The non-$\mathfrak{F}$ graph for the family of finite groups whose order is divisible by at most two different primes

Master in Algebra, Geometry and Number Theory

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Introduction

Definition 0.0.1. Let Ω be a graph. We say that a vertex x in Ω is isolated if it is not adjacent to any of the other vertices in the graph. On the other hand, we say that a vertex y of the graph Ω is universal if it is adjacent to every other vertex in the graph.

We will work with the following concept in this work:

Definition 0.0.2. Let $\mathfrak{F}$ be a class of finite groups and G a finite group. We consider the graph $\tilde{\Lambda}_{\mathfrak{F}}(G)$ whose vertices are the elements of G and where two vertices $g, h \in G$ are adjacent if and only if $\langle g, h\rangle \in \mathfrak{F}$. The $\mathfrak{F}$-graph of G, denote by $\Lambda_{\mathfrak{F}}(G)$, is the graph obtained by removing the isolated vertices from $\tilde{\Lambda}_{\mathfrak{F}}(G)$.

On the other hand, we consider the graph $\tilde{\Gamma}_{\mathfrak{F}}(G)$ whose vertices are the elements of G and where two vertices $g, h \in G$ are adjacent if and only if $\langle g, h\rangle \notin \mathfrak{F}$. We denote by $\mathfrak{I}_{\mathfrak{F}}(G)$ the set of isolated vertices of the graph $\tilde{\Gamma}_{\mathfrak{F}}(G)$. The non-$\mathfrak{F}$ graph of G, denote by $\Gamma_{\mathfrak{F}}(G)$, is the graph obtained by removing the isolated vertices from $\tilde{\Gamma}_{\mathfrak{F}}(G)$.

In this work $\pi(G)$ denotes the amount of different prime numbers dividing $|G|$. We will focus on the class $\mathfrak{F}$ of finite groups whose order is divisible by at most two different primes, i.e., those groups with $\pi(G) \leq 2$. We will be concerned specially about the non-$\mathfrak{F}$ graph of finite groups G, which we will denote by $\Gamma(G)$. The set of isolated vertices of $\Gamma(G)$ will be denoted as $I(G)$.

The concept of distance will be of great interest; the distance between two vertices x and y of a graph Ω is the length of the shortest path that connect the vertices x and y. We will also talk about the diameter of a graph; this is length of the shortest path between the most distanced vertices, i.e., the diameter of a graph Ω is the following:

$$\operatorname{diam}(\Omega) = \max_{x,y \in \Omega} \ d(x, y).$$

In order to simplify the notation, we will use $\operatorname{diam}(G)$ instead of $\operatorname{diam}(\Gamma(G))$ in this particular case.

Taking as reference the investigations explained in Section 2, we aim to study properties about some graphs associated to finite groups. In section 3 we give important definitions and preliminary results that will be useful during the work, such as the Higman Theorem among others:

Theorem 0.0.1. *Let G be a solvable group all of whose elements have prime power order. Then G has order divisible by at most two primes.*

In section 4 will be concerned about the connectivity of the non-$\mathfrak{F}$ graphs of finite groups with $\pi(G) \geq 3$. First we will prove the connectivity result for solvable groups.

Lemma 0.0.2. *Let G be a finite solvable group with $\pi(G) \geq 3$, then $\Gamma(G)$ is connected.*

Using the information about the connectivity for the graphs of finite solvable groups, we get to one of the main theorems of this work:

Theorem 0.0.3. *Let G be a finite group with $\pi(G) \geq 3$, then $\Gamma(G)$ is connected.*

Once we have proved that these graphs are connected, it seems natural to ask which is a bound for the diameter of these graphs, thing that we will study in Section 5. In order to do this we have studied first the distances between elements that hold certain properties, so that we can know how they behave when looking for the diameter. The following is a helpful result:

Lemma 0.0.4. *Let G be a solvable group such that $I(G) \neq G$. Then for all $x \notin I(G)$, $d(x,y) \leq 2$ for some y with $\pi(y) \geq 2$.*

The following is also one of the main theorems in this work, since it stablishes a connection between the diameter of a graph with the prime graph of the group. It is important because the prime graph gives us information about the orders of the elements and how many primes divide these orders in different cases. We are keen on being able to obtain this information since we are working with the cardinality of the groups generated by two elements of a given group, so it is useful to know as much as possible about the order of those elements.

Theorem 0.0.5. *Let G be a finite solvable group with $\pi(G) = 3$. If $\operatorname{diam}(G) > 4$, then the prime graph is of the form $p - r - q$.*

Using all this information, we give bounds for the diameter of non-$\mathfrak{F}$ graphs based on the characteristics of groups.

Proposition 0.0.6. *Let G be a finite group. If G is solvable and $\pi(G) \geq 4$, then $\operatorname{diam}(G) \leq 3$. If G is solvable and $\pi(G) = 3$, then $\operatorname{diam}(G) \leq 5$. If G is not solvable, then $\operatorname{diam}(G) \leq 6$.*

Our next aim is to know how sharp these bounds are. To that end, in section 5 we would like to find groups with the highest diameter as possible, so we have written a programme in GAP that computes the diameter of the non-$\mathfrak{F}$ graph of finite groups. The group with the highest diameter we have been able to find has been $G = C_7 \rtimes H$, where $H = (C_3 \times C_3) \rtimes SL(2,3)$ whose diameter is 3. Therefore, we guess that these are not the sharpest possible bounds.

Changing the focus to other properties about the non-$\mathfrak{F}$ graphs, in section 6 we have also studied the planarity of the non-$\mathfrak{F}$ graphs. We say that a graph is planar if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

Theorem 0.0.7. *Let G be a finite group. The graph $\Gamma(G)$ is planar if and only if it is an edgeless graph.*

Finally, we have also studied a bit the complement of the non-$\mathfrak{F}$ graph, the $\mathfrak{F}$-graph, and have discovered that these graphs are also connected:

Theorem 0.0.8. *Let G be a finite group. Then the $\mathfrak{F}$-graph of G is connected.*

Chapter 1

Survey

Let $\Lambda_{\mathfrak{F}}(G)$ be the $\mathfrak{F}$-graph of the group G as defined in the introduction. These graphs are of great interest, and have been studied by several authors. For instance, one of the first being studied was the commuting graph, which we will denote by $\Lambda(G)$, where vertices of the graph are non-central elements of a group G and two vertices are adjacent if the group they generate is abelian. Notice the paralelism of the commuting graph with the $\mathfrak{F}$-graph where $\mathfrak{F}$ is the class of abelian groups; the first is a special case of the second where we remove the universal vertices. Indeed, the $\mathfrak{F}$-graph and the non-$\mathfrak{F}$ graph of a group G are one the complement of the other, and the set of isolated vertices of $\tilde{\Gamma}_{\mathfrak{F}}(G)$ coincide with the set of universal vertices of $\tilde{\Lambda}_{\mathfrak{F}}(G)$. Similarly, the set of universal vertices of $\tilde{\Gamma}_{\mathfrak{F}}(G)$ is equal to the set of isolated vertices of $\tilde{\Lambda}_{\mathfrak{F}}(G)$. The commuting graphs have been studied for example in [20], where they stated the followig about simple groups:

Theorem 1.0.1. *Let G be a classical simple group defined over a field of order greater that 5. If $\Lambda(G)$ is connected then $4 \leq \operatorname{diam}(\Lambda(G)) \leq 10$.*

The connectivity of the commuting graph of finite simple groups can be compared with the connectivity of the prime graph. In order to explain this let us introduce the prime graphs first:

Definition 1.0.1. Let G be a finite group and construct its prime graph as follows: the vertices are the primes dividing the order of the group, and two vertices p and q are adjacent if and only if G contains an element of order pq.

In the papers [21] and [9] the connected components of the prime graphs of all simple non-abelian groups are studied, so in [20] Y. Segev and G. M. Seitz mention that it is easy to see that the commuting graph is connected if and only if the prime graph is connected. Therefore, the simple non-abelian groups G for which $\Lambda(G)$ is not connected are known. Later in [10] the following result was proved for finite groups:

Lemma 1.0.2. *Let G be a finite group, $\Pi(G)$ be the prime graph of G and $\Lambda(G)$ be the commuting graph of G. If $Z(G) = \{1\}$, then $\Lambda(G)$ is connected if and only if $\Pi(G)$ is connected.*

Going back to the study of the commuting graph, the Theorem 1.0.1 gave rise to thinking whether this result could be generalised for any finite group. Inspired by it, A. Iranmanesh and A. Jafarzadej in [10] presented the following conjecture:

Conjecture 1. There is a natural number b such that if G is a finite non-abelian group with $\Lambda(G)$ connected, then the diameter of $\Lambda(G)$ is smaller or equal than b.

The authors supported this conjecture with the following theorems that tell us about the connectivity and the diameter of the commuting graph of symmetric and alternating groups:

Theorem 1.0.3. *Let $\Lambda(G)$ be the commuting graph of G. For $n \geq 3$, $\Lambda(S_n)$ is connected if and only if n and $n-1$ are not primes and in this case $\operatorname{diam}(\Lambda(S_n)) \leq 5$ and this bound is sharp.*

Similarly,

Theorem 1.0.4. *Let $\Lambda(G)$ be the commuting graph of G. For $n \geq 5$, $\Lambda(A_n)$ is connected if and only if n, $n-1$ and $n-2$ are not primes and in this case $\operatorname{diam}(\Lambda(A_n)) \leq 5$ and this bound is sharp.*

However, P. Hegarty and D. Zhelezov tried to prove false the conjecture in [7] by suggesting a construction of 2-groups motivated by probabilistic methods. Although they did not succed at their goal, they inspired M. Giudici and C. Parker to prove the conjecture incorrect in [5]:

Theorem 1.0.5. *For all positive integers b, there exists a finite 2-group G such that the commuting graph of G has diameter greater than b.*

Nevertheless, M. Giudici and C. Parker belived that it might be true that the commuting graph of a finite group with trivial center is either disconnected or has diameter upper bounded by a constant, so they proposed the following cojecture:

Conjecture 2. There is an absolute constant b such that if G is a finite group with trivial center, then the commuting graph of G is either diconnected or has diameter at most b.

Motivated by this conjecture, G.L. Morgan and C.W. Parker went further in [17] and gave the upper bound for the diameter of the connected components:

Theorem 1.0.6. *Suppose that G is a finite group with trivial center. Then every connected component of the commuting graph G has diameter at most 10. In particular, if the commuting graph of G is connected, then its diameter is at most 10.*

The presented conjecture 2 was verified by C. Parker in [19] for the case where G was a solvable group with trivial center:

Theorem 1.0.7. *Suppose that G is a finite solvable group with trivial center. Then*

(i) *$\Lambda(G)$ is disconnected if and only if G is a Frobenius group or a 2- Frobenius group.*

(ii) *If $\Lambda(G)$ is connected, then $\Lambda(G)$ has diameter at most 8.*
Furthermore, there exist solvable groups G with trivial center such that $\Gamma(G)$ has diameter 8.

