where γ is again the pull-back of the space-time metric onto the world-volume. The Dirac action doesn't actually give a complete description of brane dynamics, which is instead achieved by the DBI action.

3.4 Low Energy Limits

In the previous sections we always assumed the underlying space-time to be flat, with metric $\eta_{\mu\nu}$. It turns out that moving to a curved background is quite simple: one can just replace $\eta_{\mu\nu} \to G_{\mu\nu}(X)$ in the Polyakov action, obtaining

$$S_P = -\frac{1}{4\pi l_s^2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) \quad . \tag{3.20}$$

The dynamic world-sheet metric of Polyakov action eases this generalization as there's no need to change the definition of the pull-back metric. What's significantly less trivial is proving that the space-time metric $G_{\mu\nu}(X)$ actually coincides with the graviton field that emerged from string quantization. Aside from the Feynman-Weinberg argument, this identification is supported by a path integral based observation. Expanding the metric as $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$ one sees that the partition function of the curved-background action becomes

$$Z = \int DXDge^{-S_P[G]} = \int DXDge^{-S_P[\eta]}e^{-S_P[h]} \tag{3.21}$$

where $S_P[h]$ can be shown to be the vertex operator of a graviton-string interaction. The partition function built on the "free" action $S_P[G]$ is then identified with an insertion of a graviton vertex in the free $S_P[\eta]$ theory, i.e. with an interacting theory of strings and graviton, thus confirming the original statement [11].

The action (3.20) will serve as the basis for the following discussion, where we gradually introduce the field-string interactions in low energy settings, i.e. at first order in the background fields.

3.4.1 Charged Strings and the Kalb-Ramond Field

Including a Kalb-Ramond-string interaction term, like in the graviton case, boils down to adding the vertex operator associated with $B_{\mu\nu}$ to the Polyakov action:

$$S_P = -\frac{1}{4\pi l_s^2} \int d^2\sigma \sqrt{-g} \left[G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon^{\alpha\beta} \right] \quad , \tag{3.22}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric unit 2-tensor. This term suggests that the string sweeping the world-sheet X^μ is charged under $B_{\mu\nu}$, which takes the role of "electromagnetic field" as anticipated in section 3.1. This conclusion is reached by recalling that the coupling of a charged point-particle to the gauge field A_μ is testified by the appearance of the Lorentz force in the action:

$$S = (kinetic\ terms) + \int d\tau A_\mu \dot{X}^\mu \tag{3.23}$$

which one may recognize as the pull-back of the space-time 1-form $A_1 = A_\mu dX^\mu$ onto the world-line. Analogously, the new interaction term in (3.22) can be seen as the pull-back of the 2-form B_2 onto the world sheet of a charged string. The corresponding gauge-invariant field strength is denoted as $H_3 = dB_2$.

One interesting feature of the electromagnetic field in 4-dimensional field theory is that

the Hodge duality $F_2 \to *F_2$ leads to the introduction of magnetically charged particles. Similarly, in string theories, one can dualize B_2, leading to the introduction of extended objects, i.e. branes, charged under the dual magnetic field. As we seen many times in chapter 2, a $(p+1)$-dimensional charged object naturally couples to a $(p+1)$-form gauge field which is then associated with a $(p+2)$-form field strength; for instance a 2-dimensional world-sheet couples to B_2, associated with $H_3 = dB_2$. This should apply also to the dual field, therefore we will follow this same logic in reverse to deduce the dimensionality of a magnetically charged brane.
In the most general case, the Hodge dual of a $(p+2)$-form field strength is F_{D-p-2} that will be related to a gauge field C_{D-p-3} by an exterior derivative. The objects charged under C will then be $(D-p-3)$-dimensional world-volumes, i.e. $(D-p-4)$-branes.
In $D = 10$ superstring theories the Neveu-Schwarz field couples to strings while its dual couples to branes of dimension $D - p - 4 = 5$. To distinguish them from D-branes, they are often referred to as NS5-branes. Furthermore, in the case of type II teories, the Ramond-Ramond-fields are associated with their own electrically and magnetically charged objects:

- In a type IIA theory C_1 is associated with point-like sources and, through duality, to D6-branes, whereas the C_3 RR-field is couples to 2-branes while its dual couples to D4-branes.

- In type IIB instead only the self-dual field C_4 introduces new objects (the ones related to C_2 have the same dimensions as B_2's), namely 3-branes, both electrically and magnetically charged.

Finally, it should be noted that much like magnetic monopoles in QFT, charged branes behave like topological defects. Branes are then of great interest to our research as their appearance in the dual supergravity and thus in the SymTFT will grant us the means to reconstruct the generalized symmetries of the ABJ theory as described in section 2.3.

3.4.2 The Low Energy String Action

Incorporating also the dilaton $\Phi(X)$ in the action Polyakov action through its vertex operator results in

$$S_P = -\frac{1}{4\pi l_s^2} \int d^2\sigma \sqrt{-g} \left[G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon^{\alpha\beta} + l_s^2 \Phi(X)\mathcal{R}^{(2)} \right] \tag{3.24}$$

where $\mathcal{R}^{(2)}$ is the world-sheet Ricci scalar. One immediately sees that the dilaton term disrupts Weyl invariance, yet this issue could be solved by 1-loop corrections emerging from the coupling of Φ to the graviton and Kalb-Ramond field. In order to see this one should proceed to renormalize the 2-dimensional field theory and verify under which conditions Weyl invariance is restored. We only report the result of such analysis, which is carried out in grater detailed in [11]; Weyl invariance is restored if

$$\beta_{\mu\nu}[G] = \beta_{\mu\nu}[B] = \beta[\Phi] = 0 \tag{3.25}$$

where the β functions are defined by

$$\beta_{\mu\nu}[G] = l_s^2 \mathcal{R}_{\mu\nu} + 2l_s^2 \nabla_\mu \nabla_\nu \Phi - \frac{l_s^2}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} \tag{3.26}$$

$$\beta_{\mu\nu}[B] = -\frac{l_s^2}{2} \left[\nabla^\rho + 2(\nabla^\rho \Phi)\right] H_{\rho\mu\nu} \tag{3.27}$$

$$\beta[\Phi] = -\frac{l_s^2}{2} \nabla^2 \Phi + l_s^2 \nabla_\mu \Phi \nabla^\mu \Phi - \frac{l_s^2}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} \tag{3.28}$$

with ∇ denoting a covariant derivative with respect to $G_{\mu\nu}$.
By imposing Weyl invariance we can now obtain a (one loop level) low energy effective string action by writing an action that reproduces the equations (3.25). Such an action is given by

$$S = \frac{1}{2\kappa_0^2} \int d^{26}X \sqrt{-G} e^{-2\Phi} \left[\mathcal{R} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi\right] \tag{3.29}$$

A quite peculiar result, seeing as this action, obtained on consistency grounds alone and from a single string's dynamics, drives the physics of the underlying space-time gauge fields.
The supersymmetric version of the low energy action has a more complicated expression:

Definition 9 (Supergravity) *The superstring counterpart to the low energy string action* (3.29)*, known as a Supergravity theory is made up of the following terms:*

$$S_{SuGra} = S_0 + S_1 + S_f \quad , \tag{3.30}$$

where S_0 coincides, at least for type II theories, with the low energy action for the bosonic string (in $D = 10$); S_1 describes instead the dynamics of the additional fields introduced by superstring theory. Finally S_f encodes the interactions of space-time fermions.

Since, as mentioned before, our later analysis will focus on the flux sector of a type IIA theory, we restrict our discussion to the form of S_1 for superstring theories of type II: denoting with F_q the field strengths associated to the RR-fields one gets

$$\text{Type IIA:} \qquad S_1 = -\frac{1}{4k_0^2} \int \left[\sqrt{-G}\left(|F_2|^2 + |\tilde{F}_4|^2\right) + B_2 \wedge F_4 \wedge F_4\right] \tag{3.31}$$

$$\text{Type IIB:} \qquad S_1 = -\frac{1}{4k_0^2} \int \left[\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + C_4 \wedge H_3 \wedge F_3\right] \tag{3.32}$$

with

$$\tilde{F}_4 = F_4 - C_1 \wedge H_3 \qquad \tilde{F}_3 = F_3 - C_0 \wedge H_3 \qquad \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$$
$$|F_p|^2 = F_p \wedge * F_p \tag{3.33}$$

Note that both actions contain a term that does not depend on the metric; such term is called Chern-Simons term, and will be recognizable throughout the derivation of the symmetry theory, assuming the role of a cubic anomaly in the final action. These actions are built to preserve $\mathcal{N} = 2$ supersymmetry and it can be shown that they are unique in that regard.

3.4.3 The DBI Action

As foretold when introducing D-branes as dynamical objects, the Dirac action does not provide a complete description of brane physics. A better characterization of D-brane dynamics is obtained including their interactions with the gauge fields. This will eventually result in the Dirac-Born-Infeld action (DBI).

The first step in constructing the DBI action is to derive an effective action similar to (3.29) but tailored for open strings. Through a process analogous to that used for closed strings, one can derive a 2-dimensional field theory whose renormalization generates a constraint aimed at ensuring conformal invariance of the theory. In the open string case this materializes in the equation

$$\partial_\alpha F_{\beta\gamma} \left[\frac{1}{1 - 4\pi^2 l_s^4 F^2} \right]^{\alpha\beta} = 0 \tag{3.34}$$

where $F_{\alpha\beta}$ is the field strength associated with the photon field emerging from string end-point quantization; much like B_2 for closed strings, this field charges the end-points of the string. This equation is also yielded by the so called Born-Infeld action, describing the background physics of a theory of open strings:

$$S_{BI} = -T_p \int d^{p+1}\zeta \sqrt{-\det\left(\eta_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta}\right)} \tag{3.35}$$

The DBI action, describing both the background and the branes dynamics, is then obtained replacing η with its pull-back onto the world-volume of the brane in S_{BI}:

$$S_{DBI} = -T_p \int d^{p+1}\zeta \sqrt{-\det\left(\gamma_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta}\right)} \qquad \gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \zeta^\alpha} \frac{\partial X^\nu}{\partial \zeta^\beta} \eta_{\mu\nu} \tag{3.36}$$

A further extension of the DBI action is obtained by also including the background fields emerging from closed string quantization:

$$S_{DBI} = -T_p \int d^{p+1}\zeta e^{-\Phi} \sqrt{-\det\left(\gamma_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta}\right) + B_{\alpha\beta}} \qquad \begin{cases} \gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \zeta^\alpha}\frac{\partial X^\nu}{\partial \zeta^\beta} G_{\mu\nu} \\ B_{\alpha\beta} = \frac{\partial X^\mu}{\partial \zeta^\alpha}\frac{\partial X^\nu}{\partial \zeta^\beta} B_{\mu\nu} \end{cases} \tag{3.37}$$

The fact that F_2 and B_2 only appear in the combination $2\pi l_s^2 F_2 + B_2$ suggests that the two are deeply related to one another.

3.5 Kaluza-Klein Compactification

Both bosonic and supersymmetric string theories live in space-times of rather high dimensionality, therefore, to have a possibility of describing our 4-dimensional universe in terms of oscillating strings, we need a way to "hide" some of these dimensions.
A solution to this issue is offered by the intrinsically gravitational nature of string theory: since the background space is generally curved, nothing prevents the overabundant dimensions to be so curled up that they become invisible to experiments. This conjecture is known as Kaluza-Klein compactification and relies on decomposing space-time into

$$M_D = S^{1,3} \times L_{D-4} \tag{3.38}$$

with $S^{1,3}$ being the 4-dimensional space-time solving the Einstein equations of the case[2], while L_{D-4} is a compact manifold of length scale below experimental sensitivities, that we will refer to as internal space. In vacuum conditions, where $\mathcal{R}_{\mu\nu} = 0$ one has $S^{1,3} = \mathbb{R}^{1,3}$ (Minkowski space-time) and L_{D-4} is bound to be Ricci-flat, i.e. a Calabi-Yau manifold. One can then integrate over the internal space, obtaining a 4-dimensional theory on a $S^{1,3}$ space-time.
This procedure can be simply generalized to different compactified spaces and we will make use of it later on in chapter 5 to reduce the dimensionality of the holographic dual theory and extract its topological sector.

[2] The "1, 3" superscript indicates that we are considering the space-time metric in Lorentzian signature, i.e. $(-+++)$.

4 Holographic Duals and ABJ Theories

In analogy with the well-known concept of an hologram, where a bidimensional object displays a tridimensional image, the core idea of an holographic principle is to fully describe a d-dimensional theory via a $(d+1)$-dimensional one and vice versa. Aptly switching between the two theories can then ease the analysis of both.
Originally, the idea of holographic duality was proposed as a way to probe the quantum theory of gravity, yet the implementation of the holographic principle in the physical universe is still an open challenge. Nevertheless the idea of holographic duality has proven exceptionally effective as a tool to investigate strongly coupled QFTs by mapping them to weakly coupled gravitational theories in one dimension higher. For the purpose of our research however, holography won't be used to obtain a full picture of the tridimensional QFT, but rather to isolate the topological sector encoding the symmetries of the Aharony-Bergman-Jafferis theory.

4.1 The AdS/CFT Correspondence

The most widespread and mathematically sound incarnation of holographic duality is the AdS/CFT correspondence, connecting conformal field theories (CFT) to superstring theories on an Anti-DeSitter background (AdS). This plays a part in our choice of subject, as the ABJ theory is a superconformal theory, therefore it allows to use the well established AdS_4/CFT_3 duality. After briefly reviewing the ingredients and formulation of the correspondence, we will examine how it materializes in the case at hand.

4.1.1 Conformal Field Theories

Previously, we referred to Weyl invariance as a 'conformal symmetry', meaning that it's a scale transformation that preserve angles, i.e. a dilatation. However, dilatations are not the only symmetry of a conformal field theory because they are relativistic theories, thus enjoying Poincaré invariance.

Definition 10 (CFT) *A d-dimensional conformal field theory (CFT) is defined as a theory bearing a symmetry under the conformal group, consisting of infinitesimal transformations of the form*

$$\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + (b^\mu x^2 - x^\mu b_\nu x^\nu) \quad . \tag{4.1}$$

The first two terms are easily recognized as translations and boosts/rotations, generated by

P^μ and $J^{\mu\nu}$ $\in SO(1,d)$ respectively, whereas the third term is a dilatation whose generator is indicated with D. The final term corresponds to a special conformal transformation whose counterpart in the group algebra is denoted as K^μ.

The generators of the conformal transformations obey the following algebraic relations:

$$\begin{aligned}
&[P^\mu, P^\nu] = 0 && [J^{\mu\nu}, D] = 0 \\
&[J_{\mu\nu}, J^{\rho\sigma}] = 4i\delta^{[\rho}_{[\mu}J^{\sigma]}_{\nu]} && [D, P^\mu] = iP^\mu \\
&[J^{\mu\nu}, P^\rho] = 2i\eta^{\rho[\mu}P^{\nu]} && [D, K^\mu] = -iK^\mu \\
&[J^{\mu\nu}, K^\rho] = 2i\eta^{\rho[\mu}K^{\nu]} && [K^\mu, P^\nu] = -2i\left(J^{\mu\nu} + \eta^{\mu\nu}D\right)
\end{aligned} \tag{4.2}$$

Overall we have a total of $\frac{1}{2}(d+1)(d+2)$ generators, just as much as an $SO(d+2)$ algebra. It can be shown that the group generated by (4.1) is actually isomorphic to $SO(2,d)$.
The conformal group will also need to include a discrete symmetry transformation that essentially acts as a local normalization: $x^\mu \to x^\mu/x^2$. It follows that the full conformal group in d-dimensions is

$$O(2,d) \quad .$$

In a $\mathcal{N}$-supersymmetric setting, the group has to be modified to accommodate for the supercharges Q^I_α, the R-symmetry and the so called conformal supercharges $S_\alpha \sim [K, Q_\alpha]$, that have to be included in order to close the algebra. The resulting algebra describes the superconformal group, which is often denoted as $SU(2, d-2|\mathcal{N})$.
The commutation relations (4.2) have an important consequence on the study of CFTs: the mass operator $M^2 = P^\mu P_\mu$ does not commute with all the other generators, in particular it doesn't commute with D; this means that the energy of a state is not well defined since it can be changed acting with a dilation, hindering the usual perturbative treatment of QFTs, that generally uses the mass eigenvalue to label asymptotic states.
This is where the AdS/CFT correspondence shows its might: by studying the holographic dual of a CFT one can use a perturbative approach in the weakly coupled string theory to gain insight on the non-perturbative aspects of a CFT.

