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Large-scales CMB anisotropies in Silent Universes

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Contents

Abstract

In this thesis we explore a new approach to General Relativity introduced by G. Ellis in 1971 [42], that defined new dynamical equations (called relativistic hydrodinamical equations) based on the properties of the electric and magnetic parts of the traceless part of the Riemann tensor, the Weyl tensor. Matarrese et al. in the mid-1990s [21], [22], [23], [17] found new cosmological models, called *silent universe*, for the study of irrotational dust using the relativistic hydrodinamical equations, in which the magnetic part of the Weyl tensor vanishes. The small temperature anisotropies measured in the Cosmic Microwave Background (CMB) of the order of 10^{-5} allow us to use the theory of cosmological perturbations. In particular we perturbed a Szekeres metric, in the form introduced by Goode and Wainwright in 1989, around a Friedmann-Lemaître-Robertson-Walker (FLRW) solution. This is a special silent universe that has an unique anisotropy along the z-axis. We solve the linearized Einstein Field Equations (EFE) in order to find the behaviour of the out-of-homogeneity potential and we study the phase plane analysis of this model using Ellis formalism. We study the geodesics equation for a photon path emitted on last scattering surface towards us. With the second-order solutions we compute the first and second-order CMB temperature anisotropies for the Szekeres metric. In the end we want to make explicit the expression of the first-order and second-order temperature deviation for a fully general silent metric, i.e. that contains the maximum degree of spatial anisotropy. In both cases we recover the integrated Sachs-Wolfe effect and a form that we interpret as the second-order correction of the integrated Sachs-Wolfe effect, both dependent on the direction of observation.

1 Introduction

Physical Cosmology is a branch of physics different from other fields because it has some intrinsic restrictions. First of all experimenters cannot reproduce a theorized phenomenon in a lab, in this sense they are subject to restrictions imposed by the astronomical methods that "allow one to look but not to touch" (Peebles 1993). Thus we can say that the sky is a sort of "museum lab", a place where experimenters are not active in the observational process, as in high-energy particle physics, but they must await the event, and what makes it observable is completely independent of the observer. Thus the keyword is "waiting", be ready with the best possible instrumentation to observe new events.
Nowadays, a great part of observational cosmology is the electromagnetic information of the Cosmic Microwave Background Radiation (CMBR), type Ia supernovae and high redshift galaxies. For this reason it is essential to study and to analyze the propagation of photons in a universe that is not static and perfectly homogeneous, but that it is subjected to expansion, shear and vorticity, following the rules general relativity. Only in this way we can interpret the electromagnetic data and come to conclusions about observations.

1.1 The standard model of Cosmology

We began our discussion of the standard model of cosmology focusing on the observed universe. Over enough large scales we observe the universe to be isotropic, on these scales there is no a preferential direction. Thus, we can make two hypotheses at large scales: we are in a local place in which the universe looks spherically symmetric but it is spatially inhomogeneous, or what we observe can be assumed as global, so the universe is homogeneous, i.e. it is isotropic for every observer. In this thesis we assume the second hypothesis, see [26] and [27].
Friedmann-Lemaître-Robertson-Walker (FLRW) are cosmological models in which the isotropy about the fundamental velocity u^μ is valid for every observer, i.e. the acceleration of the fluid flow is zero everywhere:

$$\dot{u}^\mu = 0. \tag{1.1}$$

In this way we can consider another symmetry. Because of these models are isotropic for every spacetime event, we can deduce that there is the invariance of physical properties in every space-like hypersurface ($t = const.$) orthogonal to the cosmological fluid element, i.e. the spatial homogeneity. These considerations implies that the cosmological tensors that describe the shear and the vorticity of the fluid flow are identically zero in every point, respectively:

$$\sigma_{\mu\nu} = 0, \qquad \omega_{\mu\nu} = 0. \tag{1.2}$$

(For the details about the meaning of kinematical quantities and General Relativistic dynamics see Section 2).
From isotropic and homogeneity assumption necessarily follows that the matter in this model must have the behaviour of a perfect fluid, so the energy flux must be zero

$$q_\mu = \mathcal{P}^\rho{}_\mu T_{\rho\nu} u^\nu = 0, \tag{1.3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and $\mathtt{h}_{\mu\nu}$ is the projection tensor that projects into the three-dimensional tangent space orthogonal to the fundamental velocity of the fluid flow. Furthermore the traceless anisotropic part of the energy-momentum tensor must be identically zero, because it contains all the information about viscosity and free-streaming,

$$\pi_{\mu\nu} = 0\,. \tag{1.4}$$

So in a FLRW universe these conditions hold everywhere.
We can define the scale factor $a(t)$ using the scalar expansion $\Theta(t)$ through the relation

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta\,. \tag{1.5}$$

The geometrical structure of FLRW can be studied using comoving coordinates $x^\mu = (t, x^i)$ we can define the line element

$$ds^2 = -dt^2 + a^2(t) f_{ij}(x^k) dx^i dx^j\,. \tag{1.6}$$

It is important to note that the metric function $f_{ij}(x^k)$ is time-independent, this fact derive directly from the constraint $\sigma_{\mu\nu} = 0$. Moreover the fundamental velocity of the cosmological fluid can be written as $u^\mu = \delta^\mu_0$.
The homogeneous hypersurfaces orthogonal to the fluid flow have the constant curvature $\kappa/a^2(t)$ in every point, where κ is a constant and can take the values $\{-1, 0, +1\}$, where respectively they refer to negatively curved, flat and positively curved cases. Thus the FLRW metric, in comoving coordinates, can be written as

$$ds^2 = -dt^2 + \gamma^{-2} a^2(t)(dx^2 + dy^2 + dz^2)\,. \tag{1.7}$$

We have defined the function

$$\gamma = 1 + \frac{1}{4}\kappa r^2\,, \tag{1.8}$$

where $r^2 = x^2 + y^2 + z^2$.
The scale factor $a(t)$ gives the information in an evolving universe about the variation of distances. Given two world lines Γ_1 and Γ_2 on the three-surface of constant time $t = t_1$, the distance d_1 measured between them along the curve $x^\mu(s) = (t_1, x^i(s))$ is given by

$$d_1 = \int_{\Gamma_1}^{\Gamma_2} \frac{a(t_1)}{\gamma} \sqrt{\delta_{ij} \frac{dx^i}{ds} \frac{dx^i}{ds}}\, ds \tag{1.9}$$

Given an other constant time three-surface $t = t_2$ we can find that the corresponding distance d_2 between the two curves along the previous world line is related to (1.9) by

$$d_2 = \frac{a(t_2)}{a(t_1)} d_1 \tag{1.10}$$

We can see that all lengths scale with the scale factor $a(t)$ and so we have, for all points in the universe, an expansion that is isotropic. For the spacial homogeneity of the model there is no center and no spatial edge, so there is not an outside universe.
The velocity of motion of the distance $d(t) = d_1 a(t)$ in a generic surface $t = const$ (where $t_1 < t$) is given by:

$$\dot{d}(t) = \dot{a}(t) d_1 = H(t) d(t)\,, \tag{1.11}$$

where we defined the Hubble parameter law $H(t) = \dot{a}(t)/a(t)$. We can interpret it as the recession rate in distances in the three-space surface $t = const$. Ellis and Rothman (1993) [28] demonstrate that Hubble rate is a "distance velocity" and it does not presuppose the exchange of information, therefore Hubble rate is not subject to the speed-of-light constraint. The most used FLRW model has two fundamental components: cold dark matter and dark energy. The non-relativistic Cold Dark Matter (CDM) has the behaviour of a perfect fluid (non collisional) and we hypothize that can be identified as a non-baryonic particle not yet experimentally observed. The other component is the Dark Energy that takes the form of a cosmological constant Λ that assume the meaning of a repulsion in Einstein's equations. This FLRW is usually called ΛCDM model.
There are different observations that shape ΛCDM: The luminosity distance and of the Ia Super Novae and gamma ray bursts, the observation of the CMB anisotropies, the Baryon Acoustic Oscillations (BAO), large-scale clustering of galaxies. Under these observations we can have different important information about the present time universe. First of all we are in an epoch of accelerated expansion, the curvature of the space can be considered flat and we can compute the principal components of the universe, that is to say respectively for dark energy, dark matter and baryonic matter. In more detail we write the *Planck 2018* results (see [64])

$$\Omega_\Lambda = 0.679 \pm 0.013 \,, \qquad \Omega_m = 0.321 \pm 0.013 \,, \qquad \Omega_b = 0.02212 \pm 0.00022 \,. \tag{1.12}$$

We can see that Ω_Λ is predominant with respect to the other parameters, therefore the current universe in an accelerated expansion epoch. Riess et al. (1998) [30] and Perlmutter et al. (1999) [30] find the evidence of the universe acceleration from the luminosity distance and the analysis of the redshift of supernovae at $z \sim 0.5$. Ω_Λ gives theoretical issues, the so-called called the "dark energy problem". The standard model ΛCDM must take into account the fact that the universe energy density must be substantially dominated by dark energy, whose nature is not understood.

We have just seen the symmetry properties of the FLRW models, the surfaces of homogeneity, given by constant time three-surfaces, and the homogeneity of the space is consequence of the isotropy in every three-space point. However we observe that when we observe smaller scales the degree of inhomogeneity increases, reaching the well-known astronomical structures. The small inhomegeneities in the cosmic fluid flow will become the today structures thanks to gravitational instability. Therefore these symmetries do not make FLRW models realistic for the observed universe. We can regain a reasonable level of reality perturbing around the FLRW solutions. This approximate method is very accurate and nowadays it is the most powerful tools in the realm of cosmology.

1.2 Cosmic Microwave Background

Theorized by G. Gamow in 1948 (see [32]), CMBR is the most important test for the thermal history of the universe. The FLRW is the space-time geometry that is more suited to CMB observations, and it is a reasonable justification to use ΛCDM as standard model.
Following Dodelson (2003) [2], in the primeval plasma, when the radiation temperature was of an order of magnitude of about $10^4 K$, there was the former formation and then the sudden ionization of neutral hydrogen by energetic photons. There is a fundamental feature in the primeval plasma: free baryons and photons were in thermodynamic equilibrium, the so called "baryon-photon" fluid. In particular there was mainly Thomson scattering of photons from free electrons, therefore the plasma was opaque. In practice the intensity of the plasma filled with photons followed the well-known black-body spectrum law:

$$I_\nu = \frac{4\pi\hbar\nu^3}{c^2\left(e^{\frac{2\pi\hbar\nu}{k_B T}} - 1\right)} . \tag{1.13}$$

The measurements of COBE/FIRAS revealed that CMB represents the best black-body spectrum ever observed in nature, see Fixsen et al. (1996) [65].
When baryons and radiation ended their interaction at $z \simeq 1100$ photons started to free stream, this is called *last scattering*. More in detail when photon-baryon interactions became negligible the universe homogeneous expansion cooled the radiation preserving the thermal spectrum. Now we can observe the CMBR spectrum and it represent the most rich information source about early universe.
The CMB was discovered by Penzias and Wilson in 1964, see [62]. They indicated a CMB temperature of $3K$ isotropic in all directions. The first CMB detections gives the information that the early universe was perfectly smooth, and the idea of the theory of the smooth Big Bang stabilized. In the radiation spectrum the main characteristic is the dipole variation, an anisotropy in the thermodynamic temperature, it is in function on the position in which the observer is pointing an antenna and has an average amplitude of $\sim 10^{-3}$. We interpret this anisotropy as the effect of earth motion relative to the CMB rest frame and another cause is the distant galaxies redshift. Then in 1992 COBE detected the presence on CMB of the quadrupole anisotropy, the data indicated a temperature deviation of about 10^{-5} around the background temperature. The observations matched with the expected gravitational perturbation to the CMBR due to the matter density fluctuations in large scales. It was the awareness that the universe was not perfectly homogeneous. These revealing anisotropies give us a large amount of information about early universe, they are crucial in order to have a subtle determination of cosmological parameters.
CMB anisotropies are divided into primary anisotropies and second anisotropies.
i)Primary anisotropies was generated before the recombination and they give information about early universe and the physics related to it. The most important example is the Sachs-Wolfe (SW) effect that we will analyze in next sections.
ii)Secondary anisotropies were generated in an epoch after than recombination, in particular at structures formation time. We will study in next sections an example of secondary anisotropies that is the Integrated Sachs-Wolfe (ISW) effect.

1.2.1 Angular power spectrum

To understand the sense of the large scale effects, which we will analyze in this thesis, we must describe the CMB field in a statistical way. We consider the temperature deviation as

$$\delta_T(\hat{\mathbf{p}}) = \frac{T(\hat{\mathbf{p}}) - T_O}{T_O} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{\mathbf{p}})\,, \tag{1.14}$$

where $\hat{\mathbf{p}}$ indicate the direction of the incoming photon, the l parameter is called the multiple moment of the angular multipole expansion and T_O is the observed temperature. We use spherical harmonics Y_{lm} to decompose the temperature deviation field because they form a complete set of orthonormal functions on a sphere that are perfectly suited to describe a distribuition on a celestial sphere. Orthonormal because they satisfy the normalization:

$$\int Y^*_{lm} Y_{l'm'}\, d\Omega = \delta_{ll'}\delta_{mm'}\,. \tag{1.15}$$

The real and imaginary part of the zeros of Y_{lm} divide the celestial sphere, in first approximation, into cells of constant solid angular size

$$\theta \simeq \frac{\pi}{l}\,. \tag{1.16}$$

It is important to remember that the first two orders of multipoles are generally neglected, because are induced by the observer and its position. The term $l = 0$ is the monopole, it defines a correction that modify the mean temperature of the sky, for a particular observer, with respect to the mean of all possible sky temperature configurations. The term $l = 1$ is the dipole, it represents the solar system barycentre motion in relation to the rest frame of the CMB. Moreover there are additional effects, aberration ad modulation effects, associated with local Lorentz boosts.

The distribution of the a_{lm} coefficients is a Gaussian distribution and its origin are the quantum fluctuations defined during inflation, see Dodelson (2003) [2]). More in detail the mean value of a_{lm}, using $\langle\cdot\rangle$ to denote ensemble average, is

$$\langle a_{lm}\rangle = 0\,. \tag{1.17}$$

Instead, the covariance $C(l) \neq 0$, in particular

$$\langle a^*_{l'm'} a_{lm}\rangle = C(l)\delta_{ll'}\delta_{mm'}\,, \tag{1.18}$$

where the quantity $C(l)$ is also called "angular power spectrum", and its value is given by the quadratic average of the a_{lm} coefficients at a fixed angular scale

$$C(l) \equiv \left\langle |a_{lm}|^2 \right\rangle = \frac{1}{2l+1} \sum_{m,m'=-l}^{l} a^*_{l'm'} a_{lm}\,. \tag{1.19}$$

It is straightforward note that the variance at high multiple moments (small angular scales) is much higher than the variance at low multipole moments (large angular scales). We can quantify this fact looking at the variance of the power spectrum

$$\frac{\Delta C(l)}{C(l)} = \sqrt{\frac{2}{2l+1}}\,, \tag{1.20}$$

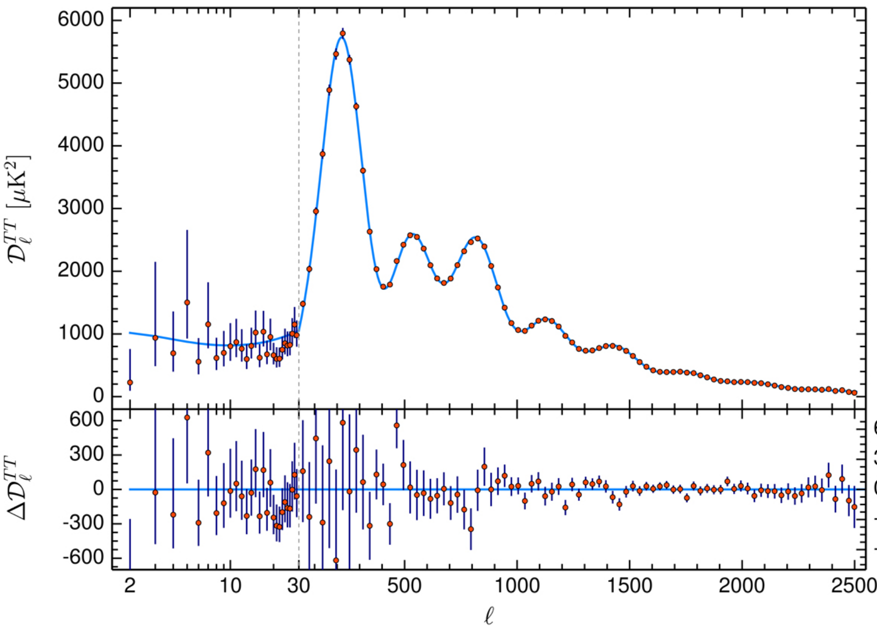


Figure 1: Angular power spectrum of temperature fluctuations in the CMB detected by Planck (2018) at different multiple moments. The red dots are the Planck's measurements, and the blue support shows the best fit of the standard model of cosmology. The lower panel shows the residuals with respect to this model. Plot from Planck Collaboration [64].

this uncertainty in the precision of the $C(l)$ is called "cosmic variance". In Figure 1 we insert the Planck measurements at different angular scales; it is easy to note the cosmic variance, at small multipoles moments we see larger error bars than at higher multipoles. Now we can write the temperature fluctuations correlation function: given two temperature anisotropies fields that point in different direction $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$, their auto-correlation is

$$\langle \delta_T(\hat{\mathbf{p}}_1)\delta_T(\hat{\mathbf{p}}_2)\rangle = \frac{1}{4\pi}\sum_{l\geq 1}^{\infty}(2l+1)C(l)\mathcal{P}_l(\cos\theta). \tag{1.21}$$

Here $\mathcal{P}_l(x)$ are the Legendre polynomials of l-order. It is easy to see that for aligned temperature fields the angular power spectrum give the whole information about the considered angular scale to the total variance of the temperature anisotropies:

$$\langle \delta_T^2(\hat{\mathbf{p}})\rangle = \frac{1}{4\pi}\sum_{l\geq 1}^{\infty}(2l+1)C(l)\,. \tag{1.22}$$

From Figure 1 we can see that the fluctuation observed have generally a relative amplitude of 10^{-5}. Therefore the experimental setup have to search for a signal of amplitude $\sim 30\mu$K on a monopole background temperature of ~ 3K. It is important to notice that at large angular scales, from $\sim 90^\circ$ to $\sim 10^\circ$ the scales of temperature anisotropies is $\delta_T \simeq 1.2\times 10^{-5}$.

1.2.2 Perturbation theory

In order to understand the large scales anisotropies of the CMB we have to introduce cosmological perturbation theory. As we have already seen in Section 1.1, the FLRW models represents a homogeneous and isotropic Universe. Following [49] we write the metric (1.6) in a more compact way, in function of the scale factor $a(t)$ and the 3-space metric of constant curvature γ_{ij}:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \, . \tag{1.23}$$

Solving Einstein Field Equations (see Section 4) we can recover the two well-known Friedmann equations [50]

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} &= \frac{1}{3}\rho + \frac{\Lambda}{3} \\ H(t) = \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} &= -p + \Lambda \, . \end{aligned} \tag{1.24}$$

In particular with ρ and p we are referring respectively to the energy density and the pressure of the cosmic fluid. In this thesis we use the value of the Hubble constant, i.e. the nowadays value of the Hubble parameter as

$$H_0 = (66.88 \pm 0.92)\,\mathrm{km/sec/Mpc}\,, \tag{1.25}$$

see Planck collaboration (2018) [64].

The fluctuations of CMB suggest that locally the Universe deviates from a perfect homogeneous and isotropic configuration. For this reasoning modern cosmology uses to perturb the metric (1.23), keeping the FLRW solutions as a rigid background. We write the metric tensor, in longitudinal gauge [51], as the sum of the FLRW metric tensor $g^{(0)}_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$, i.e.

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \, . \tag{1.26}$$

More in detail the line element of the perturbation tensor can be written as

$$h_{\mu\nu}dx^\mu dx^\nu = 2\Phi dt^2 + a^2(t)(-2\Psi\gamma_{ij} + 2\chi_{ij})dx^i dx^j \, , \tag{1.27}$$

where Φ and Ψ are the first-order gauge invariant variables called Bardeen potentials, more in detail these variables form gauge invariant expressions only to first order. χ_{ij} describes the tensor modes, in particular it is a transverse and traceless tensor

$$\nabla^i h_{ij} = \gamma^{ij} h_{ij} = 0 \, . \tag{1.28}$$

We point out that during inflation there are not vector perturbations, thus we neglected them.

Einstein Field Equations (EFE) are the relations between the perturbations of the metric and the perturbations of the energy-momentum tensor. In particular the energy-momentum tensor has the general form for an imperfect fluid flow with 4-velocity u^μ

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu} + \Pi^{\mu\nu} \, . \tag{1.29}$$

We require that the anisotropic stress tensor is traceless and flow orthogonal, i.e.

$$\Sigma^{\mu}_{\mu} = \Sigma_{\mu\nu} u^{\nu} = 0\,. \tag{1.30}$$

It contains all the information about shear and viscosity.
Considering a perfect fluid ($\Sigma_{\mu\nu}$=0), we can write the component of the of the perturbed energy momentum tensor as

$$\begin{aligned} \delta T^{00} =& \delta\rho \\ \delta T^{0i} =& (\rho^{(0)} + p^{(0)}) a^{-1} \delta u^i \\ \delta T^{ij} =& -\delta p \delta^{ij}\,, \end{aligned} \tag{1.31}$$

where the variables with the superscript $\rho^{(0)}$ and $p^{(0)}$ are background quantities. Since δT^{ij} is diagonal, from the linearization of the EFE

$$\delta G_{\mu\nu} = \delta T_{\mu\nu} \tag{1.32}$$

follows that the two Bardeen potentials are equal, $\Phi = \Psi$. This is a great simplification for gauge invariant equations that describe the dynamics of the system. Moreover we can write the energy density in terms of the density profile $\delta\rho/\rho^{(0)}$:

$$\rho = \rho^{(0)}(1+\delta)\,. \tag{1.33}$$

From equation (1.32), considering adiabatic perturbations and on scales larger than the Hubble radius we can write the following "conservation" law:

$$\dot{\zeta} = 0 \tag{1.34}$$

where we denote the physical time derivatives $(\cdot)^{\cdot} = d/dt$. The ζ parameter is known as curvature perturbation and it was first introduced by Bardeen (1983) [52], and in terms of Bardeen potentials is

$$\zeta = \frac{2}{3}\frac{H^{-1}\dot{\Phi} + \Phi}{1+w} + \Phi\,, \tag{1.35}$$

where $w = p^{(0)}/\rho^{(0)}$ is the FLRW equation of state parameter. In comoving gauge, i.e. $u^{\mu} = 0$, ζ is proportional to the perturbed spatial part of the Riemann tensor.
The perturbed EFE are linear, thus decomposing all variables in the eigenfunctions of the 3-Laplacian, the variables inside the equations can be studied separately, since they evolve in a indipendent way. We focus on the case of zero-curvature, $K = 0$, in this manner variables take the form of Fourier modes. We have already seen that during inflation there was the generation of the small deviations from homogeneity and isotropy, thus we assume that in inflationary period there was not preferred direction or position in the Universe. In particular the fluctuations must be homogeneous and isotropic, and for these reasons the Fourier modes, corresponding to each variables in the equations of motion, are untied. Now we can define the initial power spectrum $P(k)$ by

$$P(k)\delta(\mathbf{k}-\mathbf{k}') \equiv \frac{k^3}{2\pi^2}\langle \zeta(\mathbf{k}, t_{in})\zeta^*(\mathbf{k}', t_{in})\rangle\,. \tag{1.36}$$

In order to have the power spectrum containing all the information about the perturbations, also in this case, we assume the fluctuations being Gaussian.
In general we can write the power spectrum of a perturbed variable as the product of $P(k)$, the initial power spectrum, with a transfer function, usually denoted with Θ^2. For example density fluctuations can be written as

$$k^3\langle\delta(\mathbf{k},t)\delta^*(\mathbf{k}',t)\rangle = \delta(\mathbf{k}-\mathbf{k}')2\pi^2 P_\delta(k,t)\,, \tag{1.37}$$

where the density contrast power spectrum P_δ is given by

$$P_\delta(\mathbf{k},t) = \Theta_\delta^2(k,t)P(k)\,. \tag{1.38}$$

The fluctuations of the CMB temperature is different because the temperature field is a function of direction. In the previous section we have defined the CMB power spectrum (1). We have to solve EFE in order to have the evolution of the space-time and the behaviour of metric potentials, then we can derive the transfer function $\Theta_T(k,l)$ of the photon distribution through Boltzmann Equation,

$$\frac{l(l+1)}{2\pi}C(l) = \int \frac{1}{k}\Theta_T^2(k,l)P(k)\,dk\,. \tag{1.39}$$

The curvature perturbation ζ and the Bardeen potentials Φ-Ψ remain constant on super-horizon scales, i.e. $k \ll aH$. Since in radiation domination epoch and in sub-horizon scales, i.e. $k \gg aH$, Bardeen potentials decay $\propto 1/(aH)^2$ and oscillate with the density fluctuations, in particular these oscillates with constant amplitude. Writing the differential of the conformal time as

$$d\eta = \frac{dt}{a(t)}\,, \tag{1.40}$$

we can write the behaviour of the density fluctuations

$$\delta_{rad} \propto \cos(c_s k\eta)\,, \tag{1.41}$$

where c_s denotes the adiabatic speed of sound, and peculiar velocity

$$v_{rad} \propto \sin(c_s k\eta)\,. \tag{1.42}$$

Therefore in sub-horizon scales the peculiar velocity and the density fluctuations are out of phase. However, in matter domination epoch Φ and Ψ remain constant in sub-horizon scales, in this case the density profile

$$\delta_m \propto a \tag{1.43}$$

and the peculiar velocity has the proportionality

$$v_m \propto a^{1/2}\,. \tag{1.44}$$

In the end in a Λ-domination epoch both curvature fluctuations decays and there is the freeze out of the density fluctuations.