As a generalisation of the commuting graphs, in [3] T. C. Burness, A. Lucchini and D. Nemmi studied the following graphs, which they named solvable graphs: the vertices of the graph are the elements of $G \setminus R(G)$ where $R(G)$ is the solvable radical of G, and two vertices x and y are adjacent if and only if $\langle x, y \rangle$ is solvable. It is mentionable that the elements in the solvable radical are the universal vertices of the previosly defined $\mathfrak{F}$-graph $\Lambda_{\mathfrak{F}}(G)$, where $\mathfrak{F}$ is the class of solvable groups. This fact was proved by R. Guralnick, B. Kunyavskiĭ, E. Plotkin and A. Shalev in [6]. We will denote this graph by $\tilde{\Lambda}(G)$. One of the main results in this paper would be the following:

Theorem 1.0.8. *Let G be a finite insolvable group. Then $\tilde{\Lambda}(G)$ is connected and the diameter is at most* 5.

Notice that the solvable graph is a generalisation of the commuting graph. Since the class of nilpotent groups is bigger than the one of abelian groups but smaller than the one of solvable groups, in [3] the authors found interesting studying the nilpotent graph $\Lambda_{\mathfrak{N}}(G)$ constructed analogously to the commuting and solvable graphs. That is, let G be a finite group. The vertices of the nilpotent graph of G are the elements of $G \setminus I$ where I is the set of isolated vertices of the non-$\mathfrak{N}$ graph where $\mathfrak{N}$ is the class of nilpotent groups, and two vertices x and y are adjacent if and only if the group they generate is nilpotent. In the paper [2] the authors prove that the set I is the hypercenter of G, denoted by $Z_\infty(G)$, which is the last term in the upper central series of G.

In the paper [3] the following theorem is presented:

Theorem 1.0.9. *Let G be a finite non-nilpotent group. Then each connected component of the nilpotent graph $\Lambda_{\mathfrak{N}}(G)$ has diameter at most* 10.

On the prove of the theorem they mention that if x and y are adjacent vertices in the nilpotent graph of a finite group G, then those vertices have distance at most two in the commuting graph. This is because for any non-trivial element $z \in Z(\langle x, y \rangle)$ there is a path $x - z - y$ in the commuting graph. Therefore, the commuting graph and the nilpotent graph of a given finite group have the same connected components.

In this paper they also study the metacyclic graph constructed in the same way as all of the previous ones, and they check that in general it is not a connected graph either. Taking into account that the commuting, nilpotent and metacyclic graphs are not connected, but the solvable graph is connected, the following question arises: Which is the smallest class $\mathfrak{F}$ for which the $\mathfrak{F}$-graph constructed by removing the universal vertices is connected?

Let us consider now the complement of the $\mathfrak{F}$-graph, the non-$\mathfrak{F}$ graph $\Gamma_{\mathfrak{F}}(G)$ of the group G as defined in the introduction. The first time these kind of graphs were introduced was by Paul Erdös. He proposed the following question * about graphs where the vertices are elements of the group and two vertices are adjacent if and only if they do

*At the 15th Summer Research Institute of the Australian Mathematical Society, at the University of New South Wales, 13 January-14 February 1975.

not commute, also known as the non-commuting graph: Let G be such that the non-commuting graph contains no infinite complete subgraph; is there a finite bound on the cardinality of complete subgraphs of the non-commuting graph? This question was answered affirmatively by B. H. Neumann in [18]. Moreover, they proved that the class of groups whose graph contains no infinite complete subgraph coincides with the class of groups whose center has finite index.

P. Erdös' question and B. H. Neumann's answer inspired several mathematicians to ask similar question. In [1] A. Abdollahi, S. Akbari and H.R. Maimani wanted to study how the graph theoretical properties of the non-commuting graph of G had effect in the group theoretical properties of G. They also asked the following question:

Question 1. For which group property $\mathcal{P}$, if G and H are two non-abelian groups such that their non-comuting graphs are isomorphic, and G has property $\mathcal{P}$, then H has also property $\mathcal{P}$?

They also proposed the following conjectures:

Conjecture 3. Let G and H be two non-abelian finite groups such that their non-commuting graphs are isomorphic. Then, $|G| = |H|$.

Conjecture 4. Let S be a finite non-abelian simple group and G is a group such that the non-commuting graphs os S and G are isomorphic. Then, $G \cong S$.

Moreover, in the preliminary section they prove the next results:

Proposition 1.0.10. *For a non-abelian group G, the diameter of the non-commuting graph of G is equal to* 2*. In particular, the non-commuting graph of G is connected.*

There are also papers in non-nilpotent graphs i.e., the non-$\mathfrak{F}$ graph $\Gamma_{\mathfrak{F}}(G)$ where $\mathfrak{F}$ is the class of nilpotent groups. For example, in [2] A. Abdollahi and M. Zarrin proved the following theorem:

Theorem 1.0.11. *Let G be a finite non-nilpotent group. Then the non-nilpotent graph is connected and its diameter is at most* 6*. In particular, every two vertices x and y with $\pi(x) \neq \pi(y)$ are connected by a path of length at most* 4*.*

However, this result was improved by A. Lucchini and D. Nemmi in [13] where they proved the following result:

Theorem 1.0.12. *Let G be a finite group. Then the non-nilpotent graph has diameter at most* 3*.*

As far as non-solvable graphs are concerned, they were studied in the previously mentioned paper [6]. One of their main interest in this paper was to prove that the solvable radical $R(G)$ of a group G coincides with the elements $y \in G$ such that $\langle x, y\rangle$ is solvable for any $x \in G$, i.e., $R(G)$ is the set of universal vertices of the $\mathfrak{F}$-graph or the set of isolated vertices of the non-$\mathfrak{F}$ graph, where $\mathfrak{F}$ is the class of solvable groups. However, they also proved interesting results about the non-$\mathfrak{F}$ graph $\Gamma_{\mathfrak{F}}(G)$, where $\mathfrak{F}$ is the class mentioned before, such as:

Theorem 1.0.13. *Let G be a finite group. Suppose that x and y are not in $R(G)$. Then there exists $s \in G$ such that $\langle x, s\rangle$ and $\langle y, s\rangle$ are not solvable.*

This theorem will be of great interest in this work, since it proves that the non-solvable graph of a group G is connected, and that the diameter of such graph is at most 2.

Notice that non-commuting, non-nilpotent and non-solvable graphs are connected.

In order to generalize for which classes $\mathfrak{F}$ the non-$\mathfrak{F}$ graph is connected let us give the following definitions:

Definition 1.0.2. The class $\mathfrak{F}$ is said to be semiregular if $\mathfrak{I}_{\mathfrak{F}}(G)$ is a subgroup of G for every finite group G.

Definition 1.0.3. A class of groups $\mathfrak{F}$ is a formation if $\mathfrak{F}$ has the following two properties:

1. If $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$.
2. If $N_1, N_2 \trianglelefteq G$ with $N_1 \cap N_2 = 1$ and $G/N_1, G/N_2 \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

It can be proved that the classes of abelian, nilpotent or solvable groups are semiregular formations. Moreover, in the paper [14] A. Lucchini and D. Nemmi wrote the following theorem, for which we need to know that a formation $\mathfrak{F}$ is connected if the non-$\mathfrak{F}$ graph $\Gamma_{\mathfrak{F}}(G)$ is connected:

Theorem 1.0.14. *The following formations are semiregular:*

1. *The formation of finite supersolvable groups.*
2. *The formation of the finite groups with nilpotent derived subgroup.*
3. *The formation of the finite groups with Fitting length less or equal than t, for any $t \in \mathbb{N}$.*
4. *The formation of the finite groups G with $G/O_p(G)$ with Fitting length less or equal than t, for any $t \in \mathbb{N}$.*

All of these formations are connected.

In order to understand the previous theorem we might need the following definition:

Definition 1.0.4. Let G be a finite group. Let $\mathrm{Fit}(G)$ denote the Fitting subgroup of G. The upper Fitting series is

$$F_0 \leq F_1 \leq \ldots \leq F_n \ldots,$$

where $F_0 = 1$ and $F_i = \mathrm{Fit}(G/F_{i-1})$ for all $i > 0$.

The Fitting length is the length of the upper Fitting series, i.e. the number of distinct elements in the chain minus one.

That is, what they saw is that often semiregular formations are connected. Besides the ones mentioned in the previous theorems, this also occurs for formations of abelian, nilpotent or solvable groups as we could have noticed previously. This is why they adressed a more general question in [15]: suppose that $\mathfrak{F}$ is a class containig only solvable groups and groups closed under taking subgroups, and that a finite group G has the property that $\mathfrak{I}_{\mathfrak{F}}(H)$ is a subgroup of H for any $H \leq G$. Does this imply that $\Gamma_{\mathfrak{F}}(G)$ is connected?

The main theorem of that paper says the following concerning the previous question:

Theorem 1.0.15. *Let $\mathfrak{F}$ be a class with the following properties:*

1. *All the groups in $\mathfrak{F}$ are solvable.*

2. *$\mathfrak{F}$ is closed under taking subgroups.*

Suppose that G is a finite group which is minimal with respect to the following properties: G is $\mathfrak{F}$-semiregular and the graph $\Gamma_{\mathfrak{F}}(G)$ is not connected. Then G is solvable and there exists an epimorphism

$$\pi : G \to (V_1 \times \cdots \times V_t) \rtimes H,$$

where H is 2-generated and there exists a faithful irreducible H-module V with $V_i \cong V$ for $1 \leq i \leq t$ and $t = 1 + \dim_{\mathrm{End}_H(V)}(V)$.

Moreover let $\mathcal{W}$ be the set of the H-submodules of $V_1 \times \cdots \times V_t$ that are H-isomorphic to V^{t-1}. There exists one and only one $W \in \mathcal{W}$ with the property that $M = \pi^{-1}(W \rtimes H) \notin \mathfrak{F}_2$. If $g_1, g_2 \in G$ and $\langle g_1, g_2 \rangle \notin \mathfrak{F}$, then either $\langle g_1, g_2 \rangle \leq M^x$, for some $x \in G$, or H is cyclic of prime order and $\langle g_1, g_2 \rangle \leq \pi^{-1}(V_1 \times \cdots \times V_t)$.

In the previous theorem $\mathfrak{F}_2$ is the class of the finite groups G with the property that any 2-generated subgroup of G is in $\mathfrak{F}$.

What they saw with the previous theorem is that the answer to the dropped question is affirmative in every case but in the ones explained in the theorem.