4.1.2 Gravity in Anti-DeSitter Universes

As mentioned before, on the other side of the holographic correspondence we will find a gravitational theory in the form of a superstring theory. For the duality relation to work, the background space-time of such string theory will need to have isometry group identifiable with a conformal group, i.e. $O(2,d)$.
This property is held by the maximally symmetric solution to vacuum Einstein equations in presence of a negative cosmological constant (Λ):

$$\mathcal{R}_{\mu\nu} = \frac{\Lambda}{3} g_{\mu\nu} \tag{4.3}$$

Definition 11 (AdS space) *A D-dimensional Anti-DeSitter space-time AdS_D is the maximally symmetric solution to* (4.3) *and it's the Minkowskian signature analogue to an Euclidean hyperboloid. In fact it coincides with a codimension* 1 *hypersurface in* $\mathbb{R}^{1,D}$ *given by*

$$x_0^2 + x_D^2 - \sum_{i=1}^{D-1} x_i^2 = R^2 \tag{4.4}$$

where the positive sign of the time coordinate x_0 is inherited from the $\mathbb{R}^{1,D}$ metric while x_D appearing with a plus signifies that the focal points of the hyperboloid lie on the D^{th} axis.

The metric of an AdS space can then be written as that of a $\mathbb{R}^{1,(D-2)}$ space, foliated over an additional coordinate $z \in [0, \infty[$, scaled by a factor z^2:

$$ds^2(AdS_D) = R^2 \left(\frac{dz^2}{z^2} + z^2 ds^2(\mathbb{R}^{1,(D-2)}) \right) \quad . \tag{4.5}$$

Notice that this metric degenerates at both $z = 0$ and $z \to \infty$ resulting in the space having two boundaries; this will become relevant in chapter (5), where these regions are associated with the physical and symmetry boundaries of the SymTFT.
The space defined by (4.4) is visibly invariant under rotations generated by the $O(2, D-1)$ group, which is exactly what we were looking for. Observe that implementing a supersymmetric structure into the AdS theory alters the isometry group in the same way as it did the conformal group, therefore superstring theories will be connected to a supersymmetric CFT.
One then needs to construct a superstring theory on an AdS background, describing the interaction between the fields through an effective action in the form (3.30). Here we encounter an issue: on one hand we'd like to describe CFTs of various dimensions d, which would set $D = d + 1$ in order for the groups to match; on the other hand, we saw in section 3.2 that the quantization of superstring theories requires superstrings to live in 10 dimensions to preserve Lorentz invariance, which means the background would have $D = 10$. This is however only a mild inconvenience, solved thanks to Kaluza-Klein compactification. In fact, by constructing the gravitational theory on a 10-dimensional space-time of the form

$$AdS_{d+1} \times L_{10-d-1}$$

with L_{10-d-1} being a compact space, one can compactify over L, leaving with a $(d+1)$-dimensional theory. The nature of the internal space will depend on the CFT and its topology will be a crucial element of our later computations.

4.1.3 Formulation of AdS/CFT

So far we've introduced the two ends of the correspondence, now we need to investigate how the duality actually materializes. From here on out we will refer to the fields in the conformal theory as "boundary fields" (f) while the supergravity fields are called "bulk fields" (h). This is due to the fact that the CFT lives on a Minkowski-like space $\mathbb{R}^{1,d-1}$, which can be recognized as the boundary of the AdS_{d+1} background. In that sense the CFT can be interpreted as a theory living on the boundary of its dual, but still encompassing all of the physics of the latter.

Each bulk field $h(x,z)$ (where $x = (x_0, ..., x_{d-1})$ denotes the boundary coordinates while z is the extra bulk coordinate) will be associated to an operator $\mathcal{O}[f]$ and will appear in the CFT action as a source term:

$$S_{CFT}[f,h] \;=\; S^0_{CFT}[f] + \int d^dx \; h(x)\mathcal{O}[f(x)] \tag{4.6}$$

where the d-dimensional function $h(x)$ is related to the bulk field by $h(x,z) = h(x)g(z)$. This relation is chosen over the more intuitive $h(x) \equiv h(x,0)$ because the latter would clash with the fact that we will shortly take $h(x,z)$ as a solution of the equations of motion in the bulk, thus vanishing at infinity where the boundary is.

One can now write the main statement of the AdS/CFT correspondence by equating the Wilsonian action obtained integrating out the CFT's fields with the bulk action along a solution $\hat{h}$ of the equations of motion:

$$W[h(x)] \;=\; -\log \int Df \; e^{-S^0_{CFT}[f] - \int d^dx\, h\mathcal{O}[f]} \;=\; S_{AdS}[\hat{h}(x,z)] \tag{4.7}$$

From this equivalence follows that of the respective partition functions, confirming the equivalence between the two theories. Notice that, since $\hat{h}$ is bound by classical bulk dynamics while h is completely free, AdS/CFT connects an on-shell gravitational theory to an off-shell CFT.

One, crucial, missing piece of the formulation of the correspondence is how to determine which boundary operator has to be associated with each bulk field. This varies between theories but it's often achieved through symmetry based arguments. In the simple case of conserved currents of the CFT, the natural coupling is (2.7), which leads to identifying $h(x)$ with a background gauge field, meaning that the corresponding field in the bulk theory will be a dynamical gauge field. One then deduces that each global symmetry in the CFT corresponds to a gauge one in the bulk [14].

Another insight into the field/operator correspondence comes from the fact that the operator and the field must have, by consistency, the same quantum numbers with respect to the (super)conformal group. The relevant quantum numbers in this context are the AdS fields' spins (j_1, j_2) and CFT operators' conformal dimension Δ, i.e. their "charge" under dilation. The mass of the AdS field is actually defined by these quantum numbers, in particular, setting the AdS radius to $R = 1$, one gets

Field	**Spin Eignevalues**	**AdS Mass**
ϕ	$(0,0)$	$m^2 = \Delta(\Delta - d)$
A_μ	$(\frac{1}{2},\frac{1}{2})$	$m^2 = \Delta(\Delta-1)(\Delta-d+1)$
$g_{(\mu\nu)}$	$(1,1)$	$m^2 = \Delta(\Delta - d)$
$B_{[\mu\nu]}$	$(1,0)+(0,1)$	$m^2 = \Delta(\Delta-2)(\Delta-d+2)$
ψ	$(\frac{1}{2},0)+(0,\frac{1}{2})$	$m = (\Delta - d/2)$

4.2 Ortho-Symplectic 3-dimensional ABJ Theories

Killing two birds with one stone, we now provide an example of AdS/CFT dual theories by finally introducing the QFT whose symmetry structure will be probed in later chapters: a 3-dimensional superconformal theory with $\mathcal{N} = 5$ supersymmetry and $O(2N)_{2k} \times USp(2N)_{-k}$ gauge group, that was originally constructed by Aharony, Bergman and Jafferis in [4]. The same paper also provides the holographic dual type IIA superstring background, which we already stated to be the starting point of the SymTFT construction. As mentioned before, the choice of theory was driven by knowledge of the holographic background together with the fact that the same type of ABJ-theories were investigated in [5] using a completely different approach, setting a concrete expectation for the generalized symmetry we will reveal in our research.
In this chapter we give a very brief review of the construction of the theory itself, and focus more on the AdS_4 dual supergravity.

4.2.1 Construction of the $O(2N)_{2k} \times USp(2N)_{-k}$ Gauge Theory

A tridimensional ortho-symplectic theory can be constructed in two ways: it can be obtained as the projection of a type IIB string theory onto a D3-brane configuration; alternatively one can proceed by gauging a discrete symmetry in an $\mathcal{N} = 6$ supersymmetric ABJ theory of gauge group $U(M)_{2k} \times U(N)_{-2k}$, constructed in [15]. Both of these constructions provide useful information on the resulting theory.
Projecting onto a D3-brane one comes across a discrete choice of brane configurations leading to four different ortho-symplectic gauge theories, related to each other by an

$SL(2,\mathbb{Z})$ duality:

$$
\begin{aligned}
&O(2M)_{2k} \times USp(2N)_{-k} \\
&USp(2M)_k \times O(2N)_{-2k} \\
&O(2M+1)_{2k} \times USp(2N)_{-k} \\
&USp(2M)_k \times O(2N+1)_{-2k}
\end{aligned}
$$

where by $USp(2N)$ we denote the unitary symplectic group, obtained as the intersection of $U(2N)$ with the symplectic group: $USp(2N) = U(2N) \cap Sp(2N;\mathbb{C})$. The subscripts $k \in \mathbb{Z}$ correspond to the Chern-Simons level associated with the gauge field corresponding to each group. The case we are interested in is the first line, in its $M = N$ version. One can further specify the gauge group into a discrete number of variants, including

$$
SO(2N)_{2k} \times USp(2N)_{-k} \qquad \text{and} \qquad (O(2N)_{2k} \times USp(2N)_{-k})/\mathbb{Z}_2 \quad . \tag{4.8}
$$

These variants will have the same gravity dual, therefore they share the same SymTFT; the different theories are recovered collapsing the same $\mathcal{B}^{ph}_{\mathcal{T}}$ against different $\mathcal{B}^{sym}_{\mathcal{S},\mathcal{S}'}$.
From the second construction one can instead infer the degree of supersymmetry of the theory, based on the R-symmetry's behaviour along the gauging procedure. Being a 3-dimensional superconformal theory, $U(M)_{2k} \times U(N)_{-2k}$ has R-symmetry $SO(\ni)_R$ that breaks into a smaller group upon gauging the discrete symmetry, namely

$$
SO(6)_R \simeq SU(4)_R \to USp(4)_R \simeq SO(5)_R \quad ,
$$

signaling that the final ortho-symplectic theory is invariant under $\mathcal{N} = 5$ supersymmetry, thus being a superconformal theory with group $SU(2,1|5)$. The $SO(3)_f$ subgroup of the R-symmetry makes up the flavour symmetry of the ABJ theory.
It can be shown that theories of this kind have moduli spaces $\left(\mathbb{C}^4/\mathbb{D}_k\right)^N/S_N$, where $\mathbb{D}_k$ denotes the dicyclic group of order $4k$. This information suggests the supergravity background of our theory.
Thanks to the work of Mekareeya and Sacchi in [5], we know what kind of symmetries and anomaly are found in these ortho-symplectic theories, besides the flavour group. In particular the $SO(2N)_{2k} \times USp(2N)_{-k}$ variant has $\mathbb{Z}_2$ 0-form magnetic and charge conjugation symmetries. These can be shown to give birth to mixed anomalies with anomalous actions of the form

$$
S_A = \frac{2\pi i}{2} \int B_2 \wedge B_1^{(1)} \wedge B_1^{(2)} \qquad \text{with} \qquad \oint B_2 \in \mathbb{Z}_2 \quad \text{and} \quad \oint B_1^{(1,2)} \in \mathbb{Z}_2 \quad , \tag{4.9}
$$

where $B_1^{(i)}$ are the background fields associated with either the magnetic or charge conjugation symmetry while B_2 can correspond to the background field of the 1-form symmetry or to the flavour symmetry. Because it is non-Abelian, we do not expect the anomalies involving flavour symmetry to be detected by the SymTFT. Nevertheless, we expect our SymTFT to display a cubic mixed anomalous term involving two 0-form symmetries and a 1-form.

4.2.2 Holographic Dual Supergravity

As hinted above, knowing the moduli space allows to identify the background space-time of the M-theory dual with $AdS_4 \times S^7/\mathbb{D}_k$. However in the $k \ll N$ regime, i.e. in small coupling conditions, one can descend to a type IIA superstring theory by means of a S^1 compactification, resulting in the 10-dimensional background space-time

$$M_{10} = AdS_4 \times CP^3/\mathbb{Z}_2 \quad . \tag{4.10}$$

Performing a low energy limit on this superstring action provides with a supergravity theory whose full action is given by (3.30), specified by (3.29) and (3.31):

$$S_{SuGra} \propto -\int_{M_{10}} \left[\sqrt{-G}e^{-2\Phi}|H_3|^2 + \sqrt{-G}\left(|F_2|^2 + |\tilde{F}_4|^2\right) + B_2 \wedge F_4 \wedge F_4\right] + \mathcal{F}\left[G_{\mu\nu}, \Phi\right] \tag{4.11}$$

where $\mathcal{F}[G_{\mu\nu}, \Phi]$ encodes the terms depending only on dilaton and graviton fields.
As we will see in section 5.2, for the purposes of our discussion we will isolate the flux sector of this action, which is achieved by silencing the dilaton ($\Phi = 0$) and assuming a weak curvature of the background[1], yielding $\mathcal{R} \to 0$ and $\sqrt{-G} \to 1$.
This supergravity theory will admit non-vanishing quantized fluxes for the Ramond-Ramond field strengths of type IIA string theory; in particular there are $2k$ units of F_2 flux through $CP^1/\mathbb{Z}_2$ and N units of $\tilde{F}_4$ on AdS_4. The quantization of the 4-form flux is only achieved when combined with the B_2 holonomy $b_2 = c/k$ due to the fact that F_2 flowing through $CP^1/\mathbb{Z}_2$ alters F_4's equations of motion [4]. It can be shown that the allowed values for c are $-\frac{k+1}{2}$ and $\frac{k-1}{2}$, modulo $2k$.

$$\int_{CP^1/\mathbb{Z}_2} F_2 = 2k \in \mathbb{Z} \qquad\qquad \int_{CP^2} \tilde{F}_4 = f_4 \quad \text{with} \quad f_4 \mid d(f_4 + 2kb_2) = 0 \tag{4.12}$$

The former equation leads to identifying the 2-background with $2kJ$, where J denotes the Kähler form of the space.

[1]This last assumption translates on the QFT side of the duality in $N \ll k^5$, therefore our discussion will be consistent within the regime $N^{1/5} \ll k \ll N$.

5 Extracting the Symmetry TFT from an Holographic Construction

Although AdS/CFT, as presented in the previous chapter, does associate a compactified supergravity theories to SCFTs living on their boundaries, the holographic dual cannot be immediately identified with the Symmetry TFT defined in 2.3. In order to bridge the gap between AdS_4 gravity and the SymTFT corresponding to the ABJ theories, we will perform some preliminary manipulations on the 10-dimensional background before carrying out the compactification under a specific "topological limit" that isolates the part of the action that's interesting to our purpose and consistently captures the finite symmetries data. Following [2], this process is here presented in a general way and only applied to our specific case in chapter 6.

5.1 Flux Sector Action in Democratic Notation

The first thing one can notice about the supergravity action (4.11) is that it is way more than just a decoupled theory of gauge fields, with the latter coupling to both dilatons and gravitons. The same holds for the lower dimensional SuGra living in AdS, where this complexity is what allows to describe the complete dynamics of the SCFT. Nevertheless, we are interested in extracting the topological sector and in examining the physics of flat gauge fields within it. Doing so will result in a theory that complies to the definition given in section 2.3 and we will claim that it is indeed the SymTFT we seek.

The construction begins by focusing only on the so called flux sector of the supergravity action, i.e. the part depending solely on the Neveu-Schwarz and Ramond-Ramond fields. The easiest way to achieve this is to just silence the graviton and dilaton fields which, as anticipated, amounts to setting $\Phi = 0$ and $\sqrt{-G} \to 1$. The resulting ten dimensional action is

$$S_{SuGra} \propto -\int_{M_{10}} \Big[|H_3|^2 + |F_2|^2 + |\tilde{F}_4|^2 + B_2 \wedge F_4 \wedge F_4\Big] \quad . \tag{5.1}$$

In this form, the action still doesn't describe a topological theory, which is one of the defining characteristics of the symmetry theory. This point will however be addressed during the compactification process.

Despite not coinciding with the action that will be used in the dimensional reduction, (5.1) is equivalent to it. We will now review the construction of such an equivalent action in the most general case.