1.2.3 Sachs-Wolfe effects

In 1967 Sachs and Wolfe [14] predicted for the first time the existence of anisotropies due to gravitational redshift and blueshift in the cosmic microwave background. They used a general relativistic approach integrating the geodesics equation in a perturbed FLRW background. Historically, the integrated Sachs-Wolfe was first detected with the cross-correlation between the number count of radio galaxies from NVSS data and HEAO1 A1 X-ray data with the first WMAP data, see [53]. Here we want to give an inceptive idea of the SW effects following [54] and do not show in detail the calculation, in fact in Section 4 we recover the same results in a detailed and alternative way. We consider a photon described by the world line $x^\mu(\lambda)$ labelled by the affine parameter λ. We begin to write the photon null wavevector $k^\mu = dx^\mu/d\lambda$ perturbation expansion around its background value $k^{(0)\mu}$

$$k^\mu = k^{(0)\mu} + \delta k^\mu \,. \tag{1.45}$$

The photon geodesics is the solution of the well known geodesics equation

$$\frac{Dk^\mu}{D\lambda} = k^\mu{}_{;\nu} k^\nu = 0 \tag{1.46}$$

at first-order. In particular, using the metric (1.26), (1.27) in conformal time $\tilde{g}_{\mu\nu}$ (see equation (1.40)), the connection take the components

$$\Gamma^0_{00} = \Phi' \,, \qquad \Gamma^0_{0i} = \Phi_{,i} \,, \qquad \Gamma^0_{ij} = -\Psi'\delta_{ij} \,, \tag{1.47}$$

where we used the prime to denote the conformal derivatives, $(.)' = d/d\eta$.
Now we define the normalized 4-velocity truncated at first order:

$$u^\mu = u^{(0)\mu} + \delta u^\mu = \frac{1}{a}(1-\Phi, v^i) \,, \tag{1.48}$$

where the 3-velocity v^i is a first-order quantity, therefore in order to remain at first-order we can only solve the temporal part of the geodesics equation. The unperturbed solution is given by

$$k^{(0)\mu} = (1, \hat{e}^i) \,, \tag{1.49}$$

where $\hat{e}^i$ is a unity direction vector tangent to the unperturbed geodesics. Now the energy of the photon measured by a comoving observer is given by

$$\mathcal{E} = g_{\mu\nu} u^\mu k^\nu \,. \tag{1.50}$$

Denoting η_E the decoupling conformal time and η_O the comoving observation time, it can be shown that the time solution of the geodesics equation is given by

$$\delta k^0 = \delta k^0(\eta_E) - 2[\Phi(\eta_O) - \Phi(\eta_E)] + \int_{\eta_O}^{\eta_E} (\Phi' + \Psi')\, d\tau \,. \tag{1.51}$$

The redshift is defined by the ratio between the photon energy measured at decoupling time and at observer time,

$$1 + z \equiv \frac{\mathcal{E}_{\eta_E}}{\mathcal{E}_{\eta_O}}\,. \tag{1.52}$$

Recalling that the CMB observed temperature is given by the relation between redshift and emitted temperature

$$T_O = \frac{1}{1+z} T_E\,, \tag{1.53}$$

we can find the temperature deviation from the isotropy

$$\delta_T(\eta_O) = \delta_T(\eta_E) + [v^i \hat{e}_i]^{\eta_E}_{\eta_O} - \Phi_{\eta_O} + \Phi(\eta_E) + \int_{\eta_0}^{\eta_E} (\Phi' + \Psi')\, d\eta\,. \tag{1.54}$$

Let us analyze this expression term by term. The first term is equivalent to $\delta_\gamma/4$, the radiation density fluctuation in the spatial flat gauge. The next term is the Doppler effect, it is generated by the observer motion in relation to the last scattering surface. The third term is the ordinary Sachs-Wolfe (SW) effect, it is the temperature anisotropy due to the fluctuations of the potentials, in particular these fluctuations generate gravitational redshift. The integral is the integrated Sachs-Wolfe effect (ISW) and it is the temperature deviation represented by the integration along the line of sight of the gravitational potential time variation. When the gravitational potential deviates from a static behaviour it becomes relevant. In the matter dominated epoch the gravitational potential is constant, thus in the Einstein de-Sitter regime and at linear order the ISW effect vanishes. Therefore the ISW is an effect that can be generated in radiation domination (early ISW) or dark energy domination (late ISW). In particular early ISW effect is part of the primary anisotropies because the matter-radiation equality redshift $z_{eq} \simeq 3300$ is before the time of decoupling $z_{dec} \simeq 1080$. In the end the late ISW effect becomes important at redshifts $z < 1$. Moreover on large angular scales $l < 30$ the perturbations theory predicts the well-known "Sachs-Wolfe plateau", see Figure 1.

1.2.4 ISW detection

In this section we want to give an introduction on the measurement of ISW effect following [55]. In ΛCDM model the modifications of the Newtonian and curvature potentials are due to expansion. We observe that in late times there is an apparent acceleration caused by Dark Energy Ω_Λ, where it must produce a quite important signature on CMBR. Detecting ISW effect is difficult because primary anisotropies are more consistent, nearly an order of magnitude larger. Moreover ISW effect contributes at large angular scales $l < 30$, small multiple moments, therefore they are affected by cosmic variance.

The most used methods in order to detect ISW effect are the observed cross-correlation with tracers and the "fields method" (the direct comparison of temperature fields). The measurement of cross-correlated power spectra $C_{gT}(l)$ is the most used method, it uses cross-correlation function in order to detect the ISW signal. There are four different statistical methods in order to have spectra detection. The first finds the two-signals correlation, see Boughn and Crittanden (2002) [56]. The second is a model dependent method that creates a best-fit model with the measurements, see Ho et al. [57]. In practice there is the measure of how well observed signals matches the signal predicted. The third uses the χ^2 test and the fourth uses comparison between models, see Afshordi et al. (2004) [58]. The cross-correlation method has reported many detections with WMAP and Planck CMB data. One of the best measurements was find by Gannantonio et al. (2008), see [59], with a 4.5σ.

The field-to-field comparison work with the hot and cold spot of the temperature field in order to see the presence of ISW. The observable is the temperature field δ_{ISW}, in general the temperature anisotropies on large linear scales can be described as

$$\delta_{OBS} = \delta_T + \lambda\delta_{ISW} \,, \tag{1.55}$$

where δ_T is the primeval CMB temperature field, λ is the amplitude of δ_{ISW}. In order to detect the ISW signal, the best method to extrapolate information is from the large scales matter distribution. Using a bias relation, see Granett et al. (2009) [60], in this way the matter field can be estimated from galaxy surveys. Knowing the galaxy survey, the δ_{ISW} field can be approximately extracted from galaxy maps and temperature maps with

$$a_{lm}^{ISW} = \frac{C_{gT}(l)}{C_{gg}(l)} g_{lm} \,, \tag{1.56}$$

where g_{lm} are the galaxy map spherical harmonic coefficients, a_{lm}^{ISW} are the coefficients of the ISW temperature anisotropy map and $C_{gT}(l)$, $C_{gg}(l)$ are respectively the cross-correlation funtion and the autocorrelation function, see Boughn et al. (1998) [61]. In particular Granett et al. [62] detect an ISW signal with a temperature shift: for supervoids $\sim -11\mu K$ at 3.7σ, for superclusters $\sim 8\mu K$ at 2.6σ, for an average between supervoids and supercluster of $\sim 9.6\mu K$ at $> 4\sigma$.

1.3 Inhomogeneity and Silent Universes

Here we enter in the field of inhomogeneous models. The inflation theory and the perturbation theory around FLRW solutions, thus the small deviation from a metric perfectly homogeneous and isotropic, have given a first theoretical answer of the inhomogeneity observed in the universe. A scheming example is the cosmic web. As the redshift decreases the matter distribution does not remain in a homogeneous continuum, as at $z \sim 1080$, but it is forced to aggregate in regions, structures. In the observation we find clusters of galaxies forming filaments that intersect each other, leaving large void regions between the structures. In particular Hoyle and Vogeley (2002) [37], using the Point Source Catalog Survey (PSC_z) and the Updated Zwicky Catalog (UZC), detect 54 voids finding a typical scale size of about $\sim 30h^{-1}\,Mpc$ and a density contrast $\delta \sim -0.93$. So we can see that at recent epoch we have an high degree of inhomogeinity. We are out of the perturbative regime. In this case the non-linear effects of General Relativity may give an alternative answer to explain the void regions and the collapsed regions.

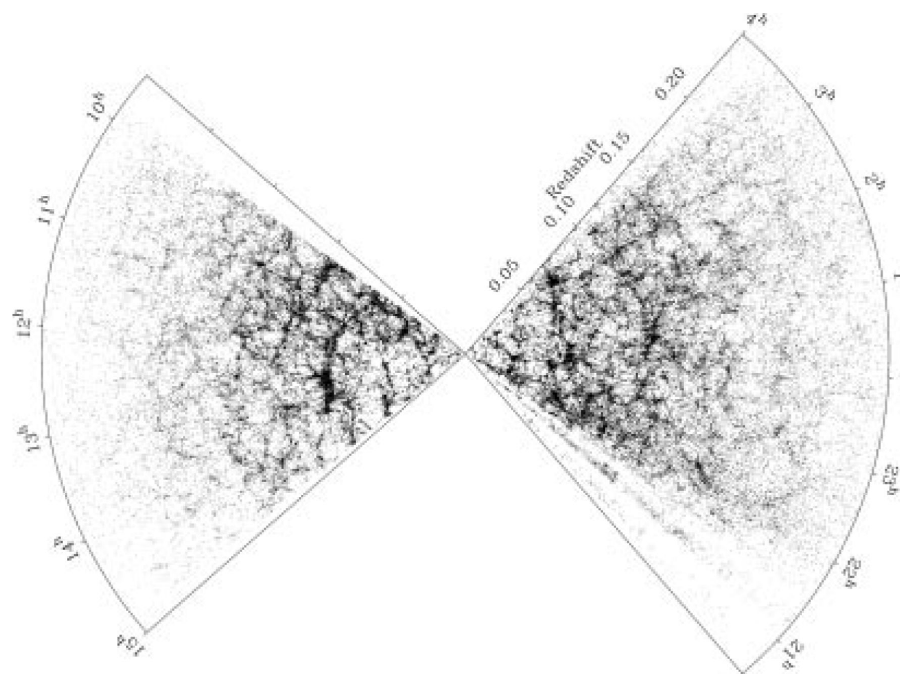


Figure 2: Projected distribution of 250 000 galaxies, where on the top strip is represented the observations in north galactic pole and on the bottom strip is represented the observations in south galactic pole. The distribution is in function of the redshift and the right ascension, taken from [41]

A new approach that covers the greatest part of the most discussed cosmological models was introduced by Matarrese et al. in 1993, see [21]. "Silent Universes" are a identification of a model in which the evolution of the cosmological fluid flow is purely local, there is not change of any typology of information, nor waves nor gravitational waves. The case of collisionless matter in cosmological perturbations in nonlinear regime in a matter-

dominated universe is easily described by relativistic irrotational dust. This assumption can be mathema-tized by the zero-vorticity kinematical restriction. This is the first assumption to identify a silent universe. The second and the most stringent condition is the assumption that the magnetic part of the Weyl tensor $H_{\mu\nu}$ vanishes, we will give a complete treatment of the meaning of $H_{\mu\nu}$ in Section 2. The motivation of this choice is important to understanding the topic. While the electric part of the Weyl tensor $E_{\mu\nu}$, the tidal force field, has a Newtonian counterpart that can be written in terms of the potential and its derivatives, the magnetic part of the Weyl tensor has not a Newtonian analogue. Neglecting the latter is equivalent to neglect the interactions of tensor mode (i.e. gravitational waves) and gives us the ability to accept generic initial conditions for the cosmological dust. In this lies the truly meaning of the designation of the name "silent" universes; Matarrese and al. (1994), heuristically with the constraint $H_{\mu\nu} = 0$, established the silence.
There is a particularity in these assumptions. For a perfect silent cosmological fluid with four-velocity u^μ there is an orthonormal tetrad $\mathbf{e}_\mu$ that is eigenframe of the tidal force $E_{\mu\nu}$ and the shear tensor $\sigma_{\mu\nu}$. In this way we can diagonalize them, finding a system of only six ODEs ruling the dynamics (if the flow is taken to be in a geodesics motion), fully characterized by the eigenvalues of the tidal force and the shear.
In the perturbation of FLRW models at first-order the magnetic tidal tensor vanishes (see [37], [38]), but at second order we get a non-zero value, this effect is called tidal induction, see [22], [23], [39].
Bruni, Matarrese and Pantano (1995) [25] have explored the dynamics of different silent models identifying the types of collapse using the phase plane analysis of the stationary points. For the Szekeres models that we will study in this thesis (see Section 3.3), they find two attracting stationary points that correspond to a pancake collapse and a spindle (or cygar) collapse (Zel'dovich 1970 [40]). In general a silent model has fluid elements that collapse in a triaxial spindle singularity. Moreover the asymptotic behavior around the stationary points, that represents FLRW universe, can be interpreted as a perturbation around the background solutions, in fact they find the directions of the growing mode and decaying mode of the perturbation in a matter dominated universe. In Section 3 we will see in detail the dynamics of silent universe and the special case of Szekeres models.

1.4 Summary of research

In this thesis we investigate the Cosmic Microwave Background temperature anisotropies of a silent model using a consistent perturbative approach.

- We present an overview of the relativistic cosmology (Section 2).
- We introduce Silent Universes and the special case of Szekeres models (Section 3).
- We study the perturbed Szekeres space-time around FLRW. We solve the linearized Einstein Field Equations in order to find the behavior of the metric inhomogeneous potential (Section 4).
- We compare the evolution of the phase-plane analysis at the first-order case and in the exact case for the Szekeres metric, and then we find the fixed points of the model (Section 4).
- Using the perturbed Jacobi equation, we find the first-order and second-order geodesics equation solutions for a propagating photon in Szekeres space-time and in a fully general silent space-time (Section 5).
- We get the temperature anisotropies at first-order and at second-order for the Szekeres metric and for a fully general silent metric (Section 5).

Throughout the thesis we choose units $c = 8\pi G = 1$.

2 Cosmological dynamics

We want to show and define an alternative approach to General Relativity proposed by Ellis G. (1971) based on the Bianchi and Ricci identities following Bertschinger (1995) [66] and Ellis (1971) [42], instead of the most used method based on the Einstein field equations and on the conservation equation. It is very useful to see the application to cosmology that followed (see for example: Ellis and Bruni 1989 [43]; Hwang and Vishniac [44]; Bruni, Dunsby and Ellis 1992 [38].
This formulation of GR is fundamental in order to understand the theory of silent universes, developed by Matarrese et al. for the first time in 1993 (see [21]).

2.1 A new approach to General Relativity

The new approach consists in dividing the four dimensional space-time into constant time three-dimensional hypersufaces. In this way each temporal coordinate has an associated spatial hypersurface in which there is a comoving observer. In a manifold point of view (see Section 5) we are dividing the space-time in time-like worldlines determined by an affine parameter λ and a position vector $\mathbf{y}$ that labels each particular worldline:

$$x^\mu(\lambda, y^i)\,. \tag{2.1}$$

We assume that exist an unique vector field that represents the velocity of the cosmic fluid element, it is defined by the relation

$$u^\mu u_\mu = -1\,, \tag{2.2}$$

so in terms of normalized comoving coordinates we can express the velocity as

$$u^\mu = (t, y^i)\,. \tag{2.3}$$

The four velocity u^μ can be seen as the tangent vector of the worldline, i.e. using the coordinates $x^\mu(\lambda, \mathbf{y})$

$$u^\mu = \frac{dx^\mu}{d\lambda}\,. \tag{2.4}$$

In this way we can split the tensor of the theory in parts that are parallel or orthogonal to the fluid flow. For this purpose we define the projection tensor

$$\mathcal{P}_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu\,. \tag{2.5}$$

It satisfies the projection conditions:

$$\mathcal{P}_{\mu\nu}u^\nu = 0\,, \qquad \mathcal{P}_\mu{}^\sigma \mathcal{P}_\sigma{}^\nu = \mathcal{P}_\mu{}^\nu\,. \tag{2.6}$$

$\mathcal{P}_{\mu\nu}$ projects, in every point, tensors in the rest space of a comoving observer identified by its four-velocity u^μ; for a generic tensor A^μ we have

$$A_\parallel = -u_\mu A^\mu\,, \qquad A^\mu_\perp = \mathcal{P}^\mu{}_\nu A^\nu\,. \tag{2.7}$$

We can think that each space-time point has a world-line with tangent vector $u^\mu(\lambda, y^i)$, so for every single world-line there will be a different decomposition of A^μ. It is straightforward to show that the decomposition for a second-rank tensor $B^{\mu\nu}$ is given by

$$B_\parallel = u_\mu u_\nu B^{\mu\nu}\,, \qquad B_{\perp\ \nu}^{\mu} = \mathcal{P}^\mu{}_\sigma \mathcal{P}\nu\rho B^{\rho\sigma}\,. \tag{2.8}$$

Using the relation (2.5) we can write the General Relativity line element in terms of the comoving 4-velocity and the projection tensor

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \mathcal{P}_{\mu\nu}dx^\mu dx^\nu - (u_\mu dx^\mu)^2\,. \tag{2.9}$$

The meaning is immediate. A comoving observer that has worldline $x^\mu(\lambda, y^i)$, given a subsequent event $x^\mu + dx^\mu$, we can identify from (2.9) an infinitesimal spatial separation

$$\delta l = \sqrt{h_{\mu\nu}x^\mu x^\nu} \tag{2.10}$$

and a infinitesimal time separation

$$\delta t = \left|u_\mu dx^\mu\right|\,. \tag{2.11}$$

The approach that must be followed is to think about a free-falling observer well defined by one worldline. This observer see itself moving along a straight line and constant velocity, thus do not feel gravitational force, however it sees the adiacent free-falling observers that, in a fully general way of thinking, follow curved lines. This effect is the well-known tidal force effect that is explained in General Relativity with the geodesics equation. The essence of geodesics equation resides in the fact that the commutator of the covariant derivatives of the 4-velocity is non-zero,

$$[\nabla_\rho, \nabla_\sigma]u^\mu = R^\mu{}_{\nu\rho\sigma}u^\nu\,. \tag{2.12}$$

This is the so called Ricci identity, and it is well defined for every 4-vector u^μ. We can see that Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, is given through the a reduction of the Riemann tensor, the Ricci tensor $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$, and thinking about the Einstein field equation

$$G_{\mu\nu} = T_{\mu\nu} \tag{2.13}$$

(where we stress the fact that in our thesis we choose units $c = 8\pi G = 1$) we can see a clear convenience, i.e. the Ricci tensor is given by the components of the energy-momentum tensor $T_{\mu\nu}$, but it doesn't allow us to extract directly the fully information carry by $R^\mu{}_{\nu\rho\sigma}$. The only method to reach the total information is finding the metric components. The method we want to show here, also called Lagrangian evolution, is based on that part of the Riemann tensor the we cannot obtain from the ricci tensor. The Weyl tensor is defined in a 4-dimensional space as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{6}Rg_{\mu[\rho}g_{\sigma]\mu}\,, \tag{2.14}$$

where the square brackets denote antysimmetrisation. We can demonstrate that this tensor obeys the same symmetries of the Riemann tensor

$$C_{[\mu\nu][\rho\sigma]} = C_{[\rho\sigma\mu\nu]}\,, \qquad C_{\mu[\nu\rho\sigma]} = 0\,. \tag{2.15}$$

The fundamental feature of the Weyl tensor is that it is trace-free

$$C^{\sigma}_{\mu\sigma\nu} = 0\,. \tag{2.16}$$

Therefore, while the Ricci tensor is the trace of the Riemann tensor that gives the information about the curvature of local sources, the Weyl tensor is the traceless part of the Riemann tensor and it contains the information of nonlocal sources. We will make a comparison between the Newtonian tidal forces and the property of the Weyl tensor. Using the Einstein field equation and the Bianchi identity

$$R^{\mu}_{\nu\rho\sigma;\lambda} = 0\,, \tag{2.17}$$

we can find a relation that join the Weyl tensor with the Energy momentum tensor

$$C_{\mu\nu\rho\sigma}{}^{;\rho} = (T_{\sigma[\nu;\mu]} + \frac{1}{3}g_{\sigma[\mu}T_{;\nu]})\,. \tag{2.18}$$

Following Ellis (1971) the Weyl tensor can be determined by the symmetric an trace-free symmetric tensors $E_{\mu\nu}$ and $H_{\mu\nu}$. We use the Levi-Civita Symbol $\epsilon_{\mu\nu\rho\sigma}$ in order to define the totally antysimmetric tensor in a space-time defined by the metric $g_{\mu\nu}$:

$$\eta_{\mu\nu\rho\sigma} = \sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\,. \tag{2.19}$$

In particular it will be useful to define projected Levi-Civita to the orthogonal tangent plane

$$\eta_{\mu\nu\rho} = \eta_{\mu\nu\rho\sigma}u^{\sigma}\,. \tag{2.20}$$

Thus we can split the Weyl tensor in

$$C_{\mu\nu\rho\sigma} = (g_{\mu\nu\alpha\beta}g_{\rho\sigma\gamma\delta} - \eta_{\mu\nu\alpha\beta}\eta_{\rho\sigma\gamma\delta})u^{\alpha}u^{\gamma}E^{\beta\delta} + (\eta_{\mu\nu\alpha\beta}g_{\rho\sigma\gamma\delta} + g_{\mu\nu\alpha\beta}\eta_{\rho\sigma\gamma\delta})u^{\alpha}u^{\gamma}H^{\beta\delta}\,, \tag{2.21}$$

where

$$g_{\mu\nu\alpha\beta} \equiv g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\,. \tag{2.22}$$

In equation (2.21) the second-rank tensor $E_{\mu\nu}$ is called the "tidal force", and we refer to it as the electric part of the Weyl tensor:

$$E_{\mu\nu} \equiv u^{\rho}u^{\sigma}C_{\mu\nu\rho\sigma}\,. \tag{2.23}$$

In order to give an intuitive idea of the meaning of the tidal force we want to see its correspondent tensor in classic gravitation. From a Newtonian point of view it takes the parts of the gravitational field ϕ, the tidal force can be written as the 3-dimensional tensor

$$E_{ij} \equiv \frac{\partial^2\phi}{\partial x^i\partial x^j} - \frac{1}{3}\mathcal{P}_{ij}\frac{\partial^2\phi}{\partial x^k\partial x_k}\,. \tag{2.24}$$

It is straightforward to show that the Poisson equation can be rewritten using the tidal force in the Newtonian case

$$\frac{\partial E_{ij}}{\partial x_j} = \frac{1}{3}\frac{\partial \rho}{\partial x^i} \,. \tag{2.25}$$

The other tensor in equation (2.21) is $H_{\mu\nu}$ and, as we anticipated in the Introduction, it will be of primary importance in order to discriminate a silent universe. It is called the magnetic part of the Weyl tensor and it is defined by

$$H_{\mu\nu} \equiv \frac{1}{2}\eta_{\alpha\beta\gamma(\mu} u^\gamma u^\delta C^{\alpha\beta}{}_{\nu)\delta} \,. \tag{2.26}$$

The magnetic tidal tensor has not a Newtonian counterpart.
Both parts of the Weyl tensor are symmetric

$$E_{[\mu\nu]} = 0 \,, \qquad H_{[\mu\nu]} = 0 \,, \tag{2.27}$$

trece-free and flow-orthogonal

$$E_{\mu\nu}u^\nu = H_{\mu\nu}u^\nu = 0 \,, \qquad \mathcal{P}^\nu_\sigma E_{\mu\nu} = E_{\mu\sigma} \,, \qquad \mathcal{P}^\nu_\sigma H_{\mu\nu} = H_{\mu\sigma} \,. \tag{2.28}$$

2.2 Relativistic hydrodynamical equations

Therefore, the new approach that we are outlining want to reach the new field equation given by (2.18). Before we must recover the kinematic quantities. Considering a particle worldline that is linked to the observer from the connection vector X^μ, we can reach the relative position of the particle with respect to the observer rest-frame simply projecting the connection into it. Defying the acceleration vector

$$\frac{Du^\mu}{D\lambda} \equiv u^\mu{}_{;\nu} u^\nu \,, \tag{2.29}$$

we can reach an expression for the relative velocity of the particle with respect to the observer

$$V^\mu \equiv v^\mu{}_\nu \mathcal{P}^\nu{}_\sigma X^\sigma \,, \tag{2.30}$$

where $v_{\mu\nu}$ is the tensor that quantifies the spatial projection of the velocity covariant derivatives

$$v_{\mu\nu} \equiv \mathcal{P}^\rho{}_\mu \mathcal{P}^\sigma{}_\nu u_{\rho;\sigma} \,. \tag{2.31}$$