This theorem can be used for example to prove that the graphs $\Gamma_{\mathfrak{F}}(G)$ are connected for any finite group G when $\mathfrak{F}$ is the class of p-groups for any prime p. However, there is an easier way to prove this as we can see in the following theorem:

Theorem 1.0.16. *Let G be a finite group with $\pi(G) \geq 2$, and $\mathfrak{F}$ the class of groups whose order is divisible by just one prime. Then $\Gamma_{\mathfrak{F}}(G)$ is connected.*

Proof. Let us assume that there exists $g \in G$ such that $\pi(g) \geq 2$. In this case the vertex g is adjacent to any vertex $h \in G$, implying that the graph $\Gamma_{\mathfrak{F}}(G)$ is connected.

Suppose then that $\pi(g) = 1$ for all $g \in G$. Let $x, y \in G$ be non-isolated vertices. If the prime dividing the order of x is different from the prime dividing the order of y, then these two vertices are clearly adjacent. On the other hand, if the prime dividing the order of these elements is the same, call it p, there exists $z \in G$ whose order is divisible by a different prime $q \neq p$, and thus z is adjacent to both x and y. Therefore, the graph is connected with diameter at most 2. $\square$

However, we cannot apply Theorem 1.0.15 to the class $\mathfrak{F}$ of groups whose order is divisible by at most two primes since it is not semiregular. Indeed, consider the group $G = D_{30}$. In the graph assosiated to this group the isolated vertices are the elements of order 3 and 5, which is not a subgroup of G. This is why we are going to focus our attention in this concrete class and prove the connectivity and more properties about it.

Notice that regarding the question asked before, although the answer is affirmative in many cases, in Theorem 1.0.15 A. Lucchini and D. Nemmi stated some counterexamples. That is, semiregularity is not a sufficient condition to ensure connectivity, and it is neither necessary as we will see in this work.

Chapter 2

Preliminaries

In this chapter we will provide some definitions and results that will show up frequently during the work.

Definition 2.0.1. Let Ω be a graph. We say that a vertex x in Ω is isolated if it is not adjacent to any of the other vertices in the graph. On the other hand, we say that a vertex y of the graph Ω is universal if it is adjacent to every other vertex in the graph.

Definition 2.0.2. Let $\mathfrak{F}$ be a class of finite groups and G a finite group. We consider the graph $\tilde{\Gamma}_{\mathfrak{F}}(G)$ whose vertices are the elements of G and where two vertices $g, h \in G$ are adjacent if and only if $\langle g, h \rangle \notin \mathfrak{F}$. We denote by $\mathfrak{I}_{\mathfrak{F}}(G)$ the set of isolated vertices of the graph $\tilde{\Gamma}_{\mathfrak{F}}(G)$. The non-$\mathfrak{F}$ graph of G, denote by $\Gamma_{\mathfrak{F}}(G)$, is the graph obtained by removing the isolated vertices from $\tilde{\Gamma}_{\mathfrak{F}}(G)$.

Definition 2.0.3. Let G be a finite group. The number $\pi(G)$ is the amount of different prime numbers dividing the cardinality of G. We say that G is a $\{p_1, \dots, p_t\}$-group if the primes $p_1, \dots, p_t$ are the ones dividing the cardinality of G.
Let $x \in G$ be an element in G. The number $\pi(x)$ is the number that indicates the amount of different prime numbers dividing the order of x.

From now on we are going to set $\mathfrak{F}$ to be the class of finite groups whose cardinality is divisible by at most two primes if not stated contrarily, i.e., those groups G such that $\pi(G) \leq 2$. In this situation, we will denote $I(G) := \mathfrak{I}_{\mathfrak{F}}(G)$ and $\Gamma(G) := \Gamma_{\mathfrak{F}}(G)$. The set $I(G)$ does not need to be a subgroup of G. For example, if we consider the group $G = D_{30}$, the isolated vertices $I(G)$ are the elements of order 3 and 5, which is not a subgroup of G.

We say that an element x is a pq-element if the primes p and q divide the order of x. Moreover, we will use the notation $x - y$ to express that the elements x and y are adjacent in a given graph.

Definition 2.0.4. Let G be a finite group and construct its prime graph as follows: the vertices are the primes dividing the order of the group, and two vertices p and q are adjacent if and only if G contains an element of order pq.

Theorem 2.0.1 (Higman)**.** *Let G be a solvable group all of whose elements have prime power order. Then G has order divisible by at most two primes.*

Proof. If G was a simple group it would be isomorphic to a cyclic group of prime order due to the fact that it is solvable, and we would be done. Suppose then that G is not simple. In this case we can ensure that G contains a normal proper non-trivial p-subgroup for some prime p (call it P'), and cannot have a normal q-subgroup greater than 1 for another prime q different from p. This is because otherwise, if we let $x \in P'$ and $y \in Q$ where Q is the supposedly normal q-subgroup, as P' and Q are normal subgroups of G, $[x, y] \leq P' \cap Q = \{1\}$, and thus the element xy would have order divisible by two different primes, which is a contradiction with the statement.

Let P be the greatest normal p-subgroup of G. If G was nilpotent, it would be equal to P since all its Sylow subgroups are normal in it and from the previous paragraph we know that the only normal one is P, thus G would be a p-group and we would be done. So assume that G is not nilpotent. In this case $P \lneqq G$. Let q be a prime such that G/P has a normal q-subgroup greater than 1, and let Q be a q-subgroup of G such that PQ/P is the greatest normal q-subgroup of G/P.

If N is a minimal normal subgroup of G (which is contained in P), the automorphism on N induced by conjugation with an element of Q does not leave any of the elements of N fixed other that 1 because otherwise there would exist $n \in N$ such that $[n, q] = 1$ and the order of nq would be divisible by the primes p and q, which is again a contradiction with the statement. It follows ([4], 334-336) that Q has no elementary abelian subgroup of order q^2, and hence if q is odd, then Q is cyclic and if $q = 2$, Q is cyclic or generalized quaternion. In any of the cases Q containes a characteristic subgroup of order q which we will denote by Z.

Next step is to prove that $C := C_G(PZ/P)$ is equal to PQ. Clearly $Z \leq Z(Q)$, which implies that $[PQ, PZ] \leq P$, and therefore $PQ \leq C$. We now want to prove that $C \leq PQ$. To that end assume there exists an element $g \in C \setminus P$ such that $\gcd(|g|, q) = 1$. As $g \in C$, the element gP commutes with any element $yP \in PZ/P$, and thus the element gy would have order divisible by two different primes, which is not possible. This means that any element of C of coprime order to q must be contained in P, i.e., C/P is a q-group normal in G/P, implying that $C/P \leq PQ/P$, and consequently $C \leq PQ$.

The action of conjugation of the elements of G/P over PZ/P implies that $\frac{G/P}{C/P} \cong G/PQ$ is a subgroup of $\mathrm{Aut}(PZ/P) \cong \mathrm{Aut}(Z)$. Therefore, G/PQ is cyclic because Z has prime order, and its order must divide $q - 1$. Moreover, G/PQ has prime power order because otherwise, as it is cyclic and consequently abelian, it would contain an element divisible by two different primes. Clearly this prime cannot be q itself. Assume there exists a third prime r such that r divides the order of G/PQ and let $x \in G$ with order r. Then the conjugation of the elements of PQ with x induce an automorphism of PQ that does not fix any element but the identity, and hence PQ is a nilpotent group ([8], Theorem 4). Note that this is not possible because we are assuming that G is not nilpotent.

All in all, we have proved that G/PQ if a p-group, and therefore the order of G is divisible by at most two different primes.

□

Proposition 2.0.2. *Let G be a solvable group with $\pi(G) \geq 3$, and $N \cong C_t^d$ a minimal normal subgroup where t is a prime number. Let $x, y \in G$ be such that two primes p, q divide the cardinality of $\langle x, y\rangle$ but t does not divide it. Then there exist $n, m \in N$ such that t divides the cardinality of $\langle xn, ym\rangle$. Consequently the three primes p, q, t divide the cardinality of $\langle xn, ym\rangle$.*

Proof. Call $S := \langle x, y\rangle N$. We can assume that N is a minimal normal subgroup of S, because if it is not there exists M a normal subgroup of S inside N such that N/M is a minimal normal subgroup of S/M and we could work in the quotient and transfer the information to the group.

So suppose that t does not divide $|\langle x, y\rangle|$ and that N is a minimal normal subgroup of S. We want to prove that there exist two elements $n, m \in N$ such that t divides the order of $\langle xn, ym\rangle$. For any $n_1, n_2 \in N$ we have that $S = \langle xn_1, yn_2\rangle N$. Moreover, $\langle xn_1, yn_2\rangle \cap N \trianglelefteq \langle xn_1, yn_2\rangle$ and $\langle xn_1, yn_2\rangle \cap N \trianglelefteq N$ because N is abelian, so $\langle xn_1, yn_2\rangle \cap N \trianglelefteq S$, which by the minimality of N implies that either $\langle xn_1, yn_2\rangle \cap N = \{1\}$ or $\langle xn_1, yn_2\rangle \cap N = N$. If there exist $n_1, n_2 \in N$ such that we are in the latest case, then $\langle xn_1, yn_2\rangle = N$ and thus t divides its cardinality as we wanted to prove. Assume, on the other hand, that for any $n_1, n_2 \in N$ the equality $\langle xn_1, yn_2\rangle \cap N = \{1\}$ holds. If $\langle xn_1, yn_2\rangle = \langle xm_1, ym_2\rangle$, then $n_1^{-1}m_1, n_2^{-1}m_2 \in \langle xn_1, yn_2\rangle \cap N$, i.e., $n_1 = m_1$ and $n_2 = m_2$. Consequently, there are $|N|^2$ complements for N in S, but by Schur-Zassenhaus we know that all the complements must be conjugate, and this is an amount of at most $|N|$ complements, which is a contradiction. Therefore, there must exist $n, m \in N$ such that t divides the cardinality of $\langle xn, ym\rangle$.

□

Proposition 2.0.3. *Let G be a finite group, $N \trianglelefteq G$ a normal subgroup, and $x \in G$. If x acts fixed point free on N, i.e., $C_N(x) = \{1\}$, then $x^N = \{xn | n \in N\}$.*

Proof. If $n \neq 1$, then $x^n = x[x, n]$ where $1 \neq [x, n] \in N$ because x acts fix point free in N and N is normal in G. Then, $x^n = xm$ where $m \in N$. Moreover, $|x^N| = |N : C_N(x)| = |N|$, which equals $|\{xn | n \in N\}|$, and thus $x^N = \{xn | n \in N\}$. □

Two other concepts that we will use during the work will be the distance and the diameter of a graph:

Definition 2.0.5. The distance between two vertices of a graph Ω is a map $d : \Omega \times \Omega \to \mathbb{N}$ where for any $x, y \in \Omega$, $d(x, y)$ is the number of edges in a shortest path connecting the vertices x and y.

Notice that the distance defined above is indeed a distance function, since it satisfies all the axioms of a metric function.