The first change we apply is to redefine the field strengths in a democratic formalism, meaning that electric and magnetic fluxes will be treated equally, allowing us to see the couplings in the final action as explicitly as possible, without Hodge dualities hindering their readability. To do so, one defines new "independent" field strengths for each Hodge dual, each to be associated to a brane-like magnetic source.

Observe that in this formalism the equation of motion of each flux is nothing but the Bianchi identity of its former Hodge dual. In presence of charged branes this results in the Bianchi identities being deformed into

$$dF^{(i)} = d * F^{(D+1-i)} = J^{(i)} + (Chern - Simons) \quad . \tag{5.2}$$

The reason we remark this is that Bianchi identities are what allows us to write the equivalent action that we will end up compactifying. More precisely we embed the identities in the action by writing it in a $(D+2)$-dimensional[1] setting that reduces to a $(D+1)$-boundary term when one enforces the Bianchi identities on it, reproducing (5.1). The auxiliary eleventh dimension is to be understood as non-compact, therefore this operation only affects the 'space-time part' of the background rather than the internal space:

$$M_{d+1} \times L_{D-d} \xrightarrow{Bianchi} M_{d+2} \times L_{D-d} \quad .$$

It follows that the topology of the internal space remains the same throughout our manipulations, ensuring that our later compactification will not assimilate unphysical information into the final theory.

We will refer to the starting point of the SymTFT derivation as "Top Action" as it has the highest dimensionality in our work; its general form is

$$S_{TOP} = \int_{M_{11}} \frac{1}{2} \sum_{i,j} \kappa_{ij} F^{(i)} \wedge dF^{(j)} + CS[\{F^{(i)}\}] - \sum_{i,j} \kappa_{ij} F^{(i)} \wedge J^{(j)} \quad , \tag{5.3}$$

where $CS[\{F^{(i)}\}]$ is a closed top-form from which the physical 10-dimensional Chern-Simons term is obtained through a total derivative. The coefficients κ_{ij} are defined by

$$\begin{aligned} &\kappa_{ij} = 0 \quad if \quad \deg F^{(i)} + \deg F^{(i)} \neq D = 10 \\ &\kappa_{ji} = (-1)^{(\deg F^{(i)}+1)(\deg F^{(j)}+1)} \kappa_{ij} \quad . \end{aligned} \tag{5.4}$$

One can recover the Bianchi identities as the EoM stemming from (5.3). The reasoning behind the top action is to make the action in democratic formalism gauge invariant on sight. Moreover, it explicitly includes dF terms, that will be involved in the enforcement of the topological limit, as well as source terms, that will eventually allow us to write

[1] Notice that, unlike in chapter 3, here we denote the dimensionality of the supertring theory as $D+1$, so that the compactification gives $D+1 \xrightarrow{KK} d+1$, with d corresponding to the superconformal theory's dimension.

the entire action in terms of $J^{(i)}$ components, clarifying which defects are involved in the SymTFT action.

Example: p-form Maxwell

To have a concrete feel of these calculations, let us consider a p-form Maxwell theory in $D+1$ dimensions:

$$S_{D+1} = \int_{M_{D+1}} \frac{1}{2e^2} f_p \wedge * f_p \quad . \tag{5.5}$$

The democratic formalism in this case sees the introduction of the following fluxes and magnetic currents:

$$\begin{aligned} F_p^{(1)} &= f_p & F_{D-p}^{(2)} &= e^{-2} * f_p \\ J_{p+1}^{(1)} &= J_{p+1}^{(m)} & J_{D-p+1}^{(2)} &= * J_{D-p+1}^{(e)} \end{aligned} \tag{5.6}$$

The original action had no Chern-Simons term, therefore, accounting for the definition of κ_{ij}, the top action of the Maxwell theory is then

$$S_{TOP} = \int_{M_{D+2}} \left[F^{(1)} \wedge dF^{(2)} - F^{(1)} \wedge J^{(2)} - (-1)^{(p+2)(D+1-p)} F^{(2)} \wedge J^{(1)} \right] \quad . \tag{5.7}$$

The EoM yielded by this action are easily verified to coincide with the ones one would associate with Maxwell's theory, i.e.

$$\begin{aligned} dF^{(2)} &= J^{(2)} & &\Longleftrightarrow & d * f &= J^{(e)} \\ dF^{(1)} &= J^{(1)} & &\Longleftrightarrow & df &= J^{(m)} \quad . \end{aligned} \tag{5.8}$$

5.2 The Topological Limit

We have already observed in the previous section that (5.1) is not topological. Therefore, in order to apply the anomaly inflow paradigm as we described it previously, one needs to define a regime within the AdS theory specifically suited to isolate the topological sector.
While holographic setups usually consider the bulk theory near $z = 0$, where the conformal physical boundary $\mathcal{B}^{ph}$ is found, the symmetry theory emerges at $z \to \infty$, i.e. near the symmetry boundary $\mathcal{B}^{sym}$ of the anti-DeSitter space. The reason behind this is that topology-related physics dominates at large scales. In this regime, one can perform a derivative expansion of the action, with higher derivatives being subleading. Upon performing the compactification, we can then neglect any derivative of the fluxes. This is known as topological limit, and it leads to a theory capturing the dynamics of flat gauge fields, which, according to the inflow paradigm, correspond to background fields of finite

symmetries of the boundary theory.

To be more precise, the Kaluza-Klein procedure will require us to decompose each of the fluxes (and currents) over the homology groups $H_q(L_{D-d}, \mathbb{Z})$ of the internal space.

$$\begin{aligned} F^{(i)} &= F_B^{(i)} + \sum_a f^{(ia)} \wedge \phi^{(a)} + \sum_b f^{(ib)} \wedge \psi^{(b)} \\ J^{(i)} &= J_B^{(i)} + \sum_a j^{(ia)} \wedge \phi^{(a)} + \sum_b j^{(ib)} \wedge \psi^{(b)} \end{aligned} \tag{5.9}$$

The background component $F_B^{(i)}$ is a closed form living in the internal space, proportional to the volume form of i-cycles. Internal components also live in L_{D-d}, whereas the external fields $f^{(ia)}$ live on the $(d+2)$-manifold generated by the AdS space and the auxilliary direction. We wrote the decomposition distinguishing closed cycles in the internal space[2], denoted with $\psi^{(a)}$ from torsional cycles, represented by the $\phi^{(a)}$ part of the sum and defined as follows:

Definition 12 (Torsion) *A cycle ϕ in a topological space L_{D-d} belongs in the torsional subgroup of $H_{\deg\phi}(L_{D-d}, \mathbb{Z})$, denoted as $Tor_{\deg\phi}(L_{D-d}, \mathbb{Z})$, if there exists a closed cycle in $H_{\deg\phi+1}(L_{D-d}, \mathbb{Z})$ whose corresponding cohomology form Φ is such that*

$$\begin{cases} d\phi \ = \ \ell\Phi \\ d\Phi = 0 \end{cases} \qquad \textit{for some } \ell \in \mathbb{Z}\ . \tag{5.10}$$

From (5.9) one can see that the kinetic term for $F^{(i)}$ in the top action, proportional to $dF^{(i)}$, is not to be neglected until the compactification has been carried out, since it produces both df terms, that will indeed be neglected, and $d\phi$ terms, that do not include any derivative of the flux, and are thus relevant to our discussion.

Another way to interpret the topological limit is that, at large distances in the AdS space, the non-flat fluxes become massive, allowing us to integrate them out of the action.

Once the topological limit is computed, the resulting $(d+2)$-dimensional action will be a function of the external components of fluxes and currents, which can be rewritten in terms of just the currents via Bianchi identities. If the calculations are done correctly the result should be nothing but a boundary term, leading to a topological theory in $(d+1)$ dimensions written in terms of the sources alone. This will be the final expression for the Symmetry TFT action:

$$S_{SymTFT} = \int_{AdS_{d+1}} \mathcal{L}[\{j^{(ia)}\}] \quad .$$

Writing everything in terms of the branes has the aforementioned benefit of bringing out the branes that one needs to project onto $\mathcal{B}^{sym}$. It also directly couples the branes, or rather the corresponding sources, thus showing which ones are involved in anomalies that

[2]Here the torsion-free part of the expansion also includes the cycles that are $\Phi \propto d\phi$.

will inflow to the underlying CFT.
Notice that throughout the whole construction knowledge of the SCFT Lagrangian was never required. This is a major advantage in using the Symmetry TFT to probe the symmetry structure of SCFTs, along with the insight it provides on the defects of the superconformal theory (by projecting them onto $\mathcal{B}^{sym}$).

Example: $AdS_5 \times S^5$ Type IIB Background

Let us now illustrate the last steps of the derivation through a simple example, starting from the compactified 6-dimensional action of a $AdS_5 \times S^5$ type IIB supergravity [2]:

$$S_{d+2} = \int_{M_6} \left[N H_3 \wedge F_3 - H_3 \wedge J_3^{(1)} - F_3 \wedge J_3^{(2)} \right] \quad , \tag{5.11}$$

where $H_3 = dB_2$ and $F_3 = dC_2$, with B_2 and C_2 being the NS and a RR field of type IIB. The constant N instead corresponds to the background flux of the Ramond-Ramond field F_5. This action leads to the following Bianchi identities for H_3 and F_3:

$$N F_3 = J_3^{(1)} \qquad , \qquad -N H_3 = J_3^{(2)} \quad . \tag{5.12}$$

Substituting these back in (5.11) leaves with an action written in terms of the sources consisting of a single quadratic term:

$$S_{d+2} = -\frac{1}{N} \int_{M_6} J_3^{(1)} \wedge J_3^{(2)} \tag{5.13}$$

The SymTFT action living in AdS_5 is then obtained recognizing the action as a total derivative exploiting closure of the currents and formally defining d^{-1} as detailed in appendix B:

$$S_{SymTFT} = -\frac{1}{N} \int_{AdS_5} d^{-1} J_3^{(1)} \wedge J_3^{(2)} = \frac{1}{N} \int_{AdS_5} B_2 \wedge dC_2 \tag{5.14}$$

This can be read as a BF gauge theory of two $\mathbb{Z}_N$ 1-form symmetries and, recalling that $J^{(1,2)}$ are δ-like functions supported on branes, it corresponds to the branes' linking number.
With an appropriate choice of boundary conditions this SymTFT can be used to probe the symmetry structure of 4-dimensional super-Yang-Mills theories with $\mathcal{N} = 4$ [1].

6 Derivation of the Symmetry TFT

With the stage completely set, we can finally begin the computation in the specific case of ABJ theories. Let us start by writing the flux sector action (5.1) in democratic formalism and preparing it for the compactification. Since the field content of a type IIA theory has no multiplicity within form-degrees, we will denote $F^{(i)} = F_i$ while the corresponding magnetic currents are denoted as $J^{(i)} = J_{i+1}$.[1] Therefore, the field content of the flux action will be:

$$
\begin{aligned}
&F^{(2)} = F_2 && F^{(6)} = *F_4 = F_6 \\
&H^{(3)} = H_3 && H^{(7)} = *H_3 = H_7 \\
&F^{(4)} = F_4 && F^{(8)} = *F_2 = F_8
\end{aligned}
$$

supplemented by the sources

$$
J_3^{(2)},\ J_4^{(3)},\ J_5^{(4)},\ J_7^{(6)},\ J_8^{(7)},\ J_9^{(8)} \quad .
$$

Eventually one moves to eleven dimensions by specifying the CS-term in (5.3) in relation to that of type IIA supergravity, which includes the cubic part of the $|\tilde{F}_4|^2$ term:

$$
\begin{aligned}
\mathcal{L}_{SuGra}|_{CS} &= -\left(\tilde{F}_4 \wedge *\tilde{F}_4\right)_{cubic} - B_2 \wedge F_4 \wedge F_4 \\
&\Downarrow \\
CS[\{F_i\}] &= -H_3 \wedge F_4 \wedge F_4 - F_2 \wedge H_3 \wedge F_6 + H_3 \wedge X_8 \quad ,
\end{aligned}
\tag{6.1}
$$

where X_8 is an higher derivative correction and is hereby neglected. The top action can then be written as

$$
\begin{aligned}
S_{TOP} = \int_{M_5 \times L_6} \Big[&-F_2 \wedge dF_8 + F_4 \wedge dF_6 + H_3 \wedge dH_7 - H_3 \wedge \left(F_2 \wedge F_6 + \frac{1}{2} F_4 \wedge F_4\right) + \\
&+ F_2 \wedge J_9^{(8)} - F_8 \wedge J_3^{(2)} - F_4 \wedge J_7^{(6)} + F_6 \wedge J_5^{(4)} - H_3 \wedge J_8^{(7)} - H_7 \wedge J_4^{(3)} \Big] \quad ,
\end{aligned}
\tag{6.2}
$$

with M_5 being a non-compact manifold having an AdS_4 boundary while $L_6 = CP^3/\mathbb{Z}_2$.

[1] To avoid confusion, the currents will often report also the form degree.

6.1 Dimensional Reduction of Supergravity in the Topological Limit

In order to compactify the top action (6.2), we anticipated while introducing the topological limit that one needs to decompose the fields in a series of homology classes of the internal space. The representative of these classes are δ-like a-forms (with a labelling the form degree) supported on the corresponding cycles in $CP^3/\mathbb{Z}_2$. In particular we denote a torsion-free cycle as ψ_a while torsional cycles and their derivatives are written as (ϕ_a, Φ_{a+1}). In appendix A we go in deeper detail on $CP^3/\mathbb{Z}_2$'s topology, meanwhile here we only report the set of internal components that we consider in our computations:

Torsional pairs:	**Torsion-free cycles:**
$(\phi_0, \Phi_1),\ (\phi_1, \Phi_2),\ (\phi_2, \Phi_3),$	$\psi_0,\ \psi_2,\ \psi_4,\ \psi_6$
$(\phi_3, \Phi_4),\ (\phi_4, \Phi_5),\ (\phi_5, \Phi_6)$	

Each torsional pair is associated with torsion degree ℓ_a, related to one another through a Poincaré-like duality between the cycles implying $\ell_a = \ell_{5-a}$ (more details in appendix A). Hence going forward we will only encounter ℓ_0, ℓ_1 and ℓ_2.
For the moment, we restrict our analysis to a subset of the torsional sector of the expansion so that the equations are sensibly lighter. We will however observe that the missing terms are of the same form as the ones contained in the restricted SymTFT, albeit involving different fields. Beside the entirety of the torsion-free sector, which includes the background terms $F_B^{(i)}$, we will neglect torsional 0- and 5-cycles as the equations for the corresponding external components should decouple from the rest. The upshot of this is that the expanded fluxes that enter the dimensional reduction are:

$$
\begin{aligned}
F_2 &= f_1^{(2)} \wedge \phi_1 + \mathbf{f}_0^{(2)} \wedge \Phi_2 + f_0^{(2)} \wedge \phi_2 \\
F_4 &= f_3^{(4)} \wedge \phi_1 + \mathbf{f}_2^{(4)} \wedge \Phi_2 + f_2^{(4)} \wedge \phi_2 + \mathbf{f}_1^{(4)} \wedge \Phi_3 + f_1^{(4)} \wedge \phi_3 + \\
&\quad + \mathbf{f}_0^{(4)} \wedge \Phi_4 + f_0^{(4)} \wedge \phi_4 \\
F_6 &= f_5^{(6)} \wedge \phi_1 + \mathbf{f}_4^{(6)} \wedge \Phi_2 + f_4^{(6)} \wedge \phi_2 + \mathbf{f}_3^{(6)} \wedge \Phi_3 + f_3^{(6)} \wedge \phi_3 + \\
&\quad + \mathbf{f}_2^{(6)} \wedge \Phi_4 + f_2^{(6)} \wedge \phi_4 + \mathbf{f}_1^{(6)} \wedge \Phi_5 \\
F_8 &= f_7^{(8)} \wedge \phi_1 + \mathbf{f}_6^{(8)} \wedge \Phi_2 + f_6^{(8)} \wedge \phi_2 + \mathbf{f}_5^{(8)} \wedge \Phi_3 + f_5^{(8)} \wedge \phi_3 + \\
&\quad + \mathbf{f}_4^{(8)} \wedge \Phi_4 + f_4^{(8)} \wedge \phi_4 + \mathbf{f}_3^{(8)} \wedge \Phi_5 \\
H_3 &= h_2^{(3)} \wedge \phi_1 + \mathbf{h}_1^{(3)} \wedge \Phi_2 + h_1^{(3)} \wedge \phi_2 + \mathbf{h}_0^{(3)} \wedge \Phi_3 + h_0^{(3)} \wedge \phi_3 \\
H_7 &= h_6^{(7)} \wedge \phi_1 + \mathbf{h}_5^{(7)} \wedge \Phi_2 + h_5^{(7)} \wedge \phi_2 + \mathbf{h}_4^{(7)} \wedge \Phi_3 + h_4^{(7)} \wedge \phi_3 + \\
&\quad + \mathbf{h}_3^{(7)} \wedge \Phi_4 + h_3^{(7)} \wedge \phi_4 + \mathbf{h}_2^{(7)} \wedge \Phi_5
\end{aligned}
$$