This quantity can be splitted in

$$v_{\mu\nu} \equiv \Theta_{\mu\nu} + \omega_{\mu\nu} \,, \tag{2.32}$$

where $\Theta_{\mu\nu} = \Theta_{(\mu\nu)}$ is the symmetric part and $\omega_{\mu\nu} = \omega_{[\mu\nu]}$ is the anti-symmetric part.
$\Theta_{\mu\nu}$ is usually called the expansion tensor and it represents the length rate in a given direction. Representing l as the length between the observer and the particle, we can quantify the isotropic part of the expansion $\Theta^\mu_\mu = \Theta$ called volume expansion, and we can reach the overall length rate as

$$\frac{Dl/D\lambda}{l} = \frac{1}{3}\Theta \,. \tag{2.33}$$

It corresponds to the well-known Hubble law in FLRW models, where $H = \Theta/3$.
$\omega_{\mu\nu}$ is called vorticity tensor and it represent a rotation that preserves all distances, i.e. a rigid rotation. We define the vorticity vector as

$$\omega_\mu = \frac{1}{2}\eta_{\mu\nu\rho}\mathcal{P}^\nu{}_\sigma u^{\rho;\sigma} \,. \tag{2.34}$$

It encodes the direction of the rotation and it gives us the possibility to write an expression for $\omega_{\mu\nu}$:

$$\omega_{\mu\nu} = \eta_{\mu\nu\rho}\omega^\rho \,. \tag{2.35}$$

Now we can decompose the covariant derivatives of the comoving flow element four velocity appearing in the spatial gradient (2.31) as

$$u_{\mu;\nu} = -u_\nu \frac{Du^\mu}{D\lambda} + \frac{1}{3}\Theta\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \,. \tag{2.36}$$

The third term, $\sigma_{\mu\nu}$, entering in this decomposition, is called shear tensor and it is trace-free; it gives the information about how the spacetime distorts the fluid flow. This distortion is volume-invariant and it is non-vanishing if the model is non-isotropic, in fact we have

seen in the Introduction (Section 1.1) that in a FLRW model, that is fully homogeneous and isotropic, the shear is zero. In other words the anisotropy direction remains unchanged while other directions change lengthening and shortening each other.
Now refocusing on the Weyl tensor, we can find the following relations for its covariant derivatives measured in a given worldline in terms of its electric and magnetic parts using the definitions of the expansion, shear and vorticity tensosr:

$$\begin{aligned}
u^\nu u^\rho C^\mu_{\nu\sigma\rho} =& \mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta E^{\alpha\beta} + \eta^{\mu\nu\alpha\beta} u_\nu \sigma_{\alpha\gamma} H^\gamma{}_\beta - 3H^\mu{}_\nu \omega^\nu \,, \\
\frac{1}{2}\mathcal{P}^\mu{}_\alpha u_\beta u^\rho \eta^{\beta\gamma\delta} C_{\gamma\delta_\sigma\rho} =& \mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta H^{\alpha\beta}{}_{;\nu} + \eta^{\mu\nu\alpha\beta} u_\nu \sigma_{\alpha\gamma} E^\gamma{}_\beta \,, \\
\mathcal{P}^{\mu\lambda}\mathcal{P}^{\nu\alpha} u^\beta C_{\alpha\beta\sigma\rho;}{}^\sigma =& \mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta \frac{DE^{\alpha\beta}}{D\lambda} + \mathcal{P}^{\alpha\nu}\eta^{\mu\beta\gamma\delta} u_\beta H_{\alpha\delta;\gamma} + \Theta H^{\mu\nu} \\
&+ 2u_\alpha H_\gamma{}^{(\mu}\eta^{\nu)\alpha\beta\gamma} \frac{Du_\beta}{D\lambda} + \mathcal{P}^{\mu\nu}\sigma^{\alpha\beta}E_{\alpha\beta} \\
&- 2E^{\alpha\nu}(\sigma^\mu{}_\alpha - \omega^\mu{}_\alpha) - E^{\alpha\mu}(\sigma^\nu{}_\alpha - \omega^\nu{}_\alpha) \,, \\
\frac{1}{2}\mathcal{P}^\mu{}_\alpha \mathcal{P}^{\nu\rho} u_\beta \eta^{\alpha\beta\gamma\delta} C_{\alpha\beta\sigma\rho;}{}^\sigma =& -\mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta \frac{DH^{\alpha\beta}}{D\lambda} + \mathcal{P}^{\alpha\nu}\eta^{\mu\beta\gamma\delta} u_\beta E_{\alpha\delta;\gamma} + \Theta E^{\mu\nu} \\
&+ 2u_\alpha E_\gamma{}^{(\mu}\eta^{\nu)\alpha\beta\gamma} \frac{Du_\beta}{D\lambda} - \mathcal{P}^{\mu\nu}\sigma^{\alpha\beta}H_{\alpha\beta} \\
&+ 2H^{\alpha\nu}(\sigma^\mu{}_\alpha - \omega^\mu{}_\alpha) + H^{\alpha\mu}(\sigma^\nu{}_\alpha - \omega^\nu{}_\alpha) \,.
\end{aligned} \tag{2.37}$$

We can see the high degree of symmetry between the magnetic tidal tensor and the force field.
The total derivatives that we used to define the acceleration vector (2.29) can be interpreted as the proper time derivatives in the rest frame of the fluid element if we choose u^μ as the velocity of the cosmological fluid flow itself. In this way equation (2.37) is very simplified for the simple reason that the projectors take a more easy configuration. Moreover the equations are fully covariant and gauge-invariant, see [43]. Now we can rewrite equation (2.18) in terms of $E_{\mu\nu}$ and $H_{\mu\nu}$, a new set of field equation for a perfect fluid introduced by Ellis (1971) [42]:

$$\begin{aligned}
\mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta E^{\alpha\beta}{}_{;\nu} =& 3H^\mu{}_\nu \omega^\nu - \eta^{\mu\nu\alpha\beta} u_\nu \sigma_{\alpha\gamma} H^\gamma{}_\beta + \frac{1}{3}\mathcal{P}^{\mu\nu}\rho_{;\nu} \,, \\
\mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta H^{\alpha\beta}{}_{;\nu} =& -3H^\mu{}_\nu \omega^\nu + \eta^{\mu\nu\alpha\beta} u_\nu \sigma_{\alpha\gamma} E^\gamma{}_\beta - (\rho + p)\omega^\mu \,, \\
\mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta \frac{DE^{\alpha\beta}}{D\lambda} =& 3E^{\alpha(\mu}\sigma^{\nu)}_\alpha - \mathcal{P}^{\alpha(\mu}\eta^{\nu)\beta\gamma\delta} u_\beta H_{\alpha\delta;\gamma} - 2u_\alpha H_\gamma{}^{(\mu}\eta^{\nu)\alpha\beta\gamma} \\
&- \Theta E^{\mu\nu} - \mathcal{P}^{\mu\nu}\sigma^{\alpha\beta}E_{\alpha\beta} - E^{\alpha(\mu}\omega^{\nu)}_\alpha - \frac{1}{2}(\rho + p)\sigma^{\mu\nu} \,, \\
\mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta \frac{DH^{\alpha\beta}}{D\lambda} =& 3H^{\alpha(\mu}\sigma^{\nu)}_\alpha + \mathcal{P}^{\alpha(\mu}\eta^{\nu)\beta\gamma\delta} u_\beta E_{\alpha\delta;\gamma} + 2u_\alpha E_\gamma{}^{(\mu}\eta^{\nu)\alpha\beta\gamma} \\
&- \Theta H^{\mu\nu} - \mathcal{P}^{\mu\nu}\sigma^{\alpha\beta}H_{\alpha\beta} - H^{\alpha(\mu}\omega^{\nu)}_\alpha \,,
\end{aligned} \tag{2.38}$$

where we denote p as the pressure of the cosmic fluid. It is fascinating how this set of equations are similar to the Maxwell equations as Ellis noted in [42]. I wrote it so that the symmetries and analogies with Maxwell equations would appear clearly. First of all these equations are exact and covariant, so independent from the coordianate system. In Newtonian theory these equation are the correspondent of the divergence and the curl of the E_{ij} defined in equation (2.24):

$$\begin{aligned} \frac{\partial E^{ij}}{\partial x^j} &= \frac{1}{3}\frac{\partial \rho}{\partial x_i} \\ \frac{\partial E^i{}_l}{\partial x^k}\eta^{jlk} &+ \frac{\partial E^j{}_l}{\partial x^k}\eta^{ilk} = 0\,. \end{aligned} \tag{2.39}$$

In order to get the "topos" of the Ellis equations, velocity gradient $u_{\mu;\nu}$, we can solve the Einstein field equations and find the covariant derivatives, or using the evolution equation of the kinematic components already defined we can write:

$$\begin{aligned} \frac{D\Theta}{D\lambda} + \frac{1}{3}\Theta^2 - \frac{Du^\mu{}_{;\mu}}{D\lambda} + 2(\omega^2 - \sigma^2) + \frac{1}{2}(\rho + 3p) = 0\,, \\ \mathcal{P}^\mu{}_\nu \frac{D\omega^\nu}{D\lambda} + \frac{1}{2}\eta^{\mu\nu\alpha\beta}u_\nu \frac{Du_{\beta;\alpha}}{D\lambda} + \frac{2}{3}\Theta\omega^\mu - \sigma^\mu{}_\nu\omega^\nu = 0\,, \\ E^{\mu\nu} + \mathcal{P}^\mu{}_\alpha \mathcal{P}^\nu{}_\beta \frac{D\sigma^{\alpha\beta}}{D\lambda} - \frac{Du^{(\nu;\mu)}}{D\lambda} + \frac{2}{3}\Theta\sigma^{\mu\nu} \\ +\sigma^{\mu\alpha}\sigma^\nu{}_\alpha + \omega^\mu\omega^\nu + \frac{1}{3}\mathcal{P}^{\mu\nu}\left(2\sigma^2 + \omega^2 - \frac{Du^\mu{}_{;\mu}}{D\lambda}\right) = 0\,. \end{aligned} \tag{2.40}$$

In particular the first equation is the well-known Raychaudhuri equation, see [45]. Adding a cosmological constant Λ and using (2.33) we can have a more intuitive form of the Raychaudhuri equation

$$3\frac{D^2 l}{D\lambda^2} = 2(\omega^2 - \sigma^2) + \frac{Du^\mu{}_{;\mu}}{D\lambda} + \frac{1}{2}(\rho + 3p) + \Lambda\,. \tag{2.41}$$

This form shows that the acceleration rate of the length between the comoving observer and a nearby particle is directly proportional to the density of the fluid flow and to the cosmological constant, therefore Λ take the parts of a constant repulsive force. Moreover we can see that the rotation scalar is acting a repulsive effect that is unaffected by gravity, the corresponding of the centrifugal force. The shear term distorts in a attractive way the fluid flow and, as could be expected, the covariant derivatives of the acceleration vector influence the distance between the worldlines.

Looking at the second dynamical equation, considering that the shear and the acceleration are zero, the variation of the vorticity can be described by a rigid flow because there is not deviations in the fluid worldlines.

In the last equation is interesting see that the only source of the shear tensor is the Electric part of the Weyl tensor. The force field take the parts of the gravitational field that induce the variation of the shear in the flow lines.

As the Einstein field equations, this set of dynamical equation must be solved only with the density and pressure evolution. We can rewrite the continuity equation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2.42}$$

in terms of the kinematical quantities, only splitting the covariant derivatives of $T^{\mu\nu}$ in the orthogonal and parallel part with respect to u^{μ}:

$$\frac{D\rho}{D\lambda} + (\rho + p)\Theta = 0\,. \tag{2.43}$$

Therefore we have defined a close set of dynamical equations for the density and the geometry of the space-time in the presence of a mass-fluid element.

3 Silent Universes

In this section we follow the analysis performed by Matarrese et. al. (1993)-(1995) [21],[17] through Lagrangian evolution, analyizing the worldline trajectory of a cold dust irrotational fluid flow element with 4-velocity u^μ, only dependent on ρ, on the velocity gradient $u^\mu{}_\nu$ and on the electric tide $E_{\mu\nu}$, in the case of $H_{\mu\nu} = 0$.

3.1 Dynamics of irrotational dust

We have seen that the density and the velocity gradient are local, moreover the evolution of $E_{\mu\nu}$ depends only on local components, therefore $H_{\mu\nu} = 0$ implies that the worldlines of the cosmological fluid flow evolve independently of each other. Barnes and Rowlingson (1989) [20] found that, imposing $\omega_{\mu\nu} = 0$ and $H_{\mu\nu} = 0$, the second equation of (2.38), that represents the H-divergence, gives rise to the equation

$$\eta^{\mu\nu\alpha\beta} u_\nu \sigma_{\alpha\gamma} E^\gamma{}_\beta = 0\,. \tag{3.1}$$

Contracting the indices with the fully antisymmetric tensor, we can find the spatial relation

$$\sigma^{i[k} E^{j]}{}_k = 0\,. \tag{3.2}$$

Therefore the representative matrix of the force field and the shear commute. So there must exist a common orthonormal tetrad for $E_{\mu\nu}$ and $\sigma_{\mu\nu}$. In this way we can write

$$E_{\mu\nu} = \sum_{s=1}^{3} E_s e_{s\mu} e_{s\nu}, \qquad \sigma_{\mu\nu} = \sum_{s=1}^{3} \sigma_s e_{s\mu} e_{s\nu} \tag{3.3}$$

where $\{E_s\}_{s=1,2,3}$ and $\{\sigma_s\}_{s=1,2,3}$ are respectively the eigenvalues of $E_{\mu\nu}$ and $\sigma_{\mu\nu}$, and $\{e_{s\mu}\}_{s=1,2,3}$ are the common eigenvectors.
Following Bruni, Matarrese and Pantano (1995) [17] there must exist a coordinate basis in which the metric is diagonal with basis vectors the eigenvectors of the force field and the shear. Moreover we can interpret $e_{s\mu}$ as the tetrad of an orthonormal hypersurface orthogonal in each point to the velocity flow field, thus we can write

$$u_\mu = -\delta^0_\mu\,, \qquad e_{s\mu} = l_\alpha \delta^s_\mu\,. \tag{3.4}$$

In this way we can write the general line element of a silent model as

$$ds^2 = -dt^2 + \sum_{s=1}^{3} l_s^2(t, \mathbf{x})(dx^s)^2\,, \tag{3.5}$$

where we used the notation for the length size l but in the above equation has a different value depending on the direction. In this way we can write the expansion rate in the the three directions $e_{s\mu}$ as

$$\frac{Dl_s/D\lambda}{l_s} = \sigma_s + \frac{1}{3}\Theta\,. \tag{3.6}$$

Moreover we can define the average length, given by

$$l = (l_1 l_2 l_3)^{1/3}\,, \tag{3.7}$$

in this way we can write the expansion rate as

$$\frac{Dl/D\lambda}{l} = \frac{1}{3}\Theta\,. \tag{3.8}$$

Now we make a differentiation of the collapses corresponding to the different configurations that the directional lengths can take.

- Point-like singularity: all three $l_s \to 0$.
- Spindle-like singularity: two $l_s \to 0$ and the third $\to \infty$.
- Pancake-like singularity: two of l_s take finite values and the other tends to zero.
- Cylinder singularity: two $l_s \to 0$ and the third takes a constant value.

We have already said that if we take the velocity field of the fluid element as in equation (2.38) the total derivatives can be written as the time derivatives, from now on we will use the time derivative notation $\dot{(.)}$.
So the fundamental values of the silent theory are the density, the expansion scalar and the eigenvalues of the shear and the force field. We can label each of these values to a component of a six dimensional phase-space position vector:

$$\mathbf{X} = (\rho, \Theta, \sigma_1, \sigma_1, E_1, E_2)\,, \tag{3.9}$$

where we uses only two eigenvalues for the shear and the force field because both are trace-free.
Matarrese et al. (1993) showed that the evolution of the flow worldlines is given by a set of six first-order ordinary differential equations giving the time evolution of the components of $\mathbf{X}$. It is is important the fact that these differential equations are ordinary, because give us the information that each fluid element is uninfluenced by the surrounding ones, and these proceed only forced by the initial conditions. In other words, defying the differential equation as $\dot{\mathbf{X}} = \mathbf{V}(\mathbf{X})$, $\mathbf{V}$ is time independent and the orbits do not intersect. Using the tidal Maxwell equations (2.38), the continuity equation (2.43) and the constraint equations

(2.40) we can write:

$$\begin{aligned}
\dot{\rho} &= -\Theta\rho\,,\\
\dot{\Theta} &= -\frac{1}{3}\Theta^2 - 2\sigma_1^2 - 2\sigma_1\sigma_2 - 2\sigma_2^2 - \frac{1}{2}\rho\,,\\
\dot{\sigma}_1 &= \frac{2}{3}\sigma_2(\sigma_1+\sigma_2) - \frac{1}{3}\sigma_1^2 - \frac{2}{3}\Theta\sigma_1 - E_1\,,\\
\dot{\sigma}_2 &= \frac{2}{3}\sigma_2(\sigma_2+\sigma_1) - \frac{1}{3}\sigma_2^2 - \frac{2}{3}\Theta\sigma_2 - E_2\,,\\
\dot{E}_1 &= E_1(\sigma_1-\sigma_2) - E_2(\sigma_1+2\sigma_2) - \Theta E_1 - \frac{1}{2}\rho\sigma_1\,,\\
\dot{E}_2 &= E_2(\sigma_2-\sigma_1) - E_1(\sigma_2+2\sigma_1) - \Theta E_2 - \frac{1}{2}\rho\sigma_2\,.
\end{aligned} \tag{3.10}$$

Defying the local average scale factor $l = (l_1 l_2 l_3)^{1/3}$ and using continuity equation in equation (3.10), we can recover the density law as function of the lengths

$$\rho = \frac{M}{l_1 l_2 l_3}\,. \tag{3.11}$$

In this equation we denote the constant averaged mass measured at a time t_0:

$$M = \rho(t_0) l(t_0)\,. \tag{3.12}$$

We can see that, when a metric length $l_s \to 0$ we get a density singularity.
From equation (3.10) we can see that, in the six dimensional phase space of the dynamics variables,

$$\nabla \cdot \mathbf{V} = -5\Theta\,. \tag{3.13}$$

Therefore the divergence $\mathbf{V}$ depends only on Θ and the system is conservative if the system is contracting $\Theta < 0$, dissipative if the system is expanding $\Theta > 0$. From Raychaudhuri equation (the second of equations (3.10)), since the velocity expansion rate is strictly negative, as soon as Θ take a negative value, the collapse is irreversible and in this case the only fixed point is the origin of the six dimensional phase space. On the other hand if the absolute value of the position vector $\mathbf{X}$ diverge also all the six dynamical variables diverge independently.
Of course the most restrictive silent universe assumption is the cancellation of $H_{\mu\nu}$. Matarrese et. al. (1994) [22] proved that in a perturbated FLRW universe the magnetic part of the Weyl tensor is generated at second order (the already mentioned "tidal induction"), so during the inhomogeneities' nonlinear evolution. If we think with a inductive reasoning we can say that silent models can be a special case of a more general model. Mutoh et. al. (1997) [46] analyzed the perturbations of silent universe and the effect of a perturbed and non-zero magnetic tidal tensor. They found that in a spindle-like collapse the perturbations of $H_{\mu\nu}$ diverge and so there is exchange of gravitational information, "the silence is broken". They dub this "quiet universe". On the other hand in a pancake-like collapse the perturbations vanish, this is in agreement that, in the Newtonian case, the configuration of the cosmological fluid tends to a pancake-like collapse.

3.2 Dimensionless Variables

Bruni, Matarrese and Pantano (1995) [17] rewrite the problem with a new set of variables in order to understand the collapsing or expanding configurations. We can perform a linear transformation in our 6-dimensional phase-space writing:

$$\begin{aligned} \sigma_+ &= \frac{1}{2}(\sigma_1+\sigma_2), \qquad & E_+ &= \frac{1}{2}(E_1+E_2)\,, \\ \sigma_- &= \frac{1}{2}(\sigma_1-\sigma_2), \qquad & E_- &= \frac{1}{2}(E_1-E_2)\,. \end{aligned} \tag{3.14}$$

With this transformation we have a new set of variables in the 6-dimensional phase-space, more specifically we denote the new position vector as $\mathbf{Y} = (\rho, \Theta, \sigma_+, \sigma_-, E_+, E_-)$ and we can write the autonomous system $\dot{\mathbf{Y}} = \mathbf{W}(\mathbf{Y})$ as

$$\begin{aligned} \dot{\rho} &= -\Theta\rho\,, \\ \dot{\Theta} &= -\frac{1}{3}\Theta^2 - 6\sigma_+^2 - 2\sigma_-^2 - \frac{1}{2}\rho\,, \\ \dot{\sigma}_+ &= \sigma_+^2 - \frac{1}{3}\sigma_-^2 - \frac{2}{3}\Theta\sigma_+ - E_+\,, \\ \dot{\sigma}_- &= -2\sigma_+\sigma_- - \frac{2}{3}\Theta\sigma_- - E_-\,, \\ \dot{E}_+ &= \sigma_-E_- - 3E_+\sigma_+ - \Theta E_+ - \frac{1}{2}\rho\sigma_+\,, \\ \dot{E}_- &= 3\sigma_-E_+ - 3E_-\sigma_+ - \Theta E_- - \frac{1}{2}\rho\sigma_-\,. \end{aligned} \tag{3.15}$$

In this case we still have the divergence of the position vector

$$\nabla \cdot \mathbf{W} = -5\Theta\,, \tag{3.16}$$

and we still recognise the only stationary point of the system as the origin of the new phase-space.

The most interesting special cases hidden in this ODEs system are the Szekeres models, see Szekeres (1975) [8]. The first possibility corresponding to these models is the case in which $\sigma_1 = \sigma_2$ and $E_1 = E_2$, or in the new variables

$$\sigma_- = 0\,, \qquad E_- = 0\,. \tag{3.17}$$

Thus the orbits of that models are restricted to a 4-dimensional phase-space, and this means that we must perform a restriction on the initial conditions. The second possibility in which we can recover Szekeres models is the case in which $\sigma_1 = \sigma_3$, $E_1 = E_3$ or $\sigma_2 = \sigma_3$, $E_2 = E_3$, in the new variables

$$\sigma_+ = \pm\frac{1}{3}\sigma_- \qquad E_+ = \pm\frac{1}{3}E_-\,. \tag{3.18}$$

In the next section we will study the dynamics of these special models. In particular in this thesis we will analyze a Szekeres type metric in order to compute the CMB temperature anisotropies.