Definition 2.0.6. The diameter of a connected component Ω of a graph $\Gamma(G)$ is the length of the shortest path between the most distanced vertices of the subgraph Ω. In case $\Gamma(G)$ is connected (which we will see it is), we will denote the diameter of the graph $\Gamma(G)$ as $\operatorname{diam}(G)$. That is,

$$\operatorname{diam}(G) = \max_{x,y \in \Gamma(G)} d(x, y).$$

Chapter 3

Connectivity and diameter

In this chapter we aim to prove that the non-$\mathfrak{F}$ graph is connected, and wish to find the sharpest upper bound for the diameter as possible for this connected graph.

Theorem 3.0.1. *Let G be a finite group with $\pi(G) \geq 3$, then $\Gamma(G)$ is connected.*

In order to prove this theorem, we are going to first prove the result for finite solvable groups.

Lemma 3.0.2. *Let G be a finite solvable group with $\pi(G) \geq 3$, then $\Gamma(G)$ is connected.*

Proof. Let G be a finite solvable group. Assume, by contradiction, that G is a solvable group with minimal order with respect to the property that $\Gamma(G)$ is not connected.

If there exists $g \in G$ such that $\pi(g) \geq 3$, then $\langle g, h \rangle \notin \mathfrak{F}$ for all $h \in G$, which implies that $\Gamma(G)$ is connected. Instead, take $g, h \in G$ such that $\pi(g) = \pi(h) = 2$. In the case where the primes dividing the orders of g and h are the same, we know that there exists a third element $a \in G$ such that $\gcd(|g|, |a|) = \gcd(|h|, |a|) = 1$ which implies that a is adjacent to both g and h, and thus they are all contained in the same connected component of $\Gamma(G)$. On the other hand, if there exists a prime number dividing the order of g but not dividing the order of h, or viceversa then clearly g and h are adjacent vertices and they are contained in the same connected component of $\Gamma(G)$. Thus, all elements of G whose cardinality is divisible by more than one prime are contained in the same connected component of $\Gamma(G)$, which we will denote by Ω.

Since we are assuming that $\Gamma(G)$ is not connected, there must exist an element $x \in G$ such that $x \notin I(G)$, $x \notin \Omega$, thus $\pi(x) = 1$. In this case there exists $y \in G$ such that x and y are adjacent vertices in $\Gamma(G)$, and we may assume that $y \notin \Omega$, thus $\pi(y) = 1$.

In this situation define the subgroup $H := \langle x, y \rangle$. As x and y are adjacent vertices, $\pi(H) \geq 3$, thus by Theorem 2.0.1 there exists an element $z \in H$ such that $\pi(z) \geq 2$, which implies that $z \in \Omega$. Clearly x is not an isolated vertex of H, and neither is z since $\pi(H) \geq 3$. Assume $H \lneqq G$. Note that $\Gamma(H)$ must be connected by the minimality of G, and as x and z are not isolated points of $\Gamma(H)$, they must be connected in $\Gamma(H)$, and consequently in $\Gamma(G)$. Moreover, $z \in \Omega$ which implies that $x \in \Omega$, and this is a contradicition.

We may therefore assume that $G = H = \langle x, y \rangle$. If G was simple, it would be a p-group, but we are assuming that $\pi(G) \geq 3$, so we can ensure the existance of a proper minimal normal subgroup N of G. As G is solvable, $N \cong C_r^d$ where $r > 0$ is a prime integer and d a natural number.

Suppose $\pi(G/N) \gneqq 2$. By Theorem 2.0.1 there exists $zN \in G/N$ such that $\pi(zN) \geq 2$, and thus zN is not an isolated point of $\Gamma(G/N)$. Moreover, xN is neither an isolated point of $\Gamma(G/N)$, otherwise we would have that $\langle xN, gN \rangle N/N \in \mathfrak{F}$ for all $gN \in G/N$. In particular, $\langle xN, yN \rangle N/N \cong G/N \in \mathfrak{F}$, which is a contradiction with our assumption. On the other hand, by the minimality of G the graph $\Gamma(G/N)$ must be connected, and consequently xN and zN are connected in that graph, and thus x and z are connected in $\Gamma(G)$. However, $\pi(zN) \geq 2$ implies that $\pi(z) \geq 2$ as well, and therefore $z \in \Omega$, then $x \in \Omega$, which is again a contradiction.

So we may assume that G/N is a $\{p, q\}$-grpup and G is a $\{p, q, r\}$-group, which implies that N is the Sylow r-subgroup of G. Note that we can choose x to be a p-element or a q-element because if it was an r-element we would have that $\pi(G) = 2$. Without loss of generality assume that x is a p-element.

We may assume that x acts fixed point free on N, i.e., $C_N(x) = \{1\}$. Otherwise, $[x, n] = 1$ for some $1 \neq n \in N$. In this case we could write $G = \langle xn, y \rangle N$, and thus the cardinality of G divides $|\langle xn, y \rangle||N|$. This implies that at least p and q divide the order of $\langle xn, y \rangle$. However, note that $\langle xn \rangle$ is a subgroup of $\langle xn, y \rangle$, and since $[x, n] = 1$, the order of xn is $|xn| = |x||n| = p^a r^b$. Therefore, $\langle xn, y \rangle \notin \mathfrak{F}$. As a consequence, the vertice y is adjacent to x and to xn, where the latest is an element of Ω since $\pi(xn) = 2$, which implies that $x \in \Omega$, a contradicition.

Suppose now that there exists an elements $z \in G$ such that pq divides $|z|$. It is clear that the primes p and q divide the cardinality of $\langle x, z \rangle$. If r divides it as well, x and z would be adjacent vertices in $\Gamma(G)$ and thus $x \in \Omega$, so we may assume that r does not divide the order of $\langle x, z \rangle$. In this case we can apply Proposition 2.0.2 to the elements $x, z \in G$, and therefore there exist $n, m \in N$ such that r divides the cardinality of $\langle xn, zm \rangle$. As pq divides the cardinality of z, note that $\langle xn, zm \rangle \notin \mathfrak{F}$ and that $xn \in \Omega$.

By Proposition 2.0.3 we know that there exists $n' \in N$ such that $xn = x^{(n')^{-1}}$, or equivalently, $x = (xn)^{n'}$. It is easy to see that there is a bijection between $\langle xn, zm \rangle$ and $\langle (xn)^{n'}, (zm)^{n'} \rangle$ which implies that they both have the same cardinality. Therefore, $(xn)^{n'} = x$ and $(zm)^{n'}$ are adjacent vertices in $\Gamma(G)$, and as the orders of $(zm)^{n'}$ and zm are the same, $(zm)^{n'} \in \Omega$, and hence $x \in \Omega$ as well. This means that there are no elements of order divisible by pq in G.

Assume now that there exists an element $g \in G$ such that qr divides the order of g. Then qr divides $|\langle x, g\rangle|$, and p divides $|\langle x, g\rangle|$ as well. Hence $\langle x, g\rangle \notin \mathfrak{F}$, but as $\pi(g) \geq 2$, g is in Ω, and as it is adjacent to x, x is in Ω as well, a contradiction. Thus, there are no elements in G whose order is divisible by qr.

By Theorem 2.0.1 there must exist an element $g \in G$ such that $\pi(g) \geq 2$, thus pr must divide the order of this element g. As a consequence this element g is adjacent with any element of G whose order is divisible by q, and therefore Ω contains all the elements whose order is divisible by q.

Let $1 \neq v$ be a q-element in G. Clearly the primes p and q divide the order of $\langle x, v\rangle$. If r does divide that order either, then x and v are adjacent vertices in $\Gamma(G)$, and as $v \in \Omega$, x is contained in Ω as well. So we may consider that r does not divide the order of $\langle x, v\rangle$. Then, by Proposition 2.0.2 there exist $n_1, n_2 \in N$ such that r divides $|\langle xn_1, vn_2\rangle|$. Clearly p and q also divide $|\langle xn_1, vn_2\rangle|$. As $vn_2 \in \Omega$, xn_1 is also contained in Ω. By Proposition 2.0.3 x and xn_1 are conjugated, implying $x \in \Omega$, a contradiction.

$\square$

Proof of Theorem 3.0.1. If G is solvable we are done by Lemma 3.0.2, so let G be a finite non-solvable group. Denote by $R := R(G)$ the solvable radical of G. It follows [6] that $R = \mathfrak{I}_{\mathfrak{S}}(G)$ where $\mathfrak{S}$ is the class of finite solvable groups. Consequently, taking $x \in G \setminus R$, there exists $y \in G$ such that $\langle x, y\rangle \notin \mathfrak{S}$, and thus $\langle x, y\rangle \notin \mathfrak{F}$ as $\mathfrak{F} \subseteq \mathfrak{S}$. As the non-solvable graph of G is connected [15], all the elements of $G \setminus R$ belong to the same connected component of $\Gamma(G)$.

Consider the following sets: $A = I(G)$, $B = I(R) \setminus I(G)$, $C = R \setminus I(R)$, $D = G \setminus R$ and note that $G = A \dot\cup B \dot\cup C \dot\cup D$.

By definition, for each $b \in B$ there exists $d \in D$ such that b and d are adjacent vertices of the graph $\Gamma(G)$. Moreover, we know that all the elements of D belong to the same connected component of $\Gamma(G)$, therefore $B \cup D \subseteq \Omega$, where Ω is a connected component of $\Gamma(G)$.

Assume that $B \neq \emptyset$. Consider $b \in B$, $d \in D$ such that b and d are adjacent vertices in $\Gamma(G)$ and define $H := R\langle d\rangle$. As R is solvable, H is solvable as well, so it is a finite proper solvable subgroups of G. Thus, by Lemma 3.0.2, $\Gamma(H)$ is connected.

Note that $b, d \in H$ and they are connected so $b, d \in H \setminus I(H)$. Moreover, $C \subseteq H \setminus I(H)$ as well. Since $\Gamma(H)$ is connected, $\{b, d, C\}$ is contained in the unique connected component of $\Gamma(H)$, so $\{b, d, C\}$ is contained in the same connected component of $\Gamma(G)$, which must be Ω since $D \subseteq \Omega$. Therefore, $\Omega = B \cup C \cup D$, and thus we conclude that $\Gamma(G)$ is connected.

We may now assume that $B = \emptyset$. This implies that $I(R) = I(G)$. As R is a finite solvable subgroup of G, $\Gamma(R)$ is connected by Lemma 3.0.2. Thus, in order to prove that

$\Gamma(G)$ is connected we just need to show that an element of $G \setminus R$ and an element of $R \setminus I(R)$ are connected in $\Gamma(G)$.