To distinguish them from the torsional component of the same degree, the external components coupled to internal Φ_a forms are written in bold; a similar convention will be used throughout the whole computation, with simple characters signaling a torsional origin while bold and tilded ones are associated with Φ and ψ terms respectively.
The magnetic currents associated with each flux are then expanded in a similar fashion:

$$\begin{aligned}
J_3^{(2)} &= j_2^{(2)} \wedge \phi_1 + \mathbf{j}_1^{(2)} \wedge \Phi_2 + j_1^{(2)} \wedge \phi_2 + \mathbf{j}_0^{(2)} \wedge \Phi_3 + j_0^{(2)} \wedge \phi_3 \\
J_4^{(3)} &= j_3^{(3)} \wedge \phi_1 + \mathbf{j}_2^{(3)} \wedge \Phi_2 + j_2^{(3)} \wedge \phi_2 + \mathbf{j}_1^{(3)} \wedge \Phi_3 + j_1^{(3)} \wedge \phi_3 + \\
&\quad + \mathbf{j}_0^{(3)} \wedge \Phi_4 + j_0^{(3)} \wedge \phi_4 \\
J_5^{(4)} &= j_4^{(4)} \wedge \phi_1 + \mathbf{j}_3^{(4)} \wedge \Phi_2 + j_3^{(4)} \wedge \phi_2 + \mathbf{j}_2^{(4)} \wedge \Phi_3 + j_2^{(4)} \wedge \phi_3 + \\
&\quad + \mathbf{j}_1^{(4)} \wedge \Phi_4 + j_1^{(4)} \wedge \phi_4 + \mathbf{j}_0^{(4)} \wedge \Phi_5 \\
J_7^{(6)} &= j_6^{(6)} \wedge \phi_1 + \mathbf{j}_5^{(6)} \wedge \Phi_2 + j_5^{(6)} \wedge \phi_2 + \mathbf{j}_4^{(6)} \wedge \Phi_3 + j_4^{(6)} \wedge \phi_3 + \\
&\quad + \mathbf{j}_3^{(6)} \wedge \Phi_4 + j_3^{(6)} \wedge \phi_4 + \mathbf{j}_2^{(6)} \wedge \Phi_5 \\
J_8^{(7)} &= j_7^{(7)} \wedge \phi_1 + \mathbf{j}_6^{(7)} \wedge \Phi_2 + j_6^{(7)} \wedge \phi_2 + \mathbf{j}_5^{(7)} \wedge \Phi_3 + j_5^{(7)} \wedge \phi_3 + \\
&\quad + \mathbf{j}_4^{(7)} \wedge \Phi_4 + j_4^{(7)} \wedge \phi_4 + \mathbf{j}_3^{(7)} \wedge \Phi_5 \\
J_9^{(8)} &= j_8^{(8)} \wedge \phi_1 + \mathbf{j}_7^{(8)} \wedge \Phi_2 + j_7^{(8)} \wedge \phi_2 + \mathbf{j}_6^{(8)} \wedge \Phi_3 + j_6^{(8)} \wedge \phi_3 + \\
&\quad + \mathbf{j}_5^{(8)} \wedge \Phi_4 + j_5^{(8)} \wedge \phi_4 + \mathbf{j}_4^{(8)} \wedge \Phi_5
\end{aligned}$$

The dimensional reduction from $D+2$ to $d+2$ dimensions is carried out by integrating the internal components over the orientifold $L_6 = CP^3/\mathbb{Z}_2$. This will produce an effective theory in which the interactions are determined by the linking and intersections of the internal cycles. In order to streamline the notation, couplings of the same nature will be denoted with the same greek letter, followed by indices signaling the degree of the forms involved; consistently with the notation used for external components, simple letter indices will correspond to torsional ϕ forms whereas bold ones indicate a closed Φ form.
The relevant terms of the expanded top action are those that carry an overall internal-form degree equal to the dimension of the internal geometry, i.e. those of $deg(*) = 6$, seeing as they are the only ones providing a non-vanishing integration.

6.1.1 Chern-Simons Term

Let us start by compactifying the Chern-Simons sector of the action, that is

$$-H_3 \wedge \left(F_2 \wedge F_6 + \frac{1}{2} F_4 \wedge F_4 \right) \tag{6.3}$$

The couplings provided by the dimensional reduction of these terms are cubic wedge-combinations of internal-forms:

$$\begin{aligned} \chi_{abc} &= \int_{L_6} \phi_a \wedge \phi_b \wedge \phi_c \qquad & \chi_{\mathbf{a}bc} &= \int_{L_6} \Phi_a \wedge \phi_b \wedge \phi_c \\ \chi_{\mathbf{ab}c} &= \int_{L_6} \Phi_a \wedge \Phi_b \wedge \phi_c \qquad & \chi_{\mathbf{abc}} &= \int_{L_6} \Phi_a \wedge \Phi_b \wedge \Phi_c \end{aligned} \tag{6.4}$$

By construction $\Phi_{a+1} \in H^{a+1}(CP^3/\mathbb{Z}_2, \mathbb{Z})$ has support on the torsional a-cycle, and has a δ-like behaviour on it. By (5.10) this implies

$$d\phi_a = \ell \Phi_{a+1} = \ell \delta(\Sigma_{a+1})\varepsilon \tag{6.5}$$

where ε is a "unit $(a+1)$-form". This means that ϕ_a will be proportional to a step function (denoted as θ) with support on the region $\Omega \quad | \quad \partial\Omega_a = \Sigma_a$.
This observation allows to specify the couplings in (6.4) as volumes of intersections between Σ's and Ω's, modulo ℓ and a combinatorial factor. Consider for instance χ_{abc}:

$$\chi_{abc} = \int_{L_6} \ell_a \theta(\Omega_a) \ell_b \theta(\Omega_b) \ell_c \theta(\Omega_c) \varepsilon_a \wedge \varepsilon_b \wedge \varepsilon_c \propto \ell_a \ell_b \ell_c Vol(\Omega_a \cap \Omega_b \cap \Omega_c) \tag{6.6}$$

where the antisymmetrization of the unit forms ε yields an overall numerical factor.
Any coupling including multiple torsional forms (ϕ) within the integral is actually vanishing. This can be seen by moving to higher dimension, specifically considering a conical manifold L_7 with boundary L_6. Due to the conical construction, L_7 has a singularity in the origin. It follows that unlike L_6, where the $\mathbb{Z}_2$ orientifold acts freely on the whole space causing torsion to appear, any torsional cycle at the boundary of L_7, trivializes when approaching the singularity. The upshot of this is that the conical manifold L_7 is torsion-free.
Applying then Stokes theorem to write the coupling in terms of such space introduces an exterior derivative, resulting in one of the original ϕ morphing into a Φ:

$$\chi_{\mathbf{a}bc} = \int_{L_6} \Phi_a \wedge \phi_b \wedge \phi_c = \int_{L_7} d(\Phi_a \wedge \phi_b \wedge \phi_c) \sim \int_{L_7} \Phi_a \wedge \Phi_b \wedge \phi_c \tag{6.7}$$

For couplings with at most one torsional form this leads to a torsion-free expression, consistent with the observation made on L_7. If instead the original coupling involved multiple ϕ-forms (as in (6.7)) the integral on the conical manifold will still display torsion, clashing with the construction of the space. The only way this contradiction can be solved is if the coupling were vanishing in the first place. Hence

$$\chi_{abc} = \chi_{\mathbf{a}bc} = 0 \quad \forall\, \mathbf{a}, a, b, c \quad . \tag{6.8}$$

Furthermore, couplings made up of only Φ forms can also be neglected because the argument of the integral can be expressed as a total derivative thanks to nilpotence of the exterior derivative:

$$\chi_{\mathbf{abc}} = \int_{L_6} \Phi_a \wedge \Phi_b \wedge \Phi_c = \int_{L_6} d\phi_{a-1} \wedge d\phi_{b-1} \wedge d\phi_{c-1} = \int_{L_6} d\left(\phi_{a-1} \wedge d\phi_{b-1} \wedge d\phi_{c-1}\right) \quad . \tag{6.9}$$

It follows that these couplings are boundary terms in a space where $\partial L_6 = \varnothing$, hence they vanish.

In the Chern-Simons sector the internal form-degree of a term is fully encoded in the components of the fluxes, therefore the selected terms are those of $\sum deg\phi + \sum deg\Phi = 6$. Combinations including two or more identical odd-degree forms are excluded by antisymmetry of the $\wedge$-product: $\phi_3 \wedge \phi_3 = (-1)^9 \phi_3 \wedge \phi_3 \iff \phi_3 \wedge \phi_3 = 0$. The surviving couplings provided by the CS sector are then:

$$\chi_{\mathbf{132}} \, , \, \chi_{\mathbf{222}} \tag{6.10}$$

with possible permutations of the indices (maintaining the "boldness" unchanged).

6.1.2 Source Terms

The source terms, that is

$$F_2 \wedge J_9^{(8)} - F_8 \wedge J_3^{(2)} - F_4 \wedge J_7^{(6)} + F_6 \wedge J_5^{(4)} - H_3 \wedge J_8^{(7)} - H_7 \wedge J_4^{(3)} \quad , \tag{6.11}$$

produce quadratic wedge-combinations of internal-forms that become quadratic couplings after carrying out the dimensional reduction. Such couplings are denoted by the letter σ:

$$\sigma_{ab} = \int_{L_6} \phi_a \wedge \phi_b \qquad \sigma_{\mathbf{a}b} = \int_{L_6} \Phi_a \wedge \phi_b \qquad \sigma_{\mathbf{ab}} = \int_{L_6} \Phi_a \wedge \Phi_b \tag{6.12}$$

where it must be $a+b=6$ for the integrand to be an internal top-form. Since they include two small ϕ, all σ_{ab} are vanishing based on the same argument used for χ couplings. Same goes for $\sigma_{\mathbf{ab}}$, as they can be written as total derivatives. The mixed couplings on the other hand bear a strict relation to the linking number between the torsional cycles associated with the two internal forms involved in the integration:

$$\sigma_{\mathbf{a}b} = \int_{L_6} d\Phi_a \wedge \phi_b = \Sigma_a \cdot_{L_6} \Omega_b = \ell_b Link_{L_6}(\Sigma_a, \Sigma_b) \tag{6.13}$$

Overall there are four quadratic couplings that, at this point in the discussion, are independent:

$$\sigma_{\mathbf{1}5} \, , \, \sigma_{\mathbf{2}4} \, , \, \sigma_{2\mathbf{4}} \, , \, \sigma_{\mathbf{3}3} \tag{6.14}$$

Note that exchanging the indices may result in a sign flip according to wedge product properties.

6.1.3 Kinetic Part

As foretold, when aiming at the construction of a SymTFT, the compactification of the kinetic terms requires some extra attention: because it depends on the derivatives of the field strengths, the kinetic part of the flux sector

$$-F_2 \wedge dF_8 + F_4 \wedge dF_6 + H_3 \wedge dH_7 \quad , \tag{6.15}$$

will be affected by the topological limit. In particular only terms where the exterior derivative acts on an internal form survive the reduction process since $df \to 0$. There is an important exception to this, represented by terms $\propto d\mathbf{f}$. In fact, it will turn out that $d\mathbf{f}_a^{(i)}$ cannot be considered subleading, as it is actually comparable to the external components $f_{a+1}^{(i)}$.

If d acts on a $\mathbf{f}$, the coupling will fall under the umbrella of $\sigma_{\mathbf{ab}}$ discussed earlier. Kinetically originated couplings in which the derivative acts on an internal form are instead denoted as τ_{ab}:

$$\tau_{ab} = \int_{L_6} \phi_a \wedge d\phi_b \qquad \tau_{\mathbf{ab}} = \int_{L_6} \Phi_a \wedge d\phi_b \tag{6.16}$$

Because the kinetic terms in S_{TOP} contains an exterior derivative, the degree of the involved cycles must add up to 5 for them to survive the integration. The presence of the exterior derivative also allows to relate the τ couplings to the linking number of the cycles: by substituting (5.10) in these expressions one notices that these coupling can be traced back to $\sigma_{a\mathbf{b}}$ and $\sigma_{\mathbf{ab}}$ respectively, giving:

$$\tau_{ab} = (-1)^{a(b+1)} \ell_b \sigma_{\mathbf{(b+1)}a} \qquad \tau_{\mathbf{ab}} = (-1)^{a(b+1)} \ell_b \sigma_{\mathbf{(b+1)a}} \tag{6.17}$$

It is then easy to conclude that couplings of the second type vanish as they are proportional to a $\sigma_{\mathbf{ab}}$-type coupling, which was also deemed as vanishing.

Although, according to (6.17), they are not independent from the quadratic couplings, they'll be treated as such for the time being in order to highlight their different origin. In particular the action will contain the couplings

$$\tau_{14} \, , \, \tau_{41} \, , \, \tau_{23} \, , \, \tau_{32} \quad . \tag{6.18}$$