In order to be consistent with the observations we can using the "dimensionless variables", see Goode (1989) [47]. We use the following transformation with the previous variables, in particular the standard density parameter is

$$\Omega = \frac{\rho}{3\Theta}\,, \tag{3.19}$$

then we have the transformation definitions

$$\begin{aligned} \Sigma_+ &= \frac{\sigma_+}{\Theta}\,, \qquad & \epsilon_+ &= \frac{E_+}{\Theta^2} \\ \Sigma_- &= \frac{\sigma_-}{\Theta}\,, \qquad & \epsilon_- &= \frac{E_-}{\Theta^2}\,. \end{aligned} \tag{3.20}$$

The most relevant case in which we can find a $\Theta = 0$ singularity is the turn-around epoch in which, in a close universe, there is the inversion between expansion and contraction. In all other cases, more specifically in the collapse analysis and in occurring singularities, it is a worthwhile choice. Now defying the 5-dimensional phase-space with the corresponding position vector $\mathbf{G} = (\Omega, \Sigma_+, \Sigma_-, \epsilon_+, \epsilon_-)$ we can write

$$\begin{aligned} &\dot{\Theta} = -\Theta^2\Big(\frac{1}{3} + 6\Sigma_+^2 + 2\Sigma_-^2 + \frac{1}{6}\Omega\Big)\,, \\ &\dot{\mathbf{G}} = \Theta\mathbf{F}(\mathbf{G})\,, \end{aligned} \tag{3.21}$$

where $\mathbf{F}$ is a subsystem of ODEs given in equation (3.15) containing the transformed variables $\{\rho, \sigma_+, \sigma_-, E_+, E_-\} \rightarrow \{\Omega, \Sigma_+, \Sigma_-, \epsilon_+, \epsilon_-\}$. From this relation we can observe that the 5-dimensional hyperplane of equation

$$\Theta = 0 \tag{3.22}$$

is an open set of stationary points in which $\mathbf{G}$ diverge on it. Of course in this case we have that the origin of the 6-dimensional phase-space is still a stationary point. The analysis of the trajectories must be done in two different subsets in which there are contraction or expansion, i.e. respectively $\Theta < 0$ or $\Theta > 0$. For this purpose we use a new dimensionless variable that take the parts of a time variable for our 5-dimensional position vector $\mathbf{G}$, we define

$$\tau = \int |\Theta|\ dt = \begin{cases} +\int \Theta\, dt & = +3\ln l \qquad \text{for} \qquad \Theta > 0 \\ -\int \Theta\, dt & = -3\ln l \qquad \text{for} \qquad \Theta < 0 \end{cases} \tag{3.23}$$

In this way the time derivatives $d\tau/dt$ must be strictly positive. In the limit in which $\tau \rightarrow \infty$ we can reach the collapse of the cosmic fluid, in fact if the average length $l \rightarrow 0$ the expansion scalar must be divergently negative $\Theta \rightarrow -\infty$. On the contrary if we have a continuous expansion we still have $\tau \rightarrow \infty$, the average length diverge, $l \rightarrow \infty$, so the expansion scalar tends to vanish, $\Theta \rightarrow 0$, see equations (3.7), (3.8).
Now we denote the τ-derivatives as $(.)' = d/d\tau$. In this way we can write for the contraction epoch the following ODEs system in the 6-dimensional phase-space $\mathcal{W} = (\Theta', \mathbf{G}')$ given by

the six dynamical equations

$$
\begin{aligned}
\Theta' &= -\Theta\left(\frac{1}{3} + 6\Sigma_+^2 + 2\Sigma_-^2 + \frac{1}{6}\Omega\right)\,,\\
\Omega' &= -\frac{1}{3}[12(3\Sigma_+^2 + \Sigma^2) - 1 + \Omega]\,,\\
\Sigma_+' &= \Sigma_+\left[\frac{1}{3} - 2(3\Sigma_+^2 + \Sigma_-^2) - \Sigma_+ - \frac{1}{6}\Omega\right] + \frac{1}{3}\Sigma_-^2 + \epsilon_+\,,\\
\Sigma_+' &= \Sigma_-\left[\frac{1}{3} - 2(3\Sigma_+^2 + \Sigma_-^2) + 2\Sigma_+ - \frac{1}{6}\Omega\right] + \epsilon_-\,,\\
\epsilon_+' &= \epsilon_+\left[\frac{1}{3} - 4(3\Sigma_+^2 + \Sigma_-^2) + 3\Sigma_+ - \frac{1}{3}\Omega\right] - \Sigma_-\epsilon_- + \frac{1}{6}\Sigma_+\Omega\,,\\
\epsilon_-' &= \epsilon_-\left[\frac{1}{3} - 4(3\Sigma_+^2 + \Sigma_-^2) - 3\Sigma_+ - \frac{1}{3}\Omega\right] - 3\Sigma_-\epsilon_+ + \frac{1}{6}\Sigma_-\Omega\,.
\end{aligned}
\tag{3.24}
$$

Now we can see that the equations that refers to the variables of the five dimensional position vector $\mathbf{G}$ are completely independent from the expansion scalar. Therefore we can only study the 5-dimensional system $\mathbf{G}' = \mathbf{F}(\mathbf{G})$

$$
\mathbf{G}' = \begin{cases} +\mathbf{F}(\mathbf{G}) & \text{for} \quad \Theta > 0 \\ -\mathbf{F}(\mathbf{G}) & \text{for} \quad \Theta < 0\,. \end{cases}
\tag{3.25}
$$

In the case of $\Theta > 0$ we have

$$
\begin{aligned}
\Theta' &= \Theta\left(\frac{1}{3} + 6\Sigma_+^2 + 2\Sigma_-^2 + \frac{1}{6}\Omega\right)\,,\\
\Omega' &= \frac{1}{3}[12(3\Sigma_+^2 + \Sigma_-^2) - 1 + \Omega]\,,\\
\Sigma_+' &= -\Sigma_+\left[\frac{1}{3} - 2(3\Sigma_+^2 + \Sigma_-^2) - \Sigma_+ - \frac{1}{6}\Omega\right] - \frac{1}{3}\Sigma_-^2 - \epsilon_+\,,\\
\Sigma_-' &= -\Sigma_-\left[\frac{1}{3} - 2(3\Sigma_+^2 + \Sigma_-^2) + 2\Sigma_+ - \frac{1}{6}\Omega\right] - \epsilon_-\,,\\
\epsilon_+' &= -\epsilon_+\left[\frac{1}{3} - 4(3\Sigma_+^2 + \Sigma_-^2) + 3\Sigma_+ - \frac{1}{3}\Omega\right] + \Sigma_-\epsilon_- - \frac{1}{6}\Sigma_+\Omega\,,\\
\epsilon_-' &= -\epsilon_-\left[\frac{1}{3} - 4(3\Sigma_+^2 + \Sigma_-^2) - 3\Sigma_+ - \frac{1}{3}\Omega\right] + 3\Sigma_-\epsilon_+ - \frac{1}{6}\Sigma_-\Omega\,.
\end{aligned}
\tag{3.26}
$$

Thus in the case of expansion we note that the sign is simply the opposite than in the contraction case, a parity symmetry driven by the time reversal transformation $\tau \to -\tau$. More in detail, the parity symmetry give us the information that the trajectories for $\Theta > 0$ and for $\Theta < 0$ are mirrored. This symmetry will make the same fixed points up and down the hyperplane $\Theta = 0$, so in the 5-dimensional phase-space the stationary points take convergent values for the $\mathbf{G}$ coordinates. Writing the Jacobian matrix of $\mathbf{G}$ we can see that the signs change with the sign of Θ, therefore the eigenvalues of the matrix change values, thus the stable points have the corresponding unstable ones and completely unstable points have the corresponding stable ones beyond the hyperplane of symmetry $\Theta = 0$. Moreover the form of the Raychaudhuri equation gives us the information that the "time" variation Θ' will be invariant under the "time-reversal" transformation. We can rewrite Raychaudhuri equation

$$\Theta' = \pm\alpha\Theta\,, \tag{3.27}$$

where

$$\alpha \equiv \left(\frac{1}{3} + 6\Sigma_+^2 + 2\Sigma_-^2 + \frac{1}{6}\Omega\right)\,. \tag{3.28}$$

This gives the solution

$$\Theta = \Theta_0 e^{\pm\alpha(\tau-\tau_0)}\,. \tag{3.29}$$

Therefore we can write the equation

$$(\tau - \tau_0) = \mp\frac{1}{\alpha}\ln[1 + \alpha\Theta_0(t - t_0)]\,. \tag{3.30}$$

This is the connection form between the physical time t and dimensionless variable τ. In this way we can write the relations for the expansion scalar, the average scale length and the various length scales along the principal axis:

$$\begin{aligned} \Theta &= \frac{\Theta_0}{1 + \alpha\Theta_0(t - t_0)}\,, \\ l_s &= l_{s0}[1 + \alpha\Theta_0(t - t_0)]^{p_s}\,, \\ l &= l_0[1 + \alpha\Theta_0(t - t_0)]^{1/3\alpha}\,, \end{aligned} \tag{3.31}$$

where

$$p_s = \frac{1}{\alpha}\left(\frac{1}{3} + \Sigma_s\right)\,. \tag{3.32}$$

In particular $\Sigma_1 = \Sigma_+ + \Sigma_-$, $\Sigma_2 = \Sigma_+ - \Sigma_-$ and $\Sigma_3 = -2\Sigma_+$. Every fixed point represents a model and the contraction or expansion is monotonic and it depends on the signs of Θ_0. We can see that the singulirity

$$t - t_0 = \begin{cases} -\frac{1}{\alpha\Theta_0} & \text{for} \qquad \Theta_0 > 0 \\ +\frac{1}{\alpha\Theta_0} & \text{for} \qquad \Theta_0 < 0\,. \end{cases} \tag{3.33}$$

depends too on the signs of Θ_0.
In the next section we want to study the special case of Szekeres models, the already seen degenerate case in which both the variables Σ_- and ϵ_- vanish.

3.3 Szekeres models

Barnes and Rowlingson (1989) [20] showed that the Szekeres solutions (see [8]) are solutions of an irrotational dust cosmological fluid element, with $H_{\mu\nu} = 0$, moreover the shear and the force field have two degenerate eigenvalues. We can interpret FLRW solutions (homogeneous and isotropic), the Kantowski-Sachs solutions (homogeneous and anisotropic) and Tolman-Bondi solutions (inhomogeneous and isotropic) as special cases of Szekeres solutions, a generalization of all three. There is a large literature on Szekeres models. More specifically Szekeres (1975) studied a finite region filled with irrotational dust collapsing. An important analysis was studied by Bonnor and Tomimura (1976) [48] where they classified various models from the different values reached by the Szekerese metric parameters. Szekeres solutions are divided in two classes:

- Class I: Solutions that formally represent the Tolman-Bondi spherically symmetric solutions, generalising them. They are primarily used for study the nonspherical collapse of an inhomogeneous dust mass distribution.
- Class II: Solutions that generalise Kantowski-Sachs and FLRW. These solutions are used for the study of cosmological models.

Goode and Wainwright (1982) [9] found that a new formulation of the Szekeres metric allows to study both Szekeres classes at the same time without modifying them. With this formulation they studied in detail the singularities of the model.
Considering $\Theta < 0$, so the collapsing epoch, we start to study the Szeekeres models dynamics imposing

$$\Sigma_- = \epsilon_- = 0\,. \tag{3.34}$$

Therefore the 5-dimensional position vector $\mathbf{G}$ is restricted to the 3-dimensional position vector $\mathbf{q} = (\Omega, \Sigma_+, \epsilon_+)$, where we can define the restricted ODEs subsystem as $\mathbf{q}' = -\mathbf{f}(\mathbf{q})$. Thus the dynamical equations take the form

$$\begin{aligned} \Theta' &= \Theta\left(\frac{1}{3} - 6\Sigma_+ - \frac{1}{6}\Omega\right)\,, \\ \mathbf{q}' &= -\mathbf{f}(\mathbf{q})\,, \end{aligned} \tag{3.35}$$

where we can explicitly define de form of $\mathbf{f}$ as

$$\begin{aligned} \Omega' &= -\frac{1}{3}\Omega(36\Sigma_+' - 1 + \Omega)\,, \\ \Sigma_+' &= \Sigma_+\left[\frac{1}{3} - \Sigma_+(1 + 6\Sigma_+) - \frac{1}{6}\Omega\right] + \epsilon_+\,, \\ \epsilon_+' &= \epsilon_+\left[\frac{1}{3} - 3\Sigma_+(4\Sigma_+ - 1) - \frac{1}{3}\Omega\right] + \frac{1}{6}\Sigma_+\Omega\,. \end{aligned} \tag{3.36}$$

Of course if the system is in a situation of expansion, $\Theta > 0$, the sign of $\mathbf{f}$ changes. We can easily calculate the divergence

$$\nabla \cdot \mathbf{f} = 1 - \frac{7}{6}\Omega + \Sigma_+(1 - 42\Sigma_+)\,. \tag{3.37}$$

Therefore we can see that the system has a well-behaviour if

$$\nabla \cdot \mathbf{f} < 0 \qquad \text{for} \qquad \Omega > \frac{6}{7}\left[1 + \Sigma_+(1 - 42\Sigma_+)\right] . \tag{3.38}$$

Thus we can see that in any point of the 3-dimensional phase-space the trajectory of the flow of $\mathbf{f}$ are converging except for the parabola in the Ω-Σ_+ plane of equation

$$\Omega = \frac{6}{7}\left[1 + \Sigma_+(1 - 42\Sigma_+)\right] . \tag{3.39}$$

Moreover it is interesting the fact that $\nabla \cdot \mathbf{f}$ is independent on ϵ_+. For the Ω factor in the right-hand side of the correspondent of the continuity equation in (3.36) we can see that the trajectories in the reduced 3-dimensional phase-space cannot cross the plane $\Omega = 0$, so we must focus on the section given by $\Omega \geq 0$. We can see that the condition in equation (3.38) are satisfied everywhere if it is satisfied the condition

$$\Omega' < 0 \qquad \text{for} \qquad \Omega > 1 - 36\Sigma_+^2 . \tag{3.40}$$

Therefore, for the condition $\Theta < 0$, the equation $\Omega = 0$ denote a convergence plane for the trajectories in the 3-dimensional phase-space with position vector $\mathbf{q}$, and we deduce that all the points on it will be attractors.
Now we can study the Jacobian matrix of the system $\mathbf{f}(\mathbf{q})$ given by:

$$J(\mathbf{q}) = \begin{pmatrix} -\frac{1}{3}(\Omega + \Sigma_+^2 - \frac{1}{3}x) & -24\Omega\Sigma_+ & 0 \\ -\frac{1}{6}\Sigma_+ & -\frac{1}{6}\Omega - \Sigma_+(12\Sigma_+ + 1) - \Sigma_+(6\Sigma_+ + 1) + \frac{1}{3} & 1 \\ \frac{1}{6}\Sigma_+ - \frac{1}{3}\epsilon_+ & \frac{1}{6}\Omega - 3\epsilon_+(8\Sigma_+ - 1) & -\frac{1}{3}x - 3\Sigma_+(4\Sigma_+ - 1) + \frac{1}{3} \end{pmatrix} , \tag{3.41}$$

Now we can find the stationary points simply imposing $\mathbf{f}(\mathbf{q}) = 0$ and solving the system. We can find the following points

Point	Ω	Σ_+	ϵ_+
D_1	1	0	0
D_2	0	0	0
D_3	0	1/6	0
D_4	0	-1/3	0
D_5	0	1/3	2/9
D_6	0	-1/12	1/32
D_7	-3	-1/3	1/6

Table 1: Stationary points for the $\mathbf{f}(\mathbf{q}) = 0$.

Now we find the values of the Jacobian for each stationary point $\{D_s\}_{s=1,\ldots,7}$ and compute the different eigenvalues $\{\lambda_r\}_{r=1,2,3}$, we impose

$$(J - \lambda_r \mathbb{1})\mathbf{v} = 0 , \tag{3.42}$$

where **v** is a generic vector on the 3-dimensional phase-space, in order to understand if the point represents a attractor, a repeller or a saddle.

We want to stress that if all $\lambda_r < 0$ the point is asymptotically stable, an attractor for the trajectories in the 3-dimensional phase-space, on the contrary if $\lambda_r > 0$ the point is unstable. Different signs denote a saddle point. The different eigenvalues values and their interpretation are given in Table 2

Point	λ_1	λ_2	λ_3
D_1	-1/3	-1/3	1/2
D_2	1/3	1/3	1/3
D_3	0	-1/2	1/2
D_4	-1	-1	-2
D_5	-1	-5/3	-2/3
D_6	1/4	-1/4	5/8
D_7	-1	-1/2	1

Table 2: Eigenvalues λ_r of the Jacobian J a in the case of expansion collapse $\Theta < 0$.

In the expansion epoch there is the transformation $\tau \to -\tau$ and the sign of the Jacobian $J(\mathbf{q})$ changes, and subsequently the divergence $\nabla \cdot \mathbf{f}$ and the eigenvalues λ_r change sign, more in detail the condition (3.39) changes the direction of the inequalities, in this way the Ω-plane is a set of repulsors points. In expansion regime or in collapse regime the λ_r have opposite sign so opposite behaviour, i.e. for $\Theta > 0$ a repulsor become an attractor and vice versa, while a saddle remains a saddle.
Now we want to interpret the stationary points $\{D_s\}_{s=1,\ldots,7}$. Of course they remain unchanged under expansion or collapse. Each stationary point represent a model, the sign of the scalar expansion Θ give us the information about the expansion of the model from an initial singularity or the collapse towards a future singularity.
In the previous section we have defined the paramaeter α and p_s, see equations (3.28), (3.32), they are useful to understand the length in $\Theta > 0$ regime or in $\Theta < 0$ regime, see Table 3. We begin with point D_1, a saddle point, it represents the well-known flat FLRW

Point	α	p_1	p_2	p_3
D_1	1/2	2/3	2/3	2/3
D_2	1/3	1	1	1
D_3	1/2	1	1	0
D_4	1	0	0	1
D_5	1	2/3	2/3	-1/3
D_6	3/8	2/3	2/3	4/3

Table 3: Parameters β and p_s for the stationary points.

model, in fact we see that the shear and the force field parameter are both zero. In point D_2 we see a Milne universe, a void without shear and force field, it is conformally flat and so locally equivalent to an open FLRW model. It is a repeller in the collapse regime (so a attractor in the expansion regime). Therefore both D_1 and D_2 have a collapse and expansion that is spherical. D_2 represents that the continuous expansion of a void is locally equivalent to a Milne universe. Point D_3 is a saddle and represents a solution with $\Omega = 0$ (vacuum) that is pure Szekeres, it do not fall in special cases. Point D_4, a repeller for $\Theta > 0$ (thus an attractor for $\Theta < 0$) is a degenerate Kasner model with $p_1 = p_2 = 0$, so non-expanding along x, y and it has a pancake singularity. Also D_5 is a repeller for $\Theta > 0$ and represents a degenerate Kasner model but now with one expansion direction and two contractiong directions. Therefore point D_4 represents the collapse of locally asymmetrical pancakes and D_5 represents the collapse of filament structures. Point D_6 is a saddle and represents an inhomogeneous expansion. Following Bonnor and Tomimura (1976) [48], there is a subclass of Szekeres models that asymptotically tend to D_6. D_7 is unphysical because of the density parameter is strictly negative. Moreover, if the shear parameter $\Sigma_+ > 0$ we can see that, during the collapse $\Theta < 0$, the fluid elements take a prolate configuration, that is the case of points D_3 and D_5, in fact we see in 3 that D_3 has a length direction that remains unchanged and D_5 has an expanding length direction during the collapse and it is easy to see in mind the filament. On the other hand if $\Sigma_+ < 0$ the fluid elements take a oblate configuration. We see that D_4 and D_6 have an oblate collapse, in fact D_4 has only a collapsing direction while the other two remain unchanged, and D_6 has a faster collapse direction.

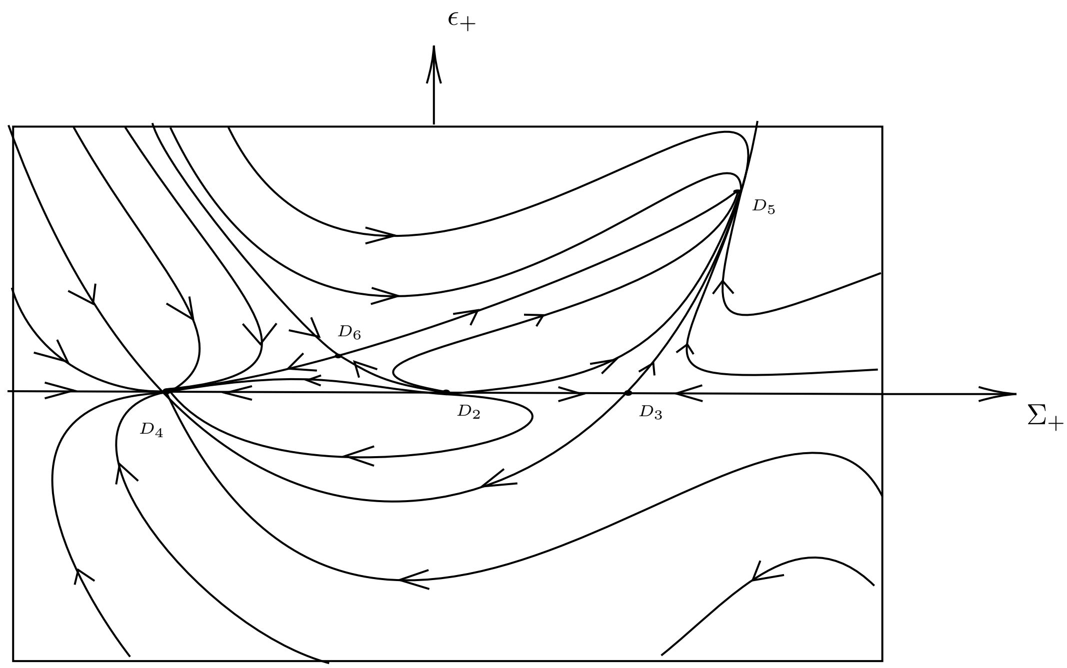


Figure 3: Σ_+-ϵ_+ plane (with $\Omega = 0$) in a collapsing regime $\Theta < 0$.

In 3.3 we show the trajectories during the collapse, $\Theta < 0$. We can see that that D_4 and D_5 are attractors, D_2 is a repeller, D_3 and D_6 are saddles. The Σ_+ axis divide in two parts the plane. Because of ϵ_+ contains the information of the electric part of the Weyl tensor, we have that the gravitational information vanish in $\epsilon_+ = 0$. As long as we are referring to a plane in which $\Omega = 0$ we have that $\rho = 0$. For this reason we interpret $\epsilon_+ = 0$ as the Minkowski space-time.

In Section 4 we will study the perturbation around FLRW of the Szekeres metric in order to study the trajectories and do the phase plane analysis.

3.4 Silence universes in Newtonian limit

If the metric perturbations are 1-dimensional there is local evolution. But there is a very interesting feature occurring if the matter distribution is in Newtonian limit. Matarrese et al. (1993) [21] showed that if the magnetic part of the Weyl tensor is null in the Newtonian limit the force field has a local Lagrangian evolution equation and the locality is broken only when there is the intersection of trajectories. Therefore, since mass element is followed by the tidal field depending locally only by density and velocity gradient, these fluid variables and tidal field must evolve indipendently. Moreover, Matarrese et al. (1994) [22] discover that $H_{\mu\nu}$ do not necessarily vanish in Newtonian limit. We follow Bertschinger and Hamilton (1994) [39] in order to get the magnetic part of the Weyl tensor in Newtonian limit. Starting from the line element

$$ds^2 = a^2(\tau)[-(a+2\phi)d\tau^2 + 2w_i d\tau dx^i + ((1-2\psi)\gamma_{ij} + 2\chi_{ij})dx^i dx^j] \,, \tag{3.43}$$

see (1.27) Section 1.2.2. The two metric potential ϕ and ψ coincide to the Bardeen potentials, respectively $-\Phi$ and Ψ. Now we compute the Weyl tensor:

$$\begin{aligned} C^0{}_{i0j} &= -\frac{1}{2}\left[D_{ij}(\psi+\phi)\dot{W}_{ij} + (\partial_\eta^2 + \nabla^2 - 2K)\chi_{ij}\right] \,, \\ C^0{}_{ijk} &= 2\nabla_{[k}W_{|i|j]} + \frac{1}{2}(\nabla^2 + 2K)\gamma_{i[j}w_{k]} + 2\nabla_{[j}\dot{\chi}_{|i|k]} \,, \\ C^i{}_{jkl} &= \gamma^i_{jmn}\gamma^{pn}_{kl}\left[C^m{}_{0p0} + (\nabla^2 - 3K)\chi^m{}_p\right] + \gamma^i{}_{jmn}\gamma^{pq}{}_{kl}\nabla_p\nabla^n\chi^m{}_q \,, \end{aligned} \tag{3.44}$$

where we used the definitions

$$D_{ij} \equiv \nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2 \,, \qquad W_{ij} \equiv \nabla_{(i}w_{j)} \,, \qquad \gamma^i{}_{jkl} \equiv 2\delta^i_{[k}\gamma_{|j|l]} \,. \tag{3.45}$$

Now using the 4-velocity of the comoving cosmological flow

$$u^\mu = (\frac{1}{a}, 0, 0, 0) \,, \tag{3.46}$$

and using the decompositions of the Weyl tensor in its electric and magnetic parts (see equations (2.23) and (2.26)) we can find the force field

$$E_{ij} = \frac{1}{2}D_{ij}(\psi+\phi) + \frac{1}{2}\dot{W}_{ij} - \frac{1}{2}(\partial_\eta^2 + \nabla^2 - 2K)\chi_{ij} \,, \tag{3.47}$$

and the magnetic tidal tensor

$$H_{ij} = -\frac{1}{2}\nabla_{(i}H_{j)} + \nabla_k\eta^{kl}{}_{(i}\dot{h}_{j)l} \,. \tag{3.48}$$

We know that in Newtonian limit $\phi = \psi$ the force field take the parts of the gravitational tidal field as we have already said in Section 2.1. However the magnetic tidal tensor have not a Newtonian correspondent expression.
This last equation demonstrates that magnetic part of the Weyl tensor does not vanish in the Newtonian limit.