Assume there exists an element $g \in R$ such that $\pi(g) = 2$. As G is not solvable, G/R is not solvable either which implies that $\pi(G/R) \geq 3$. Then there exists $x \in G \setminus R$ such that $\gcd(|g|, |x|) = 1$. Moreover, $|g|$ and $|x|$ divide the cardinality of $\langle g, x \rangle$, which implies that $\langle g, x \rangle \notin \mathfrak{F}$. Therefore, x and g are adjacent vertices in $\Gamma(G)$ which implies that $\Gamma(G)$ is connected.

In the case where $\pi(g) \leq 1$ for all $g \in R$ by Theorem 2.0.1 $\pi(R) \leq 2$, which implies that $R = I(R) = I(G)$ and thus the vertices of $\Gamma(G)$ are just the elements of $G \setminus R$ which we already know that are contained in the same connected component of $\Gamma(G)$, so it is connected.

□

Once we have proved the strong result that says that the non-$\mathfrak{F}$ graph is connected, it seems natural to try to find the smallest upper bound for the diameter.

Proposition 3.0.3. *Let G be a finite group with $\pi(G) \geq 3$. If G is solvable and $\pi(G) \geq 4$, then* $\operatorname{diam}(G) \leq 3$. *If G is solvable and $\pi(G) = 3$, then* $\operatorname{diam}(G) \leq 5$. *If G is not solvable, then* $\operatorname{diam}(G) \leq 6$.

We will use the following lemma in order to prove the stated bounds of the diameter:

Lemma 3.0.4. *Let G be a solvable group such that $I(G) \neq G$. Then for all $x \notin I(G)$, $d(x, y) \leq 2$ for some y with $\pi(y) \geq 2$.*

Proof. We are going to prove it by induction on $|G|$. Let $x \in G \setminus I(G)$. If $\pi(x) \geq 2$ the statement is clear, so assume that $\pi(x) = 1$. As $x \notin I(G)$, there exists $y \in G$ such that x and y are adjacent, i.e., $\langle x, y \rangle \notin \mathfrak{F}$. May distinguish two cases.

First assume that $\langle x, y \rangle \neq G$. By induction there exists $z \in \langle x, y \rangle$, with $\pi(z) \geq 2$, such that $d(x, z) \leq 2$.

For the second case assume that $G = \langle x, y \rangle$. Let N be a minimal normal subgroup of G, which implies that $N \cong C_r^d$ for a prime r. If $\pi(G/N) \gneqq 2$, by Theorem 2.0.1 there exists $zN \in G/N$ such that $\pi(zN) \geq 2$, so by induction $d(xN, zN) \leq 2$ and thus $d(x, z) \leq 2$. On the other hand, if $\pi(G/N) = 2$, would have that G/N is a $\{p, q\}$-group and N is an $\{r\}$-group which implies that N is the Sylow r-subgroup of G. Without loss of generality we may assume that x is a p-element.

If there exists $1 \neq n \in N$ such that $[x, n] = 1$, then in the same way as in the connectivity proof $\pi(xn) = 2$ and xn is adjacent to y, implying $d(x, xn) = 2$.

Suppose now that x acts fixed point free on N, i.e., $C_N(x) = \{1\}$. Since $\pi(G) \geq 3$, there exists an element $z \in G$ with $\pi(z) \geq 2$. If the order of this element $z \in G$ is divisible by pq, and r divides $\langle x, z\rangle$, then x and z are adjacent vertices and we are done. Contrarily, if r does not divide the order of $\langle x, z\rangle$, by Proposition 2.0.2 there exist $n_1, n_2 \in N$ such that xn_1 is adjacent to zn_2. By Proposition 2.0.3 there exists $n \in N$ such that $x = (xn_1)^n$ and therefore there exists an element $z' = (zn_2)^n \in G$ with $\pi(z') \geq 2$ such that $d(x, z') = 1$.

If otherwise, the primes q and r are the ones dividing the order of $z \in G$, then clearly $\langle x, z\rangle \notin \mathfrak{F}$, and thus $d(x, z) = 1 \leq 2$ with $\pi(z) \geq 2$.

Finally, if the order of $z \in G$ is divisible by pr, note that z is adjacent to any element whose order is divisible by q. Let $v \in G$ be a q-element. If r divides $|\langle x, v\rangle|$, then $d(x, z) \leq d(x, v) + d(v, z) = 2$, and we would be done. If otherwise r does not divide the order of $\langle x, v\rangle$, by Proposition 2.0.2 there exist $n_1, n_2 \in N$ such that $\langle xn_1, vn_2\rangle \notin \mathfrak{F}$. By Proposition 2.0.3 the vertex x is adjacent to $(zn_2)^n$ for some $n \in N$, whose order is divisible by two different primes. Therefore, we found an element $z' = (zn_2)^n$ such that $\pi(z') \geq 2$ and $d(x, z') = 1$, so we are done. □

We also need the following information about the prime graph to prove the case where G is a solvable graph with $\pi(G) = 3$:

Theorem 3.0.5. *Let G be a finite solvable group with $\pi(G) = 3$. If $\operatorname{diam}(G) > 4$, then the prime graph is of the form $p - r - q$.*

Proof. Define the set $\Sigma = \{(p_1, p_2) \mid G \text{ contains a } p_1p_2\text{-element}\}$. Our aim is to prove that if $\pi(G) = 3$ and $\operatorname{diam}(G) > 4$, then $\Sigma = \{(p, r), (r, q)\}$. To that end we will see that if $\Sigma = \{(p, q), (p, r), (r, q)\}$ or $\Sigma = \{(p, q)\}$, then $\operatorname{diam}(G) \leq 4$.

Call p, q and r the only primes dividing the order of G. If $\Sigma = \{(p, q), (p, r), (r, q)\}$, it is easy to see that $\operatorname{diam}(G) \leq 3$.

Assume that $\Sigma = \{(p, q)\}$. In this case the prime graph of G has two components: $\{p, q\}$ and $\{r\}$. Then G is Frobenius or 2-Frobenius and the prime graph of G has exactly two components, one of which consist of the primes dividing the lower Frobenius complement ([21], Corollary page 487).

We distinguish two cases:

1) G is a Frobenius group, i.e., $G = N \rtimes H$ where N is nilpotent and H acts fixed point free on N.

2) G has normal subgroups N and K such that K is a Frobenius group with Frobenius kernel N, and G/N is a Frobenius group with Frobenius kernel K/N, i.e., $K = N \rtimes K/N$ and $G/N = K/N \rtimes G/K$.

We can again divide the first case in two:

a) N is a $\{p, q\}$-group and H is an r-group.

b) N is an r-group and H is a $\{p, q\}$-group.

In case 1a all the p-elements and q-elements are isolated. Indeed, let x be a p-element, then $x \in N$. For any element $g \in N$ we have that $\langle x, g\rangle \leq N \in \mathfrak{F}$, so x is not adjacent to any of the elements in N. Let $g \in G \setminus N$. Notice that $|g|$ is a power of r because we are assuming that $\Sigma = \{(p, q)\}$. From the fact that N is a normal subgroup of G we deduce that $\langle g, x\rangle \leq \langle g\rangle P$ where P is the unique Sylow p-subgroup of N. Since $\langle g\rangle P$ is an $\{r, p\}$-group, x cannot be adjacent to any element $g \in G$, so it is isolated. Same thing happens with q-elements.

From the previous fact we deduce that the vertices of $\Gamma(G)$ can just be r-elements or pq-elements, which implies that $\mathrm{diam}(G) \leq 2$.

For the case 1b, we know that all r-elements are adjacent to every pq-element, so we need to study the non-isolated p- and q-elements. Note that some minimal normal subgroup of G must be contained in N, call it M.

Let x be a non-isolated vertex of order a power of p or a power of q. Assume x is adjacent to y, which implies by 2.0.1 that there exists $z \in \langle x, y\rangle$ which is a pq-element. Then by Proposition 2.0.2 there exist $n_1, n_2 \in M \leq N$ such that xn_1 is adjacent to zn_2. Since H acts fixed point free on N, by Proposition 2.0.3 xn_1 is a conjugate to x, so x is adjacent to an element of order equal to $|zn_2|$, which is a pq-element. This proves that all p-elements and q-elements that are not isolated are adjacent to some pq-element. Therefore, $\mathrm{diam}(G) \leq 4$.

Let us move to case 2). We also have to divide this case in two:

a) K/N is a nilpotent $\{p, q\}$-group and G/K and N are r-groups.
b) K/N is an r-group and G/K and N are $\{p, q\}$-groups.

In case 2a we can argue as in case 1b and we are done taking into account that some minimal normal subgroup of G must be contained in N.

For case 2b, the p-elements and q-elements that are inside N are isolated for the same reason as in 1a. Assume G/K contains a pq-element, call it zK. We aim to prove that any non-isolated p-element is adjacent to a pq-element. Let x be a non-isolated p-element which is outside N, consequently outside K, and assume r does not divide $\langle zN, xN\rangle N/N$, otherwise x would be adjacent to z which is a pq-element and we would be done. Since K is normal in G, K/N must contain a minimal normal subgroup of G/N, thus by Proposition 2.0.2 there must exist $k_1N, k_2N \in K/N$ such that xKk_1N is adjacent to zKk_2N, which is a pq-element. If we assume that xN does not act fixed point free over K/N, then there must exist an element $kN \in K/N$ such that $[xN, kN] = 1$, and thus the element $xNkN$ would have order divisible by p and r, which is impossible. Therefore, xN acts fixed point free over K/N, and so by Proposition 2.0.3, there exists $k'N \in K/N$ such that $xNk_1N = xN^{k'N}$. This implies that xN is adjacent to $(zNk_2N)^{(k'^{-1}N)}$, which is a pq-element, and therefore x is also adjacent to a pq-element. So as in case 1b, $\mathrm{diam}(G) \leq 4$.

Now assume that G/K does not contain a pq-element, then G/K is a p-group. This is because all Sylow subgroups of a Frobenius complement are either cyclic groups or generalized quaternions ([11], Corollary 6.17), i.e., all Sylow subgroups of G/K and K/N are cyclic or generalized quaternions. Since K/N is an r-group, it is equal to its unique Sylow r-subgroup. If it is cyclic, then G/K is abelian. This is because $\dfrac{G/K}{C_{G/K}(K/N)} \leq \mathrm{Aut}(K/N) = (\mathbb{Z}/r^b\mathbb{Z})^*$ where $|K/N| = r^b$ for some $b \in \mathbb{N}$, and since G/K acts fixed point free on K/N the centralizer $C_{G/K}(K/N)$ is trivial. Therefore if G/K was not a p-group it would contain elements of order divisible by pq, which is a contradiction. On the other hand, if K/N is a generalized quaternion, from the fact that G/K and K/N have coprime orders, we deduce that G/K is a Frobenius complement of odd order. Hence, by ([19], Lemma 2.4) any two elements of coprime order commute, so if p and q both divide the order of G/K, then there must be an element whose order is divisible by pq, which is again a contradiction. Hence, G/K must be a p-group.