6.1.4 The Dimensionally Reduced Action

At this point, one can finally write the reduced form of the top action (6.2). By account of all previous observations, the $(d+2)$-dimensional action will consist of the following terms:

$$
\begin{aligned}
S_{d+2} = \int_{M_5} & \tau_{14}\left[-f_1^{(2)} f_4^{(8)} + f_3^{(4)} f_2^{(6)} + h_2^{(3)} h_3^{(7)}\right] + \tau_{32}\left[f_1^{(4)} f_4^{(6)} - h_0^{(3)} h_5^{(7)}\right] + \\
& + \tau_{23}\left[-f_0^{(2)} f_5^{(8)} + f_2^{(4)} f_3^{(6)} + h_1^{(3)} h_4^{(7)}\right] + \tau_{41}\left[f_0^{(4)} f_5^{(6)}\right] + \sigma_{\mathbf{2}4} f_0^{(4)} d\mathbf{f}_4^{(6)} + \\
& + \sigma_{2\mathbf{4}}\left[-f_0^{(2)} d\mathbf{f}_4^{(8)} + f_2^{(4)} d\mathbf{f}_2^{(6)} + h_1^{(3)} d\mathbf{h}_3^{(7)}\right] + \sigma_{3\mathbf{3}}\left[f_1^{(4)} d\mathbf{f}_3^{(6)} - h_0^{(3)} d\mathbf{h}_4^{(7)}\right] + \\
& + \sigma_{1\mathbf{5}}\left[-f_1^{(2)} d\mathbf{f}_3^{(8)} + f_3^{(4)} d\mathbf{f}_1^{(6)} - h_2^{(3)} d\mathbf{h}_2^{(7)}\right] - \chi_{1\mathbf{32}}\left[-h_2^{(3)} \mathbf{f}_0^{(2)} \mathbf{f}_3^{(6)} - \mathbf{h}_1^{(3)} f_1^{(2)} \mathbf{f}_3^{(6)} + \right. \\
& \left. + \mathbf{h}_0^{(3)} f_1^{(2)} \mathbf{f}_4^{(6)} + \mathbf{h}_0^{(3)} \mathbf{f}_0^{(2)} f_5^{(6)} - h_2^{(3)} \mathbf{f}_1^{(4)} \mathbf{f}_2^{(4)} + \mathbf{h}_1^{(3)} \mathbf{f}_1^{(4)} f_3^{(4)} + \mathbf{h}_0^{(3)} f_3^{(4)} \mathbf{f}_2^{(4)}\right] - \\
& - \chi_{2\mathbf{22}}\left[h_1^{(3)} \mathbf{f}_0^{(2)} \mathbf{f}_4^{(6)} + \mathbf{h}_1^{(3)} f_0^{(2)} \mathbf{f}_4^{(6)} + \mathbf{h}_1^{(3)} \mathbf{f}_0^{(2)} f_4^{(6)} + \frac{1}{2} h_1^{(3)} \mathbf{f}_2^{(4)} \mathbf{f}_2^{(4)} + \mathbf{h}_1^{(3)} f_2^{(4)} \mathbf{f}_2^{(4)}\right] + \\
& + \sigma_{1\mathbf{5}}\left[f_1^{(2)} \mathbf{j}_4^{(8)} + \mathbf{f}_3^{(8)} j_2^{(2)} - f_3^{(4)} \mathbf{j}_2^{(6)} - \mathbf{f}_1^{(6)} j_4^{(4)} + f_5^{(6)} \mathbf{j}_0^{(4)} + h_2^{(3)} \mathbf{j}_3^{(7)} - \mathbf{h}_2^{(7)} j_3^{(3)}\right] + \\
& + \sigma_{2\mathbf{4}}\left[f_0^{(2)} \mathbf{j}_5^{(8)} - \mathbf{f}_4^{(8)} j_1^{(2)} + f_2^{(4)} \mathbf{j}_3^{(6)} - \mathbf{f}_0^{(4)} j_5^{(6)} + \mathbf{f}_2^{(6)} j_3^{(4)} + f_4^{(6)} \mathbf{j}_1^{(4)} - h_1^{(3)} \mathbf{j}_4^{(7)} - \right. \\
& \left. - h_5^{(7)} \mathbf{j}_0^{(3)} - \mathbf{h}_3^{(7)} j_2^{(3)}\right] + \sigma_{\mathbf{2}4}\left[\mathbf{f}_0^{(2)} j_5^{(8)} - f_4^{(8)} \mathbf{j}_1^{(2)} - \mathbf{f}_2^{(4)} j_3^{(6)} - f_0^{(4)} \mathbf{j}_5^{(6)} + f_2^{(6)} \mathbf{j}_3^{(4)} + \right. \\
& \left. + \mathbf{f}_4^{(6)} j_1^{(4)} - \mathbf{h}_1^{(3)} j_4^{(7)} - \mathbf{h}_5^{(7)} j_0^{(3)} - h_3^{(7)} \mathbf{j}_2^{(3)}\right] + \sigma_{3\mathbf{3}}\left[-f_5^{(8)} \mathbf{j}_0^{(2)} + \mathbf{f}_5^{(8)} j_0^{(2)} - f_1^{(4)} \mathbf{j}_4^{(6)} + \right. \\
& \left. + \mathbf{f}_1^{(4)} j_4^{(6)} + f_3^{(6)} \mathbf{j}_2^{(4)} - \mathbf{f}_3^{(6)} j_2^{(4)} + h_0^{(3)} \mathbf{j}_5^{(7)} - \mathbf{h}_0^{(3)} j_5^{(7)} + h_4^{(7)} \mathbf{j}_1^{(3)} - \mathbf{h}_4^{(7)} j_1^{(3)}\right]
\end{aligned}
\tag{6.19}
$$

where the wedge products between the external components are hidden for the sake of readability. A neat consistency check is done by verifying that all surviving terms are indeed top-forms of the external 5-dimensional space.

This action can be further simplified by removing some terms, namely those including fluxes of the types $f_0^{(i)}$, $f_1^{(i)}$, $\mathbf{f}_0^{(i)}$ and, consequently, $j_0^{(i)}$, $j_1^{(i)}$, $\mathbf{j}_0^{(i)}$. The reason for this can be seen by recalling that in general, a p-flux is defined as the exterior derivative of a $(p-1)$-form gauge field:

$$F_p = dA_{p-1}$$

Therefore, if one were to expand A_{p-1} in terms of the torsional pairs just like the fluxes and currents, a polynomial matching of the terms on both sides would show that the external components listed above are entirely generated by 0-form components of a gauge field

$(a_0^{(i)})$. Being scalars, the a_0 fields cannot transform under the usual gauge transformation

$$a \xrightarrow{gauge} a + d\lambda$$

as λ would be a (-1)-form for the relation to be consistent. Hence for simplicity one sets $a_0^{(i)} = 0$, thus killing the corresponding parts of fluxes and magnetic currents.
After this simplification the action is of the form:

$$\begin{aligned}
S_{d+2} \; = & \int_{M_5} \tau_{14}\left[f_3^{(4)}f_2^{(6)} + h_2^{(3)}h_3^{(7)}\right] + \tau_{23}f_2^{(4)}f_3^{(6)} + \sigma_{\mathbf{15}}\left[f_3^{(4)}d\mathbf{f}_1^{(6)} - h_2^{(3)}d\mathbf{h}_2^{(7)}\right] + \\
& + \sigma_{\mathbf{24}}f_2^{(4)}d\mathbf{f}_2^{(6)} - \chi_{\mathbf{132}}\left[-h_2^{(3)}\mathbf{f}_1^{(4)}\mathbf{f}_2^{(4)} + \mathbf{h}_1^{(3)}\mathbf{f}_1^{(4)}f_3^{(4)}\right] - \chi_{2\mathbf{22}}\mathbf{h}_1^{(3)}f_2^{(4)}\mathbf{f}_2^{(4)} + \\
& + \sigma_{\mathbf{15}}\left[\mathbf{f}_3^{(8)}j_2^{(2)} - f_3^{(4)}\mathbf{j}_2^{(6)} - \mathbf{f}_1^{(6)}j_4^{(4)} + h_2^{(3)}\mathbf{j}_3^{(7)} - \mathbf{h}_2^{(7)}j_3^{(3)}\right] + \\
& + \sigma_{\mathbf{24}}\left[f_2^{(4)}\mathbf{j}_3^{(6)} + \mathbf{f}_2^{(6)}j_3^{(4)} + f_4^{(6)}\mathbf{j}_1^{(4)} - \mathbf{h}_3^{(7)}j_2^{(3)}\right] + \\
& + \sigma_{\mathbf{24}}\left[-f_4^{(8)}\mathbf{j}_1^{(2)} - \mathbf{f}_2^{(4)}j_3^{(6)} + f_2^{(6)}\mathbf{j}_3^{(4)} - \mathbf{h}_1^{(3)}j_4^{(7)} - h_3^{(7)}\mathbf{j}_2^{(3)}\right] + \\
& + \sigma_{\mathbf{33}}\left[\mathbf{f}_1^{(4)}j_4^{(6)} + f_3^{(6)}\mathbf{j}_2^{(4)} - \mathbf{f}_3^{(6)}j_2^{(4)} + h_4^{(7)}\mathbf{j}_1^{(3)}\right]
\end{aligned} \tag{6.20}$$

6.2 The AdS_4 Topological Theory

In order to make the CS couplings between branes clearly readable in terms of the sources, along with removing the auxiliary dimension we set out to write the reduced action in terms of only the currents. To do so we will consider the fluxes' dynamics to be driven by classical equations of motion and use such relations to effectively integrate out each $f_a^{(i)}$.
One then needs to derive the equations, a task that can be carried out in two equivalent ways: the first approach consists in compactifying the known Bianchi identities for the 11-dimensional $F^{(i)}$. Alternatively one could directly compute the Euler-Lagrange equations starting from the reduced action.
Either way, it's useful to recall that, by definition, currents are closed forms, i.e. $dJ^i = 0$ $\forall\ i$. Such constraint is easily transferred to the external components of the currents by a polynomial identification, yielding the following results:

$$dj_a^{(i)} = 0 \qquad\qquad d\mathbf{j}_a^{(i)} = (-1)^{a+1}\ell_{i-a-1}j_{a+1}^{(i)} \tag{6.21}$$

Let us anticipate that we will encounter some issues with both approaches and end up choosing a simplified solution that should still provide consistent results.

6.2.1 Reduction of the Bianchi Identities

By construction, Bianchi identities of the IIA theory are derived as the equations of motion for $F^{(i)}$ induced by the $(D+2)$-dimensional top action, the general expression of which is

$$\sum_j \kappa_{ij}\left(dF^{(j)} - J^{(j)}\right) + \frac{\partial CS}{\partial F^{(i)}} = 0 \tag{6.22}$$

Substituting the values of κ_{ij} and the expression for CS, one gets the following Bianchi identities:

$$\begin{cases} dF_2 = J_3 \\ dF_4 = J_5 + H_3F_2 \\ dF_6 = J_7 + H_3F_4 \end{cases} \qquad \begin{cases} dF_8 = J_9 - H_3F_6 \\ dH_3 = J_4 \\ dH_7 = J_8 + F_2F_6 + \frac{1}{2}F_4^2 \end{cases} \tag{6.23}$$

It is possible to induce similar relations between the external components of fluxes and currents by expanding them as before; consider for instance the Ramond-Ramond flux F_4:

$$\begin{aligned} & d\left(f_3^{(4)}\phi_1 + \mathbf{f}_2^{(4)}\Phi_2 + f_2^{(4)}\phi_2 + \mathbf{f}_1^{(4)}\Phi_3 + f_1^{(4)}\phi_3 + \mathbf{f}_0^{(4)}\Phi_4 + f_0^{(4)}\phi_4\right) = \\ & = j_4^{(4)}\phi_1 + \mathbf{j}_3^{(4)}\Phi_2 + j_3^{(4)}\phi_2 + \mathbf{j}_2^{(4)}\Phi_3 + j_2^{(4)}\phi_3 + \mathbf{j}_1^{(4)}\Phi_4 + j_1^{(4)}\phi_4 + \mathbf{j}_0^{(4)}\Phi_5 + \\ & + \left(h_2^{(3)}\phi_1 + \mathbf{h}_1^{(3)}\Phi_2 + h_1^{(3)}\phi_2 + \mathbf{h}_0^{(3)}\Phi_3 + h_0^{(3)}\phi_3\right)\cdot\left(f_1^{(2)}\phi_1 + \mathbf{f}_0^{(2)}\Phi_2 + f_0^{(2)}\phi_2\right) \end{aligned}$$

At this point one would proceed with a polynomial identification with respect to the internal forms. This approach is not as straightforward as it seems because it requires knowledge of how $\wedge$-products between ϕ and Φ forms relate to each other and to the singled out internal forms: a product $\phi_a \wedge \phi_b$ might not be independent from ϕ_{a+b} or from a different combination $\phi_c \wedge \phi_d$ (such that $c+d = a+b$). Non-independent terms should then be included in the same equation, with some unknown relative numerical factors. Due to this issue, the alternative path seems more promising seeing as it relies on a purely $(d+2)$-dimensional computation, thus not requiring detailed knowledge of the cycles of $CP^3/\mathbb{Z}_2$. We will verify consistency between the two paths a posteriori.

6.2.2 Equations of Motion in $(d+2)$ Dimensions

Instead of using Bianchi Identities, which, as explained above, introduce some difficulties in their reduction and solution, one can cast the fluxes out of the action by extracting their classical equations of motion directly from the $(d+2)$-dimensional action (6.20) by means of the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial f_a^{(i)}} - d\frac{\partial \mathcal{L}}{\partial df_a^{(i)}} = 0 \qquad\qquad \frac{\partial \mathcal{L}}{\partial \mathbf{f}_a^{(i)}} - d\frac{\partial \mathcal{L}}{\partial \mathbf{f}_a^{(i)}} = 0 \tag{6.24}$$

Note that in these equations should be computed in terms of the components $f^{(i)}_{\mu_1 \cdots \mu_a}$, however we can formally derive with respect to the full a-forms thanks to the total anti-symmetrization of the action.

Computing these equations of motion produces a system of many coupled first order differential equations quadratic in f, $\mathbf{f}$ and h, $\mathbf{h}$:

$$\begin{cases}
0 = \sigma_{\mathbf{15}} d\mathbf{f}_1^{(6)} + \tau_{14} f_2^{(6)} - \chi_{\mathbf{132}} \mathbf{h}_1^{(3)} \mathbf{f}_1^{(4)} - \sigma_{\mathbf{15}} \mathbf{j}_2^{(6)} \\
0 = \tau_{14} f_3^{(4)} + \sigma_{\mathbf{24}} \mathbf{j}_3^{(4)} \\
0 = \sigma_{\mathbf{15}} d\mathbf{h}_2^{(7)} + \tau_{14} h_3^{(7)} + \chi_{\mathbf{132}} \mathbf{f}_1^{(4)} \mathbf{f}_2^{(4)} + \sigma_{\mathbf{15}} \mathbf{j}_3^{(7)} \\
0 = \tau_{14} h_2^{(3)} - \sigma_{\mathbf{24}} \mathbf{j}_2^{(3)} \\
0 = \sigma_{\mathbf{24}} d\mathbf{f}_2^{(6)} + \tau_{23} f_3^{(6)} - \chi_{\mathbf{222}} \mathbf{h}_1^{(3)} \mathbf{f}_2^{(4)} + \sigma_{\mathbf{24}} \mathbf{j}_3^{(6)} \\
0 = \tau_{23} f_2^{(4)} + \sigma_{\mathbf{33}} \mathbf{j}_2^{(4)} \\
0 = -df_2^{(4)} + j_3^{(4)} \\
0 = -df_3^{(4)} + j_4^{(4)} \\
0 = dh_2^{(3)} - j_3^{(3)} \\
0 = \chi_{\mathbf{132}} h_2^{(3)} \mathbf{f}_1^{(4)} - \chi_{\mathbf{222}} \mathbf{h}_1^{(3)} f_2^{(4)} - \sigma_{\mathbf{24}} j_3^{(6)} \\
0 = -\chi_{\mathbf{132}} \mathbf{f}_1^{(4)} f_3^{(4)} - \chi_{\mathbf{222}} f_2^{(4)} \mathbf{f}_2^{(4)} - \sigma_{\mathbf{24}} j_4^{(7)} \\
0 = \chi_{\mathbf{132}} \left[h_2^{(3)} \mathbf{f}_2^{(4)} - \mathbf{h}_1^{(3)} f_3^{(4)} \right] + \sigma_{\mathbf{33}} j_4^{(6)}
\end{cases} \tag{6.25}$$

The remaining Euler-Lagrange equations provide instead some useful direct constraints on the currents, that will allow to neglect some terms in the action:

$$\begin{cases}
\mathbf{j}_1^{(4)} = 0 \\
\mathbf{j}_1^{(3)} = 0 \\
\mathbf{j}_1^{(2)} = 0
\end{cases} \qquad\qquad \begin{cases}
j_2^{(2)} = 0 \\
j_2^{(4)} = 0 \\
j_2^{(3)} = 0
\end{cases} \tag{6.26}$$

Notice that all of these equations bear an important similarity with the Bianchi identities (6.23) in the fact that the fluxes only appear in the combinations present in the Bianchi relations; for instance the equation for a component $f_a^{(6)}$ of the RR flux F_6 will only contain quadratic terms of the form $\sim h^{(3)} f^{(4)}$. This is expected since, were the reduction of the Bianchi identities to be fully carried out, the two sets of equation should coincide, or rather (6.25) should coincide with a subset of the reduced Bianchi identities.