4 Silent metric cosmological perturbations

In this thesis we are primarily interested in evaluating the effects of light propagation for a second order perturbation of the geodesics equation in a large scale Silent cosmological model. In order to study this topic we must solve the linearized Einstein Field Equation starting from a first order perturbed metric with a FLRW background. We discuss the peculiarity of the metric with the study of the Weyl tensor and the consideration of the dynamical system in terms of the covariant variables.
Initially on all scales, and at present time on large scales ($\gg 100h^{-1}$ Mpc), large-scale structures are represented by first-order perturbation of the FLRW background geometry. We shall consider the second class Szekeres-type flat metric [8] presented in the form introduced by Goode and Wainwright [4] (1982) in a flat universe. We follow Meures and Bruni (2011), but in this case we use a first-order metric, see [5]. The first-order perturbed metric in cartesian coordinates has the form:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t)(1+2Z(t,\mathbf{x})) \end{pmatrix}, \tag{4.1}$$

where $g^{(0)}_{\mu\nu}$ is the unperturbed background metric and $h_{\mu\nu}$ is the first order perturbation. We set our coordinates in Synchronous gauge. We have done the chosen of synchronous gauge because it has the property of the existence of a set of comoving fundamental observers who fall freely without changing their spatial coordinates. The existence of a fundamental observer follows directly from the geodesic equation:

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0\,. \tag{4.2}$$

. In particular, we define the affine connection

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} (g_{\alpha\sigma,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})\,, \tag{4.3}$$

where we used the notation for the quadri-gradient as $()_{,\mu} \equiv \partial_\mu$.
It is straightforward from (4.1) that $\Gamma^i_{00} = 0$, implying that $u^i = 0$ is a geodesic. In particular fluid elements move along geodesic flow lines that are orthogonal to the cosmic time t hypersurfaces, represented by a manifold $\mathcal{H}_t$, with four-velocity:

$$u^\mu = (1,0,0,0)\,. \tag{4.4}$$

In this Section we want to find a solution for the linearization of the Einstein Field Equation including the cosmological constant Λ:

$$G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}\,. \tag{4.5}$$

We refer to matter as a representation of cold dark matter, a pressureless dust component, and the fluid flow is irrotational. For instance, the energy-momentum tensor $T^{\mu\nu}$ must have only the zero component:

$$T^{00} = \rho(t,\mathbf{x})\,. \tag{4.6}$$

We will use the following parameters for our computations: $\Omega_\Lambda = 0.75$, $H_0 = 72$ km s^{-1} Mpc$^{-1} \simeq 0.0736$ Gy and $\Omega_m = 1 - \Omega_\Lambda$. In the next parts of the thesis we will use the values of $\rho_0^{(0)} = 3H_0^2\Omega_m$ and the cosmological constant $\Lambda = 3H_0^2\Omega_\Lambda$.

4.1 Continuity Equation

In order to solve the EFE we must have the general form of the density function $\rho(t, \mathbf{x})$ in terms of the metric potential $Z(t, \mathbf{x})$. We present the continuity equation in term of the kinematic quantity expansion tensor $\Theta_{\mu\nu}$ [3]:

$$\dot{\rho} = -\Theta\rho\,, \tag{4.7}$$

where the expansion scalar is given by:

$$\Theta = u^{\mu}_{;\mu}\,. \tag{4.8}$$

We denote the time derivative $\frac{d}{dt} \equiv \dot{()}$ and the covariant derivative as $\nabla_\mu \equiv ()_{;\mu}$. Denoting the affine connection truncated to the first-order

$$\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{(0)\mu}{}_{\alpha\beta} + \Gamma^{(1)\mu}{}_{\alpha\beta}\,, \tag{4.9}$$

where we denoted the zeroth-order and first-order affine connection respectively as

$$\Gamma^{(0)\mu}{}_{\alpha\beta} = \frac{1}{2} g^{(0)\mu\sigma}(g^{(0)}_{\alpha\sigma,\beta} + g^{(0)}_{\sigma\beta,\alpha} - g^{(0)}_{\alpha\beta,\sigma}) \tag{4.10}$$

and

$$\Gamma^{(1)\mu}{}_{\alpha\beta} = \frac{1}{2} g^{(0)\mu\sigma}(h_{\alpha\sigma:\beta} + h_{\sigma\beta;\alpha} - h_{\alpha\beta;\sigma})\,. \tag{4.11}$$

Using (4.10) and (4.11) we have that the only non vanishing components are :

$$\begin{aligned} &\Gamma^{(0)0}{}_{11} = \Gamma^{(0)0}{}_{22} = \Gamma^{(0)0}{}_{33} = \dot{a}a, \quad \Gamma^{(0)k}{}_{0k} = \frac{\dot{a}}{3a} = \frac{1}{3}H \\ &\Gamma^{(1)0}{}_{33} = 2a\dot{a}Z + a^2\dot{Z}, \quad \Gamma^{(1)1}{}_{33} = -Z_x, \quad \Gamma^{(1)2}{}_{33} = -Z_y, \quad \Gamma^{(1)3}{}_{\mu 3} = Z_{,\mu} \end{aligned} \tag{4.12}$$

where we defined the Hubble rate $H(t) \equiv \dot{a}(t)/a(t)$.

With the Christoffel Symbols the continuity equation (4.7) take the form:

$$\dot{\rho} + \left(3\frac{\dot{a}}{a} + \dot{Z}(1 - 2Z)\right) = 0\,. \tag{4.13}$$

Keeping in mind that $Z \ll 1$ this form can be rearranged as:

$$\frac{\dot{\rho}}{\rho} = -\left(3\frac{\dot{a}}{a} + \frac{1}{2}\frac{(1+2Z)^{\cdot}}{(1+2Z)}\right)\,. \tag{4.14}$$

And now is straightforward to find the general solution of the density function truncated to the first order in Z:

$$\rho(t, \mathbf{x}) = \frac{m(\mathbf{x})}{a^3(t)}(1 - Z(t, \mathbf{x}))\,, \tag{4.15}$$

where $m(\mathbf{x})$ is a constant of integration that we identify as the constant in time mass function. It can be useful assuming that $\rho(t, \mathbf{x})$ can be defined through the deviation from the background function $\delta = \delta\rho/\rho^{(0)}$, where $\rho^{(0)} = \bar{\rho}_0/a^3$ is the homogeneous energy density of the background and $\bar{\rho}_0 = 3\Omega_m H_0^2$. With these definitions the continuity equation is at first order:

$$\dot{\rho}^{(0)} + 3H\rho^{(0)} + (\dot{Z} + \dot{\delta})\rho^{(0)} = 0\,. \tag{4.16}$$

Now solving the differential equation using the background continuity equation

$$\dot{\rho}^{(0)} + 3H\rho^{(0)} = 0\,, \tag{4.17}$$

we can find the result:

$$\delta(t, \mathbf{x}) = -Z(t, \mathbf{x})\,. \tag{4.18}$$

We understand that the deviation from the background is fully identifiable with the metric perturbative potential, and the study of its behaviour is the goal of the next sections.

4.2 Perturbed Einstein equations

We want to present the set of perturbed Einstein equation for the metric (4.1). For instance we define the Riemann tensor as:

$$R^{\mu}{}_{\nu\rho\sigma} \equiv \Gamma^{\mu}{}_{\nu\sigma,\rho} - \Gamma^{\mu}{}_{nu\rho,\sigma} + \Gamma^{\mu}{}_{\alpha\rho}\Gamma^{\alpha}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\rho} \,. \tag{4.19}$$

Now we are interested, in order to define the Einstein tensor, in Ricci tensor $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\nu\alpha}$ at zeroth-order and first order in the metric potential

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} \tag{4.20}$$

and we can compute the following components:

$$\begin{aligned}
&R^{(0)}_{00} = -3\frac{\ddot{a}}{a},\;\; R^{(0)}_{11} = R^{(0)}_{22} = 2\dot{a}^2 + a\ddot{a},\;\; R^{(0)}_{33} = 2\dot{a}^2 + a\ddot{a}\,,\\
&R^{(1)}_{00} = -2H\dot{Z} - \ddot{Z},\;\; R^{(1)}_{01} = -\dot{Z}_x,\;\; R^{(1)}_{02} = -\dot{Z}_y,\;\; R^{(1)}_{11} = a\dot{a}\dot{Z} - Z_{xx}\,,\\
&R^{(1)}_{22} = a\dot{a}\dot{Z} - Z_{yy},\;\; R^{(1)}_{12} = -Z_{xy},\;\; R^{(1)}_{33} = 4Z\dot{a}^2 + 2a\ddot{a}Z + 4a\dot{a}\dot{Z} + a^2\ddot{Z} - Z_{xx} - Z_{yy}\,.
\end{aligned} \tag{4.21}$$

where we use $\frac{\partial}{\partial x_i}(.) = (.)_{x_i}$ to denote the spatial partial derivatives. With the Ricci tensor we can compute the Einstein tensor with the following definition

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\,, \tag{4.22}$$

where we define the Ricci scalar

$$R = R^{\alpha}_{\alpha} = R^{(0)} + R^{(1)} = 6\left(H^2 + \frac{\ddot{a}}{a}\right) + 2\left(4a\dot{a}\dot{Z} + \ddot{Z} - Z_{xx} - Z_{yy}\right)\,. \tag{4.23}$$

Now, using (4.5) we can easily compute the set of perturbed Einstein Field Equations (EFE) truncated at linear order:

$$\begin{aligned}
&(00) \qquad G_{00} = 3H^2 + 2H\dot{Z} - \frac{Z_{xx} + Z_{yy}}{a^2} = \rho + \Lambda\\
&(01) \qquad G_{10} = G_{01} = -\dot{Z}_x = 0\\
&(02) \qquad G_{20} = G_{02} = -\dot{Z}_y = 0\\
&(11) \qquad \frac{G_{11}}{a^2} = -H^2 - 3H\dot{Z} - 2\frac{\ddot{a}}{a} - \ddot{Z} + Z_{yy} = -\Lambda\\
&(12) \qquad G_{12} = G_{21} = -Z_{xy} = 0\\
&(22) \qquad \frac{G_{22}}{a^2} = -H^2 - 3H\dot{Z} - 2\frac{\ddot{a}}{a} - \ddot{Z} + Z_{xx} = -\Lambda\\
&(33) \qquad \frac{G_{33}}{a^2} = -(1+2Z)\left(H^2 + 2\frac{\ddot{a}}{a}\right) = -\Lambda(1+2Z)\,.
\end{aligned} \tag{4.24}$$

4.3 Solution of the perturbed EFE

In this section we want to find the behavior of the metric potential and the scale factor by solving the EFE. We begin from the system of equations given in (4.24). It is clear to notice that from the off-diagonal terms (01) and (02) we have that:

$$\dot{Z}_x = \dot{Z}_y = 0\,. \tag{4.25}$$

So we can split our metric potential in:

$$Z(t, \mathbf{x}) = A(\mathbf{x}) + \psi(z, t)\,. \tag{4.26}$$

Moreover from (12) and (11),(22) we have the following relations:

$$Z_{xy} = 0, \qquad\qquad Z_{xx} = Z_{yy}\,. \tag{4.27}$$

Multiplying (33) by $a^2\dot{a}$ we get

$$\dot{a}^3 + 2\ddot{a}\dot{a}a = \Lambda a^2 \dot{a}\,, \tag{4.28}$$

and it yelds

$$(\dot{a}^2 a)^{\cdot} = \frac{\Lambda}{3}(a^3)^{\cdot} \tag{4.29}$$

that can be easily solved to:

$$\dot{a}^2 a = \frac{\Lambda}{3} + \mathcal{C}\,, \tag{4.30}$$

where identifying the moltiplication constant $\mathcal{C} = \frac{a^3\rho_0^{(0)}}{3}$ we find the first Friedmann equation for ΛCDM:

$$H^2 = \frac{\Lambda}{3} + \frac{\rho_0^{(0)}}{3a^2}\,. \tag{4.31}$$

This is a first order nonlinear equation in t, and it can be integrated, finding the equation that governs the course in time of the scaling factor

$$a(t) = \left(\frac{\rho_0^{(0)}}{\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}t\right)\,. \tag{4.32}$$

Now we can rewrite the system as:

$$\begin{aligned}
&(A) \qquad && 2H\dot{Z} + 3H^2 - 2Z_{xx} = \rho + \Lambda \\
&(B) \qquad && 2\frac{\ddot{a}}{a} + H^2 + 3H\dot{Z} + \ddot{Z} - \frac{Z_{xx}}{a^2} = \Lambda \\
&(C) \qquad && 2\frac{\ddot{a}}{a} + H^2 = \Lambda\,.
\end{aligned} \tag{4.33}$$

Now, combining the first Friedmann equation with equation (C) we can obtain the following relation:

$$2\frac{\ddot{a}}{a} - \frac{2}{3}\Lambda + \frac{\rho_0^{(0)}}{3} = 0\,, \tag{4.34}$$

thus getting

$$\frac{\ddot{a}}{a} + \frac{\rho_0^{(0)}}{6} - \frac{\Lambda}{3} = 0\,. \tag{4.35}$$

In this way we have obtained the second Friedmann equation for ΛCDM.

Now, computing algebraically the equations in the reduced sistem (4.33) $(B) + \frac{(C)}{2} - \frac{(A)}{2}$ we have the relation

$$3\frac{\ddot{a}}{a} + \ddot{Z} + 2H\dot{Z} = \Lambda - \frac{\rho}{2}\,. \tag{4.36}$$

Now, using (4.36) and (4.35) and focusing only on the first order components, we can easily get

$$a^3\ddot{Z} + 2\dot{a}a^2\dot{Z} - \frac{\rho_0^{(0)}Z}{2} + \frac{1}{2}m = 0\,, \tag{4.37}$$

where we used the relation (4.15). Using the decomposition of the metric potential (4.26) we can find:

$$a^3\ddot{\psi}(t,z) + 2\dot{a}a^2\dot{\psi}(t,z) - \frac{\rho_0^{(0)}}{2}\psi(z,t) = -\frac{1}{2}m(\mathbf{x}) + \frac{\rho_0^{(0)}}{2}A(\mathbf{x})\,. \tag{4.38}$$

We can note that the left hand side of the equation is only a function in (z,t) and the right hand side is only a function in (x,y,z), for this reason, by exclusion, every side must be equal to a function only z depending, $g(z)$. Hence we have:

$$\ddot{\psi} + 2H\dot{\psi} - \frac{\rho_0^{(0)}}{2a^3}\psi = \frac{1}{a^3}g(z)\,. \tag{4.39}$$

This is a Ordinary Differential Equation that has two homogeneous solutions, $\psi_+ + \psi_-$, and one particular solution, ψ_p that is easily findable:

$$\psi_p = -\frac{2}{\rho_0^{(0)}}g(z)\,. \tag{4.40}$$

Moreover, we have from equation (4.38) that

$$A = \frac{2g(z)}{\rho_0^{(0)}} + \frac{1}{\rho_0^{(0)}}m\,, \tag{4.41}$$

therefore

$$\psi = \psi_+ + \psi_- - \frac{1}{\bar{\rho}_0}m\,. \tag{4.42}$$

We just proved that the metric potential is independent of the function $g(z)$, so without losing generality we can set $g(z) = 0$. Thus we have all the information inside the second order homogeneous equation:

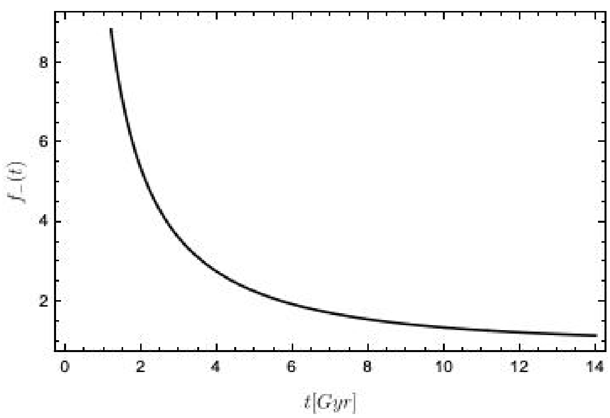
$$\ddot{\psi} + 2H\dot{\psi} - \frac{\rho_0^{(0)}}{2a^3}\psi = 0\,. \tag{4.43}$$

This ODE has two indipendent solutions

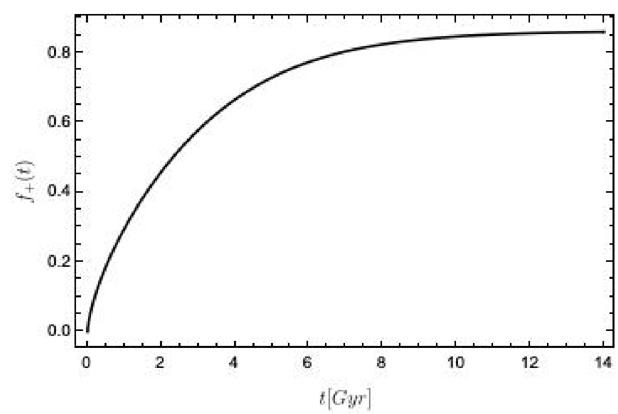
$$\psi(t,z) = f_+(t)\beta_+(z) + f_-(t)\beta_-(z)\,, \tag{4.44}$$

where the function f_+, f_- are shown in Fig.4.
These two functions represents the growing and the decaying modes of the metric potential and $\beta_+(z)$ and $\beta_-(z)$ are the integration function constant in time, depending only on z. Hypergeometric functions must be used in the integration of $f_+(t)$, we have used the notation ${}_2F_1(\cdot)$ to indicate them.

$$f_+(t) = -\frac{1}{5}\cosh\left(\frac{\sqrt{3\Lambda}t}{2}\right)\left[\sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}t\right)\left(\cosh\left(\frac{\sqrt{3\Lambda}}{2}t\right)+4\right){}_2F_1\left(1/2,5/6;11/6;-\sinh^2\left(\frac{\sqrt{3\Lambda}}{2}t\right)\right)\right.$$
$$\left.-8{}_2F_1\left(-1/2,5/6;11/6;-\sinh^2\left(\frac{\sqrt{3\Lambda t}}{2}t\right)\right)\right] m\,,$$
$$f_-(t) = \frac{\cosh\left(\frac{\sqrt{3\Lambda t}}{2}t\right)}{\sinh\left(\frac{\sqrt{3\Lambda}}{2}t\right)}\,. \tag{4.45}$$



(a) $f_-(t)$



(b) $f_+(t)$

Figure 4: Plots of the decaying mode $f_-(t)$ and the growing mode $f_+(t)$

It is interesting to note the fact that in our model, with $\Lambda \neq 0$, the growing mode $f_+(t)$ has an horizontal asymptote. On the contrary in the $\Lambda = 0$ case, in which the FLRW background is an Einstein-de Sitter model, $f_+(t) \propto a(t)$ for $t \gg 1Gyr$.
Now we have to find out the second component of the metric potential. For this purpose we consider the off diagonals components of the Einstein tensor, in particular the relations (4.27). From these it is easy to find the form of A:

$$A(\mathbf{x}) = p(z) + q(z)x + r(z)y + s(z)(x^2+y^2)\,. \tag{4.46}$$

From the reduced system (4.33) we can see that subtracting $(B)-(C)$ we can have

$$3H\dot{Z} + \ddot{Z} - \frac{Z_{xx}}{a^2} = 0 \tag{4.47}$$

and splitting Z we reach to

$$\ddot{\psi} + 3H\dot{\psi} - 2\frac{s(z)}{a^2} = 0\,, \tag{4.48}$$

thus using the second order homogeneous equation (4.43) we get its first integral

$$H\dot{\psi} + \frac{\rho_0^{(0)}}{2a^3}\psi - 2\frac{s}{a^2} = 0\,. \tag{4.49}$$

Rearranging this equation and using the general solutions of $\psi(t,z)$ we can find the exact form of $s(z)$

$$s(z) = \frac{a\dot{a}}{2}\dot{\psi}(t,z) + \frac{\rho_0^{(0)}}{4a}\psi(t,z) = B\beta_+(t)\,, \tag{4.50}$$

where $B = \frac{1}{4}(\rho_0^{(0)}\Lambda)^{1/3}$.
We can also note that, without losing generality, we can rewrite the functions $q(z)$ and $r(z)$ as

$$\begin{aligned} q(z) &= 2B\gamma(z)\beta_+(z) = 0 \\ r(z) &= 2B\omega(z)\beta_+(z) = 0\,. \end{aligned} \tag{4.51}$$

Then, substituting in (4.46)

$$A(\mathbf{x}) = p(z) + B\beta_+(z)[(x+\gamma(z))^2 + (y+\omega(z))^2 - (\gamma(z)+\omega(z))^2]\,. \tag{4.52}$$

Now rewriting this equation implementing the coordinate transformation

$$\bar{z} = \int p(z)[\gamma^2(z) + \omega^2(z)] - B\beta_+[\gamma^2(z) + \omega^2(z)]\,dz \tag{4.53}$$

we can (dropping every bars) find a more simplified form of the last member of the metric potential truncated at first order in $\beta(z)$:

$$A(\mathbf{x}) = B\beta_+(z)[(x+\gamma(z))^2 + (y+\omega(z))^2]\,. \tag{4.54}$$

We are interested in the restricted in which both $\gamma = \omega = 0$. So the final form of the metric potential is:

$$Z(t,\mathbf{x}) = f_+(t)\beta_+(z) + f_-(t)\beta_-(z) + B\beta_+(z)(x^2+y^2)\,. \tag{4.55}$$

In all the following parts of the thesis we only consider the growing solution ψ_+ considering the growing mode $\beta_+(z) = D\sin(kz)$ with $D \ll 1$ and the decaying mode identically zero, $\beta_-(z) = 0$.
Looking at the metric potential form (4.55) is natural to make a coordinates transformation from cartesian to cylindrical coordinates

$$x = r\cos(\varphi), \qquad y = r\sin(\varphi), \qquad z = z\,, \tag{4.56}$$

finding

$$Z(t,r,z) = \psi(t,z) + A(\rho,z) = f_+(t)\beta_+(z) + B\beta_+(z)r^2\,. \tag{4.57}$$

Now, we can rewrite the metric (4.1) in cylindrical coordinates

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)(1+2Z(t,r,z)) \end{pmatrix}. \tag{4.58}$$

The metric is totally independent on φ, therefore we have proved that the solution must have a z-axial symmetry.

We report in Fig. 5 the plot of the density deviation profile (4.18), in which we have a complete illustrative behavior of (4.57).

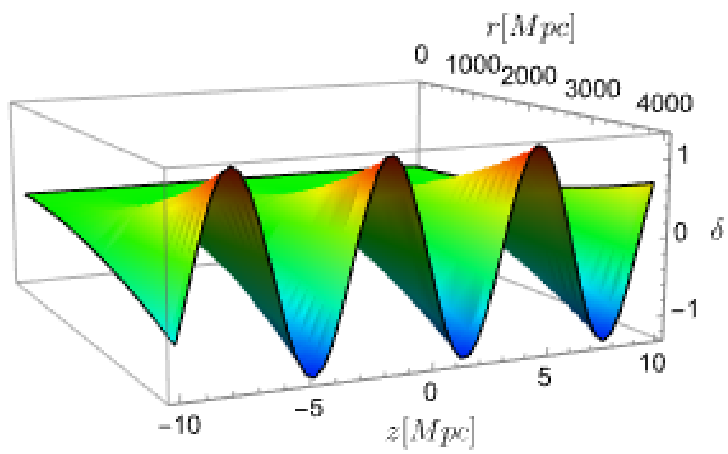


Figure 5: δ deviation profile of an under-density at the time $t_E = 3.8 \times 10^{-4}$Gyr for the perturbed metric (4.1) in the form given by Goode and Wainwright (1982). All distance are comoving and in Mpc. We force the fact that $r = (x^2 + y^2)^{1/2}$.

Now we have the complete form of the perturbed metric function, thus we can explore in detail the main aspects of photon behaviour in a silent type metric and the consequent temperature deviations.

4.4 Phase plane analysis

In this section we want to studynour model defining the different parts of the Weyl tensor and studying the dynamics of our cosmological model at perturbative order following the exact case [5]. In particular we want to use dynamics equations of silent universes following Bruni, Matarrese and Pantano (1995) [17].

We begin our treatment definying the projection tensor

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t)(1+2Z(t,\mathbf{x})) \end{pmatrix} . \tag{4.59}$$

where u_μ is the four-velocity of the irrotational fluid flow defined in equation (4.4). This tensor projects into the inertial frame of a comoving observer.
In (4.8) we have defined the expansion scalar Θ. From this we can compute the shear tensor of our model truncated at first order in the metric perturbation potential $Z(t,\mathbf{x})$, given by

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - \frac{1}{3}\Theta\mathcal{P}_{\mu\nu} + \dot{u}_{(\mu}u_{\nu)} = \begin{pmatrix} 3H+\dot{Z} & 0 & 0 & 0 \\ 0 & -\frac{1}{3}a^2\dot{Z} & 0 & 0 \\ 0 & 0 & -\frac{1}{3}a^2\dot{Z} & 0 \\ 0 & 0 & 0 & \frac{2}{3}a^2\dot{Z} \end{pmatrix} . \tag{4.60}$$

The shear tensor determines the invariant volume spacetime distortion that characterizes the relativistic fluid. We can clearly see that, considering the perfect fluid take a spherical shape and using the metric potential given in (4.55), if we had a stretching along the x, y axes then we would have a narrowing in the z axes and viceversa. Fixing this image in our mind we can recover, with an intuitive idea, the plots of the first-order photon geodesics equation solution shown in Section 2.3, see Fig. 8.
In Section 2.1 we have defined the fully antisymmetric tensor $\eta_{\mu\nu\rho\sigma}$ in a curved space-time from the well known Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$, see eqs. (2.19), (2.20).
From these relations we can define the vorticity tensor, a rigid rotation of the relativistic fluid with respect to the local inertial frame,

$$\omega_{\mu\nu} = 0\,. \tag{4.61}$$

Thus there is no non-vanishing components of the vorticity tensor. This is one of the conditions, the least tightening, that must be satisfied in order to define a silent model. This is an exact approximation from a hydrodynamical point of view, because considering an ideal irrotational fluid flow at initial time, for the well known Kelvin theorem, it must be stay irrotational in every future instant, see Raychoudhuri (1988) [19].