The vertices of $\Gamma(G)$ are pq-element, r-elements and p-elements. Note that all q-elements are contained in N, hence they are isolated. We claim that a non-isolated p-element g that is not contained in N is adjacent to an r-element or to a pq-element. If this is the case, then $\mathrm{diam}(G) \leq 4$.

In order to prove the claim let y be an element of order r. If g is adjacent to y we are done, so assume that $\langle g, y\rangle$ is a $\{p, r\}$-group. Note that the equality $gN^{K/N} = \{gN \cdot hN \mid hN \in K/N\}$ holds because of the fact that G/K acts fixed point free over K/N. This implies that $gy = g^x n$ for some $x \in G$ and $n \in N$.

On the other hand, let Q be the Sylow q-subgroup of N, and let $Z = Z(Q)$ be the center of Q. Assume $[gy, z] = 1$ for some $1 \neq z \in Z$, then $\langle g, gyz\rangle = \langle g, yz\rangle$. Note that gyz is a pq-element (because gy commutes with z, thus $|gy|$, which is divisible by p, and $|z| = q^b$, for some $b \in \mathbb{N}$, divide the order of gyz) and yz is an r-element (because if $yz \in N$ then $y \in N$ which is a contradiction since y is an r-element, thus $yz \in K \setminus N$ and it is an r-element). This implies that the three primes p, q and r divide the cardinalities of $\langle g, gyz\rangle$ and $\langle g, yz\rangle$ therefore g is adjacent to both gyz and yz and we would be done.

We may assume then that gy acts fixed point free over Z. Thus, for any $1 \neq z \in Z$ note that $1 \neq [gy, z] = [g^x n, z] = [g^x, z]^n \cdot [n, z] = [g^x, z]^n$. The last equality follows from the fact that $[N, Z] = 1$. Notice that $[N, Z] = 1$ because since N is nilpotent then $N = P \times Q$ where P and Q are the Sylow p- and q-subgroups of N, respectively. Then, the elements of Q commute with the elements of P, and consequently all the elements in Z commute with all the elements in N. Therefore, we get that $1 \neq [g^x, z]^n$ and consequently $[g^x, z] \neq 1$, which implies that g^x acts fixed point free on Z. It can be easily checked that this implies g acting fixed point free over Z as well.

Note that Q is a normal subgroup of G because for any $q \in Q$ and $g \in G$, $|q^g|$ is a q-number, thus it must be contained in Q. Since Z is characteristic in Q, Z is normal in G, and therefore it contains a minimal normal subgroup of G. Then, by Proposition 2.0.2 there exist $z_1, z_2 \in Z$ such that $\langle gz_1, yz_2 \rangle \notin \mathfrak{F}$, i.e. gz_1 is adjacent to an r-element. But from the fact that g acts fixed point free over Z we deduce that g and gz_1 are conjugate, implying that g is adjacent to an r-element.

□

Proof of Theorem 3.0.3. Assume $\pi(G) \geq 4$. Let $x, y \in G \backslash I(G)$. If $\pi(x) \geq 2$ and $\pi(y) \geq 2$, then $d(x, y) \leq 2$. If $\pi(x) = 1$, say it is a p-element, and two primes different from p divide the order of y, then x and y are adjacent vertices. If p and another prime, say q, divide the order of y, then there exists an element $z \in G$ such that its order is either qr, qt or rt (Proposition 1, [16]), where p, q, r and t are prime numbers dividing $|G|$. In any of the cases $d(x, y) \leq d(x, z) + d(z, y) = 1 + 1 = 2$. Assume finally that $\pi(y) = 1$ as well. If the prime dividing the order of y is the same as the one dividing the order of x, say they are both p-elements, then there exists an element $z \in G$ such that $|z|$ is either qr, qt or rt (Proposition 1, [16])where q, r and t are three primes different from p and from each other dividing the order of G, and so $d(x, y) \leq 2$. Finally, assume the prime dividing the order of x and the one dividing the order of y are distinct, for instance say x is a p-element and y is a q-element. There exist $z, s \in G$ such that the order of z is either pr, pt or rt, so $d(y, z) = 1$, and the order of s is either qr,qt or rt, hence $d(x, s) = 1$. Note that $d(s, z) \leq 2$. In the case where $d(s, z) = 1$, then $d(x, y) = 3$. The only case where $d(s, z) = 2$ is when both s and z are rt-elements, but in that case we have the paths $x - s - y$ and $x - z - y$ impliying that $d(x, y) = 2$. Thus, $\operatorname{diam}(G) \leq 3$.

On the other hand, assume G is a finite solvable group with $\pi(G) = 3$. We have seen in Theorem 3.0.5 that if the prime graph of G is complete or of the form $p - q$, then $\operatorname{diam}(G) \leq 4$, so assume that the prime graph is of the form $p - r - q$.

Let $\Sigma = \{g \in G \mid \pi(g) = 2\}$. By Lemma 3.0.4 for any $x \in G$ there exists $y \in \Sigma$ such that $d(x, y) \leq 2$, and $d(h, g) \leq 2$ for any $h, g \in \Sigma$. Our first claim is that if g is not an r-element, then $d(g, z) = 1$ for some $z \in \Sigma$. Indeed, note that if p divides $|g|$, then g is adjacent to an rq-element, and if q divides the order of g, then this is adjacent to a pr-element. Thus, if x and y are non-isolated vertices and x is not an r-element, then $d(x, y) \leq 5$.

Assume now that g is a non-isolated r-element that is adjacent to an element x which is not an r-element. Then $d(g, y) \leq 5$ for every non-isolated vertex $y \in G$. This is because if $\pi(x) \geq 2$, then $d(g, y) \leq 5$, so assume that x is a p-element, and let $z \in \Sigma$ such that $d(y, z) \leq 2$. There are two possibilities: the first one is that z is a qr-element which implies that $d(g, y) \leq 4$, and the second possibility is that z is a pr-element, implying that $d(g, y) \leq 5$.

For the last case let Ω be the set of non-isolated r-elements that are adjacent only to r-elements. The claim is that if $g \in \Omega$, then either there exist $a_1, a_2 \in G$ with a_1 a qr-element and a_2 with order divisible by p such that $d(g, a_1) = d(g, a_2) = 2$, or there exist $b_1, b_2 \in G$ with b_1 a pr-element and b_2 with order divisible by q, and such that $d(g, b_1) = d(g, b_2) = 2$.

In order to prove the claim assume g is adjacent to the r-element z. Let $H = \langle g, z \rangle$ and $N = O_r(H)$. We will work modulo N, and will use the notation $\overline{H} = H/N$ and $\overline{h} = hN$ for every $h \in H$. Notice that $g, z \notin N$ since $\pi(\langle n, h \rangle) \leq 2$ for all $n \in N$ and $h \in H$.

Let $\overline{A}$ be a minimal normal subgrop of $\overline{H}$. Note that if $|\overline{A}|$ is an r-number, then $N = A$, so $\overline{A}$ is either a p-group or a q-group. We will assume that it is a p-group, but it is done in the same way in the other case.

The element $\overline{z}$ acts fixed point free on $\overline{A}$. Indeed, if otherwise $[\overline{z}, \overline{a}] = 1$ for some $\overline{1} \neq \overline{a} \in \overline{A}$, the order of za is divisible by $|\overline{za}| = |\overline{z}| \cdot |\overline{a}|$ which is divisible by r and p. On the other hand, we have that $H = \langle g, z \rangle = \langle g, z \rangle A = \langle g, za \rangle A$, and since A is a $\{p, r\}$-group, q must divide $|\langle g, za \rangle|$, and since the order of za is divisible by pr, g is adjacent to a pr-element, which is a contradiction with $g \in \Omega$.

If h is an r-element and $\overline{h}$ acts fixed point free on $\overline{A}$, then h is adjacent to u, where u is an element whose order is divisible by q. This is because if we let $\overline{v}$ be an element of order q in H, by Proposition 2.0.2 there exist $\overline{a}, \overline{b} \in \overline{A}$ such that p divides $\langle \overline{ha}, \overline{vb} \rangle$. Since $\overline{h}$ acts fixed point free on $\overline{A}$, by Proposition 2.0.3 $\overline{ha} = \overline{h}^{\overline{t}} = \overline{h^t}$ for some $\overline{t} \in \overline{A}$ and so h^t is adjacent to vb, and hence h is adjacent to $(vb)^{t^{-1}}$, whose order is divisible by q.

By the previous paragraph $\overline{g}$ cannot act fixed point free over $\overline{A}$, so $[\overline{g}, \overline{a}] = \overline{1}$ for some $\overline{a} \in \overline{A}$, i.e., $[g, a] \neq N$ for some $a \in A \setminus N$. Thus, we have a path $g - z - ga$ with ga a pr-element. Moreover, $\overline{z}$ acts fixed point free over $\overline{A}$, thus we have a path $g - z - u$ with q dividing the order of u, so we have proved the claim.

Finally, let us prove that if $y_1, y_2 \in \Omega$, then $d(y_1, y_2) \leq 5$. Choose $z_1, z_2 \in \Sigma$ such that $d(y_1, z_1) = d(y_2, z_2) = 2$. In the case where z_1 is a pr-element and z_2 is a qr-element, clearly $d(z_1, z_2) = 1$, thus $d(y_1, y_2) = 5$. Thus, we need to study the case when the primes dividing the order of z_1 and z_2 are equal. If both are pr-elements, according to the claim there exists either an element u whose order is divisible by q and $d(y_1, u) = 2$, or there exists a qr-element with $d(y_1, v) = 2$. In any of the cases we have that $d(u, z_2) = d(v, z_2) = 1$, which imlpies that $d(y_1, y_2) = 5$. On the other hand, if z_1 and z_2 are qr-elements, the argument is the same, so we have proved that $\operatorname{diam}(G) \leq 5$.

Turning to the case where G is not solvable, define the following sets: $A = I(G)$, $B = I(R) \setminus I(G)$, $C = R \setminus I(R)$, $D = G \setminus R$ where $R = R(G)$ is the solvable radical of G, and note that $G = A \dot{\cup} B \dot{\cup} C \dot{\cup} D$. We also define the set $E = \{g \in G \setminus I(G) \mid \pi(g) \geq 2\}$. We have the following information:

a) The distance $d(x_1, x_2) \leq 2$ for any $x_1, x_2 \in D$, which follows from Theorem 6.4 of [6].

b) By definition, for each $b \in B$ there exists $d \in D$ such that b and d are adjacent vertices in $\Gamma(G)$. This implies that if $b \in B$ the distance $d(b, d) = 1$ for some $d \in D$.

c) Let $x \in C$, then $d(x, y) \leq 2$ for some $y \in E$. This is because there exists $z \in C$ such that $H = \langle x, z \rangle \notin \mathfrak{F}$, i.e., $\pi(H) \geq 3$. Moreover, $H \leq R\langle z \rangle$ so H is solvable, and applying Lemma 3.0.4 there exists $y \in E$ such that $d(x, y) \leq 2$.

d) Clearly $d(x, y) \leq 2$ for any $x, y \in E$.

e) Let $x \in D$ and $y \in E$, then $d(x, y) \leq 3$. In order to prove this, we may assume that $\pi(y) = \{p, q\}$. If a prime r different from p and q divides the order of x, or if $\pi(x) \geq 2$, then $d(x, y) \leq 2$, and we would be done. Therefore, we may suppose that x is a p-element. Since G is not solvable, G/R is not solvable either, thus there exists an element $z \in G \setminus R = D$ whose order is divisible by r. Hence, $d(x, z) \leq 2$ and $d(z, y) \leq 1$, implying $d(x, y) \leq 3$.