The matching between (6.25) and the reduced Bianchi identities (considering all torsionful combinations as $\propto \phi_a$ and torsionless ones as $\propto \Phi_a$) allows to fix the unknown coefficients mentioned before and incidentally provides some insight on the couplings. For instance, the first line of (6.25) needs to be matched to the Φ_5 terms of $dF_6 = J_7 + H_3F_4$ as it contains the exterior derivative of $\mathbf{f}_1^{(6)}$; this means that the following equations must be equivalent:

$$
\begin{aligned}
&0 = \sigma_{\mathbf{15}} d\mathbf{f}_1^{(6)} + \tau_{14} f_2^{(6)} - \chi_{\mathbf{132}} \mathbf{h}_1^{(3)} \mathbf{f}_1^{(4)} - \sigma_{\mathbf{15}} \mathbf{j}_2^{(6)} \\
&0 = \left(d\mathbf{f}_1^{(6)} + \ell_1 f_2^{(6)} - \mathbf{j}_2^{(6)} \right) \Phi_5 - (\phi_1 \wedge \phi_4)\, h_2^{(3)} f_0^{(4)} - (\phi_2 \wedge \phi_3) \left(h_1^{(3)} f_1^{(4)} + h_0^{(3)} f_2^{(4)} \right) - \\
&\quad - (\Phi_2 \wedge \Phi_3) \left(\mathbf{h}_1^{(3)} \mathbf{f}_1^{(4)} + \mathbf{h}_0^{(3)} \mathbf{f}_2^{(4)} \right)
\end{aligned}
\tag{6.27}
$$

Most of these terms vanish because they contain an f_0 or a similarly vanishing flux, leaving with the equivalence

$$
\sigma_{\mathbf{15}} d\mathbf{f}_1^{(6)} + \tau_{14} f_2^{(6)} - \chi_{\mathbf{132}} \mathbf{h}_1^{(3)} \mathbf{f}_1^{(4)} - \sigma_{\mathbf{15}} \mathbf{j}_2^{(6)} \iff \Phi_5 \left(d\mathbf{f}_1^{(6)} + \ell_1 f_2^{(6)} - \mathbf{j}_2^{(6)} \right) - (\Phi_2 \wedge \Phi_3) \mathbf{h}_1^{(3)} \mathbf{f}_1^{(4)} \tag{6.28}
$$

Recalling that $\tau_{14} = \ell_1 \sigma_{\mathbf{15}}$, it is easy to see that these coincide under the condition

$$
\Phi_2 \wedge \Phi_3 = \frac{\chi_{\mathbf{132}}}{\sigma_{\mathbf{15}}} \Phi_5 \quad . \tag{6.29}
$$

Doing the same for the others equations in (6.25) leads the following relations:

$$
\Phi_2 \wedge \Phi_2 = \frac{\chi_{\mathbf{222}}}{\sigma_{\mathbf{24}}} \Phi_4 \qquad \sigma_{\mathbf{15}} = \sigma_{\mathbf{24}} \qquad \sigma_{\mathbf{24}} = -\sigma_{\mathbf{33}} \tag{6.30}
$$

The later two in particular give new information on the couplings, allowing to further lower the number of parameters in our TFT.
The set of equations (6.25) does not provide a unique solution in terms of the current for all of the fluxes appearing in the action, implying that also this path is obstructed. In order to proceed with our computation we assume quadratic terms in $f, \mathbf{f}$ to be subleading. The reliability of this assumption is backed by the fact that the topological limit sets us close to the boundary of the 5-dimensional external space, i.e. at scales where the RR and NS fields become small since, as it is usually assumed in field theory,

$$
\lim_{z \longrightarrow \infty} f^{(i)}_{\mu_1 \dots \mu_p}(x, z) = 0 \quad . \tag{6.31}
$$

Truncating the equations at linear order bypasses the hindrance encountered in the previous subsection, meaning that we can straightforwardly reduce the Bianchi identities, at the cost of our final result becoming blind to higher order corrections to the fluxes. The full set of linearized Bianchi identities is summed up in the following equations:

$$\begin{cases} df_p^{(i)} = j_{p+1}^{(i)} \\ d\mathbf{f}_p^{(i)} = \mathbf{j}_{p+1}^{(i)} + (-1)^{p+1}\ell_{i-p} f_{p+1}^{(i)} \end{cases} \tag{6.32}$$

6.2.3 The SymTFT Action

The SymTFT is obtained applying Stokes theorem to the $d+2$ dimensionally reduced action, resulting in an integration domain on its boundary, that is in dimension $d+1$; to do so we need to express the action as a total derivative.
Before doing so one needs to express the TFT action in terms of the sources using (6.32). The first relation that we will use is $f_p \propto \mathbf{j}_p$, obtained from a combination the linearized Bianchi identities for the f components with closure of the currents, reading

$$df_p = j_{p+1} \propto d\mathbf{j}_p \tag{6.33}$$

up to a torsion coefficient. It is also possible to identify $\mathbf{f}_p = d^{-1}\mathbf{j}_{p+1}$. This latter identification is not as straightforward as it may seem. Observe that, stemming from an equation between exterior derivatives, $f_p \propto \mathbf{j}_p$ is completely blind to any closed component of $\mathbf{j}_p$ due to the derivation being nilpotent. Therefore if we expand the current in closed and exact components the correct result would be:

$$\mathbf{j}_p = \mathbf{j}_p^{(e)} + \mathbf{j}_p^{(c)} \tag{6.34}$$

$$f_p \propto \mathbf{j}_p^{(e)} \tag{6.35}$$

One can then consider the reduced Bianchi identities for $\mathbf{f}_p$, which equate $d\mathbf{f}$ with a combination of an external flux f and an external current, and get a relation of the form

$$d\mathbf{f}_p = \mathbf{j}_{p+1} - \ell f_{p+1} = \mathbf{j}_{p+1}^{(c)} \tag{6.36}$$

This relation is the reason why we restrained from applying the topological limit to the bold fluxes as the derivative is actually subleading only in a fine tuned setting where the current is almost purely exact. From (6.36) one can express the flux as

$$\mathbf{f}_p = d^{-1}\mathbf{j}_{p+1}^{(c)} \quad . \tag{6.37}$$

Although it is usually an ill-defined operation, in this context there is a formal way to define d^{-1}, based on the observations done in [16] and detailed in appendix B. In particular, the integration over torsional cycles of a δ-like source introduces an ambiguity in the flux, allowing to redefine it as long as its exterior derivative coincide the current.

Since we are interested in what types of brane interact in the anomaly theory, we will neglect the (c/e) superscripts when writing the SymTFT action.
Performing the substitutions (6.35) and (6.37) minimizing the number of parameters by means of the relations between the couplings found in previous sections, the action reads:

$$\begin{aligned} S_{d+2} = & \int_{M_5} \frac{\chi_{\mathbf{132}}}{\ell_1} \left[\mathbf{j}_2^{(3,e)} d^{-1}\mathbf{j}_2^{(4,c)} d^{-1}\mathbf{j}_3^{(4,c)} + d^{-1}\mathbf{j}_2^{(3,c)} d^{-1}\mathbf{j}_2^{(4,c)} \mathbf{j}_3^{(4,e)} \right] - \frac{\chi_{\mathbf{222}}}{\ell_2} d^{-1}\mathbf{j}_2^{(3,c)} \mathbf{j}_2^{(4,e)} d^{-1}\mathbf{j}_3^{(4,c)} + \\ & + \frac{\sigma_{\mathbf{24}}}{\ell_2} \left[2\mathbf{j}_2^{(4,e)}\mathbf{j}_3^{(6)} - \mathbf{j}_2^{(4,e)}\mathbf{j}_3^{(6,e)} - \mathbf{j}_2^{(4,e)}\mathbf{j}_3^{(6,c)} \right] - \frac{\sigma_{\mathbf{15}}}{\ell_1} \left[2\mathbf{j}_3^{(4,e)}\mathbf{j}_2^{(6)} - \mathbf{j}_3^{(4,e)}\mathbf{j}_2^{(6,e)} - \mathbf{j}_3^{(4,e)}\mathbf{j}_2^{(6,c)} + \right. \\ & \left. + 2\mathbf{j}_2^{(3,e)}\mathbf{j}_3^{(7)} - \mathbf{j}_2^{(3,e)}\mathbf{j}_3^{(7,e)} - \mathbf{j}_2^{(3,e)}\mathbf{j}_3^{(7,c)} \right] = \\ = & \int_{M_5} \frac{\chi_{\mathbf{132}}}{\ell_1} \left[\mathbf{j}_2^{(3,e)} d^{-1}\mathbf{j}_2^{(4,c)} d^{-1}\mathbf{j}_3^{(4,c)} + d^{-1}\mathbf{j}_2^{(3,c)} d^{-1}\mathbf{j}_2^{(4,c)} \mathbf{j}_3^{(4,e)} \right] - \frac{\chi_{\mathbf{222}}}{\ell_2} d^{-1}\mathbf{j}_2^{(3,c)} \mathbf{j}_2^{(4,e)} d^{-1}\mathbf{j}_3^{(4,c)} + \\ & + \frac{\sigma_{\mathbf{24}}}{\ell_2} \mathbf{j}_2^{(4,e)}\mathbf{j}_3^{(6)} - \frac{\sigma_{\mathbf{15}}}{\ell_1} \left[\mathbf{j}_3^{(4,e)}\mathbf{j}_2^{(6)} + \mathbf{j}_2^{(3,e)}\mathbf{j}_3^{(7)} \right] \end{aligned} \tag{6.38}$$

where we neglected any term containing a 'simple' current $j_a^{(i)}$ since they must be subleading based on the identities $df = j$. Notice that the cubic terms resemble the total derivative of $(d^{-1}\mathbf{j})^3$, except for the different coupling constants that precedes each term. Via a by-parts integration one can actually show that these constants actually coincide (neglecting L_6 boundary terms):

$$\begin{aligned} \frac{\chi_{\mathbf{132}}}{\ell_1} &= \frac{1}{\ell_1} \int_{L_6} \phi_1 \wedge \Phi_3 \wedge \Phi_2 = \frac{1}{\ell_1 \ell_2} \int_{L_6} \phi_1 \wedge d\phi_2 \wedge \Phi_2 = \\ &= -\frac{1}{\ell_1 \ell_2} \int_{\partial L_6} \phi_1 \wedge \phi_2 \wedge \Phi_2 + \frac{1}{\ell_2} \int_{L_6} \Phi_2 \wedge \phi_2 \wedge \phi_2 = \frac{\chi_{\mathbf{222}}}{\ell_2} \quad . \end{aligned} \tag{6.39}$$

The quadratic terms can instead be written as

$$\mathbf{j}_a^{(i)}\mathbf{j}_b^{(k)} = d\left(d^{-1}\mathbf{j}_a^{(i)}\mathbf{j}_b^{(k)}\right) - (-1)^a d^{-1}\mathbf{j}_a^{(i)} j_{b+1}^{(k)} \quad , \tag{6.40}$$

where the second term can be neglected since it involves a 'simple' current. All of the terms in the action are now written as exterior derivatives, allowing us to apply Stokes theorem. The final result is the following AdS_4 action:

$$S_{SymTFT} = \int_{AdS_4} \frac{\chi_{\mathbf{132}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\mathbf{j}_2^{(4)} d^{-1}\mathbf{j}_3^{(4)} - \frac{\sigma_{\mathbf{24}}}{\ell_2} d^{-1}\mathbf{j}_2^{(4)}\mathbf{j}_3^{(6)} - \frac{\sigma_{\mathbf{15}}}{\ell_1} \left(d^{-1}\mathbf{j}_3^{(4)}\mathbf{j}_2^{(6)} + d^{-1}\mathbf{j}_2^{(3)}\mathbf{j}_3^{(7)} \right) \tag{6.41}$$

This is the final (restricted) form of the Symmetry TFT action and we can use it to extract information on the generalized symmetries of the initial ABJ theory, as we will see in section 6.4. In particular we will see that the quadratic terms correspond to the BF term for the involved currents, while cubic ones are interpreted as anomalies.

6.3 Completing the Action

6.3.1 The $0, 5$-Sector

Now that the path to the SymTFT is clear, one can effortlessly include the neglected homology classes of the internal geometry. First of all, let us complete the torsional sector by including $0, 5$-cycles, meaning that the expansions will be integrated with the terms:

$$\begin{aligned} F^{(i)} &\supset f_i^{(i)} \wedge \phi_0 + \mathbf{f}_{i-1}^{(i)} \wedge \Phi_1 + f_{i-5}^{(i)} \wedge \phi_5 + \mathbf{f}_{i-6}^{(i)} \wedge \Phi_6 \\ J_{i+1}^{(i)} &\supset j_{i+1}^{(i)} \wedge \phi_0 + \mathbf{j}_i^{(i)} \wedge \Phi_1 + j_{i-4}^{(i)} \wedge \phi_5 + \mathbf{j}_{i-5}^{(i)} \wedge \Phi_6 \end{aligned} \tag{6.42}$$

Retracing the reasoning followed in previous sections, one can recognize the relevant couplings listed below:

$$\begin{gathered} \tau_{05}\ ,\ \tau_{50}\ , \sigma_{0\mathbf{6}}\ ,\ \sigma_{5\mathbf{1}} \\ \chi_{0\mathbf{24}}\ ,\ \chi_{0\mathbf{15}}\ , \chi_{\mathbf{1}\mathbf{1}4}\ ,\ \chi_{\mathbf{2}3\mathbf{1}}\ ,\ \chi_{3\mathbf{21}} \end{gathered} \tag{6.43}$$

Through Leibniz rule one can easily see that $\chi_{0\mathbf{15}}$ is actually vanishing:

$$\begin{aligned} \chi_{0\mathbf{15}} &= \int_{L_6} \phi_0 \wedge \Phi_1 \wedge \Phi_5 = \frac{1}{\ell_0} \int_{L_6} \phi_0 \wedge \Phi_1 \wedge d\phi_4 = \\ &= -\frac{1}{\ell_0} \int_{\partial L_6} \phi_0 \wedge \Phi_1 \wedge \phi_4 + \int_{L_6} \Phi_1 \wedge \Phi_1 \wedge \phi_4 = \chi_{\mathbf{11}4} = 0 \quad . \end{aligned} \tag{6.44}$$

The same holds for $\chi_{\mathbf{23}1}$, which is instead related to a $\Phi_3 \wedge \Phi_3 = 0$ form. Similar relations will later allow us to relate χ couplings and thus merge terms in the final action like we did for the cubic term of the restricted action. The $0, 5$-cycles ultimately introduce the following terms to the reduced action in 5 dimensions:

$$\begin{aligned} S_{d+2} \supset \int_{M_5} & \tau_{05} \left[-f_2^{(2)} f_3^{(8)} + h_3^{(3)} h_2^{(7)} \right] + \sigma_{0\mathbf{6}} \left[-f_2^{(2)} d\mathbf{f}_2^{(8)} + h_3^{(3)} d\mathbf{h}_1^{(7)} \right] - \\ & - \chi_{0\mathbf{24}} \mathbf{h}_1^{(3)} f_2^{(2)} \mathbf{f}_2^{(6)} - \chi_{3\mathbf{21}} \mathbf{h}_1^{(3)} \mathbf{f}_1^{(2)} f_3^{(6)} + \chi_{\mathbf{11}4} h_2^{(3)} \mathbf{f}_1^{(2)} \mathbf{f}_2^{(6)} + \\ & + \sigma_{0\mathbf{6}} \left[f_2^{(2)} \mathbf{j}_3^{(8)} - f_4^{(4)} \mathbf{j}_1^{(6)} - h_3^{(3)} \mathbf{j}_2^{(7)} \right] + \sigma_{5\mathbf{1}} \left[-f_3^{(8)} \mathbf{j}_2^{(2)} + h_2^{(7)} \mathbf{j}_3^{(3)} \right] \end{aligned} \tag{6.45}$$

By using the Bianchi identities one can then write the action in terms of the sources alone, as done in the previous section, and then move down to AdS_4:

$$\begin{aligned} S_{d+2} \supset \int_{M_5} & -\frac{\chi_{0\mathbf{24}}}{\ell_0} d^{-1} \mathbf{j}_2^{(3,c)} \mathbf{j}_2^{(2,e)} d^{-1} \mathbf{j}_3^{(6,c)} + \frac{\chi_{3\mathbf{21}}}{\ell_2} d^{-1} \mathbf{j}_2^{(3,c)} d^{-1} \mathbf{j}_2^{(2,c)} \mathbf{j}_3^{(6,e)} + \frac{\chi_{\mathbf{11}4}}{\ell_1} \mathbf{j}_2^{(3,e)} d^{-1} \mathbf{j}_2^{(2,c)} d^{-1} \mathbf{j}_3^{(6,c)} - \\ & - \frac{\sigma_{0\mathbf{6}}}{\ell_0} \mathbf{j}_4^{(4,e)} \mathbf{j}_1^{(6)} - \frac{\sigma_{5\mathbf{1}}}{\ell_0} \left[\mathbf{j}_3^{(8,e)} \mathbf{j}_2^{(2)} - \mathbf{j}_2^{(7,e)} \mathbf{j}_3^{(3)} \right] \end{aligned} \tag{6.46}$$