Now we want to follow the alternative formulation of General Relativity given by Ellis, Maartens and Maccallum in [3]. We must use the covariant variables defined above: Θ, ρ, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$. Moreover we have to define the Dynamical variables and in order to give them a formulation we must define the weyl tensor $C_{\mu\nu\rho\sigma}$. We have seen in Section 1 that Weyl tensor is the traceless part of the curvature and its explicit definition is

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{1}{2}(g_{\sigma\nu}R_{\rho\mu} + g_{\rho\mu}R_{\sigma\nu} - g_{\sigma\mu}R_{\rho\nu} - g_{\rho\nu}R_{\sigma\mu}) + \frac{1}{6}(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\mu}g_{\sigma\nu})R\,. \quad (4.62)$$

The Weyl tensor can be splitted in the symmetric traceless tidal electric tensor $E_{\alpha\beta}$ and the tidal magnetic tensor $H_{\alpha\beta}$. Using the metric (4.1) and perturbing up to the first order, $E_{\alpha\beta}$ is defined by

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\sigma}u^{\gamma}u^{\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E_{11} & E_{12} & 0 \\ 0 & E_{21} & E_{22} & 0 \\ 0 & 0 & 0 & E_{33} \end{pmatrix}, \quad (4.63)$$

where

$$\begin{aligned} E_{11} &= \frac{1}{6}\left(-a^2 + 6\dot{a}^2 + 6a\ddot{a} + 9a\dot{a}\dot{Z} + 3a^2\ddot{Z} - 3Z_{xx}\right) \\ E_{12} &= E_{21} = -\frac{1}{2}Z_{xy} = 0 \\ E_{22} &= \frac{1}{6}\left(-a^2 + 6\dot{a}^2 + 6a\ddot{a} + 9a\dot{a}\dot{Z} + 3a^2\ddot{Z} - 3Z_{yy}\right) \\ E_{33} &= \frac{1}{2}\left[2a(\ddot{a} + \dot{a}\dot{Z}) - (Z_{xx} + Z_{yy})\right]. \end{aligned} \quad (4.64)$$

Instead $H_{\alpha\beta}$ is defined by

$$H_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\gamma}{}^{\mu\nu}C_{\mu\nu\beta\delta}u^{\gamma}u^{\delta} = 0\,. \quad (4.65)$$

Thus every component of the Tidal magnetic field is identically zero. This is the second condition in order to define a silent model. In particular this restriction has an important physical meaning, the different neighborhood fluid flow elements cannot exchange gravitational information with each other, fully covering the sense of "silent universes", see Matarrese et al (1994a,b) [22], [23].
As we have already seen, Barnes and Rowlingson (1989) [20] demonstrate that the eigenframes of $\sigma_{\mu\nu}$ and $E_{\mu\nu}$ are aligned, see eq (3.3). Using a common orthonormal tetrad expanded up to the first order in Z:

$$e_{1\mu} = a\delta^1_{\mu}, \qquad e_{2\mu} = a\delta^2_{\mu}, \qquad e_{3\mu} = a(1+Z)\delta^3_{\mu}\,. \quad (4.66)$$

Because of this two tensors are traceless, we can gain the total information only from E_1, E_2 and σ_1, σ_2.
Now we can make the convenient linear transformation defined in Section 2.2:

$$\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm \sigma_2), \qquad E_{\pm} = \frac{1}{2}(E_1 \pm E_2)\,. \quad (4.67)$$

We know from the resolutions of EFE that $Z_{xx} = Z_{yy}$, so using the eigenvalues of $\sigma_{\mu\nu}$, $E_{\mu\nu}$ and the (00) component of the EFE (see equation (4.24)) we can reach the new variables

$$\sigma_+ = -\frac{1}{3}\dot{Z}, \qquad \sigma_- = 0\,. \quad (4.68)$$

and

$$E_+ = -\frac{1}{6}\rho^{(0)}\delta, \qquad\qquad E_- = 0\,. \tag{4.69}$$

Now we can clearly see that the shear and the electric part of the Weyl tensor are only first-order perturbation, this is completely in agreement with the fact that in FLWR background the shear tensor and the electric tidal tensor vanish. The set of Ordinary Differential Equations that governs the dynamics of our cosmological silent model can be seen as the evolution of a six dimensional space outlined by the variables $\{\rho, \Theta, \sigma_+, \sigma_-, E_+, E_-\}$.
We want to summarise the evolution equations that we have already defined in Section 1 without the influence of the magnetic tidal tensor $H_{\mu\nu}$. The continuity equation in terms of the scalar expansion is given in (4.7). Therfore we write the Raychaudhuri equation that defines the evolution of the expansion scalar in the presence of the cosmological constant Λ:

$$\dot{\Theta} = \Lambda - \frac{1}{3}\Theta^2 - 2\sigma^2 - \frac{1}{2}\rho\,. \tag{4.70}$$

The evolution of the shear tensor is described by

$$\dot{\sigma}_{\mu\nu} = -\sigma_{\mu\gamma}\sigma^{\gamma}_{\nu} + \frac{2}{3}\mathtt{h}_{\mu\nu}\sigma^2 - \frac{2}{3}\Theta\sigma_{\mu\nu} - E_{\mu\nu}\,, \tag{4.71}$$

and finally the evolution of the eletric part of the Weyl tensor is

$$\dot{E}_{\mu\nu} = \mathcal{P}_{\mu\nu}\sigma^{\gamma\delta}E_{\gamma\delta} - \Theta E_{\mu\nu} + 3E_{\gamma(\mu}\sigma^{\gamma}_{\nu)} - \frac{1}{2}\rho\sigma_{\mu\nu}\,. \tag{4.72}$$

Following Bruni (2011) [5] we can split our variables in the zero-order one, i.e. the background quantity, and the first-order one, i.e. the perturbed quantity that characterizes the inhomogeneities of the model. In this way we can describe the different variables as:

$$\begin{aligned} \rho &= \rho^{(0)}(1+\delta) \\ \Theta &= \Theta^{(0)} + \Theta^{(1)} \\ \sigma_+ &= \sigma_+^{(1)} \\ E_+ &= E_+^{(1)}\,, \end{aligned} \tag{4.73}$$

where $\delta \equiv \delta\rho/\rho^{(0)}$, see *Sect. 1.1.*

Using the linear transformation (4.67) we can have the ful set of first order ODEs

$$\begin{aligned} \dot{\delta} &= \Theta^{(0)}\delta + \Theta^{(1)} \\ \dot{\Theta}^{(1)} &= -\frac{2}{3}\Theta^{(0)}\Theta^{(1)} - \frac{1}{2}\rho^{(0)}\delta \\ \dot{\sigma}_+ &= -\frac{2}{3}\Theta^{(0)}\sigma_+ - E_+ \\ \dot{E}_+ &= -\Theta^{(0)}E_+ - \frac{1}{2}\rho^{(0)}\sigma_+\,, \end{aligned} \tag{4.74}$$

where we omitted the superscript denoting the first-order perturbed quantity to mark the shear and the electric tidal variables in order not to burden the notation. As we said about the new approach of GR defined in Section 2.1, with eqs. (4.74) we can re-derive the differential equations that govern the behavior for the metric potential $Z(t, \mathbf{x})$ and for the scale factor $a(t)$. Therefore we get a demonstration of the EFE alternative formulation, in which the curvature information is totally encoded in the Weyl tensor $C_{\mu\nu\rho\sigma}$.
Now we want to understand the evolution of our model as a function of cosmological parameters. In order to analyze the phase plane for our perturbed metric we want to use the cosmological parameters as variables:

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}\,, \qquad\qquad \Omega_m = \frac{\bar{\rho}}{3H^2}\,. \tag{4.75}$$

Using the notation $(.)' = (.)^{\cdot}/H$ we can rewrite the first and the second Friedmann equation for ΛCDM already find in Eqs. (4.31) and (4.35) using the new variables:

$$\begin{aligned} \Omega_\Lambda &= 1 - \Omega_m\,, \\ \Omega'_\Lambda &= 3\Omega_\Lambda(1 - \Omega_\Lambda)\,. \end{aligned} \tag{4.76}$$

Now we can rewrite the continuity equation and the shear evolution using the splitting in (4.73) and new derivatives already defined

$$\begin{aligned} \delta' &= -\frac{1}{H}(\Theta^{(0)}\delta + \Theta^{(1)}) \\ \sigma'_+ &= -2\sigma_+ + \frac{1}{2}H(1 - \Omega_\Lambda)\delta\,. \end{aligned} \tag{4.77}$$

It can be useful define in a new variable the deviation velocity, encoded in the rate of the metric potential, normalized with the expansion rate, i.e.

$$\Sigma_+ = -\frac{\Theta^{(1)}}{\Theta^{(0)}} \tag{4.78}$$

and we can easily note that $\Theta^{(1)}/\Theta^{(0)} = -\sigma_+/H$ Using this new variable the above system can be written at the first-order in Z as

$$\begin{aligned} \Omega'_\Lambda &= 3\Omega_\Lambda(1 - \Omega_\Lambda) \\ \delta' &= 3(\Sigma_+ - \delta) \\ \Sigma'_+ &= \frac{1}{2}\left[(1 - \Omega_\Lambda)\delta - \Sigma_+(1 + 3\Omega_\Lambda)\right]\,. \end{aligned} \tag{4.79}$$

Now using (4.49) and the quantity $s = \beta_+ B$ we can find the conservation equation

$$\Sigma_+ = \frac{1}{2}(\Omega_\Lambda - 1)\delta - \frac{\beta_+}{2}(\Omega_\Lambda^{1/2} - \Omega_\Lambda^{3/2})^{2/3}\,. \tag{4.80}$$

In this way we have reduce the system (4.79) in a ODE with only two dynamical equations as functions of the cosmological variable Ω_Λ

$$\begin{aligned} \Omega'_\Lambda &= 3\Omega_\Lambda(1 - \Omega_\Lambda) \\ \delta' &= -\frac{3}{2}\left[(1 - \Omega_\Lambda)\delta + \beta_+(\Omega_\Lambda^{1/2} - \Omega_\Lambda^{3/2})^{2/3}\right]\,. \end{aligned} \tag{4.81}$$

Using the exact metric proposed by Goode and Wainwright (1982) [9], that can be represented by the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + \Psi^2(t, \mathbf{x})dz^2)\,, \tag{4.82}$$

where

$$\Psi(t, \mathbf{x}) = 1 + \psi(t, z) + A(\mathbf{x})\,, \tag{4.83}$$

we can find the exact values of the cosmological variables (see[5]):

$$\begin{aligned} \tilde{\rho} &= \rho^{(0)}(1 - \frac{\psi}{\Phi})\,, \\ \tilde{\Theta} &= 3\frac{\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi}\,, \\ \tilde{\sigma}_+ &= -\frac{1}{3}\frac{\dot{\Phi}}{\Phi}\,, \\ \tilde{E}_+ &= \frac{2}{3}\frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} + \frac{1}{3}\frac{\ddot{\Phi}}{\Phi}\,, \end{aligned} \tag{4.84}$$

where we use the $\tilde{()}$ to note that the value is computed with the exact metric. The ODEs system governing the dynamics of the density and the expansion is

$$\begin{aligned} \tilde{\rho}^{\cdot} &= -\tilde{\Theta}\tilde{\rho}\,, \\ \tilde{\Theta}^{\cdot} &= -\frac{1}{3}\tilde{\Theta}^2 - 6\tilde{\sigma}^2 - \frac{1}{2}\tilde{\rho} + \Lambda\,, \\ \tilde{\sigma}^{\cdot} &= -\frac{2}{3}\Theta\tilde{\sigma}_+ + \tilde{\sigma}_+^2 - \tilde{E}_+\,, \\ \tilde{E}_+^{\cdot} &= -\tilde{\Theta}\tilde{E}_+ - 3\tilde{\sigma}_+\tilde{E}_+ - \frac{1}{2}\tilde{\rho}\tilde{\sigma}_+\,. \end{aligned} \tag{4.85}$$

Now, using the definitions (4.75) and (4.76) we can find the exact differential equation governing the density deviation

$$\tilde{\delta}' = -\frac{3}{2}\left[(1 - \Omega_\Lambda)\delta + \frac{\beta_+}{A}(1 + \tilde{\delta})(\Omega_\Lambda^{1/2} - \Omega_\Lambda^{3/2})^{2/3}\right](1 + \tilde{\delta})\,. \tag{4.86}$$

We can reach the conserved quantity that rules $\tilde{\Sigma}_+ = \tilde{\sigma}_+/H$

$$\tilde{\Sigma}_+ = \frac{1}{2}(\Omega_\Lambda - 1)\tilde{\delta} - \frac{\beta_+}{2A}(1 + \tilde{\delta})(\Omega_\Lambda^{1/2} - \Omega_\Lambda^{3/2})^{2/3}\,. \tag{4.87}$$

We have shown in Fig. 3 the plots of the phase plane analysis for the case of small underdensity $\beta_+ \ll 1$, comparing the exact solutions with the first order perturbed ones. We set the initial conditions

$$\Omega_\Lambda(t_0) \to 0\,, \qquad \delta(t_0) \to 0\,, \qquad \Sigma_+(t_0) \to 0\,. \tag{4.88}$$

In the phase plane this point represents the Einstein- de-Sitter space.
Following [24] de-Sitter (dS) space is the late time attractor for all solutions. In fact,

different values of β_+ give different trajectories in the phase plane, for this reason in the (Ω_Λ-δ) plot we can recognize the dS space in all values of δ along the vertical line $\Omega_\Lambda = 1$ and these all represents a fixed point. Moreover in the Ω_Λ-Σ_+ plot the dS space is only a fixed point, it is represented by $\Omega_\Lambda = 1$ and $\Sigma_+ = 0$.

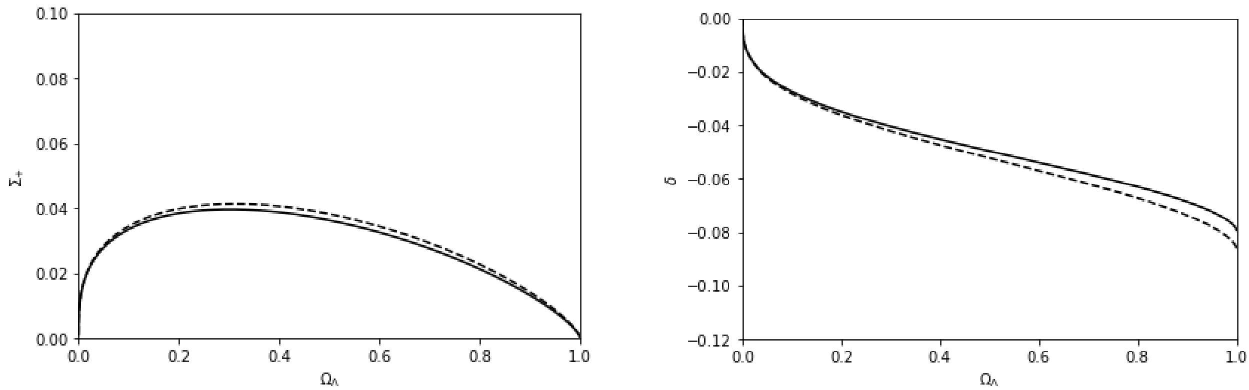


Figure 6: Evolution of the phase plane for the variables Σ_+, that contains the information of the shear, and the deviation profile δ as the background value Ω_Λ varies. We represented the case of constant under-densities with only $\beta_+ > 0$ for clarity. In every plot the continuous lines represent the exact solution, instead the dashed lines represent the first order perturbed solution.

5 CMB gravitational perturbations

We begin our discussion on CMB gravitational perturbations with a heuristic reasoning. Although the deviation from the background δ is greater than unity on scales that are enough small, the metric perturbations can be taken about small. In fact we see the logic of Sachs and Wolfe (1975) calculating the behaviour of photons to first order in this perturbations. However there is no way to know if second order expansion terms in a metric perturbation will be negligible compared to the first order terms. From an intuitive point of view, there is a great chance for effects to accumulate, because photons travel from the surface of emission (last scattering surface) to the observer. We are presenting the hypothesis that the second-order terms may be numerically large. In this section we want to investigate the behaviour of photons not only to first-order, we employ the perturbative null-geodesics equation up to the second-order to $h_{\mu\nu}$ and its derivatives.
We follow the idea that there is a manifold that represents the hypersurface of last scattering with intrinsic perturbation and metric fluctuations that are independent of each other. We follow the treatment for a second order geodesic followed by a photon presented by Pyne and Birkinshaw [7] and by Pyne and Carrol [10], we apply it to our silent metric in order to find the second order temperature deviation.

5.1 Boundary conditions

Our goal is the analysis of the temperature fluctuations seen by an observer in a perturbed silent spacetime. We refer to our perturbed metric $\bar{g}_{\mu\nu}$ (4.1) using a bar to explicit the fact that we are in physical time. We conformally transform the metric and we separate out the dependence on the scale factor $a(\eta)$:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}\,, \tag{5.1}$$

where $g^{(0)}_{\mu\nu} = \bar{g}^{(0)}_{\mu\nu}/a^2$ and $h_{\mu\nu} = \bar{h}_{\mu\nu}/a^2$. In particular our time component will be conformally transform through the relation:

$$\eta = \int \frac{1}{a(t)}\,dt\,. \tag{5.2}$$

In this way the line element in comoving coordinates and conformal time is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -d\eta^2 + dx^2 + dy^2 + (1 + 2Z(\eta, \mathbf{x}))dz^2\,. \tag{5.3}$$

Moreover we define an affine parameter $\lambda \in \mathbb{R}$ and we consider a photon path $x^\mu(\lambda)$. We consider two manifolds, $\mathcal{H}_\mathcal{O}$ defined via the hypersurface of constant conformal time for the observer $x^\mu_\mathcal{O} = (\eta_\mathcal{O}, 0, 0, 0)$, and $\mathcal{H}_E$ defined via the hypersurface of constant conformal time at the emission $x^\mu_E = (\eta_E, 0, 0, 0)$. $\mathcal{H}_\mathcal{O}$ and $\mathcal{H}_E$ are connected by the photon path, particularly what the observer sees is the intersection of its past light cone with $\mathcal{H}_E$.
Every spatial coordinate $p^i \in \mathcal{H}_E$ emits thermal radiation on direction versor $\hat{d}^i$, with temperature $T_E(p^i, \hat{d}^i)$. In particular we define the a-order separation vector as $x^{(a)\mu}(\lambda)$,

thus p^i can be decomposed in

$$p^i = \sum_{a=0}^{\infty} p^{(a)i} \,, \tag{5.4}$$

where $p^{(a)i}$ is the separation of the intersection points of the deviation path, order by order, with $\mathcal{H}_E$. Of course, for our purposes, we interpret $\mathcal{H}_E$ as the hypersurfece of last-scattering.

The photon path $x^\mu(\lambda)$ direction is controlled on the observer hypersurface by a unity vector lying in it, i.e. $\hat{e}^i \in \mathcal{H}_\mathcal{O}$. Another interpretation of $\hat{e}^i$ is to think about it as the direction on the sky in which a comoving observer is pointing an antenna. Thus $\hat{e}^i$ is the initial condition that determines the direction vector and the spatial coordinates in last-scattering surface.

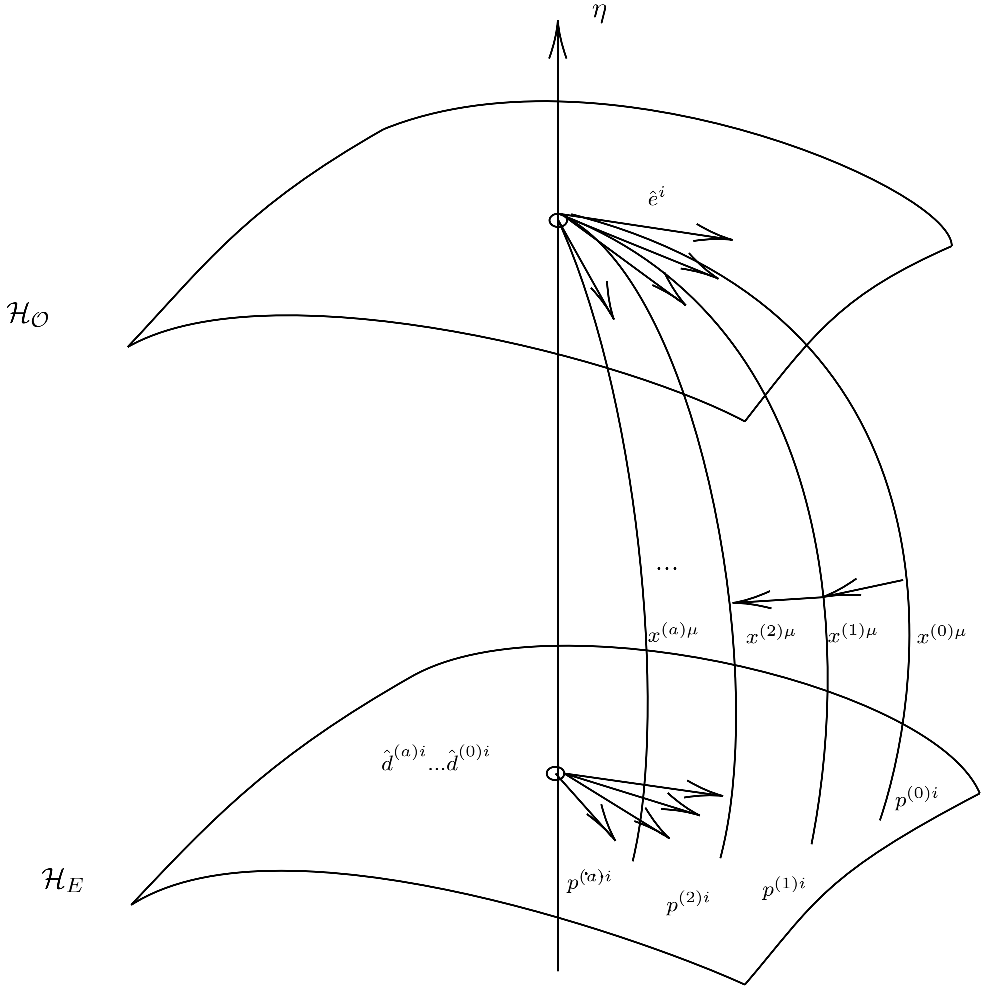


Figure 7: **Photon separation worldlines.**

We must explicit the background metric geodesics, $x^{(0)\mu}(\lambda)$. We consider a null wavevector which intersects $\mathcal{H}_{\mathcal{O}}$ at the spatial origin of the coordinates. We choose the affine parameter λ in this way:

$$\begin{aligned} k^{(0)0} &= 1 \\ g^{(0)}_{ij} k^{(0)i} k^{(0)j} &= 1\,. \end{aligned} \tag{5.5}$$

We compute the geodesic equation of the background metric $g^{(0)}_{\mu\nu}$ in order to find a λ-parameter family that satisfies the above conditions:

$$\begin{aligned} x^{(0)\mu} &= (\lambda, (\lambda_{\mathcal{O}} - \lambda)\hat{e}^i) \\ k^{(0)\mu} &= (1, -\hat{e}^i)\,. \end{aligned} \tag{5.6}$$

where $\lambda_{\mathcal{O}}$ is the affine parameter evaluated in $\mathcal{H}_{\mathcal{O}}$. Particularly we decide to point our antenna in $\hat{e}^i = (0, 0, 1)$, along the "special" comoving z-axis.
In the end, we can set, in order to find the solutions of the second-order geodesics equation, the boundary conditions

$$\begin{aligned} x^{(1)\mu}(\lambda_{\mathcal{O}}) &= x^{(2)\mu}(\lambda_{\mathcal{O}}) = 0 \\ k^{(1)i}(\lambda_{\mathcal{O}}) &= k^{(2)i}(\lambda_{\mathcal{O}}) = 0\,. \end{aligned} \tag{5.7}$$

Now, using the condition that $g_{\mu\nu} k^\mu(\lambda_{\mathcal{O}}) k^\nu(\lambda_{\mathcal{O}}) = 0$, i.e. $k^\mu(\lambda_{\mathcal{O}})$ is null wave vector in $\mathcal{H}_{\mathcal{O}}$, we have the following condition for (5.3):

$$\begin{aligned} k^{(1)0}(\lambda_{\mathcal{O}}) &= \left(\frac{1}{2} h_{00} + h_{0i} k^{(0)i} + \frac{1}{2} h_{ij} k^{(0)i} k^{(0)j}\right)_{\lambda=\lambda_{\mathcal{O}}} = Z(\lambda_{\mathcal{O}}) \\ k^{(2)0}(\lambda_{\mathcal{O}}) &= \left[\frac{3}{8}(h_{00})^2 + h_{00} h_{0i} k^{(0)i} + \frac{1}{4} h_{00} h_{ij} k^{(0)i} k^{(0)j} + \frac{1}{2}(h_{0i} k^{(0)i})^2 \right. \\ &\quad \left. -\frac{1}{8}(h_{ij} k^{(0)i} k^{(0)j})^2 \right]_{\lambda=\lambda_{\mathcal{O}}} = -\frac{1}{8} Z^2(\lambda_{\mathcal{O}})\,. \end{aligned} \tag{5.8}$$

These conditions are fundamental to study the second-order geodesics equation and to give a quantitative result that is the purposes of the next sections.