Once knowing this information we have the following cases:

Let $b_1, b_2 \in B$, then there exist $d_1, d_2 \in D$ such that by b) $d(b_1, d_1) = d(b_2, d_2) = 1$. Therefore, by a) $d(b_1, b_2) \leq d(b_1, d_1) + d(d_1, d_2) + d(d_2, b_2) \leq 1 + 2 + 1 = 4$.

Let $b \in B$ and $c \in C$, there exist $d \in D$ and $e \in E$ such that by b) $d(b, d) = 1$ and by c) $d(c, e) \leq 2$. Moreover, by e) $d(d, e) \leq 3$, so $d(b, d) \leq 6$.

Let $b \in B$ and $d \in D$. There exists $d' \in D$ such that $d(b, d') = 1$ by b), and by a) $d(d, d') \leq 2$, thus $d(b, d) \leq 3$.

Let $c_1, c_2 \in C$. There exist $e_1, e_2 \in E$ such that by c) $d(c_1, e_1) \leq 2$ and $d(c_2, e_2) \leq 2$. Moreover, by d) $d(e_1, e_2) \leq 2$, so $d(c_1, c_2) \leq 6$.

Finally, let $c \in C$ and $d \in D$. There exists $e \in E$ such that by c) $d(c, e) \leq 2$ and by e) $d(e, d) \leq 3$, so $d(c, d) \leq 5$. Therefore, in the non-solvable case $\operatorname{diam}(G) \leq 6$.

□

Chapter 4

Programme and example

Once given bounds for the diameter of the groups we would like to check if these bounds are sharp. In order to prove that we should find groups that hold the boundaries. To that end, we have written a GAP programme that finds which is the diameter of a given group.

In order to give the diameter of a group, note that it is enough to study which is the distance from the vertices $x \in \Gamma(G)$ with $\pi(x) = 1$ to the rest of the vertices. This is because for those vertices $g, h \in \Gamma(G)$ with $\pi(g) = \pi(h) \geq 2$ we already know that $d(x, y) \leq 2$. Moreover, let x be a non-isolated vertex of order p^a, and assume that the element $y = x^{p^{a-1}}$ whose order is p is not an isolated vertex. Since x is not isolated, there exists $g \in \Gamma(G)$ such that $\langle x, g\rangle \notin \mathfrak{F}$, and note that $\langle y, g\rangle \leq \langle x, g\rangle$, i.e., x is adjacent to any of the vertices that y is adjacent to, but the inverse is not true. Thus, we have that $d(y, h) \geq d(x, h)$ for any vertex h that is adjacent to y. However, it could happen that the vertex y is isolated. In this case, we should repeat the procedure in order to find an element x^b such that x^b is not an isolated point and its order is the smallest as possible.

It is also enough to choose one element among the elements of a conjugacy class whose elements have prime power order. Assume x and y are non-isolated vertices, have prime power order and are contained in the same conjugacy class. This implies that both have the same order and that there exists $g \in G$ such that $x = y^g$. As studied previously, there is a bijection between $\langle y, h\rangle$ and $\langle y^g, h^g\rangle$ for any $g \in G$. Therefore, if $d(x, h) = n$ for some $h \in H$ and $n \in \mathbb{N}$, there exists $h' \in G$ such that $d(y, h') = n$.

Taking into account this information we have written several programmes. The programme named *primeelem* computes a list of representatives of conjugacy classes whose elements have the smallest prime power order as possible. This programme has several steps. First of all we create a list called "elem" of non-isolated representatives of conjugacy classes whose order is a power of a prime. Note that if an element is isolated, then all the elements in its conjugacy class are also isolated. In the second step we create another list called "last". Using a loop "for" we study the elements x in the list "elem", and we check whether $y = x^{p^{a-1}}$ is isolated or not, p^a being the order of x. Notice that it is enough to study just the representatives of the conjugacy classes, since if v and w are in the same conjugacy class, then $v^{q^{d-1}}$ and $w^{q^{d-1}}$, where q^d is the order of v and w, are also in the same conjugacy class. In case y is not isolated, we add it to the list "last", otherwise we add x to the list "last". In each step we check whether there is an element of the conjugacy class of y or x in "last" in order to avoid having two representatives of the same conjugacy class. In the last step we check whether there are elements of the form x and x^b in "last", and if this is the case we choose the one whose order is the smallest, and we add it to the list "final". We have used several programmes in order to design this one. The first one is called *nonisolated*, and computes the non-isolated vertices of $\tilde{\Gamma}_{\mathfrak{F}}(G)$, and the second one is called *power*, and it computes the power of a prime, i.e., power(p^a) would return the value a.

We wrote another programme called *isolated* that computes the isolated vertices of a graph $\tilde{\Gamma}_{\mathfrak{F}}(G)$, and we will use this one later on. The programme named *distance* computes the maximum among the minimum distances from a given vertex x to the rest of the vertices in $\Gamma(G)$, and returns "isolated" if x is an isolated vertex. In order to do this, if x is a non-isolated vertex, we compute a list with all the vertices that are adjacent to x, and this list will be the second sublist of the list "adj", whose first component is the list $[x, 1]$ for computational reasons. Then we compute the list of vertices that are adjacent to the vertices computed before, but that are not contained in the previous list, and add a new sublist with these new vertices. We continue the procedure until we cannot find more new vertices, and then we return the length of the list "adj" minus one unit, which is the maximum among the minimum distances from the vertex x to the rest of the vertices in $\Gamma(G)$. Finally, the last programme named *diameter* returns the diameter of a given group by applying the programme "distance" to all vertices the first programme returns. Below are the used codes:

Listing 4.1: List of representatives of conjugacy classes with prime power order:

```
primeelem:=function(G)
local cl, repr, i, elem, j, a, b, x, c, d, p,z, last, k, l, w, noniso, final, v, n, m;

noniso:=nonisolated(G);
cl:=ConjugacyClasses(G);
repr:=[];

for i in [1..Length(cl)] do
a:=Representative(cl[i]);
Add(repr,a);
od;
elem:=[];
for j in [1..Length(repr)] do
b:=Order(repr[j]);
if Size(PrimeDivisors(b))=1 and repr[j] in noniso then
Add(elem, repr[j]);
fi;
```

```
od;

last:=[];
for x in elem do
c:=Order(x);
p:=PrimeDivisors(c)[1];
d:=power(c);
if x^(p^(d-1)) in noniso then
k:=0;
for z in ConjugacyClass(G,x^(p^(d-1))) do
if z in last then
k:=k+1;
fi;
od;
if k=0 then
Add(last,x^(p^(d-1)));
fi;
else
l:=0;
for w in ConjugacyClass(G,x) do
if w in last then
l:=l+1;
fi;
od;
if l=0 then
Add(last,x);
fi;
fi;
od;

final:=[];
for v in last do
m:=0;
for n in [2..Order(v)-1] do
if v^n in final or GcdInt(Order(v),n)>1 then
m:=m+1;
fi;
od;
if m=0 then
Add(final,v);
fi;
od;
return(final);
```

```
end;
quit;
```

Listing 4.2: List of the non-isolated vertices of the graph.

```
nonisolated:=function(G)
local i, nonisol, g, x, a;
nonisol:=[];
for g in G do
i:=0;
for x in G do
a:=Length(PrimeDivisors(Size(Group(g,x))));
if a>2 then
i:=1;
fi;
od;
if i>0 then
Add(nonisol,g);
fi;
od;
return(nonisol);
end;
quit;
```

Listing 4.3: Power of a prime power.

```
power:=function(b)
local p, i;
p:=PrimeDivisors(b)[1];
i:=1;
while b/p<>1 do
i:=i+1;
b:=b/p;
od;
return(i);
end;
quit;
```

Listing 4.4: List of the isolated vertices of the graph.

```
isolated:=function(G)
```

```
local i, isol, g, x, a;
isol:=[];
for g in G do
i:=0;
for x in G do
a:=Length(PrimeDivisors(Size(Group(g,x))));
if a>2 then
i:=1;
fi;
od;
if i=0 then
Add(isol,g);
fi;
od;
return(isol);
end;
quit;
```

Listing 4.5: Maximum out of the minimum distances from a given vertex to any other vertex.

```
distance:=function(G,x)
local iso, adj, max, adj2, adj3, i, indicator, g, a;
indicator:=2;
iso:=isolated(G);
adj:=[[x,1]];
max:=0;
adj2:=[x];
adj3:=[x];
if x in iso then
return("isolated");
else
while Length(adj3)>0 do
x:=adj3[1];
adj3:=[];
for i in [2..indicator] do
for g in G do
if g in adj2 then
x:=x;
else
a:=Length(PrimeDivisors(Size(Group(g,x))));
if a>2 then
Add(adj3,g);
Add(adj2,g);
```

```
fi;
fi;
od;
x:=adj[max+1][i];
od;
max:=max+1;
indicator:=Length(adj3);
Add(adj,adj3);
od;
return(max);
fi;
end;
quit;
```

Listing 4.6: Diameter of the group G.

```
diameter:=function(G)
local elem, x, diameter,a;
diameter:=[];
elems:=primeelem(G);
for x in elems do
a:=dist(G,x);
if IsInt(a) then
Add(diameter,a);
fi;
od;
return(Maximum(diameter));
end;
quit;
```