The couplings of the cubic terms can actually be shown to coincide through by parts integration, namely

$$\frac{\chi_{\mathbf{024}}}{\ell_0} = \frac{\chi_{\mathbf{114}}}{\ell_1} = \frac{\chi_{\mathbf{321}}}{\ell_2} \quad , \tag{6.47}$$

thus making the three terms add up to a total derivative. The quadratic terms can instead be rewritten using Leibniz rule, obtaining a total derivative and a vanishing term (proportional to a j). The $0,5$-cycles contribution to the SymTFT is then

$$S_{SymTFT} \supset -\frac{1}{\ell_0}\int_{AdS_4} \chi_{\mathbf{024}} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\mathbf{j}_2^{(2)} d^{-1}\mathbf{j}_3^{(6)} + \sigma_{\mathbf{06}} d^{-1}\mathbf{j}_4^{(4)}\mathbf{j}_1^{(6)} + \sigma_{\mathbf{51}}\left[d^{-1}\mathbf{j}_3^{(8)}\mathbf{j}_2^{(2)} - d^{-1}\mathbf{j}_2^{(7)}\mathbf{j}_3^{(3)}\right] \tag{6.48}$$

6.3.2 Torsionless Cycles and RR Backgrounds

We now proceed to include even the non-torsional cycles (coming from the geometry of CP^3) in the flux/current expansions, i.e the terms

$$F^{(i)} \supset F_B^{(i)} \sum_a \tilde{f}_{i-a}^{(i)} \wedge \psi_a$$

$$J_{i+1}^{(i)} \supset \sum_a \tilde{j}_{i+1-a}^{(i)} \wedge \psi_a$$

As anticipated, the presence of torsion-free cycles is signaled by tilded external components and indices (e.g. $\chi_{\mathbf{02}\tilde{\mathbf{4}}} = \int \phi_0 \Phi_2 \psi_4$).
Based on the features of the underlying supergravity observed in section 4.2.2, the background contributions will be non-vanishing only for the RR fluxes F_2 and F_4 and, being proportional to volume forms of cycles of the same degree as the flux they can be included in the sum by writing them as

$$F_B^{(2,4)} = \tilde{f}_0^{(2,4)} \wedge \psi_{2,4} \quad . \tag{6.49}$$

The appearance of non-zero $\tilde{f}_0^{(i)}$ terms does not contradict our previous statement because these are not gauge-invariant components of the flux but rather integer numbers, as one can deduce from (4.12). For $i \neq 2,4$ the 0-degree component is still set to be null. Since they are coefficients, the background components will not be expressed in terms of the sources in the final action, so that the interpretation of the terms they appear in is much clearer.
Let us now examine the couplings that the torsionless sector contains. The basic definitions for the couplings are the same as for the torsion-sector, e.g.

$$\sigma_{\tilde{a}b} = \int_{L_6} \psi_a \wedge \phi_b \quad \text{and} \quad \chi_{\tilde{a}\mathbf{bc}} = \int_{L_6} \psi_a \wedge \Phi_b \wedge \phi_c \quad . \tag{6.50}$$

Observe that any quadratic coupling related to torsionless cycles can be shown to be vanishing or intrinsically associated to vanishing external components: $\sigma_{\tilde{a}\mathbf{b}}$-type couplings (and by extension, τ-like couplings) are boundary terms in L_6 due to $d\psi_a = 0$. The other possible types of quadratic coupling, $\sigma_{\tilde{a}b}$ and $\sigma_{\tilde{a}\tilde{b}}$, are not vanishing a priori, yet the terms they are associated with always contain either a free j or $\tilde{j}$: both of these quantities are killed by the topological limit because Bianchi identities set $d(f, \tilde{f}) \sim (j, \tilde{j})$. It follows that the cubic terms will provide both the anomalies and BF terms, the latter corresponding to terms picking up a background component.
Following the same reasoning used in the previous sections one can deduce that $\chi_{\mathbf{a}\tilde{b}\tilde{c}} = \chi_{\mathbf{ab}\tilde{c}} = 0$ due to being total derivatives and $\chi_{ab\tilde{c}} = 0$ due to including multiple torsional forms. This restricts the pool of relevant cubic couplings to the following three categories:

$$\chi_{a\tilde{b}\tilde{c}} \quad , \quad \chi_{a\mathbf{b}\tilde{c}} \quad , \quad \chi_{\tilde{a}\tilde{b}\tilde{c}} \quad . \tag{6.51}$$

In term of the fluxes, the torsionless contribution to the reduced action reads:

$$\begin{aligned} S_{d+2} \supset \int_{M_5} & \chi_{1\tilde{2}\mathbf{3}} h_2^{(3)} \left[\tilde{f}_0^{(2)} \mathbf{f}_3^{(6)} + \tilde{f}_2^{(4)} \mathbf{f}_1^{(4)} \right] - \chi_{2\tilde{2}\mathbf{2}} \mathbf{h}_1^{(3)} \left[\tilde{f}_0^{(2)} f_3^{(6)} + \tilde{f}_2^{(4)} f_2^{(4)} \right] - \chi_{0\tilde{4}\mathbf{2}} \left[\mathbf{h}_1^{(3)} f_2^{(2)} \tilde{f}_2^{(6)} + \right. \\ & \left. + \mathbf{h}_1^{(3)} f_4^{(4)} \tilde{f}_0^{(4)} + h_3^{(3)} \mathbf{f}_2^{(4)} \tilde{f}_0^{(4)} \right] + \chi_{1\tilde{4}\mathbf{1}} \left[h_2^{(3)} \mathbf{f}_1^{(2)} \tilde{f}_2^{(6)} + h_2^{(3)} \mathbf{f}_3^{(4)} \tilde{f}_0^{(4)} + \mathbf{h}_2^{(3)} f_3^{(4)} \tilde{f}_0^{(4)} \right] + \\ & + \chi_{1\tilde{0}\mathbf{5}} h_2^{(3)} \tilde{f}_2^{(2)} \mathbf{f}_1^{(6)} - \chi_{4\tilde{0}\mathbf{2}} \mathbf{h}_1^{(3)} \tilde{f}_2^{(2)} f_2^{(6)} - \chi_{0\tilde{2}\mathbf{4}} h_3^{(3)} \tilde{f}_0^{(2)} \mathbf{f}_2^{(6)} - \chi_{3\tilde{2}\mathbf{1}} \mathbf{h}_2^{(3)} \tilde{f}_0^{(2)} f_4^{(6)} - \\ & - \chi_{0\tilde{2}\tilde{4}} h_3^{(3)} \left[\tilde{f}_0^{(2)} \tilde{f}_2^{(6)} + \tilde{f}_0^{(4)} \tilde{f}_2^{(4)} \right] - \chi_{2\tilde{0}\tilde{4}} \tilde{h}_3^{(3)} f_2^{(4)} \tilde{f}_0^{(4)} - \chi_{4\tilde{0}\tilde{2}} \tilde{h}_3^{(3)} f_0^{(2)} \tilde{f}_2^{(6)} - \\ & - \chi_{\tilde{0}\tilde{2}\tilde{4}} \tilde{h}_3^{(3)} \left[\tilde{f}_0^{(2)} \tilde{f}_2^{(6)} + \tilde{f}_2^{(4)} \tilde{f}_0^{(4)} \right] \quad , \end{aligned} \tag{6.52}$$

which is then rewritten in terms of the currents using linearized Bianchi identities, resulting in

$$\begin{aligned} S_{d+2} \supset \int_{M_5} & \frac{\chi_{1\tilde{2}\mathbf{3}}}{\ell_1} \mathbf{j}_2^{(3)} \left[\tilde{f}_0^{(2)} d^{-1} \mathbf{j}_4^{(6)} + d^{-1} \tilde{j}_3^{(4)} d^{-1} \mathbf{j}_2^{(4)} \right] - \frac{\chi_{2\tilde{2}\mathbf{2}}}{\ell_2} d^{-1} \mathbf{j}_2^{(3)} \left[\tilde{f}_0^{(2)} \mathbf{j}_4^{(6)} + d^{-1} \tilde{j}_3^{(4)} \mathbf{j}_2^{(4)} \right] - \\ & - \frac{\chi_{0\tilde{4}\mathbf{2}}}{\ell_0} d^{-1} \mathbf{j}_2^{(3)} \left[\mathbf{j}_2^{(2)} d^{-1} \tilde{j}_3^{(6)} + \mathbf{j}_4^{(4)} \tilde{f}_0^{(4)} \right] + \frac{\chi_{0\tilde{4}\mathbf{2}}}{\ell_0} \mathbf{j}_3^{(3)} d^{-1} \mathbf{j}_3^{(4)} \tilde{f}_0^{(4)} + \\ & + \frac{\chi_{1\tilde{4}\mathbf{1}}}{\ell_1} \mathbf{j}_2^{(3)} \left[d^{-1} \mathbf{j}_2^{(2)} d^{-1} \tilde{j}_3^{(6)} + d^{-1} \mathbf{j}_4^{(4)} \tilde{f}_0^{(4)} \right] - \frac{\chi_{1\tilde{4}\mathbf{1}}}{\ell_1} d^{-1} \mathbf{j}_3^{(3)} \mathbf{j}_3^{(4)} \tilde{f}_0^{(4)} + \\ & + \frac{\chi_{1\tilde{0}\mathbf{5}}}{\ell_1} \mathbf{j}_2^{(3)} d^{-1} \tilde{j}_3^{(2)} d^{-1} \mathbf{j}_2^{(6)} - \frac{\chi_{4\tilde{0}\mathbf{2}}}{\ell_1} d^{-1} \mathbf{j}_2^{(3)} d^{-1} \tilde{j}_3^{(2)} \mathbf{j}_2^{(6)} + \frac{\chi_{0\tilde{2}\mathbf{4}}}{\ell_1} \mathbf{j}_3^{(3)} \tilde{f}_0^{(2)} d^{-1} \mathbf{j}_3^{(6)} + \\ & + \frac{\chi_{3\tilde{2}\mathbf{1}}}{\ell_2} d^{-1} \mathbf{j}_3^{(3)} \tilde{f}_0^{(2)} \mathbf{j}_3^{(6)} + \frac{\chi_{0\tilde{2}\tilde{4}}}{\ell_0} \mathbf{j}_3^{(3)} \left[\tilde{f}_0^{(2)} d^{-1} \tilde{j}_3^{(6)} + d^{-1} \tilde{j}_3^{(4)} \tilde{f}_0^{(4)} \right] - \\ & - \frac{\chi_{2\tilde{0}\tilde{4}}}{\ell_2} d^{-1} \tilde{j}_4^{(3)} \mathbf{j}_2^{(4)} \tilde{f}_0^{(4)} - \frac{\chi_{4\tilde{0}\tilde{2}}}{\ell_1} d^{-1} \tilde{j}_4^{(3)} \tilde{f}_0^{(2)} \mathbf{j}_2^{(6)} - \\ & - \chi_{\tilde{0}\tilde{2}\tilde{4}} d^{-1} \tilde{j}_4^{(3)} \left[\tilde{f}_0^{(2)} d^{-1} \tilde{j}_3^{(6)} + d^{-1} \tilde{j}_3^{(4)} \tilde{f}_0^{(4)} \right] \quad . \end{aligned} \tag{6.53}$$

Terms related to $\chi_{a\mathbf{b}\tilde{c}}$ couplings can be paired and recognized as total derivatives of $\sim d^{-1}\tilde{j}(d^{-1}\mathbf{j})^2$, seeing as $d(d^{-1}\tilde{j})$ is vanishing. The matching between the couplings ahead of each term is again checked via by parts integration, leading to

$$\frac{\chi_{1\tilde{2}\mathbf{3}}}{\ell_1} = \frac{\chi_{2\tilde{2}\mathbf{2}}}{\ell_2} \qquad \frac{\chi_{0\tilde{4}\mathbf{2}}}{\ell_0} = \frac{\chi_{1\tilde{4}\mathbf{1}}}{\ell_1} \qquad \frac{\chi_{3\tilde{2}\mathbf{1}}}{\ell_2} = \frac{\chi_{0\tilde{2}\mathbf{4}}}{\ell_0} \qquad \chi_{1\tilde{0}\mathbf{5}} = \chi_{4\tilde{0}\mathbf{2}} \quad . \tag{6.54}$$

Similarly, since they contain two $d^{-1}\tilde{j}$, each $\chi_{a\tilde{b}\tilde{c}}$ term can be seen as a total derivative on its own.

As we mentioned before, any term containing a background component, that is either $\tilde{f}_0^{(2)}$ or $\tilde{f}_0^{(4)}$, will act as a BF term with the background becoming part of the coupling constant. There are many of these terms and they are not particularly interesting, hence in the final action they will be gathered under the symbol $BF[\{\tilde{j}_a^{(i)}\}]$.

The "torsion-free" sector of the SymTFT action is then

$$\begin{aligned} S_{SymTFT} \supset \int_{AdS_4} & \frac{\chi_{1\tilde{2}\mathbf{3}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)} \left[d^{-1}\tilde{j}_3^{(4)} d^{-1}\mathbf{j}_2^{(4)} + \tilde{f}_0^{(2)} d^{-1}\mathbf{j}_4^{(6)} \right] + \frac{\chi_{0\tilde{4}\mathbf{2}}}{\ell_0} d^{-1}\mathbf{j}_2^{(3)} \left[d^{-1}\mathbf{j}_2^{(2)} d^{-1}\tilde{j}_3^{(6)} + \right. \\ & \left. + d^{-1}\mathbf{j}_4^{(4)} \tilde{f}_0^{(4)} \right] + \frac{\chi_{0\tilde{4}\mathbf{2}}}{\ell_0} d^{-1}\mathbf{j}_3^{(3)} d^{-1}\mathbf{j}_3^{(4)} \tilde{f}_0^{(4)} + \frac{\chi_{0\tilde{2}\mathbf{4}}}{\ell_0} d^{-1}\mathbf{j}_3^{(3)} d^{-1}\mathbf{j}_3^{(6)} \tilde{f}_0^{(2)} + \\ & + \frac{\chi_{1\tilde{0}\mathbf{5}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\tilde{j}_3^{(2)} d^{-1}\mathbf{j}_2^{(6)} + \frac{\chi_{0\tilde{2}\tilde{4}}}{\ell_0} d^{-1}\mathbf{j}_3^{(3)} \left[\tilde{f}_0^{(2)} d^{-1}\tilde{j}_3^{(6)} + d^{-1}\tilde{j}_3^{(4)} \tilde{f}_0^{(4)} \right] + \\ & + \frac{\chi_{2\tilde{0}\tilde{4}}}{\ell_2} d^{-1}\tilde{j}_4^{(3)} d^{-1}\mathbf{j}_2^{(4)} \tilde{f}_0^{(4)} + \frac{\chi_{4\tilde{0}\tilde{2}}}{\ell_1} d^{-1}\tilde{j}_4^{(3)} \tilde{f}_0^{(2)} d^{-1}\mathbf{j}_2^{(6)} + \\ & + \int_{M_5} \chi_{\tilde{0}\tilde{2}\tilde{4}} d^{-1}\tilde{j}_4^{(3)} \left[\tilde{f}_0^{(2)} d^{-1}\tilde{j}_3^{(6)} + d^{-1}\tilde{j}_3^{(4)} \tilde{f}_0^{(4)} \right] \quad . \end{aligned} \tag{6.55}$$

Although we found no reason for it to be vanishing, the last line of this action is completely decoupled from the torsional sector, therefore it should be irrelevant to the scope of the SymTFT.