5.2 Formulation of the a-order geodesic equation

In the following section we study the general n-order null geodesics, in order to describe the geodesic of a photon to second-order for a perturbed metric in synchronous gauge.
Given a perturbed metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ we start with the geodesic equation:

$$\frac{dk^\mu}{d\lambda}(\lambda) + \Gamma^\mu{}_{\alpha\beta} k^\alpha(\lambda) k^\beta(\lambda) = 0\,, \tag{5.9}$$

where $k^\mu(\lambda)$ is the null wave-vector of the photon in function of some affine parameter λ, moreover it is the derivatives of the photon path, $k^\mu(\lambda) = \frac{dx^\mu}{d\lambda}(\lambda)$. In particular we express the photon path and the associated wave-vector as a series expansions in order of $h_{\mu\nu}$ and its derivatives

$$\begin{aligned} x(\lambda) &= \sum_{a=0}^{\infty} x^{(a)}(\lambda) \\ k(\lambda) &= \sum_{a=0}^{\infty} k^{(a)}(\lambda)\,. \end{aligned} \tag{5.10}$$

In the same way the affine connection can be expanded as

$$\Gamma^\mu{}_{\alpha\beta} = \sum_{a=0}^{\infty} \Gamma^{(a)\mu}{}_{\alpha\beta}\,. \tag{5.11}$$

Now, Taylor expanding the $\Gamma^{(a)\mu}{}_{\alpha\beta}$ at $x^\mu(\lambda)$ about their value at $x^{(0)\mu}(\lambda)$ and substituting it with the above expansions, we can obtain a totally equivalent geodesics equation holding for the unperturbed path $x^{(0)a}(\lambda)$:

$$\begin{aligned} \sum_{a=0}^{\infty} \Bigg[\frac{d^2 x^{(a)\mu}}{d\lambda^2} + & \left(\Gamma^{(a)\mu}{}_{\alpha\beta} + \sum_{b=1}^{\infty} \frac{1}{m!} \partial_{\sigma_1} ... \partial_{\sigma_m} \Gamma^{(a)\mu}{}_{\alpha\beta} \left(\sum_{l=1}^{\infty} x^{(l)\sigma_1} \right) ... \left(\sum_{p=1}^{\infty} x^{(p)\sigma_m} \right) \right) \cdot \\ & \times \left(\sum_{q=1}^{\infty} k^{(q)\alpha} \right) \left(\sum_{r=1}^{\infty} k^{(r)\beta} \right) \Bigg] = 0 \end{aligned} \tag{5.12}$$

Of course, at zeroth-order the photon path $x^{(0)\mu}$ is the corresponding geodesic for the FRW unperturbed metric $g^{(0)}_{\mu\nu}$.
We can see that for an a-order equation this equation can be rearrange into a the following equation for the deviation vector $x^{(a)\mu}(\lambda)$:

$$f^{(a)\mu} = \frac{d^2 x^{(a)\mu}}{d\lambda^2} + 2\Gamma^{(0)\mu}{}_{\alpha\beta} k^{(0)\alpha} k^{(a)\beta} + \partial_\sigma \Gamma^{(0)\mu}{}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} x^{(a)\sigma}\,, \tag{5.13}$$

where the forced four-vector f^{μ}, at first-order and second-order, can be written as:

$$
\begin{aligned}
f^{(1)\mu} =& -\Gamma^{(1)\mu}_{\ \ \alpha\beta} k^{(0)\alpha} k^{(0)\beta} \\
f^{(2)\mu} =& -\Gamma^{(0)\mu}_{\ \ \alpha\beta} k^{(1)\alpha} k^{(1)\alpha} - 2\Gamma^{(1)\mu}_{\ \ \alpha\beta} k^{(0)\alpha} k^{(1)\beta} - 2\partial_\sigma \Gamma^{(0)\mu}_{\ \ \alpha\beta} x^{(1)\sigma} k^{(0)\alpha} k^{(1)\beta} - \partial_\sigma \Gamma^{(1)\mu}_{\ \ \alpha\beta} x^{(1)\sigma} k^{(0)\alpha} k^{(0)\beta} \\
& - \frac{1}{2}\partial_\sigma \partial_\tau \Gamma^{(0)\mu}_{\ \ \alpha\beta} x^{(1)\sigma} x^{(1)\tau} k^{(0)\alpha} k^{(0)\beta} - \Gamma^{(2)\mu}_{\ \ \alpha\beta} k^{(0)\alpha} k^{(0)\beta} .
\end{aligned}
\tag{5.14}
$$

Using the metric (5.3) we have for the first-order force vector:

$$
\begin{aligned}
f^{(1)0} &= -Z' \\
f^{(1)1} &= Z_x \\
f^{(1)2} &= Z_y \\
f^{(1)3} &= 2Z' - Z_z .
\end{aligned}
\tag{5.15}
$$

where we used the notation for the conformal time partial derivatives $\partial(\cdot)/\partial\eta \equiv (\cdot)'$.
In order to find the second-order CMB temperature deviation we will only need its time component:

$$
f^{(2)0} = 2k^{(1)3} Z' - x^{(1)\sigma} \partial_\sigma Z' . \tag{5.16}
$$

Now looking at equation (5.13) we can see that $x^{(a)\mu}$ satisfies the system of four coupled, second order differential equations:

$$
\left(\frac{d^2}{d\lambda^2} + A\frac{d}{d\lambda} + B \right) x^{(a)} = f^{(a)} , \tag{5.17}
$$

where we made the indices implicit for ease of notation, in particular A and B are two 4×4 matrices defined by

$$
\begin{aligned}
A^{\mu}_{\ \nu} &\equiv 2\Gamma^{(0)\mu}_{\ \ \rho\nu} k^{(0)\rho} \\
B^{\mu}_{\ \nu} &\equiv \Gamma^{(0)\mu}_{\ \ \rho\sigma,\nu} k^{(0)\rho} k^{(0)\sigma} .
\end{aligned}
\tag{5.18}
$$

We can demonstrate that equation (5.13) is nothing but a perturbation of the Jacobi equation of the background spacetime (see [10]).
Given a general four-vector v^{μ} we define the covariant differentiation along the unperturbed path $x^{(0)\mu}$ as:

$$
\frac{D}{D\lambda} v^{\mu} \equiv \frac{dv^{\mu}}{d\lambda} + \Gamma^{(0)\mu}_{\ \ \alpha\beta} k^{(0)\alpha} v^{\beta} . \tag{5.19}
$$

Obviously we can obtain by differentiating a second time this form

$$
\frac{D^2}{D\lambda^2} v^{\mu} = \frac{d^2 v^{\mu}}{d\lambda^2} + \Gamma^{(0)\mu}_{\ \ \alpha\beta,\gamma} k^{(0)\alpha} k^{(0)\gamma} v^{\beta} + \Gamma^{(0)\mu}_{\ \ \alpha\beta} \frac{dk^{(0)\alpha}}{d\lambda} v^{\beta} + 2\Gamma^{(0)\mu}_{\ \ \alpha\beta} k^{(0)\alpha} \frac{dv^{\beta}}{d\lambda} + \Gamma^{(0)\mu}_{\ \ \alpha\beta} \Gamma^{(0)\beta}_{\ \ \sigma\rho} k^{(0)\alpha} k^{(0)\sigma} v^{\rho} . \tag{5.20}
$$

From the definition of the Riemann tensor (4.19) for the background metric we get

$$\frac{D^2}{D\lambda^2}v^\mu - R^{(0)\mu}{}_{\nu\rho\sigma}k^{(0)\nu}k^{(0)\rho}v^\sigma = \frac{d^2v^\mu}{d\lambda^2} + 2\Gamma^{(0)\mu}{}_{\alpha\beta}k^{(0)\alpha}\frac{dv^\beta}{d\lambda} + \Gamma^{(0)\mu}{}_{\alpha\beta,\gamma}k^{(0)\alpha}k^{(0)\beta}v^\gamma\,. \quad (5.21)$$

So we can rewrite equation (5.13) in a more compact form:

$$\frac{D^2x^{(a)}}{D\lambda^2} - \mathcal{R}(k^{(0)}, x^{(a)})k^{(0)} = f^{(a)}\,. \quad (5.22)$$

It is important to point out that equation (5.22) holds along a segment of the unperturbed path $x^{(0)\mu}$ and produces solutions for the separation segments of the perturbed paths, from $x^{(1)\mu}$ to $x^{(a)\mu}$, especially for our porpouse, to $x^{(2)\mu}$.

5.3 Perturbed Jacobi equation solution

In this section we want to perform a formal solution for equation (5.13), particularly find a form for the perturbed photon paths.
We define $P(\lambda_1, \lambda_2)$ a 4 non-singular matrix function of $\lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{R}$, and $v(\lambda)$ such that $x^a = Pv$. With these assumptions we can find, starting from (5.13), the expression

$$\ddot{v} + P^{-1}(2\dot{P} + AP)\dot{v} + P^{-1}(A\dot{P} + AP + \ddot{P})v = P^{-1}f\,, \tag{5.23}$$

where we used the notation $\dot{(\cdot)} = d/d\lambda$. Because of P is totally a general matrix we can choose it in the way this satisfies the relation

$$\dot{P} = -\frac{1}{2}AP\,. \tag{5.24}$$

For the reason that in general a matrix function do not commute with its derivatives, the solution of this differential equation is not simply an exponential. Knowing that $P(\lambda_1, \lambda_1) = \mathbb{1}$ we can write this equation as

$$P(\lambda_2, \lambda_1) = \mathbb{1} - \frac{1}{2}\int_{\lambda_1}^{\lambda_2} A(\lambda)P(\lambda, \lambda_1)\, d\lambda\,. \tag{5.25}$$

Reiterating order by order in A we can find

$$\begin{aligned} P(\lambda_2, \lambda_1) &= \mathbb{1} - \frac{1}{2}\int_{\lambda_1}^{\lambda_2} A(\lambda)\, d\lambda + \frac{1}{4}\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda} A(\lambda)A(\lambda')\, d\lambda d\lambda' + ... \\ &= \sum_{n=0}^{\infty} \frac{1}{(-2)^n}\int_{\lambda_1}^{\tau_1} ... \int_{\lambda_1}^{\tau_n} \left(A(\tau_1')...A(\tau_n')\right)\, d\tau_1'...d\tau_n' \\ &= \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!}\int_{\lambda_1}^{\lambda_2} ... \int_{\lambda_1}^{\lambda_2} \mathcal{P}\left(A(\tau_1')...A(\tau_n')\right)\, d\tau_1'...d\tau_n'\,. \end{aligned} \tag{5.26}$$

So we find that the solution of the differential equation is the path-ordered exponential

$$P(\lambda_2, \lambda_1) = \mathcal{P}\exp\left(-\frac{1}{2}\int_{\lambda_1}^{\lambda_2} A(\lambda)d\lambda\right)\,. \tag{5.27}$$

This is Synge's parallel propagator, see Synge (1960) [12]. In simple terms we have implemented a change of variables in order to remove the first derivative of v in the equation (5.23). We want to stress that the projector has the physical meaning of parallel projecting a general vector v from λ_1 to λ_2 along the geodesics.
Let us make explicit some Synge's parallel propagator's relations. First of all we have the existence of the identity

$$P(\lambda_1, \lambda_1) = \mathbb{1}\,. \tag{5.28}$$

Furthermore we have the invertibility, in fact it easy to find the relation

$$P(\lambda_2,\lambda)P(\lambda,\lambda_1) = P(\lambda_2,\lambda_1) \tag{5.29}$$

thus we can get

$$P^{-1}(\lambda_2,\lambda_1) = P(\lambda_1,\lambda_2)\,. \tag{5.30}$$

With these clarifications (5.23) can be rewritten as

$$\ddot{v} + P^{-1}\left(-\left(\frac{A}{2}\right)^2 - \frac{\dot{A}}{2} + B\right)Pv = P^{-1}v \tag{5.31}$$

and using equation (5.22)

$$\ddot{v} - (P^{-1}\mathcal{R}P)v = P^{-1}f\,. \tag{5.32}$$

This equation can be written as a first-order matrix equation

$$\frac{d}{d\lambda}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ P^{-1}\mathcal{R}P & 0 \end{pmatrix}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ P^{-1}f\,. \end{pmatrix} \tag{5.33}$$

This is a matrix differential equation that can be solved following Humi and Miller (1988) [13]. We use the method of Green's function, inter alia we define the transition matrix $U(\lambda_1,\lambda_2)$ as a Green's function for the Jacobi equation in the background, we write it as

$$U(\lambda_2,\lambda_1) = \mathcal{P}\exp\left(\int_{\lambda_1}^{\lambda_2}\begin{pmatrix} 0 & \mathbb{1} \\ P(\lambda_1,\lambda)\mathcal{R}P(\lambda,\lambda_1) & 0 \end{pmatrix} d\lambda\right)\,. \tag{5.34}$$

We want to remark that this result tells us the importance of the background curvature to the geodesics solution. So now we are able to show the solution for $x^{(a)\mu}(\lambda)$ and $k^{(a)\mu}(\lambda)$, particularly start from a fixed affine parameter λ_1. Thus defining

$$y(\lambda) = \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} P(\lambda_1,\lambda)x^{(a)}(\lambda) \\ \frac{d}{d\lambda}[P(\lambda_1,\lambda)x^{(a)}(\lambda)] \end{pmatrix} \tag{5.35}$$

and

$$s(\lambda) = \begin{pmatrix} 0 \\ P(\lambda_1,\lambda)f^{(a)}(\lambda) \end{pmatrix} \tag{5.36}$$

the formal solution is

$$y(\lambda) = U(\lambda,\lambda_1)y(\lambda_1) + \int_{\lambda_1}^{\lambda} U(\lambda,\lambda')s(\lambda')\,d\lambda'\,. \tag{5.37}$$

We can explicit our solution using the boundary conditions

$$x^{(a)\mu}(\lambda_{\mathcal{O}}) = k^{(a)i}(\lambda_{\mathcal{O}}) = 0\,, \tag{5.38}$$

where $\lambda_{\mathcal{O}}$ is the affine parameter that characterizes the observer constant conformal time hypersurface, $\mathcal{H}_{\mathcal{O}}$. Therefore we can write

$$
\begin{aligned}
x^{(a)0}(\lambda) &= (\lambda - \lambda_{\mathcal{O}})k^{(a)0}(\lambda_{\mathcal{O}}) + \int_{\lambda_{\mathcal{O}}}^{\lambda} (\lambda - \tilde{\lambda}) f^{(a)0}(\tilde{\lambda})\, d\tilde{\lambda} \\
x^{(a)i}(\lambda) &= \int_{\lambda_{\mathcal{O}}}^{\lambda} (\lambda - \tilde{\lambda}) f^{(a)i}(\tilde{\lambda})\, d\tilde{\lambda} \\
k^{(a)0}(\lambda) &= k^{(a)0}(\lambda_{\mathcal{O}}) + \int_{\lambda_{\mathcal{O}}}^{\lambda} f^{(a)0}(\tilde{\lambda})\, d\tilde{\lambda} \\
k^{(a)i}(\lambda) &= \int_{\lambda_{\mathcal{O}}}^{\lambda} f^{(a)i}(\tilde{\lambda})\, d\tilde{\lambda}\,.
\end{aligned}
\tag{5.39}
$$

What makes these solutions interesting is the fact that they are valid for any type of metric perturbation. We stress the fact that the perturbation is totally encoded in the form of the force vectors $f^{(1)}$ and $f^{(2)}$ given in (5.15) and (5.16). Thus the explicit solutions of the first order geodesics equation are for the photon path

$$
\begin{aligned}
x^{(1)0} &= (\lambda - \lambda_{\mathcal{O}})Z(\lambda_{\mathcal{O}}) - \int_{\lambda_{\mathcal{O}}}^{\lambda} Z'(\lambda - \tilde{\lambda})\, d\tilde{\lambda} \\
x^{(1)1} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} Z_x(\lambda - \tilde{\lambda})\, d\tilde{\lambda} \\
x^{(1)2} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} Z_y(\lambda - \tilde{\lambda})\, d\tilde{\lambda} \\
x^{(1)3} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} (2Z' - Z_z)(\lambda - \tilde{\lambda})\, d\tilde{\lambda}\,.
\end{aligned}
\tag{5.40}
$$

We can simply derive with respect to λ in order to find the wave vector components:

$$
\begin{aligned}
k^{(1)0} &= Z(\lambda_{\mathcal{O}}) - \int_{\lambda_{\mathcal{O}}}^{\lambda} Z'\, d\tilde{\lambda} \\
k^{(1)1} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} Z_x\, d\tilde{\lambda} \\
k^{(1)2} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} Z_y\, d\tilde{\lambda} \\
k^{(1)3} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} 2Z' - Z_z\, d\tilde{\lambda}\,.
\end{aligned}
\tag{5.41}
$$

In Fig 8 we show the solutions of equations (5.40) using the boundary conditions made explicit in Sect. 2.1 and the conformally transformed metric potential given in (4.55).
It is useful to see that the z- component of the wave vector can be written as

$$
k^{(1)3} = -k^{(0)3} \int_{\lambda_{\mathcal{O}}}^{\lambda} 2Z' + k^{(0)3} Z_z\, d\tilde{\lambda}\,, \tag{5.42}
$$

moreover the total derivative in respect to λ is:

$$\frac{dZ}{d\lambda} = Z' + k^{(0)i}\partial_i Z \,, \tag{5.43}$$

thus we get, using the boundary conditions (5.13),

$$k^{(1)3} = Z + Z(\lambda_{\mathcal{O}}) - k^{(1)0} \,. \tag{5.44}$$

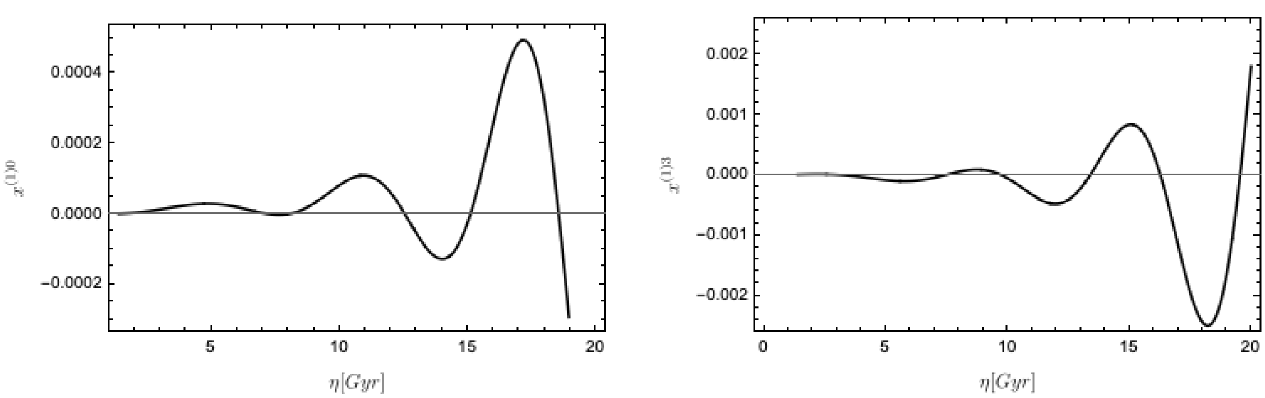
This expression for the time component of the wavevector allows us to find a more elegant form of the second order time component of the force vector given in (5.16)

$$f^{(2)0} = ZZ' - 2k^{(1)0}Z' + 2Z(\lambda_{\mathcal{O}})Z' - x^{(1)\alpha}\partial_\alpha Z' \,. \tag{5.45}$$

We will see that only second-order quantity that enter in the temperature fluctuation formula is the time component of the wavevector $k^\mu(\lambda)$ given by, using:

$$k^{(2)0} = -\frac{1}{2}Z^2(\lambda_{\mathcal{O}}) + \int_{\lambda_{\mathcal{O}}}^{\lambda} \Big[ZZ' - 2k^{(1)0}Z' + 2Z(\lambda_{\mathcal{O}})Z' - x^{1\alpha}\partial_\alpha Z'\Big]_{\lambda'} \, d\tilde{\lambda} \,. \tag{5.46}$$

We have well defined the mathematics and the fundamental steps in order to find the solution of the second-order geodesics equation. The physical quantities, that we made explicit, will be implemented in the next section in order to calculate the CMB temperature fluctuations.



(a) $x^{(1)0}(\eta)$ (b) $x^{(1)3}(\eta)$

Figure 8: Plots of the solutions of the first-order geodesics equation along special z-axis for the photon-path $x^{(1)\mu}$ for the perturbed line element (5.3).

5.4 Temperature perturbation expansion

In this section we want to show the behavior of the photon temperature fluctuations from the last scattering surface towards us. As we have done for the density profile deviation in Section 2, we define the emitted temperature as

$$T_E(p^i, \hat{d}^i) = [1 + \tau(p^i, \hat{d}^i)]T_E^{(0)} \tag{5.47}$$

where $T_E^{(0)}$ is the emitted temperature in the uniform background and $\tau(p^i, \hat{d}^i)$ is a deviation function of the same order of the metric perturbation $h_{\mu\nu}$ independent of gravitational effects. We must define the relative energy:

$$\omega = -\frac{1}{a} g_{\mu\nu} u^\mu k^\nu \,, \tag{5.48}$$

where u^μ is the conformal observer four velocity defined in (4.4) through the relation with the corresponding physical quantity $\bar{u}^\mu = a^{-1} u^\mu$. The exact CMB temperature observed in $\mathcal{O}$ is related to the emission temperature by the form:

$$T_{\mathcal{O}}(\mathbf{x}(\lambda_{\mathcal{O}}), \hat{e}^i) = \frac{\omega_{\mathcal{O}}}{\omega_E} T_E(p^i, \hat{d}^i) \,. \tag{5.49}$$

Expanding the quantities ω_E, $\hat{d}^i$ and the expansion relation given in relation (5.4) we can write

$$T_{\mathcal{O}} = \frac{\omega_{\mathcal{O}}^{(0)} + \omega_{\mathcal{O}}^{(1)} + \omega_{\mathcal{O}}^{(2)}}{\omega_E^{(0)} + \omega_E^{(1)} + \omega_E^{(2)}} [1 + \tau(p^{(0)i} + p^{(1)i}, \hat{d}^{(0)i} + \hat{d}^{(1)i})] T_E^{(0)} \,. \tag{5.50}$$

Now defying $\tilde{\omega}^{(a)} \equiv \omega^{(a)}/\omega^{(0)}$ and Taylor expanding τ we can denote the deviation temperature from the unperturbed spacetime as

$$\begin{aligned} \delta_T \equiv \left(\frac{\omega_E^{(0)}}{\omega_{\mathcal{O}}^{(0)}} \right) \frac{T_{\mathcal{O}}}{T_E^{(0)}} = \Big[1 + (\tilde{\omega}_{\mathcal{O}}^{(1)} - \tilde{\omega}_E^{(1)} + \tau) \\ \tilde{\omega}_{\mathcal{O}}^{(2)} - \tilde{\omega}_E^{(2)} + (\tilde{\omega}_E^{(1)})^2 - \tilde{\omega}_{\mathcal{O}}^{(1)} \tilde{\omega}_E^{(1)} + \tilde{\omega}_{\mathcal{O}}^{(1)} \tau - \tilde{\omega}_E^{(1)} \tau + p(1)i \frac{\partial \tau}{\partial x^i} + \hat{d}^{(1)i} \frac{\partial \tau}{\partial \hat{d}^i} \Big] \,. \end{aligned} \tag{5.51}$$

Expanding the metric perturbation and the photon wavevector we get the temperature fluctuation up to the second order perturbation:

$$
\begin{aligned}
\delta_T^{(0)} &= 1\\
\delta_T^{(1)} &= \left[\frac{1}{2}h_{ij}k^{(0)i}k^{(0)j}\right]_{\lambda=\lambda_{\mathcal{O}}} + \left[\frac{1}{2}h_{00} + h_{0i}k^{(0)i} - k^{(1)i} + \tau\right]\\
\delta_T^{(2)} &= \left[\frac{1}{2}(h_{0i}k^{(0)i})^2 - \frac{1}{8}(h_{ij}k^{(0)i}k^{(0)j})^2\right]_{\lambda=\lambda_{\mathcal{O}}}\\
&+ \left[h_{ij}k^{(0)i}k^{(0)j}\right]_{\lambda=\lambda_{\mathcal{O}}}\left[\frac{1}{2}h_{00} + h_{0i}k^{(0)i} - k^{(1)0} + \tau\right]\\
&+ \left[\frac{3}{8}(h_{00})^2 - \frac{1}{2}h_{00}k^{(1)0} + \frac{3}{2}h_{0i}h_{00}k^{(0)i} + (h_{0i}k^{(0)i})^2 - 2h_{0i}k^{(0)i}k^{(1)0}\right.\\
&+h_{0i}k^{(1)i} + (k^{(1)0})^2 - k^{(2)0} + \tau\left(\frac{1}{2}h_{00} + h_{0i}k^{(0)i} - k^{(1)0}\right) + x^{(1)0}\frac{dk^{(1)0}}{d\lambda}\\
&\left.-h_{0i}x^{(1)0}\frac{dk^(0)i}{d\lambda} + (x^{(1)i} - k^{(0)i}x^{(1)0})\left(\frac{1}{2}\frac{\partial h_{00}}{\partial x_i} + k^{(0)j}\frac{\partial h_{0j}}{\partial x_i} + \frac{\partial\tau}{\partial x^i} + \hat{d}^{(1)i}\frac{\partial\tau}{\partial\hat{d}^i}\right)\right],
\end{aligned}
\tag{5.52}
$$

where, because of the expansion of p^i, we can write

$$p^{(1)i} = x^{(1)i} - k^{(0)i}x^{(1)0}\,, \tag{5.53}$$

thus we have defined in the expression of $\delta T^{(2)}$ the firs order distance

$$\hat{d}^{(1)i} = \frac{k^{(0)i} + k^{(1)i}}{\left|k^{(0)i} + k^{(1)i}\right|} - \frac{k^{(0)i}}{\left|k^{(0)i}\right|}\,. \tag{5.54}$$