As mentioned in the introduction of this chapter, we aim to find a group with the greatest diameter as possible. We have seen in Theorem 3.0.3 that the groups that are most likely to have a bigger diameter if $\pi(G) = 3$. The biggest one we have found using the previous programme is the group $G = C_7 \rtimes H$ where $H = (C_3 \times C_3) \rtimes SL(2,3)$, whose diameter is 3. This group contains elements of order $2, 3, 4, 6, 7, 14, 21$ and 28. The presentation of the group G is the following:

$$
\begin{gathered}
G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7 \mid f_1^3, f_2^{-1} f_1^{-1} f_2 f_1 f_5^{-1} f_3^{-1} f_2^{-1}, f_3^{-1} f_1^{-1} f_3 f_1 f_2^{-1}, f_4^{-1} f_1^{-1} f_4 f_1 f_4^{-1}, \\
f_5^{-1} f_1^{-1} f_5 f_1, f_6^{-1} f_1^{-1} f_6 f_1 f_7^{-1}, f_7^{-1} f_1^{-1} f_7 f_1, f_2^2 f_5^{-1}, f_3^{-1} f_2^{-1} f_3 f_2 f_5^{-1}, f_4^{-1} f_2^{-1} f_4 f_2, f_5^{-1} f_2^{-1} f_5 f_2, \\
f_6^{-1} f_2^{-1} f_6 f_2 f_7^{-1}, f_7^{-1} f_2^{-1} f_7 f_2 f_7^{-1} f_6^{-1}, f_3^2 f_5^{-1}, f_4^{-1} f_3^{-1} f_4 f_3, f_5^{-1} f_3^{-1} f_5 f_3, f_6^{-1} f_3^{-1} f_6 f_3 f_7^{-2} f_6^{-2}, \\
f_7^{-1} f_3^{-1} f_7 f_3 f_7^{-2} f_6^{-2}, f_4^7, f_5^{-1} f_4^{-1} f_5 f_4, f_6^{-1} f_4^{-1} f_6 f_4, f_7^{-1} f_4^{-1} f_7 f_4, f_5^2, f_6^{-1} f_5^{-1} f_6 f_5 f_6^{-1}, \\
f_7^{-1} f_5^{-1} f_7 f_5 f_7^{-1}, f_6^3, f_7^{-1} f_6^{-1} f_7 f_6, f_7^3 \rangle
\end{gathered}
$$

Our aim is to find elements $x, y \in G$ such that $d(x, y) = 3 = \text{diam}(G)$. For the reasoning we have used in the explanation of the programme, we are going to consider elements in the conjugation classes of those elements that the programme *primeelem* return. For this particular example the programme *primeelem* returns the list $[f_7, f_5, f_4, f_1, f_1f_6, f_1^2f_6]$.

We are going to study each conjugation class. Using GAP it is easy to check that the element f_7 is adjacent to 378 elements. Moreover, since $|f_7| = 3$, this element is clearly adjacent to elements of order 14 and 28, which sums to a total amount of 378 elements, i.e., y is adjacent to f_7 if and only if its order is 14 or 28. The same thing happens with all the elements in its conjugation class.

A similar thing happens with the elements in the conjugation class of f_4. These elements are adjacent to 504 elements, and since $|f_4| = 7$, they are adjacent to elements of order 6, and turns out that there are just 504 elements of order 6. Thus, an element y is adjacent to an element in the conjugation class of f_4 if and only if it has order 6.

As for the case of f_5, this element, and consequently any element in its conjugacy class, is adjacent to 385 elements. Since the order of these elements is 2, they are adjacent to elements of order 21, but there are just 48 elements of that order. Using GAP we can check that these elements are adjacent to some elements of order 14 and 28.

Elements in the conjugacy class of f_1 are adjacent to 1134 elements, and among those elements we can find the ones with order 14 and 28, since $|f_1| = 3$. However, these elements are also adjacent to some elements whose order is 3 or 6. The same thing occurs for f_1f_6 and $f_1^2f_6$. An interesting thing is that the elements f_1, f_1f_6 and $f_1^2f_6$ are adjacent to the same elements, but they are not adjacent one to the other.

In order to find elements $x, y \in G$ such that $d(x, y) = 3$, it is clear that $\pi(x) = \pi(y) = 1$, otherwise $d(x, y) \leq 2$. If the prime dividing the order of x and y is the same, then $d(x, y) \leq 2$, because for any two primes dividing $|G|$ there exists an element in G whose order is divisible by those two primes. Hence, the prime dividing $|x|$, and the one dividing $|y|$ must be different. Once we know this, is seems natural to consider $x = f_7$ since it is just adjacent to elements of order 14 and 28. This implies that f_7 is adjacent to $f_4^3f_5$ whose order is 14. For the same reason we could choose $y = f_4$, since it is adjacent to elements of order 6 only, for example $f_1^2f_5$. Clearly $f_4^3f_5$ and $f_1^2f_5$ are adjacent because of their orders. Notice that neither f_7 and $f_1^2f_5$, nor f_4 and $f_4^3f_5$ are adjacent one to the other because of the reason mentioned before. So we have found the two elements we were looking for.

Chapter 5

Other properties

Once we have given an upper bound for the diameter of the non-$\mathfrak{F}$ graph, let us study more properties about this graph:

Definition 5.0.1. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

We will need the following definition in the proof of Theorem 5.0.1.

Definition 5.0.2. Let G be a finite group. The generating graph of G is the graph whose vertices are elements of G, and two vertices $g, h \in G$ are adjacent if and only if $\langle g, h\rangle = G$.

In the following theorem will prove that $\Gamma(G)$ is not planar unless all its vertices are isolated.

Theorem 5.0.1. *Let G be a finite group. The graph $\Gamma(G)$ is planar if and only if it is an edgeless graph.*

Proof. Suppose there exists $g \in G$ such that $\pi(g) \geq 3$, and let n be the order of this element g. Let $\Omega = \{x \in G \mid |x| = n\}$. The subgraph of $\Gamma(G)$ induced by elements of Ω is the complete graph K_m where $m = |\Omega|$, i.e., all the elements of Ω are adjacent to each other in $\Gamma(G)$. Moreover, since $\pi(g) \geq 3$, there exist at least $\varphi(n)$ elements in G whose order is n, the ones of the form g^a, where $\gcd(a, n) = 1$, for example. Thus, $m \geq \varphi(n) \geq \varphi(2 \cdot 3 \cdot 5) = 8$. Taking into account that K_n is planar if and only if $n < 5$, we conclude that $\Gamma(G)$ is not planar.

So we may assume that $\pi(g) \leq 2$ for every $g \in G$. Let $x, y \in G$ such that $\langle x, y\rangle \notin \mathfrak{F}$, i.e., x and y are adjacent vertices in $\Gamma(G)$, and let $H = \langle x, y\rangle$. Note that $\pi(H) \geq 3$.

Assume by contradiction that $\Gamma(G)$ is planar. If $h_1, h_2 \in H$ are adjacent in the generating graph of H, then $\langle h_1, h_2\rangle = H \notin \mathfrak{F}$, thus h_1 and h_2 are also adjacent in $\Gamma(G)$. This means that the generating graph of H is a subgraph of $\Gamma(G)$, and thus it is also planar.

Finite groups with planar generating graphs have been completely classified in [12], and from this classification we know that either $H \in \{S_3, D_6\}$ or H is nilpotent. Since $\pi(H) \geq 3$, H needs to be nilpotent. This implies that H is the direct product of its Sylow subgroups, i.e., $H \cong P_1 \times P_2 \times \ldots \times P_t$, with $t \geq 3$, $|P_i| = p_i^{d_i}$ and $|H| = p_1^{d_1} \ldots p_t^{d_t}$ with $p_1, \ldots, p_t$ prime numbers. Let $g_i \in P_i$ be such that $|g_i| = p_1$, and note that the element $(g_1, \ldots, g_t) \in P_1 \times P_2 \times \ldots \times P_t$ has order $p_1 \ldots p_t$, and thus there exists at least one element $h \in H$ such that $\pi(h) \geq 3$, which is a contradiction.

□

Let $\tilde{\Lambda}_{\mathfrak{F}}(G)$ be the complement graph of $\tilde{\Gamma}_{\mathfrak{F}}(G)$, i.e., the vertices of $\tilde{\Lambda}_{\mathfrak{F}}(G)$ are the elements of G, and two vertices are adjacent if and only if $\pi(\langle x, y\rangle) \leq 2$.

Our aim is to prove that the remaining graph $\Lambda_{\mathfrak{F}}(G)$ after removing the isolated vertices of $\tilde{\Lambda}_{\mathfrak{F}}(G)$ is connected.

First of all note that the vertices $g \in G$ with $\pi(g) \geq 3$ are isolated vertices. If $\pi(g) = 2$, then g is adjacent to all of its powers, so in particular g is adjacent to some element of prime order. So in order to study the connectivity of $\Lambda_{\mathfrak{F}}(G)$, it would be enough to study the connectivity of its subgraph $\Lambda(G)$ where the vertices are the elements of G of prime order and two elements x and y are adjacent if and only if $\pi(\langle x, y\rangle) \leq 2$.

Theorem 5.0.2. *Let G be a finite group, then $\Lambda(G)$ is connected, and consequently, $\tilde{\Lambda}_{\mathfrak{F}}(G)$ is also connected.*

Proof. Assume first that G is a solvable group, and let $N \trianglelefteq G$ be a minimal normal subgroup of G. Then $N \cong C_r^d$, where r is a prime number dividing $|G|$ and $d \in \mathbb{N}$.

All the non-trivial elements of N are universal vertices of $\Lambda(G)$, that is, they are adjacent to all the vertices of $\Lambda(G)$. This is because if $x \in G$ with $|x| = p$ and $n \in N$ with $|n| = r$, then $\langle x, n\rangle \leq \langle x\rangle N$, which implies that $\langle x, n\rangle \in \mathfrak{F}$. Therefore, for any $x, y \in G$ with prime order there exists an element $n \in N$ such that n is adjacent to both x and y, and so $d(x, y) \leq 2$.

On the other hand, assume G is not solvable. The distance from one vertex to another in the solvable graph of G is at most 5 [3]. This means that there exist $z_1, z_2, z_3, z_4 \in G$ such that $\langle x, z_1\rangle, \langle z_1, z_2\rangle, \langle z_2, z_3\rangle, \langle z_3, z_4\rangle, \langle z_4, y\rangle$ are solvable groups, and note that we can choose the elements z_i to have prime order. Indeed, if $\pi(z_i) \geq 2$ and the prime t divides the order of z_i, say a_i, then the element $z_i^{a_i/t}$ has prime order and is still adjacent to any element that z_i is adjacent to. By the remark in the previous paragraph, we get that $d(x, z_1) \leq 2$, $d(z_1, z_2) \leq 2$, $d(z_2, z_3) \leq 2$, $d(z_3, z_4) \leq 2$ and $d(z_4, y) \leq 2$, implying that $\Lambda(G)$ is connected with diameter at most 10.

□

However, if we remove the universal vertices from the $\mathfrak{F}$-graph $\Lambda_{\mathfrak{F}}(G)$, the graph is not connected anymore in general. For instance, if $G = D_{30}$ and we remove the universal vertices of $\Lambda_{\mathfrak{F}}(D_{30})$ then the graph has 15 vertices of order 2 and 8 of order 15, and the elements of order 15 are adjacent only to the elements of order 15. Therefore, the graph is not connected. An interesting question to answer would be the following: Consider the graph obtained from $\Lambda(G)$ by deleting the universal vertices. Is it connected?

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