6.4 Defects in ABJ from the SymTFT

Now that all of the contributions have been included one can put all the pieces together and write the full expression of the symmtery topological field theory action, i.e.

$$\begin{aligned} S_{SymTFT} = \int_{AdS_4} & \frac{\chi_{1\mathbf{32}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\mathbf{j}_2^{(4)} d^{-1}\mathbf{j}_3^{(4)} - \frac{\sigma_{2\mathbf{4}}}{\ell_2} d^{-1}\mathbf{j}_2^{(4)} \mathbf{j}_3^{(6)} - \frac{\sigma_{1\mathbf{5}}}{\ell_1} \left(d^{-1}\mathbf{j}_3^{(4)} \mathbf{j}_2^{(6)} + d^{-1}\mathbf{j}_2^{(3)} \mathbf{j}_3^{(7)} \right) - \\ & - \frac{1}{\ell_0} \left\{ \chi_{0\mathbf{24}} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\mathbf{j}_2^{(2)} d^{-1}\mathbf{j}_3^{(6)} + \sigma_{0\mathbf{6}} d^{-1}\mathbf{j}_4^{(4)} \mathbf{j}_1^{(6)} + \sigma_{5\mathbf{1}} \left[d^{-1}\mathbf{j}_3^{(8)} \mathbf{j}_2^{(2)} - d^{-1}\mathbf{j}_2^{(7)} \mathbf{j}_3^{(3)} \right] \right\} + \\ & + \frac{\chi_{1\tilde{2}\mathbf{3}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\tilde{j}_3^{(4)} d^{-1}\mathbf{j}_2^{(4)} + \frac{\chi_{0\tilde{4}\mathbf{2}}}{\ell_0} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\mathbf{j}_2^{(2)} d^{-1}\tilde{j}_3^{(6)} + \\ & + \frac{\chi_{1\tilde{0}\mathbf{5}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\tilde{j}_3^{(2)} d^{-1}\mathbf{j}_2^{(6)} + BF[\{\tilde{j}_a^{(i)}\}] \quad . \end{aligned} \tag{6.56}$$

This action is quite cumbersome, yet one can observe that all of the cubic anomalous terms involve the same types of objects: two 1-forms and a 2-form. It follows that they will all be associated to the same type of anomaly and therefore the same symmetry structure in the ABJ theory, although the defects would be built with respect to different fields. The upshot of this observation is that it is possible to examine the global generalized symmetries of the underlying $3d$ theory restricting the to a single anomalous term without losing information. In particular we elect to analyze the following sector of S_{SymTFT}:

$$-\int_{AdS_4} \frac{\chi_{\mathbf{024}}}{\ell_0} d^{-1}\mathbf{j}_2^{(3)} d^{-1}\mathbf{j}_2^{(2)} d^{-1}\mathbf{j}_3^{(6)} + \frac{\sigma_{\mathbf{51}}}{\ell_0} d^{-1}\mathbf{j}_3^{(8)}\mathbf{j}_2^{(2)} + \frac{\sigma_{\mathbf{24}}}{\ell_2} d^{-1}\mathbf{j}_2^{(4)}\mathbf{j}_3^{(6)} + \frac{\sigma_{\mathbf{15}}}{\ell_1} d^{-1}\mathbf{j}_2^{(3)}\mathbf{j}_3^{(7)} \quad (6.57)$$

Let us start from the quadratic couplings

$$S_{SymTFT} \supset \int_{AdS_4} d^{-1}\mathbf{j}_a^{(i)} \wedge \mathbf{j}_b^{(k)} \quad . \quad (6.58)$$

For magnetic sources $J^{(i,k)} = \delta(\mathcal{W}^{(i,k)})$ an expression of this form can be interpreted as the linking number between the support of the two extended objects. In fact the primitive $d^{-1}\mathbf{j}$ of a δ-like source is nothing but a step function supported on the region bounded by the brane, therefore the integration will count the intersection between a brane and the bulk delimited by the other, i.e. the linking number $Link(\mathcal{W}^{(i)}, \mathcal{W}^{(k)})$. Hence the appearance of terms of the form (6.58) in the action signals the linking between an a-brane and a b-brane in AdS_4.
The discrete nature of the linking number suggests that the symmetry connected to the brane should be discrete too. This intuition is confirmed by writing the quadratic term as

$$\int_{AdS_4} d^{-1}\mathbf{j}_a^{(i)} \wedge d(d^{-1}\mathbf{j}_b^{(k)}) \quad , \quad (6.59)$$

which is easily recognized as a $(b-1)$-form BF action as presented in section 2.1.2. The appearance of these terms tells us that we are dealing with higher form discrete symmetries of degrees $(a-2)$ and $(b-2)$ in the 4-dimensional theory, that will then need to be projected onto $\mathcal{B}^{sym}$. Notice that not all of these symmetries will carry over to the ABJ theory since the inflow paradigm only creates a connection between the anomalous terms of the two theories.
The anomalous cubic term, i.e.

$$S_{SymTFT} \supset \int_{AdS_4} d^{-1}\mathbf{j}_a^{(i)} \wedge d^{-1}\mathbf{j}_b^{(k)} \wedge d^{-1}\mathbf{j}_c^{(m)} \quad , \quad (6.60)$$

has a similar interpretation as the intersection between a brane and the junction between the other two, meaning that the brane-junction is charged under the flux associated to the first brane. The charge will correspond with the discrete parameter $\chi_{\mathbf{abc}}/\ell_a$ attached to the anomaly.

The anomaly in the SymTFT contains three objects of form degrees 1, 1 and 2 respectively: according to previous observation on the quadratic terms and to the characterization given in section 2.1.1, these will be the currents associated to two discrete 0-form symmetries and a 1-form discrete symmetry, resulting in the anomalous global symmetry group

$$\mathbb{Z}_{\ell_i}^{(0)} \times \mathbb{Z}_{\ell_j}^{(0)} \times \mathbb{Z}_{\ell_k}^{(1)} \quad , \tag{6.61}$$

with $\ell_{i,j,k} = 2$ being the torsional degrees appearing in the BF term of each field appearing in the anomaly.

As stated many times before, in order to see how these "inflow" to the ortho-symplectic theory one needs to select a specific gapped boundary condition $\mathcal{B}^{sym}_{\mathcal{S}}$, which select the symmetry structure $\mathcal{S}$. How to select the correct boundary conditions to achieve a specific symmetry structure is beyond the scope of our research, however we will sketch the effect this choice has on the final SDOs. Let us consider, for instance, the case in which all of the brane currents projects maintaining their freedom, i.e. the case where all of the $d^{-1}\mathbf{j}^{(i)}_a$ fields are subject to Neumann boundary conditions on $\mathcal{B}^{sym}_{\mathcal{S}}$. $d^{-1}\tilde{j}$ can then be used to construct defect operators acting in three dimensions, namely

$$\begin{aligned} U^{(2)}_\alpha(\Sigma_2) &= e^{2\pi i\alpha \int *d^{-1}\mathbf{j}^{(2)}_2} \\ U^{(3)}_\alpha(\Sigma_2') &= e^{2\pi i\alpha \int *d^{-1}\mathbf{j}^{(3)}_2} \\ U^{(6)}_\alpha(\Sigma_1) &= e^{2\pi i\alpha \int *d^{-1}\mathbf{j}^{(6)}_3} \end{aligned} \quad , \tag{6.62}$$

with the supports Σ_2, Σ_2' and Σ_1 coinciding with the projection of the brane living in AdS_4. These SDOs signal the presence of two 0-form and a 1-form discrete symmetry in the superconformal theory.

This result is in line with the results of [5] for the $SO(2N)_{2k} \times USp(2N)_{-k}$ variant of the ortho-symplectic ABJ theory: in that paper the theory was associated with a 1-form discrete symmetry $\mathbb{Z}^1_2$ and two 0-form $\mathbb{Z}_2$ symmetries associated with charge and magnetic conjugation.

If one were to choose a different $\mathcal{B}^{sym}_{\mathcal{S}}$, one or more fields would be subject to Dirichlet boundary conditions, thus "freezing" into a charged object in the projection, breaking the parts of the global group (6.61) connected to such branes. This is also in agreement with what was found in [5] seeing as any ortho-symplectic ABJ variant considered in that paper has global symmetries contained in (6.61).

7 Conclusions and Outlooks

Our research successfully lead to the construction of a theory, the SymTFT, that encodes the symmetry structure of the original ABJ theory, revealing the presence of three distinct discrete p-form symmetries (two with $p = 0$ one with $p = 1$) in the 3-dimensional QFT. The symmetries we detected are compatible with the results obtained by Mekareeya and Sacchi in [5] an it would be interesting to investigate the choice of gapped boundaries $\mathcal{B}^{sym}$ that would lead to $3d$-ABJ theories with symmetry structures coinciding with those of the different variants considered by the authors.
Beside revealing the symmetries, the method employed in our research and developed in [10][2] provides the means to also reconstruct the topological defect operators associated with them by projecting the SDOs of the SymTFT on the symmetry boundary.
These insights came at the cost of some challenges, one of which encountered when integrating out the fluxes from the $(d+2)$-dimensional action: both routes aimed at expressing the external flux components in terms of the currents were subject to some obstruction. However, a deeper understanding of the topology of the internal manifold $CP^3/\mathbb{Z}_2$ would fix the “unknown” parameters appearing in the reduced Bianchi identities, allowing to obtain a full set of exact equations of motion for the fluxes. It would be interesting to investigate if and how the $\mathcal{O}(f^2, \mathbf{f}^2)$ terms appearing in these equations affect the symmetry theory and thus the results of our analysis.

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A $CP^3/\mathbb{Z}_2$ Topology

This chapter is dedicated to reviewing the reasoning that lead to the identification of the cycles that were included in the expansion of the fluxes and currents used in the compactification procedure. This discussion is aimed at merely revealing the presence of cycles in $H_p(CP^3/\mathbb{Z}_2, \mathbb{Z})$ rather than achieving detailed information on their properties.

A.1 Torsional Cycles

To identify the torsional cycles of the orbifold that makes up the internal geometry of our supergravity background, let us start with an observation on the "unfolded" manifold CP^3, which is defined as the space of lines in $\mathbb{C}^4$ that pass through the origin and can be expressed as

$$CP^3 = S^7/U(1) \quad . \tag{1.1}$$

It can be shown that by itself the complex projective space CP^3 is torsion-free, meaning that any torsional cycle in $CP^3/\mathbb{Z}_2$ will be wrapped around the single "hole" produced by the $\mathbb{Z}_2$ orbifold of the space. As a consequence there can at most be one p-dimensional independent torsional cycle, modulo multiple windings, meaning that

$$Tor_p(CP^3/\mathbb{Z}_2, \mathbb{Z}) \subset \mathbb{Z}_2 \quad , \text{with } p \in [0, 5] \quad .$$

Therefore from here on out the internal forms ϕ will be identified by their respective form-degree.

We now need to identify which cycles are actually present in the internal space. To do so we will make use of a property known as Poincaré duality:

Theorem 2 (Poincaré Duality for Torsional Classes) *In a d-dimensional closed manifold M_d the torsional subgroups are related by*

$$Tor_p(M_d, \mathbb{Z}) \simeq Tor_{n-p-1}(M_d, \mathbb{Z}) \quad , \tag{1.2}$$

which establishes a duality relation between cycles. Dual cycles also share the same torsional degree ℓ.

As stated in [4], a torsional cycle belonging to the homology class $H_3(CP^3/\mathbb{Z}_2, \mathbb{Z})$ can be obtained by S^1-fibration of a $CP^1/\mathbb{Z}_2$ 2-cycle. One can then deduce the existence of torsional 2-cycles by means of Poincaré duality, which will share the same torsional degree as its dual ($\ell_2 = \ell_3 = 2$).

Existence of a torsional 1-cycle can be confirmed by observing that $CP^1/\mathbb{Z}_2 \subset CP^3/\mathbb{Z}_2$ actually coincides with RP^2, which is known to have a twisted geometry (often represented as a Boy surface); such a twisted surface allows to trace a non-orientable "ring" displaying torsional behaviour [20]. Using once again Poincaré duality, one deduces that there exist also torsional 4-cycles in $CP^3/\mathbb{Z}_2$, with $\ell_4 = \ell_1 = 2$.
Though it might seem counter-intuitive, the presence of points with non-trivial torsion, that is torsional 0-cycles, cannot be ruled out a priori. As a matter of fact, their appearance in $CP^3/\mathbb{Z}_2$ can be deduced from that of torsional 5-cycles. The latter existence is granted by the following corollary of the universal coefficient theorem:

Corollary 1 *For a closed connected d-manifold the torsional subgroup of $H_{d-1}(M_d, \mathbb{Z})$ is empty if the space is orientable. Otherwise, in the case of a non-orientable manifold, $Tor_{d-1}(M_d, \mathbb{Z}) = \mathbb{Z}_2$ [19].*

In the case of $CP^3/\mathbb{Z}_2$, i.e. a non-orientable 6-manifold, this means that $H_5(CP^3/\mathbb{Z}_2, \mathbb{Z})$ has torsional subgroup $Tor_5(CP^3/\mathbb{Z}_2) = \mathbb{Z}_2$. By Poincaré duality one can then confirm the presence of 0-cylces too, with $\ell_0 = \ell_5 = 2$.
These results mean that a complete expansion of the fluxes requires the inclusion of all the terms listed below:

- $p = 0$ torsional cycles associated with the couple (ϕ_0, Φ_1)
- $p = 1$ torsional cycles associated with the couple (ϕ_1, Φ_2)
- $p = 2$ torsional cycles associated with the couple (ϕ_2, Φ_3)
- $p = 3$ torsional cycles associated with the couple (ϕ_3, Φ_4)
- $p = 4$ torsional cycles associated with the couple (ϕ_4, Φ_5)
- $p = 5$ torsional cycles associated with the couple (ϕ_5, Φ_6)

A.2 Closed Cycles

A full expansion on the cycles of the internal space also needs to include cycles with trivial torsion (independent from the Φ forms belonging to torsional couples). According to CP^3 geometry, these cycles are associated with even-dimensional closed forms:

$$d\psi_{0,2,4,6} = 0 \tag{1.3}$$

The 6- and 0-forms are of clear interpretation, seeing as they are the volume form of the whole space, canonically normalized so that it trivially integrates to 1 over the full space, and a δ-like form supported on a point.

The remaining forms, ψ_4 and ψ_2, correspond to the 4-dimensional CP^2 and to the 2-dimensional $CP^1/\mathbb{Z}_2$ cycles respectively.
These same cycles are used to specify the background fluxes $F_B^{(i)}$ for the Ramond-Ramond fluxes. As mentioned before in fact, these are proportional to the volume form of an i-cycle in the internal geometry.

B On the Inverse of Exterior Derivatives

This appendix is dedicated to clarifying the meaning of the symbol d^{-1} used in the SymTFT action. This discussion is based off of the observations made on the subject by the authors of [16].
The inverse d^{-1} of the exterior derivative is a generally ill-defined operation, however in the case of our analysis, its appearance is related to the equation $d\mathbf{f} = \mathbf{j}$, which is the result of the integration over torsional cycles of $dF_p = J_{p+1}$, with the magnetic current in the right-hand-side being a δ-like function. This allows us to define d^{-1} through a cohomology argument.

Theorem 3 (Universal Coefficient) *Given a manifold M_d, the torsion cohomology group of degree $(p+1)$ is isomorphic to the space of (discrete) homomorphisms of the group of torsional p-cycles:*

$$Tor^{p+1}(M_d, \mathbb{Z}) \simeq Hom(Tor_p(M_d, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \quad . \tag{2.1}$$

An $(p+1)$-form in $Tor^{p+1}(M_d, \mathbb{Z})$ can then be understood as a map between p-cycles and phases, with each element of the same homology class corresponding to the same phase:

$$\pi_{n-1} \to e^{2\pi i \eta(\pi_p)} \tag{2.2}$$

One can then associate an $(p+1)$-form δ_{p+1} with an element in the torsional cohomology by choosing a p-form such that $dF_p = \delta_{p+1}$ and defining the phase corresponding to the p-cycle as

$$\eta(\phi_p) = \int_{\phi_p} F_p = \int_{\phi'_p} F_p + \int_{\Omega_{p+1}} \delta_{p+1} \tag{2.3}$$

where Ω_{p+1} is a chain connecting the two cycles, which belong in the same torsion class. Since the phase is independent from the cycle, this expression isn't affected by a redefinition of F_p, allowing us to identify it with a flux and write it as $F_p = d^{-1}\delta_{p+1} = d^{-1}J_{p+1}$. We can then extend this notation to the reduced relation, thus giving (6.37).