Now, using the time component of the first order force vector $f^{(1)\mu}$ computed in (5.15), we can give the first order temperature fluctuation expression for our model:

$$\delta_T^{(1)}(\lambda) = \tau(\lambda) - \mathcal{I}^{(1)}(\lambda)\,, \tag{5.55}$$

where we have defined $\mathcal{I}^{(1)}(\lambda)$ as

$$\mathcal{I}^{(1)}(\lambda) \equiv -\int_{\lambda_{\mathcal{O}}}^{\lambda} Z'\,d\tilde{\lambda}\,. \tag{5.56}$$

Therefore we have recover the well-known Sachs-Wolfe effect [14]. The second order anisotropy is given by

$$
\begin{aligned}
\delta_T^{(2)}(\lambda) =&\mathcal{I}^{(1)}\left(Z(\lambda_{\mathcal{O}}) + \mathcal{I}^{(1)} + \tau\right) - \mathcal{I}^{(2)} + \chi(\lambda) - x^{(1)0}Z'\\
&+\left[x^{(1)i} + k^{(0)i}\int_{\lambda_{\mathcal{O}}}^{\lambda}(\lambda - \tilde{\lambda})Z_z\,d\tilde{\lambda} + k^{(0)i}\Sigma(\lambda)\right]\frac{\partial\tau}{\partial x^i} + d^{(1)i}\frac{\partial\tau}{\partial d^i}\,.
\end{aligned}
\tag{5.57}
$$

In particular we define the second-order integral function

$$\chi(\lambda) \equiv \int_{\lambda_{\mathcal{O}}}^{\lambda} 2k^{(1)0} Z' - 2Z(\lambda_{\mathcal{O}}) Z' - Z Z' \, d\tilde{\lambda} \,. \tag{5.58}$$

Moreover we define the $\mathcal{I}^{(2)}(\lambda)$ in the equation (5.57) as

$$\mathcal{I}^{(2)}(\lambda) \equiv -\int_{\lambda_{\mathcal{O}}}^{\lambda} x^{(1)\alpha} \partial_{\alpha} Z' \, d\tilde{\lambda} \,. \tag{5.59}$$

Especially this term can be interpreted as a correction to the integrated Sachs-Wolfe effect that we find at first-order. The main idea of this term is that it includes the differences between the perturbations along the first-order path and the perturbations along the background path. This term is smaller than the conventional Sachs-Wolfe effect but improving the degree of observation precision it can represent a valid correction.
In the end the term $\Sigma(\lambda)$ represents the Shapiro time delay

$$\Sigma(\lambda) \equiv \int_{\lambda_{\mathcal{O}}}^{\lambda} Z d\tilde{\lambda} \,. \tag{5.60}$$

We can think about the term $k^{(0)i}\Sigma(\lambda)(\partial\tau/\partial x^i)$ as the orthogonal photon deflection due to a gravitational sources between the surface of last scattering and the observer, so a gravitational lens effect on CMBR. It can be important at smaller angular scales, in particular the deflection can give the product of the lens angle formed and the total distance travelled by the CMB photons.

5.5 Temperature anisotropies for a general silent metric

In this section we want to find the form of the CMB temperature fluctuations from a general silent metric. Following Matarrese et al. (1995) [17], the definition of a general silent metric (3.5) has three different inhomogeneous functions along the coordinate axis. Therefore we can write the general first-order perturbation of a silent metric around a FLRW background in conformal time as:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+2\xi(\eta,\mathbf{x}) & 0 & 0 \\ 0 & 0 & 1+2\upsilon(\eta,\mathbf{x}) & 0 \\ 0 & 0 & 0 & 1+2\zeta(\eta,\mathbf{x}) \end{pmatrix}, \tag{5.61}$$

where ξ, υ, ζ are the perturbed inhomogeneity functions. In order to calculate the forced vector f^μ we have to write the connection terms at zeroth, first and second-order using equation (4.3). The zeroth-orders symbols vanish in the flat background. The first-order symbols are

$$\begin{aligned}
\Gamma^{(1)0}{}_{11} &= \xi' \\
\Gamma^{(1)0}{}_{22} &= \upsilon' \\
\Gamma^{(1)0}{}_{33} &= \zeta' \\
\Gamma^{(1)1}{}_{\mu 1} &= \partial_\mu \xi \\
\Gamma^{(1)1}{}_{22} &= -\upsilon_x \\
\Gamma^{(1)1}{}_{33} &= -\zeta_x \\
\Gamma^{(1)2}{}_{\mu 2} &= \partial_\mu \upsilon \\
\Gamma^{(1)2}{}_{22} &= -\xi_y \\
\Gamma^{(1)2}{}_{33} &= -\zeta_y \\
\Gamma^{(1)3}{}_{\mu 3} &= \partial_\mu \zeta \\
\Gamma^{(1)3}{}_{11} &= -\xi_z \\
\Gamma^{(1)3}{}_{22} &= -\upsilon_z \\
\Gamma^{(1)3}{}_{\mu 3} &= \partial_\mu \zeta
\end{aligned} \tag{5.62}$$

and in the end the second-order symbols are

$$
\begin{aligned}
{\Gamma^{(2)1}}_{\mu 1} &= -2\xi\partial_\mu\xi \\
{\Gamma^{(2)1}}_{22} &= 2\xi\upsilon_x \\
{\Gamma^{(2)1}}_{33} &= 2\xi\zeta_x \\
{\Gamma^{(2)2}}_{\mu 2} &= -2\upsilon\partial_\mu\upsilon \\
{\Gamma^{(2)2}}_{22} &= 2\upsilon\xi_y \\
{\Gamma^{(2)2}}_{33} &= 2\upsilon\zeta_y \\
{\Gamma^{(2)3}}_{\mu 3} &= -2\zeta\partial_\mu\zeta \\
{\Gamma^{(2)3}}_{11} &= 2\zeta\xi_z \\
{\Gamma^{(2)3}}_{22} &= 2\zeta\upsilon_z \,.
\end{aligned}
\tag{5.63}
$$

As boundary condition we impose to point the observer antenna along a generic direction described by the unity vector $\hat{e}^i$ of the of the 3-surface orthogonal to the fluid flow 4-velocity $u^\mu = (1, 0, 0, 0)$ satisfying the following condition

$$\delta_{ij}\hat{e}^i\hat{e}^j = 1\,. \tag{5.64}$$

Therefore the background photon path and wavevector have the components

$$x^{(0)\mu}(\lambda) = (1, (\lambda_{\mathcal{O}} - \lambda)\hat{e}^i) \tag{5.65}$$

$$k^{(0)\mu} = (1, -\hat{e}^i)\,. \tag{5.66}$$

Using the same reasoning and boundary conditions that we have defined in the above sections we can find that the first-order forced vector, given in equation (5.14), for our silent metric (5.61) has the following components:

$$
\begin{aligned}
f^{(1)0} &= -[(e^1)^2\xi' + (e^2)^2\upsilon' + (e^3)^2\zeta']\,, \\
f^{(1)1} &= (e^1)^2\xi_x + (e^2)^2\upsilon_x + (e^3)^2\zeta_x - 2(e^1)k^{(0)\mu}\partial_\mu\xi\,, \\
f^{(1)2} &= (e^1)^2\xi_y + (e^2)^2\upsilon_y + (e^3)^2\zeta_y - 2(e^2)k^{(0)\mu}\partial_\mu\upsilon\,, \\
f^{(1)3} &= (e^1)^2\xi_z + (e^2)^2\upsilon_z + (e^3)^2\zeta_z - 2(e^3)k^{(0)\mu}\partial_\mu\zeta\,.
\end{aligned}
\tag{5.67}
$$

As before, we are only interested on the time dependent component of the second-order forced vector in order to compute $\delta_T^{(2)}$, we can find that

$$
\begin{aligned}
f^{(2)0} &= \partial_0 h_{ij}\hat{e}^i k^{(1)j} - 2x^{(1)\mu}\left((e^1)^2\partial_\mu\xi' + (e^2)^2\partial_\mu\upsilon' + (e^3)^2\partial_\mu\zeta'\right) \\
&= \partial_0 h_{ij}\hat{e}^i k^{(1)j} - \hat{e}^i\hat{e}^j x^{(1)\mu}\partial_\mu\partial_0 h_{ij}\,.
\end{aligned}
\tag{5.68}
$$

With equation (5.67) we can explicit the solutions of the first-order geodesics equation for the photon path and wavevector using the relations given by the equations (5.39). We can write in a compact form the first-order photon path components:

$$
\begin{aligned}
x^{(1)0} &= \frac{1}{2}h_{ij}(\lambda_{\mathcal{O}})\hat{e}^i\hat{e}^j(\lambda-\lambda_{\mathcal{O}}) - \frac{1}{2}\int_{\lambda_{\mathcal{O}}}^{\lambda} \partial_0 h_{ij}\hat{e}^i\hat{e}^j(\lambda-\tilde{\lambda})\,d\tilde{\lambda} \\
x^{(1)1} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[k^{(0)2}(k^{(0)2}\upsilon_x - k^{(0)1}\xi_y) + k^{(0)3}(k^{(0)3}\zeta_x - k^{(0)1}\xi_z) - k^{(0)1}\left(\xi' + \frac{d\xi}{d\lambda'}\right)\right](\lambda-\tilde{\lambda})\,d\tilde{\lambda} \\
x^{(1)2} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[k^{(0)3}(k^{(0)3}\zeta_y - k^{(0)2}\upsilon_x) + k^{(0)1}(k^{(0)1}\xi_y - k^{(0)2}\upsilon_z) - k^{(0)2}\left(\upsilon' + \frac{d\upsilon}{d\tilde{\lambda}}\right)\right](\lambda-\tilde{\lambda})\,d\tilde{\lambda} \\
x^{(1)3} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[k^{(0)1}(k^{(0)1}\xi_z - k^{(0)3}\zeta_x) + k^{(0)2}(k^{(0)2}\upsilon_y - k^{(0)3}\zeta_y) - k^{(0)3}\left(\zeta' + \frac{d\zeta}{d\tilde{\lambda}}\right)\right](\lambda-\tilde{\lambda})\,d\tilde{\lambda}\,.
\end{aligned}
\tag{5.69}
$$

Then the solutions of the first-order photon wavevector are:

$$
\begin{aligned}
k^{(1)0} &= \frac{1}{2}h_{ij}(\lambda_{\mathcal{O}})\hat{e}^i\hat{e}^j - \frac{1}{2}\int_{\lambda_{\mathcal{O}}}^{\lambda} \partial_0 h_{ij}\hat{e}^i\hat{e}^j\,d\tilde{\lambda} \\
k^{(1)1} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} k^{(0)2}(k^{(0)2}\upsilon_x - k^{(0)1}\xi_y) + k^{(0)3}(k^{(0)3}\zeta_x - k^{(0)1}\xi_z) - k^{(0)1}\left(\xi' + \frac{d\xi}{d\tilde{\lambda}}\right)d\tilde{\lambda}\,, \\
k^{(1)2} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} k^{(0)3}(k^{(0)3}\zeta_y - k^{(0)2}\upsilon_x) + k^{(0)1}(k^{(0)1}\xi_y - k^{(0)2}\upsilon_z) - k^{(0)2}\left(\upsilon' + \frac{d\upsilon}{d\tilde{\lambda}}\right)d\tilde{\lambda}\,, \\
k^{(1)3} &= \int_{\lambda_{\mathcal{O}}}^{\lambda} k^{(0)1}(k^{(0)1}\xi_z - k^{(0)3}\zeta_x) + k^{(0)2}(k^{(0)2}\upsilon_y - k^{(0)3}\zeta_y) - k^{(0)3}\left(\zeta' + \frac{d\zeta}{d\tilde{\lambda}}\right)d\tilde{\lambda}\,.
\end{aligned}
\tag{5.70}
$$

We interpret the integral inside $k^{(1)0}$ as the sum of the contribution to the ISW effect along each axis:

$$
\mathcal{I}^{(1)}(\lambda) = -\int_{\lambda_{\mathcal{O}}}^{\lambda} (e^1)^2\xi' + (e^2)^2\upsilon' + (e^3)^2\zeta'\,d\tilde{\lambda}\,, = (e^1)^2\mathcal{I}^{(1)}_x(\lambda) + (e^2)^2\mathcal{I}^{(1)}_y(\lambda) + (e^3)^2\mathcal{I}^{(1)}_z(\lambda)\,.
\tag{5.71}
$$

In this way we can write our components of the photon wavevector as

$$
\begin{aligned}
k^{(1)0} &= \frac{1}{2}h_{ij}(\lambda_{\mathcal{O}})\hat{e}^i\hat{e}^j + \mathcal{I}^{(1)}(\lambda) \\
k^{(1)1} &= W_x(\lambda) + k^{(0)1}(\mathcal{I}^{(1)}_x(\lambda) + \xi(\lambda_{\mathcal{O}}) - \xi) \\
k^{(1)2} &= W_y(\lambda) + k^{(0)2}(\mathcal{I}^{(1)}_y(\lambda) + \upsilon(\lambda_{\mathcal{O}}) - \upsilon) \\
k^{(1)3} &= W_z(\lambda) + k^{(0)3}(\mathcal{I}^{(1)}_z(\lambda) + \zeta(\lambda_{\mathcal{O}}) - \zeta)\,.
\end{aligned}
\tag{5.72}
$$

where $W_{s=x,y,z}$ are the sum of the first two integrals in each spatial component of the first-order photon wavevector given in equation (5.70).
Now we can write the temperature anisotropies. First-order temperature anisotropies $\delta^{(1)}_T$ is given by substituting the above relations into equation (5.52), we have

$$
\delta^{(1)}_T(\lambda) = -\mathcal{I}^{(1)}(\lambda) + \tau
\tag{5.73}
$$

So we have found that the first-order temperature deviation is the ISW effect, the great information is hidden in the expression of $\mathcal{I}^{(1)}$ equation (5.71). We can see that each metric potential perturbation evolves independently of the others. In silent universes the evolution of the cosmic fluid element is local at first-order around FLRW, i.e. each element do not change information with others elements, in fact there are no directional mixed terms in the integral of the integrated Sachs-Wolfe effect $\mathcal{I}^{(1)}$.
Now we can write the second-order time component of the photon wavevector:

$$k^{(2)0} = -\frac{1}{8}(h_{ij}k^{(0)i}k^{(0)j})^2 + I^{(2)}(\lambda) - \chi(\lambda)\,, \tag{5.74}$$

where we define

$$\mathcal{I}^{(2)}(\lambda) \equiv -\int_{\lambda_{\mathcal{O}}}^{\lambda} x^{(1)\mu}\partial_\mu h_{ij}\hat{e}^i\hat{e}^j\, d\tilde{\lambda} = (e^{(1)})^2\mathcal{I}_x^{(2)} + (\lambda)(e^{(2)})^2\mathcal{I}_y^{(2)}(\lambda) + (e^{(3)})^2\mathcal{I}_z^{(2)}(\lambda)\,, \tag{5.75}$$

where each $\{\mathcal{I}_s^{(2)}\}_{s=x,y,z}$ is given by

$$\mathcal{I}_x^{(2)}(\lambda) = -\int_{\lambda_{\mathcal{O}}}^{\lambda} x^{(1)\mu}\partial_\mu\xi'\, d\tilde{\lambda}\,, \quad \mathcal{I}_y^{(2)}(\lambda) = -\int_{\lambda_{\mathcal{O}}}^{\lambda} x^{(1)\mu}\partial_\mu\upsilon'\, d\tilde{\lambda}\,, \quad \mathcal{I}_z^{(2)}(\lambda) = -\int_{\lambda_{\mathcal{O}}}^{\lambda} x^{(1)\mu}\partial_\mu\zeta'\, d\tilde{\lambda}\,. \tag{5.76}$$

Moreover, we define the last term of the second-order photon wavevector as

$$\chi(\lambda) \equiv -2\int_{\lambda_{\mathcal{O}}}^{\lambda} \partial_0 h_{ij}\hat{e}^i k^{(1)j}\, d\tilde{\lambda}\,. \tag{5.77}$$

With these definitions we can finally write the second-order temperature deviation defined in equation (5.52), in particular we use the definition of the first-order direction vector $d^{(1)i}$ already used in equation (5.54) :

$$\begin{aligned}\delta_T^{(2)}(\lambda) =& \mathcal{I}^{(1)}(\lambda)\left(\frac{1}{2}h_{ij}(\lambda_{\mathcal{O}})\hat{e}^i\hat{e}^j + \mathcal{I}^{(1)}(\lambda) + \tau\right) \\ &- \mathcal{I}^{(2)}(\lambda) + \chi(\lambda) - x^{(1)0}\left[\xi'(e^1)^2 + \upsilon'(e^2)^2 + \zeta'(e^3)^2\right] \\ &+ \left\{x^{(1)i} + k^{(0)i}Q(\lambda) + k^{(0)i}\Sigma(\lambda)\right\}\frac{\partial\tau}{\partial x^i} + d^{(1)i}\frac{\partial\tau}{\partial d^i}\,.\end{aligned} \tag{5.78}$$

$\delta_T^{(2)}(\lambda_E)$ is the general expression describing the temperature anisotropies at second-order for a general perturbed silent metric around a flat FLRW model. We want to stress that this is nothing but the generalization of the equation (5.57). As in the previous section, we can see that there is the term $\mathcal{I}^{(2)}$, that we interpret as the second-order correction to the ISW effect but now we see explicitly the anisotropy dependence given by the metric potential perturbation. We have defined

$$\Sigma(\lambda) = \int_{\lambda_{\mathcal{O}}}^{\lambda} \xi(e^1)^2 + \upsilon(e^2)^2 + \zeta(e^3)^2\, d\lambda'\,. \tag{5.79}$$

This is the Shapiro time delay along the photon path. Moreover $Q(\lambda)$ is

$$Q(\lambda) \equiv -\frac{k^{(0)l}}{2}\int_{\lambda_{\mathcal{O}}}^{\lambda} \partial_l h_{ij}\hat{e}^i\hat{e}^j(\lambda-\tilde{\lambda})\,d\tilde{\lambda}\,. \tag{5.80}$$

We can observe that the temperature deviation at second-order is strictly dependent on the direction of observation $\hat{e}^i$. Moreover we want to write the sixth term on the second line of equation (5.52) in a more explicit form

$$\left[(e^1)^2\xi' + (e^2)^2 v' + (e^3)^2\zeta'\right]\left[\frac{1}{2}h_{ij}(\lambda_{\mathcal{O}})\hat{e}^i\hat{e}^j(\lambda-\lambda_{\mathcal{O}}) - \int_{\lambda_{\mathcal{O}}}^{\lambda}\partial_0 h_{ij}\hat{e}^i\hat{e}^j(\lambda-\tilde{\lambda})\,d\tilde{\lambda}\right]. \tag{5.81}$$

In particular, we can note an interesting feature in equation (5.78); beyond the linear order there are mixed terms between the inhomogeneous potentials. Matarrese et. al (1994a,b) [22], [23] proved that a generic second-order perturbation give rise to a non-vanishing magnetic field, they named this effect *tidal induction.* So we can interpret the presence of mixed term as the presence of non-local effects, the "breaking" of the silence due to the tidal induction.

6 Conclusions

In this work we have carried out the description of silent universes and their application on the large scales temperature anisotropies. In order to understand the dynamics of silent models, we have delved into the relativistic dynamics proposed by Ellis (1971) [42] which consists in a set of partial differential equations describing a new approach (a hydrodynamic one) to General Relativity. These equations are fully described by the magnetic and electric parts of the Weyl tensor. Silent universes have a dynamics ruled by six first-order quasi linear ODEs, which are a reduction of the Ellis equations with the constraints $p = \omega_{\mu\nu} = H_{\mu\nu} = 0$. In practice, we are talking about an irrotational dust that has a purely local dynamic; therefore, the behaviour of each flow line is fully determined in the initial time conditions. No gravitational or sound information can be exchanged between the different fluid elements after the boundary conditions given by initial time, this is the reason why Matarrese et al. called these models *silent universes*, see [21],[22],[23].
We have defined and studied the phase plane analysis, following Matarrese et. al (1995) [17], of a special case of silent universes, Szekeres models, see Szekeres (1975) [8]. Szekeres models consist in irrotational dust with vanishing magnetic field and, in particular, they have two equal eigenvalues for the shear and the force field. Goode and Wainwright (1982) [9] presented an useful formulation of the Szekeres line element within silent models. This formulation follows the general form of a silent metric given in equation (**??**). This new formulation shows that Szekeres solutions are a generalization of FLRW solutions, in fact one can find the growing and decaying modes of the perturbation around FLRW. More specifically, we have studied and solved in detail the linearized EFE following the Szekeres line element form (slightly different from the one defined by Goode and Wainwright) provided by Merues and Bruni [5]. This form creates an advantage, it allows to split the dynamics of the model in a flat ΛCDM background and in a part describing the inhomogeneous deviation from this background. We have studied the phase plane analysis of differential equations ruling the density deviation δ and the Σ_+ (encoding the shear behaviour), as a function of the Ω_Λ parameter, comparing the exact system with the first-order ones. We find the same behavior for both solutions. In particular, we have found that at first-order the dS space for the Ω_Λ-Σ_+ plane is represented by the fixed point $(1, 0)$, and for the Ω_Λ-δ plane is represented by the vertical line $\Omega_\Lambda = 1$, tracing the exact solutions.
In the second part of the thesis we have focused on finding the CMB temperature anisotropies at first-order and second-order for the perturbed Szekeres metric. In this section we followed Pyne and Carroll (1996) [7] analysis for a n-order temperature anisotropy. We have solved the perturbed geodesics equation for a photon path emitted on the surface of last scattering towards us, and we have used the method outlined by Pyne and Birkinshaw (1993) [10]. Beginning from our perturbed spacetime we have constructed null geodesics order by order using the background null geodesics. It is a generalization of the Sachs-Wolfe method for calculating the CMB fluctuation laid out in the introduction, see Section 1.2.2. In more detail we have written the photon redshift in terms of its background worldline $x^{(0)\mu}(\lambda)$ focusing on the boundary conditions; in particular, we have pointed the direction of observation along the z-axis, that is the only direction in which the out-of-homogeneity Szekeres potential is defined. Then, we have found the geodesics equation solutions to an arbitrary order, solving the perturbed Jacobi equation. In the end we have written the first-order and

second-order temperature deviation $\delta_T^{(1)}$ and $\delta_T^{(2)}$. $\delta_T^{(1)}$ presents the well-known ISW effect, while $\delta_T^{(2)}$ presents the Shapiro time delay and an integral that we have interpreted as a second-order correction of the ISW effect. In the last section we have used the first-order perturbation of a general silent metric defined by Matarrese et al. (1995) [17] using three different inhomogeneous functions, one for each axis. Using the Pyne and Carroll method, we have calculated the general form of first-order and second-order CMB temperature deviations in silent universes. We have found that the first-order ISW effect can be split in three different parts, each one following the behaviour of the anisotropy potentials. Also $\delta_T^{(2)}$ for a generic silent metric presents the second-order correction of the ISW effect and the Shapiro time delay that we have found in the previous section. The great difference from $\delta_T^{(1)}$ is the presence of mixed terms. In practice at first-order we have a pure local effect, each metric potential evolves independently of the others, but at second-order there is a mutual influence; for example, along the z-direction there are contributions of the potentials evolving along others directions. We have interpreted this fact as the effect of the tidal induction introduced by Matarrese et al. (1994a,b) [22], [23]. They showed that at first-order $H_{\mu\nu}$ vanishes, but not beyond; in fact, there are not mixed terms in $\delta_T^{(1)}$, but there are in $\delta_T^{(2)}$.
In silent models theory the most controversial assumption is that the magnetic part of the Weyl tensor vanishes, in particular, for our purpose, in the large scales CMB field, where we are forcibly neglecting the contribution of gravitational waves. Following Matarrese et al (1994a,b) we have circumvented the problem by finding non-local contributions in the second order perturbation of temperature anisotropies. But now we want to re-propose the question written by Mutoh, Hirai and Maeda (1997) [47]: "Is the attractor in silent universe still some attractor in more generic spacetimes?" This question can sprout an interesting line of research, aimed at finding a geometry that specialize the generic silent metric defined in equation (3.5).

7 Bibliography

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